LEARNERS' CONCEPTUALIZATION OF THE SINE FUNCTION, WITH SKETCHPAD AT GRADE 10 LEVEL

BY

JUGMOHAN J.H.
(8728037)

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SUPERVISOR: Professor M. de Villiers

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CONTENTS

ABSTRACT

LEARNERS' CONCEPTUALIZATION OF THE SINE FUNCTION DURING AN INTRODUCTORY ACTIVITY WITH SKETCHPAD AT GRADE 10 LEVEL

This study investigated how Grade 10 learners conceptualise an introductory activity to the sine function with *The Geometers' Sketchpad.*

In a study by Blackett and Tall (1991), the initial stages of learning the ideas of trigonometry, are described as fraught with difficulty, requiring the learner to relate pictures of triangles to numerical relationships, to cope with ratios such as sinA = opposite/hypotenuse. A computer approach might have the potential to change this by allowing the learner to manipulate the diagram and relate its dynamically changing state to the corresponding numerical concepts. The learner is thus free to focus on specific relationships, called the principle of selective construction, as stated by Blackett and Tall (1991). The use of this educational principle was put to test to analyse the understanding of Grade 10 learners' introduction to the sine function.

Data was collected from a high school situated in a middle-class area of Reservoir Hills (KZN) by means of task-based interviews and questionnaires. Given a self-exploration opportunity within *The Geometers' Sketchpad,* the study investigated learners' understanding of the sine function only within **the first quadrant**:

A) as a ratio of sides of a right-angled triangle
B) as an increasing function
C) as a function that increases from zero to one as the angle increases from 0° to 90°.
D) as a relation between input and output values
E) the similarity of triangles with the same angle as the basis for the constancy of trigonometric ratios.

The use of *Sketchpad* as a tool in answering these questions, from A) to E), proved to be a successful and meaningful activity for the learners.

From current research, it is well-known that learners do not easily accommodate or assimilate new ideas, and for meaningful learning to take place, learners ought to construct or reconstruct concepts for themselves. From a constructivist perspective the teacher cannot transmit knowledge ready-made and intact to the pupil. In the design of curriculum or learning materials it is fundamentally important to ascertain not only what intuitions learners bring to a learning context, but also how their interaction with specific learning experiences (for example, working with a computer), shapes or changes their conceptualisation.

The new ideas that the learners' were exposed to on the computer regarding the sine function, also revealed some errors and misconceptions in their mathematics. Errors and misconceptions are seen as the natural result of children's efforts to construct their own knowledge, and according to Olivier (1989), these misconceptions are intelligent constructions based on correct or incomplete (but not wrong) previous knowledge. Olivier (1989), also argues that teachers should be able to predict what errors pupils will typically make; explain how and why children make these errors and help pupils to resolve such misconceptions. In the analysis of the learners' understanding, correct intuitions as well as misconceptions in their mathematics were exposed.

DECLARATION

I, Jhansheela Jugmohan (8728037), declare that the research involved in my dissertation submitted in the partial fulfilment of the M.Ed. degree in mathematics education, entitled learners conceptualization of an introductory activity to the sine function with Sketchpad at Grade 10 level represents my own and original work.

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Ms. J. H. JUGMOHAN

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CHAPTER 1
INTRODUCTION

1.1 BACKGROUND TO THE STUDY

Curriculum reform is pushing teachers to shift their practices towards more participatory and inquiry-based methods (Pournara, 2001), where learners' meanings are given greater credence. This places greater demands on teachers since they are expected to understand learners' meanings and mediate between learners' personal meanings and public mathematical meanings. It is therefore important, according to Pournara (2001, 2), *"that we understand how learners make sense of mathematical concepts in order to support teachers in making the transition to new pedagogical approaches in the teaching of mathematics."*

My own experiences of teaching trigonometry at Grade 10-12 level and my observation of other teachers support the finding that the mathematical knowledge of secondary school learners are dominated by content- and teacher-centred pedagogies (Boaler, 1997). The difficulty of learning trigonometric functions has been emphatically stressed by learners throughout my teaching career. It has been found that mathematics teachers also find it hard to help learners make sense of this topic. However very little research has been done to explore this matter, specifically regarding the learning of sine and cosine functions.

As a teacher of mathematics in high school, I am aware of the struggles learners face when trying to understand this subject. I was looking for a way to give them a kind of visual intuition about circles and angles, which would help create concrete knowledge. The idea was to expose the learners to picture a unit circle with a right-angle in it when they thought about the sine function. It was hoped that such a dynamic sketch would help learners gain first-hand experience and conviction of relationships in trigonometry, for example, why the sine of a given angle in the first quadrant, remains constant, irrespective of the length of the corresponding sides.

This research focused on learners' conceptualisation of an introductory activity to the sine function, hence the choice of Grade 10 level.

From a constructivist perspective the teacher cannot transmit knowledge ready-made and intact to the pupil. Learners ought to construct or reconstruct concepts for themselves, as they do not easily accommodate or assimilate new ideas. Thus, for meaningful learning to take place, it is fundamentally important to consider not only what intuitions and misconceptions learners bring to a learning context, but also how their interaction with specific learning experiences (for example, working with a computer), shapes or changes their conceptualisation. Errors and misconceptions, according to Olivier (1989), are seen as the natural result of children's efforts to construct their own knowledge, and these misconceptions are intelligent constructions based on correct or incomplete (but not wrong) previous knowledge, which needs to be considered when one considers the design of the curriculum or learning materials

Many learners appear to have little understanding (Pournara, 2001) of the underlying trigonometric principles and thus resort to memorising and applying procedures and rules even though many are able to do this successfully.

The belief is that many learners have experienced difficulties with trigonometry because they have relied on memorising rules and procedures and ignored the conceptual aspects of its objects. Skemp (1976), states that this has lead many novice trigonometry learners to develop an instrumental rather than a relational understanding concentrating on trigonometry algorithms and learning 'how to' rather than 'why'.

Children do not only interpret knowledge, but they organise and structure this knowledge into large units of interrelated concepts. Olivier (1989) calls such a unit of interrelated ideas in the child's mind, a *schema.* Such schemas are valuable intellectual tools, stored in memory, and which, according to Olivier (1989) can be retrieved and utilised. Learning then basically involves the interaction between a child's schemas and new ideas. This interaction involves two interrelated processes according to Olivier (1989) called assimilation and accommodation.

Thus to understand an idea means to incorporate it into an appropriate existing schema. However, sometimes some new idea may be so different from any available schema, that it is impossible to link it to any existing schema. So assimilation and accommodation is impossible. In such a case the learner creates a new "box" and tries to memorise the idea. This, according to Olivier, is *rote learning:* because it is not linked to any previous knowledge, it is not understood; it is isolated knowledge, therefore it s difficult to remember. Such rote learning is the cause of many mistakes in mathematics as pupils try to recall partially remembered and distorted rules.

To the constructivist, learning is not as for the behaviourist, a matter of adding or stockpiling new concepts to existing ones. Rather, learning, according to Olivier (1989), leads to *changes* in our schemas.

The main motivation for this study is to address the gap in the research literature on learners' understanding on trigonometric concepts as much research has been done on various content areas of mathematics internationally and locally, however very little research has been done, locally and internationally (Pournara, 2001), in the area of trigonometry. According to Pournara (2001), a survey of Dissertation Abstracts International revealed only two masters dissertations/doctorates in the area of trigonometry in the period 1995-1999. He further states that while one may find some articles in mathematics teaching journals on methods of teaching trigonometry (e.g. Dooley, 1968; Satty, 1976; Ellery, 1980; Hirsch et al, 1991; Richardson and Sutherland, 1992), these are generally based on personal opinions and experiences rather than on empirical research, with little or no attention given to learners' thinking about trigonometry.

Data from the recent Second International Mathematics Study suggest that High school learners have not mastered the basic precalculus topics of function, graphing and problem-solving. (Waits, Demana, 1998).

Looking through recent research it is possible to note the increase of studies based on the constructivist point of view, using the computer. (Wenzelburger, 1992; Gomes-Ferreira, 1997; Borba, 1993; Hoyles, 1991,1996). Also documented are increased studies in mathematics in everyday life (Saxe, 1991; Lave, 1989; Nunes, 1993, 1996) as well as studies encompassing both contexts (Magina, 1994).

In trigonometry teaching, according to Hart (1971:22), *" there have been attempts to move away from a process-orientated style of teaching and learning which may have prevented learner understanding of important concepts."* Recent research has sought to use computer software to improve understanding of concepts.

Teacher education around the new curriculum has emphasised learner activity, participation and group-work as central aspects of classrooms (Brodie, 1998). Teachers are encouraged to *"facilitate"* learning rather than provide instruction. A *"paradigm shift"* from practices are urged, with the past being characterised as *"teacher centred"* and encouraging passive learners who engage in individualised, rote learning rather than creative and flexible thinking (National Department of Education, 1997:6-7).

In recent years, according to the NCTM (1989), mathematics educators have focused attention on rethinking the process of mathematics education at all levels. Calls for reform of mathematics education now urge teachers and faculty to improve not only the cognitive side of instruction, but also to emphasise non-cognitive issues, such as students' feelings, attitudes, beliefs, interests, expectations and motivations.

David and Machado (1996:34), also state that the teaching process might be contributing to error and failure in mathematics. The emphasis on *"formal procedures (algorithms and rules)"*, unrelated to the concept that supports them, *"inhibits the necessary flexibility of thought"* that is necessary for success in mathematics.

The importance of the use of computers in mathematics is well researched (Tall, 1989; Leinhardt, Zaslavsky et al, 1990; Duren, 1991). As Duren (1991: 23) noted, *"the availability and access to graphics software for the secondary school mathematics curriculum has provided two powerful learning modalities for students: visualisations and investigations."*. In particular, the benefits of the use of computer software on students' understanding of the function (Breidenbach, Dubinsky et al:

1992) and in developing a visual approach to transformations and graphs (Bloom, Comber et al, 1986; Dugdale, Wagner et al, 1992) have been demonstrated.

In traditional classroom teaching, when learners made errors, corrections were handed down by the teacher as an external authority. There was no way that learners could use their own abilities to evaluate and correct their own work. Von Glasersfeld (1987:14) has noted that this kind of correction is *"not completely satisfactory"* because, it denies learners the opportunity to restructure their own conceptual schemas and from a constructivist perspective, meaningful learning can only take place in the mind of the learner.

From a constructivist point of view, according to Von Glasersfeld (1987), it makes no sense to assume that any powerful cognitive satisfaction springs from simply being told that one has done something right, as long as 'rightness' is assessed by someone else. To become a source of real satisfaction, 'rightness' must be seen as the fit with an order one has established oneself." This cognitive satisfaction is gained through investigative work in learner-centred teaching, which is most effective when mediated by a computer.

1.2 RESEARCH QUESTIONS

My research questions focused on the learners' understanding of the sine function, within **the first quadrant** only given a self-exploration opportunity within *The Geometers' Sketchpad:*

A) What understanding do learners develop of the sine function as a ratio of the sides of a right triangle during an introductory activity?

B) What understanding do learners develop of the sine function as a function of an angle as independent variable during an introductory activity?

C) What intuitions and misconceptions do learners bring to the learning situation.

D) What is the role of learners' prior intuitions and misconceptions in their learning?

E) What is the mediating role of the *Sketchpad* tool?

And more specifically, given a self-exploration opportunity within *The Geometers' Sketchpad,* the study investigated the development of learners' understanding of the sine function only within **the first quadrant** with regards to the following:

F) as a ratio of sides of a right-angled triangle
G) as an increasing function
H) as a function that non-linearly increases from zero to one as the angle increases from $0°$ to $90°$.
I) as a relation between the angle aas input and a function value as output
J) the similarity of all right triangles with the same reference angle as the basis for the constancy of trigonometric ratios.

1.3 OUTLINE OF THE REPORT

Chapter 2 briefly discusses the importance of the history of mathematics for understanding how human beings or mathematicians learnt mathematics in general. More particularly, this chapter will look at the history of trigonometry and how it may possibly provide some guidelines to designing a trigonometry curriculum and the teaching and learning of trigonometry. It serves to highlight the potential socio-cultural, socio-historical role a historical perspective of mathematics can have and more importantly, to inform about broad educational and social policy, as a guideline to the curriculum, guiding principles and frames of reference.

In Chapter 3 some of the different approaches to trigonometry in the curriculum, are discussed and analysed in order to provide a background to the research. The difficulties of learning trigonometry, is also discussed.

Chapter 4 develops the theories of learning and the theoretical framework for this study. I motivate for the use of constructivism and the Van Hiele theory as a theoretical framework for this study.

Chapter 5 deals with review of the literature related to this study.

Chapter 6 describes the research methodology, methods of data collection and procedures used in analysing the data.

Chapter 7 provides an analysis and results of the research.

Chapter 8 deals with a summary of the main points of the analysis and consider the implications of my findings for the teaching and learning of trigonometry in South African schools.

CHAPTER 2

THE HISTORY OF TRIGONOMETRY

2.1 OVERVIEW

"*When we look at the history of mathematics, we see a kind of lifelike elemental rhythm. There are periods of exuberant untidy growth, when exciting, vital structures rise upon untried assumptions, and loose ends lie about all over the place. Logic and precision are not unduly honoured; because restlessness, enthusiasm, daring, and ability to tolerate a measure of confusion, are the appropriate qualities of mind at these times. Such periods are followed by pauses for consolidation, when the analysts and systematisers get to work; material is logically ordered, gaps are filled, loose ends are neatly tied up, and rigorous proofs supplied. Solemn commentators sit in judgement upon great innovators. Whole areas of mathematics are formed into deductive systems, based on sets of unproved, explicitly stated axioms. Work of this kind, at its best, is also creative: new ideas grow from the critical examination of old, and the cycle is renewed. Periods of these two kinds may overlap; or a growth period in one field may coincide with a period of consolidation in another: but the fundamental alternation would seem to subsist generally.*" (Hull,1969: 27).

The above quote refers to stages in the historical development of mathematics and might suggest that the learning of mathematics ought to be also generally structured in two such stages; firstly, an informal phase, and then secondly, a more formal phase. Indeed one of the most important reasons for studying the history of mathematics is that it may assist in the identification of possible learning stages. The purpose of this section is therefore to look briefly at the historical development of trigonometry with the idea of identifying historical stages that may be helpful with planning the curriculum.

The history of education, however, cannot answer directly routine questions in education (Fauval, 1991). Its function should be to inform prospective teachers about broad educational and social policy; and it can help in uncovering frames of reference, guiding principles, and other theoretical aspects which routine questions always raise. History of education provides the knowledge necessary for background, understanding and action in education.

Planning the curriculum involves more than choosing the facts and theories to be taught, we must also foresee in what sequence and by what methods those facts and theories should or could be taught (Polya, 1981). In this respect the "genetic principle" offers an important suggestion.

Polya (1981), states that according to the genetic principle, the learner should retrace the path followed by the original discoverers and rediscover what he has to learn. He further states that the genetic principle of teaching can be stated in various ways, for instance: In teaching a branch of science (or a theory, or a concept) we should let the learner retrace the great steps of the mental evolution of the human race. Having understood how the human race has acquired the knowledge of certain facts or concepts, we are in a better position to judge how the learner should acquire such knowledge.

It is rare to find the history of mathematics in the present curricula. Although mathematics textbooks now routinely include information about historical figures and events, that information is clearly optional and considered unnecessary in the acquisition of mathematical knowledge. The purpose of this chapter, is to highlight the use of history in the mathematics classroom and explore the evidence for the emergence of the sine function. It needs to be considered that one of the most important reasons for studying history may be in firstly understanding how mathematicians "learnt" mathematics and could therefore provide useful guidelines on how students can learn as well.

The National Council of Teachers of Mathematics (NCTM) and the Mathematical Association of America (MAA) have expressed a need for history in the mathematics classroom and produced historical materials that might be appropriate for the classroom (NCTM, 1989; Fauvel, Bekken, Johansson, & Katz, 1995). John Fauvel (1991) provided a rather comprehensive list of reasons why the history of mathematics is important and also provided a list of all the possible ways a teacher might use history in the mathematics classroom.

Fauvel (1991) clearly supports the use of history, and laments the fact that although the idea of using history has been around for a long time, few teachers have tried to incorporate history in their teaching. However, he points out that the history of mathematics should be strictly secondary to the teaching of mathematics:

Mathematics has always been presented by teachers and usually perceived by the public as an area of knowledge characterized by facts and truths, and by its total lack of connection with emotion and humane values, such as appreciating each others' contributions and work or appreciating the lengthy emergence of the theorem on mathematical beauty.

We can only extricate ourselves from this dead end by changing the basic principles of and setting new goals for mathematics teaching. Today, more than ever before, it is vital to present teachers and pupils with the development of mathematical knowledge and to show them convincingly that this knowledge will help the teachers in teaching mathematics and will instil in the pupils skills for structuring logical foundations of scientific thought. The pupil will not only assimilate a concentration of logic phrases and ways of thought that underpin the mathematical knowledge he must acquire but will also "learn" from a learning process that reflects to some extent the historical way by which humankind arrived at mathematical knowledge. With regard to the use of mathematics through history in teaching mathematics, Kleiner (1994) comments that "some pupils are enchanted by the beauty and logical structure of the mathematical phrase," and, unfortunately, "some teachers neglect the issue of the origin of the phrase or of what motivated the mathematician to present it." He suggests giving pupils investigative assignments derived from the story of a particular mathematical topic.

There are thus two issues regarding knowing the history of mathematics, trigonometry in this case, (Hull, 1969):

1) the direct use of historical material gives learners' a better cultural, socio-historical perspective on why and how trigonometry was developed.
2) and more fundamentally, gives an idea or a good guideline on how the curriculum may be structured.

Freudenthal, Polya and others argue one need not rigorously or slavishly follow history, but that understanding the historical development can help to plan and design learning activities into a curriculum.

An example of how the concept definition of a function developed, according to de Villiers, (1984):

- The first definition only appeared after the renaissance, where Jean Bernoulli in 1718 stated it as a quantity composed in any manner of a variable and any constants.
- Then Euler in 1748 stated it as an analytic expression whatsoever made up from that variable quantity and from numbers or constant quantities.
- Also Euler in 1750 stated quantities dependant on others, such that as the second change, so do the first, are said to be functions.
- Then Dirichlet in 1837, stated that if a variable y is related to a variable x so that whenever a numerical value is assigned to x, there is a rule according to which a unique value of y is determined, then y is said to be a function of the independent variable x.

These definitions include the idea of functional dependence, however the following does not:

The formal definition, according to de Villeirs (1984), that we use today, where a function is a set of ordered pairs (a,b), such that a is the element of the domain and b is the element of the range and each a belongs precisely to one ordered pair of the function and is thus uniquely related to a single element of b.

In a concept image, we have all the characteristics of a particular concept listed, but that is not a definition. A definition only selects a small subset of that, necessary and sufficient conditions, which becomes a concept definition (de Villiers, 1984). Thus the definition does not include all the properties and one of the dangers and problems of teachings is that people think that if they use the formal definition, for example the circle definition, which is a highly formalised thing that developed over many stages, but only includes a small portion of the kind of concept image that learners should have. Learners should certainly have the concept image of a ratio and of a right triangle, because without that they are going to have difficulty with solving application questions, since that is the most useful one for applications. In fact that is the context from which that particular meaning developed. The circular unit one developed for another context, which we don't even deal with at school. We don't deal with the practical aspects of periodic functions. We may if we deal with pendulums that are regular, rotating wheels, tides or the cycles of the moon etc. So one must question the idea of starting with the circle definition, because that is a limited concept image. It only abstracts and selects certain things that are useful for a certain perspective.

History illustrates this: What is meant by a function? The Greeks did not have a formal definition. Neither did Newton nor Leibniz. They knew that the dependent variable depended on the independent variable by some kind of formula. We need to ask, why did the formalisation of the function into a definition concept only develop later?

Firstly the need for more formally clarifying what a function is, arose from the dramatically increasing application from the 1600's and onwards of mathematical functions and the calculus to scientific problems of motion, forces etc. this in turn had been made possible by the development in the 1600's of the algebraic symbolism and notation, for example, $y = ax^2 + bx + c$, as well as the Cartesian co-ordinate system, which simplified the antiquated methods of the ancient Hindus, Greeks, and Arabs. On the other hand, this late development of the formalization of the function concept, also suggests that it may be conceptually a subtle and deep idea.

Trigonometry, perhaps more than any other branch of mathematics, developed according to Kennedy (1991), as the result of a continual and fertile interplay of supply and demand: the supply of applicable mathematical theories and techniques available at any given time and the demands of a single applied science namely astronomy. So intimate was the relation that not until the thirteenth century was it useful to regard the two subjects as separate entities.

Historically, what does this tell us? This tells us that trigonometry was initially used for practical applications. And what does it tell us about teaching? The problem-centered approach states that one should start with a practical problem that motivates the development of content. Historically this is how trigonometry developed. There was a practical need to build buildings, find out the time of seasons, astronomy and for that they needed some tool and the tool they developed was trigonometry. So this tells us that mathematics does not develop by itself, it develops to solve practical problems, but also theoretical problems as well.

Much of the work of abstract algebra for example, group theory, field theory, ring theory as we know them today developed to solve some problems in ordinary algebra of the real number system. Also the solving of polynomials of higher order to understand why they could not find the general form, they needed abstract algebra. There were theoretical reasons. So what does this tell us of teaching? It tells us that if we want to follow the problem-centred approach, we need to choose and select good starting problems that can similarly motivate children to see the need for trigonometry and motivate them to learn further.

In order to secure the interest of his usual readership in a German periodical, Fuhrer (1987: 127), planned to elaborate a few hints, recommendations, or suggestions on how to include historical items in the ordinary teaching of mathematics in school. It turned out that, as he feared from the beginning, that only a few of his colleagues could be talked into writing about this feature of the subject. *"It was easy to get papers on historical facts or on hypothetical developments, but everyone found it hard to write about practical ways of teaching history in the mathematics classroom."*

According to Fuhrer (1987:129), the most difficult task of the mathematics teacher *"is to show that mathematics makes, or can make sense. Further that it has guided humanity behind the curtain of the unknown, and that each mathematical perspective on reality contains several historical strands."*

The varied interaction that went on between *"theory and application also took place continually within the body of the theoretical material itself – interaction between numerical analysis and geometry.* (Kennedy: 1991:333). Algebraic considerations, in the sense of discrete operations performed on classes of objects, played an early and essential role, although "*the symbolism frequently thought of as the hallmark of algebra was not introduced until the sixteenth century. Thus the history of trigonometry, exhibits within itself, the embryonic growth of three classical divisions of mathematics: algebra, analysis and geometry.*" (Kennedy: 1991: 333).

According to Fuhrer, (1987:343), the beginnings of the development of trigonometry, are lost in prehistory. They can be thought of as the first numerical sequences correlating shadow lengths with the time of day. "*This 'trigonometry" was based on a single function, the chord of an arbitrary circular arc." Menelaus' theorem, involving either plane or spherical complete quadrilaterals, made possible the extension of this discipline to the sphere. However, other methods of passing from plane to sphere certainly competed with the Menelaus theorem and probably preceeded it.*" (Kennedy: 1991: 333).

According to Kennedy (1991: 333) these things originated in the general region of the eastern Mediterranean, were recorded by people writing in Greek, and were well established by the second century of the era. The centroid of activity then shifted to India (where the chord function was transformed into varieties of the sine), and thence it moved part of the way back. In the region stretching from Syria to Central Asia, and in the period from the ninth century through the fifteenth, the new sine function and the old shadow functions (tangent, cotangent, secant, etc.) were elaborately tabulated as sexagesimals. With this development, the first real trigonometry emerged, in the sense that only then did the object of study become the sides and angles of spherical or plane *triangle*.

According to Fuhrer (1987:59), most of the mathematics by the astronomer, Ptolemy, who wrote the *Almagest*, was based on the work of the early Greek astronomer Hipparchus. Copernicus (1473-1543) and Kepler (1571-1630), introduced their heliocentric theory of the solar system.

For the mathematician, according to Kennedy (1991: 359), the *Almagest* is of interest because of the trigonometric identities Ptolemy devised to help him in compiling his table of chords (which is roughly equivalent to the sine table).

Subsequently, as the locus of activity in astronomy moved to Europe, so also did the new trigonometry. According to Kennedy (1991:334), the same type of work that had occupied Oriental scientists – namely, the computation of tables and the discovery of functional relations between parts of triangles – was continued in the West.

" *A new a fundamental European contribution, however was the replacement of verbal rules by appropriate symbols. But the development of the infinitesimal calculus, following hard after, foreshadowed the speedy end of trigonometry as an independent and growing branch of mathematics; for with the discovery and exploitation of the complex domain the whole mass of theory was subsumed to analysis.* " (Kennedy, 1991: 334).

By the end of the eighteenth century, according to Fuhrer (1987: 209), Leonard Euler and the others had exhibited all the theorems of trigonometry as corollaries of complex function theory. As a school subject, however, especially useful for surveyors and navigators, trigonometry still keeps its separate identity.

The ever-accelerating development of trigonometry provides a ready illustration of the fact that knowledge tends to accumulate at a rate proportional to the quantity that is already at hand; broadly speaking, its growth is exponential with respect to time.

Here the account is confined to the leaders in the field of working with triangles; their predecessors and rank-and-file contemporaries operated on a more primitive level, but they created the background without which these leaders could not have existed. According to Kennedy (1991:334), knowledge of the subject did not grow steadily. It progressed, instead, by a series of discontinuous jumps. Important advances made at one time and place sometimes spread only slowly – sometimes not at all, disappearing only to be rediscovered later.

2.2 BIRTH OF THE SINE FUNCTION

In the applications of the chord function (plane as well as spherical), it is necessary to double the arc before using it as an argument in the table of chords. It would be handier to have a table in which the original arc is itself the independent variable. Eventually someone thought of calculating and using half the chord of double an arc. Once this was done, the sine function had been born. The question arises: how did they calculate these ratios? In the earliest days they actually used scale diagrams, which is the kind of *Sketchpad* approach used today. Then in the time of Euclid, they used the chord method. What did Ptolemy do? He moved beyond that, where he used the famous Ptolemy's Theorem. The use of his theorem allowed them to calculate the sine ratios far more efficiently and quickly to more decimal places.

The earliest sine tables turn up in India, where they originated. The *Surya Siddhanta* is a compendium of astronomy made up of cryptic rules in Sanskrit verse. It was composed in the fourth or fifth century A.D., but the extant version has been revised so frequently that it is difficult to say which sections are in their original form.

Thus, according to Kennedy (1991:350), the astronomers of India not only originated the sine function but also entered, however intuitively, into subjects later to be called "*difference equations*" and "*interpolation theory*."

