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Some Aspects of Semirings of Functions

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This dissertation is submitted to the School of Mathematics, Statistics and Computer Science, College of Agriculture, Engineering and Science University of KwaZulu-Natal, Durban, in fulfilment of the requirements for the degree of Master in Science.

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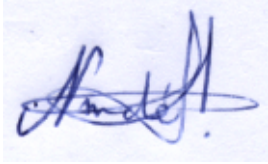
Abstract

A well known fact of classical analysis is that the Stone-Čech compactification of a Tychonoff space is achieved as the structure space of the commutative unitary ring of real valued continuous functions defined on it. An extension of this result is obtained [1] using semiring of non-negative real valued continuous functions on the Tychonoff space. The present work makes a survey of this paper and attempts to extend this to the point free domain, in which it is shown that the collection of frame homomorphisms from the frame of non-negative reals to a frame is a semiring again.

Declaration

I declare that the contents of this dissertation are original except where due reference has been made. It has not been submitted before for any degree to any other institution.

Jissy Nsonde Nsayi



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Dedication

To

My parents

and

my supervisor Partha Pratim Ghosh.

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Chapter 1

Two Constructions of Stone-Čech Compactification

This chapter is based on the paper [1], where it is shown that for a Tychonoff space, the maximal regular congruence on the semiring of non-negative real valued continuous functions on the Tychonoff space is achieved as the structure space of the semiring.

We look at the necessary facts about rings and semirings and furthermore the rings of continuous functions and the Stone-Čech compactifications, we will finish by the semirings of continuous functions and the Stone-Čech compactifications.

1.1 Necessary facts about rings

In this section we introduce a discussion of the numbers of basic concepts of rings.

Definition 1.1.1. Let $(R, +, \cdot)$ be a non-empty set equipped with two binary operations.

Then $(R, +, \cdot)$ is called a ring with identity, if

- (1) $(R, +)$ is an abelian group with identity, which we denote by 0. The element 0 is called the additive identity.
- (2) For each $a, b, c \in R$

i) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$

ii) $a \cdot (b + c) = a \cdot b + a \cdot c$

iii) $(b + c) \cdot a = b \cdot a + c \cdot a$

(3) There exists an element $1 \in R$ such that for each $a \in R$, $1 \cdot a = a \cdot 1 = a$. The element is called the identity with respect to the operation \cdot ; that is, the multiplicative identity.

We assume $1 \neq 0$.

Example 1.1.2. $(\mathbb{Z}, +, \cdot)$, $(\mathbb{Q}, +, \cdot)$, $(M_n(\mathbb{R}), +, \cdot)$, $(\mathbb{Z}_n, +, \cdot)$, are the rings.

Remark 1.1.3. If $a \cdot b = b \cdot a$ for all $a, b \in R$ then the ring R is said to be commutative.

Further if in addition, whenever $a \cdot b = 0$ either $a = 0$ or $b = 0$, then R is said to be an integral domain.

Definition 1.1.4. Let R be a commutative ring with identity and suppose $I \subseteq R$ is a non-empty subset. Then I will be called an ideal in R if

(i) $(I, +)$ is an abelian group.

(ii) For every $a \in I$ and $r \in R$ we have $r \cdot a \in I$.

Example 1.1.5. For each integer n , $n\mathbb{Z}$ is an ideal in \mathbb{Z} .

Remark 1.1.6. (i) If I is an ideal in R and $1 \in I$, then $I = R$.

(ii) If I is an ideal in a field F , then $I = \{0\}$ or $I = F$.

(iii) An ideal I is said to be a principal ideal if it is generated by one element. In other words there exists $x \in I$ such that $I = \{xr : r \in R\} = xR = Rx = \langle x \rangle$.

Definition 1.1.7. Let R be a commutative ring with identity. An ideal P of R , ($P \neq R$), is said to be a prime ideal if for each $a, b \in R$, if $a \cdot b \in P$ then $a \in P$ or $b \in P$.

Example 1.1.8. • In \mathbb{Z} the prime ideals are $\langle 0 \rangle$ and $\langle p \rangle$, for p any prime number.

• In $\mathbb{Q}[X]$ the prime ideals are $\langle 0 \rangle$ and $\langle f(X) \rangle$, for $f(X)$ any irreducible polynomial over \mathbb{Q} .

- In any integral domain R the ideal $\langle 0 \rangle$ is a prime ideal.

Definition 1.1.9. Let R a ring, an ideal $I \subset R$ is said to be proper if $I \neq (0)$ and $I \neq R$.

More generally we show:

Proposition 1.1.10. *Let R be an integral domain and I be a non-zero principal ideal. Then if I is a prime ideal it can be written as $I = xR$ for some irreducible element $x \in R$.*

Proof. Suppose $I = xR$ is a prime ideal and suppose $x = ab$. Then as I is prime it follows $a \in I$ or $b \in I$. Let us suppose $a \in I$, then $a = xr$ for some $r \in R$. It follows $x = xrb$ and so $rb = 1$, i.e b is a unit and so x is irreducible. \square

Definition 1.1.11. Let R be a commutative ring with identity. An ideal M will be called maximal if $M \neq R$ and if $M \subset I$, for any ideal I with $M \neq I$, then $I = R$.

Theorem 1.1.12. *Let R be a commutative ring with identity. Then each maximal ideal is a prime ideal.*

Proof. Let M be a maximal ideal and suppose $ab \in M$ with $b \notin M$. We show that $a \in M$. Since M is maximal it follows that $M + bR = R$.

Therefore there exist $m \in M$ and $r \in R$ such that $m + br = 1$. It follows that $a = am + abr \in M$, as $am \in M$ and $abr \in M$. Consequently M is a prime ideal as asserted. \square

Definition 1.1.13. Let $(R_1, +, \cdot)$ and $(R_2, +, \cdot)$ be rings. A homomorphism from $(R_1, +, \cdot)$ to $(R_2, +, \cdot)$ is a function $h : R_1 \rightarrow R_2$ such that

$$(1) \quad h(a + b) = h(a) + h(b) \text{ for all } a, b \in R_1;$$

$$(2) \quad h(a \cdot b) = h(a) \cdot h(b) \text{ for all } a, b \in R_1.$$

Remark 1.1.14. An injective homomorphism is a monomorphism, and a surjective homomorphism is an epimorphism. If a homomorphism is bijective, it is an isomorphism.

Remark 1.1.15. When working in a context in which all rings have a multiplicative identity, one also requires that $h(1_{R_1}) = 1_{R_2}$

Lemma 1.1.16. *Let $h : (R_1, +, \cdot) \rightarrow (R_2, +, \cdot)$ be a homomorphism of rings. Then*

(1) $h(0) = 0,$

(2) $h(-a) = -h(a)$ for each $a \in R_1.$

(3) $\text{Ker}(h) := \{x \in R_1 : h(x) = 0\}$ is an ideal of R and $\text{Ker}(h) = \{0\}$ if and only if h is a monomorphism.

1.2 Necessary facts about semirings

While a ring can be seen as a monoid of abelian groups, a semiring can be seen as a monoid of commutative monoids. More precisely:

Definition 1.2.1. Let R be a nonempty set with two binary operations $(+)$ and (\cdot) . Then $(R, +, \cdot)$ is a semiring if the following conditions are satisfied:

1) $(R, +)$ is a commutative monoid with identity 0,

$$(a + b) + c = a + (b + c),$$

$$0 + a = a + 0 = a,$$

$$a + b = b + a;$$

2) (R, \cdot) is a monoid with identity 1, henceforth $a \cdot b$ is shortened to ab .

$$(a \cdot b) \cdot c = a \cdot (b \cdot c),$$

$$1 \cdot a = a \cdot 1 = a,$$

3) multiplication distributes over addition:

$$a \cdot (b + c) = a \cdot b + a \cdot c \text{ and } (a + b) \cdot c = a \cdot c + b \cdot c$$

for $a, b, c \in R$

4) $a \cdot 0 = 0 = 0 \cdot a$ for $a \in R$.

Definition 1.2.2. a) A semiring $(R, +, \cdot)$ is said to be commutative if also (R, \cdot) is commutative, and $(R, +, \cdot)$ is called a ring if $(R, +)$ is a group.

b) A semiring $(R, +, \cdot)$ is said to be idempotent if it is one whose addition is idempotent: $a + a = a$.

c) A semiring $(R, +, \cdot)$ is said to be of characteristic m if $ma = a + a + a + a \dots + a$ (m times) equal to zero for all $a \in R$. If no such m exists we say the characteristic of the semiring R is 0 and denote it as characteristic $R = 0$ (usually we write $\text{char}(R)=0$).

Example 1.2.3. 1) By definition each ring and hence each field is a semiring.

2) $\mathbb{R}_{\geq 0}$, the set of all non-zero real numbers is a semiring with respect to usual addition and multiplication.

3) The ideals of a ring form a semiring under addition and multiplication of ideals.

4) For a topological space X , $C_{\geq 0}(X)$, the set of all non-negative real valued continuous functions defined on X , is a semiring with respect to usual addition and multiplication of functions.

Remark 1.2.4. The semiring which is not a ring is called the proper semiring, such is $(\mathbb{N}, +, \cdot)$.

Homomorphism, congruences, ideal

Definition 1.2.5. Let R_1, R_2 be two semirings, and a map $f : R_1 \rightarrow R_2$ is a semiring homomorphism if for $a, b \in R_1$

- $f(a + b) = f(a) + f(b)$
- $f(a \cdot b) = f(a) \cdot f(b)$
- $f(0) = 0$
- $f(1) = 1$

Remark 1.2.6. If such a semiring homomorphism f is surjective, injective or bijective, it is a semiring epimorphism, semiring monomorphism or semiring isomorphism respectively. Finally a semiring homomorphism f of $(R, +, \cdot)$ into $(R, +, \cdot)$ is called semiring endomorphism and semiring isomorphism of $(R, +, \cdot)$ into $(R, +, \cdot)$ is semiring automorphism.

Example 1.2.7. (Sub-semiring)

Given a subset A of a semiring R with the properties:

- i) $0, 1 \in A$
- ii) $a, b \in A \implies a + b, ab \in A,$

it is easy to verify that with the addition and multiplication of R restricted to A , A becomes a semiring in its own right; A is often called a sub-semiring of R .

Further, the inclusion map

$$\begin{aligned} \tau_A : A &\hookrightarrow R \\ x &\mapsto x \end{aligned}$$

is a semiring monomorphism.

Indeed for any semiring homomorphism $f : R \longrightarrow S$, $f(R)$ is a sub-semiring of S ; if further f be a semiring monomorphism then R is isomorphic to $f(R)$ as semirings and thus: the sub-semirings are just the ways to obtain semiring monomorphism.

Example 1.2.8. (Direct Product)

Given the semirings R and S one can define for $R \times S$:

$$\begin{cases} (r_1, s_1) + (r_2, s_2) = (r_1 + r_2, s_1 + s_2) \\ (r_1, s_1)(r_2, s_2) = (r_1 r_2, s_1 s_2) \end{cases}$$

and then $R \times S$ with these definitions along with $(0, 0)$ for the additive identity and $(1, 1)$ as the multiplicative identity becomes a semiring, often referred to as the product semiring.

Further, each of the projections $R \xleftarrow{\pi_1} R \times S \xrightarrow{\pi_2} S$ are semiring epimorphisms. If T is

any another semiring and $R \xleftarrow{f_1} T \xrightarrow{f_2} S$ be any two semiring homomorphisms then:

$$\begin{aligned} f : T &\longrightarrow R \times S \\ t &\longmapsto (f_1(t), f_2(t)) \end{aligned}$$

is a unique semiring homomorphism which makes the diagram below to commute

$$\begin{array}{ccc} & R \times S & \\ \pi_1 \swarrow & \uparrow & \searrow \pi_2 \\ R & \overset{!}{f} & S \\ f_1 \swarrow & T & \searrow f_2 \end{array}$$

Definition 1.2.9. A congruence on a semiring R is an equivalence relation $\rho \subseteq R \times R$ on R such that the following condition holds:

$$\forall (x, y), (u, v) \in \rho \implies \begin{cases} (x + u, y + v) \in \rho \\ (x.u, y.v) \in \rho \end{cases}$$

Theorem 1.2.10. If $f : R \longrightarrow S$ be any semiring homomorphism then $E_f = \{(x, y) \in R : f(x) = f(y)\}$ is a congruence on R .

Proof. Surely E_f is an equivalence relation on R .

Further: $(a, b), (c, d) \in E_f$ implies:

$$f(a + c) = f(a) + f(c) = f(b) + f(d) = f(b + d),$$

$$f(ac) = f(a)f(c) = f(b)f(d) = f(bd),$$

implying $(a + c, b + d), (ac, bd) \in E_f$.

Thus, E_f is a congruence on R . □

Theorem 1.2.11. If ρ be a congruence on R , (x/ρ) denote the equivalence class of x under ρ , then the factor set (R/ρ) is a semiring with the operations:

$$(x/\rho) + (y/\rho) = ((x + y)/\rho),$$

$$(x/\rho)(y/\rho) = ((xy)/\rho),$$

and the additive identity being $(0/\rho)$ while the multiplicative identity being $(1/\rho)$.

Furthermore,

$$\begin{aligned}\mu_\rho : R &\longrightarrow (R/\rho) \\ x &\longmapsto (x/\rho)\end{aligned}$$

is an epimorphism of semirings.

Proof. If $(x/\rho) = (x'/\rho), (y/\rho) = (y'/\rho)$ then:

$$\begin{aligned}(x, x')(x, x'), (y, y') \in \rho &\implies (x + y, x' + y'), (xy, x'y') \in \rho \\ &\implies \begin{cases} ((x + y)/\rho) = ((x' + y')/\rho) \\ ((xy)/\rho) = ((x'y')/\rho), \end{cases}\end{aligned}$$

so that the formulas:

$$\begin{aligned}((x/\rho) + (y/\rho)) &= ((x + y)/\rho), \\ ((x/\rho)(y/\rho)) &= ((xy)/\rho),\end{aligned}$$

are well-defined.

The rest is just routine verification. □

Theorem 1.2.12. *If ρ be any congruence on a semiring R and $f : R \longrightarrow S$ be a semiring homomorphism such that $\rho \subseteq E_f$ then there exists a unique homomorphism $f^* : (R/\rho) \longrightarrow S$ which makes the diagram*

$$\begin{array}{ccc} R & \xrightarrow{\nu_\rho} & (R/\rho) \\ & \searrow f & \swarrow \exists! f^* \\ & & S \end{array}$$

to commute.

Furthermore:

a) $E_{f^*} = (\nu_\rho \times \nu_\rho)(E_f)$.

b) f is an epimorphism, implies f^* is an epimorphism.

c) $E_f = \rho \iff f^*$ is a monomorphism.

Proof. The formulas: $f^*((x/\rho)) = f(x)$, is well-defined since $\rho \subseteq E_f$; clearly f^* is a homomorphism of semirings making the diagram commutes and uniqueness follows from the fact that ν_ρ is an epimorphism.

a)

$$\begin{aligned} ((x/\rho), (y/\rho)) \in E_{f^*} &\iff f^*((x/\rho)) = f^*((y/\rho)) \\ &\iff (f^* \circ \nu)(x) = (f^* \circ \nu)(y) \\ &\iff f(x) = f(y) \\ &\iff (x, y) \in E_f; \end{aligned}$$

and hence, $(\nu_\rho \times \nu_\rho)^{\leftarrow}(E_{f^*}) = E_f$;

since, ν_ρ is surjective, so also is $(\nu_\rho \times \nu_\rho)$ and hence;

$$E_{f^*} = (\nu_\rho \times \nu_\rho)((\nu_\rho \times \nu_\rho)^{\leftarrow}(E_{f^*})) = (\nu_\rho \times \nu_\rho)(E_f).$$

b) f is an epimorphism

$\iff f$ is a surjective homomorphism

\iff for each $s \in S$, there exists a $r \in R$ such that:

$$s = f(r) = f^*((r/\rho))$$

$\iff f^*$ is an epimorphism.

c) If $E_f \subseteq \rho$ then: $f^*((x/\rho)) = f^*((y/\rho)) \implies f(x) = f(y)$

$$\implies (x, y) \in E_f \subseteq \rho \implies (x, y) \in \rho \implies (x/\rho) = (y/\rho),$$

i.e, f^* is a monomorphism.

