Exploring the use of GeoGebra to enhance Grade 12 learners' knowledge construction of trigonometric 2D and 3D concepts in one school in King Cetshwayo District

by

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DECLARATION

I, Philisiwe Promise Mbatha, declare that:

1. The research work of this dissertation, except where otherwise stated, is my original work;
2. This dissertation has not been submitted for any degree or examination purposes to any other university;
3. Where words from a written source have been used, the words have been paraphrased and referenced, and where exact words from a source have been used the words have been placed inside quotation marks and referenced;
4. I have not copied and pasted any information from the internet without explicitly citing the sources and I have included appropriate references to these sources in the referenced portion of my work; and
5. I have complied with the research ethics policy of the University of KwaZulu-Natal throughout this research, having received ethics approval prior to the start of data collection and having not acted outside the conditions of the approval.

Researcher: [redacted]

____ Date: 02 January 2024

Philisiwe Promise Mbatha

Supervisor's statement

This dissertation has been submitted with/without my approval.

[redacted]___ Date: 01 January 2024

[redacted]Zanele Ngcobo

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The journey of writing this dissertation has been very challenging, but I have gained a lot as it was a rewarding exercise. I would like to extend my gratitude to a few people who have impacted my life through this journey in different ways, providing emotional, physical and spiritual support.

I thank God for the opportunity he gave me to embark on this journey and complete it without any illness.

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And, finally, to the learners who participated in this study: thank you so much.

DEDICATION

I dedicate this degree to my late grandmother, Ester Mbatha,
who always wished the best for me when she was still in this world,
and to my strong, supportive mother, Nkosikhona Mbatha,
for understanding and keeping me positive at all times on this journey.

ABSTRACT

It is evident in the literature that mathematics is the school subject that learners struggle with the most, especially the concepts of trigonometry. This study explored the effectiveness of GeoGebra as a pedagogical tool to enhance Grade 12 learners' conceptual understanding of 2D and 3D trigonometry. The study was located within the interpretive paradigm and a qualitative case study methodology was employed. Thirty Grade 12 learners were purposively selected from a high school in the King Cetshwayo District Municipality of the province of KwaZulu-Natal in South Africa. Data was collected using activity worksheets administered as a pre-test and a post-test, with lessons conducted using GeoGebra. Data was also collected using semi-structured interviews, focus group interviews and observations. The study was underpinned by APOS (Action, Process, Object and Schema) theory, which was used to analyse the mental constructions displayed by participants. To understand and explain the extent to which participants had been able to make mental constructions, a genetic decomposition model was used. The genetic decomposition model developed by Arnon et al., (2014) was used to understand and describe the extent to which participants were able to make the mental structures necessary to master a particular mathematical concept. The study found that learners' conceptual understanding of 2D and 3D trigonometry improved from the pre-test, administered before they had engaged with concepts using GeoGebra, to the post-test, which was administered after learners had integrated GeoGebra into their conceptual development. This indicates that GeoGebra may have facilitated improved knowledge construction for these learners. These findings have implications for mathematics educators, curriculum developers and further researchers, as they offer insights into the potential benefits of incorporating dynamic software tools like GeoGebra to enhance the teaching and learning of trigonometry in the high school context. Ultimately, this study contributes to the on-going discourse on effective technology integration in mathematics education and offers practical recommendations for educators seeking innovative approaches to engage and empower their students through trigonometric learning experiences.

Keywords: GeoGebra, APOS theory, 2D/3D trigonometry; mathematics education; South Africa

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CHAPTER 1: INTRODUCTION AND BACKGROUND TO THE STUDY

1.1 Introduction

This study sought to explore the use of GeoGebra to enhance grade 12 learners' mental construction in the concept of 2D and 3D trigonometry. The participants were grade 12 learners. The data was collected using tests, interviews and observation. Tests were analysed using the analytical tool called genetic decomposition with APOS stages. This chapter provides an overview of the study, explaining the background, the problem and the purpose of the study to enhance learners' understanding of the concept of 2D and 3D trigonometry by integrating GeoGebra. The rationale and motivation for the study are explained. The chapter presents the research objectives and questions that guided the study. The selection of the research site and participants is described. The significance of the study and ethical considerations are discussed, and key terms used in the study are defined.

Mathematics is a science that works with logic, quantity and sequence. According to the South African Mathematics Curriculum and Assessment Policy Statement (CAPS) (DBE, 2011), for FET phase, mathematics is a distinctly human activity that humans across the world have engaged in for thousands of years. It further states that mathematics enables a deeper understanding of the physical, social and economic realities around us; in addition, mathematical problem-solving promotes creative thinking. Because of this, competency in mathematics is seen as a key driver of national development and mathematics education is one of the national priorities articulated in South Africa's National Development Plan (NDP, 2011).

Although mathematics is considered to be such an important subject, mathematics concepts can be particularly difficult for learners to grasp. Low mathematics results continue to be a cause for concern. A variety of instructional strategies, some of which use digital technologies, have shown potential to enable learners to construct a deeper understanding of mathematics concepts. This study explores the possibility of enhancing Grade 12 learners' understanding of 2D and 3D trigonometry concepts in mathematics using GeoGebra. This software assists learners to explore conjectures and develop a sense of mathematical inquiry through active engagement. GeoGebra uses a hands-on approach and has been found to deepen users' understanding of trigonometry and cultivate a sense of curiosity and problem-solving skills that extend beyond the classroom. The researcher hopes that this study will encourage educators to embrace this innovative tool, paving the way for a generation of mathematically literate learners equipped with the skills and mind-set needed for success in an increasingly technology-driven world.

1.2 Background of the study

In South Africa, learners have struggled with mathematics for many years, with South Africa ranking among the poorest performing countries in international assessments (Spaull, 2013). Although there have been many interventions to address the crisis of learner performance in mathematics, the trend persists. For example, the latest Trends in International Mathematics Science Study (TIMSS, 2019), stated that South Africa was ranks among the five low-performing countries even after it showed improvement in performance (Mullis & Martin, 2017). Alarmingly, there has been evidence showing a decline in learner performance with trigonometric concepts compared to other concepts (Madonsela et al., 2020). This indicates that mathematics teachers need new and innovative strategies to ensure that learners learn mathematics with understanding.

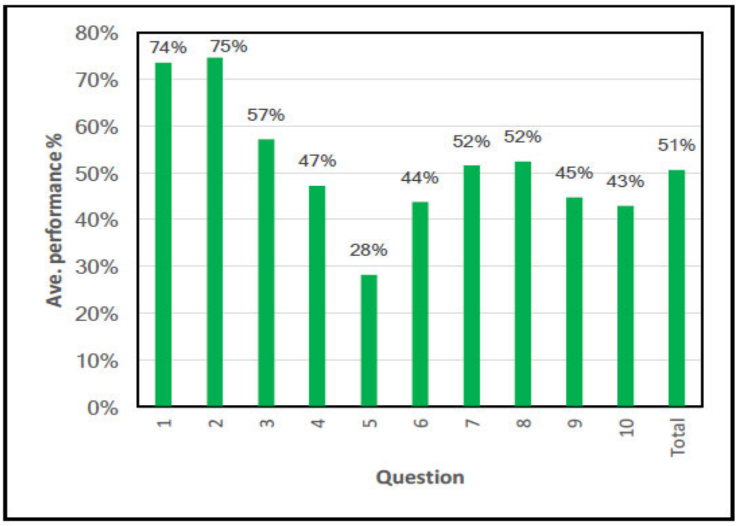
The South African Curriculum and Assessment Policy Statement (CAPS) for FET mathematics give trigonometry the following weighting in the curriculum: 40% in Grade 10; 50% in Grade 11; and 40% in Grade 12 (DBE, 2012). The heavy weighting allocated to trigonometric concepts indicates that this is considered an important area of mathematics that learners need to grasp. As failing to understand trigonometric concepts would result in overall poor performance in the subject, attention needs to be given to enhance the teaching and learning of these concepts. In response to this, this study focuses on strategies that could potentially enhance the teaching of mathematical concepts specifically, for 2D and 3D trigonometry.

In light of the poor performance by learners in mathematics at large, the incorporation of technology software such as GeoGebra has been advocated to improve the teaching and learning of mathematical concepts (Atan et al., 2010; Shadaan, 2013; Bayaga et al., 2019; Mthethwa, 2020; Tamam et al., 2021). A number of studies have been conducted on the use of such teaching aids for other areas of mathematics; this study aims to contribute to this discourse. GeoGebra has been found to enhance learners' thinking in the area of geometry (Rajagopal et al., 2015; Tay, 2003; Meng & Idris, 2012) but little has been done to investigate its effectiveness in enhancing the learning of trigonometric concepts.

It is against this backdrop that the researcher deemed it necessary to explore the use of GeoGebra to enhance learners understanding of 2D and 3D trigonometric concepts. This study, therefore, hopes to contribute to bridging the gap in the literature. This study also contributes a new dimension by drawing on the work of Dubinsky (1986), who advocates that topics that are found to be challenging to learners need to be analysed, by means of research, to enable alternative teaching strategies to be developed that are tailored to address specific challenges. In response to this, this study first analysed learners' mental constructions of 2D and 3D trigonometric concepts, then investigated the extent to which the use of GeoGebra enhanced learners' mental constructions.

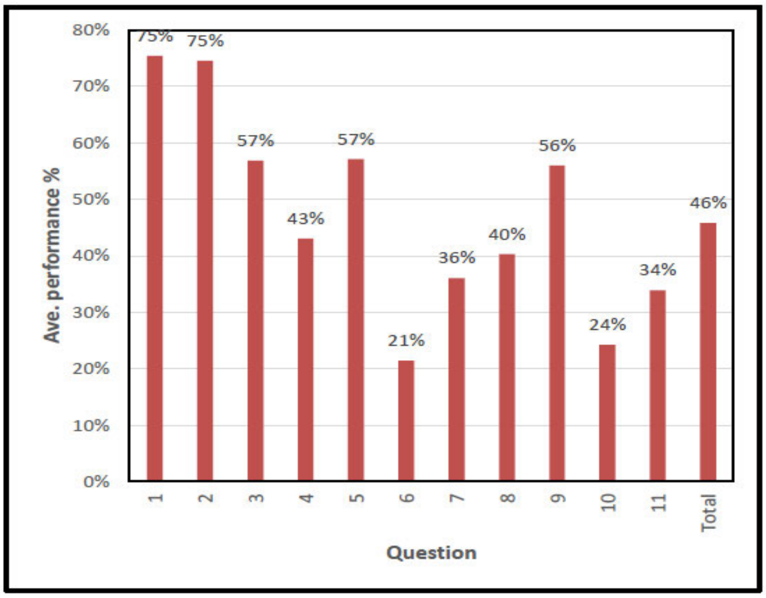
1.3 Statement of the problem

The literature demonstrates that mathematics has been, and continues to be, a challenging subject for learners (Puteh & Masri, 2006; Voogt, 2008). In recent years, trigonometry concepts and especially 2D and 3D concepts have been identified as the most challenging mathematical concepts for learners to comprehend (Siyepu, 2015). This has been confirmed by an analysis of Grade 12 results by exam moderators in South Africa (DBE, 2020; DBE, 2021). This is illustrated in Figures 1 and 2 below, showing the Grade 12 results for 2020 and 2021. Questions 5 and 6, in the 2020 report, and Questions 6, 7 and 8, in the 2021 report, use 2D and 3D trigonometry concepts.



Q	Topic/s
1	Data Handling
2	Data Handling
3	Analytical Geometry
4	Analytical Geometry
5	Trigonometry
6	Trigonometry
7	Trigonometry
8	Euclidean Geometry
9	Euclidean Geometry
10	Euclidean Geometry

Figure 1.1 Department of Basic Education diagnostic report (2020, p.195)



Q	Topic(s)
1	Data Handling
2	Data Handling
3	Analytical Geometry
4	Analytical Geometry
5	Trigonometry
6	Trigonometry
7	Trigonometry
8	Trigonometry
9	Euclidean Geometry
10	Euclidean Geometry
11	Euclidean Geometry

Figure 1.2 Department of Basic Education diagnostic report (2021, p.198)

Figures 1.1 and 1.2 demonstrate that trigonometry was the area in which learners performed the poorest; performance was worst for questions that involved 2D and 3D trigonometric concepts. Moreover, a comparison of learners' performance on questions involving 2D and 3D trigonometry from 2020 to 2021 reveals a decline in learner performance. While this decline can be attributed to challenges associated with the COVID-19 pandemic, the 2019 results for questions relating to 2D and 3D trigonometry were below 40%, as indicted in the Department of Basic Education diagnostic report (DBE, 2019), confirming that learners have persistently found this topic difficult to comprehend. Some of the findings stated in the diagnostic report of 2020 (DBE, 2020) showed that learners struggled to remember some of the trigonometric content taught in Grades 10 and 11. In the area of trigonometric 2D and 3D concepts, the DBE states that learners were unable to visualise what was required to be solved and recommended that teachers integrate strategies that would assist learners to be able to solve 2D and 3D problems effectively (DBE, 2020) and find innovative ways to expose learners to complex and problem-solving types of questions (DBE, 2021).

The challenges learners encounter with 2D and 3D concepts has also been noted in the literature. For example, Ismail and Rahman (2017) conducted a study that examined the thinking and reasoning of leaners when solving 2D and 3D geometry questions; they found that learners displayed low levels of reasoning. In addition, the findings in the literature and the analysis of Grade 12 results by the South African Department of Basic Education (2020; 2021), it is evident that learners experience difficulty comprehending 2D and 3D concepts. To date, however, the focus of the majority of studies has been on geometry, and limited studies have been done to explore how learners' understanding of 2D and 3D trigonometric can be enhanced. For example Bayaga et al., (2019) conducted a study to evaluate the efficiency of applying GeoGebra software as a teaching tool to enhance the understanding of Euclidean geometry circle theorems by Grade 11 learners in a rural area with a high rate of poverty. It was found that GeoGebra yielded positive result in terms of improving logical thinking, confidence, interaction, engagement, connection, reasoning, creativity and understanding.

Knowledge of 2D and 3D concepts has stronger application to real life than some of the more abstract areas of mathematics; thus, helping learners to develop their conceptual understanding of such concepts can not only benefit their performance but also their understanding of mathematics in real life contexts. Therefore, an exploration of tools to enhance learners' understanding of difficult mathematical concepts has value. Dubinsky (1997, cited in Ndlovu, 2014) emphasises the importance of analysis by means of research that poses problems for learners to solve. On this basis, the researcher deemed it necessary to first investigate learners' mental constructions of 2D and 3D trigonometry concepts before exploring of the use GeoGebra to enhance learners' understanding of these concepts.

1.4 Purpose of the study

The purpose of this study was to explore the use of GeoGebra to enhance Grade 12 learners' knowledge of trigonometric 2D and 3D concepts. Dubinsky (1997, cited in Ndlovu, 2014) argues that concepts that learners find difficult to comprehend need to be analysed by means of research in order for alternative teaching strategies to be devised. The study focuses on learners, because enhancing their mental constructions of the concept would lead to improved performance related to the concept.

The incorporation of technological tools to enhance learning has been advocated for in literature. In South Africa, the Department of Basic Education, through its inclusion in education policy and the White Paper on e-Education (2004) of DBE has supported the incorporation of technological tools to enhance learning, especially in mathematics. While there are various tools that could be used to enhance learners' understanding, there has been evidence backed by literature arguing for GeoGebra as a tool that enhances learners' understanding of mathematical concepts. GeoGebra is considered to be an alternative that could potentially aid educators teaching trigonometric to enable learners to easily comprehend the concepts (Goos et al., 2003; Duncomb, 2011; Stols & Kriek, 2011; Stols, 2012). For example, Elsever (2011) conducted a study with the aim of determining the effect of using GeoGebra on students' achievements in trigonometric functions and graphs. After administering pre and post-tests, the findings showed that the application of GeoGebra in teaching and learning trigonometric functions did enhance learners' performance in trigonometry.

1.4.1 Motivation for selecting trigonometric 2D and 3D

In mathematics, 3D trigonometry involves the application of the trigonometric skills developed from 2D dimensional triangles. The rules of sine and cosine apply, as well as the theorem of Pythagoras and trigonometric ratios, which are used to find the missing sides or angles in two- and three-dimensional shapes. In the South African school system, trigonometry is taught from Grade 10 to 12, although learners learn some of the foundational concepts in earlier grades. Learners start to learn about the definition of trigonometric ratios of sine, cosine, and tan in right angled triangles in Grade 10, in grade 11 the definition involves 2D and extends the definition to Cartesian plane of 0 to 360 degrees in a deep level of the content and use special angles, also sine, cosine and area rule are done in grade 11. Problems that deal with three dimensions start to be introduced in Grade 12. As discussed in the statement of the problem (Section 1.3), diagnostic reports have shown that learners find trigonometric concepts difficult to understand and thus perform poorly on assessments, which contributes to poor performance in mathematics in general. Furthermore, complex problems and problem-solving types of questions aim to enhance learners' reasoning in mathematics. Thus, if learners perform poorly when solving complex problems it suggests that their reasoning is not well-developed. It is therefore critical that instructional strategies for developing learners' logical reasoning are explored to enhance the teaching and learning of mathematical concepts.

1.4.2 Background to GeoGebra and justification for its selection for the study

Markus Hohenwater developed GeoGebra as part of his master's degree in mathematics education and computer science at the University of Salzburg in Austria in 2001 (Preiner 2008). After receiving the European Academic Software Award EASA in 2002, Hohenwarter continued working on GeoGebra to make it more usable and functional. In addition to developing Geogebra further, Hohenwarter obtained a DOC scholarship from the Austrian Academy of Sciences, which enabled him to complete a PhD on pedagogical applications of GeoGebra in Austrian secondary schools (Preiner, 2008). According to Hohenwarter (2005) (cited in Preiner, 2008), GeoGebra won several more software and media awards across

Europe, including Austria, Germany, and France. As of 2006, GeoGebra has been developed at Florida Atlantic University, USA, where Hohenwarter has been involved in a math and science partnership teacher training project funded by the National Science Foundation. As part of a further extension and enhancement of GeoGebra, a dynamically linked spreadsheet was implemented as well as a computer algebra extension, further advancing the software toward its goal of being a versatile and easy-to-use software package that can be used by students and teachers worldwide for a wide variety of grade levels and mathematical content (Hohenwarter, 2005).

Researchers are using GeoGebra in research and teachers are using it to enhance learners understanding of maths concepts. Teachers and learners can use this software to explain, explore and model mathematical concepts and the relationships between them (Hohenwarter & Jones, 2007). Geogebra helps users to create activities that provide multiple representations of mathematics concepts that are dynamically linked. The software is free: there is no cost involved for learners to use the software on home computers or at school and it works across multiple platforms.

Bedada (2021) argue that the use of this software is believed to have a positive impact on learners' attitudes, beliefs, and perceptions. On this basis, this study integrates Geogebra into the teaching of trigonometric 2D and 3D concepts. It is the hope of the researcher that teachers will use this study to enhance their students' understanding of the concepts of trigonometric 2D and 3D.

Geogebra is one of the most well-known programmes that have been evident in the literatures that it used to integrate mathematical concepts in education. There are other technological tools, such as 3D Cabri, that are in use, but Geogebra is more common. Therefore this recommends that technology should be used by schools on a daily basis. The proposal that the integration of technology should be regarded as part of teachers' professional development (Trigueros & Lozano, 2012) should be considered since some teachers have access to technological tools in their schools but lack the knowledge to use them. Content workshops on how to use technological tools could enable this.

Moreover, the versatility of Geogebra enables teachers to use the software at all grade levels from secondary school to the tertiary level and for a wide range of different mathematical topics. Fuchs and Hohenwarter (2005) state that the integration of educational software into 'traditional' teaching requires a minimum of technical equipment in the classroom and therefore, could be implemented by any teacher who is willing to integrate it with their everyday teaching of mathematics.

In addition, a plethora of research that has been done using GeoGebra in areas such as Euclidean geometry has found that learners were more motivated and positive when learning with GeoGebra and, as a result, showed an incline in performance (e.g. Uwurukundo et al., 2020). Thus, GeoGebra has the potential to foster student-centred, active learning.

There are many technological tools available to enhance the teaching and learning of mathematics, and the choice of the most appropriate technological tool can be difficult (Ruthven et al., 2004). The positiveness and effectiveness of GeoGebra has generated enthusiasm for choosing this tool to use in the topic of trigonometric 2D and 3D and many researchers have produced positive results.

1.5 Rationale and motivation for the study

Bertram and Christiansen (2014, cited by Dube, 2019) describe a rationale or motivation for a study as the reasons or inspiring factors, such as personal, professional or academic that motivates the researcher to undertake a study. This study's rational encompasses all three areas.

1.5.1 Academic motivation

The rationale for this study is based on anecdotal evidence, the Department of Basic Education moderators' reports (DBE, 2020; 2021) and scholarly literature. There is evidence that learners find geometry and trigonometry concepts difficult to conceptualize; especially, concepts that require application and proof, such as 2D and 3D concepts. While there is a plethora of research exploring and designing interventions to enhance learners' understanding of geometry concepts, there is a dearth in the area of trigonometry.

Ngcobo et al. (2019; 2020) explored learners' mental constructions of the sine and cosine rules. Their findings revealed that learners were operating at the action level (with reference to APOS theory) which suggests that their understanding had not developed beyond recalling facts. The authors make a strong argument that learners' mental constructions of concepts need to be analysed by means of research and that learners need to be encouraged to engage with what they write in order to enhance their understanding. However, the authors do not explain how this can be achieved. This study, then, aimed to explore how GeoGebra could be used to enhance learners' conceptions of certain trigonometric concepts. First, the study explored learners' conception of trigonometry 2D and 3D, through a review of literature and also by generating new data. After engaging with learners' conceptions of trigonometric 2D and 3D and determining the level at which they were operating, GeoGebra was introduced with the aim of enhancing learners' conceptualisation.

1.5.2 Professional Motivation

In my teaching experience at the school where I currently teach at Quintile 3 School, I have never witnessed teachers encouraging each other to use technological tools, such as GeoGebra. This is different at suburban schools because the researcher has once worked in one of the suburban schools, and probably because generally, such schools have greater technological resources, thus teachers have opportunities to explore teaching using these tools. Having been exposed to teaching at both suburban schools and rural schools, I developed an interest in exploring the use of technological tools such as GeoGebra at rural schools. Therefore, I decided to explore the use of GeoGebra to enhance the teaching of trigonometric 2D and 3D concepts at a single school.

Furthermore, this study aims to enhance learners' knowledge of trigonometric 2D and 3D by using GeoGebra software; it is the hope of the researcher that this study will motivate teachers to integrate technology into teaching and learning. Simply because this world is revolving around technology and evolving technology for our learners is the safe way to the future.

The Department of Basic Education has implemented some ways to improve learner performance involving technology. For example, in 2015 the Gauteng provincial MEC of Education injected ICT resources in the form of smart boards, tablets and laptops to all non-fee-paying secondary schools as part of the education pillar of Operation Phakisa, which aims to transform education by appropriately integrating ICT (Makadidze, 2020). This implies that technology education should be implemented across all districts to help improve learners' performance in mathematics.

1.5.3 Personal experience

The researcher is a qualified mathematics teacher with ten years' experience. The researcher is employed at a school that does not have computers and thus cannot incorporate GeoGebra into teaching and learning. However, the researcher experienced working with GeoGebra previously while working at a suburban school that had computers. She found that learners enjoyed the lessons that used software, where they explored different graphs of functions, more than they did lessons that that used tradition teaching approaches. The researcher was thus motivated to undertake this study with the hope of emphasising the usefulness of technological tools such as GeoGebra.

1.6 Objectives of the study

This study aimed to explore the use of GeoGebra to enhance learners' mental constructions of 2D and 3D trigonometric concepts. Guided by this aim, the researcher formulated the following objectives:

(a) to explore learners' levels of mental construction of 2D and 3D trigonometry concepts, as guided by APOS theory with and without GeoGebra.
(b) to explore factors contributing to learners' construction of knowledge of 2D and 3D trigonometric concepts with and without GeoGebra; and
(c) to explore the extent to which GeoGebra enhanced learners' conceptualisation of 2D and 3D trigonometry concepts

1.7 Research questions

The following research questions guided this investigation into the use of GeoGebra to enhance learners' knowledge construction of trigonometric 2D and 3D concepts:

(a) What level of mental construction of 2D and 3D trigonometric concepts do learners demonstrate with and without Geogebra?
(b) Why are learners able, or not able, to construct knowledge of 2D and 3D trigonometric concepts with and without GeoGebra?
(c) To what extent does GeoGebra enhance learners' mental construction of 2D and 3D trigonometric concepts?

1.8 Selection of research site and participants

The study was conducted at a school in eSikhawini, located in the King Cetshwayo District of the South African province of KwaZulu-Natal. In the past, this district has had one of the lowest past rates in the province (DBE, 2021). The school was identified because the researcher was involved in collaborative teaching at the school. While the school is located in township area and it is categorised as a technical school it has the resources required to use GeoGebra. The school falls in Quintile 4, quintile in South African schools means, how schools are categorised based on areas and paying and non-paying school fees, whereas quintile 1-3 are schools that does not charge for a school fees and are usually in rural areas and quintile 4-5 are schools that are allowed to charge school fees and most of these schools are located in townships and suburbs, therefore that simply implies that the school is in township and its population is 1035 Combining teachers, learner and non-teaching staff. The school has principal, 5 departmental heads, 28 teachers and 8 non-teaching staff members. The researcher is not employed at the school but collaborates with mathematics teachers at this school as part of a cluster teaching initiative.

The targeted population for this study was Grade 12 learners taking pure mathematics. Thirty learners were selected from the 97 Grade 12 learners taking this subject. The sample size was influenced by the number of computers available for using GeoGebra at the school.

To ensure that learners with a range of abilities was selected, in terms of academic performance, selection was guided by the performance level descriptors corresponding to the seven indicators used by Department of Basic Education (CAPS, 2011) to promote learners: Level 1: 0-29%; Level 2: 30-39%; Level 3: 40-49%; Level 4: 50-59%; Level 5: 60-69%; Level 6: 70-79%; and Level 7: 80-100%. Four learners were randomly selected from each of the first 6 levels. Six learners were selected from Level 7 because Level 7 incorporate two groups, i.e. 80-90% and 91-100%. Learners scoring at Level 6 and above are considered to be high achievers, those at Levels 4 and 5 are considered average; and those at Levels 1, 2 and 3 are considered to be low achievers. Participants were selected on the basis of their 2022 Term 1 results. These results reflected their performance in mathematics in general, but did not reflect their competence with trigonometry concepts, as trigonometry concepts were taught or tested in Term 1.

1.9 Significance of the study

South Africa has repeatedly been ranked among the countries with the lowest pass rate in mathematics (Spaull, 2013). Many mathematics teachers put in enormous effort to making a positive contribution toward changing this grim reality. But while teachers of mathematics work extra hours every day in the hope that learners will achieve good results, the diagnostics analysis of the Department of Basic Education (2020; 2021) reveals that their hopes have not been realised. It is the wish of the researcher to contribute to closing one of the gaps that plays a role in learners' poor performance in mathematics. A key purpose of this study is to contribute to improving mathematics results in South Africa, which in turn will contribute to improved overall academic performance and success.

This study provides teachers with an opportunity to explore the effectiveness of GeoGebra in the teaching of trigonometric 2D and 3D concepts and potentially expand their teaching strategies. The research on mathematical 2D and 3D concepts using GeoGebra is limited, however; this study augments the current literature. While Ngcobo et al. (2019; 2020) and others have developed the genetic decomposition needed to analyse learners' mental constructions, this study goes further to explore a tool that could potentially be used to enhance learners' conceptualisation. The learners who participated in this study have gained experience and knowledge in working with GeoGebra which will benefit them as they progress to higher education institutions. They have visualized shapes, made coherent deductions and rapid calculations and reflected on this.

The findings of this study can benefit mathematics teachers by providing ways to enhance the teaching and learning of trigonometry and other areas of mathematics. The study also motivates teachers to incorporate software such as GeoGebra into their teaching to improving the performance of learners. The findings of this study also enable teachers to explore learners' conceptualization of trigonometric 2D and 3D concepts and analyze their mental construction to determine the levels at which their learners are operating.

1.10 Ethical consideration

Ethical clearance for this study was granted by the University of KwaZulu-Natal (HSSREC/00004156/2022). Permission to conduct the study at the school was obtained from the school principal. Assent and consent forms were given to learners and their parents or guardians to request their voluntary participation and assure them that learners' privacy would be protected through anonymity and the confidentiality of data.

1.11 Definition of key terms used in this study

GeoGebra is multiplatform mathematics software for all levels of education from basic education through higher education that works dynamically with geometry, algebra, tables, graphing, spread sheets, statistics and calculus (Hewsons 2009; Hohenwarter & Lavicza, 2009).

Trigonometry is an area of mathematics that deals with the relationships between the sides and angles of triangles. Trigonometry combines algebra, geometry and graphical reasoning and serves as the basis of calculus, physics, architecture and quantity surveying (Puteh, 2017). The three-dimensional trigonometric ratios sine, cosine, and tangent are used to calculate angles and lengths in right triangles. Trigonometry is an area of mathematics in which many students experience significant difficulty in learning (Adamek, 2005; Kutluca & Baki, 2009; Tatar, 2008).

Conceptual understanding refers to an integration and functional grasp of mathematics ideas that enables learners to connect with the designed task (Kilpatrick, 2001).

Technology refers to the science or knowledge practiced to solve problems or invent useful tools. Mackenzie and Wajcman (1985) define technology as the "integration of physical objects or artifacts or the process of making the object and the meaning associated with the objects" (p.3).

Technology integration refers to the use of technology such as computers, smartboards, calculators, data projectors, software and the internet in teaching and learning (Makandidze, 2020).

APOS theory is a framework for the process of learning complex mathematical concepts (Wayer, 2010). The theory is based on four mental constructs (action, process, object and schema: APOS) which explain the levels of knowledge construction an individual progresses through in order to formulate a schema for the particular concepts. APOS theory is discussed in detail in the theoretical framework chapter.

ACE Teaching Cycle consists of three components: Activities, Classroom discussions and Exercises. Activities are tasks that encourage students to make the necessary constructions, as outlined in the genetic decomposition, through reflective abstraction. These activities are completed in small groups during brief periods of time in class. Classroom discussion is led by the instructor, during which students are given time to reflect on their work and collaborate as an entire class. Exercises are standard problems designed to reinforce the activities and classroom discussion (Arnon et al., 2014). ACE teaching styles are explained in detailed in the theoretical framework chapter.

Genetic decomposition

Genetic decomposition refers to the structured set of mental constructs which might describe how the concept can develop in the mind of an individual (Brijlall et al., 2013).

Learners and students

In this study learners and student referred to school going learners.

Levels and stages

In this chapter level and stage referred to a point, period or step in a process development

1.12 Assumptions and limitations

This study was based on the assumption that learners' responses indicate their level of knowledge construction in other words, their conceptual understanding and that their responses are an accurate reflection of their capabilities, in the context of 2D and 3 D trigonometric concepts. This assumption comes with some limitations.

Since this study used GeoGebra software, and some learners lacked the necessary technological skills to use it effectively, it was important that learners were work shopped on key functions of using a computer prior to data collection. In addition, instead of learners working independently while practicing using GeoGebra, team tasks were used so that learners could assist each other during the practise stage. Team tasks aligned with the ACE teaching style were used for this purpose.

Another limitation was that there were not enough computers available for each participant to be assigned a computer for the study activities. To address this, learners of mixed abilities were paired. Pairing learners of mixed abilities aligns with cooperative learning which is advocated when adopting ACE teaching style.

As not all of the Grade 12 learners at the school participated in the study, it was not possible to conduct the study during normal teaching hours. Thus, in consultation with the participants, parents and principal, the study was conducted after school.

The power dynamics between the researcher and the participants was also a potential limitation to the study. As the researcher was also the teacher of the participants, it was possible that learners might feel obligated to participate in the study. To ensure that learners did not feel coerced to participate in the study or that refusal to participate might jeopardise their marks, the purpose of the study was explained to learners and they were ensured that their performance during the study did not factor into marks used towards their progression. The decision to conduct the study after school hours helped to mitigate this risk, as learners already viewed after-hours tuition as being for the purpose of developing their mathematical knowledge.

1.13 Overview of the dissertation

This dissertation consists of seven chapters.

Chapter One has provided the background of the study, statement of the problem, purpose of the study, research question, significance of the study and a brief research methodology and the theory that underpins this study, the background of GeoGebra. The chapter also laid out assumptions and definitions of key terms.

Chapter Two reviews the literature related to using GeoGebra to improve learners' understanding of 2D and 3D trigonometry. It discusses the role of technology in mathematics, the limitations to integrating technology in mathematics and teachers and learners' beliefs and attitudes toward using technology in mathematics and, specifically, the use of GeoGebra in mathematics teaching. It explores the complexities that hinder learners' construction of knowledge in trigonometric concepts and conceptual understanding in mathematics.

Chapter Three presents the theoretical framework that informs and guides the study. This study uses ACE (activity, classroom discussion and exercise) teaching styles which are framed by APOS (action, process, object and schemas) theory to explore and enhance learners' construction of trigonometric 2D and 3D concepts. This chapter also presents the genetic decomposition used to analyse learners' written work using APOS stages.

Chapter Four discusses the qualitative research methodology used in this study. The research design, research style, sample and population selection, and development of research instruments are discussed. The data collection procedures and analysis strategies adopted in this study are presented.

Chapter Five presents the results of data collected through the pre-test using genetic decomposition with APOS levels.

Chapter Six presents the results of data collected from the post-test after using GeoGebra. The analysis used genetic decomposition with APOS levels.

Chapter Seven summarises the major findings of the study, discusses the limitations and delimitations of the study, presents the implications of the study and makes recommendations for further research.

1.14 Conclusion

This chapter has provided the background to the study and stated the problem that the study responds to: the challenge learners' experience with learning 2D and 3D trigonometric concepts. The purpose and rationale of the study have been explained and the objectives, research question and site and participant selection have been explained. The significance of the study, and ethical considerations, were also engaged. The next chapter reviews the literature relevant to the focus of this research and delineates the scope of the study.

CHAPTER 2: LITERATURE REVIEW

2.1 Introduction

The purpose of this chapter is to review the literature about the use of GeoGebra to enhance learners' understanding of 2D and 3D trigonometry. According to the literature, mathematics is a subject that many students find it difficult (Centre for Development & Enterprise, 2004; Ndlovu, 2014; Vilakazi, 2020). A possible reason for this difficulty may lie in the way mathematics is taught, says the Centre for Development and Enterprise (2004). Therefore several researchers have attempted to find the most effective way to teach mathematics to overcome learners' difficulties on both the national and international level (Biyela, 2008; Barkartsas et al., 2009; Li & Ma, 2010). The issue of the difficulty of mathematics is also confirmed in analyses of learner performance in South Africa, such as the Annual National Assessment report (2014), the Trends in International Mathematics and Science Study (TIMSS,2019); and the Department of Basic Education diagnostic report (DBE, 2021).

The poor mathematics performance of learners had prompted researchers and scholars to carry out a variety of studies. Some researchers have attempted to identify factors that contribute to low performance among learners (Mji & Makgato, 2006; Bansilal et al, 2014; Taylor, 2021). Other scholars have focussed on exploring interventions to enhance learner performance in mathematics (Burney, 2015; Adler, 2015; Umugiraneza, 2018; Brijlall et al., 2022). Despite these efforts, poor performance persists. Further research is thus needed in this field.

It is within these parameters that this study was undertaken to explore the use of GeoGebra to enhance learner performance in trigonometry particularly, their understanding of 2D and 3D concepts. While a plethora of research has paid particular attention to improving learner performance in algebra and geometry, limited attention has been given to trigonometry especially the area of 2D and 3D concepts despite the evidence (presented in Chapter One) that many learners find trigonometry of 2D and 3D concepts challenging. This study thus contributes to filling this gap in the literature and augments the exploration of alternative instructional strategies to enhance learner performance in mathematics.

2.2 Trigonometry in the South African curriculum

The curriculum is essentially what a learner encounters, studies, practices, and masters in a given learning environment (DBE, 2012). This involves making decisions about what should be taught, how it should be taught, and when it should be taught (DBE, 2012). School curriculums are drafted by curriculum experts appointed by the government and required to be followed by schools at the formal level of education (Dube 2019).

Trigonometry is a branch of mathematics that deals with the relationship between the sides of triangles and the angles formed by the vertices of those triangles (Tatira, 2020). Charles (2015), Guhl (2009) and Mensah (2017) explain that the word trigonometry comes from the Greek 'trigonon', which refers to 'triangle', and 'metron', which refers to the science of measurement. They further state that trigonometry, as a study of triangles, originated with the early Egyptians; Leonard Euler is credited with being the originator of the current form of trigonometry. Trigonometry is a very important component of mathematics but is also regarded as one of the most difficult areas for learners to master, as confirmed in the literature and by analysis of Grade 12 results, discussed in Chapter One.

In the South African schooling system, trigonometry is taught from Grade 10 to Grade 12, although learners start to learn foundational concepts, such as those related to right-angle triangles, earlier than this. Key concepts, such as the definitions of the trigonometric ratios of sine, cosine, and tan in right-angled triangles, the definition of the Cartesian plane of 0° to 360° and special angles, are introduced in Grade 10. Trigonometry forms approximately 40% of the entire content to be covered; thus, it is a significant topic and learners need to pass it to get good results in mathematics as a whole. Among the topics covered are 2D and 3D trigonometry, which are taught from Grade 10 to Grade 12.

The Curriculum and Assessment Policy Statement document (CAPS, 2011) mathematics FET phase, states that the order of topics is not prescriptive but ensures that part of trigonometry is taught in the first term of each year, and more than the six topics of paper 1 and 2 that are taught in the first two terms. Trigonometry is given more time than any other topics in mathematics. Makandidze (2020) notes that trigonometry is allocated a major share of 43 to 53 marks out of 150 marks on the final examination for Grades 11 and 12. This indicates that trigonometry is given a high level of importance. The importance of the topic of 2D and 3D trigonometry is also evident because it is covered not only under trigonometry but also under Euclidean geometry. In this study, however, the focus is on the 2D and 3D concepts that are covered under trigonometry.

There are several reasons trigonometry is given such importance. Through trigonometry, learners develop their ability to memorize concepts and problem-solve (Hill, 2015). According to Simons and Wibawa (2021), learners must be familiar with a number of sophisticated mathematical concepts in order to be successful in trigonometry, including the association between numbers and sides of triangles representing length and the measurement of triangles, which are necessary for studies such as engineering, physics, construction, and design. A good understanding of trigonometry is crucial for the solving of advanced mathematics tasks, since it equips learners with comprehensive knowledge of the necessary mathematical concepts (Koyunkaya, 2016).

The importance of trigonometry in the curriculum globally is noted by Kissane et al., (2009), who posit that trigonometry is not a recent addition to the mathematics curriculum globally but has been a long-standing component of the secondary school curriculum. Trigonometry has a significant place in the curriculum of many countries, even though the curricula may vary from country to country at secondary school level (Delice & Roper, 2006). Thompson (2008) states that in the United States trigonometry constitutes a critical component of mathematics as it is needed in various field of careers, such as engineering and construction, which are considered to be of scarce skills. Thus learners intending to pursue such careers must pass trigonometry at school level.

In a study of Ghanaian senior high school learners' errors in the learning of trigonometry, Mensah (2017) also confirms that trigonometry is major component of the Ghanaian curriculum, although learners find it difficult to understand. Makandedze (2020) argues that the importance of trigonometry as a topic should not only be stated in the curriculum, but mathematics teachers should also emphasise it in the process of teaching and learning and show its relevance to real life so that learners see its benefit beyond the classroom level.

2.3 Learners' difficulties understanding trigonometric concepts

Trigonometry is an important area of mathematics which improves learners' thinking skills (Dündar & Yaman, 2015), but learners find it difficult to comprehend. The evidence of this is found in the diagnostic reports published by the South African Department of Basic Education (DBE, 2020; 2021) at the end of every year after the release of Grade 12 results. The scholarly literature also bears witness to this. For example, Bornstein (2017) states that learners and even teachers have trouble articulating and justifying trigonometric concepts. Similar sentiments are echoed by Madonsela et al., (2020), whose findings revealed that learners have knowledge gaps when solving problems related to 2D and 3D; they recommend interventions to be implemented by teachers to enhance learners' understanding of these concepts. One of the reasons trigonometry is perceived as difficult is that most trigonometry concepts are taught from an algorithmic approach, without necessarily emphasizing conceptual understanding (de Villiers & Jugmohan, 2012). But thoroughly teaching the concept before having learners attempt to solve problems is very significant. For example the (DBD, 2019) diagnostic analysis reported that teachers are using previous question papers for teaching and learning instead of thoroughly teach the concepts and use previous question papers thereafter.

According to Nugroho and Wlandari (2017), language plays an important role in teaching and learning new concepts. Therefore, for learners to understand mathematics, they need to comprehend the mathematical language used. As purported by Tall (1981, cited in Ndlovu, 2014), concept definition is critical in the formation of the concept image, meaning that having the correct concept definition is the key to constructing the correct concept image; therefore, language plays a crucial role in the formation of the concept in the learner's mind. Chikiwa (2015) posits that the fact that trigonometry uses technical mathematical terms that are not usually used in everyday lives of learners and teachers poses a challenge in the conceptualisation of its concepts. Mensah (2017) found that learners' errors in trigonometry arose from a lack of basic mathematical knowledge, such as how to compute ratios. Madonsela et al., (2020) found that weak basic mathematical knowledge hindered learners' understanding of trigonometric concepts.

2.3.1 Overview of studies about trigonometry

Kissane and Kemp (2009) of Murdoch University in Australia, studied how technology can be used to teach and learn trigonometry (graphic calculators, computer applets). As a result of using technology, learners were able to draw graphs of trigonometric functions quicker and understand their basic features more easily. As a result of the study, learners were able to write graphs with more freedom when they used applets on computers to illustrate periodicity and amplitude.

Khuzwayo (2019) conducted a study in one of the South African schools in order to explore the learning of trigonometric identities by Grade 11 mathematics students. In response to the learners' responses, some demonstrated a profound understanding of trigonometric ratios while others demonstrated confusion. In addition, the study found that learners had difficulty factoring trigonometric equations and simplifying trigonometric expressions when using algebraic manipulations.

Gur (2009) conducted a study with 140 Grade 10 learners and 6 Grade 10 teachers at Balikesir University exploring learners' types of errors, misconceptions and obstacles with trigonometry. The study found that the most common errors that learners made were the improper use of equations, incorrect order of operations and values and place of sin, cosine, misused data, misinterpreted language, and logical and mechanical errors. These misconceptions and errors are commonly found across most areas of mathematics, but especially those areas that learners find most difficult to comprehend such as trigonometry.

These studies indicate that learners do experience challenges in comprehending geometric concepts; however interventions such as teaching using technology like GeoGebra have yielded positive results (Atan et al., 2010; Shadaan, 2013; Bayaga et al., 2019; Mthethwa, 2020; Tamam et al., 2021). It is within these parameters that this study explored the use of GeoGebra to enhance the teaching of trigonometric 2D and 3D concepts, as these concepts have proven to be particularly challenging for learners. Dubinsky (1997) emphasises that topics that prove to be challenging for learners need to be explored by means of research in order to design alternative instructional strategies to enhance learning.

2.4 Teaching and learning of 2D and 3D mathematical concepts

Trigonometric 3D and 2D is one of the areas of mathematics that is most relevant to real life. This area can be related to many things around us, especially the regular 3D forms of most human-made structures. Bobo (2010) argues that the teaching of 3D concepts allows learners to relate their learning more clearly to the world around them while also avoiding misconceptions about the differences between 2D and 3D shapes.

Although schools use the 2D drawing of nets to develop learners' understanding of 3D shapes, our understanding of 2D shapes comes from our experience with 3D realities in everyday life. For this reason, I believe that learners should be introduced to 3D shapes before they are taught about 2D shapes. It is different when it comes to calculations because learners understand how to calculate 2D shapes more easily than they do 3D shapes. Ping and Hau (2016) argue that 3D models that allow learners to touch and feel 3D shapes enable learners to better understand features of 3D shapes such as vertices, surface and edges. Gal and Linchevski (2010) argue that as learners prefer to visualize material rather than working with text, the teaching and learning of trigonometric 2D and 3D should be more visual; learners' visualization should be the first priority because, if they cannot visualize shapes, it will be difficult for them to state a shape's characteristics and, as a result, they will wrongly select formulae for their calculations.

