ON THE GEOMETRY OF CR-MANIFOLDS

by

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Abstract

We study two classes of CR-submanifolds in Kählerian and cosymplectic manifolds. More precisely, we compare the geometry of CR-submanifolds of the above two underlying smooth manifolds. We derive expressions relating the sectional curvatures, the necessary and sufficient conditions for the integrability of distributions. Further, we study totally umbilical, totally geodesic and foliation geometry of the CR-submanifolds of both spaces and found many interesting results. We prove that, under some condition, there are classes CR submanifold in cosymplectic space forms which are in the classes extrinsic spheres. Examples are given throughout the thesis.

KEYWORDS: Riemannian manifold, CR-submanifold, Kählerian manifold, nearly Kählerian manifold, cosymplectic manifold, totally umbilical, totally geodesic.

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List of Symbols

$\overline{M}$ - n-dimensional connected differentiable manifold
$\overline{\nabla}$ - Linear connection on $\overline{M}$
$\overline{g}$ - Riemannian metric on $\overline{M}$
$[X, Y]$ - Lie bracket
$T\overline{M}$ - Tangent bundle to $\overline{M}$
M - Submanifold of $\overline{M}$
TM - Tangent bundle to M
$TM^{\perp}$ - Normal bundle to M
g - Riemannian metric on M
$Tr(h)$ - Trace of h
μ - Mean curvature on M
D - Distribution on $\overline{M}$
$D^{\perp}$ - Complementary distribution to D
J - Almost complex structure on $\overline{M}$
Ω - 2-form of Hermitian manifold $\overline{M}$
S^6 - Six-sphere
$\oplus$ - Orthogonal direct sum
η - 1-form on $\overline{M}$
$M^{-m}(c)$ - Complex space form
$\overline{c}$ - sectional curvature on $\overline{M}$

Contents

Chapter 1

Introduction

The study of CR-submanifolds of a Kählerian manifold was initiated by Bejancu [2] and later studied by many other authors, including among others, [1], [3], [11] and [15]. In Bejancu's definition (See [2] for details), the tangent bundle of any CR-submanifold splits into a complex part, of constant dimension and a totally real part that is orthogonal to the first one. Precisely, let $\overline{M}$ be an almost Hermitian manifold and let J be an almost complex structure on $\overline{M}$. A real submanifold M of $\overline{M}$ is called a CR-submanifold if there exist a differentiable distribution D and its orthogonal complementary distribution $D^{\perp}$ in TM satisfying: $JD_x = D_x$ and $JD_x^{\perp} \subset TM_x^{\perp}$, for each $x \in M$, where $TM^{\perp}$ is the normal space to M.

Later on, the definition was extended to other ambient spaces (See [10] and [12] and references therein for details), which gave rise to a large body of literature. The purpose of this dissertation is to further the study of CR-submanifolds by comparing those of Hermitian manifolds to selected non-Hermitian manifolds. We use Kählerian, nearly Kählerian and cosymplectic manifolds as the ambient manifolds. The rest of the dissertation is organized as follows; Chapter 2 mainly contains the basic concepts needed in other parts of the dissertation. Precisely, we introduce Riemannian manifolds, distributions on a manifold and lastly Kählerian manifolds. In Chapter 3, we introduce CR-submanifolds of almost Hermitian manifolds. We start by obtaining the CR-structures of M. We study the integrability of distributions in a nearly Kählerian manifold and the abtained results are compared to the existing ones. We also compare the curvatures of $\overline{M}$ and M in a nearly Kählerian manifold.In chapter 4 we study CR-submanifolds of non-Hermitian manifold. We use cosymplectic manifold as our ambient manifold and is compared to nearly Kählerian manifold. The following are examined under CR-submanifolds of a cosymplectic manifold: CR-structures, integrability of distributions, geodesics, invariant submanifolds, foliation, parallel ϕ structures, totally umbilical submanifolds. Also normal ϕ structures are examined. Finally, Chapter 5 winds up the study by summarizing our major findings.

Chapter 2

Preliminaries

2.1 Riemannian manifolds

Let $\overline{M}$ be a real n-dimensional connected differentiable manifold and $\overline{\nabla}$ be a linear connection on $\overline{M}$. Let us denote by $\Gamma(\Xi)$, the set of smooth sections of a vector bundle Ξ. Let p be a point on $\overline{M}$, then, the torsion tensor T of a linear connection $\overline{\nabla}$ is a tensor field of type $(1, 2)$ defined by

$$T(X, Y) = \overline{\nabla}_X Y - \overline{\nabla}_Y X - [X, Y], \tag{2.1}$$

for any X, $Y \in \Gamma(T\overline{M})$, where $[X, Y] = X(Y) - Y(X)$ is the Lie bracket of X and Y.

The curvature tensor $\overline{R}$ of a linear connection $\overline{\nabla}$ is a tensor field of type $(1, 3)$ defined by

$$\overline{R}(X, Y)Z = \overline{\nabla}_X \overline{\nabla}_Y Z - \overline{\nabla}_Y \overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]} Z, \tag{2.2}$$

for any X, Y, $Z \in \Gamma(T\overline{M})$.

Definition 2.1.1. Let $\overline{M}$ be a smooth m-dimensional manifold. A $(0, 2)$-type tensor field $\overline{g}$ is said to be a Riemannian metric on $\overline{M}$ if the following conditions are satisfied

(i) $\overline{g}$ is symmetric, i.e., $\overline{g}(X, Y) = \overline{g}(Y, X)$ for any X, $Y \in \Gamma(T\overline{M})$.

(ii) $\overline{g}$ is positive definite, i.e, $\overline{g}(X, X) > 0$, for any $X \in \Gamma(T\overline{M})$.

(iii) $\overline{g}(X, X) = 0$ if and only if $X = 0$ for any $X \in \Gamma(T\overline{M})$.

A smooth manifold (M, g) endowed with a Riemannian metric g is called *Riemannian manifold.*

Just as in Euclidean geometry, if p is a point in a Riemannian manifold (M, g), we define the length or norm of any tangent vector $X \in \Gamma(T\overline{M})$ to be

$$||X||_g^2 := g(X, X).$$

Given a Riemannian manifold (M, g) and a chart $(U, x^i)_{1 \leq i \leq n}$ consider the functions $g_{ij} : U \longrightarrow \mathbb{R}$ mapping $p \longmapsto g_{ij}(p) := g(\frac{\partial}{\partial x^i}|_p, \frac{\partial}{\partial x^j}|_p)$.

Note that, for each $p \in U$, $(g_{ij}(p))_{1 \leq i,j \leq n}$ is an $n \times n$ matrix that is

- Symmetric: $g_{ij}(p) = g_{ji}(p)$.
- Positive definite: $g_{ij}(p)(v^i, v^j) > 0$, for any $(v^1, \cdots, v^n) \neq 0$. This means that $(g_{ij}(p))$ is an invertible matrix.

These functions g_{ij} are called the local representations of the Riemannian metric g with respect to the coordinate (U, x^i).

Definition 2.1.2. The tensor field S of type $(0, s)$ or (r, s) is said to be parallel with respect to the linear connection $\overline{\nabla}$ if, for any $X \in \Gamma(T\overline{M})$,

$$\overline{\nabla}_X S = 0,$$

Definition 2.1.3. A linear connection $\overline{\nabla}$ on $\overline{M}$ is said to be a *metric* or *the Levi Civita connection* if $\overline{g}$-compatible or $\overline{g}$ is parallel with respect to $\overline{\nabla}$, i.e. $\overline{\nabla}\overline{g} = 0$. This is equivalent to, for any X, Y, $Z \in \Gamma(T\overline{M})$,

$$X(\overline{g}(Y, Z)) = \overline{g}(\overline{\nabla}_X Y, Z) + \overline{g}(Y, \overline{\nabla}_X Z).$$

If the connection $\overline{\nabla}$ is Levi-Civita, then $T = 0$, and therefore,

$$[X, Y] = \overline{\nabla}_X Y - \overline{\nabla}_Y X. \tag{2.3}$$

In this case we have the following relation called *Koszul's formula*

$$\begin{aligned} 2\overline{g}(\overline{\nabla}_X Y, Z) = {} & X(\overline{g}(Y, Z)) + Y(\overline{g}(X, Z)) - Z(\overline{g}(X, Y)) - \overline{g}(X, [Y, Z]) \\ & - \overline{g}(Y, [X, Z]) + \overline{g}(Z, [X, Y]). \end{aligned} \tag{2.4}$$

Let $\overline{R}$ be the curvature tensor fields of the smooth manifold $\overline{M}$. The Riemannian curvature is a tensor field of type $(0, 4)$ defined by

$$\overline{R}(X, Y, U, V) = \overline{g}(R(X, Y)U, V), \tag{2.5}$$

for any X, Y, U, $V \in \Gamma(T\overline{M})$. It satisfies the following properties

(i) $\overline{R}(X, Y, U, V) + \overline{R}(Y, X, U, V) = 0.$

(ii) $\overline{R}(X, Y, U, V) + \overline{R}(X, Y, V, U) = 0.$

(iii) $\overline{R}(X, Y, U, V) = \overline{R}(U, V, X, Y).$

(iv) $\overline{R}(X, Y, U, V) + \overline{R}(Y, U, X, V) + \overline{R}(U, X, Y, V) = 0.$

Note that a manifold has zero curvature if and only if it is flat, that is, locally isometric to Euclidean space. As an example on the geometric objects (connection, curvature tensor) recalled above, we have the following.

Example 2.1.1. Let us consider the circle centered at the origin with radius $r > 0$, that is, $\overline{M} = S^1(r) = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = r^2\}$ and parametrized by

$$x = r\cos\varphi \text{ and } y = r\sin\varphi.$$

Where $0 \leq \varphi \leq 2\Pi$, by direct calculations, we have

$$dx = \cos\varphi dr - r\sin\varphi d\varphi \text{ and } dy = \sin\varphi dr + r\cos\varphi d\varphi.$$

In the new coordinates the Riemannian metric, G, is obtained as follows

$$\begin{aligned} G &= (dx)^2 + (dy)^2 \\ &= (\cos\varphi dr - r\sin\varphi d\varphi)^2 + (\sin\varphi dr + r\cos\varphi d\varphi)^2 \\ &= (dr)^2 + r^2(d\varphi)^2. \end{aligned}$$

By letting $G = (\overline{g}_{ik})$, the Cartesian and polar forms of G are respectively given below

$$G = \begin{pmatrix} \overline{g}_{xx} & \overline{g}_{xy} \\ \overline{g}_{yx} & \overline{g}_{yy} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$G = \begin{pmatrix} \overline{g}_{rr} & \overline{g}_{r\varphi} \\ \overline{g}_{\varphi r} & \overline{g}_{\varphi\varphi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}.$$

Using (2.4) we obtain

$$2\overline{g}(\overline{\nabla}_{\partial\varphi}\partial\varphi, \partial r) = -\partial r\overline{g}(\partial\varphi, \partial\varphi) = -2r\partial r,$$

from which $\overline{g}(\overline{\nabla}_{\partial\varphi}\partial\varphi, \partial r) = -r\partial r$. Similarly, $\overline{g}(\overline{\nabla}_{\partial\varphi}\partial\varphi, \partial\varphi) = 0$. Hence

$$\overline{\nabla}_{\partial\varphi}\partial\varphi = -r\partial r.$$

Using the same method as above, we derive

$$\overline{\nabla}_{\partial\varphi}\partial r = \overline{\nabla}_{\partial r}\partial r = 0 \text{ and } \overline{\nabla}_{\partial r}\partial r = r\partial\varphi.$$

Then, calculating the curvature tensor $\overline{R}$ using (2.2) we obtain

$$\begin{aligned}
\overline{R}(\partial r, \partial r)\partial r &= \overline{\nabla}_{\partial r}\overline{\nabla}_{\partial r}\partial r - \overline{\nabla}_{\partial r}\overline{\nabla}_{\partial r}\partial r - \overline{\nabla}_{[\partial r, \partial r]}\partial r = 0.\\
\overline{R}(\partial r, \partial r)\partial\varphi &= \overline{\nabla}_{\partial r}\overline{\nabla}_{\partial r}\partial\varphi - \overline{\nabla}_{\partial r}\overline{\nabla}_{\partial r}\partial\varphi - \overline{\nabla}_{[\partial r, \partial r]}\partial\varphi = 0.\\
\overline{R}(\partial r, \partial\varphi)\partial r &= \overline{\nabla}_{\partial r}\overline{\nabla}_{\partial\varphi}\partial r - \overline{\nabla}_{\partial\varphi}\overline{\nabla}_{\partial r}\partial r - \overline{\nabla}_{[\partial r, \partial\varphi]}\partial r = -\overline{\nabla}_{\partial\varphi}\partial\varphi(r)\\
&= r^2\partial r.\\
\overline{R}(\partial r, \partial\varphi)\varphi &= \overline{\nabla}_{\partial r}\overline{\nabla}_{\partial\varphi}\partial\varphi - \overline{\nabla}_{\partial\varphi}\overline{\nabla}_{\partial r}\partial\varphi - \overline{\nabla}_{[\partial r, \partial\varphi]}\partial\varphi = -\overline{\nabla}_{\partial r}\partial r(r)\\
&= -r^2\partial r.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\overline{R}(\partial\varphi, \partial r)\partial r = -r^2\partial r, &\quad \overline{R}(\partial\varphi, \partial r)\partial\varphi = r^2\partial\varphi\\
\overline{R}(\partial\varphi, \partial\varphi)\partial\varphi = 0 \quad \text{and} &\quad \overline{R}(\partial\varphi, \partial\varphi)\partial\varphi = 0.
\end{aligned}$$

2.2 Submanifolds of a Riemannian manifold

Let $\overline{M}$ be an n-dimensional Riemannian manifold and let M be a m-dimens ional manifold of $\overline{M}$. Then M becomes a Riemannian submanifold of $\overline{M}$ with Riemannian metric induced by the Riemannian metric on $\overline{M}$. Let $TM^{\perp}$ denote the normal bundle to M and g both metrics on M and $\overline{M}$. Also let $\overline{\nabla}$ and ∇ denote the Levi-Civita connections on $\overline{M}$ and M respectively. Then, the Gauss and wiengaten equations are respectively given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \ \ \forall X, Y \in \Gamma(TM), \tag{2.6}$$

$$\overline{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \ \ \forall V \in \Gamma(TM^{\perp}), \tag{2.7}$$

where $\nabla_X Y$, $A_V X \in \Gamma(TM)$, and $h(X, Y)$, $\nabla^{\perp}_X V \in \Gamma(TM^{\perp})$. Further, h is a symmetric bilinear form called the *second fundamental form* of M and A_V is a linear operator known as shape operator and satisfying

$$g(h(X, Y), V) = g(A_V X, Y), \ \ \forall X, Y \in \Gamma(TM). \tag{2.8}$$

The covariant derivative of h is given by

$$(\nabla_X h)(Y, Z) = \nabla^{\perp}_X(h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z), \tag{2.9}$$

for any X, Y, $Z \in \Gamma(TM)$.