Conversely, if f^* is a monomorphism, then:

$$(x, y) \in E_f \implies f(x) = f(y) \implies f^*((x/\rho)) = f^*((y/\rho))$$

$$\implies (x/\rho) = (y/\rho) \implies (x, y) \in \rho.$$

□

Corollary 1.2.13. *The quotient semirings of a semiring R are precisely (up to isomorphism) all the epimorphic images of R .*

Proof. If $f : R \rightarrow S$ be an epimorphism, taking $\rho = E_f$ above, it follows that $S \simeq R/E_f$; this completes the proof. □

Definition 1.2.14. 1) A congruence ρ is cancellative if and only if

$$(x + z, y + z) \in \rho \implies (x, y) \in \rho$$

2) A congruence is regular if and only if

i) It is cancellative

ii) There exist $e_1, e_2 \in R$ such that $(a + e_1a, e_2a) \in \rho, (a + ae_1, ae_2) \in \rho$.

Remark 1.2.15. Evidently each cancellative congruence on a semiring R with unity is regular. □

Proposition 1.2.16. *Let R be a semiring and A be a non-empty subset of $R \times R$. An arbitrary intersection of congruences on R is a congruence on R , and hence there is a smallest congruence (with respect to set inclusion) containing A .*

N.B

The smallest congruence will usually be called the congruence generated by A , and will be denoted by (A) .

In what follows, for A a non-empty subset of $R \times R$ we also set

$$A^{-1} = \{(x, y) : (y, x) \in A\},$$

$$A^1 = A \quad \text{and} \quad A^n = A^1 \circ A^{n-1} \quad \forall n \geq 2,$$

where for any two non-empty subsets B, C of $R \times R$, $B \circ C$ denotes the usual compositions of the relations, i.e, $B \circ C = \{(x, z) \in R \times R : (\exists y \in R)((x, y) \in C \quad \text{and} \quad (y, z) \in B)\}$.

We further set

$[A] = \{(xa + y, xb + y) : (a, b) \in A \cup A^{-1}; x, y \in R\} \cup \{(ax + y, bx + y) : (a, b) \in A \cup A^{-1}; x, y \in R\} \cup \Delta_R$ where Δ_R is the diagonal of $R \times R$

$A^* = \bigcup_{n=1}^{\infty} A^n$ and

$\tilde{A} = \{(x, y) : \exists z \in R \text{ such that } (x + z, y + z) \in A\}$

Then the following theorem can be established, see [1].

Theorem 1.2.17. *For any non-empty subset A of $R \times R$,*

$$(A) = [A]^*.$$

Definition 1.2.18. Let R_1, R_2 two semirings and $\psi : R_1 \rightarrow R_2$ be a homomorphism. Then the set

$$\rho_\psi = \{(x, y) \in R_1 \times R_1 : \psi(x) = \psi(y)\}$$

is called the Kernel of ρ .

Theorem 1.2.19. *If $\psi : R_1 \rightarrow R_2$ be a homomorphism, then the Kernel ρ_ψ is a congruence on R_1 . If R_2 is additively cancellative, then ρ_ψ is cancellative.*

Moreover if R_1 has the unity element then ρ_ψ is regular. Finally if ψ is onto, then the quotient semiring R_1/ρ_ψ is isomorphism to R_2 .

Proof. Let $\psi : R_1 \rightarrow R_2$ be a homomorphism

The Kernel ρ_ψ is a congruence on R_1 , because: For all $x, y, z \in R_1$

$(x, y), (u, v) \in \rho_\psi, \implies \psi(x) = \psi(y) \text{ and } \psi(u) = \psi(v) \implies \psi(x) + \psi(u) = \psi(y) + \psi(v) \implies \psi(x + u) = \psi(y + v)$, Then $(x + u, y + v) \in \rho_\psi$.

and $\psi(x).\psi(u) = \psi(y).\psi(v) \implies \psi(x.u) = \psi(y.v)$, then $(x.u, y.v) \in \rho_\psi$.

R_2 is additively cancellative ($\forall a, b, c \in R_2, a + c = b + c \implies a = b$), then ρ_ψ is cancellative, because:

For all $x, y, z \in R_1, (x + z, y + z) \in \rho_\psi \implies \psi(x + z) = \psi(y + z) \implies \psi(x) + \psi(z) = \psi(y) + \psi(z) \implies \psi(x) = \psi(y)$ because $\psi(x), \psi(y), \psi(z) \in R_2$, then $(x, y) \in \rho_\psi$.

The fact that R_1 has the unity element and ρ_ψ is cancellative on the semiring R_1 , it is obvious that ρ_ψ is regular.

The operations on the quotient set R_1/ρ_ψ are defined as follows: for $\rho_\psi(a), \rho_\psi(b) \in R_1/\rho_\psi$, $\rho_\psi(a) + \rho_\psi(b) = \rho_\psi(a + b)$ and $\rho_\psi(a) \cdot \rho_\psi(b) = \rho_\psi(a \cdot b)$.

These operations are well defined and with these operations R_1/ρ_ψ is a semiring. The mapping $\rho_\psi(a) \mapsto \psi(a)$ of R_1/ρ_ψ to R_2 is the required isomorphism. \square

Definition 1.2.20. Let $(R, +, \cdot)$ be a semiring. A nonempty set $I \subseteq R$ is a right (respectively, left) ideal of R if:

- 1) I is a subsemiring,
- 2) for all $i \in I$ and $a \in R$ we have $ia \in I$ (respectively, $ai \in I$).

Definition 1.2.21. Let R be a semiring. A non-empty subset I of R is said to be an ideal of R if I is simultaneously a right and left ideal of R .

Remark 1.2.22. We warn that if R is a ring, a semiring-ideal I is not necessarily a ring-ideal, because the semiring-ideal I is only a subsemiring i.e. a submonoid rather than a subgroup of $(R, +)$. However if the ring R is finite, every semiring-ideal is also a ring-ideal.

1.3 Rings of Continuous Functions and the Čech compactification

Rings of Continuous Functions

Let X be a topological space and $C(X)$ be the function space consisting of all continuous functions from X in \mathbb{R} , the real line (with the usual metric topology) see [7].

We begin by imposing the structures on the collection \mathbb{R}^X of all functions from X into the set \mathbb{R} of real numbers.

We define on \mathbb{R}^X the following operation:

For any $f, g \in \mathbb{R}^X$ and $x \in X$.

$$(1) (f + g)(x) = f(x) + g(x).$$

$$(2) (f \cdot g)(x) = f(x) \cdot g(x).$$

(3) Define $r(x) = r$ for all $x \in X$ and $r \in \mathbb{R}$. These are the constants functions. The special constant functions $1(x)$ and $0(x)$ are the multiplicative and additive identities in \mathbb{R}^X .

$$(4) (-f)(x) = -f(x).$$

(5) If $f(x) \neq 0$ for all $x \in X$ then we may define f^{-1} by

$$f^{-1}(x) = \frac{1}{f(x)}.$$

All the ring axioms are verified. So \mathbb{R}^X is a ring and actually a commutative ring. It is immediate that any constant function other than the additive identity is invertible.

In addition to having these structures, \mathbb{R}^X also has a natural order structure, with the partial order defined by

$$f \geq g \iff f(x) \geq g(x)$$

for all $x \in X$.

Clearly, for every h , $f + h \geq g + h \iff f \geq g$, $f \geq 0$ and $g \geq 0$ implies $f + g \geq 0$ and $f \cdot g \geq 0$. Therefore \mathbb{R}^X is a partially ordered ring.

The positive cone is the set $\{f : f \geq 0\}$.

The absolute value, given by $|f|(x) = |f(x)|$ is an operator mapping \mathbb{R}^X onto its positive cone.

With the absolute value operator defined, we can put a lattice structure on \mathbb{R}^X as well;

- Join: $f \vee g = 2^{-1}(f + g + |f - g|)$.
- Meet: $f \wedge g = f + g - (f \vee g)$.

The set of all continuous functions from the topological space X into the topological space \mathbb{R} is denoted by $C(X)$.

The sum of continuous functions is continuous and so is the product. Furthermore if $f \in C(X)$, then is so $-f$. We have defined a ring structure on \mathbb{R}^X so that $C(X)$ inherits that structure

and forms a commutative ring itself, so it is a subring of \mathbb{R}^X . The constant function $1 \in C(X)$ and its unity element.

Since taking the absolute value of a continuous function is again continuous, $C(X)$ is a sublattice of \mathbb{R}^X . As a result of meet and join, we may consider $C(X)$ as a lattice-ordered ring of continuous functions.

Remark 1.3.1. Any subring of $C(X)$ is called a ring of continuous functions over X . This subring may or may not be a sublattice of $C(X)$.

$C^*(X)$, the subset of $C(X)$ consisting of all bounded continuous functions, is also closed under all of the algebraic operations (ring-theoretic or lattice-theoretic). So $C^*(X)$ is a lattice-ordered subring of $C(X)$.

The Stone-Čech compactification

We show that the maximal ideal spectrum of the rings of continuous functions is a Hausdorff compactification of X with the property that any continuous function from X to any compact Hausdorff space has a unique extension to the maximal ideal spectrum.

The proof is provided in the following sections

All our topological spaces are Tychonoff spaces.

Here is a relevant list of definitions.

Definition 1.3.2. • The spectrum of a commutative ring R , denoted by $Spec(R)$, is defined to be the set of all prime ideals of R . It is commonly augmented with a topology, the Zariski topology.

For any subset A of R , we define the variety of A to be the set

$$V(A) = \{P \in Spec(R) : A \subset P\} \subset Spec(R)$$

and take as closed sets in $Spec(R)$ all subsets of the form $V(A)$.

Obviously one could also consider maximal ideal spectrum $\mathcal{M}(R) = \{M \in Spec(R) : M \text{ is a maximal ideal}\}$, surely it is a subspace of $Spec(R)$ and hence is indeed a T_1 -space.

- For $f \in C(X)$, $Z(f) = \{x \in X : f(x) = 0\}$ is the zero-set of f , $X \setminus Z(f) = \{x \in X : f(x) \neq 0\}$ is co-zero set of f .

Obviously, X is Tychonoff, if and only if $Z(X) = \{Z(f) : f \in C(X)\}$ is a base for its closed sets, if and only if $\{X \setminus Z(f) : f \in C(X)\}$ is a base for the topology.

Firstly one needs to observe a few facts about ideals and the zero sets $Z(X)$ of X .

Lemma 1.3.3. a) *Each member of $Z(X)$ is a closed G_δ -set.*

b) *$Z(X)$ is closed under countable intersections and finite unions with finite unions distributing over arbitrary intersections, i.e.,*

$$i) Z_1, Z_2 \in Z(X) \Rightarrow Z_1 \cup Z_2 \in Z(X),$$

$$ii) (\forall n \geq 0)(Z_n \in Z(X)) \Rightarrow Z = \bigcap_{n \geq 0} Z_n \in Z(X),$$

iii) $Z \in Z(X), Z_n \in Z(X)$ for each $n \geq 0$ imply:

$$Z \cup \left(\bigcap_{n \geq 0} Z_n \right) = \bigcap_{n \geq 0} (Z \cap Z_n).$$

Proof. a) If $Z \in Z(X)$ then there exists a $f \in C(X)$ such that:

$$Z = Z(f) = f^{-1}\{0\} \tag{1.1}$$

$$= \bigcap_{n \geq 0} \{x \in X \mid |f(x)| < \frac{1}{n}\} \tag{1.2}$$

proving the statement.

b) Let for each $n \geq 0$, $f_n \in C(X)$ and $Z_n = Z(f_n)$.

$$i) \text{ Clearly, } Z_1 \cup Z_2 = Z(f_1) \cup Z(f_2) = Z(f_1 f_2).$$

ii) Without any loss of generality we might as well take $f_n \in C(X)$ to be such that

$$x \in X \Rightarrow 0 \leq f_n(x) \leq 1$$

for it not, then $g_n \in C(X)$ where:

$$g_n(x) = \min\{|f_n(x)|, 1\}, x \in X \text{ and } Z(g_n) = Z(f), n \geq 0.$$

Now, using Weirstrass M-test, the series:

$$f(x) = \sum_{n \geq 0} \frac{f_n(x)}{2^n}, x \in X,$$

converges uniformly, and hence $f \in C(X)$; furthermore,

$$f(x) = 0 \iff (\forall n \geq 0)(f_n(x) = 0), i.e$$

$$Z(f) = \bigcap_{n \geq 0} Z(f_n).$$

The distributivity is obvious.

□

In particular, $Z(X)$ is a bounded distributive lattice and one can define a filter/ ultrafilter on $Z(X)$.

Recall, a lattice is a set L with a partial order such that every set $\{x, y\}$ has a supremum $x \vee y$ and an infimum $x \wedge y$; if further there exists a largest element 1 and a smallest element 0 then the lattice is said to be bounded. Furthermore, if any of the following (equivalent) equations hold:

$$* a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c), \text{ for all } a, b, c \in L$$

$$* a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c), \text{ for all } a, b, c \in L$$

then the lattice is a bounded distributive lattice.

If L be a bounded distributive lattice a filter F is a subset of L such that:

$$\text{a) } y \geq x \in F \Rightarrow y \in F$$

$$\text{b) } x, y \in F \Rightarrow x \wedge y \in F.$$

An ultrafilter on L is a filter U which is not contained in any other filter than L or itself.

Having shown that the collection of zero sets $Z(X)$ is a bounded distributive lattice, one calls a filter (resp, an ultrafilter) on $Z(X)$ a z -filter (resp, a z -ultrafilter) on the space X .

Now, an ideal I of $C(X)$ is said to be a z -ideal, if and only if, for any $f \in C(X)$, if $Z(f) = Z(g)$ for some $g \in I$, then $f \in I$.

We could state this much more simply, once we consider the function

$$\begin{aligned} Z : C(X) &\longrightarrow Z(X) \\ f &\longmapsto Z(f) \end{aligned}$$

an ideal I of $C(X)$ is a z -ideal, if and only if $I = Z^{\leftarrow} Z(I)$, where for any function $h : P \rightarrow Q$ and any subset $T \subseteq Q$, $h^{\leftarrow}(T) = \{p \in P : h(p) \in T\}$, the set of pre-images for h for $T \subseteq Q$.

The z -ideals play an important role in this theory and we need to describe them more closely, but before such we need to consider a few more facts pertinent to prime ideals of a commutative unitary ring.

Proposition 1.3.4. *If K be a commutative unitary ring, $S \subseteq K$ be a set closed with respect to multiplication, i.e. ,*

$$x, y \in S \Rightarrow xy \in S$$

and I be any ideal of K disjoint from S , then:

i) there exists an ideal P of K containing I and disjoint from S such that: if Q be any ideal of K containing I and disjoint from S then $Q \subseteq P$.

ii) the ideal P asserted to exist in (i) is prime.

Proof. Let \mathcal{U} be the set of all ideals of K containing I and disjoint from S , partially ordered by \subseteq because every chain in \mathcal{U} has an upper bound in \mathcal{U} namely the union in \mathcal{U} of the chain, it follows from Zorns lemma that there exists a maximal element $p \in \mathcal{U}$, proving (i).

Let $x, y \in K$ such that $y \notin P$ and $x \notin P$; from the maximality of P , there exist $p, q \in P, r, s \in K$ such that: $p + rx, q + sy \in S$ and as S is multiplicatively closed:

$(p + rx)(q + sy)pq + psy + rsxy \in S$, and using the fact that $S \cap P = \emptyset$, $xy \notin P$.

Therefore P is prime ideal. □

Corollary 1.3.5. *If K is a commutative unitary ring, I is an ideal of K , $a \in K$ such that for each $n \geq 0$, $a^n \notin I$, then there exists a prime ideal P such that $a \notin P \supseteq I$, and P is maximal with respect to not containing a and containing I .*

In particular

$$\bigcap \{P : P \supseteq I \text{ and } P \text{ is a prime ideal}\} = \{x \in K : (\exists n \geq 0)(x^n \in I)\}$$

Proof. Take $S = \{a^n : n \geq 0\}$ in 1.3.4 □

Returning back to z -ideals;

Proposition 1.3.6. *The set of z -ideals of $C(X)$ is closed under arbitrary intersection and every z -ideal in $C(X)$ is an intersection of prime ideals.*

Proof. If I be a z -ideals of $C(X)$ then:

$$\begin{aligned} I &\subseteq \bigcap \{P : P \supseteq I \text{ and } P \text{ is a prime ideal}\} = \{f \in C(X) : (\exists n \geq 0)(f^n \in I)\} \\ &\subseteq I \end{aligned}$$

because $Z(f^n) = Z(f)$ and $I = Z^{\leftarrow} Z(I)$. □

Proposition 1.3.7. *There is a one-to-one correspondence between the z -filters on X and the z -ideals of $C(X)$.*

Furthermore this correspondence restricts to a correspondence between the z -ultrafilters on X and the maximal ideals of $C(X)$.