2.5 Overview of studies about the teaching and learning of 2D and 3D concepts in mathematics

In South Africa, the Department of Basic Education recommends that learners be exposed to solving 3D problems (DBE, 2021). One departmental report revealed that the incorrect use of operations was observed most often when learners were solving 3D problems (DBE, 2017). Similarly, Brown (2005) found that learners commonly misused the order of operations and the value and place of sine and cosine in trigonometry when working with 2D and 3D. Brown (2005) found that many learners had an incomplete or fragmented understanding of the three major ways to view sine and cosine: as coordinates of a point on the unit circle, as horizontal and vertical distances that are graphical entailments of those coordinates, and as ratios of sides of a reference triangle. These findings attest to the fact that the challenges learners experience with learning 2D and 3D concepts has been an on-going problem. However, little has been said about interventions teachers have explored to enhance learners' understanding of the concepts and this study hopes to bridge that gap.

2.5.1 Overview of studies on the learning of 2D and 3D concepts in geometry

A plethora of research has explored the issues relating to the teaching and learning of 2D and 3D concepts in geometry. For example, Ping and Hua (2016) conducted a study in geometric 2D and 3D with Grade 3, 4 and 5 learners at a school in Asia with the aim of assessing the effectiveness of applying teaching aids in the classroom through playing and learning sessions and mastering the geometry of 2D and 3D shapes. A pre-test was conducted to generate baseline data, where learners were asked to define 2D shapes before the teacher guided them to determine the correct shapes. On the post-test, learners were tested for 2D and 3D where examination study was conducted to define critical thinking of the learners' level. Their findings revealed that the 'playing while learning' method nurtured students' interest in learning and motivation to explore the content of education with the guidance of a teacher. Moreover, they found that learners were active and able to state the characteristics of the objects.

A recent study conducted by Lelinge and Svensson (2020) contributes to the knowledge of how teachers' awareness through collaboration and understanding of the conditions necessary to develop grade 4 students' abilities to discern two- and three-dimensional shapes in mathematics. They have used collaboration opportunities to apply professional classroom instructions and activities to enhance students' knowledge of two- and three-dimensional shapes. The study involved 3 teachers and 2 lecturers and 14 Grade 4 learners. A pre-test and post-test were administered to assess changes in learners' understanding of the similarities and differences between 2D and 3D shapes. On the pre-test, none of the learners answered all 8 questions correctly. Some said the test was difficult; others said it was easy. On the post-test results, none of the learners had less than 5 correct answers for all 2D shapes, whereas in the pre-test all learners got only one question correct for 2D shapes. Hence this study implies that choosing the appropriate method or tool for teaching and learning does influence learners' learning performance either in a positive or negative way. Therefore, teachers need an accurate assessment of their learners' capabilities to enable them to choose appropriate teaching and learning methods. Teaching topics that are applicable in real life, such as 2D and 3D trigonometry and geometry, require teachers to work with models or technological tools such as GeoGebra to help learners clearly visualise and understand the concept before they attempt to solve a problem that uses this concept.

2.6 Challenges encountered in the teaching of 2D and 3D mathematics concepts

It is not easy to teach 2D and 3D concepts in mathematics without the use of practical approaches; this is especially true with 3D problems. While the teacher may begin by drawing a 2D representation of a 3D shape, some learners might identify that it is a representation of a 3D object, while others might only perceive it as a 2D image that uses shapes combined together. For example, from anecdotal experience, if a teacher draws triangular pyramid (as shown in Figure 2.1a) and asks learners to name the number of triangles contained in the pyramid, learners sometimes fail to see the number of triangles present. They may count three triangles: the triangles on the left, the triangle on the right and the triangle on the bottom. This shows that the learner views the diagram as 2D shape. Very few will see that there is a triangle that is in the front joining the other three triangles to make a 3D shape. This is why learners should be given a chance to explore 3D shapes using technological tools for better visualization before attempting to solve problems.

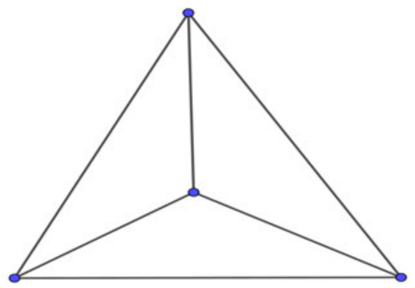


Figure 2.1a: Triangular pyramid

Figure 2.1a it shows the net drawing of triangular pyramid. A net can help learners and teachers who do not have access to tools such as computers for accessing GeoGebra. However, in the Further Education and Training (FET) phase (Grades 10 to 12) in South Africa, learners are just given the shape without being reminded of the net, because construction is not part of curriculum for this phase.

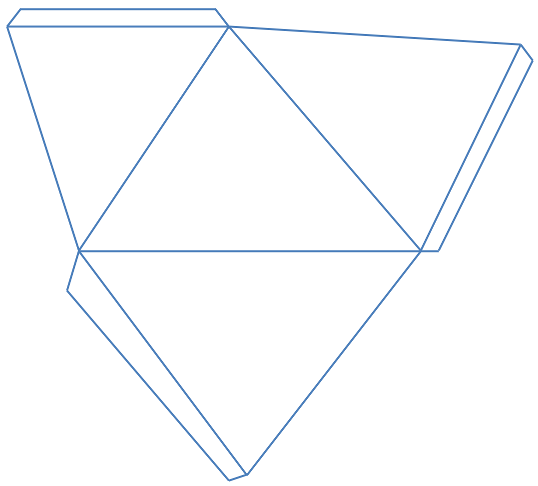


Figure 2.1b Net of a triangular pyramid

It is argued by Bos (2007) that complex mathematical tasks may be difficult for the teacher to demonstrate manually; however, this difficulty can be alleviated by the inclusion of technological tools when teaching the concept. GeoGebra can assist learners to visualize shapes before they begin problem-solving. In addition, Atagana (2009) states that learners find trigonometric rules and concepts related to 2D shapes difficult to understand (DBE, 2017).

Therefore, more time should be invested in the teaching and learning of trigonometric 2D and 3D.

2.7 The role of technology in the teaching and learning of mathematics

As indicated by Bos (2007) and many other researchers, technology has proven useful in the teaching and learning of mathematics. The aim of this study is to explore the use of GeoGebra to enhance learners' mental construction of trigonometric 2D and 3D concepts.

Mthethwa et al., (2020) state that educational technologies have played a transformational role in teaching and learning worldwide, and that South Africa have begun to invest hugely in transforming its education sector to achieve effective teaching and learning. Transformation is being aided by the convergence of technology in teaching and learning, leading to visible efforts to update the relevance and application of educational technologies to create a more productive, technology-driven, knowledge-based economy (DBE, 2015). Ndlovu (2014) notes that technologies like GeoGebra, Blackboard and Moodle have been integrated into South African schools to improve the educational environment.

In addition to Voogt (2008) stated that, technology is useful because it serves as an object of education, influencing learning objectives and content, and also as a medium for enhancing the teaching and learning process. While authors such as Kim et al., (2017) agree that technology is crucial to teaching and learning especially for mathematics they also argue that the benefit of using technology is not only to improve learner performance but also to motivate learners to learn, because an interest in learning can improve performance. Extending this argument, Bester and Brand (2013) and Segal and Stupel (2015) argue that computer technology assists learners to make meaning of the learning material and has the potential to improve the quality of teaching.

For example in a study by Ertekin (2014), he examined how teaching analytical geometry using Cabri 3D affected trainee teachers' ability to calculate equations, identify normal vectors, and draw graphs of planes. Study results revealed that students taught using the software were significantly more successful to identify the equations of special planes and their normal vectors and to draw graphs than those who were not taught using the software.

On the other hand, Todorova (2015) investigated students' conceptual understanding of the dot product of vectors in the context of a dynamic geometry environment (DGE). In the study, it was discovered that DGE offers students multiple representations, which allow them to gain deeper knowledge about a specific mathematical concept rather than observing one static representation only. Various technological tools have indeed been proven to be useful in the teaching and learning of mathematics, which is why this study was designed to explore how GeoGebra can be used to help learners improve their mental understanding of trigonometric 2D and 3D concepts, which are particularly challenging for students.

2.7.1 Challenges integrating technology into the teaching and learning of mathematics

The use of technology in teaching to enhance learning has been recommended in the literature; however, the complexities associated with this also have been brought to light. For example, Nkhwalume (2013) found that a number of factors affected teachers' use of technology in mathematics teaching and learning in Botswana, but three themes emerged that were of particular concept: lack of adequate access to computers, inadequate contact time for teaching mathematics, and lack of administrative support. The lack of time was also alluded by Sicilia (2005), who mooted that that the most common challenge reported by all teachers was the lack of time to plan technology lessons, explore various internet sites, or look at various aspects of educational software. Mthethwa et al., (2020) extend the debate to include issues associated with the actual use of such technologies: for example, the lack of software available at rural schools. Researchers at the Centre of Implementing Technology in Education (2015) in the US found same challenges facing schools and districts in the integration of technology into mathematics education.

The shortage of technological resources in the context of South Africa was revealed in a study by Makandidze (2020), who found that even schools in the so-called urban provinces such as Gauteng had limited ICT resources and experienced connectivity challenges when teaching using technology. This shows that the shortage of technology resources is not only a hindering factor to rural schools, as mentioned by Mthethwa et al., (2020). While tools like GeoGebra have proven to be useful for enhancing the teaching of mathematics, when schools do not have access to computers it becomes a challenge to explore their usefulness in the teaching of mathematics. The issue of connectivity is also prevalent. Outside the borders of South Africa, a study conducted in Botswana revealed that teachers complained about technical problems such as connectivity and malfunctioning computers it hinder the process of teaching and learning using technology tools (Nkhwalume, 2013).

In a study conducted in Nigeria, Owofala et al., (2019) identified a different challenge to the use of technology in the teaching and learning of mathematics: teacher competence. The study found that with the introduction of ICT, teachers encountered numerous challenges because they lacked the necessary ICT skills themselves; they reported that less than 10% of teachers in Nigerian primary and secondary schools were computer literate. They also stated that the majority of pre-service STEM teachers seemed not to possess the required confidence to use ICT facilities for teaching and learning. This is an indication that even if the technological tools are available, the user or implementer also needs to have the necessary expertise for them to be used effectively. These findings demonstrate that while technology has proven useful to enhance teaching and learning, technical and human resource factors may pose a serious challenge to their effective implementation.

Moreovver, some things in the world we live in, has pros and cons, and so does the integration of technology in mathematics education. For example the integration of technology in mathematics might come with the limitation of the technology fluency of learners in class. In addition, the implementation of technology in education will require a teacher to have a backup plan due to possible system failure since we are living in a country that is facing an electricity crisis. Moreover, as Geogebra is a technology software, it also comes with limitations. For example in the study conducted by Wassie and Zergaw (2019) where they presented the potential affordance challenges and limitations of using Geogebra in mathematics education with University students in the Bahir University. They have listed the following limitations of the use of Geogebra: 1) some of the commands used in the input bar are not user-friendly, especially for those with no prior programming experience. 2) It lacks correct representation of graphs of discontinuous in an automatic way. 3) The embedded manual requires an internet connection. This restricts its use in places where there is no internet connection.

This study did not explore the factors hindering the teaching of mathematics using technology, but rather explored the use of GeoGebra to enhance learners' mental construction of trigonometric 2D and 3D. Therefore, it was critically important to reveal the challenges associated with the use of technological tools that could pose a threat to the study in order to proactively plan alternative measures to minimise the possible limitations.

2.7.2. The challenges the curriculum addresses/identifies and sufficiently prepare learners in mathematics

The guideline for inclusive teaching and learning 2010, state that the different challenges/barriers faced by learners in schools for example for example the barrier with language and communication, it state that learners find themselves in the situations where the school's language of teaching and learning is either rarely or never used at home, thereby compromising effective learning. The guideline for inclusive teaching and learning therefore came up with guidelines to address language and communication challenges to prepare learners by stating that learners have to learn their home language and at least one additional South African official language. The issue is a very crucial thing even in mathematics, it all start with instructions sometimes learners omit instructions and jump into answering the question without the clear understanding of what a question is asking for, for example, there are mathematics terminologies in mathematics that learner could find it hard to differentiation such as evaluate and simplify. Therefore if a learner will not understand the language in which the school use in teaching and learning mathematics that will be a challenge to a learner which can contribute in a learner failing mathematics.

In addition, the guideline for inclusive teaching and learning identified another challenge that contribute into the number of low pass rate in schools especially in mathematics which is negative attitude towards subjects. Teachers in schools are dealing with negative attitude especially in mathematics. If a learner come in class with the attitude in the subject for instance being told that mathematics is the difficult subject, therefore it will be difficult for him because his mind is loaded with fear and that will lower his energy to even give himself a chance or time to deal with it. Creating a classroom environment that encourages curiosity exploration and positive attitude towards mathematics can reduce anxiety and increase learner confidence. The guideline for inclusive teaching and learning 2010 state factors to overcome negative attitude such as viewing learners in the positive light and there should be determined effort to establish what every learner's real strengths are, for the purpose of development and that schools should be welcoming environment for all learners. The national curriculum statement adopts an inclusive approach. The NSC state that to prepare learners teaching and learning and assessments activities will be adapted at lesson plan level for learners who need specific additional.

The national diagnostic report on learner performance provide teachers, subject advisors and curriculum planners with a picture of learner performance in each of key subject

(DOE,2023.p.4) and misconceptions are identified during marking and suggestions for improvements are also provided. Therefore ongoing assessment regular formative assessment provide immediate feedback to both learners and teachers about understanding and areas needing improvement, it encourage learners to reflect on their own learning processes and progress fosters self-regulation and independence in learning . Standardized test and final exams evaluate cumulative knowledge and proficiency, helping to identify long term learning gaps and overall progress.

2.7.3. Teachers beliefs' about integrating technology into mathematics

Han and Carpenter (2014), beliefs about mathematics are cognitive components of attitudes toward mathematics. Goldin (2002) claims that beliefs are internal representations whereby individuals attribute truth and validate information based on what they believe, and these beliefs are usually stable. Additionally, beliefs are inbuilt within individuals and can be difficult to change (Ether & Leftwich, 2010; Aiken, 2004). One's beliefs are a determining factor in what one is willing to implement or not in terms of technology; teachers' beliefs play a crucial role in determining whether or not they will use available technologies in their classrooms. While in this study the focus was not on teachers' beliefs, but rather on the use of a particular technologic tool (GeoGebra), the researcher deemed it critical to review literature related to teachers' beliefs about using technological tools in the teaching and learning of mathematical concepts, as these might influence the implementation process.

A study conducted by Jones (2017) on teachers' beliefs and technology in a Montessori classroom identified different beliefs that teachers held regarding the role of teaching using technology. One educator stated that she was very comfortable using technology at the high school level and considered it important given the prevalence of new technologies in society. She held the view that, as a teacher, it is important to be familiar with using a computer in order to be competitive. Moreover Jones' (2017) study highlighted the positive and negative attitudes of teachers towards the use of technology. Teachers' negative attitudes emanated from a view it was difficult to maintain a balance between human interaction and computer time; they were not opposed to using technology, but felt it was important to not let it replace all human interaction between teacher and learners.

Tatli et al., (2019) conducted a study investigating the effects of using Web 2.0 tools in teaching within the context of a technology acceptance model (TAM) on teacher candidates' attitudes towards using instructional materials and technology in the classroom environment. It was found that using Web 2.0 was perceived as useful for attracting the attention of learners to learn and the candidates developed positive attitudes towards the use of technology in education. The researchers reported that the teacher candidates who participated in the research stated that it is vital to use technology because it brings benefits to the learning environment and to teachers and also helps them prepare instructional materials. Before the training, candidates had a negative attitude toward using technological tools, believing that the use of technology in the classroom was a waste of time as it distracted students or restricted their imagination; however, by the end of the study they had developed a positive attitude towards technology integration.

What can be concluded from these studies is that the view of teachers regarding the use of technology in the teaching and learning process is important.

2.7.4. Learners' beliefs about integrating technology into mathematics

It is students' beliefs about mathematics that influence their achievement in mathematics. In addition to cognitive, emotional and behavioural aspects of attitude, students who lack confidence about mathematics express negative attitudes (Di Martino & Zan, 2007). Moreover, learners' negative beliefs contribute to the difficulties they face in learning mathematics. Negative conceptions may diminish students' interest and may have a major impact on their learning and performance (Amirali, 2010; Whitin, 2007). Mathematics educators, Leder and Grootenboer, (2005) and White et al., (2006) stated that learners' beliefs shape the cognitive and affective elements of learning mathematics.

There is limited literature on learners' beliefs and attitudes about the use of technology in learning mathematics. Nevertheless, the mathematical disposition of learners can be influenced by using technological tools that may help them develop a positive belief about using the tools for learning mathematics. The use of technology for learning mathematics may affect students' beliefs depending on the extent of use and their ability to use the tools for independent learning. The concept of conceptual beliefs governs the content of mathematics and the processes of memorization, imagination, logical thinking, and problem-solving (Belbase, 2020). The development of positive beliefs may occur when learners have positive learning experiences. It is also possible that they may think that technology is the cause of more distractions in learning (Beckman, 2015). Learners who have experience using laptops or other technological tools in the classroom may have positive beliefs and attitudes toward learning mathematics or other disciplines with technology, which also indicates that one's beliefs about technological tools for learning mathematics may be associated with one's self-efficacy toward the use of the tools in the long term (Gudek, 2019).

Belbase (2020) conducted a study with the aims to explore the beliefs of learners about learning mathematics using technology and how their learning experiences may affect their beliefs. He found that participants' beliefs about the use of the technology for understanding the mathematical meaning were not strongly aligned: about 34% disagreed with the statement that technology helps one understand math concepts better than one would without using it, while a similar percentage around 37% agreed, and the remaining 29% were neutral to this view.

Many governments believe that technology can positively impact student learning, leading them to incorporate technology into schools and universities (Hew & Brush, 2007). According to Belbase (2020), technology may not assist students in understanding mathematical problem-solving because it is direct and quick. In contrast, students may be able to learn concepts more quickly when they have learned them before using technology. In spite of the fact that learners may sometimes believe that technology helps them understand mathematical meanings and concepts, this may not always be true.

Moreover, there is scepticism around how technologies such as calculators and computers help students make meaning of mathematical process and develop conceptual understanding, although they may support the visualization and concretization of many mathematical concepts (Utterberg & Lundin, 2017). The process of creating associations and making meaning is a part of learning (Marshall, 2002). However, the mere presence of technological tools such as calculators and computers does not facilitate students' learning: they must be used as tools for making sense of mathematical concepts, and not just for computing, graphing or demonstrating (Shifflet & Weilbacher, 2015).

2.8 Integrating GeoGebra into the teaching and learning of mathematics

GeoGebra is a tool that has been advocated for by many researchers in mathematics towards the enhancement of the teaching of mathematics (Tay, 2003; Meng & Idris, 2012; Rajagopal, et al., 2015). A lot has been said in Chapter One about its usefulness in mathematics. It is free, it does not require an internet connect, the main reason for integrating GeoGebra into mathematics is to improve the understanding and performance of learners. Its benefits in this regard are evident in the literature: for example, Bayaga et al., (2019) found that GeoGebra yielded positive results in terms of improving logical thinking, confidence, interaction, engagement, connection, reasoning, creativity and understanding. This section explores the integration of GeoGebra into mathematics and then, specifically, into trigonometry and the area of 2D and 3D concepts which is the focus of this study.

2.8.1 The utility of GeoGebra in the teaching and learning of mathematics

GeoGebra has been recognised as a tool that can help teachers to design effective instructional lessons to enhance learners understanding (Li, 2007). Prodromou (2015), in a study exploring the use of GeoGebra in the teaching of probability, found that GeoGebra as a teaching tool had the potential to improve learners' understanding of mathematical concepts. Li (2007) reported that 73% of the learners participating in a study indicated that they found GeoGebra to be a very useful technology for learning. While abstract mathematics has proven to be challenging for learners, Venkataraman (2012) posits that by using GeoGebra the learning of abstract concepts becomes more manageable and helps learners to visualize related concepts. While advocating for the use of GeoGebra in the teaching and learning of mathematics, Atan et al., (2010) raises a critical point that: it is the teacher who plays a critical role in motivating learners in a mathematics classroom through adopting innovative teaching strategies.

The aforementioned studies argue for the importance of GeoGebra in enhancing learners' understanding of mathematical concepts such as geometry, abstract mathematics and probability. However, there is death of literature about how GeoGebra can be used to enhance learners' understanding of trigonometric 2D and 3D concepts although there is evidence that learners find trigonometric concepts difficult to understand. This study, therefore, aims to contribute toward filling that gap by exploring the use of GeoGebra to enhance learners' understanding of trigonometric 2D and 3D concepts.

Although a plethora of literature advocates for the effective use of GeoGebra in the teaching of mathematics, other scholars such as Stols and Kriek (2011), Guven (2012) and Lin et al., (2013) have highlighted challenges that have the potential to hinder the effective use of GeoGebra to enhance teaching and learning. Saralar & Iinsworth (2017, cited in Uwurukundo et al., 2020) found that while middle school teachers experienced challenges when using GeoGebra, they still expressed positive attitudes about using GeoGebra and wanted to continue using it. Reporting on studies that investigate the effectiveness of integrating GeoGebra into mathematics in secondary school, Awurukundo et al., (2020) indicate that 80% of the studies shows that teaching and learning using GeoGebra was found to be beneficial and only 20% show no benefit. There is thus evidence that this technology can be useful to enhance learners' conceptions in mathematics, although there are factors that could hinder its effectiveness.

2.8.2 The integration of GeoGebra into the teaching of trigonometry

In the mathematics curriculum, trigonometry is one of the topics that is very important yet very difficult (Khuzwayo, 2019), requiring more time for learners to comprehend it. It is thus an area that requires more interventions, such as the integration of technological tools to help learners understand concepts or enhance their understanding of concepts (Powers & Blubaugh, 2005). While there are various studies advocating for the use of GeoGebra in the teaching of mathematical concepts, limited research has been done to explore the use of GeoGebra in the teaching of trigonometric 2D and 3D concepts specifically.

Elsever (2011) conducted a study with the aim of determining the effect of using GeoGebra on students' achievements in trigonometric functions and graphs. After administering pre- and post-test, the findings showed that the application of GeoGebra in the teaching and learning of trigonometric functions did enhance learners' performance in trigonometry. Similar findings were evident in a study by Rahman and Puteh (2017) that explored the use of GeoGebra in the teaching of trigonometry for under-achieving learners. Their findings showed that, after being exposed to the use of GeoGebra, learners were attracted to self-learning progress. therefore having learners that are attracted in self-learning can easily develop conceptual understanding of mathematics because they cannot be force to learn but do it with love. The researcher further argues that good learning material should enhance not only the learners' achievement but also learners' motivation.

Another study was conducted by Makandidze, (2020) that explored Grade 11 learners' understanding of trigonometric functions using GeoGebra and employing group work and collaboration. These were also used by Hertel and Cullen (2011) during the sketching of trigonometric functions using the dynamic geometry capabilities of GeoGebra. The study revealed a statistically significant growth from pre-test to post-test that was attributed to the use of the software.

In the South African and African contexts, researchers such as Bayaga (2019), Mthethwa, (2020) and Mokotjo and Makgalwa (2021) also advocate for the use of GeoGebra. Mokotjo and Makgalwa (2021) extended the argument to include teachers' perspectives on the use of GeoGebra. The findings of their study showed that teachers were enthusiastic about implementing GeoGebra in their teaching as they saw its impact in terms of enhancing learners' understanding of mathematical concepts. Although the findings showed that teachers were enthusiastic, the authors raised the need for professional development for teachers to enable effective teaching using GeoGebra. In this study, the focus is not on teacher development but rather exploring the use of GeoGebra to enhance learners' mental construction of trigonometric 2D and 3D concepts.

Teaching Mathematics for Understanding (2018) has reinforced the importance of incorporating technology into mathematics classrooms in order to enhance students' understanding of mathematical concepts. The proper integration of technology in mathematics education improves mathematics teaching and learning (Dick & Hollebrands, 2011). Accordingly, GeoGebra was employed in this study to improve learners' understanding of 2D and 3D trigonometry.

2.8.3 The integration of GeoGebra in the teaching and learning of 2D and 3D mathematical concepts

In real-life contexts, people are more familiar with 3D shapes than with 2D shapes simply because most of the things in real life exist in 3D form and are concrete: they can be touched and manipulated. It is different for 2D objects drawn on a plane. Also, the drawing and stating of properties and doing calculations is very different for 3D and 2D shapes. In a study by Ismail and Rahman (2017), it was found that learners did better in 2D than in 3D on all levels of Van Hiele's (1986) geometric thinking. This can be attributed to the fact that while mathematics relates to real life, the teaching of it is mainly abstract and learners rely on memorisation of facts which can easily be recalled with 2D shapes. This simply means that when learners memorise facts it doesn't mean they actually comprehend the related concepts. Linchevski (2010) states that learners prefer to visualize material rather than working with text in studies that focussed on classifying shapes. In light of these understandings, this study explored the possibility of assisting learners with visualizing, analysing and making calculations using 2D and 3D concepts by integrating GeoGebra software.

Ismail and Rahman (2017) conducted a study to examine the geometric thinking of young children who worked with GeoGebra to learn 2D and 3D geometry. The geometrical thinking of learners was best elaborated by the Van Hiele (1986) model. In that study, geometry was integrated to support the development of geometric thinking in 2D and 3D geometry. The study was conducted with a sample of 30 learners. A group pre-test and post-test quasi experimental research design was employed. During weeks 1 and 2, learners learned 2D and 3D mathematics in s conventional setting without technology interventions, and then they wrote a pre-test. During week 3, learners experienced learning using GeoGebra for the first time. They explored 2D and 3D shapes using GeoGebra, and then wrote a post-test. The results of the post-test showed that the learners' geometric thinking after the intervention was maintained at the highest level for the visualization of 3D geometry and analysis of 2D geometry. The overall performance of the students showed improvement in geometric thinking after using the GeoGebra software.

These studies show that working with GeoGebra can be more effective in terms of visualization and analysing learners' understanding. This shows that learners perform better when they are engaged in the process of learning rather than being in a passive role while concepts are explained to them. The more learners are involved and allowed to explore as part of their learning, the easier it becomes for them to understand. GeoGebra is one of the teaching tools that allows learners to be engaged and to explore. Visualizing something makes it is easy for learners to understand what they have learnt. For example, Lelinge and Svensson (2020) state that teaching of the geometrical concept is closely related to the ability to discern visual representations in different forms. This is supported by Duval (2006), who claims that several repetitions of representations are required to develop deeper mathematical knowledge. Thus, technology integration should be encouraged in schools and the Department of Basic Education should support this more actively. The success of GeoGebra has also been advocated for by numerous researchers (Ismail & Rahman, 2017; Bayaga et al., 2019; Mthethwa et al., 2020), who argue that working with technology in mathematics is one of the most effective ways to enhance learners' capabilities and also state that working with GeoGebra is sufficient to improve learners' visualization, analysis and informed deduction.

2.8.4 The complexities of using GeoGebra in the teaching and learning of mathematics

It is human nature that when starting something new one can experience fear of the unknown until it becomes familiar, and one regains one's confidence. In this regard, Uddin (2011) notes that learners or teachers without previous programming experience may find difficulty using algebraic commands in GeoGebra, Uddin (2011) further states that some software need specific syntax when commands are to be input; for instance, when entering a quadratic function, GeoGebra requires the user to use a “^” sign and an Asterix “*” for multiplication or product when entering exponents. Mackrell (2012) also notes that, if GeoGebra were used to introduce symbolic algebra, the fact that the software utilizes letters to label geometric objects would be a problem, especially to learners who are not familiar with the software. In light of this, additional training time is needed since mathematical symbols may differ from those learners and teachers are already familiar with. Mackrell (2012) further notes that dynamic numbers must be represented symbolically and that this may create problems with learners who struggle with symbolic algebra.

A class, as a whole, will include learners with different needs and intellectual abilities. Some might become more motivated with the use of GeoGebra, while others might fall behind. Erkek and Isiksa-Bostan (2016) conducted a study with GeoGebra to determine whether GeoGebra was advantageous in the process of argumentation. The results indicated that there were some disadvantages to using GeoGebra in argumentation. All learners were not able to progress at the same pace while using GeoGebra and interpreting the drawings; as a result it was difficult to work with all of the learners together and facilitate group discussions (Erkek & Isiksal-Bostan 2016), having learners lagging behind in teaching and learning makes it more difficult to work with them in discussions. Discussion needs two or more people; if you have one learner that is working at the same pace as the teacher but others are left behind, discussion will not be effective.

The difference in learners’ pace may be even greater when integrating GeoGebra at rural schools. For example, Stols and Kriek (2011), Guven (2012) and Lin et al., (2013) note that in rural schools in South Africa and other developing nations electronic technologies have not been widely affordable and utilized in mathematics classrooms. Moreover, Dikovic (2009) cautions that learners may feel embarrassed or stop trying when they experience difficulty entering algebraic commands in the input box when using the software for the first time; some methodological approaches such as independent exploration and experimentation would not be appropriate for such learners.

This study does not focus on exploring the use of the technical features of GeoGebra to teach, but rather focuses on the use of GeoGebra to enhance learners’ mental construction of 2D and 3D concepts. It was deemed necessary to explore what the literature says about challenges arising with the use of GeoGebra in the teaching of mathematics to ensure that such challenges are addressed before this tool was used in this study.

2.9 Enhancing learners' construction of knowledge in mathematics

Martinez et al., (2005) pointed out that a true understanding of mathematics occurs when learners progress through phases of action (physical and mental), abstraction (in which actions become mentally entrenched so that they can be analysed and implemented), and reflection (the deliberate process by which one evaluates their actions). Throughout these phases, learners develop increasingly sophisticated mental models of the abstraction (Martínez et al., 2005). Learners must interact with their environment in a variety of ways to develop a working understanding of mathematics concepts and to explore, manipulate, compare, arrange, and rearrange real objects to develop their innate logical sense (Mudaly, 2016). Construction of mathematical knowledge therefore entails developing a good grasp of mathematical concepts. This requires more than just knowledge of facts. Thus, the development of learners' concepts requires the use of innovative tools and teaching strategies. In this context, this study aimed to explore the use of GeoGebra to enhance learners' construction of knowledge of trigonometric 2D and 3D concepts.

2.9.1 Conceptual understanding of mathematics

A conceptual understanding of mathematics involves understanding the foundational concepts that underlie the algorithms performed in mathematics (Andamon & Tan, 2018).According to Andamon and Tan (2018), conceptual understanding of mathematics is a critical component of expertise. Andomon and Tan (2018) conducted a study investigating the conceptual understanding and attitude of learners in mathematics. They found that the level of conceptual understanding demonstrate the skills transferring knowledge of the students and the complete understanding in mathematics language. Simply because the resource linked the previous learning with the new learning, therefore the knowledge was transferred such that the knowledge was extended in one context to another context.

Conceptual understanding in mathematics is one of the most important things a learner needs to have as a foundation. Learners without conceptual understanding understand isolated facts and methods. Without conceptual understanding the learner is more likely to struggle when a problem is presented in a different way than it was presented in class. For example, a learner who memorised the properties of an isosceles triangle as 'base angles are equal', due to be influenced by the normal classroom traditions of representing the triangle as shown in the figure below, when the angle opposite to side *w* is given, may find it easy to determine that it will be equal to the angle opposite to side *c*; but when the angle of the opposite side a, is given, they may experience difficulty finding the base angles.

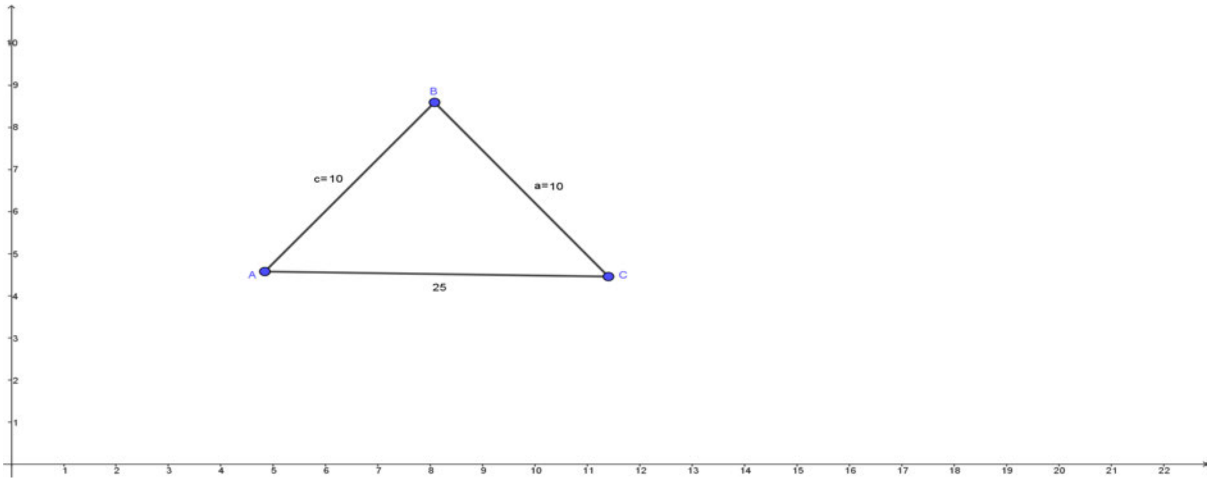


Figure 2.2: Isosceles triangle

A lack of conceptual understanding results in learners not being able to reason mathematically (Ndlovu and Brijlall, 2015); as a result, it can be common to hear learners say that they understand the mathematics when taught in the classroom, but when they sit for examinations, they do not understand it. This is because they lack the conceptual understanding to be able to apply it to a new context. In many cases, learners have the innate ability to be successful in mathematics yet believe that they are incapable of success because they have encountered obstacles to developing their conceptual understanding of mathematics (Beghetto & Baxter, 2012; Tambychik & Meerah, 2010). A diagnostic analysis conducted by the South African Department of Basic Education (DBE, 2019) found that the reason learners did not perform well during the examinations was that they lacked conceptual understanding. This means that the teaching they experienced did not facilitate the necessary construction of knowledge that is required for the development of conceptual understanding.

In this context, this study aimed to explore the use of GeoGebra to enhance learners' understanding of 2D and 3D trigonometric concepts with the hope of developing learners' conceptual understanding of the taught concepts. Development of conceptual understanding supersedes the memorisation of facts. The Principles and Standards for School Mathematics (PSSM), in United State (National Council of Teachers of Mathematics, 2000, cited in Bosse

& Bahr, 2008) argues that learners who memorise facts or procedures without understanding often are not sure when and how to use what they know, and such learning is often quite fragile. Bosse and Bahr (2008) further argue that if learners learn mathematical concepts before they learn procedures, they are more likely to understanding the procedures when they learn them; however, if they learn procedure first, they are less likely to ever learn the concepts underlying them.

Having a sound understanding of concepts, content and processes is vital for making sense of mathematics; therefore, learners need to first make sense of the concepts before engaging in problem-solving. According to Francisco (2013), providing justifications backed by mathematical reasoning is one of the key aspects of making sense of mathematics. Bosse and Bahr (2008), explain conceptual understanding in contrast to procedural knowledge is adaptable, adjustable, transferable and applicable to other situations and is born of development connections that make mathematics sensible and meaningful. They argue that such understanding is the essence of mathematical thinking and only true kind of understanding that leads students to see a bigger picture, rather than relying on memorization of facts.

This implies that teachers must develop concepts with learners to enable them to build their own mental constructions; this makes mathematics more understanding and meaningful to learners. Similarly, in the teaching and learning of 2D and 3D concepts, enabling learners' construction of fundamentals is critical before engaging in problem solving to build the conceptual understanding of the learner. Therefore, to enhance learners' conceptualisation of difficult mathematics concepts, it is important that innovative approaches are included. It is within these parameters that this study explored learners' conceptions of 2D and 3D aspects of trigonometry using GeoGebra to enhance their mental understanding of the concepts.

2.9.2 Complexities that hinder learners' construction of mathematics concepts

During the process of teaching and learning mathematics, learners typically face many obstacles because problem solving in mathematics is a complex skill; even when they know how to answer the question, a careless error in computation may result in an incorrect solution (Ibrahim & Mat, 2010).

Ibrahim and Mat (2010) stated that the mastery of basic mathematical knowledge is essential in concept development. Without a clear understanding of basic concepts and skills learned during the early stages, the learning process will become more difficult at the next stage.

Learners generally make errors when solving mathematically concepts due to not having constructed concepts fully at an earlier stage of learning. Tall (2008) refers to these previously taught concepts as ‘met before’.

Studies conducted by Ndlovu (2014) and Ndlovu and Brijalall (2015) mention a number of factors that hinder construction of knowledge in the learning of mathematical concepts. These include incorrect use of mathematical language and mathematical notation as a result of not understanding its meaning. Notation in mathematics is critically important: for example, 30° is not the same as 30cm; when a learner calculates an angle and leave the answer as 30, the meaning is different than 30°. However, with poor conceptualisation of concepts, learners tend to use notation or mathematical language loosely, thus hindering knowledge construction of the concept.

Prakitipong and Nakamura (2006) pointed out that in the process of problem solving there are two kinds of obstacles that hinder learners from constructing the necessary knowledge to solve the problem. The first one refers to the problems in linguistic fluency and conceptual understanding that correspond with the level of simple reading and understanding meaning of problems. The second obstacle is the problem in mathematical processing that consists of transformation, process skills and encoding answers. Learners must interpret the meaning of the question before they proceed to mathematical processing to obtain a solution. In the process of teaching and learning, teachers play a critical role in the process of learners’ acquisition of conceptual understanding. The barriers to knowledge construction are exacerbated by the misconceptions that learners might have constructed about a concept.

Yasin (2017) defines ‘misconception’ as a way of misunderstanding the concept. Malambo (2021) purports that a misconception may merely be implied in one’s written or verbalized reasoning, thereby requiring close analysis of the written text or verbalized views for detection. Therefore, to address misconceptions it is important to understand how misconceptions manifest. However, to enhance learners’ construction of knowledge it is imperative to understand the concepts that learners are struggling to comprehend and thus explore ways to enhance their understanding. The researcher also noted that learners were performing poorly with trigonometric concepts from the diagnostic reports (DBE) and in her teaching experience, and particularly with the 2D and 3D concepts; the study thus aimed to explore the use of GeoGebra to enhance learners’ mental construction of the concept

2.9.3 Learners’ mental construction of trigonometric concepts

Individuals build their mathematical knowledge in different ways, but the whole thought process is cognitive (Ndlovu, 2014). Hence, learners construct new knowledge for understanding from the existing knowledge they have. Hiebert and Carpenter (1992) argue that learning mathematics with understanding involves making connections between ideas, and these connections are seen as facilitating the transfer of prior knowledge to new situations. Hence, Jojo (2011) claims that mathematics is understood when its mental representations are part of a representational network. Therefore, there should be connection of concepts through prior knowledge and new knowledge for the mental construction of mathematics concepts. An understanding of a trigonometry concept is based on the development of schema relevant to it; thus, solving mathematics problems requires taking examples, cognitively reconstructing them, and representing them in a mathematically meaningful way (Ngcobo et al., 2019). This requires the teacher to know the capabilities of the learners they are working with so that they can choose appropriate teaching strategies to facilitate learners' constructing a mathematical understanding of the concept. As trigonometry is one of the most difficult mathematics topics, where learners often struggle to construct understanding, it is worth exploring the integration of technologies that may aid construction of concepts into teaching and learning (Atan et al., 2010; Shadaan, 2013; Bayaga et al., 2019; Mthethwa, 2020; Tamam et al., 2021). On this basis, this study explored the use of GeoGebra to facilitate learners' construction of 2D and 3D trigonometry concepts.

A number of studies have explored learners' or undergraduate students' mental constructions of mathematical concepts; however, the majority of these have explored the mental construction of algebra and calculus concepts (e.g. Trigueros et al., 2010; Ndlovu, 2013; Maharaj, 2014, Ndlovu & Brijalall, 2015; Kadzunga et al., 2017; Chagwiza, 2019; Vilakazi, 2021). While the aforementioned studies explored learners' mental constructions of mathematical concepts, none engaged with trigonometric concepts. A review of the literature suggests that this critical area of mathematics where many learners struggle has not being researched extensively. The researcher came across four studies exploring learners' mental construction of trigonometric concepts (i.e. Martines & Castro 2013; Nabie et al., 2018; Ngcobo et al., 2019; Madonsela et al., 2020).

Ngcobo et al., (2019) conducted a study with 30 Grade 12 learners at a school in South Africa that explored the mental constructions made when solving for the unknown properties of triangles in trigonometry. The researchers used the genetic decomposition to reveal the mental constructions that learners had made or failed to make and to discuss the impact of the analysis in relation to the development of the concepts assessed. They found that the types of mental structures demonstrated by the learners, suggested that schema development is a necessary prerequisite for Grade 12 learners to develop the conceptual understanding required for solving problems involving triangles. It was further evident that having the genetic decomposition for problems involving 2D shapes as a data analytic tool assists immensely with explaining the mental constructions made and also helps in understanding the difficulties learners have with a concept. It was further revealed that learners' mental constructions of the solution of triangles were generally at the action stage, in terms of APOS theory.

In a study with undergraduate students at a mid-size public university in Puerto Rico, Martinez and Castro (2013) investigating students' mental construction of the sine and cosine functions and their inverses. They observed that students showed difficulty in construction and using an inverse trigonometric function. The 11 students interviewed failed at one point or another to construct and apply an inverse trigonometric function, thus showing a lack of having made the necessary mental constructions predicted in the genetic decomposition.

A similar study conducted by Nabie et al., (2018) in the Northern Region of Ghana but only focusing on pre-service teachers' perceptions and knowledge of trigonometric concepts revealed that the pre-services teachers had not made the necessary mental constructions of trigonometric concepts. The findings suggested that pre-service teachers find learning of trigonometry concepts to be abstract, difficult, and tedious and had limited conceptual knowledge of basic trigonometry concepts. As a result, more than 50% of them were unable to construct and reconstruct the appropriate mental structures for meaningful understanding and to solve problems.

A study conducted by Madonsela et al., (2020) revealed the alignment between the genetic decomposition and learners' mental constructions of three-dimensional trigonometric concepts. However, it further revealed that learners' constructions had only progressed to the action stage, meaning that learners depended on working out the solution explicitly, step-by-step, to solve the problem indicating that the learners' knowledge was grounded in rules rather than on conceptual understanding.

While these studies explored learners' mental constructions of trigonometric concepts and revealed the difficulties that learners and undergraduate students experienced in this regard, none of these studies explored how learners' mental constructions could be enhanced, which was an aim of the current study.

2.10 Conclusion

This chapter has reviewed the scholarly literature about the use of GeoGebra to enhance learners' mental construction of trigonometric 2D and 3D concepts. The literature throws light on the need for educators to modify their pedagogy for the teaching of trigonometric 2D and 3D concepts. The literature confirms that GeoGebra is a tool that has the potential to motivate learners and improve their understanding of mathematics concepts. Technological tools like GeoGebra that are visual may help learners to visualize or create mental pictures of trigonometric 2D and 3D shapes so that their engagement with abstract descriptions of shapes can be backed up by meaningful interpretations from those visuals. The beliefs of learners and teachers, which are essential to the uptake of new technological tools, were discussed. Moreover, while GeoGebra and other technological tools have great potential for benefit, complexities in the context in which they are implemented may cause them to hinder the construction of knowledge in teaching. These challenges have been discussed in this chapter. The next chapter presents the theoretical framework adopted for this study.

