Spanning Trees: Eigenvalues, Special Numbers, and the Tree-Cover Ratios, Asymptotes and Areas of Graphs

by

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As the candidate's supervisor I have approved this dissertation for submission.

Dr. Paul Winter

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As the candidate's co-supervisor I have approved this dissertation for submission.

Professor Simon Mukwembi

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Preface

The study described in this dissertation was carried out in the School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Durban, from February 2014 to November 2014, under the supervision of Dr. Paul August Winter and co-supervision of Professor Simon Mukwembi.

These studies represent original work by the author and have not otherwise been submitted in any form for any degree or diploma to any tertiary institution. Where use has been made of the work of others it is duly acknowledged in the text.

Abstract

This dissertation deals with spanning trees associated with graphs. The number of spanning trees of a graph can be found by considering the eigenvalues of the Laplacian matrix associated with that graph. Special classes of graphs are also considered, such as fan and wheel graphs, where their spanning tree numbers are connected to special numbers, like the Lucas and Fibonacci numbers. We use the eigenvalues of the complete graph and its associated circulant matrix to create a unit-trigonometric equation which generates a sequence and diagram similar to that of the famous Farey sequence. A new ratio is introduced: the tree-cover ratio involving spanning trees and vertex coverings and is motivated by the fact that such a ratio, associated with complete graphs, has the asymptotic convergence identical to that of the secretary problem. We use this ratio to introduce the idea of tree-cover asymptotes and areas and determine such values for known classes of graphs. This ratio, in communication networks, allows for the investigation of the outward social connectivity from a vertex covering to the rest of the network when a large number of vertices are involved.

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Declaration 1: Plagiarism

I, Fadekemi Janet Adewusi declare that:

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Declaration 2: Publications

Publication 1

Winter P.A and Adewusi F. J. *Tree-Cover Ratios of Graphs With Asymptotic Convergence Identical to that of the Secretary Problem.*

Published in: Advances in Mathematics: Scientific Journal. 3(1) (2014), pages 47-61.

(50 % contribution from each author).

Paper based on chapter 7 of thesis.

Publication 2

Preprint Archive Paul August Winter, Carol Lynne Jessop and Fadekemi Janet Adewusi, *The Complete Graph: Eigenvalues, Trigonometrical Unit-Equations with associated t-Complete-Eigen Sequences, Ratios, Sums and Diagrams.*

($33\frac{1}{3}$% contribution from each author).

Published as preprint in viXra:1411.0315 (2014)

Accepted for publication in Journal for Mathematics and System Science, USA.

Paper based on chapter 5 of thesis.

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F. J. Adewusi

November 2014

Dedication

To

God Almighty;

My children; Bolu, Dieko, and Semilore.

Contents

List of symbols

Symbol	Meaning
$G = (V, E)$	Graph with vertex set V and edge set E
$V(G)$ or V	Vertex set of G
$E(G)$ or E	Edge set of G
$H \leq G$	H is a subgraph of G
$A \square B$	The Cartesian product of A and B
n, $\lvert V(G) \rvert$	Order of G
m, $\lvert E(G) \rvert$	Size of G
$deg(v)$	Degree of vertex v
$\delta(G)$	Minimum of vertex degrees of G
$\Delta(G)$	Maximum of the vertex degrees of G
$Q(G)$	Average degree of G
$diam(G)$	Diameter of G
$\mathfrak{r}(G)$	Radius of G
$\epsilon(v)$	Eccentricity of vertex, v
$C(G)$	Center of G
$U \subseteq V$	Set U is a subset of set V
$\mathbb{Z}$	Set of integers
$x \in S$	x belongs to set S
$x \notin S$	x does not belongs to S
$\det(A)$, $\lvert A \rvert$	Determinant of matrix A
$\lvert \vec{x} \rvert$	Absolute value of vector x
$\sum_{i=1}^{m} a_i$	Sum of values of a_i as i runs from 1 to m
$\prod_{i=1}^{n} a_i$	Product of values of a_i as i runs from 1 to n
K_n	Complete graph on n vertices
C_n	Cycle graph on n vertices
$K_{p,q}$	Complete bipartite graph of order $p + q$
P_n	Path graph on n vertices
$S(n, 1)$ or S_n	Star graph with k rays of length 1
$S_k(n, r)$	Star graph with k rays of length r

W_n	Wheel graph on $n+1$ vertices
F_n	Fan graph on $n+1$ vertices
L_n	Ladder graph on n vertices
$\mathbb{Z}_p$	Set of integers modulo p
$GF(n)$	Galois Fields modulo n
$E_1 \oplus E_2$	Ring sum of sets E_1 and E_2
$\binom{n}{i}$	Binomial coefficient
$A(G)$, A	Adjacency matrix of G
$B(G)$, B	Incidence matrix of G
$L(G)$, L	Laplacian of G
$\hat{L}$	Reduced Laplacian of G
B^T	Transpose of B
$\sigma(G)$	Spectrum of G
$P(G)$	Characteristic polynomial of G
λ_i	Laplacian eigenvalues of G
μ_i	Adjacency eigenvalues of G
$t(G)$	Number of spanning trees of G
$\mathcal{F}_n$	Fibonacci sequence
$\mathcal{L}_n$	Lucas numbers
$\mathbb{P}_n(r)$	Probability of selecting r objects from a set of n objects
$tc(G)_s$	Tree-cover ratio of G wrt set S
$Asyp(G)_s$	Tree-cover asymptote wrt S
$Ar(\xi)$	Tree-cover area of classes of graph, ξ
$ht(G)$	Tree-cover height of G
$\rho(G)$	Hall ratio of G
$\alpha(G)$	Independence number of G
$\tau(G)$	Vertex cover number of G
$I_R(G)$	Independence ratio of G

Chapter 1

Introduction

Graphs are distinct structures that consist of points or vertices, connected by edges. Different real life situations can be modelled using graphs. Examples of such are representations of hydrocarbons in chemistry, distributions in transportation, links in communication networks, and the study of neural networks and species interactions in biology.

There has been an increased interest in the study of graphs and networks in the past few decades, particularly amongst mathematicians, computer scientists and engineers. Leonard Euler (1736), was the first to study graph theory in his paper published on the Königsberg bridge problem encountered by the then eighteenth century people of Königsberg, the capital of Prussia in the German Empire.

The last two decades have also witnessed increased interest in spectral graph theory involving the adjacency and Laplacian matrices associated with a graph. During that era, matrix theory and linear algebraic properties were used to efficiently analyze the adjacency matrices of graphs. For example, the "algebraic connectivity " of a graph was given as a measure of the connectivity of the graph and claimed to be connected to the adjacency eigenvalues by Bousquet-M'elouy

M. and Weller [7]. The second smallest eigenvalue and the largest eigenvalue of the Laplacian are also very important parameters as measures of connectivity, as studied by Bojan Monar in Beineke et. al. [12].

Many developments in spectral graph theory have mostly involved inputs of geometry in recent times. One of these is the recent progress in expander graphs and eigenvalues initiated by the problems in communication networks by Alon and Spencer [1]. Here, they showed that any k-regular graph whose adjacency eigenvalues lie between -0.9 and 0.9, is well connected. The explicit construction of expander graphs due to Gabber and Galil in [16], is based on Laplacian eigenvalues and isoperimetric properties of graphs. However, there is no doubt that both eigenvalues are crucial to the fundamental understanding of the connectivity in graphs.

The aim of this dissertation is to consider the connectivity properties of some special graphs based on properties such as their eigenvalues, their number of spanning trees, and the cardinality of their (minimum) vertex-cover. With these properties, we introduce a new concept of tree-cover ratios, asymptotes and areas of graphs. In this dissertation, only simple undirected connected finite graphs are considered.

The number of spanning trees was first studied by Gustav Kirchoff, a German physicist [24], in his quest to study electrical networks. He described the connection between the number of spanning trees in terms of linear algebraic terms [24]. Several other proofs for the number of spanning trees are also known. For example, Temperly [32], for a graph of order n, gave it as $\det[L(G) + \frac{1}{n^2}J]$, where J is an $n \times n$ matrix whose elements are 1 and $L(G)$ is the graph Laplacian. Kal'man and Chelnokov [23], gave it as $\frac{1}{n}\prod_{k=2}^{n}\lambda_k$, where λ_k are the Laplacian eigenvalues of the n-vertex graph.

Similarly, the number of spanning trees for some special classes of graphs has been considered by many authors. Arthur Cayley [2], a British mathematician, was the first to use the term "tree", in his quest for the study of the number of carbon atoms in a given saturated hydrocarbons, and he gave the number of spanning trees for the complete graph K_n as n^{n-2}. Hilton [20], gave the number of spanning trees of labelled wheels, fans and baskets, in terms of the Fibonacci and Lucas numbers. Moreso, different graph ratios have been considered in recent years. For example, Buckley in [8], gave the central radius ratio asymptote as 1. Winter and Jessop describe the eigen-pair ratio based on the adjacency matrix in [37] having asymptote on interval $[-1, 0]$, and the Hall ratio with asymptote infinity, was described by Gabor in [15].

This dissertation is structured into eight chapters under the following organisation: Chapter 2 details the definitions and terminologies in graphs required for better understanding of this dissertation. In addition, it also introduces the special graphs which will be used in this dissertation. Chapter 3 introduces basic linear algebra terms used in graph theory and necessary for the understanding of this dissertation. Chapter 4 introduces and discusses the eigenvalues of some special graphs. Here, both the adjacency eigenvalues and the Laplacian eigenvalues are considered. In Chapter 5, we consider the t-complete eigen sequence generated from the adjacency eigenvalues of the complete graph, which is similar to the Farey sequence. In Chapter 6, we discuss spanning trees of the special graphs introduced in Chapter 2 in terms of the Laplacian eigenvalues discussed in Chapter 4, or in terms of special numbers, namely, the Fibonacci sequence and Lucas numbers, for some special class of graphs using the Kirchoff's matrix tree theorem.

In Chapter 7, we consider the spanning trees discussed in Chapter 6 and the (minimum) vertex covering sets of these graphs, to introduce their tree-cover ratio, asymptotes and areas. This ratio, in communication networks, allows for the investigation of the outward social connectivity from a vertex covering with the rest of the network, when a large number of vertices are involved. Among them is the complete graph K_n, which is a graph theoretical interpretation for the secretary problem, and the gambling problem with social decision making, as described in [38]. This provided a motivation for the study of the tree-cover ratio, asymptotes and areas. In Chapter 8, conclusions are drawn.

Chapter 2

Graph Theory Terminologies

2.1 Introduction

The purpose of this chapter is to define the most important terms that will be used in this dissertation and to present motivations for our study, as well as provide relevant backgrounds. Hence, the graph theoretical terminology of Harris, Hirst and Mossinghoff [19] will be adopted. Terms not defined in this chapter will be defined in subsequent chapters, as the need arises.

2.2 Graph Terminologies

Definition 1 *A graph, $G = (V, E)$, consists of a non-empty finite set, $V(G)$, of elements called vertices and a possibly empty finite set, $E(G)$, of 2-subsets of V called edges.*

The number of vertices in G, denoted as n, is called its *order* while the number of edges in G, denoted as m, is called the *size* of G. Let $e = \{u, v\}$ be an edge of G. Then, u and v are called the *end vertices* of the edge e, while e is said to be *incident* with u and v, and we say that u and v are *adjacent* or *neighbours.*

We could simply write e as $e = uv$ in place of $\{u, v\}$. Two edges are said to be *incident* or *adjacent* if they have a common end vertex.

The *degree* of a vertex, v, denoted as $deg(v)$, is the number of edges incident with v. An *isolated* vertex is a vertex of degree zero while an *end* or *leaf* vertex of G is a vertex of degree one. A graph with no edges is called a *null or empty graph* while a graph with only one vertex is called a *trivial graph*. The *minimum degree of* G, denoted by $\delta(G)$, is the minimum of the vertex degrees of G and the *maximum degree of* G, denoted by $\Delta(G)$, is the maximum of the vertex degrees of G. The *average degree of* G, denoted as $Q(G)$, is defined as

$$Q(G) = \frac{\sum_{i=1}^{n} deg(v_i)}{n} = \frac{2m}{n},$$

where $V(G) = \{v_1, v_2, \ldots, v_n\}$.

A *walk*,

$$W := v_0, e_0, v_1, e_1, v_2, \ldots, e_{k-1}, v_k,$$

in a graph G, is an alternating sequence of vertices and edges such that for $i = 0, 1, 2, 3, \ldots, k-1$, $e_i = v_i v_{i+1} \in E(G)$. We say that W is a *closed walk* if $v_0 = v_k$. The length of W is k, i.e., the number of edges in W. The walk that begins at v_0 and ends at v_k is called a v_0-v_k walk. A *trail* is a walk in which all edges are distinct, while a *path* is a walk in which all vertices are distinct. A *cycle* is a closed path while a *circuit* is a closed trail.

Let u, v be vertices in G. The distance $d(u, v)$ between u and v in G is defined as the *length* of a shortest u-v path. The maximum distance between a vertex v and any vertex u farthest from v in G is called the *eccentricity* of v, denoted as

$\epsilon(v)$, and is given by,

$$\epsilon(v) = \max_{u \in V(G)} d(v, u).$$

The maximum distance between any two vertices in G is called the *diameter* of G, denoted as *diam(G)*, and can also be given as

$$diam(G) = \max_{v \in V(G)} \epsilon(v).$$

Vertices u and v are said to be *connected* if there is a u-v path in G. The graph G is *connected* if every pair of vertices in G is connected. The *center of* G, denoted as $C(G)$, is the set of vertices in G with minimum eccentricity. The *radius of* G, $\mathfrak{r}(G)$, is the minimum of all the eccentricities of vertices in G.

Definition 2 (Subgraphs) *Let $G = (V, E)$ be a graph. The graph $H = (U, F)$ is a subgraph of the graph G if $U \subseteq V$ and $F \subseteq E$. If $U = V$, then H is called a spanning subgraph of G. If H is a subgraph of graph G, we write $H \leq G$.*

Definition 3 (Trees and Spanning Trees) *A tree, denoted as T, is a connected graph with no cycles. A spanning subgraph of a graph that is a tree is called a spanning tree.*

Definition 4 (Cartesian Product of Two Graphs) *Let $G = (V_1, E_1)$ and $H = (V_2, E_2)$ be two connected simple graphs with $V_1 \cap V_2 = \emptyset$. The Cartesian product of G and H, denoted as $G \Box H$, is the graph with vertex set $V_1 \times V_2$, where vertices (v_1^1, v_2^1) and (v_1^2, v_2^2) are adjacent in $G \Box H$ if either (i) $v_1^1 = v_1^2$ and v_2^1 is adjacent to v_2^2 in H* or *(ii) $v_2^1 = v_2^2$ and v_1^1 is adjacent to v_1^2 in G.*

2.3 Special Types of Graphs

We now introduce several types of connected graphs which may arise in many real life applications and will often be used as examples in this thesis.

A *cycle*, denoted as C_n, for $n \geq 3$, is a graph consisting of n vertices, $V(C_n)=\{v_1, v_2, \ldots, v_n\}$, and edges $E(C_n)=\{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}$, where $|E(C_n)| = n$. A *path* of order n, denoted as P_n, is the graph obtained from C_n by removing one edge. Here, $|E(P_n)| = n - 1$. A *complete graph* of order n, K_n, is a graph with n vertices in which any two vertices are adjacent. The size of K_n is $|E(K_n)| = \frac{n(n-1)}{2}$. A graph is *k-regular* if all its vertices have degree k. For example, the null graph is 0-regular, K_n is $(n-1)$-regular, and the cycle graph is 2-regular.

A graph is *bipartite* if its vertex set V can be partitioned into two disjoint sets V_1 and V_2, called *partite sets*, such that every edge in the graph has one end in V_1 and the other end in V_2. A *complete bipartite graph*, denoted as $K_{p,q}$, is a bipartite graph with partite sets V_1 and V_2, of cardinalities p and q respectively, such that every vertex in V_1 is adjacent to every vertex in V_2. The size of $K_{p,q}$ is given as $|E(K_{p,q})| = pq$, where $p + q = n$, for an n-vertex graph. The *complete split bipartite graph* is the complete bipartite graph with equal partitions of vertices, denoted as $K_{\frac{n}{2},\frac{n}{2}}$.

According to Jordan [22], if T is a tree, then either $diam(T) = 2\mathfrak{r}(T)$ and $C(T)$

contains exactly one vertex or $diam(T) = 2\mathfrak{r}(T) - 1$ and $C(T)$ consists of two adjacent vertices. In this dissertation, we will deal with special trees with only one center vertex. The commonly known *star graph*, denoted as S_n, is a tree of order n with maximum diameter 2. Let $e = uv$ be an edge of G. By subdividing the edge e, we mean removing e from G and adding a new vertex w together with edges uw and wv to G. A *subdivided star* is a graph obtained from the star graph, S_n, by a sequence of edge subdivisions.

Let $S_k(n,r)$ be a subdivided star of order n, whose center contains only one vertex, say $C(S_k(n,r)) = \{v_1\}$. A *branch* of $S_k(n,r)$, denoted as $B[S_k(n,r)]$ is a $v_1 - v_2$ path in $S_k(n,r)$, where v_2 is an end vertex. The *ray* of $S_k(n,r)$, denoted as $r[S_k(n,r)] = r$, is the number of branches in $S_k(n,r)$. In this dissertation, we deal with subdivided stars with branches of the same length and we call such a subdivided star the *star graph with k rays of length r*, denoted as $S_k(n,r)$. Below is a star graph of order 7 and center vertex v_1. The branches are the paths $v_1 - v_7$, $v_1 - v_6$ and $v_1 - v_5$. Therefore, the star graph is of ray 3.

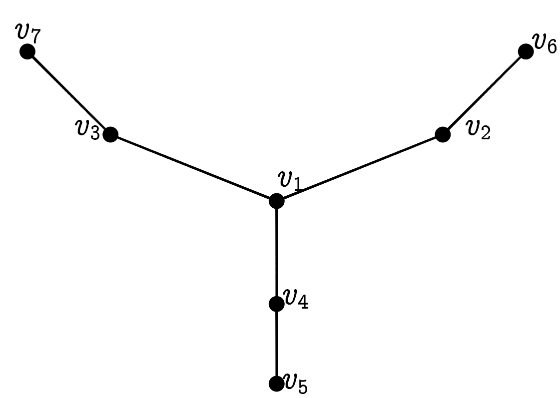


Figure 2.1: Star graph with 3 rays of length 2, $S_3(7,2)$.

A *wheel*, W_n, is the graph obtained by taking a cycle C_n and adding a new vertex v which is joined to every vertex of C_n. The *rim* of W_n is the cycle C_n, the *spokes* are the edges incident with v, while the *hub* is the vertex v. A *Fan*, F_n, is the graph obtained by taking a path P_n and adding a new vertex v which is joined to every vertex of P_n. Both W_n and F_n have order $n+1$ for convenience in proofs. A *Ladder*, L_n, of order on n , is the Cartesian product, $P_2 \Box P_{\frac{n}{2}}$, of P_2 and $P_{\frac{n}{2}}$. A *Sun graph*, SN_n, of order n, is the graph obtained by taking $C_{\frac{n}{2}}$ and attaching exactly one vertex to each vertex of $C_{\frac{n}{2}}$.

2.4 Summary

This chapter provided the basics on graph terminology necessary to understand the remaining parts of this dissertation. We specified the meaning of subgraphs, spanning subgraphs, trees, branches, rays, and the Cartesian product of two graphs as would be needed for better understanding of the subsequent chapters. Furthermore, we studied the different types of classes of simple finite connected graphs termed "special graphs", that are commonly found in many real-life applications, and would serve mainly as examples in this dissertation. To further understand the properties of these graphs, we must take into account an understanding of basic linear algebra in the next chapter.

Chapter 3

Representations of Graphs

3.1 Introduction

In solving graph problems, it is easier to represent graphs algebraically. One such algebraic tool is the use of matrix theory, which is used in the representation and structural study of graphs. In this chapter, basic important linear algebra concepts necessary for the study and description of graphs are considered.

3.2 Basic Linear Algebra

This section is based on pages 112-125 of [11].

3.2.1 Brief Definitions

Definition 5 *Let $S = \{s_1, s_2, \ldots, s_n\}$ be a finite set. A group $(S, *)$ is a pair consisting of a set, S, and a binary operation $*$ satisfying the following properties.*

i. S is closed under $*$, i.e., for $s_1, s_2 \in S$, $s_1 * s_2 \in S$.

ii. S is associative under $*$, i.e., $(s_1 * s_2) * s_3 = s_1 * (s_2 * s_3)$ for all $s_1, s_2, s_3 \in S$.

iii. There is a unique identity element of the group, e, such that for any element $s \in S$, $e * s = s = s * e$.

iv. Every element $s \in S$ has an inverse element $s^{-1} \in S$, such that $s * s^{-1} = e = s^{-1} * s$.

If in addition to the above properties for all $s_1, s_2 \in S$, $s_1 * s_2 = s_2 * s_1$, then the group is said to be *abelian* or *commutative*.