Beginning with the ninth century, the number of people working in trigonometry increased markedly. Astronomers lived and travelled widely over a region reaching from India to Spain: the Iranian plateau, Iraq, Syria, Egypt, North Africa, and Spain. Indian scientific books were the first to receive the attention of Moslem scholarship; some were translated into doggerel Arabic verse in imitation of the Sanskrit slokas. Later the available Greek works were translated. The sine function was quickly adopted in preference to the chord. In fact, the etymology of the word "sine" indicates the wide variation in background of those who dealt with the function it designates. The Indians called the function *ardhajya*, Sanskrit for "half chord". This was shorteed to *jya* and transliterated into three Arabic characters, *jhb*. This can be read as *jayb*, Arabic for "pocket" or "gulf." It was so read by Europeans, who translated it into Latin *sinus,* whence English "sine."

2.3 MORE FUNCTIONS AND TABLES

The subject matter of the previous section is primarily geometrical. Its development, according to Kennedy (1991: 353), was accompanied by an accumulation of numerical and computational materials and techniques. By the ninth century, tables of the "(horizontally) extended shadow" were common.

Al-Biruni, a great scientist who lived in Central Asias in the eleventh century, wrote an exhaustive treatise on shadow lore. Among Orientals, he asserts, it was customary to use a gnomon of a handspan of length. Rarely tabulated, but explicitly defined and applied in Sanskrit as well as Arabic works, were relations called the "hypotenuse of the shadow."

According to Kennedy (1991:355), by the end of the ninth century A.D. and probably well before then, all six of the common modern trigonometric functions were well established, and the identities connecting them were in full application.

To summarise, my historical overview suggests the following that is relevant for the teaching of trigonometry:
The practical problem is one aspect, however the other aspect is the idea that the function definition and function concept also developed later and also the function concept there was a need for it people were beginning more and more to apply mathematics in sciences to things that involved periodic function and for that they needed trigonometry.

So again the motivation for the development of the function definition was also from practical consideration but from different kind of practical consideration than when it was originally. Originally, it was about land surveying when simple triangulation and simple the 3-4-5 triangle was adequate but by the time it came to the renaissance, the type of practical considerations were investigating the pendulum, investigating the movement of the planets around the sun, all these were periodic, and for that they needed to come up with a more abstract definition and the function concept became more developed and the Greeks did not have any sort of formal, written definition. What does this tell us? It tells us that if we were to follow a historical approach, then maybe we should not start with the function definition because it may be more subtle for children to understand. But this is an empirical question that can only be answered by further research by only setting up a comparative study by maybe setting up a research where I let learners follow one approach and the other a different approach and see if there is going for one to the other.

CHAPTER 3

TEACHING TRIGONOMETRY

3.1 INTRODUCTION

The basic idea on which the whole of trigonometry is based is that triangles (and other objects) can have the same shape but different sizes, and that if the angles of two triangles are equal, then the corresponding sides are in proportion. Such triangles are then said to be *similar triangles*. Trigonometry starts with right-angled triangles, for which the side lengths are related by *Pythagoras' Theorem*.

In a right-angled triangle, the size of any angle is related to the ratio of the lengths of any two sides by the trigonometric functions. The basic functions are sine, cosine, and tangent. These functions are based on the similarity of triangles that have a right angle and one other angle in common.

3.2 TWO METHODS OF INTRODUCING TRIGONOMETRY

In this chapter some of the different approaches to trigonometry in the curriculum are discussed in order to provide a background to the research.

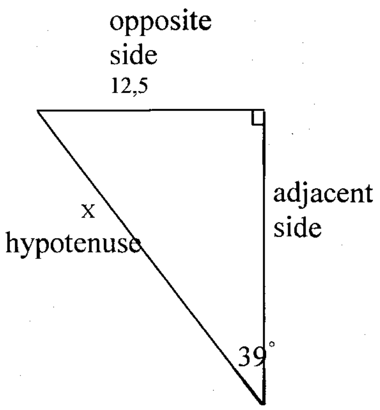
In countries such as Canada (De Kee et al, 1996), the United States (Satty, 1976), Australia (Willis, 1966), the United Kingdom (Collins, 1973) and in South Africa, school trigonometry has traditionally been introduced by means of ratios and right angled triangles. The introduction of the "new maths" in the 1960's called for a shift in school trigonometry, from a ratio to a function approach with particular emphasis on the unit circle. The unit circle has come to embody the function approach (Pournara, 2001) to the extent that one frequently reads about "unit circle approaches" rather than "function approaches" to trigonometry.

According to Kendal and Stacey (1996: 7), *"since the advent of the "new mathematics", two methods of introductory trigonometry have been used in Victorian schools: the ratio method and the unit circle method".*

Originally, introductory trigonometry was taught by the ratio method, where trigonometric functions are defined as the ratio of pairs of sides in a right-angled triangle. *"From the early 1960s, an alternative "modern" method was advocated by some educationalists"* (Trende, 1962; Willis, 1967) *'as a more desirable way for students to learn and understand the topic. This approach, known as the unit circle method, defined cosine and sine as the x and y co-ordinates of a point on a unit circle"*. Currently, there are textbooks promoting each way of teaching trigonometry and a few try to blend the two approaches.

3.2.1 THE RATIO METHOD

The ratio method arises from observation, probably known to the Babylonians 3000 years ago, that the similarity properties of triangles can be used to find lengths and angles of triangles, and hence unknown lengths and angles in a variety of other figures. It is naturally allied to measurement and surveying. For the ratio method, the trigonometry functions are defined as the ratios of lengths of the sides in right-angled triangles. For example, the sin of an angle is defined as the ratio of the lengths of the "opposite side" to the length of the hypotenuse. Learners are often taught to remember the definitions of the ratios perhaps using a mnemonic such as SOHCAHTOA (Sine = Opposite + Hypotenuse etc).



Find the length of the hypotenuse, x.

$$\sin 39^\circ = \frac{length\ \ of\ \ opposite}{length\ \ of\ \ hypotenuse}$$

$$\sin 39^\circ = \frac{1.25}{x}$$

$$x = 12.5 \div \sin 39^\circ$$

$$x = 12.5 \div 0.629$$

$$x = 19.873$$

Figure 3.1: Naming the sides of a Right-Angled Triangle

The ratio approach is generally used for solving triangles, that is, tasks where learners are given information about some of the sides and angles of a triangle and are required to determine the lengths of the other sides or the sizes of the other angles.

3.2.2 THE UNIT CIRCLE METHOD

When first introduced as part of the "new maths", the unit circle method emphasised the nature of the trigonometry functions "*as functions taking real numbers to real numbers*" (Kendal and Stacey:1992: 77). One of its virtues was seen to be that no reference to angles or triangles was required. Practical applications to measurement were not the primary motivator although the solution of triangles was noted as *"an interesting and useful outcome"* (Dooley, 1968).

Kendal (1992) describes how unit circle approaches have evolved since they were first introduced. Three different unit circle methods are described. Kendal (1992) refers to these as *function of a real variable, angle-based definition,* and *scale factor technique.*

The trigonometry functions were defined as functions of a real variable. The angle for example, is defined as the length (a real number) along the circumference of a circle of radius 1, and the sine is the y-coordinate. This definition avoided the undefined notion of an angle. *"One of the aims of new maths was to use mathematical language more precisely, so this was thought to be a desirable feature. Cosine and tangent are similarly defined as lengths"* (Kendal and Stacey: 1992:89).

Whereas the ratio definitions arise naturally from applications to mensuration and surveying, the unit circle definitions lean more naturally to applications to periodic phenomena, such as simple harmonic motion. Both of these sources of applications are important. The question here is, not which definition should be used, as learners learn more advanced -mathematics, they need both.

FUNCTION OF A REAL VARIABLE

The first unit circle method defines the trigonometric functions as functions of a real variable. According to Kendal (1992), learners had difficulty understanding these definitions. This, combined with the need for angle-based definitions to solve triangles, lead to the second unit circle method.

ANGLE-BASED DEFINITION

The only significant mathematical difference between the two definitions is that the angle is measured in radians in the first method and in degrees in the second method. However, there are vast conceptual differences because learners work with reference triangles, derived from the unit circles in the angle-based method. In using these reference triangles, learners must focus on the lengths of the sides of triangles rather than arc lengths as in the previous method.

SCALE FACTOR METHOD

This method is derived from the angle-based definition and also requires the use of "reference triangles". There are two important differences (Pournara, 2001), between the scale factor method and the angle-based method. Firstly, the scale factor method does not require learners to transpose equations, hence the algebraic demands are reduced. The second difference lies in the way the learner works with the two triangles. In the angle-based method, the learner looks for corresponding sides and sets up equivalent ratios. In the scale factor method, (Pournara, 2001), the learner views each triangle as a whole and treats the one triangle as an enlargement of the other, hence the term *scale factor*.

3.3 THE INTERNATIONAL DEBATE

There exists only two research studies (Pournara, 2001), documented in the literature that compare the ratio and function approaches, one conducted in Australia (Kendal, 1992) and the other in Canada (De Kee et al, 1996).

Teachers and academics were involved in a heated debate following the call from the "new maths" for function approaches to trigonometry, as to the most appropriate method of teaching trigonometry. De Kee (1996), Kendal and Stacey (1996) and Markel (1982), maintain that a ratio approach is best. Others preferred a function approach based on the unit circle. For, example, Dooley (1968) argued for the function-of-a-real-variable method because it does not depend on angles or triangles, whereas Willis (1966) proposed the angle-based method because of learners' difficulties in working with functions of a real variable in the context of circular functions. Yet others proposed an approach that combined both methods (Satty, 1976). However, according to Pournara (2001), most of the debate seems to have been based on personal preferences and the individual experiences of participants in the debate with little reference to empirical research on teaching and learning trigonometry.

3.4 THE FUNCTION VS. UNIT CIRCLE APPROACH

The question may be asked: What do you mean by sine function?

Quite often the term sine function is used synonymous with the circle definition. This is rather limiting of the idea of the sine function because the sine function can also develop within the right-triangle orientation, in fact although the Greeks did not formalise the function concept or did not use $\frac{y}{r}$ and $\frac{x}{r}$, did not mean that they did not intuitively understand the sine function. Indeed, they must have otherwise they would not be able to compile the sine tables. Basically how do we understand a function? We understand a function as something which relates input to output values, by some form of rule and then of course how do we represent functions? We represent functions by some kind of formula for example, $y = \sin x$ or by $\sin\theta = \frac{opposite}{hypotenuse}$. So, strictly speaking, it's incorrect to restrict the sine function term to only the circle definition (de Villiers, in conversation, 2004). We talk about the sine function within the right triangle context as well, it exists and it's plausible that the Greeks certainly knew that.

How else do we also represent a function? We also represent it by tables. Tables of values are very old. Another way is of representing functions, is by using a graph. The Cartesian graph did not exist at the time of the Greeks because they did not have the co-ordinate system. But that does not mean that the function concept did not exist. Maybe it was not explicated and they did not formulate it, but they certainly had an intuitive definition. This is the whole distinction by David and Tall between concept image and concept definition. The fact that there is no concept definition, that the Greeks had no concept definition, does not mean that the did not have a reasonable concept image (de Villiers, in conversation, 2004). Newton and Leibniz did not have a concept definition, but they had a good understanding of what a function is, even though they did not have a formal definition.

When the function approach is used in this research, it is referring to the unit circle approach. When that terminology is used, its not meant to imply that the ratio method does not include the implicit ideas of the function approach as well. The function value as a ratio, also changes as θ changes, so it has the idea of variability, functional dependence. The fact the learners in this study did tables is one method of representing a function. The three methods of representing a function are: graphically, tables and a formula and translating from one to the other.

History provides some very interesting aspects about the learning of trigonometry. It tells us that if we were to follow a historical approach, then maybe we should not start with the function definition because it may be more subtle for children to understand. Originally, mathematics, especially trigonometry was used in land surveying when simple triangulation and the 3-4-5 triangle was adequate. The Greeks did not have any sort of formal, written definition. Consider the fact that Newton, Leibnitz and others did not have todays formal set-theoretic definition of this. All they knew is that some dependant variable depends on an independent variable by some kind of formula.

The function concept was only developed later. One has to ask, "*Why was it developed later?*" (De Villiers, 2004, in conversation). We also need to note that the Greeks did not have a function concept as well. Maybe the function concept is more difficult, as discussed in chapter 2.

Why do people argue that we need to use the unit circle approach? Euclid, 300BC, saw trigonometry as part of geometry. From the 1600's onwards, "*people battled with the trigonometric/algebraic function*". Why did this happen? Why was there a need here? At this stage, none of the Greeks had the co-ordinate system. The late, historical development of the coordinate system suggest that it may not be such an easy idea, (De Villiers, 2004, in conversation). First the coordinate system developed from physics, mechanics, astronomy and so on and the more involved problems that involved periodic motion required the use of functions and thus the need for further trigonometry.

By the time it came to the renaissance, the type of practical considerations were investigating the pendulum, investigating the movement of the planets around the sun, all these were periodic, and for that they needed to come up with a more abstract definition and the function concept became more developed.

The idea for the function definition and function concept also developed later. Regarding the function concept, there was a need for it as people were beginning more and more to apply mathematics in sciences to things that involved periodic function and for that they needed trigonometry. So again the motivation for the development of the function definition was also from practical consideration but from different kind of practical consideration than when it was originally.

We can conclude that this view of function developed late and thus maybe we should not start there. From history, we note it took 2000 years for the function approach to trigonometry to develop, then maybe it's a lot more subtle.

3.5 DIFFICULTIES IN THE LEARNING OF TRIGONOMETRY

As a mathematics educator, my experiences of teaching trigonometry at Grades 10-12 level, at an advantaged school, as well as my experience as a teacher having taught for a while at a disadvantaged school, it seemed to me that there were many possible sources of learners' difficulties in trigonometry, and these relate to the curriculum, to teaching and learning and to assessment. The present curriculum documents (Pournara, 2001) reflect a poorly conceived trigonometry curriculum that does not pay sufficient attention to a notion of trigonometric function nor develop appropriate links between trigonometric ratio and trigonometric function. That's quite correct all I was looking at really was liking at trigonometric ratio and then one or two of these questions touch, here involving of functions, also some estimate, reading off table where they can read output value, do they understand it as a function, they have develop some understand, not perfect ideas but my purpose was do they have some understand. Current assessment practices, particularly at Grade 12 level, reward procedural and rule-based thinking (Pournara, 2001), and hence teaching in trigonometry and thus may not reveal learners' poor conceptual understanding in this area. These factors, combined with learners' inability to perform algebraic manipulations such as factorising and solving equations, and their under-developed spatial skills (Pournara, 2001), lead to generally poor performance and difficulty in trigonometry.

Also what needs to be considered is that the initial stages of learning the ideas of trigonometry are fraught with difficulty, requiring the learner to relate pictures of triangles to numerical relationships, to cope with ratios such as $\sin A = \frac{opposite}{hypotenuse}$ and to algebraically manipulate the symbols involved in such relationships.

According to Hart (1981:44), ratios in general prove to be extremely difficult for children to comprehend, and some modern texts have responded to the perceived difficulties by introducing the sine of an angle not as a ratio, but as the opposite side length in a right-angled triangle with unit hypotenuse, which must be recognised with the triangle rotated into any position.

Further difficulties occur, as the learner must conceptualise what happens as the right-angled triangle changes size in two essentially different geometric and dynamic ways, according to Blackett and Tall (1991:13):

- *as an acute angle in the triangle is increased and the hypotenuse remains fixed, so the opposite side increases and the adjacent side decreases,*
- *as the angles remain constant, the enlargement of the hypotenuse by a given factor changes the other two sides by the same factor*

According to Blackett and Tall (1991:15), *"the traditional approach uses pictures in two different ways, each of which has its drawbacks".* Rough sketches of triangles may give the impression that the numerical procedures are the only way to get accurate results, downgrading the role of pictures in the minds of the learners. On the other hand, if learners draw an accurate diagram, this focuses on the production of one static picture rather than the visualisation of dynamically changing relationships.

A computer approach (Blackett and Tall, 1991) can change all this by allowing the child to manipulate the picture and relate its dynamically changing state to the corresponding numerical concepts. It therefore has the potential of improving understanding. This ability to use the computer to carry out certain arduous constructions whilst the child can focus on specific relationships, we call the principle of selective construction. Blackett and Tall (1991) believe this to be one of the most powerful educational principles for the use of the new technology.

CHAPTER 4

THEORETICAL FRAMEWORK : THEORIES OF LEARNING RELATED TO MATHEMATICS

The purpose of this chapter is twofold: a) to examine and outline two opposing learning theories, which will illustrate different approaches to handling learners' understanding as well as their misconceptions in mathematics and b) to discuss the theoretical framework for this study.

4.1 LEARNING THEORIES

In the past years, there have been various perspectives that have been presented and adopted on the teaching and learning of mathematics. Some learning theories that have influenced mathematics teaching and learning in South African classrooms are Behaviourism and Constructivism.

4.1.1 THE BEHAVIOURIST THEORY

The behaviourist or connectionist theory of learning relates to an empiricist philosophy of science, that all knowledge originates in experience. The traditional empiricist motto is *"There is nothing in the mind that was not first in the senses."* Hence, according to Olivier (1998), it is assumed that a person can obtain direct and absolute knowledge of any reality, because, through the senses, the image of that reality corresponds exactly with the reality.

Behaviourism therefore assumes that pupils learn what they are taught , or at least some subset of what they are taught, because, according to Olivier (1998), it is assumed that *"knowledge can be transferred intact from one person to another."*

The behaviourists see learning as the forming of habit, based on reinforcement. The more times a stimulus-induced response is elicited, the longer the learning (response) will be retained. This suggests rote-learning, drill and practice are important factors in the learning of mathematical knowledge.

Behaviourists see the minds of learners as empty, waiting to be filled by knowledge, transmitted by their teachers. The learners are seen as "*a sponge absorbing the mathematical structures invented by others* (Clements and Battista, 1990: 33). Behaviourists, therefore, believe that knowledge is taken directly from experience, and that a pupil's current knowledge is unnecessary to learning.

This type of learning does not allow for application of knowledge. According to Penchalia (1997), skills acquired in this manner are not transferable and children become mathematically illiterate. Furthermore, (1992:26), argues that school mathematics is an activity having its own goals and means and cannot be "*simply transplanted into another activity.*"

The organisation of learning, according to behaviourist principles, must proceed from the simple to the complex, and exercise through drill and practice.

From a behaviourist perspective, errors and misconceptions are not important, because it does not consider pupils' current concepts as relevant to learning. This perspective is succinctly put by Gagne(1983):

"The effects of incorrect rules of computation, as exhibited in faulty performance, can most readily be overcome by deliberate teaching of correct rules...This means that teachers would best ignore the incorrect performances and set about as directly as possible teaching the rules for correct ones.

4.1.2 CONSTRUCTIVISM

Constructivism is where the learners construct meaning, that their understanding is dependant on their pre-knowledge, and that even correct teaching leads to misconceptions. The example that Olivier (1998) uses that learners develop the misconception that multiplication makes bigger because their experience is limited to whole numbers, so when they move to decimals and fractions, teachers must be aware of the fact that they have developed this misconception that ALL multiplication makes bigger. Then it is important that they must be given experiences, which conflict with their learning thus far. This is far more important than the rules, for example create conflict by giving them $10 \times \frac{1}{2}$.

The constructivist perspective on learning (eg. Piaget, 1970; Skemp, 1979) assumes that concepts are not taken directly from experience, but from a person's ability to learn from and what he learns from an experience depends on the quality of the ideas that he is able to bring to that experience. According to Olivier (1989), "*knowledge does not simply arise from experience. Rather it arises from the interaction between experience and our current knowledge structures.*"

The learner is therefore not seen as passively receiving knowledge from the environment, *it is not possible that knowledge can be transferred ready-made and intact from one person to another.* The child is an active participant in the construction of his or her own knowledge. This construction activity, according to Olivier (1989), *involves the interaction of a child's existing ideas and new ideas, i.e. new ideas are interpreted and understood in the light of that child's own current knowledge, built up out of his previous experience.*" Children do not only interpret knowledge, but they organise and structure this knowledge into large units of interrelated concepts. Olivier(1989), calls such a unit of interrelated ideas in the child's mind a *schema.* Such schemas are valuable intellectual tools, stored in memory, and which can be retrieved and utilized. Learning then basically involves the interaction between a child's schema and new ideas.

However, sometimes, some new idea may be so different from any available schema, that it is impossible to link it to any existing schema. In such a case the learner creates a new "box" and tries to memorise the idea. This, according to Olivier (1989), is *rote-learning*: because it is not linked to any previous knowledge and is not understood, it is isolated knowledge, therefore it is difficult to remember. Such rote learning is the cause of many mistakes in mathematics as pupils try to recall partially remembered and distorted rules.

To the constructivist, learning is not, as for the behaviourists, a matter of adding, of stockpiling new concepts to existing ones. Rather, learning leads to "*changes*" in our schemas.

It is clear that the character of a pupil's existing schemas will determine what he or she learns from experience or information and how it is understood. At the heart of

the constructivist approach to teaching is an awareness of the interaction between a child's current schemas and learning experiences, to look at learning from the perspective of the child, for the teacher to put himself or herself into the child's shoes (Olivier, 1989), by considering the mental process by which new knowledge is acquired. Because knowledge cannot be transferred ready-made, to support the child to construct his or her *own* knowledge, discussion, communication, reflection and negotiation are essential components of a constructivist approach to teaching.
From a constructivist perspective, misconceptions are crucially important to learning and teaching, because misconceptions form part of a pupil's conceptual structure that will interact with new concepts, and influence new learning, mostly, according to Olivier (1989), in a negative way, because misconceptions *generate* errors.

The theory is based on a view that "*knowledge is made and not given – it is constructed by an active cognising subject rather than transmitted by a teacher or text*" (Adler 1992: 29).

According to (Nickson and Noddings, 1997), since learners are "*internally motivated, they interpret and adjust information to their personal mathematical schemes, thereby constructing their own mental representations of situations and concepts.*" The person's ability to learn depends on the ideas he or she brings to the experience. The learner's pre-existing knowledge will influence the type of knowledge gained (Muthukrishna and Rocher, 1999).

According to Olivier (1989: 18), "*errors and misconceptions are considered an integral part of the learning process.*" The process of accommodation and assimilation allows new information to become a part of the existing schema of the learner. Socio-constructivists contend that learning is a social, as well as an individual activity.

Olivier (1989), points out that there is an "*awareness of interaction between a child's current schema and learning experience.*" The child's point of view is taken into consideration. Steffe and Killion (1986), state that "*mathematics teaching consists primarily of the mathematical interaction between teacher and children.*" Pupils can communicate their ideas and interpretations with each other. An active, self-reliant attitude to learning is inculcated within the learner through discovery, negotiation and reflection. Many learners develop their own methods, rather than relying on the methods taught at school.

Murray, Human and Olivier (1989), conducted research that showed that learners preferred to use their own methods of computation, which proved to be highly successful. Murray et al (1989), attributed this result to the fact that *"when children used methods that they had constructed for themselves, they chose only those they really understand."*

4.1.3 VYGOTSKY AND SOCIO-CULTURAL THEORY

Social relationships are fundamentally important for learning for Vygotsky that he claims: *"all higher mental functions are internalised social relationships"* (Vygotsky, 1981, in Wertsch and Stone, 1986, p166). The notion of *"higher mental functions"* includes voluntary attention, logical memory and concept formation. Vygotsky argues that the direction of learning is from the social to the individual.

Piaget and some constructivists argue the opposite: learning goes from individual to the social. Prof M. de Villiers (in conversation) makes the following good point: *"One also needs to consider that Vygotsky was a socialist/Marxist in the old USSR and it was politically and ideologically correct to place a primary emphasis on the social-cultural context."* Vygotsky argues that learning is first social, which is dominant, and that the individual comes afterwards.

Vygotsky's socio-cultural theory is not very useful to my study. There was no social interaction between individual learners, nor was there much interaction between the learners and myself, nor did I investigate an artifact unfamiliar to their background. Thus I cannot comment on social or cultural interaction. The reason I look at constructivism is because I was looking at the way in which individual learners constructed knowledge.

For learning to take place, the learner must reconstruct and transform external, social activity into internal, individual activity through a process of internalisation (Wertsch and Stone, 1986). The formation of such consciousness, according to Wertsch and

Stone depends on social interaction and on *"mastering semiotically mediated processes and categories."*

4.1.4 VAN HIELE THEORY

4.1.4.1 INTRODUCTION

The Van Hiele theory originated in the respective doctoral dissertations of Dina van Hiele-Geldof and her husband Pierre van Hiele at the University of Utrecht, Netherlands in 1957. Dina unfortunately died shortly after the completion of her dissertation, and Pierre was the one who developed and disseminated the theory further in later publications. While Pierre's dissertation mainly tried to explain why pupils experienced problems in geometry education (in this respect it was **explanatory** and **descriptive**), Dina's dissertation was about a teaching experiment and in that sense is more **prescriptive** regarding the ordering of geometry content and learning activities of pupils (de Villiers, 1996). The most obvious characteristic of the theory is the distinction of five discrete thought levels in respect to the development of pupils' understanding of geometry. Four important characteristics of the theory are summarised as follows by (de Villiers, 1996):

- **fixed order** - The order in which pupils progress through the thought levels is invariant. In other words, a pupil cannot be at level *n* without having passed through level *n-1*.
- **adjacency** - At each level of thought that which was intrinsic in the preceding level becomes extrinsic in the current level.
- **distinction** - Each level has its own linguistic symbols and own network of relationships connecting those symbols.
- **separation** - Two persons who reason at different levels cannot understand each other.

The main reason for the failure of the traditional geometry curriculum (de Villers, 1996) was attributed by the Van Hieles to the fact that the curriculum was presented at a higher level than those of the pupils; in other words they could not understand the teacher nor could the teacher understand why they could not understand!

The Van Hiele theory distinguishes between five different levels of thought. The general characteristics of each level, according to (de Villiers, 1996) can be described as follows:

Level 1: Recognition

Pupils visually recognize figures by their global appearance. They recognize triangles, squares, parallelograms, and so forth by their shape, but they do not explicitly identify the properties of these figures.

Level 2: Analysis

Pupils start analysing the properties of figures and learn the appropriate technical terminology for describing them, but they they do not interrelate figures or properties of figures.

Level 3: Ordering

Pupils logically order the properties of figures by short chains of deductions and understand the interrelationships between figures (eg. class inclusions).

Level 4: Deduction

Pupils start developing longer sequences of statements and begin to understand the significance of deduction, the role of axioms, theorems and proof.