Proof. Firstly, let $z\text{-}\mathcal{I}dl(X)$ be the set of all z -ideals of $C(X)$ and $z\text{-}\mathcal{F}il(X)$ be the set of all z -filters on X . We assert that:

- a) for any ideal I of $C(X)$, $Z(I) = \{Z(f) : f \in I\}$ is a z -filter on X and

b) for any z -filter \mathcal{F} on X , $Z^{-}(\mathcal{F}) = \{f \in C(X) : Z(f) \in \mathcal{F}\}$ is a z -ideal of $C(X)$.

Given these assertions, we have the functions:

$$\begin{array}{ccc} & Z & \\ & \curvearrowright & \\ z - \mathcal{I}dl(X) & & z - \mathcal{F}il(X) \\ & \curvearrowleft & \\ & Z^{-} & \end{array}$$

such that Z is surjective and for any z -ideal, I of $C(X)$, $Z^{-}Z(I) = I$, implying Z to be a bijection and its inverse being Z^{-} .

Let $m(X)$ be the set of all maximal ideals of $C(X)$ and $z\text{-}\mathcal{U}\mathcal{F}il(X)$ be the set of all z -ultrafilters of X .

We shall now show:

c) $M \in m(X) \Rightarrow Z(M) \in z\text{-}\mathcal{U}\mathcal{F}il(X)$;

d) $\mathcal{U} \in z\text{-}\mathcal{U}\mathcal{F}il(X) \Rightarrow Z^{-}(\mathcal{U}) \in m(X)$,

and hence completely prove the proposition, modulo the proofs of the statements (a), (b), (c) and (d).

Proof of (a) Let I be any ideal of $C(X)$,

- obviously, $Z(I) = \{Z(f) : f \in I\} \subseteq Z(X)$;
- if $f \in C(X), g \in I$ such that $Z(g) \subseteq Z(f)$ then

$$\begin{aligned} Z(fg) &= Z(g) \cup Z(f) = Z(f), \\ \text{and } gf \in I, &\text{ implies } Z(f) \in Z(I), \end{aligned}$$

- if $f, g \in I$ then $f^2 + g^2 \in I$ and

$$Z(f^2 + g^2) = Z(f) \cap Z(g) \in Z(I).$$

Hence $Z(I)$ is a filter on $Z(X)$, i.e, $Z(I)$ is a z -filter on X .

Proof of (b) Let \mathcal{F} be a z -filter on X ,

- if $f, g \in Z^{\leftarrow}(\mathcal{F})$ then $Z(f), Z(g) \in \mathcal{F}$ and

$$Z(f + g) \supseteq Z(f) \cap Z(g),$$

implying that $Z(f + g) \in \mathcal{F} \Rightarrow f + g \in Z^{\leftarrow}(\mathcal{F})$,

- if $f \in Z^{\leftarrow}(\mathcal{F}), g \in C(X)$ then $Z(f) \in \mathcal{F}$ and

$$Z(fg) = Z(f) \cup Z(g) \supseteq Z(f),$$

implying $Z(fg) \in \mathcal{F} \Rightarrow fg \in Z^{\leftarrow}(\mathcal{F})$.

- finally, $Z^{\leftarrow}ZZ^{\leftarrow}(\mathcal{F}) = Z^{\leftarrow}(\mathcal{F})$, implying that $Z^{\leftarrow}(\mathcal{F})$ is a z -ideal of $C(X)$.

Proof of (c) If $M \in m(X) \subseteq z\text{-Idl}(X)$, $Z(M) \in z\text{-Fil}(X)$, and if $\mathcal{G} \in z\text{-Fil}(X)$ such that $Z(M) \subseteq \mathcal{G}$ then:

$$M = Z^{\leftarrow}Z(M) \subseteq Z^{\leftarrow}(\mathcal{G}),$$

and then the maximality of M forces $M = Z^{\leftarrow}(\mathcal{G})$ implying $Z(M) = ZZ^{\leftarrow}(\mathcal{G}) = \mathcal{G}$

Hence $Z(M)$ is a z -ultrafilter on X .

Proof of (d) If $\mathcal{U} \in z\text{-UFil}(X) \subseteq z\text{-Fil}(X)$, $Z^{\leftarrow}(\mathcal{U}) \in z\text{-Idl}(X)$ and if I be any ideal of $C(X)$ such that $Z^{\leftarrow}(\mathcal{U}) \subseteq I$ then:

$$\mathcal{U} = ZZ^{\leftarrow}(\mathcal{U}) \subseteq Z(I)$$

so that using maximality of \mathcal{U} , $\mathcal{U} = Z(I)$, implying:

$$\begin{aligned} I \supseteq Z^{\leftarrow}(\mathcal{U}) &= Z^{\leftarrow}Z(I) \supseteq I \\ &\Rightarrow I = Z^{\leftarrow}(\mathcal{U}) \end{aligned}$$

and hence, $Z^{\leftarrow}(\mathcal{U}) \in m(X)$.

□

Furthermore we have:

Proposition 1.3.8. [2]

$m(X)$ is a compact T_1 -space, and Hausdorff if X is Tychonoff.

Proposition 1.3.9. For each $x \in X$, $M_x = \{h \in C(X) : h(x) = 0\}$ is a maximal ideal of $C(X)$

Proof. For $x \in X$, the map

$$\begin{aligned} M_x : C(X) &\longrightarrow \mathbb{R} \\ h &\longmapsto h(x) \end{aligned}$$

is a surjective homomorphism with $\ker h = M_x$. Then, according to the first isomorphism theorem $C(X)/M_x \simeq \mathbb{R}$ which is a field. Therefore M_x is maximal. \square

z -Ideals are indeed algebraic invariants, i.e, z -ideals are preserved under isomorphisms. More precisely: if $\sigma : C(X) \longrightarrow C(Y)$ be an isomorphism of rings then:

$$I \in z\text{-}\mathcal{I}dl(X) \implies \sigma(I) \in z\text{-}\mathcal{I}dl(Y).$$

Proposition 1.3.10. a) For $f, g \in C(X)$, if f belongs to every maximal ideal to which g belongs then $Z(g) \subseteq Z(f)$; i.e, for $f, g \in C(X)$, $M \in m(X)$:

$$(g \in M \implies f \in M) \implies (Z(g) \subseteq Z(f)).$$

b) For any ideal I of $C(X)$, $I \in z\text{-}\mathcal{I}dl(X)$, if and only if, for any $f \in C(X)$:

$$(\exists g \in I) [(\forall M \in m(X))(g \in M \implies f \in M)] \implies f \in I$$

c) If $\sigma : C(X) \longrightarrow C(Y)$ be an isomorphism of rings, and $I \in z\text{-}\mathcal{I}dl(X)$ then $\sigma(I) \in z\text{-}\mathcal{I}dl(Y)$.

Proof. a) Note that for each $x \in X$, $M_x = \{h \in C(X) : h(x) = 0\}$ is a maximal ideal of $C(X)$.

Now, for any $x \in Z(g)$, since $g \in M_x$, from the hypothesis about f , $f \in M_x$, i.e, $f(x) = 0 \implies x \in Z(f)$.

Hence: $Z(g) \subseteq Z(f)$.

b) Let I be a z -ideal of $C(X)$, $f \in C(X)$ such that there exists a $g \in I$ with the property that f belongs to every maximal ideal to which g belongs.

By (a): $Z(g) \subseteq Z(f)$, and hence, $Z(f) \in Z(I)$ implying $f \in Z^+Z(I) = I$.

Conversely, suppose the statement is true for an ideal I of $C(X)$, i.e, for any $f \in C(X)$, if there exists a $g \in I$ such that f belongs to every maximal ideal of $C(X)$ to which g belongs, then $f \in I$.

Choose and fix any $f \in C(X)$ such that $Z(f) \in Z(I)$, i.e $\exists g \in I$, $Z(f) = Z(g)$.

If $M \in m(X)$ such that $g \in M$ then $Z(f) = Z(g) \in Z(M)$ and hence $f \in Z^+Z(M) = M$, since maximal ideals are z -ideals.

Hence, from the given property of I , $f \in I$.

Therefore, $I \in z\text{-}Idl(X)$.

c) Let $\sigma : C(X) \rightarrow C(Y)$ be an isomorphism of rings, $I \in z\text{-}Idl(X)$. Obviously $\sigma(I)$ is an ideal of $C(Y)$.

Let $f \in C(Y)$ be such that there exists a $g \in \sigma(I)$ such that f belongs to any $M \in m(Y)$ whenever $g \in M$.

Hence $\sigma^{-1}(f) \in C(X)$ has the property: $\sigma^{-1}(g) \in I$ such that for any $M \in m(X)$, $\sigma^{-1}(g) \in M \Rightarrow \sigma^{-1}(f) \in M$.

Hence by (b), $\sigma(I) \in z\text{-}Idl(Y)$.

□

The next series of propositions highlight some of the important necessary properties of maximal ideals and prime ideals.

Proposition 1.3.11. $M \in m(X)$, if and only if, for any $f \in C(X)$, if $Z(f) \cap Z(g) \neq \emptyset$ for each $g \in M$, then $f \in M$.

Proof. (\Rightarrow) Let $f \in C(X)$ such that $Z(f) \cap Z(g) \neq \emptyset$, for each $g \in M$. Then $\{Z(f)\} \cup Z(M)$ is a filter base on $Z(X)$ and hence generates a filter \mathcal{G} ; by 1.3.7, $Z(M) \in z\text{-}UFil(X)$ and hence $\mathcal{G} = Z(M)$, i.e, $Z(f) \in \mathcal{G} = Z(M)$, implying $f \in M$.

(\Leftarrow) Let I be an ideal of $C(X)$ with $M \subseteq I$. If $f \in I$, then since $Z(I)$ is a z -filter, $Z(f) \cap Z(g) \neq \emptyset$, for any $g \in M$, and hence, $f \in M$, implying $I = M$.

Hence $M \in m(X)$ □

Proposition 1.3.12. *If $I \in z\text{-I}dl(X)$ then the following are equivalent:*

(a) I is prime ideal

(b) $I \supseteq P$, for some prime ideal P

(c) $fg = 0 \Rightarrow f \in I$ or $g \in I$

(d) for any $f \in C(X)$, there exists a $g \in I$ on which f does not change sign i.e. ,

$$Z(g) \subseteq [f \geq 0] \quad \text{or} \quad Z(g) \subseteq [f \leq 0].$$

Proof. (a) \Rightarrow (b) \Rightarrow (c): trivial.

(c) \Rightarrow (d) If $f \in C(X)$ then $(f \vee 0)(f \wedge 0) = 0$ and hence by (c), $f \vee 0 \in I$ or $f \wedge 0 \in I$.

(d) \Rightarrow (a) Let $f, g \in C(X)$ such that $fg \in I$, and let $h = |f| - |g|$.

by (d), there exists a $u \in I$ such that either $Z(u) \subseteq [h \geq 0]$ or $Z(u) \subseteq [h \leq 0]$.

If $Z(u) \subseteq [h \geq 0] = [|f| - |g|]$ then: $Z(u) \cap Z(f) \subseteq Z(g) \cap Z(u)$,

and hence:

$$\begin{aligned} Z(u) \cap Z(fg) &= Z(u) \cap (Z(f) \cup Z(g)) \\ &= Z(u) \cap Z(g); \end{aligned}$$

since $u, fg \in I$, $Z(u) \cap Z(g) \in Z(I)$, implying further that $Z(g) \in Z(I)$ and hence $g \in I$.

On the other hand, if $Z(u) \subseteq [h \leq 0] = [|g| \leq |f|]$ then we would have obtained similarly, $f \in I$.

Hence: $fg \in I \Rightarrow f \in I$ or $g \in I$, i.e I is prime. □

Theorem 1.3.13. *Every prime ideal of $C(X)$ is contained in a unique maximal ideal.*

Proof. See [7] page 29 □

Theorem 1.3.14. $m(X)$ the maximal ideal spectrum of $C(X)$ is a Hausdorff compactification of X with the property that any continuous function from X to any compact Hausdorff space has a unique extension to $m(X)$.

In particular, $m(X)$ is the Stone-Ćech compactification of X .

Proof. $m(X)$ is endowed with the Hull Kernel Topology

- $m_f = \{M \in m(X) : f \in M\}$, $f \in C(X)$ are basic closed sets
- for any $\mathcal{D} \subseteq m(X)$:

$$cl_{m(X)}(\mathcal{D}) = \{M \in m(X) : M \supseteq \bigcap \mathcal{D}\}.$$

It is shown that $m(X)$ with this topology is compact Hausdorff if X be Tychonoff and

$$\begin{aligned} \eta : X &\longrightarrow m(X) \\ x &\longmapsto M_x \end{aligned}$$

is continuous, injective and $\eta : X \rightarrow \eta(X)$ with $X \simeq \eta(X)$, i.e η is an embedding , also $cl_{m(X)}(\eta(X)) = m(X)$; because

(i) continuous.

$$m_f = \{M_x \in m(X) : f \in M_x\} \quad f \in C(X), \text{ a closed set of } m(X)$$

$$x \in \eta^{-1}(m_f) \Leftrightarrow \eta(x) \in m_f \Leftrightarrow M_x \in m_f \Leftrightarrow f(x) = 0 \Leftrightarrow Z(f) = \{x \in X : f(x) = 0\}$$

$\eta^{-1}(m_f) = Z(f)$ Since $Z(f)$ is a closed set according to the definition , so $\eta^{-1}(m_f)$ is a closed set of X , thus η is continuous.

(ii) injective.

In fact, for all $x, y \in X$, if $x \neq y$, then there exists $f \in C(X)$; $f(x) = 0, f(y) = 1$, which implies that $M_x \neq M_y$

(iii) $\{\eta(x) : x \in Z(f)\}$ is open.

$$\text{In fact, } \eta(Z(f)) = \{\eta(x) : f(x) = 0\} = \{M_x : f(x) = 0\} = \{M_x : f \in M_x\} = m_f \cap \eta(X)$$

So basic open sets of X are open in $\eta(X)$ with $m_f = \{M \in m(X) : f \in M\}$.

iv) Let's show that $cl_{m(X)}(\eta(X)) = m(X)$.

$$\eta(X) = \{M_x : x \in X\}$$

$$cl_{m(X)}\eta(X) = cl_{m(X)}\{M_x : x \in X\} = \{M \in m(X) : M \supseteq \bigcap_{x \in X} M_x\}$$

$\bigcap_{x \in X} M_x = (0)$, because X is Tychonoff.

$$cl_{m(X)}\eta(X) = \{M \in \mathcal{M}(X) : M \supseteq (0)\} = m(X)$$

So $\eta(x)$ is dense in $\mathcal{M}(X)$.

Hence $m(X)$ is a Hausdorff compactification of X .

It remains to show the extension property:

if Y be any compact Hausdorff space and $f : X \rightarrow Y$ be continuous then there exists a unique continuous map $\hat{f} : m(X) \rightarrow Y$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & m(X) \\ & \searrow f & \downarrow \hat{f} \\ & & Y \end{array}$$

commutes.

Obviously, if $g, h : m(X) \rightarrow Y$ are continuous such that $g \circ \eta_X = f = h \circ \eta_X$, then g, h agree on a dense subset of $\mathcal{M}(X)$ and hence, $g = h$. Thus, it is enough to produce one such continuous functions.

Let $f : X \rightarrow Y$ be a continuous map from X to the compact Hausdorff space Y . Obviously compactness of Y forces: $m(Y) = \{M_y : y \in Y\}$.

Now let $M_0 \in m(X)$; obviously: $C(f)^\leftarrow(M_0) = \{g \in C(Y) : g \circ f \in M_0\}$ is prime ideal of $C(Y)$, and hence by theorem 1.3.13 there exists a unique $y_0 \in Y$, such that $C(f)^\leftarrow(M_0) \subseteq M_{y_0}$; define : $\hat{f}(M_0) = y_0$. Clearly, for any $x \in X$: $C(f)^\leftarrow(M_x) = \{g \in C(Y) : g \circ f(x) = 0\} = M_f(x)$; and thus, $\hat{f}(M_x) = f(x)$, i.e $\hat{f} \circ \eta_x = f$.

So it remains to prove the continuity of \hat{f} .