A *field*, $(S, +, \star)$, on the other hand, is a set S together with two binary operations $+$ and $\star$ satisfying properties $p1$ to $p3$.

$p1$. S is an abelian group under $+$.

$p2$. $S - \{0\}$ is an abelian group under $\star$, where 0 is the additive identity element.

$p3$. S satisfies the distributive property, i.e., given $s_1, s_2, s_3 \in S$,

$$s_1 \star (s_2 + s_3) = (s_1 \star s_2) + (s_1 \star s_3) \quad \text{and}$$

$$(s_1 + s_2) \star s_3 = (s_1 \star s_3) + (s_2 \star s_3).$$

For example, the set of integers modulo p, $\mathbb{Z}_p$, for p prime is a field, and $(\mathbb{Z}_p, +, \star)$ is called the *Galois Field*. In graph theory, the Galois field modulo 2, represented as $GF(2)$, is of special interest because it consists of only 0's and 1's. An illustration is describe below.

$+$	0	1
0	0	1
1	1	0

$\star$	0	1
0	0	0
1	0	1

In general, it should be observed that apart from satisfying the properties (i)-(iv), it also satisfies the properties $p1$ to $p3$.

3.2.2 Vector Spaces Associated With Graphs

Many physical properties of natural phenomena are well represented as vector spaces. Since graphs are intuitively points that are either connected by lines or not, their structural study can be well described in terms of their geometrical properties, as described by Beezer in [3].

A *vector space*, $\mathcal{V}$, defined over a field $\mathcal{F}$, consists of objects called vectors which satisfy the following properties.

(i) If $v_1, v_2, v_3 \in \mathcal{V}$, then $v_1 + v_2 \in \mathcal{V}$. Also, $v_1 + v_2 = v_2 + v_1$, $(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$.

(ii) There exists the additive identity element $\mathbf{0} \in \mathcal{V}$, such that for any $v \in \mathcal{V}$, $v + \mathbf{0} = v$.

(iii) For any $v \in \mathcal{V}$, $\exists -v \in V$ such that $v + (-v) = \mathbf{0}$.

(iv) For any $v \in \mathcal{V}$ and a scalar $c \in \mathcal{F}$, $cv \in \mathcal{V}$.

(v) For any $v_1, v_2, \in \mathcal{V}$ and $c_1, c_2 \in \mathcal{F}$, $c_1(v_1 + v_2) = c_1v_1 + c_1v_2$, $(c_1 + c_2)v_1 = c_1v_1 + c_2v_1$, and $c_1(c_2v_1) = (c_1c_2)v_1$.

(vi) For all elements $v \in \mathcal{V}$, $1.v = v$ is satisfied, where 1 is the multiplicative identity element.

The elements of field $\mathcal{F}$ are called *scalars*. A *vector* is a matrix with only one row called a *row vector* or one column called a *column vector*. A subspace of a vector space $\mathcal{V}$ is a subset $\mathcal{H}$ of $\mathcal{V}$ which satisfy the properties below.

(i). The zero vector of $\mathcal{V}$ is in $\mathcal{H}$.

(ii). For each elements $u, v \in \mathcal{H}$, $u + v \in \mathcal{H}$.

(iii). For each $u \in \mathcal{H}$ and scalar $c \in \mathcal{F}$, $u.c \in \mathcal{H}$.

Consider an $n \times n$ matrix K whose entries are $k_{ij} \in \mathbb{R}$ for $i, j \in \{1, 2, \ldots, n\}$. We can rewrite K as column vectors, $K = [\mathbf{c}_1\, \mathbf{c}_2\, \ldots\, \mathbf{c}_n] \in \mathbb{R}^n$ or row vectors, $K = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_n \end{bmatrix} \in \mathbb{R}^n$, where $\mathbf{c}_i = \begin{bmatrix} k_{1i} \\ k_{2i} \\ \vdots \\ k_{ni} \end{bmatrix}$ and $\mathbf{r}_i = [k_{i1}\, k_{i2}\, \ldots\, k_{in}]$. The subspaces $\mathbf{c}_i$ and $\mathbf{r}_i$ in $\mathbb{R}^n$ are the *column space* and *row space* of K respectively. The *null space* or *kernel* of K is defined as the set of all vectors $\mathbf{v}$, which maps to the zero vector, $\mathbf{0}$, i.e.,

$$Ker(K) = \{\mathbf{v} \in \mathbf{V} | K\mathbf{v} = \mathbf{0}\}.$$

The *minor*, M_{ij}, of the entry in the ith row and jth column of K, is the determinant of the square matrix obtained from K by deleting its ith row and its jth column. The $i \times i$ *principal minor* of K is the minor obtained from K by deleting $(n-i)$ rows and its corresponding $(n-i)$ columns; for some $1 \leq i \leq n$. A *cofactor*, C_{ij}, is given as $C_{ij} = (-1)^{i+j} M_{ij}$. The maximum *dimension* (number of rows and columns) of a non-zero minor of K is its *rank*.

3.2.3 Galois Fields Representation of Graphs

Given a graph of order n and size m, with edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$, and vertex set, $V(G) = \{v_1, v_2, \ldots, v_n\}$, any subgraph, $G_k \leq G$, can be represented by an m-tuple,

$$G_k = (g_1, g_2, \ldots, g_m),$$

where,

$$g_i = \begin{cases} 1 & \text{if} \quad e_i \in E(G_k), \\ 0 & \text{if} \quad e_i \notin E(G_k), \end{cases}$$

for $i = 1, \ldots, m$.

Example 1 *Consider the graph G below.*

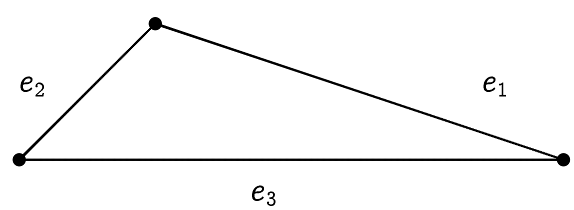


Figure 3.1: Graph, G

Then, Figure 3.2 shows two subgraphs of G and their corresponding 3-tuple representation.

Other possible subgraphs of G are $(0,0,1), (1,0,0), (0,1,0), (0,0,0), (1,1,1), (1,1,0)$, where $(0,0,0)$ represents the null graph and $(1,1,1)$ is the graph G itself. In the following corollary, we make use of the m-tuple representation of graphs to determine the number of subgraphs in a given graph.

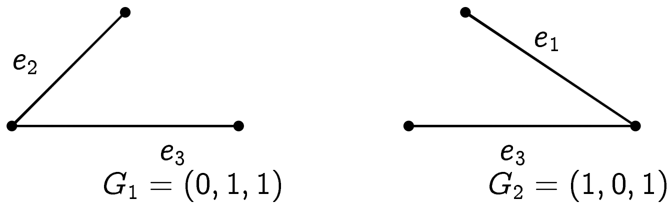


Figure 3.2: Subgraphs G_1 and G_2

Corollary 1 *The number of possible subgraphs of order n associated with any graph, G of order n, is 2^m, where m is the size of the graph.*

Proof

Let G be a graph with edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. Then every subgraph G_k of G is represented by one m-tuple $(g_1, g_2, \ldots, g_m)$ where $g_i = \begin{cases} 1 & \text{if} \quad e_i \in E(G_k), \\ 0 & \text{otherwise,} \end{cases}$

for $i = 1, 2, \ldots, m$. Since for each $i = 1, 2, \ldots, m$, there are two choices for g_i, by the Multiplication Principle, there are $2 \times 2 \times \ldots \times 2 = 2^m$ m-tuples of this form. Hence, G has 2^m subgraphs, as desired. $\square$

In [11], these tuples representing subgraphs have been used as vectors to construct a vector space $\mathcal{V}$ over the Galois field, $[GF(2), +, \star]$, where addition and multiplication is modulo 2. In particular, $\mathcal{V}$ consists of all 2^m m-tuples representing all subgraphs of G. In $\mathcal{V}$, addition of vectors is component wise. Scalar multiplication is as follows. Let $a \in \{0, 1\}$ and $G_k \leq G$ be represented by $(g_1, g_2, \ldots, g_m)$. Then,

$$a \star (g_1, g_2, \ldots, g_m) = \begin{cases} (g_1, g_2, \ldots, g_m) & \text{if} \quad a = 1, \\ (0, 0, \ldots, 0) & \text{if} \quad a = 0. \end{cases}$$

The result in Corollary 1 could be used to find the number of spanning trees of a complete graph. For example, the complete graph, K_4 has 6 edges so that it has $2^6 = 64$ subgraphs. A spanning tree will require 3 edges, so we remove all subgraphs of sizes 0,1,2,4,5 and 6 and the subgraphs of size 3 that have triangles. Thus the total number of spanning trees will be $64-(1+6+15+15+6+1+4) = 64-48 = 16$ which gives the number of spanning trees as expected from known results of [2]. One may wish to generalize this method to find the number of spanning trees on a complete graph but it would be cumbersome. A more elegant proof will be presented in Chapter 6.

3.3 Matrix Representations of Graphs

Generally, apart from the Galois field representation, the connectivity property of graphs is mostly represented in terms of matrices. In this section, we study structures of graphs using their adjacency, incidence and Laplacian matrices.

Definition 6 (Adjacency Matrix) *Let $G = (V, E)$ be a graph with vertex set, $V = \{v_1, v_2, \ldots, v_n\}$. Then, the adjacency matrix of G, denoted as $A = A(G)$, is defined as the $n \times n$ matrix, $A(G) = [a_{ij}]$ whose entries are given as:*

$$a_{ij} = \begin{cases} 1 & \text{if} \quad v_iv_j \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $a_{ij} = a_{ji}$ since we are considering undirected graphs.

Let G be a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. Let $e = v_iv_j$ be an edge of G with $i < j$. Here and in the sequel, we will say that v_i is the *tail* of e and v_j is the *head* of e.

Definition 7 (Incidence Matrix) *Let G be a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{e_1, e_2, \ldots, e_m\}$. The incidence matrix of G, denoted by $B = B(G) = [b_{ij}]$, is the $n \times m$ matrix whose (i,j)th entry is given by*

$$b_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is the head of } e_j, \\ -1 & \text{if } v_i \text{ is the tail of } e_j, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 8 (Laplacian) *The Laplacian of a graph G is an $n \times n$ matrix, denoted as $L = L(G) = [l_{ij}]$, where*

$$l_{ij} = \begin{cases} d_i, & \textit{if} \quad i = j, \\ -a_{ij} & \textit{if} \quad i \neq j, \end{cases}$$

with d_i denoting the degree of vertex v_i and a_{ij} is the (i,j)th entry of the adjacency matrix of G.

The graph Laplacian can be given as

$$L(G) = D(G) - A(G),$$

where $D(G)$ is the diagonal matrix of the degrees of G and $A(G)$ is the adjacency matrix of the graph. Basically, the relationship between the adjacency and incidence matrices together with the graph Laplacian, can be used to find important properties of the graph. This will be considered in Chapter 5. For convenience, we use A for adjacency matrix, B for incidence matrix, and L for Laplacian.

3.3.1 Relationship Between Adjacency Matrix, Incidence Matrix and the Graph Laplacian.

Theorem 1 *Let B be an $(n \times m)$ incidence matrix and L an $(n \times n)$ Laplacian matrix given as $D - A$. Then*

$$BB^T = L.$$

Proof

Let

$$B = \begin{bmatrix} B_{11} & B_{12} & \dots & B_{1m} \\ B_{21} & B_{22} & \dots & B_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ B_{n1} & B_{n2} & \dots & B_{nm} \end{bmatrix} \quad \text{and} \quad B^T = \begin{bmatrix} B_{11} & B_{21} & \dots & B_{n1} \\ B_{12} & B_{22} & \dots & B_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ B_{1m} & B_{2m} & \dots & B_{nm} \end{bmatrix}.$$

Then, for $i, j \in \{1, 2, \dots, n\}$, we have

$$\begin{aligned} [BB^T]_{ij} &= [B_{i1}, B_{i2}, \dots, B_{im}] \,.\, [B_{1j}{}^T, B_{2j}{}^T, \dots, B_{nj}{}^T] \\ &= [B_{i1}, B_{i2}, \dots, B_{im}] \,.\, [B_{1j}, B_{2j}, \dots, B_{nj}]^T \\ &= \sum_{k=1}^{m} B_{ik} B_{kj}{}^T \\ &= \sum_{k=1}^{m} B_{ik} B_{jk}. \end{aligned}$$

If $i = j$, then

$$\sum_{k=1}^{m} B_{ik} B_{ik} = \sum_{k=1}^{m} [B_{ik}]^2 = deg(v_i).$$

It follows that the $diag(BB^T) = [deg(v_1), deg(v_2), \dots, deg(v_n)]$. Since BB^T is a symmetric matrix, each of its entry consists of only -1, 0 and $deg(v_i)$ on the diagonals. Thus, it can be expressed as

$$[BB^T]_{ij} = \begin{cases} deg(v_i) & \text{if} \quad i = j, \\ -1 & \text{if} \quad v_i v_j \in E, \\ 0 & \text{if} \quad v_i v_j \notin E. \end{cases} \tag{3.1}$$

Since $[BB^T]_{ij}$ is an $n \times n$ matrix, we can split it into two matrices containing one for the diagonals and the other for the 0's and the -1's such that it can be represented as:

$$\begin{aligned} [BB^T]_{ij} &= \begin{cases} deg(v_i) & \text{if} \quad i = j, \\ -a_{ij} & \text{if} \quad i \neq j \end{cases} \\ &= D - A \\ &= L, \end{aligned} \tag{3.2}$$

where D is the diagonal matrix, A is the adjacency matrix and L is the graph Laplacian. □

Definition 9 (Reduced Laplacian)

Let the cofactor of $[BB^T]_{ij}$ be given as

$$C_{ij} = (-1)^{i+j} \det \hat{L}, \tag{3.3}$$

where $\hat{L}$ is obtained from $[BB^T]_{ij} = L$ by deleting the ith row and the jth column. Then $\hat{L}$ is called the reduced Laplacian.

Let the adjoint of L (transpose of the matrix of cofactors of L), be given as L^{**}. By elementary matrix theory, it is known that

$$[LL^{**}] = \det L.I_n \tag{3.4}$$

where I_n is the $n \times n$ identity matrix of L. Since L is symmetric, it is diagonalizable. Hence, the sum of entries in its every row and column is zero. By the use of elementary row operations, an upper triangular matrix with a column of 0's is created by taking the first column and then replacing with this column plus the sum of all the other columns.

It is a well known fact that the determinant of an upper (or lower) triangular matrix equals the product of its diagonal entries. Hence, with a zero on the diagonal, the determinant of L is zero.

From (3.4), it implies that

$$LL^{**} = \vec{0},$$

that is

$$\text{Nullspace}(L) = \text{kernel}(L).$$

We can also say that

$$\dim[\text{kernel}\,(L)] = 1.$$

Hence, kernel(L) is the span of $(1, 1, \ldots, 1)$ and L^{**} is of the form

$$L^{**} = aJ = a \begin{bmatrix} 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 1 \end{bmatrix}, \tag{3.5}$$

where a is a positive real number.

3.4 Summary

This chapter presented an overview of the basic linear algebra concepts for the structural approach to representation of graphs. The purpose of this study was to describe the various methods of representing connected graphs. We started by giving brief definitions involving groups and fields, and then generalizing it to the Galois fields representation of graphs modulo 2. Based on the analysis of this chapter, the number of possible subgraphs associated with graphs was evaluated as 2^m in Corollary 1, where m is the size of the graph. The matrix representations involving the incidence matrix, adjacency matrix and the graph Laplacian were considered and will be used in the next chapter for evaluating the eigenvalues of graphs.

Chapter 4

Eigenvalues Associated With Graphs

4.1 Introduction

Graphs are better interpreted with linear algebraic properties in terms of their matrix representation compared to the Galois field representation. Of particular interest in this chapter is the eigenvalues of the adjacency and Laplacian matrices, which are useful for obtaining important information about the properties of the graph.

Definition 10 *Let K be an $n \times n$ matrix. The real number λ is called an eigenvalue of K if there exists a non-zero vector v such that*

$$Kv = \lambda v. \tag{4.1}$$

Every non-zero vector v satisfying (4.1) *is called an eigenvector of K associated with the eigenvalue λ.*

The *characteristic polynomial*, $P(K, \lambda)$, of K is defined as

$$P(K, \lambda) = \det(K - \lambda I_n) = (-1)^n[\lambda^n + c_1\lambda^{n-1} + \ldots + c_{n-2}\lambda^2 + c_{n-1}\lambda + c_n].$$

The zeros or roots of the characteristic polynomial are the eigenvalues of K.

Let K be a symmetric matrix. The following simple observations are well known and we will state them without proof.

Observation 1: K is diagonalizable.

Observation 2: Let P be a non-singular matrix and D a diagonal matrix such that $K = P^{-1}DP$. Then

(i) $K^k = P^{-1}D^kP$ for any positive integer k.

(ii)

$$Tr(K^k) = Tr(D^k) = \sum_{i=1}^{n} \lambda_i^k, \tag{4.2}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of K.

4.2 Adjacency Eigenvalues

The *eigenvalues* of G are defined as its eigenvalues of the adjacency matrix, A.

Definition 11 *Let G be a graph of order n. Then the set of eigenvalues, $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_{n-1} \geq \mu_n$ of G together with their multiplicities, is called the spectrum of G, denoted as $\sigma(G)$.*

We state the following useful theorem without proof. Its proof can be found in any standard graph theory text.

Theorem 2 *Let G be a connected graph with adjacency matrix A. Let A^k be the kth power of A and A_{ij}^k be the (i,j)th entry of A^k. Then, A_{ij}^k is the number of distinct walks joining vertices v_i and v_j of length k.*

4.2.1 Properties of the Adjacency Spectrum

For undirected graphs, the adjacency matrices are symmetric, i.e., $A = A^T$, where A^T is the transpose of A. But, A is diagonalizable since it is symmetric. Then, $A = P^{-1}DP$, where

$$D = \begin{bmatrix} \mu_1 & 0 & \dots & 0 \\ 0 & \mu_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mu_n \end{bmatrix}.$$

Fact 1: From (4.2) and the fact that $Tr(A) = 0$, we get

$$Tr(A) = \sum_{i=1}^{n} \mu_i = 0.$$

Fact 2: Recall from Theorem 2 that the A_{ii} entries of A^2_{ij} is the number of $v_i - v_j$ walks of length 2 in G. Hence, $Tr(A^2) = \sum_{i=1}^{n} deg(v_i)$. This, in conjunction with the Handshaking Lemma [28] and (4.2) gives

$$Tr(A^2) = \sum_{i=1}^{n} \mu_i^2 = 2m.$$

Fact 3: Let $tr(G)$ be the number of triangles in G. Consider the diagonal entry a^3_{ii} in A^3. Then by Theorem 2, a^3_{ii} is equal to the number of $v_i - v_i$ walks in G of length 3. Any two $v_i - v_i$ walks v_i, u, w, v_i and v_i, w, u, v_i form one and only one triangle. Furthermore, in $Tr(A^3)$, each triangle is counted three times. Therefore, $Tr(A^3) = 6tr(G)$. This, together with (4.2), gives

$$\sum_{i=1}^{n} \mu_i^3 = Tr(A^3) = 6tr(G).$$

In the theorem that follows, we state and prove some well-known elementary properties of some of the co-efficients of the characteristic polynomial of a graph.

Theorem 3 *Let A be the adjacency matrix of a graph G of size m and eigenvalue μ. Let $P(A, \mu) = (-1)^n[\mu^n + c_1\mu^{n-1} + \ldots + c_{n-2}\mu^2 + c_{n-1}\mu + c_n]$ be its characteristic polynomial. Then*

(i) $c_1 = 0$.

(ii) $-c_2 = m$.

(iii) $c_n = (-1)^n \det A = (-1)^n \prod_{i=1}^n \mu_i$.

Proof

The $i \times i$ principal minor of A is the determinant of the $i \times i$ submatrix obtained from A by deleting $(n-i)$ rows and its corresponding $(n-i)$ columns. It can be observed that the coefficients c_i's in $P(A, \mu)$ are the sums of the $i \times i$ principal minors of A. Hence, we have that

(i) For c_1, the 1×1 submatrices of A are the zeros along the diagonal of A. Thus, $c_1 = 0$.

(ii) Every edge in A can be represented as the non-zero 2×2 principal minor of A, represented as $\det\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. Since each principal minor is unique for each adjacent vertices in G and other principal minors of A for $i = 3, 4, \ldots, n$ are all zeros, then $c_2 = -m$.

(iii) c_n is the $n \times n$ principal minor of A. Since there is only one $n \times n$ submatrix equivalent to A itself, then,

$$c_n = (-1)^n \det A,$$

where $(-1)^n$ can be obtained by expanding the determinant along the diagonal. Recall that

$$P(A, \mu) = \det(A - \mu I) = (\mu_1 - \mu)(\mu_2 - \mu) \ldots (\mu_n - \mu).$$

Since μ is a variable, we can set $\mu = 0$ so that

$$P(A, \mu) = \det(A) = \mu_1 \mu_2 \ldots \mu_n,$$

as desired. $\square$

In the following theorem, we state without proof, a generalised formular by Sachs [29] for the coefficients c_i of the characteristic polynomial of a graph G.