Level 5: Rigor

The learner can work in a variety of axiomatic systems such as those for the non-Euclidean geometries, and different systems can be compared Students at this level can compare different axiom systems (non-Euclidean geometry can be studied). Geometry is seen in the abstract with a high degree of rigor, even without concrete examples. They can analyse the consequences of and manipulate different axioms and definitions. The learner understands the formal aspects of deductions.

The majority of high school geometry courses is taught at Level 3. The van Hieles also identified some characteristics of their model, including the fact that a person must proceed through the levels in order, that the advancement from level to level depends more on content and mode of instruction than on age, and that each level has its own vocabulary and its own system of relations. The van Hieles proposed sequential phases of learning to help students move from one level to another.

Phase1: Inquiry/Information

At this initial stage the teacher and the students engage in conversation and activity about the objects of study for this level. Observations are made, questions are raised, and level-specific vocabulary is introduced.

Phase2: Directed Orientation

The students explore the topic through materials that the teacher has carefully sequenced. These activities should gradually reveal to the students the structures characteristic at this level.

Phase 3: Explication

Building on their previous experiences students express and exchange their emerging views about the structures that have been observed. Other than to assist the students in using accurate and appropriate vocabulary, the teacher's role is minimal. It is during this phase that the level's system of relations begins to become apparent.

Phase 4: Free Orientation

Students encounter more complex tasks - tasks with many steps, tasks that can be completed in more than one way, and open-ended tasks. They gain experience in resolving problems on their own and make explicit many relations among the objects of the structures being studied.

Phase 5: Integration

Students are able to internalize and unify relations into a new body of thought. The teacher can assist in the synthesis by giving "global surveys" of what students already have learned.

4.1.5 THIS STUDY

This research is informed by two fundamental assumptions. Firstly, learners actively construct meaning as they engage with mathematics. Secondly, knowledge construction must occur individually first, and then to the social.

Vygotsky does not provide adequate tools for investigating learners' mathematical thinking. The Van Hiele and constructivist theory, however, provides an appropriate tool for the investigation of learners' thinking in this study in trigonometry. An attempt is made below to conjecture what levels 1, 2, 3, 4 and 5 for trigonometry would be.

4.1.5.1 VAN HIELE

Level 1: Visualisation

The learner is able to recognise a right-angled triangle in different orientations and distinguish between scalene and isosceles right-angled triangles, right-angled triangle in the unit circle. The ability to recognise the opposite side, adjacent side, rectangular sides and the hypotenuse of a triangle is also visualisation.

Level 2: Analysis

In a right-angled triangle, the discovery that, for a fixed angle, the ratios between any two sides remain the same, irrespective of the size of the triangle (which is the concept of similarity). Students are able to solve practical and theoretical problems related to right triangles.

Level 3: Definition

The aforementioned discoveries become formalised as definitions in terms of ratios of sides of right triangles. Understanding of the increasing or decreasing nature of the trigonometric functions within the first quadrant develop, as well as of their non-linear nature. The understanding of inverse, for example $\sin x = \frac{1}{2} \quad \therefore x = ?$ develops.

Level 4: Circle Definition

Understanding of the abstract definition of trigonometry develops in terms of the unit circle and in terms of the trigonometric function. The unit circle to be defined as function, which is independent of the right-angled triangle, and the trigonometric functions are extended into the other three quadrants. Understanding of the periodicity and the graphical representation develops.

Level 5: Spherical Trigonometry

This is not relevant for this study, but to take it further for progression, this level could include spherical trigonometry, which is the trigonometry on the sphere, but could be extended to other surfaces, as well as hyperbolic functions, for example, sinh, cosh, etc.

4.1.5.2 CONSTRUCTIVISM

Constructivism is where the learners construct meaning and their understanding is dependant on their pre-knowledge, and that even correct teaching leads to misconceptions. Concepts are not taken directly from experience, but from a person's ability to learn from and what he learns from an experience. The learning depends on the quality of the ideas that he is able to bring to that experience.

CHAPTER 5

REVIEW OF STUDIES ON TRIGONOMETRY AND TEACHING

5.1 RESEARCH STUDIES

De Kee et al (1996), made use of in-depth qualitative interviews with five Canadian learners who were the equivalent of Grade 11 level in South Africa. De Kee explored the learners' understandings of sine and cosine as they relate to both trigonometric ratios and functions. Overall, her findings showed that the learners had difficulties with both approaches but were more comfortable with the ratio approach. They found the work on functions of real variables confusing. De Kee et al as quoted in Pournara (2001) identified four concept images of *sin* and *cos* revealed by the learners:

- A procedure whereby the lengths of two sides of a right angled triangle are divided by each other, thus producing the sine or cosine of the angle.
- The *sin* or *cos* functions of a calculator.
- The typical undulating curves of the *sin* and *cos* functions.
- The Cartesian coordinates of a point. Learners referred to these as the *sin* or *cos of the point*, rather than referring to arc length or angle.

In a study by Kendal (1992), where the scale factor method and ratio method of introducing trigonometry was compared, it was found that learners who were taught by the ratio method were more successful in solving problems involving the solution of triangles. However, Kendal and Stacey (1996), argue that the focus of introductory trigonometry in Australia is the solution of right-angled triangles and therefore the method used to introduce trigonometry should support this goal. They acknowledge that the study did not investigate conceptual development in learners nor the extent to which either method laid foundations for future work in trigonometry.

Pournara (2001) observes that in recent years the focus in school mathematics has shifted from formalist approaches with their emphasis on mathematical rigour, to approaches that prioritise mathematical meaning. Hirsch et al (1991) argue that the general curriculum changes demand a shift in focus in school trigonometry – from an emphasis on ratios and triangles to a focus on trigonometric functions and modelling. However their call for a function approach to trigonometry is not related to the unit circle (Pournara, 2001). Their focus is on the undulating, periodic *sin* and *cos* curves that provide tools for analysing periodic phenomena and hence applications in modelling. They argue that this type of function approach will broaden and deepen learners' understanding of the function concept in general and hence strengthen connections with algebraic functions.

5.2 SYMBOLS AS PROCESSES AND OBJECTS

Pournara (2001), focused on ways of working with trigonometric ratio and trigonometric function, as well as the ways in which learners see these as both processes and objects.

The notion of "procept" (Gray and Tall, 1992, 1994), derived from process and concept, provides a starting point for seeing symbols in different ways. A procept is a "*cognitive construct, in which the symbol can act as a pivot, switch from a focus on process to compute and manipulate, to a concept that may be thought about as a manipulable entity*" (Tall et. al, 2000, in press)

There are many examples of procepts in mathematics, for example, Pournara (2001), mentions ¾ stands for the process of division and the notion of fraction; 3x+2 represents an expression as the object and the process of multiplying 3 by x and then adding 2. Gray and Tall (1992) consider all the trigonometric ratios to be "precepts". For them the symbol *sin A= opposite/hypotenuse* involves both the process of dividing the lengths of two sides, and the product, which is the ratio of the two lengths. The symbol *opposite/hypotenuse (*without *sinA)* can itself be seen as a process or as an object (Pournara, 2001). As a process, it indicates a method for calculating the ratio. As an object, it represents a ratio that can be used in other calculations. The symbol *sin A* can be seen either as a ratio or a function. It can be seen as a ratio because it is equivalent to *opposite/hypotenuse*, but it can also be seen as a function, it bears no relation to the fraction *opposite/hypotenuse.* Within each of these possibilities – ratio and function- the symbol can be seen as a process or an object. Thus, according to Pournara (2001), *sinA* can be seen in four different ways as summarised in Table 1.1.

	PROCESS	OBJECT
***Sin A* seen as a ratio**	A process for calculating a ratio	A ratio describing the relationship between the hypotenuse and the side opposite *A*.
***Sin A* seen as a function**	A process whereby *sin* operates on *A* to produce an answer	The result of *sin* operating on *A*. e.g. the coordinates of a point: *(rcosA, rsinA)*

Sfard (2000) argues that the introduction of a symbol constitutes the *"conception"* of a mathematical object and not its birth. The symbol *sin A* can be viewed in multiple ways. Firstly as ratio or as function then as process or as object. These views influence and are influenced by the operations that learners perform with and on the symbol *sin A*.

5.3 THE IMPACT OF METHODS AND PROCEDURES ON LEARNERS' CONCEPTIONS OF RATIO AND FUNCTION

Trigonometry is a sub-domain of school mathematics that lends itself to procedures and methods (Pournara, 2001). Currently learners can score high marks in the trigonometry section of the grade 12 examination if they apply correctly the procedures they have been taught. Thus the role of procedures in the teaching and learning of trigonometry should not be downplayed. However, Pournara (2001), argues that some procedures are better than others in supporting a conceptual understanding of trigonometric principles.

Pournara (2001), found that learners' conceptions of trigonometric ratio are closely tied to the methods they use, particularly their methods for solving triangles. He stated that in some cases, learners appeared to treat the ratio simply as a step in the procedure for solving triangles. The first step of the procedure is to set up a ratio of two sides – "what I want over what I know" – and this he stated reflected the way in which they worked with the ratio.

Procedures and methods for solving trigonometric tasks provide an efficient means of solving problems (Pournara, 2001), but if learners do not understand the meaning behind the procedures, they may not be able to execute the procedures successfully. Methods and procedures, he states are therefore both necessary and problematic. He further states that learners need to *"reappropriate these procedures on a personal level and they do so through participation in the mathematical culture of the classroom."* Without the appropriate participation, the procedures will have no meaning to the learners.

5.4 THE METAPHOR OF A CONVERTER

The metaphor of the trigonometric operator as a *converter* (Pournara, 2001), is one possible means for helping learners to shift orientations. Learners need to see *sin* operating on an angle and converting it to a ratio. The idea of a converter, according to Pournara (2001), may also help to deal with the cognitive discontinuity where the learners expect the input and output numbers to be the same type of number. The notion of a converter, according to Pournara (2001), has many physical applications and is embodied in the slider-crank mechanism (Atkins et al, 1994), which converts between linear and rotary motion.

Children's toys provide an excellent illustration of how circular movement is translated into vertical and horizontal movement. The "popper" which consists of a dome-shaped chamber on wheels is an illustrative example. The rotation of this popper can be related to the trigonometric circle: when the wheel axel hits the spring-loaded mechanism, it has rotated through 90° and is at its maximum displacement. This illustrates the conversion of circular movement to linear motion – a change in angle (of rotation of the axel) produces a change in vertical distance (of spring-loaded mechanism and balls). In a similar way, Pournara (2001), states, *tan* converts an angle of 41° to a ratio of 0.87. The *tan*-button (or more correctly, its second-function) can also be used to convert from a ratio of 0.87 to an angle of 41°. Similarly, if the triangle contains an angle of 41°, then the ratio of the vertical to the horizontal side is 0.87. This notion of a converter (Pournara, 2001), may help learners when solving triangles to see the relationship between the angle and the ratio of the sides.

5.5 THE SCALE FACTOR METHOD FOR SOLVING TRIANGLES

Pournara (2001) suggests the scale factor method for solving triangles. He states that although the class method is very efficient, the role of the ratio is lost in the algebraic manipulation and so the procedure may become a meaningless sequence of steps that do not support a deeper understanding of one of the fundamental principles of trigonometry – the relationship between the angle and the ratio of the sides.

Determine the length of AB.

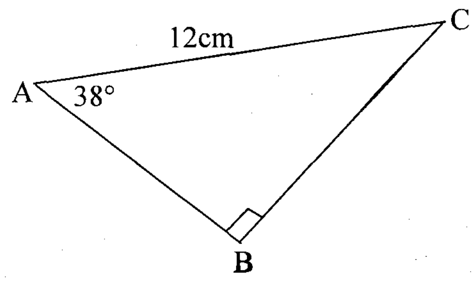


Fig. 5.1 Triangle task for scale factor method

He suggests that there are two advantages of using the ratio as a scale factor: it promotes a structural conception of ratio, and it requires that the learners shift between function and ratio orientations.

Pournara (2001), states that this approach makes explicit the equivalence of $\cos 38°$ and $\frac{AB}{AC}$. In doing so, it helps learners to see the ratio as an object – a scale factor that gives the proportion of the sides. And yet the method still requires learners to work with an operational notion of ratio in doing the multiplication. The only algebraic manipulation required in this method, he states, is to isolate the unknown in the ratio. And this manipulation may not be essential because learners can reason "what over 12 gives me 0.788" and then carry out the multiplication without actually doing the algebraic manipulation.

Another advantage of this approach, he states, is that it avoids the need for the reciprocal ratios in the introductory stages of trigonometry.

5.6 MISCONCEPTIONS IN MATHEMATICS

When learning takes place, the new ideas that the learners' are exposed to, needs to be linked to the learners' previous conceptions. It is the learners existing conceptions that determine what he/she learns and how it is understood. In the analysis of the learners' understanding, correct intuitions as well as misconceptions in their mathematics were exposed.

Olivier (1989), states that if one understands the general principles of cognitive functioning from a constructivist perspective, one will realise that, for the most part, children do not make mistakes because they are stupid – their mistakes are *rational* and *meaningful* efforts to cope with mathematics. These mistakes, he states, are derivations from what they have been taught. These derivations may be objectively wrong and illogical, but psychologically, from the child's perspective, they make a lot of sense (Ginsburg, 1977).

Mathematics is a cumulative subject and any new learning depends on previous learning. The following points need to be further considered (Olivier, 1989):

- correct new learning depends on previous correct learning
- incorrect new learning is often the result of previous incorrect learning
- incorrect new learning is mostly the result of previous correct learning

Every misconception has a legitimate origin in previous correct learning – each misconception is correct for some earlier task, as performed in some earlier domain of the curriculum.

The source of misconceptions is mostly an *overgeneralisation* of previous knowledge (that was correct in an earlier domain), to an extended domain (where it is not valid). A schema, states Olivier (1989), acquired early and developed well is highly resistant to change. Children do not easily accommodate new ideas when necessary, i.e. change their present schemas, but rather assimilate new ideas into existing schemas, which means that the new idea must to a certain extent be *distorted* to be "like" a previous idea (Olivier, 1989). Traditionally the university blames the high school for poor teaching, the high school blames senior primary, who blames junior primary, who blames the home....Where does the problem really lie? Either earlier learning must be changed, states Olivier (1989), so that pupils' ideas will not later have to be changed, or special effort must be made later, to prevent or remediate children's misconceptions. Neither, according to Olivier (1989), are easy.

Children do not only interpret knowledge, but they organise and structure this knowledge into large units of interrelated concepts. Olivier (1989) calls such a unit of interrelated ideas in the child's mind, a *schema.* Such schemas are valuable intellectual tools, stored in memory, and which, according to Olivier (1989) can be retrieved and utilised. Learning then basically involves the interaction between a child's schemas and new ideas. This interaction involves two interrelated processes (Olivier, 1989):

1. *Assimilation:* if some new, but recognisable familiar idea is encountered, this new idea can be incorporated directly into an existing schema that is very much like the new idea. In this process, the new idea contributes to our schemas by expanding existing concepts, and by forming new distinctions through differentiation.
2. *Accommodation:* Sometimes a new idea might be quite different from existing schemas; we may have a schema, which is relevant, but not adequate to assimilate the new idea. Then it is necessary to reconstruct and reorganise our schema. Such reconstruction leaves previous knowledge intact, as part or subset or special case of the new, modified schema.

Thus to understand an idea means to incorporate it into an appropriate existing schema. However, sometimes some new idea may be so different from any available schema, that it is impossible to link it to any existing schema. So assimilation and accommodation is impossible. In such a case the learner creates a new "box" and tries to memorise the idea. This, according to Olivier, is *rote learning:* because it is not linked to any previous knowledge, it is not understood; it is isolated knowledge, therefore it s difficult to remember. Such rote learning is the cause of many mistake in mathematics as pupils try to recall partially remembered and distorted rules.

To the constructivist, learning is not, as for the behaviourist, a matter of adding, of stockpiling new concepts to existing ones. Rather, learning, according to Olivier (1989), leads to *changes* in our schemas.

For early learning, e.g. "multiplication makes bigger" originates from the early teaching of multiplication as repeated addition. This leaves us with the fundamental didactical dilemma. If we continue to introduce multiplication via repeated addition, we create a strong and resistant, but incomplete meaning of multiplication that will come to conflict with later meanings of multiplication (Olivier, 1989). On the other hand repeated addition is probably the best introductory meaning available for multiplication, so one has little choice but to continue this way. The notion of decimals before fractions is an interesting possibility, states Olivier (1989). But it is not possible to teach decimals before whole numbers, or algebra before arithmetic! There is little to change, we must accept that early learning may, through overgeneralization, lead to misconceptions.

From a constructivist perspective the teacher cannot transmit knowledge ready-made and intact to a pupil. Errors and misconceptions are seen as the natural result of children's efforts to reconstruct their own knowledge, and these misconceptions are intelligent constructions based on correct or incomplete (but not wrong) previous knowledge. Misconceptions therefore cannot be avoided.

5.7 THE LEARNING OF MATHEMATICS

The introduction to the sine function, using *Sketchpad* provided a unique way of introducing this section and allowed for experimentation, questioning, reflecting and discovering. Mathematics is effectively learned only by experimenting, questioning, reflecting, discovering, inventing and discussing (Ahmed:1987). Any use of a system, which denies learners the opportunity for reflection, discussion and the posing of their own questions must be seriously questioned. Grouws (1998: 6) has for example, pointed out that, *"What is needed is instruction that focuses on development and the meaning of ideas, with a goal of achieving diversity of learner thought and the use of a variety of solution methods and techniques*". In this introduction to the sine function using *Sketchpad,* it was aimed to introduce the sine function to facilitate this.

From a practical point of view, teachers should be on their guard against designing lessons, which aim to develop skills strictly through repetitive practice. This practice, according to Artigue (1991*), may have no meaning or to encourage learners to apply* *another person's algorithms without giving thought to why they might work.* Generally, lessons, which, are designed to give learners time to reflect and think, are rare, yet thinking and reflecting are at the heart of doing mathematics. If, according to Artigue (1991: 171*),* we see doing mathematics as a prerequisite to understanding new concepts then we have to encourage children about *"to think about what they are doing"* and if we wish to develop *the "application of mathematics"* then we have to encourage children to *"think about what it is they are about to do and the contexts in which the mathematics applies."* Furthermore the more solutions and strategies learners see and discuss (Ahmed,1987), the more likely they are to develop a real appreciation of mathematics at their own level'

5.8 THE TEACHING OF MATHEMATICS WITH A COMPUTER

5.8.1 MICROWORLDS

A microworld, which I tried to create in my study, is technically described as a computer environment, which, in this case was provided by *Sketchpad*, which embodies a domain of knowledge, to provide access to ideas and phenomena that are not easily found in other media by learners. A microworld represents mathematical concepts in a peculiar way that can be close or far away from the school mathematics for example, Hoyles and Noss (1993: 84) had observed that *"learners frequently construct and articulate mathematical relationships which are general within the microworld yet are interpretable and meaningful only by reference to the specific (computational) setting."*

Microworlds can be viewed as computational environments, which 'embody' mathematical ideas. Conventional wisdom asserts that by exploration and use of the computer tools and reflection upon computer feedback learners come to understand the mathematical structures and relationships, which have been planted according to a priori learning objectives. Yerushalmy et. al.(1990), suggests however, that mathematical learning tends not to be unproblematic. Key issues of debate centre on the degree of explicitness and timing of pedagogical intervention whilst maintaining a climate of learner decision-making and exploration. For them, a microworld consists of software together with carefully sequenced sets of activities on and off the computer, organised in pairs, groups and whole classes designed with both the mathematics in mind and the learners' developing conceptions of that mathematics.

Simulations, microworlds and modelling are powerful implementations, which have enormous potential for the enrichment of learning processes. Each in its own way is capable of offering a computer environment, which supports exploration of the users ideas. Exploration may happen at different levels. The nature of the software and the knowledge domain of the user are likely to determine the kinds of exploration that can take place.

The notion of a computer-based microworld for exploring mathematics in a classroom situation appears to be the most attractive. The attraction lies in the ability to focus upon a limited number of related concepts. Exploration of these concepts can take place without the user having to waste time and effort in overcoming difficulties presented by the computer language used; yet at the same time some access to the computer language is allowed in order to change relationships or rules. It could be argued that a small program such as the calculator, which generates a sequence from a given rule, is a microworld in its simplest form.

5.8.2 VISUAL REASONING

"The main aim of the mathematics department is to provide interesting lessons for all learners, in order to develop their mathematical skills and knowledge. A central way of achieving understanding of mathematics is by talking, reading and writing about it. In order to do this we must provide learners with the appropriate mathematics vocabulary and the appropriate stimulus for the use of language to take place" (Cox, Gammon et al. 1993:9).

Cox, Gammon, (1993), believe that the ease with the computer produces a visual image of the function, and the need to retain a picture of this image, pushes the learners into talking and describing, and hence using *"appropriate mathematical language."* Recent research in mathematics and especially in trigonometry, has shown that the function concept is most difficult to understand (Pournara, 2001). By using *Sketchpad,* or computers it makes it possible to represent visual trigonometry or mathematics with an amount of structure not offered by any other medium. Graphic computer screen representations of mathematical objects and relationships allow for direct visual action on these objects (rather, their representatives) and observation of the ensuing changes in the represented relationships; moreover, the situation can be inverted: it is possible to also investigate the question which actions will lead to a given change in the relationships. The result of such action often can be dynamically implemented; actions can be repeated at liberty, with or without changing parameters of the action and conclusions can be drawn on the basis of the feedback given by the computer program. The power of the computer for learning visual reasoning in mathematics derives from these possibilities.

Like many, Cox et. al. (1993) were impressed by the potential of technology to make visual representations of mathematics widely available. At the same time they were aware of learners difficulties with graphs described in the mathematics education literature. Rather than approach learner difficulties as "*misconceptions to be uprooted*", they approached them as ideas they could change in the normal course of learning and instruction, and as indications of "*conventions in which their training blinds them.*"

The computer itself was an important part of the environment. Cox et. al. have mentioned "*we would conjecture that the process of active graphing was instrumental in enabling some children to come to a fuller understanding of their experiment, their data and their graph. The graphs that the children generated were judged according to new criteria. Instead of being aesthetically pleasing drawings or pictures, they were imbued with a sense of purpose; the graph could tell them how to proceed with the experiment. Children could discover points, which did not seem to make sense according to the trend in the rest of the data. They could see gaps in their own data which, ought perhaps to be filled by further experimentation and they could predict results from outside the current range of data in preparation for the next experiment.*" This meaningful interaction with the graph and the data seemed to enable some children to develop a better understanding of their graphs and so eventually be able to give a fuller interpretation of their meaning.

Recent research on visualisation is concerned with the effects of a visual versus a symbolic approach and how learners relate both (Eisenberg, Dreyfus, 1989). *There are studies, which show the positive effects of visualising in mathematical concept formation* (Bishop, 1989) and give convincing arguments for emphasising visual components in the introduction of concepts in school. "*But there are dangers in doing this carelessly because visual presentations have their own ambiguities* "(Goldenberg, 1988:122).

Tall (1991) reports on using the computer to encourage visually based concept formation in calculus. Tall stresses that the goal is not only to provide solid visual intuitive support, but to sow the seeds for understanding the formal subtleties that occur later. This implies that the learners learn to reason visually with the details of screen representations of concepts such as function, secant, tangent, gradient etc. Kaput (1989) has used concrete visual computer representations to build on natural actions in the learners' world with the aim of supporting the learning and application of multiplicative reasoning, ratio and proportion. In particular, he aims to tie the visually concrete and enactive operations on objects on the screen with more formal and abstract representations of these operations. Thus learners' visual operations are directly used in the learning process. Yerushalmi and Chazan (1990) have given learners the opportunity to empirically generate visual information about geometrical constructions and to infer conjectures from such information. Again, this cannot be done without visually based action (to generate the geometric information) and visually based cognitive activity to infer a conjecture. Shama and Dreyfus (1991) have used computer screen presentations of linear programming situations to allow learners to develop their own solution strategies. For this purpose also, learners need to analyse the problems in terms of the visually presented information and thus to give a visual basis to their strategies. All of these projects thus aim for detailed analysis of the relationships contained in the visual screen presentation and for reasoning based on such analysis.

In computerised learning environments it is possible to directly address and overcome some of the problems associated with visualisation, mainly those related to lack of flexibility in the learners thinking; it is also possible to transfer a large measure of control over the mathematical actions to the learner; but the potential of computers for visual mathematics does not by itself solve the more important problems which were mentioned in the introduction. In every case, visual representations need to be carefully constructed and their cognitive properties for the learner need to be investigated in detail. The adaptation and correction of features of these visual representations on the basis of learner reaction to them is an integral part of the development, and in some cases has been reported in the literature. Tall's choice of local straightness rather than a limiting process for the derivative is a case in point. Similarly, Kaput describes how he has found dissonances between learners' visual experience and the semantic structure of the situation being modelled and has consequently designed a way to avoid such difficulties. These difficulties associated with visual representations can be overcome, but only if they are systematically searched for, analysed and dealt with. In this endeavour, the design of learner activities within the learning environment plays at least as important a part as the design of the computerised environment itself (Dreyfus & Halevi, 1990).

5.8.3 COMPUTER-AIDED INSTRUCTION

In many schools today, the phrase 'computer aided instruction' means making the computer teach the child. One might say the computer is being used to program the child. "*In my vision, the child programs the computer and, in so doing, both acquires a sense of mastery over a piece of the most modern technology and establishes an intimate contact with some of the deepest ideas .... from mathematics (*Ainley: 302)*"*.

This quotation, according to Ainley, is at least 17 years old (by Seymour Papert, 1980), in the introduction to his book *Mindstorms.* What dates it most clearly for me are the word 'instruction' and the reference to programming, both of which are somewhat out of fashion in educational discussion. What does not feel at all out of date is the tension expressed between computers being seen primarily as rigid and mechanistic tools for teaching and as tools for learning. It is also interesting to note that the teacher does not appear. Although the developments in technology have been enormous, ambiguity about the computer's role, and the teacher's role in a computer-classroom, continue to be causes of anxiety for many practitioners.

According to Ainley, the above scenario is a very complex situation, in at least two ways: 1) a lack of clarity about relative roles of teacher and computer (and, of course, learner) is only one of a long list of factors which affect the extent and quality of the use of computers in mathematics classrooms. Issues to do with access to appropriate hardware and software, curriculum constraints and assessment requirements, attitudes to technology and management issues at both classroom and school level are all extremely significant. Even when high levels of access are available, and curriculum pressures relaxed, teachers' confidence in integrating technology within their existing classroom practice remains a key issue.

5.8.4 STUDIES ON MATHEMATICS AND THE COMPUTER

Here, in studies on mathematics and computers I will first review general studies and thereafter, the ability of the computer to contribute to mathematics will be explained.