Choose and fix any $M_0 \in m(X)$ and let $\hat{f}(M_0) = y_0$. Since $Z(Y)$ is a base for closed subsets of Y and $\{X \setminus Z(f) : f \in C(Y)\}$ is a base for the topology of Y , for any open set containing y_0 ,

there exist $g, h \in C(Y)$ such that :

$$y_0 \in X \setminus Z(g) \subseteq Z(h) \subseteq U.$$

Clearly $gh = 0$ and

$$\begin{aligned} y_0 \in X \setminus Z(g) &\Leftrightarrow g(y_0) \neq 0 \\ &\Leftrightarrow g \notin M_{y_0} \\ &\Leftrightarrow g \notin C(f)^{\leftarrow}(M_0) \\ &\Leftrightarrow g \circ f \notin M_0 \\ &\Leftrightarrow M_0 \notin m_{g \circ f}, \end{aligned}$$

i.e $M_0 \in m(X) \setminus m_{g \circ f}$.

Now:

$$\begin{aligned} L \in m(X) \setminus m_{g \circ f} &\Leftrightarrow L \notin m_{g \circ f} \\ &\Leftrightarrow g \circ f \notin L \\ &\Leftrightarrow g \notin C(f)^{\leftarrow}(L) \\ &\Rightarrow h \in C(f)^{\leftarrow}(L) \quad (C(f)^{\leftarrow}(L) \text{ is prime and } gh = 0) \\ &\Rightarrow h \in M_{\widehat{f}(L)} \\ &\Leftrightarrow \widehat{f}(L) \in Z(h) \\ &\Rightarrow \widehat{f}(L) \in U; \end{aligned}$$

i.e $\widehat{f}(m(X) \setminus m_{g \circ f}) \subseteq U$, proving continuity . □

1.4 Semirings of Continuous Functions and the Čech compactification

Reminder

It is well known that a proper ideal I in a ring R becomes maximal if and only if the quotient ring R/I does not contain any proper ideal other than the trivial one. Following is the semiring

analogue to this result.

Theorem 1.4.1. *A proper regular congruence ρ on a semiring R is maximal, in the sense that there does not exist any proper regular congruence σ on R with $\rho \subsetneq \sigma$, if and only if the quotient semiring R/ρ has no proper regular congruence, other than the diagonal.*

It was shown in [5] that every proper regular congruence on a semiring can be extended to a maximal congruence.

To prove this theorem it is sufficient to prove the following lemma.

Lemma 1.4.2. *Let ρ_0 be a regular congruence on the semiring R . Then there can be established an order preserving bijection on the set of all regular congruences on R containing ρ_0 onto the set of all regular congruence on the semiring R/ρ_0 .*

Proof. For an arbitrary regular congruence ρ on R containing ρ_0 , set

$$\sigma_\rho = \{(\rho_0(x), \rho_0(y)) : (x, y) \in \rho\}.$$

Then it is easy to see that σ_ρ is a cancellative congruence on R/ρ_0 , which is also regular because so is ρ .

On the another hand for any regular congruence σ on R/ρ_0 , we can see that

$$\rho_\sigma = \{(x, y) \in R \times R : (\rho_0(x), \rho_0(y)) \in \sigma\}.$$

is a regular congruence on R containing ρ_0 . Since for all x, y in R and for any regular congruence ρ on R containing ρ_0 ,

$$(x, y) \in \rho \iff (\rho_0(x), \rho_0(y)) \in \sigma_\rho \iff (x, y) \in \rho_{\sigma_\rho},$$

we have $\rho_{\sigma_\rho} = \rho$. This clearly indicates that $\rho \rightarrow \sigma_\rho$ is a desired order preversing bijection. \square

Henceforth we assume that every semiring consists of at least two elements and also contains the unity.

Let R be such a semiring and let $W(R)$ be the set of all maximal regular congruence on R ,

then it is clear that $W(R)$ is non-empty.

For any subset \mathcal{A} of $W(R)$, we set,

$$h^*(\mathcal{A}) = \{\rho \in W(R) : \rho \supset \cap \mathcal{A}\}$$

We observe that the set $h^*(\mathcal{A})$ satisfy the conditions:

- 1) $\mathcal{A} \subseteq h^*(\mathcal{A})$
- 2) $h^*(h^*(\mathcal{A})) = h^*(\mathcal{A})$
- 3) $\mathcal{A} \subseteq \mathcal{B} \implies h^*(\mathcal{A}) \subseteq h^*(\mathcal{B})$
- 4) $h^*(\mathcal{A} \cup \mathcal{B}) = h^*(\mathcal{A}) \cup h^*(\mathcal{B})$

Definition 1.4.3. [15] A Kuratowski closure operator on a set X is a function from the power set of X to itself i.e. $cl : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ which satisfies the following conditions for all sets $A, B \subseteq X$:

- (1) $A \subseteq cl(A)$.
- (2) $X \subseteq Y \implies cl(X) \subseteq cl(Y)$.
- (3) $cl(cl(A)) = cl(A)$.
- (4) $cl(A \cup B) = cl(A) \cup cl(B)$.

Actually, a closure operator satisfies (1)-(3); the last condition gives Kuratowski closure operator and hence topology.

Thus we are led to:

Theorem 1.4.4. h^* is a Kuratowski closure operator on $W(R)$.

Definition 1.4.5. For R and for $W(R)$,

$$h^*(\mathcal{A}) = \{\rho \in W(R) : \rho \supset \cap \mathcal{A}\}$$

for \mathcal{A} a subset of $W(R)$ determines a topology on $W(R)$, equipped with the topology induced by h^* , is called the structure space of the semiring R . The topology of the structure space $W(R)$ is known as the hull-kernel topology.

We shall note that for each x, y in a semiring, we define

$$W(R)(x, y) = \{\rho \in W(R) : (x, y) \in \rho\}$$

We simply write $W(x, y)$ instead $W(R)(x, y)$. It is easily verifiable that for $x_1, x_2, y_1, y_2 \in R$

$$W(x_1, y_1) \cup W(x_2, y_2) = W(x_1x_2 + y_1y_2, x_1y_2 + x_2y_1)$$

and hence $\{W(x, y) : x, y \in R\}$ is a base for the closed sets of a topology on $W(R)$.

It can further be shown that the topology induced by the base $\{W(x, y) : x, y \in R\}$ is the same as that induced by h^* .

We conclude this section with the following theorem.

Theorem 1.4.6. *$W(R)$ is a compact space.*

To prove the compactness of $W(R)$ it is sufficient to prove the following lemma.

Lemma 1.4.7. *Let $B \subset R \times R$ be such that $(\tilde{B}) = R \times R$. Then there exists a finite subset F of B with the property $(\tilde{F}) = R \times R$*

Proof. Since $(\tilde{B}) = R \times R$, we have $(0, 1) \in (\tilde{B})$. Consequently, there exists $r \in R$ with $(r, 1 + r) \in (B)$.

The equality $(B) = [B]^*$ hold (see 1.2.17) therefore there exist points $x_1, x_2, x_3, \dots, x_n$ in R for a suitable $n \in \mathbb{N}$ such that $(r, x_1), (x_1, x_2), \dots, (x_n, 1 + r)$ each belongs to $[B]$.

This is clearly indicates from the definition of $[B]$ that there exists a finite subset F of B with $(r, x_1), (x_1, x_2), \dots, (x_n, 1 + r)$ each belongs to $[F]$. Which implies that $(r, 1 + r)$ belongs to $[F]^* = (F)$.

Hence $(0, 1)$ belongs to (\tilde{F}) so that $(\tilde{F}) = R \times R$. □

Congruence on the semiring $C_+(X)$

In this section we consider each topological space X to be Tychonoff. For a real number r , \underline{r} denotes the constant function on X such that $\underline{r}(x) = r$ for all $x \in X$ and we shall denote the

subset $\{f \in C(X) : f \geq 0\}$ of $C(X)$ by $C_+(X)$.

Then, it trivially follows that $C_+(X)$ is a semiring with respect to the operations $+$ and \cdot defined as in $C(X)$. The semiring $C_+(X)$ also contains the unity element $\underline{1}$.

Definition 1.4.8. For any f, g in $C_+(X)$, we define

$$E(f, g) = \{x \in X : f(x) = g(x)\}$$

and call it the equaliser set of f and g .

Thus we have a function $E : C_+(X) \times C_+(X) \rightarrow \mathcal{P}(X)$, and hence adopt similar notations for function to describe direct images or pre-images of subsets.

Remark 1.4.9. Let $E(f, g)$ for $f, g \in C_+(X)$:

- 1) $E(f, g) = \mathfrak{Z}(X)$, where $\mathfrak{Z}(X)$ is a zero set in X .
- 2) Every member of $\mathfrak{Z}(X)$ is of the form $E(f, g)$.

Theorem 1.4.10. Let ρ be a cancellative regular congruence on $C_+(X)$. Then

$$E(\rho) = \{E(f, g) : (f, g) \in \rho\}$$

is a z -filter on X .

Proof. (See [1]) We first assert that for any (f, g) in ρ , $E(f, g)$ is non-empty. If not, then for some (f, g) in ρ , $E(f, g) = \emptyset$.

Since for every positive real r and h, k in $C_+(X)$, we have

$$(h + \underline{r}, k + \underline{r}) \in \rho \iff (h, k) \in \rho,$$

without loss of generality we may assume that $Z(f) = Z(g) = \emptyset$, where $Z(f)$ denotes the zero-set of f in X .

Let

$$V_1 = \{x \in X : f(x) > g(x)\}$$

$$V_2 = \{x \in X : f(x) < g(x)\}$$

then V_1, V_2 are disjoint open sets in X with $X = V_1 \cup V_2$. Define h_1, h_2 in $C_+(X)$ as follows:

$$h_1(x) = \begin{cases} g(x), & \text{if } x \in V_1 \\ 0, & \text{if } x \in V_2 \end{cases} \quad \text{and} \quad h_2(x) = \begin{cases} 0, & \text{if } x \in V_1 \\ f(x), & \text{if } x \in V_2 \end{cases}$$

Since ρ is cancellative and $f - h_1, g - h_1, f - h_2, g - h_2$ are all members of $C_+(X)$ and $(f, g) \in \rho$, we must have

$$(f - h_1, g - h_1) \in \rho \quad \text{and} \quad (f - h_2, g - h_2) \in \rho$$

consequently

$$((f - h_1)(g - h_2), (g - h_1)(f - h_2)) \in \rho$$

since $(g - h_1)(f - h_2) = \underline{0}$ and the function $(f - h_1)(g - h_2)$, being strictly positive on X , has a multiplicative inverse in $C_+(X)$, $(\underline{1}, \underline{0}) \in \rho$ and hence $\rho = C_+(X) \times C_+(X)$, then we have a contradiction

The rest of the proof is a routine verification. □

Theorem 1.4.11. *If \mathcal{F} be a z -filter on X , then*

$$E^{\leftarrow}(\mathcal{F}) = \{f, g \in C_+(X) : E(f, g) \in \mathcal{F}\}$$

is a cancellative congruence on $C_+(X)$.

Definition 1.4.12. A cancellative congruence ρ on $C_+(X)$ is a z -congruence if and only if $\rho = E^{\leftarrow}E(\rho)$.

According to the theorems 1.4.10 and 1.4.11, it is clear that any maximal regular congruence on $C_+(X)$ is a z -congruence.

Remark 1.4.13. An arbitrary z -filter \mathcal{F} on X can be expressed as $\mathcal{F} = E(\rho)$ for a z -congruence ρ on $C_+(X)$, indeed one can take $\rho = E^{\leftarrow}(\mathcal{F})$.

The above remark leads to the following theorem

Theorem 1.4.14. *i) If ρ is a maximal cancellative regular congruence on $C_+(X)$, then $E(\rho)$ is a z -ultrafilter on X .*

ii) If \mathcal{F} is a z -ultrafilter on X , then $E^\leftarrow(\mathcal{F})$ is a maximal cancellative regular congruence on $C_+(X)$.

iii) Furthermore $\rho \mapsto E(\rho)$ establishes a bijection on the set of all maximal regular congruences on $C_+(X)$ onto the set of all z -ultrafilters on X .

The above theorem shows that there is a one to one correspondence between a maximal cancellative regular congruence and z -ultrafilters.

Definition 1.4.15. A regular congruence ρ on $C_+(X)$ is called fixed if

$$\bigcap \{E(f, g) : (f, g) \in \rho\} \neq \emptyset$$

and free is otherwise.

Below is a complete description of the fixed maximal regular congruences on $C_+(X)$:

The set of all fixed maximal regular congruences on $C_+(X)$ is precisely $\{\rho_x : x \in X\}$, where

$$\rho_x = \{(f, g) \in C_+(X) \times C_+(X) : f(x) = g(x)\}.$$

ρ_x, ρ_y are distinct whenever x, y are distinct.

Proposition 1.4.16. *For each $x \in X$, the mapping*

$$\begin{aligned} \rho_x : C_+(X) &\longrightarrow \mathbb{R}_+ \\ f &\longmapsto f(x) \end{aligned}$$

is an isomorphism of the quotient semiring $C_+(X)/\rho_x$ onto the semiring \mathbb{R}_+ , i.e. $\mathbb{R}_+ \simeq C_+(X)/\rho_x$ where \mathbb{R}_+ denotes the semiring of non-negative real numbers with usual addition and multiplication,

Proof. For $x \in X$, the map

$$\begin{aligned} \rho_x : C_+(X) &\longrightarrow \mathbb{R}_+ \\ f &\longmapsto f(x) \end{aligned}$$

is a surjective homomorphism with $\ker f = \rho_x$. Then, according to the first isomorphism theorem $C_+(X)/\rho_x \simeq \mathbb{R}_+$ which is a field. \square

Theorem 1.4.17. $W(C_+(X))$ is a compact Hausdorff space.

Proof. Since $C_+(X)$ contains the unity element $\underline{1}$, in view of the Theorem 1.4.6 it follows that $W(C_+(X))$ is compact.

Now let us show that $W(C_+(X))$ is Hausdorff space.

Let ρ_1, ρ_2 be two distinct elements of $W(C_+(X))$, then by Theorem 1.4.14 it follows that $E(\rho_1)$ and $E(\rho_2)$ are distinct z -ultrafilters on X and hence there exist (f_i, g_i) in ρ_i for $i = 1, 2$ such that $E(f_1, g_1) \cap E(f_2, g_2) = \emptyset$

Since X is a Tychonoff space, we can find zero-sets Z_1, Z_2 in X such that:

$$Z_1 \cup Z_2 = X \text{ and } Z_i \cap E(f_i, g_i) = \emptyset \text{ for } i = 1, 2.$$

Choose h_1, h_2 in $C_+(X)$ such that $Z_i = Z(h_i)$ for $i = 1, 2$, then we have $(h_i, \underline{0})$ no belong in ρ_i for $i = 1, 2$ and hence ρ_1 is not in $W(h_1, \underline{0})$ implies $\rho_i \in W(C_+(X) \setminus W(h_i, \underline{0}))$ for $i = 1, 2$

Thus $W(C_+(X) \setminus W(h_i, \underline{0}))$ is a neighbourhood of ρ_i in $W(C_+(X))$ for each $i = 1, 2$.

Observe that $E(h_1.h_2, \underline{0}) = Z(h_1.h_2) = Z_1 \cup Z_2 = X$ and hence $h_1.h_2 = \underline{0}$

Thus $(h_1.h_2, \underline{0}) \in \rho$ for each ρ in $W(C_+(X))$, consequently

$$\rho \in W(h_1.h_2, \underline{0}) = W(h_1, \underline{0}) \cup W(h_2, \underline{0}), \forall \rho \in W(C_+(X)).$$

Hence

$$(W(C_+(X)) \setminus W(h_1, \underline{0})) \cap (W(C_+(X)) \setminus W(h_2, \underline{0})) = \emptyset.$$

Therefore we conclude that $W(C_+(X))$ is Hausdorff space. \square

Theorem 1.4.18. The map

$$\begin{aligned} \eta_X : X &\longrightarrow W(C_+(X)) \\ x &\longmapsto \rho_x \end{aligned}$$

is a dense embedding.

Proof. • η_X is one to one because for $x, y \in X$ if $x \neq y$, as X is Tychonoff then there exists $f \in C_+(X)$; $f(x) = 0, f(y) = 1$ which implies that $\rho_x \neq \rho_y$, then $\eta_X(x) \neq \eta_X(y)$.