Theorem 4 *Let $\mathcal{H}_i$ denote the collection of i-vertex subgraphs of G whose components are edges or cycles. If $P(G, \lambda) = \sum_i c_i \lambda^{n-1}$ is the characteristic polynomial of G, then $c_i = \sum_{H \in \mathcal{H}_i} (-1)^{c(H)} 2^{y(H)}$, where $c(H)$ is the number of components of H, and $y(H)$ is the number of components that are cycles in H.*

$\square$

4.3 Laplacian Eigenvalues

Apart from the adjacency eigenvalues, $\mu(A)$, the Laplacian eigenvalue, $\lambda(L)$ or simply denoted as λ_i in this thesis, is another important concept in graph theory basically because of its application to the evaluation of the number of spanning trees of graphs which will be discussed in Chapter 5.

4.3.1 Properties of the Laplacian Eigenvalues

Lemma 1 *Let G be a graph. The Laplacian eigenvalues of G, written as $\lambda_i(L)$, has values* $\lambda_i \geq 0$.

Proof

Let $\mathbf{v}$ be a non-zero orthonormal eigenvector of L with eigenvalue λ. Then, $L\mathbf{v} = \lambda\mathbf{v}$ and so

$$\mathbf{v}^T \lambda \mathbf{v} = \mathbf{v}^T L \mathbf{v}, \tag{4.3}$$

which can be written as

$$\lambda(\mathbf{v}^T \mathbf{v}) = \mathbf{v}^T L \mathbf{v}.$$

Since $\mathbf{v}^T \mathbf{v} = 1$,

$$\lambda = \mathbf{v}^T L \mathbf{v}.$$

Recall that from (3.2), the Laplacian can be expressed as:

$$[L]_{ij} = [BB^T]_{ij} = [B^T B]_{ij},$$

where B is the incidence matrix of G which is symmetric. Hence, (4.3) becomes

$$\mathbf{v}^T L \mathbf{v} = \mathbf{v}^T (B^T B) \mathbf{v} = (\mathbf{v}^T B^T)(B\mathbf{v}).$$

It follows that

$$(B\mathbf{v})^T (B\mathbf{v}) = (B\mathbf{v}).(B\mathbf{v}) = \sum_{i=1}^{n} (B\mathbf{v})^2 \geq 0,$$

which proves the lemma. $\square$

Lemma 2 *Zero is an eigenvalue of L.*

Proof

Let $\mathbf{v}$ be the column vector, $\mathbf{v} = [1\,1\,\ldots\,1]^T$. Then $L\mathbf{v} = \mathbf{0} = 0.\mathbf{v}$ since each row associated with vertex v_i has $-d_i$ and sum of $1's$ equivalent to the sum of neighbours of v_i, for $i = 1, 2, \ldots, n$.

Hence, $\lambda = 0$ is an eigenvalue of L. $\square$

Lemma 3 *If G is k-regular, then*

$$(-1)^n \det L = \prod_{i=1}^{n} \lambda_i = \prod_{i=1}^{n} (k - \mu_i).$$

Proof

Recall from Theorem 3 that $(-1)^n \det L = \prod_{i=1}^{n} \lambda_i$. Hence, using the fact that $L = D - A$ from (3.2) in conjunction with Theorem 3 completes the proof. $\square$

4.4 Laplacian Eigenvalues of Some Special Graphs

In this section, we state and prove by derivation some well known Laplacian eigenvalues of the special graphs discussed in Chapter 2.

Complete Graph, K_n

Theorem 5 *The Laplacian eigenvalues of the complete graph K_n are* 0 *and* n*. The multiplicities are 1 and* $n - 1$ *respectively.*

Proof

A complete graph is an $(n-1)$-regular graph whose adjacency matrix is given as:

$$A(K_n) = \begin{bmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{bmatrix}_{n \times n}.$$

The Laplacian can be written as:

$$L(K_n) = \begin{bmatrix} n-1 & 0 & 0 & \dots & 0 \\ 0 & n-1 & 0 & \dots & 0 \\ 0 & 0 & n-1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & n-1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{bmatrix}$$

$$= \begin{bmatrix} n-1 & -1 & -1 & \dots & -1 \\ -1 & n-1 & -1 & \dots & -1 \\ -1 & -1 & n-1 & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \dots & n-1 \end{bmatrix}$$

We can re-write $L(K_n)$ as

$$L(K_n) = \begin{bmatrix} n & 0 & 0 & \dots & 0 \\ 0 & n & 0 & \dots & 0 \\ 0 & 0 & n & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & n \end{bmatrix} - \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 1 \end{bmatrix}.$$

That is

$$L(K_n) = nI_n - J,$$

where J is a matrix whose elements are all ones. Thus,

$$L(K_n) - nI_n = \begin{bmatrix} -1 & -1 & -1 & \dots & -1 \\ -1 & -1 & -1 & \dots & -1 \\ -1 & -1 & -1 & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \dots & -1 \end{bmatrix} = -J.$$

But,

$$\det(L(K_n) - nI_n) = \begin{vmatrix} -1 & -1 & -1 & \dots & -1 \\ -1 & -1 & -1 & \dots & -1 \\ -1 & -1 & -1 & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \dots & -1 \end{vmatrix} = 0.$$

Therefore, n is an eigenvalue of $L(K_n)$. By expressing $(-J)$ in echelon form, we have that:

$$(-J) = \begin{bmatrix} -1 & -1 & -1 & \dots & -1 \\ -1 & -1 & -1 & \dots & -1 \\ -1 & -1 & -1 & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \dots & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

Thus, $(-J)$ is a rank one matrix of eigenvector $\vec{1} = [111\dots1]^T$. This means that it has only one non-zero eigenvalue, n and the remaining eigenvalues as zeros of multiplicity $(n-1)$. The eigenvalues of $L(K_n)$ are reversed when nI is added to $(-J)$. Therefore, we obtain the eigenvalues of $L(K_n)$ as 0 of multiplicity 1 and n of multiplicity $n-1$ which completes the proof. $\square$

Complete Bipartite Graph, $K_{p,q}$

Theorem 6 *Let $K_{p,q}$ be the complete bipartite graph of order n. Then, the Laplacian eigenvalues of $K_{p,q}$ are 0, $p+q$, p and q with respective multiplicities 1,1, $q-1$ and $p-1$.*

Proof

Consider a complete bipartite graph, $K_{p,q}$, of order n. since the adjacency matrix of $K_{p,q}$ is given as:

$$A(K_{p,q}) = \begin{bmatrix} 0 & \cdots & 0 & 1\cdots & 1 \\ \vdots & \ddots & \vdots & \ddots & \\ 0 & \cdots & 0 & 1\cdots & 1 \\ 1 & \cdots & 1 & 0\cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & \cdots & 1 & 0\cdots & 0 \end{bmatrix},$$

its Laplacian can be expressed as:

$$L(K_{p,q}) = \begin{bmatrix} q & \cdots & 0 & -1\cdots & -1 \\ \vdots & \ddots & \vdots & \ddots & \\ 0 & \cdots & q & -1\cdots & -1 \\ -1 & \cdots & -1 & p\cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ -1 & \cdots & -1 & 0\cdots & p \end{bmatrix}_{n\times n}.$$

In block matrix form, we can re-write $L(K_{p,q})$ as

$$L(K_{p,q}) = \left[\begin{array}{c|c} D_q & -J \\ \hline -J^T & D_p \end{array}\right],$$

where D_q is a $(p \times p)$ diagonal matrix containing q in its diagonals and D_p is a $(q \times q)$ diagonal matrix containing p in its diagonals. By reducing the Laplacian to its echelon form, we have that

$$L(K_{p,q}) = \begin{bmatrix} q & \cdots & 0 & -1\cdots & -1 \\ \vdots & \ddots & \vdots & \ddots & \\ 0 & \cdots & q & -1\cdots & -1 \\ -1 & \cdots & -1 & p\cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ -1 & \cdots & -1 & 0\cdots & p \end{bmatrix} \sim \begin{bmatrix} 1 & \cdots & 1 & 0\cdots & -p \\ \vdots & \ddots & \vdots & \ddots & \\ 0 & \cdots & q & p\cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0\cdots & -p \\ 0 & \cdots & 0 & 0\cdots & 0 \end{bmatrix}.$$

Hence, $L(K_{p,q})$ has a rank of $n-1$, implying that there are $n-1$ non zero eigenvalues containing p of multiplicity $q-1$, and q of multiplicity $p-1$ on the diagonals, and one zero eigenvalue. Therefore, the characteristic equation is then expressed as:

$$\begin{aligned} P(L) &= \lambda^n + b_{n-1}\lambda^{n-1} + \ldots + b_1\lambda + b_0 \\ &= \lambda(\lambda-\lambda_1)(\lambda-\lambda_2)\ldots(\lambda-\lambda_{n-1}), \quad \text{since } \lambda_n = 0 \\ &= \lambda^n - (\sum_{i=i}^{n}\lambda_i)\lambda^{n-1} + \ldots + \lambda\prod_{i=1}^{n-1}\lambda_i = 0. \end{aligned}$$

It then follows that

$$\lambda - \lambda^n = 0,$$

and so

$$\lambda = \lambda^n.$$

Hence, n is also a Laplacian eigenvalue of $L(K_{p,q})$ of multiplicity 1. Thus, the eigenvalues of $K_{p,q}$ are 0, $p+q$, p and q with respective multiplicities 1,1, $q-1$ and $p-1$, as required. □

Star graph

The star graph, S_n, is the complete bipartite graph, $K_{1,n-1}$, so we can get the Laplacian eigenvalues directly from the previous section. The adjacency matrix denoted as $A(S_n)$, is given as

$$A(S_n) = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{(n\times n)}.$$

Thus, the Laplacian matrix, denoted $L(S_n)$, can be expressed as follows:

$$L(S_n) = \begin{bmatrix} n-1 & -1 & -1 & \cdots & -1 & -1 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & 0 & \cdots & 1 & 0 \\ -1 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}_{n\times n}.$$

By substituting for $p = 1$ and $q = n - 1$ in the Theorem 6, it suffices to show that the Laplacian eigenvalues of S_n are n and 0 of multiplicity 1 each; and 1 of multiplicity $n - 2$ as required.

4.4.1 Graphs With Teoplitz Laplacian Matrices

In this section, we state and provide proofs for the Laplacian eigenvalues of special graphs whose adjacency and Laplacian matrix are Teoplitz matrices. Teoplitz matrices are matrices named after Otto Teoplitz, a German mathematician, in which each descending diagonal from left to right is constant.

Here, we use the idea by Yueh and Cheng in [39] to derive the Laplacian eigenvalues, λ_i of L, from the eigenvalue problem $L\mathbf{v} = \lambda\mathbf{v}$.

Theorem 7 *The eigenvalues, λ_i, associated with the homogeneous difference equation of a Teoplitz matrix is given as :*

$$\lambda_j = a + 2\sqrt{bc}\Big[\cos(\frac{\pi j}{n})\Big], \quad j = 0, 1, 2, \ldots, n-1 \tag{4.4}$$

Proof

Let G be a graph whose Laplacian can be represented by the Teoplitz matrix as follows:

$$L(G) = \begin{bmatrix} a+\alpha & c & 0 & 0 & \cdots & 0 & \beta \\ b & a & c & 0 & \cdots & 0 & 0 \\ 0 & b & a & c & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a & c \\ \gamma & 0 & 0 & 0 & \cdots & b & a+\delta \end{bmatrix}_{n\times n} . \tag{4.5}$$

Let the eigenvector associated to $L(G)$ be $\vec{v}$ such that $L\vec{v} = \lambda\vec{v}$. Then:

$$\begin{bmatrix} a+\alpha & c & 0 & 0 & \cdots & 0 & \beta \\ b & a & c & 0 & \cdots & 0 & 0 \\ 0 & b & a & c & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a & c \\ \gamma & 0 & 0 & 0 & \cdots & b & a+\delta \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{n-1} \\ v_n \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{n-1} \\ v_n \end{bmatrix} . \tag{4.6}$$

By changing (4.6) to system of equations, we obtain sets of difference equations:

$$(a+\alpha)v_1 + cv_2 + \beta v_n = \lambda v_1 \tag{4.7}$$

$$bv_1 + av_2 + cv_3 = \lambda v_2 \tag{4.8}$$

$$bv_2 + av_3 + cv_4 = \lambda v_3 \tag{4.9}$$

$$bv_{n-2} + av_{n-1} + cv_n = \lambda v_{n-1} \tag{4.10}$$

$$\gamma v_1 + bv_{n-1} + (a+\delta)v_n = \lambda v_n. \tag{4.11}$$

The general form of the difference equations (4.7) to (4.11) above can be expressed as

$$\begin{aligned} &bv_{k-1} + av_k + cv_{k+1} = \lambda v_k + F_k \\ &bv_{k-1} + (a-\lambda)v_k + cv_{k+1} = F_k \\ &cv_{k+2} + (a-\lambda)v_{k+1} + bv_k = F_{k+1}, \end{aligned} \tag{4.12}$$

where $k \in \mathbb{Z}^+$ and $0 < k \leq n$. Therefore, by dividing (4.12) by c, we obtained the following.

$$v_{k+2} + \frac{(a-\lambda)}{c}v_{k+1} + (\frac{b}{c})v_k = \frac{F_{k+1}}{c},$$

where, F_k is the inhomogeneous part which is also a function of k.

By re-writing (4.7) and comparing with (4.12), we obtain the equation for $k=1$ as

$$(a-\lambda)v_1 + cv_2 = -\alpha v_1 - \beta v_n,$$

$$bv_0 = -\beta v_n - \alpha v_1 = 0.$$

Similarly, by re-writing (4.11) and comparing with (4.12), we obtain for $k=n$,

$$bv_{n-1} + (a-\lambda)v_n = -\delta v_n - \gamma v_1,$$

$$cv_{n+1} = -\gamma v_1 - \delta v_n = 0.$$

Therefore, the non-homogeneous part of (4.12) is given as:

$$F_k = \begin{cases} -(\beta v_n + \alpha v_1) & \text{if} \quad k = 1, \\ -(\gamma v_1 + \delta v_n) & \text{if} \quad k = n, \\ 0 & \text{otherwise.} \end{cases}$$

By solving the homogeneous part of (4.12),

$$v_{k+2} + \frac{(a - \lambda)}{c} v_{k+1} + (\frac{b}{c}) v_k = 0, \quad \text{for } k \neq 1,\, n,$$

it follows that the characteristic equation is given as

$$r^2 + (\frac{a - \lambda}{c}) r + \frac{b}{c} = 0. \tag{4.13}$$

Solutions of the difference equations can be of the form:

$$v_k = \begin{cases} t_1 r_1^k + t_2 r_2^k & \text{if} \quad r_1 \neq r_2, \\ t_1 r^k + t_2 k r^k & \text{if} \quad r_1 = r_2 = r, \\ 0 & \text{otherwise,} \end{cases}$$

where t_1 and t_2 are arbitrary constants. But $r_1 \neq r_2$, therefore solutions of v_k are of the form $v_k = t_1 r_1^k + t_2 r_2^k$. Let the initial conditions be $v_0 = v_n = 0$. Then, when $k = 0, v_0 = 0$,

$$v_0 = t_1 r_1^0 + t_2 r_2^0 = 0,$$

so that

$$t_1 = -t_2.$$

When $k = n, v_n = 0$,

$$v_{n+1} = t_1 r_1^n + t_2 r_2^n = 0. \tag{4.14}$$

By substituting t_1 for $-t_2$ in (4.14), it follows that

$$(\frac{r_1}{r_2})^n = 1.$$

Hence, a representation in complex form is

$$\frac{r_1}{r_2} = \exp(i\frac{2\pi j}{n}). \tag{4.15}$$

Moreso, we can write (4.13) as

$$r^2 + (\frac{a-\lambda}{c})r + \frac{b}{c} = (r - r_1)(r - r_2), \tag{4.16}$$

so that

$$r_1 r_2 = \frac{b}{c} \quad \text{and} \quad r_1 + r_2 = -(\frac{a-\lambda}{c}). \tag{4.17}$$

By substituting (4.15) into (4.17), we have that:

$$r_2^2 \exp(\frac{2\pi i j}{n}) = \frac{b}{c},$$

which can also be expressed as

$$\Big[r_2 \exp(\frac{\pi i j}{n})\Big]^2 = \frac{b}{c},$$

so that by taking the square root of both sides and dividing through by $\exp(\frac{\pi i j}{n})$, we obtain

$$r_2 = \sqrt{\frac{b}{c}}\Big[\exp(-\frac{\pi i j}{n})\Big].$$

Thus, by substituting for r_2 in (4.15), we can obtain r_1 as follows:

$$r_1 = r_2 \exp(\frac{2\pi i j}{n}) = \sqrt{\frac{b}{c}}\Big[\exp(\frac{\pi i j}{n})\Big].$$

By substituting for r_1 and r_2 in (4.17), we get

$$\sqrt{\frac{b}{c}}\Big\{\exp(\frac{\pi i j}{n})\Big\} + \sqrt{\frac{b}{c}}\Big\{\exp(-\frac{\pi i j}{n})\Big\} = \frac{\lambda - a}{c},$$

so that

$$\sqrt{bc}\Big[\exp(\frac{\pi ij}{n}) + \exp(-\frac{\pi ij}{n})\Big] + a = \lambda,$$

and

$$\sqrt{bc}\Big[2\cos(\frac{\pi j}{n})\Big] + a = \lambda,$$

which proves the theorem. □

Bojan Mohar in [12] gave the Laplacian eigenvalues of the path, cycle and ladder graphs in terms of the sine of some angles without proof. Here, we derived the Laplacian eigenvalues of the path, cycle and ladder from Theorem 7.

Path Graph, P_n

Theorem 8 *The Laplacian eigenvalues of P_n are:*

$$\lambda_j(P_n) = 4\sin^2(\frac{\pi j}{2n}), \quad j = 0, 1, \ldots, n-1.$$

Proof

The path graph's adjacency and Laplacian matrices are examples of Teoplitz matrices. The adjacency matrix is given as

$$A(P_n) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}_{n\times n},$$

and the Laplacian matrix is

$$L(P_n) = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{bmatrix}_{n\times n}.$$

By comparing with (4.5), we have that $a = 2$, $b = c = -1$, $\alpha = \delta = -1$, and $\beta = \gamma = 0$. Hence, the Laplacian eigenvalues of P_n is given as

$$\begin{aligned} \lambda_j(P_n) &= a + 2\sqrt{bc}\cos(\frac{\pi j}{n}) \\ &= a + 2b\cos(\frac{\pi j}{n}). \end{aligned}$$

Since $b = c$ and $\cos(\frac{\pi j}{n}) = 1 - 2\sin^2(\frac{\pi j}{2n})$, we have that

$$\begin{aligned} \lambda_j(P_n) &= 2 - 2\cos(\frac{j}{n})\pi = 2\left[1 - \cos(\frac{j}{n})\pi\right] \\ &= 4\sin^2(\frac{\pi j}{2n}), \end{aligned} \tag{4.18}$$

for $j = 0, 1, \dots, n-1$, and the theorem is proved. □

Ladder Graph, L_n

Similar to the path graph is the Laplacian of the Ladder graph stated by Bojan in [12]. We derive the Laplacian eigenvalues from Theorem 8 using a lemma by Bojan in [12].

Theorem 9 *The Laplacian eigenvalues of the ladder graph is given as:*

$$\lambda_j(L_n) = 4\sin^2\left(\frac{\pi i}{4}\right) + 4\sin^2\left(\frac{\pi j}{n}\right), \quad \forall\ i, j = 0, 1, \dots, n-1.$$

To derive the Laplacian eigenvalues of L_n, we use a lemma in [12] without proof.