A common teaching strategy in mathematics, according to Dugdale (1992), is the use of graphical representations, mostly on the blackboard, but also on worksheets, textbooks, homework assignments or written examinations. Since microcomputers are more and more accessible, there exists a new powerful tool to represent graphs and functions and thus study mathematics. The study by Dugdale (1992), is based on the *"development of graphical environments with computers"*, which enable learners to discover and acquire function concepts in Algebra and Trigonometry at high school level.

The approach and rationale behind Dugdale's study was attempted in my study as well as the approach to the teaching and learning strategies discussed here which is based on a constructivist point of view which describes human beings as builders of theory and structures *(*Balacheff, 199b, Schoenfeld, 1987*)*. My study involved the sine function, which, according to Dugdale (1992), *the function concept is a central one in mathematics because of its potential to tie together seemingly unrelated subjects like geometry, algebra and trigonometry.* It is also a very complex concept which has various sub-concepts associated (Dreyfus, 1990: 33*)*. In spite of efforts to teach functions by means of multiple representations, high school learners show limited concept images of functions (Vinner, Dreyfus, 1989).

Wenzelburger, (1990) states that, *"the computer plays an important role in mathematics education, since it is considered a valuable tool to aid in the teaching learning process in mathematics."* Tedious and complex computations can be done on

the computer. The learners remain free to concentrate on essential aspects of concepts. Carefully designed graphing software used thoughtfully presents new opportunities to teach functions successfully. Such software, according to Goldenberg, (1988), makes use of *"multiple linked representations."*
According to Tall: (1985, 1987), *"Computer environments seem to be an ideal tool to build a curriculum from a constructivist point of view, which allows learners to make transitions between algebraic and geometric representations."* Pea (1987) puts computers in the context of *"interactive cognitive technologies."* Computers can provide functions that promote mathematical thinking. *"They fulfil the process functions of being a tool to integrate different mathematical representations."*

Garancon et.al(1983)., undertook a study to explore the role of the computer in a specific activity. Their aim was to introduce the idea of line graphs in two ways, one making use of the computer, and one relying on more traditional resources. The conjecture was that children who had used the computer would be better able to produce their own graphs by hand, and to interpolate from them. Garancon et.al(1983), gave the whole class a pre-test which presented a table of data and a hand-drawn line graph of a child's growth, and asked them to recognise specific points, and to interpolate. The results from this pre-test were used to establish a baseline of skills, and to divide the class into two groups of matched pairs.

On reflection, Garancon et.al(1983), conjectured that the children were *"able to interpolate, handle scale, plot points and construct sensibly scaled axes because they did not attempt to teach them these skills"*. They see this as an example of what Hewitt (1994), refers to as *functionalisation, "the process by which skills reach a level at which we are able to function with them automatically, when they are encountered in contexts in levels subordinated to other tasks."*

More traditional approaches to teaching line graphs would necessarily begin by teaching construction skills: constructing suitably scaled axes and plotting points. If attention is focused on these, it is difficult for children to keep in mind the context and purpose for which the graph is being drawn. Indeed, the skills of constructing graphs are often taught in isolation from meaningful context, and so appear to children to be an end in themselves.

Using computers allows children to have control: to select the data, which is appropriate for their work, and to produce graphical images of that data quickly and easily. Garancon et.al (1983), state that "*their experience suggests that, given this opportunity, young children's ability to work with line graphs is far greater than is generally understood.*"

Another related study by Garancon et.al (1983), focused on "*a functional approach to the teaching of early algebra*" and made extensive use of computer- assisted graphical representations as tools for solving a range of problems. The aim was to uncover areas of ease/difficulty experienced by seventh graders in learning how to produce, interpret and modify graphs. They worked in pairs at a computer during approximately 25 problem-solving sessions. Garancon et. al. (1983), describes the ways in which one pair of learners coped with the two types of infinity they encountered in a dynamic graphing environment that plotted intervals of discrete points rather than continuous curves. In addition to helping learners became aware of the use of graphical representations as problem-solving tools, "*the environment provided a rich context for learning about density of points, infinity, continuity and other issues that tend to be ignored until calculus*".

McDermot et. al.(1987), conjectured that the computer played a significant role in enabling children to gain access to work with line graphs, and furthermore in allowing children to build on their intuitive understanding to construct for themselves the skills required to draw such graphs by hand. It seemed possible that being able to produce graphs without needing to worry about the problems of scaling axes and plotting points, freed children to focus attention on using the graph in a meaningful way. Also, experiencing a number of examples of similar graphs enabled them to assimilate some features of the use of scale, which they were then able to use to produce their own graphs. One feature of the software seemed to be potentially important here: if the size of the frame within which the graph is drawn is changed, the scale is altered to fit the new frame. McDermot et. al.(1987), had a sense that this might be powerful in implicitly drawing children's attention to significant features of the graph, which did not change under these conditions.

Although there appears to be considerable difference in the results McDermot et. al.(1987), had obtained and those reported by these two papers, it is worth pointing out two factors which they recognize as having considerable significance. The children in their project class were carrying out within the context of a project they had been closely involved with for some weeks. The data, according to McDermot et. al.(1987), they were working with was, although artificial in the sense that it referred to imaginary children, real and meaningful to them. This would not be the case for learners in either of the studies referred to above. Secondly, the line graphs the children produced were ones in which the appearance of the graph matched the phenomenon which was being graphed; the graph goes up as the child grows up. Kerslake (1981) suggests that "*graphs of this type are the easiest for children to interpret, and it is not clear whether Padilla's or Swatton's test items contained graphs of this kind.*"

In traditional classroom teaching, corrections are handed down by external authority. There is no way that learners can use their own abilities to correct their own work. Von Glasersfeld (1987), has noted that this kind of correction is not completely satisfactory: From the constructivist point of view, it makes no sense to assume that any powerful cognitive satisfaction springs from simply being told that one has done something right, as long as "rightness " is assessed by someone else. To become a source of real satisfaction, "rightness" must be seen as the fit with an order one has established oneself.

Dugdale (1992) has pointed out the principles, which should be followed in designing learning environments for mathematics, which I tried to use in my study:

- ❖ the environment should consist of a "working model" of the concepts to be learned, in which the mathematics is intrinsic. Learners should be able to explore and manipulate this working model.
- ❖ Direct meaningful feedback should be provided which the learners themselves can interpret in order to diagnose and correct their own errors.
- ❖ This environment should include a set of inherently-interesting problems which can be explored by learners of varying abilities and inclinations.

This type of learning, according to Edwards, (1991:119), is constructivist, in that the learner must build upon his or her existing knowledge, and the microworld provides the tools needed to correct and refine this knowledge. These environments also have the potential to allow learners more independent and self-directed exploration of mathematical patterns, in which learners can go beyond the goals of the game and continue to satisfy their own desire to find meaning and order in their educational experiences.

What is significant about much of the learners' activities in a computer environment is the very much reduced, traditional role of the teacher. It is not by design or a conscious act on the part of the teacher to stay more in the background, it appears a thing to do under the circumstances, which come to prevail. Linked to this role change of the teacher is an equal and opposite role change of the learner. "*Comparisons of computer use and conventional instruction reveal from 39 to 88% reduction in time taken to complete a task*" (Kulik, et al. 1983:24). This, may be due to the software itself, how content is presented and solutions perused, or it may simply be due to an increased work rate by the learners. A novelty effect may also contribute to an increased work rate. On the face of it there seems to be sufficient evidence to support the use of computers as instructional aids, but we should not overestimate their effectiveness for learning, neither should we equate reduced time on task with an increase in conceptual knowledge.

To use a microcomputer for learning mathematics only when it is clear that there are no better ways may be an oversimplification in the light of the above comments. According to Yerushalmy, *"The use of computers where there is some control over graphic output is an area where it is difficult to argue that there are any better ways of learning. Functions and their graphs, raw numerical data and bar or pie charts, scatter diagrams or just manipulating shapes, all fall into this category."* The essence of this work lies in the control which the user has over the computer environment, the control being exercised by the teacher in demonstration-mode or by learners in a workshop mode. Learners can now draw graphs accurately, superimpose one another, change parameters to see the effect, zoom in, zoom out, 'see' a limiting value, understand what it is to talk about a point of inflexion. All manner of things can be presented in an interesting way, so that learners feel that they need to know more about what is going on. A balance needs to be maintained between what is explored, appreciated and expressed using computers and how mathematics is encouraged, expressed and refined.

Proponents of computer-based group-work suggest potential benefits include: the externalisation of ideas, through interaction, the consideration of alternative perspectives, a greater diversity of skills and knowledge enabling exchange of information and ideas and increased attentiveness and on-task behaviour. *"Research has indicated higher levels of discussion in computer-based mathematical environments as compared to paper and pencil environments"* (Healy et al. 1990). Research studies into learning resulting from computer-based group-work have however produced conflicting evidence.

5.8.5 NEGATIVE FACTORS IN COMPUTER IMPLEMENTATION

Some of the factors at present militating against computers realising their full potential are (Yerushalmy, 1994):

- lack of potential in managing the resource
- identification of areas of the curriculum which can be enhanced by the use of computers
- integration into non-computer maths work
- status of mathematical programming and choice of languages

There is also the very real danger that an overuse of computer algorithms for solving problems will delay or even prevent some of the mathematical thinking.

CHAPTER 6

RESEARCH DESIGN AND METHODOLOGY

6.1 METHODOLOGICAL FRAMEWORK

Research in mathematics education that focus on trigonometry, in particular learners' thinking about trigonometry, is limited. This study has thus been chosen so it provides clear details rather than generalities (Erickson, 1986) because it explores issues that have not yet been widely researched. The methods of data collection were determined by the research questions and thus it was decided to use the method of qualitative analysis by means of one-to-one task-based interviews (Goldin, 2000) and interview schedules. This method makes it possible to document the high level of information that individual learners reveal about their sense making of situations and contexts. This method is also beneficial to the researcher as it allows him or her greater control to observe and take note of, how each learner went through the task sheet.

As Novak and Gowin (1984:12) stated: "*For this reason most psychologists prefer to do research in the laboratory, where variation in events can be rigidly prescribed or controlled. This approach clearly increases the chances for observing regularities in events and hence for creating new concepts.*" By reducing the number of external variables, one narrows the focus, giving generalizations based on findings during task-based interviews greater credibility. Such findings, however, might be able to dictate future classroom practice. The sample chosen was a purposive one (Cohen and Manion, 1994), which is one that is selected by the researcher subjectively. The researcher attempts to obtain a sample that appears to him/her to be representative of the population and will usually try to ensure that a range from one extreme to the other is included.

6.2 THE SAMPLE

This research is based on a case study of a class of 15-16 year old learners from a school situated in a middle-class area of Reservoir Hills(KZN). The aim was to obtain insight into how learners at Grade 10 understand different aspects of the sine function. It was expected to see similarities and differences between the learners and it was hoped that these would illuminate different aspects of learning and provide a deeper and more complete understanding of the important issues surrounding the understanding of the sine function.

The school was chosen due to the convenience of having easy access to the computer laboratory and arrangements could easily be made to interview the learners. The intention was to investigate the learners' intuitive understanding of the sine function and interaction with the software. These learners were selected at random by their mathematics teacher who chose every fifth learner appearing in her mark-book. They were selected from a group of 123 learners in February 2003. These learners were of different ability levels and no screening was done in this respect. Three girls and three boys were chosen. The purpose of the research was explained to the learners before the research was carried out. The learners were given a letter to inform their parents/guardians of the research and to obtain permission to participate. They were given the opportunity to withdraw at any stage of the research process. They were very enthusiastic and happy to be part of the research.

This was the beginning of the academic year for the learners and no tests and exams were written by them as yet and thus could not be commented upon. Grade 10 learners were ideal for this study as the questions were suited to their level of understanding, taking into account the topics dealt with in their previous year. Learners were not exposed to Trigonometry before and this was their first experience with an introduction to trigonometric concepts. This was well within the capabilities of the Grade 10 learners.

The school itself is situated in Reservoir Hills, which is a predominantly Indian suburb south of Durban. The residents of Reservoir Hills are generally those of the middle to high social class. The school at which the research was carried out, was previously administered by the ex-House of Delegates. There were a larger number of Indian learners at this school as compared to learners from other race groups. All the learners selected were Indian.

Learners were not previously exposed to the use of computers in Mathematics and therefore also not the use of *Sketchpad.* Thus the learners involved were brought together for a period of 60 minutes in order to familiarise themselves with the general use and application of this software before the resumption of the actual investigation. The fact that the learners were not exposed to *Sketchpad* did not affect the experiment because minimal knowledge was expected from the learners about the software. Each learner was made to feel at ease before the interview commenced, in order to ensure that they would respond in a way that would reflect their understanding of the task provided.

6.3 THE INTERVIEW AND MICROTEACHING EXPERIMENT

This study relates to action research, as it was more a teaching experiment done in an interview setting. This study was also different, as it was not done in a full classroom setting but also using an interview format. A microteaching experiment was designed, using psychological interview techniques to study how each child experiences and conceptualises each activity. The objective was to see if some learning took place, and to analyse the nature and quality of that learning. Have the learners managed to form some concepts of the sine ratio and function? What is the nature and quality of their understanding? What intuitions and misconceptions do learners bring to the learning situation and what is their role in their learning? To what extent did *Sketchpad* assist in their conceptualization?

The learner interviews were structured, task-based interviews (Goldin, 2000). One of the salient features of such interviews, is that the interviewer and interviewee(s) interact in relation to a task(s) that is/are presented by the interviewer in a pre-planned way. This method of interview is according to Goldin (2000), particularly well-suited for exploring conceptual understanding, complex problem solving and the construction of mathematical meaning. The "*structured mathematical environment*" according to Goldin (2000), can be controlled to some extent but also adapted where necessary. Two important advantages of structured, task-based interviews are that they provide access to the learners' *processes* of thinking about a predetermined task, and consequently provide opportunities to investigate more complex mathematical topics in greater depth.

An important point to consider is that, although the interview setting provides a means of exploring learner thinking in a controlled and systematic way that is not possible in the classroom, the interview setting is not the classroom setting. A significant difference is in the power relations between interviewer and interviewees that is considerable different to the power relations between the teacher and learners. The interactions between the interviewer and interviewees are also different. In a task-based interview setting, the focus is generally on learners' thinking and so the correctness or not of the answer may be of little consequence to the interviewer. In classroom interactions, however, the teacher's focus is to usually obtain the correct answer and so the teacher provides the learner with appropriate feedback to indicate whether their responses are correct or not. The researcher, playing the role of the interviewer, continuously probed the learners' understanding but did not necessarily reveal whether their responses were correct or not.

The purpose in selecting interviews as the main means of collecting data, was to gain deeper insight into learners' initial understanding of the sine function, in order to make inferences about their thinking at a particular point in time.

It must also be pointed out that learning took place in the interview, as can be evidenced in the analysis in chapter 7. It is also possible that the probing by the researcher influenced the learners' thinking and that this may have lead to learning. When and how learning took place and how it impacted on learner thinking in the interview is a very important aspect of the analysis in Chapter 7.

6.4 THE INTERVIEW TASK

The interview focused on three types of tasks: procedural, conceptual and applied.

Procedural: tasks that are generally solved by applying a particular method, which is usually taught by the teacher, for example, the majority of tasks in the existing textbooks.

Conceptual: tasks that probe learners' understanding of the fundamental principles of mathematics.

Applied: tasks that require learners to make use of their knowledge of mathematical principles to solve the task successfully.

A pilot interview was carried out. Several adaptations were made thereafter. The researcher chose different types of tasks. It was intended to explore the extent to which learners could work correctly with trigonometric concepts; also if learners understood the algorithms they learnt and practised and if they could apply their existing knowledge to a unique situation.

The introductory task to the sine function that the learners had to work through, was based on a circle within a Cartesian-coordinate system. This sketch was presented ready-made to the learners, although the task of constructing it for themselves might have been an interesting task on its own. The decision to present the diagram to them was based on the following reasons:

- it would take each learner a long time to figure out how to construct a dynamic circle within a Cartesian coordinate system because they were not familiar with *Sketchpad.*

- The construction of the sketch was not one of the objectives of this experiment. So presenting the construction to them did not affect the essence of the experiment.

All measurements were clearly visible on the screen of the computer, so that learners could easily view any changes that might have taken place.

At the commencement of the interview (when the tape recording began), learners were put at ease by the researcher. They were asked whether they understood the task and if they had any questions at present.

The empirical part of this research focused on the following major research questions: Given a self-exploration opportunity within *The Geometer's Sketchpad,* have the learners gained, in my opinion, some understanding of the sine function in the **first quadrant**, during a first introductory activity:

- as a ratio of sides
- as an increasing function
- as a function with a variable (non-linear) rate of change
- as a function that increases from 0 to 1 as the angle increases
- as a relation between input and output values

In order to evaluate their understanding of the last category above (functional relationship), this study checked to see if learners could estimate the value of the sine function for an angle. Conversely, it also checked to see if the learners could determine the input value from a given output value, that is, the size of an angle from a given sine value.

The learners in this study were introduced to the sine function in a purely mathematical way, without a real-world context. This situation may go against outcomes-based education, which proposes that learning should start with a problem in the real-world and then move on to the more theoretical, abstract aspects. In a modelling approach, scale drawing could first be used to solve the problem and introducing similarity (constant ratios) of corresponding sides as the basis of trigonometry. Thereafter, the formal definition of the circle may be introduced. In this study, however, my purpose was not to investigate modelling, but to concentrate on the learning of the sine function during a more formal stage, using *Sketchpad*.

The interview protocol that follows was redesigned after a trial run. This is what it finally looked like:

Research Question 1.1: Do learners understand the sine function as a relationship between input and output values (can they fill in the ratio and $\sin\theta$ in the table?)

no → Probe and redirect thinking

yes

Research Question 1.2: Can learners see $\sin\theta$ as a ratio of two sides ie. $\frac{y}{r}$?

no → Probe and redirect thinking

yes

↓ ↓

Research Question 2, 3 and 5: Can learners see that $\sin\theta$ is independent of r?

no → Probe and redirect thinking

yes

↓ ↓

Research Question 4: Can learners see the similarity of triangles with the same angle as the basis for the constancy of trigonometric ratios

no → Probe and redirect thinking

yes

↓ ↓

Research Question 6: Can learners approximate the value of a sine function for an angle not included in the data or vice versa? (using the table or *Sketchpad*)

no → Probe and redirect thinking

yes

Each interview was approximately 60 to 90 minutes long and each was audio taped. Although these questions were structured around the critical questions, it also allowed for variation in expected responses from the learners, and further probing was done in particular cases. Learners wrote out the answers to questions at each step of the experiment.

In the final stage, the data was analysed. This required the systematic grouping and summarising of the responses, which provided a coherent organising framework that explained the way each learner produced meaning whilst working through the tasks provided.

6.5 The study

The tool used for the task of creating visual intuition was the *Geometer's Sketchpad.* At first, I used *Sketchpad,* showing learners some of the basic tools of the program such as how to make points and lines and how to measure angles. The learners were given ready-made sketches, which they manipulated as required. Here it was very clear about how the y-coordinate and x- coordinate of a point changes as the central angle changes.

The learners needed guidance initially in getting to know *Sketchpad.* To build their intuition, they needed to observe, reflect and conjecture about their experiments. They were provided with a series of questions for them to explore. They were then asked to generalize from their findings.

They were "rediscovering" the law of sines for themselves when they manipulated the sketch, measured sides, and computed ratios. It was not an original discovery in the strictest sense because it's been known for thousands of years, but it provided learners with a sense of ownership and a taste for deeper investigation. Yet it felt like discovery to them when they realized that from their own calculations it came out a constant ratio.

In this study I also partially took on the role of the teacher for the processes of guiding learners through the task. On the other hand, I also assumed the role of researcher during and after the problem-solving session and analysing the results. My interest was in what the learners did and their conceptual understanding, not analysing the learning objectives.

The 6 learners first task was to complete a set of tables for $r = 1, r = 2, r = 3$ and $r = 4$ for example, see Table 6.1. This they completed individually. The learners were then each interviewed. The interview protocol on pages 63 and 64, was based on the interview schedule (appendix A) and was merely a guideline to important questions.

Relationship between $\sin\theta$ and $\frac{y}{r}$

$r = 1$

θ	$\frac{y}{r}$	$\sin\theta$
10°		
20°		
30°		
40°		
50°		
60°		
70°		
80°		
90°		

Table 6.1

6.6 Completing the table

The researcher took each learner quickly through a brief session, which described the clicking and dragging to use *Sketchpad*:

- POINT: Move the mouse until the tip of the cursor is over the desired object
- CLICK: Press and release the mouse button quickly
- DRAG: Point at the object you wish to drag, then press and hold down the mouse button. Move the mouse to drag the object, then release the mouse button.

The 6 learners were asked questions to ensure that they understood exactly what was expected of them. The learners seemed to quickly grasp the clicking and dragging operations of *Sketchpad*. All the learners had computer literacy and home computers, which they used for projects and assignments for school (but not previously in mathematics). They also referred to playing computer games, and thus their ability to use the mouse was good.

After the introduction, they were asked to complete a set of tables (see Table 6.1) for $r=1, r=2, r=3$ and $r=4$. The sketch they used in order to complete the table is given in Figure 6.1.

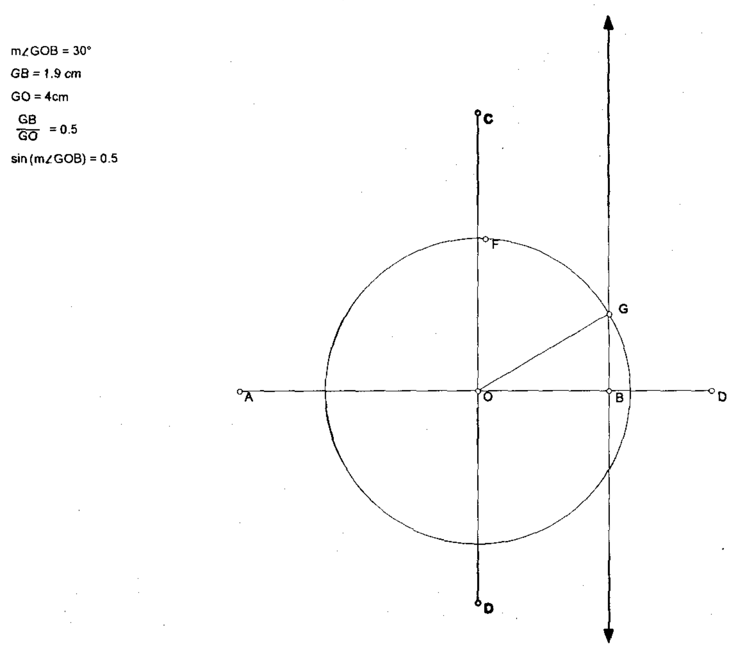


Figure 6.1: The Geometer's Sketchpad Screen.

Learners initially found the information displayed on the upper left hand corner a little confusing as the labels in the sketch did not match those of the table. In retrospect, it might have been better to have used the variables y and r in the sketch as well as for the variables in the upper left corner.

To enable the learners to correctly identify the relevant variables, the researcher then lead each learner by questioning as follows:

Omika: *What has to be* $10°$*?*
Researcher: *What do you think* $10°$ *represents?*
Omika: *An angle.*
Researcher: *Now looking at the diagram on the screen, are there any angles we are dealing with?*
Omika: *Yes ..... these two (pointing them out on the screen).*

Researcher: *So which one do you think has to be* $10°$ *?*

Omika: *This one ..... (correctly pointing it out)...... $G\hat{O}D$.*

The learners first completed table 6.1. Based on the information in this table and the exposure to the software, they were then further interviewed (see question 1 in the interview schedule, appendix A). Before completing the table for $r = 2$, they were also interviewed and required to complete questions 2 in the interview schedule (appendix A). Thereafter, they then completed the rest of the tables, were interviewed and probed for their understanding and answered questions in the interview schedule (appendix A).

6.7 Transcripts of interview

The transcripts of the interview in this research, is very important as it forms the primary source of data for the analysis of the learners' understanding. The interviews were audio-taped and the transcription of the interviews were done completely by the researcher. It is important that one does not assume that a transcript is an accurate reflection of the interview as there is a great deal of information in the interview situation that an audio-recording cannot capture, for example, learners' emotions, the power relations between the interviewer and the interviewee, physical movement and facial expression. An important consideration in transcripts of interviews, is that it is not a written down version of an audio-recording, it is an interpretation of the audio-recording. There is a great deal of information, such as intonation, length of pauses, etc. that cannot be captured easily in a transcript. The transcriber makes decisions about the manner of information that is transcribed by giving meaning from the tone of the speaker on the tape.

CHAPTER 7

ANALYSIS & RESULTS

7.1 Introduction

In this chapter, I focus on how learners engage, conceptualise and visualise the sine function while working with *The Geometer's Sketchpad* during a formal circle definition introduction. Also discussed will be how learners' procedures impacted on their thinking.

7.2 Theoretical Framework

The Van Hiele model of geometric thought and the aspects of the existence of levels, properties of levels and movement from one level to the next as well as constructivism is used as a framework. Tall and Vinner (1992) and their notion of concept image and concept definition are also used in the analysis.

7.3 A Ratio Orientation

According to Pournara (2001), the mathematical symbol most central to a ratio orientation is the right-angled triangle. Other mathematical elements, he states, include: definitions of trigonometric ratios in terms of the lengths of sides of a right-angled triangle; the relationships between the ratios – particularly the quotient ratios such as $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and the inverse ratios such as $\operatorname{cosec}\theta = \frac{1}{\sin\theta}$; and typical Grade 10 tasks where learners are given a point in the Cartesian plane and are asked to determine values of particular trigonometric ratios and expressions involving these ratios.

Such problems, he states, usually require learners to set up a right-angled triangle and to make use of the Theorem of Pythagoras. The angle is backgrounded in a ratio orientation and it merely serves as a reference point to locate the opposite and adjacent sides of the triangle, and must be positioned in the triangle before the opposite and adjacent sides are assigned. Thereafter the angle plays no further part in the problem.

An important point made by him is that the use of the phrase *"the sine ratio"* may cause misunderstandings on the part of learners. This statement is often used for simplicity, however, in referring to *"the sine ratio"* we hide the role of the angle and so when learners work with notation such as $\sin 30°$, they may not see $\sin 30°$ as a single object, a ratio. They tend to treat sin and $30°$ separately (Pournara, 2001). They know that $30°$ is an angle, so they treat sin as the ratio, hence *"the sin ratio"*. He further goes on to say that such misconceptions are reinforced when we speak of *"the sin of an angle"* – if *sin* is a ratio and $30°$ is an angle, then learners see no problem in the *"sin of* $30°$*"* as *"the ratio of an angle."*

7.4 A function orientation

A functional orientation to trigonometry is based on a notion of input-processing-output, similar to algebraic functions (Pournara, 2001). A strong function orientation, he states, makes explicit that the process links the input to the output, and vice versa, whereas a weak function orientation does not make the connection explicit.