- η_X is continuous because:

Recall that $\rho = \{W(f, g) : f, g \in C_+(X)\}$ is a base for the closed sets of $W(C_+(X))$

$$x \in \eta_X^{-1}(\rho) \iff \eta_X(x) \in \rho \iff \rho_x \in W(C_+(X)) \iff f(x) = g(x) \iff E(f, g) = \{x \in X : f(x) = g(x)\} = \mathfrak{Z}(X)$$

As $Z(X)$ is a closed set according to the definition, so $\eta_X^{-1}(\rho)$ is a closed set of X

- η_X is dense because for each A of X , note that

$$\begin{aligned} \eta_X(A) &= \{\rho : \rho \supset \cap \eta_X(A)\} \\ &= \{\rho : \rho \supset \{(f, g) : f(x) = g(x), \forall x \in A\}\} \\ &= \{\rho : \rho \supset \{(f, g) : f(x) = g(x), \forall x \in \overline{A}\}\} \end{aligned}$$

This equality implies that

$$\begin{aligned} \overline{\eta_X(X)} &= \{\rho : \rho \supset \{(f, f) : f \in C_+(X)\}\} \\ &= W(C_+(X)) \end{aligned}$$

$$\text{and } \overline{\eta_X(A)} \cap \eta_X(X) = \eta_X(\overline{A}) \quad \forall \quad A \subset X.$$

□

Since for X is a Tychonoff space, $W(C_+(X))$ is a Hausdorff space, by the above theorem 1.4.18, we have shown that $(\eta_x, W(C_+(X)))$ is a Hausdorff Compactification of X .

Now we shall show that $(\eta_X, W(C_+(X)))$ is the Stone-Ćech compactification of X . We first prove the following:

Theorem 1.4.19. *Let X, Y be two Tychonoff spaces, $f : X \longrightarrow Y$ be a continuous map. Then there exists a continuous function $f^W : W(C_+(X)) \longrightarrow W(C_+(Y))$ such that $\eta_Y \circ f = f^W \circ \eta_X$; that is, such that the square below commutes:*

$$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\eta_X \downarrow & & \downarrow \eta_Y \\
W(C_+(X)) & \xrightarrow{f^W} & W(C_+(Y))
\end{array}$$

Proof. For any ρ in $W(C_+(X))$ define $f^W(\rho)$ in $W(C_+(Y))$ by,

$$\forall h, g \in W(C_+(Y)), \quad (h, g) \in f^W(\rho) \quad \text{if and only if} \quad (h \circ f, g \circ f) \in \rho$$

Recall that a basic closed set of $W(C_+(Y))$ is of the form $\{W(h, g) : h, g \in W(C_+(Y))\}$ and also we have

$$\begin{aligned}
(f^W)^{-1}(W(h, g)) &= \{(f^W)^{-1}(\sigma) : \sigma \in W(h, g)\} \\
&= W(h \circ f, g \circ f)
\end{aligned}$$

which is a basic closed set in $W(C_+(X))$ because $h \circ f, g \circ f$ belong to $C_+(X)$. Thus f^W is continuous.

Finally for all $x \in X$,

$$\begin{aligned}
(f^W \circ \eta_X)(x) &= f^W(\rho_X), \\
&= \{(h, g) \in C_+(Y) \times C_+(Y) : (h \circ f, g \circ f) \in \rho_X\} \\
&= \{(h, g) \in C_+(Y) \times C_+(Y) : (h \circ f)(x) = (g \circ f)(x)\} \\
&= \{(h, g) \in C_+(Y) \times C_+(Y) : h(f(x)) = g(f(x))\} \\
&= \rho_{f(x)} = (\eta_Y \circ f)(x).
\end{aligned}$$

This completes the proof. □

Theorem 1.4.20. *If Y be any compact Hausdorff space and $f : X \rightarrow Y$ be continuous, then there exists a unique continuous map $f^* : W(C_+(X)) \rightarrow Y$ such that the following diagram commutes*

$$\begin{array}{ccc}
X & \xrightarrow{\eta_X} & W(C_+(X)) \\
& \searrow f & \downarrow f^* \\
& & Y
\end{array}$$

Now we conclude this chapter with the following corollary:

Corollary 1.4.21. *For a Tychonoff space X , $W(C_+(X))$ is the Stone-Čech compactification of X , i.e. $W(C_+(X)) \simeq \beta X$.*

Proof. Let $f : X \rightarrow Y$ be an arbitrary continuous function. Then by the above theorem 1.4.19 there is a continuous extension f^* of the function f over $W(C_+(X))$ such that $f^* \circ \eta_X = f$. This shows that $W(C_+(X))$ is the Stone-Čech compactification of X . □

Chapter 2

The Point Free World

In this chapter we describe the Isbell duality and describe the “point free” analogue of topological spaces, namely the locales. We refer to P.T Johnstone [8], and A.Pultr [13] for further details.

2.1 The first facts

Frames

Definition 2.1.1. A frame is a bounded lattice L , which is complete and in which meets distribute over arbitrary joins,

i.e

$$a \wedge (\bigvee S) = \bigvee_{s \in S} (a \wedge s)$$

We shall denote the bottom of L by \perp or 0 and the top by \top or 1 .

Example 2.1.2. 1) Any finite distributive lattice is a frame.

2) Any complete Boolean Algebra is a frame.

Distributive lattice with $0, e$

every x has complement $\sim x$: $x \wedge (\sim x) = 0$; $x \vee (\sim x) = e$.

Fact: $a \wedge b \leq c \iff b \leq (\sim a) \vee c$

(\Leftarrow) $a \wedge b \leq a \wedge ((\sim a) \vee c) = (a \wedge \sim a) \vee (a \wedge c) = a \wedge c \leq c$.

(\Rightarrow)

$$\begin{aligned} (\sim a) \vee c \geq (\sim a) \vee (a \wedge b) &= (\sim a \vee a) \wedge ((\sim a) \vee b) \\ &= e \wedge (\sim a \vee b) \\ &= \sim a \vee b \geq b. \end{aligned}$$

And

$$\begin{aligned} x \wedge \bigvee_i x_i \leq z &\iff \bigvee_i x_i \leq (\sim x) \vee z \\ &\iff x_i \leq (\sim x) \vee z \text{ for each } i \\ &\iff x \wedge x_i \leq z \text{ for each } i \\ &\iff \bigvee_i x \wedge x_i \leq z \text{ for each } i. \end{aligned}$$

Thus $x \wedge \bigvee_i x_i = \bigvee_i x \wedge x_i$.

- 3) Any complete chain i.e totally ordered set is frame.
- 4) The topology of any topological space is a frame.
- 5) $2 = \{\perp, \top\}$ the 2 element frame.

Example 2.1.3. Given a frame L , $M \subset L$ closed under finite meets in L , arbitrary joins in L , then M is frame: it is complete with join and meet restricted from L . M is called a subframe of L .

Definition 2.1.4. Let L and F be two frames. A frame homomorphism is a map $f : L \rightarrow F$ which preserves finite meets and arbitrary joins i.e:

$$f(a \wedge b) = f(a) \wedge f(b) \text{ for all } a, b \text{ in } L,$$

$$f(\bigvee S) = \bigvee_{s \in S} f(s) \text{ for all } S \subseteq L,$$

in particular $f(\top) = \top, f(\perp) = \perp$

Example 2.1.5. 1) For finite distributive lattices, any lattice homomorphism is a frame homomorphism,

2) For each spaces X, Y , the map $f : X \rightarrow Y$ induces a frame homomorphism

$$\begin{aligned} \mathcal{O}f : \mathcal{O}Y &\longrightarrow \mathcal{O}X \\ U &\longmapsto f^{\leftarrow}(U), \end{aligned}$$

3) For complete Boolean Algebras, every complete Boolean homomorphism (complete preserve arbitrary joins) is a frame homomorphism.

Lemma 2.1.6. *The composites of frame homomorphisms are homomorphisms.*

We denote by $\mathcal{F}rm$ the category in which the objects are frames and the arrows are the frames homomorphisms, while $\mathcal{L}oc = \mathcal{F}rm^{op}$ and shall be called the category of locales.

Heyting Algebra

Definition 2.1.7. A lattice L (looked upon as a category) is a Heyting algebra, if and only if for any $a \in L$ the functor $a \wedge - : L \longrightarrow L$ has a right adjoint $a \Rightarrow - : L \longrightarrow L$

Remark 2.1.8. 1) Reconciling with the classical definition:

- for each $a, x, y \in L, a \wedge x \leq y \iff x \leq (a \Rightarrow y)$.

2) Since right adjoints preserve limits, we have:

- if L has a top element then: $(a \Rightarrow 1) = 1$;
- if $y, z \in L$ then: $(a \Rightarrow (y \wedge z)) = (a \Rightarrow y) \wedge (a \Rightarrow z)$.

3) Since left adjoints preserve co-limits, we have:

- if $y, z \in L$ then: $a \wedge (y \vee z) = (a \wedge y) \vee (a \wedge z)$, i.e, L is a distributive lattice.

4) From the unit of the adjunction it follows that for each $x \in L$: $x \leq (a \Rightarrow x)$ and equivalently

$$x \wedge (a \Rightarrow x) = x \quad (2.1)$$

5) From the co-unit of the adjunction it follows that for each $x \in L$: $a \wedge (a \Rightarrow x) \leq x$, and thus:

$$\begin{aligned} a \wedge (a \Rightarrow x) \leq a \wedge x &= a \wedge (a \wedge (a \Rightarrow x)) \quad (\text{from(2.1)}) \\ &\leq a \wedge (a \Rightarrow x) \end{aligned}$$

$$\text{i.e., } a \wedge (a \Rightarrow x) = a \wedge x \quad (2.2)$$

6) Conversely, given the preservation properties in (2) and the equations (2.1) and (2.2) it follows that for any $a \in L$: $(a \wedge -) \dashv (a \Rightarrow -)$ implying that L is a Heyting algebra.

7) If further, L be a complete lattice and a Heyting algebra, then:

$$(a \Rightarrow x) = \bigvee \{y \in L : a \wedge y \leq x\}.$$

Proposition 2.1.9. *Let L be a Heyting algebra*

$$a) \ a \Rightarrow (b \vee c) \geq (a \Rightarrow b) \vee (a \Rightarrow c),$$

$$b) \ p \leq q \implies (a \Rightarrow p) \leq (a \Rightarrow q),$$

$$c) \ (a \vee b) \Rightarrow c = (a \Rightarrow c) \wedge (b \Rightarrow c),$$

$$d) \ p \leq q \implies (q \Rightarrow a) \leq (p \Rightarrow a),$$

$$e) \ (a \wedge b) \Rightarrow c \geq (a \Rightarrow c) \vee (b \Rightarrow c).$$

Proof. (a) and (b) follow from the above remarks and so the proof omitted.

c)

$$\begin{aligned}
x \leq (a \vee b) \Rightarrow c &\iff x \wedge (a \vee b) \leq c \\
&\iff (x \wedge a) \vee (x \wedge b) \leq c \\
&\iff (x \wedge a) \leq c, (x \wedge b) \leq c \\
&\iff x \leq a \Rightarrow c, x \leq b \Rightarrow c \\
&\iff x \leq (a \Rightarrow c) \wedge (b \Rightarrow c)
\end{aligned}$$

Thus we have shown that $(a \vee b) \Rightarrow c = (a \Rightarrow c) \wedge (b \Rightarrow c)$

d) We are going to use the axiom (c) and we have $p \vee q = q$ because $p \leq q$ and by the way $p \vee q \Rightarrow a = q \Rightarrow a$ and according to the last one (c) $(p \Rightarrow a) \wedge (q \Rightarrow a) = q \Rightarrow a$.
Therefore we have $q \Rightarrow a \leq p \Rightarrow a$.

e) $((a \wedge b) \Rightarrow c) \wedge [(a \Rightarrow c) \vee (b \Rightarrow c)] = [((a \wedge b) \Rightarrow c) \wedge (a \Rightarrow c)] \vee [((a \wedge b) \Rightarrow c) \wedge (b \Rightarrow c)]$

According to the axiom (c), we are getting

$$((a \wedge b) \vee a \Rightarrow c) \vee ((a \wedge b) \vee b \Rightarrow c) = (a \Rightarrow c) \vee (b \Rightarrow c)$$

We have proved $((a \wedge b) \Rightarrow c) \wedge [(a \Rightarrow c) \vee (b \Rightarrow c)] = (a \Rightarrow c) \vee (b \Rightarrow c)$

Then we can conclude that $(a \wedge b) \Rightarrow c \geq (a \Rightarrow c) \vee (b \Rightarrow c)$.

□

Definition 2.1.10. Let L be a lattice with 0 and let $a \in L$; a^* is a pseudocomplement of a if $a^* = a \longrightarrow 0$.

We shall denote

$$a^* = \bigvee \{x \in L : a \wedge x = 0\}.$$

Specialising the above results to pseudocomplements yield:

Lemma 2.1.11. Let L be a frame and $a, b \in L$. The following are true:

a) $0^* = 1$ and $1^* = 0$

b) $b \leq b^{**}$

$$c) a \leq b \implies b^* \leq a^*$$

$$d) (a \vee b)^* = a^* \wedge b^*$$

$$e) (a \wedge b)^* \geq a^* \vee b^*.$$

From Remark 3 above, complete Heyting algebras obviously satisfy the infinite distributive law and hence are frames.

Conversely, given any frame, for any element a of the frame the formula

$$\phi_a(x) = \bigvee \{y \in L : a \wedge y \leq x\}$$

provides a right adjoint for $a \wedge -$, and hence is a complete Heyting algebra.

2.2 Isbell Duality

Let X and Y be topological spaces then $\mathcal{O}X$ and $\mathcal{O}Y$ are respective topologies, evidently for any continuous map $f : X \rightarrow Y$ between spaces one gets a frame homomorphism $\mathcal{O}f : \mathcal{O}Y \rightarrow \mathcal{O}X$ which takes $U \in \mathcal{O}Y$ to $f^{\leftarrow}(U) \in \mathcal{O}X$.

We shall now describe a functor in the opposite direction. Let L be a frame and let:

1) $\Sigma L = \mathcal{F}rm(L, 2)$ the set of all frame homomorphisms from L to 2 .

2) For $a \in L$, let $\Sigma_a = \{\theta \in \Sigma L : \theta(a) = 1\}$.

Then

Lemma 2.2.1. $\tau = \mathcal{O}\Sigma L = \{\Sigma_a : a \in L\}$ is indeed a topology on ΣL .

Proof.

a) $\Sigma_i = \{\theta \in \Sigma L : \theta(i) = i, \text{ for more } i \in I\} = \Sigma L \in \tau$

b) Let $\Sigma_a, \Sigma_b \in \tau$ then $\Sigma_a \cap \Sigma_b = \{\theta \in \Sigma L : \theta(a) = 1\} \cap \{\theta \in \Sigma L : \theta(b) = 1\} = \{\theta \in \Sigma L : \theta(a) = 1 \text{ and } \theta(b) = 1\} = \{\theta \in \Sigma L : \theta(a) \wedge \theta(b) = 1\} = \{\theta \in \Sigma L : \theta(a \wedge b) = 1\} = \Sigma_{a \wedge b} \in \tau$.

- c) Let $\Sigma_{a_i} \in \tau$, for some $i \in I$ then $\bigcup \{\theta \in \Sigma L : \theta(a_i) = 1 \text{ for more } i \in I\} = \{\theta \in \Sigma L : \bigvee_{i \in I} \theta_i(a) = 1\} = \{\theta \in \Sigma L : \theta(\bigvee a_i) = 1\} = \Sigma_{\bigvee a_i}$ for some $i \in I$. \square

Usually when we will talk about the topological space ΣL , we shall endow it with this topology $\mathcal{O}\Sigma L$, this space is also called the spectrum of L .

Proposition 2.2.2. *If L and M are frames, and $f : L \rightarrow M$ is a frame homomorphism, we define $\Sigma f : \Sigma M \rightarrow \Sigma L$ as: $\Sigma f(\theta) = \theta \circ f$. Then Σf is continuous.*

Proof. For $a \in L$, $\theta \in (\Sigma f)^{\leftarrow}(\Sigma_a)$

$$\iff \theta \circ f \in \Sigma_a$$

$$\iff (\theta \circ f)(a) = 1$$

$$\iff \theta(f(a)) = 1$$

$$\iff \theta \in \Sigma_{f(a)}$$

Then $(\Sigma f)^{\leftarrow}(\Sigma_a) = \Sigma_{f(a)}$ \square

This describes the "Spectrum" functor $\Sigma : \mathcal{L}oc \rightarrow \mathcal{T}op$.

Our aim is to show the map $\mathcal{O} : \mathcal{T}op \rightarrow \mathcal{L}oc$ is right adjoint to the map $\Sigma : \mathcal{L}oc \rightarrow \mathcal{T}op$, refer to [9] for right adjoint details.