Lemma 2.2.1. *Let $\overline{R}$ and R be the curvature tensors of $\overline{M}$ and M, respectively. Then $\overline{R}$ and R are related by the following equation,*

$$\begin{aligned}\overline{R}(X,Y)Z &= R(X,Y)Z - A_{h(Y,Z)}X + A_{h(X,Z)}Y \\ &\quad + (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z). \end{aligned} \tag{2.10}$$

By comparing the tangential and normal components of (2.10) we respectively obtain the following equations.

$$\{\overline{R}(X,Y)Z\}^{\top} = R(X,Y)Z + A_{h(X,Z)}Y - A_{(Y,Z)}X \tag{2.11}$$

and

$$\{\overline{R}(X,Y)Z\}^{\perp} = (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z), \tag{2.12}$$

for any X, Y, $Z \in \Gamma(TM)$.

Definition 2.2.1. Let $\{E_1, \ldots, E_m\}$ be orthonormal basis in a tangent bundle to M, then

$$Tr(h) = \sum_{i=1}^{m} h(E_1, E_i),$$

is called the *trace* of h and is independent of the basis.

Lemma 2.2.2. *[15] Let $\overline{M}$ be a smooth manifold and M be a submanifold of $\overline{M}$. Let $R^{\perp}$ be the curvature tensor of the normal bundle of M. Then, the curvature tensors R and $R^{\perp}$ are related by the following equation called Ricci equation*

$$\overline{g}(R(X,Y)U,V) = \overline{g}(R^{\perp}(X,Y)U,V) + \overline{g}([A_U, A_V]X,Y), \tag{2.13}$$

for any X, Y, U, $V \in \Gamma(TM)$.

Definition 2.2.2. Let $\overline{M}$ be a Riemannian manifold, then a submanifold M of $\overline{M}$ is said to be totally umbilical if there exist a normal vector field μ, called *mean curvature vector*, such that

$$h(X,Y) = g(X,Y)\mu, \tag{2.14}$$

for $X, Y \in \Gamma(TM)$. If $\mu = 0$ then M is said to be minimal.

Definition 2.2.3. The submanifold M of h is said to be totally geodesic in $\overline{M}$ if its second fundamental form h vanishes identically on M, i.e., $h = 0$.

2.3 Distributions on a manifold

An m-dimensional distribution on a manifold $\overline{M}$ is defined by the following map

$$D : x \to D_x \subset T_x\overline{M},$$

for any $x \in \overline{M}$. A vector field X on $\overline{M}$ belongs to D if $X_x \in \Gamma(D_x)$ for each $x \in \overline{M}$. The distribution is said to be integrable if for all vector fields X, $Y \in \Gamma(D)$ we have $[X, Y] \in \Gamma(D)$.

Definition 2.3.1. Let $\overline{M}$ be a n-dimension smooth manifold. A submanifold M of $\overline{M}$ is said to be an integral manifold of D if every point $x \in M$, D_x coincides with the tangent space to M at x. If there exists no integral manifold of D containing M, then M is called a leaf of D.

Definition 2.3.2. Let ∇ be a linear connection on a smooth manifold $\overline{M}$. The distribution D is said to be parellel with respect to ∇ if we have

$$\nabla_X Y \in \Gamma(D),$$

for any $X \in \Gamma(TM)$ and $Y \in \Gamma(D)$.

2.4 Kählerian manifolds

Definition 2.4.1. Let $\overline{M}$ be a manifold, then $\overline{M}$ is called an almost complex manifold if there is an almost complex structure J on $\overline{M}$ which is a tensor of type $(1, 1)$ such that, for every $X \in \Gamma(T\overline{M})$, we have

$$J^2X = -X.$$

Theorem 2.4.1. *Every almost complex manifold $\overline{M}$ is of even dimension.*

Definition 2.4.2. Let $(\overline{M}, J)$ be an almost complex manifold. Then the *Nijenhuis tensor* of J is defined by

$$[J, J](X, Y) = [JX, JY] - [X, Y] - J[JX, Y] - J[X, JY], \tag{2.15}$$

for any X, $Y \in \Gamma(T\overline{M})$.

Note that $\overline{M}$ becomes a complex manifold if the Nijenhuis tensor of J vanishes identically on $\overline{M}$.

A Hermitian metric on an almost complex manifold $\overline{M}$ is a Riemannian metric g satisfying

$$\overline{g}(JX, JY) = \overline{g}(X, Y),$$

for any X, $Y \in \Gamma(T\overline{M})$.

The triple $(\overline{M}, J, \overline{g})$ is called almost Hermitian manifold. The fundamental 2-form Ω of an almost Hermitian manifold $\overline{M}$ is defined by

$$\Omega(X, Y) = \overline{g}(X, JY),$$

for any X, $Y \in \Gamma(T\overline{M})$.

Definition 2.4.3. We say $(\overline{M}, J, \overline{g})$ is *Kählerian manifold* if we have $\mathrm{d}\Omega = 0$, i.e., the almost complex structure J is parallel and *nearly Kählerian manifold* if

$$(\overline{\nabla}_X J)X = 0, \tag{2.16}$$

for any $X \in \Gamma(T\overline{M})$.

Replacing X with $X+Y$ in (2.16) we obtain that $\overline{M}$ is a nearly Kählerian manifold if and only if

$$(\overline{\nabla}_X J)Y + (\overline{\nabla}_Y J)X = 0, \tag{2.17}$$

for any X, $Y \in \Gamma(T\overline{M})$.

As an example of an almost Hermitian (or Kählerian) manifold we have the following.

Example 2.4.1. Consider $\overline{M} = \mathbb{R}^4$. Let $x = (x^1, x^2, x^3, x^4) \in \mathbb{R}^4$ and define the metric on $\mathbb{R}^4$ by

$$\overline{g}_{ij} = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Let $T\mathbb{R}^4 = \mathrm{Span}\{\partial x^1, \partial x^2, \partial x^3, \partial x^4\}$ and the almost complex structure J is defined by

$$J\partial x^1 = -\partial x^2, \quad J\partial x^2 = \partial x^1, \quad J\partial x^3 = -\partial x^4 \;\text{ and }\; J\partial x^4 = \partial x^3.$$

In the matrix form, we have

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

and we have

$$J^2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = -\mathbb{I}.$$

It is easy to check that

$$\overline{g}(J\partial x^i, J\partial x^j) = \overline{g}(\partial x^i, \partial x^j).$$

Hence $\overline{g}(JX, JY) = \overline{g}(X, Y)$. Thus $(\mathbb{R}^4, J, \overline{g})$ is an almost Kählerian manifold. Using (2.4) we obtain

$$\begin{aligned} &\overline{g}(\overline{\nabla}_{\partial x^1}\partial x^1, \partial x^1) = \overline{g}(\overline{\nabla}_{\partial x^1}\partial x^1, \partial x^3) = \overline{g}(\overline{\nabla}_{\partial x^1}\partial x^1, \partial x^4) = 0, \\ &\overline{g}(\overline{\nabla}_{\partial x^1}\partial x^1, \partial x^2) = 1, \end{aligned}$$

and hence $\overline{\nabla}_{\partial x^1}\partial x^1 = \partial x^2$. Also

$$\begin{aligned} &\overline{g}(\overline{\nabla}_{\partial x^2}\partial x^2, \partial x^2) = \overline{g}(\overline{\nabla}_{\partial x^2}\partial x^2, \partial x^3) = \overline{g}(\overline{\nabla}_{\partial x^2}\partial x^2, \partial x^4) = 0, \\ &\overline{g}(\overline{\nabla}_{\partial x^2}\partial x^2, \partial x^1) = -1, \end{aligned}$$

which give $\overline{\nabla}_{\partial x^2}\partial x^2 = -\partial x^1$. Similarly,

$$\overline{\nabla}_{\partial x^3}\partial x^3 = \partial x^4 \text{ and } \overline{\nabla}_{\partial x^4}\partial x^4 = -\partial x^3.$$

Putting these pieces together, we obtain

$$(\overline{\nabla}_{\partial x^i} J)\partial x^j \neq 0.$$

This means that $(\mathbb{R}^4, J, \overline{g})$ is not a Kählerian manifold.

Definition 2.4.4. A semi-Riemannian metric on a manifold $\overline{M}$ is a family $\overline{g}$ non-degenerate symmetric bilinear forms

$$\overline{g}_p = T_p\overline{M} \times T_p\overline{M} \longrightarrow \mathbb{R}, \quad p \in \overline{M},$$

such that the function $\overline{g}(X, Y) : p \longmapsto \overline{g}(X(p), Y(p))$ is smooth for all smooth vector fields X, Y on $\overline{M}$. Thus $(\overline{g}, \overline{M})$ is called semi-Riemannian manifold.

Definition 2.4.5. *[9]* A *space form* is a complete connected semi-Riemannian manifold of constant curvature.

It is known that if a $2n$-dimensional Kählerian manifold $\overline{M}$ is of constant curvature, then, $\overline{M}$ is flat provided $n > 1$ (see [4] for more details). This tells us that the notion of constant curvature for Kählerian manifolds is not essential. Therefore, the notion of constant holomorphic sectional curvature was introduced for Kählerian manifolds. For this purpose, we first state the notion of holomorphic section as follows:

Consider a tangent vector U at a point p of a smooth Kähler manifold $(\overline{M}, J, \overline{g})$. Then, the pair (U, JU) generates a plane π (since JU is obviously orthogonal to U, i.e., $\overline{g}(U, JU) = 0$) element called a *holomorphic section*, whose curvature K is given by

$$K = \frac{\overline{g}(\overline{R}(U, JU)JU, U)}{(\overline{g}(U, U))^2}.$$

K is called the *holomorphic sectional curvature* with respect to U. Now, if K is independent of the choice of U at a point, then $K = c$, a constant.

A simply connected complete Kähler manifold of constant sectional curvature c is called a *complex space-form*, and is denoted by $\overline{M}(c)$. Now, if $c > 0$, $c = 0$ or $c < 0$, then $\overline{M}(c)$ can be identified with the complex projective space $\mathbb{C}P^n(c)$, complex plane $\mathbb{C}^n$ or the open ball D^n in $\mathbb{C}^n$. The curvature tensor $\overline{R}$ of a simply connected complete Kählerian space form $\overline{M}(c)$ is given by

$$\begin{aligned}\overline{R}(X, Y)Z = \frac{c}{4}\{&\overline{g}(Y, Z)X - \overline{g}(X, Z)Y + \overline{g}(Z, JY)JX - \overline{g}(Z, JX)JY \\ &+ 2\overline{g}(X, JY)JZ\},\end{aligned} \tag{2.18}$$

for any X, Y, $Z \in \Gamma(T\overline{M})$.

Lemma 2.4.2. *A simply connected complete $2n$-dimensional Kählerian space form $\overline{M}(c)$ is Einstein.*

Proof. By contracting the equation 2.18, one obtains

$$\overline{\mathrm{Ric}}(X, Y) = \frac{n+1}{2} c\overline{g}(X, Y),$$

for any X, $Y \in \Gamma(TM)$, where $\overline{\mathrm{Ric}}$ is the Ricci tensor associated with $\overline{g}$. $\square$

Remark 2.4.1. It is clear that every Kählerian manifold is nearly Kählerian but the converse is not true. The nearly Kählerian six-sphere S^6 is an example of a nearly Kählerian manifold that is not Kählerian.

Proposition 2.4.3. *Let $\overline{M}$ be a nearly Kählerian manifold. Then the Nijenhuis tensor of J is given by*

$$[J,J](X,Y) = 4J(\overline{\nabla}_Y J)X, \tag{2.19}$$

for any X, $Y \in \Gamma(T\overline{M})$.

Proof. Using the fact that $\overline{\nabla}$ is a torsion free connection on $\overline{M}$ and (2.15) we derive

$$\begin{aligned}
[J,J](X,Y) &= \overline{\nabla}_{JX}JY - \overline{\nabla}_{JY}JX - \overline{\nabla}_X Y + \overline{\nabla}_Y X \\
&- J[\overline{\nabla}_{JX}Y - \overline{\nabla}_Y JX] - J[\overline{\nabla}_X JY - \overline{\nabla}_{JY}X] \\
&= (\overline{\nabla}_{JX}J)Y + J(\overline{\nabla}_{JX}Y) - (\overline{\nabla}_{JY}J)X - J(\overline{\nabla}_{JY}X) \\
&+ J(\overline{\nabla}_X JY) - J((\overline{\nabla}_X J)Y) - J(\overline{\nabla}_Y JX) + J((\overline{\nabla}_Y J))Y \\
&- J(\overline{\nabla}_{JX}Y) + J(\overline{\nabla}_Y JX) - J(\overline{\nabla}_X JY) + J(\overline{\nabla}_{JY}X) \\
&= (\overline{\nabla}_{JX}J)Y - (\overline{\nabla}_{JY}J)X - J((\overline{\nabla}_X J))Y + J((\overline{\nabla}_Y J)X).
\end{aligned} \tag{2.20}$$

Using the fact that $(\overline{\nabla}_{JY}J)X = J((\overline{\nabla}_X J)Y)$ and substituting into (2.20) we derive

$$\begin{aligned}
[J,J](X,Y) &= 2(\overline{\nabla}_Y X + J(\overline{\nabla}_X JX) - \overline{\nabla}_X Y - J(\overline{\nabla}_X JY)) \\
&= 2\{J(\overline{\nabla}_Y JX - J(\overline{\nabla}_Y X)) - J(\overline{\nabla}_X JY - J(\overline{\nabla}_X Y))\} \\
&= 2J((\overline{\nabla}_Y J)X - (\overline{\nabla}_X J)Y) = 4J((\overline{\nabla}_Y J)X),
\end{aligned}$$

for any X, $Y \in \Gamma(T\overline{M})$. □

Chapter 3

CR-submanifolds of almost Hermitian manifolds

This Chapter focuses on CR-submanifolds of Hermitian manifold. We introduce the idea of CR-submanifold M of $\overline{M}$ and also study CR-structures. Finaly we consider a nearly Kählerian as an example of Hermitian manifold and we study its geometrical properties.

3.1 CR-submanifolds and CR-structures

Definition 3.1.1. *[15]* Let $(\overline{M}, J, \overline{g})$ be a $2n$-dimensional almost Hermitian manifold. Let M be a m-dimensional submanifold of $\overline{M}$. Then M is called a *complex holomorphic submanifold* if T_xM is invariant by J, i.e., we have

$$J(T_xM) = T_xM$$

for each $x \in M$, also M is called a anti-invariant submanifold of $\overline{M}$ if we have

$$J(T_xM) \subset T_xM^{\perp}$$

for each $x \in M$.