A function orientation focuses on three aspects: the angle, the trigonometric operator (e.g. *sin, cos, tan*) and the function value. This orientation is dependant on an understanding that the trigonometric operator maps an angle to a real number in a many-to-one relationship. The trigonometric operator, according to Pournara (2001), is seen as exactly that – an operator. In the function definition, function values are not defined in terms of the sides of a triangle. He further goes on to say that a function orientation is more likely to promote a dynamic view of trigonometry than would a ratio orientation because a function orientation assumes that the independent variable – the angle in this case – can take on many values and the resulting function value reflects clearly the effect of changing the angle.

The mathematical elements of a function orientation include the notions of periodicity, amplitude, asymptotes and discontinuity; as well as the representation of trigonometric functions by means of table, equation, or graph (Pournara, 2001). It is possible that South African learners may develop a distorted view of trigonometric functions because the trigonometry curriculum places a great deal of emphasis on algebraic solutions of trigonometric equations and only studies the graphs of *sine,* *cosine* and *tangent.* As a result, learners may develop a function orientation that is limited to the graphical representation of these functions. Pournara (2001), argues that this is limited if learners are to develop a broader understanding of functions and hence be able to draw links between trigonometric functions and linear, quadratic, cubic and exponential functions.

7.5 Methods and Procedures

In this section I discuss the methods learners used to solve the interview tasks and how their answers relate to their conceptions of trigonometric ratio and function (see appendix A).

7.5.1 Research Question 1.1: Do learners understand the sine function as a relationship between input and output values and as a ratio of sides of a right-angled triangle

After the tables for $r = 1$ was completed, the learner was asked interview question 1.1 (appendix A):

"What do you notice about the y and r value respectively as the angle θ increases?"

The reader is reminded that the table only referred to the first quadrant. This question tested if learners, for a given r, could observe and understand that y increases in the first quadrant as the angle increases.

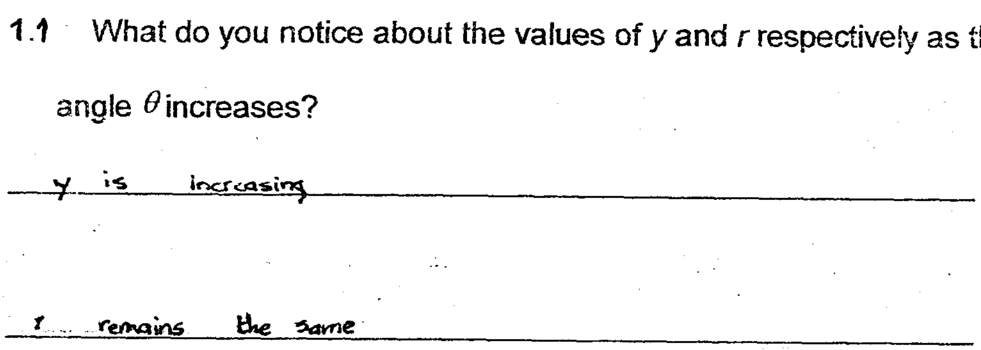
The majority of learners (four) made a correct observation in this regard, while two learners could not. Their responses are now discussed in more detail in two categories below.

7.5.1.1 Category 1 – y increases and r stays the same

Four learners felt that y increases and r remains the same, for example:

Perusha, was accurate in her answer when she replied: *"As it increases, y gets higher and r stays the same."* Nadeem and Mayuri referred to the table when answering this question: Nadeem actually mistakes the $\frac{y}{r}$ in his completed table with the y and r value respectively: (looking at table) *'they increase as well.* " When questioned about y and r, he realised that they are not angles, but line segments. He then asked if he could first try on the computer. He correctly dragged the angle and answered: "*r value remains the same, and the y value increases.*"

Mayuri referrred to her completed tables and replied: *y and r changes as the angle increases, but the spaces in between them is not equal.*" When asked what she meant by the spaces in between, she replied referring to the table: "*like if this is 10....20* (pointing at the angles)*.....then this increases by like 20* (referring to the ratios), *but not all the time.*" She further described, for the r value: "*there are changes happening....the ratio is going further up from the y ....the triangle gets different.....and the degrees and the ratio change.*" When asked directly about the r and y value now, as the angle changes, she responded: "(smiling)*....y is increasing yes, and r is remaining the same.*" Figure 7.1 gives her answer in her interview schedule:


1.1 What do you notice about the values of y and r respectively as the angle θ increases?

y is increasing

r remains the same

Figure 7.1 Mayuri's answer to interview question 1.1

When first asked this question, Vishen was initially confused:

Researcher	Ok, Vishen, what do you notice about the y value and the r value as the angle θ increases
Vishen	I don't understand.

The researcher further probed by focusing his attention on only one variable:

Researcher	Ok, what do you notice about the y value as the angle increases?
Vishen	y is that (pointing to y in the diagram)
Researcher	Yes
Vishen	The angle is straightening up and becoming a straight line. And the angle O becomes a straight line, OC. There won't be a triangle.

The researcher further questioned Vishen to clarify the answer:

Researcher	Ok, now, what do you notice about the *y* value as the angle is increasing?
Vishen	*y* value? Increasing or decreasing?
Researcher	You must tell me, what do you notice about the *y* value as the angle is increasing?
Vishen	As the angle is going up or down?
Researcher	As the angle is going up.
Vishen	Can I check it?
Researcher	Yes.
Vishen	So the *y* value is GH.
Researcher	Yes.

Vishen, further referred to the diagram on *Sketchpad* and by manipulating and appropriate questioning, correctly answered this question:

Vishen	So when it goes up (dragging point up), the *y* value is increasing.
Researcher	What do you notice about the *r* value?
Vishen	So the *r* value is OG, as the angle increases?
Researcher	As the angle increases
Vishen	(working on the computer) Stays the same. When the angle increases *r* stays the same.

It's interesting that two learners, namely Vishen and Perusha referred back to *Sketchpad* when answering this question, whereas two learners, Suren and Omika relied on their memory, and got it wrong. The other two learners, Mayuri and Nadeem, referred to the table of values they had completed.

In conclusion, four out of six learners correctly observed and understood that, for a given *r* value, *y* increases in the first quadrant as the angle increases. Thus they were successful in the conjectured level 1 of the Van Hiele Theory. The four learners who answered correctly, referred to *Sketchpad* (Vishen and Perusha) and the table (Mayuri and Nadeem). Suren and Omika who replied that both increase, not realising that r did not increase, may have made a slip but was unfortunately not probed further by the interviewer. It is also noted that they did not check their responses using either *Sketchpad* or the table.

7.5.1.2 Category 2 - Both y and r increase

Two learners felt that both y and r increase.

Example: Suren was quite convinced that both y and r increase, for example, he responded by saying that: *"……as the degrees get higher, they both increase……y and r increase."*

Also Omika was direct in her answer: *"the values increase y and r".*

These learners have clearly not yet mastered Level 1 of the Van Hiele Theory, namely correct visual observation of the displayed lengths. Without correct observation, it is difficult to discover the generalization of constant ratio, which is characteristic of Van Hiele level 2.

7.5.2 Research Question 1.2: Can learners see $\sin\theta$ as a ratio of two sides $\frac{y}{r}$?

For interview question 1.2: *"What do you notice about the values of* $\frac{y}{r}$ *and* $\sin\theta$ *in each table?"* the learners were more confident and seemed clear about what was asked.

7.5.2.1 $\frac{y}{r}$ and $\sin\theta$ are the same

All the learners correctly observed and understood that $\frac{y}{r}$ and $\sin\theta$ are the same or almost the same.

Suren went straight to the table: *"oh you mean this part here?* (pointing to the table)."

It is interesting to note that some learners picked up on the small differences in the decimal displays. Even though the second decimal differed in only a few cases in the table, the learners did not say that they were just equal.

Mayuri immediately replied: "(looking at the table), *they are almost the same.*"
Vishen immediately answered even without looking at the table: *"They are the same."* When questioned why he did not refer to the sketch on the computer screen, he said that he knew that from when he completed the table and did not need to look.

Perusha also replied immediately that: *"Answer for* $\frac{y}{r}$ *and* $\sin\theta$ *are almost the same."*

Also Omika replied: *"...its like the same....its not exactly the same, some are exactly the same.....but some are like below or above the value."*
Mayuri, Perusha and Omika answered that the values were almost the same as their values differed in one decimal in some cases. For example, see below Perusha's table:

θ	$\frac{y}{r}$	$\sin\theta$
10°	0,17	0,18
20°	0,34	0,34
30°	0,50	0,49
40°	0,64	0,65
50°	0,77	0,77
60°	0,86	0,86
70°	0,94	0,94
80°	0,99	0,99
90°	1,00	1,00

Fig 7.2: Perusha's Table

Nadeem in addition observed that: "*As the angle increases,* $\frac{y}{r}$ *and* $\sin\theta$ *increase as well.*"

Perhaps interestingly, Nadeem's mathematics teacher picked him out as being "*the one who gets the best marks amongst them all.*"

To summarise: This question demonstrated that all six out of the six learners, with the aid of a visual representation *(Sketchpad)*, were able to correctly deduce that $\frac{y}{r}$ and $\sin\theta$ are the same. This is not to say that they could not have discovered in the same way using paper and pencil. All the learners who were interviewed therefore achieved Van Hiele Level 2, called Analysis, where through observation and experimentation, the learners observed the equality between the "sine of an angle" and the displayed co-ordinates.

7.5.3 Research Question 2: Can learners see that $\sin\theta$ is independent of r ?

The main purpose of the following question was to establish if the learners were able to make a conjecture regarding their observations, and generalise that the ratios would remain unchanged for $r = 2$. The learners were asked question 2 in the interview schedule (appendix A):

"*What do you think will happen to the above ratios if we increase r to 2? Why?*

The reader must note that this question was asked after the first table for $r = 1$ and interview question 1 were completed and before $r = 2$ was drawn.

7.5.3.1 Category 1 – the ratio $\frac{y}{r}$ will increase

Initially all learners replied that the ratio $\frac{y}{r}$ will increase.

Perusha answered this question by stating that: "*....the circle will increase...er, the I suppose the answers for the ratios will increase. The total degrees will get higher by probably 2.* $\frac{y}{r}$ *will get higher.*"

A similar response was given by Vishen: "*.... so I think if we change r to 2, the sin value and the* $\frac{y}{r}$ *value will increase proportionate to that.*"

Mayuri also stated that: *"Each figure will increase by about 4 .... each ratio ... around about."* In a similar vein Suren stated: *"The values will increase...."*

To summarise, only two of the six learners were able to conjecture that the sine of a given angle will remain constant, irrespective of *r*. The two learners did not see the relationship initially, but actually went back to correct their answers after completing the second table. According to the levels of geometric thought in the Van Hiele Theory, they had achieved Level 3, called Informal Deduction (Ordering), where learners can establish interrelationships of properties within and among figures and definitions are meaningful. Informal arguments are given and followed.

7.5.4 Research Question 3: Can learners see that $\sin\theta$ is independent of r, as an increasing function, as a function with a variable rate of change and can they generalise?

For interview question 3 (appendix A): *"For any given angle, what do you notice about the corresponding values of $\frac{y}{r}$ in each table for $r = 2$, $r = 3$ and $r = 4$?"*, the learners now easily went to their tables and answered without further hesitation. This can be evidenced by their responses as shown below:

7.5.4.1 Category 1 – they are the same or almost the same

Omika replied: *"The values are similar, its' either one below or one above."* So she correctly refers to the degree of accuracy in the decimals. Also Perusha answered: *"They are almost the same."* This again takes into account the correct values of decimals.

Nadeem responded: *".... they are the same. The values of $\frac{y}{r}$ in each table is the same."* He did not even look at his table to answer this question. When asked to explain this, he replied: *"I don't know......I am not sure. I assume, I take it for ranted. The $\frac{y}{r}$ in each table, I know, is equal, from completing the table, I remember."*

Suren took a little time to answer: *"....explain please...(*silence, smiling, thinking, looking at the table*)......oh.....they are the same.....the both corresponding values are equal."*

However, after completing table for $r = 2$, two learners correctly observed that the ratio $\frac{y}{r}$, for a given angle for interview question 1.2, will remain the same, since the y and r value increases proportionally.

Nadeem and Omika were able to understand this after actually completing the second table. Nadeem, after an initial surprise, asked if he could change his answer to this question: *"Now I realise that the $\frac{y}{r}$ will still have the same ratio, because when you increase the r to 2, y will increase as well."* Below is what Nadeem wrote as his answer in his interview schedule:

1.2. What do you think will happen to the above ratios if we increase r to 2? Why?

sin θ & $\frac{y}{r}$ Increase. The circle is getting larger.

(Reflection) The ratios stay the same because when you increase the r to 4 y u increase as well. Therefore when the radius is 3 and then when it is l $\frac{y}{r}$ will still has the same ratio.

Fig. 7.3: Nadeem's changed Answer to Interview Question 1.2

Omika (quite surprised), after completing the second table, also realised that her earlier answer was wrong: *"Won't it be the same.......because every time I hold it at 10° for example, I notice both the values are the same."*

It seems both Nadeem and Omika understood the question but were only able to create a link with the aid of *Sketchpad.* They were surprised, but convinced by their visual discovery. Earlier, without *Sketchpad,* they could not make the cognitive leap required to answer this question.

Its important to note that if learners are given the time to think, they eventually do come up with an answer. I found that the learners continually asked questions, as above, for explanation of a question, to elaborate or to re-read the question with them. They seemed to be very dependant on the researcher for the direction of their cognitive processes. I found that, if the learners were given time and probed further about their thinking, this gave them space to think, even to correct themselves and come up with their own answers. The learners wanted immediate gratification and often didn't have the patience and persistence that is one of the pre-requisites in problem-solving. This also came through when they read a question. They did not read it carefully enough and rather read what they expected the question to ask.

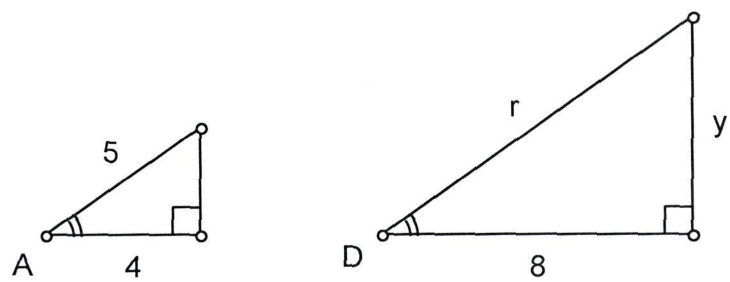
This can be seen from Vishen's response to the same question above: *"As* $\frac{y}{r}$ *increases, the value of sine increases with it."* This was not answering the question. He just seemed to home in on the $\frac{y}{r}$ and felt that the answer had to do something with this. When asked to comment on the $\frac{y}{r}$ for each angle, his thought pattern was directed and he correctly answered: *"They don't increase by much.* $r = 3$ *and* $r = 2$ *are equal, but* $r = 4$ *changes by 1, except for* $90°$*, which is 1 in each case."* He does not give a general answer, for example, stating that they are almost equal, but is very specific for where the changes occur.

Mayuri also followed the same process to answering this question: *"....(looking at tables)...they are not all the same only* $90°$ *all over is the same....three points are the same....".* It's also interesting to note that, at this point, she corrected the ratio for $30°$: *"Can I check this one?"* She found that her value did not fit in with the pattern that she detected. When asked, what made her do that, she replied: *"Because all of these are the same ....the ratios.....* $\frac{y}{r}$ *and the sin......I wanted to come up with the same."*

To summarise, all six out of the six learners answered correctly that the for any given angle, the corresponding values of $\frac{y}{r}$ were the same in each table for $r = 2$, $r = 3$ and $r = 4$. All six of the six learners achieved Level 2.

7.5.5 Research Question 4: Can learners see or apply the similarity of triangles with the same angle as the basis for the constancy of trigonometric ratios?

For the Question 4 in the interview schedule (appendixA) see below:



"Find the ratio $\frac{y}{r}$ *for the second right triangle above (the angles at A and D are equal."*

The purpose of this question was to establish if learners were able to:

1) use the Theorem of Pythagoras
2) use scale in the appropriate way

It was found that five out of six learners, after further probing, were able to eventually obtain $\frac{y}{r} = \frac{6}{10}$, whereas one learner did not succeed.

7.5.5.1 Suren's Method

Suren appeared to initially confuse angles with lengths, and attempted to use angles of a triangle to solve for lengths as shown by the following excerpt.

Researcher	Now Suren, for no. 4 ......(turn the page), we have two triangles here. Look at the two triangles ......I want you to find the ratio $\frac{y}{r}$ for the second right triangle below (the angles at vertices A and D are equal).
Suren	Eg. The y value.... $\frac{90°}{r}$ ......how do you work them out?
Researcher	If you look at your diagram, what do you see?
Suren	I can see a 90° angle here and another angle here.

Researcher	Ok, now you require $\frac{y}{r}$
Suren	I think y is 90°, it's an angle.
Researcher	You think y is an angle?
Suren	Ya, because it is 90°
Researcher	Will 90° be an answer for an angle or a length?
Suren	It's a length ….. it can be both.

After some more probing by the researcher, Suren realised that *y* and *r* were lengths, but still could not find the ratio, for example:

Researcher	Can it be both?
Suren	No it can't be both, it can only be an angle.
Researcher	Why?
Suren	Because it is degrees
Researcher	So when we are talking about *y* and *r*, are we talking about two lengths or two angles?
Suren	No two lengths (confidently).
Researcher	So can you find the ratio $\frac{y}{r}$, in any way?….(long silence)….. A method?
Suren	Not that I know of…….(long silence)…….(thinking)

The researcher next tried to focus Suren's attention on the similarity between the triangles to see if he would notice that the corresponding sides were in the same ratio. This helped him to see that the scale factor was 2.

Researcher	Ok look, you got another triangle. Can you make a relationship with the other triangle?
Suren	They both are the same, only one is larger than the other….oh, and they have values. They both have 90°…… one is 5 for the r value…….(silence)
Researcher	Will the r value in the second one be the same as well?
Suren	No, it will be higher.
Researcher	Why?
Suren	Because the triangle is bigger. (Silence) …….it will be 10.
Researcher	Why do you say 10?
Suren	Because I see this is doubled (pointing to 4 in the first triangle and 8 in the second), so I just suppose it will be doubled here too.

However, Suren still could not find the unknown length of the one rectangular side of the smaller triangle, so the researcher told him that it was 3. This enabled him to finally arrive at the correct answer using the scale factor of 2.

Researcher	......(pause)...Ok, what if the *y* is 3 (small triangle)
Suren	(Enthusiastically)....so y will be 9, because 8, 9, 10 (pointing to large triangle) and 3, 4, 5 (pointing to the smaller triangle).
Researcher	Look carefully at the diagram.
Suren	.... Oh, it will be 6
Researcher	Why will it be 6?
Suren	Because all the other sides are doubled.
Researcher	Good, so $\frac{y}{r}$ is?
Suren	It could be half....no it is doubled.....that's $\frac{6}{10}$.

7.5.5.2 Mayuri's Method

Mayuri quickly identified the right angles and the different measurements. She correctly answered the proportional lengths. She did not suggest that she go to the computer in this question as Suren did previously. She realised that she has two lengths and needs to use the Theorem of Pythagoras, but did not remember it and tried an incorrect version of the Theorem.

Mayuri	So we got two lengths, ok that's 10 that 8 ......er.......only **angles** of a triangle = 180° ....Theorem of Pythagoras?
Researcher	Ok, good
Mayuri	But I don't know it, can't remember it. ....

The researcher eventually gave Mayuri the answer and she immediately used the correct scale to obtain the correct answer.

7.5.5.3 Nadeem's Method

Nadeem did not relate the second triangle to the first and immediately gave up, saying he can't do it:

Nadeem	..... (Silence) ...... $\frac{r}{y}$, oh you got that....can't work it out.

On further probing, he observed the proportional relationship between the two triangles. When asked about working out the third side of the smaller triangle, he realised that the theorem of Pythagoras could be used, but could not recall it.

Researcher	You have two lengths there, so how do you work out the third side using the two given sides?
Nadeem	Is it Pythagoras ...... the Theorem of Pythagoras.
Researcher	Yes, so ......
Nadeem	We did it in Std 6 – last year we didn't do it at all ....can't remember.

The researcher then gives him 3 as the answer and he immediately answered $\frac{y}{r} = \frac{6}{10}$.

Nadeem further asked to simplify the answer because he stated that the ratio was needed and tried to work to decimal form, like those values of $\frac{y}{r}$ generated by the computer.

Nadeem	(simplifies $\frac{3}{5}$ to 0.6)

7.5.5.4 Vishen's Method

Vishen asked if he could use the computer, but quickly realised that he could obtain the answer by relating it to the given triangle. He correctly answered the proportional lengths and recognised that he needed to use Pythagoras Theorem to find the third side, but did not recall it. He suggested he go to his completed tables in order to answer this as follows:

Vishen	*r* is the hypotenuse? Can I use the Pythagoras Theorem?
Researcher	What do you think?
Vishen	Yes, you use Pythagoras to find the hypotenuse. But I don't remember........(silence)
Researcher	Can you use Pythagoras to find the other side of a right angled triangle, given the hypotenuse and another side?
Vishen	Our teacher showed us how to find the hypotenuse, he did not show us how to find the other side.
Researcher	Ok
Vishen	Will the information be in the table?

7.5.5.5 Perusha's Method

Perusha was the only learner to finally not see the relationship and did not see the second triangle as related to the first. She tried to go back to the computer to answer this question. She felt confused by the given lengths and tried to divide the angle by the given length, but intuitively realised that this was wrong, because she stated that

90 is an angle and 8 is a length. She knew that she had finally not got the correct answer, even though the researcher did not indicate this:

Perusha	Ok, (looking to computer), must I get the $\frac{y}{r}$. (trying to get the sketchpad triangle look almost the same as the second triangle)………(long silence)……..is there a *y* there?
Researcher	You want to know if there is a *y* there? So ……what do you think?
Perusha	(smiling)….I think this is bad.
Researcher	(laughing together)…… you think this is bad?….Ok……alright…….anything else?
Perusha	(smiling)……..I think this is very bad.
Researcher	(laughing), Ok………show me $\frac{y}{r}$. Yes….anyway in which you could find your *y* or *r* value?

She correctly stated that she was not sure what to do with the information in the diagram:

Perusha	The 8 is confusing me.
Researcher	The 8 is confusing you. What do you think its there for?
Perusha	It is the x-axis. I am thinking $\frac{8}{90°}$.
Researcher	Ok, you are happy with your answer?

Perusha further tried to answer this question, knowing that she had not correctly answered:

Perusha	The 90° you can't divide it by a normal number.
Researcher	Ok, because 90 is a
Perusha	Degree
Researcher	And 8 is …..
Perusha	A length
Researcher	Any other way?
Perusha	Because there is 5, I suppose it will be something over 5 ……but I don't know what's the something.
Researcher	And ……..
Perusha	8 divided by 4 will give you the y.
Researcher	Ok, all right Perusha……that was interesting

In retrospect, this question was unnecessarily complicated by not giving the third side, as the intention was really whether they would recognise the similarity and use the scale factor to correctly determine the ratio.

7.5.5.6 Omika's Method

Omika initially did not know what 4 and 8 represented. She quickly realised the relationship between the two triangles and correctly names the missing lengths. She hastily answered $\frac{y}{r}$ is $\frac{10}{8}$, but on further probing, she realised the error and stated that she was not good at geometry:

Researcher	Now Omika, for no. 4. Find the ratio $\frac{y}{r}$ for the second right triangle below (the angles at vertices A and D are equal).
Omika	(silence)......what is 4 and 8?
Researcher	The length of the side of the triangle, the x value.
Omika	....(silence)..... $\frac{10}{8}$

She hastily answered $\frac{10}{8}$. But upon further probing by the researcher:

Researcher	You said 10 so quickly, why did you say 10?
Omika	'Cause there is 5 and 4 here (pointing to small triangle) and 2 times 4 is 8, so 5 times 2 is 10.
Researcher	So your $\frac{y}{r}$ is $\frac{10}{8}$?
Omika	(silence)....... Is this $\frac{y}{r}$?
Researcher	Which is your $\frac{y}{r}$?
Omika	This is the value for y and this for r. (pointing them out correctly in the diagram).
Researcher	Ok ......
Omika	(smiling) I am not too good with geometry.

Upon further encouragement, Omika correctly gave the r values but still got the ratio incorrect. Upon further probing, she correctly answered the question.

7.5.5.6 (a) Category 1 - Learners obtained $\frac{y}{r} = \frac{6}{10}$

This question gave the learners some difficulty. They did not know what to find and how to go about it. Three learners asked whether they could use the computer: Suren: *"Can I use the computer? Must I use the computer."* When asked what he thought, he replied: *"I...I'll use the values (pointing to the diagram).* Also Vishen asked: *".....(silence)...no computer?",* even though the triangles were clearly presented to them, they needed to relate this with the computer, as up to now, they were involved with the computer.

Confusion between angles and lines were apparent in 3 learners, for example:

When it was clear that values represent length, learners seemed to need to clarify that.

Omika said: *"What is 4 and 8?"*

Mayuri stated: "*...can y be* $90°$*?*" even though it clearly represents the length.

Mayuri enquired: "*....can y be* $90°$*?*"

Also Vishen enquired: *"now I must find y....must I find out how many degrees?"*

Suren stated: *"I think y is* $90°$*, it's an angle."* When asked if he thinks y is *an angle.* He replied: "Yes, because it is an angle."

7.5.5.6 (b) Category 2 – Did not succeed

When Perusha was asked for the value of $\frac{y}{r}$, she replied: *"I am thinking* $\frac{90°}{8}$*"*, when asked if she was happy with the answer, she replied: *"How am I going to divide* $\frac{90°}{8}$*?.....the* $90°$*, you can't divide it by a normal number...because* $90°$ *is an angle and 8 is a length."* So she was aware that she needed two quantities of the same kind to form a ratio. Also, its quite clear that Perusha wanted to get the fraction in simplest form, the form that she got to in the tables. She wanted to divide and convert the answer to a decimal. Also very important is that she felt that $y = 90°$. She was not the only one.

Perusha tried to use the Theorem of Pythagoras, to find *y* in the first triangle and then by proportion find the next *y*. She did not use the fact that the triangles are similar, so the ratio $\frac{y}{r} = \frac{3}{5}$.