Lemma 4 *If λ_i are the Laplacian eigenvalues of the graph G_i and λ_j are the eigenvalues of the graph G_j, then the eigenvalues of the Cartesian product $G_i \Box G_j$ is given as $\lambda_i + \lambda_j$ for $1 \leq i, j \leq n$.*

Proof of Theorem

The ladder graph adjacency and Laplacian matrices are also examples of Teoplitz matrices. The adjacency matrix of the ladder graph is given as

$$A(L_n) = \left[\begin{array}{c|c} T & I \\ \hline I & T \end{array}\right],$$

where I is an $(\frac{n}{2} \times \frac{n}{2})$ identity matrix and T is a leading $(\frac{n}{2} \times \frac{n}{2})$ diagonal Teoplitz matrix given below.

$$T = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}_{(\frac{n}{2} \times \frac{n}{2})} .$$

The Laplacian matrix can be expressed as below.

$$L(L_n) = \left[\begin{array}{c|c} P & -I \\ \hline -I & P \end{array}\right]_{n \times n},$$

where P can be expressed as:

$$P = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 3 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 3 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 3 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix}_{(\frac{n}{2}\times\frac{n}{2})} .$$

Since L_n is a Cartesian product of P_2 and $P_{\frac{n}{2}}$, it follows from Lemma 4 that

$$\begin{aligned} \lambda_j(L_n) &= \lambda_j(P_2 \Box P_{\frac{n}{2}}) \\ &= \lambda_j(P_2) + \lambda_j(P_{\frac{n}{2}}). \end{aligned}$$

Since $\lambda_i(P_n) = 4\sin^2(\frac{\pi j}{2n})$, we have that

$$\begin{aligned} &= 4\sin^2\left(\frac{\pi i}{2(2)}\right) + 4\sin^2\left(\frac{\pi j}{2(\frac{n}{2})}\right) \\ &= 4\sin^2\left(\frac{\pi i}{4}\right) + 4\sin^2\left(\frac{\pi j}{n}\right), \end{aligned}$$

for all $i, j = 0, 1, \ldots, n-1$, as required. □

Cycle Graph, C_n

Here, we also derived the cycle Laplacian eigenvalues using Theorem 7.

Theorem 10 *The Laplacian eigenvalues of the cycle is given as:*

$$\lambda_j(C_n) = 4\sin^2(\frac{\pi j}{n}), \quad j = 1, \ldots, n$$

The cycle graph has the adjacency matrix as

$$A(C_n) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}_{n\times n},$$

and the Laplacian matrix as

$$L(C_n) = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ -1 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix}_{n\times n} = 2I - A(C_n).$$

By comparing $L(C_n)$ with (4.5), we have that $a = 2$, $b = c = -1$, $\alpha = \delta = 0$ and $\beta = \gamma = -1$. Since a path is obtained from C_n by removing an edge, then the Laplacian eigenvalues of C_n is given as

$$\lambda_j(C_n) = a + 2b\cos(\frac{2j}{n})\pi.$$

From $b = c$, we get

$$\lambda_j(C_n) = 2 - 2\cos(\frac{2j}{n})\pi = 2\Big[1 - \cos(\frac{2j}{n})\pi\Big].$$

It follows using the fact that $\cos(2u) = 1 - 2\sin^2 u$, that

$$\begin{aligned}\lambda_j(C_n) &= 2\Big[1 - (1 - 2\sin^2(\frac{\pi j}{n}))\Big] \\ &= 2\Big[2\sin^2(\frac{\pi j}{n})\Big] \\ &= 4\sin^2(\frac{\pi j}{n}),\end{aligned}$$

for $j = 1, \ldots, n.$, which proves the theorem. □

According to Lee et. al. [26], the wheel and fan graphs are also special cases of the Teoplitz matrix with Laplacian eigenvalues $1 + 4\sin^2(\frac{\pi j}{n})$, and $1 + 4\sin^2(\frac{\pi j}{2n})$ respectively. We state first the Laplacian as a theorem and then give proofs from the general Teoplitz formulae in Theorem 7.

Wheel Graph, W_n

Theorem 11 *The Laplacian eigenvalues of the W_n are* 0*,* $n+1$ *and* $[1 + 4\sin^2(\frac{\pi j}{n})], \quad j = 1, \ldots, n-1.$

The wheel graph, W_n is such that the adjacency matrix can be expressed as

$$A(W_n) = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 1 & \cdots & 0 & 1 \\ 1 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 1 & 0 & \cdots & 1 & 0 \end{bmatrix} = \left[\begin{array}{c|c} 0 & J \\ \hline J^T & A(C_n) \end{array}\right].$$

Here, J is a $1 \times n$ matrix and $A(C_n)$ is the corresponding adjacency matrix of the cycle associated with the wheel graph.

The Laplacian matrix is given as:

$$L(W_n) = \begin{bmatrix} n & -1 & -1 & \cdots & -1 & -1 \\ -1 & 3 & -1 & \cdots & 0 & -1 \\ -1 & -1 & 3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & 3 & \cdots & 3 & -1 \\ -1 & -1 & 0 & \cdots & -1 & 3 \end{bmatrix} = \left[\begin{array}{c|c} n & -J \\ \hline (-J)^T & L(C_n) + I_n \end{array} \right].$$

Recall from Lemma 2 that 0 is an eigenvalue. The remaining eigenvalues, denoted as $\lambda_i, i = 1, \ldots, n$, can be obtained as follows:

$$\lambda_i(W_n) = \lambda_i(L(C_n + I_n)).$$

Here, $a = 3, b = c = -1, \beta = \gamma = -1$ and $\alpha = \delta = 0$. Hence, we have that:

$$\lambda_i(W_n) = a + 2b\cos(\frac{2j}{n})\pi.$$

By substituting for $a = 3$ and $b = c = -1$, we have that

$$\lambda_i(W_n) = 3 - 2\cos(\frac{2\pi j}{n}) = 1 + 2\Big[1 - \cos(\frac{2\pi j}{n})\Big].$$

It follows that

$$\lambda_i(W_n) = 1 + 4\sin^2(\frac{\pi j}{n}).$$

This completes the proof. $\square$

Since $\lambda_i(C_n) = 4\sin^2(\frac{\pi j}{n})$, we can express the Laplacian eigenvalues of the wheel in terms of the cycle graph as follows

$$\lambda_i(W_n) = 1 + \lambda_i(C_n).$$

Fan Graph, F_n

Theorem 12 *The Laplacian eigenvalues of the F_n are* 0*,* $n+1$ *and* $[1 + 4\sin^2(\frac{\pi j}{2n}), \quad j = 1, \ldots, n-1.$

Proof

The fan graph, F_n, has the adjacency matrix:

$$A(F_n) = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 1 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} = \left[\begin{array}{c|c} 0 & J \\ \hline J^T & A(P_n) \end{array}\right],$$

and Laplacian as below.

$$L(F_n) = \begin{bmatrix} n & -1 & -1 & \cdots & -1 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ -1 & -1 & 3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & 3 & \cdots & 3 & -1 \\ -1 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix} = \left[\begin{array}{c|c} n & -J \\ \hline (-J)^T & L(P_n) + I_n \end{array}\right].$$

Similarly with that of the wheel, 0 is an eigenvalue and the remaining corresponding eigenvalues are obtained from:

$$\lambda_j(F_n) = \lambda_j(L(P_n) + I_n).$$

Hence by comparison, $a = 3$, $b = c = -1$, $\alpha = \delta = -1$, $\beta = \gamma = 0$.

$$\lambda_j(F_n) = a + 2b\cos(\frac{\pi j}{2n}).$$

From $a = 3$ and $b = c = -1$, we have that

$$\lambda_j(F_n) = 3 - 2\cos(\frac{\pi j}{2n}) = 1 + 2\Big[1 - \cos(\frac{\pi j}{2n})\Big].$$

Since $1 - \cos(\frac{\pi j}{2n}) = 2\sin^2(\frac{\pi j}{2n})$, it follows that

$$\lambda_j(F_n) = 1 + 4\sin^2(\frac{\pi j}{2n}),$$

for $j = 1, \ldots, n$, which completes the proof. $\square$

Since $\lambda_i(P_n) = 4\sin^2(\frac{\pi j}{2n})$, we can express the Laplacian eigenvalues of the fan as follows

$$\lambda_i(F_n) = 1 + \lambda_i(P_n).$$

4.5 Summary

This chapter gives an in-depth overview of the adjacency and Laplacian eigenvalues of the special graphs described earlier in Chapter 2. The properties of the adjacency spectrum were studied using the characteristic polynomial which contains important information about the graph especially the determinant, trace and eigenvalues of the matrix. Furthermore, based on the relationship between the adjacency and Laplacian matrices described in Chapter 3, the Laplacian eigenvalues of the special graphs were evaluated. These results will be used in Chapter 5 in the determination of their various number of spanning trees. Moreover, the derived generalised eigenvalues of the Teoplitz matrix will be used for the derivation of the adjacency spectrum of the complete graph in the next chapter.

Chapter 5

t-Complete Eigen Sequences

5.1 Introduction

The complete graph is often used in verifying certain new graph theoretical definitions because of its strong connectivity and rigid property. In this chapter, we use the adjacency spectrum of the complete graph to generate trigonometric unit equations involving sum of terms in the form of cosines for n odd, which gives rise to the t- complete eigen sequences, similar to the famous Farey sequence. The main results in this chapter are presented in Theorem 13 and Corollaries 2 and 3. This results have been submitted for publication.

5.2 Formation of the t-Complete Eigen Sequence

Generally, the complete graph is a circulant graph. Circulant matrices are special cases of the Teoplitz matrices whose remaining $n-1$ rows are cyclic permutations of the first row.

Lemma 5 *(Jessop, [21]) Let*

$$A = \begin{bmatrix} c_0 & c_1 & c_2 & \dots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \dots & c_{n-3} & c_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_2 & c_3 & c_4 & \dots & c_0 & c_1 \\ c_1 & c_2 & c_3 & \dots & c_{n-1} & c_0 \end{bmatrix}$$

be an $n \times n$ circulant matrix, such that $AV_j = \lambda V_j$, where λ_j and V_j are the eigenvalue and eigenvector respectively. Then, the eigenvectors of A are given as:

$$V_j = [1, \rho_j, \rho_j^2, \dots, \rho_j^{n-1}]^T,\ j = 0, 1, \dots, n-1,$$

where $\rho_j = \exp\left(\frac{2\pi i j}{n}\right)$ are the nth roots of unity and $i = \sqrt{-1}$.
The corresponding eigenvalues are given as:

$$\lambda_j = c_0 + c_1\rho_j + c_2\rho_j^2 + \dots + c_{n-1}\rho_j^{n-1}.$$

Lemma 6 *Let the adjacency matrix of the complete graph, K_n on n vertices be given as:*

$$A(K_n) = \begin{bmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{bmatrix}_{n \times n} .$$

Then, its eigenvalues for all j, where $j = 0, 1, \dots, n-1$, are:

$$
\begin{aligned}
\lambda_j &= e^{\frac{2\pi ij}{n}} + e^{\frac{4\pi ij}{n}} + \ldots + e^{\frac{2(n-1)\pi ij}{n}} \\
&= \left[\cos\left(\frac{2\pi j}{n}\right) + i\sin\left(\frac{2\pi j}{n}\right)\right] + \left[\cos\left(\frac{4\pi j}{n}\right) + i\sin\left(\frac{4\pi j}{n}\right)\right] + \ldots \\
&+ \left[\cos\left(\frac{2(n-1)\pi j}{n}\right) + i\sin\left(\frac{2(n-1)\pi j}{n}\right)\right] \\
&= \sum_{k=1}^{n-1}\cos\left(\frac{2\pi jk}{n}\right) + i\sum_{k=1}^{n-1}\sin\left(\frac{2\pi jk}{n}\right).
\end{aligned}
$$

The proofs for Lemmas 5 and 6 can be found in (Jessop [21]).

Using the above lemmas and the fact that the eigenvalues of the adjacency matrix associated with the complete graph are $n-1$ of multiplicity 1 and -1 of multiplicity $n-1$, we have the following theorem.

Theorem 13 *Let $t \in \mathbb{Z}^+$. Then*

$$
\begin{aligned}
&2\left[\cos\left(\frac{\pi}{2t+1}\right) + \cos\left(\frac{3\pi}{2t+1}\right) + \ldots + \cos\left(\frac{(2t-1)\pi}{2t+1}\right)\right] \\
&= 2\sum_{r=1}^{t}\cos\left(\frac{\pi(2t-2r+1)}{2t-1}\right) = 1.
\end{aligned}
\tag{5.1}
$$

Proof

For $j = 0$, Lemma 6 yields the eigenvalue $n-1$. Thus for $j \neq 0$, the eigenvalues are -1, i.e., for $j \neq 0$,

$$
\lambda_j = \sum_{k=1}^{n-1}\cos\left(\frac{2\pi jk}{n}\right) + i\sum_{k=1}^{n-1}\sin\left(\frac{2\pi jk}{n}\right) = -1.
$$

Considering $n = 2t + 1$ for $j \neq 0$, we have that:

$$\begin{aligned}
\sum_{k=1}^{n-1} \sin\left(\frac{2\pi jk}{n}\right) &= \sum_{k=1}^{2t} \sin\left(\frac{2\pi jk}{2t+1}\right) \\
&= \sin\left(\frac{2\pi j}{2t+1}\right) + \sin\left(\frac{4\pi j}{2t+1}\right) + \ldots + \sin\left(\frac{4t\pi j}{2t+1}\right) \\
&= \left[\sin\left(\frac{2\pi j}{2t+1}\right) + \ldots + \sin\left(\frac{2t\pi j}{2t+1}\right)\right] + \left[\sin\left(\frac{2(t+1)\pi j}{2t+1}\right) + \ldots + \sin\left(\frac{4t\pi j}{2t+1}\right)\right] \\
&= [A] + [B],
\end{aligned}$$

where A has the first t terms and B the next t terms. By adding the first term of A and the last term of B, it follows that:

$$\begin{aligned}
&\sin\left(\frac{2\pi}{2t+1}\right) + \sin\left(\frac{4\pi t}{2t+1}\right) \\
&= \sin\left(\frac{(2t+1)\pi}{2t+1} - \frac{(2t-1)\pi}{2t+1}\right) + \sin\left(\frac{(2t+1)\pi}{2t+1} + \frac{(2t-1)\pi}{2t+1}\right) \\
&= \sin\left[\frac{(2t-1)\pi}{2t+1}\right] - \sin\left[\frac{(2t-1)\pi}{2t+1}\right] = 0.
\end{aligned}$$

Generally, adding the rth term of A and the $t-(r-1)th$ term of B, where $r = 1, 2, \ldots t$, yields:

$$\begin{aligned}
&\sin\left(\frac{2\pi r}{2t+1}\right) + \sin\left(\frac{2\pi(2t-r+1)}{2t+1}\right) \\
&= \sin\left(\frac{(2t+1)\pi}{2t+1} - \frac{(2t-2r+1)\pi}{2t+1}\right) + \sin\left(\frac{(2t+1)\pi}{2t+1} - \frac{(2t-2r+1)\pi}{2t+1}\right) \\
&= \sin\left[\frac{(2t-2r+1)\pi}{2t+1}\right] - \sin\left[\frac{(2t-2r+1)\pi}{2t+1}\right] = 0; \quad r = 1, 2, \ldots t.
\end{aligned}$$

Therefore, for $j \neq 0$,

$$\sum_{k=1}^{n-1} \sin\left(\frac{2\pi jk}{n}\right) = 0,$$

and then,

$$\lambda_j = \sum_{k=1}^{2t-1} \cos\left(\frac{2\pi jk}{n}\right) + i\sum_{k=1}^{n-1} \sin\left(\frac{2\pi jk}{n}\right) = \sum_{k=1}^{2t-1} \cos\left(\frac{2\pi jk}{2t}\right) = -1.$$

Now,

$$\begin{aligned}
\sum_{k=1}^{2t-1} \cos\left(\frac{2\pi jk}{2t}\right) &= \left[\cos\left(\frac{2\pi}{2t+1}\right) + \cos\left(\frac{4\pi}{2t+1}\right) + \ldots + \cos\left(\frac{2\pi t}{2t+1}\right)\right] \\
&= +\left[\cos\left(\frac{2\pi(t+1)}{2t+1}\right) + \cos\left(\frac{2\pi(t+2)}{2t+1}\right) + \ldots + \cos\left(\frac{4\pi t}{2t+1}\right)\right] \\
&= A + B,
\end{aligned}$$

where A has the first t terms and B the next t terms. By adding the first term of A and the last term of B, we have:

$$\begin{aligned}
&\cos\left(\frac{2\pi}{2t+1}\right) + \cos\left(\frac{4\pi t}{2t+1}\right) \\
&= \cos\left(\frac{(2t+1)\pi}{2t+1} - \frac{(2t-1)\pi}{2t+1}\right) + \cos\left(\frac{(2t+1)\pi}{2t+1} + \frac{(2t-1)\pi}{2t+1}\right) \\
&= 2\cos\left[\frac{(2t+1)\pi}{2t+1}\right]\cos\left[\frac{(2t-1)\pi}{2t+1}\right] \\
&= -2\cos\left[\frac{(2t-1)\pi}{2t+1}\right].
\end{aligned}$$

By adding the t-th term of A and the first term of B yield

$$\begin{aligned}
&\cos\left(\frac{2\pi t}{2t+1}\right) + \cos\left(\frac{2\pi(t+1)}{2t+1}\right) \\
&= \cos\left(\frac{(2t+1)\pi}{2t+1} - \frac{\pi}{2t+1}\right) + \cos\left(\frac{(2t+1)\pi}{2t+1} + \frac{\pi}{2t+1}\right) \\
&= -2\cos\left[\frac{\pi}{2t+1}\right].
\end{aligned}$$

Adding the second term of A and the second to the last term of B yields:

$$\begin{aligned}
&\cos\left(\frac{2\pi(2)}{2t+1}\right)+\cos\left(\frac{2\pi(2t-2+1)}{2t+1}\right)\\
&=\cos\left[\frac{4t\pi}{2t+1}\right]+\cos\left[\frac{(2t-1)2\pi}{2t+1}\right]\\
&=\cos\left(\frac{(2t+1)\pi}{2t+1}-\frac{(2t-3)\pi}{2t+1}\right)+\cos\left(\frac{(2t+1)\pi}{2t+1}+\frac{(2t-3)\pi}{2t+1}\right)\\
&=-2\cos\left[\frac{(2t-3)\pi}{2t+1}\right].
\end{aligned}$$

Generally, adding the rth term of A and the $t-(r-1)$th term of B, for $r=1,2,\ldots t$, yields

$$\begin{aligned}
&\cos\left(\frac{2\pi r}{2t+1}\right)+\cos\left(\frac{2\pi(2t-r+1)}{2t+1}\right)\\
&=\cos\left(\frac{(2t+1)\pi}{2t+1}-\frac{(2t-2r+1)\pi}{2t+1}\right)+\cos\left(\frac{(2t+1)\pi}{2t+1}+\frac{(2t-2r+1)\pi}{2t+1}\right)\\
&=-2\cos\pi\cos\left[\frac{(2t-2r+1)\pi}{2t+1}\right]\\
&=-2\cos\left[\frac{(2t-2r+1)\pi}{2t+1}\right],\quad r=1,2,\ldots t.
\end{aligned}$$

Thus,

$$2\sum_{r=1}^{t}\cos\left[\frac{(2t-2r+1)\pi}{2t+1}\right]=1,\quad t\in\mathbb{Z}^+.$$

This implies that

$$2\left[\cos\left(\frac{\pi}{2t+1}\right)+\cos\left(\frac{3\pi}{2t+1}\right)+\cos\left(\frac{5\pi}{2t+1}\right)+\ldots+\cos\left(\frac{(2t-1)\pi}{2t+1}\right)\right]=1,\quad t\in\mathbb{Z}^+,$$

which is equivalently

$$\left[\cos\left(\frac{\pi}{2t+1}\right)+\cos\left(\frac{3\pi}{2t+1}\right)+\cos\left(\frac{5\pi}{2t+1}\right)+\ldots+\cos\left(\frac{(2t-1)\pi}{2t+1}\right)\right]=\frac{1}{2},\quad t\in\mathbb{Z}^+,$$

and the theorem is proven. $\square$

From Theorem 13, we can generate the following trigonometric unit-equations having t terms involving $\cos(\frac{a}{2t+1})\pi$, where $\frac{a}{2t+1}$ involves all "odd" rational numbers in the interval $(0,1)$, i.e., a is also odd. There will be exactly t such odd rational numbers forming a t-sequence:

$$t=1; \qquad 2\Big[\cos(\frac{\pi}{3})\Big] = 1 \Longrightarrow d = \frac{1}{3}$$

$$t=2; \qquad 2\Big[\cos(\frac{\pi}{5}) + \cos(\frac{3\pi}{5})\Big] = 1 \Longrightarrow d = \frac{1}{5}, \frac{3}{5}$$

$$t=3; \qquad 2\Big[\cos(\frac{\pi}{7}) + \cos(\frac{3\pi}{7}) + \cos(\frac{5\pi}{7})\Big] = 1 \Longrightarrow d = \frac{1}{7}, \frac{3}{7}, \frac{5}{7}$$

$$t=4; \qquad 2\Big[\cos(\frac{\pi}{9}) + \cos(\frac{3\pi}{9}) + \cos(\frac{5\pi}{9}) + \cos(\frac{7\pi}{9})\Big] = 1 \Longrightarrow d = \frac{1}{9}, \frac{3}{9}, \frac{5}{9}, \frac{7}{9},$$

where d is the sequence consisting of the odd rational numbers between 0 and 1. For each t, we therefore associate the t-sequence of odd rational terms, each term belonging to the interval $(0,1)$ and having the form $\frac{a}{2t+1}$, containing t-terms:

$$\frac{1}{2t+1}, \frac{3}{2t+1}, \frac{5}{2t+1}, \frac{7}{2t+1}, \ldots, \frac{2t-1}{2t+1}; \quad t \in \mathbb{Z}^+.$$

This sequence has similarities to the **Farey sequence**. The Farey sequence of order n is the sequence, FY_n, of completely reduced fractions between 0 and 1, which; when in lowest terms, have denominators less than or equal to n, arranged in order of increasing size [18].