CHAPTER 3: THEORETICAL FRAMEWORK

3.1. Introduction

This study explored the use of GeoGebra to enhance learners' understanding of trigonometry of 2D and 3D concepts. In the previous chapter, the literature relevant to this study was presented and discussed. The theoretical framework that guided this study is APOS theory, which is described in detail in this chapter. APOS theory is a framework for understanding learning processes, in particular the learning of complex mathematical concepts (Wayer, 2010). A key component of the framework is the theoretical perspective that describes all mathematical conceptions as actions, processes, objects and schemas.

This study used this theoretical framework (APOS), which is aligned with the ACE (action, classroom discussion and exercises) teaching style, to explore and enhance learners' knowledge construction of trigonometry of 2D and 3D concepts using GeoGebra. As a result, the APOS stages played a major role in the genetic decomposition goosed to analyse learners' written work. Moreover, a genetic decomposition refers to the structured set of mental constructs which might describe how the concept can develop in the mind of an individual (Brijlall, Jojo & Maharaj, 2013). The genetic decomposition designed for this study is discussed in detail in this chapter. The researcher designed it based on the understanding of the concept of 2D and 3D to track the development of the necessary mental structures in the concept of 2D and 3D trigonometry before and after the use of GeoGebra.

The aim of the study was to explore the use of GeoGebra to enhance learners' understanding of trigonometry in 2D and 3D. APOS emphasises that the teaching of mathematics should help learners construct mental structures and build new structures as they engage with increasingly difficult problems (Chamberlian & Vidakovic, 2015). For example, when teaching mathematical concepts with the aim of guiding learners to learn how to construct mental representations to solve difficult problems, APOS theory identifies key elements in how learning takes place. Therefore, APOS theory and the ACE teaching style were used in the study as a framework for research in mathematics education, as advocated by Asiala et al. (2007).

In addition, as per APOS theory, mathematical understanding involves the creation of mental structures through sequences of actions and processes on mathematical objects, leading to mathematical schemas. APOS theory, with the use of GeoGebra in teaching 2D and 3D trigonometry, created a powerful learning environment. Learners were actively involved in actions and processes, manipulating mathematical objects in an interactive way, leading to the construction of strong mathematical schemas. This approach promotes a deeper and more meaningful understanding of trigonometry. The construction of mathematical knowledge is discussed in detail in the next section.

3.2. Mathematical knowledge and its construction

Teachers play a significant role in facilitating learners' conceptual development in mathematics through a range of pedagogical approaches. This is supported by Bingimlas (2009), who stated that several studies indicate that majority of teachers have competence and confidence in using computers in the classroom, as their teaching strategy to improve mathematics performance and understanding. However, other studies (e.g Bansilal et al., 2014; Pournara et al., 2015) questioned the competence and confidence of teachers in utilising the relevant instructional strategies such as technological tools like GeoGebra to improve learners' understanding. This simply implies that if a teacher is not competent enough about the content and a good selection of the instructional method to be used for learning, it is likely that learners will not understand the concepts

In light of this, the researcher wondered if the use of GeoGebra in the teaching and learning of 2D and 3D trigonometry could be facilitated using the ACE teaching style to help learners understand these concepts. Furthermore, Ndlovu (2014) argued that the level of knowledge construction achieved by learners in learning mathematics can be explained by their mental constructions, but Gama (2015) argued that learners' conceptions of mathematics are shaped by the nature of mathematics, the character of mathematical activities, and the essence of mathematics education. As a result, it is very significant that a teacher develop activities that will help learners develop their mental structures and teaching strategies that also help learners understand the concept clearly, as mathematics is seen as the development of the rational and logical mind.

Based on Douglas (1996, p.12), learners construct knowledge of mathematics based on three major themes: composition, structure, and status. A mathematical knowledge composition includes mathematical knowledge representations, formulas and algorithms. In this conception, Reid et al., (2003) argue that students view mathematical activities like numbers and calculations as basic arithmetic, without making any important advances. A mathematical knowledge structure refers to the understanding that mathematical knowledge is either interconnected or is composed of concepts that differ from each other. In the latter case, mathematics is regarded as a collection of non-related methods and applications with no practical application (Reid et al., 2003). Depending on how mathematics is understood, it may either be viewed as rigid or as dynamic. From a view of mathematics as rigid, students view mathematics as a subject governed by rules and laws like physics that does not allow for creativity,

In light of this, it is apparent that the understanding of mathematics that teachers convey to learners will impact the way learners construct mathematical knowledge. For example, presenting mathematics as a body of facts leads to learners constructing procedures without understand their meaning, such as understanding the conceptual knowledge. In addition, when a teacher presents mathematics as a collection of unrelated content, it will result in learners constructing knowledge outside its domain thus making no connection to real life. For example, for 2D and 3D concepts, learners could learn the sine rule and area rule etc but fail to make the connection that these can be used to determine the angles and areas of triangles. In Douglas' (1996) view, how a concept is presented to a learner determines how knowledge is constructed in their minds. On this basis, I argue that by modifying instructional strategies learners' knowledge construction can be enhanced. This study explored the enhancement of learners' mental construction through integrating the use of GeoGebra in the teaching of 2D and 3D trigonometric concepts.

3.3 Theories of understanding

There have been several theories that attempt to explain how a mathematical concept is conceived in the mind of an individual, including the APOS theory used in this study. The theory of understanding presented by Skemp (1987) identifies three types of understanding: instructional understanding, relational understanding and formal understanding. A person with instrumental understanding is capable of applying a memorized rule to solve a problem without understanding why it works. Using mathematical relations as a basis for deducing specific rules or procedures is considered relational understanding. To have formal understanding, one must be able to connect mathematical symbolism with relevant mathematical ideas and to combine these ideas into chains of logical reasoning.

Additionally, the theories of understanding that emphasise that understanding develop as a result of making connections between concepts. In this study, the researcher adopts understanding of mathematical concepts as discussed within the APOS theory emphasises understanding what might happen in the individual mind when they are learning a concept in 2D and 3D of trigonometry, i.e. how does one come to make the connection between concepts as alluded in the theories above.

Moreover a person's understanding of something can only be achieved by assimilation into a schema, according to Skemp (1987). According to him, a schema consists of multiple connected concepts constructed by abstracting invariant properties from sensory motor input or from another concept. Concepts are then linked by transformations or relations. In addition, the more schemas we have, the better our chances of coping with the unexpected. This implies that understanding of the mathematics concepts is much better to learners that have different ways of solving mathematics problems; so that, when encountering a difficult question they are better able to switch and make connections between concepts and methods to solve the problem.

The more knowledge and experience a person has of a concept, the better they will be able to deal with subsequent concepts they encounter in mathematics. Hiebert and Carpenter (1992) argue that to learn mathematics with understanding, one must connect ideas; these connections facilitate the transfer of prior knowledge to novel situations. Therefore, when teachers are teaching a new concept for understanding to learners, they should consider the learners' prior knowledge as the foundation on which they will build new knowledge, because learners cannot reconstruct schemas if they encounter situations for which their existing schemas are not appropriate (Skemp, 1987).

While not disputing Skemp's theory, Wiggins (1993) offers a different perspective. He defines understanding as a cyclic process where one reflects on the taught concepts and critiques their elements; this leads to rethinking and relearning the whole process. It is based on this view that Pirie and Kieren (1994) believe that mathematical comprehension is a dynamic process that occurs within a person, within a topic, and within a particular environment, where individuals constantly organize their own knowledge structures rather than acquiring static categories of knowledge. Hence, understanding is not static but dynamic, changing as the individual interacts with the world around them or a new environment they encounter. For example, a learner's mental constructions of understanding can be influenced differently by different strategies employed by teachers. Kastberg (2002) argues that mathematics teachers know that students' understanding of mathematical concepts changes over time. As learners learn and develop their beliefs both inside and outside of the classroom, it is not always clear how their understanding will change their mental structures for better or worse as they engage with different teaching strategies. Consequently, Pirie and Kieren (1994) suggest that understanding evolves in an orderly manner through processes involving different levels of abstraction. Therefore this study aimed to explore the use of GeoGebra a novel teaching strategy that departs from traditional strategies to enhance learners' understanding of 2D and 3D concepts in trigonometry.

Furthermore on understanding, Wiggins and McTighe (1998) propose that understanding consists of several interrelated components, namely: explanation, interpretation, contextual application, perspective, empathy and self-knowledge. While each individual is unique, and their conceptual development is grounded in their lived experience in a particular environment, these aspects work together in the development of an individual understands of the concept.

Similarly, teaching and learning without building a strong conceptual foundation leads to learners memorizing procedure and problem types; when they encounter new questions, they may be unable to apply concepts to a new situation. A learner's logical understanding is what enables them to communicate mathematically and be understood by others, according to Skemp (1987). Therefore a learner who shows logical understanding when solving mathematics problems is likely to be conceptualising the problems. A learner who can explain his/her ideas and justify their understanding by making connections and inferences can also deal with complex mathematics problems by applying what they have learned.

Furthermore, Skemp (1987) found that the relationship between procedural knowledge and problem types degrades rapidly, leaving the learner being able to match the problem with the concept (Kastberg, 2002). This means that when teaching is instrumental it serves the short-term process of enabling the learner to solve a particular problem, but does not assist the learner in building a deeper understanding of the concept. A diagnostic analysis conducted by the South African Department of Basic Education (2019) found that teachers were using previous exam papers to teach rather than teaching learners to understand concepts; as a result, learners were failing to make connections between concepts and were unable to generalise their knowledge to solve similar problems.

The APOS theory is discussed in the next section.

3.4. APOS theory

APOS theory is a constructivist theory of how mathematical concepts are learned. It was developed by David Tall (2008), a British mathematician and mathematics educator. The theory aims to describe the mental processes and structures that individuals use when learning and understanding mathematical concepts. Furthermore, Dubinsky and McDonald (2001) stated that the APOS theory is derived from the hypothesis that mathematics develops through four phases, characterized by the progressive development of distinct mental structures: actions, processes, objects, and schemas (APOS).

The APOS theory was selected as the theoretical foundation of this study since it describes the learning process of complex mathematical concepts (Weyer, 2010). In this study, APOS was used to analyse the mental constructions of learners made when learning 2D and 3D of trigonometry. It also informed the design of the activities used in the study to facilitate learners' mental construction of trigonometry concepts and schema.

APOS provides a theoretical basis that contributes to understanding how learners think or construct mental structures necessary for mathematics concepts. As per APOS theory (Maharaj, 2010), if a student understands a mathematical concept, he or she has a corresponding mental structure. In accordance with this theory, learning activities can be designed to help students construct a mental structure to help them comprehend the concept (Dubinsky & McDonald 2001). According to Asiala et al. (2004), understanding a mathematical concept begins with the manipulation of previously constructed mental or physical objects; actions are then interiorised to form processes, and processes are encapsulated to create objects. After a student understands a mathematical concept, they reflect on the processes they used to construct that knowledge and use those processes to construct new knowledge (Dubinsky, 1991).

APOS theory proposes that individuals must have an appropriate mental structure based in actions, processes, objects and schemas to understand specific mathematical concepts (Ndlovu, 2014). Therefore it is significant that teachers know the abilities of the learners they are working with in terms of the prior knowledge before introducing new concepts, to help them determine which strategies to use to support learners' construction of the necessary mental structures. Therefore, APOS was deemed suitable to analyse learners' mental construction of 2D and 3D trigonometry in this study.

3.4.1. Suitability of APOS theory for this study

It is evident in the literature that mathematics is a particularly challenging subject due to its complexity. In APOS theory, Dubinsky and McDonald (2001) suggest that an individual must develop appropriate mental structures to understand mathematical concepts. In addition, Dubinsky and McDonald (2001) state that research based on this theory indicates that it is important to identify the mental structures required for the understanding of a concept so that suitable learning activities can be designed that support the development of these mental structures.

As the aim of this study was to describe the nature of the learners' mental constructions and enhance these where necessary, and the APOS theory of learning has been found effective for identifying and explaining the nature of individuals' mental structures, APOS was selected as the most appropriate theoretical framework for the study.

In this study, learners' mental constructions for trigonometry of 2D and 3D concepts were identified using APOS theory, and then learning activities were designed and implemented employing GeoGebra software and the APOS-aligned ACE teaching style to enable learners to build on their prior mental constructions to construct and enhance understanding of these concepts. Participants' responses were analysed using the APOS stages, which are discussed in the next section.

3.4.2 Stages of development of mental structures identified in APOS theory

APOS theory understands mathematical knowledge as being constructed in four distinct stages, characterised by action, process, object and schema (APOS).

3.4.2.1 Action stage

Arnon et al. (2014) contend that concepts first emerge as externally directed transformations of previously conceived objects. An action is a physical or mental manipulation that transforms an object by responding to an external cue. In the action stage, the learner must not only be able to explicitly perform each step of the transformation, but also follow specific instructions or rules. At this stage, the learner thinks through the problem step by step and sees only one step at a time (Vilakazi, 2021).

An example of working at the action stage in 2D and 3D trigonometry can be found on the activity sheet used for the pre- and post-tests in this study (Appendix A), where the learner is asked the question *" What is your understanding of a cyclic quad?"*. In a question of this kind, the learner is required to recall basic knowledge that was taught in the previous grade. The figure that is displayed is intended to provide an external cue to trigger the learner's understanding of a cyclic quadrilateral. A learner in Grade 12 is expected to know how to explain theorems and solve problems related to cyclic quadrilaterals. Being able to recall assists the learner to invoke different rules and select the correct rule to be used in the subsequent questions.

3.4.2.2. Process stage

Mental construction, progresses from the 'action' stage to the 'process' stage. In this stage, the individual performs the same action as in the action stage, but entirely in their minds (Dubinsky & McDonald, 2001). The learners can visualize performing transformations in their minds without having to explicitly perform each step (Chagwiza, 2019). Vilakazi (2021) claims that the defining characteristic of a process is that it enable the individual to describe, or reflect upon, the steps of the transformation without having to write them down. When a mental process is constructed, the learner can also visualise it in reverse, possibly creating a new process that is the reversal of the original process.

During his study of sequences and series, Chagwiza (2019) found that students working at the process level could describe changes in basic sequences or series as a result of applying a transformation without performing each step of the transformation. A key characteristic of the process level is the student's control over the object's transformation. Chagwiza (2019) stated that the interiorization of action into a process occurs as the learner reflects on it. In the process of learning, the student becomes able to observe the behaviour of the functional values of a variable over an interval without having to evaluate the sequence or series for explicit values.

An example of a problem targeting the process stage in this study is Question 1.2 of the test, where the learner is asked to explain how they would find the area of triangle DAB without solving the problem. This question requires the learner to work at the process stage as, at this stage, the learner can describe or reflect upon the steps of transformation without performing the steps (Arnon et al., 2014).

3.4.2.3 Object stage

The learners learn that transformations act on totality, and they can construct such transformations explicitly or in their imaginations as they become aware of a process as a whole. It is at this point that the process is encapsulated into a cognitive object (Arnon et al., 2014). In addition, as the learner reflects on all the processes, and begins to think of them as a whole in representative contexts and has the flexibility to work in a variety of representative contexts, they encapsulate the processes, converting them into an object. At this point, the student can reflect on the operations applied to a particular process, realise those transformations and also construct such transformations.

An example of a problem that requires the learner to work at the object is Question 1.5 of the test, where a learner is asked to calculate the area of triangle DAB. A learner should apply the area rule for 3D shapes, not 2D shapes. The action of calculating the area of a 2D triangle has been encapsulated in totality to apply to the area of a 3D shape using the area rule.

3.4.2.4 Schema stage

An individual's actions, processes and objects gradually become interconnected in their mind, forming a coherent framework called a schema. Asiala et al., (2004) state that a schema for a mathematical concept is an individual‘s collection of actions, processes, objects and other schemas linked consciously or unconsciously in a coherent framework in the individual's mind. Possani et al. (2010) argue that the formation of new connections between new and previous actions, processes, objects and schemas leads to the formation of new schemas that are a collection of cognitive objects, their connections and their internals process that operates these processes. Specifically, an individual's schema for a given mathematical concept is their collection of actions, processes, objects and other schemas, linked together by some general principle or relationship to form a framework that may be brought to bear upon a problem situation involving that concept (Arnon et al., 2014).

The construction of a schema allows a learner to take a more creative and active role in solving a mathematical problem as they are able to interpret more complex information and negotiate a series of processing steps (Jojo, 2011). A learner who has constructed a schema is able to formulate the problem and often develop a suitable model that facilitates its solution; they are able to identify and apply relevant tools and knowledge in unfamiliar problem situations (Vilakazi, 2021). In this study, learners with completed schema for 2D trigonometry concepts would be able to solve 3D trigonometry problems.

3.4.3. Mental mechanisms used in APOS stages

The mental structures described in APOS theory (action, process, object and schema) are developed progressively: the construction of one mental structure is dependent on the construction of the previous one. Dubinsky (1991) describes five mental mechanisms that lead to the construction of the mental structures: interiorisation, encapsulation, coordination, reversal and generalization. Other mental mechanisms are discussed in the next section.

3.4.3.1 Interiorisation

Interiorisation refers to a learner moving beyond a reliance on external cues to solve a problem. An individual can solve mathematical problems by building internal processes without going through the step-by-step process of understanding perceived phenomena. According to Sfard (1991), during the interiorisation stage (which she terms 'internalisation'), learners become familiar enough with the process that they can execute it through mental representation, for example, in 2D and 3D trigonometry. A learner at the action stage will be able to use sine or cosine rules when required to find the length of triangle ABC by recalling the conditions to choose the correct rule and reflecting on the process of mental manipulation. Once the mental structures of the learner enable them to use the rule to solve the problem without performing all the steps for solving the problem, the learner has interiorised the action into a process. Or, when provided with a diagram of a cyclic quad, a learner would be able to provide a coherent definition of a cyclic quad, demonstrating that they had interiorised the action into a process. Hence, interiorisation allows one to become conscious of an action, reflect on it and combine it with other actions to form new ones (Dubinsky, 1991).

3.4.3.2 Encapsulation

Encapsulation refers to the ability to apply actions to processes and view them as one process. Here, the construction of mathematical understanding is extended from one level to another, and new forms of processes derived from the previous ones are constructed to form objects (Jojo, 2011). This reflects Sfard's notion of 'reification', which involves the ability of the learner to imagine the outcome of a process in their mind's eye as a distinct, enduring entity (Sfard & Lynchevsky, 1994).

In this study, Questions 2.2 and 2.3 of the test required learners to determine the values of x for which the area of triangle PQR would be maximized and calculate the length PR if the area is maximum. These questions integrate trigonometry concepts with calculus concepts. For a learner to answer these questions, they must be very proficient at the process stage and have built mental structures that encapsulate these processes into objects (object stage). A learner who uses an area rule of trigonometry, and can also see that for the question to be answered the calculus concept of derivatives must be used to find length when the area is maximized, has constructed the process as a totality. Dubinsky et al. (2005) argues that, once a learner becomes aware of the process as a totality and realises that a transformation can act on that totality and that they can actually construct such a transformation, they have encapsulated the process into an object. Moreover, once a mathematical concept is well understood, students reflect on the processes used to construct that mathematical knowledge and use those processes to construct new forms of knowledge (Ndlovu, 2014).

3.4.3.3 Coordination

Coordination occurs when two or more processes are coordinated to form a new process (Jojo, 2011). An example of coordination in this study was Question 1.4 of the test, where learners were given two joint scalene triangles that share one side, and were asked to find the angle of the second triangle in order to find length AD, where the information given for the second triangle is incomplete. To find the angle, the learner must first find the common side using the triangle DCB, which gives the learner enough information to find the angle in the first triangle. Arnon et al., (2014) argue that, to compose two functions f and g to obtain *(f)(g),* the two function objects must be de-encapsulated to the processes that gave rise to them; these processes are then coordinated by applying the process of f to the elements obtained by applying the process of g.

3.4.3.4 Reversal

Reversal is the ability to reverse the thought processes of previous internalized processes (Mutambara, 2018). Chagwiza (2019) states that a process that has been encapsulated forms a mental object that, if needed, can be de-encapsulated back to its underlying process.

In this study of 2D and 3D trigonometry, de-encapsulation took place in shapes, from 3D to 2D, for example the 3D shape can be reverse to a 2D shape when, for instance, the four triangles of a triangular pyramid are flattened into 2D net; when the net is assembled, it becomes a 3D shape as shown if figure 2.1a and 2.1b in chapter 2.

3.4.3.5 Generalisation

Generalization is the ability to apply existing schemas to a wider range of contexts (Mutambara, 2018). In this study, an example of generalization is in question 1 item 5, when the learner is required to calculate the area of triangle DAB; when the lengths AB and AD form angle A, an included angle, they are able to see that they have to use an area rule to find the area of triangle DAB.

All of these mechanisms need to develop, or be constructed, in the mind of a learner to learn mathematics concepts. Their responses can then be analysed using genetic decomposition assisted by APOS stages. Genetic decomposition is discussed next.

3.5. Genetic decomposition

Genetic decomposition is a hypothetical model that describes the mental structures and mechanisms that a student may need to construct in order to learn a particular mathematical concept (Arnon et al., 2014). According to Maharaj (2010), a genetic decomposition of a concept is a structured set of mental constructs that attempts to describe how the concept might develop in the mind of an individual. Furthermore, a genetic decomposition postulates the particular actions, processes and objects that play a role in the construction of a mental schema for dealing with a given mathematical situation (Arnon et al., 2014). Arnon et al., (2014) also describe how concepts are constructed mentally.

Dubinsky (2001) states that the purpose of theoretical analysis is to plan the specific mental constructions that are needed for a student to make the necessary mental construction and this gives rise to the construction of a genetic decomposition of the concept. This is helpful for monitoring learning progress and can guide lesson designs. Furthermore genetic decomposition consists of a description of the actions that the student must perform on existing mental objects, and a description of how those actions are internalized into processes (Arnon et al., 2014). Genetic decomposition can include descriptions of premise structures that individuals must build up beforehand.

Possani et al. (2010) argue that, if a student fails to perform the necessary mental structures required for genetic decomposition, this will lead to revision and modification of the genetic decomposition or educational strategy. In this study, the mental structures and mechanisms that Grade 12 learners need to develop in order to learn trigonometric 2D and 3D concepts are described. The genetic decomposition for this study is presented below in the next sub-section.

3.5.1. Designing genetic decomposition

Arnon et al. (2014) further state that genetic decomposition does not explain the processes that occur in the individual's mind, or provide an exclusively theoretical analysis of how mathematics is learned; rather, it predicts whether a person will use a particular structure when asked to do so. While it does not explain what is going on in the mind, it is a diagnostic tool that describes how the concept was constructed in the mind of an individual (Brijlall & Ndlovu, 2013).

This study explored the use of Geogebra to enhance learners‘ mental construction of trigonometric 2D and 3D concepts with the use of Geogebra. Therefore, this study provides a tool and appropriate activities to enhance the learners‘ understanding of these concepts. The structured pre- and post-test was designed to assist learners to make the mental structures required to enable their understanding of 2D and 3D trigonometry.

For a learner to be considered as understanding the concept of 2D and 3D trigonometry: A genetic decomposition of 2D and 3D trigonometry concepts was constructed as follows

Action stage

Given a circle with a quad inside, the learner performs the action of conjoining the figure to describe a cyclic quad. The action of constructing a four-sided figure inside the circle with all vertexes touching the circumference leads to the construction of the cyclic quad. Below is the description of the necessary mental structures that a learner at the action stage should make.

- The action of naming the parts of the circle and parts of the quad is conceived as using external cues;
- The action of performing step-by-step calculations to determine the solution;
- The action to recall and invoke the conditions and rules of sine and cosine;
- The action to calculate the sum of angles in a triangle.

Process stage

A learner performing the same actions performed at the action stage, but solve the problem without explicitly writing down all the steps has interiorised the actions into a process and is thus operating at the process stage.

The following is a description of the mental structures that a learner at the process stage is required to make for 2D and 3D trigonometric concepts.

- The learner reflects and begins to think about the properties of all cyclic quads without the need to construct the diagram;
- The learner reflects on the given formula and substitutes values without the need to transform the equation to determine the subject of the formula;
- The action of performing step-by-step calculations is interiorised and some steps are skipped when solving 2D and 3D problems;
- The application of trig ratios to solve a problem in 2D and 3D.

Object stage

A learner at the object stage can reflect on the operation applied at the process stage and become aware of the process in general, invoking different concepts within the subject matter. The mental structures that a learner at the process stage is required to make for concept in 2D and 3D trigonometry are as follows:

- Integrate other mathematical concepts within the subject, is used to process stage in totality and encapsulate to object stage;
- consider more than one shape to define the area formula in 3D;
- Apply the necessary actions and processes to de-encapsulate a 3D shape to 2D and apply necessary formulas to determine the area, cosine and sine rules.

Described are all necessary mental structures required for a learner to make in order to understand 2D and 3D of trigonometry using the theoretical framework that will be discussed in the next section below.

3.6. Research framework

A theoretical framework is a structure included in or supported by research studies that presents and explains theories that explain why the research problem under study exists (USC Libraries, 2014). This research references APOS theory as its theoretical framework. APOS theory emphasizes the cognitive growth of individuals as they construct the knowledge necessary to learn mathematical concepts. As proposed by (Asiala et al., 2004), this framework comprises three components: theoretical analysis, instructional treatment, and observations and assessment of student learning. In theoretical analysis, statements can be made about mental structures that can be used to study math concepts. These constructions can be fostered by instructional treatment. Evaluation and observation seek to determine if the structure is being constructed and how much the student has actually learned (Asiala et al., 1997).

Figure 3.1 below shows the relationship among these three components of the framework. Dubinsky (2004) advocates that research based on APOS be organised using the three key components in a research cycle, as illustrated in Figure 3.1.

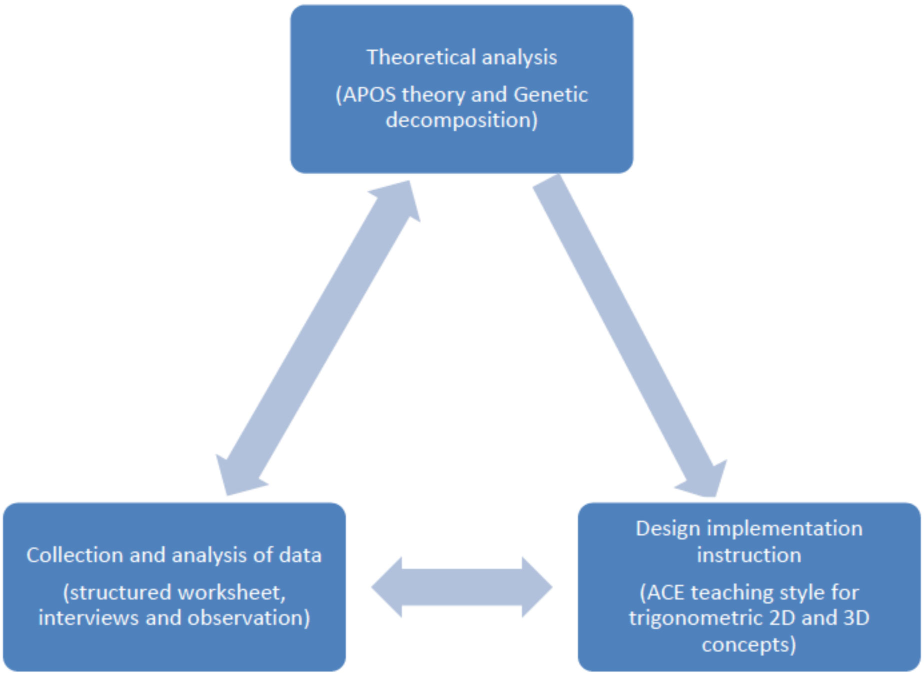


Figure 3.1 Framework for the research cycle (modified from Asiala et al., 2004)

Developing an understanding of 2D and 3D in trigonometry requires a theoretical analysis of the concepts related to the mental structures individuals make. According to Arnon et al. (2014), the ability of a learner to comprehend a concept is affected by the teacher's knowledge of that concept. Using a preliminary genetic decomposition of the concepts, teachers can describe the mental structures the learners require and design activities to aid them in developing the necessary mental structures. In contrast to Hobden (2006), who argues that each individual constructs their own unique view of the world based on their experiences and schemas, Ndlovu (2014) argues that students should be allowed to process concepts until they can process them into objects that contribute to the development of schemas that are capable of adapting or assimilating new schemas into existing schemas in order to develop knowledge. This means that individual experiences of learning are important and this is not a once-off process. Understanding of concepts also requires that learners should reflect on the procedure reiteratively.

To select the best teaching strategies, a teacher needs to know their learner's abilities. Arnon et al. (2014) suggest that cooperative learning, group discussions and lectures are pedagogical strategies that can assist learners in making the necessary mental constructions. Learners tend to understand better if they explain or discuss the concept with each other. In light of this, co-operative learning was used as part of the ACE teaching style implemented in this study. The teachers' strategies and structured activities used should be design in a manner that assists learners to develop the necessary mental structures. The following sub-sections discuss more mental structures required to assist a learner to comprehend mathematical concepts.

3.6.1. Theoretical analysis

Dubinsky (2000) stated that a theoretical analysis provides mental structures that can be used to learn a specific math topic. This study used APOS theory to describe and analyse the use of GeoGebra to enhance learners' understanding in 2D and 3D trigonometry. APOS theory, as the fundamental theory guiding the study, was used as a lens in the design of the study and the lesson and the collection and analysis of data.

Dubinsky (2000) stated that "the purpose of the theoretical analysis is to propose the specific mental constructions, the initial genetic decomposition, through which a student might learn the concept under construction" (p. 12). This clarifies the reasons for using the theoretical analysis presented in Figure 3.1.

Arnon et al. (2014) state the three components of a research cycle namely: theoretical analysis design, implementation of instruction, and data collection and analysis influence each other, with theoretical analysis leading to the creation and implementation of instruction through the design of activities designed to promote mental constructions, and data collection and analysis through the implementation of instruction. APOS theory was used in this study to frame the theoretical analysis and genetic decomposition that guided the design of the tests.

Genetic decomposition defines what it means to comprehend concepts in mathematics and how students can construct mental structures and mechanisms to learn a specific mathematical concept. Zingiswa (2011) indicates that, to determine whether students are forming the mental structures indicated by the genetic decomposition, lesson designs should be drafted using activity sheets that assist students in performing them. This means teachers should know their learners' abilities so they can plan strategies to assist learners to construct mental structures.

The final step is to implement instruction, resulting in the collection of data that should be analysed within the theoretical framework. In this study, the researcher designed the pre-test and post-test to develop and enhance the learners' mental construction in 2D and 3D of trigonometry concepts using GeoGebra. Analysis of the results is presented in Chapters 5 and 6.

3.6.2. Design and implementation of instruction

Arnon et al. (2014) explain that the design corresponds to the genetic decomposition that explains mental constructions individuals may make in order to grasp a concept. The genetic decomposition presented in Section 3.5.1 described the mental constructions the learners might make as they develop an understanding of trigonometric 2D and 3D concepts. Implementation of instruction was done using the ACE teaching style. As illustrated in the genetic decomposition diagram, the ACE teaching style supports the development of mental constructions (Arnon et al., 2014; Ndlovu & Chiromo, 2019). The ACE teaching style approach used for teaching trigonometric 2D and 3D concepts is discussed next sub-section.

3.6.2.1 ACE Teaching Style

The ACE teaching cycle is a pedagogical approach based on APOS theory. This iterative cycle has three components: activities (A), classroom discussions (C), and exercises (E) (Arnon et al., 2014). These components are useful tools for the development of mental constructions. Several studies have demonstrated the effectiveness of the ACE teaching style in helping students build mental structures for mathematical concepts (Asiala et al., 1996; Weller et al., 2011; Arnon et al., 2014). The style assists learners with constructing and reconstructing a concept until it is fully understood (Ndlovu et al., 2019). In this study, this teaching style was used with the aim of helping learners to reconstruct 2D and 3D concepts, resulting in a deeper understanding.

The ACE teaching style incorporates collaborative learning which allows students to share ideas and clarify their understanding (National Council of Teachers of Mathematics, 2000). Listening to others' explanations gives learners the opportunity to develop their own understanding. With the ACE approach, the exchange of ideas and clarification of mutual understandings take place in on-going discussions and classrooms groups (Chagwiza, 2019). Activities such as classroom and group discussions help students build a logical train of thought.

Furthermore, when using the ACE teaching style, the discussion gives learners an opportunity to see their mistakes and see what they may have misunderstood, because sharing ideas allowed them to better understand the concepts being discussed. This teaching style is learner-centred and was well-suited for this study because it was the learners' first time using GeoGebra and this allowed them to have more time to discuss what they had learnt, which helped them to comprehend the concept. Chagwiza (2019) stated that the ACE learning cycle promotes social and communication skills by creating a classroom atmosphere that emphasizes exchanges of ideas. The components of the ACE teaching style are discussed below.

3.6.2.2 Activities

Arnon et al. (2014) propose that tasks should focus on promoting reflective abstraction rather than getting the right answer. The tasks used for activities are attached in Appendix A and discussed in Chapter Four.

Arnon et al., (2014) advocate that students should work collaboratively in small groups for the planned activities. While Ndlovu (2014) does not dispute the value of working collaboratively, she cautions that some students may dominate in groups while others hold back. To address this, in her study learners first worked independently before coming together to work collaboratively, to promote reflective abstraction rather than emphasis on the correctness of the answer. In this study, the same approach adopted by Ndlovu (2014) was used, where learners first worked independently on the activities before working co-operatively. As part of the constructivist model, learners worked in groups on activities which helped them develop correct cognitive structures. In this study, before the implementation of the ACE instructional approach, the mental constructions that learners already possessed were explored by means of research using structured tests and analysed using the designed genetic decomposition. Since the aim was to explore and enhance the learners' mental constructions, the ACE instructional approach was implemented, then GeoGebra software was used where learners worked on a class activity of 2D and 3D of trigonometry, then wrote a structured test.

3.6.2.3 Classroom discussion

The second part of the ACE teaching cycle is classroom discussion. This phase included all learners and teacher-led class discussions. Class discussion was introduced after the fourth lesson to allow learners to gain some experience using GeoGebra before engaging in discussion. The discussions revolved around learners' work with 2D and 3D trigonometric concepts using GeoGebra software. Questions were presented to a sample of 30 students who volunteered to participate in the study. In the interview process, some students were asked to explain how they came up with their solutions. Audio recordings of interviews were made and later transcribed, as well as all written material collected.

3.6.2.4 Exercises

The third phase of the cycle consisted of exercises designed to reinforce the activity and classroom discussion phases. A relatively standard set of exercises was designed to promote computer activity and classroom discussion. Genetic decomposition suggested that the exercises might support the ongoing development of mental structures (Dubinsky et al., 2013). For homework assignments, Arnon et al. (2014) propose that learners be given fairly standardised problems to support continued mental constructions. In this study, the exercises were in the form of an open book test where learners worked individually. The focus was not on the correctness of the answer but on applying what they had learnt about trigonometric 2D and 3D concepts while using GeoGebra. At the end of the lessons, learners wrote a post test that was analysed.

3.6.3 Collection and analysis of data

The main aim of this phase in the research cycle is to understand whether students made the necessary mental constructions described in the genetic decomposition and to understand the extent to which learners learnt the concepts taught (Arnon et al., 2014). While the research framework does not prescribe the methods to be used for collecting and analysing data, it was important that these methods aligned with the objective of the study. To ensure this, structured tests, semi-structured interviews were used. These are discussed in detailed in Chapter Four.

3.7. Conclusion

This chapter has presented APOS theory, as the theoretical framework of the study, and genetic decomposition, a theoretical took used to analyse the mental constructions made by a learner in line with APOS theory. In the classroom, ACE teaching style was used for teaching and learning; at first learners worked individually and after a few lessons they work in groups to enable discussion. Mental mechanisms were also discussed in this chapter, to understand how learners move from one phase to another and in which phase of APOS they operate. The next chapter discusses the research design and methodology used in this study.

CHAPTER 4: RESEARCH DESIGN AND METHODOLOGY

4.1. Introduction

The previous chapter discussed APOS theory, the theoretical framework that underpins this study that describes the progression of mental structures that are made during the learning of mathematical concepts (Ndlovu, 2014).

This chapter provides the methodology of the study. As for methodology, it is the method by which researchers select suitable data collection methods and analysis procedures to answer a specific research question (McMillan & Schumacher, 2014). Methodology aims to enable the most comprehensive understanding possible of the process itself, rather than the results of scientific research (Cohen et al., 2007; Ndlovu, 2014).

This chapter explains the research approach and plan, describes the target population and research setting, and discusses how the data was collected and analysed to answer the research questions. A qualitative case study design was selected. Thirty Grade 12 learners were selected to participate in this study using purposive sampling. Data was collected by testing learners were tested before and after ACE-style teaching using GeoGebra software. The issue of trustworthiness and ethical considerations are also discussed in this chapter.

4.2. Research paradigm

Taylor and Medina (2013) define a paradigm as a system of beliefs, or a worldview, that guides research and informs practice. According to Schwardt (2014), a paradigm is a shared worldview that guides the way problems are solved as well as represents the beliefs and values that inform a discipline.

According to Kivunja & Kuyini (2007), paradigms are important because they provide beliefs and indicate which scholars in a particular field influence what should be studied, how it should be studied and interpreted. Research paradigms influence the research questions researchers ask and the methods they use to answer them (Morgan, 2007).

In this study, the interpretive paradigm was used. As Golafshani (2003) contends, the interpretive paradigm is characterized by multiple data collection methods to ensure the validity of the collected data. According to Rabtee and Miller (1999), participants serve as informants for researchers in the interpretive paradigm. Participants can describe their own perspective on reality while informing the researcher, allowing the researcher to understand their behaviour better. In this study, 4 of the 30 participants participated in semi-structured interviews so that the researcher could hear their perspectives, rather than only collecting data through participants' written responses. Participants were encouraged to explain the meaning of what they had written in their own way and describe how they had experienced the use of GeoGebra for the teaching and learning of trigonometric 2D and 3D concepts.

A paradigm differs from another in both its ontology and epistemology, which corresponds to its assumption of truth and knowledge (Scotland, 2012). In contrast, epistemology focuses on how humans create, acquire, and transmit knowledge (Scotland 2012). This study aimed to improve learners' understanding of 2D and 3D trigonometry using GeoGebra. According to Schultz (1962, cited in Ndlovu, 2014), interpretive researchers believe that reality is constructed intersubjectively through meanings, and understanding is acquired through social and experiential means. Thus, interpretive researchers aim to understand individual perceptions, share meaning, and develop insights about observed cases (Bryman, 2008).

Denzin (2002) stated that an interpretive approach should draw from material that comes from the world of lived experiences and incorporates prior understandings. In this study, the learners' prior knowledge of trigonometric 2D and 3D concepts was incorporated since learners had learnt the concepts first without the use of GeoGebra. Thereafter, I integrated the use of GeoGebra into teaching of the same concepts, to ascertain from learners how GeoGebra assisted or did not assist them in improving their mathematical understanding of trigonometry in 2D and 3D.

As this study focused on exploring and enhancing learners' knowledge with regard to key concepts, the interpretive paradigm was selected to guide the methodological framework of this study. In an interpretive perspective, Bertram and Christiansen (2014) argue that research is primarily about understanding how people make sense of the situations they are living in. The purpose of this study was to understand how learners construct knowledge and how the use of GeoGebra software in the teaching and learning of mathematical concepts influences their conceptual understanding.

Research using the interpretive paradigm enables researchers to understand the world from the perspective of their participants' perceptions and experiences (Thanh & Thanh, 2015). Cohen et al. (2007) argue that the learner makes a room in the mind to accommodate the new knowledge in addition to existing knowledge. The aim of this study, to explore and enhance the already existing knowledge in the mind of the learner, thus aligns with the interpretive paradigm. In addition, the interpretive paradigm was found to be suitable for this study because it aims to understand the subjective human experience of individual participants. The study sought to reveal the nature of individual learners' mental constructions in the process of learning; the interpretive paradigm allows for the examining and understanding of an individual's world (Cohen et al., 2011). The interpretive paradigm fits very well with this study because interviews, observations and written work were employed as data collection instruments.

Cohen (2011) indicates that the interpretive approach involves analysing written work, conducting interviews and observing situations. In this study, data was collected initially using a structured pre-test to explore the existing knowledge learners had constructed; thereafter, GeoGebra was used to improve learners' existing knowledge. Then interviews were used to understand how participants constructed the meaning of their responses. As Kawulich (2012) argues, the interpretive paradigm sees research as an attempt to understand and describe human nature.

4.3. Research design and style

In research, a design provides specific direction for the procedures that will be used (Creswell, 2014). In terms of research design, it can be described as the procedure by which a researcher will systematically collect and analyse the data required to answer the research questions (Bertram & Christiansen, 2014; Lewis, 2015). Using a qualitative design, the study examined how individuals make sense of a particular situation (Christensen et al., 2015). The case study approach was selected as the research style.

4.3.1 Qualitative research design

According to Creswell (2016), qualitative research examines and understands the meanings individuals or groups place on social or human problems. This approach allows for a range of methods for collecting data. Asiala et al., (2004) explain that the qualitative approach requires two aspects to be determined: the theoretical perspective that will guide the research, and the methodology for collecting and analysing data.

Cresswell (2009) argues that the theoretical perspective provides three aspects: a total lens for the study of the problem, an advocacy point of view that shapes the type of questions asked, and a guide for how data will be collected and analysed. The theoretical framework chosen for this study is APOS theory, which was discussed in Chapter 3. APOS stages were used to analyse learners' written work in this study.

Hence the participants' written work and views in the interview assisted me in gathering information about learners' conceptions of trigonometry using GeoGebra. Dube (2019) stated that qualitative approaches utilise data to gain an understanding of problems and develop solutions by understanding the underlying reasons, opinions and motivations. Green and Thorogood (2018) suggest that qualitative research assists researchers in generating useful knowledge about a topic at different levels, from individual perceptions to perception systems. Therefore, by using this approach, the aim was to enhance the understanding of learners by using GeoGebra to the concept of 2D and 3D of trigonometry.

The qualitative approach emphasises the multiplicity of social realities as experienced by individuals (Gama, 2015). People are different, and the way each individual thinks and reasons is different; therefore, different strategies should be presented to the learners. In this study, 30 written work of learners and four learners that were interviewed, were a main source of information to answer the research questions was the participants; their view of how they interacted with GeoGebra to enhance their understanding of concepts.

This study aimed to explore learners' mental constructions and the use of GeoGebra to enhance learners' understanding of 2D and 3D of trigonometry. Since the learners did have prior knowledge of the concepts of 2D and 3D of trigonometry, I hoped that this previously constructed knowledge would be enhanced by the use of GeoGebra software. However, it was the participants' experiences and their perspectives that revealed the extent to which their knowledge construction was enhanced.

It is noted by Niranjan (2013) that qualitative research methodology enables for the use of various strategies for collecting data, as well as allowing the voice of the participants to be heard. In line with this, this research used different methods for collecting data, including tests, interviews, activity sheets and observations. Qualitative research makes use of different ways of collecting data to obtain detailed data that can provide greater insight into a phenomenon. As a result, a qualitative design was determined to be appropriate for this study because the focus of the study was based on a description of learners' mental constructions and the process to enhance their knowledge construction using GeoGebra.

The focus of the study was to explore learners' mental constructions and the use of GeoGebra towards enhancing their construction of knowledge; thus, the qualitative approach allowed me to understand the construction of knowledge from the participants' perspectives. Qualitative research therefore offers an opportunity to understand people's perceptions of their natural environment and speaks directly to participants. In addition, it allows the researcher to conduct the study with participants in a natural setting (Cohen et al., 2011). This study was conducted with learners at their school, which is a natural setting of learning, and it was conducted by someone they had engaged with in their day-to-day learning process.

4.3.2 Case study research style

In this study, a case study research style was chosen. Qualitative case studies assist in exploring a phenomenon within a specific context by using various lenses and data sources to reveal the many facets of the phenomenon (Baxter & Jack, 2008). A case study examines and investigates contemporary real-life phenomena by analysing a limited number of conditions and events in a context (Dube, 2019).