Proposition 2.2.3. *For a frame $L \in \mathcal{L} : L \longrightarrow \mathcal{O}\Sigma L$*

$$a \longmapsto \Sigma_a$$

is a frame homomorphism and a surjection.

Proof. The proof is straightforward. \square

Proposition 2.2.4. *For any $f : L \rightarrow \mathcal{O}X$, there exists a unique $\hat{f} : X \rightarrow \Sigma L$ such that the following diagram commutes:*

$$\begin{array}{ccc}
 L & \xrightarrow{\epsilon_L} & \mathcal{O}\Sigma L \\
 & \searrow f & \swarrow \mathcal{O}\hat{f} \\
 & & \mathcal{O}X
 \end{array}$$

i.e $\mathcal{O}\widehat{f} \circ \epsilon_L = f$.

Proof. For a frame L , ϵ_L is homomorphism and surjective. For each $x \in X$, define $\widehat{x} : L \rightarrow 2$ which is defined by $\widehat{x}(a) = 1$, iff $x \in f(a)$. Then:

- $\widehat{x}(a \wedge b) = 1$
 - $\iff x \in f(a \wedge b) = f(a) \cap f(b)$
 - $\iff x \in f(a), x \in f(b)$
 - $\iff \widehat{x}(a) = 1 = \widehat{x}(b)$
 - $\iff \widehat{x}(a) \wedge \widehat{x}(b) = 1$
 - $\implies \widehat{x}(a \wedge b) = \widehat{x}(a) \wedge \widehat{x}(b)$
- $\widehat{x}(\bigvee S) = 0$
 - $\iff x \notin f(\bigvee S) = \bigcup_{s \in S} f(s)$
 - $\iff x \notin f(s), s \in S$
 - $\iff \widehat{x}(s) = 0, s \in S$
 - $\iff \bigvee_{s \in S} \widehat{x}(s) = 0$
 - $\implies \widehat{x}(\bigvee S) = \bigvee_{s \in S} \widehat{x}(s)$.

Define $\widehat{f} : X \rightarrow \Sigma L$ by, $\widehat{f}(x) = \widehat{x}$. Then: $x \in \widehat{f}^{-1}(\Sigma_a)$

$$\iff \widehat{f}(x) = \widehat{x} \in \Sigma_a$$

$$\iff \widehat{x}(a) = 1$$

$$\iff x \in f(a)$$

$$\implies f(a) = \widehat{f}^{-1}(\Sigma_a) \in \mathcal{O}X.$$

Hence \widehat{f} is continuous.

Further, if $h : X \rightarrow \Sigma L$ be any continuous function such that $\mathcal{O}h \circ \epsilon_L = f$ then for any $a \in L : f(a) = h^{-1}(\Sigma_a)$, so that for any $x \in X$,

$$x \in f(a) = h^{-1}(\Sigma_a) \iff h(x) \in \Sigma_a \iff h(x)(a) = 1 \iff \widehat{x}(a) = 1,$$

implying that $\widehat{x} = h(x)$, further implying $h = \widehat{f}$. □

Corollary 2.2.5. $\mathcal{T}op \begin{array}{c} \xrightarrow{\mathcal{O}} \\ \perp \\ \xleftarrow{\Sigma} \end{array} \mathcal{L}oc$, the open set functor is left adjoint to the spectrum functor.

Furthermore:

- a) The co-unit of the adjunction is $\epsilon : \mathcal{O}\Sigma \longrightarrow 1_{\mathcal{L}oc}$ given by the components ϵ_L for each frame L ; $\epsilon_L : L \longrightarrow \mathcal{O}\Sigma L$ with $\epsilon_L(a) = \Sigma_a$.
- b) the unit of the adjunction is $\eta : 1_{\mathcal{T}op} \longrightarrow \Sigma\mathcal{O}$ given by the components η_X for each space X ;

$$\begin{aligned} \eta_X : X &\longrightarrow \Sigma\mathcal{O} \\ x &\longmapsto \widehat{x} \end{aligned}$$

where $\widehat{x}(U) = 1$, iff $x \in U, U \in \mathcal{O}X$.

- c) the adjunction isomorphisms are for each space X and each locale L :

$$\begin{array}{ccc} \theta_{X,L} : \mathcal{T}op & \longrightarrow & \mathcal{L}oc = \mathcal{F}rm(L, \mathcal{O}X) \\ & g \longmapsto & \mathcal{O}g \circ \epsilon_L \\ \Sigma f \circ \eta_X & \longleftarrow & f \end{array}$$

Proof. Routine computation from 2.2.4. □

Proposition 2.2.6. For a frame, the co-unit $L, \epsilon_L : L \rightarrow \mathcal{O}\Sigma L$ is an isomorphism if and only if ϵ_L is strictly order preserving, i.e. $a < b \implies \Sigma_a \subseteq \Sigma_b$.

Definition 2.2.7. A point in the frame L can be viewed as a frame homomorphism $h : L \rightarrow 2$.

A frame L is said to be spatial, if and only if it has enough points in the sense that for every pair of distinct elements of the frame L , there exists a point of L separating the elements, i.e., more precisely:

$$a, b \in L, a < b \implies (\exists \theta \in \Sigma L)(\theta(a) = 0 < 1 = \theta(b)).$$

Thus, it turns out that L is a frame, if and only if ϵ_L is an isomorphism of frames, i.e. L is isomorphism of some topological space.

We shall now characterise the situation when each unit $\eta_X : X \longrightarrow \Sigma \mathcal{O}X$ is a homeomorphism of topological spaces.

Definition 2.2.8. Let L be a frame

- 1) A filter (resp ideal) of L is an up-set (resp down-set) which is closed under finite meets (resp joins).
- 2) A prime filter (resp ideal) of L is a filter (resp ideal) F (resp I) such that $x \vee y \in F \implies x \in F$ or $y \in F$ (resp $x \wedge y \in I \implies x \in I$ or $y \in I$).
- 3) A completely prime filter (resp ideal) of L is a filter (resp ideal) F (resp I) such that $\bigvee S \in F \implies S \cap F \neq \emptyset$ (resp $\bigwedge S \in I \implies S \cap I \neq \emptyset$).
- 4) An element $b \neq 1$ in a lattice L is prime if for any $x, y \in L$, $x \wedge y \leq b \implies x \leq b$ or $y \leq b$.

Clearly, for any $b \in L$, $\uparrow b = \{x \in L : x \geq b\}$ (resp, $\downarrow b = \{x \in L : x \leq b\}$) is a filter (resp, ideal) of L and $b \in L$ is a prime element of L if and only if $\downarrow b$ is prime ideal of L .

Well, every element in a frame need not be a prime element; for instance, in a topological space X if we consider the topology $\mathcal{O}X$ then:

$U \in \mathcal{O}X$ is a prime element of $\mathcal{O}X$

\iff for any open set $V, W : V \cap W \subseteq U \iff V \subseteq U$ or $W \subseteq U$

\iff for any closed set $F, G : X \setminus U \subseteq F \cup G \iff X \setminus U \subseteq F$ or $X \setminus U \subseteq G$

\iff $X \setminus U$ is an irreducible closed set, where a closed subset D of a topological space X is said to be irreducible, if and only if for every finite closed cover \mathcal{C} of D , there exists one $C \in \mathcal{C}$ such that $D \subseteq C$. For instance:

- if X is any infinite set with co-finite topology, the irreducible closed sets are either the singleton sets or else X itself, and hence the open sets which are prime elements of $\mathcal{O}X$ are actually either the co-singleton sets or the empty set;
- if we have a T_1 -space then the irreducible closed sets are precisely the singletons.

However, a topological space X is said to be sober, if and only if the only irreducible closed subsets of X are the closures of singletons. Hence, for a sober space X , the prime elements of X are precisely the open sets of the form $U_x = X \setminus Cl_X\{x\}$.

Theorem 2.2.9. *Let L be a frame, and b be a prime element of L . Then the map*

$$f : L \longrightarrow 2$$

$$x \longmapsto \begin{cases} 1, & \text{if } x \not\leq b \\ 0, & \text{otherwise} \end{cases}$$

is a frame homomorphism.

Proof. •

$$\begin{aligned} f(x \wedge y) = 1 &\iff x \wedge y \not\leq b \\ &\iff x \not\leq b, y \not\leq b \\ &\iff f(x) = 1 = f(y) \\ &\iff f(x \wedge y) = f(x) \wedge f(y) \end{aligned}$$

•

$$\begin{aligned} f(\bigvee S) = 1 &\iff \bigvee S \not\leq b \\ &\iff \exists s \in S : s \not\leq b \\ &\iff \exists s \in S : f(s) = 1 \\ &\iff \bigvee_{s \in S} f(s) = 1 \end{aligned}$$

□

We shall write b as: $b = \bigvee\{x \in L : f(x) = 0\}$.

Theorem 2.2.10. *For any frame L , there exists a bijection between the following sets:*

a) ΣL the set of all frame homomorphisms from L to 2 .

b) *The completely prime filters of L .*

c) *The completely prime ideals of L .*

d) *The prime elements of L .*

Proof. (See [13] or [8])

□

Definition 2.2.11. A topological space X is irreducible if and only if any finite closed cover is trivial. Equivalently X is irreducible if $X \neq \emptyset$ and X is not the union of two closed subsets different from X .

A closed subspace is irreducible if and only if it is an irreducible space with its subspace topology.

Theorem 2.2.12. *$U \in \mathcal{O}X$ is a prime element if and only if $X \setminus U$ is irreducible.*

Theorem 2.2.13. *For any space X and $x \in X$, $cl\{x\}$ are irreducible closed sets.*

In what follows, by the theorem below we will show that η_X satisfy these axioms is a homeomorphism

Theorem 2.2.14. *For any space X*

a) *η_X is continuous and open.*

b) *η_X is surjective if and only if $Cl\{x\}, x \in X$ are precisely the irreducible closed subsets of X .*

c) *η_X is one to one if and only if $Cl\{x\} = Cl\{y\} \implies x = y$*

Proof. The proof is straightforward.

□

Definition 2.2.15. A Kolmogorov space or T_0 space is sober if and only if the irreducible closed subsets are precisely the closure of singletons.

Let $\mathcal{F}ix(\eta)$ (respectively, $\mathcal{F}ix(\epsilon)$) denoted the spaces X (respectively, frames L) for which η_X (respectively, ϵ_L) is a homeomorphism of topological spaces (respectively, isomorphism of frames). We have shown that $\mathcal{F}ix(\eta)$ is the class of sober topological spaces and $\mathcal{F}ix(\epsilon)$ is the class of spatial frames. These define full subcategories $\mathcal{S}ob$ and $\mathcal{S}p\mathcal{F}rm$ and the adjunction yield:

$$\begin{array}{ccc}
 & \mathcal{O} & \\
 \mathcal{T}op & \begin{array}{c} \xrightarrow{\quad} \\ \perp \\ \xleftarrow{\quad} \end{array} & \mathcal{F}rm \\
 \uparrow & \Sigma & \uparrow \\
 \mathcal{F}ix(\eta) & \begin{array}{c} \xrightarrow{\quad} \\ \perp \\ \xleftarrow{\quad} \end{array} & \mathcal{F}ix(\epsilon)^{op} \\
 & \Sigma &
 \end{array}$$

2.3 Frames are Algebraic over Sets

On Monadicity of Frames

The notion of a monoid can be internalised within any category \mathbb{A} with finite products. More precisely, an internal monoid of \mathbb{A} is a triplet (M, m, e) where M is an object of \mathbb{A} , $m : M \times M \rightarrow M$ is a binary operation on M and $e : 1 \rightarrow M$ is a unary operation on M such that the following equations hold good:

$$\begin{array}{ccccc}
 M \times M \times M & \xrightarrow{m \times 1_M} & M \times M & \xleftarrow{\langle e_M, 1_M \rangle} & M \\
 \downarrow 1_M \times m & & \downarrow m & \parallel & \\
 M \times M & \xrightarrow{m} & M & & \\
 \uparrow \langle 1_M, e_M \rangle & & \parallel & & \\
 M & & & &
 \end{array}$$

In this formulation one could also ask the role of “ \times ”, the terminal object and hence the nullary operation. Appropriate answers to this was provided by Jean Bénabou [6] and Saunders MacLane [10] giving birth to the notion of monoidal categories. Incidentally, both Bénabou and MacLane called them categories with multiplication; the present name is due to Eilenberg.

Definition 2.3.1. A monoidal category is provided by the 6-tuple $(\mathbb{A}, \otimes, I, \alpha, \lambda, \rho)$ where:

- a) \mathbb{A} is a category,
- b) $\otimes : \mathbb{A} \times \mathbb{A} \longrightarrow \mathbb{A}$ is a bi-functor,
- c) I is an object of \mathbb{A} ,
- d) for each $A, B, C \in \mathbb{A}_0$ there are isomorphisms

$$(A \otimes B) \otimes C \xrightarrow{\alpha_{A,B,C}} A \otimes (B \otimes C)$$

natural in A, B, C thereby yielding the natural isomorphism:

$$\alpha : - \otimes (- \otimes -) \longrightarrow (- \otimes -) \otimes -,$$

- e) for each $A \in \mathbb{A}_0$ there are isomorphisms:

$$A \otimes I \xrightarrow{\lambda_A} A \xleftarrow{\rho_A} I \otimes A,$$

natural in A , yielding natural isomorphisms:

$$- \otimes I \xrightarrow{\lambda} - \xleftarrow{\rho} I \otimes -,$$

and there are subject to the following coherence conditions:

- i) Coherence Pentagon

$$\begin{array}{ccc}
 ((A \otimes B) \otimes C) \otimes D & \xrightarrow{\alpha = \alpha_{A \otimes B, C, D}} & (A \otimes B) \otimes (C \otimes D) \\
 \downarrow \alpha_{A,B,C} \otimes 1_D = \alpha \otimes 1_D & & \searrow \alpha = \alpha_{A,B,C \otimes D} \\
 (A \otimes (B \otimes C)) \otimes D & \xrightarrow{\alpha = \alpha_{A, B \otimes C, D}} & A \otimes ((B \otimes C) \otimes D) \\
 & & \nearrow 1_A \otimes \alpha = 1_A \otimes \alpha_{B,C,D} \\
 & & A \otimes (B \otimes (C \otimes D))
 \end{array}$$

ii)

$$\begin{array}{ccc}
 (A \otimes I) \otimes B & \xrightarrow{\alpha = \alpha_{A,I,B}} & A \otimes (I \otimes B) \\
 \searrow \lambda_A \otimes 1_B = \lambda \otimes 1_B & & \swarrow 1_A \otimes \rho = 1_A \otimes \rho_B \\
 & A \otimes B &
 \end{array}$$

iii)

$$\begin{array}{ccc}
 I \otimes I & \xlongequal{\quad} & I \otimes I \\
 \searrow \rho_I & & \swarrow \lambda_I \\
 & I &
 \end{array}$$

i.e., $\lambda_I = \rho_I$.

While every category with finite product is an example – take $\otimes := X, I = 1, \alpha, \lambda, \rho$ are the usually isomorphisms, the following is also another example.

Example 2.3.2. Consider the functor category $\mathbb{A}^{\mathbb{A}}$ of all endo-functors on \mathbb{A} and then put:

- $\otimes :=$ the composition of functors,
- $I := 1_{\mathbb{A}}$,
- α, λ, ρ are equalities, and we get a monoidal category.

One can then copy the notion of a monoid in any monoidal category:

Definition 2.3.3. Given any monoidal category $(\mathbb{A}, \otimes, I, \alpha, \lambda, \rho)$ any internal monoid of \mathbb{A} is a triplet (M, m, e) where M is an object of \mathbb{A} , $m; M \otimes M \rightarrow M$ is a monoidal binary operation on M , $e : I \rightarrow M$ is a monoidal nullary operation on M , such that the following equations are true:

$$\begin{array}{ccccc}
 (M \otimes M) \otimes M & \xrightarrow{\alpha = \alpha_{M,M,M}} & M \otimes (M \otimes M) & & \\
 \downarrow m \otimes 1_M & & \downarrow 1_M \otimes m & & \\
 M \otimes M & & M \otimes M & & \\
 \nearrow e \otimes 1_M & & \nwarrow m & & \nwarrow 1_M \otimes e \\
 I \otimes M & \xrightarrow{\rho = \rho_M} & M & \xleftarrow{\lambda = \lambda_M} & M \otimes I
 \end{array}$$

Surely in all categories with finite products the monoids from 2.3.2 are precisely the internal monoids; in the monoidal category of endo-functors above the monoids are precisely the monads. More precisely:

Definition 2.3.4. A monad on \mathbb{A} is a triplet (T, μ, η) where $T : \mathbb{A} \rightarrow \mathbb{A}$ is an endo-functor on \mathbb{A} , $\mu : T^2 \rightarrow T$ and $\eta : 1_{\mathbb{A}} \rightarrow T$ are natural transformations such that:

$$\begin{array}{ccccc}
 T^3 & \xrightarrow{T\mu} & T^2 & \xleftarrow{T\eta} & T \\
 \mu T \downarrow & & \downarrow \mu & & \parallel \\
 T^2 & \xrightarrow{\mu} & T & & \\
 \eta T \uparrow & & \parallel & & \\
 T & & & &
 \end{array}$$

commutes.