Definition 3.1.2. *[15]* Let M be a submanifold of $\overline{M}$. Then M is called a *CR-submanifold* of $\overline{M}$ if there exists a distribution. $D : x \mapsto D_x \subset T_xM$ on M satisfying

(i) $J(D_x) = D_x$ for each $x \in M$.

(ii) $J(D^{\perp}) \subset T_xM^{\perp}$ for each $x \in M$.

Then, the tangent bundles of $\overline{M}$ and M are respectively decomposed as

$$T\overline{M} = TM \oplus TM^{\perp} \text{ and } TM = D \oplus D^{\perp},$$

where $\oplus$ is a orthogonal direct sum. For vector fields $X \in \Gamma(TM)$, $V \in \Gamma(TM^{\perp})$ we have

$$JX = \phi X + \omega X, \tag{3.1}$$

and

$$JV = BV + CV, \tag{3.2}$$

where ϕX and ωX are tangential and normal components of JX respectively and BV and CV are tangential and normal components of JV respectively.

Theorem 3.1.1. *The submanifold M of $\overline{M}$ is a CR-submanifold if and only if*

$$rank(\phi) = constant \;\; and \;\; \omega \circ \phi = 0.$$

Proof. Suppose M is a CR-submanifold of an almost Hermitian manifold $\overline{M}$. Denote P and Q the projections morphisms of TM to D and $D^{\perp}$ respectively. For any $X \in \Gamma(TM)$

$$X = PX + QX,$$

which on applying J leads to

$$JX = JPX + JQX = \phi X + \omega X,$$

where $\phi X = JPX$ and $\omega X = JQX$, which are the tangential and normal components of JX. Then, it follows that $rank(\phi) = 2p$ and thus, every almost Hermitian manifold is even dimensional and from (3.1) we can see by inspection that $\omega \circ \phi = 0$.

Conversely, suppose that $rank(\phi) = constant$ and $\omega \circ \phi = 0$. Let the distribution D be defined by $D_x = \mathrm{Im}.\phi_x$ for each $x \in M$ and let $X = \phi Y \in \Gamma(D)$, then

$$JX = J\phi Y = \phi^2 Y + (\omega \circ \phi)Y = \phi^2 Y \in \Gamma(Im.\phi) \subseteq \Gamma(D),$$

so D is an invariant distribution. Let denote by $D^{\perp}$ the complementary orthogonal distribution to D in TM. Then, D is an anti-invariant distribution. Since for any $X \in \Gamma(D^{\perp})$ and $Y = U + W$ where $U \in \Gamma(D)$ and $W \in \Gamma(D^{\perp})$ we obtain

$$g(JX, Y) = -g(X, JY) = -g(X, JU) - g(X, JW) = 0.$$

Thus M is a CR-submanifold of $\overline{M}$. $\square$

Lemma 3.1.2. *Let $\overline{M}$ be a manifold and M be a submanifold of $\overline{M}$. The f-structure on TM is given by*

$$\phi^3 + \phi = 0.$$

Proof. Using (3.1) and (3.2), for any $X \in \Gamma(TM)$ we obtain

$$J^2X = J(JX) = J(\phi X + \omega X) = \phi^2 X + B\omega X + C\omega X.$$

So

$$-X = \phi^2 X + B\omega X \text{ and } C\omega X = 0.$$

Therefore

$$\phi^2 = -I - B\omega \text{ and } C \circ \omega = 0.$$

Using the fact that $\phi X = JPX$ and $JP = PJ$ we obtain that

$$\phi^2 X = \phi(\phi X) = JP(JPX) = P^2 J^2 X = -PX,$$

Applying ϕ we get

$$\phi^3 = \phi(\phi^2) = -\phi P = -JP^2 = -JP = -\phi,$$

from which our assertion follows. □

Definition 3.1.3. *[15]* Suppose that M is a CR-submanifold of an almost Hermitian manifold $\overline{M}$, then

(i) $J(D) = D$

(ii) $J(D^\perp) \subset TM^\perp$

$JD^\perp \subset \mathrm{T}M^\perp$ implies that there exist δ, a complementary distribution to $J(D^\perp)$ such that

$$TM^\perp = JD^\perp \oplus \delta.$$

It is easy to see that δ is invariant with respect to J, i.e., $J\delta = \delta$.

Lemma 3.1.3. *Let $\overline{M}$ be a manifold and M be a submanifold of $\overline{M}$. The f-sructure on $TM^\perp$ is given by*

$$C^3 + C = 0.$$

Proof. Let P and Q be projections morphism of $TM^{\perp}$ to $J(D^{\perp})$ and δ respectively, for any $V \in \Gamma(TM^{\perp})$ we have

$$V = PV + QV,$$

where PV and QV are tangential and normal part of JV.

$$JV = JPV + JQV = BV + CV.$$

So

$$-V = J^2V = J(BV + CV) = \omega BV + BCV + C^2V.$$

Therefore

$$-I - \omega B = C^2 = -I - \omega B \ \text{ and } \ B \circ C = 0.$$

Using the fact that $CV = JQV$ we derive

$$C^2V = C(CV) = CJQV = (JQ)^2V = Q^2J^2V = -QV,$$

which implies that

$$C^2V + V + \omega BV = 0.$$

Applying C we obtain

$$C^3 + C = 0.$$

□

Definition 3.1.4. Let M be a CR-submanifold of an almost Hermitian manifold $\overline{M}$. Then the Nijenhuis tensor field of ϕ is given by

$$[\phi, \phi](X, Y) = [\phi X, \phi Y] + \phi^2[X, Y] - \phi([X, \phi Y]) - \phi([\phi X, Y]). \tag{3.3}$$

Proposition 3.1.4. *Let M be a CR-submanifold of an almost Hermitian manifold $\overline{M}$, Then*

$$[J, J](X, Y) = [\phi, \phi](X, Y) - Q[X, Y] - \omega([\phi X, Y] + [X, \phi Y]). \tag{3.4}$$

Proof. Using (2.15) and (3.1) we derive

$$[J, J](X, Y) = -[X, Y]+[\phi X, \phi Y]-\phi([\phi X, Y]+[X, \phi Y])-\omega([\phi X, Y]+[X, \phi Y]). \tag{3.5}$$

Applying (3.3) in (3.5) we obtain

$$[J, J](X, Y) = -[X, Y] + [\phi, \phi](X, Y) - \phi^2[X, Y] - \omega([\phi X, Y] + [X, \phi Y]),$$

which reduces to

$$\begin{aligned}[J, J](X, Y) &= [\phi, \phi](X, Y) - [X, Y] + P[X, Y] - \omega([\phi X, Y] + [X, \phi Y]) \\ &= [\phi, \phi](X, Y) - Q[X, Y] - \omega([\phi X, Y] + [X, \phi Y])\end{aligned}$$

for any X, $Y \in \Gamma(D)$. □

Theorem 3.1.5. *Let M be a CR-submanifold of an almost Hermitian manifold $\overline{M}$. Then the distribution D is integrable if and only if*

$$[J, J](X, Y)^{\top} = [\phi, \phi](X, Y)$$

for any X, $Y \in \Gamma(D)$.

Proof. Suppose that D is integrable, then $Q[X, Y] = 0$ and $\omega([\phi X, Y] + [X, \phi Y]) = 0$ which reduses (3.4) to

$$[J, J](X, Y)^{\top} = [\phi, \phi](X, Y).$$

Conversely, suppose

$$[J, J](X, Y)^{\top} = [\phi, \phi](X, Y).$$

Then the tangential component of (3.4) becomes

$$Q[X, Y] = 0,$$

which implies that

$$[X, Y] = P[X, Y].$$

Therefore $[X, Y] \in \Gamma(D)$. Thus D is integrable. □

Theorem 3.1.6. *Let M be a CR submanifold of an almost Hermitian manifold $\overline{M}$. Then the distribution D is integrable if and only if*

(i) $[J, J](X, Y)^{\perp} = 0$.

(ii) $Q[\phi, \phi](X, Y) = 0$ *for any X, $Y \in \Gamma(D)$.*

Proof. Suppose that D is integrable, then $[\phi X, Y]$ and $[X, \phi Y] \in \Gamma(D)$ for any X, $Y \in \Gamma(D)$ and $\phi X \in \Gamma(D)$. So

$$Q([\phi X, Y] + [X, \phi Y]) = 0 \text{ and } \omega([\phi X, Y] + [X, \phi Y]) = 0.$$

Then the tangential component of (3.4) reduces to

$$[J, J](X, Y)^{\perp} = -Q[X, Y] - \omega([\phi X, Y] + [X, \phi Y]) = 0,$$

which proves (i). It follows from (3.3) that $[\phi, \phi]$(X, Y) $\in \Gamma(D)$ which implies that

$$Q[\phi, \phi](X, Y) = 0,$$

which proves (ii).
Conversely, suppose that (i) and (ii) holds.

$$\omega([\phi X, Y] + [X, \phi Y]) = 0,$$

if and only if

$$Q([JX, Y] + [X, JY]) = 0.$$

Let $Y = JY$, then

$$Q([JX, JY] - [X, Y]) = 0.$$

Hence

$$Q([J, J](X, Y)^{\top}) = 0.$$

On the other hand by (3.3) we have

$$Q([J, J](X, Y)^{\top}) = Q[\phi, \phi](X, Y) - Q[X, Y],$$

which implies that

$$Q[X, Y] = 0.$$

Hence $[X, Y] \in \Gamma(D)$. So D is integrable. □

3.2 Nearly Kählerian manifolds

3.2.1 Integrability

Let M be a CR-submanifold of a nearly Kählerian manifold $\overline{M}$. Then by using (2.6), (2.7), (2.15) and (2.17) we derive

$$[JX, Y] + [X, JY] = J[J, J](X, Y) + J[X, Y] - J[JX, JY]. \tag{3.6}$$

Now,

$$\begin{aligned}
[JX, JY] &= \overline{\nabla}_{JX} JY - \overline{\nabla}_{JY} JX \\
&= (\overline{\nabla}_{JX} J)Y + J(\overline{\nabla}_{JX} Y) - (\overline{\nabla}_{JY} J)X - J(\overline{\nabla}_{JY} X) \\
&= -\overline{\nabla}_Y J^2 X + J(\overline{\nabla}_Y JX) + (\overline{\nabla}_X J^2 Y) - J(\overline{\nabla}_X JY) \\
&+ J(\overline{\nabla}_{JX} Y - \overline{\nabla}_{JY} X) \\
&= \overline{\nabla}_Y X - \overline{\nabla}_X Y + J(\overline{\nabla}_Y JX) - J(\overline{\nabla}_X JY) \\
&+ J(\overline{\nabla}_{JX} Y - \overline{\nabla}_{JY} X).
\end{aligned} \tag{3.7}$$

Applying J to (3.7) we derive

$$\begin{aligned}
J[JX, JY] &= -J[X, Y] - [Y, JX] - \overline{\nabla}_{JX}Y + [X, JY] \\
&+ \overline{\nabla}_{JY}X + \overline{\nabla}_{JY}X - \overline{\nabla}_{JX}Y \\
&= -J[X, Y] + [JX, Y] + [X, JY] + 2\overline{\nabla}_{JY}X \\
&- 2\overline{\nabla}_{JX}Y \\
&= -J[X, Y] + [JX, Y] + [X, JY] + 2\nabla_{JY}X \\
&+ 2h(X, JY) - 2\nabla_{JX}Y - 2h(JX, Y). \qquad (3.8)
\end{aligned}$$

Substituting (3.8) into equation (3.6) we obtain

$$\begin{aligned}
[JX, Y] + [X, JY] &= \frac{1}{2}J([J, J](X, Y)) + J([X, Y]) + \nabla_{JX}Y - \nabla_{JY}X \\
&+ h(JX, Y) - h(X, JY). \qquad (3.9)
\end{aligned}$$

Taking into account that ∇ is a torsion-free connection then from (3.9) we obtain

$$\begin{aligned}
h(X, JY) - h(JX, Y) &= \frac{1}{2}J([J, J](X, Y)) + J([X, Y]) + \nabla_Y JX \\
&- \nabla_X JY \qquad (3.10)
\end{aligned}$$

for any X, $Y \in \Gamma(D)$.

Theorem 3.2.1. *Let M be a CR-submanifold of a nearly Kählerian manifold $\overline{M}$. Then the distribution D is integrable if and only if the following conditions are satisfied*

(i) $h(X, JY) = h(JX, Y)$.

(ii) $[J, J](X, Y) \in \Gamma(D)$ *for any X, $Y \in \Gamma(D)$.*

Proof. Suppose that D is integrable. Then $JX = \phi X$ and $JY = \phi Y$. Comparing tangential and normal components of (3.9) we have

$$\begin{aligned}
[JX, Y] + [X, JY] &= J[X, Y] + \nabla_{JX}Y - \nabla_{JY}X \\
\text{and } \; h(JX, Y) - h(X, JY) &= \frac{1}{2}J([J, J](X, Y)). \qquad (3.11)
\end{aligned}$$

Since

$$[J, J](X, Y) = [J, J](X, Y)^{\top} + [J, J](X, Y)^{\perp}, \qquad (3.12)$$

taking the tangential component of (3.4) and using Theorem 3.1.5 we obtain

$$[J, J](X, Y) = [J, J](X, Y)^{\top} = [\phi, \phi](X, Y).$$

Applying J we have

$$J[J, J](X, Y) = J[\phi, \phi](X, Y).$$

Since D is integrable and by Theorem 3.1.6 we have

$$[\phi, \phi](X, Y) = 0,$$

which implies that

$$J[J, J](X, Y) = 0,$$

and substituting into (3.11), we obtain

$$h(JX, Y) - h(X, JY) = 0,$$

so

$$h(JX, Y) = h(X, JY) \text{ and } [J, J](X, Y) \in \Gamma(D).$$

Conversely, suppose that equation (i) and (ii) holds, then for any X, $Y \in \Gamma(D)$ we must show that $[X, Y] \in \Gamma(D)$. From (3.10) we have

$$J[X, Y] = \nabla_X JY - \nabla_Y JX - \frac{1}{2}J[J, J](X, Y).$$

Let $Z \in \Gamma(D^{\perp})$, then there exist $V \in \Gamma(TM^{\perp})$ such that $Z = JV$.

$$\begin{aligned} g([X, Y], Z) &= g([X, Y], JV) = -g(J[X, Y], V) \\ &= -g(\nabla_X JY, V) + g(\nabla_Y JX, V) + \frac{1}{2}g(J[J, J](X, Y), V) = 0. \end{aligned}$$

Therefore $[X, Y] \in \Gamma(D)$. Hence D is integrable as required. □

From Theorem 3.2.1 we deduce the following Theorem and a Corollary.

Theorem 3.2.2. *Let M be a CR-submanifold of a nearly Kählerian manifold $\overline{M}$. Then the distribution D is integrable if and only if*

(i) $(\overline{\nabla}_X J)Y \in \Gamma(D)$.