According to Cohen et al. (2007), case studies provide unique examples of real people in real situations, allowing for a deeper understanding. In Dube's (2019) argument, a case study describes what it is like to be in a particular situation. In a similar vein, Cohen et al. (2011) argue that case studies aim to capture participants' reality and their perceptions of the particular situation under investigation. The aim of a case study, according to Bertram and Christiansen (2014), is to describe what it is like to be in a particular situation, as well as participants' experiences and thoughts regarding a particular phenomenon. In this way, the case study allows the researcher to examine data within a specific context. Hence, Ndlovu (2014) states that understanding a student's experience with learning a concept is essential. This study hoped to enhance learners' understanding of concepts with GeoGebra.

Research using a case study style has the advantage of allowing researchers to examine real situations and compare their views directly to phenomena of actual occurrences (Flyvbjerg, 2006). The main benefit of a case study, according to Josefsson (2016), is that it provides a detailed analysis of individual cases. Additionally, a case study can provide insight that illuminates meaning, enhancing the readers' experience by complementing experiments. Using the case study approach, I was able to collect detailed information from participants. In this study, the case in the case study was to explore the phenomena which is Geogebra software in teaching and learning trigonometry of 2D and 3D concept, with the hope that it will enhance the learners' understanding of the concept. Since literature has evident that mathematics is a complex subject and figures 1.1 and 1.2 in chapter one of the DOE diagnostic analysis also show that learners are not doing well in this section. Therefore it was a cause of concern that I should undertake this study to explore the phenomena so that it contribute to instruction strategies in teaching and learning of trigonometry of 2D and 3D concept.

Due to the fact that it provides means of exploring the learners' experiences, the case study was deemed appropriate for this study. In addition, a case study allows the researcher to select those with rich knowledge of the phenomena to participate in the study. In this study, selecting Grade 12 learners as the participants allowed me to make an in-depth analysis of their mental constructions of concepts they had been studying since Grade 11.

For the information gathered through a case study to be trustworthy, multiple methods should be used for data collection. The learners' mental constructions were explored through a variety of lenses to identify multiple facets of the phenomena, as recommended by Baxter and Jack (2008). This study collected data from learners' written responses; after this phase of data collection, semi-structured interviews were conducted to verify the understanding of learners' construction of meaning derived from an analysis of their written responses. The in-depth analysis of the learners' mental constructions revealed their knowledge of the concepts and their thoughts about the use of GeoGebra software to enhance their conceptual understanding.

4.4 Sample selection

Creswell (2012) states that identifying the population and locations to be studied are the first step in the research process. In qualitative research, specific cases events, or actions are collected in order to clarify and deepen understanding (Denscombe, 1998; Neuman, 2006).

'Population' refers to the total number of defined classes of people, events or things. A sampling process involves selecting representative individuals from a whole population. By sampling, the researcher can select a reasonable number of cases and materials to study, reducing the scope of the study (Dube, 2019). Samples of the population used for larger studies are generally larger, while samples of the population used for detailed studies are smaller, more defined and more accessible (Scotland, 2012).

The process of sampling involves selecting people, attitudes, events or behaviours to include in a study as well as determining how many people, individuals, groups or things to observe. According to Fagerholt et al. (2010), sample size determines what's examined in a particular study. By sampling, inferences can be made about a population or generalisations can be made about existing theories.

There are two types of sampling methods: random (probabilistic) sampling and targeted (non-probabilistic) (Bertram & Christiansen, 2014). Random sampling refers to the process of making each item in a population equally likely to be selected as part of the sample; with targeted sampling, all representatives of the population are not given an equal chance of being selected. According to Strydom and Delport (2005) qualitative studies usually employ non-probability sampling. This study employed purposive sampling, which is a form of targeted, or non-probability, sampling, because it allowed for the deliberate selection of Grade 12 learners doing pure mathematics as this sample had engaged with the concepts central to the study since Grade 11 but had not used GeoGebra before, and were thus ideally situated to provide data that could answer the research questions.

While purposive sampling was used to select the participants, convenience sampling was used to select the study site. I am a high school mathematics teacher teaching mathematics to Grade 10 and 12 learners. The site that was selected was another school in the area where I teaches. The participants thus lived in the same area as my area but did not attend the same school.

4.4.1 Selection of the site

The school is situated in a township in the King Cetshwayo District Municipality of the KwaZulu-Natal province of South Africa. The learners enrolled at the school were boys and girls of African identity. The school is a well-resourced technical school that is well-known in the area as it offers technical subjects such as civil engineering, mechanical and electrical technology subjects that aren't available at other schools in the area. Class size does not exceed 30 learners per class at the school.

The school was selected on the basis of convenience. The convenient sampling procedure was used to select the school, as it is near the school where I teaches and will minimize my travelling cost. In addition I had a working relationship with the school as my school is in the same cluster as the school that was selected, and I had collaborated with teachers at the school to provide extra classes**.** The school was also selected on the basis of having the resources needed to complete the study. In particular, the study required learners to have access to computers. The school had 22 working computers, which was close to the number (30) of learners selected for the study. The restriction of the class size to 30 made it feasible for me to control and manage the class during teaching and learning.

Although the school had resources such as computers, software and other technologies, these were used for technical subjects, not for pure mathematics. Whilst many of the learners were disciplined and self-motivated, and many of the teachers were committed and hard-working, the pass rate had been below 100% for the previous three years. This resulted in a situation where the school was open to considering alternative teaching strategies to enhance the teaching and learning of mathematics. The principal welcomed initiatives at the school that could benefit the learners.

4.4.2 Sampling of participants

In purposive sampling, participants are selected based on their relevance to research; therefore, researchers decide which individuals to include in their sample on this basis (Babbie, 2007; Scotland, 2012; Bertram & Christiansen, 2014). Purposive sampling was used in this study so the researcher could collect comprehensive information about participants' experiences with 2D and 3D trigonometry concepts in GeoGebra.

In purposive sampling, the researcher includes cases or participants in the sample because they believe that they warrant inclusion. The target population for this study was Grade 12 learners that take pure mathematics as a subject. A group of 30 learners that met this criterion were selected from the school. . Scotland 2012 started that samples of the population used for detailed studies, are smaller, more defined and more accessible. Therefore this study sample size of 30 learners were used or chosen because 30 is the enough number of learners that can give me the data that is required for the study and that it will enable me to give detail information from learners on how the integration of Geogebra enhanced their understanding.

It was initially planned that 6 learners would be interviewed; unfortunately, this did not go as planned. The GeoGebra-based lessons took place in fourth term and then learners wrote a post-test two days before their final exams. It was difficult to meet with the learners after that: they were either at home or busy with other subjects. As a result, I was able to interview only 4 of the 6 learners that had been selected on the day that she came to the school for the interviews.

4.5 Data collection procedures

It has been noted by Cohen et al. (2007) that using a single method of collecting information is not adequate in qualitative research. Diverse data collection methods complement each other to answer the research questions and increase the dependability of the results (Thuzini, 2011). Also, different data collection techniques may be needed in order to obtain answers to the research questions.

Mason (2002) offers five reasons why qualitative studies need to incorporate methodology: to examine various aspects of a phenomenon in order to determine how they are related; to obtain answers to various research questions; to consider research question from various perspectives; to add more depth or breadth to the analysis; and to triangulate and confirm the data.

In this study, multiple methods were used to collect data. Tests and interviews were used to triangulate and bolster the trustworthiness of the findings and add depth and breadth to the analysis. The methods used for the collection and analysis of data in this study are shown in the table below and discussed thereafter.

Table 4.1 Data collection and analysis

Research question	Method of data generation	Data collection instrument	Method of data analysis
1. What level of mental construction of 2D and 3D trigonometric concepts do learners demonstrate with and without Geogebra?	Pre-test with a class of 30 learners Semi-structure interviews with 4 learners purposely selected	Structured activity sheet Audio recorder	Data transcription and deductive analysis using APOS constructs
2. Why are learners able or not able to construct knowledge of trigonometric 2D and 3D concepts with and without GeoGebra?	Semi-structure interviews with 4 learners purposely selected	Audio recorder	Deductive and inductive analysis

3. To what extent does GeoGebra enhance learners' knowledge construction for 2D and 3D trigonometric concepts?	Post-test with the class of 30 learners Focus group interview with 4 learners	Structured activity sheet Audio recorder	Deductive and inductive analysis

4.5.1. Phases of data collection

This study was conducted in two phases. The first phase involved exploring learners' mental constructions of trigonometric 2D and 3D concepts. A pre-test was administered in the form of a structured worksheet. Ndlovu (2014) stated that a structured worksheet can model how meaningful learning can take place. After data from the structured tests was analysed and responses categorised in terms of the constructs of APOS theory, semi-structured interviews were conducted with 4 participants (2 learners were unable to participate due to reasons explained in Section 4.4.2). As explained earlier, the selection of the participants for the interview was purposely based on the categories of constructs evident in the analysis of responses. The purpose of conducting the semi-structured interviews was to verify learners' level of knowledge construction that had been identified in their written responses. Maharaj (2014) advises that learners' responses should be verified by engaging them verbally about what they have written. The interviews were audio-recorded.

The second phase involved the teaching of the selected concepts using GeoGebra and the ACE teaching style. The study took place over a period of 4 weeks, with 4 lessons per week. Each lesson took a maximum of 1 hour. To capture learners' learning progress, observation was used. A revolving camera was used as an observation tool to capture observant and non-observant activities during the learning process. Following the lesson, a post-test was administered. A focus group was then conducted with the 4 learners that were interviewed individually in the first phase. The choice to use focus group interviews rather than individual interviews was made to allow learners not only to reflect on the written responses but also to reflect on the learning process, and thus to enhance classroom discussion while teaching and learning with GeoGebra so as to eventually enhance their understanding of the concepts.

4.5.2 Data collection instruments

Data collection involves gathering specific information, testing hypotheses and evaluating outcomes by using a systematic approach. Through the use of multiple research methods, social scientists can gain a deeper understanding of human behaviour by studying it from more than one perspective (Cohen et al., 2018). Accordingly, this study used multiple data collection instruments. A structured test was used in the testing process, the interview questions were used for the interview that was audio recorded and a camera was used during the teaching and learning process.

4.5.2.1 Written pre-test and post-test

As part of this study, testing was used as a data collection method. Cohen et al. (2007) describe different ways in which tests can be used: for assessing performance in a variety of school subjects and for assessing ability, potential and language proficiency. The purpose of using tests in this study was to explore learners' mental constructions of trigonometric 2D and 3D concepts before implementing an instructional design using GeoGebra (pre-test) and to explore the extent to which GeoGebra enhanced learners' mental constructions after teaching (post-test). The same activity sheet was used for the pre- and post-test. The terms 'pre-test' and 'post-test' thus refer to the tool used to investigate learners' mental constructions.

While some scholars (e.g., Arnon et al., 2014; Chagwiza, 2019) suggest exercises be administered in the form of homework, Ndlovu and Chiromo (2019) advocate for exercises to be administered in the classroom so that the researcher can ensure that the mental constructions evident in the learners' response can be attributed to the learner's own thinking process. Arnon et al., (2014) state that exercises can support the development of learners' mental constructions suggested by genetic decomposition and also guide learners to apply what they have been taught and use related mathematical ideas. For this reason, the researcher opted to administer the exercises in the classroom. A pre-test (Appendix A) was conducted in the form of a structured activity sheet during the first lesson. A post-test exercise was administered after the researcher finished the GeoGebra-based lessons.

4.5.2.2 Semi-structured individual interviews

Christiansen et al. (2010) describe an interview as a conversation between an interviewer and interviewee, but note that the researcher sets the agenda and asks pertinent questions. There are different types of interviews: the structured interview; the unstructured interview; the non-directive interview; and the focused interview. This study used semi-structure interviews.

Semi-structured interviews consist of open-ended questions based on the topic I wishes to explore. When conducting a semi-structured interview, I can ask the respondent to elaborate on their original answer or to follow their line of thinking. Participants are able to express themselves from the point of view of their own interpretation of the world through a semi-structured interview (Cohen, Manion & Morrison, 2007). The interview was therefore a suitable data collection method for this study, which aimed to explore and enhance learners' understanding of trigonometric 2D and 3D concepts using GeoGebra.

In this study, it was important to understand learners' mental constructions from their own perspectives and not only from my interpretation of their written responses. A very significant aspect of this study was the learners' viewpoints about their mental constructions of concepts; for this reason, semi-structured interviews were used where the interview was guided and informed by the learners' responses.

Bertram (2004) says that interviews are good tools for collecting information about someone's knowledge. A number of benefits are highlighted by Alshenqeeti (2014) when using interviews to collect data: they allow for a more focused discussion and follow-up questions; individuals may provide information during interviews that they would not provide in a group setting; interviews can provide context and story; and interviewers can observe respondents' non-verbal behaviour during interviews.

As purported by Maharaj (2014) and Ndlovu and Brijlall (2015), learners need to be allowed to engage with what they write to understand their thought processes. In this study, semi-structured interviews helped the researcher to get more understanding about the way learners construct their understanding of the concepts of 2D and 3D in trigonometry. The interviews provided rich data from learners' points of views and helped answer the research questions. The interviews also helped the researcher to analyse learners' reflections and views on the use of GeoGebra in trigonometry of 2D and 3D.

The semi-structured interview was selected as a data collection instrument for this study because the purpose of the interviews was to understand and interpret the participants' responses. This is supported by Kvale (1996), who agrees that interviews in qualitative research help us understand what the subject sees and make sense of their experience. Most questions were open-ended, designed to clarify and probe participants' responses to gain deep information. The semi-structured interview included a list of questions prepared for guidance.

All interview sessions were audio-recorded, as stated on the consent form. This helped to capture information that could not be captured through notetaking during the interviews. The interviews took place after learners had written the pre-test and post-test. Four learners were interviewed; the other 2 who had been selected were not available. These questions were structured for pre-test and post-test and the session of interview took place after writing the pre-test and post-test. The duration of the interview was 45 minutes.

Merg (2012) addresses limitations that researchers need to be aware of when collecting data. The first one is that time constraints can be a factor for the interviewer and interviewee. In this study, the interviews did not impact the school day as it was conducted after school hours. Additionally, respondents' interviews can restrict the scope and sample of data collection. Also, the results of many interviews may contradict one another or be difficult to analyse. More on the analysis is discussed in the next chapter. It should be noted that interviewees may be biased or have only a limited perspective on performance issues or themes. To mitigate these limitations, I had to conduct the interviews of pre-test and post-test having learners to have both their written work in front of them and give each learner time to reflect on their work and discus their experience during the test based on questions asked.

The interview session was audio-recorded, as stated on the consent form. This helped to capture information that could not be captured through notetaking during the interviews. The interviews took place after learners had written the pre-test. Four learners were interviewed; the other 2 who had been selected were not available. These questions were structured for pre-test and the session of interview took place after writing the pre-test. The duration of the interview was 45 minutes. Questions were open ended, I asked questions and each learner was given an opportunity to respond to the question openly based on what they have experienced in the pre-test. This was done till all interview questions in appendix 2 were completed.

4.5.2.3 Focus group interviews

Qualitative interviews are defined by Sewell (2002) as attempts to clarify the meaning of people's experiences and understanding of their world before attempting scientific explanations. The aim of conducting focus group interviews was to obtain information from learners that had been purposely selected and engage in dialogue with the learners asking all the structured interview questions. Using the focus group interview help to get the qualitative data, because learners have to state their perceptions and opinions about the use of GeoGebra whether it did or did not enhance their understanding in the concept of 2D and 3D of trigonometry. I asked questions in an interactive group setting. A single focus group was held. The learners discussed the question or topic indicated by me. All structured research questions were asked and answered by all 4 learners that where interviewed. Findings are presented in the next chapter.

This interview took place after the learners wrote a post-test and the learners' scripts were marked and provided to learners as feedback. The interview took place with the four learners, the very same learners that were interviewed in the first interview. I asked questions in an interactive group setting. A single focus group was held. The learners discussed the question or topic indicated by me. All structured research questions were asked and answered by all 4 learners that were interviewed. The interview took 45 minutes and the interview was audio recorded.

After the semi-structured and focus group interviews were conducted, I then transcribed the interview questions and responses of the interviewer (researcher/myself) and interviewees (learners). The findings are presented in the next chapter.

4.5.2.4Observation

According to Kumar (2022), an observation method involves observing a phenomenon and recording information about its nature in order to gather data. According to Morrison (1993, p.80), observations provide data on, namely: 1) A physical setting (for example, its structure and environment); 2)A human setting (e.g., the organization and characteristics of the groups or individuals who are being observed; for instance: gender, class) that needs to be considered; 3) A description of the interactional settings (e.g., formal or informal, planned or unplanned, verbal or non-verbal, etc.) and 4) the programme setting (e.g., the organization of resources, pedagogic styles and curricula).

Therefore, in this study observation was based on observing Grade 12 learners interacting with GeoGebra. I conducted lessons using GeoGebra with a hope of enhancing learner's understanding of2D and 3D trigonometry concepts. During teaching and learning, I used the ACE teaching style to facilitate discussion while the learners were using GeoGebra. The use of the ACE teaching style and GeoGebra to collect data is discussed in this section below.

a) ACE teaching style

Arnon et al. (2014) describe the ACE teaching style as a pedagogical strategy with three components: activity, classroom discussion and exercise. The activity represents the first step in a learning cycle, during which learners work together in teams to undertake tasks designed to create the mental structures suggested by the genetic decomposition. The focus of these tasks is not on arriving at a correct answer, but on promoting reflective abstraction. The ACE teaching style was discussed in detail in Chapter 3.

Learners first wrote a pre-test before teaching took place. After the analysis of the pre-test, activities were designed to help learners improve their mental structures during teaching and learning. The teaching and learning were designed to be learner-centred. Learners were free to discuss and express their thoughts aloud during teaching and learning. Sometimes they worked in group discussions doing activities. Learners were active in helping each other and were not afraid to ask for assistance when they encountered a challenge.

I began by engaging learners about 2D shapes to access their prior knowledge. The learners started constructing these 2D shapes in GeoGebra and used these drawn shapes to answer questions. The focus was to allow learners to first explore and construct the shapes using GeoGebra to help them see how these shapes come about. They explored how angles are formed using the software. After this, the class activity was based on a 3D shape activity. Drawing of 2D and 3D in GeoGebra is done in different ways. For 3D, they drew in 3D graphics which has many toolbars for learners to explore such as using the rotation tool to rotate the 3D shape to explore all triangles in a triangular pyramid. Classroom discussion was also facilitated, which is the second phase in the ACE teaching style.

b) Observation of Classroom interaction with GeoGebra software

GeoGebra was the most significant tool used in this study. It was the first time the participants used GeoGebra, and they were very excited. Their excitement sometimes resulted in them not following instructions, as they wanted to explore every tool bar especially for 3D trigonometry. Observing the rotation of the 3D shape seemed to be interesting to learners. This integration of GeoGebra in the classroom helped me to describe and understand events as they actually happened in the classroom setting to assess whether learners found it positive to learn with GeoGebra. The level of excitement was an indication to the researcher that learners enjoyed exploring mathematics using GeoGebra.

In addition, learners' responses during teaching and learning using GeoGebra revealed to me that the learners had not thought that there could be something that could make mathematics fun and exciting, as they found with GeoGebra. Teaching and learning were learner-centred. The literature showed that learners became motivated and positive when learning with GeoGebra, and experienced an improvement in performance (Awurukundo et al., 2020).

In this study, there was a substantial improvement in learners' performance from the pre-test to the post-test, which will be discussed in depth in Chapter 5, and participants were found to be motivated and eager to interact. Their experience of working with GeoGebra was discussed during the semi-structured interviews.

4.6. Data analysis

According to Roulston and Shelton (2015), qualitative data can be analysed using both inductive and deductive methods. Inductive analysis entails categorizing data and identifying patterns between categories. As a result of data analysis, these categories, themes, and patterns emerge. In a deductive approach, researchers start with a set of categories and use them to organize data. In this study, data analysis was done using both deductive and inductive approaches.

For deductive approach, APOS (ACTION-PROCESS-OBJECT-SCHEMA) was used as the analytical framework in this study. According to Dubinsky (2000), this data analysis method examines interview transcripts and APOS provides an account of student successes and failures. The constructs of APOS theory guided the analysis of the learners' knowledge constructions of trigonometric 2D and 3D concepts in the pre-test and post-test.

In addition to analysing emerging patterns, the inductive approach was used to generate themes to provide depth in answering the research questions. Since inductive analysis entails categorizing and identifying patterns between categories and a researcher collect data first to generate hypothesis, therefore in this study, I started collecting data by conducting a pre-test before integrating Geogebra software. The learners' written responses were analysed and it was revealed from the learners written work that different instructional teaching strategy is needed to enhance learners understanding of the concept of 2D and 3D trigonometry.

4.7. Authenticity and trustworthiness of the study

In qualitative research, Guba and Lincoln (1994) suggested that different terms should be used instead of validity and reliability. Thus, we used two terms in this study: authenticity and trustworthiness. While validity and reliability are vital to the quality of quantitative research, in qualitative research, trustworthiness and dependability of the findings are used to ensure the quality of the research.

As it has been stated in this study that data collection took place in two phases, therefore the data collection for the study involved multiple methods to ensure authenticity of the study. More of the issue of authenticity of this study is debated under triangulation below. This study established trustworthiness and dependability through the use of triangulation. Triangulation refers to the way of surveying and mapping using different methods (Creswell, 1999). Neuman (2006) argues that triangulation means that something is better viewed from multiple angles. Denzin (2002) distinguishes five types of triangulation: triangulation of data, including time, space, and people; namely: 1). investigator triangulation, which involves multiple investigators in the data collection process; 2). joint-level triangulation, which uses two or more levels of analysis from the individual, 3). interactive level and collective levels; 4). theoretical triangulation, which uses multiple theoretical schemes in interpreting phenomena; and 5). methodological triangulation; which involves multiple data collection methods, such as interviews, observations, questionnaires and documents. Therefore this study used methodological triangulation to explore learners' mental constructions and the use of GeoGebra to enhance learners' construction of knowledge. Triangulation was used where data was collected using tests, interviews (semi-structured and focus group) and observations using a variety of instruments, such as structured activity sheets for the tests, audio recorder for the interviews and revolving camera for classroom observations. Data from these multiple sources enabled me to triangulate data to ensure trustworthiness. While methodological triangulation was the focus in this study, it also accounts for the variability over time as the study methods were implemented in different phases during the data collection process. Also, to ensure that the data collection instruments yield the data required, the tests, interview questions and classroom discussion activities were checked and moderated by the supervisor as well as other senior teachers involved in the teaching of extra classes in the cluster before being administered to learners. The data was collected over two phases, with both phases aiming to provide data to respond to the phenomena under study.

Lauckner et al. (2012) perceive dependability as the ability of the study to account for variability over time. Dependability was ensured even though there was a change to research design in terms of the number of learners that participated in the interviews. The plan was to interview six learners after the pre-test and post-test, I only interviewed four learners due to absenteeism. The structured interview questions were constructed to help me to answer research questions and responses were audio-recorded and transcribed. Therefore dependability was ensured by conduction of the planned pre-test and post-test and interviews.

4.8. Ethical considerations

According to Maree (2007) and Cohen et al. (2007), it is crucial to highlight ethical dilemmas when human subjects are included in research. I was permitted to carry out this study after applying for ethical clearance from the University of KwaZulu-Natal, obtaining permission from the principal of the selected school, informed consent from the parents and assent from the minor participants. Discussed below is the ethical consideration procedure:

The ethical consideration procedure was ensured before, during and after data collection:

a). Gate keeper permission was looked for

b). All participants were issued a letter of contest

c). Parents/guardians were issued a letter of contest

d). The researcher was transparent with regards to participants freedom of withdrawal

e). Participants confidentiality was ensured through the use of pseudonyms

f). Full details of participants and school understudy were not provided

g). Participants were issued copies of the findings of the study

h). Participants were constantly reminded of their role and rights

4.8.1 Ethical clearance

Obtaining ethical clearance from the study's ethics committee ensures that participants will not suffer harm (Scott, 2013). Research permission was granted by the University of KwaZulu-Natal in May 2022, after the researcher applied late for ethical clearance in January 2022. The clearance certificate (HSSREC/00004156/2022) is found in Appendix

4.8.2 Gaining access to the research site

I requested and obtained permission (Appendix B) from the principal of the selected school to conduct the study with Grade 12 learners at the school.

4.8.3 Informed consent

Considering that participants were the source of the data in this study, and that the study is intended to benefit them, Thuzini (2011) believes it is important to discuss the study's purpose with participants.

Participation in this study was voluntary and this was explained to learners. Learners were also told that the information gathered would not have any impact on their school marks but would be used for research purposes only.

When conducting research, obtaining informed consent from participants is important to ensure that participants are not coerced to participate in the study. I drafted an informed consent form (Appendix C) that explained the purpose and design of the study and addressed participants' right to withdraw, confidentiality, methodological rigour and fairness. The letter explained the procedures that would be followed during the research and provided contact details for relevant personnel at the University.

While some of the Grade 12 participants were adults who could give consent directly, for those under 18, letters were sent to their parents or guardians to obtain their informed consent. Marshall and Rossman (2011) argue that individual privacy and protection are important when reporting on research. To protect participants' identities and the name of the study site, pseudonyms were used.

4.9. Limitations of the study

I encountered several challenges while conducting the study.

As I am also a teacher, I had to leave my learners while conducting the study at a different school, but not using national hours stipulated by DOE but after school hours, that is when the study took place. The impact of this was minimised by making arrangements with my colleagues at the school to combine classes while she was away during the data collection process. This arrangement was approved by her departmental head.

Another limitation was that, while there were 30 participants, there were only 22 working computers available. To address this, some learners were asked to work in pairs. Care was taken to ensure that learners working in pairs were of different capabilities in terms of performance. Pairing was done on the basis of the results of the pre-test. This enabled some degree of collaboration, which is recommended by APOS theory. The shortage of computers created a level of challenge for learners who had to share, because one had to watch while the other drew and explored different triangles using GeoGebra. Thus, one learner of the pair gained more experience and understanding of the concept while the other gained less understanding because they didn't have the opportunity to use the software themselves. Even though they were alternating but at the time one is using the laptop the other gain less experience.

Another limitation was the issue of power dynamics. Although my position during the study was that of a researcher and not a teacher at the school, because I was a teacher within the same cluster of schools and partnered with the school for an extra lessons programme, the participants considered me to be one of the teachers. This introduced the possibility that they would be less willing to speak openly to me when responding to interview questions because I was not their teacher: learners sometimes pay less attention if someone other than their usual teacher teaches them; also, they might have been concerned that the results of the test would influence their marks. While it was impossible to ensure that power dynamics had no impact, care was taken to minimise the possible impact. For example, the principal came to observe the lesson sometimes, which encouraged them to focus. Also, before the data collection process started, the purpose of the study was explained to the participants and their right to withdraw anytime was clearly explained. This was supported by Neuman, (2006) in the statement that says that participants can acknowledge their rights as participants by reading and signing the declaration and all the learners under 18 had to sign assent form. Moreover, I worked alone with the learners during the data collection process which was different from the format of the extra classes, where learners were taught by a group of teachers. This made it difficult for me to engage with each learner individually. Sometimes they will change drawings and unable to fix the drawing and I had to assist them one by one whilst there was limited time per day which was one hour lesson. Also, the study was conducted in the computer lab which was a different venue than that used for conducting extra classes.

4.10. Conclusion

In this chapter, the interpretive paradigm, qualitative design, and case study style used in this study have been presented. A research site was selected using convenience sampling, which is described in this chapter. To select participants, a purposeful sampling method was used. The methods and instruments used to collect and to analyse the data were described. The trustworthiness and reliability of the data, ethical issues, and limitations of this study were discussed. The results of the study are presented in the next chapter.

CHAPTER 5: DATA PRESENTATION, ANALYSIS AND FINDINGS OF THE PRE-TEST

5.1. Introduction

Chapter 4 discussed the methodological framework used to answer the research questions that guided this study and described the paradigm within which the study was positioned.

One of the data collection instruments developed for this study was an activity sheet that was used as both a pre-test, before learners were introduced to GeoGebra, and as a post-test, after they had worked with GeoGebra, to track the development of their mental constructions. The questions were designed to gain in-depth information.

APOS theory was used in combination with genetic decomposition to analyse learners' responses to the pre-test to answer the research questions. This chapter presents the analysis of data related to the first two research questions:

- What are the learners' mental constructions of trigonometric 2D and 3D concepts?
- Why are learners able or not able to construct knowledge of trigonometric 2D and 3D concepts?

Chapter six will present the data analysis of the post-test so as the researcher will be able to answer the research question three.

The test consisted of 5 main questions, each with sub-questions (called 'items'), with 18 sub-questions in total.

Questions 1 and 3, and Questions 2 and 5 had been designed to assess the same mental construction to avoid estimation bias and also to give learners exposure to the way that 2D and 3D trigonometry exam questions are designed. While learners' responses to all of the questions were assessed, it was decided that analysis of Questions 1 and 2 would be seen satisfactory, and Questions 3 and 5 could be excluded from the analysis. This chapter thus presents the analysis of learners' responses to Questions 1, 2 and 4.

Question 1 was designed to integrate Euclidean geometry and 2D trigonometry. Learners were required to define and explain their understanding of the concepts and explain how they would solve a problem. The questions required few calculations. Question 1 was aligned with Levels

1 and 2 of Bloom's (1954) taxonomy, since in education it is crucial that teachers balance the levels of Bloom's taxonomy when designing an assessment.

Question 2 required learners to choose rules and integrate concepts within the subject. For example, they were asked to determine the value when the area is maximum, which involves integration of the area rule from trigonometry and application of calculus (optimization). These questions required learners to explore the relationships between topics and apply step-by-step methods to solve the problems. The items in Question 2 were aligned with Levels 3 and 4 of Bloom's (1954) taxonomy.

Question 3 of the test was not analysed as it was designed to elicit the same mental structures as Question 1 as stated above.

Question 4 required learners to differentiate between the usage of rules and the Pythagoras theorem and trig ratios to solve mathematics problems. The items in Question 4 were aligned with Levels 2 and 3 of Bloom's (1954) taxonomy.

Question 5 of the test was not analysed as it was designed to elicit the same mental structures as question 2 and 5 as explained above.

To analyse learners' responses, Ndlovu's (2014) categorisation system was used:

- Category 1 indicates responses that were correct, indicating that the learner had made the mental constructions necessary to solve the particular problem;
- Category 2 indicates an incomplete response in which the learner demonstrated a knowledge of procedures but failed to reach a correct final solution;
- Category 3 indicates responses where the learner attempted to solve the problem but displayed no mental constructions;
- Category 4 indicates instances where the learner lacked the necessary prerequisite knowledge to be able to attempt the problem.

The analysis was tabulated using Ndlovu's (2014) categories as follows:

CATEGORY	1	2	3	4
RESPONSES	Provide a correct and complete explanation	Partially correct	Incorrect answer	Not attempted

NO OF RESPONSES	

Responses falling in Category 1 indicate that the learner is operating at either the action level or process level. At the action level, a learner will show all the steps for solving the problem explicitly, while at the process level they will perform the same action but without writing all the steps down as they will have done some of them mentally. A learner operating at object level will be able to integrate the mathematics topics within and be able to use the area rule of 3D than the area formula of 2D shapes.

This chapter presents analysis of the learners' responses for the pre-test, only.

5.2. Analysis of learners' written responses for Question 1 of the pre-test

Question 1 was designed to assess a learner's mental construction of using cosine, sine and area rule of trigonometry. A learner able to use these rules correctly shows an understanding of the concepts of 2D and 3D trigonometry.

The following diagram was given for Question 1. The diagram shows a four-sided figure inside a circle; as all vertices touch the circle, the diagram shows a cyclic quadrilateral.

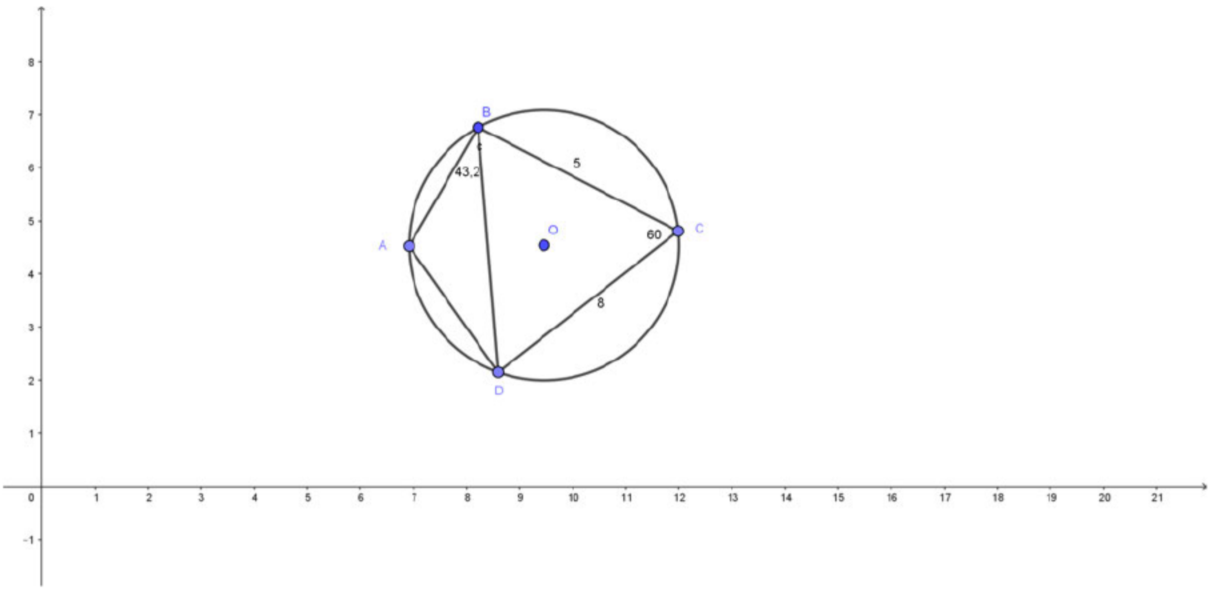


Figure 5.1: Diagram used for Question 1 modified from (DBE Exemplar 2014, P.9)

Question 1 had 5 sub-questions ('items').

5.2.1. Question 1, Item 1: *What is your understanding of a cyclic quad*?

Item 1 explored learners' definition of the cyclic quad concept. Tall (1981, cited in Ndlovu, 2014) states that concept definition assists in the construction of the concept image. Grade 12 learners were expected to be able to answer this question correctly because cyclic quads are covered in Grade 11.

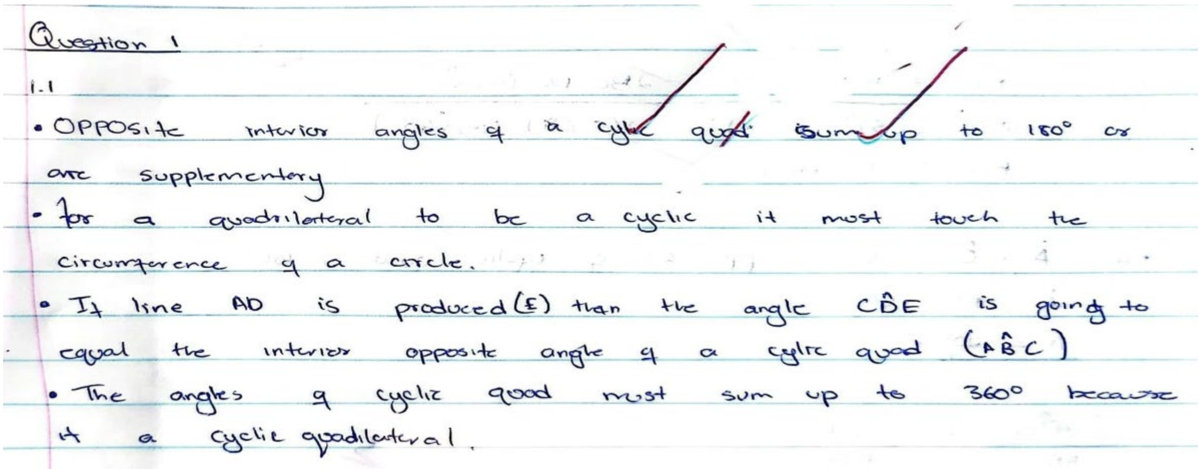
According to APOS theory, a learner operating at the action level should be able to name the parts of the circle and parts of a quad. Table 5.1 demonstrates that only twenty-three learners had constructed a conceptual understanding of a cycle quad (responses in Category 1). Sixteen out of twenty-three had developed action level and seven responded at the process level of APOS. Seven learners in category 3 showed poor conceptual understanding of this item.

Table 5.1: Analysis of learners' responses to Item 1 (Question 1)

CATEGORY	**1**	**2**	**3**	**4**
RESPONSES	Provide the correct and complete explanation	Partially correct	Incorrect answer	Not attempted

NO OF RESPONSES	23	0	7	0

The analysis, as shown in Table 5.1, indicates that twenty-three learners provided the correct concept definition of cyclic quad. However, in terms of their mental constructions, they displayed an action conception of the concept because, in their responses, they included irrelevant concepts or, instead of providing a coherent definition, listed the characteristics of a cyclic quad as separate facts. The genetic decomposition indicated that a learner who is able to name some of the parts of a circle or quad is operating at the action level. This is illustrated by Ndabenhle's written response, below.


Question 1

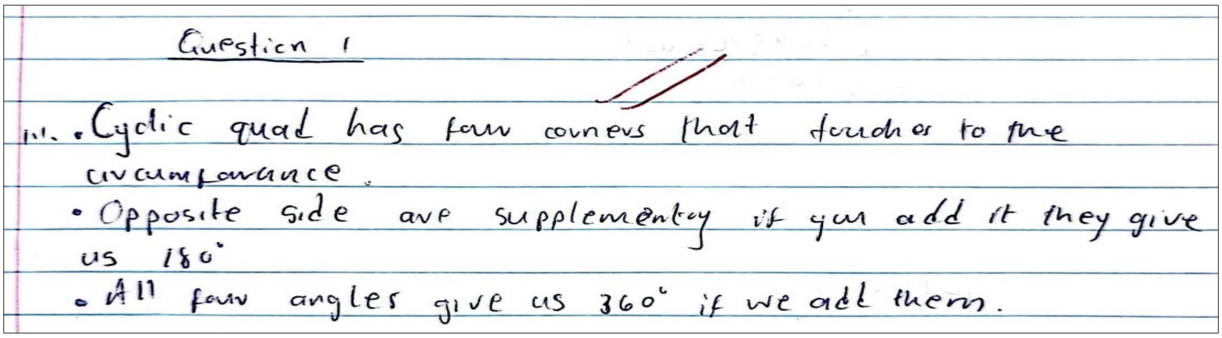
1.1

- Opposite interior angles of a cyclic quad sum up to 180° or are supplementery
- for a quadrilateral to be a cyclic it must touch the circumference of a circle.
- If line AD is produced (E) than the angle CD̂E is going to equal the interior opposite angle of a cyclic quad (AB̂C)
- The angles of cyclic quad must sum up to 360° because it a cyclic quadilateral.

Extract 1: Ndabenhle's written response to Item 1 (Question 1)

It is evident in this example that, while this learner had recalled some of the characteristics that define a cyclic quad, the concept had not been interiorised as they left out important facts or included irrelevant facts such as how a circle and a quad connect to each other so as to be called a circle quad. Ndabenhle demonstrated that he had confused the point and vertices/corner of a quad, because his concept definition of a vertex was in conflict with the definition of a point; as a result, this conflict seemed to prevent him from making the necessary mental construction to correctly understand what a circle quad is. While his response indicated that he knew parts of a cyclic quad, he struggled to construct a coherent definition.

Of the twenty-three learners, seven showed interiorisation of the action of defining a cyclic quad into a process as they were able to explain how the quad and circle connect to be called a cyclic quad. This was evident as they provided a concise definition of the cyclic quad, as illustrated by Thabo's response, below:



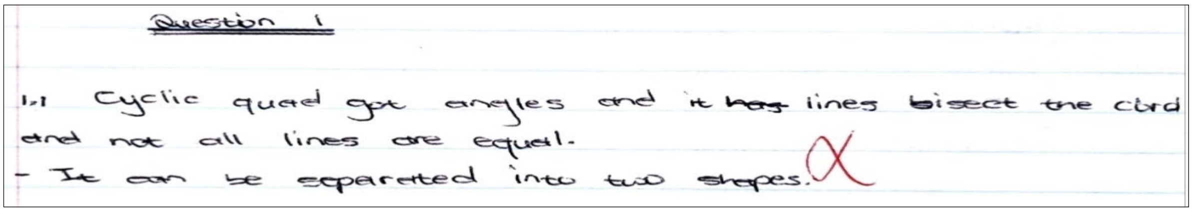
Question 1

1.1. • Cyclic quad has four conners that touch or to the circumforance.
• Opposite side are supplementary if you add it they give us 180°
• All four angles give us 360° if we add them.

Extract 2: Thabo's written response to Item 1 (Question 1)

Furthermore, it is evident that Thabo used the properties of a cyclic quad to construct a coherent definition as his response shows his understanding that a cycle quad does not end in four-sided figures with vertexes touching the circle.

Although the learners whose responses fell into Category 3 had attempted to solve the problem, instead of explaining the cyclic quad they stated theorems relating to cyclic quads—such as the theorem about opposite angles of a cyclic quad. Others appeared to state any theorem that came in to mind, as illustrated in the example below.



Question 1

1.1 Cyclic quad got angles and it ~~has~~ lines bisect the cord and not all lines are equal.
- It can be separated into two shapes.

Extract 3: Tebogo's written response to Item 1 (Question 1)

Tebogo's response fell into Category 3 as he gave an incorrect definition of a cyclic quad. Instead of defining a cyclic quad, he provided some facts about a theorem and combined them to define a cyclic quad. This revealed that Tebogo wrote what came into his mind about circle geometry in terms of the nine theorems taught in Grade 11.

5.2.2. Question 1, Item 2: *Explain how you would find the area of triangle DAB*

In Item 2, learners were asked to explain how to find the area of a triangle with reference to triangle DAB in Figure 5.1. Item 2 explored learners' mental constructions of areas of triangles. Knowledge of 2D geometry is critical in understanding trigonometric 2D and 3D concepts. Item 2 focused on learners explaining the process rather than carrying out the procedure. However, knowledge of the procedure is critical in the understanding of the process. Maharaj

(2014) and Ndlovu (2015) argue that learners generally do not engage with what they write but asking them to explain the thought process helps them to engage with the concepts. In line with this, Item 2 aimed to engage learners about their thought processes.

For this item, a learner operating at the APOS action level should have explained in steps how to find the area of triangle DAB. Table 5.2, below, shows that only five learners in Category 1 understood the concept and their written responses showed that they were operating at the action level; seven learners in Category 2 had not fully developed action level; while fifteen learners in Category 3 and three learners in Category 4 struggled with the conceptual understanding needed to reach the action level.