The following provides enough examples of a monad:

Theorem 2.3.5. Given any adjunction $\mathbb{A} \begin{array}{c} \xrightarrow{F} \\ \perp \\ \xleftarrow{U} \end{array} \mathbb{B}$, F is left adjoint U with unit $\eta : 1_{\mathbb{A}} \rightarrow UF$ and co-unit $\epsilon : FU \rightarrow 1_{\mathbb{B}}$ the triplet $(T = UF, \mu = U\epsilon F, \eta)$ is monad on \mathbb{A} .

Proof. refer [9]. □

However, these are in some sense the only ways to obtain monads and we describe this part of the story.

Definition 2.3.6. Given any monad $\mathcal{J} = (T, \mu, \eta)$ on \mathbb{A} , a \mathcal{J} -algebra on \mathbb{A} is a pair (A, a) where A is an object of \mathbb{A} , $a : T(A) \rightarrow A$ is an arrow of \mathbb{A} (often called the \mathcal{J} -algebra structure on A) such that:

$$\begin{array}{ccc}
 T^2(A) & \xrightarrow{T(a)} & T(A) \\
 \mu_A \downarrow & & \downarrow a \\
 T(A) & \xrightarrow{a} & A
 \end{array}$$

commutes.

If (A, a) and (B, b) are \mathcal{J} -algebra on \mathbb{A} then a \mathcal{J} -algebra homomorphism from (A, a) to (B, b) is an arrow $f : A \rightarrow B$ of \mathbb{A} preserving the \mathcal{J} -algebra structure i.e. the diagram below commutes.

$$\begin{array}{ccc} T(A) & \xrightarrow{T(f)} & T(B) \\ a \downarrow & & \downarrow b \\ A & \xrightarrow{f} & B \end{array}$$

We shall denote this by: $(A, a) \xrightarrow{f} (B, b)$.

$\mathbb{A}^{\mathcal{J}}$:= the category of \mathcal{J} -algebras and \mathcal{J} -algebra homomorphisms.

Obviously, one has the forgetful functor:

$$U^{\mathcal{J}} : \begin{array}{ccc} \mathbb{A}^{\mathcal{J}} & \longrightarrow & \mathbb{A} \\ (A, a) & \longmapsto & A \\ f \downarrow & & \downarrow f \\ (B, b) & \longmapsto & B \end{array}$$

which forgets the \mathcal{J} -algebra structure on A .

However, there is indeed the “ \mathcal{J} -algebra” functor which is granted by:

Theorem 2.3.7. *Given any monad $\mathcal{J} = (T, \mu, \eta)$ on \mathbb{A} the following statements are true:*

a) *for any object A of \mathbb{A} , $(T(A), \mu_A)$ is a \mathcal{J} -algebra on A .*

b) *for any arrow $A \xrightarrow{f} B$ of \mathbb{A} :*

$$(T(A), \mu_A) \xrightarrow{T(f)} (T(B), \mu_B)$$

is a \mathcal{J} -algebra homomorphism.

c) *For any object A of \mathbb{A} , any \mathcal{J} -algebra (B, b) and any arrow $f : A \rightarrow B$ there exists a*

unique \mathcal{J} -algebra homomorphism $f^* : (T(A), \mu_A) \longrightarrow (B, b)$ such that: $f = U^{\mathcal{J}}(f^*) \circ \eta_A$.

$$\begin{array}{ccc}
 A & \xrightarrow{\eta_A} & T(A) = U^{\mathcal{J}}(T(A), \mu_A) \\
 & \searrow f & \swarrow \text{---} U^{\mathcal{J}}(f^*) \text{---} \\
 & & B = U^{\mathcal{J}}(B, b)
 \end{array}$$

d) $U^{\mathcal{J}} : \mathbb{A}^{\mathcal{J}} \longrightarrow \mathbb{A}$ has a left adjoint:

$$F^{\mathcal{J}} : \quad \mathbb{A} \longrightarrow \mathbb{A}^{\mathcal{J}}$$

$$\begin{array}{ccc}
 A \vdash & \longrightarrow & (T(A), \mu_A) \\
 \downarrow f & & \downarrow T(f) \\
 B \vdash & \longrightarrow & (T(B), \mu_B)
 \end{array}$$

($F^{\mathcal{J}}$ is called the free \mathcal{J} -algebra functor).

e) The adjunction $\mathbb{A} \begin{array}{c} \xrightarrow{F^{\mathcal{J}}} \\ \perp \\ \xleftarrow{U^{\mathcal{J}}} \end{array} \mathbb{A}^{\mathcal{J}}$ where $F^{\mathcal{J}}$ is left adjoint $U^{\mathcal{J}}$ has the unit $\eta : 1_{\mathbb{A}} \longrightarrow U^{\mathcal{J}}F^{\mathcal{J}}$ and the co-unit $\epsilon^{\mathcal{J}} : F^{\mathcal{J}}U^{\mathcal{J}} \longrightarrow 1_{\mathbb{A}^{\mathcal{J}}}$ with $\epsilon_{(a,b)}^{\mathcal{J}} = b$.

f) The monad on \mathbb{A} induced by $\mathbb{A} \begin{array}{c} \xrightarrow{F^{\mathcal{J}}} \\ \perp \\ \xleftarrow{U^{\mathcal{J}}} \end{array} \mathbb{A}^{\mathcal{J}}$ where $F^{\mathcal{J}}$ is left adjoint $U^{\mathcal{J}}$ is \mathcal{J} .

Proof. refer to [9]. □

Having established the fact that adjunctions yield monads and vice-versa we proceed to the relationship between adjunctions inducing the same monad.

Theorem 2.3.8. Given the adjunction $\mathbb{A} \begin{array}{c} \xrightarrow{F} \\ \perp \\ \xleftarrow{U} \end{array} \mathbb{B}$ where F is left adjoint U with co-unit $\epsilon : FU \longrightarrow 1_{\mathbb{B}}$ and unit $\eta : 1_{\mathbb{A}} \longrightarrow UF$ inducing the monad $\mathcal{J} = (t = UF, \eta, \mu = U\epsilon F)$ on \mathbb{A}

there exists a unique functor $K : \mathbb{B} \longrightarrow \mathbb{A}^{\mathcal{J}}$ defined by:

$$K : \quad \mathbb{B} \longrightarrow \mathbb{A}^{\mathcal{J}}$$

$$\begin{array}{ccc} B & \longmapsto & (U(B), U_{\epsilon_B}) \\ \downarrow f & & \downarrow U(f) \\ C & \longmapsto & (U(C), U_{\epsilon_C}) \end{array}$$

such that the following diagram commutes

$$\begin{array}{ccc} & \mathbb{A}^{\mathcal{J}} & \\ F^{\mathcal{J}} \nearrow & & \searrow U^{\mathcal{J}} \\ \mathbb{A} & & \mathbb{A} \\ & \dashrightarrow K & \\ F \searrow & & \nearrow U \\ & \mathbb{B} & \end{array}$$

(K is called the comparison functor).

The comparison functor obviously compares \mathbb{B} with \mathcal{J} -algebras and thus:

Definition 2.3.9. Given a functor $\mathcal{U} : \mathbb{B} \longrightarrow \mathbb{A}$ with a left adjoint, we say that \mathbb{B} is monadic over \mathbb{A} via \mathcal{U} or \mathcal{U} is monadic, if and only if the comparison functor is an isomorphism of categories.

Alternative usage: “ algebraic ” instead of “ monadic”.

It turns out that all the equationally defined algebra are monadic over sets by the usual forgetful functor. The following produces a complete characterisation of monadicity.

Theorem 2.3.10. Beck’s Precise Monadicity Theorem (BPMT)

The following statements are equivalent for any functor $\mathcal{U} : \mathbb{B} \longrightarrow \mathbb{A}$ with a left adjoint:

- a) \mathcal{U} is monadic,

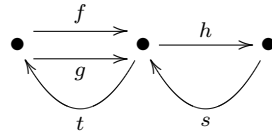
b) \mathcal{U} creates co-equalisers of every all pairs in \mathbb{B} whose \mathcal{U} -image has a absolute co-equaliser in \mathbb{A} ,

c) \mathcal{U} creates co-equalisers of all pairs in \mathbb{B} whose \mathcal{U} -image has a split co-equaliser in \mathbb{A} .

The statement of BPMT demands explanation of certain terms:

Definition 2.3.11. a) A parallel pair or pair in a category is a pair of arrows with same domain and co-domain.

b) A split coequaliser in a category is a diagram of the form:



such that: $hf = hg, hs = 1, ft = 1, gt = sh$.

c) An absolute coequaliser in a category is a coequaliser which is preserved by any functor.

d) A functor $\mathcal{U} : \mathbb{B} \rightarrow \mathbb{A}$ creates coequalisers for the pair $A \begin{matrix} \xrightarrow{f} \\ \xrightarrow{g} \end{matrix} B$ if and only if the following is true:

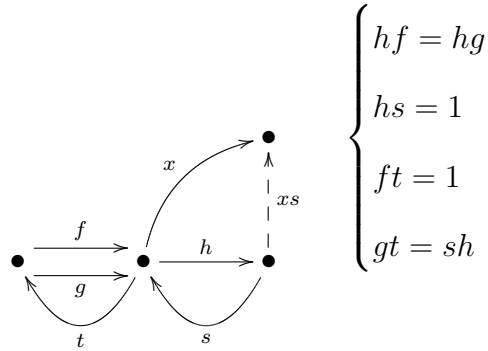
if: the coequaliser of $\mathcal{U}(A) \begin{matrix} \xrightarrow{\mathcal{U}(f)} \\ \xrightarrow{\mathcal{U}(g)} \end{matrix} \mathcal{U}(B)$ exists in \mathbb{A} and is :

$$\mathcal{U}(A) \begin{matrix} \xrightarrow{\mathcal{U}(f)} \\ \xrightarrow{\mathcal{U}(g)} \end{matrix} \mathcal{U}(B) \xrightarrow{p} Z$$

then: there exists a unique arrow $h : B \rightarrow C$ such that $A \begin{matrix} \xrightarrow{f} \\ \xrightarrow{g} \end{matrix} B \xrightarrow{h} C$ is the coequaliser of f and g and $\mathcal{U}(h) = p$.

Remark 2.3.12. Given the split coequaliser, if $xf = xg$ then:

$xsh = xgt = xft = x$, and if y is any arrow with $yh = x \implies xs = yhs = h$.



Hence h is the coequaliser of f and g ; also since this is completely defined by certain equations it is preserved by any functor.

In other words: split coequalisers are absolute coequalisers which are special coequalisers.

BPMT there asserts: this monadic, if and only if it creates some good coequalisers.

We shall use this to deduce:

Theorem 2.3.13. *The category \mathcal{Frm} of frames and homomorphisms is monadic over \mathcal{Sets} under the usual forgetful functor.*

Surely every frame is a bounded (meet) semi-lattice and we shall first show:

Theorem 2.3.14. *\mathcal{Frm} is monadic over the category $\mathcal{BSemLat}$ of bounded semi-lattices and their homomorphisms under the usual forgetful functor.*

Proof. Recall the category $\mathcal{BSemLat}$ has:

- object are bounded semi-lattices; i.e. $(B, \wedge, 0, 1)$ where $(B, \wedge, 1)$ is a commutative monoid, 0 is a nullary with: $x \wedge 0 = 0$;
- arrows are bounded semi-lattice homomorphisms, i.e. if $(B, \wedge, 0, 1)$ and $(C, \wedge, 0, 1)$ are bounded semi-lattices then a function $f : B \rightarrow C$ such that: $f(x \wedge y) = f(x) \wedge f(y)$, $f(0) = 0$, $f(1) = 1$.

Obviously each frame is a bounded semi-lattice, giving the forgetful function

$$\mathcal{U} : \mathcal{Frm} \longrightarrow \mathcal{BSemLat}.$$

Step 1 \mathcal{U} has a left adjoint:

Choose any bounded semi-lattice $(B, \wedge, 0, 1)$.

Let $\mathcal{DB} :=$ the set of all down-sets of B , i.e. subsets $X \subseteq B$ such that: $y \leq x \in X \Rightarrow y \in X$, and \mathcal{D} be a distribution lattice.

Clearly:

- for each $x \in B, \downarrow x = \{y \in B : y \leq x\} \in \mathcal{DB}$,
- $B, \{0\} \in \mathcal{DB}$,
- if $X, Y \in \mathcal{DB}$ then $X \cap Y \in \mathcal{DB}$,
- if $S \subseteq B$ is any subset of B and

$$\tilde{S} = \{x \in B : (\exists s \in S)(x \leq s)\} = \bigcup_{s \in S} (\downarrow s).$$

Then:

- $\tilde{S} \supseteq S$
- \tilde{S} is a down-set
- if $T \in \mathcal{DB}, S \subseteq T$ then:

$$\begin{aligned} x \in \tilde{S} &\iff (\exists s \in S)(x \leq s) \\ &\implies (\exists s \in T)(x \leq s) \\ &\implies x \in T, \end{aligned}$$

implying $\tilde{S} \subseteq T$.

Thus, given family $(X_i)_{i \in I}$ of down-sets in B ,

$$\bigvee_{i \in I} X_i = \left(\bigcup_{i \in I} \tilde{X}_i \right), \tag{2.3}$$

showing \mathcal{DB} to be a complete lattice with meet as intersection, the joins defined by (2.3), B as the largest element and $\{0\}$ as the smallest element.

If $(X_i)_{i \in I}$ is any family of down-sets of B and $Y \in \mathcal{DB}$ then:

– for each $i \in I$,

$$Y \cap X_i \subseteq Y \cap \bigvee_{i \in I} X_i \implies \bigvee_{i \in I} (Y \cap X_i) \subseteq Y \cap \bigvee_{i \in I} X_i$$

–

$$\begin{aligned} y \in Y \cap \bigvee_{i \in I} X_i &\iff y \in Y \text{ and } y \in \bigcup_{i \in I} \tilde{X}_i \\ &\iff y \in Y \text{ and } (\exists i \in I)(\exists x \in X_i)(y \leq x) \\ &\implies y \in Y \cap X_i \subseteq \bigvee_{i \in I} (Y \cap X_i), \end{aligned}$$

implying:

$$Y \cap \bigvee_{i \in I} X_i = \bigvee_{i \in I} (Y \cap X_i)$$

Hence \mathcal{DB} is a frame.

Further, for the function:

$$\begin{aligned} \downarrow: B &\longrightarrow \mathcal{UDB} \\ x &\longmapsto (\downarrow x) \end{aligned}$$

•

$$\begin{aligned} x \leq (\downarrow b) \cap (\downarrow c) &\iff x \leq b \text{ and } x \leq c \\ &\iff X \leq \downarrow (b \wedge c) \end{aligned}$$

i.e $(\downarrow b) \cap (\downarrow c) = \downarrow (b \wedge c)$,

• $\downarrow 1 = B$,

• $\downarrow 0 = \{0\}$,

implying that $\downarrow: B \rightarrow \mathcal{U}DB$ is a homomorphism of bounded semi-lattices.

