I confirm that the student has carefully and thoroughly carried out the corrections by the two examiners.
O.T. Mewomo (Supervisor)

Self-adaptive inertial algorithms for approximating solutions of split feasilbility, monotone inclusion, variational inequality and fixed point problems

by

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As the candidate's supervisor, I have approved this dissertation for submission

Prof. O. T. Mewomo

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Dedication

This dissertation is dedicated to the Almighty Allah, my wife, parents and siblings.

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Abstract

In this dissertation, we introduce a self-adaptive hybrid inertial algorithm for approximating a solution of split feasibility problem which also solves a monotone inclusion problem and a fixed point problem in p-uniformly convex and uniformly smooth Banach spaces. We prove a strong convergence theorem for the sequence generated by our algorithm which does not require a prior knowledge of the norm of the bounded linear operator. Numerical examples are given to compare the computational performance of our algorithm with other existing algorithms.

Moreover, we present a new iterative algorithm of inertial form for solving Monotone Inclusion Problem (MIP) and common Fixed Point Problem (FPP) of a finite family of demimetric mappings in a real Hilbert space. Motivated by the Armijo line search technique, we incorporate the inertial technique to accelerate the convergence of the proposed method. Under standard and mild assumptions of monotonicity and Lipschitz continuity of the MIP associated mappings, we establish the strong convergence of the iterative algorithm. Some numerical examples are presented to illustrate the performance of our method as well as comparing it with the non-inertial version and some related methods in the literature.

Furthermore, we propose a new modified self-adaptive inertial subgradient extragradient algorithm in which the two projections are made onto some half spaces. Moreover, under mild conditions, we obtain a strong convergence of the sequence generated by our proposed algorithm for approximating a common solution of variational inequality problems and common fixed points of a finite family of demicontractive mappings in a real Hilbert space. The main advantages of our algorithm are: strong convergence result obtained without prior knowledge of the Lipschitz constant of the the related monotone operator, the two projections made onto some half-spaces and the inertial technique which speeds up rate of convergence. Finally, we present an application and a numerical example to illustrate the usefulness and applicability of our algorithm.

Contents

Declaration

This dissertation has not been submitted to this or any other institution in support of an application for the award of a degree. It represents the author's own work and where the work of others has been used, proper reference has been made.

Abd-semii Oluwatosin-Enitan Owolabi

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Contributed papers from the dissertation

Papers from the dissertation submitted and still in the refereeing process.

1. **A.O. Owolabi,** A. Taiwo, L.O. Jolaoso and O.T. Mewomo, A self-adaptive hybrid inertial algorithm for Split Feasibility Problems in Banach spaces, Submitted to Acta Mathematica Scientia.

2. **A-O-E Owolabi** T.O. Alakoya, A. Taiwo and O.T. Mewomo, A new inertial-projection algorithm for approximating common solution of variational inequality and fixed point problems of multivalued mappings, Submitted to Inverse Problems and Imaging.

3. **A-O-E-Owolabi**, A. Taiwo, L.O. Jolaoso and O.T. Mewomo, An inertial algorithm with line search technique for solving monotone inclusion and fixed point problems in Hilbert spaces, Submitted to Mathematica Slovaca

CHAPTER 1

Introduction

1.1 Background of study

Let E_1 and E_2 be real Banach spaces, C and Q be nonempty closed convex subsets of E_1 and E_2 respectively and $A : E_1 \to E_2$ be a bounded linear operator. The Split Feasibility Problem (SFP) is defined as follows:

$$\text{Find } \; x \in C \; \text{ such that } \; Ax \in Q. \tag{1.1.1}$$

The concept of the SFP was introduced in finite dimensional Hilbert spaces by Censor and Elfving [22] in 1994. The problem has recently attracted much attention from several researchers due to its application in modelling inverse problems which arise from signal processing, radiotherapy and data compression(see, for example [6, 21, 23, 68, 89, 91]).

Let C be a nonempty closed convex subset of a real Hilbert space H, $A : H \to H$ be a nonlinear operator, $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ be the norm and inner product on H respectively. The Variational Inequality Problem (VIP) is defined as follows:

$$\text{Find } \; \bar{x} \in C \; \text{ such that } \; \langle A\bar{x}, x - \bar{x} \rangle \geq 0. \tag{1.1.2}$$

We denote the solution set of (1.1.2) by VI(C, A). The problem (1.1.2) was proposed in the early 1960's by Stampacchia [93] to study some certain problems relating to partial differential equations. Eventually, applications of VIP was found in several fields such as optimization, nonlinear programming, mechanics, among others. Consequently, it has been studied in both finite and infinite dimensional spaces by many authors. Furthermore, various algorithms such as extragradient method, subgradient extragradient, among others have been proposed and modified by several authors for solution of VIPs (see, for instance [55, 25, 72, 73]).

The Monotone Inclusion Problem (MIP) which is known as an essential generalisation of VIP is defined as follows :

$$\text{Find} \quad z \in H \quad \text{such that} \quad 0 \in (A+B)z, \tag{1.1.3}$$

where $A : H \to H$ and $B : H \to 2^H$ are monotone and set-valued maximal monotone operators respectively. Several methods have been proposed and improved by numerous researchers for solving MIP (see, for instance [32, 34, 35, 70]).

There are several efficient methods for finding solutions of VIPS and MIPs. The fixed point method which is one of the most efficient methods for solving VIPs and MIPs, have been employed by numerous researchers using different techniques.

A point $x \in H$ is said to be a fixed point of T if

$$Tx = x, \tag{1.1.4}$$

where H is a real Hilbert space, and $T : H \to H$ is a nonlinear operator. For a given multi-valued mapping $T : X \to 2^X$, where X is a nonempty set, the point $x \in X$ is said to be a fixed point of T, if $x \in Tx$. We denote the set of fixed points of T by $F(T)$. The fixed point theory is one of the most dynamic areas of research which plays significant roles in many theoretical and applied field of mathematics such as nonlinear analysis. Furthermore, it has broad applications which can be employed in proving the existence and uniqueness of solutions of various mathematical problems. Thus, it is known as kernel of modern nonlinear analysis. As a result of the development of various effective methods employed by researchers for computing fixed points, the importance of application of fixed point theory has significantly increased. The uniqueness and existence of a fixed point plays vital roles in many areas of mathematics. For instance, given an initial value problem

$$\begin{cases} \frac{dx(t)}{dt} = f(t, x(t)), \\ x(t_0) = x_0. \end{cases} \tag{1.1.5}$$

The problem (1.1.5) can be solved by finding the solution of the equation

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds. \tag{1.1.6}$$

In order to establish the existence of the solution of nonlinear differential equation (1.1.5), we consider the operator $T : c([a, b]) \to c([a, b])$ defined by

$$Tx = x_0 + \int_{t_0}^{t} f(s, x(s))ds. \tag{1.1.7}$$

If x is a solution of the problem (1.1.6), then x is a fixed point of T. Hence, finding a solution of (1.1.5) is tantamount to finding a fixed point of T.

It is our intention in this dissertation to further develop, generalize and contribute to the study of fixed point theory to optimization and fixed point problems of nonlinear operators in Hilbert and Banach spaces.

1.2 Research motivation

The SFP introduced by Censor and Elfving [22] in 1994 is defined as follows:

$$\text{Find} \quad x^* \in C \quad \text{such that} \quad Ax^* \in Q. \tag{1.2.1}$$

Let H be a real Hilbert space, F be a strictly convex, reflexive smooth Banach space, J_F denotes the duality mapping on F, C and Q be non-empty closed convex subsets of H and F, respectively. The following algorithm was proposed by Alsulami and Takahashi [4] in 2015: For any $x_1 \in H$,

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)P_C(x_n - rA^*J_F(I-P_Q)Ax_n), \quad n \geq 1, \tag{1.2.2}$$

where P_C is the orthogonal projection. It was proved that for some $a, b \in \mathbb{R}$, if $0 < a \leq \alpha_n \leq b < 1$ and $0 < r\|A\|^2 < 2$, where $0 < r < \infty$ and $\{\alpha_n\} \subset [0,1]$, then $\{x_n\}$ weakly converges to $\omega_0 = \lim_{n\to\infty} P_{C\cap A^{-1}Q}x_n$, where $w_0 \in C \cap A^{-1}Q$. Furthermore, they introduced the following Halpern's type iteration in order to obtain strong convergence result. Let $\{t_n\}$ be a sequence in H such that $t_n \to t \in H$ and $x_1, t_1 \in H$,

$$\begin{cases} \nu_n = \lambda_n t_n + (1-\lambda_n)P_C(x_n - rA^*J_F(I-P_Q)Ax_n), \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)\nu_n, \quad n \geq 1, \end{cases} \tag{1.2.3}$$

where $0 < r < \infty$ and $\{\alpha_n\} \subset (0,1)$. It was proved that the sequence $\{x_n\}$ defined by (1.2.3) converges strongly to a point $\omega_0 \in C \cap A^{-1}Q$, for some $\omega_0 = P_{C\cap A^{-1}Q}t_1$, $\forall\, a, b \in \mathbb{R}$ if $0 < r\|A\|^2 < 2$, $\lim_{n\to\infty}\lambda_n = 0$, $\sum_{n=1}^{\infty}\lambda_n = \infty$, and $0 < a \leq \alpha_n \leq b < 1$.

Recently, Suantai et al. 2019 [94] considered the following modified SFP:

$$\text{Find} \quad x \in F(T) \cap C \quad \text{such that} \quad Ax \in Q. \tag{1.2.4}$$

Clearly, when $F(T) = C$, then (1.2.4) reduces to (1.2.1). Under some suitable conditions, Suantai et al. [94] proved weak and strong convergence theorems using Mann's iteration and Halpern's type iteration process respectively for solving the modified SFP (1.2.4), where $T : C \to C$ is a nonexpansive mapping.

Motivated by the above results, we study the following modified SFP:

$$\text{Find} \quad x \in F(T) \cap C \quad \text{such that} \quad Ax \in B^{-1}(0), \tag{1.2.5}$$

where $B : E_2 \to 2^{E_2^*}$ is a maximal monotone operator. Also, We introduce a self-adaptive hybrid iterative algorithm for approximating a solution of problem (1.2.5) in p-uniformly convex and uniformly smooth Banach spaces. Our algorithm is designed such that its implementation does not require a prior knowledge of the norm of the bounded linear operator.

Next, we consider the MIP. One of the most famous methods for solving MIP is the forward-backward splitting method which have been modified and improved by several authors (for instance, see [103, 109]). The forward-backward splitting method and Armijo-line search technique have recently attracted much attention by several authors and has been modified in various forms to prove the weak convergence of the sequence generated by it (see [1, 11, 16, 30, 34, 35, 59]).

In order to obtain a strong convergence result, Thong and Cholamjiak [103] proposed a modified forward-backward splitting method (which we present in Chapter 4 of this dissertation) for solving the MIP.

Motivated by the above results and the current research interest in this direction, we propose an inertial algorithm with Armijo-line search technique for approximating solutions of MIP (where A is a Lipschitz continuous and monotone operator and B is a maximal monotone operator) and common fixed points of a finite family of demimetric mappings in real Hilbert spaces. Our algorithm is designed such that its convergence does not require the prior estimate of the Lipschitz constant of A in the MIP and we obtain a strong convergence theorem for the sequence generated by our algorithm. We further present some numerical examples to illustrate the performance of our method as well as comparing it with some related methods in the literature.

Another important problem that we consider in the study is the VIP. Several methods such as extragradient, subgradient, subgradient-extragradient, among others have been proposed and developed by many authors (see, for instance [2, 55, 25] and the references therein) for approximating the solution of VIP. Recently, Thong and Hieu [106] introduced the following viscosity-type subgradient extragradient algorithm for finding an element $q \in F(S) \cap VI(C, A)$:

Algorithm 1.2.1.
Let $S : H \to H$ be a $\lambda-$ demicontractive mapping and $I - S$ be demiclosed at zero, $A : H \to H$ be monotone and Lipschitz continuous and $f : H \to H$ be a contraction mapping. Let $x_1 \in H$, $\lambda_1 > 0$ and $\mu \in (0, 1)$. Compute x_{n+1} as follows:

Step 1. Calculate

$$y_n = P_C(x_n - \lambda_n Ax_n).$$

Step 2. Compute

$$z_n = P_{T_n}(x_n - \lambda_n Ay_n),$$

where $T_n = \{x \in H : \langle x_n - \lambda_n Ax_n - y_n, x - y_n\rangle \leq 0\}$.

Step 3. Compute

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)\big[(1 - \beta_n)z_n + \beta_n Sz_n\big],$$

and

$$\lambda_{n+1} = \begin{cases} \min\{\frac{\mu\|x_n - y_n\|}{\|Ax_n - Ay_n\|}, \lambda_n\}, & \text{if } \; Ax_n - Ay_n \neq 0, \\ \lambda_n & \text{otherwise.} \end{cases}$$

Set $n := n + 1$ and go to **Step1.**

Motivated by the work of Thong and Hieu [106] and the current research interest in this direction, in this dissertation, we propose a modified inertial viscosity subgradient-extragradient algorithm with self-adaptive step-size in which each of the two projections is made onto an half space. Furthermore, we prove that the sequence generated by our algorithm converges strongly to a common solution of VIP and FPP of a finite family of multivalued demicontractive mappings. Finally, we present an application and a numerical example to illustrate the usefulness and applicability of the algorithm.

1.3 Statement of problem

The following are the problems studied in this dissertation.

• Let E_1 and E_2 be p-uniformly convex and uniformly smooth real Banach spaces, C and Q be nonempty closed convex subsets of E_1 and E_2 respectively, E_1^* and E_2^* be the duals of E_1 and E_2 respectively, $T : C \to C$ be a Bregman weak relatively nonexpansive mapping, $A : E_1 \to E_2$ be a bounded linear operator with $A^* : E_2^* \to E_1^*$, $B : E_2 \to 2^{E_2^*}$ be a maximal monotone operator. We consider the following problem:

$$\text{Find } \ x \in F(T) \cap C \ \text{ such that } \ Ax \in B^{-1}(0).$$

• Let H be a real Hilbert space, C be a nonempty closed convex subset of H, $S_i : H \to H$ be a finite family of demimetric mappings with a constant l_i, for $i = 1, 2, \cdots, m$, $A : H \to H$ be Lipschitz continuous and monotone, $B : H \to 2^H$ be a maximal monotone mapping. We also consider the following problem:

$$\text{Find } \ x \in \Gamma = (A + B)^{-1}(0) \cap \bigcap_{i=1}^{m} F(S_i),$$

where $F(S_i)$ denotes the fixed point of S_i.

• Let C be a nonempty closed convex subset of a real Hilbert space H, $A : C \to H$ be monotone and Lipschitz continuous, B be maximal monotone, $CB(H)$ be families of nonempty closed bounded subsets of H, $S_i : H \to CB(H)$ be a multivalued demicontractive mapping, for each $i = 1, 2, \cdots, m$. Finally, we consider the following problem:

$$\text{Find } \ x \in \Gamma = VI(C, A) \cap \bigcap_{i=1}^{m} F(S_i), \tag{1.3.1}$$

where $F(S_i)$ is a fixed point of S_i.

1.4 Objectives

The main objectives of this dissertation are to:

(i) review some useful results on SFP, MIP and VIP,

(ii) propose, study and apply self-adapive inertial algorithms for approximating common solutions of SFP in Banach spaces, MIP and VIP in Hilbert spaces,

(iii) establish strong convergence of the sequences generated by our proposed algorithms,

(iv) study finite family of demimetric mappings and demicontractive mappings in Hilbert spaces,

(v) apply our results to certain optimization problems,

(vi) give some numerical examples to illustrate the performance of our methods as well as comparing them with some related methods in the literature.

1.5 Organization of dissertation

Subsequent chapters of this dissertation are organised as follows:

In Chapter 2, we recall some basic definitions and relevant preliminaries that are needed to establish our main results in this dissertation.

In Chapter 3, we propose and study a self-adaptive hybrid inertial algorithm for approximating solution of split feasibility problems in Banach spaces. Furthermore, we give numerical examples to show the efficiency of our proposed algorithm.

In Chapter 4, we introduce a new inertial algorithm with Armijo-line search technique for finding common solutions of monotone inclusion problems and fixed point of a finite family of demimetric mappings in a real Hilbert space. We prove strong convergence theorem of the sequence generated by our proposed iterative scheme. Also, we give an application and numerical examples to illustrate the effectiveness of our iterative algorithm.

In Chapter 5, we propose a new modified viscosity subgradient extragradient algorithm with inertial extrapolation. We state and prove that the sequence generated by our iterative scheme converges strongly to a common solution of variational inequality problems and fixed point of a finite family of demicontractive mappings in a real Hilbert space. Finally, we illustrate the usefulness of our proposed algorithm by giving numerical example.

In Chapter 6, we present the conclusion of our study. Furthermore, we highlighted the contributions of our study to knowledge and finally discuss some areas of future research.

CHAPTER 2

Preliminaries

In this chapter, we recall some basic definitions, notions and results that will be required to establish our main results.

2.1 Hilbert spaces

Definition 2.1.1. Let H be a complex vector space. An inner product on H is a function $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{C}$ such that(for all $x, y, z \in H, \mu, \lambda \in \mathbb{C}$):

(i) $\langle y, x \rangle = \overline{\langle x, y \rangle}$,

(ii) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0$,

(iii) $\langle \mu x, y \rangle = \mu \langle x, y \rangle$,

(iv) $\langle \mu x + \lambda y, z \rangle = \mu \langle x, z \rangle + \lambda \langle y, z \rangle$.

We observe from (i), (iii) and (iv) that the following properties hold:

(i) $\langle x, \mu y \rangle = \bar{\mu} \langle x, y \rangle$,

(ii) $\langle x, \mu y + \lambda z \rangle = \bar{\mu} \langle x, y \rangle + \bar{\lambda} \langle x, z \rangle$.

It is known that if the range of $\langle \cdot, \cdot \rangle$ is $\mathbb{R}$, then the pair $(H, \langle \cdot, \cdot \rangle)$ is called a real inner product space. Also, we note that an inner product on H gives rise to a norm defined as $\|x\| = \sqrt{\langle x, x \rangle}$. If the inner product space is complete with respect to this norm (that is every Cauchy sequence in H converges to a point in H), then it is called a Hilbert space.

2.1.1 Examples of Hilbert spaces

The following are examples of Hilbert spaces:

(i) $\mathbb{C}^n$ with inner product $\langle z, w\rangle = \sum_{i=1}^{n} z_i \bar{w}_i$ is a Hilbert space over $\mathbb{C}$, where $z = (z_1, z_2, \cdots, z_n)$ and $w = (w_1, w_2, \cdots, w_n)$.

(ii) $\mathbb{R}^n$ with inner product $\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$ is a Hilbert space over $\mathbb{R}$, where $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$.

(iii) $L^2[a, b]$ are Hilbert spaces with respect to the inner product

$$\langle f, g\rangle = \int f\bar{g},$$

where the integral is taken over an appropriate domain.

Lemma 2.1.1. *(Cauchy-Schwarz inequality) Let $(H, \langle\cdot,\cdot\rangle)$ be an inner product space, then*

$$|\langle x, y\rangle| \leq \|x\|\|y\|, \quad \forall \quad x, y \in H. \tag{2.1.1}$$

Definition 2.1.2. Let H be a Hilbert space and $C \subseteq H$. C is said to be convex if $(1-\sigma)x + \sigma y \in C$ for all $x, y \in C$ and for all $\sigma \in [0, 1]$.

Remark 2.1.2. An intersection of arbitrary family of convex subsets is convex.

Definition 2.1.3. Let X be convex subset of a Hilbert space H and $f : X \to \mathbb{R}$ be a mapping. f is said to be convex if

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y), \forall \lambda \in [0, 1] \text{ and } x, y \in X. \tag{2.1.2}$$

Lemma 2.1.3. *Let $C \subseteq H$ be convex, then its closure and interior are convex subsets.*

2.1.2 Some important nonlinear operators in Hilbert spaces

Definition 2.1.4. Let $A : H \to H$ be a nonlinear operator. Then A is

(i) Lipschitz continuous if for all $L > 0$,

$$\|Ax - Ay\| \leq L\|x - y\|, \quad \forall x, y \in H;$$

if $0 \leq L < 1$, then A is a contraction mapping,

(ii) monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in H;$$

(iii) $\tau-$ inverse strongly monotone($\tau - ism$) if for all $\tau > 0$,

$$\langle Ax - Ay, x - y\rangle \geq \tau\|Ax - Ay\|^2, \quad \forall x, y \in H;$$

(iv) $\beta-$strongly monotone if for all $\beta > 0$,

$$\langle Ax - Ay, x - y\rangle \geq \beta\|x - y\|^2, \quad \forall x, y \in H.$$

Remark 2.1.4. We note that every τ-inverse strongly monotone is monotone and $\frac{1}{\tau}$-Lipschitz continuous.

Definition 2.1.5. A multivalued mapping $A : H \to 2^H$ is said to be monotone if $\forall\ u \in A(x), v \in A(y)$,

$$\langle x - y, u - v\rangle \geq 0, \quad \forall x, y \in H.$$

Definition 2.1.6. Let $G(A)$ be the graph of A defined by $G(A) = \big\{(x, u) \in H \times H : u \in Ax\big\}$. A multivalued mapping $A : H \to 2^H$ is maximal if the graph $G(A)$ is not contained in the graph of any other monotone operator.

Clearly, a multi-valued monotone mapping is said to be maximal if and only if for any $(x, u) \in H \times H, (y, v) \in G(A), \langle x - y, u - v\rangle \geq 0$ implies $u \in Ax$.

Definition 2.1.7. Let H be a real Hilbert space H and $A : H \to 2^H$ be a multivalued mapping. Then the effective domain of A, denoted by $D(A)$ is defined as:

$$D(A) = \{x \in H \ : Ax \neq \emptyset\}.$$

Definition 2.1.8. Let $A : H \to 2^H$ be a maximal monotone operator. The resolvent operator of A denoted by J_r^A is defined by $J_r^A = (I + rA)^{-1}$, where $r > 0$ and I is the identity operator on H.

It is known that J_r^A is single-valued, firmly nonexpansive and nonexpansive (see [54]).

Definition 2.1.9. Let H be a real Hilbert space. A mapping $A : H \to H$ is called

(i) nonexpansive if

$$\|Ax - Ay\| \leq \|x - y\|, \quad \forall\ \ x, y \in H;$$

(ii) quasi-nonexpansive if $F(A) \neq \emptyset$ such that

$$\|Ax - Ay\| \leq \|x - y\|, \quad \forall\ \ x \in H, y \in F(A);$$

(iii) firmly nonexpansive if

$$\langle Ax - Ay, x - y\rangle \geq \|Ax - Ay\|, \quad \forall\ \ x, y \in H;$$

(iv) averaged if there exists a constant $\mu \in (0, 1)$ such that

$$A := (1 - \mu)I + \mu S,$$

where I is the identity operator and $S : H \to H$ is a nonexpansive mapping.

(v) α-strictly pseudocontractive if there exists a constant $\alpha \in [0, 1)$ such that

$$\|Ax - Ay\|^2 \leq \|x - y\|^2 + \alpha\|(I - A)x - (I - A)y\|^2, \quad \forall\, x, y \in H.$$

Remark 2.1.5. Obviously, from Definition 2.1.9(v), if $\alpha = 0$, then A is a nonexpansive mapping. Furthermore, it is known that firmly nonexpanive mappings are averaged and averaged mappings are nonexpansive mappings.

Definition 2.1.10. A mapping $T : H \to H$ is said to be a demicontractive mapping if there exists a constant $k \in [0, 1)$ and $F(T) \neq \emptyset$ such that

$$\|Tx - y\|^2 \leq \|x - y\|^2 + k\|x - Tx\|^2, \quad \forall\;\; x \in H, y \in F(T).$$

The following is an example of a demicontractive mapping.

Example 2.1.6. Let H be the real line and $C = [-1, 1]$. Let $T : C \to C$ be defined by

$$Tx = \begin{cases} \frac{3}{5}x\sin\left(\frac{1}{x}\right), & \text{if } \ x \neq 0, \\ 0 & \text{if } \ x = 0. \end{cases}$$

It is obvious that $F(T) = \{0\}$. We have for all $x \in C$

$$\begin{aligned} \|Tx - 0\|^2 &= \left\|\frac{3}{5}x\sin\left(\frac{1}{x}\right)\right\|^2 \\ &\leq \left\|\frac{3}{5}x\right\|^2 \\ &\leq \|x\|^2 \\ &\leq \|x - 0\|^2 + k\|Tx - x\|^2. \end{aligned}$$

Hence, T is k-demicontractive for all $k \in [0, 1)$.

Definition 2.1.11. Let H be a real Hilbert space. A mapping $T : H \to H$ is called a demimetric mapping if there exists a constant $k \in (-\infty, 1)$ and $F(T) \neq \emptyset$ such that

$$\|Tx - y\|^2 \leq \|x - y\|^2 + k\|x - Tx\|^2, \quad \forall\, x \in H, y \in F(T).$$

Next, we give an example of a demimetric mapping.

Example 2.1.7. Let H be a real line and $C = [-1, 1]$. Let $T : H \to H$ be defined on C by

$$Tx = \begin{cases} \frac{x}{3}cosx, & \text{if } \ x \neq 0, \\ 0, & \text{if } \ x = 0. \end{cases} \tag{2.1.3}$$

It is clear that $F(T) = \{0\}$, thus, for all $x \in C$, we have

$$\begin{aligned} \|Tx - 0\|^2 &= \left\|\frac{x}{3}\cos x\right\|^2 \\ &\leq \left\|\frac{x}{3}\right\|^2 \\ &\leq \|x\|^2 \\ &\leq \|x - 0\|^2 + k\|Tx - x\|^2. \end{aligned}$$

Thus, T is k-demimetric for all $k \in (-\infty, 1)$.

2.2 Metric projection

In this section, we briefly discuss the characterization of the metric projection in Hilbert spaces. This operator plays vital roles in establishing our results.

Definition 2.2.1. Let H be a Hilbert space and C be a nonempty closed convex subset of H. If for all $x \in H$, there exists a point $y \in C$ such that

$$\|x - y\| = \inf\{\|x - z\| : z \in C\}, \tag{2.2.1}$$

then, we say that y is a metric projection of x onto C and it is denoted by $P_C x$.

Definition 2.2.2. Let C be a nonempty closed convex subset of a real Hilbert space H, P_C be the metric projection onto C. It is known that P_C satisfies the following properties:

(i) $\langle x - y, P_C x - P_C y\rangle \geq ||P_C x - P_C y||^2$, for every $x, y \in H$;

(ii) for $x \in H$ and $z \in C$, $z = P_C x$ if and only if

$$\langle x - z, z - y\rangle \geq 0, \quad \forall y \in C; \tag{2.2.2}$$

(iii) for $x \in H$ and $y \in C$,

$$||y - P_C(x)||^2 + ||x - P_C(x)||^2 \leq ||x - y||^2. \tag{2.2.3}$$

2.2.1 Examples of metric projection

The following are examples of a metric projection:

(i) Suppose C is the range of a $m \times n$ matrix A with full column rank, then

$$P_C x = A(A^* A)^{-1} A^* x \tag{2.2.4}$$

is the metric projection onto C, where A^* is the adjoint of A.

(ii) Let $C = [a, b]$ be a closed rectangle in $\mathbb{R}^n$, where $a = (a_1, a_2, \cdots, a_n)^T$ and $b = (b_1, b_2, \cdots, b_n)^T$. The metric projection with the i^{th} coordinate denoted by $(P_C x)_i$ is given by

$$(P_C x)_i = \begin{cases} a_i, & x_i < a_i, \\ x_i, & x_i \in [a_i, b_i], \\ b_i, & x_i > b_i, \end{cases}$$

for $1 \leq i \leq n$.

(iii) Let $C = \{y \in H : \langle \alpha, y\rangle = \beta\}$ be a hyperplane with $\alpha \neq 0$, then the metric projection onto C is given by

$$P_C x = x - \frac{\langle \alpha, x\rangle - \beta}{\|\alpha\|^2}\alpha, \quad \forall\ \alpha \in \mathbb{R}.$$

2.3 Geometric properties of Banach spaces

In this section, we give some geometric properties of Banach spaces required in this dissertation. Throughout this section, we denote the dual of the normed linear space E by E^*.

Definition 2.3.1. Let E be a Banach space. The space of all continuous linear functionals on E with respect to the operator norm

$$\|f\| = \sup_{\|x\|_E=1} |f(x)|, \quad \forall \ \ x \in E, \tag{2.3.1}$$

is called the dual space of E and it is denoted by E^*.

The dual space of E^* is known as the bidual space of E and it is denoted by E^{**}.

2.3.1 Smooth spaces

Definition 2.3.2. [28] Let E be a normed linear space. E is called smooth if for every $x \in E$, there exists a unique element $x^* \in E^*$ such that $\|x^*\| = 1$, $\|x\| = 1$, and $\langle x, x^* \rangle = \|x\|$.

Definition 2.3.3. [28] Let E be a normed linear space with dim $E \geq 2$, then the modulus of smoothness of E is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\begin{aligned}\rho_E(\tau) &= \sup\left\{\frac{\|x-y\| + \|x+y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau\right\}\\ &= \sup\left\{\frac{\|x-\tau y\| + \|x+\tau y\|}{2} - 1 : \|x\| = 1 = \|y\|\right\}.\end{aligned}$$

Definition 2.3.4. A normed linear space E is said to be uniformly smooth if for any $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in E$, $\|x\| = 1, \|y\| \leq \delta$ then

$$\|x + y\| + \|x - y\| < 2 + \epsilon\|y\|.$$

Proposition 2.3.1. *[28] A normed linear space E is uniformly smooth if and only if*

$$\lim_{\tau \to 0^+} \frac{\rho_E(\tau)}{\tau} = 0.$$

Definition 2.3.5. Let E be a Banach space. E is said to be q-uniformly smooth if there exists a constant $D_q > 0$ such that

$$\rho_E(\tau) \leq D_q \tau^q, \quad \text{for} \quad q > 1 \quad \text{and} \quad \tau > 0.$$

2.3.2 Uniformly convex spaces

Definition 2.3.6. [28] A normed linear space E is said to be uniformly convex if for any $\epsilon \in (0, 2]$, there exists a $\delta(\epsilon) > 0$ such that $\|x\| \leq 1, \|y\| \leq 1$ and $\|x - y\| \geq \epsilon$, then

$$\|\frac{x+y}{2}\| \leq 1 - \delta, \quad \forall \ \ x, y \in E.$$

Also, A normed linear space E is said to be uniformly convex if for any $\epsilon \in (0, 2]$, there exists a $\delta(\epsilon) > 0$ such that $\|x\| = 1, \|y\| = 1$ and $\|x - y\| \geq \epsilon$, then $\|\frac{x+y}{2}\| \leq 1 - \delta$,for all $x, y \in E$.