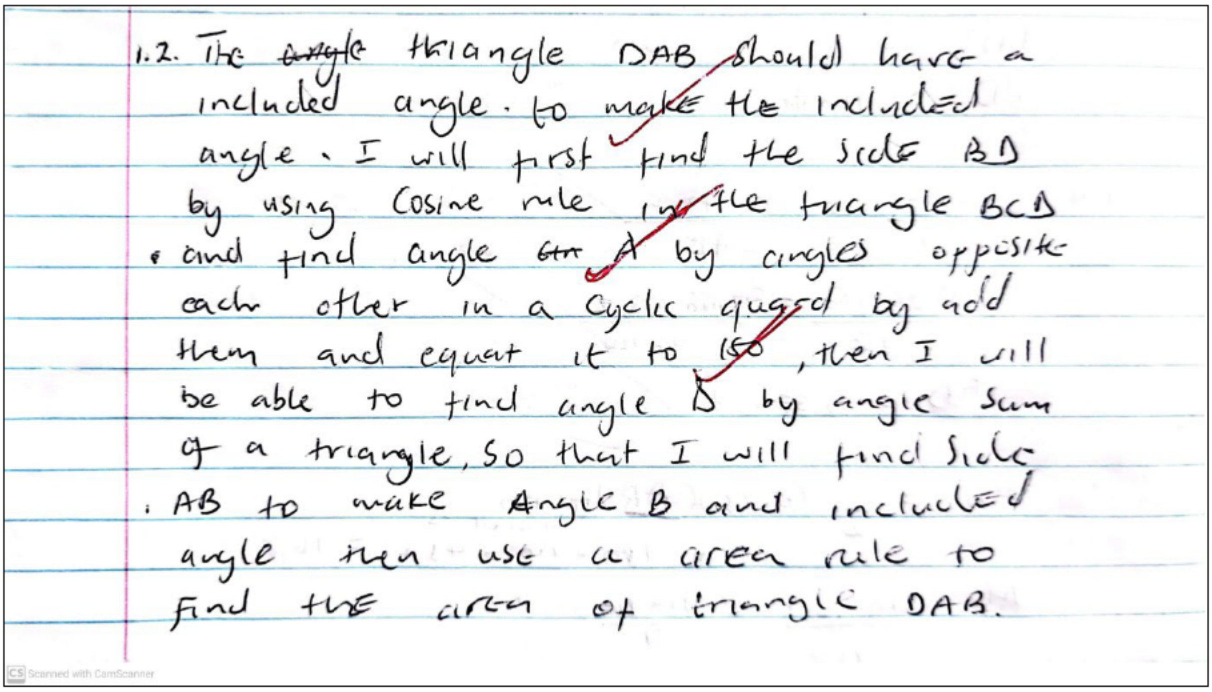
This item explored whether action had been interiorized to process stage. Learners at the process level are able to describe, or reflect upon, the steps of the transformation mentally without actually performing those steps explicitly (Vilakazi, 2021). This item required learners to explain how they would solve the problem, without actually solving it. Fifteen learners attempted to answer the question but failed to explain how they would find the area of triangle DAB, as shown in Category 3 of Table 5.2; three learners in Category 4 were unable to attempt it. Seven learners tried to explain how they would solve the area of triangle DAB' however, their explanations were considered partially correct because, while some parts of the explanation related to finding the area of a triangle, the explanation showed gaps in their knowledge constructed. Also, they focused on explaining the procedure to be followed to calculate the area but did not state the rules that need to be used. For example, some stated that they needed to find the included angle of triangle DAB, which is D, by first finding angle A of triangle DAB without mentioning the rules to be used for that particular step. Even with explaining the procedure, some procedures were incomplete. For example, after explaining the need to find the included angle, they could not articulate to explain how these sides would be found by statin the correct rule. Their explanations were incomplete and some procedures were explained incorrectly because some learners used the sides of triangle DCB as sides that form an included angle because two sides of triangle DCB are known. Their explanations were wrong because they used the sides of triangle DCB instead of the sides of triangle DAB.

Table 5.2: Analysis of learners' responses to Item 2 (Question 1)

CATEGORY	1	2	3	4

RESPONSES	Provide the correct and complete explanation	Partially correct	Incorrect answer	Not attempted
NO OF RESPONSES	5	7	15	3

As illustrated in Table 5.2, only five learners in Category 1 provided a correct response. An example of one of the correct responses is shown below.


1.2. The ~~angle~~ triangle DAB should have a included angle. to make the included angle. I will first find the side BD by using Cosine rule in the triangle BCD and find angle ~~Ctr~~ A by angles opposite each other in a cyclic quad by add them and equat it to 180, then I will be able to find angle D by angle sum of a triangle, so that I will find side AB to make Angle B and included angle then use a area rule to find the area of triangle DAB.

Extract 4: Mzwandile's written work for Item 2 (Question1)

Mzwandile's response fell into Category 1. It is evident that he had grasped the concept well. He provided the explanation without having to perform step-by-step procedures, thus showing that he had interiorized the action of calculating the area of a 3D triangle into a process. This was evident as he explained the process of finding the area of the triangle. In addition, being able to differentiate between the processes involved to find the area of a 3D triangle and a 2D triangle shows that his process conception was well-developed and that he was transitioning to encapsulate the process into an object. The concept of the area of a 3D triangle had been conceptualized as the whole entity, in which other processes and actions could be performed as he differentiated between the area rule and formula $A=\frac{1}{2}\times b\times h$ for calculating the area of a 2D triangle.

In APOS theory, a learner is considered to be operating at the process level if they are able to explain the step-by-step process for solving the problem without writing the steps down. In Mzwandile's response, he successfully gave a detailed explanation of his process for the solution method.

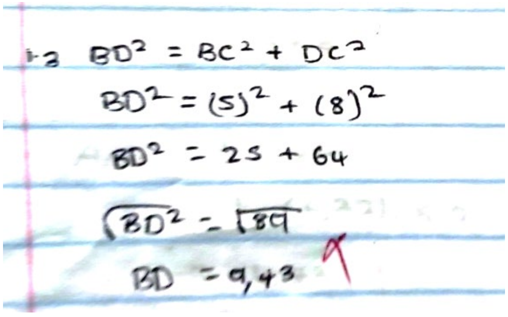
5.2.3. Question 1, Item 3: *Calculate the length of BD*

In Item 3, learners were asked to calculate the length of BD, with reference to Figure 5.1. This required a learner to invoke the rule of cosine based on the information given in the diagram, which is a condition of an included angle. However, the analysis showed that some of the learners who did provide a correct answer (Category 3) failed to recall the condition to invoke the correct rule. This indicated that they had not reached the action level.

Table 5.3: Analysis of learners' responses to Item 3 (Question 1)

CATEGORY	1	2	3	4
RESPONSES	Provide the correct and complete explanation	Partially correct	Incorrect answer	Not attempted
NO OF RESPONSES	5	6	14	5

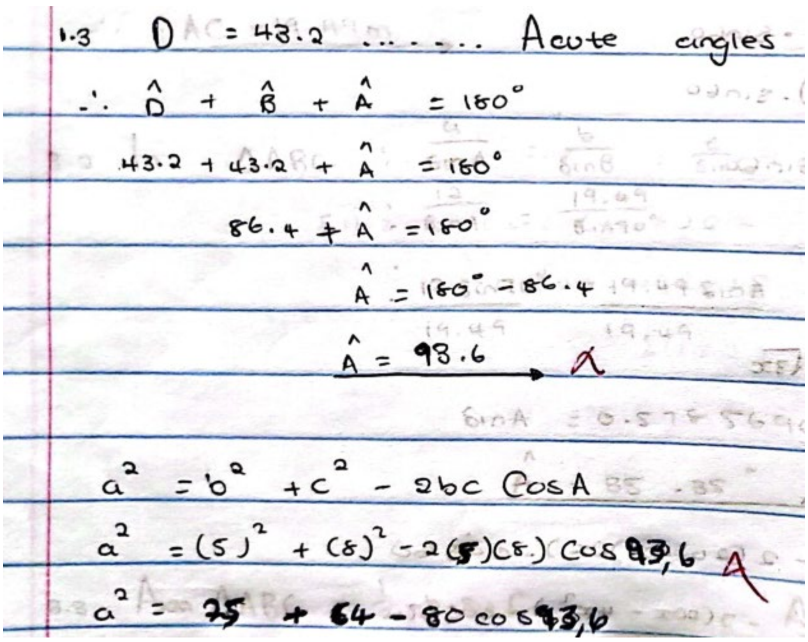
Thirteen of the learners whose answers were incorrect (Category 3) used the theorem of Pythagoras to solve the length of BD. However, the theorem of Pythagoras is only used to find the sides of a right-angled triangle, and the triangle in the diagram is not a right-angled triangle. The fact that these learners used the theorem of Pythagoras indicated that they had failed to construct the relevant mental structures to invoke the correct cosine rule to be used in this situation. An example of one of these learners' responses is shown in Extract 5, below.



Extract 5: Thabiso's written work for Item 3 (Question 1)

Thabiso demonstrated a limited understanding of how to apply the Pythagoras theorem by using it to find a missing side but ignoring the condition that the theorem is only applicable to right-angled triangles, while in this case BD is not the side of a right-angled triangle.

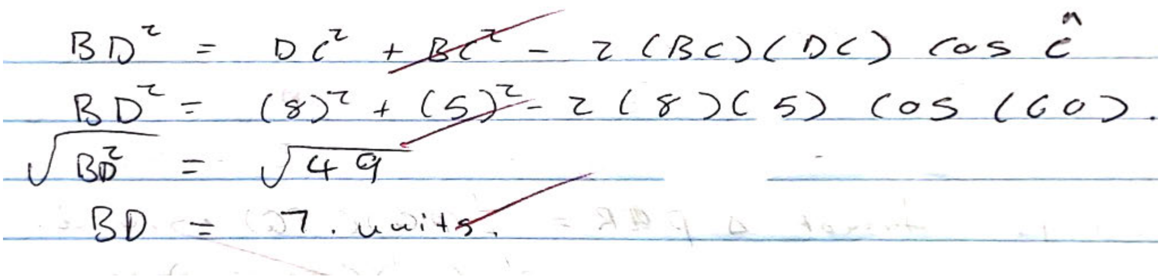
Another learner who gave an incorrect response (Category 3) assumed that triangle DAB was an isosceles triangle and used the properties of an isosceles to attempt to solve it. He said that angle D and angle B were equal and, on this basis, used the angle sum of a triangle to find angle A. Thereafter, he used angle A as an included angle with side lengths but not of not triangle DAB but of the triangle opposite to triangle DAB using a cosine rule to find side BD, as shown in Extract 6, below.



Extract 6 Thabo's written response for Item 3 (Question 1)

Although Thabo invoked the correct rule to be used, the response showed that he did not understand the concept of the included angle. Tall (2008) also found that the properties of an isosceles triangle elicited misconceptions, where individuals assumed that a triangle sides that appeared visually to be of the equal length was an isosceles, without first proving this.

A different pattern was seen with responses that fell into Category 1, in which all five learners provided complete and correct responses and all of them were found to be operating at the action level. An example of one learner's written work is shown in Extract 7, below.



Extract 7 Mzwandile's written work for Item 3 (Question1)

Mzwandile's answer demonstrates that he was able to apply a correct mathematical operation to the problem following conventional procedures. His response also shows that he had grasped the rules of trigonometry and knew how and when to apply them, as did the other learners in Category 1.

Analysis of the learners' mental constructions revealed that the four learners who had provided the correct response were operating at the action level in terms of identifying the correct rule to solve a trigonometric 3D problem. Having the visual representation of the diagram triggered the action to be performed. As shown in Mzwandile's response above, the action of using cosine rule to calculate the side of any triangle was performed step by step. Dubinsky (1991) states that the ability to perform stepwise actions is indicative of the action level of APOS theory. All of these learners had to first extract all the variables needed to perform the procedure for calculating DB and substitute the values given. As can be seen in Mzwandile' response, the preceding step triggered the next step to be performed and, thus, no step could be skipped to arrive at the answer.

5.2.4. Question 1, *Item 4: Calculate the side AD*

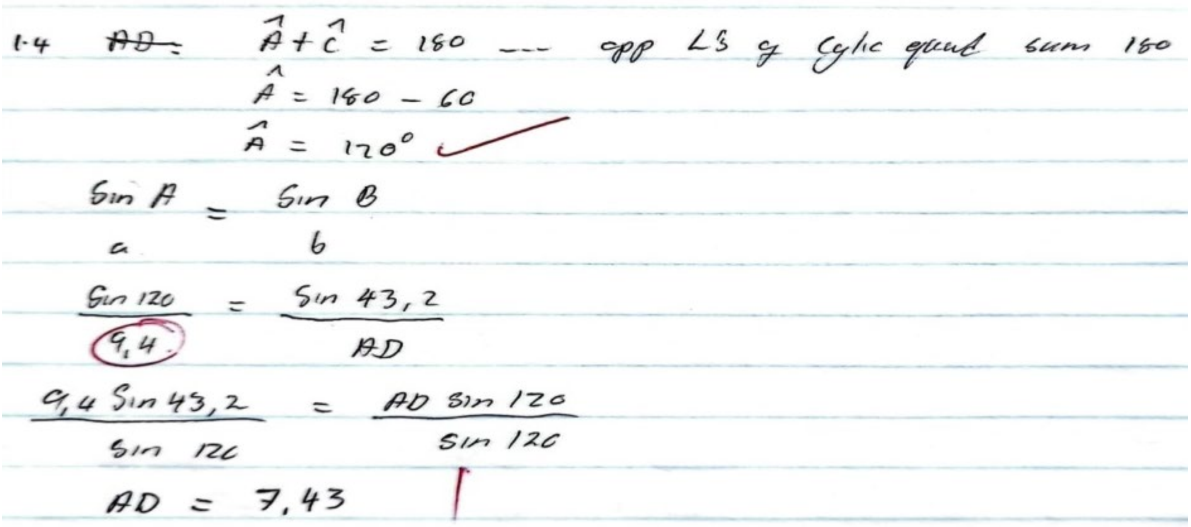
While Item 3 invoked the concept of the cosine rule, Item 4 invoked the concept of the sine rule. The two items required learners to calculate the sides of a triangle, thus structuring the question so that learners could identify when and how to use the cosine and sine rules.

A learner operating at the action level for this item would be able to select the correct rules and complete all the steps to answer the question correctly. A learner operating at the process level would do the same but would not write out all of the steps as they would be able to do them mentally. Table 5.4 shows that, of the four learners who demonstrated that they had an action-level conception, none demonstrated that they had reached the process level in their written work for this question.

Table 5.4: Analysis of learners' responses for Item 4 (Question 1)

CATEGORY	1	2	3	4
RESPONSES	Provide the correct and complete explanation	Partially correct	Incorrect answer	Not attempted
NO OF RESPONSES	4	5	14	7

A similar pattern was observed with Item 4 as was seen with Item 3: most of the learners failed to recall the rule to be used to calculate AD. Also following the pattern of responses for Item 3, learners who gave partially correct answers (Category 2) demonstrated that they were able to invoke the correct rules but lacked the procedural knowledge to solve the problem. For example, they were able to find angle A which is an opposite angle with side BD in a triangle, therefore the sine rule will be used to find side AD, however they failed. What hindered their progress was that they extracted incorrect values to be substituted into the formula, as shown in Extract 8, below:

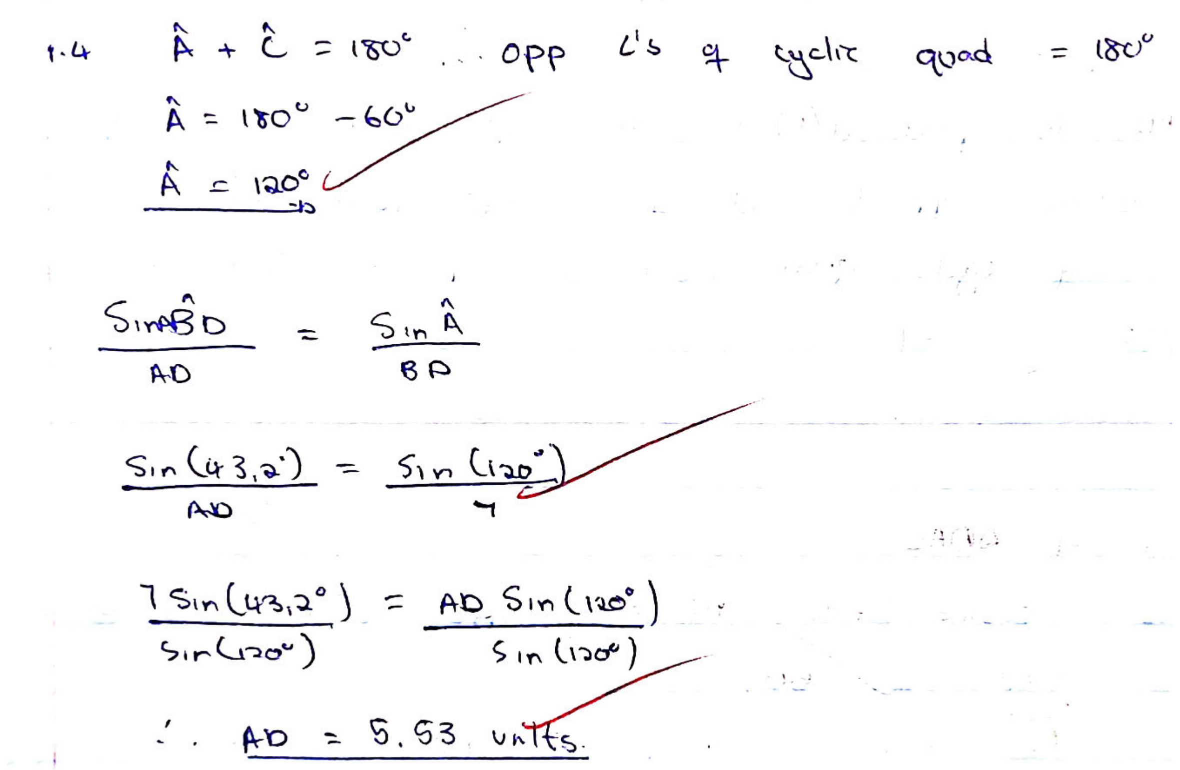


Extract 8 Ndabenhle's written work for Item 4 (Question 1)

Ndabenhle recalled the correct formula to use to calculate angle A. However, he failed to substitute the correct values to solve for AD. Other learners with partially correct responses (Category 2) also correctly recalled the rule to be used, but failed to identify the correct values to be used, suggesting that they had not reached the action stage.

Learners with completely incorrect responses (Category 3), however, failed to invoke the correct rule to be used. Their errors were similar to those in Item 3, because they used formulas that applied to right angle triangles as they tried to use trigonometry ratios to calculate AD. Also in item 4 they commonly used theorem of Pythagoras while the triangle is not a right angled triangle such as extract 5 of Thabiso written work. Ndlovu and Brijlall (2015) stated that when rules are memorised without conceptual understanding, cognitive conflict may arise.

Analysis of responses where the learners provided correct answers (Category 1) revealed that they had recalled the correct rule and invoked the correct steps to be performed to determine AD. The extract below illustrates this.



Extract 9 Nhlakanipho’s written work for Item 4 (Question 1)

As was the case with Category 1 responses for Item 3, none of the learners’ responses in Category 1 showed interiorization of the action stage into a process (as evident in Nhlakanipho’s response, above) as all of the learners wrote out their step-by-step calculations to determine AD. This indicated that they still relied on external cues to solve the problem: the preceding step informed the next step. DeVries and Arnon (2004) indicate that, at the process stage, an individual can invert processed in their head without the need for external cues. In this study, responses for the pre-test showed that learners had not interiorized the action of solving trigonometric 2D and 3D problems into a process.

Item 5 focused on calculating the area of triangle DAB which is more the same as item 2 where learners were asked to explain how they will find the area of triangle DAB. In mathematics learners find it hard to deal with word problems; this is evident in DBE (2021) in the probability section of counting principle. Learners lack to understand and translate word problems into numbers. Therefore item 2 was designed to develop the learners’ comprehension skills just before embarking in problem solving. Learners in mathematics find it hard to deal with the problem when there is a slight change from what or how questions are normally asked. This is seen in the extract below.

5.2.5. Question 1, Item 5: *Calculate the area of triangle DAB*

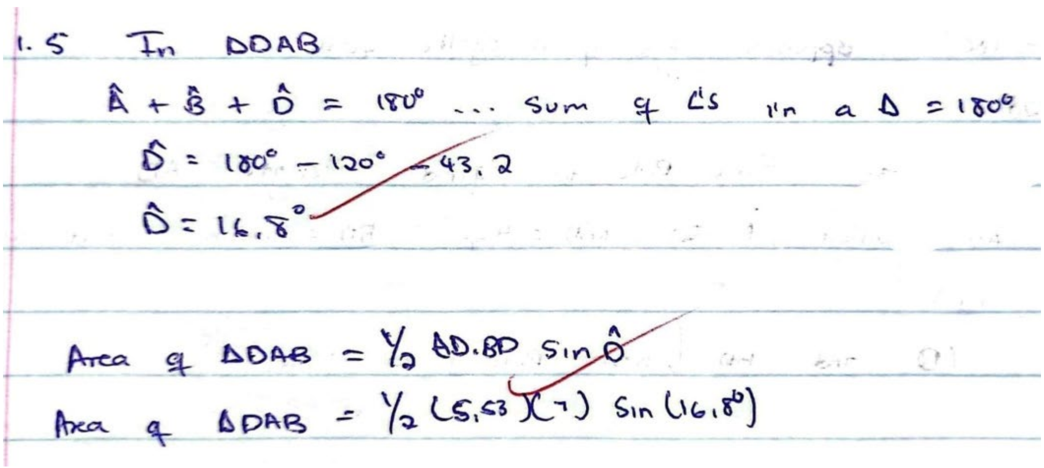
While Items 3 and 4 focused on calculating parts of the given triangle, Item 5 focused on calculating the area of that triangle. Thus, the solutions to Items 3 and 4 were intended to guide the participant towards solving Item 5.

Table 5.5 indicates that four learners' responses were correct (Category 1). Learners were required to use the area rule of 3D shapes. As stipulated in the genetic decomposition, a learner who is able to use the area rule for 3D is considered to be operating at the object level of APOS. Thus, the four learners who gave correct responses were found to be operating at the object level. Learners whose responses fell into other categories were found to be struggling with conceptual and procedural knowledge of 3D area still.

Table 5.5: Analysis of learners' responses of item 5 (Question 1)

CATEGORY	1	2	3	4
RESPONSES	Provide the correct and complete explanation	Partially correct	Incorrect answer	Not attempted
NO OF RESPONSES	4	6	9	11

A learner who was unable to solve Items 3 and 4 would not be able to solve Item 5 because it uses the lengths determined in Items 3 and 4 to form an included angle of D so that a learner can use an area rule. In addition, a learner showing the encapsulation of a 2D area to calculate the area of a 3D shape indicates that they are operating at the object level of APOS. Learners were able to make connections between the formula for finding the area of a 2D shape and that of a 3D shape and perform the necessary actions and processes to determine the area of triangle using an area rule of trigonometry. This is illustrated in Extract 10 from Nhlakanipho's written work, shown below. His action of remembering to use the correct rule (the area rule) instead of the area formula of A=$\frac{1}{2} \times b \times h$ indicated that he was operating at the object level of APOS theory.



Extract 10 Nhlakanipho's written work for Item 5 (Question 1)

5.3 Analysis of learners' written responses for Question 2 of the pre-test

Question 2 integrates calculus concepts and trigonometric concepts as it focuses on optimization in 2D and 3D of trigonometric concepts. The diagram in Figure 5.2 was given for Question 2. The question had 3 items for participants to solve.

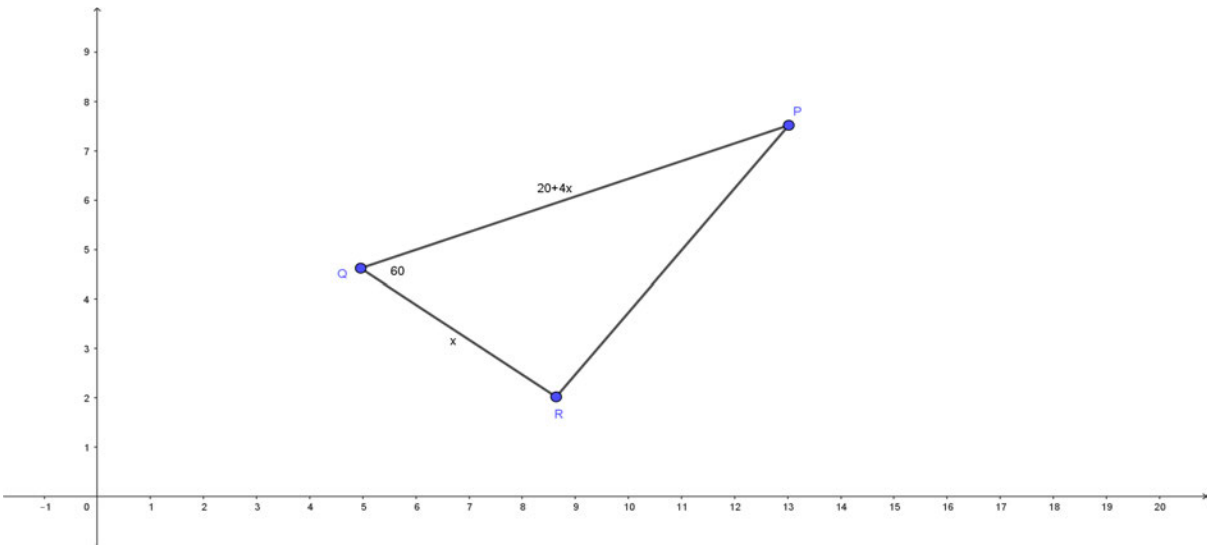


Figure 5.2: Diagram for Question 2 from (DBE February / March 2016, p.9)

5.3.1. Question 2, Item 1: *Show that the area of* $\Delta PQR = \sqrt{3}x - \sqrt{3}x^2$

Item 1 of Question 2 explored learners' mental construction of applying proofs using the concept of the area rule.

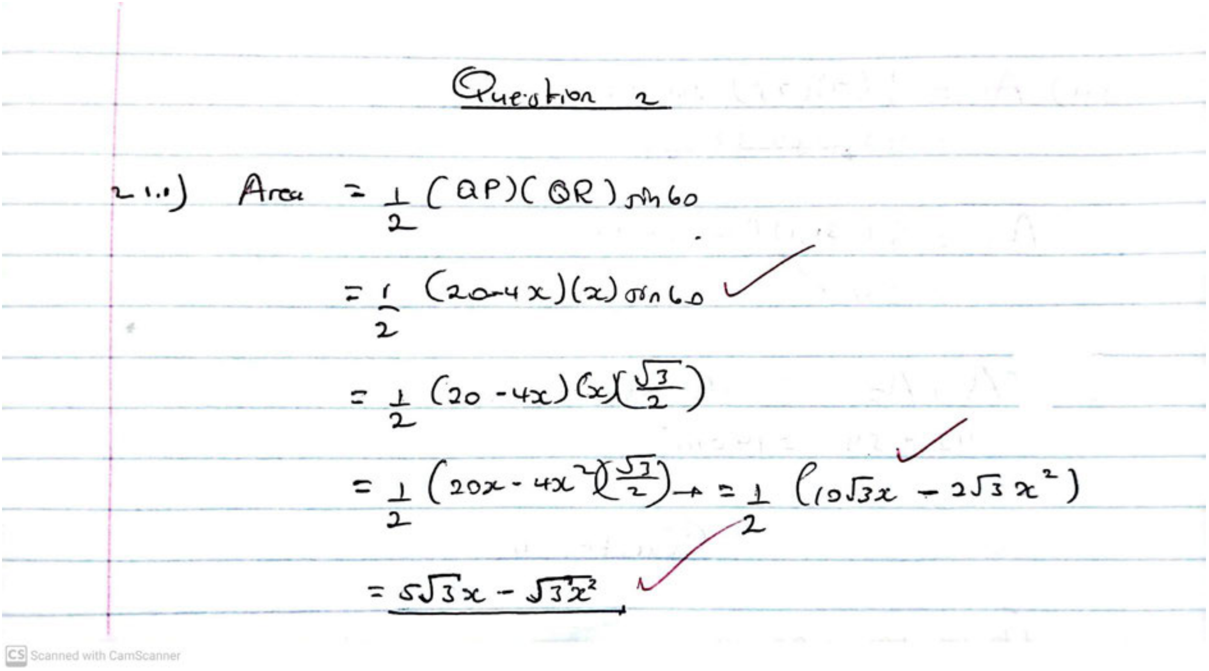
Table 5.6: Analysis of learners' responses of item 1 (Question 2)

CATEGORY	1	2	3	4
RESPONSES	Provide the correct and complete explanation	Partially correct	Incorrect answer	Not attempted
NO OF RESPONSES	5	3	17	5

The majority of learners whose responses fell into Category 3 demonstrated that they were unable to retrieve the correct formula to be used to show the area of the given triangle. Instead of recalling the formula for the area, they recalled the cosine rule, which was not applicable in this case. This indicates that when these learners lacked understanding they could not sift through their knowledge to identify the correct information needed to solve a particular question. In three of the Category 3 responses, recall of the formula was triggered by the word 'area' in the question; however, instead of invoking the correct formula these learners recalled the formula for calculating the area of a 2D triangle (A=$\frac{1}{2} \times b \times h$). Since the question asked learners to show that ΔPQR=$\sqrt{3x}$-$\sqrt{3}x^2$, these three learners were unable to proceed using the formula A= $\frac{1}{2} \times b \times h$, as there was no base or height given. The challenges encountered by these learners indicated that they had not made the mental constructions of using an area rule.

Learners whose responses fell into Category 2 were able to recall the correct formula but failed to substitute the correct values or failed to carry out the required procedures to solve the problem, which highlights that having knowledge of isolated facts is not always enough to be able to solve the problem.

Learners whose responses fell into Category 1 provided a correct, detailed response. These five learners were able to recall the rule and correctly illustrated the step-by-step process to show that Area=$5\sqrt{3}x - \sqrt{3}x^2$. This is illustrated in Extract 11, which shows Lungile's written work.


Question 2

2.1.1) Area $= \frac{1}{2}(QP)(QR)\sin 60$

$= \frac{1}{2}(20-4x)(x)\sin 60$

$= \frac{1}{2}(20-4x)(x)\left(\frac{\sqrt{3}}{2}\right)$

$= \frac{1}{2}(20x-4x^2)\left(\frac{\sqrt{3}}{2}\right) \rightarrow = \frac{1}{2}(10\sqrt{3}x - 2\sqrt{3}x^2)$

$= 5\sqrt{3}x - \sqrt{3}x^2$

Extract 11: Lungile's written work for Item 1 (Question 2)

Lungile's response, which fell into Category 1, showed her conceptual understanding and that she had reflected on the given formula and substituted values without the need to transform the equation to determine the subject of the formula. All five learners in Category 1 were operating at the object level.

A learner in extract 12 with one other learner showed all steps in their written responses and the other three learners got the answer correct but some of the steps did not appear in their workings.

5.3.2. **Question 2, Item 2:** ***Determine the value of x for which the area of ΔPQR will be in maximum***

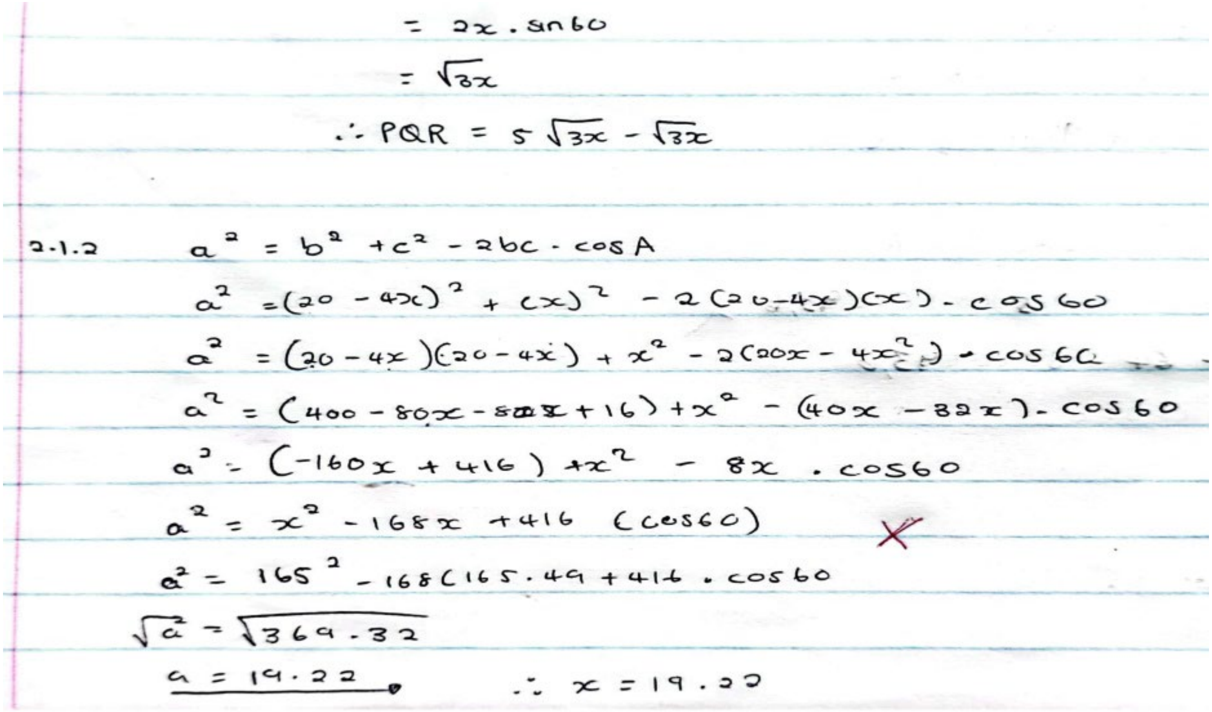
For Item 2, learners were expected to use the concept of maximum and minimum and integrate it while solving problems in trigonometric 2D and 3D concepts. The table below illustrates that only three learners' responses fell into Category 1, demonstrating that they understood the concept and were able to integrate the calculus concept into trigonometry. The genetic decomposition in Chapter 3 indicated that a learner that can integrate other topic of maths within maths are used to process level in totality and encapsulate to object level, meaning they are operating at object level. Therefore all the three learners whose responses fell into Category 1 were found to be operating at the object level of APOS. Learners in Categories 2, 3 and 4 had not developed the conceptual understanding of integrating concepts.

Table 5.7: Analysis of learners' responses to Item 2 (Question 2)

CATEGORY	1	2	3	4
RESPONSES	Provide the correct and complete explanation	Partially correct	Incorrect answer	Not attempted
NO OF RESPONSES	3	5	15	7

Table 5.7 above shows that only three learners were able to integrate by deriving the equation to arrive at the correct solution (Category 1). Five learners saw that they had to derive but derived incorrectly (Category 2) and, therefore, had not reached the object level. Twenty-two learners did not know how to approach the problem (Categories 3 and 4).

The main challenge evident in responses that fell into Category 3 was the integration of calculus with trigonometry concepts; thus, learners failed to see that they needed to first determine the derivative of the area to find the value of x. Khuzwayo (2019) notes that if mathematics is learnt in a compartmentalised way, learners will be unable to integrate concepts. It is evident with learners whose responses fell into Category 3 that, when dealing with trigonometry concepts, they were unable to draw from other taught concepts to solve the given problem. This question aimed to ascertain learners' encapsulation of trigonometric 2D and 3D concepts; thus, the findings show that learners whose responses fell into Category 3 had not made the necessary mental construction. This is illustrated in Thabiso's response, below, where he applied the cosine rule even in contexts where it was not applicable. This suggests he struggled to relate concepts from different areas of mathematics to each other.



$= 2x \cdot \sin 60$

$= \sqrt{3x}$

$\therefore PQR = 5\sqrt{3x} - \sqrt{3x}$

2.1.2 $a^2 = b^2 + c^2 - 2bc \cdot \cos A$

$a^2 = (20 - 4x)^2 + (x)^2 - 2(20-4x)(x) \cdot \cos 60$

$a^2 = (20 - 4x)(20 - 4x) + x^2 - 2(20x - 4x^2) \cdot \cos 60$

$a^2 = (400 - 80x - 80x + 16) + x^2 - (40x - 32x) \cdot \cos 60$

$a^2 = (-160x + 416) + x^2 - 8x \cdot \cos 60$

$a^2 = x^2 - 168x + 416 \ (\cos 60)$

$a^2 = 165^2 - 168(165.49 + 416 \cdot \cos 60$

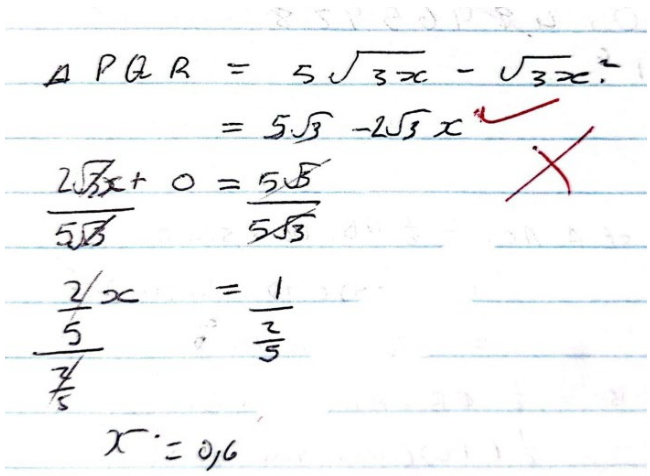
$\sqrt{a^2} = \sqrt{369.32}$

$a = 19.22$ $\therefore x = 19.22$

Extract 12 Thabiso's written work for Item 2 (Question 2)

While the given shape is labelled PQR, Thabiso continuously used a, b and c to name the sides, ignoring the information given. Failure to extract the correct information in a given problem indicates a lack of understanding of the procedures to be followed and to make meaning. In his final answer, he calculated one side and replaced it with *x,* showing that he did not understand the rule he was using. The failure of the learners in Category 3 to integrate concepts concur with the findings of Siew et al. (2016) that learners have difficulty in coming up with solutions to questions that require application and understanding of algebraic expressions to perform complex processes.

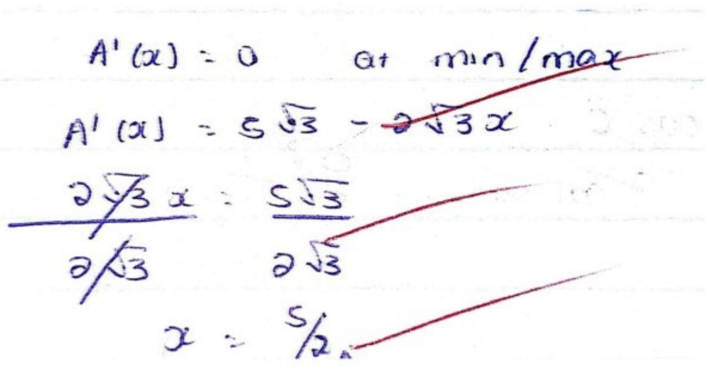
Of those learners in Category 2 that had an idea that they had to derive but failed to carry out their calculations to the final step, an example is shown in Extract 13, below.



Extract 13: Philani's written work for Item 2 (Question 2)

It is clear that Philani had an idea how to solve the problem, because his first step when he derived the equation showed that he understood the integration of topics within the subject. By making x the subject of the formula, however, he revealed that he lacked the algebraic skill to solve the problem; this was evident when he divided by $5\sqrt{3}$ instead of by $2\sqrt{3}$. The learners in Category 2 failed to perform the action to determine the subject of the formula, and therefore had not developed the necessary mental structures to reach the object level of APOS.

Three learners were able to carry out the problem completely and correctly (Category 1). This is illustrated in Nhlakanipho's written work, shown in Extract 14, which reveals his conceptual understanding and his understanding of topic integration. He was thus operating at the object level as he was able to integrate calculus into trigonometry. The responses of the other two learners in this category also revealed that they were operating at the object level.



Extract 14: Nhlakanipho's written work for Item 2 (Question 2)

5.3.3. **Question 2, Item 3:** ***Calculate the length of PR if the area of ΔPQR as a maximum***

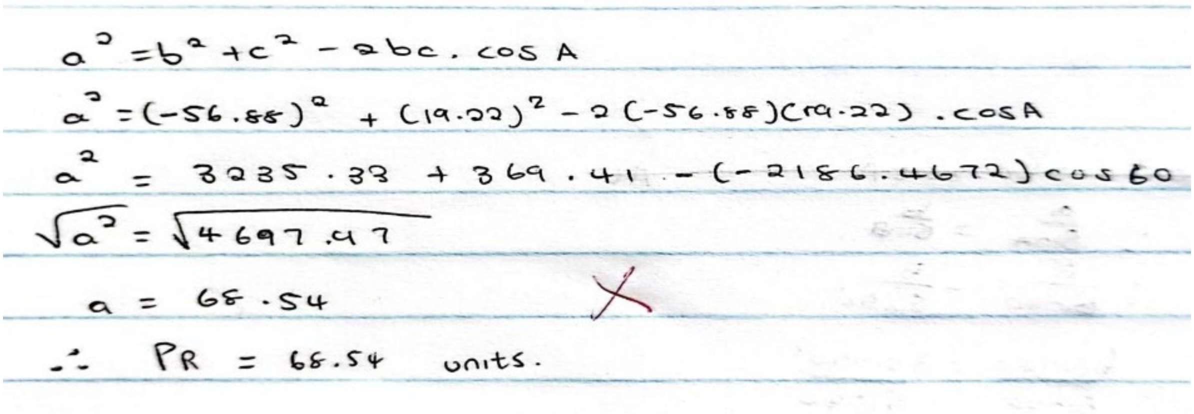
Similarly to Item 2, Item 3 required the integration of calculus and trigonometric concepts.

The majority of the learners (20 out of 30) did not attempt to solve this problem (Category 4). Of the 10 who attempted it, the majority (9) failed to solve it (Category 3); only one learner was able to provide the correct answer (Category 1). If a learner failed to answer the previous question they would not be able to answer this question. Thus, many learners could not even attempt it.

Table 5.8: Analysis of learners' responses to Item 3 (Question 2)

CATEGORY	1	2	3	4
RESPONSES	Provide the correct and complete explanation	Partially correct	Incorrect answer	Not attempted
NO OF RESPONSES	1	0	9	20

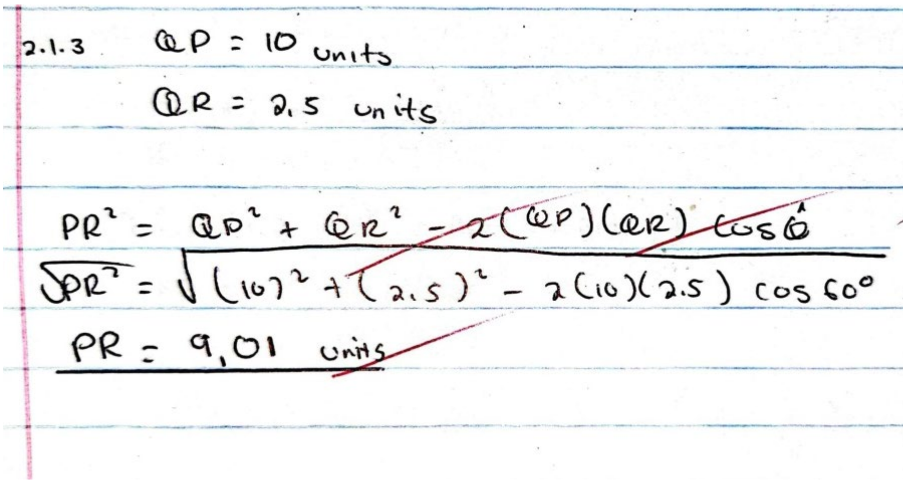
It is interesting to note that while the nine learners who answered the question incorrectly (Category 3) failed to solve the problem, they were able to recall the formula to be used. However, their failure to carry out the procedure correctly suggests that their choice of the formula was based on guessing. This is also based on the analysis of the previous question where learners in Category 3 tried to apply cosine rule for every question. Although they recalled the formula, four failed to extract the correct facts as they used a, b and c to name the sides instead of P, Q and R. This is illustrated by Thabo's response, shown in Extract 15 below.



Extract 15 Thabo's written work for Item 3 (Question 2)

As illustrated in Extract 15, while the correct formula was recalled, the variables used and the values substituted were incorrect, thus leading to the incorrect answer. What is noticeable is that, as noted by Maharaj (2014), that learners did not engage with what they wrote and thus the lack of development of the necessary mental constructions meant that the concept had not been constructed mentally.

While most of the learners were found to not have the required mental structures to solve Item 5, Mzwandile's response (Category 1) showed all the necessary mental construction and conceptual understanding. His work shows that he was operating at the process stage, because some of the steps were skipped and the solution was correct.



Extract 16: Mzwandile's written work for Item 3 (Question 2)

5.4 Analysis of learners' written responses for Question 4 of the pre-test

In this question, the aim was to explore learners' mental construction of integrating trigonometric concepts. The application of trigonometric ratios and cosine rule was needed to solve the three items in this question.

The diagram given for Question 4 is shown in Figure 5.3.