The sequence derived from the eigenvalues of the complete graph, K_n is called the **t-Complete-Eigen Sequence**.

Corollary 2 *The sum of the terms of the t-complete eigen sequence:*

$$\frac{1}{2t+1}, \frac{3}{2t+1}, \frac{5}{2t+1}, \frac{7}{2t+1}, \ldots, \frac{2t-1}{2t+1},$$

is given by:

$$\sum_{r=1}^{t} \frac{2t-2r+1}{2t+1} = \frac{t^2}{2t+1}; \quad t \in \mathbb{Z}^+.$$

Proof

By writing each t-sequence down twice, with the second reversed, we obtain:

$$S : \frac{1}{2t+1}, \frac{3}{2t+1}, \frac{5}{2t+1}, \frac{7}{2t+1}, \ldots, \frac{2t-1}{2t+1}$$

$$S' : \frac{2t-1}{2t+1}, \frac{2t-3}{2t+1}, \frac{2t-5}{2t+1}, \frac{2t-7}{2t+1}, \ldots, \frac{1}{2t+1}$$

Adding corresponding terms, we obtain a double sum of the terms of the sequence as:

$$\sum_{r=1}^{t} \frac{2t-2r+1}{2t+1} = \frac{1}{2}\left(\frac{2t^2}{2t+1}\right) = \frac{t^2}{2t+1}; \quad t \in \mathbb{Z}^+,$$

which proves the corollary. □

5.3 t-Complete-Eigen Ratio

Let t_{K_n} be the ratio formed by dividing each term of the t-complete eigen sequence by t to obtain a new sequence below

$$\frac{1}{t(2t+1)}, \frac{3}{t(2t+1)}, \frac{5}{t(2t+1)}, \frac{7}{t(2t+1)}, \ldots, \frac{2t-1}{t(2t+1)}; \quad t \in \mathbb{Z}^+.$$

Then,

$$\sum_{r=1}^{t} \frac{2t-2r+1}{t(2t+1)} = \frac{1}{t}\left(\frac{t^2}{2t+1}\right) = \frac{t}{2t+1}; \quad t \in \mathbb{Z}^+,$$

which converges to $\frac{1}{2}$, as t increases. Hence, $\frac{t^2}{t(2t+1)}$ is the *t-Complete-Eigen Ratio* of $\frac{t^2}{(2t+1)}$, denoted as t_{K_n}, which converges to the constant $\frac{1}{2}$.

Corollary 3 *Let t_{K_n} be the t-Complete-Eigen Ratio. Then, the asymptotic value of t_{K_n} as $t \to \infty$ is given as*

$$t_{K_n}^{\infty} = \lim_{t\to\infty} \sum_{r=1}^{t} \frac{2t-2r+1}{t(2t+1)} = \lim_{t\to\infty} \frac{t^2}{t(2t+1)} = \frac{1}{2}.$$

Similarly from Theorem 13,

$$t_{K_n}^{\infty} = \sum_{r=1}^{t} \cos\left[\frac{(2t-2r+1)\pi}{2t+1}\right] = \frac{1}{2}.$$

Thus,

$$t_{K_n}^{\infty} = \lim_{t\to\infty} \sum_{r=1}^{t} \frac{2t-2r+1}{t(2t+1)} = \sum_{r=1}^{t} \cos\left[\frac{(2t-2r+1)\pi}{2t+1}\right] = \frac{1}{2}.$$

5.4 Total t-Complete Eigen Sequence

Given the sequence

$$S : \frac{1}{2t+1}, \frac{3}{2t+1}, \frac{5}{2t+1}, \frac{7}{2t+1}, \dots, \frac{2t-1}{2t+1},$$

we associate the *mirror image unit-pair* partner belonging to the *unit t-complete eigen sequence*:

$$S'' : \frac{2t}{2t+1}, \frac{2t-2}{2t+1}, \frac{2t-4}{2t+1}, \frac{2t-6}{2t+1}, \dots, \frac{2}{2t+1},$$

of the form $\frac{c}{2t+1}$, where c is even. The sum of the corresponding pairs of terms of S and S'' yields:

$$\begin{aligned} \frac{2t-2r+1}{2t+1} + \frac{2r}{2t+1} &= \frac{2t-2r+1+2r}{2t+1} \\ &= \frac{2t+1}{2t+1} = 1 \ (r = 1, 2, 3, \dots, t). \end{aligned}$$

Thus, $(\frac{2t-2r+1}{2t+1}; \frac{2r}{2t+1})$ are unit mirror pairs. The union of S and S'' yields the **total t-complete eigen sequence** as follows:

$$S \cup S'' = \frac{1}{2t+1}, \frac{2}{2t+1}, \frac{3}{2t+1}, \frac{4}{2t+1}, \ldots, \frac{2t-1}{2t+1}, \frac{2t}{2t+1},$$

so that

$$\sum_{k=1}^{2t} \frac{k}{2t+1} = t.$$

By joining neighbours and unit mirror pairs, a diagram for $t = 3$ similar to the Farey sequence diagram is created in Figure $\boxed{5.1}$.

The average degree of the vertices of the complete graph on $n = 2t + 1$ vertices

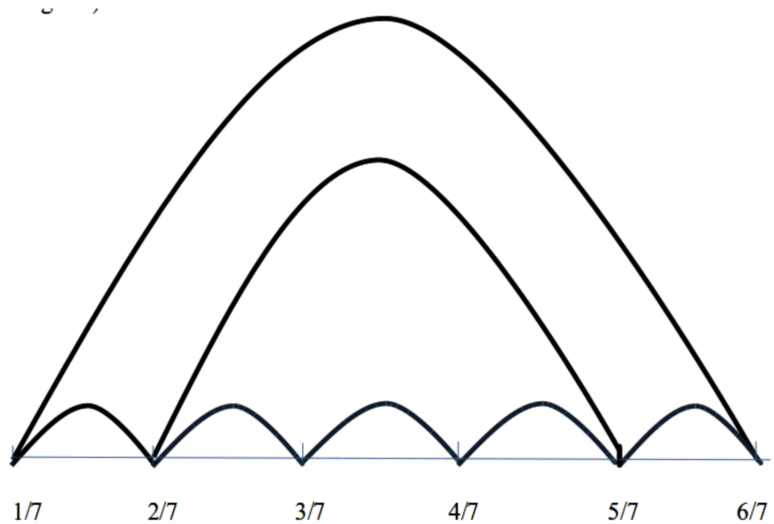


Figure 5.1: Diagram for total t-complete eigen sequence for $t = 3$.

is $n - 1 = (2t + 1) - 1 = 2t$. The first t-complete eigen sequence arises when $n = 3$ and $t = 1$.

5.5 Summary

In this chapter, the trigonometric unit equations as regards adjacency spectrum of K_n were formed, such that for each parameter t considered, a t-complete eigen sequence of odd rational terms were generated for the interval $(0, 1)$ and having the form $\frac{a}{2t+1}$. We also showed that the sum of the terms of this sequence is $\frac{t}{2t+1}$, and it converges to $\frac{1}{2}$ as t goes to infinity. Moreover, by joining the unit mirror pairs, the diagram similar to the Farey sequence is obtained for $t = 3$.

Chapter 6

Spanning Trees of Special Graphs

6.1 Introduction

The aim of this chapter is to study the methods of evaluating the number of spanning trees of the special graphs discussed earlier in Chapter 2. The number of spanning trees of the connected simple graphs considered in this dissertation are well known and are not original. Methods to be considered include the use of Kirchoff's Matrix Tree Theorem [24] and the Kelman and Chelnokov formular [23] based on the evaluated Laplacian eigenvalues in Chapter 4 and special numbers involving Fibonacci and Lucas numbers.

6.2 Spanning Trees Using Eigenvalues

We consider the matrix tree theorem useful for the evaluation of the number of spanning trees of any given graph.

Theorem 14 (Kirchoff's Matrix Tree Theorem) *([24], [23]) Let $L(G)$ be the Laplacian matrix of a connected simple graph, G, of order n. Then the num-*

ber of spanning trees in G, denoted as $t(G)$, is given as:

$$t(G) = \frac{1}{n} \prod_{k=1}^{n-1} \lambda_k. \tag{6.1}$$

Proof

Let the n-eigenvalues of the Laplacian, L, of G be $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{n-1} \geq \lambda_n = 0$. Recall that from (3.2) and (3.3) respectively,

$$L = BB^T \quad \text{and} \quad C_{kk} = \det(\hat{L}).$$

This implies that the entries of L were obtained from the correponding dot products of B and B^T. Let B_k be a $(n-1) \times m$ submatrix B_k obtained from B by deleting its kth row. Recall from (3.5) that

$$Adjoint\ of\ L = aJ,$$

where $a \in \mathbb{R}$ and J is a matrix whose entries are 1. Then, the fact that the cofactors of L are all equal implies that:

$$C_{kk} = \det(\hat{L}) = \det(B_k B_k^T). \tag{6.2}$$

Let B_k^s be obtained as all possible $(n-1) \times (n-1)$ sets, s of $(n-1)$-combinations submatrices of B_k using the Cauchy-Binet formula [34], we have

$$\begin{aligned} C_{kk} &= \det(B_k B_k^T) \\ &= \sum_s \det(B_k^s) \times \det(B_k^s)^T. \end{aligned} \tag{6.3}$$

But the determinant of any $(n-1)\times(n-1)$ submatrix of B is either ± 1 or zero since each row of B contains $+1, -1$ and zero and sums up to zero. Hence,

$$\begin{aligned} C_{kk} &= \det(B_k B_k^T) \\ &= \sum_s \det(B_k^s) \times \det(B_k^s)^T \\ &= \sum_s \Big[\det(B_k^s)\Big]^2, \text{ since } \det(B_k^s) = \det(B_k^s)^T \\ &= \sum_s \Big[\det(B_k^s) = 0\Big] + \sum_s \Big[\det(B_k^s) = (\pm 1)^2\Big]. \end{aligned}$$

This implies that C_{kk} is the number of invertible $(n-1)\times(n-1)$ submatrices of B_k^s. Since $C_{kk} = \det \hat{L}$, then the number of spanning trees of G, $t(G)$, is C_{kk}, i.e.,

$$C_{kk} = t(G) = \det(\hat{L}). \tag{6.4}$$

However, since all the cofactors are equal to the determinant of the reduced Laplacian, $\hat{L}$ and the coefficients, b_k in the characteristic polynomial also equals the sum of the cofactors of order k, then we have

$$\begin{aligned} \lambda(L) &= (\lambda-\lambda_1)(\lambda-\lambda_2)\ldots(\lambda-\lambda_n) \\ &= \lambda(\lambda-\lambda_1)(\lambda-\lambda_2)\ldots(\lambda-\lambda_{n-1}) \quad \text{since } \lambda_n = 0 \\ &= \lambda^n + b_{n-1}\lambda^{n-1} + \ldots + b_1\lambda + b_0, \end{aligned}$$

where,

$$b_k = \sum_k C_{kk}.$$

Hence,

$$\begin{aligned} b_k &= t(G) + t(G) + \ldots + t(G) \quad (n \text{ times}) \\ &= n\ t(G). \end{aligned}$$

From Theorem 3, a generalisation for a Laplacian gives:

$$\prod_{k=1}^{n-1} \lambda_k = \sum_{k=1}^{n} \det \hat{L_k} = b_k.$$

Thus, the number of spanning trees can be expressed as:

$$\begin{aligned} t(G) &= \frac{1}{n} b_k \\ &= \frac{1}{n} \prod_{k=1}^{n-1} \lambda_k, \end{aligned}$$

which completes the proof of the theorem. □

6.3 Number of Spanning Trees of Some Classes of Graphs

Complete Graph, K_n

This was first proved by Arthur Cayley in 1889 [2]. From Theorem 5, the non-zero Laplacian eigenvalue of a complete graph is n of multiplicity $n-1$. Hence, the number of spanning trees in a complete graph, $t(K_n)$, is given as:

$$\begin{aligned} t(K_n) &= \frac{1}{n} \prod_{i=1}^{n-1} \lambda_i \\ &= \frac{1}{n} \overbrace{[n.n.n \ldots n]}^{(n-1)\text{times}} \\ &= \frac{1}{n} [n^{n-1}] \\ &= n^{n-2}. \end{aligned}$$

Complete Bipartite Graph, $K_{p,q}$

In this section, we give an interesting proof using [24] based on linear algebraic properties.

Theorem 15 *[6] The number of spanning trees of $K_{p,q}$ is given as*

$$\begin{aligned} t(K_{p,q}) &= det(L^{**}) = 1.p^{q-1}.q^{p-1} \\ &= p^{q-1}q^{p-1}. \end{aligned}$$

Proof

Recall that from (4.4)

$$L(K_{p,q}) = \left[\begin{array}{c|c} D_q & -J \\ \hline -J^T & D_p \end{array}\right],$$

where D_q is a $(p \times p)$ diagonal matrix containing q in its diagonals and D_p is a $(q \times q)$ diagonal matrix containing p in its diagonals. Let the reduced Laplacian matrix, L^*, obtained by removing the first row and first column, be

$$L^*(K_{p,q}) = \left[\begin{array}{c|c} D_q^* & -J^* \\ \hline (-J^*)^T & D_p^* \end{array}\right].$$

Here, $D_q^* = pI_{D_q*}$, I_{D_q*} is an $(q-1) \times (q-1)$ identity matrix and $(-J)^*$ is an $(p-1) \times (q-1)$ matrix. To obtain the number of spanning trees, our aim is to find the determinant of the reduced Laplacian matrix.

Let P be given as

$$P = \begin{bmatrix} I & O \\ (\frac{1}{p})I & I \end{bmatrix},$$

so that

$$PL^* = \begin{bmatrix} I & O \\ (\frac{1}{p})I & I \end{bmatrix} \begin{bmatrix} D_q^* & (-J)^* \\ (-J^*)^T & D_p^* \end{bmatrix} = \begin{bmatrix} D_q^* & -J^* \\ O & D_p^* \end{bmatrix},$$

where,

$$D_p^* = \begin{bmatrix} q-1/p & -1/p & \dots & -1/p \\ -1/p & q-1/p & \dots & -1/p \\ \vdots & \vdots & \ddots & \vdots \\ -1/p & -1/p & \dots & -1/p \end{bmatrix}_{p\times p}.$$

In echelon form, D_p^* becomes

$$D_p^{**} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 0 & q & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & q \end{bmatrix}_{p\times p}.$$

Let

$$L^{**} = PL^* = \begin{bmatrix} D_q^* & -J^* \\ O & D_p^{**} \end{bmatrix}.$$

By taking λ to be an eigenvalue, we have that

$$\det(L^{**} - \lambda I_{p+q}) = 0.$$

By substututing for L^{**} and I_{p+q} in $\det(L^{**} - \lambda I_{p+q}) = 0$, it follows that:

$$\det(L^{**} - \lambda I_{p+q}) = \det\Big[\begin{pmatrix} D_q^* & -J^* \\ O & D_p^{**} \end{pmatrix} - \lambda I_{p+q} \Big] = 0$$

$$= \det\Big[\begin{pmatrix} D_q^* & -J^* \\ O & D_p^{**} \end{pmatrix} - \begin{pmatrix} D_\lambda^* & O \\ O & D_\lambda^* \end{pmatrix} \Big] = 0$$

$$= \det\Big[\begin{pmatrix} D_q^* - \lambda I_{p-1} & -J^* \\ O & D_p^{**} - \lambda I_{q-1} \end{pmatrix} \Big] = 0$$

$$= \det(D_q^* - \lambda I_{p-1}) \times \det(D_p^{**} - \lambda I_{q-1}) = 0.$$

This implies that if λ is an eigenvalue of D_q^* or D_p^{**}, then $\det(D_q^* - \lambda I_{p-1}) = 0$ or $\det(D_p^{**} - \lambda I_{q-1}) = 0$. Hence, λ is an eigenvalue of L^{**}. But the eigenvalue of D_q^* is p of multiplicity $(q-1)$ and the eigenvalue of D_p^{**} is q of multiplicity $p-1$. Hence, the eigenvalues of L^{**} are $1, p^{q-1}$, and q^{p-1}. This theorem follows by an application of the Kirchoff's Matrix Tree Theorem. $\square$

Path Graph, P_n

The path graph P_n has the adjacency matrix and Laplacian given as below whose number of spanning trees is well known to be 1. Here, we give an interesting proof using difference equations.

$$A(P_n) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}_{n\times n}.$$

The Laplacian matrix can be given as:

$$L(P_n) = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{bmatrix}_{n\times n}.$$

By removing the first row and its corresponding column, we obtain the reduced Laplacian matrix as

$$\hat{L}(P_n) = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{bmatrix}_{(n-1)\times(n-1)}.$$

When $n = 1$,

$$\det \hat{L}(P_1) = 1.$$

When $n = 2$,

$$\det \hat{L}(P_2) = |1| = 1.$$

Similarly, when $n = 3$,

$$\det \hat{L}(P_3) = \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} = 1.$$

Let $k < n$. By mathematical induction, and assuming it is true for any $k < n$, by expanding $\det \hat{L}(P_n)$ along the first row, we have that

$$\det \hat{L}(P_n) = 2 \det \hat{L}(P_{n-1}) + \det \left[\begin{array}{c|c} -1 & T \\ \hline \mathbf{0} & \hat{L}(P_{n-2}) \end{array} \right],$$

where $T = [-1, 0, 0 \dots, 0]$ and $\mathbf{0}$ is the zero vector.

Hence,

$$\det \hat{L}(P_n) = 2 \det \hat{L}(P_{n-1}) - \det \hat{L}(P_{n-2}).$$

Let $\det \hat{L}(P_k)$ be F_k for $k < n$.

Then,

$$F_k = 2F_{k-1} - F_{k-2}.$$

This can be re-written as:

$$F_{k+2} - 2F_{k+1} + F_k = 0.$$

The solution to the difference equation is of the form $F_k = \alpha x^k$, so that the associated characteristic equation is given as

$$x^2 - 2x + 1 = 0,$$

and having solutions $x = 1$(twice).

Therefore, $F_k = \alpha$(constant). With the initial conditions of the difference equation being $F_1 = F_2 = 1$, then $\det \hat{L}(P_n) = 1$. Hence, the number of spanning trees of the path graph P_n is always 1. Thus,

$$\begin{aligned} t(P_n) &= \frac{1}{n}\prod_{i=1}^{n-1}\Big[\lambda_j(P_n)\Big]_{j=1,\ldots,n-1} \\ &= \frac{1}{n}\prod_{i=1}^{n-1}\Big[4\sin^2(\frac{\pi j}{2n})\Big]_{j=1,\ldots,n-1} = 1, \end{aligned}$$

where $\lambda_j(P_n) = 0$ for $j = n$.

Ladder Graph, L_n

The ladder graph, L_n, on n vertices have the Laplacian matrix $L(L_n)$. Let the vertices along the top of L_n be $(u_1, u_2, \ldots, u_t)$ and vertices along the bottom of L_n be $(v_1, v_2, \ldots, v_t)$ for $t = \frac{n}{2}$. We derive the formula for the number of spanning trees of the ladder as stated in [31]. We first give a lemma which will help us in proving the theorem.

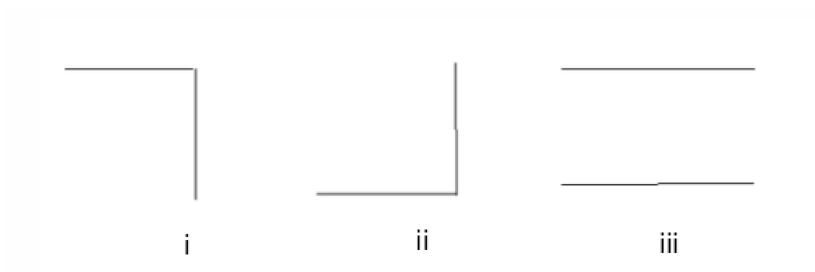
Lemma 7 *Every spanning tree of L_n that contain both $\{u_i, u_{i+1}\}$ and $\{v_i, v_{i+1}\}$ does not contain $\{u_{i+1}, v_{i+1}\}$, for $i = 2, \ldots, \frac{n}{2}$. Let L_{n-2} be the ladder graph* *obtained by removing any two opposite end vertices of L_n. Then, by adding any of $(i)-(iii)$ below to any spanning tree of L_{n-2}, we create a spanning tree of L_n.*

(i) A graph consisting of the edges $\{u_{i-1}, u_i\}$ and $\{u_i, v_i\}$.

(ii) A graph consisting of the edges $\{u_i, v_i\}$ and $\{v_{i-1}, v_i\}$.

(iii) A graph consisting of the edges $\{u_{i-1}, u_i\}$ and $\{v_{i-1}.v_i\}$.

The graphs (i) to (iii) are shown in the graphs below.