Definition 2.3.7. [28] Let E be a normed linear space with dim $E \geq 2$, then the modulus of convexity of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : \|x\| = \|y\| = 1; \epsilon = \|x - y\| \right\}.$$

Theorem 2.3.2. *A normed linear space E is unifomly convex if and only if $\delta_E(\epsilon) > 0$, for all $\epsilon \in (0, 2]$.*

Definition 2.3.8. Let $p > 1$ be a real number. A normed linear space E is p-uniformly convex if there exists a constant $C_p > 0$ such that

$$\delta_E(\epsilon) \geq C_p \epsilon^p, \quad \text{for any} \ \ \epsilon \in (0, 2].$$

Definition 2.3.9. [28] A normed linear space E is called strictly convex if for all $x, y \in E$, $x \neq y$, with $\|x\| = \|y\| = 1$, then

$$\|\sigma x + (1 - \sigma)y\| < 1, \quad \forall \ \ \sigma \in (0, 1).$$

2.3.3 Reflexive Banach spaces

Definition 2.3.10. Let E^* be the dual of a Banach space E and E^{**} be the dual of E^*. Then there exists a mapping $J : E \to E^{**}$ defined by

$$J(x) = \Psi_x \in E^{**}, \quad \forall \ \ x \in E,$$

known as a canonical mapping (or canonical embedding), where $\Psi_x : E^* \to \mathbb{R}$ is given by

$$\langle \Psi_x, g \rangle = \langle g, x \rangle, \quad \text{for every} \ \ g \in E^*.$$

Hence, $\langle J(x), g \rangle \equiv \langle g, x \rangle$, for every $g \in E^*$. If the canonical mapping $J(x)$ is an onto mapping, then E is called reflexive. Hence a reflexive Banach space is the one in which the canonical mapping is onto.

It is generally known that the canonical mapping has the following properties:

(i) J is isometry i.e. $\|Jx\| = \|x\|$, $\forall \ x \in E$,

(ii) J is linear.

Remark 2.3.3.

(i) Every uniformly smooth space is smooth.

(ii) E is uniformly smooth if and only if E^* is uniformly convex.

(iii) Every uniformly convex space is reflexive.

(iv) If the dual space E^* is reflexive, then E is reflexive.

(v) Every uniformly convex space is strictly convex.

The following remark can be deduced from Remarks 2.3.3(ii), 2.3.3(iv) and 2.3.3(iii).

Remark 2.3.4. Every uniformly smooth space is reflexive.

2.3.4 Some basic notions in Banach spaces

Definition 2.3.11. The sub-differential of a function f is a function $\delta f : E \to 2^{E^*}$ defined by

$$\delta f(x) = \{x^* \in E^* : f(y) \geq f(x) + \langle y - x, x^* \rangle, \quad \forall \quad y \in E\}.$$

Definition 2.3.12. Let $p > 1$, the generalized duality mapping $J_E^p : E \to 2^{E^*}$ is defined by

$$J_E^p = \left\{x^* \in E^* : \langle x, x^* \rangle = \|x\|^p, \|x^*\| = \|x\|^{p-1}\right\}.$$

If $p = 2$, we have $J_E^2 = J$ and it is called the normalized duality mapping on E.

Remark 2.3.5. It is known that when E is uniformly smooth, then J_E^p is norm to norm uniformly continuous on bounded subsets of E and E is smooth if and only if J_E^p is single valued. Also, when E is reflexive and strictly convex then $J_E^p = (J_{E^*}^q)^{-1}$ is one-to-one and surjective, where $J_{E^*}^q$ is the duality mapping of E^*, for $1 < q \leq 2 \leq p$ and $\frac{1}{p} + \frac{1}{q} = 1$. (see [28]). Furthermore, it is known that if E is p-uniformly convex and uniformly smooth, then its dual space E^* is q-uniformly smooth and uniformly convex.

Definition 2.3.13. The duality mapping J_E^p is said to be weak-to-weak continuous if

$$x_n \rightharpoonup x \Rightarrow \langle J_E^P x_n, y \rangle \to \langle J_E^p x, y \rangle, \quad \text{for any} \quad y \in E.$$

It is known that $l_p (p > 1)$ has such property, but $L_p (p > 2)$ does not share this property.

Definition 2.3.14. [14] A function $f : E \to \mathbb{R} \cup \{+\infty\}$ is said to be

(i) proper if its effective domain $D(f) = \{x \in E : f(x) < +\infty\}$ is non-empty,

(ii) convex if $f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda) f(y), \ \forall \ \lambda \in (0, 1), x, y \in D(f)$,

(iii) lower semi-continuous at $x_0 \in D(f)$, if $f(x_0) \leq \lim_{x \to x_0} \inf f(x)$.

Definition 2.3.15. Let $x \in int(D(f))$, for any $y \in E$, the directional derivative of f at x denoted by $f^0(x,y)$ is defined by

$$f^0(x,y) := \lim_{t \to 0^+} \frac{f(x+ty) - f(x)}{t}. \tag{2.3.2}$$

If the limit at $t \to 0^+$ in (2.3.2) exists for each y, then the function f is said to be Gâteaux differentiable at x. In this case $f^0(x,y) = \langle \nabla f(x), y \rangle$, where $\nabla f(x)$ is the value of the gradient of f at x.

Definition 2.3.16. Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and convex function. The Bregman distance denoted as $\Delta_f : domf \times domf \to [0, +\infty)$ is defined as

$$\Delta_f(x,y) = f(y) - f(x) - \langle f^{'}(x), y - x \rangle, x, y \in E. \tag{2.3.3}$$

Note that $\Delta_f(x,y) \geq 0$ (see [12, 75]). It is worthy to note that the duality mapping J_E^p is actually the derivative of the function $f_p(x) = \frac{1}{p}\|x\|^p$ for $2 \leq p < \infty$. Hence, if $f = f_p$ in (2.3.3), the Bregman distance with respect to f_p now becomes

$$\begin{aligned}
\Delta_p(x,y) &= \frac{1}{q}\|x\|^p - \langle J_E^p x, y \rangle + \frac{1}{p}\|y\|^p \\
&= \frac{1}{p}\left(\|y\|^p - \|x\|^p\right) + \langle J_E^p x, x - y \rangle \\
&= \frac{1}{q}\left(\|x\|^p - \|y\|^p\right) - \langle J_E^p x - J_E^p y, x \rangle.
\end{aligned}$$

It is generally known that the Bregman distance is not a metric as a result of absence of symmetry, but it possesses some distance-like properties which are stated below:

$$\Delta_p(x,y) = \Delta_p(x,z) + \Delta_p(z,y) + \langle z - y, J_E^p x - J_E^p z \rangle, \tag{2.3.4}$$

and

$$\Delta_p(x,y) + \Delta_p(y,x) = \langle x - y, J_E^p x - J_E^p y \rangle.$$

The relationship between the metric and Bregman distance in p-uniformly convex space is as follow:

$$\tau\|x - y\|^p \leq \Delta_p(x,y) \leq \langle x - y, J_E^p x - J_E^p y \rangle, \tag{2.3.5}$$

where $\tau > 0$ is some fixed number.

Definition 2.3.17. Let C be a non-empty closed convex subset of a Banach space E. The Bregman projection denoted by Π_C is defined as

$$\Pi_C x = \arg \min_{y \in C} \Delta_p(y,x), \forall\ x \in E.$$

Definition 2.3.18. Let E be a Banach space and let C be a nonempty closed convex subset of E. Then the metric projection is given by

$$P_C x = \arg \min_{y \in C} \|y - x\|, \forall\ x \in E.$$

The Bregman projection is the unique minimizer of the Bregman distance and it is characterized by the following variational inequalities (see [85, 86]):

$$\langle J_E^p(x) - J_E^p(\Pi_C x), z - \Pi_C x\rangle \leq 0,\ \forall z \in C, \tag{2.3.6}$$

from which we have

$$\Delta_p(\Pi_C x, z) \leq \Delta_p(x, z) - \Delta_p(x, \Pi_C x),\ \forall z \in C. \tag{2.3.7}$$

The metric projection which is also the unique minimizer of the norm distance is characterized by the following variational inequality:

$$\langle J_E^p(x - P_C x), z - P_C x\rangle \leq 0,\ \forall z \in C. \tag{2.3.8}$$

Definition 2.3.19. Let E be a p-uniformly convex Banach space. The function V_p : $E \times E \to [0, \infty]$ associated with $f_p(x) = \frac{1}{p}\|x\|^p$ by

$$V_p(x, \bar{x}) = \frac{1}{p}\|x\|^p - \langle x, \bar{x}\rangle + \frac{1}{q}\|\bar{x}\|^q,\ x \in E,\ \bar{x} \in E^*, \tag{2.3.9}$$

where $V_p(x, \bar{x}) \geq 0$. It then follows that

$$V_p(x, \bar{x}) = \Delta_p(x, J_{E^*}^q(\bar{x})),\ \forall x \in E,\ \bar{x} \in E^*.$$

The following inequality was proved by Chuasuk et al [29]:

$$V_p(x, \bar{x}) + \langle J_{E^*}^q(\bar{x}) - x, \bar{y}\rangle \leq V_p(x, \bar{x} + \bar{y}),\ \forall x \in E, \bar{x},\ \bar{y} \in E^*.$$

Furthermore, V_p is convex in the second variable, and thus, for all $z \in E$, $\{x_i\}_{i=1}^N$, and $\{t_i\}_{i=1}^N \subset (0, 1)$, $\sum_{i=1}^N t_i = 1$ we have (see [88])

$$\Delta_p\left(z, J_{E^*}^q\left(\sum_{i=1}^N t_i J_E^p(x_i)\right)\right) = V_p\left(z, \left(\sum_{i=1}^N t_i J_E^p(x_i)\right)\right) \leq \sum_{i=1}^N t_i \Delta_p(z, x_i). \tag{2.3.10}$$

Let C be a non-empty, closed and convex subset of a smooth Banach space E and let $T : C \to C$ be a mapping. A point $x^* \in C$ is called an asymptotic fixed point of T if a sequence $\{x_n\}_{n\in\mathbb{N}}$ exists in C and converges weakly to x^* such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote the set of all asymptotic fixed points of T by $\hat{F}(T)$. Moreover, a point $x^* \in C$ is said to be a strong asymptotic fixed point of T if there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in C which converges strongly to x^* such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote the set of all strong asymptotic fixed points of T by $\tilde{F}(T)$. It follows from the definitions that $F(T) \subset \tilde{F}(T) \subset \hat{F}(T)$.

Definition 2.3.20. [81] Let C be a nonempty closed convex subset of a Banach space E. Let T be a mapping such that $T : C \to E$. T is said to be

(i) nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for each $x, y \in C$,

(ii) quasi-nonexpansive if $\|Tx - Ty^*\| \leq \|x - y^*\|$ such that $F(T) \neq \emptyset$, $\forall x \in C$ and $y^* \in F(T)$.

Definition 2.3.21. [82, 29] Let $T : C \to E$ be a mapping. T is said to be

1. Bregman nonexpansive if

$$\Delta_p(Tx, Ty) \leq \Delta_p(x, y), \quad \forall x, y \in C,$$

2. Bregman quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$\Delta_p(Tx, y^*) \leq \Delta_p(x, y^*), \quad \forall x \in C, y^* \in F(T),$$

3. Bregman weak relatively nonexpansive if $\tilde{F}(T) \neq \emptyset$, $\tilde{F}(T) = F(T)$ and

$$\Delta_p(Tx, y^*) \leq \Delta_p(x, y^*) \quad \forall x \in C, y^* \in F(T),$$

4. Bregman relatively nonexpansive if $F(T) \neq \emptyset$, $\hat{F}(T) = F(T)$ and

$$\Delta_p(Tx, y^*) \leq \Delta_p(x, y^*) \quad \forall x \in C, y^* \in F(T).$$

From the definitions, it is evident that the class of Bregman quasi-nonexpansive maps contains the class of Bregman weak relatively nonexpansive maps. The class of Bregman weak relatively nonexpansive maps contains the class of Bregman relatively nonexpansive maps.

2.4 Some important nonlinear problems

In this section, we briefly introduce and review some existing results on nonlinear problems that are studied in this work.

2.4.1 Monotone inclusion problems

Several problems emanating from applied sciences such as filtration theory and quantum mechanics can be modeled mathematically as an operator equation which can be considered as the sum of two monotone operators, see for instance [3] and references therein. We denote the set of solutions of MIP (1.1.3) by Γ, i.e $\Gamma = (A + B)^{-1}(0)$ and assume $\Gamma \neq \emptyset$. Problem (1.1.3) and related optimization problems have been studied with various iterative algorithms proposed for approximating their solutions by several authors in Hilbert, Banach and Hadamard spaces (see, for instance [10, 32, 45, 47, 50, 70, 76, 96, 98, 100, 109]).

Martinez [67] first introduced the Proximal Point Algorithm (PPA) for finding the zero point of a maximal monotone operator B. The sequence generated by PPA is defined as follows:

$$x_{n+1} = J_{r_n}^B x_n,$$

where $0 < r_n < \infty$, $J_{r_n}^B = (I + r_n B)^{-1}$ is the resolvent operator of B and I is the identity mapping. This algorithm was eventually modified by Rockafellar [83] to the following PPA with errors:

$$x_{n+1} = J_{r_n}^B x_n + e_n,$$

where $\{e_n\}$ is an error sequence. It was proved that if $e_n \to 0$ such that $\sum_{n=1}^{\infty} \|e_n\| < +\infty$, $B^{-1}(0) \neq \emptyset$ and $\liminf_{n\to\infty} r_n > 0$, then the sequence $\{x_n\}$ converges weakly to a solution of a zero point of B. Moudafi and Théra [70] further introduced the following iterative algorithm for solving problem (1.1.3):

$$\begin{cases} x_n = J_r^B v_n, \\ v_{n+1} = tv_n + (1-t)x_n - \mu(1-t)Ax_n, \end{cases} \tag{2.4.1}$$

where $t \in (0,1)$, $r > 0$, A is Lipschitz continuous and strongly monotone and B is maximal monotone. They proved that the sequence $\{x_n\}$ generated by the iterative algorithm converges weakly to an element in $(A+B)^{-1}(0)$.

More so, Nakajo and Takahashi [74] introduced the following hybrid projection method and proved a strong convergence theorem for approximating zeros of maximal monotone operators in Hilbert spaces:

$$\begin{cases} x_0 = x \in H, \\ y_n = J_{r_n}^B (x_n + f_n), \\ C_n = \{z \in H : \|y_n - z\| \leq \|x_n + f_n - z\|\}, \\ Q_n = \{z \in H : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{cases} \tag{2.4.2}$$

for all $n \in \mathbb{N} \cup \{0\}$, $r_n \subset (0,\infty)$. Furthermore, they proved that $x_n \to z_0 = P_{B^{-1}(0)} x_0$, (where $P_{B^{-1}(0)}$ is the metric projection on $B^{-1}(0)$ if $\liminf_{n\to\infty} r_n > 0$ and $\lim_{n\to\infty} \|f_n\| = 0$. Moreover, Yuying and Plubtieng [116] improved Algorithm (2.4.2) and introduced a new hybrid projection method which is defined below, for finding the zero point of MIP.

$$\begin{cases} x_0, z_0 \in C, \\ y_n = \alpha_n z_n + (1-\alpha_n) x_n, \\ z_{n+1} = J_{r_n}(y_n - r_n A y_n), \\ C_n = \{z \in C : \|z_{n+1} - z\|^2 \leq \alpha_n \|z_n - z\|^2 + (1-\alpha_n)\|x_n - z\|^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{cases} \tag{2.4.3}$$

where $n \geq 0$, $\{\alpha_n\}$ and $\{r_n\}$ are sequences of positive real numbers with $\alpha_n \in [0,\beta]$ for some $\beta \in [0, \frac{1}{2})$ and $0 < r_n < 2\alpha$. It was proved that the sequence $\{x_n\}$ converges to a point $p = P_{(A+B)^{-1}(0)}(x_0)$. They further proved that the sequence $\{x_n\}$ converges to

$p = P_{(A+B)^{-1}(0)}(x_0)$ using the following algorithm known as shrinking projection method:

$$\begin{cases} x_0, z_0 \in C, \\ y_n = \alpha_n z_n + (1-\alpha_n)x_n, \\ z_{n+1} = J_{r_n}^B(y_n - r_n A y_n), \\ C_{n+1} = \{z \in C : \|z_{n+1} - z\|^2 \leq \alpha_n \|z_n - z\|^2 + (1-\alpha_n)\|x_n - z\|^2\}, \\ x_{n+1} = P_{C_{n+1}}(x_0). \end{cases} \tag{2.4.4}$$

Furthermore, the following modified splitting method with Armijo line search technique was introduced by Tseng [109]:

Algorithm 2.4.1. Let $X \subseteq H$ be a closed convex set, choose x_0 arbitrarily in X, given the current iterate, calculate the next iterate through the rule:

$$y_n = (Id + r_n B)^{-1}(Id - r_n A)x_n,$$

where r_n is selected as the largest $r \in \{\lambda, \lambda m, \lambda m^2, ...\}$ satisfying the following:

$$r\|Ax_n - Ay_n\| \leq \sigma \|x_n - y_n\|, \tag{2.4.5}$$

where $0 < \sigma < 1$, $\lambda > 0$ and $0 < m < 1$ are constraints. Set

$$x_{n+1} = P_X(y_n - r_n(Ax_n - Ay_n)). \tag{2.4.6}$$

It was proved that the sequence generated by Algorithm 2.4.1 converges weakly to an element in the solution of MIP (1.1.3), where B is maximal monotone and A is Lipschitz continuous.

On the other hand, Alvarez and Attouch [5] proposed the following modified PPA of inertial form:

$$\begin{cases} y_n = x_n + \mu_n(x_n - x_{n-1}), \\ x_{n+1} = J_{\lambda_n}^B y_n, \qquad n \geq 1, \end{cases} \tag{2.4.7}$$

where $\{\mu_n\} \subset [0,1)$, $\{\lambda_n\}$ is non-decreasing and

$$\sum_{n=1}^{\infty} \mu_n \|x_n - x_{n-1}\|^2 < \infty, \quad \forall \mu_n < \frac{1}{3}. \tag{2.4.8}$$

It was proved that Algorithm (2.4.7) converges weakly to a zero of B.

Recently, Moudafi and Oliny [71] introduced the following inertial PPA for approximating the zero point problem of the sum of two monotone operators:

$$\begin{cases} y_n = x_n + \mu_n(x_n - x_{n-1}), \\ x_{n+1} = J_{\lambda_n}^B(y_n - \lambda_n A x_n), n \geq 1, \end{cases} \tag{2.4.9}$$

where $B : H \to 2^H$ is maximal monotone and A is Lipschitz continuous. They proved that the sequence generated by Algorithm (2.4.9) converges weakly if $\lambda_n < \frac{2}{L}$, where L is the Lipschitz constant of A.

Moreover, the following inertial forward-backward algorithm was introduced by Lorenz and Pock [59]:

$$\begin{cases} y_n = x_n + \mu_n(x_n - x_{n-1}), \\ x_{n+1} = J^B_{\lambda_n}(y_n - \lambda_n A y_n), n \geq 1, \end{cases} \tag{2.4.10}$$

where $\{\lambda_n\}$ is a positive real sequence. Algorithm (2.4.10) differs from Algorithm (2.4.9) since the operator A was evaluated at the inertial extrapolate y_n. It was also proved to converge weakly to a solution of the problem (1.1.3)

It is highly desirable to obtain strong convergence than weak convergence, thus, Thong and Cholamjiak [103] proposed a new modified forward-backward splitting method (which will be presented in Chapter 4) to obtain strong convergence.

2.4.2 Variational inequality problems

The VIPs are known to have wide applications in various mathematical problems such as equilibrium problems, optimization problems, fixed point problems, among others. We denote the solution set of the VIP (1.1.2) by $VI(C, A)$.

In the early 1960's, Stampacchia [93] and Fichera [37] introduced the theory of VIP. The Problem (1.1.2) is a fundamental problem which has a wide range of applications in applied field of mathematics such as network equilibrium problems, complementary problems, optimization theory and systems of nonlinear equations (see [38, 47, 48, 50, 53, 97]). Under suitable conditions, there are generally two main approaches to finding the solutions of VIP (1.1.2). These are projection method and regularisation method. In this study, we are concern with the projection method. Recently, several authors have studied and proposed many iterative algorithms for approximating the solution of VIP and related optimization problems, (see [24, 25, 26, 27, 49, 52, 96, 98, 99]) and the references therein. The VIP is widely known to be equivalent to the following fixed point equation:

$$x^* = P_C(I - \mu A)x^*, \tag{2.4.11}$$

for, $\mu > 0$, where P_C is the metric projection from H onto C. The iteration formula which is an extension of the projection gradient method (2.4.11) can be defined as follows :

$$x_{n+1} = P_C(I - \mu A)x_n, \tag{2.4.12}$$

where $\mu \in \left(0, \frac{2\alpha}{L^2}\right)$ and $A : H \to H$ is $\alpha-$ strongly monotone and $L-$Lipschitz continuous.

One of the most famous methods for finding the solutions of VIP (1.1.2) is the extragradient method, which was proposed by Korpelevich [55] and it is given as follows:

Algorithm 2.4.2. (Extragradient Method (EgM))

$$\begin{cases} x_0 \in C, \\ y_n = P_C(x_n - \mu A x_n), \\ x_{n+1} = P_C(x_n - \mu A y_n), \end{cases} \tag{2.4.13}$$

where $\mu \in \left(0, \frac{1}{L}\right)$, $A : C \to \mathbb{R}^n$ is monotone and $L-$ Lipschitz continuous and $C \subseteq \mathbb{R}^n$ is a closed convex subset. If the solution set VI(C, A) is nonempty, then the sequence $\{x_n\}$ generated by EgM converges weakly to an element in VI(C, A). In recent years, the EgM has received great attention by numerous authors who have developed it in various ways (see for instance [61, 65, 107]). It is obvious that the EgM requires the computation of two projections from H onto closed convex subset C per iteration. However, projection onto an arbitrary closed convex set C is often very difficult to compute. In order to overcome this barrier, some authors have developed several iterative algorithms, some of these algorithms are given below:

In 2000, Tseng [109] proposed the following iterative scheme:

Algorithm 2.4.3.

$$\begin{cases} x_0 \in H, \\ y_n = P_C(x_n - \mu A x_n), \\ x_{n+1} = y_n - \mu(A y_n - A x_n), \end{cases} \tag{2.4.14}$$

where A is a monotone and Lipschitz continuous operator and $\mu \in \left(0, \frac{1}{L}\right)$. Clearly, Tseng's method requires one projection to be computed per iteration and hence has an advantage in computing projection over extragradient method. Furthermore, Censor et al. [25] introduced a new method which involves the modification of one of the projections by replacing it with a projection onto an half space. This method is called the subgradient extragradient method and is defined as follows:

Algorithm 2.4.4. (The Subgradient Extragradient Method(SEgM))

$$\begin{cases} x_0 \in H, \\ y_n = P_C(x_n - \mu A x_n), \\ Q_n = \{z \in H : \langle x_n - \mu A x_n - y_n, z - y_n \rangle \leq 0\}, \\ x_{n+1} = P_{Q_n}(x_n - \mu A y_n). \end{cases} \tag{2.4.15}$$

Censor et al. [25] proved that provided the solution set VI(C, A) is nonempty, the sequence $\{x_n\}$ generated by SEgM converges weakly to an element $p \in VI(C, A)$, where $p = \lim_{n\to\infty} P_{VI(C,A)} x_n$. Also, Maingé and Gobinddass [62] obtained a result which relates to weak convergence algorithm by using only a single projection by means of a projected reflected gradient-type method [65] and inertial term for finding solution of VIP in a real Hilbert space.

Another important problem in fixed point theory is the Fixed Point Problem (FPP), which is defined as follows:

$$\text{Find a point} \quad x^* \in C \quad \text{such that} \quad Sx^* = x^*, \tag{2.4.16}$$

where $S : C \to C$ is a nonlinear operator. If S is a multivalued mapping, i.e., $S : C \to 2^C$, then $x^* \in C$ is called a fixed point of S if

$$x^* \in Sx^*. \tag{2.4.17}$$

We denote the set of fixed points of S by $F(S)$. In this study, we also focus on finding a common solution of FPP and VIP (1.1.2), i.e. find a point $\hat{x}$ such that

$$\hat{x} \in F(S) \cap VI(C, A). \tag{2.4.18}$$

Takahashi and Toyoda [102] studied (2.4.18) and proposed an algorithm given by a sequence $\{x_n\}$ generated by the following iterative scheme:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n SP_C(x_n - \mu Ax_n), \tag{2.4.19}$$

where $S : C \to C$ is a nonexpansive mapping, $A : C \to H$ is an inverse strongly monotone operator and P_C is the metric projection onto C. Generally, the major disadvantage with the algorithm above is that it does not work whenever A is only a Lipschitz continuous and monotone mapping. In this case, inspired by Korpelevich's idea of extragradient, Nadezhkina and Takahashi [73] proposed the following iterative scheme for finding an element $\hat{x} \in F(S) \cap VI(C, A)$:

$$\begin{cases} x_0 \in C, \\ y_n = P_C(x_n - \mu_n Ax_n), \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n SP_C(x_n - \mu_n Ay_n), \end{cases} \tag{2.4.20}$$

where $\alpha_n \in (0, 1)$, $\mu_n \in \left(0, \frac{1}{L}\right)$ and $S : C \to C$ is a nonexpansive mapping. It was proved that the sequence $\{x_n\}$ generated by (2.4.20) converges weakly to $\hat{x} \in F(S) \cap VI(C, A)$. Furthermore, Censor et al. [25] studied and proposed the following Subgradient Extragradient Method(SEgM) for finding the common solution of VIP and FPP for nonexpansive mapping:

$$\begin{cases} x_0 \in H, \\ y_n = P_C(x_n - \mu Ax_n), \\ Q_n = \{z \in H : \langle x_n - \mu Ax_n - y_n, z - y_n \rangle \leq 0\}, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)SP_{Q_n}(x_n - \mu Ay_n). \end{cases} \tag{2.4.21}$$

They proved that the sequence $\{x_n\}$ generated by (2.4.21) converges weakly to a solution $u^* \in F(S) \cap VI(C, A)$.

It is highly desirable and more important to obtain strong convergence of an iterative algorithm than weak convergence as emphasised by Bauschke and Combettes [13]. Hence, authors always work towards developing algorithms which give strong convergence result.

In 2006, Nadezhkina and Takahashi [72] proposed the following iterative algorithm which is a combination of hybrid type method and extragradient type method:

$$\begin{cases} x_0 \in C, \\ y_n = P_C(x_n - \mu_n A x_n), \\ z_n = (1-\alpha_n)x_n + \alpha_n S P_C(x_n - \mu_n A y_n), \\ C_n = \{u \in C : \|z_n - u\| \leq \|x_n - u\|\}, \\ Q_n = \{u \in C : \langle x_n - u, x_n - x_0 \rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \tag{2.4.22}$$

where $\alpha_n \in (0,1)$ and $\mu_n \in \left(0, \frac{1}{L}\right)$. The sequence $\{x_n\}$ was proved to converge strongly to a solution of (2.4.18).

Motivated by the iterative scheme proposed by Nadezhkina and Takahashi [72], the following iterative algorithm was introduced by Zeng and Yao [117], for finding $\bar{x} \in F(S) \cap VI(C, A)$:

Algorithm 2.4.5.

$$\begin{cases} x_0 \in C, \\ y_n = P_C(x_n - \mu_n A x_n), \\ x_{n+1} = \alpha_n x_0 + (1-\alpha_n) S P_C(x_n - \mu_n A y_n), \end{cases} \tag{2.4.23}$$

where $S : C \to C$ is nonexpansive. It was proved that the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to a point $P_{F(S)\cap VI(C,A)}x_0$ if $\{\mu_n\}$ and $\{\alpha_n\}$ satisfy the following conditions :

(i) $\{\alpha_n\} \subset (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\{\mu_n\} \subset (0, 1-\sigma)$ for some $\sigma \in (0,1)$,

provided $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

An inertial extrapolation was first introduced by Polyak [79] to solve the smooth convex minimization problem. The inertial method involves a two-step iteration in which the next iterate is defined by making use of the previous two iterates and this generally increases the rate of convergence of iterative algorithms. The inertial type algorithm has been studied and modified in various forms by many authors (see [16, 104, 105]). For example, an inertial hybrid proximal-extragradient algorithm which is a combination of hybrid proximal extragradient and inertial type algorithm for a maximal monotone operator was proposed by Bot and Csetnek [17]. Dong et al. [33] proposed an algorithm which incorporates inertial term to EgM. This algorithm is known as Inertial Extragradient Method (IEGM) and is defined as follows:

$$\begin{cases} x_0, x_1 \in H, \\ w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = P_C(w_n - \lambda A w_n), \\ x_{n+1} = (1-\beta_n)w_n + \beta_n P_C(w_n - \lambda A y_n). \end{cases} \tag{2.4.24}$$

It was proved under suitable conditions that the sequence $\{x_n\}$ converges weakly to $p \in VI(C, A)$.

2.4.3 Split feasibility problems

We denote the set of solution of SFP (1.1.1) by $\Omega = C \cap A^{-1}(Q) = \{x^* \in C : Ax^* \in Q\}$. Several authors have studied and proposed iterative algorithms for solving SFP (1.1.1).

The following popular algorithm, called the CQ algorithm was proposed by Byrne [19] to approximate the solution of SFP (1.1.1) in real Hilbert spaces:

$$x_{n+1} = P_C\left(x_n - \mu A^*(I - P_Q)Ax_n\right), \quad \forall n \geq 1, \tag{2.4.25}$$

where

$$\mu \in \left(0, \frac{2}{\|A\|^2}\right), \tag{2.4.26}$$

P_C and P_Q denote the metric projections of H_1 onto C and H_2 onto Q, respectively. It was proved that the sequence $\{x_n\}$ generated by (2.4.25) converges weakly to a solution of SFP provided the step size μ satisfies the condition (2.4.26). As a result of the CQ algorithm, several iterative algorithms have been introduced for solving SFP in Hilbert and Banach spaces, (see for example, [101, 43]). The following algorithm was proposed by Schöpfer et al. [85] for solving SFP in p-uniformly convex real Banach spaces which are also uniformly smooth:

$$x_{n+1} = \Pi_C J^q_{E_1^*}\left[J^p_{E_1} - \mu_n A^* J^p_{E_2}\big(Ax_n - P_Q(Ax_n)\big)\right], \tag{2.4.27}$$

$\forall x \in E_1$ $n \geq 1$, where Π_c and J are the Bregman projection and the duality mapping respectively.
In 2014, Algorithm 2.4.27 was modified by Wang [110] and studied the following Multiple-sets Split Feasibility Problem (MSSFP): Find $x \in E_1$ satisfying

$$x \in \bigcap_{i=1}^{r} C_i, \; Ax \in \bigcap_{j=r+1}^{r+s} Q_j, \tag{2.4.28}$$

where s and r are two given integers, $C_i, i = 1, \cdots, r$ is a nonempty closed convex subset of E_1, and $Q_j, j = r+1, ..., r+s$ is a closed convex subset in E_2. Using the idea of [74], he proved strong convergence of the sequence generated by the following algorithm: for any initial guess x_0, let x_n be defined recursively by

$$\begin{cases} y_n = T_n x_n \\ D_n = \big\{u \in E : \Delta_p(y_n, u) \leq \Delta_p(x_n, u)\big\} \\ E_n = \big\{u \in E : \langle x_n - u, J_E^p(x_0) - J_E^p(x_n)\rangle \geq 0\big\} \\ x_{n+1} = \Pi_{D_n \cap E_n}(x_0). \end{cases} \tag{2.4.29}$$

For each $n \in \mathbb{N}$, T_n is defined by

$$T_n(x) = \begin{cases} \Pi_{C_{i(n)}},\ 1 \leq i(n) \leq r, \\ J_{E_1^*}^q[J_{E_1}^p(x) - \mu_n A^* J_{E_2}^p(I - P_{Q_{j(n)}})Ax],\ r+1 \leq i(n) \leq r+s, \end{cases}$$

$i : \mathbb{N} \to I$ is the cyclic control mapping, $i(n) = n \mod (r+s) + 1$, where μ_n satisfies

$$0 < \mu \leq \mu_n \leq \left(\frac{q}{C_q \|A\|^q}\right)^{\frac{1}{q-1}}, \tag{2.4.30}$$

where C_q is a uniform smoothness constant.