7.5.5.6 (c) Confusion between angles and lengths

Also particularly revealing was the fact that two learners attempted to use angles of a triangle in order to solve for a length. For example Suren asked: *"Must I use the triangle* $= 180°$ *...?"* When asked if we were talking about angles, he replied: *"No we are talking about lengths........so ....you can say* $180° - (8+10) =$ *so* $180 - 18$ *will give you y."* Later, he realised that he is incorrect: "$180° -$ *is an angle! Oh ......*

so no ..... (silence) ...... (thinking) ...... there could be a way to do it, but I don't know."

Three learners Vishen, Mayuri and Nadeem realised that they needed to use the Theorem of Pythagoras to calculate the third side but could not recall it.

To summarise, in this question, Suren was the only learner who initially confused angles and lengths. The other learners at this point did not confuse the angles and lengths.

Suren, Omika and Perusha did not mention the Theorem of Pythagoras as instrumental in calculating the third side in the triangle. However when told that the third side was 3, Suren quickly saw the proportional relationship and correctly answered the question.

Five out of the six learners, that is Suren, Omika, Vishen, Muyuri and Nadeem correctly noticed and used the scale factor between the two triangles to find the ratios.

7.5.6 Research Question 5.1: Can learners see that $\sin\theta$ is independent of r?

For the question: "*Do you think this ratio $\frac{y}{r}$, for a given angle will always remain constant irrespective of how r changes?"*

All six learners correctly answered that the ratio will remain constant and three learners were able to explain their observation in terms of proportionality.

When each learner was asked the reason for their answer above, they answered as follows:

Vishen: " When r is doubled for example, I notice that y is also doubled.. $\frac{y}{r}$ is then constant."

Vishen is using the scale ratio. Omika uses the scale factor:

Omika: "Er, if you want the triangle to be bigger, you can times it by 2 and the number will get doubled or if you want it 3 times bigger, it will always be times by 3 ......The number will be bigger, but if you divide it like you want it 3 times bigger, you can divide it by 3 and you will still get the same number."

Nadeem: ".......(silence)......it will stay the same if y increases with r."

When the researcher asked Nadeem if the ratio will still remain the same if $r = 2.1$ or if $r = \pi$, he replies: "Will the ratio?.....can I see on the computer?"
He correctly clicked by dragging a point and replied "yes" by noting the constant ratios on the screen. The researcher then asked him to explain or justify his answer. Nadeem replied: "It is the same because of the same answer as that one (pointing to the equal ratios on the screen). Because as r increases, the y increases the same. So when you find $\frac{y}{r}$, it will always be the same......y decreases and r decreases but they do not decrease the same......because they are different numbers. I am not too sure. Can I try something else here?" The researcher said "yes".
Nadeem replied: (dragging a point)......"Now see, its 2 right and then 0,7, now bring it up and that 3, that 1.1. So this went up 1cm and that went up 0.3......0.4 cause see 7.7 and 8,8,10,11....so they are not actually increasing and decreasing in the same amount." When the researcher probed further, Nadeem replied: "I am looking for a word that says it decreases ......it decreases in a certain manner......"
To which the researcher stated: Proportionally? And Nadeem replied: (happily) "ya proportionally, it increases proportionally."

Suren: "Because we did all these three tables and, for each angle the value of $\frac{y}{r}$ is the same."
Mayuri: "Yes, because as r is increasing the other values are increasing in proportion."

To summarise, it can be concluded that it seems as if the learners generalised their previous experience of filling in tables for $r = 1$ to $r = 4$ to $\frac{y}{r}$ being constant for ANY fixed angle. According to the conjectured levels of geometric thought in the Van Hiele Theory, they have achieved Van Hiele Level 2. Here the emphasis is on ratio.

7.5.7 Research Question 5.2 and 5.3

This question was intended to check whether their generalization from integer values of *r* extended to decimals and irrational numbers.

Interview question 5.2 read: *What if r = 2.1? or r = π? Will it still be the same for a given angle?*

Interview question 5.3 asked: *Why?*

Five out of six learners correctly answered "yes" to this question and three logically explained it in terms of proportional increase. For example:

Vishen, Omika and Mayuri uses scale factor ratios to correctly answer:

Vishen: Yes, as the ratios increased in the past exercises, if one ratio had to change, as in *y*, the *r* will change. It will be in proportion with it……(thinking)…..when we are talking about it will remain the same, we are talking about it will change, but in the same proportion. The number will double, but it will stay in the same multiples."

Omika: "Yes, the number will always double the amount of the other ………increase in the same way…..ya …OK"

Mayuri: "Yes, because as *r* is increasing the other values are increasing in proportion."

Suren and Nadeem, two of the five correct responses, did not give a logical explanation but appealed to empirical evidence, the one referring to the tables and the other using *Sketchpad.* For example Suren uses the table to correctly answer this question:

Suren: "Yes, because all these tables have equal values for each angle. I noticed when filling them in."

Nadeem however, uses *Sketchpad* to correctly answer this question:

Nadeem: Yes, (checks by dragging a point), they will be equal for all values.

The one learner, Perusha seemed to have misinterpreted the question. In retrospect the interviewer ought to have probed further.

Perusha: "No, because only if you change the size of the circle, then only will the radius change."

This was a significant finding, as five out of the six learners surprisingly generalised successfully and confidently to ANY (positive) numerical value of r. It seemed that Vishen, Omika and Mayuri had a good idea that the sides changed in proportion. According to the levels of geometric thought in the Van Hiele Theory, they have achieved Van Hiele Level 2.

7.5.8 Research Question 6: Learners' understanding of trigonometric function, as a relation between input and output values

For interview question 6, the learners used different methods to solve the problem. Three of the learners used the table and three of them used the computer.
Three learners obtained $30°$ and three learners did not.

In Interview Question 6, the learners firstly had to find an angle, given a ratio in their table. In the following two questions, they had to find values not in their table, namely, the sin of an angle and finally to estimate the angle, given a ratio:

6. Answer the following questions:

6.1 If $\sin(angle) = \frac{1}{2}$ then $angle =$ ________?

6.2 $\sin 35° =$ ________?

6.3 Estimate the value of the $angle$ if $\frac{y}{r} = 0,55$

7.5.8.1 Research Question 6.1: Finding an angle given a ratio

Suren initially was confused:

Researcher	Answer the following question. If sin (angle)= $\frac{1}{2}$, then angle =
Suren	$\frac{1}{2}$. ....(silence)....I don't understand this question.....if we are talking about a triangle....then two $90°$ will give you 180.
Researcher	So what answer would you write there?
Suren	..... (Silence)......mmmmm

The researcher further questioned and Suren then realised he could read it off from the table:

Researcher	Another way of writing half is?
Suren	0,5 ......so its this ......(looking at the table) the angle is $30°$. Oh yeh.

Suren easily went to the table that he had completed earlier, and read off angle as 30° if the ratio = 0,5.

Mayurii and Nadeem also used the computer as Suren had done to obtain their answers.

7.5.8.1(a) Category 1 – Learners obtained 30°

After a while, Nadeem asked if he could use the computer. He correctly dragged the point to the ratio 0,5 and easily read off the angle = $30°$.

Mayuri also found the answer in the same way: She asked if she could check it on the computer and dragged the point to "*make it to half*" and read off $30°$.

Suren referred to the table for this answer. After he clarified that half is 0,5 he went to his table and correctly read off the angle as $30°$.

Perusha was confused for the question if sin(angle)=0.5, then the angle measurement was 0.5.

7.5.8.1(b) Category 2 – Learners obtained some other answer

Vishen reasoned the following way: "*the angle is equal to* $45°$ *....... Because when the angle increases to* $90°$*, it is 1. Half of* $90°$ *is* $45°$*.*" Vishen is therefore implicitly assuming here that the sine function is a linear function. Given their background at grade 10 level where they've been largely exposed to linear functions. This has an important implication for teaching. Teachers should make learners' explicitly aware of the non-linear behaviour of the sine function.

Omika and Perusha had difficulty interpreting the question, and it does not seem as if they had made a clear distinction between input and output values. They seemed to be confusing this question with the earlier conclusion that $\sin\theta = \frac{y}{r}$. This can be evidenced by their answers to this question. Perusha replied: "*if sin(angle)*$=\frac{1}{2}$, *the angle* $=\frac{1}{2}$ *....or 0.5.*

Omika also shared the same idea: "*the answer is* $\frac{1}{2}$ *...... these are the same... (pointing to* $\frac{y}{r}$ *and* $\sin\theta$ *in the table), so this will be the same too.*"

7.5.9 Research Question 6.2: Finding an ratio, given a angle

The next question $\sin 35° = ?$ required that they now find a ratio for a value not in their tables. The learners used different methods to calculate the answer.
Five learners correctly obtained approximately 0.57 and one obtained some other answer.

7.5.9.1 Category 1 – Learners obtained 0.57 (approximately)

Five out of six learners obtained the correct answer. Three learners for example, Mayuri, Suren and Nadeem used *Sketchpad* by dragging the point until the angle was 35° and reading off the answer 0,57.
Suren initially attempted to use the table to calculate $\sin 35° = ?$ but then realised that there was no information on $\sin 35° =$. He then reverted to using *Sketchpad* to drag and read off:

Researcher	Good, what do you think $\sin 35° =$?
Suren	.... Can look in these tables here?.......I can't tell you.
Researcher	Why?
Suren	Because there is no information here telling you of 35° .... , oh here it is here.....
Researcher	So?
Suren	Go to 35° (dragging point) 0.58

Two students, Vishen and Omika used the table and estimation to answer this question, for example, Vishen explained: "*Ok 0,57, because 30 and 40 have a 14 number difference, in terms of the ratio. So the answer is 0,57 using the ratio.*"

Omika also used the table to get the answer: *"0,58 or something ...... 15 divided by 2 is* $7\frac{1}{2}$. *Then 0,57 since we know for* 30° *you get 0,5."*

It is interesting to note that the two learners also assumed as Vishen had done previously that the sine function was linear, but in their case, it worked reasonably well as the sine function is approximately linear in this particular short interval. This is a good example of incorrect reasoning leading to a correct answer and teachers should realise the importance of probing learners' reasoning more deeply.

7.5.9.2. Category 2 – some other answer

Perusha replied that $\sin 35° = 35°$. When asked for a reason, she said that she made this conclusion from the fact that $\sin\theta = \frac{y}{r}$ in all the tables. She apparently confused this with the first part where the conclusion of her task was $\frac{y}{r} = \sin\theta$. So she mistakenly equated the angle and the ratio and does not seem to yet distinguish clearly between input and output values.

In hindsight, the researcher should have used the opportunity to probe further whether her answer was just a careless mistake or a misconception regarding the distinction between input and output values. Given that she responds similiarly in the previous question, gives some indication that she may not yet have a good conceptualisation.

7.5.10 Research Question 6.3: Estimation of an angle not in the tables

For the final question, the learners were required to estimate the value of the angle if $\frac{y}{r}$=0,55. All six learners obtained approximately 33°. They worked this out using the tables or the computer, for example:

Both Mayuri and Nadeem did not use the computer to estimate the value $\frac{y}{r} = 0.55$. They both made a calculated guess based on the previous answers. Their answer was correct.

Both Vishen and Perusha got the final answer correct, but they used the computer to check the angle if the ratio was 0.55.

7.5.10.1 Category 1 – angle = 33°

Suren immediately uses *Sketchpad* to answer correctly:

Researcher	Good! Estimate the value of the angle if $\frac{y}{r} = 0.55$.
Suren	…. (Silence)……(rereading the question)…..you check it out in the computer?
Researcher	Ok….
Suren	$33°$ ….. your angle is $33°$.

Nadeem used estimation.

Researcher	Ok, estimate the value of the angle if $\frac{y}{r} = 0.55$.
Nadeem	So I must take a guess?
Researcher	Must you guess?
Nadeem	Because it says estimate. Or must I use the computer?
Researcher	Ok, what do you think you will do?
Nadeem	……. (silence)……because it says estimate I will do it myself.
Researcher	Ok.
Nadeem	…….. (silence)……
Researcher	What are you thinking?
Nadeem	I am thinking how to work it out.
Researcher	Ok

Nadeem's method was revealing…

Nadeem	If 0.5 is 30 and 0.57 is 35 , so 0.55 will be, I am just like……. I am like blocked. ……(silence)……if I have to estimate, I will say $34°$
Researcher	How did you get $34°$?
Nadeem	Mmmmm, like for every 0.02, there is $1°$, I am just saying for example, so then 0.56, it will be 33. I tried using 1.5.
Researcher	You settle for that answer?
Nadeem	Just now …….. I will leave it at $34°$

Its interesting to note that he also incorrectly assumed a linear relationship and thus used incorrect reasoning to get a correct answer.

Omika guessed $35°$ since: "... 0,5 is $30°$, so I guessed $35°$." The researcher unfortunately neglected to probe further how she arrived at the answer.

Mayuri incorrectly first estimated the answer and wrote it down. Thinking that she was over, the researcher thanked her for her participation in the research. She then asked: *"Can I check it?"* Curious, the researcher asked her why she did not check it in the first place. She replied: *"Cause you had to estimate...."* This clearly indicates that she was trying to follow the instructions and only used the computer to check her written out answer.

This is what Mayuri did:

Researcher	Nice, Ok, estimate the value of the angle if $\frac{y}{r} = 0{,}55$.
Mayuri	.... (silence).....you are finding the degrees?
Researcher	Yes, (after a long silence), try
Mayuri	$50°$ ....0.50?........(silence)...... $28°$.
Researcher	You think x will be $28°$. Why?
Mayuri	Because here 0.5 for $30°$, so 0.55 should be for $28°$.

Initially Mayuri thought that the sine function was decreasing. Upon further probing, she arrived at the correct answer.

Next she asked if she can check it and she correctly used *Sketchpad* by dragging to get the answer:

Mayuri	Can I check it?
Researcher	You want to check it? .... why did you not check it in the first place?
Mayuri	'Cause you had estimate ......
Researcher	Oh ......Ok ..... so then but then what did you realise about your estimation?
Mayuri	Now or ....
Researcher	Now

Mayuri	That I was wrong.
Researcher	So what would you go with.... The computer or what you had?
Mayuri	The computer...computer learning is never wrong.... you put in the wrong input, it will give you the wrong output.
Researcher	So what is your final answer?
Mayuri	$33°$, can I change it?
Researcher	Change it ...... OK, Thank you Mayuri.

To summarise, the learners were very proficient in the use of the computer and *Sketchpad* at this stage and were able to use it appropriately and correctly, reading off the angle when required, as in interview question 6.1 or the ratio, as in interview question 6.2, and to approximate correctly, as required in interview question 6.3. The learners clearly had achieved Van Hiele Level 3, they were able to deduce ratios when angle were given and angles when ratios were given and informal arguments could be followed. However, some used incorrect reasoning to get a correct answer.

CHAPTER 8

CONCLUSIONS AND RECOMMENDATIONS

8.1 INTRODUCTION

The focus in this study was Learners' Conceptualisation of an Introductory Activity to the Sine Function with Sketchpad at Grade 10 level. In this chapter, the findings from the interviews and interview schedules are summarised. Further, some issues and difficulties in trigonometry are discussed and recommendations are made.

8.2 SUMMARY OF FINDINGS

8.2.1 Overall Findings

This research has yielded some valuable results in terms of the teaching and learning of trigonometry, functions and the whole of mathematics in general. The teaching (and learning) approach used in the empirical research seemed to provide learners a greater, and more meaningful, understanding of the sine function. This study concentrated mainly on the introduction of the sine function to learners as a means of personal discovery.

It has provided substantial evidence that the participants of this study did not have big problems in learning how to use *The Geometer's Sketchpad* (GSP). Their past experiences with computers and previous knowledge about computers helped them to feel comfortable with the use of GSP in their study.

The visual function of GSP helped them to bring about a better understanding of the abstract questions that were asked in the interview and the questionnaire. In the beginning of the course, the use of GSP and the learners understanding seemed to be separate. But as they became more familiar with GSP, they were finally able to solve most of the problems. The computer approach can change, by allowing the child to manipulate the picture and relate its dynamically changing state to the corresponding numerical concepts (Blackett and Tall, 1991: 146). It therefore has the potential of improving understanding. This ability to use the computer to carry out certain arduous constructions whilst the learner can focus on specific relationships, is referred to by

Blackett and Tall as the "*principle of selective construction*". They believe this to be one of the most powerful educational principles for the use of the new technology.

The following statistics reveal the significant level of success that the learners obtained in each of the interview questions:

Interview Question 1.1: Four (67%) of the learners interviewed correctly observed and understood that as the angle increases y increases and r stays the same. The one learner (33%) may have made a careless slip as she may have assumed that $\frac{y}{r}$ and not y and r separately was referred to in the question. Also that they may have not made a keen observation of the question.

Interview Question 1.2: All of the learners interviewed observed and correctly understood $\sin\theta$ as a ratio of sides.

Interview Question 2: None of the learners interviewed at this stage of the interview, could conjecture, without the use of *Sketchpad* or tables that the sine of a given angle would be independent of the radius. It must be noted that this question was posed to them just after they completed the first table for $r = 1$ and interview question 1.

Interview Question 3: All six learners of the learners interviewed correctly observed and understood, after completing the tables, that the sine of a given angle is independent of r. Three learners gave correct explanations in terms of scale factor.

Interview Question 4: Five (83%) of the learners interviewed understood the similarity of triangles with a given angle as the basis for the constancy of the sine of a given angle.

However learners struggled to recall the Theorem of Pythagoras and use it to calculate the third side. None of the learners were able to use the Theorem of Pythagoras. This question was unnecessarily complicated by not giving the third side of the triangle as the aim was not to test Pythagoras, but to test their understanding and recognition of the underlying similarity.

Interview Question 5.1: All the learners understood that the sine of an angle will always remain constant irrespective of how r changes. Three learners explained this in terms of the proportional increase of both y and r and three referred back to the tables and stated that since the values did not change in the tables, they would therefore not change for all values.

Interview Question 5.2: Five (83%) of the learners interviewed were convinced that the sine of an angle would remain constant even for decimal and irrational values of the radius. This is quite a remarkable result as they were confident that, for any r the sine of a given angle will be constant. They seemed to be satisfied by the tables that they completed and understood the underlying similarity of the triangles.

Interview Question 5.3: Five (83%) of the learners interviewed could correctly generalise from their findings, where they used the completed tables and underlying similarity.

Interview Question 6.1: Five (83%) of the learners interviewed could correctly read off the ratio for a given angle from their table. They correctly dragged and read off the value. Those who obtained some other answer misinterpreted the $\frac{1}{2}$ in $\sin(angle)=\frac{1}{2}$, to be half an angle. These learners answered $45°$ as they reasoned that when an angle increases to $90°$, it is 1 and half of $90°$ is $45°$, so they gave an answer of $45°$. Good sensible reasoning, though wrong. One of the misconceptions that learners have is that they have the belief, particularly at grade 10 level, that ALL functions are linear. This came out very nicely where they had to estimate an angle given a ratio. All of them used the assumption that the sine function is linear. They got the right answer because within the small interval, it is approximately linear. As a teacher, it's very important that one is aware that the learners are using incorrect reasoning. So it's very important the one should be alerted that this is a very big problem. One learner made the mistake when he used the very large interval, where he said that in the interval $0°$ to $90°$ where the function goes from 0 to 1, then $\sin(angle)=\frac{1}{2}$, will be $45°$, because it is half-way, because he is using his linear model of the sine function. Teachers should be aware of this and should develop strategies to alert hem to the fact that the sine function is not linear, over a small interval, yes, but over a larger interval, learners are bound to make mistakes. Learners should develop that and its known to be a big problem. It is not only applicable to the sine function, but the quadratic function as well.

Interview Question 6.2: Five (83%) of the learners interviewed could correctly determine the ratio of an angle not given in their completed tables. The learners correctly dragged the angle to 35° and read off the corresponding ratio.
Interview Question 6.3: All of the learners could estimate an angle for a ratio not included in the table.

Thus, the use of GSP and understanding seem to go together for better results of understanding mathematics. At the end of the study, the learners' attitude changed dramatically in the favour of this type of exercise in their daily classrooms. They looked back and found that it is convenient and easy to explore the trigonometric questions with the aid of a dynamic sketch.

Teachers think that when learners get right answers in a test, that they have deep understanding but what has occurred in reality, is that they learner has just learnt to play the game. Learners also get right answers with completely incorrect reasoning. One thing this research is showing is the pervasiveness of the idea that all functions are LINEAR. Telling learners the correct answer will not help, activities should be designed to place learners in cognitive conflict. One needs to be concerned about the teachers, and their perceptions, as more harm may be done in this manner.
This has important implications for teaching. We need to do important follow-up activities for a deeper understanding.

The computer environment is significant in changing the traditional mathematical environment. Freed from performing mathematical techniques, the problem solver can now focus on mathematical meaning, methods and explanations. By combining various representations of mathematical problems, teachers can invent new ones.

Using computers as an integral part of mathematics instruction has several important benefits. Computers encourage generalization because learners are able to investigate many special cases quickly and gain the necessary evidence to make these generalizations. Computers enable the user using it to become a powerful problem-solving tool. Computers permit learners to access problems, which are more interesting, realistic, and broader in scope than those typically found in standard textbooks. Computers encourage learners to become active partners in the learning process. I believe that computer usage in high school mathematics classes should be much more than enrichment or supplementary. We need to exploit what the computer has to offer learners on a regular basis in our mathematics classrooms. Computers can and should be used to enhance much of the mathematical curriculum rather than providing only supplementary or enrichment topics for a limited number of learners.

8.2.2 Misconceptions and Constructivism

It is clear that the character of a learner's existing schemas will determine what s/he learns from experience or information and how it is understood (Olivier, 1989). He further states that at the heart of a constructivist approach to teaching is the awareness of the interaction between a child's current schemas and learning experiences, to look at learning from the perspective of the child, by considering the mental processes by which new knowledge is acquired. Because knowledge cannot be transferred ready-made, to support the child to construct his/her *own* knowledge, discussion, communication, reflection and negotiation are essential components of a constructivist approach to teaching.

From a constructivist perspective, misconceptions are crucially important to learning and teaching, because, according to Olivier (1989), misconceptions form part of a pupil's conceptual structure that will interact with new concepts, and influence new learning, mostly in a negative way, because misconceptions *generate* errors, for example in this study, when learners were asked to calculate a ratio for an angle that was not in their completed tables (see analysis). Given their mathematics experience at Grade 10, they obtained the correct answer by incorrect reasoning, assuming that the sine function was linear. This worked out well for them, as in a short interval, the sine function is approximately linear.

8.2.3 Van Hiele Theory

At Van Hiele Level 1 it would appear appropriate to provide children with ready-made sketches, for example, quadrilaterals or a unit circle, which they can easily manipulate and first investigate visually. Next they could start using the measure features of the software to analyse the properties to enable them to reach Level 2. Then it would be appropriate to ask them how they would construct dynamic figures themselves or generalise, thus assisting the transition to Level 3.

In this study, it would appear that lots of learners started on Van Hiele level 1, where they relied on visualisation and *Sketchpad.* Later the relationship moved more to the analysis part than visualisation as it progressed from the first difficulty, which were, for example, recognising angles and lengths, Van Hiele level 1 (visualisation), to looking at properties embedded (Van Hiele level 2) and then eventually to the generalisation that occurred in Question 5, where the learners were asked about the sine of an angle when $r = 2$ to $r = 4$, decimal and even for $r = \pi$ all the learners seemed happy that for a fixed angle, it does not matter what value r assumed. That seems to indicate that the learners have progressed to Van Hiele level 3 where they made a generalisation from particular cases that were documented in the tables for $r = 1$ and so on, to the general case that it doesn't matter for the value of r, the sine of a fixed angle will always be a constant ratio. That is significant for a first introduction of about 90 minutes it seems to be have been quite successful.

8.2.4 Learners' understanding of ratio

Learners worked with the ratio in different contexts. Initially Suren did not know the meaning of the term "ratio". For example when Suren, was asked to "find the ratio $\frac{y}{r}$ for the second right triangle:

Researcher	Now Suren, for no. 4 ……(turn the page), we have two triangles here. Look at the two triangles……I want you to find the ratio $\frac{y}{r}$ for the second right triangle below (the angles at vertices A and D are equal).
Suren	Can I use the computer? Must I use the computer?
Researcher	You want to use the computer?
Suren	No I am asking you, er….. , does it relate to the computer?
Researcher	What do you think?
Suren	I……I'll use the values (pointing to the diagram)
Researcher	Yes, so you think this will relate to the computer?
Suren	Not if I need to start here……(thinking)…..what do you mean by ratio?

It is clear that some learners also did not fully understand ratio and proportion. Learners could not link the word "ratio" to the relationship between two sides of a triangle and thus could not explain why its value increased or decreased as the angle changed.

Also Suren confused y for the angle $90°$, but after further probing, comes to the conclusion that y and r represents two lengths but one will be concerned if he completely understood it.

Many learners have problems in trigonometry because of a poor conceptualisation of the underlying similarity. Suren also initially had problems with similarity. He seemed stuck, so the researcher questioned him:

Researcher	Ok look, you got another triangle. Can you make a relationship with the other triangle?
Suren	They both are the same, only one is larger than the other….oh, and they have values. They both have 90°……one is 5 for the r value…….(silence)

He does not make a link with the second triangle, so the researcher further probes:

Researcher	Will the r value in the second one be the same as well?
Suren	No, it will be higher
Researcher	Why?
Suren	Because the triangle is bigger. (Silence)…….it will be 10.
Researcher	Why do you say 10?
Suren	Because I see this is doubled (pointing to 4 in the first triangle and 8 in the second), so I just suppose it will be doubled here too.
Researcher	Have you answered the question?
Researcher	Ok, what if the y is 3 (small triangle)
Suren	(enthusiastically)….so y will be 9, because 8, 9, 10 (pointing to large triangle) and 3, 4, 5 (pointing to the smaller triangle).
Researcher	Look carefully at the diagram.
Suren	….oh, it will be 6
Researcher	Why will it be 6?
Suren	Because all the other sides are doubled.
Researcher	Good, so $\frac{y}{r}$ is?
Suren	It could be half….no it is doubled…..that's $\frac{6}{10}$.

Learners made use of the theorem of Pythagoras. These tasks reinforce an operational view of ratio since learners focus on individual sides of a triangle. Thus learners did not view their answers as ratios of an angle, but as common fractions. Understood in this manner, the numerator and denominator are treated as separate entities that have meaning independent of each other. This weakens the development of a concept of ratio.

Also Mayuri is not sure about the ratio $\frac{y}{r?}$.