(ii) $h(X, JY) = h(JX, Y)$ *for any* X, $Y \in \Gamma(D)$.

Proof. Suppose that D is integrable then $[J,J](X,Y) = [\phi,\phi](X,Y) \in \Gamma(D)$. By Proposition 2.4.3 $[J,J](X,Y) = 4J(\overline{\nabla}_X J)Y$, which implies that $\overline{\nabla}_X Y \in \Gamma(D)$. Part (ii) follows from Theorem 3.2.1. The converse follow from Theorem 3.2.1. $\square$

Corollary 3.2.3. *Let M be a CR-submanifold of a nearly Kählerian manifold $\overline{M}$. If the distribution D is integrable, then*

(i) $[J,J](X,U)^\top \in \Gamma(D^\perp)$ *for any* $X \in \Gamma(D)$ *and* $U \in \Gamma(D^\perp)$.

(ii) $h(X,JY) = h(JX,Y)$.

Proof. Suppose that D is integrable. Let $X \in \Gamma(D)$ and $U \in \Gamma(D^\perp)$, then by (2.6), (2.7) and Proposition 2.4.3 we derive

$$\begin{aligned}(\overline{\nabla}_U J)X &= -(\overline{\nabla}_X J)U = -\overline{\nabla}_X JU + J(\overline{\nabla}_X U)\\ &= A_{JU}X - \nabla^\perp_X JU + J\nabla_X U + Jh(X,U).\end{aligned}$$

Applying J we obtain

$$\begin{aligned}J(\overline{\nabla}_U J)X &= JA_{JU}X - J\nabla^\perp_X JU - \nabla_X U - h(X,U)\\ &= \phi A_{JU}X + \omega A_{JU}X - B\nabla^\perp_X JU - C\nabla^\perp_X JU\\ &\quad - \nabla_X U - h(X,U). \qquad (3.13)\end{aligned}$$

Taking the normal and tangential component of (3.13) we obtain

$$\begin{aligned}[J,J](X,Y)^\top &= \phi A_{JU}X - B\nabla^\perp_X JU - \nabla_X U\\ \textit{and}\;\; [J,J](X,Y)^\perp &= -C\nabla^\perp_X JU - h(X,U) + \omega A_{JU}X.\end{aligned}$$

Hence, $[J,J](X,Y)^\top \in \Gamma(D^\perp)$. Finally, (ii) follows from Theorem 3.2.1. $\square$

Theorem 3.2.4. *Let M be a CR-submanifold of a nearly Kählerian manifold $\overline{M}$. Suppose the following conditions are satisfied*

$$g(h(X,Y),JZ) = 0, \qquad (3.14)$$

for any X, $Y \in \Gamma(TM)$, $Y \in \Gamma(D)$, $Z \in \Gamma(D^\perp)$ and

$$g([J,J](X,Y),W) = 0 \qquad (3.15)$$

for any X, $Y \in \Gamma(D)$ and $W \in \Gamma(D^\perp)$. Then M is a CR-product of $\overline{M}$.

Proof. Using (3.14) and (2.6) we derive

$$\begin{aligned} 0 = g(h(X,Y), JZ) &= g(\overline{\nabla}_X Y, JZ) - g(\nabla_X Y, JZ) = g(\overline{\nabla}_X Y, JZ) \\ &= -g(J\overline{\nabla}_X Y, Z). \end{aligned} \tag{3.16}$$

Since $\overline{M}$ is a nearly Kählerian manifold, then

$$J\overline{\nabla}_X Y = \overline{\nabla}_X JY + \overline{\nabla}_Y JX - J\overline{\nabla}_Y X. \tag{3.17}$$

Substituting (3.17) into (3.16) and using (2.6) and (2.7) we derive

$$\begin{aligned} 0 &= -g(\overline{\nabla}_Y X, JZ) - g(\overline{\nabla}_X JY, Z) - g(\overline{\nabla}_Y JX, Z) \\ &= -g(h(X,Y), JZ) - g(\nabla_X JY, Z) + g(A_{JX} Y, Z) \\ &= -g(\nabla_X JY, Z), \end{aligned}$$

for any $Y \in \Gamma(D)$ and $X,\ Z \in \Gamma(D^{\perp})$. Thus we have $\nabla_X JY \in \Gamma(D)$. On the other hand by using (2.19), (2.6), (3.14) and (3.15) we derive

$$\begin{aligned} 0 &= g(4J(\overline{\nabla}_Y J)X, W) \\ &= g(\overline{\nabla}_X JY, JW) - g(\overline{\nabla}_X Y, W) \\ &= g(\nabla_X JY + h(X, JY), JW) \\ &- g(\nabla_X Y + h(X,Y), W) \\ &= g(\nabla_X JY, JW). \end{aligned}$$

Hence, $\nabla_X JY \in \Gamma(D)$ and thus D is parallel. Our assertion follows from Theorem 3.2.1. □

3.2.2 Curvatures

In this section we compare the curvature tensors of $\overline{M}$ and M.

Using (2.11), and , for any $X,\ Y,\ Z,\ W \in \Gamma(TM)$, we have

$$\begin{aligned} g([\overline{R}(X,Y)Z]^{\top}, W) = g(R(X,Y)Z, W) &+ g(A_{h(X,Z)}Y, W) \\ &- g(A_{h(Y,Z)}X, W), \end{aligned}$$

then

$$\begin{aligned} \overline{R}(X,Y,Z,W) = R(X,Y,Z,W) &+ g(h(Y,W), h(Y,Z)) \\ &- g(h(X,W), h(Y,Z)). \end{aligned} \tag{3.18}$$

Suppose D is integrable then by Theorem 3.2.1 we have

$$h(X, JY) = h(JX, Y),$$

for any $X, Y \in \Gamma(D)$. Taking $X = JX$ in this equation, one has $h(JX, JY) = -h(X, Y)$.

Thus equation (3.18) becomes

$$\begin{aligned}\overline{R}(X, JX, JY, Y) &= R(X, JX, JY, Y) + g(h(JX, Y), h(X, JY)) \\ &\quad - g(h(X, Y), h(JX, JY)) \\ &= R(X, JX, JY, Y) + ||h(JX, Y)||^2 + ||h(X, Y)||^2. \quad (3.19)\end{aligned}$$

If we assume that M is totally geodesic, then (3.19) reduces to

$$\overline{R}(X, JX, JY, Y) = R(X, JX, JY, Y).$$

Let $\overline{H}(X) = \overline{R}(X, JX, JX, X)$ and $H(X) = R(X, JX, JX, X)$ be the holomorphic sectional curvatures of $\overline{M}$ and M, respectively. We therefore observe the following.

Theorem 3.2.5. *Let M be a CR-submanifold of a Kählrian space form $\overline{M}(c)$ with an integrable distribuion D. Then, the holomorphic sectional curvatures $\overline{H}$ and H of $\overline{M}$ and M, respectively, satisfy*

$$\overline{H}(X) \geq H(X), \quad \forall X \in \Gamma(D), \quad ||X||_g = 1,$$

and the equality holds if M is D-totally geodesic.

Chapter 4

CR-submanifolds of non-Hermitian manifold

In this Chapter we study CR-submanifolds of non-Hermitian manifolds. We shall consider a cosymplectic manifold $\overline{M}$ and study Geometric properties. We compare the results of a cosymplectic manifold $\overline{M}$ to a nearly Kählerian manifold in Chapter 3.

4.1 Cosymplectic manifolds

Let $\overline{M}$ be an almost contact smooth manifold and let $(\overline{\phi}, \xi, \eta, \overline{g})$ be its almost contact metric structure. Thus $\overline{M}$ is a odd dimensional differentiable manifold and $\overline{\phi}$ is a $(1,1)$ tensor field, ξ is a space-like vector field, called the structure vector field and η is a 1-form on $\overline{M}$, such that

$$\begin{aligned}
&\overline{\phi}^2 X = -X + \eta(X)\xi, \;\; \eta(X) = \overline{g}(X, \xi), \;\; \overline{\phi}(\xi) = 0, \\
&\eta \circ \overline{\phi} = 0, \;\; \eta(\xi) = 1 \;\text{ and }\; \overline{g}(\overline{\phi}X, \overline{\phi}Y) = \overline{g}(X, Y) - \eta(X)\eta(Y),
\end{aligned}$$

for any vector fields $X, Y \in \Gamma(T\overline{M})$. Then, $\overline{M}$ is called cosymplectic manifold if

$$(\overline{\nabla}_X \overline{\phi})Y = 0, \quad \forall\, X, Y \in \Gamma(T\overline{M}). \tag{4.1}$$

Replacing Y with ξ in (4.1) we have

$$\overline{\nabla}_X \xi = 0, \quad \text{for all;}\, , X \in \Gamma(T\overline{M}). \tag{4.2}$$

Definition 4.1.1. Let M be a Riemannian submanifold tangent to the structure vector field ξ isometrically immersed in a cosymplectic manifold $\overline{M}$, then M is called *contact CR-submanifold* if there exist a differential distribution $D : x \to D_x \subset T_xM$ such that

(i) $\overline{\phi}D_x \subset D_x$, for each x in M.

(ii) $\overline{\phi}D^{\perp} \subset T_xM^{\perp}$, for each x in M.

Note that $\overline{\phi}D^{\perp} \subset T_xM^{\perp}$ implies that there exist a complimentary orthogonal distribution μ to $\overline{\phi}D^{\perp}$ in $TM^{\perp}$ such that $TM^{\perp} = \overline{\phi}D^{\perp} \oplus \mu$.

Consider the following decomposition of TM,

$$TM = D \oplus D^{\perp} \oplus \{\xi\},$$

where $\{\xi\}$ is a line bundle spanned by ξ.

Let P and Q be projection morphisms of TM to D and $D^{\perp}$ respectively, for any $X \in \Gamma(TM)$ we obtain

$$X = PX + QX + \eta(X)\xi \tag{4.3}$$

Then applying $\overline{\phi}$ to (4.3) we obtain

$$\overline{\phi}X = \phi X + \omega X, \tag{4.4}$$

where $\phi X = \overline{\phi}PX$ and $\omega X = \overline{\phi}QX$. Computing ϕ^2 we obtain

$$\phi^2 X = \phi(\phi X) = \phi\overline{\phi}PX = \overline{\phi}P\overline{\phi}PX = P^2\overline{\phi}^2 X = P(-X + \eta(X)\xi) = -PX.$$

Applying ϕ we obtain

$$\phi^3 X = \phi\phi^2 X = -\phi PX = -\overline{\phi}P^2 X = -\overline{\phi}PX = -\phi X.$$

Therefore, we have the following.

Lemma 4.1.1. *Let $\overline{M}$ be a cosymplectic manifold and M be a submanifold of $\overline{M}$. Then ϕ is an f-structure in TM, i.e.,*

$$\phi^3 + \phi = 0.$$

Let P and Q be projections morphisms of $TM^{\perp}$ to $\overline{\phi}D^{\perp}$ and μ, for any $V \in \Gamma(TM^{\perp})$

$$V = PV + QV,$$

where PV is a tangential component and QV is a normal component of $TM^{\perp}$, then applying $\overline{\phi}$ we have

$$\overline{\phi}V = \overline{\phi}PV + \overline{\phi}QV = BV + CV,$$

where $BV = \overline{\phi}PV$ and $CV = \overline{\phi}QV$. Computing C^2 we obtain

$$C^2V = CCV = \overline{\phi}Q(\overline{\phi}QV) = \overline{\phi}^2QV = Q(-V + \eta(V)\xi) = -QV.$$

Applying C we derive

$$C^3V = CC^2V = \overline{\phi}Q(-QV) = -\overline{\phi}QV = -CV.$$

Therefore, we have the following.

Lemma 4.1.2. *Let $\overline{M}$ be a cosymplectic manifold and let M be a submanifold of M. Then C is an f-structure in $TM^{\perp}$, i.e.,*

$$C^3 + C = 0.$$

The f-structures on TM and $TM^{\perp}$ for both cosymplectic and Kählerian manifold are similar.

Lemma 4.1.3. *Let $\overline{M}$ be a cosymplectic manifold then*

$$C^2 + \omega B = -I \quad and \quad \phi B + BC = 0. \tag{4.5}$$

Proof. Let $V \in \Gamma(TM^{\perp})$, then

$$\overline{\phi}^2V = \overline{\phi}\,\overline{\phi}V = \overline{\phi}(BV + CV) = \phi(BV) + \omega(BV) + B(CV) + C(CV).$$

So

$$-V = \phi BV + \omega BV + BCV + C^2V. \tag{4.6}$$

Comparing tangential and normal components of (4.6) we obtain

$$-V = C^2V + \omega BV \quad \text{and} \quad \phi BV + BCV = 0,$$

from which our assertion follows. □

Let X be a unit vector which is orthogonal to ξ. We say that X and ϕX span a $\overline{\phi}$-sections. If the sectional curvature c of all $\overline{\phi}$-sections is independent of X, we say M is of pointwise constant $\overline{\phi}$-sectional curvature. If a cosymplectic manifold $\overline{M}$ is of pointwise constant $\overline{\phi}$-sectional curvature c, then the curvature tensor has the following form (see [7, 14])

$$\begin{aligned}\overline{R}(X,Y)Z = \frac{c}{4}\{&g(Y,Z)X - g(X,Z)Y + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+ \eta(Y)g(X,Z)\xi - \eta(X)g(Y,Z)\xi + \overline{g}(\overline{\phi}Y,Z)\overline{\phi}X \\ &- \overline{g}(\overline{\phi}X,Z)\overline{\phi}Y - 2\overline{g}(\overline{\phi}X,Y)\overline{\phi}Z\}. \end{aligned} \tag{4.7}$$

As an example of a cosymplectic manifold, we have the following.