2.5 Some important iterative methods

In this section, we present some notable and important iterative schemes for approximating the fixed point of nonlinear mappings and optimization problems.

2.5.1 Picard iteration

Lemma 2.5.1. *(Banach contraction principle) Let (X, ρ) be a complete metric space, $\sigma \in [0, 1)$ and $S : X \to X$ be a contraction mapping, i.e.*

$$\rho(Sx, Sy) \leq \sigma\rho(x, y), \quad \forall \quad x, y \in X,$$

then,

(i) there exists a unique fixed point $\bar{x} \in X$,

(ii) the Picard iteration given by

$$x_{n+1} = Sx_n, \quad \textit{for} \ \ x_0 \in X \quad \textit{and} \ \ n = 1, 2, \cdots \tag{2.5.1}$$

converges to $\bar{x} \in X$.

2.5.2 Krasnoselskii iteration

Let the Picard iteration formula be replaced by the following sequence:

$$x_{n+1} = \frac{(I+S)x_n}{2}, \quad \forall \;\; n \geq 0, \text{for} \;\; x_0 \in C, \tag{2.5.2}$$

then, the iterative sequence (2.5.2) converges to the unique fixed point. Generally, suppose T is a nonexpansive mapping and X is a normed linear space, then the following recursive formula which is a generalization of (2.5.2) was given by Schaefer [84]:

$$x_{n+1} = \mu S x_n + (1-\mu)x_n, \quad \text{for} \;\; x_0 \in C, \quad n \geq 0, \tag{2.5.3}$$

where $\mu \in (0,1)$. The iterative sequence (2.5.3) is called the Krasnoselskii iteration. Clearly, if $\mu = 1$, the Krasnoselskii iteration reduces to Picard iteration. Clearly, the Krasnoselskii iteration corresponds to the Picard iteration for the averaged operator $S_\mu = \mu S + (1-\mu)I$, where I is the identity operator.

2.5.3 Mann iteration

Another notable iterative scheme for approximating the fixed points of nonlinear mappings is the Mann iteration. This was introduced by Mann [60] and is given as follows:

$$x_{n+1} = \gamma_n S x_n + (1-\gamma_n)x_n, \quad \text{for} \;\; x_0 \in C, \quad n \geq 0, \tag{2.5.4}$$

where $\{\gamma_n\} \subset (0,1)$ satisfies the following:

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$,

(ii) $\lim_{n\to\infty} \gamma_n = 0$.

Obviously, Mann iteration reduces to Krasnoselskii iteration if $\gamma_n = \mu$ and aslo reduces to Picard iteration (2.5.1) if $\gamma_n = 1$.

2.5.4 Ishikawa iteration

Ishikawa [42] modified and developed Mann iteration to an iterative scheme which generates a sequence $\{x_n\}$ given by

$$x_{n+1} = (1-\lambda_n)x_n + \lambda_n S\big[(1-\tau_n)x_n + \tau_n S x_n\big], \quad x_0 \in C, \quad n = 0, 1, 2, \cdots, \tag{2.5.5}$$

where the sequences $\{\tau_n\}_{n=1}^{\infty}$ and $\{\lambda_n\}_{n=1}^{\infty}$ are in $(0,1)$ satisfying the following:

(i) $\lim_{n\to\infty} \tau_n = 0$;

(ii) $\sum_{n=1}^{\infty} \lambda_n \tau_n = \infty$;

(iii) $0 \leq \lambda_n \leq \tau_n \leq 1$.

Alternatively, (2.5.5) can be written as:

$$\begin{cases} y_n = (1-\tau_n)x_n + \tau_n S x_n, x_{n+1} = (1-\lambda_n)x_n + \lambda_n S y_n. \end{cases} \tag{2.5.6}$$

Thus, it is known as a dual step Mann iteration. We note that when $\tau_n = 0$, the Ishikawa iteration scheme (2.5.5) reduces to Mann iteration. Moreover, for a Lipschitz pseudo-contractive mapping in a Hilbert space, Mann iteration may fail to converge while the Ishikawa iteration can converge. However, under suitable conditions such as comnpactness of either the operator S or the subset C, it will converge to a fixed point of β-strictly pseudocontractive maps.

2.5.5 Halpern iteration

An explicit iterative algorithm which generates a sequence through the recursive formula

$$x_{n+1} = (1-\beta_n)Sx_n + \beta_n v, \quad n \geq 0, \text{ for } x_0 \in C, \tag{2.5.7}$$

where $\{\beta_n\}$ is a sequence in $(0,1)$ and $v \in C$, was introduced by Halpern [40]. This iterative method, called the Halpern iteration is used for finding the fixed points of a nonlinear operator $S : C \to C$.

2.5.6 Viscosity iteration

Moudafi [69] introduced viscosity iterative scheme which is defined as follows: Let x_0 be an initial point, define sequence $\{x_t\}$ by

$$x_t = \frac{\epsilon_t}{1+\epsilon_t} f(x_t) + \frac{1}{1+\epsilon_t} S x_t, \tag{2.5.8}$$

such that

(i) $\lim_{t\to+\infty} \left| \frac{1}{\epsilon_t} - \frac{1}{\epsilon_t - 1} \right| = 0$,

(ii) $\sum_{t=1}^{+\infty} \epsilon_t = +\infty$,

(iii) $\lim_{t\to+\infty} \epsilon_t = 0$,

where $\{\epsilon_t\} \subset (0,1)$, f is a contraction mapping and S is a nonexpansive self-mapping. For all x_0, the sequence $\{x_t\}$ converges strongly to $\bar{x} \in F(S)$, which is a unique solution of the variational inequality

$$\langle (I-f)\bar{x}, x - \bar{x} \rangle \geq 0, \quad \forall \ x \in F(S). \tag{2.5.9}$$

CHAPTER 3

On Inertial Hybrid Algorithm For Solving Split Feasibility Problems in Banach Spaces

3.1 Introduction

In this chapter, we propose and study a self-adaptive hybrid inertial algorithm for finding solutions of SFP which also solves MIP and FFP in p-uniformly convex and uniformly smooth Banach spaces. We prove a strong convergence of the sequence generated by our proposed algorithm which does not require a prior knowledge of the norm of the bounded linear operator. We give numerical examples to compare the computational performance of our algorithm with other existing algorithms.

In 2015, Alsulami and Takahashi [4] proposed the following algorithm for approximating the solution of SFP (1.1.1): For any $x_1 \in H$,

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)P_C(x_n - rA^* J_F(I-P_Q)Ax_n), \quad n \geq 1. \tag{3.1.1}$$

It was proved that for some $a, b \in \mathbb{R}$ if $0 < a \leq \alpha_n \leq b < 1$ and $0 < r\|A\|^2 < 2$, where $0 < r < \infty$ and $\{\alpha_n\} \subset [0,1]$, then $\{x_n\}$ weakly converges to $\omega_0 = \lim_{n\to\infty} P_{C\cap A^{-1}Q}x_n$, where $w_0 \in C \cap A^{-1}Q$, H is a real Hilbert space, F is a strictly convex, reflexive smooth Banach space, J_F denotes the duality mapping on F, C and Q are non-empty closed convex subsets of H and F respectively. Furthermore, they introduced the following Halpern-type iteration in order to obtain strong convergence result: Let $\{t_n\}$ be a sequence in H such that $t_n \to t \in H$ and $x_1, t_1 \in H$,

$$\begin{cases} \nu_n = \lambda_n t_n + (1-\lambda_n)P_C(x_n - rA^* J_F(I-P_Q)Ax_n), \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)\nu_n, \quad n \geq 1, \end{cases} \tag{3.1.2}$$

where $0 < r < \infty$ and $\{\alpha_n\} \subset (0,1)$. It was proved that the sequence $\{x_n\}$ defined by

(3.1.2) converges strongly to a point $\omega_0 \in C \cap A^{-1}Q$, for some $\omega_0 = P_{C\cap A^{-1}Q}t_1$, $\forall\, a, b \in \mathbb{R}$ if, $0 < r\|A\|^2 < 2$, $\lim\limits_{n\to\infty} \lambda_n = 0$, $\sum\limits_{n=1}^{\infty} \lambda_n = \infty$, and $0 < a \le \alpha_n \le b < 1$.

Recently, Suantai et al. 2019 [94] considered the following modified SFP:

$$\text{Find} \quad x \in F(T) \cap C \quad \text{such that} \quad Ax \in Q. \tag{3.1.3}$$

Clearly, when $F(T) = C$, then (3.1.3) reduces to (1.1.1). Suantai et al. [94] proved the following weak and strong convergence theorems using Mann's iteration and Halpern-type iteration process, respectively for solving SFP and FPP for nonexpansive mappings.

Theorem 3.1.1. *Let H be a Hilbert space, F be a strictly convex, reflexive and smooth Banach space, C and Q be non-empty, closed and convex subsets of H and F, J_F be the duality mapping on F, P_Q and P_C denote the metric projections of F on Q and H on C, respectively. Let $T : C \to C$ be nonexpansive mapping. Suppose $\Gamma \neq \emptyset$, where $\Gamma = F(T) \cap C \cap A^{-1}Q$, for $x_1 \in C$, define $\{x_n\}$ by*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)TP_C\left(x_n - \gamma_n \frac{f(x_n)}{\|g(x_n)\|^2 + \|x_n - Tx_n\|^2} g(x_n)\right), \tag{3.1.4}$$

where $g(x_n) = A^* J_F(I - P_Q)Ax_n$, $f(x_n) = \frac{1}{2}\|(I - P_Q)Ax_n\|^2$, $\{\gamma_n\} \subset (0, 4)$, $\forall n \in \mathbb{N}$ which satisfies the following conditions:

1. $\liminf_{n\to\infty} \gamma_n(4 - \gamma_n) > 0$,
2. $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$,

then $\{x_n\}$ weakly converges to $\omega_0 \in \Gamma$, where $\omega_0 = \lim\limits_{n\to\infty} P_\Gamma x_n$.

Theorem 3.1.2. *Let H be a Hilbert space, F be a strictly convex, reflexive and smooth Banach space, C and Q be non-empty, closed and convex subsets of H and F, J_F be the duality mapping on F, P_Q and P_C denote the metric projections of F on Q and H on C respectively. Let $T : C \to C$ be nonexpansive mapping. Suppose $\Gamma \neq \emptyset$, where $\Gamma = F(T) \cap C \cap A^{-1}Q$. Let $x_1 \in C$, $\{t_n\}$ be a sequence in C such that $t_n \to t$, and let $\{x_n\}$ be a sequence defined by*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)\left(\lambda_n t_n + (1-\lambda_n)TP_C\left(x_n - \gamma_n \frac{f(x_n)}{\|g(x_n)\|^2 + \|x_n - Tx_n\|^2} g(x_n)\right)\right), \tag{3.1.5}$$

where $g(x_n) = A^* J_F(I - P_Q)Ax_n$, $f(x_n) = \frac{1}{2}\|(I - P_Q)Ax_n\|^2$, $\{\gamma_n\} \subset (0, 4)$, $\lambda_n \subset (0, 1)$, $\{\alpha_n\} \subset (0, 1)$, $\forall n \in \mathbb{N}$ which satisfy the following conditions:

1. $\liminf_{n\to\infty} \gamma_n(4 - \gamma_n) > 0$,
2. $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$,
3. $\lim_{n\to\infty} \lambda_n = 0$ *and* $\sum_{n=1}^{\infty} \lambda_n = \infty$,

then $\{x_n\}$ *strongly converges to* $w_0 \in \Gamma$.

Motivated by the above results, in this chapter, we study the following modified SFP in Banach spaces: Let E_1 and E_2 be p-uniformly convex and uniformly smooth real Banach spaces, C and Q be non-empty closed convex subsets of E_1 and E_2 respectively, and $A : E_1 \to E_2$ be a bounded linear operator with $A^* : E_2^* \to E_1^*$. Let $T : C \to C$ be a mapping, and $B : E_2 \to 2^{E_2^*}$ be a maximal monotone operator:

$$\text{Find } \; x \in F(T) \cap C \; \text{ such that } Ax \in B^{-1}(0), \tag{3.1.6}$$

where $B : E_2 \to 2^{E_2^*}$ is a maximal monotone operator. Obviously, the SFP (3.1.6) is more general than (1.1.1) and (3.1.3). When $B = \partial i_Q$, a maximal monotone operator and the subdifferential of the indicator function on Q, then (3.1.6) reduces to (3.1.3).

3.2 Preliminaries

In this section, we recall some basic notions and lemmas which will be useful in establishing our results in this chapter. We denote the strong and weak convergence of the sequence $\{x_n\}$ to a point x by $x_n \to x$ and $x_n \rightharpoonup x$ respectively.

Definition 3.2.1. Let C be a nonempty closed convex subset of a real Banach space E and $T : C \to E$ be a mapping. T is said to be Bregman weak relatively nonexpansive if $\tilde{F}(T) \neq \emptyset$ and $\tilde{F}(T) = F(T)$ with

$$\Delta_p(Tx, y^*) \leq \Delta_p(x, y^*) \quad \forall x \in C, y^* \in F(T).$$

We recall that the Bregman distance satisfy the following properties:

$$\Delta_p(x, y) = \Delta_p(x, z) + \Delta_p(z, y) + \langle z - y, J_E^p x - J_E^p z\rangle, \tag{3.2.1}$$

and

$$\Delta_p(x, y) + \Delta_p(y, x) = \langle x - y, J_E^p x - J_E^p y\rangle.$$

We also recall the following useful relation between metric and Bregman distance in p-uniformly convex space:

$$\tau\|x - y\|^p \leq \Delta_p(x, y) \leq \langle x - y, J_E^p x - J_E^p y\rangle, \tag{3.2.2}$$

where $\tau > 0$ is some fixed number.

The Bregman projection is the unique minimizer of the Bregman distance and is characterized by the following variational inequalities (see [85, 86]):

$$\langle J_E^p(x) - J_E^p(\Pi_C x), z - \Pi_C x\rangle \leq 0, \; \forall z \in C, \tag{3.2.3}$$

from which we have

$$\Delta_p(\Pi_C x, z) \leq \Delta_p(x, z) - \Delta_p(x, \Pi_C x), \; \forall z \in C. \tag{3.2.4}$$

The metric projection which is also the unique minimizer of the norm distance is characterized by the following variational inequality:

$$\langle J_E^p(x - P_C x), z - P_C x\rangle \leq 0, \ \forall z \in C. \tag{3.2.5}$$

Definition 3.2.2. Let E be a p-uniformly convex Banach space, the function $V_p : E \times E \to [0, \infty]$ associated with $f_p(x) = \frac{1}{p}\|x\|^p$ is given by

$$V_p(x, \bar{x}) = \frac{1}{p}\|x\|^p - \langle x, \bar{x}\rangle + \frac{1}{q}\|\bar{x}\|^q, \ x \in E, \ \bar{x} \in E^*, \tag{3.2.6}$$

where $V_p(x, \bar{x}) \geq 0$.

It then follows that

$$V_p(x, \bar{x}) = \Delta_p(x, J_{E^*}^q(\bar{x})), \ \forall x \in E, \ \bar{x} \in E^*.$$

Definition 3.2.3. Let E be a smooth, strictly convex and reflexive Banach space, and $A : E \to 2^{E^*}$ be a maximal monotone operator. We define a mapping $Q_r^A : E \to D(A)$ by (see [95])

$$Q_r^A(x) = (I + r(J_E^p)^{-1}A)^{-1}(x), \text{ for all } \ x \in E \ \text{ and } \ r > 0.$$

Q_r^A is called the metric resolvent of A. Obviously, for all $r > 0$, we have

$$0 \in J_E^p(Q_r^A(x) - x) + rAQ_r^A(x), \tag{3.2.7}$$

and $F(Q_r^A) = A^{-1}(0)$. Furthermore, for all $x, y \in E$ and by the monotonicity of A, we can show that

$$\langle Q_r^A(x) - Q_r^A(y), J_E^p(x - Q_r^A(x)) - J_E^p(y - Q_r^A(y))\rangle \geq 0. \tag{3.2.8}$$

From (3.2.7), we have for all $x, y \in E$

$$\frac{J_E^p(x - Q_r^A(x))}{r} \in AQ_r^A(x), \tag{3.2.9}$$

and

$$\frac{J_E^p(y - Q_r^A(y))}{r} \in AQ_r^A(y). \tag{3.2.10}$$

Since A is monotone, then we obtain (3.2.8) from (3.2.9) and (3.2.10). This implies that for all $x \in E$, $t \in A^{-1}(0)$, and whenever $A^{-1}(0) \neq \emptyset$, we have

$$\langle Q_r^A(x) - t, J_E^P(x - Q_r^A(x))\rangle \geq 0. \tag{3.2.11}$$

The following lemmas are also useful in establishing our main results in this chapter.

Lemma 3.2.1. *[111] Let $x, y \in E$. If E is a q-uniformly smooth Banach space, then there exists a $D_q > 0$ such that*

$$\|x - y\|^q \leq \|x\|^q - q\langle y, J_E^q(x)\rangle + D_q\|y\|^q. \tag{3.2.12}$$

Lemma 3.2.2. *[57] Let C be a non-empty, closed and convex subset of a reflexive, strictly convex and smooth Banach space E, $x_0 \in C$ and $x \in E$, then the following assertions are equivalent:*

1. $x_0 = \Pi_C(x)$;
2. $\langle z - x_0, J_E^p(x_0) - J_E^p(x) \geq 0\rangle, \quad \forall z \in C.$

Furthermore, for all $y \in C$, we have

$$\Delta_p(\Pi_C(x), y) + \Delta_p(x, \Pi_C(x)) \leq \Delta_p(x, y).$$

Lemma 3.2.3. *[75] Let E be a smooth and uniformly convex real Banach space. Let $\{x_n\}$ and $\{y_n\}$ be two bounded sequences in E. Then $\lim_{n\to\infty} \Delta_p(x_n, y_n) = 0$ if and only if*

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

Lemma 3.2.4. *[111] Let $q \geq 1$ and $r > 0$ be two fixed real numbers, then a Banach space E is uniformly convex if and only if there exists a continuous, strictly, increasing and convex function $g : \mathbb{R}^+ \to \mathbb{R}^+, g(0) = 0$ such that for all $x, y \in B_r$ and $0 \leq \alpha \leq 1$,*

$$\|\alpha x + (1-\alpha)y\|^q \leq \alpha\|x\|^q + (1-\alpha)\|y\|^q - W_q(\alpha)g(\|x-y\|), \tag{3.2.13}$$

where $W_q := \alpha^q(1-\alpha) + \alpha(1-\alpha)^q$ and $B_r := \{x \in E : \|x\| \leq r\}$.

3.3 Main results

In this section, we present our inertial technique for solving the modified SFP (3.1.6) in Banach spaces. We also prove a strong convergence result for the sequence generated by our algorithm.

Algorithm 3.3.1. Let E_1, E_2 be p-uniformly convex and uniformly smooth real Banach spaces, C and Q be non-empty closed convex subsets of E_1 and E_2 respectively, and $A : E_1 \to E_2$ be a bounded linear operator with $A^* : E_2^* \to E_1^*$. Let $T : C \to C$ be a Bregman weak relatively nonexpansive mapping, and $B : E_2 \to 2^{E_2^*}$ be a maximal monotone operator. Let $\Gamma = F(T) \cap C \cap A^{-1}(B^{-1}(0)) \neq \emptyset$. Also, Let $\{\alpha_n\}$ be a sequence in (0,1) with $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$, $x_0, x_1 \in C = C_1 = H_1$, $\{\theta_n\}$ be a real sequence such that $-\theta \leq \theta_n \leq \theta$, for some $\theta > 0$ and $r_n > 0$. Assuming the $(n-1)th$ and nth iterates have been constructed, we calculate the next iterate $(n+1)th$ via the formula

$$\begin{cases} w_n = J_{E_1^*}^q \left[J_{E_1}^p(x_n) + \theta_n(J_{E_1}^p(x_n) - J_{E_1}^p(x_{n-1}))\right], \\ v_n = \Pi_C J_{E_1^*}^q \left[J_{E_1}^p(w_n) - \mu_n A^* J_{E_2}^p(I - Q_{r_n}^B)Aw_n\right], \\ u_n = J_{E_1^*}^q \left[\alpha_n J_{E_1}^p(v_n) + (1-\alpha_n)J_{E_1}^p(Tv_n)\right], \\ C_n = \{u \in E_1 : \Delta_p(u, u_n) \leq \Delta_p(u, w_n)\}, \\ H_n = \{u \in E_1 : \langle x_n - u, J_{E_1}^p(x_1) - J_{E_1}^p(x_n)\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap H_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases} \tag{3.3.1}$$

where μ_n is a positive number satisfying

$$\mu_n^{q-1} = \begin{cases} \frac{q\|(I-Q_{r_n}^B)Aw_n\|^p}{D_q\|A^*J_{E_2}^p(I-Q_{r_n}^B)Aw_n\|^q}, & \text{if } Aw_n \neq Q_{r_n}^B Aw_n, \\ \epsilon, & \text{if } Aw_n = Q_{r_n}^B Aw_n, \end{cases} \tag{3.3.2}$$

for any $\epsilon > 0$.

Note that the step size defined in (3.3.2) does not require a prior knowledge or estimate of the operator norm $\|A\|$. This is very important because in practice, it is very difficult to estimate the norm of bounded linear operators (for simple estimate, see [87]).

Next, we prove some necessary results which will be used to establish our main theorem.

First, we show that the sequence $\{x_n\}$ generated by Algorithm 3.3.1 is well-defined.

Lemma 3.3.2. *Let $\{x_n\}$ be generated by (3.3.1), then $\{x_n\}$ is well-defined.*

Proof. We need to show that $C_n \cap H_n$ is a non-empty closed and convex set $\forall n \geq 1$. It is obvious that H_n is closed and convex while C_n is closed. So we show that C_n is also convex. Observe that

$$\Delta_p(u, u_n) \leq \Delta_p(u, w_n)$$

is equivalent to

$$\langle J_{E_1}^p(w_n) - J_{E_1}^p(u_n), u\rangle \leq \frac{1}{q}\left(\|w_n\|^p - \|u_n\|^p\right).$$

Hence C_n is a half space and so convex. This implies that $C_n \cap H_n$ is closed and convex for $n \in \mathbb{N}$. Furthermore, we need to show that $C_n \cap H_n$ is non-empty. It is sufficient to show that $\Gamma \subset C_n \cap H_n$. Let $x^* \in \Gamma$, then,

$$\begin{aligned} \Delta_p(x^*, u_n) &= \Delta_p\left(x^*, J_{E_1^*}^q\left[\alpha_n J_{E_1}^p v_n + (1-\alpha_n)J_{E_1}^p T v_n\right]\right) \\ &\leq \alpha_n \Delta_p(x^*, v_n) + (1-\alpha)\Delta_p(x^*, Tv_n) \\ &\leq \alpha_n \Delta_p(x^*, v_n) + (1-\alpha)\Delta_p(x^*, v_n) \\ &= \Delta_p(x^*, v_n). \end{aligned} \tag{3.3.3}$$

Also from Lemma 3.2.1 and (3.2.6), we have

$$\begin{aligned} \Delta_p(x^*, v_n) &= \Delta_p\left(x^*, \Pi_C J_{E_1^*}^q\left[J_{E_1}^p(w_n) - \mu_n A^* J_{E_2}^p(I - Q_{r_n}^B)Aw_n\right]\right) \\ &\leq \Delta_p\left(x^*, J_{E_1^*}^q\left[J_{E_1}^p(w_n) - \mu_n A^* J_{E_2}^p(I - Q_{r_n}^B)Aw_n\right]\right) \\ &= V_p(x^*, [J_{E_1}^p(w_n) - \mu_n A^* J_{E_2}^p(I - Q_{r_n}^B)Aw_n]) \\ &= \frac{\|x^*\|^p}{p} - \langle x^*, J_{E_1}^p w_n\rangle + \langle x^*, \mu_n A^* J_{E_2}^p(I - Q_{r_n}^B)Aw_n\rangle \\ &+ \frac{1}{q}\|J_{E_1}^p w_n - \mu_n A^* J_{E_2}^p(I - Q_{r_n}^B)Aw_n\|^q \\ &\leq \frac{\|x^*\|^p}{p} - \langle x^*, J_{E_1}^p w_n\rangle + \mu_n\langle Ax^*, J_{E_2}^p(I - Q_{r_n}^B)Aw_n\rangle + \frac{1}{q}\|J_{E_1}^p w_n\|^q \\ &- \mu_n\langle Aw_n, J_{E_2}^p(I - Q_{r_n}^B)Aw_n\rangle + \frac{D_q\mu_n^q}{q}\|A^* J_{E_2}^p(I - Q_{r_n}^B)Aw_n\|^q \end{aligned}$$

$$
\begin{aligned}
&= \frac{\|x^*\|^p}{p} - \langle x^*, J_{E_1}^p w_n\rangle + \frac{1}{q}\|J_{E_1}^p w_n\|^q + \mu_n\langle Ax^* - Aw_n, J_{E_2}^p(I - Q_{r_n}^B)Aw_n\rangle \\
&+ \frac{D_q\mu_n^q}{q}\|A^* J_{E_2}^p(I - Q_{r_n}^B)Aw_n\|^q \\
&= \Delta_p(x^*, w_n) + \mu_n\langle Ax^* - Q_{r_n}^B Aw_n + Q_{r_n}^B Aw_n - Aw_n, J_{E_2}^p(I - Q_{r_n}^B)Aw_n\rangle \\
&+ \frac{D_q\mu_n^q}{q}\|A^* J_{E_2}^p(I - Q_{r_n}^B)Aw_n\|^q \\
&= \Delta_p(x^*, w_n) + \mu_n\langle Ax^* - Q_{r_n}^B Aw_n, J_{E_2}^p(I - Q_{r_n}^B)Aw_n\rangle \\
&- \mu_n\langle Aw_n - Q_{r_n}^B Aw_n, J_{E_2}^p(I - Q_{r_n}^B)Aw_n\rangle + \frac{D_q\mu_n^q}{q}\|A^* J_{E_2}^p(I - Q_{r_n}^B)Aw_n\|^q.
\end{aligned}
$$

From (3.2.11), we have

$$
\begin{aligned}
\Delta_p(x^*, v_n) &\leq \Delta_p(x^*, w_n) - \mu_n\langle (I - Q_{r_n}^B)Aw_n, J_{E_2}^p(I - Q_{r_n}^B)Aw_n\rangle \\
&+ \frac{D_q\mu_n^q}{q}\|A^* J_{E_2}^p(I - Q_{r_n}^B)Aw_n\|^q \\
&= \Delta_p(x^*, w_n) \\
&- \mu_n\left\{\|(I - Q_{r_n}^B)Aw_n\|^p - \frac{D_q\mu_n^{q-1}}{q}\|J_{E_1}^p(I - Q_{r_n}^B)Aw_n\|^q\right\}. \qquad (3.3.4)
\end{aligned}
$$

Hence, from (3.3.2), we have

$$\Delta_p(x^*, v_n) \leq \Delta_p(x^*, w_n).$$

This implies that

$$\Delta_p(x^*, u_n) \leq \Delta_p(x^*, w_n).$$

So $\Gamma \subset C_n$, for all $n \in \mathbb{N}$. Since $x_{n+1} = \Pi_{C_n\cap H_n}x_1$, then $\langle J_{E_1}^p x_1 - J_{E_1}^p x_{n+1}, v - x_{n+1}\rangle \leq 0$, $\forall v \in C_n \cap H_n \subset C$. In particular, for $x^* \in \Gamma$, we have $\langle J_{E_1}^p x_1 - J_{E_1}^p x_{n+1}, x^* - x_{n+1}\rangle \leq 0$. This implies that $\Gamma \subset H_n$ for all $n \in \mathbb{N}$. So we obtain that $\Gamma \subset C_n \cap H_n$ for all $n \in \mathbb{N}$. Therefore, $C_n \cap H_n$ is non-empty and thus $x_{n+1} = \Pi_{C_n\cap H_n}x_1$ is well-defined. □

Lemma 3.3.3. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.3.1. Then*

(i) $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$,

(ii) $\lim_{n\to\infty}\|x_n - w_n\| = 0$,

(iii) $\lim_{n\to\infty}\|Tv_n - v_n\| = 0$,

(iv) $\lim_{n\to\infty}\|x_n - v_n\| = 0$,

(v) $\lim_{n\to\infty}\|A^* J_{E_2}^p(I - Q_{r_n}^B)Aw_n\| = 0$.

Proof. (i) Let $w \in \Gamma$. Since $\Gamma \subset C_n \cap H_n$, $\forall n \geq 1$ and $x_{n+1} = \Pi_{C_n\cap H_n}x_1$, it follows that

$$\Delta_p(x_{n+1}, x_1) \leq \Delta_p(w, x_1), \forall\, n \geq 1.$$

Thus $\{\Delta_p(x_{n+1}, x_1)\}$ is bounded.
We observe that $x_{n+1} \in H_n$ and by (3.2.3), we have

$$\langle x_n - x_{n+1}, J_{E_1}^p(x_n) - J_{E_1}^p(x_1)\rangle \le 0,$$

also by (3.2.4), we have

$$\Delta_p(x_{n+1}, x_n) \le \Delta_p(x_{n+1}, x_1) - \Delta_p(x_n, x_1), \ \forall n \ge 1, \tag{3.3.5}$$

which implies that

$$\Delta_p(x_n, x_1) \le \Delta_p(x_{n+1}, x_1) - \Delta_p(x_{n+1}, x_n).$$

Thus

$$\Delta_p(x_n, x_1) \le \Delta_p(x_{n+1}, x_1),$$

therefore, $\{\Delta_p(x_n, x_1)\}$ is a bounded monotone nondecreasing sequence.

Hence $\lim_{n\to\infty} \{\Delta_p(x_n, x_1)\}$ exists.