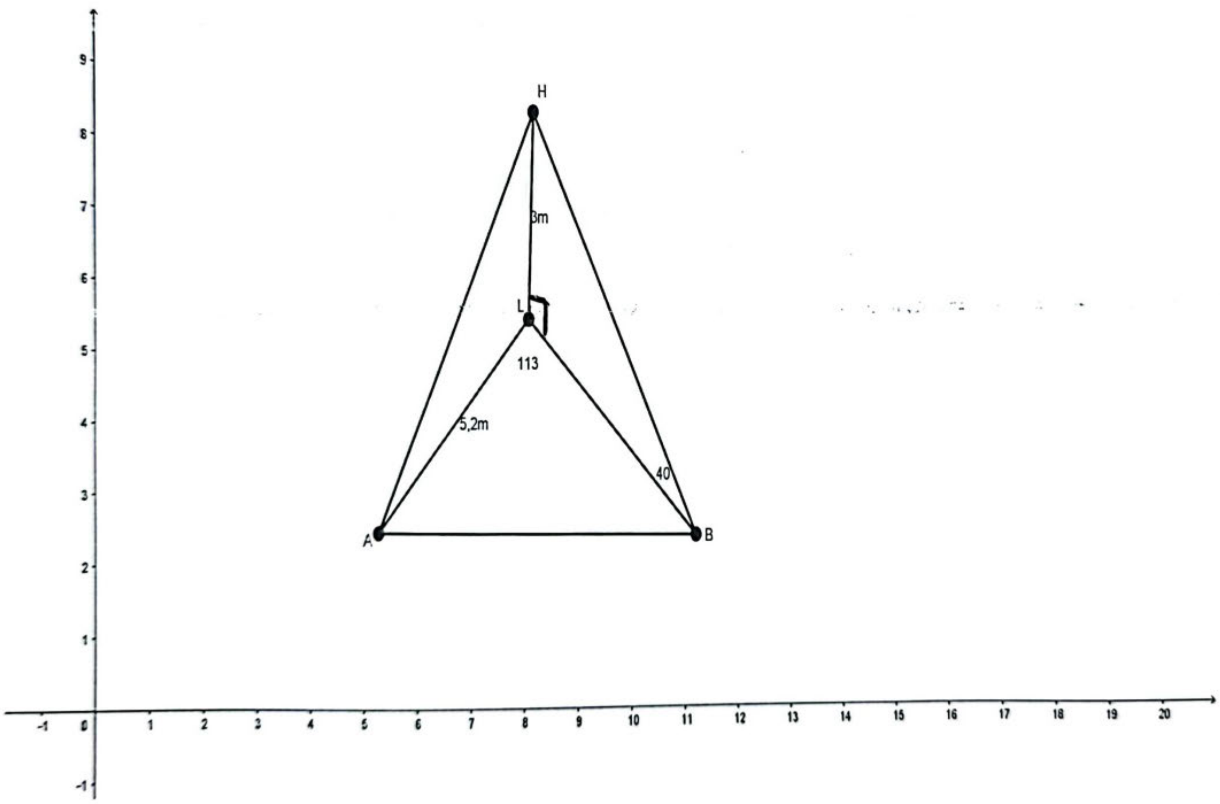


Figure 5.3: Diagram used for Question 4 from (DBE November 2008, p.7)

5.4.1. Question 4, Item 1: *Calculate the length of LB*

Item 1 aimed to ascertain learners' mental constructions of calculating the length of LB.

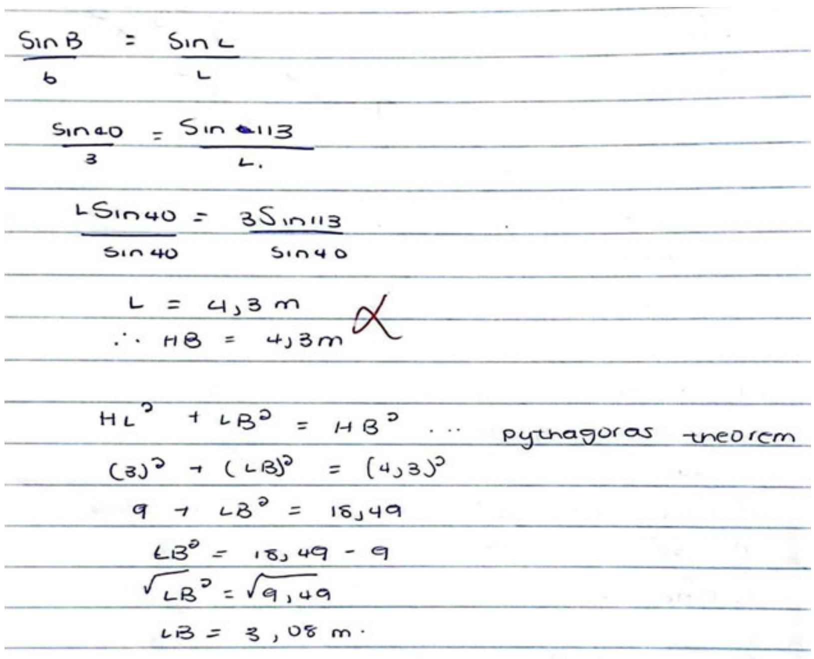
Table 5.9 indicates that there were nine learners in Category 1 who were able to understand the concept and twenty-one learners across the other three categories who showed no conceptual understanding.

Table 5.9: Analysis of learners' responses of item 1 (Question 4)

CATEGORY	1	2	3	4
RESPONSES	Provide the correct and complete explanation	Partially correct	Incorrect answer	Not attempted
NO OF RESPONSES	9	4	14	3

For this item, a learner was considered to be acting at the action level if they were able to choose the correct rule and show all their steps to arrive at the solution. A learner who used the correct rules and was able to arrive at the correct answer without explicitly working through each step had gone further in their development and was considered to be working at the process level. The nine learners who provided complete and correct answers (Category 1) demonstrated that they had developed the necessary mental structures. Of these nine learners, seven were found to be operating at the action level and two at the process level. Those whose responses were incomplete (Category 2) or incorrect (Category 3), or who didn't attempt to answer the question (Category 4), were found to have not developed the action conception needed for this item.

The majority of the learners (17 of 30) demonstrated that they did not comprehend the concept of three dimensions in this question. These learners failed to break down the diagram to see that LB is part of a right-angled triangle for which the Pythagorean theorem and trigonometric ratios could be applied. As was the case with Question 2, responses that fell into Category 3 showed that learners had used other formulas that came to mind without interrogating the meaning of the question to understand what was being asked. This is seen in Thabiso's response, shown below.



Extract 17 Thabiso's written response for Item 1 (Question 4)

Retnowati and Maulidya (2018) argue that solving trigonometric problems is considered complex because it involves not only trigonometric ratios but also other knowledge bases. As shown in the findings of this study, learners whose answers fell into Category 3 failed to identify the knowledge bases needed to solve the problem and thus opted for number grabbing and guessing the formula to be used. The use of the sine rule without first attempting to find the angle DLH so that there would be one unknown also revealed that the learners failed to show the action of choosing the correct rule. The learners' responses revealed that they were confused about when to use trig ratios or rules. This shows that they had failed to understand the basic parts of conditions that assist one to recall the correct rule.

5.4.2. **Question 4, Item 2:** ***Hence or otherwise, calculate the length of AB***

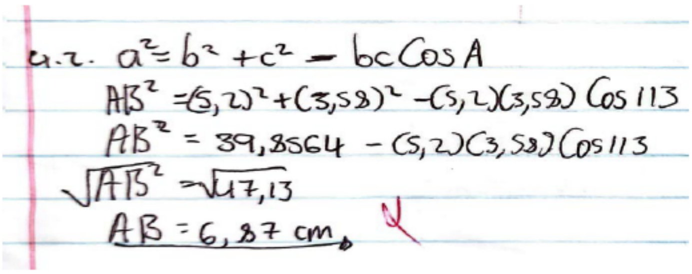
Similarly to Item 1, in Item 2 learners were asked to calculate the length of AB.

Figure 5.10 shows the number of learners in different categories indicating their understanding or poor understanding of the concept. Only eight learners were able to understand what need to be done to answer this item. The other twenty-two learners showed that they lacked an understanding of the concept.

Table 5.10: Analysis of learners' responses for Item 2 (Question 4)

CATEGORY	1	2	3	4
RESPONSES	Provide the correct and complete explanation	Partially correct	Incorrect answer	Not attempted
NO OF RESPONSES	8	0	17	5

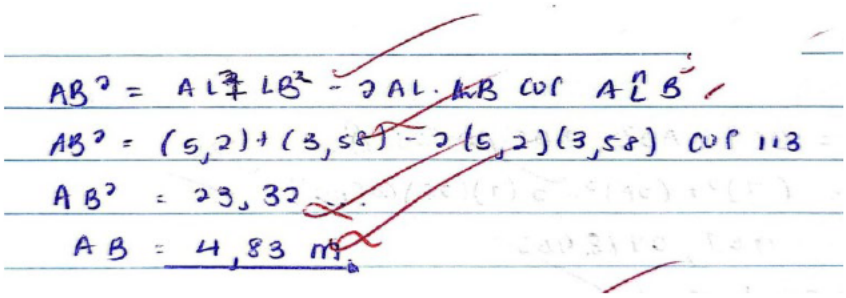
The responses of learners in Category 3 revealed common errors that led them to the wrong answer. While they seemed to have an idea of the formula to be used, they had failed to memorize the formula, with the result that they used correct values in the incorrect formula, thus failing to solve the problem. Also, a basic knowledge of how to make one variable the subject of the formula seemed to be lacking among the learners in Category 3. As mentioned above, a failure to break down the questions indicates that learners are failing to recall the actions to be performed. This is illustrated in the extract below.



Extract 18 Philani’s written work for Item 2 (Question 4)

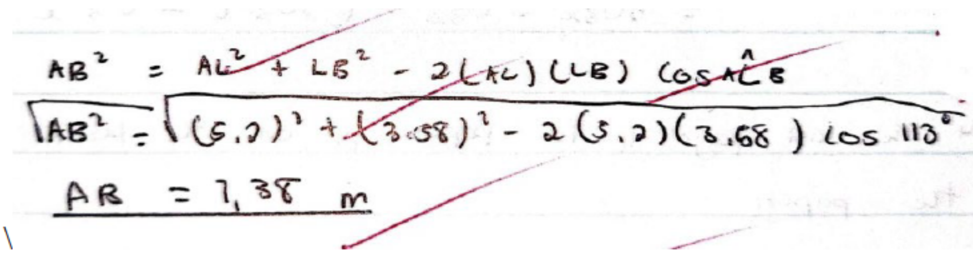
As noted in Extract 18, above, Philani used the correct values in an incorrect formula; thus, the action of reflecting on which formula to use had not been interiorized into a process.

Extract 19, below, shows Nhlakanipho’s response, which falls into Category 2. Nhlakanipho was able to choose the correct rule to be used and his substituted values are correct, but in step 2 he forgot to square the values, resulting in error for the rest of the calculations. Despite the fact that he forgot to square the values, he was found to be working at the process stage as he was able to skip some of the steps in arriving at the solution.



Extract 19: Nhlakanipho's written work for Item 2 (Question 4)

Eight of the learners were able to provide the correct answer (Category 1). Most of these learners revealed that they could reflect on the given formula and substitute values without the need to transform the equation to determine the subject of the formula; they were also able to skip some steps. This indicates that these learners were operating at the process stage in totality. This is seen in Mzwandile's work, below.



Extract 20: Mzwandile's written work for Item 2 (Question 4)

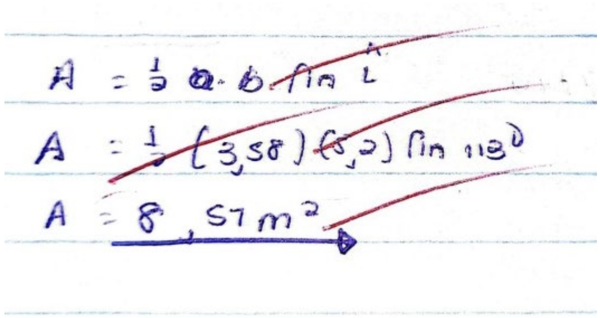
5.4.3. Question 4, Item 3: *Determine the of area ΔABL*

Table 5.11: Analysis of learners' responses to Item 3 (Question 4)

CATEGORY	1	2	3	4
RESPONSES	Provide the correct and complete explanation	Partially correct	Incorrect answer	Not attempted
NO OF RESPONSES	5	7	16	5

This question required a learner to notice that the item's diagram is not a right angled triangle and therefore have to use sine rule to find the area of triangle. Thus, learners needed to perform the action of identifying the formula, reflect on the formula and perform the process of calculating the area. The findings showed that learners in Categories 2 and 3 had not yet developed the necessary mental constructions as they failed to identify the correct formula. While the word 'area' triggered the formula for finding the area of a 2D triangle, the learners did not reflect on the concept of finding the area of a 2D shape to apply it to finding the area of a trigonometric 3D shape.

The five learners in Category 1 were able to show the necessary mental construction of the concept successfully. This is illustrated in Nlakanipho's work, shown in Extract 21.



Extract 21: Nhlakanipho's written work for Item 3 (Question 4)

The learners' action of choosing the correct area rule for 3D shapes indicates that were operating at the object stage of APOS.

5.5 Conclusion

This chapter has presented the analysis of learners' mental constructions of trigonometric 2D and 3D concepts during the pre-test. The purpose of this was to explore their mental constructions before they engaged with GeoGebra. The findings revealed critical aspects about learners' mental constructions of 2D and 3D concepts. It was observed that, while the majority was operating at the action stage, there were some learners who had interiorised the concepts, showing they had developed to the process level in their mental constructs. Contradictory to the findings of Ngcobo et al., (2020) about learners' mental constructions of the solution of triangles, which found learners to only be operating at the action stage or below the action stage, this study showed some evolution of learners' mental constructions for trigonometric 2D and 3D concepts beyond the action level.

It was observed that the majority of learners were operating at the action stage, this indicated that learners need more assistance so that they understand the concept much better, so that the mental construction of the concept can proceed to the next step. It was also observed that many learners in the tables above especially table 5.8 and 5.9 were in category 3 and 4 which is for incorrect and not attempted. The common reasons for having majority of learners in category 3 and 4 was observed in their written responses, was that learners could not differentiate when, how and what formula they had to use especially when they have to calculate the area for example they will use A= $\frac{1}{2} \times b \times h\ instead\ of\ A = bsSinA\ \ vise\ verser.$ The second common reason why learners could not provide the appropriate mental construction required was the failure to integrate mathematic concepts. Those that provided the correct written work it was simply because they understand the concept with or without the integration of Geogebra and they were very few of them.

The next chapter analyses learners' responses on the post-test, which was administered after the researcher taught learners 2D and 3D concepts using GeoGebra, to elicit whether GeoGebra did or did not enhance the learners' understanding of the concept, to answer the third research question.

CHAPTER 6: DATA PRESENTATION, ANALYSIS AND FINDINGS OF POST-TEST

6.1 Introduction

Chapter 5 presented the analysis and findings of the pre-test. This chapter presents the analysis and findings of the post-test that was completed by learners after GeoGebra was integrated into their lessons. The researcher taught learners 2D and 3D trigonometry concepts using GeoGebra following the ACE teaching style discussed in Chapter Four. This chapter begins with a description of the activities that were designed and implemented that integrated GeoGebra as the instructional tool.

This chapter presents the analysis of data related to the third research question:

> *To what extent does GeoGebra enhance learners' mental construction of 2D and 3D trigonometric concepts?*

APOS theory was used in combination with genetic decomposition to analyse learners' responses to the post-test to answer this research question.

The test consisted of 5 main questions, each with sub-questions (called 'items'), with 18 sub-questions in total. Questions 1 and 3, and Questions 2 and 5 had been designed to assess the development of the same mental construction as a means to avoid estimation bias and also to give learners exposure to the way that 2D and 3D trigonometry exam questions are designed. While learners' responses to all of the questions were assessed, it was decided that analysis of Questions 1 and 2 would be seen satisfactory, and Questions 3 and 5 could be excluded from the analysis. This chapter thus presents the analysis of learners' responses to Questions 1, 2 and 4.

For the analysis of learners' responses, Ndlovu's (2014) categorisation system was used:

- Category 1 indicates responses that were correct, indicating that the learner had made the mental constructions necessary to solve the particular problem;
- Category 2 indicates an incomplete response in which the learner demonstrated a knowledge of procedures but failed to reach a correct final solution;

- Category 3 indicates responses where the learner attempted to solve the problem but displayed no mental constructions;
- Category 4 indicates instances where the learner lacked the necessary prerequisite knowledge to be able to attempt the problem.

The analysis was tabulated using Ndlovu's (2014) categories as follows:

CATEGORY	**1**	**2**	**3**	**4**
RESPONSES	Provided a correct and complete explanation	Partially correct	Incorrect answer	Not attempted
NO OF RESPONSES				

Responses that fall into Category 1 indicate that the learner is operating at either the action level or the process level. At the action level, a learner will show all the steps for solving the problem explicitly, while at the process level they will perform the same action but without writing all the steps down, as they will have done some of the calculations mentally. A learner operating at the object level will be able to integrate the mathematics topics and be able to use the area rule for 3D shapes rather than the area formula of 2D shapes.

6.2 Instructional activities using GeoGebra

The aim of implementing GeoGebra was to enhance learners' mental constructions of trigonometric 2D and 3D concepts. The analysis of learners' mental constructions during the pre-test was presented in the previous chapter. The overarching finding from the pre-test responses was that learners were mainly operating at the action stage in terms of APOS theory. To integrate GeoGebra into teaching and learning, the researcher designed activity sheets to address knowledge gaps identified from the pre-test responses, such as correct use of the rules for solving different triangles and knowledge of and correct use of trigonometric ratios. Therefore the first activity on the post-test involves construction of triangles, as shown in figure 3 below.

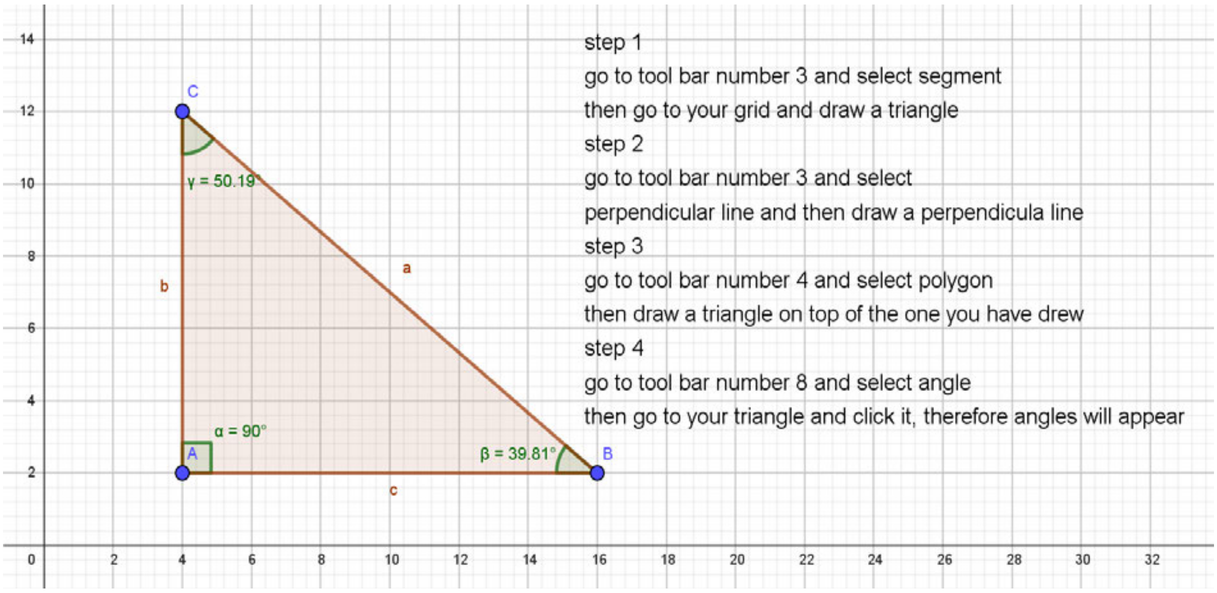


Figure 6.1: Right-angled triangle

The figure above shows a right-angled triangle drawn using the GeoGebra software. The aim of teaching the learners the construction of a right-angled triangle was to emphasise the importance of using trig ratios and the theorem of Pythagoras whenever they see a triangle where one of the three angles is 90 degrees. Another purpose was to familiarise them with the right-angled triangle so that they would be able to differentiate it from other triangles.

Subsequently activities required the drawing of other triangles, as shown below.

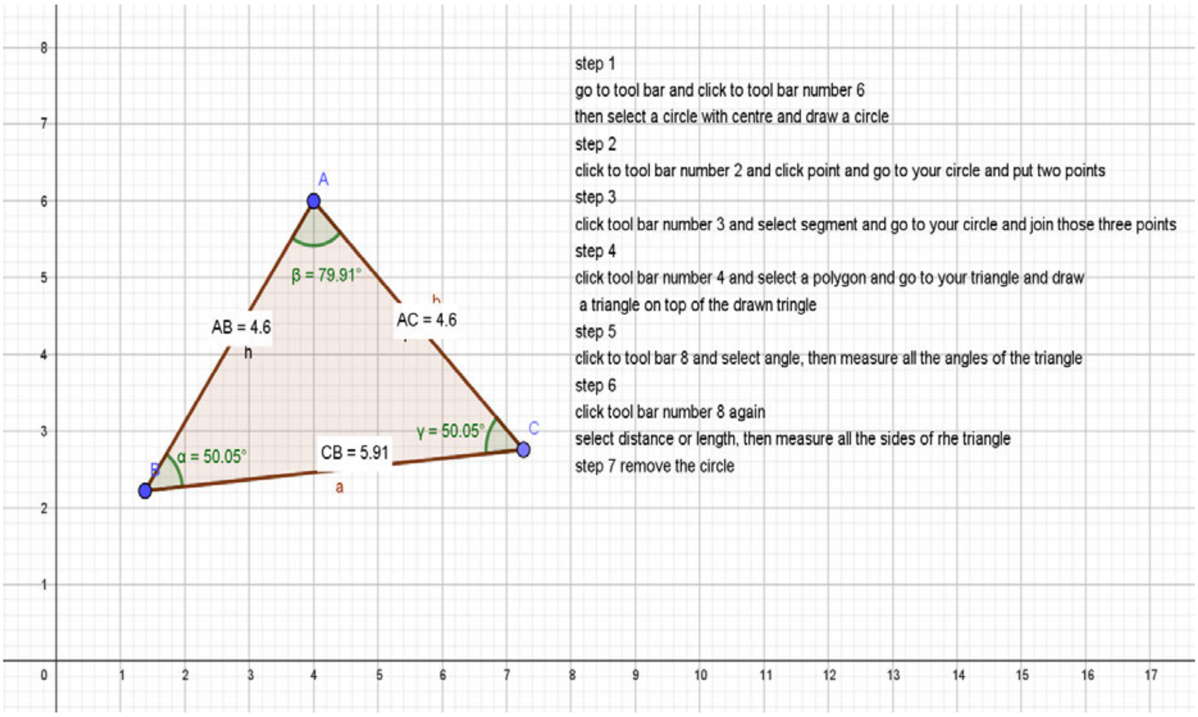


Figure 6.2: Isosceles triangle

The aim of this exercise was to allow learners to explore when trigonometric ratios cannot be applied to solve the angles of the triangle, so that they would learn to recognise the differences between triangles before applying rules to solve lengths and angles of the triangles.

The next figure below will now emphasise the importance of two triangles attached by one common side, to illustrate that having two triangles sharing a common side does not mean that they have the exactly the same properties.

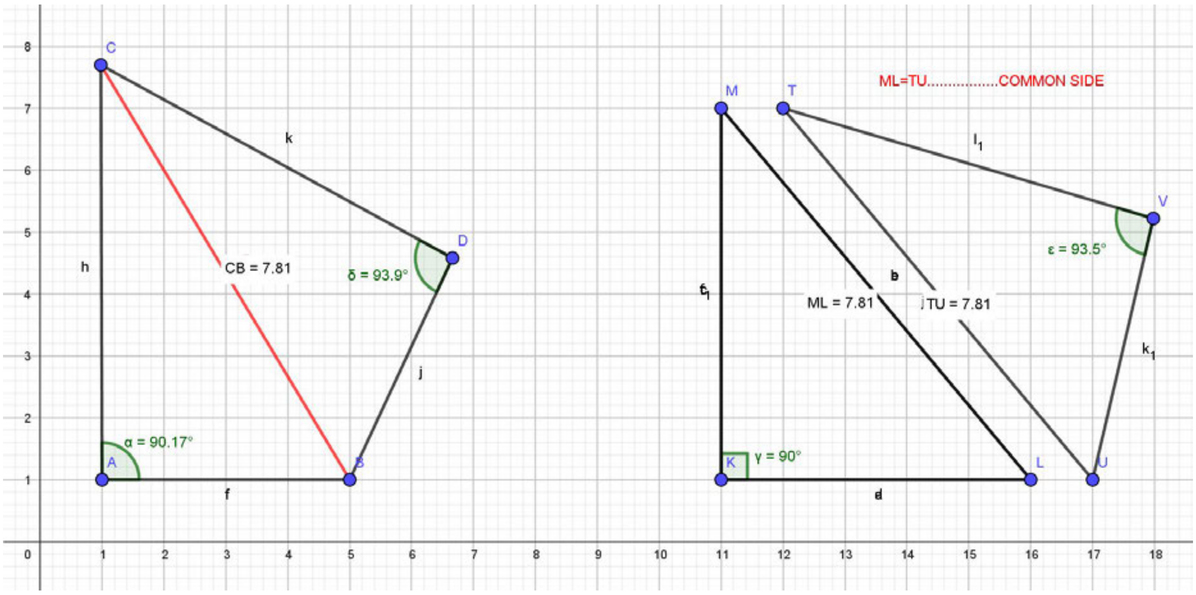


Figure 6.3: Two joined triangles

In their responses to the pre-test, the learners displayed difficulty interpreting joint triangles. The question requires learners to solve the area of triangle ABC. Learners need to use the information from triangle CDB, as this triangle has more information than triangle ABC. However, the learners failed to use the information from triangle CDB to assist them to find the area and length of triangle ABC. The common side of these triangles is equal. But when the triangles were joined, learners thought that the length of a common side worked only for one triangle and, in most cases, they thought it worked for the triangle that had more information. Therefore, it was important to make sure that the learners were able to interpret diagrams correctly before embarking on calculations. In line with this, as discussed in Chapter 2, Brown (2006) found that many learners had a fragmented or incomplete understanding of the three major ways to view sine and cosine: as coordinates of points on a unit circle, as horizontal and vertical distances resulting from those coordinates, and as ratios of sides of a reference triangle. Therefore, strengthening learners' understanding of 3D diagrams is very important. To address this, Figure 6.4 below was drawn and discussed with learners in class.

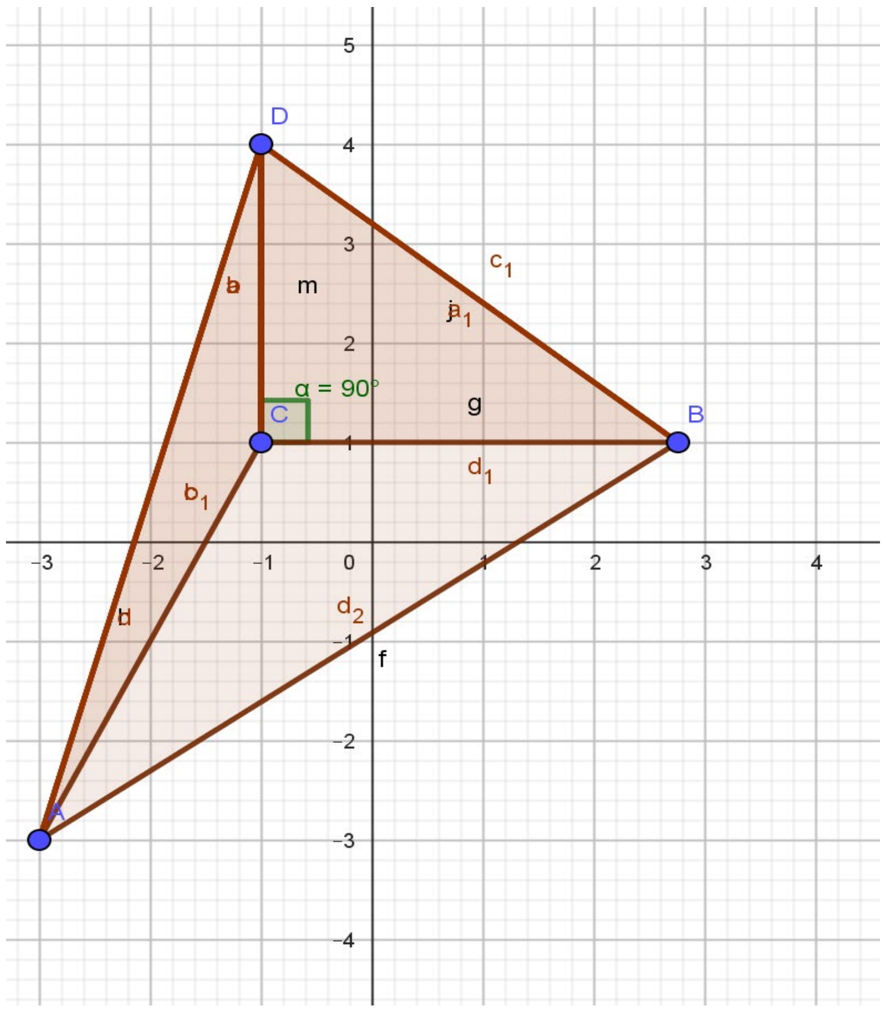


Figure 6.4: The joined nets of four triangles to form 3D

The diagram is of a 3D shape: a right-angled triangular pyramid. As it was observed that, on the pre-test, the learners struggled to work with 3D problems, this activity was designed to help learners see how these kind of diagrams that may appear on exam papers—may actually help them understand and interpret 3D shapes so that they will be able to solve the problem. The advantage of using GeoGebra to model 3D shapes is that it allows the shape to rotate to show all sides, heights and faces of triangles, allowing the learners to see the different triangles that make up a particular pyramid or diagram.

6.3 Learners' interaction with the activities using GeoGebra

Figure 6.5: Learners interacting with GeoGebra

The above figure shows the learners interacting with GeoGebra software, where they were drawing and exploring 3D shapes using the rotation tool bar to view all sides and angles. Learners found it difficult to work with GeoGebra during the first three days of the lessons because, for some, it was their first experience using computers for learning purposes. However, the researcher allowed them to explore and make mistakes. Eventually, they grasped how to construct shapes using GeoGebra. On the pre-test, learners constructed 3D shapes using pen and paper. On the post test, the construction of 3D images was done using GeoGebra, where the learners could rotate the diagram to identify the 2D faces of the shape. For example, in the case of a triangular pyramid, they could identify the correct number and type of triangles.

6.4. Analysis of learners' written responses for the post-test

After the instructional activities, the post-test was administered using questions similar to those on the pre-test. As with the pre-test, each topical question had sub-questions which, for the purpose of this study, are referred to as 'items'.

6.4.1 Analysis of learners' responses to Question 1 of the post-test

Question 1 explored learners' understanding of a cyclic quadrilateral using the following diagram. There were 5 items for this question.

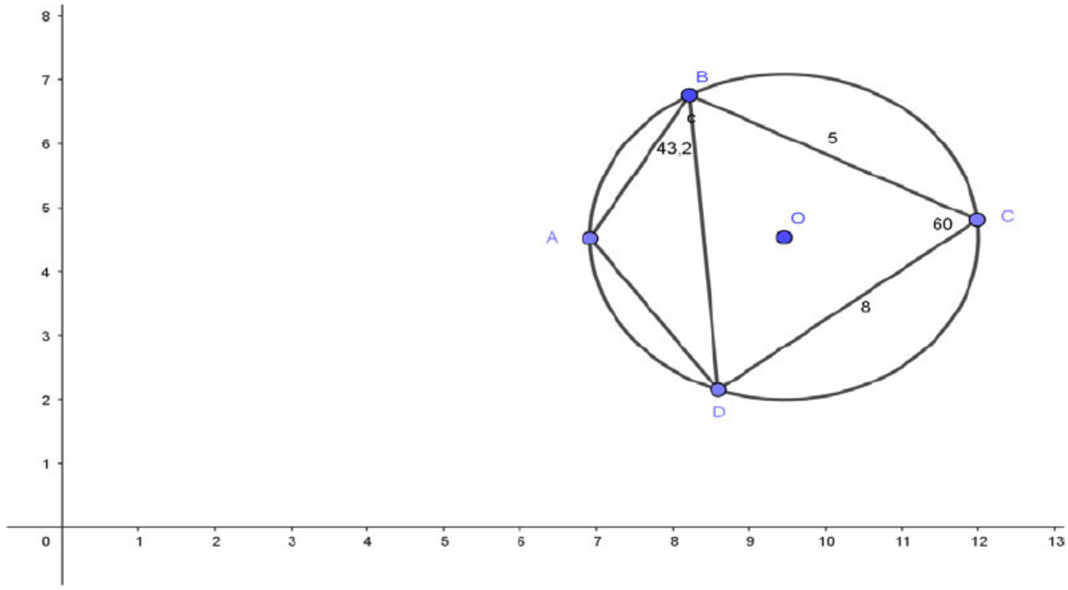


Figure 6.6: Diagram of a cyclical quadrilateral used for Question 1 modified from (DBE Exemplar 2014, P.9)

6.4.1.1 Question 1, Item 1: What is your understanding of a cyclic quad?

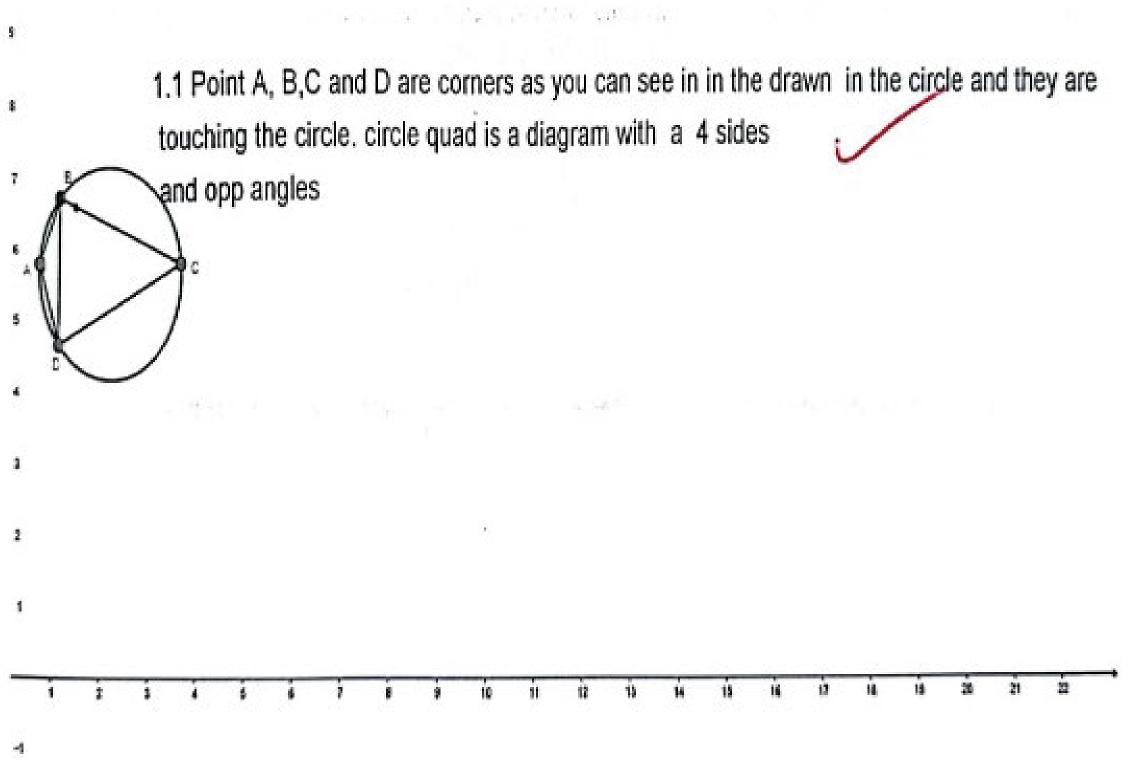
Item 1 explored whether learners had developed the conception of the definition of a cyclic quad. A learner in Grade 12 would be expected to be able to explain what a cyclic quad is and should have constructed a process-level conception of the definition of a cyclic quad, enabling them to define it without needing to construct a diagram. Those who first constructed a diagram were considered to have developed an action-level conception of the definition of a cyclic quad.

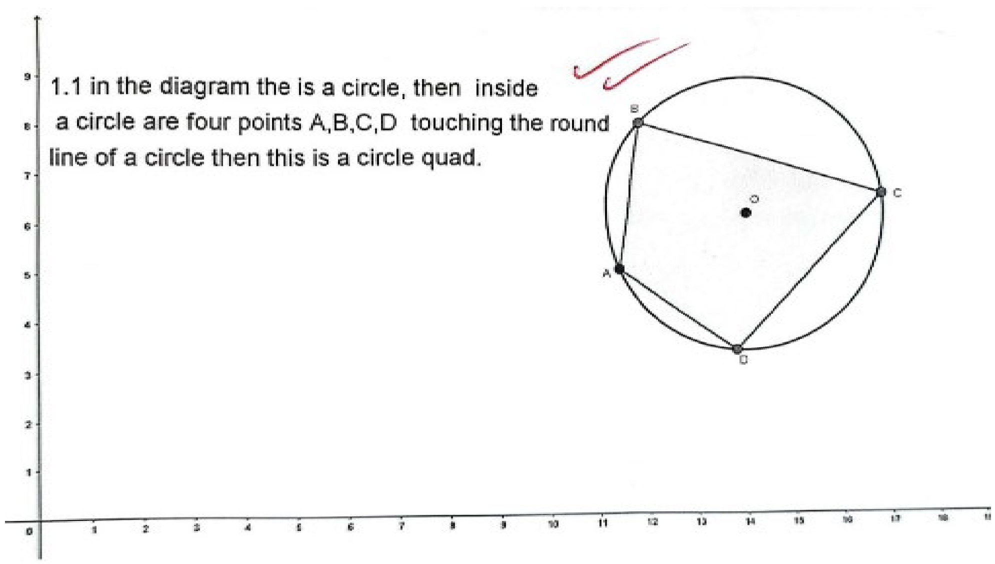
Table 6.1: Analysis of learners' responses for Item 1 (Question 1)

CATEGORY	1	2	3	4
RESPONSES	Provide a correct and complete explanation	Partially correct	Incorrect answer	Not attempted
NO OF RESPONSES	27	2	1	0

While on the pre-test 23 learners had provided a correct definition of a cyclic quad on the post-test 27 were able to do this. While on the pre-test 7 learners were unable to provide the correct definition (partially correct or incorrect), on the post-test there were only 3 who could not. While on the pre-test 1 learner was unable to attempt to explain what a circle quad is, on the post-test all learners were able to attempt this.

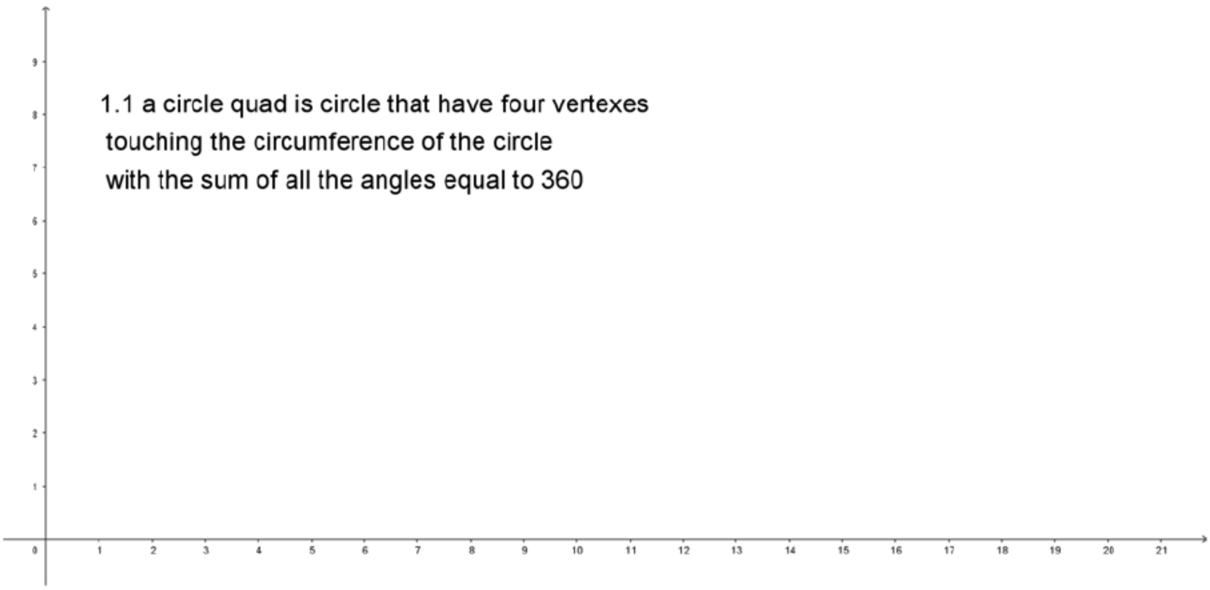
In terms of learners' mental constructions, it was evident that while more learners were able to provide the definition on the post-test, they were still operating at the action stage as they needed to construct the diagram before defining the cyclic quad, as shown in the extracts below.





Extract 1: Ndabenhle (top) and Tebogo's (bottom) responses to Item 1

Although Ndabenhle and Tebogo were considered to be at the action stage in terms of their responses on the pre-test, a difference was noted in the way they defined the cyclic quad on the post-test: it was concise and to the point, using correct mathematical terms, while on the pre-test they had also included the properties of a cyclic quad in their definition. Their definitions on the pre-test were not direct: they did not explain what parts of the quad should touch the circle and at what points on the circle. As illustrated in the extracts above, when they did the post-test they were able to show with the diagram the parts of the quad that touched the circle and illustrate points as vertices, such as indicated in the diagram above that by point A, B, C and D they mean vertices. Therefore, the learners naming the parts of the circle and the parts of a quad was conceived as external cues and thus triggered the construction of the definition, displaying an action conception of the definition of a cyclic quad. In contrast to the pre-test, where it was evident that learners had not progressed beyond the action conception for defining a cyclic quad, on the post-test as it was evident that some learners had interiorised the action into a process as they were able to define the cyclic quad without having to construct the diagram, as illustrated in the extract below.



Extract 2: Thabo's response to Item 1

On the pre-test, Thabo had already displayed an understanding of a cyclic quad, but his definition was accompanied by a diagram. On the post-test, however, he was able to define it without the use of a diagram and incorporate correct mathematical terminology, such as vertices and circumference, into his definition. This was similar to Tebogo's response: he was able to name the parts of a circle and a quad and illustrate with a diagram what he was talking about and how the two relate to be called a cyclic quad. Thus, 25 of the 27 learners were found to be at the action stage, with 2 learners having progressed to the process stage.

6.4.1.2 Question 1, Item 2: Explain how you will find the area of triangle DAB

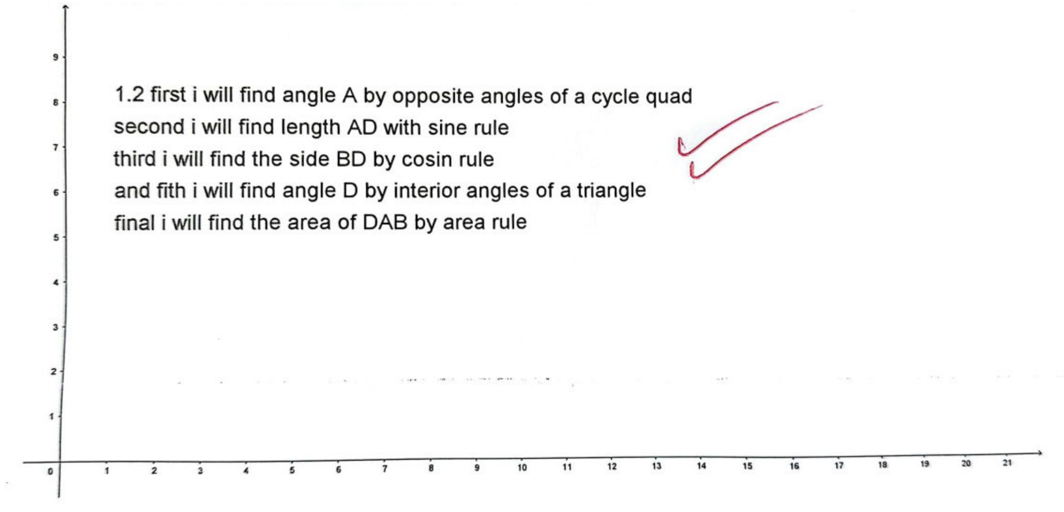
Item 2 required learners to find the area of triangle DAB.