Example 4.1.1. Let $\overline{M}$ be the three dimensional manifold defined by, $\overline{M} = \{(x,y,z) \in \mathbb{R}^3, z = x^2+y^2\}$. Consider vector fields

$$e_1 = \partial z, \quad e_2 = 2y\partial x + 2x\partial y + \partial z \quad \text{and} \quad e_3 = 2x\partial x - 2y\partial y,$$

which are linearly independent. Let $\overline{g}$ be a Riemannian metric define by

$$\overline{g}(e_1,e_3) = \overline{g}(e_1,e_2) = \overline{g}(e_2,e_3) = 0.$$
$$\overline{g}(e_1,e_1) = \overline{g}(e_2,e_2) = \overline{g}(e_3,e_3) = 1.$$

Let η be the 1-form defined by $\eta(Z) = g(Z,e_1)$ for any $Z \in \overline{M}$. Let $\overline{\phi}$ be the $(1,1)$-tensor field define by

$$\overline{\phi}(e_2) = e_3, \quad \overline{\phi}(e_3) = -e_2 \quad \text{and} \quad \overline{\phi}(e_1) = 0.$$

Then using the linearity of $\overline{\phi}$ and $\overline{g}$ we have

$$\eta(e_1) = 0, \quad \overline{\phi}^2 Z = -Z \oplus \eta(Z)e_1 \quad \text{and} \quad \overline{g}(\overline{\phi}Z,\overline{\phi}W) = \overline{g}(Z,W) - \eta(Z)\eta(W)$$

for any $Z, W \in \overline{M}$. So $e_1 = \xi$, $(\overline{\phi},\xi,\eta,\overline{g},)$ defines an almost contact metric structure on $\overline{M}$. Let $\overline{\nabla}$ be the Levi-Civita connection on $\overline{M}$ with respect to the metric $\overline{g}$. Then $[e_1,e_i] = 0$ for $i = 1,2,3$ and

$$[e_2,e_3] = [2y\partial x + 2x\partial y + \partial z, 2x\partial x - 2y\partial y] = 8(y\partial x - x\partial y).$$

Also

$$\overline{g}([e_2,e_3],e_1) = 0, \quad \overline{g}([e_2,e_3],e_2) = 16(y^2-x^2) \quad \text{and} \quad \overline{g}([e_2,e_3],e_3) = 32xy.$$

Using (2.4) to compute $\overline{\nabla}$ we obatain

$$\begin{aligned}&\overline{\nabla}_{e_1}e_1 = \overline{\nabla}_{e_1}e_2 = \overline{\nabla}_{e_1}e_3 = \overline{\nabla}_{e_2}e_1 = \overline{\nabla}_{e_3}e_1 = 0, \\ &\overline{\nabla}_{e_2}e_2 = 16xye_2 + 24(x^2-y^2)e_3, \ \overline{\nabla}_{e_2}e_3 = -24(x^2-y^2)e_2 + 16xye_3, \\ &\overline{\nabla}_{e_3}e_2 = 8(x^2-y^2)e_2 - 48xye_3, \ \overline{\nabla}_{e_3}e_3 = 8(x^2-y^2)e_3 + 48xye_2.\end{aligned}$$

From the above connections we obtain

$$\begin{aligned}
(\overline{\nabla}_{e_2}\overline{\phi})e_2 &= \overline{\nabla}_{e_2}\overline{\phi}(e_2) - \overline{\phi}(\overline{\nabla}_{e_2}e_2)\\
&= -24(x^2-y^2)e_2 + 16xye_3 - 16xye_3 + 24(x^2-y^2)e_2 = 0,\\
(\overline{\nabla}_{e_2}\overline{\phi})e_3 &= \overline{\nabla}_{e_2}\overline{\phi}(e_3) - \overline{\phi}(\overline{\nabla}_{e_2}e_3)\\
&= -16xye_2 - 24(x^2-y^2)e_3 + 16xye_2 + 24(x^2-y^2)e_3 = 0,\\
(\overline{\nabla}_{e_3}\overline{\phi})e_2 &= \overline{\nabla}_{e_3}\overline{\phi}(e_2) - \overline{\phi}(\overline{\nabla}_{e_3}e_2)\\
&= 8(x^2-y^2)e_3 + 48xye_2 - 8(x^2-y^2)e_3 - 48xye_2 = 0,\\
(\overline{\nabla}_{e_3}\overline{\phi})e_3 &= \overline{\nabla}_{e_3}\overline{\phi}(e_3) - \overline{\phi}(\overline{\nabla}_{e_3}e_3)\\
&= -8(x^2-y^2)e_2 + 48xye_3 + 8(x^2-y^2)e_2 - 48xye_3 = 0,\\
(\overline{\nabla}_{e_1}\overline{\phi})e_1 &= (\overline{\nabla}_{e_1}\overline{\phi})e_2 = (\overline{\nabla}_{e_1}\overline{\phi})e_3 = (\overline{\nabla}_{e_2}\overline{\phi})e_1 = (\overline{\nabla}_{e_3}\overline{\phi})e_1 = 0,
\end{aligned}$$

from which it follows that $\overline{M}$ is a cosymplectic manifold.

4.2 Integrability of distributions

Theorem 4.2.1. *Let $\overline{M}$ be a cosymplectic manifold. The distribuion $D\oplus\{\xi\}$ is integrable if and only if*

$$h(X,\phi Y) = h(\phi X, Y),$$

for all X, $Y \in \Gamma(D\oplus\{\xi\})$.

Proof. Suppose that $D\oplus\{\xi\}$ is integrable, then $\overline{\phi}X = \phi X$ and $\overline{\phi}Y = \phi Y$, which implies that $\overline{\phi}[X,Y] = \phi[X,Y]$ i.e. $\omega[X,Y]=0$. Using (2.6) we derive

$$\begin{aligned}
\overline{\phi}[X,Y] &= \overline{\phi}\,\overline{\nabla}_X Y - \overline{\phi}\,\overline{\nabla}_Y X\\
&= \overline{\nabla}_X\overline{\phi}Y - \overline{\nabla}_Y\overline{\phi}X\\
&= \overline{\nabla}_X\phi Y - \overline{\nabla}_Y\phi X\\
&= \nabla_X\phi Y + h(X,\phi Y) - \nabla_Y\phi X - h(Y,\phi X). \qquad (4.8)
\end{aligned}$$

Comparing the tangential and normal components of (4.8) we have

$$\overline{\phi}[X,Y] = \nabla_X\phi Y - \nabla_Y\phi X \quad \text{and} \quad h(X,\phi Y) = h(\phi X, Y).$$

Conversely, suppose that $h(X, \phi Y) = h(\phi X, Y)$ for all $X,\ Y \in \Gamma(D \oplus \{\xi\})$. Then using (2.6) we derive

$$\begin{aligned}
\overline{\phi}[X, Y] &= \overline{\phi}\ \overline{\nabla}_X Y - \overline{\phi}\ \overline{\nabla}_Y X \\
&= \overline{\nabla}_X \overline{\phi} Y - \overline{\nabla}_Y \overline{\phi} X \\
&= \nabla_X \phi Y - \nabla_Y \phi X + h(X, \phi Y) - h(\phi X, Y) \\
&= \nabla_X \phi Y - \nabla_Y \phi X,
\end{aligned}$$

which implies that $\overline{\phi}[X, Y] \in \Gamma(TM)$. So $\omega[X, Y] = 0$, which implies that

$$Q[X, Y] = 0.$$

Thus $[X, Y] \in \Gamma(D \oplus \{\xi\})$ from which our assertion follows. $\square$

The result in Theorem 4.2.1 is similar to the one found in [15, Theorem 3.2] in the case where the ambient manifolds are Sasakian.

Lemma 4.2.2. *Let M be a contact CR-submanifold of cosymplectic manifold $\overline{M}$ with the structure vector field ξ tangent to M. Then, for any $X, Y \in \Gamma(TM)$*

$$(\nabla_X \phi) Y = A_{\omega Y} X + Bh(X, Y), \tag{4.9}$$

$$\textit{and}\ \ (\nabla^{\perp}_X \omega) Y = -h(X, \phi Y) + Ch(X, Y). \tag{4.10}$$

Proof. Let $X, Y \in \Gamma(TM)$. Using (2.6) and (2.7) we derive

$$\begin{aligned}
(\nabla_X \phi) Y &= \nabla_X \phi Y - \phi(\nabla_X Y) \\
&= \overline{\nabla}_X \phi Y - h(X, \phi Y) - \phi \nabla_X Y \\
&= \overline{\nabla}_X \overline{\phi} Y - \overline{\nabla}_X \omega Y - h(X, \phi Y) - \overline{\phi}\ \overline{\nabla}_X Y + \omega \nabla_X Y \\
&= \overline{\nabla}_X \overline{\phi} Y - \overline{\nabla}_X \omega Y - h(X, \phi Y) - \overline{\phi}\ \overline{\nabla}_X Y + \overline{\phi} h(X, Y) + \omega \nabla_X Y \\
&= A_{\omega Y} X - \nabla^{\perp}_X \omega Y - h(X, \phi Y) + \overline{\phi} h(X, Y) + \omega \nabla_X Y \\
&= A_{\omega Y} X - (\nabla^{\perp}_X \omega) Y - h(X, \phi Y) + \overline{\phi} h(X, Y),
\end{aligned}$$

which implies that

$$\begin{aligned}
(\nabla_X \phi) Y + (\nabla^{\perp}_X \omega) Y = A_{\omega Y} X - h(X, \phi Y) + Bh(X, Y) \\
+ Ch(X, Y).
\end{aligned} \tag{4.11}$$

Our assertions follows from (4.11). $\square$

Equations (4.9) and (4.10) were also found by Uddin and Ozel in [13] when they were working on totally umbilical submanifolds in a cosymplectic manifold.

Lemma 4.2.3. *Let M be a contact CR-submanifold of cosymplectic manifold $\overline{M}$ with the structure vector field ξ tangent to M. Then, for any $V \in \Gamma(TM^{\perp})$*

$$(\nabla_X B)V = A_{CV}X - \phi A_V X, \tag{4.12}$$

$$\text{and } (\nabla^{\perp}_X C)V = -\omega A_V X + h(X, BV). \tag{4.13}$$

Proof. Let $V \in \Gamma(TM^{\perp})$, then

$$-\overline{\phi} A_V X = -\phi A_V X - \omega A_V X. \tag{4.14}$$

Also

$$\begin{aligned}
-\overline{\phi} A_V X &= \overline{\phi}\ \overline{\nabla}_X V - \overline{\phi}\nabla^{\perp}_X V \\
&= \overline{\nabla}_X \overline{\phi} V - \overline{\phi}\nabla^{\perp}_X V \\
&= \overline{\nabla}_X BV - A_{CV}X + \nabla^{\perp}_X CV - B\nabla^{\perp}_X V - C\nabla^{\perp}_X V \\
&= \nabla_X BV + h(X, BV) - A_{CV}X + (\nabla^{\perp}_X C)V - B\nabla^{\perp}_X V \\
&= (\nabla_X B)V + B\nabla^{\perp}_X V + h(X, BV) - A_{CV}X + (\nabla^{\perp}_X C)V \\
&\quad - B\nabla^{\perp}_X V. \qquad (4.15)
\end{aligned}$$

By equating (4.14) and (4.15) we obtain

$$-\phi A_V X - \omega A_V X = (\nabla_X B)V + h(X, BV) - A_{CV}X + (\nabla^{\perp}_X)V. \tag{4.16}$$

The assertion follows from (4.16). □

Lemma 4.2.4. *Let M be a contact CR-submanifold of cosymplectic manifold $\overline{M}$ with the structure vector field ξ tangent to M.*

$$A_{\omega X} Y = A_{\omega Y} X. \tag{4.17}$$

Proof. Let X ,$Y \in \Gamma(D^{\perp})$, $Z \in \Gamma(TM)$ and using (4.9) and (2.8) we obtain

$$\begin{aligned}
0 = g((\nabla_Z \phi)X, Y) &= g(A_{\omega X} Z + Bh(Z, X), Y) \\
&= g(Z, A_{\omega X} Y) + \overline{g}(\overline{\phi} h(Z, X), Y) \\
&= g(Z, A_{\omega X} Y) - \overline{g}(h(Z, X), \overline{\phi} Y) \\
&= g(Z, A_{\omega X} Y) - \overline{g}(h(Z, X), \omega Y) \\
&= g(A_{\omega X} Y - A_{\omega Y} X, Z).
\end{aligned}$$

Since g is non-degenerate, then

$$A_{\omega X}Y - A_{\omega Y}X = 0,$$

from which the assertion follows. □

The equation (4.17) coincides with the relation (4.1) in [15] in the case where the ambient manifold is Kählerian manifold.

Lemma 4.2.5. *Let M be a contact CR-submanifold of cosymplectic manifold $\overline{M}$ with the structure vector field ξ tangent to M. Then the distribution $D^\perp$ is always integrable.*

Proof. Let X, $Y \in \Gamma(D^\perp \oplus \{\xi\})$, then it suffices to show that $\phi[X,Y] = 0$. Using (4.9) we obtain

$$\begin{aligned}\phi[X,Y] = \phi\nabla_X Y - \phi\nabla_Y X &= (\nabla_Y\phi)X - (\nabla_X\phi)Y \\ &= -A_{\omega Y}X - Bh(X,Y) + A_{\omega X}Y + Bh(Y,X) = 0,\end{aligned}$$

which implies that $[X,Y] \in \Gamma(D^\perp \oplus \{\xi\})$. Thus $D^\perp$ is integrable. □

Definition 4.2.1. A submanifold M is said to be totally geodesic if

$$h(X,Y) = 0,$$

for $X, Y \in \Gamma(TM)$.

Definition 4.2.2. A submanifold M is said to be mixed totally geodesic if

$$h(X,Y) = 0, \text{ for } X \in \Gamma(D) \text{ and } Y \in \Gamma(D^\perp).$$

Proposition 4.2.6. *A necessary and sufficient condition for integral submanifold $M^\perp$ of $D^\perp$ to be mixed totally geodesic in M is that*

$$h(X,Y) \in C(TM^\perp),$$

for all $X \in \Gamma(D)^\perp$ and $Y \in \Gamma(D)$.

Proof. Let X, $Z \in \Gamma(D^\perp)$, $Y \in \Gamma(D)$, then

$$g((\nabla_X\phi)Z, Y) = g(\nabla_X \phi Z - \phi\nabla_X Z, Y) = 0.$$

By the use of (4.9) we obtain

$$g(A_{\omega Z}X + Bh(X,Z), Y) = (\nabla_X\phi)Z = 0,$$

if and only if

$$g(A_{\omega Z}X, Y) = g(h(X,Y), \omega Z) = 0.$$

Consider $TM^\perp = \overline{\phi}D^\perp \oplus \nu$ where ν is the complimentary and invariant distribution to $\overline{\phi}D^\perp$ and $\omega Z \in \Gamma(\overline{\phi}D^\perp)$. Since $h \in \Gamma(TM^\perp)$ and $g(h(X,Y), \omega Z) = 0$, then $h(X,Y) \in \Gamma(\nu)$ which proves our assertion □

4.3 Invariant submanifolds

Definition 4.3.1. A submanifold M of a cosymplectic manifold $\overline{M}$ is invariant if

$$\overline{\phi}X = \phi X \ \text{ and } \ \overline{\phi}V = BV$$

for any $X \in \Gamma(TM)$ and $V \in \Gamma(TM^{\perp})$, that is $\omega X = 0$ and $CV = 0$.