From (3.3.5), we have $\lim_{n\to\infty} \Delta_p(x_{n+1}, x_n) = 0$. Thus, using Lemma 3.2.3

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.3.6}$$

(ii) Since $J_{E_1}^p$ is uniformly continuous on bounded subsets of E_1, we have from (3.3.6) that

$$\lim_{n\to\infty} \|J_{E_1}^p(x_{n+1}) - J_{E_1}^p(x_n)\| = \lim_{n\to\infty} \|J_{E_1}^p(x_n) - J_{E_1}^p(x_{n-1})\| = 0.$$

From (3.3.1), we have

$$w_n = J_{E_1^*}^q \left(J_{E_1}^p(x_n) + \theta_n(J_{E_1}^p(x_n) - J_{E_1}^p(x_{n-1}))\right),$$

then

$$J_{E_1}^p w_n = J_{E_1}^p x_n + \theta_n(J_{E_1}^p(x_n) - J_{E_1}^p(x_{n-1}),$$

which gives

$$\|J_{E_1}^p(w_n) - J_{E_1}^p(x_n)\| = |\theta_n|\|J_{E_1}^p(x_n) - J_{E_1}^p(x_{n-1})\|.$$

Therefore

$$\lim_{n\to\infty} \|J_{E_1}^p(w_n) - J_{E_1}^p(x_n)\| = 0.$$

Since $J_{E_1^*}^q$ is also uniformly continuous on bounded subsets of E_1^*, then we have

$$\lim_{n\to\infty} \|x_n - w_n\| = 0. \tag{3.3.7}$$

(iii) From (3.3.6) and (3.3.7), we obtain

$$\begin{aligned}\|x_{n+1} - w_n\| &= \|x_{n+1} - x_n + x_n - w_n\| \\ &\le \|x_{n+1} - x_n\| + \|x_n - w_n\| \to 0 \text{ as } n \to \infty.\end{aligned}$$

Note that from the construction of C_n, we have that

$$\Delta_p(x_{n+1}, u_n) \leq \Delta_p(x_{n+1}, w_n) \to 0 \text{ as } n \to \infty,$$

therefore by Lemma 3.2.3, we have

$$\lim_{n\to\infty} \|x_{n+1} - u_n\| = 0.$$

Again, since $\|x_n - u_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - u_n\|$, it then follows that

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.3.8}$$

It follows from (3.3.7) and (3.3.8) that

$$\lim_{n\to\infty} ||w_n - u_n\| = 0. \tag{3.3.9}$$

Using Lemma 3.2.4, we have

$$\begin{aligned}
\Delta_p(x^*, u_n) &= \Delta_p\left(x^*, J^q_{E_1^*}\left[\alpha_n J^p_{E_1} v_n + (1-\alpha_n) J^p_{E_1} T v_n\right]\right)\\
&= V_p\left(x^*, \alpha_n J^p_{E_1} v_n + (1-\alpha_n) J^p_{E_1} T v_n\right)\\
&= \frac{1}{p}\|x^*\|^p - \langle x^*, \alpha_n J^p_{E_1} v_n\rangle - \langle x^*, (1-\alpha_n) J^p_{E_1} T v_n\rangle + \frac{1}{q}\|\alpha_n J^p_{E_1} v_n + (1-\alpha_n) J^p_{E_1} T v_n\|^q\\
&\leq \frac{1}{p}\|x^*\|^p - \alpha_n\langle x^*, J^p_{E_1} v_n\rangle - (1-\alpha_n)\langle x^*, J^p_{E_1} T v_n\rangle + \frac{1}{q}\alpha_n\|v_n\|^p + \frac{(1-\alpha_n)}{q}\|T v_n\|^p\\
&\quad -\frac{W_q(\alpha_n)}{q} g\left(\|J^p_{E_1} v_n - J^p_{E_1} T v_n\|\right) \qquad (3.3.10)\\
&= \alpha_n \frac{1}{p}\|x^*\|^p + (1-\alpha_n)\frac{1}{p}\|x^*\|^p - \alpha_n\langle x^*, J^p_{E_1} v_n\rangle - (1-\alpha_n)\langle x^*, J^p_{E_1} T v_n\rangle\\
&+ \frac{1}{q}\alpha_n\|v_n\|^p + \frac{(1-\alpha_n)}{q}\|T v_n\|^p - \frac{W_q(\alpha_n)}{q} g\left(\|J^p_{E_1} v_n - J^p_{E_1} T v_n\|\right)
\end{aligned}$$

$$\begin{aligned}
&= \alpha_n\left\{\frac{1}{p}\|x^*\|^p - \langle x^*, J^p_{E_1} v_n\rangle + \frac{1}{q}\|v_n\|^p\right\} + (1-\alpha_n)\left\{\frac{1}{p}\|x^*\|^p - \langle x^*, J^p_{E_1} T v_n\rangle + \frac{1}{q}\|T v_n\|^p\right\}\\
&\quad -\frac{W_q(\alpha_n)}{q} g\left(\|J^p_{E_1} v_n - J^p_{E_1} T v_n\|\right)\\
&= \alpha_n \Delta_p(x^*, v_n) + (1-\alpha_n)\Delta_p(x^*, T v_n) - \frac{W_q(\alpha_n)}{q} g\left(\|J^p_{E_1} v_n - J^p_{E_1} T v_n\|\right)\\
&\leq \alpha_n \Delta_p(x^*, v_n) + (1-\alpha_n)\Delta_p(x^*, v_n) - \frac{W_q(\alpha_n)}{q} g\left(\|J^p_{E_1} v_n - J^p_{E_1} T v_n\|\right)\\
&= \Delta_p(x^*, v_n) - \frac{W_q(\alpha_n)}{q} g\left(\|J^p_{E_1} v_n - J^p_{E_1} T v_n\|\right)\\
&\leq \Delta_p(x^*, w_n) - \frac{W_q(\alpha_n)}{q} g\left(\|J^p_{E_1} v_n - J^p_{E_1} T v_n\|\right). \qquad (3.3.11)
\end{aligned}$$

Hence, from (3.2.1) and (3.2.2), we get

$$\begin{aligned}\frac{W_q(\alpha_n)}{q} g\left(\|J^p_{E_1} v_n - J^p_{E_1} T v_n\|\right) &\leq \Delta_p(x^*, w_n) - \Delta_p(x^*, u_n)\\ &= \Delta_p(u_n, w_n) + \langle u_n - w_n, J^p_{E_1} x^* - J^p_{E_1} u_n\rangle\\ &\leq \langle u_n - w_n, J^p_{E_1} u_n - J^p_{E_1} w_n\rangle + \langle u_n - w_n, J^p_{E_1} x^* - J^p_{E_1} u_n\rangle\\ &= \langle u_n - w_n, J^p_{E_1} x^* - J^p_{E_1} w_n\rangle. \end{aligned} \tag{3.3.12}$$

From (3.3.9), we have

$$\lim_{n\to\infty} \frac{W_q(\alpha_n)}{q} g\left(\|J^p_{E_1} v_n - J^p_{E_1} T v_n\|\right) = 0,$$

which implies

$$\lim_{n\to\infty} g\left(\|J^p_{E_1} v_n - J^p_{E_1} T v_n\|\right) = 0.$$

By the property of mapping g, we obtain

$$\lim_{n\to\infty} \|J^p_{E_1} v_n - J^p_{E_1} T v_n\| = 0.$$

Since $J^q_{E_1^*}$ is uniformly continuous on bounded subsets of E_1^*, we have

$$\lim_{n\to\infty} \|T v_n - v_n\| = 0. \tag{3.3.13}$$

(iv) From Algorithm 3.3.1, we have that,

$$J^P_{E_1} u_n - J^P_{E_1} v_n = (1-\alpha_n)(J^P_{E_1} T v_n - J^P_{E_1} v_n).$$

Since $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$ and from (3.3.13), we have

$$\lim_{n\to\infty} \|J^P_{E_1} u_n - J^P_{E_1} v_n\| = 0,$$

hence

$$\lim_{n\to\infty} \|u_n - v_n\| = 0.$$

Since $\|w_n - v_n\| \leq \|w_n - u_n\| + \|u_n - v_n\|$, then from (3.3.9) we have

$$\lim_{n\to\infty} \|w_n - v_n\| = 0. \tag{3.3.14}$$

Therefore, from (3.3.7), we have

$$\lim_{n\to\infty} \|v_n - x_n\| = 0.$$

(v) From (3.3.4), we have

$$\mu_n \left\{ \|(I - Q_{r_n}^B)Aw_n\|^p - \frac{D_q \mu_n^{q-1}}{q} \|A^* J_{E_2}^p (I - Q_{r_n}^B)Aw_n\|^q \right\} \leq \Delta_p(x^*, w_n) - \Delta_p(x^*, v_n)$$
$$\begin{aligned} &= \Delta_p(v_n, w_n) \\ &+ \langle v_n - w_n, J_{E_1}^p x^* - J_{E_1}^p v_n \rangle \\ &\leq \langle v_n - w_n, J_{E_1}^p v_n - J_{E_1}^p w_n \rangle \\ &+ \langle v_n - w_n, J_{E_1}^p x^* - J_{E_1}^p v_n \rangle \\ &= \langle v_n - w_n, J_{E_1}^p x^* - J_{E_1}^p w_n \rangle. \end{aligned} \tag{3.3.15}$$

It follows from (3.3.14) that

$$\lim_{n\to\infty} \left(\|(I - Q_{r_n}^B)Aw_n\|^p - \frac{D_q \mu_n^{q-1}}{q} \|A^* J_{E_2}^p (I - Q_{r_n}^B)Aw_n\|^q \right) = 0. \tag{3.3.16}$$

From the choice of μ_n in (3.3.2), we have

$$\mu_n^{q-1} < \frac{q\|(I - Q_{r_n}^B)Aw_n\|^p}{D_q \|A^* J_{E_2}^p (I - Q_{r_n}^B)Aw_n\|^q} - \epsilon, \tag{3.3.17}$$

for small $\epsilon > 0$. This implies that

$$\frac{D_q \mu_n^{q-1} \|A^* J_{E_2}^p (I - Q_{r_n}^B)Aw_n\|^q}{q} < \|(I - Q_{r_n}^B)Aw_n\|^p - \frac{\epsilon D_q \|A^* J_{E_2}^p (I - Q_{r_n}^B)Aw_n\|^q}{q}.$$

Then we have

$$\frac{\epsilon D_q \|A^* J_{E_2}^p (I - Q_{r_n}^B)Aw_n\|^q}{q} < \|(I - Q_{r_n}^B)Aw_n\|^p - \frac{D_q \mu_n^{q-1} \|A^* J_{E_2}^p (I - Q_{r_n}^B)Aw_n\|^q}{q}.$$

Therefore from (3.3.16), we have

$$\lim_{n\to\infty} \frac{\epsilon D_q}{q} \|A^* J_{E_2}^p (I - Q_{r_n}^B)Aw_n\|^q = 0,$$

hence

$$\lim_{n\to\infty} \|A^* J_{E_2}^p (I - Q_{r_n}^B)Aw_n\| = 0. \tag{3.3.18}$$

Also from (3.3.16), we have that

$$\lim_{n\to\infty} \|(I - Q_{r_n}^B)Aw_n\| = 0. \tag{3.3.19}$$

□

Now, we present a strong convergence theorem for solving the SFP (3.1.6) using Algorithm 3.3.1.

Theorem 3.3.4. *Let E_1, E_2 be p-uniformly convex and uniformly smooth Banach spaces, C and Q be non-empty closed convex subsets of E_1 and E_2 respectively, and $A : E_1 \to E_2$ be a bounded linear operator with $A^* : E_2^* \to E_1^*$. Let $T : C \to C$ be a Bregman weak relatively nonexpansive mapping, and $B : E_2 \to 2^{E_2^*}$ be a maximal monotone operator. Suppose $\Gamma = F(T) \cap C \cap A^{-1}(B^{-1}(0)) \neq \emptyset$. Then, the sequence $\{x_n\}$ generated by Algorithm 3.3.1 converges strongly to $u \in \Gamma$, where $u = \Pi_\Gamma x_1$.*

Proof. We have already shown in Lemma 3.3.3(i) that $\lim\limits_{n\to\infty} \Delta_p(x_n, x_1)$ exists. Next, we show that $x_n \to \bar{x} \in \Gamma$. Let $m, n \in \mathbb{N}$, then

$$\Delta_p(x_m, x_n) = \Delta_p(x_m, \Pi_{C_{n-1}\cap H_{n-1}} x_1) \leq \Delta_p(x_m, x_1) - \Delta_p(x_n, x_1) \to 0.$$

Therefore by Lemma 3.2.4, we get that $\|x_m - x_n\| \to 0$ as $m, n \to \infty$. Thus $\{x_n\}$ is a Cauchy sequence in C. Since C is closed and convex, it implies that there exists $\bar{x} \in C$ such that $x_n \to \bar{x}$ as $n \to \infty$. Since $\|x_n - v_n\| \to 0$, $\|Tv_n - v_n\| \to 0$ and T is a Bregman weak relatively nonexpansive mapping, then $\bar{x} \in F(T)$. More so, since $\|x_n - w_n\| \to 0$, then $w_n \to \bar{x}$ and by the linearity of A, we have $Aw_n \to A\bar{x}$. Also from (3.3.19), $Q_{r_n}^B Aw_n \to A\bar{x}$. Since $Q_{r_n}^B$ is a resolvent metric of B for $r_n > 0$, then for all $n \in \mathbb{N}$, we have

$$\frac{J_{E_2}^P(Aw_n - Q_{r_n}^B Aw_n)}{r_n} \in BQ_{r_n}^B Aw_n.$$

So for all $(s, s^*) \in B$, we have

$$0 \leq \langle s - Q_{r_n}^B Aw_n, s^* - \tfrac{J_{E_2}^P(Aw_n - Q_{r_n}^B Aw_n)}{r_n}\rangle.$$

It follows from (3.3.19) that for all $(s, s^*) \in B$, we have

$$0 \leq \langle s - A\bar{x}, s^* - 0\rangle.$$

Since B is maximal monotone, then it implies that $A\bar{x} \in B^{-1}(0)$, hence $\bar{x} \in A^{-1}(B^{-1}0)$. Therefore, $\bar{x} \in \Gamma$.

Finally, we show that $\bar{x} = \Pi_\Gamma x_1$. Suppose there exists $\bar{y} \in \Gamma$ such that $\bar{y} = \Pi_\Gamma x_1$. Then

$$\Delta_p(\bar{y}, x_1) \leq \Delta_p(\bar{x}, x_1). \tag{3.3.20}$$

We have shown in Lemma 3.3.2 that $\Gamma \subset C_n \; \forall n \geq 1$, then $\Delta_p(x_n, x_1) \leq \Delta_p(\bar{x}, x_1)$. By the lower semi-continuity of the norm, we have

$$\begin{aligned}\Delta_p(\bar{x}, x_1) &= \frac{\|\bar{x}\|^p}{q} - \langle J_{E_1}^p \bar{x}, x_1\rangle + \frac{\|x_1\|^p}{p} \\ &\leq \liminf_{n\to\infty} \left\{ \frac{\|\bar{x}\|^p}{q} - \langle J_{E_1}^p x_n, x_1\rangle + \frac{\|x_1\|^p}{p} \right\} \\ &= \liminf_{n\to\infty} \Delta_p(\bar{x}, x_1).\end{aligned}$$

$$\leq \limsup_{n\to\infty} \Delta_p(\bar{x}, x_1) \leq \Delta_p(\bar{y}, x_1). \tag{3.3.21}$$

Combining (3.3.20) and (3.3.21) we have $\Delta_p(\bar{y}, x_1) \leq \Delta_p(\bar{x}, x_1) \leq \Delta_p(\bar{y}, x_1)$. This implies $\bar{x} = \bar{y}$ and $\bar{x} = \Pi_\Gamma x_1$. Hence $x_n \to \bar{x} = \Pi_\Gamma x_1 \in \Gamma$. This completes the proof.

□

The following are consequences of our results.

(i) Taking $B = \partial i_Q$ which is a maximal monotone operator, then $Q^B_{r_n} = P_Q$ (the metric projection on Q). Thus, we obtain the following result from Theorem 3.3.4 which improve the corresponding results of Suantai et al. [94].

Corollary 3.3.5. *Let E_1, E_2 be p-uniformly convex and uniformly smooth real Banach spaces, C and Q be non-empty closed convex subsets of E_1 and E_2 respectively, and $A : E_1 \to E_2$ be a bounded linear operator with $A^* : E_2^* \to E_1^*$ and let $T : C \to C$ be a Bregman weak relatively nonexpansive mapping. Suppose $\Gamma = SFP \cap F(T) \neq \emptyset$. Then, the sequence $\{x_n\}$ generated by the following algorithm converges strongly to $u \in \Gamma$, where $u = \Pi_\Gamma x_1$.*

Algorithm 3.3.6. Let $\{\alpha_n\}$ be a sequence in (0,1), $x_1 \in C = C_1 = Q_1$, θ_n be a real sequence such that $-\theta \leq \theta_n \leq \theta$, for some $\theta > 0$. Assuming the $(n-1)th$ and nth iterates have been constructed, we calculate the next iterate $(n+1)th$ via the formula

$$\begin{cases} w_n = J^q_{E_1^*}\left[J^p_{E_1}(x_n) + \theta_n(J^p_{E_1}(x_n) - J^p_{E_1}(x_{n-1}))\right], \\ v_n = \Pi_C J^q_{E_1^*}\left[J^p_{E_1}(w_n) - \mu_n A^* J^p_{E_2}(I - P_Q)Aw_n\right], \\ u_n = J^q_{E_1^*}\left[\alpha_n J^p_{E_1}(v_n) + (1-\alpha_n)J^p_{E_1}(Tv_n)\right], \\ C_n = \{u \in E_1 : \Delta_p(u, u_n) \leq \Delta_p(u, w_n)\}, \\ H_n = \{u \in E_1 : \langle x_n - u, J^p_{E_1}(x_1) - J^p_{E_1}(x_n)\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap H_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases} \tag{3.3.22}$$

where μ_n is a positive number satisfying

$$\mu_n^{q-1} = \begin{cases} \frac{q\|(I-P_Q)Aw_n\|^p}{D_q\|A^* J^p_{E_2}(I-P_Q)Aw_n\|^q}, & \text{if } Aw_n \neq P_Q Aw_n, \\ \epsilon, & \text{if } Aw_n = P_Q Aw_n, \end{cases} \tag{3.3.23}$$

for any $\epsilon > 0$.

(ii) Taking $E_1 = H_1$ and $E_2 = H_2$, where H_1 and H_2 are real Hilbert spaces, we obtain the following result which improve the results of Byrne [19].

Corollary 3.3.7. *Let H_1, H_2 be real Hilbert spaces, C and Q be non-empty closed convex subsets of H_1 and H_2 respectively, and $A : H_1 \to H_2$ be a bounded linear operator. Let $T : C \to C$ be a Bregman weak relatively nonexpansive mapping, and $B : H_2 \to 2^{H_2}$ be a maximal monotone operator. Suppose $\Gamma = F(T) \cap C \cap A^{-1}(B^{-1}(0)) \neq \emptyset$. Then, the sequence $\{x_n\}$ generated by the following algorithm converges strongly to $u \in \Gamma$, where $u = P_\Gamma x_1$.*

Algorithm 3.3.8. Let $\{\alpha_n\}$ be a sequence in (0,1), $x_1 \in C = C_1 = Q_1$, θ_n be a real sequence such that $-\theta \leq \theta_n \leq \theta$, for some $\theta > 0$ and $\lambda_n > 0$. Assuming the $(n-1)th$ and nth iterates have been constructed, we calculate the next iterate $(n+1)th$ via the formula

$$
\begin{cases}
w_n = x_n + \theta_n(x_n - x_{n-1}),\\
v_n = P_C(w_n - \mu_n A^*(I - \mathrm{Q}^B_{r_n})Aw_n),\\
u_n = \alpha_n v_n + (1-\alpha_n)Tv_n,\\
C_n = \{u \in H_1 : \|u_n - u\|^2 \le \|w_n - u\|^2,\\
H_n = \{u \in H_1 : \langle x_n - u, x_1 - x_n\rangle \ge 0\},\\
x_{n+1} = P_{C_n \cap H_n} x_1, \quad \forall n \in \mathbb{N},
\end{cases}
\tag{3.3.24}
$$

where μ_n is a positive number satisfying

$$
\mu_n = \begin{cases}
\frac{2\|(I-\mathrm{Q}^B_{r_n})Aw_n\|^2}{\|A^*(I-\mathrm{Q}^B_{r_n})Aw_n\|^2}, & \text{if } Aw_n \neq \mathrm{Q}^B_{r_n}Aw_n,\\
\epsilon, & \text{if } Aw_n = \mathrm{Q}^B_{r_n}Aw_n,
\end{cases}
\tag{3.3.25}
$$

for any $\epsilon > 0$.

3.4 Numerical examples

In this section, we present two numerical examples to illustrate the performance of our method as well as comparing it with some related methods in the literature.

Example 3.4.1. Let $E_1 = E_2 = \mathbb{R}^m$ and A be a $m \times m$ randomly generated matrix. Let $C = \{x \in \mathbb{R}^m : \langle a, x\rangle \ge b\}$, where $a = (1, -5, 4, 0, \dots, 0) \in \mathbb{R}^m$ and $b = 1$. Then

$$
\Pi_C(x) = P_C(x) = \frac{b - \langle a, x\rangle}{\|a\|_2^2} a + x.
$$

Let $B : \mathbb{R}^m \to 2^{\mathbb{R}^m}$ be defined by $B(x) = \{2x\}$, and $T = P_C$. We take $\theta_n = \frac{3}{7n}$, $r_n = \frac{1}{2n}$, and $\alpha_n = \frac{n}{5n+1}$. Then our Algorithm (3.3.1) becomes

$$
\begin{cases}
w_n = x_n + \frac{3}{7n}(x_n - x_{n-1}),\\
v_n = P_C(w_n - \mu_n A^*(I - \mathrm{Q}^B_{r_n})Aw_n),\\
u_n = \frac{n}{5n+1} v_n + \frac{4n+1}{5n+1} P_C(v_n),\\
C_n = \{u \in E_1 : \|u_n - u\|^2 \le \|w_n - u\|^2\},\\
H_n = \{u \in E_1 : \langle x_n - u, x_1 - x_n \ge 0\},\\
x_{n+1} = P_{C_n \cap H_n} x_1, \quad \forall n \in \mathbb{N},
\end{cases}
$$

where μ_n is chosen as defined by (3.3.2) and $Q^B_{r_n}(Aw_n) = \left(\frac{n}{n+1}\right) Aw_n$ for all $n \ge 1$. We choose various values of m as follows:

Case I: $m = 10$, Case II: $m = 20$, Case III: $m = 50$, Case IV: $m = 40$,

and use $\frac{\|x_{n+1}-x_n\|_2^2}{\|x_2-x_1\|_2^2} < 10^{-6}$ as the stopping criterion. Thus, we plot the graph of $\|x_{n+1}-x_n\|_2^2$ against number of iteration in each case and compare the computation results of our algorithm with Algorithm 3.1.1 and 3.1.2 of Alsulami and Takahashi [4]. We found that Algorithm 3.3.1 performs better in terms of number of iterations and CPU time-taken for computation than both Algorithms 3.1.1 and 3.1.2. The computation result can be seen in Figure 3.1 and Table 3.1.

Example 3.4.2. In this second example, we consider the infinite-dimensional space and compare our Algorithm 3.3.1 with Algorithms 3.1.4 and 3.1.5 of Suantai et al. [94]. Let $E_1 = E_2 = E_3 = L^2([0, 2\pi])$ with norm $||x||^2 = \int_0^{2\pi} |x(t)|^2 dt$ and inner product $\langle x, y\rangle = \int_0^{2\pi} x(t)y(t)dt,\ x, y \in E$. Suppose $C := \{x \in L^2([0, 2\pi]) : \int_0^{2\pi}(t^2 + 1)x(t)dt \leq 1\}$ and $Q := \{x \in L^2([0, 2\pi]) : \int_0^{2\pi} |x(t) - \sin(t)|^2 \leq 16\}$ are subsets of E_1 and E_2 respectively. Define $A : L^2([0, 2\pi]) \to L^2([0, 2\pi])$ by $A(x)(t) = \int_0^{2\pi} \exp^{-st} x(t)dt$ for all $x \in L^2([0, 2\pi])$ and let $B = \partial i_Q$, subdifferential of the indicator function on Q, then $Q_{r_nB} = P_Q$. Let $T(x)(t) = \int_0^{2\pi} x(t)dt$ and choose $\theta_n = \frac{1}{2(n+1)}$ and $\alpha_n = \frac{5n}{8n+7}$. Then our Algorithm 3.3.1 becomes :

$$\begin{cases} w_n = x_n + \frac{1}{2(n+1)}(x_n - x_{n-1}), \\ v_n = \Pi_C(w_n - \mu_n A^*(I - P_Q)Aw_n), \\ u_n = \frac{5n}{8n+7}v_n + \frac{3n+7}{8n+7}T(v_n), \\ C_n = \{u \in E_1 : \Delta_p(u, u_n) \leq \Delta_p(u, w_n)\}, \\ H_n = \{u \in E_1 : \langle x_n - u, x_1 - x_n \geq 0\}, \\ x_{n+1} = P_{C_n \cap H_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where μ_n is chosen as defined by (3.3.2) for all $n \geq 1$. We choose various values of the initial point as follows:
Case (i): $x_1 = 2t\exp(5t)$, $x_0 = \frac{t^2}{2}$,
Case (ii): $x_1 = t^2\cos(2\pi t)$, $x_0 = \exp(2t)$,
Case (iii): $x_1 = \frac{3}{7}\sin(4t)$, $x_0 = 2t\sin(3t)$,
Case (iv): $x_1 = 5t\cos(2\pi t)$, $x_0 = 2\cos(3\pi t)$.

Using $\frac{\|x_{n+1}-x_n\|^2}{\|x_2-x_1\|^2} < 10^{-4}$ as stopping criterion, we plot the graph of $\|x_{n+1} - x_n\|^2$ against number of iteration and compare the computation results of our algorithm with Algorithm 3.1.4 and 3.1.5 of Suantai et al. [94]. We found out that our Algorithm 3.3.1 also perform better than Algorithm 3.1.4 and 3.1.5 in terms of number of iterations and cpu-time. The computational results can be seen in Table 3.2 and Figure 3.2.

Table 3.1: Computation result for Example 3.4.1.

		Algorithm 3.3.1	Algorithm 3.1.2	Algorithm 3.1.3
Case I $m = 10$	CPU time (sec) No. of Iter.	$7.7660e-4$ 10	0.0024 144	0.0027 88
Case II $m = 20$	CPU time (sec) No. of Iter.	$7.8468e-4$ 10	0.0013 150	0.0011 91
Case III $m = 50$	CPU time (sec) No. of Iter.	$7.2380e-4$ 10	0.0063 155	0.0064 94
Case IV $m = 100$	CPU time (sec) No. of Iter.	$7.2747e-4$ 10	0.0057 159	0.0061 97

Table 3.2: Computation result for Example 3.4.2.

		Algorithm 3.3.1	Algorithm 3.1.4	Algorithm 3.1.5
Case I	CPU time (sec) No. of Iter.	2.9098 12	13.5328 34	3.1893 25
Case II	CPU time (sec) No. of Iter.	2.0835 10	18.7739 35	5.8123 26
Case III	CPU time (sec) No. of Iter.	2.3470 12	7.5584 34	4.4957 25
Case IV	CPU time (sec) No. of Iter.	2.0908 10	6.0053 27	3.0760 20

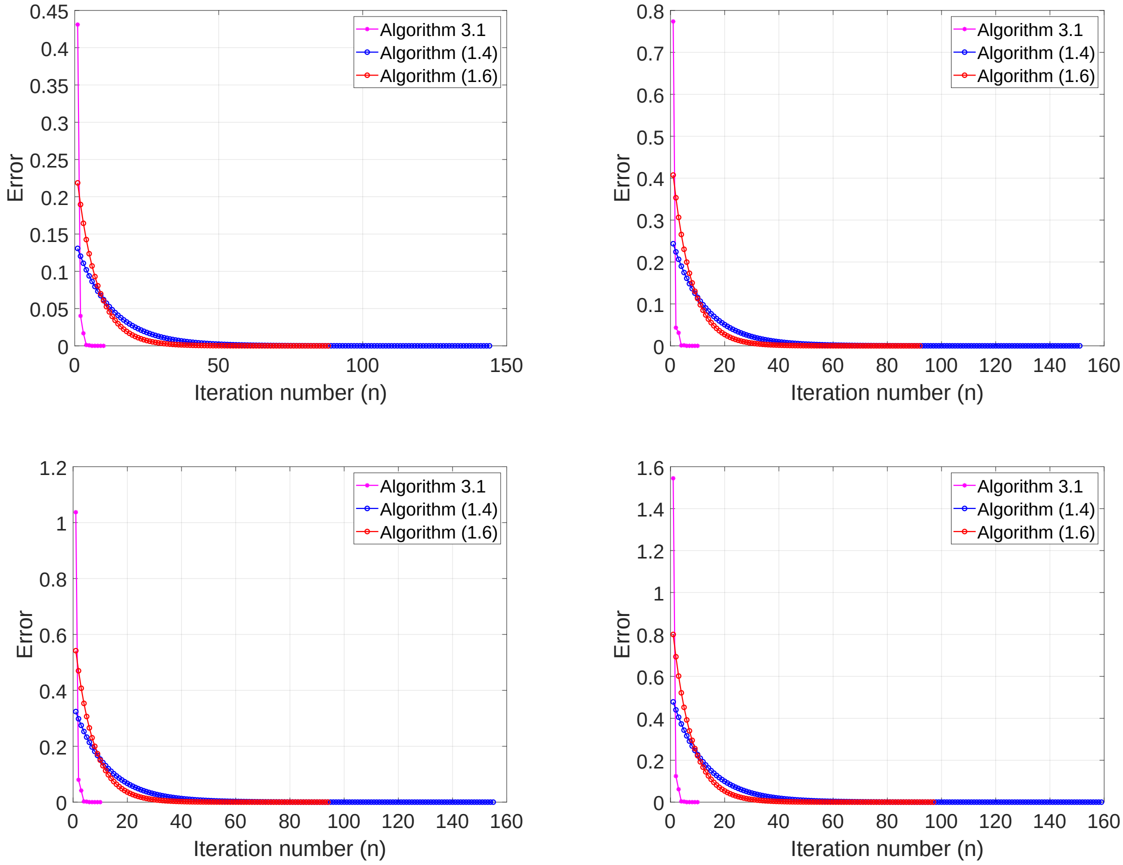

Figure 3.1: Example 3.4.1: Top Left Case I; Top Right: Case II; Bottom Left: Case III; Bottom Right: Case IV.