. $\square$

Theorem 16 *The number of spanning trees of the Ladder graph, L_n, is given as:*

$$t(L_n) = \frac{\sqrt{3}}{6}\Big[(2+\sqrt{3})^n - (2-\sqrt{3})^n\Big],$$

$$= 4\prod_{i,j=1}^{n-1}\Big[\sin^2(\frac{\pi i}{4}) + \sin^2(\frac{\pi j}{n})\Big].$$

Proof

Let

$$L(L_n) = \begin{bmatrix} 2 & \cdots & 0 & -1\cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \\ 0 & \cdots & 2 & 0\cdots & -1 \\ -1 & \cdots & 0 & 2\cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & -1 & 0\cdots & 2 \end{bmatrix}_{n\times n} = \left[\begin{array}{c|c} P & -I \\ \hline -I & P \end{array}\right],$$

where I is the identity matrix, and P can be expressed as

$$P = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 3 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 3 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix}_{(\frac{n}{2}\times\frac{n}{2})}.$$

By deleting the last row and its corresponding last column of $L(L_n)$, we obtain the reduced Laplacian below

$$\hat{L}(L_n) = \begin{bmatrix} 2 & \cdots & 0 & -1\cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \\ 0 & \cdots & 2 & 0\cdots & 0 \\ -1 & \cdots & 0 & 2\cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots-1 & 0 & 0\cdots & 3 \end{bmatrix}_{(n-1)\times(n-1)}.$$

Let the det $\hat{L}(L_n) = \mathbb{L}_n$. From Lemma 7, it follows that

$$\mathbb{L}_n = 3\mathbb{L}_{n-1} + \mathbb{S}_{n-1}, \tag{6.5}$$

where $\mathbb{S}_{n-1}$ is the number of spanning trees that contain $\{u_{(\frac{n}{2})-1}, v_{(\frac{n}{2})-1}\}$ and $3\mathbb{L}_{n-1}$ is the number of spanning trees obtained by adding any of the graphs in (i) to (iii) above to form a spanning tree.

Similarly, by adding either of (i) and (ii) to a spanning tree of L_n which contains $\{u_{(\frac{n}{2})}, v_{(\frac{n}{2})}\}$, we obtain $2\mathbb{L}_{n-1}$, and by adding (iii) to a spanning tree of $\mathbb{L}_n$ that contain $\{u_{(\frac{n}{2})-1}, v_{(\frac{n}{2})-1}\}$ and then deleting the $\{u_{(\frac{n}{2})-1}, v_{(\frac{n}{2})-1}\}$, we obtain $\mathbb{S}_{n-1}$. Thus,

$$\mathbb{S}_n = 2\mathbb{L}_{n-1} + \mathbb{S}_{n-1}. \tag{6.6}$$

By subtracting (6.6) from (6.5), we have

$$\begin{aligned} \mathbb{L}_n - \mathbb{S}_n &= 3\mathbb{L}_{n-1} + \mathbb{S}_{n-1} - 2\mathbb{L}_{n-1} - \mathbb{S}_{n-1} \\ &= \mathbb{L}_{n-1}. \end{aligned} \tag{6.7}$$

This implies that

$$\mathbb{S}_n = \mathbb{L}_n - \mathbb{L}_{n-1},$$

so that

$$\mathbb{S}_{n-1} = \mathbb{L}_{n-1} - \mathbb{L}_{n-2}.$$

Thus, (6.5) becomes

$$\mathbb{L}_n = 4\mathbb{L}_{n-1} - \mathbb{L}_{n-2}.$$

Solving the difference equation, we obtain the associated charateristic equation $x^2 - 4x + 1 = 0$ with roots $x = 2 \pm \sqrt{3}$. Hence, the solution of the difference equation becomes:

$$\mathbb{L}_n = \alpha(2+\sqrt{3})^n + \beta(2-\sqrt{3})^n.$$

For $n = 0$, we get $\mathbb{L}_0 = 0$, and $\alpha + \beta = 0$, implying that $\alpha = -\beta$.

For $n = 1$, we have $\mathbb{L}_1 = 1$, so that $\alpha(2+\sqrt{3}) + \beta(2-\sqrt{3}) = 1$.

By solving for α and β, we obtain $\alpha = \frac{1}{2\sqrt{3}}$ and $\beta = -\frac{1}{2\sqrt{3}}$,

so that

$$\mathbb{L}_n = \frac{1}{2\sqrt{3}}(2+\sqrt{3})^n - \frac{1}{2\sqrt{3}}(2-\sqrt{3})^n,$$

or equivalently as

$$\mathbb{L}_n = \frac{\sqrt{3}}{6}\Big[(2+\sqrt{3})^n - (2-\sqrt{3})^n\Big].$$

This completes the proof. $\square$

Cycle Graph, C_n

The number of spanning trees of the cycle graph is also known to be n. Here, we confirmed this using solutions of difference equations.

Theorem 17 *The number of spanning trees of C_n is n.*

Proof

The adjacency matrix of the cycle graph, C_n, can be expressed as

$$A(C_n) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}_{n\times n} .$$

Also, its Laplacian matrix can be given as

$$L(C_n) = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ -1 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix}_{n\times n}.$$

Therefore, the reduced Laplacian matrix obtained by removing the first row and first column is given as:

$$\hat{L}(C_n) = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix}_{(n-1)\times(n-1)}.$$

The number of spanning trees of C_n can be expressed as

$$\det \hat{L}(C_n) = 2 \det \hat{L}(C_{n-1}) + \det \left[\begin{array}{c|c} -1 & T \\ \hline 0 & \hat{L}(C_{n-2}) \end{array}\right], \tag{6.8}$$

where $T = [-1, 0, 0, \dots, 0]$. Hence,

$$\det \hat{L}(C_n) = 2 \det \hat{L}(C_{n-1}) - \det \hat{L}(C_{n-2}). \tag{6.9}$$

But,

$$\det \hat{L}(C_2) = |2| = 2,$$

and

$$\det \hat{L}(C_3) = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3.$$

Let $\det \hat{L}(C_n)$ be Y_n. By substituting for Y_n in (6.9) above, we have that

$$Y_n - 2Y_{n-1} + Y_{n-2} = 0.$$

Let $Y_n = \alpha y^n$ be a solution to the difference equation. Then the characteristic equation is given as

$$y^2 - 2y + 1 = 0,$$

and having solutions $y = 1$(twice). Hence, the solution to the difference equation becomes

$$Y_n = \alpha(1)^n + \beta n(1)^n = \alpha + \beta n.$$

For $n = 2$, we have $Y_2 = 2$, and so $2 = \alpha + 2\beta$.

For $n = 3$, we get $Y_3 = 3$, so that $3 = \alpha + 3\beta$.

By solving the two equations for both α and β, we have that: $\alpha = 0$ and $\beta = 1$. Hence, the solution to the difference equation is given as:

$$Y_n = n$$

Therefore, the number of spanning trees of the cycle, C_n is n, as desired. □

Note: The sun graph, SN_n, on n vertices has the number of vertices, $\frac{n}{2}$ of its cycle $C_{\frac{n}{2}}$ as the number of spanning trees, $t(SN_n)$.

Star Graph, S_n

The star graph is known to have the number of spanning trees as 1 since it is a tree, whose Laplacian, denoted as $L(S_n)$, can be expressed as follows

$$L(S_n) = \begin{bmatrix} n-1 & -1 & -1 & \cdots & -1 & -1 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

The number of spanning trees of S_n, denoted $t(S_n)$ is the determinant, det $\hat{L}(S_n)$. But,

$$\hat{L}(S_n) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}_{n \times n}.$$

Hence, $t(S_n) = \det I_{n-1} = 1$, since the reduced Laplacian matrix is the identity matrix, I_n. Also,

$$\begin{aligned} t(S_n) &= \frac{1}{n} \prod_{i=1}^{n-1} \Big[\lambda(S_n) \Big] \\ &= \frac{1}{n} \Big[n.(1)^{n-2} \Big] = 1. \end{aligned}$$

6.4 Spanning Trees of Graphs With Special Numbers

6.4.1 Fibonacci and Lucas Numbers

The number of spanning trees can be expressed in terms of special sequences of numbers called the Fibonacci numbers, denoted as $\mathcal{F}_n$, and the Lucas numbers, denoted as $\mathcal{L}_n$ [20]. The numbers are derived as the sum of the preceeding pair of terms. The set of the first ten Fibonacci and Lucas numbers respectively, can be expressed as follows [27] .

n	1	2	3	4	5	6	7	8	9	10
$\mathcal{F}_n$	1	1	2	3	5	8	13	21	34	55
$\mathcal{L}_n$	1	3	4	7	11	18	29	47	76	123

Thus, these sequences, for $n \geq 3$, can be defined as

$$\begin{aligned} \mathcal{F}_n &= \mathcal{F}_{n-2} + \mathcal{F}_{n-1} \\ \mathcal{L}_n &= \mathcal{L}_{n-2} + \mathcal{L}_{n-1} \quad |_{n\geq 3}, \end{aligned} \tag{6.10}$$

or for $n \geq 1$ as

$$\begin{aligned} \mathcal{F}_{n+2} &= \mathcal{F}_{n+1} + \mathcal{F}_n \\ \mathcal{L}_{n+2} &= \mathcal{L}_{n+1} + \mathcal{L}_n \quad |_{n\geq 1}. \end{aligned} \tag{6.11}$$

From (6.10) and (6.11) above, we get

$$\begin{aligned} \mathcal{F}_n &= \mathcal{F}_{n-2} + \mathcal{F}_{n-1} = \mathcal{F}_{n+2} - \mathcal{F}_{n+1} \\ \mathcal{F}_{n+2} + \mathcal{F}_{n-2} &= (\mathcal{F}_{n+1} + \mathcal{F}_n) + (\mathcal{F}_n - \mathcal{F}_{n-1}) \\ &= 2\mathcal{F}_n + \mathcal{F}_{n+1} - \mathcal{F}_{n-1}. \end{aligned} \tag{6.12}$$

But if $\mathcal{F}_n = \mathcal{F}_{n-2} + \mathcal{F}_{n-1}$, it implies that

$$\mathcal{F}_{n+1} = \mathcal{F}_{n-1} + \mathcal{F}_n \Longrightarrow \mathcal{F}_n = \mathcal{F}_{n+1} - \mathcal{F}_{n-1},$$

and hence,

$$\mathcal{F}_{n+2} + \mathcal{F}_{n-2} = 2\mathcal{F}_n + \mathcal{F}_n = 3\mathcal{F}_n.$$

Therefore,

$$\mathcal{F}_{n+2} = 3\mathcal{F}_n - \mathcal{F}_{n-2}. \tag{6.13}$$

From the table above, it is obvious that

$$\mathcal{L}_n = \mathcal{F}_{n-1} + \mathcal{F}_{n+1}.$$

From (6.10) and (6.11),we obtain

$$\mathcal{F}_{n-2} + \mathcal{F}_{n-1} = \mathcal{F}_{n+2} - \mathcal{F}_{n+1},$$

$$\mathcal{F}_{n+2} - \mathcal{F}_{n-2} = \mathcal{F}_{n+1} + \mathcal{F}_{n-1} = \mathcal{L}_n.$$

By subtracting $\mathcal{F}_{n-2}$ from both sides, we have that

$$\begin{aligned} \mathcal{F}_{n+2} + \mathcal{F}_{n-2} &= 3\mathcal{F}_n - 2\mathcal{F}_{n-2} \\ \mathcal{L}_n &= 3\mathcal{F}_n - 2\mathcal{F}_{n-2}. \end{aligned} \tag{6.14}$$

From (6.10) and (6.14), it follows that

$$\begin{aligned} \mathcal{F}_{n+2} + \mathcal{F}_{n-2} &= 3\mathcal{F}_n - 2\mathcal{F}_{n-2} \\ \mathcal{L}_{2n} &= 3\mathcal{F}_{2n} - 2\mathcal{F}_{2n-2} \quad |_{n\geq 3}. \end{aligned} \tag{6.15}$$

Different formulae are known for the number of spanning trees of the wheel and fan graphs. Sedlacek in 1969 [30] was the first to give explicit formulae for spanning trees of finite graphs. Here, we derived the number of spanning trees in terms of Fibonacci and Lucas numbers following an argument due to Hilton [20].

Wheel Graph, W_n

Theorem 18 *The number of spanning trees of the wheel graph, W_n, is given as:*

$$\begin{aligned} t(W_n) &= \mathcal{L}_{2n} - 2, \quad n \geq 1, ([20], [31]) \\ &= \left(\frac{3+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{3-\sqrt{5}}{2}\right)^{n+1} - 2, \quad n \geq 1, ([27]) \\ &= \prod_{j=1}^{n-1}\left[1 + 4\sin^2(\frac{\pi j}{n})\right], \quad j = 1, 2, \ldots, n-1. \end{aligned}$$

Proof

By the use of the Matrix Tree Theorem [24], we choose a vertex, say the hub, and eliminate its corresponding row and column to get the reduced Laplacian matrix, denoted as $\hat{L}(W_n)$, given generally as

$$\hat{L}(W_n) = \begin{bmatrix} 3 & -1 & 0 & 0 & \cdots & 0 & -1 \\ -1 & 3 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 3 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 3 & -1 \\ -1 & 0 & 0 & 0 & \cdots & -1 & 3 \end{bmatrix}_{n\times n}.$$

The reduced Laplacian matrix is such that it has 3's on the diagonal, -1's on the upper and lower parts of the diagonal, -1's in the extreme upper and lower parts of the matrix, and 0's elsewhere.

Let the reduced Laplacian matrix without the extreme -1's in the upper right and lower left be given as

$$A_n^* = \begin{bmatrix} 3 & -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 3 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 3 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 3 & -1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 3 \end{bmatrix}_{n\times n}.$$

By the principle of mathematical induction, the determinants of A_n for any given $n = 1, 2, \ldots, k$, is described below.

$$|A_1^*| = |3| = 3 = \mathcal{F}_4 = \mathcal{F}_{2(1)+2},$$

$$|A_2^*| = \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} = 8 = \mathcal{F}_6 = \mathcal{F}_{2(2)+2}.$$

Hence, the result is true for $n = 1$ and $n = 2$. Assuming it is true for any positive integer $n > k$, we have,

$$|A_n^*| = 3|A_{n-1}^*| - (-1)\begin{vmatrix} -1 & -1 & 0\ldots & 0 & 0 \\ 0 & 3 & -1\ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0\ldots & 3 & -1 \\ 0 & 0 & 0\ldots & -1 & 3 \end{vmatrix}_{(n-1)\times(n-1)}$$

$$= 3|A_{n-1}^*| + (-1)\begin{vmatrix} 3 & -1 & 0\ldots & 0 & 0 \\ -1 & 3 & -1\ldots & 0 & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0\ldots & 3 & -1 \\ 0 & 0 & 0\ldots & -1 & 3 \end{vmatrix}_{(n-2)\times(n-2)}$$

$$= 3|A_{n-1}^*| - |A_{n-2}^*|.$$

By the use of the inductive hypothesis,

$$\begin{aligned} |A_n^*| &= 3|A_{n-1}^*| - |A_{n-2}^*| \\ &= 3\mathcal{F}_{2(n-1)+2} - \mathcal{F}_{2(n-2)+2} \\ &= 3\mathcal{F}_{2n} - \mathcal{F}_{2n-2} \\ &= \mathcal{F}_{2n+2} \qquad (\text{for} \quad n \geq 1). \end{aligned}$$

Hence, the determinant of the reduced Laplacian matrix, $\hat{L}_n$, is given as:

$$\det(\hat{L}_n) = \begin{vmatrix} 3 & -1 & 0 & 0 & \cdots & 0 & -1 \\ -1 & 3 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 3 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 3 & -1 \\ -1 & 0 & 0 & 0 & \cdots & -1 & 3 \end{vmatrix}_{n\times n}$$

$$= 3|A^*_{n-1}| + |B^*_{n-1}| + (-1)(-1)^{n+1}|C^*_{n-1}|,$$

where,

$$|A^*_{n-1}| = \begin{vmatrix} 3 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 3 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 3 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 3 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 3 \end{vmatrix}_{(n-1)\times(n-1)},$$

$$|B^*_{n-1}| = \begin{vmatrix} -1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 3 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 3 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 3 & -1 \\ -1 & 0 & 0 & \cdots & -1 & 3 \end{vmatrix}_{(n-1)\times(n-1)},$$

$$|C^*_{n-1}| = \begin{vmatrix} -1 & 3 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 3 & -1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 3 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & -1 & 3 \\ -1 & 0 & 0 & 0 & \cdots & 0 & -1 \end{vmatrix}_{(n-1)\times(n-1)}.$$

Solving for each of the determinants:

$$|B^*_{n-1}| = (-1)\begin{vmatrix} 3 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 3 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 3 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 3 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 3 \end{vmatrix}_{(n-2)\times(n-2)} + \begin{vmatrix} 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 3 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 3 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 3 & -1 \\ -1 & 0 & 0 & \cdots & -1 & 3 \end{vmatrix}_{(n-2)\times(n-2)}$$

$$= (-1)|A^*_{n-2}| + (-1)^{n-1} \begin{vmatrix} -1 & 0 & 0 & \cdots & 0 & 0 \\ 3 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 3 & -1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -1 \end{vmatrix}.$$

Thus,

$$\begin{aligned} |B^*_{n-1}| &= (-1)|A^*_{n-2}| + (-1)^{n-1}(-1)^{n-2} \\ &= (-1)|A^*_{n-2}| + (-1)^{2n-3} \\ &= (-1)|A^*_{n-2}| - 1 \\ &= -\mathcal{F}_{2(n-2)+2} - 1 \\ &= -\mathcal{F}_{2n-2} - 1. \end{aligned}$$

Similarly,

$$|C^*_{n-1}| = (-1)^{n-1} + (-1)(-1)^{n-2} \begin{vmatrix} 3 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 3 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 3 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 3 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 3 \end{vmatrix}_{(n-2)\times(n-2)},$$

so that

$$\begin{aligned} |C^*_{n-1}| &= (-1)^{n-1} + (-1)(-1)^{n-2}|A^*_{n-2}| \\ &= (-1)^{n-1} + (-1)^{n-1}|A^*_{n-2}| \\ &= (-1)^{n-1}\Big[1 + |A^*_{n-2}|\Big] \\ &= (-1)^{n-1}\Big[1 + \mathcal{F}_{2n-2}\Big]. \end{aligned}$$

It follows that,

$$\begin{aligned}
\det(\hat{L}_n) &= 3|A^*_{n-1}| + |B^*_{n-1}| + (-1)^{n+2}|C^*_{n-1}| \\
&= 3|A^*_{n-1}| + (-\mathcal{F}_{2n-2} - 1) + (-1)^{n+2}(-1)^{n-1}(1 + \mathcal{F}_{2n-2}) \\
&= 3\mathcal{F}_{2n} - \mathcal{F}_{2n-2} - 1 + (-1)^{2n+1}(1 + \mathcal{F}_{2n-2}) \\
&= 3F_{2n} - \mathcal{F}_{2n-2} - 1 + (-1)(1 + \mathcal{F}_{2n-2}) \\
&= 3\mathcal{F}_{2n} - \mathcal{F}_{2n-2} - 1 + -1 - \mathcal{F}_{2n-2} \\
&= 3\mathcal{F}_{2n} - 2\mathcal{F}_{2n-2} - 2 \\
&= \mathcal{L}_{2n} - 2,
\end{aligned}$$

as desired. □

Hence, the number of spanning trees of a wheel can be evaluated using the matrix tree theorem to be $\mathcal{L}_{2n} - 2$, where $\mathcal{L}_{2n}$ is the $(2n)$th Lucas number.

Fan Graph, F_n

Similarly as in the case of the wheel graph, we state and prove by derivation the number of spanning trees of the fan graph in terms of Fibonacci numbers.