Let M be a contact CR-submanifold of cosymplectic manifold $\overline{M}$ with the structure vector field ξ tangent to M. If M is invariant, then by putting $Y = \xi$ in (4.9), one has

$$Bh(X, \xi) = -\phi \nabla_X \xi, \ \ \forall X \in \Gamma(TM).$$

This means that $Bh(\cdot, \xi) \in \Gamma(D^{\perp})$. By the first relation in (4.5), we have, $\omega Bh(X, \xi) = -h(X, \xi)$. Therefore,

$$h(X, \xi) = 0, \ \ \forall\, X \in \Gamma(TM). \tag{4.18}$$

Proposition 4.3.1. *Let M be a contact CR-submanifold of a cosymplectic manifold $\overline{M}$ with the structure vector field ξ tangent to M. If M is invariant, then it is cosymplectic.*

Proof. Let $\overline{M}$ be a cosymplectic and suppose M is an invariant submanifold of $\overline{M}$, then $\overline{\phi}X = \phi X$ and $\overline{\phi}V = BV$ for all $X \in \Gamma(TM)$ and $V \in \Gamma(TM^{\perp})$.

$$\begin{aligned} \overline{\phi}h(X, Y) &= \overline{\phi}\,\overline{\nabla}_X Y - \overline{\phi}\nabla_X Y \\ &= \overline{\nabla}_X \overline{\phi} Y - \phi \nabla_X Y \\ &= \nabla_X \phi Y + h(X, \overline{\phi}Y) - \phi \nabla_X Y \\ &= h(X, \overline{\phi}Y) + (\nabla_X \phi)Y. \end{aligned}$$

Therefore, we have

$$(\nabla_X \phi)Y = \overline{\phi}h(X, Y) - h(X, \overline{\phi}Y). \tag{4.19}$$

Since M is invariant, $\omega X = 0$, $CV = 0$, and by the use of (4.9) and (4.10) we have

$$h(X, \phi Y) = 0, \ \text{ and } \ (\nabla_X \phi)Y = Bh(X, Y).$$

Let $Y = \phi Y$, then $h(X, \phi^2 Y) = h(X, \overline{\phi}^2 Y) = 0$. That is $h(X, -Y + \eta(Y)\xi) = 0$ This is equivalent to

$$h(X, Y) = \eta(Y)h(X, \xi).$$

By (4.18), $h(X,\xi)=0$, and we have $h(X,Y)=0$. Therefore (4.19) become

$$(\nabla_X\phi)Y=\overline{\phi}h(X,Y)-h(X,\overline{\phi}Y)=0,$$

from which our assertion follows. $\square$

Lemma 4.3.2. *Let M be a contact CR-submanifold of a cosymplectic manifold $\overline{M}$ with the structure vector field ξ tangent to M. If M is invariant, then,*

(i) M is totally geodesic.

(ii) The distribution $D\oplus\{\xi\}$ is integrable.

Proof. The proof follows straightforward from Proposition 4.3.1. $\square$

4.4 Foliations

Definition 4.4.1. *[15]* A CR submanifold is said to be mixed foliate if it is mixed totally geodesic and $h(\phi X,Y)=h(X,\phi Y)$ for all X, $Y\in\Gamma(D)$.

Lemma 4.4.1. *let M be mixed foliate CR submanifold of a cosymplectic manifold $\overline{M}$. Then*

$$A_V\phi+\phi A_V=0, \tag{4.20}$$

for any vector field $V\in\Gamma(TM^{\perp})$.

Proof. Let X, $Y\in\Gamma(TM)$, then

$$g(\phi X,QY)=\overline{g}(\overline{\phi}PX,QY)=-\overline{g}(PX,\overline{\phi}QY)=-g(PX,\omega Y)=0,$$

which implies that $\phi X\in\Gamma(D)$. Also

$$h(X,\phi Y)=h(PX+QX+\eta(X)\xi,\phi Y)=h(PX,\phi Y).$$

Since $h(X,\phi Y)=h(\phi X,Y)$ for all X, $Y\in\Gamma(TM)$ and for all $V\in\Gamma(TM^{\perp})$, then

$$\overline{g}(h(X,\phi Y),V)=\overline{g}(h(\phi X,Y),V),$$

which leads to

$$g(\phi A_VX+A_V\phi X,Y)=0.$$

Therefore

$$\phi A_V+A_V\phi=0,$$

which completes the proof. $\square$

Equation (4.20) was also derived in [15] but the authors were dealing with an almost cosymplectic manifold with a Kählerian integral submanifold.

Theorem 4.4.2. *If M is a mixed foliate non-trivial CR submanifold, i.e., neither an invariant submanifold nor anti-invariant submanifold of a cosymplectic space form, then $c \geq 0$.*

Proof. Let X, $Y \in \Gamma(D)$ and $Z \in \Gamma(D)^{\perp}$. Using (2.9) we obtain

$$\begin{aligned}(\nabla_X h)(Y,Z) &= \nabla^{\perp}_X(h(Y,Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)\\ (\nabla_Y h)(X,Z) &= \nabla^{\perp}_Y(h(X,Z)) - h(\nabla_Y Z, X) - h(X, \nabla_Y Z).\end{aligned}$$

Taking the difference of these two equations we obtain

$$\begin{aligned}(\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z) = {} & \nabla^{\perp}_X(h(Y,Z)) - \nabla^{\perp}_Y(h(X,Z))\\ & - h([X,Y],Z) + h(X, \nabla_Y Z)\\ & - h(Y, \nabla_X Z).\end{aligned} \tag{4.21}$$

Using the fact M is a mixed foliate submanifold, (4.21) reduces to

$$(\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z) = h(X, \nabla_Y Z) - h(Y, \nabla_X Z). \tag{4.22}$$

Let $V \in \Gamma(TM^{\perp})$ such that $Z = \overline{\phi}V = BV$ then

$$\overline{\nabla}_Y Z = \overline{\nabla}_Y \overline{\phi} V = \overline{\phi}(\overline{\nabla}_Y V) = \overline{\phi}(-A_V Y + \nabla^{\perp}_Y V) = -\phi A_V Y + B\nabla^{\perp}_Y V. \tag{4.23}$$

Substituting (4.23) into (4.22) we obtain

$$\begin{aligned}(\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z) &= -h(X, \phi A_V Y) + h(Y, \phi A_V X)\\ &= h(\phi Y, A_V X) + h(X, A_V \phi Y).\end{aligned} \tag{4.24}$$

By letting $X = \phi Y$ and substituting into (4.24) we obtain

$$\begin{aligned}(\nabla_{\phi Y} h)(Y,Z) - (\nabla_Y h)(\phi Y,Z) &= h(\phi Y, A_V \phi Y) + h(\phi Y, A_V \phi Y)\\ &= 2h(\phi Y, A_V \phi Y).\end{aligned}$$

Comparing the normal components of (2.10) and (4.7), we obtain

$$\begin{aligned}&(\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z)\\ &\qquad = \frac{c}{4}\left(\overline{g}(\overline{\phi}Y,Z)\omega X - \overline{g}(\overline{\phi}X,Z)\omega Y - 2\overline{g}(\overline{\phi}X,Y)\omega Z\right),\end{aligned}$$

which is equivalent, by letting $X = \phi Y$, to

$$2h(\phi Y, A_V \phi Y) = -\frac{c}{2}\overline{g}(\overline{\phi}^2 Y, Y)\omega\overline{\phi}V.$$

Taking the inner product of (4.4) with V, we obtain

$$\begin{aligned}-\frac{c}{2}\overline{g}(Y,Y)\overline{g}(\omega\overline{\phi}V,V) &= 2\overline{g}(h(\phi Y, A_V\phi Y),V) = 2\overline{g}(A_V\phi Y, A_V\phi Y)\\ &= 2\|A_V\phi Y\|^2.\end{aligned}$$

So

$$-\frac{c}{2}\overline{g}(Y,Y)\overline{g}(-V,V) = 2\overline{g}(A_V\phi Y, A_V\phi Y),$$

which reduces to

$$\frac{c}{2}\overline{g}(Y,Y)\overline{g}(V,V) = 2\overline{g}(A_V\phi Y, A_V\phi Y) \geq 0.$$

Since $\overline{g}(Y,Y) > 0$ and $\overline{g}(V,V) > 0$ then $\frac{c}{2} \geq 0$, which implies that $c \geq 0$. $\square$

This means that, if M is a mixed foliate non-trivial CR submanifold, i.e., neither an invariant submanifold nor anti-invariant submanifold of a cosymplectic space form, M can be identified with a sphere, whereas the Proposition 4.5 in [15] tells us that M is an open ball in $\mathbb{C}^{\dim M}$.

Corollary 4.4.3. *Let M be a mixed foliate CR submanifold of a complex space form $M^{-m}(c)$. If $c < 0$ then M is a invariant submanifold or anti-invariant submanifold of a cosymplectic space form $\overline{M}(c)$.*

Proof. From Proposition 4.4.2 we have

$$\frac{c}{2}\overline{g}(Y,Y)\overline{g}(V,V) = 2\overline{g}(A_V\phi Y, A_V\phi Y) \geq 0.$$

If $c < 0$, then

$$\overline{g}(Y,Y)\overline{g}(V,V) \leq 0,$$

which implies that

$$\overline{g}(Y,Y)\overline{g}(V,V) = 0.$$

Which implies that

$$\overline{g}(Y,Y) = 0 \text{ or } \overline{g}(V,V) = 0,$$

therefore $Y = 0$ or $V = 0$ which implies that $TM = \{0\}$ or $TM^{\perp} = \{0\}$. Hence $\overline{\phi}X = \omega X$ or $\overline{\phi}X = \phi X$ for all X, $Y \in \Gamma(TM)$ and $V \in \Gamma(TM^{\perp})$. Thus M is an invariant or an anti-invariant submanifold of $M^{-m}(c)$. $\square$

4.5 Parallel ϕ structures

Definition 4.5.1. Let M be an n-dimensional CR submanifold of a complex m-dimensional cosymplectic manifold $\overline{M}$. If $\nabla_X \phi = 0$ for any $X \in \Gamma(TM)$, then the f-structure ϕ is said to be parallel.

Definition 4.5.2. Let M be a submanifold of a cosymplectic manifold $\overline{M}$ and let

$$D = TM \oplus \overline{\phi}(TM).$$

If D defines a differentiable distribution of TM then M is said to be a generic submanifold of $\overline{M}$.

Proposition 4.5.1. *Let M be an n-dimensional generic submanifold of a complex m-dimensional cosymplectic manifold $\overline{M}$. If thef-structure ϕ on M is parellel, then M is locally a Riemannian direct product $M^{\perp} \times M^{\top}$, where $M^{\top}$ is a totally geodesic invariant submanifold of $\overline{M}$ of a complex dimnsional $n-m$ and $M^{\perp}$ is an anti-invariant submanifold of $\overline{M}$ of a real dimensional $2m-n$.*

Proof. Let $X, Y \in \Gamma(TM)$ and using the fact that M is a generic submanifold of $\overline{M}$ we obtain

$$\overline{\phi}h(X,Y) = Bh(X,Y) + Ch(X,Y) = Bh(X,Y).$$

By the use of (4.9) and the fact that the f-structure ϕ is parallel we obtain

$$Bh(X,Y) = -A_{\omega Y}X.$$

So

$$\overline{\phi}h(X,Y) = -A_{\omega Y}X. \tag{4.25}$$

Let $Y = \phi Y$, then (4.25) reduces to

$$\overline{\phi}h(X,\phi Y) = -A_{\omega\phi Y}X = 0.$$

Hence

$$h(X,\phi Y) = 0.$$

Similarly using (4.10) we obtain

$$(\nabla^{\perp}_X \omega)Y = -h(X,\phi Y) + Ch(X,Y) = -h(X,\phi Y) = 0.$$

Let $Y \in \Gamma(D)^{\perp}$. Then

$$\phi\nabla_X Y = \nabla_X(\phi Y) - (\nabla_X \phi)Y = 0.$$

Therefore the distribution $D^{\perp}$ is parallel. Similarly D is also parallel. Consequently, $M^{\top}$ is a leaf of D and $M^{\perp}$ is a leaf of $D^{\perp}$. Since $h(X,\phi Y) = 0$ for any $X, Y \in \Gamma(TM)$, $M^{\top}$ is totally geodesic in $\overline{M}$. Hence M is locally a Riemannian direct product $M^{\top} \times M^{\perp}$. $\square$

4.6 Totally contact umbilical submanifolds

Definition 4.6.1. *[15]* A submanifold M of $\overline{M}$ is said to be *totally contact umbilical* if

$$h(X,Y) = \{g(X,Y) - \eta(X)\eta(Y)\}\mu - \eta(Y)h(\xi,X) - \eta(X)h(\xi,Y)$$

for $X, Y \in \Gamma(TM)$ and μ is the mean curvature vector field of M. If $\mu = 0$, then M is totally contact geodesic.

Theorem 4.6.1. *Let M be a contact umbilical CR submanifold of a cosymplectic manifold $\overline{M}$ with ξ tangent to M. If $\dim D^{\perp} > 1$, then M is totally contact geodesic in $\overline{M}$.*

Proof. We first show that $B\mu = 0$, where μ is a mean curvature vector of M. By Lemma (4.2.4) we have

$$A_{\omega X}Y = A_{\omega Y}X. \tag{4.26}$$

for any X, $Y \in \Gamma(D^{\perp})$. Let M be an almost contact umbilical non-trivial CR submanifold of a cosymplectic manifold $\overline{M}$. Suppose that $dimD^{\perp} > 1$. Then from (4.26) we have

$$A_{\omega X}B\mu = A_{\omega B\mu}X. \tag{4.27}$$

Taking the inner product of (4.27) with X, we obtain

$$g(A_{\omega X}B\mu, X) = g(A_{\omega B\mu}X, X). \tag{4.28}$$

Computing the left hand side of (4.28) we obtain

$$\begin{aligned} g(A_{\omega X}B\mu, X) &= g(h(B\mu, X), \omega X) \\ &= g((g(B\mu, X) - \eta(B\mu)\eta(X))\mu - h(\xi, B\mu)\eta(X) \\ &- h(\xi, X)\eta(B\mu), \omega X) \\ &= g(B\mu, X)g(\mu, \omega X) - \eta(B\mu)\eta(X)g(\mu, \omega X) \\ &- \eta(X)g(h(\xi, B\mu), \omega X) - \eta(B\mu)g(h(\xi, X), \omega X). \end{aligned} \tag{4.29}$$

Recall that if M is totally geodesic then $h(\xi, X) = 0$. So (4.29) reduces to

$$g(A_{\omega X}B\mu, X) = g(B\mu, X)g(\mu, \omega X). \tag{4.30}$$

Since $\eta(X) = 0$, for any $X \in \Gamma(D^{\perp})$. Computing the right hand side of (4.28), we obtain

$$\begin{aligned} g(A_{\omega B\mu}X, X) &= g(h(X,X), \omega B\mu) \\ &= g((g(X,X) - \eta(X)\eta(X))\mu - h(\xi, X)\eta(X) \\ &- h(\xi, X)\eta(X), \omega B\mu) \\ &= g(X,X)g(\mu, \omega B\mu). \end{aligned} \tag{4.31}$$

By equating (4.30) and (4.31) we obtain

$$g(B\mu, X)g(\mu, \omega X) = g(X,X)g(\mu, \omega B\mu),$$

from which follows

$$g(g(B\mu, X)\mu + g(\mu, \omega B\mu)X, X) = 0.$$

From the above relation, we deduce that

$$g(\mu, \omega B\mu)X + g(B\mu, X)\mu = 0.$$

Since $\dim D^{\perp} > 1$, we can choose $X \neq 0$ such that

$$g(B\mu, X) = 0.$$

Since $B\mu, X \in D^{\perp}$ and g is a Riemannian along $D^{\perp}$ then we have

$$g(B\mu, X) = 0,$$

which implies that $B\mu = 0$. On the other hand using (4.12) we obtain

$$\begin{aligned} g((\nabla_X B)\mu, Y) &= g(A_{C\mu}X - \phi A_{\mu}X, Y) \\ &= g(A_{C\mu}X, Y) + g(A_{\mu}X, \phi Y) \\ &= g(h(X,Y), C\mu) + g(h(X, \phi Y), \mu) \\ &= g(X,Y)g(C\mu, \mu) + g(X, \phi Y)g(\mu, \mu) \\ &= g(X, \phi Y)g(\mu, \mu). \end{aligned} \tag{4.32}$$

Putting $Y = \phi X$ in (4.32), we obtain

$$g(X, \phi^2 X)g(\mu, \mu) = 0,$$

which implies that

$$g(X,X)g(\mu,\mu) - g(\omega X, \omega X)g(\mu, \mu) = 0.$$

Since M is non-trivial, we can choose an $X \in \Gamma(D)$ such that $\omega X = 0$. So $\mu = 0$ and hence M is totally contact geodesic. $\square$

This result means in fact that a contact umbilical non-trivial CR submanifold M of a cosymplectic manifold $\overline{M}$ is minimal. Theorem 4.6.1 is a contact analogous of the one found in [15] for Kählerian manifolds.