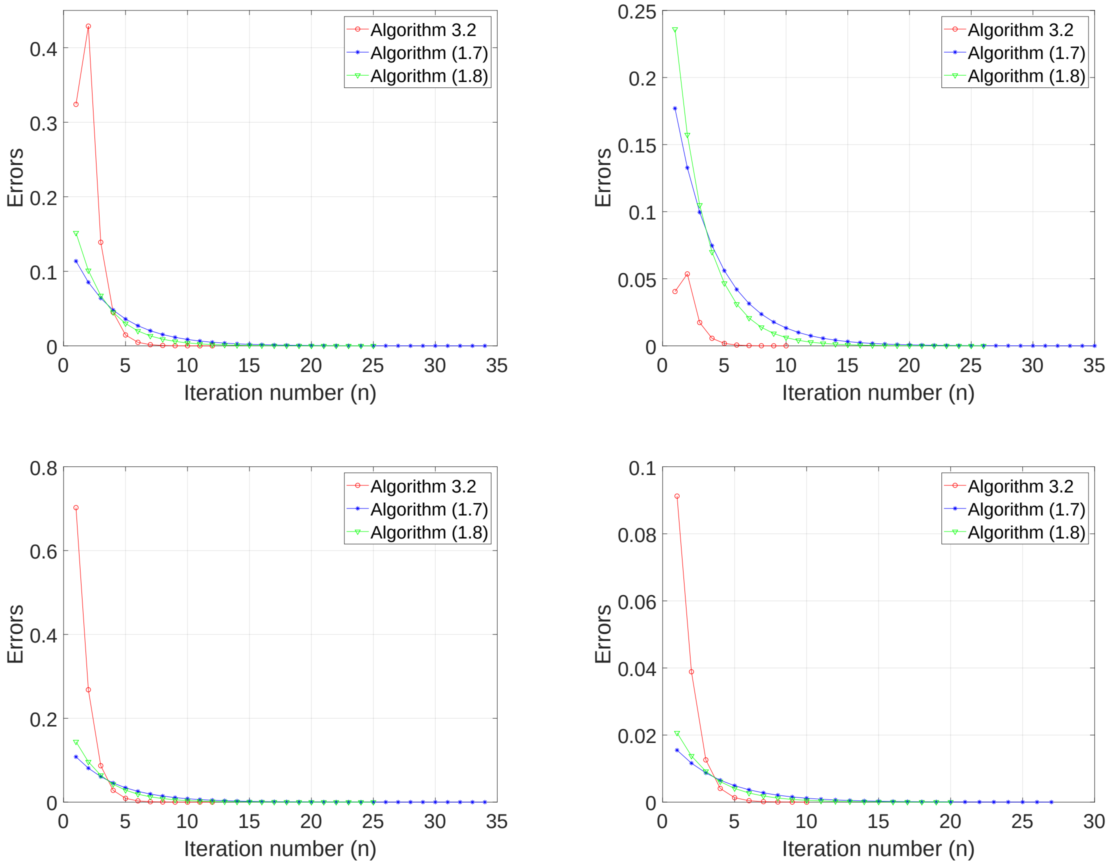

Figure 3.2: Example 3.4.2: Top Left Case I; Top Right: Case II; Bottom Left: Case III; Bottom Right: Case IV.

CHAPTER 4

Approximation of Common Solutions of Monotone Inclusion and Fixed Point Problems of Demimetric Mappings in Hilbert Spaces

4.1 Introduction

In this chapter, we study the common solutions of MIP and FPP of a finite family of demimetric mappings.

The MIP (1.1.3) have been studied by several authors and various iterative algorithms have been modified and improved (see, for instance [70, 34, 35, 109, 116]). One of the most famous methods for finding solution of MIP is the forward-backward splitting method which have been developed in various forms.

The forward-backward splitting method and Armijo-line search technique have recently attracted much attention by several authors and has been modified in various forms to prove the weak convergence of the sequence generated by it (see [1, 11, 16, 30, 34, 35, 59])

In [103], Thong and Cholamjiak obtained strong convergence for the following modified forward-backward splitting method:

Algorithm 4.1.1. Let $x_0 \in H$, C be a nonempty closed convex subset of H, $\lambda > 0, \tau \in (0,2), \sigma \in (0,1)$ and $m \in (0,1)$. Then calculate x_{n+1} as follows:

Step 1 : Calculate

$$y_n = J_{r_n}^B (I - r_n A) x_n,$$

where $J_{r_n}^B = (I - r_n B)^{-1}$ and r_n is selected to be the largest $r \in \{\lambda, \lambda m, \lambda m^2, ...\}$ satisfying

$$r\langle Ax_n - Ay_n, x_n - y_n \rangle \leq \sigma \|x_n - y_n\|^2.$$

If $y_n = x_n$, stop and hence y_n is a solution of Γ. Otherwise

Step 2 : Calculate

$$z_n = x_n - \tau\mu_n c_n,$$

where

$$c_n = x_n - y_n - r_n(Ax_n - Ay_n),$$

and

$$\mu_n = (1-\sigma)\frac{\|x_n - y_n\|^2}{\|c_n\|^2}.$$

Step 3 : Calculate

$$x_{n+1} = (1-\theta_n)z_n + \theta_n f(x_n).$$

Set $n := n+1$ and go to **Step 1**

It was proved that the sequence $\{x_n\}$ generated by Algorithm 4.1.1 converges strongly to an element r of the solution set Γ of the MIP (1.1.3), where $r = P_\Gamma \circ f(r)$.

Inspired by the above result and in the current research interest in this direction, we propose an inertial algorithm with Armijo-like search technique for approximating solutions of MIP (where A is a Lipschitz continuous and monotone operator and B is a maximal monotone operator) and common fixed points of a finite family of demimetric mappings in real Hilbert spaces. Our algorithm is designed so that its convergence does not require the prior estimate of the Lipschitz constant of A in the MIP and we obtain a strong convergence theorem for the sequence generated by our algorithm. We further present some numerical examples to illustrate the performance of our method as well as comparing it with some related methods in the literature.

Subsequent sections of this chapter are organised as follows: In Section 4.2, we recall some basic definitions and lemmas that are relevant in establishing our main results. In Section 4.3, we prove some lemmas that are useful in establishing the strong convergence of our proposed algorithm and also prove the strong convergence theorem for the algorithm. In Section 4.4, we give an application and some numerical examples to illustrate the performance of our method as well as comparing it with some related methods in the literature.

4.2 Preliminaries

In this section, we will recall some basic notions and useful lemmas which will be needed in the sequel.

Definition 4.2.1. A mapping $A : H \to H$ is said to be l-demimetric if there exists $l \in (-\infty, 1)$ and $F(A) \neq \emptyset$ such that

$$\|Ax - t\|^2 \leq \|x - t\|^2 + l\|x - Ax\|^2, \ \forall x \in H, t \in F(A).$$

Definition 4.2.2. Let C be a nonempty closed convex subset of a real Hilbert space H, the *metric projection* $P_C : H \to C$ is defined, for each $x \in H$, as the unique element $P_C x \in C$ such that

$$||x - P_C x|| = \inf\{||x - y|| : y \in C\}.$$

It is known that P_C is nonexpansive and has the following properties:

(i) $\langle x-y, P_Cx-P_Cy\rangle \geq ||P_Cx-P_Cy||^2$, for every $x,y\in H$;

(ii) for $x\in H$ and $z\in C$, $z=P_Cx \Leftrightarrow$

$$\langle x-z, z-y\rangle \geq 0, \quad \forall y\in C; \tag{4.2.1}$$

(iii) for $x\in H$ and $y\in C$,

$$||y-P_C(x)||^2+||x-P_C(x)||^2 \leq ||x-y||^2. \tag{4.2.2}$$

Lemma 4.2.1. *[7] Let C be a nonempty closed convex subset of a real Hilbert space H and $A: C\to H$ be a mapping on H, $B: H\to 2^H$ be a maximal monotone operator. Then we have*

$$F(J_r^B(I-rA)) = (A+B)^{-1}(0), \forall r>0,$$

where $F(J_r^B(I-rA))$ is the set of the fixed points of $J_r^B(I-rA)$.

Lemma 4.2.2. *[46, 63] Let $\{\alpha_n\}$ and $\{\delta_n\}$ be sequences of non-negative real numbers such that*

$$\alpha_{n+1} \leq (1-\delta_n)\alpha_n+\beta_n+\gamma_n, \quad n\geq 1,$$

where $\{\delta_n\}$ is a sequence in $(0,1)$ and $\{\beta_n\}$ is a real sequence. Assume that $\sum_{n=0}^{\infty}\gamma_n<\infty$. Then the following results hold:

(i) If $\beta_n\leq \delta_n M$ for some $M\geq 0$, then $\{\alpha_n\}$ is a bounded sequence.

(ii) If $\sum_{n=0}^{\infty}\delta_n=\infty$ and $\limsup_{n\to\infty}\frac{\beta_n}{\delta_n}\leq 0$, then $\lim_{n\to\infty}\alpha_n=0$.

Lemma 4.2.3. *[18] Let $A: H\to H$ be a Lipschitz continuous and monotone mapping and $B: H\to 2^H$ be a maximal monotone operator. Then $A+B$ is a maximal monotone mapping.*

Lemma 4.2.4. *[2] Let H be a real Hilbert space, then the following inequalities hold:*

(i) $\|x-y\|^2=\|x\|^2-2\langle x,y\rangle+\|y\|^2, \quad x,y\in H,$

(ii) $\|x+y\|^2\leq \|x\|^2+2\langle y, x+y\rangle, \qquad x,y\in H.$

Lemma 4.2.5. *[99] Let $x_i\in H, (1\leq i\leq m), \sum_{i=1}^m \alpha_i=1$, where $\{\alpha_i\}\subseteq(0,1)$. Then*

$$\left\|\sum_{i=1}^m \alpha_i x_i\right\|^2=\sum_{i=1}^m \alpha_i\|x_i\|^2-\sum_{i=j=1, i\neq j}^m \alpha_i\alpha_j\|x_i-x_j\|^2.$$

4.3 Main result

In this section, we assume that $\Gamma = (A+B)^{-1}(0) \cap \bigcap_{i=1}^{m} F(S_i) \neq \emptyset$, where $A : H \to H$ is a L-Lipschitz continuous and monotone mapping, $B : H \to 2^H$ is a maximal monotone operator and $S_i : H \to H$ is a finite family of demimetric mappings with constant l_i for $i = 1, 2, ...m$. Let $f : H \to H$ be σ contraction with constant $\sigma \in (0,1)$, D be a bounded operator with co-efficient $\rho > 0$, such that $0 < \xi < \frac{\rho}{\sigma}$ and let $\{\alpha_n\}$, $\{\epsilon_n\}$, $\{\beta_{n,i}\}$ be non-negative sequences such that $0 < a \leq \epsilon_n,\ \beta_{n,i},\ \alpha_n \leq b < 1$, and $\theta \geq 3$. We propose the following algorithm for approximating the common solutions of MIP and FPP of a finite family of demimetric mappings in real Hilbert spaces.

Algorithm 4.3.1.
Step 0: Select initial guess $x_0, x_1 \in H$ and set $n = 1$.
Step 1: Given the $(n-1)th$ and nth iterates, choose θ_n such that $0 \leq \theta_n \leq \tilde{\theta}_n$ with $\tilde{\theta}_n$ defined by

$$\tilde{\theta}_n = \begin{cases} \min\left\{\frac{n-1}{n+\theta-1}, \frac{\epsilon_n}{||x_n - x_{n-1}||}\right\}, & if \quad x_n \neq x_{n-1}, \\ \frac{n-1}{n+\theta-1} & otherwise. \end{cases} \tag{4.3.1}$$

Step 2: Compute

$$y_n = J_{r_n}^B(w_n - r_n A(w_n)),$$

where $w_n = x_n + \theta_n(x_n - x_{n-1})$ and r_n is selected to be the largest $r \in \{\lambda, \lambda p, \lambda p^2, \dots\}$ satisfying

$$r\langle Aw_n - Ay_n, w_n - y_n\rangle \leq \sigma\|w_n - y_n\|^2, \quad \text{where } \lambda > 0, \ \ p \in (0,1) \ \text{ and } \ \sigma \in (0,1). \tag{4.3.2}$$

If $y_n = w_n$, then $y_n \in (A+B)^{-1}(0)$. In this case, set $y_n = z_n$ and go to Step 4, otherwise do Step 3.
Step 3 : Compute

$$z_n = w_n - \tau\mu_n c_n, \quad \forall\, \tau \in (0,2),$$

where

$$c_n = w_n - y_n - r_n(Aw_n - Ay_n),$$

and

$$\mu_n = (1-\sigma)\frac{\|w_n - y_n\|^2}{\|c_n\|^2}.$$

Step 4 : Compute

$$x_{n+1} = \alpha_n \xi f(x_n) + (1 - \alpha_n D)\left[\beta_{n,0} z_n + \sum_{i=1}^{m} \beta_{n,i} S_i z_n\right].$$

Set $n := n+1$ and go to **Step 1**.

Remark 4.3.2: We note that the step size r_n defined in Algorithm 4.3.1 is well defined and $\min\left\{\lambda, \frac{\sigma p}{L}\right\} \leq r_n \leq \lambda$ (see [103]).

The following assumptions will be useful to establish our main result:

$(C1)$ $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$;

$(C2)$ $\liminf_{n\to\infty}(\beta_{n,0} - l)\beta_{n,i} > 0$ for all $i = 1, 2, ...m$, where $l = \max_{1\le i\le m}\{l_i\}$;

$(C3)$ $\epsilon_n = o(\alpha_n)$ i.e, $\lim_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0$ $\left(\text{for example } \epsilon_n = \frac{1}{(n+1)^2}, \ \alpha_n = \frac{1}{n+1}\right)$.

Remark 4.3.3: From (4.3.1) and (C3), we have $\lim_{n\to\infty} \theta_n\|x_n - x_{n-1}\| = 0$ and $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$.

The following lemmas will be useful to prove the strong convergence of the sequence generated by our proposed algorithm.

Lemma 4.3.2. *Let $\{x_n\}$ be a sequence generated by Algorithm 4.3.1. Then we have*

$$\|z_n - t\|^2 \le \|w_n - t\|^2 - \frac{2-\tau}{\tau}\|w_n - z_n\|^2, \quad \forall \ t \in \Gamma \ \textit{ and } \ n \ge 0.$$

Proof. From (4.3.2), we have

$$\begin{aligned}
\langle c_n, w_n - t\rangle &= \langle c_n, w_n - y_n\rangle + \langle c_n, y_n - t\rangle \\
&= \langle w_n - y_n - r_n(Aw_n - Ay_n), w_n - y_n\rangle + \langle w_n - y_n - r_n(Aw_n - Ay_n), y_n - t\rangle \\
&= \|w_n - y_n\|^2 - \langle r_n(Aw_n - Ay_n), w_n - y_n\rangle + \langle w_n - y_n - r_n(Aw_n - Ay_n), y_n - t\rangle \\
&\ge \|w_n - y_n\|^2 - \sigma\|w_n - y_n\|^2 + \langle w_n - y_n - r_n(Aw_n - Ay_n), y_n - t\rangle. \quad (4.3.3)
\end{aligned}$$

Since $y_n = J^B_{r_n}(w_n - r_nAw_n)$, thus $w_n - r_nAw_n \in y_n - r_nBy_n$. Hence

$$\frac{1}{r_n}(w_n - y_n - r_nAw_n) \in By_n.$$

Since $0 \in (A+B)t$, $Ay_n + v_n \in (A+B)y_n$, then by Lemma 4.2.3, we obtain $A+B$ is a maximal monotone mapping. Hence, we get

$$\frac{1}{r_n}\langle y_n - t, r_nAy_n + w_n - y_n - r_nAw_n\rangle \ge 0.$$

Which implies that

$$\langle y_n - t, w_n - y_n - r_n(Aw_n - Ay_n)\rangle \ge 0. \qquad (4.3.4)$$

Thus, from (4.3.3) and (4.3.4), we have

$$\langle c_n, w_n - t\rangle \ge (1-\sigma)\|w_n - y_n\|^2.$$

Also,

$$\begin{aligned}
\|z_n - t\|^2 &= \|w_n - \tau\mu_nc_n - t\|^2 \\
&= \|w_n - t\|^2 + \tau^2\|\mu_nc_n\|^2 - 2\tau\mu_n\langle w_n - t, c_n\rangle \\
&\le \|w_n - t\|^2 + \tau^2\|\mu_nc_n\|^2 - 2\tau\mu_n(1-\sigma)\|w_n - y_n\|^2. \qquad (4.3.5)
\end{aligned}$$

From the Algorithm, we have

$$\mu_n\|c_n\|^2 = (1-\sigma)\|w_n - y_n\|^2,$$

thus

$$2\tau\mu_n(1-\sigma)\|w_n - y_n\|^2 = 2\tau\|\mu_n c_n\|^2,$$

hence, we have

$$\begin{aligned}\|z_n - t\|^2 &\le \|w_n - t\|^2 + \tau^2\|\mu_n c_n\|^2 - 2\tau\|\mu_n c_n\|^2 \\ &= \|w_n - t\|^2 - \frac{2-\tau}{\tau}\|\tau\mu_n c_n\|^2. \end{aligned} \tag{4.3.6}$$

Since $z_n = w_n - \tau\mu_n c_n$. Hence

$$w_n - z_n = \tau\mu_n c_n. \tag{4.3.7}$$

By substituting (4.3.7) in (4.3.6), we have

$$\|z_n - t\|^2 \le \|w_n - t\|^2 - \frac{2-\tau}{\tau}\|w_n - z_n\|^2. \tag{4.3.8}$$

□

Lemma 4.3.3. *Let $\{x_n\}$ be a sequence generated by Algorithm 4.3.1. Then*

$$\|w_n - y_n\|^2 \le \frac{1+L^2\lambda^2}{[(1-\sigma)\tau]^2}\|w_n - z_n\|^2.$$

Proof. Since A is a monotone mapping, then we have

$$\begin{aligned}\|c_n\|^2 &\le \|w_n - y_n - r_n(Aw_n - Ay_n)\|^2 \\ &= \|w_n - y_n\|^2 + r_n^2\|Aw_n - Ay_n\|^2 - 2r_n\langle w_n - y_n, Aw_n - Ay_n\rangle \\ &\le (1+L^2\lambda^2)\|w_n - y_n\|^2,\end{aligned}$$

which implies that

$$\frac{1}{(1+L^2\lambda^2)\|w_n - y_n\|^2} \le \frac{1}{\|c_n\|^2}. \tag{4.3.9}$$

Hence

$$\frac{1-\sigma}{1+L^2\lambda^2} \le \mu_n = (1-\sigma)\frac{\|w_n - y_n\|^2}{\|c_n\|^2}. \tag{4.3.10}$$

Therefore,

$$\begin{aligned}\|w_n - y_n\|^2 &= \frac{1}{1-\sigma}\mu_n\|c_n\|^2 = \frac{1}{1-\sigma}\|\tau\mu_n c_n\|^2.\frac{1}{\tau^2}.\frac{1}{\mu_n} \\ &= \frac{1}{1-\sigma}\|w_n - z_n\|^2.\frac{1}{\mu_n\tau^2}.\end{aligned} \tag{4.3.11}$$

From (4.3.10) and (4.3.11), we have

$$\|w_n - y_n\|^2 \le \frac{1+L^2\lambda^2}{[(1-\sigma)\tau]^2}\|w_n - z_n\|^2. \tag{4.3.12}$$

□

Lemma 4.3.4. *The sequence* $\{x_n\}$ *generated by Algorithm* (4.3.1) *is bounded.*

Proof. Let $t \in \Gamma$. This implies that $J_{r_n}^B(I - r_n A)t = t$. Then,

$$\begin{aligned} \|w_n - t\| &= \|x_n + \alpha_n(x_n - x_{n-1}) - t\| \\ &\leq \|x_n - t\| + \alpha_n\|x_n - x_{n-1}\|. \end{aligned} \tag{4.3.13}$$

Furthermore, let $v_n = \beta_{n,0}z_n + \sum_{i=1}^m \beta_{n,i}S_i z_n$ and by Lemma 4.2.5, we have

$$\begin{aligned} \|v_n - t\|^2 &= \left\|\beta_{n,0}z_n + \sum_{i=1}^m \beta_{n,i}S_i z_n - t\right\|^2 \\ &\leq \beta_{n,0}\|z_n - t\|^2 + \sum_{i=1}^m \beta_{n,i}\|S_i z_n - t\|^2 - \sum_{i=1}^m \beta_{n,0}\beta_{n,i}\|z_n - S_i z_n\|^2 \\ &\leq \beta_{n,0}\|z_n - t\|^2 + \sum_{i=1}^m \beta_{n,i}(\|z_n - t\|^2 + l_i\|z_n - S_i z_n\|^2) - \sum_{i=1}^m \beta_{n,0}\beta_{n,i}\|z_n - S_i z_n\|^2 \end{aligned}$$

$$\begin{aligned} &\leq \beta_{n,0}\|z_n - t\|^2 + \sum_{i=1}^m \beta_{n,i}\|z_n - t\|^2 + \sum_{i=1}^m \beta_{n,i}l\|z_n - S_i z_n\|^2 - \sum_{i=1}^m \beta_{n,0}\beta_{n,i}\|z_n - S_i z_n\|^2 \\ &= \|z_n - t\|^2 - \sum_{i=1}^m (\beta_{n,0} - l)\beta_{n,i}\|z_n - S_i z_n\|^2. \end{aligned} \tag{4.3.14}$$

Then from condition (C2), we have

$$\|v_n - t\| \leq \|z_n - t\|. \tag{4.3.15}$$

Moreover, from (4.3.8), we can obtain

$$\|z_n - t\| \leq \|w_n - t\|. \tag{4.3.16}$$

From (4.3.13), (4.3.15) and (4.3.16), we get

$$\begin{aligned} \|x_{n+1} - t\| &= \|\alpha_n \xi f(x_n) + (1 - \alpha_n D)v_n\| \\ &\leq \|\alpha_n(\xi f(x_n) - Dt) + (1 - \alpha_n D)(v_n - t)\| \\ &\leq \alpha_n\|\xi f(x_n) - Dt\| + (1 - \alpha_n\rho)\|v_n - t\| \\ &\leq \alpha_n\big[\|\xi(f(x_n) - f(t)) + (\xi f(t) - Dt)\|\big] + (1 - \alpha_n\rho)\|v_n - t\| \\ &\leq \alpha_n\xi\sigma\|x_n - t\| + \alpha_n\|\xi f(t) - Dt\| + (1 - \alpha_n\rho)\big[\|x_n - t\| + \theta_n\|x_n - x_{n-1}\|\big] \\ &= (1 - \alpha_n(\rho - \xi\sigma))\|x_n - t\| + \alpha_n\|\xi f(t) - Dt\| + (1 - \alpha_n\rho)\theta_n\|x_n - x_{n-1}\| \\ &= (1 - \alpha_n(\rho - \xi\sigma))\|x_n - t\| + (\rho - \xi\sigma)\alpha_n\bigg\{\frac{\|\xi f(t) - Dt\|}{\rho - \xi\sigma} \\ &+ \left(\frac{1 - \alpha_n\rho}{\rho - \xi\sigma}\right)\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|\bigg\}. \end{aligned}$$

It is known that $\sup_{n\geq 1}\left(\frac{1-\alpha_n\rho}{\rho-\xi\sigma}\right)\beta_{n,0}\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|$ exists by Remark 3.3. Suppose

$$M := \max\left\{\frac{\|\xi f(t) - Dt\|}{\|\rho - \xi\sigma\|}, \sup_{n\geq 1}\left(\frac{1-\alpha_n\rho}{\rho-\xi\sigma}\right)\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|\right\}.$$

Thus, we have

$$\|x_{n+1} - t\| \leq (1 - \alpha_n(\rho - \xi\sigma))\|x_n - t\| + \alpha_n(\rho - \xi\sigma)M. \tag{4.3.17}$$

Hence, by (4.3.17) and Lemma 4.2.2(i), $\{\|x_n - t\|\}$ is bounded and thus $\{x_n\}$ is bounded. □

Lemma 4.3.5. *Let $\{x_n\}$ be a sequence generated by Algorithm 4.3.1. Let $s_n = \|x_n - t\|^2$, $\tilde{a}_n = \frac{2\alpha_n(\rho-\xi\sigma)}{1-\alpha_n\xi\sigma}$,*
$b_n = \frac{1}{2(\rho-\xi\sigma)}\big(2\langle\xi f(t) - Dt, x_{n+1} - t\rangle + \alpha_n M_1\big)$, for some $M_1 > 0$ and $c_n = \frac{\theta_n\|x_n - x_{n-1}\|}{1-\alpha_n\xi\sigma}M_2$, where
$M_2 = \sup_{n\geq 1}\bigg((1-\alpha_n\rho)^2(\|x_n - t\| + \|x_{n-1} - t\|) + 2((1-\alpha_n\rho)^2\|x_n - x_{n-1}\|\bigg)$ and $t \in \Gamma$. Then the inequality below holds:

$$s_{n+1} \leq (1 - \tilde{a}_n)s_n + \tilde{a}_n b_n + c_n.$$

Proof. It is already known that

$$\begin{aligned}\|w_n - t\|^2 &= \|x_n + \theta_n(x_n - x_{n-1})\|^2\\ &= \|x_n - t\|^2 + 2\theta_n\langle x_n - t, x_n - x_{n-1}\rangle + \theta_n^2\|x_n - x_{n-1}\|^2.\end{aligned} \tag{4.3.18}$$

By Lemma 4.2.4(i), we have

$$2\langle x_n - t, x_n - x_{n-1}\rangle = -\|x_{n-1} - t\|^2 + \|x_n - t\|^2 + \|x_n - x_{n-1}\|^2, \tag{4.3.19}$$

and substituting (4.3.19) into (4.3.18), we obtain

$$\begin{aligned}\|w_n - t\|^2 &= \|x_n - t\|^2 + \theta_n(-\|x_{n-1} - t\|^2 + \|x_n - t\|^2 + \|x_n - x_{n-1}\|^2) + \theta_n^2\|x_n - x_{n-1}\|^2\\ &\leq \|x_n - t\|^2 + \theta_n(\|x_n - t\|^2 - \|x_{n-1} - t\|^2) + 2\theta_n\|x_n - x_{n-1}\|^2.\end{aligned} \tag{4.3.20}$$

From Lemma 4.2.4(ii), we have

$$\begin{aligned}\|x_{n-1} - t\|^2 &= \|\alpha_n(\xi f(x_n) - Dt) + (1 - \alpha_n D)(v_n - t)\|^2\\ &\leq (1-\alpha_n\rho)^2\|v_n - t\|^2 + 2\alpha_n\langle\xi f(x_n) - Dt, x_{n+1} - t\rangle.\end{aligned} \tag{4.3.21}$$

From (4.3.15), (4.3.16), (4.3.18), we have

$$\begin{aligned}\|x_{n-1} - t\|^2 &\leq (1-\alpha_n\rho)^2\|w_n - t\|^2 + 2\alpha_n\langle\xi f(x_n) - Dt, x_{n+1} - t\rangle\\ &= (1-\alpha_n\rho)^2\bigg(\|x_n - t\|^2 + \theta_n(\|x_n - t\|^2 - \|x_{n-1} - t\|^2) + 2\theta_n\|x_n - x_{n-1}\|^2\bigg)\\ &+ 2\alpha_n\langle\xi f(x_n) - Dt, x_{n+1} - t\rangle\\ &= (1-\alpha_n\rho)^2\|x_n - t\|^2 + \theta_n(1-\alpha_n\rho)^2(\|x_n - t\|^2 - \|x_{n-1} - t\|^2)\\ &+ 2\theta_n(1-\alpha_n\rho)^2\|x_n - x_{n-1}\|^2 + 2\alpha_n\langle\xi f(x_n) - Dt, x_{n+1} - t\rangle\\ &\leq (1-\alpha_n\rho)^2\|x_n - t\|^2 + \theta_n(1-\alpha_n\rho)^2(\|x_n - t\| + \|x_{n-1} - t\|)\|x_n - x_{n-1}\|\\ &+ 2\theta_n(1-\alpha_n\rho)^2\|x_n - x_{n-1}\|^2 + 2\alpha_n\langle\xi f(x_n) - Dt, x_{n+1} - t\rangle.\end{aligned} \tag{4.3.22}$$

Furthermore,

$$\begin{aligned}2\langle \xi f(x_n) - Dt, x_{n+1} - t\rangle &= 2\langle \xi(f(x_n) - f(t)) + \xi f(t) - Dt, x_{n+1} - t\rangle\\ &\le 2\xi\sigma\|x_n - t\|.\|x_{n+1} - t\| + 2\langle \xi f(t) - Dt, x_{n+1} - t\rangle\\ &\le \xi\sigma(\|x_n - t\|^2 + \|x_{n+1} - t\|^2) + 2\langle \xi f(t) - Dt, x_{n+1} - t\rangle.\end{aligned} \tag{4.3.23}$$

By substituting (4.3.23) into (4.3.22), we obtain

$$\begin{aligned}\|x_{n+1} - t\|^2 \le& \big[(1-\alpha_n\rho)^2 + \alpha_n\xi\sigma\big]\|x_n - t\|^2 + \theta_n(1-\alpha_n\rho)^2(\|x_n - t\| + \|x_{n-1} - t\|)\|x_n - x_{n-1}\|\\ &+ 2\theta_n(1-\alpha_n\rho)^2\|x_n - x_{n-1}\|^2 + \alpha_n\xi\sigma\|x_{n+1} - t\|^2 + 2\alpha_n\langle \xi f(t) - Dt, x_{n+1} - t\rangle\\ &= \big(1 - \alpha_n(2\rho - \xi\sigma)\big)\|x_n - t\|^2 + (\alpha_n\rho)^2\|x_n - t\|^2 + \theta_n\Bigg[(1-\alpha_n\rho)^2(\|x_n - t\|\\ &+ \|x_{n-1} - t\|) + 2(1-\alpha_n\rho)^2\|x_n - x_{n-1}\|\Bigg]\|x_n - x_{n-1}\| + \alpha_n\xi\sigma\|x_{n+1} - t\|^2\\ &+ 2\alpha_n\langle \xi f(t) - Dt, x_{n+1} - t\rangle\end{aligned}$$

$$\begin{aligned}&\le \big(1 - \alpha_n(2\rho - \xi\sigma)\big)\|x_n - t\|^2 + \alpha_n\xi\sigma\|x_{n+1} - t\|^2\\ &+ \theta_n\Bigg[(1-\alpha_n\rho)^2(\|x_n - t\| + \|x_{n-1} - t\|) + 2(1-\alpha_n\rho)^2\|x_n - x_{n-1}\|\Bigg]\|x_n - x_{n-1}\|\\ &+ \alpha_n\big(2\langle \xi f(t) - Dt, x_{n+1} - t\rangle + \alpha_n M_1\big).\end{aligned}$$

For some $M_1 \ge 0$. Thus

$$\begin{aligned}\|x_{n+1} - t\|^2 &\le \frac{\big(1 - \alpha_n(2\rho - \xi\sigma)\big)}{1 - \alpha_n\xi\sigma}\|x_n - t\|^2 + \frac{\theta_n}{1 - \alpha_n\xi\sigma}\|x_n - x_{n-1}\|M_2\\ &+ \frac{\alpha_n(2\langle \xi f(t) - Dt, x_{n+1} - t\rangle + \alpha_n M_1)}{1 - \alpha_n\xi\sigma}\\ &= \left(1 - \frac{2\alpha_n(\rho - \xi\sigma)}{1 - \alpha_n\xi\sigma}\right)\|x_n - t\|^2 + \frac{\theta_n}{1 - \alpha_n\xi\sigma}\|x_n - x_{n-1}\|M_2\\ &+ \frac{2\alpha_n(\rho - \xi\sigma)}{1 - \alpha_n\xi\sigma} \times \frac{\big(2\langle \xi f(t) - Dt, x_{n+1} - t\rangle + \alpha_n M_1\big)}{2(\rho - \xi\sigma)}.\end{aligned} \tag{4.3.24}$$

Hence, we obtain the desired result.