Mayuri	Ok, they are both 90°, they are equal.... their angles are all equal......4 and 8 are double, so this will be 10 because that is 5. Can y be 90?
Researcher	Is y an angle or a length?
Mayuri	A length...oh they both are lengths (smiling)
Researcher	So $\frac{y}{r?}$
Mayuri	So we got two lengths, ok that's 10 that 8 ......er.......only **angles** of a triangle = 180° ....Theorem of Pythagoras?

Nadeem also does not know what to do with the information. He recognises the use of the Theorem of Pythagoras, but does not remember it:

Researcher	You have two lengths there, so how do you work out the third side using the two given sides?
Nadeem	Is it Pythagoras ...... the Theorem of Pythagoras.
Researcher	Yes, so ......
Nadeem	We did it in Std 6 – last year we didn't do it at all ....can't remember.

Vishen also recognised the use of the Theorem of Pythagoras in the problem, but could not recall it. For these learners, the class method was a hinderance because they could not remember it but did not have an alternate method. Nor did these learners attempt to reconstruct the class method or make up their own. They knew that a method existed and they believed that they had to remember it and reproduce it but they were unable to do so.

The first activity, illustrates *"pseudostructural conception"*, because, according to Sfard and Linchevski (1994), *" learners were able to calculate the ratios for different angle measurements"*.

They gave the ratio in decimal form.

Researcher	......(long pause)....Ok fine, if I give you this as 3.
Nadeem	Then multiply by 2. so $\frac{y}{r}=\frac{6}{10}$. Simplify it?
Researcher	Yes!..(smiling)... ....Simplify it, you want to simplify it or .....
Nadeem	Ya .....$\frac{3}{5}$. You want the ratio.
Researcher	You happy with $\frac{3}{5}$?
Nadeem	No, I want to get like an answer. Like a number not a fraction.
Researcher	So a fraction is not a number?
Nadeem	No it is a number, but they asked for the ratio. Like the table.
Researcher	Ok .....
Nadeem	(simplifies $\frac{3}{5}$ to 0.6)

Here the ratio was seen as the length of the sides of the triangle. This suggests that the learners did not fully understand the meaning of the ratio and its relationship to the angle or to the sides of the triangle. It seems many learners did not understand that the process of dividing the length of the opposite side by the length of the hypotenuse is equivalent to keying in an angle and pressing the *sin*-button. Thus they were unable to take the output from the calculator and relate it to the appropriate sides of the triangle.

8.3 What makes trigonometry difficult to learn?

There are a variety of factors that make trigonometry difficult for learners. Some of the factors are: poor understanding of trigonometric notation, difficulties in the use of the calculator, a poor concept of ratio and difficulties with algebraic manipulation. Pournara (2001), summarises many of the issues into three categories:
Inadequate pre-knowledge, confusing the ratio of sides with the actual length of sides, and the need to understand the conversion between angle and ratio.

8.3.1 Inadequate pre-knowledge

According to Pritchard and Simpson (1999:81), *"trigonometry is the confluence of a number of streams of mathematical difficulty."* Trigonometry is the place where number, algebra, function, geometry and measurement meet and it may expose learners' difficulties in some or all of these areas simultaneously (Pournara, 2001). While trigonometric concepts are only introduced at grade 10 level, the foundations are being laid in earlier years. The data gathered from the interviews and interview schedules suggests that the learners' lack the necessary pre-knowledge for trigonometry. The pre-knowledge, according to Pournara (2001), assumes the ratio approach to trigonometry and includes knowledge and skills such as:

- concepts of angles, distance and ratio
- ability to use the calculator for basic arithmetic operations
- estimation and approximation skills, including rounding to a required number of decimal places
- the ability to work accurately with a ruler, protractor and compass
- knowledge of the Theorem of Pythagoras and ability to apply it appropriately
- ability to solve simple linear equations

In chapters 6 and 7, there are several instances where learners did not have this pre-knowledge. For example, learners used lengths as angles of a triangle and they confused ratios for the actual lengths of the sides of the triangle. Further learners did not round off to the correct number of decimals. Many learners were unable to formulate the Theorem of Pythagoras correctly for a given triangle. Without this pre-knowledge, (Pournara, 2001), it is unlikely that learners will be able to develop a conceptual understanding of trigonometry.

8.3.2 Working with a ratio and sin as decimals

On the Sketchpad screen display, the values for $\frac{y}{r}$ and $\sin\theta$ were in decimal form.

When learners work with the ratio in decimal form, they may have got confused because they have only one number – and hence the length of only one side in their view.

In ratio tasks where the learners are not allowed to use a calculator, there is confusion between the *ratio* of sides and *actual lengths* of sides of triangles. For example, according to Pournara (2001), given $\sin\theta = \frac{5}{13}$, the learner is likely to draw a triangle. In the original ratio, the values 5 and 13 represent the ratio of the sides. However, $\sin\theta = \frac{5}{13}$, represented in a triangle, were treated by the learners as the length 5 the actual length of the vertical side, and 13 as the actual length of the hypotenuse. If the learner determines the length of the horizontal side, they treat the answer (12) as the actual length of the horizontal side and may even write 12cm.

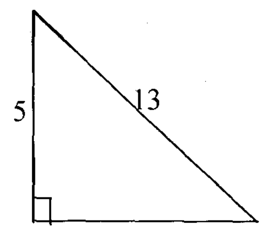


Fig. 8.1 Triangle showing ratio allocated to sides

Pournara (2001), stated that he wonders *"whether learners see any difference between the 'lengths' of sides in this task and the lengths of the sides of the triangle, where they are given the actual measurements, including units."*

It is then not surprising that learners have difficulty relating trigonometric ratios to right-angled triangles.

8.3.3 Converting between angle and ratio

It is found that learners experience difficulty with another aspect of trigonometry where the input and output are different types of numbers (compare question 6 in the nterview schedule). For example, consider $\sin 35°$: the input is an angle, measured in degrees while the output can be seen either as a ratio or simply a decimal number, depending on the orientation that is adopted (Pournara, 2001).

Tall et al, *(in press)*, stated that this situation, where the input and output numbers are not the same type of numbers, is an example of *cognitive discontinuity.*

In trigonometry, the difference between the input number and the output number is more critical as not only are they different types of numbers, but the y – values is also measured in different units.

Researcher	Nice, Ok, estimate the value of the angle if $\frac{y}{r} = 0,55$.
Mayuri	…….. (silence)…..you are finding the degrees?

Mayuri asked: "*….you are finding the degrees?*" She may have recognised this as the question clearly asks for the angle. She subsequently answers this question correctly.

Here Nadeem was also confused about which value represented the angle and which value represented the ratio.

Researcher	How would you find sin 35°.
Nadeem	Use the computer again?
Researcher	Use the computer again.
Nadeem	(dragging point)…… is sin the angle?

Some learners clearly distinguished between input and output values, and could use the inverse function, for example Vishen clearly saw the fact that in question 6.1 the angle was required and in question 6.2, the question required the ratio.

Researcher	Now, $\sin 35° = ?$
Vishen	$\frac{y}{r} = \frac{35}{1}$? Here you want the ratio? Question 6.1 you wanted the angle?
Researcher	Yes

Perusha clearly treated the answer as an angle because there was an angle in the question. It seems that she was working with the assumption that the input and output should be in the same units. She however, correctly answered the last question, since it clearly stated *"estimate the value of the angle"* , thus she dragged the terminal ray until the ratio is 0.55 and correctly read off the corresponding angle in degrees.

Researcher	Ok, $\sin 35° =$ ________
Perusha	0.57°
Researcher	Ok Perusha, estimate the value of the angle if $\frac{y}{r} = 0.55$.
Perusah	The angle? (going to the computer)….ok, that the angle right?
Researcher	Right.
Perusha	And the radius, can I use the computer?
Researcher	Hmmmm…..
Perusha	(dragging, stopped)…..0.55, then it is 33°

8.3.4 LEARNERS' MISCONCEPTIONS

Errors of misconception should not be treated as terrible things to be uprooted, since this may confuse the child and shake his or her confidence in his previous knowledge. Instead, making errors, according to Olivier (1989), is best regarded as part of the process of learning. We should create a classroom atmosphere that is tolerant of errors and misconceptions, and exploit them as *opportunities* to enhance learning. In this regard direct teaching ("telling") of missing concepts will not do. Rather teachers should help pupils to connect new knowledge to previous learning. Swan (1983), Nesher (1987) and Olivier (1988a) describe a teaching approach that is designed to expose children's misconceptions and provide a feedback mechanism that leads to *cognitive conflict.* Discussion, communication, reflection and negotiation of meaning are essential features of a successful approach to resolve pupils' misconceptions, states Olivier (1989).

8.4 RECOMMENDATIONS

In this section, I make recommendations at three levels: classroom practice, the broader curriculum and research.

8.4.1 Classroom Practice

8.4.1.1. Computer Software

This research has yielded some valuable results in terms of the teaching and
learning of trigonometry , functions and the whole of mathematics in general.
The teaching (and learning) approach used in the empirical research seemed to provide learners a greater, and more meaningful, understanding of the sine function. This study concentrated mainly on the introduction of the sine function to learners as a means of personal discovery.

It has provided substantial evidence that the participants of this study did not have big problems in learning how to use *The Geometer's Sketchpad* (GSP). Their past experiences with computers and previous knowledge about computers helped them to feel comfortable with the use of GSP in their study.

The visual function of GSP helped them to bring about a better understanding of the abstract questions that were asked in the interview and the interview schedule. In the beginning of the course, the use of GSP and the learners understanding seemed to be separate. But as they became more familiar with GSP, they were finally able to solve most of the problems. The computer approach can change, by allowing the child to manipulate the picture and relate its dynamically changing state to the corresponding numerical concepts (Blackett and Tall, 1991: 146). It therefore has the potential of improving understanding. This ability to use the computer to carry out certain arduous constructions whilst the learner can focus on specific relationships, is referred to by Blackett and Tall as the *"principle of selective construction"*. They believe this to be one of the most powerful educational principles for the use of the new technology.

Thus, the use of GSP and understanding seem to go together for better results of understanding mathematics. At the end of the study, the learners' attitude changed dramatically in the favour of this type of exercise in their daily classrooms. They looked back and found that it is convenient and easy to exploe the trigonometric questions with the aid of a dynamic sketch.

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This has important implications for teaching. We need to do important follow-up activities for a deeper understanding.

The computer environment is significant in changing the traditional mathematical environment. Freed from performing mathematical techniques, the problem solver can now focus on mathematical meaning, methods and explanations. By combining various representations of mathematical problems, teachers can invent new ones.

Using computers as an integral part of mathematics instruction has several important benefits. Computers encourage generalization because learners are able to investigate many special cases quickly and gain the necessary evidence to make these generalizations. Computers enable the user using it to become a powerful problem-solving tool. Computers permit learners to access problems, which are more interesting, realistic, and broader in scope than those typically found in standard textbooks. Computers encourage learners to become active partners in the learning process. I believe that computer usage in high school mathematics classes should be much more than enrichment or supplementary. We need to exploit what the computer has to offer learners on a regular basis in our mathematics classrooms. Computers can and should be used to enhance much of the mathematical curriculum rather than providing only supplementary or enrichment topics for a limited number of learners.

8.4.1.2. Ratio and function orientation

Learners also revealed gaps in their knowledge, especially their ratio and function orientation needs to be improved and thus teachers must help learners shift from implicit to explicit notions of ratio and function. It seems that learners could easily be assisted to develop a strong function orientation and thus an explicit notion of function. The teacher needs to make the input-process-output mechanism explicit so that learners can use it (Pournara, 2001). This may be done with the aid of a calculator, showing how it takes an input and operates on it to produce the output. The notion of a function machine (Tall et al, 2000) may also be useful here.

It is also important that learners are able to shift between ratio and function orientations in order to solve trigonometric tasks. Pournara (2001), states that teachers need to make ratio and function orientations explicit to learners. He further states that learners need to understand that $\sin 35°$ can be seen as a ratio or as a function, and that the orientation which they adopt, will depend on the task, or sub-task, at hand.

Teachers need to make learners know that there are two orientations, both equally valid, then they can make use of this resource in their thinking. As this happens, the orientations become explicit tools that learners can draw on consciously. There is a problem here though. Pournara (2001), states that in making the orientations explicit, they become objects of attention and therefore may become too visible (Lave and Wenger, 1991) which leads to the dilemma of transparency (Adler, 2001). In making the orientations visible, learners may focus on the orientations as ends, not means. Thus they may see the orientations, but not see them to the trigonometry (Pournara, 2001). Through continuous use of the orientations in a variety of different tasks, learners become familiar with them (Pournara, 2001), and ultimately the orientations become implicit again.

8.4.1.3 Classroom Strategies

The results of this research, has important implications for classroom use of textbooks as well. First teachers should become aware of the potential gaps in level in parts of the textbook and of the low level of thinking with which most of the exercises can be completed. Teachers must help develop strategies to help students to get as much as possible from the available textbooks. Some suggested strategies are given below:

a) The teacher can use the text as a follow-up to more exploratory activities in trigonometry.
b) Encourage learners to talk about trigonometric concepts, and to develop expressive language.
c) The teacher should be alert to possible misconceptions formed as a result of limited visual examples. To help students fill in level 0 understanding of trigonometric concepts, textbook presentations should be supplemented by manipulative models, and/or the computer.
d) To help students progress to level 1 thought, the teacher can supplement the one or few examples in the text by encouraging students to test many examples, with drawing or on the computer to determine if properties are true or false.
e) To help students progress to level 2 thought, the teacher can raise the level required in many routine exercises by asking "Why?", or "Explain your answer."
f) The teacher can revise or supplement tests to reflect higher levels of thinking.

8.4.2 Curriculum

Hirsch et al (1991) propose a trigonometry curriculum that is built around the graphing calculator. I would like to further propose that the use of computer software like *The Geometers' Sketchpad* or a graphing calculator to assist in the learning of this subject. This will not be possible in South Africa because of the cost of the graphing calculator and a computer. However, it may be possible to use the scientific calculator to the same benefit.

Have calls for change in texts in recent years produced any changes? Are there now any commonly used texts which involve more level 1 and level 2 thinking, and which are more consistent with the Van Hiele model? In particular, it would be interesting to look at some of the more innovative, less frequently used texts. The teacher's guide might be more explicit in identifying Van Hiele levels of parts of the text, and in helping teachers plan instruction to fill in levels and lead to a higher level of thinking. More attention should be given to selection of visual examples in lessons involving level 1 thought. More opportunity should be provided for students to advance to and use level 1 thinking. There are different ways of teaching concepts in mathematics. We need to be innovative in trying to use methods that will aid the understanding of mathematics, making it easier to visualise that which is abstract.

If the chalk and talk technique does not work we need to try other methods to assist the teaching and learning of mathematics. When the learner discovers something it is easier to recall and apply the concept as compared to just taking results for granted. Computer aided software like *Sketchpad*, provide visuals and easy to use techniques to enhance discovery learning.

8.4.3 Shortcomings of my research

- The learners had some familiarity with computers the findings may not necessarily generalise to learners who were not familiar with computers
- The study was done in one school, which was a reasonably homogenous group, with only six learners. It is therefore not possible to generalise the findings to larger, especially, non-homogenous groups.
- The study used task-based interviews with individual learners, which is very different from a classroom context. This was just an introductory activity and what needed is a longitudinal study, that is, learners engage using *Sketchpad* over time and to assess if they can develop a sound understanding of the unit circle definition.
- The study did not focus on modelling trigonometry functions and how this might motivate learners to learn trigonometry and aid conceptualisation.

8.4.4 Further research

- Further research would indicate whether similar results could be obtained within a classroom of learners plus non-homogeneous groups, instead of a one-to-one interview.

- Perhaps more relevant to the present classroom situation in many South African schools, an investigation can be carried out to ascertain whether these results are also true for the scientific or graphing calculator environment.

- Further research needs to be carried out in order to determine whether examination and test results improve if learners are exposed to these types of environments.

- To what extent is the analytical framework developed in this research useful in similar studies.

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APPENDIX A

LEARNERS' CONCEPTUALIZATION OF THE SINE FUNCTION DURING AN INTRODUCTORY ACTIVITY WITH SKETCHPAD AT GRADE 10 LEVEL

INTERVIEW SCHEDULE

1.1 What do you notice about the values of y and r respectively as the angle θ increases?

____

____

1.2 What do you notice about the values of $\frac{y}{r}$ and $\sin\theta$ in table 1?

____

____

2. What do you think will happen to the above ratios if we increase r to 2? Why?

____

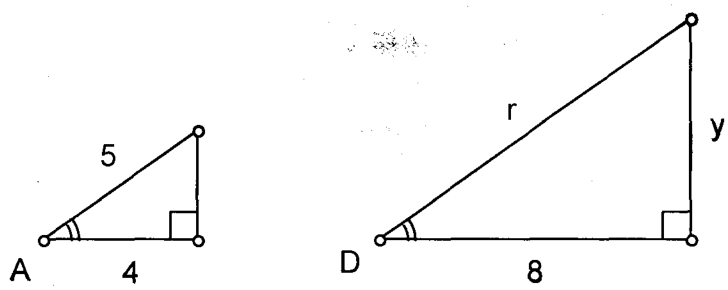
____

3. For any given angle, what do you notice about the corresponding values of $\frac{y}{r}$ in each table for $r = 2, r = 3, r = 4$?

____

____

4.



Find the ratio $\frac{y}{r}$ for the second right triangle below. (The angles at vertices A and D are equal). ____________________

5.1 Do you think this ratio $\frac{y}{r}$, for a given angle, will always remain constant irrespective of how r changes? ______________

5.2 What if $r = 2.1$? Or if $r = \pi$? Will it still be the same for a given angle?

____

5.3 Why? (Explain or justify your reasoning.)

____

____

6. Answer the following questions:

6.1 If $\sin(angle) = \frac{1}{2}$ then $angle =$ _______?

6.2 $\sin 35° =$ _______?

6.3 Estimate the value of the *angle* if $\frac{y}{r} = 0{,}55$

____

$r = 3$

θ	$\frac{y}{r}$	$\sin\theta$
10°		
20°		
30°		
40°		
50°		
60°		
70°		
80°		
90°		

$r = 4$

θ	$\frac{y}{r}$	$\sin\theta$
10°		
20°		
30°		
40°		
50°		
60°		
70°		
80°		
90°		

APPENDIX B

Researcher	Ok,Suren, what do you notice about the y and r value respectively as the angle θ increases?
Suren	….. (silence)…..as the degrees get higher, they both increase…….y and r increase
Researcher	What do you notice about the values of $\frac{y}{r}$ and $\sin\theta$ in each table?
Suren	Or you mean this part here? (pointing to the table). The first one is the same, the next one decreases by one and the next two, the rest are the same…no, er, these three are one lower and the next two are the same.
Researcher	What do you think will happen to the above ratios if we increase r to 2? Why?
Suren	The values will increase…it's like the same thing here (pointing to the first table), the first one stays the same, …..er…it just goes higher and lower.
Suren now completed the table by increasing r to 2, 3 and 4 respectively.	
Researcher	For any given angle, what do you notice about the corresponding values of $\frac{y}{r}$ in each table for r=2, r=3 and r=4?
Suren	Explain please ……(silence, smiling, thinking, looking at the table) …..oh…..they are the same….the both corresponding values are equal.
Researcher	Now Suren, for no. 4 ……(turn the page), we have two triangles here. Look at the two triangles ……I want you to find the ratio $\frac{y}{r}$ for the second right triangle below (the angles at vertices A and D are equal).
Suren	Can I use the computer? Must I use the computer?
Researcher	You want to use the computer?
Suren	No I am asking you, er ….. , does it relate to the computer?
Researcher	What do you think?
Suren	I ……I'll use the values (pointing to the given diagram)
Researcher	Yes, so you think this will relate to the computer?
Suren	Not if I need to start here ……(thinking) …..what do you mean by ratio?

Researcher	The ratio $\frac{y}{r}$, look at the figures you have filled in your table …… the length of y over the length of r in simplest form.
Suren	Eg. The y value…. $\frac{90°}{r}$ ……how do you work them out?
Researcher	If you look at your diagram, what do you see?
Suren	I can see a 90° angle here and another angle here.
Researcher	Ok, now you require $\frac{y}{r}$
Suren	I think y is 90°, it's an angle.
Researcher	You think y is an angle?
Suren	Ya, because it is 90°
Researcher	Will 90° be an answer for an angle or a length?
Suren	It's a length ….. it can be both.
Researcher	Can it be both?
Suren	No it can't be both, it can only be an angle.
Researcher	Why?
Suren	Because it is degrees
Researcher	So when we are talking about y and r, are we talking about two lengths or two angles?
Suren	No two lengths (confidently).
Researcher	So can you find the ratio $\frac{y}{r}$, in any way?….(long silence)….. A method?
Suren	Not that I know of…….(long silence)…….(thinking)
Researcher	Ok look, you got another triangle. Can you make a relationship with the other triangle?
Suren	They both are the same, only one is larger than the other….oh, and they have values. They both have 90°…… one is 5 for the r value…….(silence)
Researcher	Will the r value in the second one be the same as well?
Suren	No, it will be higher
Researcher	Why?

Suren	Because the triangle is bigger. (Silence) ....it will be 10.
Researcher	Why do you say 10?
Suren	Because I see this is doubled (pointing to 4 in the first triangle and 8 in the second), so I just suppose it will be doubled here too.
Researcher	Have you answered the question?
Suren	Must I use the triangle = $180°$ or must I use something else?
Researcher	What of the triangle = $180°$?
Suren	Angles
Researcher	Are we talking about angles here?
Suren	No we are talking about the lengths ......so ....you can say $180° - (8+10) =$ so $18 - 180$ will give you y.
Researcher	Oh, these are two lengths, 180 - .... is
Suren	An angle! .... oh....so no..... (silence)......(thinking)......there could be a way to do it, but I don't know it.....(long silence).
Researcher	......(pause)...Ok, what if the y is 3 (small triangle)
Suren	(Enthusiastically)....so y will be 9, because 8, 9, 10 (pointing to large triangle) and 3, 4, 5 (pointing to the smaller triangle).
Researcher	Look carefully at the diagram.
Suren	.... oh, it will be 6
Researcher	Why will it be 6?
Suren	Because all the other sides are doubled.
Researcher	Good, so $\frac{y}{r}$ is?
Suren	It could be half....no it is doubled.....that's $\frac{6}{10}$.
Researcher	Do you think this ratio $\frac{y}{r}$, for a given angle, will always remain constant irrespective of how r changes?
Suren	Er...what do you mean by constant?
Researcher	What do you think constant means?
Suren	Like repeating
Researcher	You feel constant means repeating?
Suren	Oh, staying the same.

Researcher	What if r=2.1? or if $r=\pi$? Will it still be the same for a given angle?
Suren	Yes
Researcher	Why? (Explain)?
Suren	Because we did all these three tables and they all are equal.
Researcher	Answer the following questions. If sin (angle)= $\frac{1}{2}$, then angle =
Suren	$\frac{1}{2}$. ….(silence)….I don't understand this question…..if we are talking about a triangle….then two $90°$ will give you 180.
Researcher	Then how are they related to half? Are they related in any way?
Suren	Ya, because half of 180 is $90°$.
Researcher	So what answer would you write there?
Suren	….. (Silence)……mmmmm
Researcher	Another way of writing half is?
Suren	0,5 ……so its this ……(looking at the table) the angle is $30°$. Oh yeh.
Researcher	Good, what do you think $\sin 35° =$?
Suren	……. Can look in these tables here?…….I can't tell you.
Researcher	Why?
Suren	Because there is no information here telling you of $35°$…….. , oh here it is here, $20°$.
Researcher	Is that 35
Suren	Oh no its 0.35. (Smiling), a ratio…….not an angle…….check it out on the computer……..?
Researcher	So?
Suren	Go to $35°$ (dragging point) ……. 0.58
Researcher	Good! Estimate the value of the angle if $\frac{y}{r} = 0.55$.
Suren	…. (Silence)……(rereading the question)…..you check it out in the computer?
Researcher	Ok….
Suren	$33°$ ….. your angle is $33°$.
Researcher	Thank you Suren
Suren	Ok, it's a pleasure!

Researcher	Ok, Mayuri, what do you notice about the values of y and r respectively as the angle θ increases?
Mayuri	……. (silence)……y and r changes as the angle increases , but the spaces in between them is not equal.
Researcher	The spaces inbetween them?
Mayuri	(referring to the table)….like if this is 0.1…0.34….(pointing at the ratios)….then this increases by like 2 …..here 0,6….. is less than 0.53?…..er….no…is 0.6 is larger than 0.53….ya!
Researcher	……ooh…ok…What do you notice about the r value? If you increase the angle?
Mayuri	There are changes happening ….. the ratio is going further up from the y……… the triangle gets different……..and the degrees and the ratio change
Researcher	Ok, what do you notice about the y value as the angle increases?
Mayuri	y is increasing……ya (smiling)
Researcher	And r?
Mayuri	r is remaining the same
Researcher	What do you notice about the values of $\frac{y}{r}$ and $\sin\theta$ in each table?
Mayuri	(looking at the table)……its not the same…….they both increase
Researcher	For each angle ……for each angle, what do you notice about $\frac{y}{r}$ and $\sin\theta$?
Mayuri	They are almost the same.
Researcher	Ok, What do you think will happen to the above ratios if we increase r to 2? Why?
Mayuri	(looking at table)………Each figure will increase about 4……each ratio…..around about.

Mayuri now completed the table by increasing r to 2, 3 and 4 respectively.	
Researcher	For any given angle, what do you notice about the corresponding values of $\frac{y}{r}$ in each table for r=2, r=3 and r=4.
Mayuri	(Looking at tables).....they not all the same......only 90° all over is the same......certain points are the same....three points.....its different because of the ratio.....can I check this one?....(corrects the ratio for 30°).
Researcher	What made you do that?
Mayuri	Because all of these are the same ....the ratios ......$\frac{y}{r}$ and the sin ..... I wanted to come up with the same.
Researcher	Ok, coming back to our question, what do you notice about the corresponding values of $\frac{y}{r}$ in each table for r=2, r=3 and r=4.
Mayuri	In not all cases, they are the same figures. I think when the ratio is a whole number, they may be the same.
Researcher	Now Mayuri, for no. 4 , we have two triangles here. Find the ratio $\frac{y}{r}$ for the second right triangle below (the angles at vertices A and D are equal). Tell me what you see.
Mayuri	Ok, they are both 90°, they are equal.... their angles are all equal......4 and 8 are double, so this will be 10 because that is 5. Can y be 90?
Researcher	Is y an angle or a length?
Mayuri	A length...oh they both are lengths (smiling)
Researcher	So $\frac{y}{r}$?
Mayuri	So we got two lengths, ok that's 10 that 8 ......er.......only **angles** of a triangle = 180° ....Theorem of Pythagoras?
Researcher	Ok, good
Mayuri	But I don't know it, can't remember it. ....
Researcher	....(long pause)....how are the sides related?
Mayuri	...5-4=1? No, not so easy....I don't know, I'll go with 1.