Theorem 19 *The number of spanning trees of the fan graph, F_n, is given as*

$$\begin{aligned}
t(F_n) &= \mathcal{F}_{2n}, \quad n \geq 1, ([20], [27]) \\
&= \frac{1}{2\sqrt{5}}\Big[(3+\sqrt{5})^{n+1} - (3-\sqrt{5})^{n+1}\Big], \quad n \geq 0, ([5]) \\
&= \prod_{j=1}^{n-1}\Big[1 + 4\sin^2(\frac{\pi j}{2n})\Big], \quad j = 1, 2, \ldots, n-1.
\end{aligned}$$

Proof

The reduced Laplacian of the fan graph, F_n, is expressed as:

$$\hat{L}(F_n) = \begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 3 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 3 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 3 & -1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix}_{n\times n}.$$

The determinant of $\hat{L}(F_n)$ can be expressed as

$$\det \hat{L}(F_n) = \begin{vmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 3 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 3 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 3 & -1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 \end{vmatrix}_{n\times n}$$

$$= 2|T_{n-1}| + \begin{bmatrix} -1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 3 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 3 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 3 & -1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix}_{(n-1)\times(n-1)}$$

$$= 2|T_{n-1}| + (-1) \begin{bmatrix} 3 & -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 3 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 3 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 3 & -1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix}_{(n-2)\times(n-2)},$$

where

$$T_n = \begin{bmatrix} 3 & -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 3 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 3 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 3 & -1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix}_{n\times n},$$

so that we obtain the difference equation:

$$\det \hat{L}(F_n) = 2|T_{n-1}| - |T_{n-2}|, \quad n \geq 3. \tag{6.16}$$

Using the principle of mathematical induction, the determinants of T_n for any given positive integer $n > k$ are as follows.

$$|T_1| = |2| = 2 = \mathcal{F}_3 = \mathcal{F}_{2(1)+1},$$

$$|T_2| = \left| \begin{smallmatrix} 3 & -1 \\ -1 & 2 \end{smallmatrix} \right| = 5 = \mathcal{F}_5 = \mathcal{F}_{2(2)+1},$$

$$|T_3| = \left| \begin{smallmatrix} 3 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{smallmatrix} \right| = 13 = \mathcal{F}_7 = \mathcal{F}_{2(3)+1}.$$

By inductive hypothesis, assuming that it is true for $n > k$, we have

$$|T_n| = 3|T_{n-1}| - (-1) \begin{vmatrix} -1 & -1 & 0\ldots & 0 & 0 \\ 0 & 3 & -1\ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0\ldots & 3 & -1 \\ 0 & 0 & 0\ldots & -1 & 3 \end{vmatrix}_{(n-1)\times(n-1)}$$

$$= 3|T_{n-1}| + (-1) \begin{vmatrix} 3 & -1 & 0\ldots & 0 & 0 \\ -1 & 3 & -1\ldots & 0 & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0\ldots & 3 & -1 \\ 0 & 0 & 0\ldots & -1 & 3 \end{vmatrix}_{(n-2)\times(n-2)}$$

$$= 3|T_{n-1}| - |T_{n-2}|.$$

Hence,

$$\begin{aligned} |T_n| &= 3|T_{n-1}| - |T_{n-2}| \\ &= 3\mathcal{F}_{2(n-1)+1} - \mathcal{F}_{2(n-2)+1} \\ &= 3\mathcal{F}_{2n-1} - \mathcal{F}_{2n-3}. \end{aligned} \tag{6.17}$$

Recall from (6.13) that $3\mathcal{F}_{2n-1} - \mathcal{F}_{2n-3} = \mathcal{F}_{2n+1}$. It follows that for $n \geq 1$,

$$|T_n| = \mathcal{F}_{2n+1}.$$

By substituting for $|T_i|$ in (6.16), we have that

$$\begin{aligned} \det \hat{L}(F_n) &= 2|T_{n-1}| - |T_{n-2}| \\ &= 2\mathcal{F}_{2(n-1)+1} - \mathcal{F}_{2(n-2)+1} \\ &= 2\mathcal{F}_{2n-1} - \mathcal{F}_{2n-3} \\ &= \mathcal{F}_{2n}, \quad n \geq 1. \end{aligned} \tag{6.18}$$

Also, $|T_n|$ satisfies

$$|T_n| - 3|T_{n-1}| + |T_{n-2}| = 0.$$

This implies that F_{2n} also satisfies the recurrence relation

$$\mathcal{F}_{2n} - 3\mathcal{F}_{2n-1} + \mathcal{F}_{2n-2} = 0.$$

The associated characteristic equation is given as $x^2 - 3x + 1 = 0$, with roots $x = \frac{3\pm\sqrt{5}}{2}$. Let $f_i = \mathcal{F}_{2n}$. Hence, we have

$$\mathcal{F}_{2n} = f_i = \alpha(\frac{3+\sqrt{5}}{2})^i + \beta(\frac{3-\sqrt{5}}{2})^i.$$

For $i = 0$, we get $f_0 = \mathcal{F}_0 = 0$ so that $\alpha + \beta = 0$, i.e., $\alpha = -\beta$.

For $i = 1$, we obtain $f_1 = \mathcal{F}_2 = 1$ and so $\alpha(\frac{3+\sqrt{5}}{2}) + \beta(\frac{3-\sqrt{5}}{2}) = 1$.

Thus,

$$\alpha(\frac{3+\sqrt{5}}{2}) - \alpha(\frac{3-\sqrt{5}}{2}) = 1.$$

Therefore,

$$\alpha = \frac{1}{\sqrt{5}} \quad \text{and } \beta = -\frac{1}{\sqrt{5}}.$$

We conclude that

$$F_{2n} = \frac{1}{2\sqrt{5}}\Big[(3+\sqrt{5})^{n+1} - (3-\sqrt{5})^{n+1}\Big].$$

This proves the theorem. $\square$

We summarize all the results proved above in Table 1 below.

Graph, G	$t(G)$
K_n	n^{n-2}
$K_{p,q}$	$p^{q-1}q^{p-1}$
P_n	1
C_n	n
S_n	1
W_n	$\mathcal{L}_{2n} - 2 = \left(\frac{3+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{3-\sqrt{5}}{2}\right)^{n+1} - 2$
F_n	$\mathcal{F}_{2n} = \frac{1}{2\sqrt{5}}\left[(3+\sqrt{5})^{n+1} - (3-\sqrt{5})^{n+1}\right]$
L_n	$\frac{\sqrt{3}}{6}\left[(2+\sqrt{3})^n - (2-\sqrt{3})^n\right]$

Table 1: Number of spanning trees, $t(G)$ of special graphs

6.5 Summary

In this chapter, we studied the different methods of obtaining formulae for the number of spanning trees, $t(G)$, of the special graphs, G, discussed in the previous chapters. We further studied methods involving the use of the graphs' Laplacian eigenvalues and the representations of the wheel and fan graphs in terms of Fibonacci and Lucas numbers. In the next chapter, we will introduce a new concept of a ratio, namely, tree-cover ratios, asymptotes and areas of classes of graphs, involving spanning trees and vertex coverings.

Chapter 7

Tree-Cover Ratios, Asymptotes and Areas of Graphs

7.1 Introduction

This chapter is entirely original and is based on the paper, Winter and Adewusi [38]. In this chapter, we describe the tree-cover ratios, asymptotes and areas of classes of graphs. This involves spanning trees and minimum vertex covering sets of the graphs discussed in Chapter 5. This was motivated by the fact that such a ratio, associated with a complete graph, has an asymptotic convergence which is identical to that of the secretary problem.

7.2 Ratios

Different types of ratios and their asymptotic values connected to graphs have been investigated in recent times. In this section, we study some of the known graph ratios and provide the motivation for the tree-cover ratio.

The Hall Ratio

Cropper et al. in [9] were the first to study the Hall ratio of a graph which was motivated by the problems of listing coloring. The asymptotic values of Hall ratios of graphs had also been studied by Gabor in [15] and by Cropper and Gyárfás in [10].

Definition 12 (Independent Set)

Let G be a graph of order n. Let $U \subseteq V$, such that no two vertices in U are adjacent. Then, U is said to be an **Independent Set** *of G. The cardinality of the largest independent set of G is called the* **Independence Number** *of G, denoted by $\alpha(G)$.*

Definition 13 *The ratio between $\alpha(G)$ and $|V|$, denoted as $I_R(G)$, is called the* **Independence Ratio** *of G.*

That is,

$$I_R(G) = \frac{\alpha(G)}{|V|}.$$

The **Hall Ratio** of G, denoted as $\rho(G)$, is defined as the ratio of the number of vertices and the independence number maximized over all subgraphs of G [15], i.e.,

$$\rho(G) = \max\left\{\frac{|H|}{\alpha(H)};\ H \subseteq G\right\}.$$

Definition 14 (Vertex Cover)

Let $G = (V, E)$ be a graph and $V' \subseteq V$. We say that V' is a vertex cover if for every edge $e = uv \in E$, $u \in V'$ or $v \in V'$ or both $u, v \in V'$. A **Minimum Vertex**

Cover *is the vertex cover with the smallest cardinality and the cardinality of the minimum vertex cover is called the* **Vertex Cover Number** *of G, denoted as $\tau(G)$.*

By definition,

$$\alpha(G) + \tau(G) = |V|.$$

For the complete graph K_n, $\alpha(K_n) = 1$ and $\tau(K_n) = n - 1$, so that the asymptotic convergence of the Hall ratio of K_n is given as

$$\lim_{n\to\infty} \rho(K_n) = \lim_{n\to\infty} \left\{ \max \left(\frac{n}{1}\right) \right\} = \infty.$$

Central Radius Ratio

The central radius ratio, described by Buckley [8], is characterised by the closeness of some sets of vertices to other vertices in the graph.

Let $G = (V, E)$ be a graph and $v_1, v_2 \in V$. The distance, $d(v_1, v_2)$, between v_1 and v_2, is the length of a shortest $v_1 - v_2$ path in G.

The **Eccentricity** of any vertex, v in G, denoted as $\epsilon(v)$, is the largest out of all the distances, $d(v, v_i)$ between v and any other vertex v_i in G. The **Radius** of G, denoted as $\mathfrak{r}(G)$, is the minimum of all the eccentricities of vertices in G.

That is,

$$\mathfrak{r}(G) = \min_{v\in V} \epsilon(v).$$

Definition 15 (Center of a Graph) *[8] The center of G, $C(G)$, is the set of all vertices in G with minimum eccentricity. The **Central Radius Ratio**, $c(G)$ is defined as*

$$c(G) = \frac{|C(G)|}{|V(G)|}, \quad 0 < c(G) \leq 1.$$

The graphs with $c(G) = 1$ are said to be self-centered. The complete graph K_n has all vertices connected to every other vertex in it. Hence, for any $v_i, v_j \in V(K_n)$, $d(v_i, v_j) = 1$ and $\epsilon(K_n) = 1$. The central ratio, $c(K_n)$, is then $\frac{n}{n} = 1$ and the asymptotic convergence of $c(K_n)$ is 1.

Edge Expansion Ratio

The edge expansion ratio, also known as the Cheeger constant, as explained by Alon and Spencer in [1], is a measure of the degree of a graph. They argued that by taking all possible nonempty subsets $\mathtt{S} \subseteq V$ of order at most $\frac{n}{2}$, and edge boundary of $\mathtt{S}$ (i.e., the number of "out" edges from $\mathtt{S}$), denoted as $\delta(\mathtt{S})$, then the edge expansion ratio, $e_x(G)$, is the value of the smallest ratio between $\mathtt{S}$ and $\delta(\mathtt{S})$.

Definition 16 *([1]) The edge expansion ratio, $e_x(G)$, of a graph G of order n is given as the ratio*

$$e_x(G) = \min_{0<|S|\leq\frac{n}{2}} \frac{|\delta(\mathtt{S})|}{|\mathtt{S}|}.$$

For the complete graph on n vertices, $|\mathtt{S}(K_n)| = \frac{n}{2}$ and $|\delta(\mathtt{S})(K_n)| = \frac{n}{2} \cdot \frac{n}{2} = \frac{n^2}{4}$, so that $e_x(K_n) = \frac{n}{2}$. Since this is a function of n, the idea of asymptotes can be considered as n becomes large. Thus,

$$\lim_{n\to\infty} e_x(K_n) = \infty.$$

The Secretary Problem

We use the historical discussion and assumptions of the problem by T.S. Ferguson in [13], where n denotes the number of applicants applying for the secretary job, with the objective of maximixing the probability of selecting the best applicant.

Let n be the number of candidates and r the selected candidates, where, $r \in \{1, 2, \ldots, n\}$. Assume that the probability of selecting r candidates from n candidates is $\mathbb{P}_n(r)$ and that $\mathbb{S}_r$ denotes the event of success of selecting r candidates. Let x_n be the order of arrival of the candidates. Since the candidates arrive in a random order, then x_n is uniformly distributed. Hence, the conditional probability of selecting the best candidate given the order of arrival is

$$\mathbb{P}_n(\mathbb{S}_r|x_n = i) = \begin{cases} 0, & \text{if} \quad i < r; \\ \frac{r-1}{i-1} & \text{if} \quad i \geq r. \end{cases}$$

This implies that if the best applicant i is less than r, then i is rejected. But given that $i \geq r$, then i is selected if and only if the best applicant among the remaining $i-1$ applicants, is among the first $r-1$ that were rejected.

Therefore,

$$\begin{aligned} \mathbb{P}_n(\mathbb{S}_r) &= \sum_{i=1}^{n} \mathbb{P}(x_n).\mathbb{P}(\mathbb{S}_r|x_n = i) \\ &= \sum_{i=r}^{n} \frac{1}{n}.\frac{r-1}{i-1} \\ &= \frac{r-1}{n} \sum_{i=r}^{n} \frac{1}{i-1}. \end{aligned}$$

By the use of Riemann Sum, we have that

$$\mathbb{P}_n(\mathbb{S}_r) = x \int_x^1 \frac{1}{t} dt = -x \ln x,$$

By differentiating $\mathbb{P}_n(\mathbb{S}_r)$, we get

$$\mathbb{P}'_n(\mathbb{S}_r) = -1 - \ln x,$$

where

$$x = \lim_{n \longrightarrow \infty} \frac{r}{n}, \quad t = \frac{i}{n}.$$

Hence, at the point where $\mathbb{P}_n' = 0$, $x = e^{-1}$. This implies that as n becomes very large, the probability of selecting the best candidate tends to e^{-1}.

We now provide a graph theoretical variation of the secretary problem with convergent ratio identical to the cut-off number e^{-1}, and we use it to motivate for the definition of a tree cover ratio of classes of graphs.

Gambling problem with social decision making and guaranteed win

We have n gamblers each coming to the casino with 1 million dollars. We assume that these individuals do not know each other, and they agree to the conditions of the game determined by the casino. The casino guarantees that a pair will leave with 2 million dollars each, and selects 1 participant randomly, say n_i.

This n_i is given 2 million (so he/she has a total of 3 million dollars) by the casino and n_i must decide who to share the 3 million dollars with by social interaction with the other $n - 1$ participants. This is done with exactly one spanning tree which he/she arbitrarily selects. Only n_i and the casino knows who has been selected. This participant must decide who he/she "likes" the most through the spanning tree. The casino then selects, randomly, an individual (other than n_i) from the remaining $n - 1$ participants. This individual, say n_k, must decide, through communication involving all possible spanning trees determined by the

$n-1$ participants, if he/she has been chosen by n_i.

A correct guess, i.e., a perfect match (in terms of the individual chosen by n_i) means that both n_i and n_k walk away with 2 million dollars each and the game is over. If n_k is correct (in terms of not being chosen by n_i), he keeps his million, remains in the game but cannot play to win and is an inactive participant, and then the casino selects the next participant. Otherwise, if n_k is wrong (he believes he has been chosen by n_k, but was not), he loses a million and the casino then selects a next participant (with n_k remaining as part of the communication spanning trees but cannot be chosen again-an inactive participant).

The last case is when n_k is wrong (he believes he has been chosen when he has not been chosen by n_i). In this case, since there must be a perfect match, the casino makes the changes as per $7(iv)$ below.

Conditions:

1. All n individuals are communicationally linked by edges of a complete graph, and have not known each other before the game.

2. Every individual can communicate with the others with a two way directed edge- i.e., we have a complete digraph, G, representing their connections.

3. Individual n_i is selected at random by the casino. This individual then interacts with all the $n-1$ others by selecting any one of the possible spanning

trees: either directly to each of the individuals (a star graph= a spanning tree), connecting from n_i to the remaining $n-1$ vertices. Or, for example, via all possible paths to n_j, i.e., through discussing with the individuals along each path to n_j.

4. Once n_i has been found through a spanning tree, the individual n^* he wants to share the money with, the money is kept to this individual n^*. The probability of finding a match will be:

$$P = \frac{1}{t(K_n)} = \frac{1}{n^{n-2}},$$

where $t(K_n) = n^{n-2}$ is the possible number of spanning trees of the complete graph.

5. We now remove n_i and work with the complete subgraph, H of G, induced by the remaining $n-1$ active individuals (vertices, which is a covering set of G), and select a n_k^1 (as a leader) randomly, such that each of the $n-1$ individuals in this subgraph have interacted with n_i through some spanning tree.

6. This individual leader n_k^1 interacts with the remaining $n-2$ vertices using all possible spanning trees on the set of $n-1$ vertices and decides if he has or has not been chosen by n_i (conforming or contradicting n_i's choice).

7. This individual n_k^1 must go through all possible spanning trees $t(K_{n-1})$ before making a decision.

(i.) If n_k^1 decides through social interaction with others that he is chosen/not chosen by n_i and is correct (a perfect match), the pair n_i and n_k^1 walk away with 2 million dollars each and the game ends.

(ii.) If n_k^1 is correct (a match) in the "n_i has not chosen me" case, then the contestant keeps his million and the casino selects the next participant other than n_k^1 (n_k^1 cannot be chosen again but remains in the game as a communicator or inactive participant or vertex).

(iii.) If n_k^1 is wrong (a non- match) by saying that he has been chosen, when in fact he has not been chosen. He loses the million to the casino and the casino proceed randomly to the next active vertex n_k^2 in the subgraph H, excluding n_k^1.

(iv.) If n_k^1 is wrong by saying that he has not been chosen by n_i, when in fact he has been chosen by n_i, then $n_k^1 = n^*$. If this is the last contestant, then the casino declares a perfect match and the game ends. Otherwise, he keeps his million and the casino swaps him with an arbitrary active participant n_k^j (n_k^j becomes inactive but keeps his million), and the casino proceed randomly to the next active vertex n_k^2 in the subgraph H (the participant now know that he is n^* so will eventually be a perfect matched with n_i). The new leader n_k^2 selected randomly by the casino, interacts with the other active individuals in the same way n_k^1 did and decides if he is chosen or not by n_i.

(v) The game stops when a perfect match is found. If no perfect match has been found after $n-2$ contestants in the set of $n-1$ contestants, then the last contestant allows for a perfect match by default.

For each of the $n-1$ vertices in H, we have $(n-1)^{n-3}$ spanning trees, so that the probability of arriving at a perfect match of n_i with n^* will be according to the following theorem.

Theorem 20 *The probability of arriving at a perfect match of n_i with n^* through spanning trees in the gambling problem above is*

$$\left(\frac{n-1}{n}\right)^{n-2} = \frac{|S|t(H(S))}{t(K_n)}.$$

Proof

This probability of a perfect match through the spanning trees is given as: (probability of selecting (n_i)) $\times$ (probability of n_k^1 in H having a perfect match with n^* through $t(K_{n-1})$ spanning trees OR n_k^2 having a perfect match with n^* through $t(K_{n-1})$ spanning trees OR ... OR n_k^{n-1} having a perfect match with n^* through spanning trees).

Mathematically, this can be expressed as:

$$
\begin{aligned}
\mathbb{P}_n &= \frac{1}{t(K_n)} \times \overbrace{\Big[t(K_{n-1}) + t(K_{n-1}) + \ldots + t(K_{n-1})\Big]}^{(n-1)\text{times}} \\
&= \frac{1}{t(K_n)} \times (n-1)t(K_{n-1}) \\
&= \frac{(n-1)(n-1)^{n-3}}{n^{n-2}} = \Big(\frac{n-1}{n}\Big)^{n-2} \\
&\equiv \frac{|S|t(H(S))}{t(K_n)},
\end{aligned}
$$

where S is the number of vertices in the minimum vertex covering set of K_n, $H(S)$ is the complete subgraph induced by the remaining $n-1$ vertices and $t(K_n)$ is the number of spanning trees of K_n. □

Corollary 4 *The probability ratio* $\left(\frac{n-1}{n}\right)^{n-2} = \frac{|S|t(H(S))}{t(K_n)}$ *of the gambling problem converges to* e^{-1}*. (Same as the probability of selecting the best applicant in the secretary problem).*

Proof

Let

$$q = \left(\frac{n-1}{n}\right)^{n-2}.$$

Then,

$$\ln q = \frac{\ln(1-\frac{1}{n})}{\frac{1}{n-2}}.$$

As $n \to \infty$, $\ln q$ tends to -1. Hence, q tends to e^{-1} as n goes to ∞. □

The ratio $\frac{|S|t(H(S))}{t(K_n)}$ involving spanning trees and vertex cover, S, with its convergence property, provides the motivation for the definition of the tree-cover ratio and asymptotes of classes of graphs presented below.

7.3 Tree Cover Ratios and Asymptotes

Definition 17 (Tree Cover Ratio)

Let $t(G)$ be the number of spanning trees of a graph, G. Let S be a minimum vertex cover of G and $H(S)$ be the subgraph of G induced by S. Then, the **tree cover ratio** *of G with respect to S is defined as*

$$tc(G)_s = \frac{|S|t(H(S))}{t(G)},$$

where $|S|$ is the cardinality of S.

The following cases might arise:

(i) If $H(S)$ is connected, then $t(H(S))$ is the number of spanning trees.

(ii) If $H(S)$ consists of n isolated vertices, then $t(H(S))$ is defined as $t(H(S)) = 1$.

(iii) If $H(S)$ is disconnected, a spanning forest may be considered involving the components of $H(S)$. This case would not be considered in the dissertation.