□

Lemma 4.3.6. *Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by Algorithm 4.3.1 and $\{x_{n_k}\} \subset \{x_n\}$. If*

$$\lim_{n\to\infty} \|x_n - y_n\| = 0$$

and $x_{n_k} \rightharpoonup t$, then $t \in \Gamma = (A+B)^{-1}(0) \cap \bigcap_{i=1}^m F(S_i)$.

Proof. Let $u - Av \in Bv$ and $y_{n_k} = (I + r_{n_k} B)^{-1}(I - r_{n_k} A)x_{n_k}$. Thus we have

$$(I - r_{n_k} A)x_{n_k} \in (I + r_{n_k} B)y_{n_k},$$

hence,

$$\frac{1}{r_{n_k}}(x_{n_k} - y_{n_k} - r_{n_k} Ax_{n_k}) \in By_{n_k}.$$

Since B is maximal monotone, we have

$$\langle u - Av - \frac{1}{r_{n_k}}(x_{n_k} - y_{n_k} - r_{n_k} Ax_{n_k}), v - y_{n_k}\rangle \geq 0,$$

which implies that

$$\begin{aligned}
\langle u, v - y_{n_k}\rangle &\geq \langle Av + \frac{1}{r_{n_k}}(x_{n_k} - y_{n_k} - r_{n_k} Ax_{n_k}), v - y_{n_k}\rangle \\
&= \langle Av - Ax_{n_k}, v - y_{n_k}\rangle + \langle \frac{1}{r_{n_k}}(x_{n_k} - y_{n_k}), v - y_{n_k}\rangle \\
&= \langle Av - Ay_{n_k}, v - y_{n_k}\rangle + \langle Ay_{n_k} - Ax_{n_k}, v - y_{n_k}\rangle + \langle \frac{1}{r_{n_k}}(x_{n_k} - y_{n_k}), v - y_{n_k}\rangle \\
&\geq \langle Ay_{n_k} - Ax_{n_k}, v - y_{n_k}\rangle + \langle \frac{1}{r_{n_k}}(x_{n_k} - y_{n_k}), v - y_{n_k}\rangle.
\end{aligned}$$

We note that $r_n > \frac{\sigma m}{L}$ and $\lim_{n\to\infty} \|x_n - y_n\| = 0$. Hence we have

$$\langle u, v - t\rangle = \lim_{k\to\infty} \langle u, v - y_{n_k}\rangle \geq 0. \tag{4.3.25}$$

From (4.3.25) and Lemma 4.2.3, we can conclude that $0 \in (A + B)t$, which implies $t \in \Gamma$. $\square$

Theorem 4.3.7. *Let C be a nonempty closed convex subset of a real Hilbert space H, $A : C \to H$ be a Lipschitz continuous and monotone mapping and $B : H \to 2^H$ be a maximal monotone operator. Let $S_i : H \to H$ be a finite family of demimetric mappings with constant l such that $l = \max\{l_i\}$ and $I - S_i$ are demiclosed at zero, for $i = 1, 2, ..., m$. Let $\Gamma = (A + B)^{-1}(0) \cap \bigcap_{i=1}^{m} F(S_i) \neq \emptyset$ and $f : H \to H$ be a σ contraction , where $\sigma \in (0, 1)$ and D is a bounded operator with co-efficient $\rho > 0$ such that $0 < \xi < \frac{\rho}{\sigma}$. Let $\{x_n\}$ be generated by Algorithm 4.3.1 and suppose Assumptions $(C1), (C2)$ and $(C3)$ are satisfied. Then the sequence $\{x_n\}$ converges strongly to a solution $t = P_\Gamma(I - D + \xi f)(t)$ which is also a unique solution of the variational inequality*

$$\langle (D - \xi f)t, t - x\rangle \leq 0, \quad x \in \Gamma. \tag{4.3.26}$$

Proof. Let $t \in \Gamma$ and $\Phi_n = ||x_n - t||^2$. Suppose there exists $n_0 \in \mathbb{N}$ such that Φ_n is monotonically non-increasing for all $n \geq n_0$. Since Φ_n is bounded, then it is convergent and thus $\Phi_n - \Phi_{n+1} \to 0$ as $n \to \infty$. From (4.3.14), (4.3.16), (4.3.20) and (4.3.21), we have

$$\begin{aligned}
\|x_{n+1} - t\|^2 &\leq (1 - \alpha_n\rho)^2 \|v_n - t\|^2 + 2\alpha_n \langle \xi f(x_n) - Dt, x_{n+1} - t\rangle \\
&\leq (1 - \alpha_n\rho)^2 \left\{ \|z_n - t\|^2 - \sum_{i=1}^{m} (\beta_{n,0} - l)\beta_{n,i} \|z_n - S_i z_n\|^2 \right\} \\
&+ 2\alpha_n \langle \xi f(x_n) - Dt, x_{n+1} - t\rangle \\
&\leq (1 - \alpha_n\rho)^2 \left\{ \|x_n - t\|^2 + \theta_n(\|x_n - t\|^2 - \|x_{n-1} - t\|^2) + 2\theta_n \|x_n - x_{n-1}\|^2 \right. \\
&\left. - \sum_{i=1}^{m} (\beta_{n,0} - l)\beta_{n,i} \|z_n - S_i z_n\|^2 \right\} + 2\alpha_n \langle \xi f(x_n) - Dt, x_{n+1} - t\rangle.
\end{aligned}$$

Thus

$$\begin{aligned}
(1 - \alpha_n\rho)^2 \sum_{i=1}^{m} (\beta_{n,0} - l)\beta_{n,i} \|z_n - S_i\|^2 &\leq (1 - \alpha_n\rho)^2 \|x_n - t\|^2 \\
&+ \theta_n (1 - \alpha_n\rho)^2 (\|x_n - t\|^2 - \|x_{n-1} - t\|^2) \\
&+ 2\theta_n (1 - \alpha_n\rho)^2 \|x_n - x_{n-1}\|^2 + 2\alpha_n \langle \xi f(x_n) - Dt, x_{n+1} - t\rangle \\
&- \|x_{n+1} - t\|^2 \\
&\leq \Phi_n - \Phi_{n+1} + \alpha_n M_3 + \theta_n (1 - \alpha_n\rho)^2 (\Phi_n - \Phi_{n-1}) \\
&+ 2\theta_n (1 - \alpha_n\rho)^2 \|x_n - x_{n-1}\|^2 \\
&+ 2\alpha_n \langle \xi f(x_n) - Dt, x_{n+1} - t\rangle \to 0.
\end{aligned}$$

Hence, by condition $(C2)$, we have

$$\lim_{n\to\infty} \|z_n - S_i z_n\| = 0. \tag{4.3.27}$$

Furthermore,

$$\begin{aligned}
\|v_n - z_n\| &= \|\beta_{n,0} z_n + \sum_{i=1}^{m} \beta_{n,i} S_i z_n - z_n\| \\
&\leq \beta_{n,0} \|z_n - z_n\| + \sum_{i=1}^{m} \beta_{n,i} \|S_i z_n - z_n\| \to 0.
\end{aligned}$$

This implies that

$$\lim_{n\to\infty} \|v_n - z_n\| = 0.$$

From the definition of w_n and Remark 4.3.3, we obtain

$$\begin{aligned}
\|w_n - x_n\| &= \|x_n + \theta_n(x_n - x_{n-1}) - x_n\| \\
&= \theta_n \|x_n - x_{n-1}\| \to 0, \quad \text{as} \quad n \to \infty.
\end{aligned} \tag{4.3.28}$$

Furthermore, from Lemma 4.3.2 and (4.3.15), we have

$$\begin{aligned}
\|x_{n+1}-t\|^2 &\leq (1-\alpha_n\rho)^2\|v_n-t\|^2+2\alpha_n\langle \xi f(x_n)-Dt, x_{n+1}-t\rangle\\
&\leq (1-\alpha_n\rho)^2\|z_n-t\|^2+2\alpha_n\langle \xi f(x_n)-Dt, x_{n+1}-t\rangle\\
&\leq (1-\alpha_n\rho)^2\Big\{\|w_n-t\|^2-\frac{2-\tau}{\tau}\|w_n-z_n\|^2\Big\}+2\alpha_n\langle \xi f(x_n)-Dt, x_{n+1}-t\rangle\\
&\leq (1-\alpha_n\rho)^2\Big\{\|x_n-t\|^2-\frac{2-\tau}{\tau}\|w_n-z_n\|^2+\theta_n(\|x_n-t\|^2-\|x_{n-1}-t\|^2)\\
&+2\theta_n\|x_n-x_{n-1}\|^2\Big\}+2\alpha_n\langle \xi f(x_n)-Dt, x_{n+1}-t\rangle.
\end{aligned}$$

Hence

$$\begin{aligned}
(1-\alpha_n\rho)^2\frac{2-\tau}{\tau}\|w_n-z_n\|^2 &\leq \Phi_n-\Phi_{n+1}+\alpha_nM_3+\theta_n(1-\alpha_n\rho)^2(\Phi_n-\Phi_{n-1})\\
&+ 2\theta_n(1-\alpha_n\rho)^2\|x_n-x_{n-1}\|^2+2\alpha_n\langle \xi f(x_n)-Dt, x_{n+1}-t\rangle\to 0,
\end{aligned}$$

for some $M_3>0$. Hence

$$\lim_{n\to\infty}||w_n-z_n||=0. \tag{4.3.29}$$

Therefore from Lemma 4.3.3, we obtain

$$\lim_{n\to\infty}||w_n-y_n||=0. \tag{4.3.30}$$

Consequently from (4.3.28), we get

$$\lim_{n\to\infty}||x_n-y_n||=0. \tag{4.3.31}$$

Moreover,

$$\begin{aligned}
\|x_n-z_n\| &= \|x_n-w_n+w_n-z_n\|\\
&\leq \|x_n-w_n\|+\|w_n-z_n\|\to 0.
\end{aligned} \tag{4.3.32}$$

Also,

$$\begin{aligned}
\|v_n-x_n\| &= \|v_n-z_n+z_n-x_n\|\\
&\leq \|v_n-z_n\|+\|z_n-x_n\|\to 0.
\end{aligned}$$

From $(C1)$, we obtain

$$\begin{aligned}
\|x_{n+1}-v_n\| &= \|\alpha_n\xi f(x_n)+(1-\alpha_nD)v_n-v_n\|\\
&= \alpha_n\|\xi f(x_n)-Dv_n\|\to 0.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|x_{n+1}-x_n\| &= \|x_{n+1}-v_n+v_n-x_n\|\\
&\leq \|x_{n+1}-v_n\|+\|v_n-x_n\|\to 0.
\end{aligned} \tag{4.3.33}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \bar{x} \in H$. From Lemma 4.3.6 and (4.3.31), we have that $\bar{x} \in (A+B)^{-1}(0)$. Also from (4.3.32), we see that $z_{n_k} \rightharpoonup \bar{x}$, where $\{z_{n_k}\} \subset \{z_n\}$. Since $I - S_i$ are demiclosed at zero, it follows from (4.3.27) that $\bar{x} \in F(S_i)$ for $i = 1, 2, \dots, m$. Hence $\bar{x} \in \Gamma := (A+B)^{-1}(0) \cap \bigcap_{i=1}^{m} F(S_i)$.

Next, we show that $\{x_n\}$ converges strongly to x^*, where $x^* = P_\Gamma(I - D + \xi f)x^*$ is a unique solution of the variational inequality

$$\langle (D - \xi f)x^*, x^* - x_n \rangle \leq 0, \quad x \in \Gamma.$$

To do this, we prove that $\limsup_{n\to\infty} \langle (D - \xi f)x^*, x^* - x_n \rangle \leq 0$. Choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{j\to\infty} \langle (D - \xi f)x^*, x^* - x_n \rangle = \lim_{j\to\infty} \langle (D - \xi f)x^*, x^* - x_{n_j} \rangle.$$

Since $x_{n_j} \rightharpoonup \bar{x}$, using (4.2.1), we have that

$$\begin{aligned} \limsup_{j\to\infty} \langle (D - \xi f)x^*, x^* - x_n \rangle &= \lim_{j\to\infty} \langle (D - \xi f)x^*, x^* - x_{n_j} \rangle \\ &= \langle (D - \xi f)x^*, x^* - \bar{x} \rangle \\ &= \langle x^* - (I - (D - \xi f))x^*, x^* - \bar{x} \rangle \leq 0. \end{aligned} \tag{4.3.34}$$

Now, using Lemma 4.2.2, Lemma 4.3.5 and (4.3.34), we obtain that $\|x_n - x^*\| \to 0$, which implies that $\{x_n\}$ converges strongly to x^*.

Now, suppose $\{\|x_n - t\|^2\}$ is not monotonically decreasing. Select some n_0 large enough. Let $\Psi : \mathbb{N} \to \mathbb{N}$ be defined by

$$\Psi(n) = \max\{l \in \mathbb{N} : l \leq n : \Psi_l \leq \Psi_{l+1}\}, \quad \forall \;\; n \geq n_0.$$

Obviously, Ψ is non-decreasing, where $\Psi(n) \to \infty$, as $n \to \infty$ and

$$0 \leq \|x_{\Psi(n)} - t\| \leq \|x_{\Psi(n)+1} - t\|, \quad \forall \;\; n \geq n_0.$$

Moreover, we have

$$\limsup_{n\to\infty} \langle (D - \xi f)t, t - x_{\Psi(n)} \rangle \leq 0. \tag{4.3.35}$$

From (4.3.24), we obtain

$$\begin{aligned} \|x_{\Psi(n)+1} - t\|^2 \leq & \left(1 - \frac{2\alpha_{\Psi(n)}(\rho - \xi\sigma)}{1 - \alpha_{\Psi(n)}\xi\sigma}\right) \|x_{\Psi(n)} - t\|^2 \\ & + \frac{2\alpha_{\Psi(n)}(\rho - \xi\sigma)}{1 - \alpha_{\Psi(n)}\xi\sigma} (2\langle \xi f(t) - Dt, x_{\Psi(n)+1} - t \rangle + \alpha_{\Psi(n)} M) \\ & + \frac{\theta_{\Psi(n)} M_2 \|x_{\Psi(n)} - x_{\Psi(n)-1}\|}{1 - \alpha_{\Psi(n)}\xi\sigma}, \end{aligned} \tag{4.3.36}$$

for some $M > 0$, where

$$M_2 = \sup\Big((1 - \alpha_{\Psi(n)}\rho)^2 (\|x_{\Psi(n)} - t\| + \|x_{\Psi(n)-1} - t\|) + 2(1 - \alpha_{\Psi(n)}\rho)^2 \|x_{\Psi(n)} - x_{\Psi(n)-1}\| \Big).$$

Since $\|x_{\Psi(n)} - t\|^2 \le \|x_{\Psi(n)-1} - t\|^2$, hence from (4.3.36), we obtain,

$$0 \le \left(1 - \frac{2\alpha_{\Psi(n)}(\rho - \xi\sigma)}{1 - \alpha_{\Psi(n)}\xi\sigma}\right)\|x_{\Psi(n)} - t\|^2 + \frac{2\alpha_{\Psi(n)}(\rho - \xi\sigma)}{1 - \alpha_{\Psi(n)}\xi\sigma}(2\langle \xi f(t) - Dt, x_{\Psi(n)+1} - t\rangle + \alpha_{\Psi(n)}M)$$
$$+ \frac{\theta_{\Psi(n)}M_2\|x_{\Psi(n)} - x_{\Psi(n)-1}\|}{1 - \alpha_{\Psi(n)}\xi\sigma} - \|x_{\Psi(n)} - t\|^2.$$

Thus,

$$\begin{aligned}\frac{2\alpha_{\Psi(n)}(\rho - \xi\sigma)}{1 - \alpha_{\Psi(n)}\xi\sigma}\|x_{\Psi(n)} - t\|^2 \le{}& \frac{2\alpha_{\Psi(n)}(\rho - \xi\sigma)}{1 - \alpha_{\Psi(n)}\xi\sigma}(2\langle \xi f(t) - Dt, x_{\Psi(n)+1} - t\rangle + \alpha_{\Psi(n)}M) \\ &+ \frac{\theta_{\Psi(n)}M_2\|x_{\Psi(n)} - x_{\Psi(n)-1}\|}{1 - \alpha_{\Psi(n)}\xi\sigma}.\end{aligned}$$

Hence,

$$\begin{aligned}\|x_{\Psi(n)} - t\|^2 \le{}& 2\langle \xi f(t) - Dt, x_{\Psi(n)+1} - t\rangle + \alpha_{\Psi(n)}M_4 \\ &+ \frac{\theta_{\Psi(n)}M_2\|x_{\Psi(n)} - x_{\Psi(n)-1}\|}{2\alpha_{\Psi(n)}(\rho - \xi\sigma)}.\end{aligned}$$

Since $\{x_{\Psi(n)}\}$ is bounded and $\alpha_{\Psi(n)} \to 0$, as $n \to \infty$, therefore from (4.3.35) and Remark 4.3.3, we have

$$\lim_{n\to\infty} \|x_{\Psi(n)} - t\| = 0.$$

Consequently, for all $n \ge n_0$, we have

$$0 \le \|x_n - t\|^2 \le \max\{\|x_{\Psi(n)} - t\|, \|x_{\Psi(n)+1} - t\|^2\} = \|x_{\Psi(n)+1} - t\|^2.$$

Thus, $\|x_n - t\| \to 0$ as $n \to \infty$. This implies that $x_n \to t$. □

4.4 Application and numerical examples

4.4.1 Application to split feasibility problems

Let H_1 and H_2 be real Hilbert spaces, C and Q be nonempty, closed and convex subsets of H_1 and H_2 respectively. Let $L : H_1 \to H_2$ be a bounded linear operator. The SFP for H_1 and H_2 is defined as finding

$$x \in C \text{ such that } Lx \in Q. \tag{4.4.1}$$

We denote the solution set of SFP (4.4.1) by Ω. The SFP was first introduced in finite dimensional Hilbert space by Censor and Elfving [22] and has received much attention from many researchers due to its application in signal processing, radiotherapy, data compression and many others, (see, for example, [21, 49, 96] and references therein).

The SFP (4.4.1) is equivalent to the following variational inequality problem [58]: find $x \in C$ such that

$$\langle L^*(I - P_Q)Lx, y - x\rangle \geq 0, \ \forall y \in C, \tag{4.4.2}$$

where P_Q is the metric projection from H_2 onto Q. It is worthy to mention that the operator $L^*(I - P_Q)L$ is $\frac{1}{2||L||^2}$-inverse strongly monotone (ism) and the normal cone N_C at $x \in C$ defined by

$$N_C(x) = \{y \in H_1 : \langle y, z - x\rangle \leq 0, \quad \forall z \in H_1\}$$

is maximal monotone. Equivalently, (4.4.2) can be rewritten as the following inclusion problem: find $x \in C$ such that

$$0 \in (L^*(I - P_Q)Lx + N_C(x)).$$

Note that the resolvent $J^{N_C} = P_C$ is the metric projection from H_1 onto C. It is easy to show that $L^*(I - P_Q)L$ is $2||L||^2$-Lipschitzian. Thus, by setting $A = L^*(I - P_Q)L$ and $B = N_C$ in Theorem 4.3.7, we obtain the following result for approximating a common solution of SFP and common fixed point of a finite family of demimetric mappings in Hilbert spaces.

Theorem 4.4.1. *Let H_1 and H_2 be real Hilbert spaces, C and Q be nonempty, closed and convex subsets of H_1 and H_2 respectively. Let $L : H_1 \to H_2$ be a bounded linear operator. For $i = 1, 2, \ldots, m$, let $S_i : H_1 \to H_1$ be a finite family of l_i-demimetric mappings with $l = \max_{1\leq i\leq m} \{l_i\}$ and $I - S_i$ are demiclosed at zero. Suppose*

$$\Gamma = \Omega \cap \bigcap_{i=1}^{m} F(S_i) \neq \emptyset.$$

Let $f : H_1 \to H_1$ be a σ contraction, where $\sigma \in (0, 1)$ and D be a bounded operator with co-efficient $\rho > 0$ such that $0 < \xi < \frac{\rho}{\sigma}$. Assume that Assumptions $(C1), (C2)$ and $(C3)$ are satisfied, then the sequence $\{x_n\}$ generated by the following algorithm converges strongly to a solution $t = P_\Gamma(I - D + \xi f)(t)$ which is also a unique solution of the variational inequality

$$\langle (D - \xi f)t, t - x\rangle \leq 0, \quad x \in \Gamma. \tag{4.4.3}$$

Algorithm 4.4.2.
Step 0: Select initial guess $x_0, x_1 \in H_1$ and set $n = 1$.
Step 1: Given the $(n-1)th$ and nth iterates, choose θ_n such that we have $0 \leq \theta_n \leq \tilde{\theta}_n$, with $\tilde{\theta}_n$ defined by

$$\tilde{\theta}_n = \begin{cases} \min\left\{\frac{n-1}{n+\theta-1}, \frac{\epsilon_n}{||x_n - x_{n-1}||}\right\}, & if \quad x_n \neq x_{n-1}, \\ \frac{n-1}{n+\theta-1} & otherwise. \end{cases} \tag{4.4.4}$$

Step 2: Compute

$$y_n = P_C(w_n - r_n L^*(I - P_Q)Lw_n),$$

where $w_n = x_n + \theta_n(x_n - x_{n-1})$ and r_n is selected to be the largest $r \in \{\lambda, \lambda p, \lambda p^2, \dots\}$ satisfying

$$r\langle L^*(I-P_Q)Lw_n - L^*(I-P_Q)Ly_n, w_n - y_n\rangle \leq \sigma\|w_n - y_n\|^2, \text{ where } \lambda > 0,\ p \in (0,1) \text{ and } \sigma \in (0,1). \tag{4.4.5}$$

If $y_n = w_n$, then $y_n \in \Omega$. In that case, set $y_n = z_n$ and go to Step 4, otherwise do Step 3.
Step 3 : Calculate

$$z_n = w_n - \tau\mu_n c_n, \quad \forall\ \tau \in (0,2),$$

where

$$c_n = w_n - y_n - r_n[L^*(I - P_Q)Lw_n - L^*(I - P_Q)Ly_n],$$

and

$$\mu_n = (1-\sigma)\frac{\|w_n - y_n\|^2}{\|c_n\|^2}.$$

Step 4 : Calculate

$$x_{n+1} = \alpha_n \xi f(x_n) + (1 - \alpha_n D)\Bigg[\beta_{n,0} z_n + \sum_{i=1}^{m} \beta_{n,i} S_i(z_n)\Bigg].$$

Set $n := n+1$ and go to Step 1.

4.4.2 Numerical examples

In this section, we present some numerical examples to illustrate the performance of our method as well as comparing it with some related methods in the literature.

Example 4.4.3. Let $H = L_2([0,1])$ with $\|x\| = \left(\int_0^1 |x(t)|^2 dt\right)^{\frac{1}{2}}$ and inner product $\langle x, y\rangle = \int_0^1 x(t)y(t)dt\ \forall x, y \in H$. We take $C = \{x \in L_2([0,1]) : \int_0^1 (t^2+1)x(t)dt \leq 1\}$, we define $A : C \to H$ by $Ax = \frac{x(t)}{2}$ and $B : H \to 2^H$ by $Bx = \{2x\}$ for all $x \in L_2([0,1])$. Also, for $i = 1, 2, \dots, 10$, we define $S_i : L_2([0,1]) \to L_2([0,1])$ by

$$S_i x = -(i+1)x.$$

It is easy to check that $F(S_i) = \{0\}$ and S_i is $\frac{i}{i+2}$-demimetric for $i \in \mathbb{N}$. Clearly, $\Gamma = \{0\}$. Also, we define $f(x) = \frac{x}{8}$, $Dx = x$, $\xi = 1.2$, $\alpha = 5$, $\alpha_n = \frac{1}{9n+12}$, $\beta_{n,i} = \frac{3n}{5(7n+3)}$, $\beta_{n,0} = \frac{n+3}{7n+3}$, $\epsilon_n = \frac{1}{(5n+1)^2}$. Using $\|x_{n+1} - x_n\|^2 < 10^{-5}$ as stopping criterion, we compare our Algorithm 4.3.1 with Algorithm 4.1.1 of [103] using the following starting points and plot the graph of error ($\|x_{n+1} - x_n\|^2$) against the number of iterations:

$$\begin{aligned}
&\text{Case 1}: \quad x_0 = t^2\exp(t), \quad x_1 = t+3\\
&\text{Case 2}: \quad x_0 = \frac{t^3+2t-1}{4}, \quad x_1 = \frac{t^3-1}{5}\\
&\text{Case 3}: \quad x_0 = \exp(-t)\sin(t), \quad x_1 = t+3\\
&\text{Case 4}: \quad x_0 = \frac{t^2}{5}, \quad x_1 = \frac{t^3+2t-1}{4}.
\end{aligned}$$

The numerical computations are carried out using MATLAB 2015(a). The numerical results can be seen in Table 4.1 and Figure 4.1.

Table 4.1: Numerical results.

		Alg. 4.3.1 (Algorithm 3.1)	Alg. 4.1.1(Algorithm 2.1) [103]
Case Ia	CPU time (sec) No of Iter.	1.7306 4	3.4817 15
Case Ib	CPU time (sec) No. of Iter.	1.6187 3	2.6164 9
Case Ic	CPU time (sec) No of Iter.	1.7335 4	3.4591 15
Case Id	CPU time (sec) No of Iter.	1.5895 3	2.8829 10

Example 4.4.4. Let $H = \mathbb{R}^N$, $A : \mathbb{R}^N \to \mathbb{R}^N$ be defined by $A(x) = \frac{x}{2}$, for $x \in \mathbb{R}^N$ and $B : \mathbb{R}^N \to 2^{\mathbb{R}^N}$ be the normal cone at a point x in the unit ball $C := \{x \in H : ||x|| \leq 1\}$ defined by

$$B(x) = N_C(x) = \{y \in H : \langle y, z - x\rangle \leq 0, \quad \forall\ z \in H\}.$$

Obviously,

$$P_C(x) = \begin{cases} \frac{x}{||x||}, & ||x|| > 1, \\ x, & ||x|| \leq 1. \end{cases}$$

For $i = 1, 2, \ldots, 5$, we define the mapping $S_i : H \to H$ by $S_i x = \frac{x}{2i}$ which is 0-demimetric and $F(S_i) = \{0\}$. Clearly, $\Gamma = (A+B)^{-1}(0) \cap \bigcap_{i=1}^{5} F(S_i) = \{0\}$. Let $f(x) = \frac{x}{16}$, $D(x) = x$, for all $x \in \mathbb{R}^N$, $\beta_{n,0} = \frac{2n+1}{5n+7}$, $\beta_{n,i} = \frac{3n+6}{25n+35}$ for $n \in \mathbb{N}$. We choose the parameters $\alpha = 3$, $\lambda = 2$, $p = 0.5$, $\xi = 1, \sigma = 0.9$, $\alpha_n = \frac{1}{n+1}$ and $\epsilon_n = \frac{1}{(n+1)^2}$. The initial points x_0, x_1 are generated randomly by $x_0 = -2 \times rand(N, 1)$ and $x_1 = 0.5 \times rand(N, 1)$ where $N = 10, 50, 100, 200$. Using $\|x_{n+1} - x_n\|^2 < 10^{-4}$ as stopping criterion, we compare our Algorithm 4.3.1 with the non-inertial version by taking $\theta_n = 0$ in Algorithm 4.3.1. We also plot the graph of error $(\|x_{n+1} - x_n\|^2)$ against the number of iterations. The numerical result can be found in Figure 4.2 and Table 4.2.

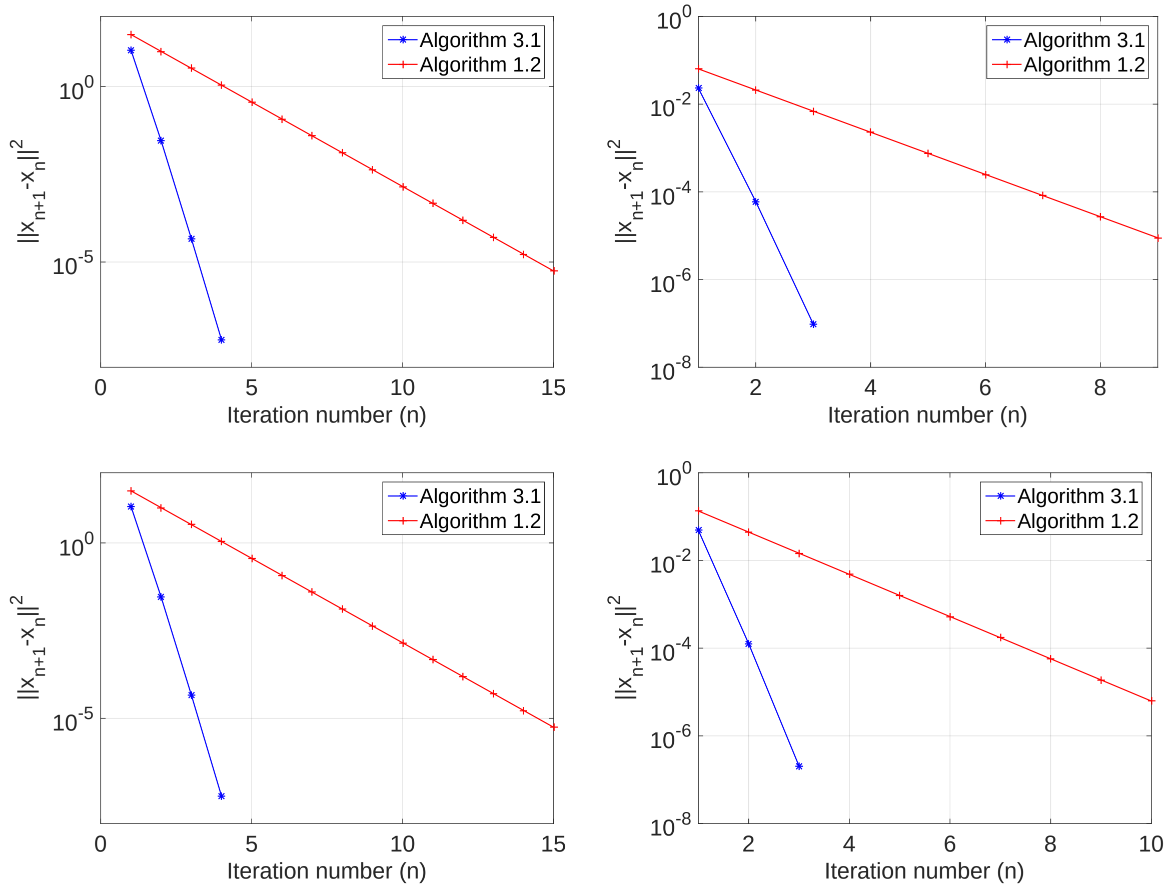

Figure 4.1: Top left: Case Ia; Top right: Case Ib; Bottom left: Case Ic; Bottom right: Case Id.