Table 6.2: Analysis of learners' responses to Item 2 (Question 1)

CATEGORY	1	2	3	4
RESPONSES	Provided a correct and complete explanation	Partially correct	Incorrect answer	Not attempted
NO OF RESPONSES	17	5	7	1

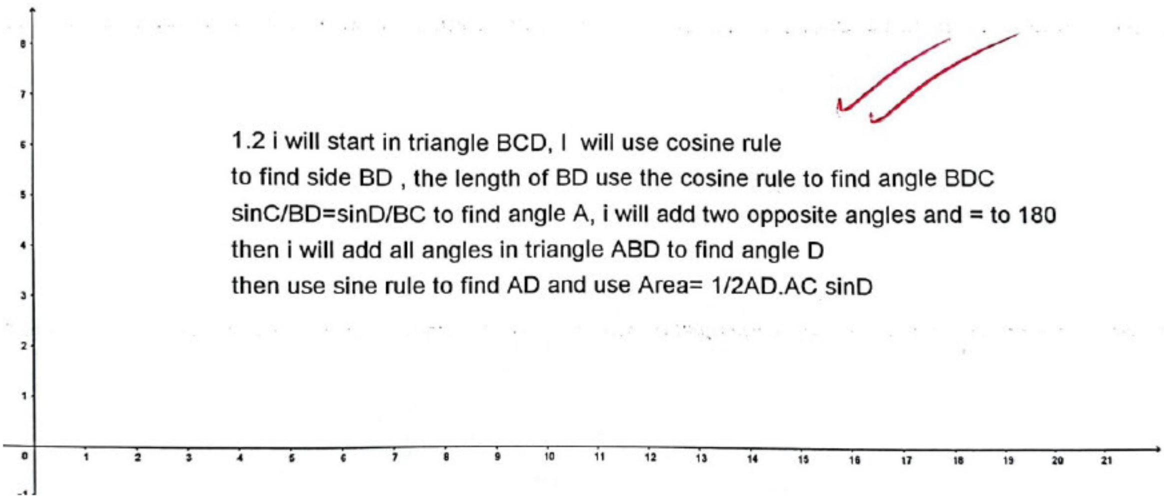
Learners at the action stage would perform step-by-step calculations to determine the area and their explanations would focus on explaining the steps they took to calculate the area. However, those at the process stage would have performed the action in their minds and thus would explain the critical features of finding the area without articulating step-by-step procedures. A learner at an object level would consider the triangle DAB to be the whole entity and would use its components to explain how to calculate the area.

The results of the post-test, after learners had been exposure to GeoGebra, were similarly to those of pre-test for the majority of the learners, indicating that they had not progressed beyond the action stage. However, while on the pre-test 15 learners had been unable to perform the step-by-step procedures or even articulate them in words and thus were found to have not yet reached the action stage, on the post-test 17 learners were categorised as operating at the action stage. Furthermore, 5 learners who had given incorrect responses on the pre-test and 7 learners who had only been able to partially explain how they would find the area of triangle DAB on the pre-test demonstrated an evolution of their mental constructions, as on the post-test they were able to construct the required mental structures and articulate the procedures without first having to explicitly write them down, as illustrated in the extract below:



Extract 3: Mnotho's response to Item 2

As mentioned above, there was no evidence of evolution of learners' mental constructions from the action stage to the process stage, because the learners who were operating at the action stage on the pre-test continued to display action-level conceptions on the post-test for the solving of problems with triangles, as illustrated in the extract below.


1.2 i will start in triangle BCD, I will use cosine rule
to find side BD , the length of BD use the cosine rule to find angle BDC
sinC/BD=sinD/BC to find angle A, i will add two opposite angles and = to 180
then i will add all angles in triangle ABD to find angle D
then use sine rule to find AD and use Area= 1/2AD.AC sinD

Extract 4: Mzwandile's response to Item 2

In Mzwandile's explanation, he even identified the formulas or rules that needed to be used at each step to execute the procedures for calculating the area, thus displaying development of procedural knowledge, not only conceptual understanding.

6.4.1.3 Question 1, Item 3: Calculate BD

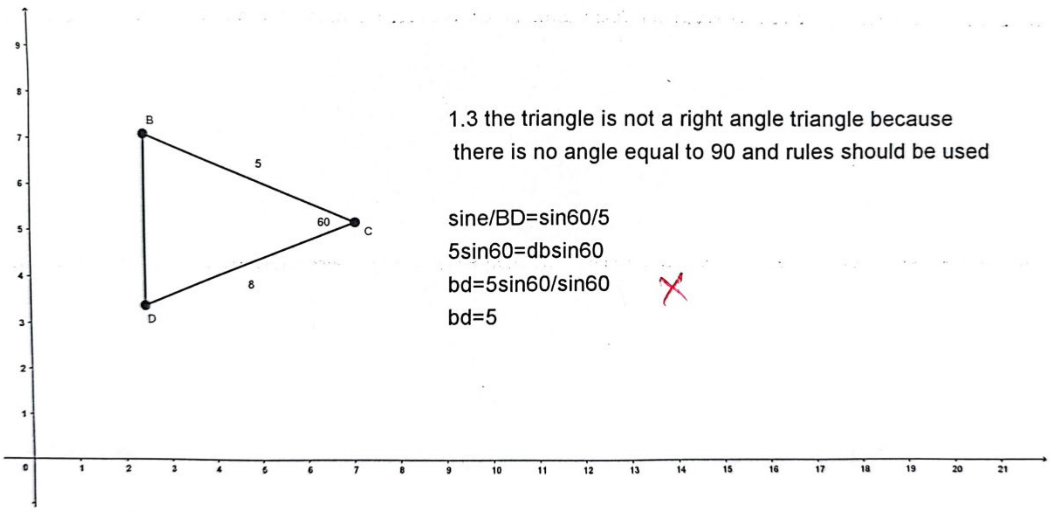
In Item 3, the question was extended to ask learners to determine the length of the side. The aim was to explore learners' conceptual understanding, as they needed to make a connection between the area of a triangle and the sides of a triangle.

Table 6.3: Analysis of learners' responses to Item 3 (Question 1)

CATEGORY	1	2	3	4
RESPONSES	Provided a correct and complete explanation	Partially correct	Incorrect answer	Not attempted
NO OF RESPONSES	19	3	8	0

Table 6.3 shows that 19 learners made the necessary mental structures, while 3 had not fully developed the required conceptual understanding and 8 had not yet developed conceptual understanding. Fourteen of the 19 learners in Category 1 were found to be operating at the action level, while 5 were operating at the process level. On the pre-test, there were 5 learners who did not attempt to solve the question; however, on the post-test, all learners attempted to answer the question. Even though 4 of the 5 learners who had been unable to attempt to solve

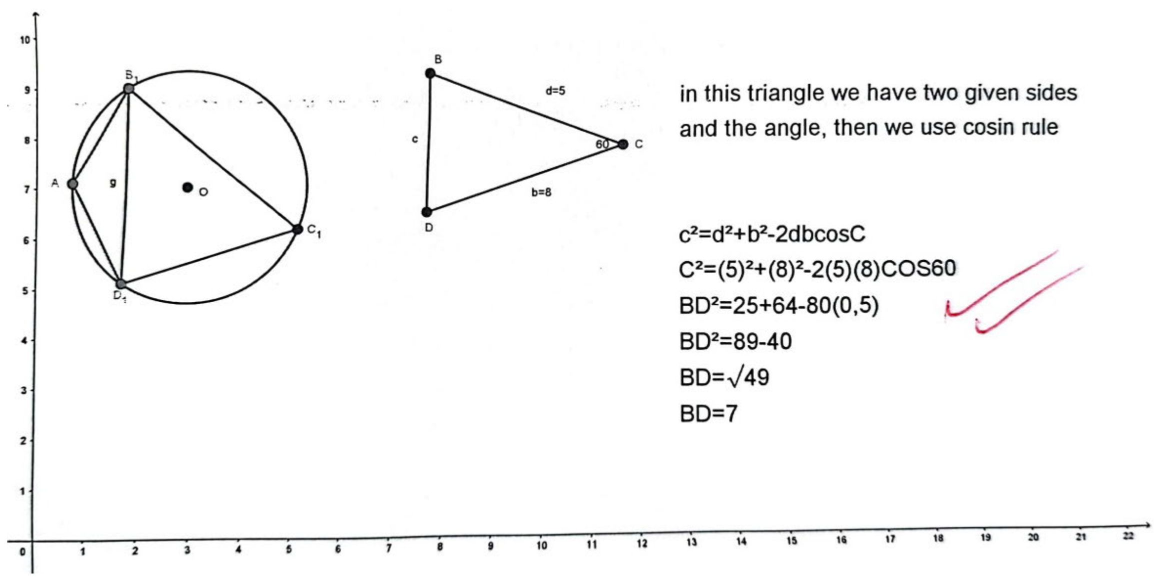
this item still provided an incorrect answer, their ability to attempt the question on the post-test is an indication that there was some evolution of the concept as these learners invoked the correct formula, but used incorrect values when substituting into the formula or used the incorrect answer calculated in Item 2. In addition, they were able to identify when or when not to use trigonometric ratios and attempted to use other related formulas, as shown in the extract below.



Extract 5: Thabiso's response to Item 3

Although Thabiso used the incorrect rule, he was able to identify that BD is a common side to both triangles and decided to deconstruct the diagram, as he identified which triangle had the all the necessary information needed to calculate DB. On the pre-test, he had used Pythagoras' theorem but, as noted in his response, was able to identify that the triangle is not a right angle triangle and use the sine rule; even though it was not the correct rule, because he was able to see that rules were required in the case of a triangle that is not a right-angled triangle, this indicates that his mental construction had developed understanding that a triangle that is not a right angled triangle uses rules not a theorem of Pythagoras.

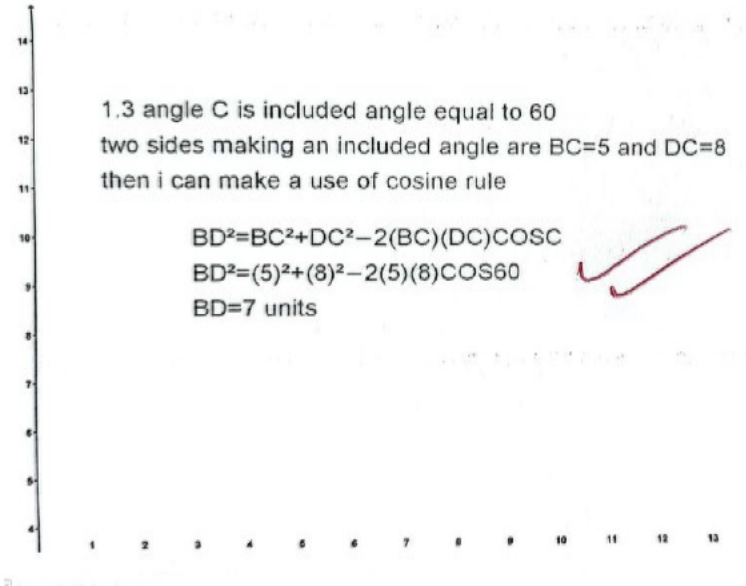
In addition, it was observed on the pre-test that the main challenge learners had was identifying the correct formula or correct values to be substituted. On the pre-test, the learners did not seem to know when or where they should or shouldn't use trigonometric ratios to solve the problems. While the challenge still persisted in their responses to the post-test, it was evident that the majority of learners had grasped the concept as more had provided the correct answer and even those with incorrect answers were able to identify the correct formula to be used. As illustrated in the extract below, one of the learners who had been unable to identify and use the correct formula on the pre-test provided a correct solution on the post-test.



Extract 6 Thabo's response to Item 3

Thabo's response to Item 3 showed development of his understanding from his response to this question on the pre-test. His response clearly shows improvement in the action of choosing the correct rule and using relevant information in terms of the substitutions has. His misconception and assumption that the triangle is an isosceles, on the pre-test, had been rectified through the construction of different triangles using GeoGebra. It could be argued that using GeoGebra thus helped the learners construct knowledge of when or not when to use trigonometric ratios and to recall other formulas applicable to solving non-right-angled triangles.

On the post-test, a few learners showed interiorisation of the action stage to a process, as shown in the extract below.



Extract 7: Mzwandile's response to Item 3

Mzwandile started by explained the key features which then invoked the formula to use. As it could be noted without the need to deconstruct the two triangles, he identified the correct values to be substituted and determined the value of DB.

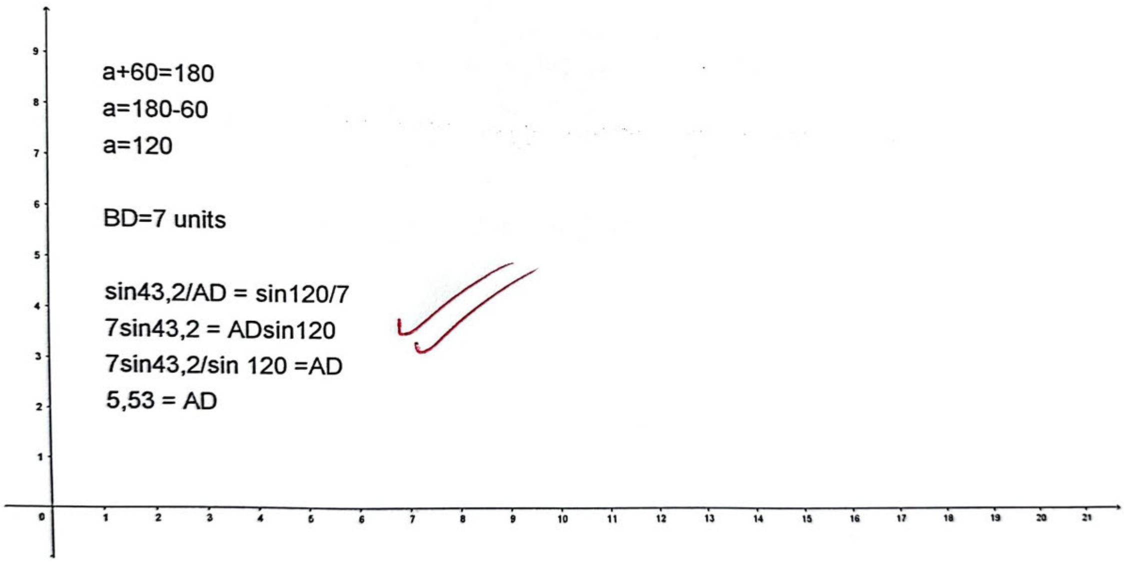
6.4.1.4 Question 1, Item 4: Calculate AD

Similarly to Item 3, on Item 4 learners were required to show their mental constructions for solving the question and making connection between the area of the triangle and the sides of the triangle.

Table 6.4: Analysis of learners' responses for Item 4 (Question 1)

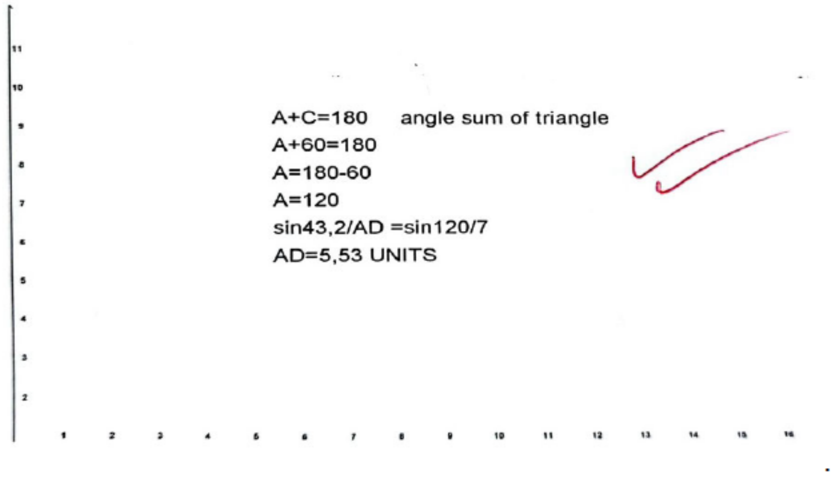
CATEGORY	1	2	3	4
RESPONSES	Provide a correct and complete explanation	Partially correct	Incorrect answer	Not attempted
NO OF RESPONSES	18	4	8	0

Similar to the findings for Item 3, there were learners who were found to have developed to the process stage on the post-test. Also, while 4 learners had not been able to attempt the questions on the pre-test or identify the correct rule to use, on the post-test all of learners were able to attempt the questions and identify the correct rule to use. Also, more learners provided the correct response to Item 4 on the post-test than on the pre-test, and some of the learners who had failed to solve Item 4 on the pre-test provided the correct response on the post-test. In the analysis of learners' mental constructions, it was observed that they were operating at the action stage, as illustrated in the extract below:



Extract 8: Ndabenhle's response to Item 4

Invoking the correct formula triggered the other action than needed to be performed in order to find the value of AD. Using the properties of a cyclic quad, Ndabenhle calculated angle *a,* showing that he was able to make a connection between the two concepts. While explicitly performing the step-by-step calculations to determine the value of AD is considered to indicate the action stage, his ability to integrate the knowledge of the properties of the cyclic quad and the solution of the triangle showed encapsulation of the process into an object; thus, evolution of Ndabenhle's mental constructions was observed on the post-test. Thus, for Item 4, Nhlakanipho was another learner whose mental constructions were found to have evolved since the pre-test, as shown in the extract below.



Extract 9: Nhlakanipho's response to Item 4

6.4.1.5. Question 1, Item 5: Calculate the area of triangle DAB

Items 2 and 5 asked the same question, but in Item 2 learners had to explain how they would find the area of triangle DAB, while in Item 5 learners had to calculate the area of triangle DAB. These questions were designed in this way to develop a balance understanding between problem solving skill and comprehensive skill. Some learners are able to solve a problem but find it difficult to explain how they arrived at the solution; thus, teaching learners to be able to explain how they solve problems can help them to be able to explain to and help each other. While on Item 2, 17 learners had given correct and complete responses, on Item 5, 18 were able to solve the problem. The additional learner who gave the correct and complete response on Item 5 as they were 17 learners who got item 2 correct and 18 learners got item 5 correct, therefore that one additional one learner failed to explain how he will find the area of triangle DAB but was able to execute the correct calculations on Item 5, which verifies the claim above.

Table 6.5 shows that only 18 learners were able to make the necessary mental constructions for Item 5 on the post-test.

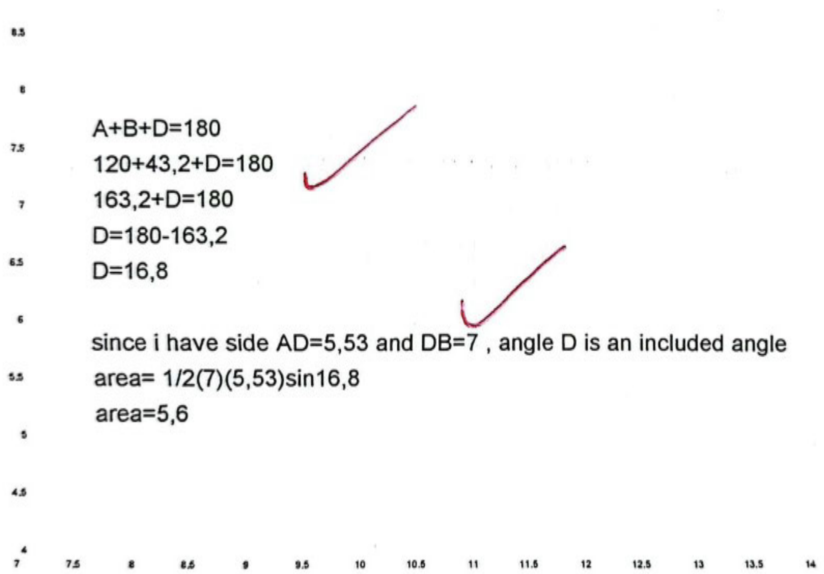
Table 6.5: Analysis of learners' responses to Item 5 (Question 1)

CATEGORY	1	2	3	4
RESPONSES	Provide the correct and complete explanation	Partially correct	Incorrect answer	Not attempted
NO OF RESPONSES	18	2	10	0

Item 5 explored whether learners could differentiate between the area formula for 2D shapes and the area rule for a 3D triangle. For this item, a learner who was able to use the 3D area rule instead of the area formula for 2D would be considered to be operating at the object level of APOS. In Table 6.5, it is evident that 18 learners had made the necessary mental constructions and had used the correct rule, which is the area rule for 3D. Two learners had tried to solve the problem but failed: they chose the correct rule but substituted wrong values. Ten learners were found to still be struggling with conceptual understanding.

In addition, Table 6.5 shows that more learners (18) were in Category 1 than in other categories, which shows that more learners understood the concept and could make the necessary mental constructions than they had before. On the pre-test, 11 learners had not even attempted to answer this item (Category 4): of these, 1 had progressed to Category 2 and 10 had progressed to Category 3. Despite failing to arrive at the correct solution, the attempts made by these learners enable an educator to provide targeted assistance. Of the 9 learners whose responses fell into Category 3 on the pre-test, 8 had progressed to Category 1 and 1 had progressed to Category 2 by the post-test.

Nhlakanipho was a learner in Category 1 who consistently showed a sound construction of the concept. Extract 12, below, illustrates the coherence in Nhlakanipho's response.



Extract 12 Nhlakanipho's response to Item 5

Nhlakanipho was one of the learners in Category 1 whose understanding of the concept had not changed from the pre-test to the post-test: on both, he demonstrated a sound conceptual understanding. His response indicated that his process level understanding had been encapsulated to the object level because he was able to choose correct rule and showed the necessary steps to answer the question. Being able to use the area rule of 3D, instead of the 2D formula A= $\frac{1}{2} \times b \times h$, indicated that he was at the object level of APOS.

6.4.2 Analysis of learners' written responses to Question 3 of the post-test

Question 2 of the post-test was based on the integration of the calculus concept of optimization in 2D and 3D trigonometry. Figure 6.2 shows the diagram used for Question 2. This question had the same three items that were used on the pre-test.

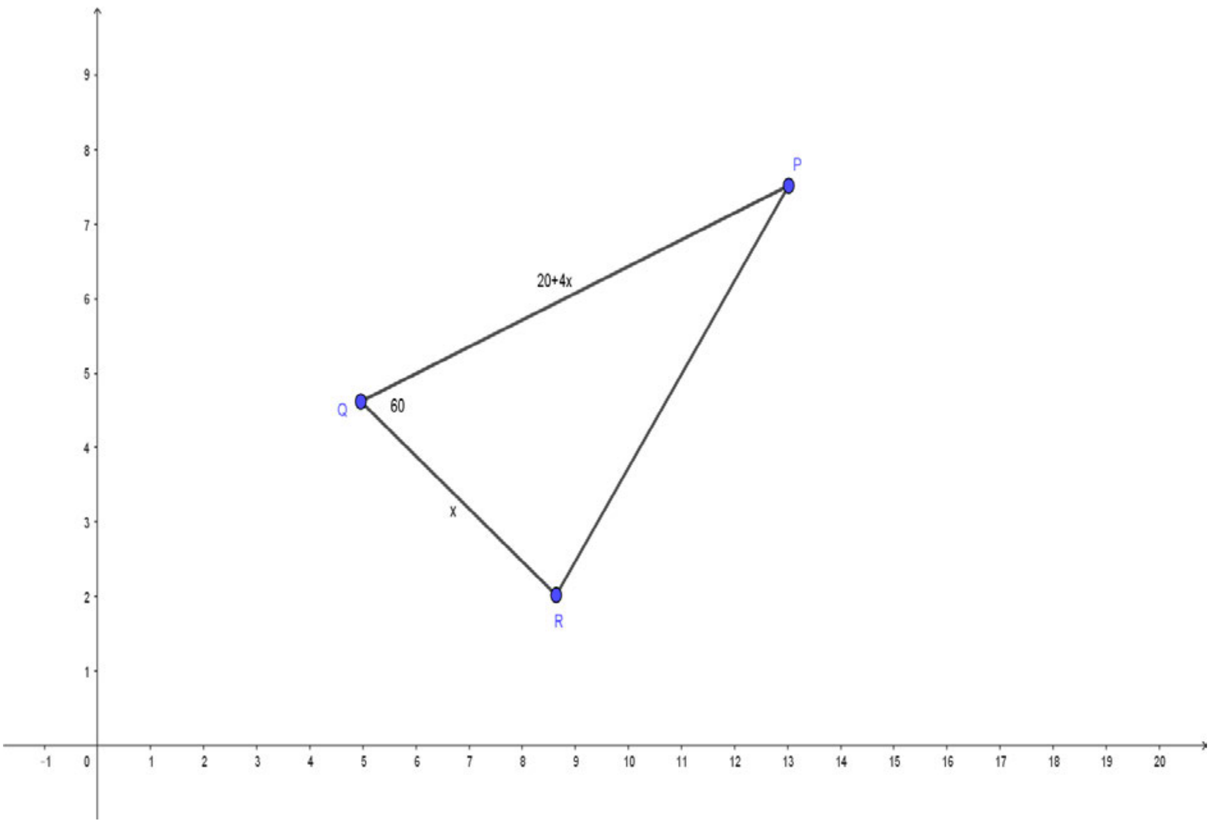


Figure 6.7: Diagram used for Question 2, requiring application of the area rule from (DBE February /March 2016, p.9)

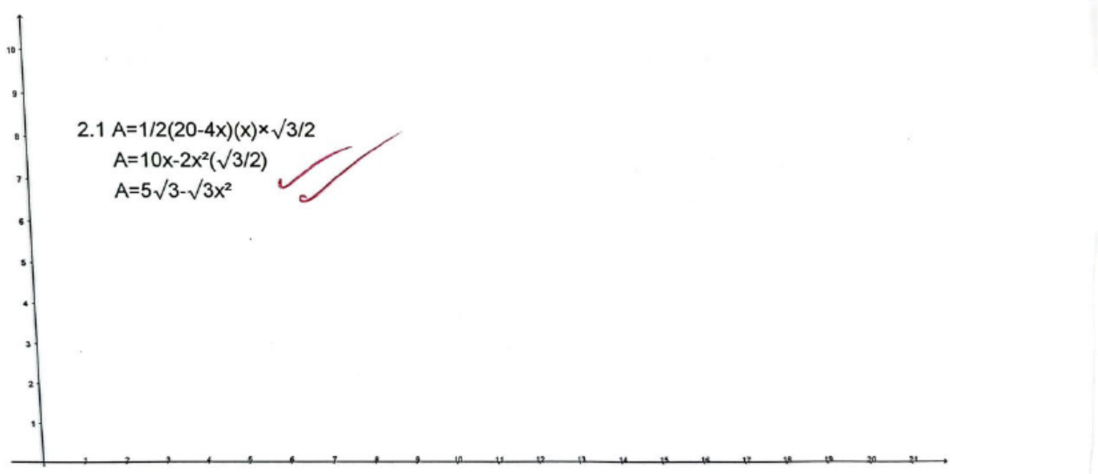
6.4.2.1 Question 2, Item 1: Show that the area of $\Delta PQR = \sqrt{3}x - \sqrt{3}x^2$

Item 1 of Question 2 required learners to show that the area of ΔPQR=$\sqrt{3}$x-$\sqrt{3}$x^2 by using the area rule, as the triangle included an angle that was not 90 degrees.

Table 6.6: Analysis of learners' responses to Item 1 (Question 2)

CATEGORY	1	2	3	4
RESPONSES	Provide a correct and complete explanation	Partially correct	Incorrect answer	Not attempted
NUMBER OF RESPONSES	23	4	2	1

For this item, learners were expected to show the application of rules to show the area of a given triangle. This goes beyond the recall of rules which happens at the action stage and requires application, which is considered to be at the object stage. Table 6.6 shows that 23 learners were able to provide necessary mental constructions. Therefore this item was strictly design for the development of integration within mathematics concepts. Additionally, all 14 learners were able to see that they needed to derive the equation given before performing any other actions, as illustrated below in Lungile's response.



Extract 12: Lungile's response to Item 2

As can be seen in the table above that, 18 learners are added in category 1 of the post-test from other categories compared to Table 5.6 category 1 of the pre-test, where there were 5 learners. This showed evolution of their mental constructions on the post-test, as more arrived at the correct answer on the post-test than did on the pre-test.

6.4.2.2 Question 2, Item 2: Determine the value of x for which the area of ΔPQR will be in maximum

A similar trend was observed for Item 2. Item 2 required the integration of calculus, algebra and trigonometry. According to the genetic decomposition presented in Chapter 3, those learners demonstrating that they had constructed the action and process concept of identifying integrated concepts needed to solve Item 2 were considered to be operating at the object level. The table below shows that 14 learners showed the action and process conceptions required to qualify as having reached the object level.

Table 6.7: Analysis of learners' responses of item 2 (Question 2)

CATEGORY	1	2	3	4
RESPONSES	Provide a correct and complete explanation	Partially correct	Incorrect answer	Not attempted
NUMBER OF RESPONSES	14	5	8	3

While more learners showed the change of their mental constructions between the pre and post-test it was observed that the challenges that had been evident in learners' responses on the pre-test persisted in their responses on the post-test: for example, on the pre-test, one common mistake was the use of an incorrect formula; another was that when asked to prove an equation, learners would use the given equation and substitute the value to area instant of keeping the A(area) and substitute the lengths and angle. Newman (1983) refers to as failure to comprehend the question or failure to carry out a mental transformation from the words of the question to the selection of an appropriate mathematical strategy. This is illustrated in the extract below.

2.2 area=1/2absinC
1/2(20-4x)(x)sin60=5√3x-√3x
(10-2x)(x)sin60=5√3x-√3x
(10x-2x²)√3/2=5√3x-√3x
(10x-2x²)(√3)=10√3x-2√3x
17,32x-3,46x²=10√3x-2√3x
-3,46x²=10√3x-2√3x-17,32x
-3,4x²/2=-2√3x/
1,73x=1,31x
0=1,31x-1,73x

Extract 13: Thabiso’s response to Item 2

While Thabiso invoked the correct strategy, i.e., determining the correct formula to be used, he failed to comprehend what was required and to carry out the mental transformation. This resulted in him not seeing that what he had on the left-hand side was the expanded version of the right-hand side. He thus failed to apply the correct process skills demanded by the strategy. Some learners who failed to solve this question on the pre-test showed evolution of their mental constructions on the post-test, as shown in the extract below:

2.1.2. area of triangle PQR=5√3x -√3x²
area of triangle PQR'=5√3 -2√3x
area of triangle PQR=0 AT maximum
5√3 -2√3x=0
2x=5
x=2,5

Extract 14: Nhlakanipho’s response to Item 2

Nhlakanipho identified the correct strategy to be used, which he had failed to do on the pre-test, and correctly applied the process skills demanded by the strategy to determine the maximum value.

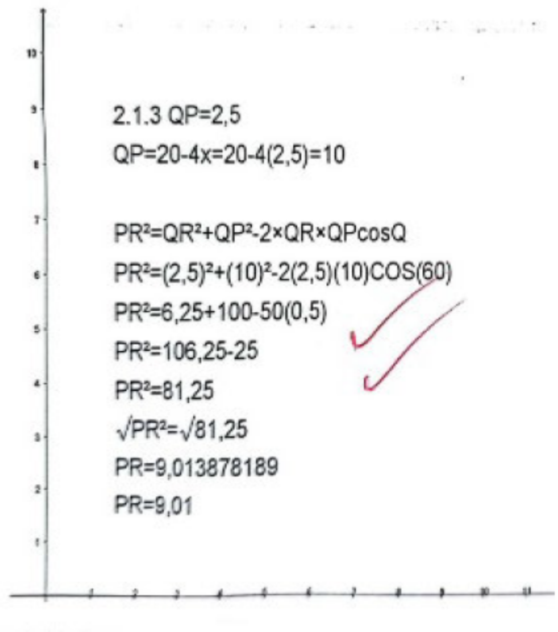
6.4.2.3 Question 2, Item 3: Calculate the length of PR if the area of ΔPQR will be in maximum

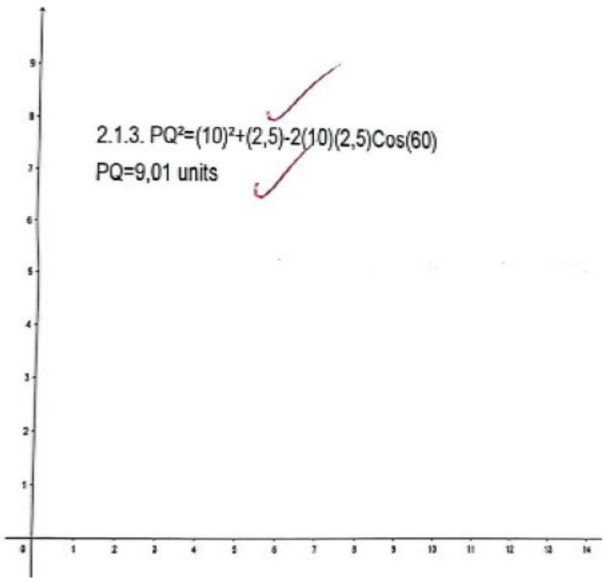
Item 3 requires a learner to find the length of PR by using the value of x found in Item 2.

Table 6.8: Analysis of learners' responses of item 3 (Question 2)

CATEGORY	1	2	3	4
RESPONSES	Provide a correct and complete explanation	Partially correct	Incorrect answer	Not attempted
NO OF RESPONSES	13	6	7	4

Table 6.8 shows that 13 learners provided the correct answer. In terms of their mental constructions, it was observed that 7 were operating at the action stage while 6 were operating at the process stage.





Extract 15(a) Philasande's written work Extract 15(b) Mzwandile's written work

The response shown in Extract 15(a) illustrates an action-level conception, while the response in Extract 15(b) illustrates a process-level conception. Persistent errors observed on the pre-test were a failure to invoke the relevant concepts and correct formulas. It was interesting to note that one learner who had provided the correct answer for Item 2 failed to solve Item 3, as shown in the extract below:

2.1.3. x=0,76

$q^2=p^2+r^2-2pr\cos 60$

$q^2=(0,76)^2+(16,96)^2-2(0,76)(16,96)\cos 60$

$q^2=275,33$

$q=16,59$

Extract 16: Thabo's response to Item 3

Thabo's work shows the correct method, the correct choice of rule and correct manipulation of numbers; however, he made an error in his substitution of the value of *x.* While this might be considered to be a slip, in terms of his mental constructions this shows that while procedures were carried out correctly meaning was not constructed, because if he had conceptualised the concept, he would have identified that his solution did not make sense in relation to the question being solved.

6.4.3 Analysis of learners' written responses to Question 4 of the post-test

Question 4 involved the application of trigonometric ratios and the cosine rule to solve 2D and 3D problems in trigonometry. Similar items were administered on the pre-test and analysed in the previous chapter.

The following diagram was provided for Question 4:

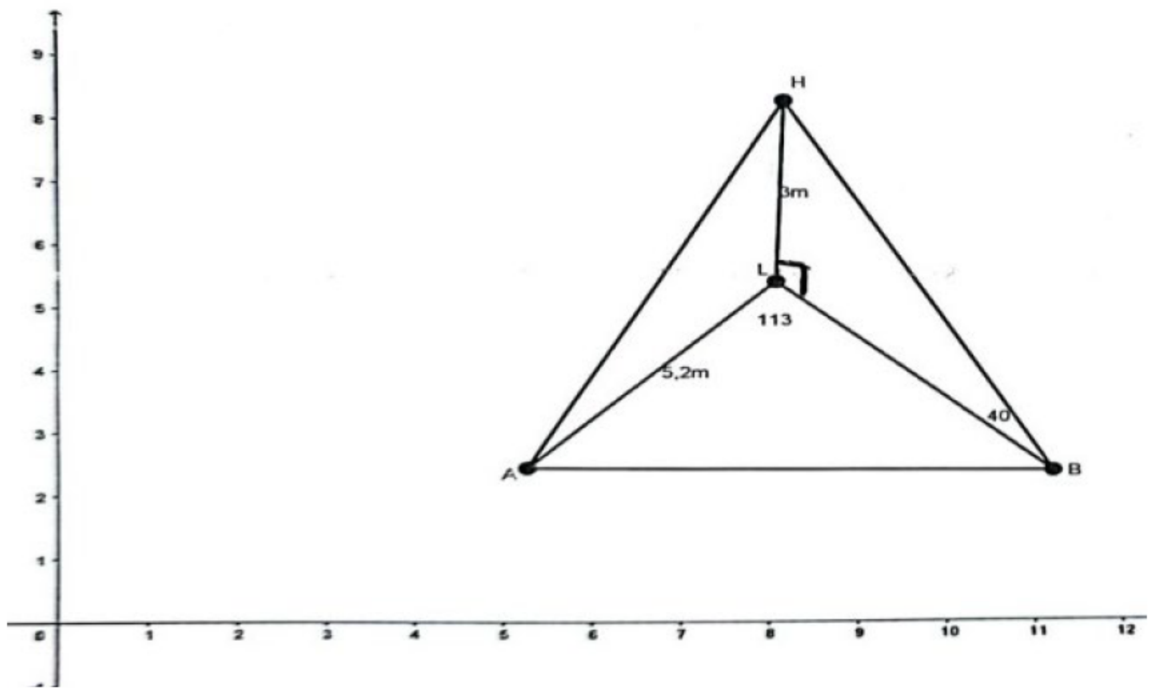


Figure 6.8: Diagram used for Question 4 to explore learners' application of trigonometric ratios and cosine and sine rules from (DBE November 2008, p.7)

6.4.3.1 Question 4, Item 1: Calculate the length of LB

Item 1 of Question 4 requires a learner to apply trigonometric ratios in the solution of a triangle.

Table 6.9: Analysis of learners' responses to item 1 (Question 4)

CATEGORY	1	2	3	4
RESPONSES	Provide a correct and complete explanation	Partially correct	Incorrect answer	Not attempted
NO OF RESPONSES	23	4	3	0

As shown in Table 6.9, 23 learners correctly determine the length of LB, indicating that they were able to identify the correct strategy and apply the correct process skills to execute the strategy. Further analyses showed that, of the 23 learners, 13 were operating at the action stage and 10 had interiorised the action to a process. None of the learners showed encapsulation of the process to an object. On the pre-test, only 9 learners displayed the action conception, as they performed step-by-step calculations to determine the length of LB. On the post-test, all 9 of these learners showed evolution of their mental constructions as they were among the 10 who showed interiorisation of the action to a process.

Another interesting observation was that, on the post-test, while there were learners who failed to provide the correct solution, all the learners attempted the question. The confidence demonstrated by the learners attempting to answer the items suggests that, while some had not constructed the concept fully enough to make the necessary mental constructions, they had constructed it partially. The fragmented knowledge constructed was evident as the learners continued to make certain mistakes. While on the post-test learners evoked the correct rule, their retrieval mechanism failed to locate the correct schema. For example, learners evoked the correct ratio to be use: e.g., $\tan\theta = \frac{opposite}{adjacent}$; however, in the substitution of values, they confused the values and wrote $\tan\theta = \frac{adjacent}{opposite}$, thus leading to a failure to carry out the mental transformation required to solve the problem.

While on the pre-test the learners who had been found to have made some mental constructions were only able to find the length of LB using trigonometric ratios, on the post-test it was found that learners were able to integrate the two and realise that the trigonometric ratio and sine rule could be used to solve for the length of LB, as shown in the extract below.

4.1 triangle HLB is a right angle
40+90+H=180
H=50

sinH/h =isnB/b
sin50/LB =sin40/3
LB=sin50/sin40/3
LB= 3,58

Extract 17: Thabiso's response to Item 1

4.1 trig ratios
tan40= opposite/adjacent
tan40= 3/LB
LB tan40=3
LB=3/tan40
LB= 3,56

Extract 18: Nonhle's response to Item 1

Thabiso was able to integrate procedures to evoke the application of the sine rule to solve for the length of LB. On the other hand, Nonhle evoked the concept using trigonometric ratios. As the triangle where LB lies is a right-angled triangle, there are multiple ways to determine the length of the triangle.

6.4.3.2 Question 4, Item 2: Hence or otherwise, calculate the length of AB

Item 2 of Question 4 required the learner to find the value of AB. While it might seem both Items 1 and 2 required learners to calculate the length of the side of the triangle, the questions aimed to explore learners' understanding of the rules to be used and when the rules apply. For example, to determine the length of AB, trigonometric ratios cannot be used since the triangle is not a right-angled triangle, neither does the sine rule apply.

Table 6.10: Analysis of learners' responses of item 2 (Question 4)

CATEGORY	1	2	3	4
RESPONSES	Provide a correct and complete explanation	Partially correct	Incorrect answer	Not attempted
NO OF RESPONSES	22	5	3	0

Fourteen learners in Category 1 were found to be operating at the action level because they were able to show all the steps correctly; as indicated in the genetic decomposition, a learner who relies on one step to get the next step is operating at the action level. Eight of the 22 learners in Category 1 were found to be operating at the process level because they had been able to skip some steps.

Learners in Category 2 evoked the correct rule and but failed to carry out the procedures correctly, which resulted in the wrong value for AB as they had arrived at the wrong value for LB in Item 1, as shown in the extract below.

4.2 since LB=4

$AB^2=(5,2)^2+(4)^2-2(5,4\times4)COS113$

$AB^2=59,23$

AB=7,69m

Extract 19: Ndabenhle's response to Item 2

Although these learners failed to determine the value of AB, they demonstrated that they knew the procedures to be performed.

6.4.3.3 Question 3, Item 3: Determine the area ΔABL

Item 3 required the learners to use the area rule of 3D because the triangle given was not a right-angled triangle.

Table 6.11: Analysis of learners' responses of item 3 (Question 4)

CATEGORY	1	2	3	4
RESPONSES	Provide a correct and complete explanation	Partially correct	Incorrect answer	Not attempted
NO OF RESPONSES	22	6	2	0

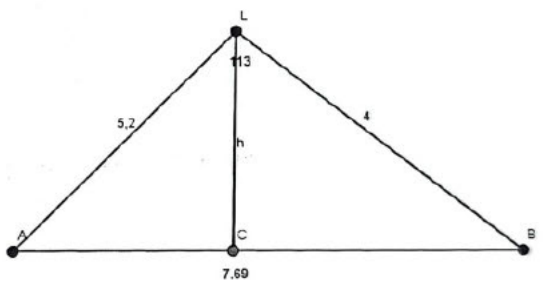
While, as with Items 1 and 2, the majority of the learners provided the correct solution and all the learners attempted the question, one learner failed to determine the area of triangle ABL on the post-test while he had solved this on the pre-test. Thabani constructed a height to form a right-angled triangle and used the formula A=$\frac{1}{2} \times h \times b$, as shown in the extract below.

construct height
AL= 5,2 and AB=7,69

construct height now half of AB=3,845

r²=x²+y²
3,845²=5,2²+y²
y=3,500
h=3,500

Area= 1/2×b×h
Area=1/2(7,69)(3,500)
Area=13,46m²



Extract 20: Thabani's response to Item 3

It was clear that ,as Thabani considered which procedures to perform, he held a misconception that any line constructed from one angle is a perpendicular bisector. It could be argued that while alternative instructions could be given to enhance learners' mental constructions, it is important that learners' underlying misconceptions are first addressed because they may continue to impede their construction of knowledge. As the triangle was scalene, Thabani's method did not help him. However, in terms of his decision to construct a height, his mental structures show that he had an understanding of when to use $A=\frac{1}{2} \times h \times b$ and $A=\frac{1}{2} \times absinC$.