Definition 18 (Tree Cover Asymptotes) *Let the tree cover ratio be defined as a function on the order n of the graph, i.e.,*

$$tc(G)_s = \frac{|S|t(H(S))}{t(G)} = f(n).$$

Then, the tree cover asymptote of $f(n)$, denoted as $Asyp(G)_s$ with respect to S, is defined as

$$Asyp(G)_s = \lim_{n \longrightarrow \infty} f(n).$$

The asymptote describes the behaviour of the tree cover ratio of the graph when the order is very large.

7.3.1 An Ideal communication problem and tree-cover asymptote.

In [7], the communication problem is to select a minimal set of placed sensor devices in a service area so that the entire service area is accessible by the minimal set of sensors. Finding the minimal set of sensors is modelled as a vertex-cover problem, where the vertex-cover set facilitates the communications between the sensors. The tree-cover asymptote may therefore have application where communication is involved in networks with a large number of vertices, i.e., in extreme networks.

If $H(S)$, in the tree-cover definition, is connected, and M represents the vertices of G not in S, then each vertex of M is connected directly by an edge (an out-edge) to a vertex of $H(S)$ which is part of a spanning tree. Thus, the ease of communication between vertices of $H(S)$ and M through the out edges, involving spanning trees, may be represented by this tree-cover ratio – the "ideal" case, involving large number of vertices, being when this tree-cover asymptotic ratio of e^{-1} is the smallest (and positive)- which we believe is the case of the complete graphs.

This tree-cover ratio, in communication networks, allows for the investigation of the outward social connectivity from a vertex covering, with the rest of the network when large number of vertices are involved.

7.3.2 Examples of Tree Cover Ratios and Asymptotes

Complete graph

Let K_n be a complete graph on n vertices. Any subset of $n-1$ vertices of K_n is a minimum vertex cover of K_n. Thus, $|S| = n-1$ and $t(K_{n-1}) = (n-1)^{n-3}$. Hence, the tree cover ratio of K_n is given as

$$\begin{aligned} tc(K_n)_s &= \frac{|S|t(K_{n-1})}{t(K_n)} \\ &= \frac{(n-1)(n-1)^{n-3}}{n^{n-2}} \\ &= \Big(\frac{n-1}{n}\Big)^{n-2}. \end{aligned}$$

Note that

$$\Big(\frac{n-1}{n}\Big)^{n-2} = \Big(\frac{n-1}{n}\Big)^{n} \times \frac{1}{(1-\frac{1}{n})^2} \approx \Big(\frac{n-1}{n}\Big)^{n}.$$

Therefore, for very large n, the tree cover asymptotes of K_n can be evaluated as

$$\begin{aligned} Asyp(K_n)_s &= \lim_{n\to\infty} \Big(\frac{n-1}{n}\Big)^{n} \\ &= \lim_{n\to\infty} \Big[\sum_{i=0}^{\infty} \binom{n}{i} \Big(\frac{-1}{n}\Big)^{n}\Big] \\ &\approx \lim_{n\to\infty} \Big[\frac{[1-\frac{1}{n}]}{2!} - \frac{[1-\frac{3}{n}+\frac{2}{n^2}]}{3!} + \dots\Big] \\ &= \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \dots \\ &= e^{-1}. \end{aligned}$$

Thus, the tree-cover asymptote for K_n is $Asyp(K_n)_s = e^{-1}$.

Complete split bipartite graph

Consider a complete split bipartite graph, $K_{\frac{n}{2},\frac{n}{2}}$, of order n, whose number of spanning trees, $t(K_{\frac{n}{2},\frac{n}{2}})$, is $(\frac{n}{2})^{n-2}$. The minimum vertex cover consists of $|S| = \frac{n}{2}$ elements such that $t(H(S)) = 1$.

Hence, the tree cover ratio of the complete split bipartite graph is given as

$$\begin{aligned} tc(K_{\frac{n}{2},\frac{n}{2}})_s &= \frac{|S|t(H(S))}{t(K_{\frac{n}{2},\frac{n}{2}})} \\ &= \frac{\frac{n}{2}.1}{(\frac{n}{2})^{n-2}} \\ &= (\frac{2}{n})^{n-3}. \end{aligned}$$

It follows that the tree cover asymptote of $K_{\frac{n}{2},\frac{n}{2}}$ is

$$Asyp(K_{\frac{n}{2},\frac{n}{2}})_s = \lim_{n\to\infty}\left[(\frac{2}{n})^{n-3}\right] = 0.$$

Cycle graph

The cycle on n vertices has n spanning trees. For n even, the minimum vertex cover, S consists of $\frac{n}{2}$ isolated alternating vertices. Thus, $t(H(S)) = 1$ and the tree cover ratio of C_n is given as:

$$\begin{aligned} tc(C_n)_s &= \frac{|S|t(H(S))}{t(C_n)} \\ &= \frac{\frac{n}{2}.1}{n} = \frac{1}{2}. \end{aligned}$$

It follows that the tree cover asymptotes of C_n is

$$Asyp(C_n)_s = \lim_{n\to\infty}\left(\frac{1}{2}\right) = \frac{1}{2}.$$

For n odd, $|S| = \frac{n}{2} + 1$ but the number of spanning forest has to be considered for $t(H(S))$.

Sun graph

The sun graph, SN_n, on n vertices has $\frac{n}{2}$ spanning trees. The minimum vertex set S consists of all the vertices of $C_{\frac{n}{2}}$.

Thus, $|S| = \frac{n}{2}$, $t(SN_{\frac{n}{2}}) = \frac{n}{2}$, and the tree-cover ratio is

$$tc(SN_n)_s = \frac{|S|t(SN_{\frac{n}{2}})}{t(SN_n)} = \frac{\frac{n}{2} \cdot \frac{n}{2}}{\frac{n}{2}} = \frac{n}{2},$$

so that the tree-cover asymptotes of $tc(SN_n)_s$ is,

$$Asyp(SN_n) = \lim_{n \to \infty} (\frac{n}{2}) = \infty.$$

Path graph

From (6.3), $t(P_n) = 1$. For n even, the minimum vertex cover, S, consists of $\frac{n}{2}$ vertices while for n odd, S consists of $\frac{n-1}{2}$ vertices.

Hence,

$$tc(P_n)_s = \frac{|S|t(H(S))}{t(P_n)} = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{n-1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

It follows that $Asyp(P_n)_s = \infty$.

Wheel graph

Recall that the wheel graph, W_n, is the join between C_n and K_1. For n even, W_n has odd order and the minimum vertex cover of W_n consists of the alternating vertices of C_n and the center vertex K_1. But the number of spanning trees of W_n on $n+1$ vertices is given as $(\frac{3+\sqrt{5}}{2})^{n+1} + (\frac{3-\sqrt{5}}{2})^{n+1} - 2$. Then,

$$\begin{aligned} tc(W_n)_s &= \frac{|S|t(H(S))}{t(W_n)} \\ &= \frac{n+1}{(\frac{3+\sqrt{5}}{2})^{n+1} + (\frac{3-\sqrt{5}}{2})^{n+1} - 2}. \end{aligned}$$

As n increases,

$$tc(W_n)_s \approx \frac{n+1}{2(\frac{3}{2})^{n+1}}.$$

The tree cover asymptote of W_n is

$$\begin{aligned} Asyp(W_n)_s &= \lim_{n\to\infty} \left[\frac{n+1}{2(\frac{3}{2})^{n+1}}\right] \\ &= \lim_{n\to\infty} \frac{n+1}{3(1.5)^n} = 0. \end{aligned}$$

For n odd, a disconnected subgraph $H(S)$ is obtained, which is not considered in this thesis.

Ladder graph

Recall that the ladder graph, L_n, of order n, can be obtained as the Cartesian product of two path graphs one of which has only one edge. Let the vertices in the upper path be $P' = \{u_1, u_2, \ldots, u_{\frac{n}{2}}\}$ and lower path be $P'' = \{v_1, v_2, \ldots, v_{\frac{n}{2}}\}$. Then, the minimum vertex cover, S, can be obtained by taking alternating vertices from P' and P''. Hence, $S = \{u_1, v_2, \ldots, u_{\frac{n}{2}-1}, v_{\frac{n}{2}}\}$ and $|S| = \frac{n}{2}$. Since $t(H(S)) = 1$ and $t(L_n) = \frac{\sqrt{3}}{6}\Big[(2+\sqrt{3})^n - (2-\sqrt{3})^n\Big]$, it follows that the tree cover ratio of L_n

is given as:

$$
\begin{aligned}
tc(L_n)_s &= \frac{|S|t(H(S))}{t(L_n)} \\
&= \frac{\frac{n}{2}}{\frac{\sqrt{3}}{6}\Big[(2+\sqrt{3})^n - (2-\sqrt{3})^n\Big]} \\
&= \frac{n\sqrt{3}}{(2+\sqrt{3})^n - (2-\sqrt{3})^n}.
\end{aligned}
$$

For n large,

$$
tc(L_n)_s \approx \frac{n\sqrt{3}}{2(\sqrt{3})^n}.
$$

Thus, the tree cover asymptotes of L_n is given as

$$
\begin{aligned}
Asyp(L_n)_s &= \lim_{n\to\infty}\left[\frac{n\sqrt{3}}{2(\sqrt{3})^n}\right] \\
&= \lim_{n\to\infty}\left[\frac{n}{2}(\sqrt{3})^{1-n}\right] = 0.
\end{aligned}
$$

Fan graph

From (19), $t(F_n) = \frac{1}{2\sqrt{5}}\Big[(3+\sqrt{5})^{n+1} - (3-\sqrt{5})^{n+1}\Big]$. For n even, the minimum vertex cover, S, consists of $\frac{n}{2}+1$ vertices while for n odd, S consists of $\frac{n+1}{2}$ vertices. Hence, for n large,

$$
tc(F_n)_s = \frac{|S|t(H(S))}{t(F_n)} \approx \begin{cases} \frac{(n+2)}{2}\left(\frac{\sqrt{5}}{5}\right)^n, & \text{if } n \text{ is even,} \\ \frac{(n+1)}{2}\left(\frac{\sqrt{5}}{5}\right)^n, & \text{if } n \text{ is odd.} \end{cases}
$$

It follows that $Asyp(F_n)_s = 0$.

Star graph

The star graph, S_n, consists of the center vertex as the minimum vertex cover. Thus, $|S| = 1$, $t(H(S)) = 1$, and $t(S_n) = 1$. Hence, the tree cover ratio and asymptote respectively are given as:

$$tc(S_n)_s = \frac{|S|t(H(S))}{t(F_n)} = 1,$$

and

$$Asyp(S_n)_s = 1.$$

Star graph of k rays of length 2

The minimum vertex cover S of the star graph of k rays of length 2 consists of vertices of distance 1 from the center of the graph. Its cardinality is $|S| = \frac{n-1}{2}$, so that the tree cover ratio is

$$tc(S_k(n,2))_s = \frac{\frac{n-1}{2}.1}{1} = \frac{n-1}{2}.$$

It follows that the tree cover asymptotes is

$$Asyp(S_k(n,2))_s = \lim_{n\to\infty}\left[\frac{n-1}{2}\right] = \infty.$$

We summarize all the results above in the next theorem.

Theorem 21 *The tree cover ratios and asymptotes of the following graphs are given below.*

Graphs, G	$tc(G)_s$	$Asyp(G)_s$
K_n	$\left(\frac{n-1}{n}\right)^{n-2}$	e^{-1}
$K_{\frac{n}{2},\frac{n}{2}}$	$(\frac{n}{2})^{n-1}$	0
C_n(n even)	$\frac{1}{2}$	$\frac{1}{2}$
P_n	$\frac{n}{2}$(n even), $\frac{n-1}{2}$(n odd)	∞
W_n	$\frac{n+1}{(\frac{3+\sqrt{5}}{2})^{n+1}+(\frac{3-\sqrt{5}}{2})^{n+1}-2}$, ($n$ even)	0
L_n	$\frac{n\sqrt{3}}{(2+\sqrt{3})^n-(2-\sqrt{3})^n}$	0
F_n	$\frac{\sqrt{5}(n+2)}{(3+\sqrt{5})^{n+1}-(3-\sqrt{5})^{n+1}}$($n$ even), $\frac{\sqrt{5}(n+1)}{(3+\sqrt{5})^{n+1}-(3-\sqrt{5})^{n+1}}$($n$ odd)	0
$S_{(n,1)}$	1	1
$S_k(n,2)$	$\frac{n-1}{2}$	∞
SN_n	$\frac{n}{2}$	∞

Table 2: Tree-cover ratios and asymptotes of classes of graphs

7.4 Tree-Cover Areas of Graphs

Definition 19 *Let ξ be the class of all special graphs considered above and $tc(G)_s = \frac{|S|t(H(S))}{t(G)} = f(n)$, for each $G \in \xi$. Then the* **tree-cover area,** *denoted as $Ar(\xi)$, is defined as*

$$Ar(\xi) = \frac{2m}{n}\int f(n)dn.$$

where $\frac{2m}{n}$ is the average degree of the graph and $\int f(n)dn$ represents the **tree-cover height** *of the graph, denoted as $ht(G)$.*

7.4.1 Examples of tree-cover areas of graphs

1. K_n

$$Ar(K_n) = (n-1)\int \left(\frac{n-1}{n}\right)^{n-2} dn.$$

2. $K_{\frac{n}{2},\frac{n}{2}}$

$$Ar(K_{\frac{n}{2},\frac{n}{2}}) = \frac{n}{2}\int\left(\frac{2}{n}\right)^{n-3} dn.$$

3. C_n

$$Ar(C_n) = \frac{2n}{n}\int\frac{1}{2}dn.$$

4. SN_n

$$Ar(SN_n) = 2\int ndn.$$

5. P_n

$$Ar(P_n) = \begin{cases} \frac{2(n-1)}{n}\int\frac{n}{2}\,dn & \text{if } n \text{ is even,} \\ \frac{n-1}{n}\int\frac{n}{2}\,dn & \text{if } n \text{ is odd.} \end{cases}$$

6. S_n

$$Ar(S_n) = \frac{2(n-1)}{n}\int dn.$$

7. $S_k(n,2)$

$$Ar(S_k(n,2)) = \frac{2(n-1)}{n}\int\frac{n-1}{2}dn.$$

8. L_n

$$Ar(L_n) = \frac{3n-4}{n}\int\frac{n\sqrt{3}}{(2+\sqrt{3})^n-(2-\sqrt{3})^n}dn.$$

9. F_n

$$Ar(F_n) = \begin{cases} \frac{2(2n-1)}{n} \int \frac{\sqrt{5}(n+2)}{(3+\sqrt{5})^{n+1}-(3-\sqrt{5})^{n+1}} dn & \text{if } n \text{ is even,} \\ \frac{2(2n-1)}{n} \int \frac{\sqrt{5}(n+1)}{(3+\sqrt{5})^{n+1}-(3-\sqrt{5})^{n+1}} dn & \text{if } n \text{ is odd.} \end{cases}$$

10. W_n

$$Ar(W_n) = \frac{4n}{n+1} \int \frac{n+1}{(\frac{3+\sqrt{5}}{2})^{n+1} + (\frac{3-\sqrt{5}}{2})^{n+1} - 2} dn.$$

By the use of Trapezoid rule, we obtained the following tree cover heights and areas for the range $n = 2$ to $n = 6$.

ξ	$ht(\xi)$	$Ar(\xi)$
K_n	2.4372	12.1860
$K_{\frac{n}{2},\frac{n}{2}}$	2.2260	6.6780
C_n	1.5000	3.0000
P_n, n even	8.0000	12.0000
P_n, n odd	8.0000	6.0000
S_n	4.0000	6.6667
L_n	0.2576	0.3965
F_n, n even	0.1787	0.5754
F_n, n odd	0.2284	0.7343
W_n	0.2364	0.7820

Table 3: Tree-cover heights and areas for classes of graphs

The tree-cover area could be used as alternative confirmation of the use of complete graphs as representations in the ideal communication network problem described in [7]. It follows from Table 3 above that a placed sensor device in a service area modelled as a complete graph has higher degree of accessibility to other set of sensors than other connected graphs considered.

7.5 Summary

In this chapter, we combined the concept of the number of spanning trees, $t(G)$, together with the vertex cover number, S, to derive explicit formulas for the tree cover ratios, and then use these formulas to evaluate the asymptotes of these graphs as n becomes large. However, for the star graph of k rays of length 2, the asymptote as n becomes large is not defined for n even. This concept was motivated by the graph theoretical interpretations of the secretary and gambling problems which converges to the same asymptotes as the complete graph. We defined the notions of a tree-cover, asymptotes and area, and specifically showed that for regular classes of graphs, the complete graph has the largest tree-cover area indicating its robustness and strong connectivity property.

Chapter 8

Conclusion

The main objective of this dissertation is to study the parameters of graphs such as its eigenvalues (adjacency and Laplacian) and the number of spanning trees of special classes of simple connected graphs, G, using the Kirchoff's matrix tree theorem and to apply these formulas in the evaluation of the newly introduced concept of tree-cover ratios, asymptotes and areas [38].

Moreover, to achieve this objective, basic graph terminologies such as subgraphs, spanning subgraphs, trees, branch, rays, and the Cartesian product of two graphs were defined for better understanding of the other chapters. Furthermore, the different types of classes of simple finite connected graphs termed "special graphs", that are commonly found in many real-life applications, were studied and were also used mainly as examples in this dissertation.

In particular, we presented an overview of the basic linear algebra concepts for the structural approach to representation of graphs. The purpose of this study was to describe the various methods of representing connected graphs. We started by giving brief definitions involving groups and fields, and then generalizing it to the Galois fields representation of graphs modulo 2. Based on this Galois representation, we evaluated the number of possible subgraphs associated with graphs as 2^m, where m is the size of the graph. Here, we showed the significance of these subgraphs with an example on K_4, that the number of spanning trees, $t(G) \leq 2^m$. Furthermore, we gave an overview of matrix representations involving the incidence matrix, adjacency matrix and the graph Laplacian which are instrumental for evaluating the eigenvalues of graphs.

However, with the use of the properties of the adjacency spectrum, we studied the characteristic polynomial which contains important information about the graph especially the determinant, trace and eigenvalues of the matrix. In addition, the relationship between the adjacency and Laplacian eigenvalues for k-regular graphs was found to be

$$\prod_{i=1}^{n} \lambda_i = \prod_{i=1}^{n} (k - \mu_i),$$

where λ_i is the Laplacian eigenvalues and μ_i is the adjacency eigenvalues, for $i = 1, 2, \ldots, n$. Moreso, the derived generalised eigenvalues of the Teoplitz matrix was found to be as

$$\lambda_j = a + b\sqrt{bc}\Big[\cos\left(\frac{\pi j}{n+1}\right)\Big],$$

for a, b and c constants.

Further, the trigonometric unit equations based on adjacency spectrum of K_n were investigated and formed, such that for each parameter t considered, a t-complete eigen sequence of odd rational terms were generated for the interval $(0,1)$ and having the form $\frac{a}{2t+1}$. We also showed that the sum of the terms of this sequence is $\frac{t^2}{2t+1}$, and it converges to $\frac{1}{2}$ as t goes to infinity. Additionally, by joining the unit mirror pairs, we obtained the diagram similar to the Farey sequence for $t = 3$.

However, apart from the adjacency eigenvalues, the other work which was done in this dissertation has been on the Laplacian eigenvalues and its use for the derivation of the number of spanning trees of some "special graphs" in Table 1. We showed that the wheel and fan graphs' numbers of spanning trees can be represented in terms of Lucas numbers and the Fibonacci sequence respectively.

Most importantly, we combined the concept of the number of spanning trees, $t(G)$, together with the vertex cover number, S, to derive explicit formulas for the tree cover ratios, and then used these formulas to evaluate the asymptotes of these graphs as n becomes large. For the star graph with k rays of length 2, the asymptote as n becomes large is not defined for n even. This concept was motivated by the graph theoretical interpretations of the secretary and gambling problems which converges to the same asymptotes, e^{-1} as the complete graph. We specifically showed that for regular classes of graphs, the complete graph has the largest tree cover area indicating its robustness and strong connectivity property as shown in Tables 2 and 3.

In future research work, we propose the evaluation of the number of spanning trees using the Galois fields modulo 2 by combinatorial and algorithmic methods. Also, the tree-cover ratios of graphs involving disconnected subgraphs, $H(S)$ of G can be investigated where the number of spanning forest, $t(H(S))$, is evaluated. Future research may also involve considering the tree cover ratio of the complements of the classes of graphs discussed here. Also, the idea of the reciprocal of the tree cover ratio,

$$[tc(G)]^{-1} = \frac{t(G)}{|S|t(H(S))},$$

can be used as alternative measure for the study of graph connectivity. Moreso, it should be observed that the complete graph K_n has many untapped special qualities which would be interesting to explore. It would also be interesting to find a lower bound on tree-cover asymptotes for connected graphs. We mention here that it is conceivable that the complete graph has the smallest value amongst all connected graphs with a positive tree-cover asymptote.

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