Table 4.2: Computation result for Example 4.4.4.

		Algorithm 4.3.1(Algorithm 3.1)	Non-inertial Alg.
$N = 10$	No of Iter. CPU time (sec)	6 0.0066	13 0.0128
$N = 50$	No of Iter. CPU time (sec)	6 0.0052	13 0.0156
$N = 100$	No of Iter. CPU time (sec)	6 0.0147	12 0.2740
$N = 200$	No of Iter. CPU time (sec)	7 0.0191	10 0.3118

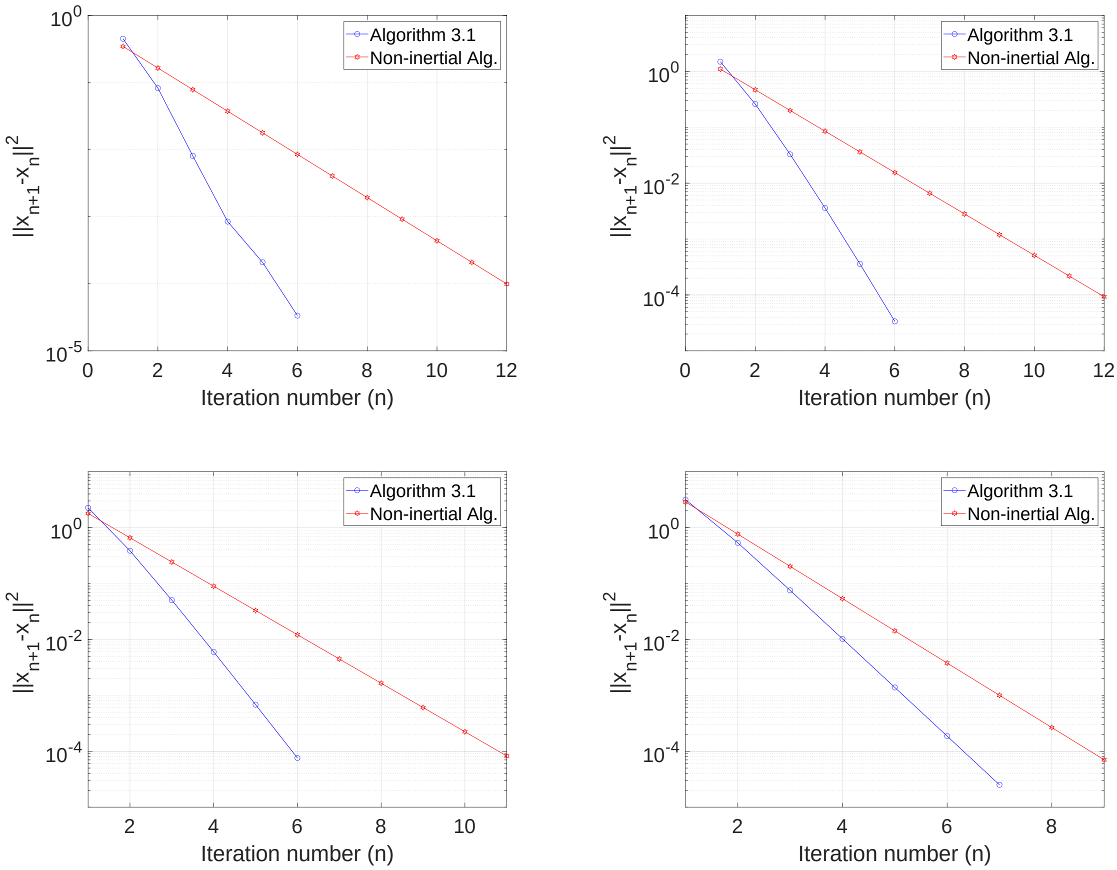

Figure 4.2: Example 4.4.4, Top Left: $N = 10$; Top Right: $N = 50$, Bottom Left: $N = 100$; Bottom Right: $N = 200$.

CHAPTER 5

Approximation of Common Solutions of Variational Inequality and Fixed Point Problems of Multivalued Demicontractive Mappings in Hilbert Spaces

5.1 Introduction

In this chapter, we study the common solutions of VIP and FPP of multivalued demicontractive mappings in a real Hilbert space.

Let H be a real Hilbert space, C a nonempty closed convex subset of H, $A : H \to H$ a monotone and $L-$Lipschitz continuous, and $f : H \to H$ a contraction mapping. For each $i = 1, 2, \ldots, m$, let $S_i : H \to CB(H)$ be a multivalued demicontractive mapping with constant k_i. We consider the following problem: Find a unique element $t \in \Gamma$ such that

$$t = P_\Gamma \circ f(t), \tag{5.1.1}$$

where $\Gamma = VI(C, A) \cap \bigcap_{i=1}^{m} F(S_i) \neq \emptyset$ is the solution set.

Furthermore, we propose a modified inertial viscosity subgradient extragradient algorithm with self-adaptive step-size in which each of the two projections is made onto an half space. Also, we prove that the sequence generated by our algorithm converges strongly to a common solution of VIP and FPP of a finite family of multivalued demicontractive mappings. We present an application and a numerical example to illustrate the efficacy and applicability of the algorithm.

We organise the remaining sections of this chapter as follows: In Section 5.2, we recall some basic definitions and give some lemmas that are useful in establishing our main result. In Section 5.3, we prove some lemmas used to establish the strong convergence of our proposed algorithm and thereafter we prove a strong convergence theorem for the algorithm. Lastly, in Section 5.4 we present an application and numerical example to illustrate the advantage, performance and behaviour of our method.

5.2 Preliminaries

In this section, we recall some basic notions and useful lemmas that will be needed to establish our main results in this chapter. Throughout this chapter, C is a nonempty closed convex subset of a real Hilbert space H. A point x^* is said to be a weak cluster point of sequence $\{x_n\}$, if there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges weakly to x^*. We denote the set of all weak cluster point of sequence $\{x_n\}$ by $\omega_w(x_n)$.

Definition 5.2.1. Let C be a nonempty closed convex subset of a real Hilbert space H. The normal cone to C at $x \in C$ is defined as:

$$N_c(x) = \big\{v \in H : \langle v, y - x\rangle \leq 0, \forall y \in C\big\}.$$

Definition 5.2.2. Let D be a nonempty subset of H. D is said to be proximal if there exists $y \in D$ such that

$$\|x - y\| = d(x, D), \quad x \in H.$$

Definition 5.2.3. Let $CC(H), CB(H)$ and $P(H)$ be the family of nonempty closed convex subset of H, nonempty closed bounded subsets of H and nonempty proximal bounded subsets of H respectively. The Hausdorff metric on $CB(H)$ is defined as follows:

$$H(A, B) := \max\big\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\big\}, \quad \forall A, B \in CB(H).$$

Let $S : H \to 2^H$ be a multivalued mapping. An element $x \in H$ is said to be a fixed point of S if $x \in Sx$. We say that S satisfies the endpoint condition if $Sp = \{p\}$, for all $p \in F(S)$. For multivalued mappings $S_i : H \to 2^H$ $(i \in \mathbb{N})$ with $\cap_{i=1}^{\infty} F(S_i) \neq \emptyset$, we say S_i satisfies the common endpoint condition if $S_i(p) = \{p\}$ for all $i \in \mathbb{N},\ p \in \cap_{i=1}^{\infty} F(S_i)$.

Definition 5.2.4. Let $S : H \to CB(H)$ be a multivalued mapping. S is said to be

(i) nonexpansive if
$$H(Sx, Sy) \leq \|x - y\|, \quad \forall x, y \in H,$$

(ii) quasi-nonexpansive if $F(S) \neq \emptyset$ such that
$$H(Sx, Sp) \leq \|x - p\|, \quad \forall x \in H, p \in F(S),$$

(iii) $\alpha-$demicontractive if $F(S) \neq \emptyset$ such that
$$H(Sx, Sp)^2 \leq \|x - p\|^2 + \alpha d(x, Sx)^2, \quad \forall x \in H, p \in F(S), \quad \alpha \in [0, 1).$$

Clearly, the class of demicontractive mappings includes the class of nonexpansive and quasi-nonexpansive mappings.

Let $S : H \to P(H)$ be a multivalued mapping. The best approximation operator for S denoted by $P_s(x)$ is defined as

$$P_s(x) := \{y \in Sx : \|x - y\| = d(x, Sx)\}.$$

It is easy to prove that $F(S) = F(P_S)$, where P_S satisfies the endpoint condition. An example of best approximation operator P_S which is nonexpansive, where S is not necessarily a nonexpansive mapping, was given by Song and Cho [92].

Let $S : H \to CB(H)$ be a multivalued mapping. The multivalued mapping $I - S$ is said to be demiclosed at zero if for any sequence $\{x_n\} \subset H$ which converges weakly to p and the sequence $\{\|x_n - u_n\|\}$ converges strongly to 0, where $u_n \in Sx_n$, then $p \in F(S)$.

Lemma 5.2.1. *[2, 66] Let H be a real Hilbert space, $\lambda \in \mathbb{R}$, then $\forall x, y \in H$, we have*

(i) $\|x + y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2$;

(ii) $\|x - y\|^2 = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2$;

(iii) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$;

(iv) $\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2$.

Lemma 5.2.2. *[100] Suppose $\{a_n\}$ and $\{b_n\}$ are sequences of nonnegative real numbers, such that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n b_n, \quad \forall n \geq 0,$$

and the following conditions are satisfied:

(i) $\{\alpha_n\} \subset (0, 1)$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\limsup_{n\to\infty} b_n \leq 0$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 5.2.3. *[56] Let $B : H \to H$ be a monotone and L-Lischiptz continuous mapping on a nonempty closed convex subset C, let $\{x_n\}$ be a sequence in H, and $S = P_C(I - \mu A)$. If $x_n \rightharpoonup q$ and $x_n - Sx_n \to 0$. then $q \in VI(C, A) = F(S)$.*

Lemma 5.2.4. *[51] Let $x_i \in H, (1 \leq i \leq m), \sum_{i=1}^{m} \alpha_i = 1$, where $\{\alpha_i\} \subseteq (0, 1)$. Then*

$$\|\sum_{i=1}^{m} \alpha_i x_i\|^2 = \sum_{i=1}^{m} \alpha_i \|x_i\|^2 - \sum_{i=j=1, i\neq j}^{m} \alpha_i \alpha_j \|x_i - x_j\|^2.$$

Lemma 5.2.5. *[39] Let C be a nonempty closed convex subset of a real Hilbert space H. Let $z \in C$ and $x \in H$. Then*

(i) $z = P_C x$ *if and only if* $\langle x - z, z - y\rangle \geq 0, \ \forall \ y \in C$;

(ii) $\|P_C x - y\|^2 \leq \|x-y\|^2 - \|x - P_C x\|^2, \quad \forall \ \ y \in C;$

(iii) $\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle \ \ \forall \ \ y \in C.$

Lemma 5.2.6. *[51] Let $\{a_n\}$ be a sequence of real numbers. Let $\{a_{n_i}\}$ be a subsequence of $\{a_n\}$ such that $a_{n_i} \leq a_{n_i+1}$, $\forall \ i \in \mathbb{N}$. Let $\{m_k\}$ be an integer defined by*

$$m_k = \max\{j \leq k : a_j < a_{j+1}\}. \tag{5.2.1}$$

Then $\{m_k\}$ is a nondecreasing sequence, where $\lim_{n\to\infty} m_n = \infty$ and $\forall \ k \in \mathbb{N}$, we have

$$a_k \leq a_{m_k+1} \quad \text{and} \quad a_{m_k} \leq a_{m_k+1}.$$

Lemma 5.2.7. *[107] Suppose $T : H \to H$ is β-demicontractive, $\mu \in (0, 1-\beta)$ and $T_\mu = (1-\mu)I + \mu T$, then for all $x \in H$, we have:*

(i) *$F(T)$ is a nonempty closed convex subset of H;*

(ii) *$F(T) = F(T_\mu)$;*

(iii) $\|T_\mu x - z\|^2 \leq \|x - z\|^2 - \frac{1}{\mu}(1-\beta-\mu)\|(I - T_\mu)x\|^2, \forall z \in F(T).$

5.3 Main result

In this section, we prove the strong convergence theorem for the sequence generated by our algorithm. Let H be a real Hilbert space, C a nonempty closed convex subset of H and $A : H \to H$ a monotone and $L-$Lipschitz continuous. For each $i = 1, 2, \ldots, m$, let $S_i : H \to CB(H)$ be a multivalued demicontractive mapping with constant k_i such that $I - S_i$ is demiclosed at zero, $S_i(p) = \{p\}$ for all $p \in \cap_{i=1}^m F(S_i)$, and $k = \max\{k_i\}$. We denote our solution set by $\Gamma = VI(C, A) \cap \bigcap_{i=1}^m F(S_i) \neq \emptyset$. Assume $f : H \to H$ is a $\rho-$contraction mapping with $\rho \in (0, 1)$. Let $\{\beta_{n,i}\}$ be sequence of nonnegative real numbers such that $\{\beta_{n,i}\} \subset (0, 1)$ and $\sum_{i=0}^m \beta_{n,i} = 1$. The following conditions are needed to obtain our result:

Condition A :

(A1) $\liminf_{n\to\infty}(\beta_{n,0} - k)\beta_{n,i} > 0, \quad \forall \ \ i = 1, 2, \cdots, m.$

(A2) $\lambda_1 > 0, \mu \in (0, 1), \{\alpha_n\} \subset (0, 1), \lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^\infty \alpha_n = \infty$.

(A3) Let $\theta \geq 3$ and $\{\epsilon_n\}$ be a nonnegative sequence such that $0 < d \leq \epsilon_n$.

(A4) $\epsilon_n = o(\alpha_n)$, i.e. $\lim_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0$ $\big($for example $\alpha_n = \frac{1}{n+1}, \epsilon_n = \frac{1}{(n+1)^2}\big)$.

Condition B :

(B1) Let C be defined by

$$C = \{x \in H : h(x) \leq 0\},$$

where $h : H \to \mathbb{R}$ is a bounded convex and subdifferentiable function. In addition, h is bounded on bounded sets.

(B2) For any $x \in H$, there exists at least one subgradient $\xi \in \delta h(x)$ which can be calculated, where

$$\delta h(x) = \{z \in H : h(u) \geq h(x) + \langle u - x, z\rangle, \quad \forall \quad u \in H\}.$$

Our algorithm is presented as follows:

Algorithm 5.3.1.
Step 0: Select $x_0, x_1 \in H, \lambda_1 > 0, \mu \in (0,1)$ and set $n = 1$.
Step 1: Given the $(n-1)th$ and nth iterates, choose θ_n such that $0 \leq \theta_n \leq \tilde{\theta}_n$ with $\tilde{\theta}_n$ defined by

$$\tilde{\theta}_n = \begin{cases} \min\left\{\frac{n-1}{n+\theta-1}, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } \ x_n \neq x_{n-1}, \\ \frac{n-1}{n+\theta-1} & \text{otherwise.} \end{cases} \tag{5.3.1}$$

Step 2 : Compute

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

Step 3 : Compute

$$y_n = P_{C_n}(w_n - \lambda_n A w_n),$$

where

$$C_n = \big\{w \in H : h(w_n) + \langle \xi_n, w - w_n\rangle \leq 0\big\},$$

and $\xi_n \in \delta h(w_n)$. If $y_n = w_n$, then set $w_n = z_n$ and go to Step 5, otherwise go to Step 4.

Step 4 : Compute

$$z_n = P_{T_n}(w_n - \lambda_n A y_n),$$

where

$$T_n = \big\{w \in H : \langle w_n - \lambda_n A w_n - y_n, w - y_n\rangle \leq 0\big\}.$$

Step 5 : Compute

$$\lambda_{n+1} = \begin{cases} \min\{\frac{\mu\|w_n - y_n\|}{\|Aw_n - Ay_n\|}, \lambda_n\}, & \text{if } \ Aw_n - Ay_n \neq 0 \\ \lambda_n, & \text{otherwise.} \end{cases} \tag{5.3.2}$$

Step 6 : Compute

$$\begin{cases} v_n = \beta_{n,0} z_n + \sum_{i=1}^m \beta_{n,i} u_{n,i}, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) v_n, \end{cases}$$

where $u_{n,i} \in S_i z_n$. Set $n := n+1$ and go to **Step 1**.

Remark 5.3.2. From (5.3.1) and (A4), we have $\lim_{n\to\infty} \theta_n \|x_n - x_{n-1}\| = 0$ and $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$.

Remark 5.3.3. Observe that the sequence $\{\lambda_n\}$ generated by (5.3.2) is monotonically non-decreasing and (see [113])

$$\lim_{n\to\infty} \lambda_n = \lambda \geq \min\left\{\lambda_1, \frac{\mu}{L}\right\}.$$

Lemma 5.3.4. *Assume $w_n = y_n = x_n$ and $x_n \rightharpoonup x^*$, then we have $x^* \in VI(C, A)$.*

Proof. If $w_n = y_n = x_n$ and $x_n \rightharpoonup x^*$, then we have $x_n = P_C(x_n - \lambda_n A x_n)$. We need to show that $x_n \in C$. From the definition of C_n in the Algorithm, we have

$$h(x_n) + \langle \xi_n, x_n - x_n\rangle \leq 0,$$

which implies that $h(x_n) \leq 0$. Thus $x_n \in C$. By Lemma 5.2.5(i), we have

$$\langle x_n - \lambda_n A x_n - x_n, x_n - y\rangle \geq 0, \quad \forall \ \ y \in C.$$

Thus we have

$$\lambda_n \langle A x_n, y - x_n\rangle \geq 0, \quad \forall \ \ y \in C. \tag{5.3.3}$$

By taking the limit in (5.3.3), using the fact that A is monotone and $\lambda_n \to \lambda$, then we have

$$\langle A x^*, y - x^*\rangle \geq 0, \quad \forall \ \ y \in C.$$

Thus $x^* \in VI(C, A)$. □

Lemma 5.3.5. *Let $\{x_n\}$ be a sequence generated by Algorithm 5.3.1, then $\{x_n\}$ is bounded.*

Proof. Let $t \in \Gamma$, then we have

$$\begin{aligned}
\|w_n - t\| &= \|x_n + \theta_n(x_n - x_{n-1}) - t\| \\
&\leq \|x_n - t\| + \theta_n\|x_n - x_{n-1}\| \\
&\leq \|x_n - t\| + \alpha_n \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|.
\end{aligned} \tag{5.3.4}$$

From Remark 5.3.2, it is known that $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$, then there exists a constant $M_1 > 0$ such that $\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \leq M_1$, for all $n \geq 1$. Thus from (5.3.4), we have

$$\|w_n - t\| \leq \|x_n - t\| + \alpha_n M_1. \tag{5.3.5}$$

Also,

$$\begin{aligned}
\|z_n - t\|^2 &= \|P_{T_n}(w_n - \lambda_n A y_n) - t\|^2 \\
&\leq \|(w_n - \lambda_n A y_n) - t\|^2 - \|(w_n - \lambda_n A y_n) - z_n\|^2 \\
&\leq \|w_n - t\|^2 - 2\lambda_n\langle w_n - \lambda_n A y_n - t, A y_n\rangle - \|w_n - z_n\|^2 \\
&+ 2\lambda_n\langle w_n - \lambda_n A y_n - z_n, A y_n\rangle \\
&= \|w_n - t\|^2 + 2\lambda_n\langle t - z_n, A y_n\rangle - \|w_n - z_n\|^2 \\
&= \|w_n - t\|^2 + 2\lambda_n\langle t - y_n, A y_n - At\rangle + 2\lambda_n\langle t - y_n, At\rangle + 2\lambda_n\langle y_n - z_n, A y_n\rangle \\
&- \|w_n - z_n\|^2 \\
&\leq \|w_n - t\|^2 + 2\lambda_n\langle y_n - z_n, A y_n\rangle - \|w_n - z_n\|^2 \\
&= \|w_n - t\|^2 + 2\lambda_n\langle y_n - z_n, A y_n\rangle - \|w_n - y_n\|^2 - 2\langle w_n - y_n, y_n - z_n\rangle - \|y_n - z_n\|^2 \\
&= \|w_n - t\|^2 - \|w_n - y_n\|^2 - \|y_n - z_n\|^2 + 2\langle w_n - \lambda_n A y_n - y_n, z_n - y_n\rangle.
\end{aligned} \tag{5.3.6}$$

By applying Cauchy-Schwartz inequality and from the definition of λ_{n+1} and T_n, we obtain

$$\begin{aligned}\langle w_n - \lambda_n A y_n - y_n, z_n - y_n\rangle &= \langle w_n - \lambda_n A w_n - y_n, z_n - y_n\rangle + \langle \lambda_n A w_n - \lambda_n A y_n, z_n - y_n\rangle \\ &\leq \lambda_n \langle A w_n - A y_n, z_n - y_n\rangle \\ &\leq \mu \frac{\lambda_n}{\lambda_{n+1}} \|w_n - y_n\| \|z_n - y_n\| \\ &\leq \mu \frac{\lambda_n}{2\lambda_{n+1}} (\|w_n - y_n\|^2 + \|z_n - y_n\|^2). \qquad (5.3.7)\end{aligned}$$

From (5.3.6) and (5.3.7), we get

$$\begin{aligned}\|z_n - t\|^2 &\leq \|w_n - t\|^2 - \|w_n - y_n\|^2 - \|y_n - z_n\|^2 + \mu \frac{\lambda_n}{\lambda_{n+1}} (\|w_n - y_n\|^2 + \|z_n - y_n\|^2) \\ &= \|w_n - t\|^2 - \left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}}\right) \|w_n - y_n\|^2 - \left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}}\right) \|y_n - z_n\|^2 \\ &= \|w_n - t\|^2 - \left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}}\right) \left(\|w_n - y_n\|^2 + \|y_n - z_n\|^2\right). \qquad (5.3.8)\end{aligned}$$

Indeed,

$$\lim_{n\to\infty} \left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}}\right) = 1 - \mu > 0. \qquad (5.3.9)$$

Thus, there exists $n_0 \geq 0$ such that for all $n \geq n_0$, we have $1 - \mu \frac{\lambda_n}{\lambda_{n+1}} > 0$. Hence, from (5.3.8) we have

$$\|z_n - t\|^2 \leq \|w_n - t\|^2, \qquad (5.3.10)$$

which implies that

$$\|z_n - t\| \leq \|w_n - t\|. \qquad (5.3.11)$$

Also, by applying Lemma 5.2.4, we have

$$\begin{aligned}\|v_n - t\|^2 &= \left\|\beta_{n,0} z_n + \sum_{i=1}^m \beta_{n,i} u_{n,i} - t\right\|^2 \\ &\leq \beta_{n,0}\|z_n - t\|^2 + \sum_{i=1}^m \beta_{n,i}\|u_{n,i} - t\|^2 - \sum_{i=1}^m \beta_{n,0}\beta_{n,i}\|z_n - u_{n,i}\|^2 \\ &= \beta_{n,0}\|z_n - t\|^2 + \sum_{i=1}^m \beta_{n,i}(u_{n,i}, S_i t)^2 - \sum_{i=1}^m \beta_{n,0}\beta_{n,i}\|z_n - u_{n,i}\|^2 \\ &\leq \beta_{n,0}\|z_n - t\|^2 + \sum_{i=1}^m \beta_{n,i} H(S_i z_n, S_i t)^2 - \sum_{i=1}^m \beta_{n,0}\beta_{n,i}\|z_n - u_{n,i}\|^2 \\ &\leq \beta_{n,0}\|z_n - t\|^2 + \sum_{i=1}^m \beta_{n,i}\Big(\|z_n - t\|^2 + k_i d(z_n, S_i z_n)^2\Big) - \sum_{i=1}^m \beta_{n,0}\beta_{n,i}\|z_n - u_{n,i}\|^2 \\ &\leq \beta_{n,0}\|z_n - t\|^2 + \sum_{i=1}^m \beta_{n,i}\|z_n - t\|^2 + \sum_{i=1}^m \beta_{n,i} k\|z_n - u_{n,i}\|^2 - \sum_{i=1}^m \beta_{n,0}\beta_{n,i}\|z_n - u_{n,i}\|^2\end{aligned}$$

$$= \|z_n - t\|^2 - \sum_{i=1}^{m} (\beta_{n,0} - k)\beta_{n,i}\|z_n - u_{n,i}\|^2. \tag{5.3.12}$$

Thus, by condition (A1), we have

$$\|v_n - t\|^2 \le \|z_n - t\|^2, \tag{5.3.13}$$

which implies that

$$\|v_n - t\| \le \|z_n - t\|. \tag{5.3.14}$$

From (5.3.14), (5.3.11) and (5.3.5), we have for all $n \ge n_0$

$$\begin{aligned}
\|x_{n+1} - t\| &= \|\alpha_n f(x_n) + (1-\alpha_n)v_n - t\| \\
&= \|\alpha_n(f(x_n) - t) + (1-\alpha_n)(v_n - t)\| \\
&\le \alpha_n\|f(x_n) - t\| + (1-\alpha_n)\|v_n - t\| \\
&\le \alpha_n\|f(x_n) - f(t)\| + \alpha_n\|f(t) - t\| + (1-\alpha_n)\|v_n - t\| \\
&\le \alpha_n\rho\|x_n - t\| + \alpha_n\|f(t) - t\| + (1-\alpha_n)\|z_n - t\| \\
&\le \alpha_n\rho\|x_n - t\| + \alpha_n\|f(t) - t\| + (1-\alpha_n)\Big(\|x_n - t\| + \alpha_n M_1\Big) \\
&= \alpha_n\rho\|x_n - t\| + \alpha_n\|f(t) - t\| + (1-\alpha_n)\|x_n - t\| + \alpha_n(1-\alpha_n)M_1 \\
&\le \alpha_n\rho\|x_n - t\| + \alpha_n\|f(t) - t\| + (1-\alpha_n)\|x_n - t\| + \alpha_n M_1 \\
&\le \big[1 - \alpha_n(1-\rho)\big]\|x_n - t\| + \alpha_n\|f(t) - t\| + \alpha_n M_1 \\
&= \big[1 - \alpha_n(1-\rho)\big]\|x_n - t\| + \alpha_n(1-\rho)\left[\frac{\|f(t) - t\|}{1-\rho} + \frac{M_1}{1-\rho}\right] \\
&\le \max\left\{\|x_n - t\|, \frac{\|f(t) - t\|}{1-\rho} + \frac{M_1}{1-\rho}\right\} \\
&\vdots \\
&\le \max\left\{\|x_{n_0} - t\|, \frac{\|f(t) - t\|}{1-\rho} + \frac{M_1}{1-\rho}\right\}.
\end{aligned}$$

Thus, the sequence $\{x_n\}$ is bounded, then it follows that $\{w_n\}$, $\{y_n\}$ and $\{z_n\}$ are bounded. □

Theorem 5.3.6. *Let H be a real Hilbert space, C a nonempty closed convex subset of H and $A : C \to H$ a monotone and $L-$Lipschitz continuous mapping. Then the sequence $\{x_n\}$ generated by Algorithm 5.3.1 converges strongly to $t \in \Gamma$, where $t = P_\Gamma \circ f(t)$.*

Remark 5.3.7. By Lemma 5.2.7(i), we know that $\cap_{i=1}^m F(S_i)$ is a closed convex subset and $VI(C, A)$ is also a closed convex subset. It follows that the solution set Γ is a closed convex subset. Hence $P_\Gamma \circ f : H \to H$ is a contraction mapping. Thus by Banach contraction principle, there exists a unique element $t \in H$ such that $t = P_\Gamma \circ f(t)$ and

$$\langle f(t) - t, z - t\rangle \le 0, \quad \forall z \in \Gamma.$$

We divide the proof of Theorem 5.3.6 into the following lemmas:

Lemma 5.3.8. *Let $\{x_n\}$ be a sequence generated by Algorithm 5.3.1, then the following inequality holds:*

$$
\begin{aligned}
&(1-\alpha_n)\left(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|w_n-y_n\|^2+(1-\alpha_n)\left(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|y_n-z_n\|^2\\
&+(1-\alpha_n)\sum_{i=1}^{m}(\beta_{n,0}-k)\beta_{n,i}\|z_n-u_{n,i}\|^2\\
&\leq \|x_n-t\|^2-\|x_{n+1}-t\|^2+\alpha_n M_3.
\end{aligned}
$$

Proof. From (5.3.5), we have

$$
\begin{aligned}
\|w_n-t\|^2 &\leq \big(\|x_n-t\|+\alpha_n M_1\big)^2\\
&=\|x_n-t\|^2+\alpha_n\big(2M_1\|x_n-t\|+\alpha_n M_1^2\big)\\
&\leq \|x_n-t\|^2+\alpha_n M_2, \quad \text{for some} \quad M_2>0. \qquad (5.3.15)
\end{aligned}
$$

By using (5.3.8), (5.3.12), (5.3.15) and Lemma 5.2.1(iv), we obtain

$$
\begin{aligned}
\|x_{n+1}-t\|^2 &= \|\alpha_n f(x_n)+(1-\alpha_n)v_n-t\|^2\\
&=\|\alpha_n(f(x_n)-t)+(1-\alpha_n)(v_n-t)\|^2\\
&=\alpha_n\|f(x_n)-t\|^2+(1-\alpha_n)\|v_n-t\|^2-\alpha_n(1-\alpha_n)\|f(x_n)-v_n\|^2\\
&\leq \alpha_n\|f(x_n)-t\|^2+(1-\alpha_n)\|v_n-t\|^2\\
&\leq \alpha_n\|f(x_n)-t\|^2+(1-\alpha_n)\left(\|z_n-t\|^2-\sum_{i=1}^{m}(\beta_{n,0}-k)\beta_{n,i}\|z_n-u_{n,i}\|^2\right)\\
&\leq \alpha_n\|f(x_n)-t\|^2+(1-\alpha_n)\Bigg\{\|w_n-t\|^2-\left(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\right)(\|w_n-y_n\|^2+\|y_n-z_n\|^2)\\
&-\sum_{i=1}^{m}(\beta_{n,0}-k)\beta_{n,i}\|z_n-u_{n,i}\|^2\Bigg\}\\
&=\alpha_n\|f(x_n)-t\|^2+(1-\alpha_n)\|w_n-t\|^2-(1-\alpha_n)\left(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|w_n-y_n\|^2\\
&-(1-\alpha_n)\left(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|y_n-z_n\|^2-(1-\alpha_n)\sum_{i=1}^{m}(\beta_{n,0}-k)\beta_{n,i}\|z_n-u_{n,i}\|^2\\
&\leq \alpha_n\|f(x_n)-t\|^2+\|w_n-t\|^2-(1-\alpha_n)\left(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|w_n-y_n\|^2\\
&-(1-\alpha_n)\left(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|y_n-z_n\|^2-(1-\alpha_n)\sum_{i=1}^{m}(\beta_{n,0}-k)\beta_{n,i}\|z_n-u_{n,i}\|^2.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&(1-\alpha_n)\left(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|w_n-y_n\|^2+(1-\alpha_n)\left(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|y_n-z_n\|^2\\
&+(1-\alpha_n)\sum_{i=1}^{m}(\beta_{n,0}-k)\beta_{n,i}\|z_n-u_{n,i}\|^2
\end{aligned}
$$

$$
\begin{aligned}
&\leq \|w_n - t\|^2 - \|x_{n+1} - t\|^2 + \alpha_n \|f(x_n) - t\|^2 \\
&\leq \|x_n - t\|^2 - \|x_{n+1} - t\|^2 + \alpha_n (\|f(x_n) - t\|^2 + M_2) \\
&\leq \|x_n - t\|^2 - \|x_{n+1} - t\|^2 + \alpha_n M_3, \qquad (5.3.16)
\end{aligned}
$$

for some $M_3 > 0$. □

Lemma 5.3.9. *Let $\{x_n\}$ be a sequence generated by Algorithm 5.3.1, then the following inequality holds $\forall n \geq n_0$:*

$$
\|x_{n+1}-t\|^2 \leq \big(1-(1-\rho)\alpha_n\big)\|x_n-t\|^2+(1-\rho)\alpha_n\left[\frac{2}{1-\rho}\langle f(t)-t, x_{n+1}-t\rangle+\frac{3M}{1-\rho}.\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|\right].
$$