Lemma 4.6.2. *In general, there are no totally geodesic submanifold of a cosymplectic manifold which are totally umbilical.*

Proof. Let M be a totally umbilical submanifold of a cosymplectic manifold $\overline{M}$ then for every X, $Y \in \Gamma(TM)$

$$h(X,Y) = g(X,Y)\mu,$$

which implies that

$$\mu = h(\xi,\xi) = -\nabla_\xi \xi,$$

and this does not vanish in general. □

Remark 4.6.1. $\nabla_\xi \xi = 0$ if and only if the totally geodesic submanifold M of a cosymplectic manifold $\overline{M}$ is totally geodesic. As an example to this, we have the invariant totally geodesic submanifold.

Theorem 4.6.3. *Let M be a contact umbilical CR submanifold of a cosymplectic space form $\overline{M}(c)$ with ξ tangent to M. Then, the mean curvature vector μ satisfies the following partial differential equations*

$$\nabla_X \mu = 0,$$

for any $X \in \Gamma(TM)$.

Proof. Equating the normal components of (2.10) and (4.7), we obtain

$$\begin{aligned}(\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z) = \frac{c}{4}\{&\overline{g}(\overline{\phi}Y,Z)\omega X - \overline{g}(\overline{\phi}X,Z)\omega Y \\ &- 2\overline{g}(\overline{\phi}X,\omega Y)\omega Z\}.\end{aligned} \tag{4.33}$$

Since M is totally contact umbilical, $\nabla_X \xi = 0$ and $(\nabla_X \eta)Y = 0$, then we have

$$(\nabla_X h)(Y,Z) = \{g(Y,Z) - \eta(Y)\eta(Z)\}\nabla_X \mu. \tag{4.34}$$

Also since $\nabla_Y \xi = 0$ and $(\nabla_Y \eta)X = 0$, we have

$$(\nabla_Y h)(X,Z) = \{g(X,Z) - \eta(X)\eta(Z)\}\nabla_Y \mu. \tag{4.35}$$

Substituting (4.35) and (4.34) into (4.33) we obtain

$$\begin{aligned}&\{g(X,Z)-\eta(Y)\eta(Z)\}\nabla_X\mu-\{g(Y,Z)-\eta(X)\eta(Z)\}\nabla_Y\mu\\&=\frac{c}{4}\{\overline{g}(\overline{\phi}Y,Z)\omega X-\overline{g}(\overline{\phi}X,Z)\omega Y-2\overline{g}(\overline{\phi}X,\omega Y)\omega Z\}.\end{aligned}\tag{4.36}$$

If we let $Y=\xi$, then (4.36) reduces to

$$-\{g(X,Z)-\eta(X)\eta(Z)\}\nabla_\xi\mu=0,$$

which implies that

$$\nabla_\xi\mu=0.\tag{4.37}$$

If $X,Y,Z\neq\xi\in\Gamma(TM)$, then (4.36) reduces to

$$\begin{aligned}g(X,Z)\nabla_X\mu-g(Y,Z)\nabla_Y\mu=\frac{c}{4}\{&\overline{g}(\overline{\phi}Y,Z)\omega X-\overline{g}(\overline{\phi}X,Z)\omega Y\\&-2\overline{g}(\overline{\phi}X,\omega Y)\omega Z\}.\end{aligned}\tag{4.38}$$

If $X,Z\in\Gamma(D)$ and $Y\in\Gamma(D^\perp)$, then (4.38) reduces to

$$-g(X,Z)\nabla_X\mu=\frac{c}{4}(\overline{g}(\overline{\phi}X,Z)\omega Y),$$

if and only if

$$g(X,Z)\nabla_X\mu=\frac{c}{4}\overline{g}(\overline{\phi}X,Z)\omega Y,$$

if $X=Z$, then

$$\nabla_X\mu=0.$$

If $X\in\Gamma(D)$ and $Y,Z\in\Gamma(D^\perp)$, then (4.36) reduces to

$$g(Y,Z)\nabla_Y\mu=0,$$

which implies that $\nabla_Y\mu=0$ which completes the proof. □

As a corollary to the Theorem 4.6.3, we have the following.

Corollary 4.6.4. *Let M be a contact umbilical CR submanifold of a cosymplectic space form $\overline{M}(c)$ with ξ tangent to M. Then, M is an extrinsic sphere.*

Proposition 4.6.5. *Let $\overline{M}$ be a cosymplectic manifold and M be a submanifold of $\overline{M}$. If M is a totally contact geodesic CR-submanifold of $\overline{M}$, then the distributions D and $D^\perp$ are parallel.*

Proof. Suppose that M is totally contact geodesic. Let $X \in \Gamma(D)$ and $Z \in \Gamma(TM)$. Then

$$(\nabla_Z \omega)X = \nabla_Z \omega X - \omega(\nabla_Z X) = -\omega(\nabla_Z X).$$

Using (4.10) we obtain

$$\begin{aligned}\omega(\nabla_Z X) &= -(\nabla_Z \omega)X \\ &= h(Z, \phi X) - Ch(Z, X) \\ &= \eta(Z)h(\phi X, \xi) + \eta(\phi X)h(Z, \xi) \\ &\quad - C(\eta(Z)h(X, \xi) + \eta(X)h(Z, \xi)). \end{aligned} \tag{4.39}$$

Let $V \in \Gamma(TM^{\perp})$, then taking the inner product of (4.39) with V, we derive

$$\begin{aligned} g(\omega(\nabla_Z X), V) &= \eta(Z)g(h(\phi X, \xi), V) \\ &\quad - \eta(Z)g(Ch(X, \xi), V) - \eta(X)g(Ch(Z, \xi), V) \\ &= \eta(Z)g(A_W \xi, \phi X) - \eta(Z)g(A_{qW}\xi, X) \\ &\quad - \eta(X)g(A_{qW}\xi, Z) \\ &= 0, \end{aligned}$$

if $A_W \xi \in \Gamma(D^{\perp})$. On the other hand for any $Y \in \Gamma(D^{\perp})$, we obtain

$$(\nabla_Z \phi)Y = \nabla_Z \phi Y - \phi(\nabla_Z Y) = -\phi(\nabla_Z Y).$$

Using (4.9) we derive

$$\begin{aligned} \phi(\nabla_Z Y) &= -(\nabla_Z \phi)Y \\ &= -A_{\omega Y} Z - Bh(Z, Y) \\ &= -A_{\omega Y} Z - B(\eta(Z)h(Y, \xi) + \eta(Y)h(Z, \xi)) = -A_{\omega Y} Z. \end{aligned} \tag{4.40}$$

Let $X \in \Gamma(TM)$, then taking the inner product of (4.40) with X, we obtain

$$\begin{aligned} g(A_{\omega Y} Z, X) &= -g(h(X, Z), \omega Y) \\ &= -g(\eta(X)h(Z, \xi) + \eta(Z)h(X, \xi), \omega Y) \\ &= -\eta(X)g(A_{\omega Y}\xi, Z) - \eta(Z)g(A_{\omega Y}\xi, X) \\ &= 0, \end{aligned}$$

if $A_{\omega Y} X \in \Gamma(D)$. Therefore the distributions $D^{\perp}$ and D are parallel. $\square$

4.7 Curvatures

Let $Y = \overline{\phi}X$ and $Z = \overline{\phi}Y$, then using (4.7) we derive

$$
\begin{aligned}
\overline{R}(X, \overline{\phi}X)\overline{\phi}Y = \frac{c}{4}\{&\overline{g}(\overline{\phi}X, \overline{\phi}Y)X - \overline{g}(X, \overline{\phi}Y)\overline{\phi}X - \eta(X)\overline{g}(\overline{\phi}X, \overline{\phi}Y)\xi \\
&+ \overline{g}(\overline{\phi}^2X, \overline{\phi}Y)\overline{\phi}X - \overline{g}(\overline{\phi}X, \overline{\phi}Y)\overline{\phi}^2X - 2\overline{g}(\overline{\phi}X, \overline{\phi}X)\overline{\phi}^2Y\} \\
= \frac{c}{4}\{&\overline{g}(\overline{\phi}X, \overline{\phi}Y)X - \overline{g}(X, \overline{\phi}Y)\overline{\phi}X - \eta(X)\overline{g}(\overline{\phi}X, \overline{\phi}Y)\xi \\
&- \overline{g}(X, \overline{\phi}Y)\overline{\phi}X + \eta(X)\overline{g}(\xi, \overline{\phi}Y)\overline{\phi}X + \overline{g}(\overline{\phi}X, \overline{\phi}Y)X \\
&- \eta(X)\overline{g}(\overline{\phi}X, \overline{\phi}Y)\xi + 2\overline{g}(\overline{\phi}X, \overline{\phi}X)Y - 2\eta(Y)\overline{g}(\overline{\phi}X, \overline{\phi}X)\xi\} \\
= \frac{c}{4}\{&2\overline{g}(\overline{\phi}X, \overline{\phi}Y)X - 2\overline{g}(X, \overline{\phi}Y)\overline{\phi}X - 2\eta(X)\overline{g}(\overline{\phi}X, \overline{\phi}Y)\xi \\
&+ 2\overline{g}(\overline{\phi}X, \overline{\phi}X)Y - \eta(Y)\overline{g}(\overline{\phi}X, \overline{\phi}X)\xi\}.
\end{aligned}
$$

This leads to

$$
\begin{aligned}
\overline{R}(X, \overline{\phi}X, \overline{\phi}Y, Y) &= g(\overline{R}(X, \overline{\phi}X)\overline{\phi}Y, Y) \\
&= \frac{c}{4}\{2g(\overline{\phi}X, \overline{\phi}Y)g(X, Y) - 2g(X, \overline{\phi}Y)g(\overline{\phi}X, Y) \\
&- 2\eta(X)g(\overline{\phi}X, \overline{\phi}Y)g(\xi, Y) + 2g(\overline{\phi}X, \overline{\phi}X)g(Y, Y) \\
&- \eta(Y)g(\overline{\phi}X, \overline{\phi}X)g(\xi, Y)\}. \qquad (4.41)
\end{aligned}
$$

Let $Y = \phi Y$ then (4.41) reduces to

$$
\begin{aligned}
\overline{R}(X, \overline{\phi}X, \overline{\phi}Y, Y) = \frac{c}{4}\{&-2g(\overline{\phi}X, Y)g(X, \phi Y) + 2g(\overline{\phi}X, \phi Y)g(X, Y) \\
&+ 2g(\overline{\phi}X, \overline{\phi}X)g(\overline{\phi}Y, \overline{\phi}Y)\}. \qquad (4.42)
\end{aligned}
$$

On the other hand,

$$
\overline{R}(X, \overline{\phi}X, \overline{\phi}Y, Y) = R(X, \overline{\phi}X, \overline{\phi}Y, Y) + \|h(\overline{\phi}X, Y)\|^2 + g(h(X, Y), h(\overline{\phi}X, \overline{\phi}Y)). \qquad (4.43)
$$

If M is totally contact geodesic, the second fundamental form h reduces to

$$
h(X, Y) = -\eta(Y)h(\xi, X) - \eta(X)h(\xi, Y).
$$

This means that $h(\overline{\phi}X, Y) = -\eta(Y)h(\xi, \overline{\phi}X)$ and $h(\overline{\phi}X, \overline{\phi}Y) = 0$. Therefore (4.43) becomes

$$
\overline{R}(X, \overline{\phi}X, \overline{\phi}Y, Y) = R(X, \overline{\phi}X, \overline{\phi}Y, Y) + |\eta(Y)|\|h(\overline{\phi}X, \xi)\|^2. \qquad (4.44)
$$

If the distribution $D \oplus \{\xi\}$ is integrable, then, by (4.2.1), $h(\overline{\phi}X, \xi) = 0$ and therefore

$$\overline{R}(X, \overline{\phi}X, \overline{\phi}Y, Y) = R(X, \overline{\phi}X, \overline{\phi}Y, Y). \tag{4.45}$$

Let $\overline{H}(X) = \overline{R}(X, \overline{\phi}X, \overline{\phi}X, X)$ and $H(X) = R(X, \overline{\phi}X, \overline{\phi}X, X)$ be the holomorphic sectional curvatures of $\overline{M}$ and M, respectively. We therefore have the following.

Theorem 4.7.1. *Let M be a totally contact geodesic CR-submanifold of cosymplectic manifold $\overline{M}$ with the structure vector field ξ tangent to M. Then, the holomorphic sectional curvatures $\overline{H}$ and H of $\overline{M}$ and M, respectively, satisfy*

$$\overline{H}(X) \geq H(X), \;\; \forall X \in \Gamma(D \oplus \{\xi\}), \;\; ||X||_g = 1,$$

and the equality holds if the distribution $D \oplus \{\xi\}$ is integrable.

We extend the three-dimensional manifold $\overline{M}$ in Example 4.1.1 to a five-dimensional manifold.