6.5 Conclusion

This chapter has presented the analyses of learners' mental constructions after the intervention to enhance learners' mental construction using GeoGebra. Many of the previous studies that have explored learners' mental constructions of different mathematical concepts (e.g., Ndlovu, 2014; Maharaj; 2014, Madonsela et al., 2020) have focused exclusively on explaining learners' mental constructions. However, Dubinsky (1997) emphasises the importance of exploring alternative instructional strategies to enhance learners' mental constructions. This study has responded to this by incorporating GeoGebra. During the first phase, learners' mental constructions of trigonometric 2D and 3D concepts were analysed; the findings were presented in the previous chapter. Thereafter, instructional strategies using GeoGebra were implemented with the aim of enhancing learners' mental constructions.

As shown in the analysis above, on the pre-test some learners were unable even to attempt to answer Question 1, Item 2 and Question 2, Item 1, but for the other items of questions in the post-test, the learners had enough confidence to attempt to answer the question even if their answer was wrong. This showed that the implementation of GeoGebra had assisted learners to construct additional knowledge of the concepts. More learners provided correct answers for all the items on the post-test than had on the pre-test. Furthermore, while on the pre-test the learners who solved questions correctly were mostly operating at the action stage, on the post-test that some learners that had been found to be operating at the action stage on the pre-test were found to have evolved to process conception, with a few progressing even further to display object conception.

The findings in this chapter concur with Arnon et al. (2014), who posit that the APOS stages are cumulative (the development of one stage depends on the development of the previous stage), as it was observed that the learners who were found to have not made any mental constructions on the pre-test were found to have mental constructions that had evolved to the action stage by the time of the post-test, while those whose mental constructions had been at the action stage at the time of the pre-test were found to have evolved to the process stage. In the next chapter, the synthesise of the findings and response to research questions is presented.

CHAPTER 7: FINDINGS, RECOMMENDATIONS AND LIMITATIONS

7.1 Introduction

This study is about exploring the use of GeoGebra to enhance Grade 12 learners' mental construction of 2D and 3D trigonometric concepts. A case study and qualitative methods were used to conduct the research through the lens of an interpretive paradigm.

Chapters 5 and 6 presented the analysis and findings of the pre-test and post-test. This chapter synthesises the findings to answer the research questions. The learners' interview responses, their mental constructions and the factors that contribute to their complexities that hindered them from making the necessary mental constructions are presented in this chapter. The chapter also discusses the implications of the findings for the teaching and learning of 2D and 3D trigonometric concepts. The contributions of the study, as well as its limitations, are discussed. Finally, the chapter presents recommendations for educators and curriculum managers and for further research.

Guided by the phenomenon under study, three research questions informed the data collection process. The three research questions were:

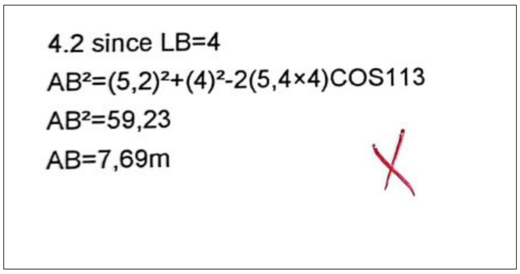
- What are the learners' levels of mental constructions of 2D and 3D trigonometric concepts?
- Why were learners able, or not able, to construct knowledge of 2D and 3D trigonometric concepts?
- To what extent, if any, does GeoGebra enhance learners' mental construction of 2D and 3D trigonometric concepts?

The pre-test and post-test were designed to reveal the development of learners' mental constructions of 2D and 3D trigonometric concepts after being taught using GeoGebra. Learners' mental construction of 2D and 3D concepts is discussed next.

7.2 Learners' mental construction of knowledge in 2D and 3D of trigonometry

The findings show that before being exposed to any alternative instructional strategy such as GeoGebra, it is evident in the pre-test of learners score marks that the majority of participants had difficulties in conceptual understanding as a result they could not made the required necessary mental constructions of 2D and 3D trigonometry. In terms of APOS theory, this means that most of the learners were operating below the action stage, with a few showing that their mental constructions were approaching the action stage. It was also evident in the learners' written work that the majority had difficulties in dealing with the procedural knowledge necessary to perform step-by-step calculations. This was shown by the number of learners that either were unable to attempt to solve a problem or attempted to solve the problem but failed to arrive at the correct answer. It was observed on the pre-test that the majority of learners failed to invoke the correct rules needed to solve the problem.

After GeoGebra was implemented as an alternative instructional strategy, however, the majority of the learners were found to be operating at the action stage, indicating that they had developed the knowledge of the procedures needed to solve the problem, while a few had progressed even further to the process and object levels. Furthermore, it was noted that after the intervention almost all of the learners were confident enough in their knowledge to attempt to solve the problem, while those who attempted the problem but arrived at an incorrect response demonstrated that they had, in fact, invoked the correct rule. The most frequent reason for their incorrect solution was that they had substituted incorrectly or had answered previous questions incorrectly where subsequent questions relied on the answers to the previous questions. For example, for Question 4, Item 1, a learner found the value of LB to be 4, which was incorrect. This value was required for Item 2, resulting in an incorrect answer to Item 2, as well, as shown in the extract below.



4.2 since LB=4

$AB^2=(5,2)^2+(4)^2-2(5,4\times4)COS113$

$AB^2=59,23$

AB=7,69m

Extract 1: Ndabenhle's written response to Question 4, Item 2

The findings identified several difficulties that learners experienced that prevented them from developing the required mental structures. These are discussed next.

7.3 Difficulties that hindered learners' development of the necessary mental constructions

Tall (2008) states that the construction of new knowledge is built on previously constructed knowledge. In this study, learners had encountered 2D trigonometric concepts in the previous grade and had been taught both 2D and 3D trigonometric concepts in Grade 12. However, the study found that learners struggled to make the mental constructions purported by APOS theory to be necessary for the eventual construction of schema. The findings revealed that learners had several knowledge gaps which hindered their development of the necessary mental constructions.

7.3.1. The use of incorrect formulas

It was evident from learners' responses, discussed in Chapter 5, that most did not understand how and when to apply a formula correctly. The learners showed inadequate knowledge constructions to be able to choose the correct rule, especially when they had to apply the area rule for 3D shapes: they used the rule for calculating the area of a 2D shape instead. For example, to calculate the length of a side of an oblique triangle, they used the theorem of Pythagoras instead of the sine or cosine rule. In addition, for a problem where they had to determine the area of an oblique triangle that was not an isosceles or equilateral triangle, they used the formula area $= \frac{1}{2} \times b \times h$ which can only be used if a triangle is isosceles or equilateral, where the height is constructed to form two right angled triangles—instead of using the sine or cosine rules. This was verified during an interview, as illustrated in the extract below.

Researcher*: In the pre-test, you could not solve this question, but in the post-test you were able to use the correct rule. Can you explain what changed?*

FB1*: I just got confused: I could not differentiate which formula to use. Bekunzima nje [it was difficult] because there was a word 'area': I just use the formula I remembered. But in the post-test, angazi [I do not know]: I can visualise the diagram and recall the formula to be used. I think practising different orientation of the triangles helps to differentiate between 2D and 3D and helps with recalling the formula.*

The majority of learners' mental schemas did not incorporate differentiation of triangles, and it is difficult for someone who cannot differentiate between triangles to apply the correct formula for a particular triangle. Differentiation of triangles is taught as early as primary school, where learners learn to identify and name the different triangles. In later grades, they learn to state the properties of a triangle. Yet this study found, as demonstrated in Chapters 5 and 6, that learners at the Grade 12 level were struggling to differentiate triangles and, thus, were unable to identify the correct strategy to use to solve for the area. These findings concur with Tall (2008), who states that the previous knowledge learnt has either a positive or a negative impact on constructing knowledge for a new concept. Ndlovu (2014) emphasises that when a concept is not well understood, and its meaning is not adequately constructed, it will become detrimental to future learning.

When learners were interviewed after the post-test, the researcher showed them their answers to the pre-test and asked them to explain what they had done wrong. They were able to reflect on their thought processes and identify what they had done wrong on the pre-test, as shown in the extract below.

***Researcher**: Please explain your answer to this question [pointing to Item 1.5].*

***FB4**: I used the Area rule to calculate that area,*

***Researcher:** Is that the correct formula?*

***FB4:** Yes, because I was finding the area. But I think the angle that I used, I got it wrong. So… mmmh … that's why the answer was wrong when I was doing the previous questions. If my answer was correct **there**, my answer was going to be correct **here**.*

The findings of the study showed that, after exposure to GeoGebra as an alternative instructional strategy, learners were able to either invoke the correct formula or engage with what they had written to identify where they had gone wrong, thus showing the evolution of their mental constructions to the level of a schema.

7.3.2 Difficulty with language and terminology

Ndlovu (2015) stated that concept definition has an impact on the concept image constructed, meaning that if the definition of the concept is wrong, the concept image will be incorrect. In other words, the language used to describe concepts should be correct in concept definition for learners to construct the correct mathematical concept image.

In this study, the findings showed that learners' use of mathematical terminology was incorrect. For example, learners had difficulty describing points or vertices when they had to define a cyclic quad. Learners frequently used points instead of vertices to define what a circle quad is;

while both are mathematic terms, vertices are a correct reference when defining a cyclic quad. On the post-test, the diagrams they drew revealed that what they had referred to as point were actually the vertices, meaning that without a concrete object to refer to they had not interiorised the definitions. The use of concrete objects can thus minimize or eliminate some of the difficulties that hinder the learners' mental construction of 2D and 3D trigonometric concepts. The next section discusses how the integration of GeoGebra enhanced learners' mental constructions.

7.4 GeoGebra as an alternative instructional strategy to enhance learners' mental constructions

In terms of the extent to which GeoGebra enhanced learners' mental constructions, the findings showed that after the implementation of GeoGebra there was evidence of the evolution of learners' mental constructions. While most learners were found to have not constructed even the action conception of 2D and 3D trigonometric concepts, after teaching and learning was implemented using GeoGebra, most learners were found to be operating at the action stage. The findings before the implementation of GeoGebra coincide with the findings of a study conducted by Madonsela et al. (2020) that found that learners struggled to make the necessary mental constructions. Madonsela et al.'s (2020) study did not explore whether learners' mental constructions might develop more effectively if an alternative instructional strategy was used, while this study showed that GeoGebra did assist the evolution of learners' mental constructions. Table 7.1 below showed the learners' mental construction before and after the implementation of GeoGebra.

Table7.1: Levels of learners' mental constructions displayed on Pre-test Question 1(APOS levels)

Items	Action	Process	Object
1	16	7	-
2	4	1	-
3	5	-	-
4	4	-	-
5	-	-	4

Table 7.2: Level of learners' mental constructions displayed on Post-test Question 1 (APOS levels)

Items	Action	Process	Object
1	25	2	-
2	17	-	-
3	14	5	-
4	12	6	-
5	-	-	18

In Table 7.2, Item 5 shows that 18 learners were operating at the object level, because Item 5 uses the answers from Items 3 and 4, which were designed to assess learners' development of action and process level mental constructions. This was the case on both the pre-test and the post-test. As Item 5 uses the solutions from Items 3 and 4, a learner who is using an action or process conception will be able to answer Item 5 correctly, indicating that they are operating at the object level. The genetic decomposition indicates that a learner who is able to use the area rule, rather than the area formula of $A = \frac{1}{2} \times b \times h$, is operating at the object level; thus, Item 5 requires a learner to use the area rule using values determined in Items 3 and 4.

Furthermore, Tables 7.1 and 7.2 show that, while the majority of learners were found to be operating at the action stage during both the pre-test and the post-test, during the pre-test learners who had arrived at the correct solution demonstrated action conception, but the majority of the learners were not able to perform the necessary procedures. On the post-test, however, a number of learners demonstrated that they had made the necessary mental constructions—some at the action level, with several others progressing further to the process and object stages, showing interiorisation of action into a process and encapsulating the process into an object. Since the items were developmental, Item 5 was designed to illustrate the object conception as it built on Items 3 and 4, which required learners to demonstrate the action conception and process conception. At the time of the pre-test, before the implementation of GeoGebra, learners demonstrated that they were operating at the action stage on Items 3 and 4, with no learners showing interiorisation of the action into a process. After the implementation of GeoGebra, however, it was found that 12 learners were operating at the action stage, with 6 learners showing the interiorisation of the action to a process. For Item 5, while on the pre-test only a few learners showed encapsulation of the process into an object, on the post-test many learners were found to be operating at the object stage.

Similar trends were observed in the analysis of Questions 2 (Tables 7.3 and 7.4) and Question 4 (Tables 7.5 and 7.6), as shown and discussed below.

Table 7.3: Levels of learners' mental constructions displayed on Pre-test Question 2 (APOS levels)

Items	Action	Process	Object
1	-	-	5
2	-	-	3
3	-	1	-

Table 7.4: Levels of learners' mental constructions displayed on Post-test Question 2 (APOS levels)

Items	Action	Process	Object
1	-	-	23
2	-	-	14
3	7	6	-

Question 2 of the paper required learners to integrate calculus into trigonometry. As shown in Tables 7.3 and 7.4, learners showed evolution of their mental constructions after the implementation of GeoGebra.

For Item 1, a learner who is able to recall the correct rule to use and how to use it is considered to be at the object level, such as being able to use the area rule instead of area formula $A = \frac{1}{2} \times b \times h$.

Item 2 of Question 2, on both the pre-test and the post-test, was designed to assess the development of learners' mental construction of how calculus integrates with trigonometry. Item 2 assessed those learners who demonstrated the action and process level of identifying the integrated concepts needed to solve Item 2 as operating at the object level. For this item, a learner needed to a ply derivation, which is the integration of calculus concept, to be able to find the solution. The purpose of Item 2 was thus to assess the development of integration within the subject. All 15 learners were able to see that they needed to derive the equation given before performing any other actions.

Table 7.5: Levels of learners' mental construction displayed on Pre-test Question 4 (APOS levels)

Item	Action	Process	Object
1	7	2	-
2	5	3	-
3	-	-	5

Table 7.6: Levels of learners' mental constructions displayed on Post-test Question 4 (APOS levels)

Items	Action	Process	Object
1	13	10	-
2	14	8	-
3	-	-	22

Tables 7.5 and 7.6 above show learners' mental constructions for Question 4 of the pre-test and post-test. Item 1 indicates that, of those learners who understood the concept, most demonstrated action conception on both the pre-test and post-test. This implies that procedural knowledge requires one step to be completed correctly before the next step can be taken.

Item 3 of Question 4 linked back to Item 1, which meant that if learner failed to solve Item 1 they would not be able to solve Item 3, because Item 3 uses the solution of Item 1. Most of the learners who had demonstrated action and process concepts for Item 1 were able to solve Item 3, where they were required to use an area rule to calculate the area of triangle ABL. As the genetic decomposition stated that learners who are able to use the area rule rather than the formula of $A = \frac{1}{2} \times b \times h$, are operating at the object level, therefore 22 learners were able to find item 3 using the area rule.

In addition to evidence that was derived from the analysis of learners' written responses, interview responses confirmed that learners found that the inclusion of GeoGebra as an instructional strategy assisted them in learning 2D and 3D trigonometric concepts, as illustrated in the extracts below.

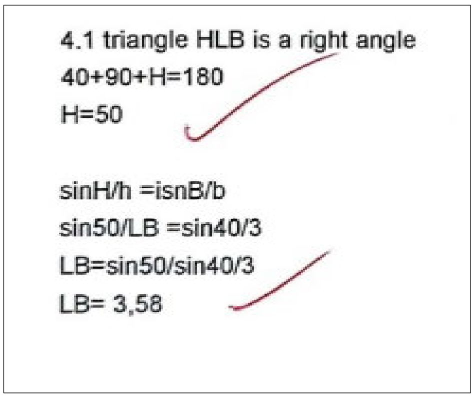
Researcher: What can you say about the inclusion of GeoGebra in your learning of 2D and 3D trigonometry?

Ndebenhle (FB2): "*Ah… it helped me understand a lot more about 2D and 3D by making me look at shapes in the very different ways and know that… just like a right-angle triangle: it made me realise that the hypotenuse side is the longest side in that triangle. Ya, it just helped me in those basic things. Like the angles in a triangle—how these angles add up to 180. And other different shapes of triangles. Also, as you can see, I did very badly in the previous test—but now I know which formulas to use and when.*"

What FB2 said concurs with what FB1 said (in Section 7.4.1): that being able to visualise shapes in different orientations assisted him to recall the correct formula and not confuse 2D and 3D triangles. Drawing from the extract above, this confirmed that learners had been having difficulty differentiating triangles and choosing the correct formula to use, but after the implementation of teaching using GeoGebra learners could differentiate between triangles and thus invoke the formula to be used. This was explained by one of the learners as follows:

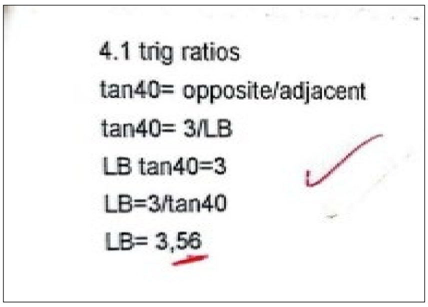
Nhlakanipho) (FB3): "*For me, GeoGebra did assist me in understanding of 2D and 3D because now I know how to answer question of 2D and 3D in different ways than before. Before, I had just one way of answering the question, but now I can understand how I am going to answer it and now I can decide which formula to use when.*"

Nhlakanipho (FB3) was one of the learners who demonstrated having made all the necessary mental constructions and was found to have encapsulated the process into an object on certain items, such as Item 5 of Question 1. However, he indicated that before the implementation of GeoGebra he had only been to solve problems using one method and if he had forgotten the rule he would have provided an incorrect answer, but after the implementation of GeoGebra he was able to solve problems using more than one method, as shown in the following extract shows 2 learners' written work on the post-test using the sine rule and trigonometric ratios.



4.1 triangle HLB is a right angle
40+90+H=180
H=50

sinH/h =isnB/b
sin50/LB =sin40/3
LB=sin50/sin40/3
LB= 3,58



4.1 trig ratios
tan40= opposite/adjacent
tan40= 3/LB
LB tan40=3
LB=3/tan40
LB= 3,56

Extract 2: Thabiso (left) and Nonhle's (right) written work for item 1 of question 4

Another critical issue raised by learners during the interview was that through GeoGebra they were able to visualise 2D and3D shapes and their properties, which was not possible when looking at a shape drawn on paper.

Ndabenhle (FB2): *GeoGebra helped to realise that shapes in question papers might look the way you think that they look, but they actually don't. As we saw in GeoGebra, when we draw certain shapes like, for instance, a triangle might look in a familiar way, but when we saw it in GeoGebra using, like, the 3D view, it looked in a different way. And you can see that the formula* $\frac{1}{2} base\ x\ height$ *cannot work in determine the area of this shape and … aaah …you remember the formula to use.*

Thabiso (FB1): *Before I got to learn about GeoGebra and everything, I didn't know how to differentiate between 2D and 3D on the question paper, on exam or any paper that we write at that time. But when I got to GeoGebra I learnt that 3D and 2D are different shapes, and sometimes in 3D we may find the right-angle triangle ,and that in question with 2D there are no vertical poles that are there, it helped to understand of how am I going to answer questions in 3D and questions in 2D.*

As evident in the extracts above that, learners' challenges emanated from an inability to differentiate between 2D and 3D in trigonometry which then hindered their ability to invoke the correct formula. However, with the inclusion of GeoGebra, learners could rotate the shapes and be able to identify the properties and thus invoke the correct formula to be used.

The study focused on exploring learner's mental constructions before and after implementing GeoGebra to ascertain the extent to which GeoGebra could enhance learners' mental constructions. The findings showed that before the implementation of GeoGebra learners' mental constructions had not reached the action level in most cases; the few who displayed action conception relied on step-by-step procedural understanding. As a result, learners struggled to solve abstract problems. Ngcobo et al. (2019) argues that learners operating at the process and object stage can solve abstract problems and provide or deal with proofs meaning that without the development of process and object stage learners cannot handle proofs. It was revealed in this study that before the implementation of GeoGebra in Question 2 item 1 where learners were required to show that the area of the triangle PQR=$5\sqrt{3x} - \sqrt{3}x^2$ some used incorrect rules, such as the sine rule or the cosine rule, to prove the area. However, after the implementation of GeoGebra, it was found as Ngcobo et al. (2019) had found that there were learners who could solve Item 1 of Question 2 because their mental constructions had evolved from action to process, or from process to object.

7.5. Implications of the findings

Dubinsky (1997) argues that concepts that learners struggle with should be explored by means of research and alternative instructional tools should be designed to enhance learners' mental constructions. This study explored specific challenges and GeoGebra proved to be an alternative instructional strategy that was effective in enhancing learners' mental constructions of 2D and 3D trigonometric concepts.

Teachers need a clear understanding of the abilities of their learners to enable them to plan and design strategies to assist learners to construct the required mental structures. Madonsela (2017) states that understanding the mental construction learners make can assist teachers in designing alternative teaching strategies to assist learners in constructing mental constructions that are appropriate for the development of conceptual understanding. In addition, teachers should encourage group discussions and let learners explain problems to themselves. Arnon et al. (2014) argues that some pedagogical strategies are required to assist learners with making the necessary mental constructions such as cooperative learning, group discussion and lecturing. Learners tend to understand better if they explain or discus the concept among themselves; thus, co-operative learning plays a crucial role, while teachers can provide immediate feedback, and students can revise their understanding based on interactive feedback loops.

Many studies have advocated for the introduction of GeoGebra as an instructional tool. For example, Bayaga et al. (2019) found that GeoGebra yielded positive results in terms of improving logical thinking, confidence, interaction, engagement, connection, reasoning, creativity and understanding. This study has shown that GeoGebra is also useful for enhancing learners' mental constructions in the area of trigonometry.

7.6 Contributions of this study

This is not the first study to explore learners' mental constructions of trigonometric concepts. Ngcobo et al. (2019) and Madonsela et al. (2020) are some of the scholars that have contributed to this field and have provided the genetic decomposition to analyse learners' mental constructions. However, this study adds to the knowledge in this field by providing a genetic decomposition of both 2D and 3D trigonometric concepts. In addition, while some studies have advocated for alternative instructional strategies to be introduced to enhance learners' mental constructions, none of them have actually implemented an alternative instructional strategy to explore the evolution of learners' mental constructions. This study, however, not only advocates for alternative instructional strategies but demonstrates that GeoGebra can be an effective alternative instructional tool to enhance learners' mental constructions of trigonometric 2D and 3D concepts.

7.7 Limitation of the study

The study explored the use of GeoGebra to enhance Grade 12 learners' understanding of 2D and 3D concepts in trigonometry. This exploration was done with 30 learners at one school in King Cetshwayo District Municipality in KwaZulu-Natal, South Africa. While the aim of the study was not to generalise the findings, additional studies will need to be done to confirm the veracity of the findings.

As the study was conducted at a school other than the one where the researcher teaches, in order to limit bias, she worked with some of the participants of the study in an extra-curricular programme developed by the cluster. This meant that learners still viewed her as a teacher, which might have prompted them to respond in certain ways to try to please her, thinking that this might impact their marks. This limitation was minimised by assuring learners that their performance on the test would not contribute to their school-based marks. Also, when the study was completed, the findings were shared with the participants to assure them that their marks were not included in their school-based marks.

The pre- and post-test were designed such that several questions built on the previous questions. This meant that if a learner answered one question incorrectly, it would be impossible for them to answer the subsequent question correctly as it depended on a value obtained in the previous question. This limited the learner from making necessary mental constructions for the later questions.

Another limitation was that while originally six learners had agreed to participate in the interviews, two later withdrew and only four learners participated in the interviews. While interviewing only four learners did not compromise the findings in any way, the researcher had hoped to elicit more interpretations by learners of the implementation of GeoGebra as an instructional tool.

Implementation of GeoGebra required computers; the shortage of computers in the computer lab meant that some learners had to share a computer. As a result, during activities that had been designed for learners to do individual work, some had to work in pairs. Thus, while the analysis of learners' responses for the pre-test is presented in terms of individual learners' mental constructions, as learners were working in pairs it was not easy to distinguish between learners' mental constructions. For the post-test, however, this issue was eliminated by borrowing staff laptops to enable all the learners to write the post-test individually.

The next section presents recommendations for teachers, for curriculum managers and for further researcher.

7.8 Recommendations

This study recommends the integration of digital pedagogies teaching tool such as GeoGebra, into mathematics to enhance learners' understanding of mathematic concepts. Specific recommendations are made for educators and curriculum managers and for further research.

7.8.1. Recommendation for educators

As numerous scholars have demonstrated that mathematics is challenging for learners (see Dubinsky, 1997; Brijlall, 2009; Ndlovu, 2015), mathematical concepts need to be analysed by means of research in order to develop alternative instructional strategies to enhance learners' mental constructions. In this study, a genetic decomposition was created for 2D and 3D trigonometric concepts (presented in Chapter 3) that teachers can use to analyse learners' mental constructions. It is recommended that mathematics teachers use this genetic decomposition to analyse their learners' mental constructions so that they can pinpoint where misconceptions are creating barriers to their mental construction and address these.

This study has demonstrated that GeoGebra can aid the development of learners' mental construction of 2D and 3D trigonometric concepts. It is recommended that teachers incorporate this valuable alternative instructional strategy into their teaching to enhance learners' understanding of these concepts. Teachers can develop their proficiency with GeoGebra by attending workshops or online training sessions to enable them to create more engaging and effective learning experiences for their students. Collaborating with colleagues to create GeoGebra-based lessons and share knowledge and expertise about using GeoGebra is also recommended. In addition, it is recommended that teachers encourage learners to explore GeoGebra on their own. This can be done by assigning GeoGebra tasks that require learners to manipulate geometric objects and observe the effects, fostering a sense of discovery and independent learning.

7.8.2 Recommendation for curriculum managers

Bedada (2021) reported that GeoGebra is believed to have a positive impact on learners' attitudes, beliefs and perceptions. The findings of this study have provided evidence that GeoGebra is helpful in the teaching of 2D and 3D trigonometric concepts. Curriculum managers are thus encouraged to consider the inclusion of such instructional strategies at staff development workshops and when procuring instructional resources for schools.

7.8.3 Recommendations for further research

This was a small-scale study conducted with learners from one school who had been exposed to digital technologies because their school is a technical school. It is thus recommended that a bigger study be conducted with learners who have not been exposed to using software to explore the veracity of the findings of this small-scale study.

Furthermore, the use of GeoGebra to enhance learners' mental constructions of other mathematical concepts, such as algebra, could be explored.

7.9 Conclusion

This study has demonstrated that an alternative instructional strategy can potentially enhance learners' mental constructions. It was found that prior to the implementation of GeoGebra, most learners were operating below the action level, with only a few operating at the action and process level. However, after learners had worked with 2D and 3D trigonometric concepts using GeoGebra, the majority were found to be operating at the action level, with some having advanced to the process and object levels. While schema development was still lacking among many learners, it was evident that their mental constructions had evolved. It was also found that the learners had knowledge gaps that hindered the development of their mental constructions and that alternative instructional strategies thus needed to be employed to bridge these knowledge gaps so that learners would be able to make the necessary mental constructions. The use of GeoGebra, as an alternative instructional strategy, enabled learners to view 2D and 3D shapes in different orientations which resulted in observable evolution of their mental constructions.

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APPENDICES

Appendix 1: Activity sheet (Test)

Appendix 1

Activity sheet

The questions are extracted from grade 12 past matric papers

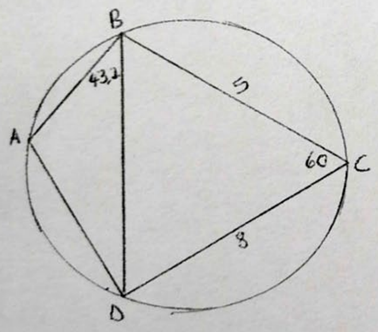
Instructions

Read the following instructions carefully before answering the questions

1. Answer all the questions
2. Clearly show all calculations
3. You may use an approved scientific calculator
4. If necessary round of answers correct ton TWO decimal places, unless stated otherwise
5. Diagrams are not necessarily drawn to scale
6. Write neatly and legibly

Question 1

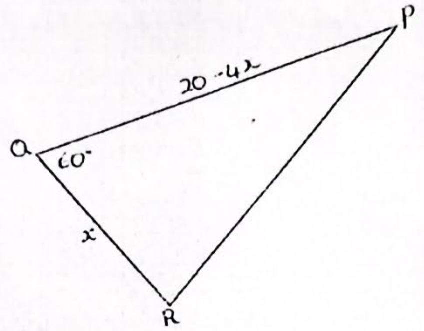
Given that ABCD is a cyclic quadrilateral of a circle BC=5 units, DC=8units, C=60 and ABC=43,2



1.1. What is your understanding of a cyclic quad? (3)
1.2. Explain how you will find the area of ΔDAB (3)
1.3. Calculate BD (3)
1.4. Calculate AD (3)
1.5. Calculate the area of ΔDAB (3)

Question 2

2.1 In the diagram below ΔPQR is drawn with PQ=20-4X, RQ=X and $\hat{Q}$=60



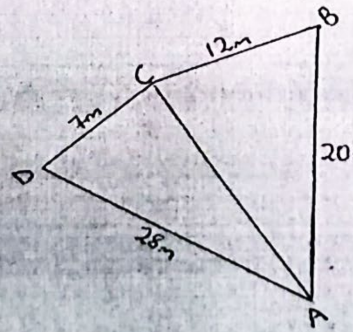
2.1.1 Show that the area of ΔPQR=5√3x-√3x^2 (2)

2.1.2. Determine the value of x for which the area of ΔPQR will be in maximum. (3)

2.1.3. Calculate the length of PR if the area of ΔPQR is a maximum (3)

Question 3

3. A piece of land has a form of a quadrilateral ABCD with AB=20m BC=12m, CD=7m and AD=28, $\hat{D}$=110 . The owner decides to divide the land into two plots by erecting a fence from A to C



3.1. Calculate the length of the fence AC correct to one decimal place (2)

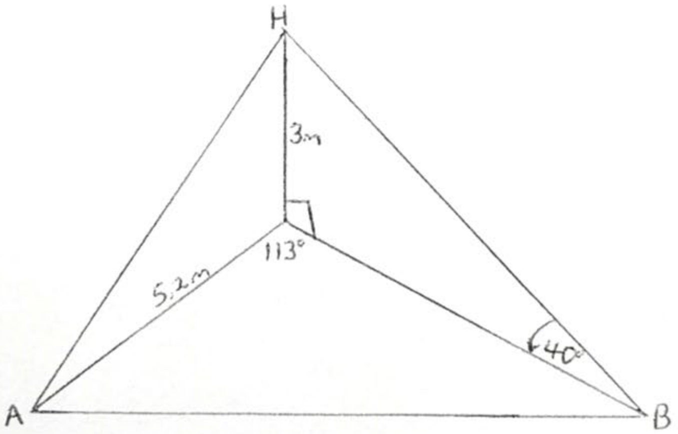
3.2. Calculate the size of BAC correct to the nearest degree (2)

3.3. Calculate the size of $\hat{D}$ correct to the nearest degree (3)

3.4. Calculate the area of the entire piece of land ABCD, correct to one decimal place (3)

Question 4

4 A, B and L are points on the same horizontal plane HL is a vertical pole of length 3metres, AL =5,2cm, the angle ALB=113 and the angle of elevation of H from B is 40



4.1. Calculate the length of LB (3)

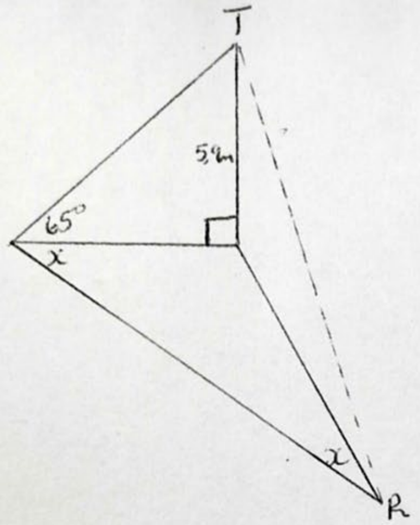
4.2. Hence or otherwise, calculate the length of AB (4)

4.3. determine the area ΔABL (4)

Question 5

5 in the sketch, the points P, Q and R, are in the horizontal plane TP is a vertical tower, 5,9 metres high. The angle of elevation of T from Q is 65

PQR=PRQ=X



5.1. Calculate the length of PQ to the nearest metre (3)

5.2. Hence show that RQ=6cosx (5)

5.3. It further given that X=42, calculate the area of ΔPQR (4)

Appendix 2: Interview schedule after pre-test (semi-group interview)

In line with APOS theory interviews are meant to verify what transpired in the Written work thus broad interview questions are set however more questions Will be informed about what transpired in the written responses

1. Explain how you solve the question

2. Is there any other way you could have solved this question?

3. In the case there are questions not answered (Why the question was not Answered? Can you attempt it now?

4. in the case where there are questions attempted but the answer is wrong (The Answer to this question is not correct:

   a) Can you tell me where you encountered the problem?

   b) What is your understanding of 2D and 3D? Do you think if we Deconstruct the question e.g have the triangles separately you can solve it?

Appendix 3: Interview Schedule after post-test (Focus group interviews)

Exploring the use of GeoGebra to enhance grade 12 learners' knowledge of Trigonometric 2D and 3D.

Broad questions would be asked and questions for probing will be informed by The discussion

1. Compare your performance in the pre- test and post-test is there any Improvement?

2, why you think you did /did not do better in the post test compared to pretest?

3. Before writing the post-test we use Geogebra to teach 2D and 3D. Comment On how it assisted you/not assisted you to enhance your understanding and Solving of 2D and 3D

4. To what extent if any does GeoGebra enhance your knowledge construction of 2D and 3D? Trigonometric concept

Appendix 4: Permission letter to school principal

To: The Principal
Name of the school: Tisand Technical

Year: 2022

REQUEST FOR PERMISSION TO USE GRADE 12 LEARNERS AND COMPUTERS TO ENHANCE LEARNERS MENTAL CONSTRUCTION OF KNOWLEDGE IN TRIGONOMETRIC 2D AND 3D USING GEOGEBRA.

My name is Philisiwe Mbatha who is a teacher of Mathematics at Matamzana Dube Secondary School. I am currently registered and working on a full research thesis with the University of Kwazulu Natal. The title of the thesis is to **explore the use of geogebra to enhance grade 12 learners knowledge construction of trigonometric 2D and 3D concepts: Case of one school at King Cetshwayo district.**

I am asking for the permission to use grade 12 learners and computers. The lessons will take place after school, session will take one hour for a period of four weeks.

My contact details are as follows:

Email: philsiwe.mbatha@gmail.com

Cell phone: 0679009484

My supervisor is Dr Zanele Ngcobo She is a lecturer in the School of Education, College of Humanities, Edgewood Campus, University of KwaZulu-Natal

My supervisor's contact details are:

Email: NgcoboA2@ukzn.ac.za

Phone number:0724011275

You may also contact the Research Office at:

University of KwaZulu-Natal

Humanities and Social Sciences Research Ethics

Ms. Mariette Snyman (HSSREC Research Office)

Tel +27312604557

Email: HSSREC@ukzn.ac.za

[redacted]

30/08/2021

------------------------------------ ---------------------------

(Researcher's signature) (Date)

DECLARATION

I, [redacted] ________ mgenge P.Z. ________ (NAME and SIGNATURE)

Principal on this day of 30 month 09 2021, hereby grant permission to go ahead with the research in the above-mentioned School following the terms of reference noted in this request letter.

TISAND TECHNICAL HIGH SCHOOL
PRIVATE BAG X8533 ESIKHAWINI 3887
2021-09-30
TEL: 035 710 2007 / 035 796 1040
FAX: 086 615 2380
CELL: 082 870 5199

Appendix 5: Parent consent / learner assent letter

Consent letter

....

School of Education, College of Humanities, University of KwaZulu-Natal,

Edgewood Campus

Dear Participant

INFORMED CONSENT LETTER

My name is Philisiwe Mbatha I am a master's candidate studying at the University of KwaZulu-Natal, Edgewood campus, South Africa. I am interested in exploring the use of Geogebra to enhance your knowledge construction in trigonometric 2D and 3D to gather the information, I am interested in asking you some questions.

Please note that:

- Your confidentiality is guaranteed as your inputs will not be attributed to you in person, but reported only as a population member opinion.
- The interview may last for about 45 minutes to 1 hour.
- Any information given by you cannot be used against you, and the collected data will be used for purposes of this research only.
- Data will be stored in secure storage and destroyed after 5 years.
- You have a choice to participate, not participate or stop participating in the research. You will not be penalized for taking such an action.
- Your involvement is purely for academic purposes only, and there are no financial benefits involved.
- If you are willing to be interviewed, please indicate (by ticking as applicable) whether or not you are willing to allow the interview to be recorded by the following equipment:

Equipment	Willing	Not willing
Audio equipment	✓	
Photographic equipment	✓	
Video equipment	✓	

I can be contacted at:

Email: philsiwe.mbatha@gmail.com

Cell: 0679009484

My supervisor is Dr Z Ngcobo who is located at the School of Education, Edgewood campus, University of KwaZulu-Natal (UKZN).

Contact details: Room CU 150, Main Tutorial Building, Edgewood Campus, UKZN.

email: ngcoboA2@ukzn.ac.za

Phone number: 031 2603784

You may also contact the Research Office through:

Ms. Mariette Snyman (HSSREC Research Office)
Tel: 031 260 8350
Email: Snymanm@ukzn.ac.za

Thank you for your contribution to this research.

DECLARATION

I. Nhlakanipho Zwane .... (full names of participant) hereby confirm that I understand the contents of this document and the nature of the research project, and I consent to participating in the research project.

I understand that I am at liberty to withdraw from the project at any time, should I so desire.

SIGNATURE OF PARTICIPANT **DATE**

[redacted]............................ 04/10/2022

SIGNATURE OF PARENT (If participant is a minor) **DATE**

[redacted].......................... 04/10/2022

Appendix 6: Learner assent form (isiZulu)

INCWADI YOMZALI

Arboretum

Central village

Pigeon wood-street

3900

2 October 2022

Mzali

Ngiyakubingelela, ngiwuthisha ofundisa eMatamzana High School. Ngenza ucwaningo mayelana nesifundo sezibalo (Mathematics). Abafundi abenzi kahle kulesisifundo iminyaka eminingi. Lolu Cwaningo luzobheka izimbangela ezenza behluleke ukuqonda izibalo nemuphumela emimbi kulesifundo.Umntwana wakho uyacelwa ukuba abe ingxenye yalolu cwaningo. Igama lengane lizovikeleka kanti futhi ngeke aphazamiseke ezifundweni zakhe. Konke kuzoba imfihlo, futhi awukho umholo ozotholakala. Imiphumela yocwaningo izosiza abaphethe ezemfundo ukuze ekugcineni kusizakale umntwana.

Ozithobayo

Mbatha Philisiwe P

Signiture.[redacted].....

Date..2/10/2022..........

NB: khetha okukodwa (Tick one)

[X] Ngiyavuma (Agree)

[ ] Angivumi (Disagree)

DECLARATION BY LEARNERS

PART ONE

I..Nhlakanipho Zwane........ (Learner's name)[redacted].......

Signature ..04/10/2022.............Date..

NB: TICK ONE

[X] Agree

[ ] Disagree

Appendix 7: Ethical clearance letter

UNIVERSITY OF KWAZULU-NATAL™
INYUVESI YAKWAZULU-NATALI

20 May 2022

Philisiwe Promise Mbatha (221116169)
School Of Education
Edgewood Campus

Dear PP Mbatha,

Protocol reference number: HSSREC/00004156/2022
Project title: Exploring the use of Geogebra to enhance grade 12 learners knowledge construction of trigonometric 2D and 3D concepts in one school in King Cetshayo District
Degree: Masters

Approval Notification – Expedited Application

This letter serves to notify you that your application received on 03 May 2022 in connection with the above, was reviewed by the Humanities and Social Sciences Research Ethics Committee (HSSREC) and the protocol has been granted **FULL APPROVAL.**

Any alteration/s to the approved research protocol i.e. Questionnaire/Interview Schedule, Informed Consent Form, Title of the Project, Location of the Study, Research Approach and Methods must be reviewed and approved through the amendment/modification prior to its implementation. In case you have further queries, please quote the above reference number. PLEASE NOTE: Research data should be securely stored in the discipline/department for a period of 5 years.

This approval is valid until 20 May 2023.
To ensure uninterrupted approval of this study beyond the approval expiry date, a progress report must be submitted to the Research Office on the appropriate form 2 - 3 months before the expiry date. A close-out report to be submitted when study is finished.

All research conducted during the COVID-19 period must adhere to the national and UKZN guidelines.

HSSREC is registered with the South African National Research Ethics Council (REC-040414-040).

Yours sincerely,

[redacted]

Professor Dipane Hlalele (Chair)

/dd

Humanities and Social Sciences Research Ethics Committee
Postal Address: Private Bag X54001, Durban, 4000, South Africa
Telephone: +27 (0)31 260 8350/4557/3587 **Email:** hssrec@ukzn.ac.za **Website:** http://research.ukzn.ac.za/Research-Ethics

Founding Campuses: Edgewood | Howard College | Medical School | Pietermaritzburg | Westville

INSPIRING GREATNESS

Appendix 8: Turn it in report

Turnitin Originality Report

Processed on: 27-Dec-2023 5:02 PM SAST
ID: 2265108435
Word Count: 47781
Submitted: 1

Enhancing learners' mental constructions using Geogebra By Philisiwe Mbatha

Similarity Index	Similarity by Source	
16%	Internet Sources:	12%
	Publications:	7%
	Student Papers:	5%

1% match ()
Makandidze, Lancelot Sibanengi. "Exploring learners' understanding of trigonometric functions using GeoGebra software : a case of grade 11 Mathematics learners at a school in Tshwane South District", 2020

1% match ()
Mutambara, illias Hamufari Natsai., "An APOS analysis of the understanding of vector space concepts by Zimbabwean in-service Mathematics teachers.", 2018

1% match (Shashidhar Belbase. "Early Undergraduate Emirati Female Students' Beliefs about Learning Mathematics Using Technology", European Journal of Educational Research, 2020)
Shashidhar Belbase. "Early Undergraduate Emirati Female Students' Beliefs about Learning Mathematics Using Technology", European Journal of Educational Research, 2020

< 1% match ()
Mushipe, Melody. "Effects of integrating GeoGebra into the teaching of linear functions on Grade 9 learners' achievement in Mopani district, Limpopo Province", 2016

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Appendix 9: The editing certificate

CERTIFICATE OF PROFESSIONAL EDITING

I, Barbara L. Louton, do hereby declare that I am a professional editor with a Bachelor of Arts in Professional Writing and seventeen years of experience as an editor, researcher and writer.

I declare that I was contracted by Philisiwe Promise Mbatha, a Master of Education candidate under the supervision of Prof Zanele Ngcobo in the School of Education at the University of KwaZulu-Natal, to complete a professional edit of her dissertation:

Exploring the use of Geogebra to enhance Grade 12 learners' understanding in 2D and 3D of trigonometry in one school in King Cetshwayo district

I declare that I have completed a two-stage professional edit of the document, addressing structural and logical issues, the clarity and flow of language, and correcting grammatical, spelling and formatting errors. Changes were tracked and comments were left for the client, who then make further revisions which were then edited.

Disclaimer:

Responsibility for the originality and accuracy of the material presented in the edited document lies with the client. I have not verified the originality or accuracy of statements, quotations or citations and references presented in the thesis. Where I have detected inaccuracies I have rectified them or reported them to the client. In addition, the client was free to make further changes to the edited material after the edit was complete.

I can be contacted at:

Cell: 073 766 1139

Email: bellway@gmail.com

Barbara L Louton	[redacted]	31 December 2023
Name	Signed	Date