Proof. By Lemma 5.2.1(i), we have that

$$
\begin{aligned}
\|w_n - t\|^2 &= \|x_n + \theta_n(x_n - x_{n-1}) - t\|^2 \\
&= \|x_n - t\|^2 + 2\theta_n \langle x_n - t, x_n - x_{n-1}\rangle + \theta_n^2 \|x_n - x_{n-1}\|^2 \\
&\leq \|x_n - t\|^2 + 2\theta_n \|x_n - t\| \|x_n - x_{n-1}\| + \theta_n^2 \|x_n - x_{n-1}\|^2. \qquad (5.3.17)
\end{aligned}
$$

From (5.3.13), (5.3.10), (5.3.17) and Lemma 5.2.1(iii), we have

$$
\begin{aligned}
\|x_{n+1} - t\|^2 &= \|\alpha_n f(x_n) + (1-\alpha_n) v_n - t\|^2 \\
&= \|\alpha_n (f(x_n) - f(t)) + (1-\alpha_n)(v_n - t) + \alpha_n (f(t) - t)\|^2 \\
&\leq \|\alpha_n (f(x_n) - f(t)) + (1-\alpha_n)(v_n - t)\|^2 + 2\alpha_n \langle f(t) - t, x_{n+1} - t\rangle \\
&\leq \alpha_n \|f(x_n) - f(t)\|^2 + (1-\alpha_n)\|v_n - t\|^2 + 2\alpha_n \langle f(t) - t, x_{n+1} - t\rangle \\
&\leq \alpha_n \rho^2 \|x_n - t\|^2 + (1-\alpha_n)\|v_n - t\|^2 + 2\alpha_n \langle f(t) - t, x_{n+1} - t\rangle \\
&\leq \alpha_n \rho \|x_n - t\|^2 + (1-\alpha_n)\|v_n - t\|^2 + 2\alpha_n \langle f(t) - t, x_{n+1} - t\rangle \\
&\leq \alpha_n \rho \|x_n - t\|^2 + (1-\alpha_n)\|w_n - t\|^2 + 2\alpha_n \langle f(t) - t, x_{n+1} - t\rangle \\
&\leq \alpha_n \rho \|x_n - t\|^2 + (1-\alpha_n)\|x_n - t\|^2 + (1-\alpha_n) 2\theta_n \|x_n - t\| \|x_n - x_{n-1}\| \\
&\quad + (1-\alpha_n)\theta_n^2 \|x_n - x_{n-1}\|^2 + 2\alpha_n \langle f(t) - t, x_{n+1} - t\rangle \\
&\leq \big(1 - (1-\rho)\alpha_n\big)\|x_n - t\|^2 + 2\theta_n \|x_n - t\| \|x_n - x_{n-1}\| + \theta_n^2 \|x_n - x_{n-1}\|^2 \\
&\quad + 2\alpha_n \langle f(t) - t, x_{n+1} - t\rangle \\
&= \big(1 - (1-\rho)\alpha_n\big)\|x_n - t\|^2 + (1-\rho)\alpha_n \frac{2}{1-\rho}\langle f(t) - t, x_{n+1} - t\rangle \\
&\quad + \theta_n \|x_n - x_{n-1}\| \big(2\|x_n - t\| + \theta_n \|x_n - x_{n-1}\|\big) \\
&\leq \big(1 - (1-\rho)\alpha_n\big)\|x_n - t\|^2 + (1-\rho)\alpha_n \frac{2}{1-\rho}\langle f(t) - t, x_{n+1} - t\rangle \\
&\quad + 3M\theta_n \|x_n - x_{n-1}\| \\
&\leq \big(1 - (1-\rho)\alpha_n\big)\|x_n - t\|^2 \\
&\quad + (1-\rho)\alpha_n \left[\frac{2}{1-\rho}\langle f(t) - t, x_{n+1} - t\rangle + \frac{3M}{1-\rho}.\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|\right], \qquad (5.3.18)
\end{aligned}
$$

where $M := \sup_{n\in\mathbb{N}}\{\|x_n - t\|, \theta\|x_n - x_{n-1}\|\} > 0, \quad \forall n \geq n_0$. □

Lemma 5.3.10. *The sequence $\{\|x_n - t\|^2\}$ converges to zero.*

Proof. **Case 1:** There exists $N \in \mathbb{N}$ such that $\|x_{n+1} - t\|^2 \leq \|x_n - t\|^2$, $\forall n \geq N$. This implies that
$\lim_{n\to\infty} \|x_n - t\|^2$ exists. Then from (5.3.9), (5.3.16) and by conditions (A1) and (A2), we have

$$\lim_{n\to\infty} \|w_n - y_n\| = 0, \tag{5.3.19}$$

$$\lim_{n\to\infty} \|y_n - z_n\| = 0, \tag{5.3.20}$$

and

$$\lim_{n\to\infty} \|z_n - u_{n,i}\| = 0 \quad \forall\, i = 1, 2, \ldots, m. \tag{5.3.21}$$

It then follows from (5.3.19) and (5.3.20) that

$$\lim_{n\to\infty} \|w_n - z_n\| = 0. \tag{5.3.22}$$

Furthermore,

$$\begin{aligned}
\|v_n - z_n\| &= \left\| \beta_{n,0} z_n + \sum_{i=1}^{m} \beta_{n,i} u_{n,i} - z_n \right\| \\
&\leq \beta_{n,0} \|z_n - z_n\| + \sum_{i=1}^{m} \beta_{n,i} \|u_{n,i} - z_n\| \to 0.
\end{aligned}$$

This implies that

$$\lim_{n\to\infty} \|v_n - z_n\| = 0. \tag{5.3.23}$$

Also, from Remark 5.3.2

$$\begin{aligned}
\|w_n - x_n\| &= \|x_n + \theta_n (x_n - x_{n-1}) - x_n\| \\
&= \theta_n \|x_n - x_{n-1}\| \to 0,
\end{aligned}$$

which implies that

$$\|w_n - x_n\| \to 0, \quad \text{as} \quad n \to \infty. \tag{5.3.24}$$

From (5.3.19) and (5.3.24), we obtain

$$\|x_n - y_n\| \leq \|x_n - w_n\| + \|w_n - y_n\| \to 0.$$

Hence, we have

$$\|x_n - y_n\| \to 0, \quad \text{as} \quad n \to \infty. \tag{5.3.25}$$

Also, from (5.3.20) and (5.3.25), we have

$$\|z_n - x_n\| \leq \|z_n - y_n\| + \|y_n - x_n\| \to 0.$$

Hence,

$$\|z_n - x_n\| \to 0, \quad \text{as} \quad n \to \infty. \tag{5.3.26}$$

Moreover, from (5.3.23) and (5.3.26), we have

$$\|v_n - x_n\| \leq \|v_n - z_n\| + \|z_n - x_n\| \to 0.$$

Thus,

$$\lim_{n\to\infty} \|v_n - x_n\| = 0. \tag{5.3.27}$$

Next, we need to show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. By (5.3.27), we obtain

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|\alpha_n f(x_n) + (1-\alpha_n)v_n - x_n\| \\ &\le \alpha_n \|f(x_n) - x_n\| + (1-\alpha_n)\|v_n - x_n\| \\ &\le \alpha_n \|f(x_n) - x_n\| + \|v_n - x_n\| \to 0,\end{aligned}$$

which implies that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{5.3.28}$$

It is known that the sequence $\{x_n\}$ is bounded. Hence, there exists a subsequence $\{x_{n_j}\}$ which converges weakly to $z \in H$ such that

$$\limsup_{n\to\infty} \langle f(t) - t, x_n - t\rangle = \lim_{j\to\infty} \langle f(t) - t, x_{n_j} - t\rangle = \langle f(t) - t, z - t\rangle. \tag{5.3.29}$$

Since $\lim_{n\to\infty} \lambda_n = \lambda, \|w_n - x_n\| \to 0$, and $\|w_n - y_n\| \to 0$, then by Lemma 5.2.3, we obtain $z \in VI(C, A)$. Also $z_{n_j} \rightharpoonup z$ by (5.3.26), then by the demiclosedness of $I - S$ and (5.3.21), we have that $z \in F(S_i), \ \ \forall\, i = 1, 2, m$, which implies that $z \in \cap_{i=1}^{m} F(S_i)$. Hence, it follows that $z \in \Gamma$. Furthermore, by Remark 5.3.7, we have for all $z \in \Gamma$,

$$\limsup_{n\to\infty} \langle f(t) - t, x_n - t\rangle = \lim_{j\to\infty} \langle f(t) - t, x_{n_j} - t\rangle = \langle f(t) - t, z - t\rangle \le 0. \tag{5.3.30}$$

Hence by (5.3.28), (5.3.29) and (5.3.30), we obtain

$$\begin{aligned}\limsup_{n\to\infty} \langle f(t) - t, x_{n+1} - t\rangle &\le \limsup_{n\to\infty} \langle f(t) - t, x_{n+1} - x_n\rangle + \limsup_{n\to\infty} \langle f(t) - t, x_n - t\rangle \\ &= \langle f(t) - t, z - t\rangle \le 0. \end{aligned} \tag{5.3.31}$$

By (5.3.18), (5.3.31) and Lemma 5.2.2, we get $x_n \to t$.

Case 2: There exists a subsequence $\{\|x_{n_j} - t\|^2\}$ of $\{\|x_n - t\|^2\}$ such that $\|x_{n_j} - t\|^2 \le \|x_{n_j+1} - t\|^2$, $\forall j \in \mathbb{N}$. We have in this case by Lemma 5.2.6, there exists a nondecreasing sequence $\{m_k\}$ and $\lim_{k\to\infty} m_k = \infty$ such that

$$\|x_{m_k} - t\|^2 \le \|x_{m_k+1} - t\|^2, \|x_k - t\|^2 \le \|x_{m_k} - t\|^2. \tag{5.3.32}$$

From (5.3.16), we have

$$\begin{aligned}&(1-\alpha_{m_k})\left(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|w_{m_k} - y_{m_k}\|^2 + (1-\alpha_{m_k})\left(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|y_{m_k} - z_{m_k}\|^2 \\ &+ (1-\alpha_{m_k})\sum_{i=1}^{m}(\beta_{m_k,0} - k)\beta_{n,i}\|z_{m_k} - u_{m_k,i}\|^2 \le \|x_{m_k} - t\|^2 - \|x_{m_k+1} - t\|^2 + \alpha_{m_k} M_3 \\ &\le \alpha_{m_k} M_3.\end{aligned}$$

From $\lim_{n\to\infty}\alpha_n = 0$ and (5.3.9),we have

$$\lim_{k\to\infty}\|w_{m_k} - y_{m_k}\| = 0,$$

$$\lim_{k\to\infty}\|z_{m_k} - u_{m_k,i}\| = 0,$$

and

$$\lim_{k\to\infty}\|y_{m_k} - z_{m_k}\| = 0.$$

Following similar argument as in Case 1, we have that $\|x_{m_k} - y_{m_k}\| \to 0$, as $k \to \infty$ and

$$\limsup_{k\to\infty}\langle f(t) - t, x_{m_k+1} - t\rangle \leq 0. \tag{5.3.33}$$

From (5.3.18), we have for all $k \geq n_0$

$$\begin{aligned}\|x_{m_k+1} - t\|^2 &\leq \big(1-(1-\rho)\alpha_{m_k}\big)\|x_{m_k} - t\|^2 \\ &\quad + (1-\rho)\alpha_{m_k}\left[\frac{2}{1-\rho}\langle f(t) - t, x_{m_k+1} - t\rangle + \frac{3M}{1-\rho}.\frac{\theta_{m_k}}{\alpha_{m_k}}\|x_{m_k} - x_{m_k-1}\|\right].\end{aligned} \tag{5.3.34}$$

Then, it follows from (5.3.32) and (5.3.34) that

$$\begin{aligned}\|x_{m_k+1} - t\|^2 &\leq \big(1-(1-\rho)\alpha_{m_k}\big)\|x_{m_k+1} - t\|^2 \\ &\quad + (1-\rho)\alpha_{m_k}\left[\frac{2}{1-\rho}\langle f(t) - t, x_{m_k+1} - t\rangle + \frac{3M}{1-\rho}.\frac{\theta_{m_k}}{\alpha_{m_k}}\|x_{m_k} - x_{m_k-1}\|\right].\end{aligned} \tag{5.3.35}$$

Thus from (5.3.35), we obtain

$$\|x_{m_k+1} - t\|^2 \leq \frac{2}{1-\rho}\langle f(t) - t, x_{m_k+1} - t\rangle + \frac{3M}{1-\rho}.\frac{\theta_{m_k}}{\alpha_{m_k}}\|x_{m_k} - x_{m_k-1}\|.$$

Hence, by Remark 5.3.2 and (5.3.33), we obtain

$$\limsup_{k\to\infty}\|x_{m_k+1} - t\| \leq 0. \tag{5.3.36}$$

By (5.3.32) and (5.3.36), we have

$$\limsup_{k\to\infty}\|x_k - t\| \leq 0, \tag{5.3.37}$$

which implies that $x_k \to t$. Hence, the proof is complete. □

By the properties of the best approximation operator, we obtain the following consequent result.

Corollary 5.3.11. *Let H be a real Hilbert space, C a nonempty closed convex subset of H and $A : C \to H$ a monotone and $L-$Lipschitz continuous mapping. For each $i = 1, 2, \ldots, m$, let $S_i : H \to P(H)$ be a multivalued mapping such that P_{S_i} is k_i-demicontractive with $k = \max\{k_i\}$ and suppose $I - P_{S_i}$ is demiclosed at zero. Let $\Omega = VI(C, A) \cap \bigcap_{i=1}^m F(P_{S_i}) \neq \emptyset$ be the solution set. Assume $f : H \to H$ is a $\rho-$contraction mapping such that $\rho \in (0, 1)$. Let $\{x_n\}$ be a sequence generated as follows:*

Algorithm 5.3.12.
Step 0: Select $x_0, x_1 \in H, \lambda_1 > 0, \mu \in (0,1)$ and set $n = 1$.
Step 1: Given the $(n-1)th$ and nth iterates, choose θ_n such that $0 \leq \theta_n \leq \tilde{\theta}_n$ with $\tilde{\theta}_n$ defined by

$$\tilde{\theta}_n = \begin{cases} \min\left\{\frac{n-1}{n+\theta-1}, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } \ x_n \neq x_{n-1}, \\ \frac{n-1}{n+\theta-1} & \text{otherwise.} \end{cases} \tag{5.3.38}$$

Step 2 : Compute

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

Step 3 : Compute

$$y_n = P_{C_n}(w_n - \lambda_n A w_n)$$

where

$$C_n = \{w \in H : h(w_n) + \langle \xi_n, w - w_n \rangle \leq 0\},$$

and $\xi_n \in \delta h(w_n)$. If $y_n = w_n$, then set $w_n = z_n$ and go to Step 5, otherwise go to Step 4.

Step 4 : Compute

$$z_n = P_{T_n}(w_n - \lambda_n A y_n),$$

where

$$T_n = \{w \in H : \langle w_n - \lambda_n A w_n - y_n, w - y_n \rangle \leq 0\}.$$

Step 5 : Compute

$$\lambda_{n+1} = \begin{cases} \min\{\frac{\mu\|w_n - y_n\|}{\|Aw_n - Ay_n\|}, \lambda_n\}, & \text{if } \ Aw_n - Ay_n \neq 0 \\ \lambda_n, & \text{otherwise.} \end{cases} \tag{5.3.39}$$

Step 6 : Compute

$$\begin{cases} v_n = \beta_{n,0} z_n + \sum_{i=1}^m \beta_{n,i} u_{n,i}, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) v_n, \end{cases}$$

where $u_{n,i} \in P_{S_i}(z_n)$. Set $n := n+1$ and go to **Step 1**.

Let $\{\beta_{n,i}\}$ be sequence of nonnegative real numbers such that $\{\beta_{n,i}\} \subset (0,1)$ and $\sum_{i=0}^m \beta_{n,i} = 1$. Suppose Condition A and Condition B are satisfied. Then the sequence $\{x_n\}$ generated by Algorithm 5.3.12 converges strongly to $t \in \Omega$, where $t = P_\Omega \circ f(t)$.

Proof. Since P_{S_i} satisfies the common endpoint condition and $F(S_i) = F(P_{S_i})$ for each $i = 1, 2, \ldots, m$, then the result follows from Theorem 5.3.6.

□

5.4 Application and numerical example

5.4.1 Convex minimization problem

In this section, we apply our result to solve Convex Minimisation Problem (CMP). Let C be a nonempty closed convex subset of a real Hilbert space H. The constrained CMP is defined as follows:

$$\text{Find } \; x^* \in C, \text{such that } \; \phi(x^*) = \min_{x \in C} \phi(x), \tag{5.4.1}$$

where ϕ is a real-valued convex function. The solution set of the Problem (5.4.1) is denoted by $\arg\min_{x\in C} \phi(x)$. For details on CMP and related optimization problems, see [1, 8, 9, 45, 46, 78].

Lemma 5.4.1. *[108] Let C be a nonempty closed convex subset of a real Hilbert space H and $\phi : H \to \mathbb{R}$ be a convex function. If ϕ is differentiable, then z is a solution of the Problem (5.4.1) if and only if $z \in VI(C, \nabla\phi)$.*

By applying Theorem 5.3.6 and Lemma 5.4.1, we obtain the following result.

Theorem 5.4.2. *Let H be a real Hilbert space and $\phi : H \to \mathbb{R}$ be a differentiable convex function. Suppose $\nabla\phi$ is $\alpha-$ism. Let $\{x_n\}$ be a sequence generated by the Algorithm defined as follows:*

Algorithm 5.4.3.
Step 0: Select $x_0, x_1 \in H, \lambda_0 > 0$ and set $n = 1$.
Step 1: Given the $(n-1)th$ and nth iterates, choose θ_n such that $0 \le \theta_n \le \tilde{\theta}_n$ with $\tilde{\theta}_n$ defined by

$$\tilde{\theta}_n = \begin{cases} \min\left\{\frac{n-1}{n+\theta-1}, \frac{\epsilon_n}{||x_n - x_{n-1}||}\right\}, & if \quad x_n \neq x_{n-1}, \\ \frac{n-1}{n+\theta-1} \quad otherwise. \end{cases} \tag{5.4.2}$$

Step 2 : Compute

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

Step 3 : Compute

$$y_n = P_{C_n}(w_n - \lambda_n \nabla\phi w_n),$$

where

$$C_n = \big\{w \in H : h(w_n) + \langle \xi_n, w - w_n\rangle \le 0\big\},$$

and $\xi_n \in \delta h(w_n)$. If $y_n = w_n$, then set $w_n = z_n$ and go to Step 5, otherwise go to Step 4.

Step 4 : Compute

$$z_n = P_{T_n}(w_n - \lambda_n \nabla\phi y_n),$$

where

$$T_n = \big\{w \in H : \langle w_n - \lambda_n\nabla\phi w_n - y_n, w - y_n\rangle \le 0\big\}.$$

Step 5 : Compute

$$\lambda_{n+1} = \begin{cases} \min\{\frac{\mu\|w_n - y_n\|}{\|\nabla\phi w_n - \nabla\phi y_n\|}, \lambda_n\}, & \text{if } \nabla\phi w_n - \nabla\phi y_n \neq 0 \\ \lambda_n, & \text{otherwise.} \end{cases} \tag{5.4.3}$$

Step 6 : Compute

$$\begin{cases} v_n = \beta_{n,0} z_n + \sum_{i=1}^{m} \beta_{n,i} u_{n,i}, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) v_n, \end{cases}$$

where $u_{n,i} \in S_i z_n$ and $\sum_{i=0}^{m} \beta_{n,i} = 1$. Set $n := n + 1$ and go to **Step 1**.

If Conditions A and B are satisfied, then the sequence $\{x_n\}$ generated by Algorithm 5.4.3 converges strongly to $\bar{x} \in \Gamma = \arg\min_{x \in C} \phi(x) \cap \bigcap_{i=1}^{m} F(S_i) \neq \emptyset$, where $\bar{x} = P_\Gamma \circ f(\bar{x})$.

Proof. By setting $A = \nabla\phi$ in Theorem 5.3.6 and applying Lemma 5.4.1, we obtain the desired result. □

5.4.2 Numerical example

In this section, we present a numerical example to illustrate our method.

Example 5.4.4. Let $H = \ell_2(\mathbb{R})$, where $\ell_2(\mathbb{R}) := \{x = (x_1, x_2, \ldots, x_n, \ldots), x_i \in \mathbb{R} : \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$, $\|x\|_2 = (\sum_{i=1}^{\infty} |x_i|^2)^{\frac{1}{2}}\ \forall\ x \in \ell_2(\mathbb{R})$. Let $A : H \to H$ be mapping defined by $Ax = 3x$ for every $x \in H$, $C := \{x \in H : \|x\|_2 \leq 4\}$ and $h(x) = \|x\|^2$. It is well known that

$$P_C(x) = \begin{cases} \frac{4x}{\|x\|_2} & \text{if } \|x\|_2 > 4, \\ x & \text{if } \|x\|_2 \leq 4. \end{cases}$$

Let $S_i : \ell_2(\mathbb{R}) \to \ell_2(\mathbb{R})$ be defined for $i = 1, 2, \cdots, 10$ by

$$S_i x = \{-\frac{(i+2)}{3} x\}.$$

It is easy to check that S_i is $\frac{i-1}{i+5}$-demicontractive. Obviously, $\Gamma = VI(C, A) \cap \bigcap_{i \in \mathbb{N}} F(S_i) = \{0\}$, for $i = 1, 2, \cdots, 10$. Let $f(x) = \frac{x}{2}, \alpha_n = \frac{1}{n+1}, \epsilon_n = \frac{1}{(n+1)^2}, \theta = 3, \mu = 0.6, \lambda_1 = 0.9, \beta_{n,0} = \frac{n}{2n+1}$ and $\beta_{n,i} = \frac{n+1}{10(2n+1)}$,
$i = 1, 2, \cdots, 10$. We consider the following three different cases as starting points:
Case I: $x_0 = (3, -\frac{3}{2}, \frac{3}{4}, \cdots)$ and $x_1 = (2, \frac{2}{5}, \frac{2}{25}, \cdots)$,
Case II: $x_0 = (-3, 1, -\frac{1}{3}, \cdots)$ and $x_1 = (-1, \frac{1}{10}, -\frac{1}{100}, \cdots)$,
Case III: $x_0 = (2, \frac{1}{2}, \frac{1}{8} \cdots)$ and $x_1 = (3, \frac{1}{3}, \frac{1}{27}, \cdots)$,
Case IV: $x_0 = (1, \frac{1}{2}, \frac{1}{4} \cdots)$ and $x_1 = (-\frac{1}{5}, \frac{1}{15}, -\frac{1}{45}, \cdots)$.

Using MATLAB R2019(b), we compare the performance of Algorithm 5.3.1 with non-inertial form of Algorithms 5.3.1 and Algorithm 1.2.1. The stopping criterion used for our computation is $|x_{n+1} - x_n|_2 < 10^{-6}$. We plot the graphs of errors against the number of iterations in each case. The figures and numerical results are shown in Figure 5.1 and Table 5.1, respectively.

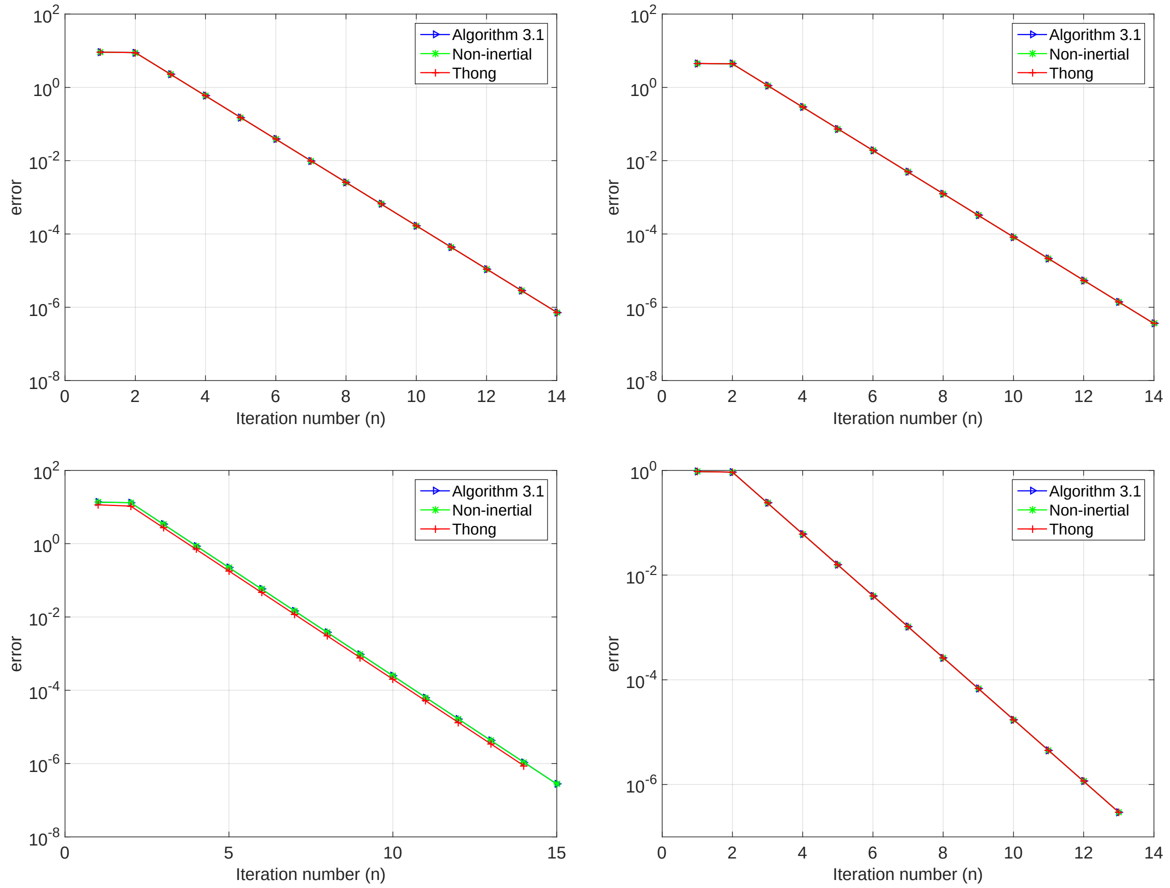

Figure 5.1: Top left: Case Ia; Top right: Case Ib; Bottom left: Case Ic; Bottom right: Case Id.

Table 5.1: Numerical results.

		Alg. 5.3.1	Non-inertial	Alg. 1.2.1 Thong
Case Ia	CPU time (sec)	0.3464	0.1038	0.0160
	No of Iter.	14	14	14
Case Ib	CPU time (sec)	0.1204	0.0834	0.1056
	No. of Iter.	14	14	14
Case Ic	CPU time (sec)	0.0747	0.0078	0.0172
	No of Iter.	15	15	14
Case Id	CPU time (sec)	0.0138	0.0017	0.0024
	No of Iter.	13	13	13

CHAPTER 6

Conclusion, Contribution to Knowledge and Future Research

6.1 Conclusion

In this dissertation, we proposed and studied iterative schemes for approximating common solutions of SFP, MIP and FPP in p-uniformly convex Banach spaces which are also uniformly smooth. We also proposed and studied iterative schemes for approximating common solutions of MIP, FPP and VIP in real Hilbert spaces. In Chapter 3, we proved a strong convergence theorem for approximating common solutions of SFP, MIP and FPP for the class of Bregman weak relatively nonexpansive mapping in p-uniformly convex Banach spaces which are also uniformly smooth without prior knowledge of the norm of the bounded linear operator. We gave some numerical examples to illustrate the performance of our method as well as compared it with some related methods in the literature. In Chapter 4, we established strong convergence of the sequence generated by our proposed algorithm for approximating common solutions of MIP and FPP of a finite family of single-valued demimetric mappings in a real Hilbert space. Our Algorithm is proposed in such a way that it does not require Lipschitz constant of the associated mapping. Furthermore, we applied our result to solve SFP and gave some numerical examples to illustrate the performance of our method as well as comparing it with the non-inertial version and some related methods in the literature. In Chapter 5, we obtained a strong convergence theorem for approximating common solutions of VIP and FPP of a finite family of multivalued demicontractive mappings in a real Hilbert space without prior knowledge of the Lipschitz constant of the associated monotone operator. We gave an application and a numerical example to illustrate our algorithm.

6.2 Contribution to knowledge

(1) Our iterative scheme (3.3.1) generalises and extends some recent results in the literature. For instance, the results obtained in [4] and [94] are extended to approximation of common solutions of SFP, MIP and FPP of Bregman weak relatively nonexpansive mapping in Banach spaces.

(2) In [103], the authors established a strong convergence theorem for solving MIP, while in Chapter 4 of this dissertation, we stated and proved strong convergence theorem for approximating solution of MIP and common fixed point of a finite family of demimetric mappings. Thus, we can deduce from this that our result in Chapter 4 improves and extends the results obtained in [103].

(3) Thong and Hieu [106] proposed some extragradient viscosity-type iterative algorithms for approximating a common solution of VIP and a set of fixed points of a demicontractive mapping. In Chapter 5, we introduced a modified inertial subgradient extragradient algorithm with self-adaptive step-size for finding a common solution of VIP and fixed points of a finite family of demicontractive mappings. Moreover, we obtained a strong convergence result which extends the results of Thong and Hieu [106].

(4) Three research articles obtained from this dissertation have been submitted to ISI and Scopus indexed mathematics journals for possible publication.

6.3 Future research

The problems considered and studied in this dissertation offer many opportunities for future research. The geometric properties of Banach spaces play a key role in the results obtained in Chapter 3. We hope in the future to investigate more on the geometric properties of Banach spaces, so that we can extend the results in Chapter 4 and 5 to Banach spaces. Part of our future research will also be to extend the results obtained in this dissertation to more useful and important spaces, for example Hadamard spaces.

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