Example 4.7.1. Let $\overline{M} = \{(x, y, z, s, t) \in \mathbb{R}^5, z^2 = x^2 + y^2\}$ and consider the vector fields

$$e_1 = \partial z, \; e_2 = 2y\partial x + 2x\partial y + \partial z, \; e_3 = 2x\partial x - 2y\partial y, \; e_4 = \partial t, \text{ and } e_5 = \partial s.$$

Then, its easy to check that the above vectors are linearly independent at each point of $\overline{M}$. Let $\overline{g}$ be a Riemannian metric define by

$$\overline{g}(e_i, e_j) = 1, \text{ for } i = j \;\text{ otherwise }\; \overline{g}(e_i, e_j) = 0.$$

Let η be the 1-form defined by $\eta(Z) = \overline{g}(Z, e_1)$ for any $Z \in \Gamma(T\overline{M})$. Let $\overline{\phi}$ be the $(1, 1)$-tensor field defined by

$$\overline{\phi}(e_3) = -e_2, \; \overline{\phi}(e_2) = e_3, \; \overline{\phi}(e_4) = e_5, \; \overline{\phi}(e_5) = -e_4, \text{ and } \overline{\phi}(e_1) = 0.$$

Then using the linearity of $\overline{\phi}$ and $\overline{g}$ we have

$$\eta(e_1) = 0, \; \overline{\phi}^2 Z = -Z + \eta(Z)e_1 \text{ and } \overline{g}(\overline{\phi}Z, \overline{\phi}W) = \overline{g}(Z, W) - \eta(Z)\eta(W)$$

for any $Z, W \in \Gamma(T\overline{M})$. Thus for $e_1 = \xi$, $(\overline{\phi}, \xi, \eta, \overline{g},)$ defines an almost contact metric structure on $\overline{M}$. Let $\overline{\nabla}$ be the Levi-Civita connection on $\overline{M}$ with respect to the metric $\overline{g}$. Then $[e_1, e_i] = 0$ for $i = 1, 2, 3, 4, 5.$ and

$$[e_2, e_4] = 0, \;\; [e_2, e_5] = 0, \;\; [e_3, e_4] = 0, \;\; [e_3, e_5] = 0, \;\; [e_4, e_5] = 0 \;\text{ and}$$
$$[e_2, e_3] = 8(y\partial x - x\partial y).$$

By using (2.4) to compute $\overline{\nabla}$ we obatain

$$\begin{aligned}
&\overline{\nabla}_{e_1}e_1 = \overline{\nabla}_{e_1}e_2 = \overline{\nabla}_{e_1}e_3 = \overline{\nabla}_{e_2}e_1 = \overline{\nabla}_{e_3}e_1 = \overline{\nabla}_{e_1}e_4 = 0.\\
&\overline{\nabla}_{e_1}e_5 = \overline{\nabla}_{e_2}e_4 = \overline{\nabla}_{e_2}e_5 = \overline{\nabla}_{e_3}e_4 = \overline{\nabla}_{e_3}e_5 = \overline{\nabla}_{e_4}e_1 = 0.\\
&\overline{\nabla}_{e_4}e_2 = \overline{\nabla}_{e_4}e_3 = \overline{\nabla}_{e_4}e_4 = \overline{\nabla}_{e_4}e_5 = \overline{\nabla}_{e_5}e_1 = \overline{\nabla}_{e_5}e_2 = 0.\\
&\overline{\nabla}_{e_5}e_3 = \overline{\nabla}_{e_5}e_4 = \overline{\nabla}_{e_5}e_5 = 0.\ \overline{\nabla}_{e_2}e_2 = 16xye_2 + 24(x^2-y^2)e_3.\\
&\overline{\nabla}_{e_2}e_3 = -24(x^2-y^2)e_2 + 16xye_3,\ \overline{\nabla}_{e_3}e_2 = 8(x^2-y^2)e_2 - 48xye_3.\\
&\overline{\nabla}_{e_3}e_3 = 8(x^2-y^2)e_3 + 48xye_2.
\end{aligned}$$

From these connections we obtain

$$\begin{aligned}
(\overline{\nabla}_{e_2}\overline{\phi})e_2 &= \overline{\nabla}_{e_2}\overline{\phi}(e_2) - \overline{\phi}(\overline{\nabla}_{e_2}e_2)\\
&= -24(x^2-y^2)e_2 + 16xye_3 - 16xye_3 + 24(x^2-y^2)e_2 = 0,\\
(\overline{\nabla}_{e_2}\overline{\phi})e_3 &= \overline{\nabla}_{e_2}\overline{\phi}(e_3) - \overline{\phi}(\overline{\nabla}_{e_2}e_3)\\
&= -16xye_2 - 24(x^2-y^2)e_3 + 16xye_2 + 24(x^2-y^2)e_3 = 0,\\
(\overline{\nabla}_{e_3}\overline{\phi})e_2 &= \overline{\nabla}_{e_3}\overline{\phi}(e_2) - \overline{\phi}(\overline{\nabla}_{e_3}e_2)\\
&= 8(x^2-y^2)e_3 + 48xye_2 - 8(x^2-y^2)e_3 - 48xye_2 = 0,\\
(\overline{\nabla}_{e_3}\overline{\phi})e_3 &= \overline{\nabla}_{e_3}\overline{\phi}(e_3) - \overline{\phi}(\overline{\nabla}_{e_3}e_3)\\
&= -8(x^2-y^2)e_2 + 48xye_3 + 8(x^2-y^2)e_2 - 48xye_3 = 0,\\
(\overline{\nabla}_{e_1}\overline{\phi})e_1 &= (\overline{\nabla}_{e_1}\overline{\phi})e_2 = (\overline{\nabla}_{e_1}\overline{\phi})e_3 = (\overline{\nabla}_{e_2}\overline{\phi})e_1 = (\overline{\nabla}_{e_1}\overline{\phi})e_4 = (\overline{\nabla}_{e_1}\overline{\phi})e_5 = 0,\\
(\overline{\nabla}_{e_2}\overline{\phi})e_4 &= (\overline{\nabla}_{e_2}\overline{\phi})e_5 = (\overline{\nabla}_{e_3}\overline{\phi})e_4 = (\overline{\nabla}_{e_3}\overline{\phi})e_5 = (\overline{\nabla}_{e_4}\overline{\phi})e_1 = (\overline{\nabla}_{e_4}\overline{\phi})e_2 = 0,\\
(\overline{\nabla}_{e_4}\overline{\phi})e_3 &= (\overline{\nabla}_{e_4}\overline{\phi})e_4 = (\overline{\nabla}_{e_4}\overline{\phi})e_5 = (\overline{\nabla}_{e_5}\overline{\phi})e_1 = (\overline{\nabla}_{e_5}\overline{\phi})e_2 = (\overline{\nabla}_{e_5}\overline{\phi})e_3 = 0,\\
(\overline{\nabla}_{e_5}\overline{\phi})e_4 &= (\overline{\nabla}_{e_5}\overline{\phi})e_4 = (\overline{\nabla}_{e_5}\overline{\phi})e_5 = 0.
\end{aligned}$$

Thus, $\overline{M}$ satisfies $(\overline{\nabla}_X\overline{\phi})Y = 0$ and $\overline{\nabla}_X\xi = 0$ for $\xi = e_1$. Hence $\overline{M}$ is a cosymplectic manifold.

Consider a submanifold M of $\overline{M}$ defined by

$$M = \{x, y, z, s, t \in \overline{M} : x = y, x > 0\}.$$

The tangent space $TM = \text{Span}\{\partial x - \partial y, \partial x + \partial y, \partial z, \partial t\}$. Let vector fields be

$$Z_1 = \partial z,\ Z_2 = \partial x - \partial y\ Z_3 = \partial x + \partial y, \text{ and } Z_4 = \partial t.$$

Since $e_2 = 2y\partial x + 2x\partial y + \partial z$, and $e_3 = 2x\partial x - 2y\partial y$ from Example 4.1.1

then

$$\partial y = \frac{x}{2(x^2+y^2)}e_2 - \frac{y}{2(x^2+y^2)}e_3 - \frac{x}{2(x^2+y^2)}e_1$$
$$\text{and } \partial x = \frac{y}{2(x^2+y^2)}e_2 + \frac{x}{2(x^2+y^2)}e_3 - \frac{y}{2(x^2+y^2)}e_1.$$

So representing Z_2 and Z_3 in terms of e_i and using the fact that $x = y$, we obtain

$$Z_2 = \partial x - \partial y = \frac{1}{2x}e_3 \text{ and } Z_3 = \partial x + \partial y = \frac{1}{2x}e_2 - \frac{1}{2x}e_1.$$

Thus,

$$[Z_2, Z_3] = \frac{1}{4x^2}[e_3, e_2] = -\frac{1}{x^2}e_3.$$

Computing $\overline{\nabla}$ we obtain

$$\overline{\nabla}_{Z_1}Z_1 = \overline{\nabla}_{Z_1}Z_2 = \overline{\nabla}_{Z_1}Z_3 = \overline{\nabla}_{Z_1}Z_4 = \overline{\nabla}_{Z_2}Z_1 = \overline{\nabla}_{Z_2}Z_4 = 0.$$
$$\overline{\nabla}_{Z_3}Z_1 = \overline{\nabla}_{Z_3}Z_4 = \overline{\nabla}_{Z_4}Z_1 = \overline{\nabla}_{Z_4}Z_2 = \overline{\nabla}_{Z_4}Z_3 = \overline{\nabla}_{Z_4}Z_4 = 0.$$
$$\overline{\nabla}_{Z_2}Z_2 = \frac{1}{4x^2}\overline{\nabla}_{e_3}e_3 = \frac{1}{4x^2}(8(x^2-y^2)e_3 + 48xye_2) = 12e_2,$$
$$\overline{\nabla}_{Z_2}Z_3 = \frac{1}{4x^2}\overline{\nabla}_{e_3}e_2 - \frac{1}{4x^2}\overline{\nabla}_{e_2}e_3 = -\frac{1}{4x^2}(-48x^2)e_3 - \frac{1}{4x^2}(16x^2)e_3 = 8e_3.$$
$$\overline{\nabla}_{Z_3}Z_2 = \frac{1}{4x^2}\overline{\nabla}_{e_2}e_3 = \frac{1}{4x^2}(16x^2)e_3 = 4e_3.$$
$$\overline{\nabla}_{Z_3}Z_3 = \frac{1}{4x^2}\overline{\nabla}_{e_2}e_2 = \frac{1}{4x^2}(16x^2)e_2 = 4e_2.$$

Using (2.4) to compute ∇ we obtain

$$\nabla_{Z_1}Z_1 = \nabla_{Z_3}Z_1 = \nabla_{Z_1}Z_2 = \nabla_{Z_1}Z_3 = \nabla_{Z_1}Z_4 = \nabla_{Z_3}Z_4 = 0.$$
$$\nabla_{Z_4}Z_1 = \nabla_{Z_2}Z_4 = \nabla_{Z_4}Z_4 = \nabla_{Z_4}Z_2 = \nabla_{Z_4}Z_3 = 0.$$
$$\nabla_{Z_3}Z_2 = -\frac{1}{4x^3}Z_2 - \frac{1}{2x^3}Z_3,\ \nabla_{Z_3}Z_3 = \frac{1}{2x^3}Z_2 - \frac{1}{2x^3}Z_3.$$
$$\nabla_{Z_2}Z_2 = -\frac{1}{4x^3}Z_2 + \frac{3}{4x^3}Z_3,\ \nabla_{Z_2}Z_3 = -\frac{3}{4x^3}Z_2 - \frac{1}{2x^3}Z_3.$$

Using $\overline{\nabla}$ and ∇ to compute h we obtain

$$h(Z_1,Z_1)=h(Z_2,Z_4)=h(Z_1,Z_2)=h(Z_3,Z_1)=h(Z_1,Z_3)=h(Z_3,Z_4)=0.$$
$$h(Z_1,Z_4)=h(Z_4,Z_1)=h(Z_2,Z_1)=h(Z_4,Z_2)=h(Z_4,Z_3)=h(Z_4,Z_4)=0.$$
$$h(Z_2,Z_2)=\frac{1}{8x^4}e_3-(\frac{3}{8x^4}-12)e_2+\frac{3}{8x^4}e_1,$$
$$h(Z_2,Z_3)=(8+\frac{3}{8x^4})e_3+\frac{1}{4x^4}e_2-\frac{1}{4x^4}e_1.$$
$$h(Z_3,Z_2)=(4+\frac{1}{8x^4})e_3+\frac{1}{4x^4}e_2-\frac{1}{4x^4}e_1,$$
$$h(Z_3,Z_3)=\frac{3}{4x^4}e_3+(4+\frac{1}{4x^4})e_2-\frac{1}{4x^4}e_1.$$

Hence, M is not totally geodesic. Also M is not invariant since for $Z_2,Z_3\in\Gamma(TM)$ we obtain

$$\bar{\phi}[Z_2,Z_3] \;=\; -\frac{1}{x^2}\bar{\phi}(e_3)=\frac{1}{x^2}e_2\neq-\frac{1}{x^2}\phi(e_3)=\phi[Z_2,Z_3].$$

Thus $\bar{\phi}[Z_2,Z_3]\neq\phi[Z_2,Z_3]$.

Chapter 5

Conclusion and Perspectives

In conclusion, our study of Hermitian manifolds has revealed that the condition $h(JX,Y) = h(X,JY)$ is crucial for the distributions to be integrable under nearly Kaehlerian manifolds. In a cosymplectic manifold $D^{\perp}$ is always integrable without any condition. In a non-Hermitian manifold using a cosymplectic manifold as our ambient manifold we saw that $h(\phi X,Y) = h(X,\phi Y)$ was crucial for $D \oplus \{\xi\}$ to be integrable. Theorems 3.2.5 and 4.7.1 give partially the same result but different in general because they are based on different hypotheses. If a submanifold of a cosymplectic manifold is invariant then it is also a cosymplectic manifold and all invariant submanifolds are totally geodesic and the the distribution $D \oplus \{\xi\}$ is integrable on them. We also obtain crucial result that a mixed foliated CR-submanifold of a cosymplectic manifold of a complex space form has a curvature that is greater or equal to zero and if the curvature is less than zero then a CR-submanifold of a cosypmlectic manifold of a complex form space is said to be invariant or anti-invariant. We also saw that in a cosymplectic manifold $\overline{M}$ if a CR-submanifold M of $\overline{M}$ is almost contact umbilical and the dimension of $D^{\perp}$ is greater than one, then M is totally contact geodesic in $\overline{M}$ and the minimal submanifold of M. We also saw that in general there are no totally geodesic submanifold of a cosymplectic manifold which are totally umbilical.

As perspectives, we would like to investigate the null subspaces which are contact CR in nearly Kählerian and cosymplectic manifolds. The study might need more information on the ambient spaces. The latter must be semi-Riemannian manifolds. Although the semi-Riemannian concept generalizes the Riemannian one, some topological obstructions may appear throughout the study. Therefore, a particular attention must be paid to Semi-Riemannian nearly Kählerian and cosymplectic manifolds before investigating its null subspaces.

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