Researcher	…(long pause)…..Ok what if I give you 3. Then?
Mayuri	y = 6 (enthusiastically) $\frac{y}{r}=\frac{6}{10}$…… can't we simplify it?
Researcher	What do you think?
Mayuri	(silence)………(smiling)….leave it……it won't make a difference
Researcher	So if it won't make a difference, is it better to leave it or simplify it?
Mayuri	Leave it (laughing) …..
Researcher	Right, Thank you, now do you think this ratio $\frac{y}{r}$, for a given angle, will always remain constant irrespective of how r changes?
Mayuri	yes
Researcher	What if r=2.1? or if $r=\pi$? Will it still be the same for a given angle?
Mayuri	I think it should.
Researcher	Why? (Explain or justify your reasoning.)
Mayuri	Because as r is increasing the other values are increasing in proportion.
Researcher	Ok Mayuri, in no.6, Answer the following questions: if sin (angle)= $\frac{1}{2}$, then the angle = _______?
Mayuri	………(silence) …..sin?
Researcher	Tell me what you are thinking.
Mayuri	……….(Silence) ……can I just check it here?
Researcher	Check it.
Mayuri	Make it half (dragging point to half)………er …….30° !
Researcher	V good. Now what is $\sin 35° = ?$
Mayuri	…… (silence)……mmmm….change to 35° …..is it right?
Researcher	Is it right (smiling) …….what do you think?
Mayuri	No, because sin is not degrees ……now?
Researcher	Now? What do you think now?
Mayuri	Now …..(having obtained an angle of 35°.)…..it is 0,57.
Researcher	Nice, Ok, estimate the value of the angle if $\frac{y}{r}=0{,}55$.
Mayuri	…….. +(silence)…..you are finding the degrees?
Researcher	Yes, (after a long silence), try

Mayuri	50° ……..0.50?……..(silence)…… 28°.
Researcher	You think x will be 28°. Why?
Mayuri	Because here 0.5 for 30°, so 0.55 should be for 28°.
Researcher	You mean 0.55 should be less than, the answer should be less than 30°?
Mayuri	….er…half is 0.5……er….it should be 0.58………Ok, 0.58.
Researcher	0.58, you happy with that?
Mayuri	(Nodding her head) …..Yes.
Researcher	Ok, thank you very much, Mayuri.
Mayuri	Can I check it?
Researcher	You want to check it? …….. why did you not check it in the first place?
Mayuri	'Cause you had estimate ……
Researcher	Oh ……Ok ….. so then but then what did you realise about your estimation?
Mayuri	Now or ……..
Researcher	Now
Mayuri	That I was wrong.
Researcher	So what would you go with…. The computer or what you had?
Mayuri	The computer…computer learning is never wrong…. you put in the wrong input, it will give you the wrong output.
Researcher	So what is your final answer?
Mayuri	33°, can I change it?
Researcher	Change it …… OK, Thank you Mayuri.

Researcher	Ok, Nadeem, what do you notice about the values of y and the r respectively as the angle θ increases?
Nadeem	(silence)....can I see the table?
Researcher	You want to see the table ..... Ok
Nadeem	They increase as well (writes angles y and r increase)
Researcher	Nadeem, do you think y and r are angles?
Nadeem	...(silence).....er.....no they are lines......
Researcher	So what do you notice about y and r as the angle increases?
Nadeem	(Turns to computer) Can I try?
Researcher	Yes, you should.
Nadeem	(Drags to make angle larger) r value remains he same, and the y value increases.
Researcher	Ok Good, What do you notice about the values of $\frac{y}{r}$ and $\sin\theta$ in each table?
Nadeem	(Immediately without looking at the table).They increase....... As the angle increases, $\frac{y}{r}$ and $\sin\theta$ increase as well.
Researcher	Ok, Nadeem, if you look at the question it says: What do you notice about the values of $\frac{y}{r}$ and $\sin\theta$ in each table?
Nadeem	(looking at the table, then smiling) they are the same.
Researcher	You are smiling ...Why?
Nadeem	No ....., because, when you read the question over and over, then you realise what they are really asking.
Researcher	What do you think will happen to the above ratios if we increase r to 2? Why?
Nadeem	Must I try it out or just give you an answer?
Researcher	Ok, but first, what do you think will happen?
Nadeem	Er .....increase. The circle is getting bigger.
Nadeem then continued to complete the table by increasing r to 2, halfway through the second table, he said (surprised):	

Nadeem	Now I realise that the $\frac{y}{r}$ will still have the same ratio, because when you increase the r to 2, for example, y will increase as well.
Researcher	That's a very good observation.
Nadeem now completed the table by increasing r to 2, 3 and 4 respectively.	
Researcher	For any given angle, what do you notice about the corresponding values of $\frac{y}{r}$ in each table for r=2, r=3 and r=4?
Nadeem	…(silence)….they are the same. The values of $\frac{y}{r}$, in each table is the same.
Researcher	I find it interesting, you know, when I ask you a question on the table, you don't look at it to answer. Why is that?
Nadeem	I don't know ……. I am not sure. I assume, I take it for granted. The $\frac{y}{r}$ in each table, I know, is equal from the completion of the table, I remember. They are not talking about y alone and r alone? They are not talking about the ratio. I know that r is increasing, and y will increase too.
Researcher	So….
Nadeem	It will increase …. It will remain the same. The value of $\frac{y}{r}$ in each table is the same.
Researcher	Now Nadeem, for no. 4. Find the ratio $\frac{y}{r}$ for the second right triangle below (the angles at vertices A and D are equal).
Nadeem	The second triangle?
Researcher	Hmmm…
Nadeem	….. (Silence) …… $\frac{r}{y}$, oh you got that….can't work it out.
Researcher	Ok, tell me what you notice?
Nadeem	Its almost the same as the circle. Because you can use this as the r and this is the y (pointing to r and y respectively).
Researcher	What else do you have?

Nadeem	The 8. ……(long silence)
Researcher	What else do you have?
Nadeem	The small triangle.
Researcher	The small triangle …..Now why did you not look at the small triangle earlier?
Nadeem	Because we had to solve for the second right triangle.
Researcher	……. Ohhh …. Now you do not feel like the first triangle is related in some way to the second?
Nadeem	Ya, ……, the angles at vertices A and D are equal.
Researcher	So now, why do you think the triangle was placed there?
Nadeem	Mmmm, to help you out ……, but also to confuse you (smiling)…. (Both laugh)
Researcher	(smiling)…..Anyway, you feel that this triangle will help you?
Nadeem	Do you times it by two? Because 4 and 8 so 5 and 10, r will be 10.
Researcher	So continue…
Nadeem	That's why I say, I forgot to work out a triangle.
Researcher	You have two lengths there, so how do you work out the third side using the two given sides?
Nadeem	Is it Pythagoras …… the Theorem of Pythagoras.
Researcher	Yes, so ……
Nadeem	We did it in Std 6 – last year we didn't do it at all ……..can't remember…(thinking)……(long pause)…..
Researcher	……(long pause)….Ok fine, if I give you this as 3.
Nadeem	Then multiply by 2. so $\frac{y}{r} = \frac{6}{10}$. Simplify it?
Researcher	Yes!..(smiling)… ….Simplify it, you want to simplify it or …..
Nadeem	Ya …..$\frac{3}{5}$. You want the ratio.
Researcher	You happy with $\frac{3}{5}$?
Nadeem	No, I want to get like an answer. Like a number not a fraction.
Researcher	So a fraction is not a number?
Nadeem	No it is a number, but they asked for the ratio. Like the table.

Researcher	Ok …..
Nadeem	(simplifies $\frac{3}{5}$ to 0.6)
Researcher	Right, Thank you, now do you think this ratio $\frac{y}{r}$, for a given angle, will always remain constant irrespective of how r changes?
Nadeem	……. (silence)……..it will stay the same if y increases with the r.
Researcher	What if r=2.1? or if r= π ? Will it still be the same for a given angle?
Nadeem	What's π ?
Researcher	Ok π is an irrational number. You may be familiar with it as $\frac{22}{7}$. It can be approximated to 3.14…….. it is a non-terminating and non-repeating decimal.
Nadeem	Will the ratio? …… can I see on the computer?
Researcher	Yes
Nadeem	(Checks by dragging the point) yes
Researcher	Why? (Explain or justify your reasoning.)
Nadeem	It is the same because same answer as that one. Because r increases the y increases the same. So when you find $\frac{y}{r}$, it will always be the same. …… y decreases and r decreases but they do not decrease the same. Because they are different numbers. I am not too sure. Can I just try something else here?
Researcher	Ok …….
Nadeem	(dragging point)…. Now see, its 2 right and that 0.7. now bring it up and that's 3, that's 1.1 so this went up 1cm and that went up 0.3…..0.4. 'cause see 0.7 and 8.9,10 and 11.. so they are not actually increasing and decreasing in the same amount.
Researcher	So what are you looking for?
Nadeem	I am looking for a word that says it decreases ……. I can't get the word
Researcher	Proportionally …… it decreases proportionally?
Nadeem	Ya

Researcher	Ok Nadeem, in no.6 it says answer the following question. If $\sin angle = \frac{1}{2}$ then $angle =$ ______?
Nadeem	…… (long silence) ……. Can I try something out here?
Researcher	Ok
Nadeem	(Working on computer) …… the angle is $30°$.
Researcher	Good, why do you say that?
Nadeem	Because a half is 0.5. it says angle $30°$.
Researcher	How would you find sin 35°.
Nadeem	Use the computer again?
Researcher	Use the computer again.
Nadeem	(dragging point)…… is sin the angle?
Researcher	Sin value indicates the y value divided by the r value of the angle.
Nadeem	(using the computer)…..0.57
Researcher	Ok, estimate the value of the angle if $\frac{y}{r} = 0.55$.
Nadeem	So I must take a guess?
Researcher	Must you guess?
Nadeem	Because it says estimate. Or must I use the computer?
Researcher	Ok, what do you think you will do?
Nadeem	……. (silence)……because it says estimate I will do it myself.
Researcher	Ok.
Nadeem	…….. (silence)……
Researcher	What are you thinking?
Nadeem	I am thinking how to work it out.
Researcher	Ok
Nadeem	If 0.5 is 30 and 0.57 is 35 , so 0.55 will be, I am just like……. I am like blocked. ……(silence)……if I have to estimate, I will say $34°$
Researcher	How did you get $34°$?
Nadeem	Mmmmm, like for every 0.02, there is $1°$, I am just saying for example, so then 0.56, it will be 33. I tried using 1.5.
Researcher	You settle for that answer?
Nadeem	Just now …….. I will leave it at $34°$

Researcher	Ok, you happy with that?
Nadeem	I am happy with that.
Researcher	Ok, thank you, Nadeem.
Nadeem	Ok. Thanks.

Researcher	Ok, Vishen, what do you notice about the y value and the r value as the angle θ increases
Vishen	I don't understand.
Researcher	Ok, what do you notice about the y value as the angle increases?
Vishen	y is that (pointing to y in the diagram)
Researcher	Yes
Vishen	The angle is straightening up and becoming a straight line. And the angle O becomes a straight line, OC. There won't be a triangle.
Researcher	Ok, now, what do you notice about the y value as the angle is increasing.
Vishen	Y value? Increasing or decreasing?
Researcher	You must tell me, what do you notice about the y value as the angle is increasing?
Vishen	As the angle is going up or down?
Researcher	As the angle is going up.
Vishen	Can I check it?
Researcher	Yes
Vishen	So the y value is GH.
Researcher	Yes
Vishen	So when it goes up (dragging point up), the y value is increasing.
Researcher	What do you notice about the r value?
Vishen	So the r value is OG, as the angle increases?
Researcher	As the angle increases
Vishen	(working on the computer) Stays the same. When the angle increases r stays the same.

Researcher	What do you notice about the values of $\frac{y}{r}$ and $\sin\theta$ in each table?......
Vishen	They are the same.
Researcher	What do you think will happen to the above ratios if we increase r to 2? Why?
Vishen	When we increase just say, 10 to 20, it was 0.17 to 0.34. so this also changed to 0.17 to 0.35? Why? Can I check it?
Researcher	Ok, check it again.
Vishen	(checks using the computer) for $20°$ it is 0.34 also.
Researcher	What do you think will happen to the above ratios if we increase r to 2? Why?
Vishen	When we increase just say, 10 to 20, it was 0.17 to 0.34. so this also changed to 0.17 to 0.34. so I think if we change r to 2, the sine value and the r. $\frac{y}{r}$ value will increase proportionate to that.
Vishen now completed the table by increasing r to 2, 3 and 4 respectively.	
Researcher	For any given angle, what do you notice about the corresponding values of $\frac{y}{r}$ in each table for r=2, r=3 and r=4?
Vishen	As $\frac{y}{r}$ increases the value of sine increases with it.
Researcher	So the $\frac{y}{r}$ for each angle?
Vishen	They don't increase by much. r=4.2 ad r= 2 are equal, but r=5 changes by 1, except for $90°$, which is 1 in each case.
Researcher	Now Vishen, for no. 4 Find the ratio $\frac{y}{r}$ for the second right triangle below (the angles at vertices A and D are equal).
Vishen	(silence)......no computer?
Researcher	Will you have to use the computer?
Vishen	(rereading the question), ok this is 10
Researcher	How did you get that?
Vishen	Because from the information earlier, when this doubles, the other

	doubles also. So for 4 it will be 8 and therefore 5 will be 10. now I must find y? must I find out how many degrees?
Researcher	Can y be in degrees?
Vishen	Ya, it's a right-angled triangle.
Researcher	y and r represent length or angles?
Vishen	Length
Researcher	So
Vishen	r is the hypotenuse? Can I use the Pythagoras Theorem?
Researcher	What do you think?
Vishen	Yes, you use Pythagoras to find the hypotenuse. But I don't remember........(silence)
Researcher	Can you use Pythagoras to find the other side of a right angled triangle, given the hypotenuse and another side?
Vishen	Our teacher showed us how to find the hypotenuse, he did not show us how to find the other side.
Researcher	Ok
Vishen	Will the information be in the table?
Researcher	What do you think?
Vishen	I think yes.
Researcher	Why do you think yes?
Vishen	Because I found this and that in the table. (pointing to tables)
Researcher	Ok, looking at the smaller figure, can you find y there?
Vishen	.....(long silence).....(thinking)..........can't....don't know
Researcher	....(long pause)....Ok,....What if I tell you it is 3.
Vishen	Then this will be 6. So $\frac{y}{r} = \frac{6}{10}$.
Researcher	Right, Thank you, now do you think this ratio $\frac{y}{r}$, for a given angle, will always remain constant irrespective of how r changes?
Vishen	So if r changes like to 20, then 6 changes to like 12. is that what you are asking? Yes I think, if r doubles, then y doubles.
Researcher	What if r=2.1? or if r=π? Will it still be the same for a given angle?

Vishen	We are talking about? $\frac{y}{r}$? What is π? π is the circumference?
Researcher	π is $\frac{circumference}{diameter}$. Its represented by?
Vishen	22?.....$22\frac{1}{2}$?
Researcher	The irrational number ....$\frac{22}{7}$, or the also represented by 3.14....
Vishen	Ok, so if it is any number, 2.1 or π, will the ratio still be the same? I think yes.
Researcher	Why? (Explain or justify your reasoning.)
Vishen	As the ratios increased in the past exercises, if one ratio had to change, as in y, the r will change. It will be in proportion with it ....(thinking). When we are talking about it will remain the same, are we talking about it will change, but in the same proportion. The number will double, but it will still stay in the same multiples.
Researcher	Yes, Vishen, in no.6 it says answer the following question. If the sin(angle) = $\frac{1}{2}$, then angle = ______?
Vishen	0,5
Researcher	Then the angle = 0.5?
Vishen	Er........(silence).......if sin is half....then the angle........you are asking how many degrees the angle will be?
Researcher	Yes......Yes
Vishen	(working with the computer)......mmmm, if the angle is equal to half......45°
Researcher	45° why do you say 45°?
Vishen	Because when the angle increases to 90°, it is 1. half of 90° is 45°, if $\frac{y}{r}$=0.5.
Researcher	Now, $\sin 35°$=?
Vishen	$\frac{y}{r}=\frac{35}{1}$? Here you want the ratio? Question 6.1 you wanted the angle?

Researcher	Yes
Vishen	Ok, 0.57, because 30 and 40 have a 14 number difference, in terms of the ratio. So the answer is 0.57 using the ratio. Is it right?
Researcher	Yes, now Vishen, estimate the value of x if $\frac{y}{r} = 0.55$.
Vishen	Angle? 0.55. not 55° (looking at the table), will it be 35°?
Researcher	Why do you say 35°?
Vishen	If you increase the ratio by 5, will the angle grow or shrink?
Researcher	If you increase the ratio by 5, you want to know if the angle will grow or shrink?
Vishen	Can I test it out on the computer?
Researcher	Yes
Vishen	I could have tested the other ones out also?
Researcher	What do you think? Why did you not do that earlier?
Vishen	I thought I was not allowed to use the computer?
Researcher	Why did you think that.
Vishen	Because for one of these questions I was not allowed to use the computer. So I thought that for all the others, I could not use the computer. So can I go back and check my answer?
Researcher	What do you think?
Vishen	Yes
Researcher	What are you checking?
Vishen	When the ratio becomes 0.55 I am checking the value of the angle. .... Its 33°. Is it right?
Researcher	Yes ...... Thank you Vishen
Vishen	Thank you.

Researcher	Ok, Perusha, what do you notice about the values of y and r respectively as the angle θ increases?

Perusha	(silence)……ok, as it increases, y gets higher and r stays the same.
Researcher	What do you notice about the values of $\frac{y}{r}$ and $\sin\theta$ in each table?……
Perusha	Answer for $\frac{y}{r}$ and $\sin\theta$ are almost the same.
Researcher	What do you think will happen to the above ratios if we increase r to 2? Why?
Perusha	R to 2? You mean….increase. the circle will increase….er, the , I suppose the answers for the ratios will increase. The total degrees will get higher by probably 2. $\frac{y}{r}$ will get higher.
Researcher	Why do you think the ratios will increase?
Perusha	If the radius gets higher all the other ones, the ratios will increase also.
Perusha now completed the table by increasing r to 2, 3 and 4 respectively.	
Researcher	For any given angle, what do you notice about the corresponding values of $\frac{y}{r}$ in each table for r=2, r=3 and r=4?
Perusha	They are almost the same.
Researcher	Now Perusha, for no. 4 ……(turn the page), we have two triangles here. Find the ratio $\frac{y}{r}$ for the second right triangle below (the angles at vertices A and D are equal).
Perusha	Ok, (looking to computer), must I get the $\frac{y}{r}$. (trying to get the sketchpad triangle look almost the same as the second triangle)………(long silence)……..is there a y there?
Researcher	You want to know if there is a y there? So ……what do you think?
Perusha	(smiling)….I think this is bad.
Researcher	(laughing together)…… you think this is bad?….Ok……alright…….anything else?
Perusha	(smiling)……..I think this is very bad.
Researcher	(laughing), Ok………show me $\frac{y}{r}$. Yes….anyway in which you could

	find your y or r value?
Perusha	Something to do with the D…I know ……What will happen if we draw the whole circle around?
Researcher	What will happen if you draw the circle around it? Ok, you think it will make it easier?
Perusha	No, because I wont be able to move the radius up and down the page.
Researcher	Ohhh.. Ok….
Perusha	The 8 is confusing me.
Researcher	The 8 is confusing you. What do you think its there for?
Perusha	It is the x-axis. I am thinking $\frac{8}{90°}$.
Researcher	Ok, you are happy with your answer?
Perusha	No, Ok …..anything you can think of?
Researcher	Ok …… you can write your answer you feel that is right.
Perusha	How am I going to divide $\frac{90°}{8}$?
Researcher	You want to know how to divide it? Is that the problem?
Perusha	No, it is too hard to divide.
Researcher	Ok, its too hard to divide …..so you say you don't want to write $\frac{90°}{8}$ because it is too hard to divide?
Perusha	The 90° you can't divide it by a normal number.
Researcher	Ok, because 90 is a
Perusha	Degree
Researcher	And 8 is …..
Perusha	A length
Researcher	Any other way?
Perusha	Because there is 5, I suppose it will be something over 5 ……but I don't know what's the something.
Researcher	And ……..
Perusha	8 divided by 4 will give you the y.
Researcher	Ok, all right Perusha……that was interesting
Perusha	It was bad.

Researcher	It was bad? Why because you feel ……
Perusha	Because I didn't get the answer.
Researcher	Oh, ok …..Now, do you think this ratio $\frac{y}{r}$, for a given angle, will always remain constant irrespective of how r changes?
Perusha	No
Researcher	What if r=2.1? or if $r=\pi$? Will it still be the same for a given angle?
Perusha	……(silence) ya it will.
Researcher	Why? (Explain or justify your reasoning.)
Perusha	'Cause the r remains the same. Ok, as you change the size of the circle …..r will not remain the same.
Researcher	So…. Is that the reason?
Perusha	No …….because if you only change the size of the circle then only will the radius change.
Researcher	Ok, Perusha, now for number 6. answer the following questions: if sin (angle) = $\frac{1}{2}$, then angle =
Perusha	……..(silence)…..0.5
Researcher	Ok, $\sin 35° =$ ________
Perusha	0,57°
Researcher	Ok Perusha, estimate the value of x if $\frac{y}{r} = 0.55$.
Perusah	The angle? (going to the computer)….ok, that the angle right?
Researcher	Right.
Perusha	And the radius, can I use the computer?
Researcher	Hmmmm…..
Perusha	(dragging, stopped)…..0.55, then it is 33°
Researcher	Ok, good, thank you very much Perusha

Researcher	Ok, Omika, what do you notice about the y and r value as the angle θ increases?
Omika	Mmmm, the values increase, y and r.

Researcher	What do you notice about the values of $\frac{y}{r}$ and $\sin\theta$ in each table?
Omika	Mmmmmm, its like the same.....it's not exactly the same, some are exactly the same......but some are like below or above the value.
Researcher	What do you think will happen to the above ratios if we increase r to 2? Why?
Omika	Mmmmm, the ratios will be more than what they are.(after completing table 2)...wont it be the same.....because every time I hold it to $10°$ for example, I notice both the values are the same.
Omika now completed the table by increasing r to 2, 3 and 4 respectively.	
Researcher	For any given angle, what do you notice about the corresponding values of $\frac{y}{r}$ in each table for r=2, r=3 and r=4?
Omika	The values are similar, its either one below or one above.
Researcher	Now Omika, for no. 4. Find the ratio $\frac{y}{r}$ for the second right triangle below (the angles at vertices A and D are equal).
Omika	(silence)......what is 4 and 8?
Researcher	The length of the side of the triangle, the x value.
Omika	....(silence)..... $\frac{10}{8}$
Researcher	You said 10 so quickly, why did you say 10?
Omika	'Cause there is 5 and 4 here (pointing to small triangle) and 2X4 is 8, so 5X2 is 10.
Researcher	So your $\frac{y}{r}$ is $\frac{10}{8}$?
Omika	(silence)....... Is this $\frac{y}{r}$?
Researcher	Which is your $\frac{y}{r}$?
Omika	This is the value for y and this for r. (pointing them out correctly in the diagram).
Researcher	Ok ......
Omika	(smiling) I am not too good with geometry.

Researcher	That's Ok, just tell me what you are thinking, its not a test.
Omika	I am not sure, maybe, er …..r is 10
Researcher	So you know r is 10?
Omika	(smiling), but I don't know what y is………. I have no idea what to do.
Researcher	You have no idea what to do? …….Ok……..
Omika	I know $\frac{10}{8}$ is not right. But I know r is 10………I'll just guess anything……this is 4 and this is 5 (pointing to small triangle), then, this is 6 or it will be 3.
Researcher	So what you think it will be?
Omika	It looks small………I mean it looks smaller than this one, so I will go with 3 and this one will be 6.
Researcher	So………..
Omika	So $\frac{y}{r}=\frac{6}{10}$
Researcher	Right, Thank you, now do you think this ratio $\frac{y}{r}$, for a given angle, will always remain constant irrespective of how r changes?
Omika	Ya, Yes
Researcher	What if r=2.1? or if $r=\pi$? Will it still be the same for a given angle?
Omika	……I am not sure. What is π?
Researcher	π is the irrational number ….. you may be familiar with it as $\frac{22}{7}$ or 3.14…..
Omika	Right, …… right
Researcher	What if r=2.1? or if $r=\pi$? Will it still be the same for a given angle? Why?
Omika	Mmmm, the number will always be double the amount of how you want it?
Researcher	Of what? How you mean?
Omika	Er, if you like want the triangle to be bigger, you can times it by 2 and the number will get doubled or if you want it 3 times bigger, it

	will always be times by 3. …….. The number will be bigger, but if you divide it like you want it 3X bigger, you can divide it by 3 and you will still get the same number.
Researcher	Ok, Omika
Researcher	Now for number 6. answer the following questions: if sin (angle) = $\frac{1}{2}$, then angle =
Omika	$\frac{1}{2}$? $\frac{1}{2}$.
Researcher	Why……
Omika	These are the same…….(pointing to $\frac{y}{r}$ and $\sin\theta$ in the table), so this will be the same too.
Researcher	You happy with the answer?
Omika	You want angles, degrees……50° ?, cos its half.
Researcher	$\sin 35° = ?$
Omika	$\frac{35}{100}$? Because its degrees……….(looking at tables)……..for 30° you get 0,5.
Reasearcher	So $35° = ?$
Omika	0,58 or something.
Researcher	You estimated?
Omika	No, 15 divided by 2 is $7\frac{1}{2}$. Then 57.5………0.57
Researcher	Estimate the value of the angle if $\frac{y}{r} = 0.55$.
Omika	35°.
Reasearcher	Why do you say 35°.?
Omika	Because 0.5 is 30, so I just guessed 35°.
Researcher	Ok! Thank you, Omika
Omika	(smiling)…..Thank you

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Relationship between $\sin\theta$ **and** $\frac{y}{r}$.

$r = 1$

θ	$\frac{y}{r}$	$\sin\theta$
10°		
20°		
30°		
40°		
50°		
60°		
70°		
80°		
90°		

Relationship between $\sin\theta$ **and** $\frac{y}{r}$.

$r = 2$

θ	$\frac{y}{r}$	$\sin\theta$
10°		
20°		
30°		
40°		
50°		
60°		
70°		
80°		
90°		