Finally, suppose that L is any frame and $f: B \rightarrow \mathcal{U}(L)$ is a homomorphism of bounded semi-lattices.

define:

$$f^* : \mathcal{D}B \rightarrow L$$

$$X \mapsto \bigvee f(X)$$

such that the following diagram commutes.

$$\begin{array}{ccc}
 B & \xrightarrow{\quad \downarrow \quad} & \mathcal{U}DB \\
 & \searrow f & \swarrow \mathcal{U}(f^*) \\
 & & \mathcal{U}(L)
 \end{array}$$

Then:

- if $X, Y \in \mathcal{D}B$ then

$$\begin{aligned}
 & \left(\bigvee f(X) \right) \wedge \left(\bigvee f(Y) \right) \\
 &= \bigvee_{x \in X} (f(x) \wedge f(Y)) \\
 &= \bigvee_{x \in X} \bigvee_{y \in Y} (f(x) \wedge f(y)) \\
 &= \bigvee_{x \in X} \bigvee_{y \in Y} f(x \wedge y) \quad (\because f: B \rightarrow \mathcal{U}(L) \text{ is in } \mathcal{BSemLat}) \\
 &\leq \bigvee_{r \in X \cap Y} f(r) = \bigvee f(X \cap Y) \\
 &\leq \left(\bigvee f(X) \right) \wedge \left(\bigvee f(Y) \right)
 \end{aligned}$$

implying $f^*(X \cap Y) = f^*(X) \wedge f^*(Y)$.

- if S is any subset of B then:

$$\begin{aligned}
f^*(\tilde{S}) &= \bigvee f(\tilde{S}) \\
&= \bigvee f(\{x \in B : (\exists s \in S)(x \leq s)\}) \\
&= \bigvee (\{f(x) : (\exists s \in S)(x \leq s)\}) \\
&\leq \bigvee_{s \in S} f(s) = \bigvee f(S) \\
&\leq \bigvee f(\tilde{S}) \quad (\because S \subseteq \tilde{S})
\end{aligned}$$

implying $f^*(\tilde{S}) = \bigvee f(S)$.

- if $(X_i)_{i \in I}$ is any family of down-sets of B then:

$$\begin{aligned}
f^*\left(\bigvee_{i \in I} X_i\right) &= f^*\left(\bigcup_{i \in I} X_i\right) \\
&= \bigvee f\left(\bigcup_{i \in I} X_i\right) \\
&= \bigvee \left(\bigcup_{i \in I} f(X_i)\right) \\
&= \bigvee_{r \in \bigcup_{i \in I} X_i} f(r) \\
&= \bigvee_{i \in I} \bigvee_{r \in X_i} f(r) = \bigvee_{i \in I} \left(\bigvee f(X_i)\right) \\
&= \bigvee_{i \in I} f^*(X_i).
\end{aligned}$$

Hence, $f^* : \mathcal{D}B \rightarrow L$ is a frame homomorphism such that $f^*(\downarrow x) = f(x)$.

Finally, if $\mathcal{D}B \xrightarrow[h]{g} L$ are frame homomorphisms which agree on principal down-sets, i.e.

$$h(\downarrow x) = g(\downarrow x), \quad x \in B$$

then for any down set X , since $X = \bigvee_{x \in X} (\downarrow x)$:

$$\begin{aligned} h(X) &= h\left(\bigvee_{x \in X} (\downarrow x)\right) \\ &= \bigvee_{x \in X} h(\downarrow x) \\ &= \bigvee_{x \in X} g(\downarrow x) = g(X), \end{aligned}$$

i.e. $g = h$, showing $\downarrow: B \rightarrow \mathcal{U}DB$ to be epimorphism of bounded semi-lattices.

Hence, $f^*: \mathcal{D}B \rightarrow L$ is the unique frame homomorphism such that $\mathcal{U}(f^*) \circ \downarrow = f$.

Therefore $(\mathcal{D}B, \downarrow: B \rightarrow \mathcal{U}DB)$ is the universal arrow from B to \mathcal{U} .

Since each bounded semi-lattice produces a universal arrow there is a left adjoint to $\mathcal{U}: \mathcal{F}rm \rightarrow \mathcal{B}Sem\mathcal{L}at$, namely the down set frame functor:

$$\mathcal{D}: \mathcal{B}Sem\mathcal{L}at \longrightarrow \mathcal{F}rm$$

$$\begin{array}{ccc} B & \xrightarrow{\quad} & \mathcal{D}B \\ f \downarrow & & \downarrow \mathcal{D}f \\ C & \xrightarrow{\quad} & \mathcal{D}C \end{array}$$

where:

$$\mathcal{D}f: \mathcal{D}B \rightarrow \mathcal{D}C$$

$$X \mapsto \bigvee_{x \in X} \downarrow(f(x)).$$

Thus $\mathcal{B}Sem\mathcal{L}at \overset{\mathcal{D}}{\rightleftarrows} \mathcal{F}rm$ where \mathcal{D} is left adjoint \mathcal{U} with unit $\downarrow: 1_{\mathcal{B}Sem\mathcal{L}at} \rightarrow \mathcal{U}\mathcal{D}$.

Obviously, the co-unit of the adjunction is $\bigvee: \mathcal{D}\mathcal{U} \rightarrow 1_{\mathcal{F}rm}$, where for any frame L ,

$$\begin{aligned} \bigvee_L: \mathcal{D}\mathcal{U}L &\rightarrow L \\ X &\mapsto \bigvee X \end{aligned}$$

Step 2 $\mathcal{U} : \mathcal{Frm} \rightarrow \mathcal{BSemLat}$ satisfies the condition:

given any pair $L \begin{smallmatrix} \xrightarrow{f} \\ \xrightarrow{g} \end{smallmatrix} M$ of frame homomorphisms such that the underlying homomorphism of bounded semi-lattices admit a coequaliser:

$$\mathcal{U}(L) \begin{smallmatrix} \xrightarrow{\mathcal{U}(f)} \\ \xrightarrow{\mathcal{U}(g)} \end{smallmatrix} \mathcal{U}(M) \longrightarrow B ,$$

there exists a unique frame structure on B such that $h : M \rightarrow B$ becomes a frame homomorphism as B a coequaliser of f and g .

Firstly, the coequalisers of $\mathcal{BSemLat}$ are obviously determined by congruences, so that the bounded semi-lattice is actually: B/ρ , where ρ is the smallest congruence of bounded semi-lattices generated by:

$$R = \{(f(x), g(x)) : x \in L\}.$$

In other words R is a bounded meet semilattice of the frame $M \times M$.

We assert that ρ is indeed a sub-frame of $M \times M$.

Towards this: it is enough to show that R is closed under arbitrary joins.

But:

$$\begin{aligned} \bigvee_{i \in I} (f(x_i), g(x_i)) &= \left(\bigvee_{i \in I} f(x_i), \bigvee_{i \in I} g(x_i) \right) \\ &= \left(f \left(\bigvee_{i \in I} x_i \right), g \left(\bigvee_{i \in I} x_i \right) \right) \in R \\ &\quad (\because f, g \text{ are frame homomorphisms}) \end{aligned}$$

and hence ρ is a sub-frame of $M \times M$.

Consequently $B = M/\rho$ is a frame and the rest follows.

Hence from BPMT, \mathcal{U} is monadic. □

Having shown 2.3.14 we observe the forgetful functors:

$$\mathcal{Frm} \xrightarrow{\text{forget}} \mathcal{BSemLat} \xrightarrow{\text{forget}} \mathcal{Set}$$

of which both are monadic and hence by BPMT again their composite is monadic, proving 2.3.13.

2.4 Free Frames, Coproduct of Frames and the Frame of Reals.

Sublocales, Congruences, Nuclei

We therefore look at the various ways of elucidating the structure of the frame homomorphic surjection $f : A \rightarrow B$ (so B is a frame homomorphic image of A). For this section, we advice the reader for more information to read [13].

Definition 2.4.1. Let $f : A \rightarrow B$ and $g : A \rightarrow C$ two surjective homomorphisms.

Then we write $f \sqsubseteq g$ if and only if there is a map $l : B \rightarrow C$ such that $f = lg$.

This is a preorder.

It is natural to define sublocales of the frame L as the surjective frame homomorphisms $h : L \rightarrow M$.

Remark 2.4.2. The sublocales f, g are equivalent if $f \sqsubseteq g$ and $g \sqsubseteq f$.

The ensuing partial ordered set will be denoted by $\mathcal{S}(L)$.

Definition 2.4.3. A congruence on the frame L is an equivalence relation ε which is a subframe of $L \times L$.

We shall denoted $\mathcal{C}L$ the set of all congruence.

Definition 2.4.4. A nucleus on the frame L is map $\mu : L \rightarrow L$ such that for any $a, b \in L$:

- a) $a \leq \mu(a)$ (Extensionality),
- b) $\mu\mu(a) = \mu(a)$ (Idempotence),
- c) $\mu(a \wedge b) = \mu(a) \wedge \mu(b)$.

We shall denote by $\mathcal{N}L$ the collection of all nuclei on the frame L .

Lemma 2.4.5. For a nucleus $\mu : L \rightarrow L$.

- 1) If $a_i = \mu(a_i), i \in I$ then $\mu(\bigwedge a_i) = \bigwedge a_i$.

2) If $b = \mu(b)$ then for any $a, \mu(a \rightarrow b) = (a \rightarrow b)$.

Proof. 1) $\mu(\wedge a_i) \leq \wedge \mu(a_i) = \wedge a_i$, then $\mu(\wedge a_i) = \wedge a_i$.

2) $x \leq (a \rightarrow b) \implies x \wedge a \leq b \implies \mu(x) \wedge \mu(a) \leq \mu(b) = b \implies \mu(x) \wedge a \leq b$

because for any $a, \mu(a) = a$ this implies $\mu(x) \leq (a \rightarrow b)$.

Then $\mu(a \rightarrow b) = (a \rightarrow b)$.

□

Theorem 2.4.6. If $\mu \in \mathcal{NL}$ then $\mu(L)$ is a frame with

$$\mu(a) \bigwedge_{\mu(L)} \mu(b) = \mu(a \wedge b)$$

$$\bigvee_i^{\mu(L)} \mu(a_i) = \mu(\bigvee_i \mu(a_i)),$$

and $\mu : L \rightarrow \mu(L)$ is a surjective frame homomorphism.

Proof. In fact with the operations \bigwedge, \bigvee , defined on $\mu(L)$, we show:

i) It is complete lattice, $\mu(0) \leq \mu(a) \leq 1 = \mu(1)$.

- Let for each $i \in I, \mu(a_i)$ be taken

a) $\mu(a_i) \leq \bigvee_i \mu(a_i) \leq \mu(\bigvee_i \mu(a_i))$, so that $\mu(\bigvee_i \mu(a_i))$ is an upper bound of $\{\mu(a_i) : i \in I\}$ in $\mu(L)$.

b) Let for each $i \in I, \mu(a_i) \leq \mu(p) \implies \bigvee_i (\mu(a_i)) \leq \mu(p) \implies \mu(\bigvee_i \mu(a_i)) \leq \mu(p)$

$\mu(\bigvee_i \mu(a_i)) \leq \mu(p) = \text{supremun of } \{\mu(a_i) : i \in I\} \text{ in } \mu(L)$.

- $\mu(L)$ is closed under intersections.

ii) $\mu : L \rightarrow \mu(L)$,

- preserves finite meets.

- Preserves finite joins because

$$\bigvee_i^{\mu(L)} \mu(a_i) = \mu\left(\bigvee_i a_i\right) \quad \text{because} \quad \bigvee_i^{\mu(L)} \mu(a_i) = \mu\left(\bigvee_i \mu(a_i)\right) \geq \mu\left(\bigvee_i a_i\right)$$

$$\text{and} \quad a_i \leq \bigvee_i a_i \implies \mu(a_i) \leq \mu\left(\bigvee_i a_i\right) \implies \bigvee_i^{\mu(L)} \mu(a_i) \leq \mu\left(\bigvee_i a_i\right).$$

$$\text{So} \quad \mu\left(\bigvee_i a_i\right) \leq \bigvee_i^{\mu(L)} \mu(a_i) \leq \mu\left(\bigvee_i a_i\right).$$

•

$$\mu(a) \wedge \bigvee_i^{\mu(L)} \mu(b_i) = \mu(a) \bigwedge_{\mu(L)} \mu\left(\bigvee_i b_i\right) = \mu\left(a \wedge \bigvee_i b_i\right) = \mu\left(\bigvee_i (a \wedge b_i)\right) = \bigvee_i^{\mu(L)} \mu(a \wedge b_i).$$

- It is obvious that the map $\mu : L \rightarrow \mu(L)$ is onto.

□

Theorem 2.4.7. i) If $\xi \in \mathcal{C}L \implies \mu_\xi \in \mathcal{N}L$ with μ_ξ define as:

$$\begin{aligned} \mu_\xi : L &\longrightarrow L \\ x &\longmapsto \bigvee (x/\xi). \end{aligned}$$

ii) If $\mu \in \mathcal{N}L \implies \xi_\mu = \{(x, y) : \mu(x) = \mu(y)\} \in \mathcal{C}L$.

iii) The map define as following:

$$\begin{aligned} \mathcal{C}L &\longrightarrow \mathcal{N}L \\ \xi &\longmapsto \mu_\xi \end{aligned}$$

is an isomorphism of poset.

iv) $\mathcal{N}L$ is a complete lattice.

Proof. i) $\xi \in \mathcal{C}L$ let us show that $\mu = \mu_\xi \in \mathcal{N}L$.

In fact:

- $x \leq \mu(x)$ is true,

- $\bigvee(x/\xi) \in (x/\xi)$,

- $\mu(\mu(x)) = \mu(x)$,

- $x \leq y \implies \mu(x) \leq \mu(y)$.

In fact: $x \leq y \implies x\xi x'$ and $y\xi y' \implies x' \wedge y' \in (x/\xi)$

$$\mu(x) \wedge \mu(y) = \bigvee\{p \wedge q : p\xi x, q\xi y\} = \bigvee(x/\xi) = \mu(x) \implies \mu(x) \leq \mu(y).$$

- $\mu(x \wedge y) = \mu(x) \wedge \mu(y)$.

In fact: $\mu(x \wedge y) = \bigvee(x \wedge y)/\xi$,

$$\begin{aligned} \text{and } \mu(x) \wedge \mu(y) &= (\bigvee(x/\xi)) \wedge (\bigvee(y/\xi)) = (\bigvee(\{p : p\xi x\})) \wedge (\bigvee(\{q : q\xi y\})) \\ &= \bigvee(\{p \wedge q : p\xi x, q\xi y\}) \leq \bigvee t : t\xi(x \wedge y). \end{aligned}$$

So $\mu(x) \wedge \mu(y) \leq \mu(x \wedge y) \leq \mu(x) \wedge \mu(y)$.

ii) $\mu \in \mathcal{NL}$ let us show that $\xi = \xi_\mu = \{(x, y) \in L \times L : \mu(x) = \mu(y)\} \in \mathcal{CL}$.

In fact: $(x, y), (x', y') \in \xi$,

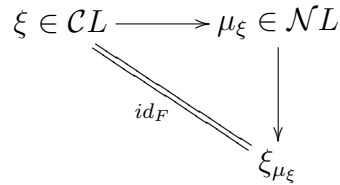
$$\mu(x \wedge x') = \mu(x) \wedge \mu(x') = \mu(y) \wedge \mu(y') = \mu(y \wedge y') \implies (x \wedge x', y \wedge y') \in \xi,$$

and

$$(x_i, y_i) \in \xi, \quad \mu(\bigvee_i x_i) = \bigvee_i^{\mu(L)} \mu(x_i) = \bigvee_i^{\mu(L)} \mu(y_i) = \mu(\bigvee_i y_i)$$

this implies that $(\bigvee_i x_i, \bigvee_i y_i) \in \xi$.

iii) Let the following diagrams we have:



$$(x, y) \in \xi_{\mu_\xi} \iff \mu_\xi(x) = \mu_\xi(y) \iff \bigvee(x/\xi) = \bigvee(y/\xi) \iff x\xi y \iff (x, y) \in \xi$$

and

$$\begin{array}{ccc}
 \mu \in \mathcal{NL} & \longrightarrow & \xi_\mu \in \mathcal{CL} \\
 & \searrow \text{id}_F & \downarrow \\
 & & \mu_{\xi_\mu}
 \end{array}$$

$$\mu_{\xi_\mu}(x) = \bigvee (x/\xi_\mu) = \bigvee \{y : \mu(x) = \mu(y)\} = \bigvee \{\mu(y) : \mu(x) = \mu(y)\} = \mu(x)$$

Thus the isomorphism between \mathcal{CL} and \mathcal{NL} .

$$\text{iv) } \xi \subseteq \xi' \implies \mu_\xi(x) \leq \mu_{\xi'}(x) \implies \mu \leq \mu_{\xi'}.$$

□

Definition 2.4.8. A subset S of a frame L is a sublocale set if and only if

- a) $A \subseteq S \implies \bigwedge A \in S$,
- b) $b \in S \implies (a \rightarrow b) \in S$.

$\mathcal{S}'(L)$ is the set of sublocale sets of L ordered by \subseteq .

Proposition 2.4.9. $\mu \in \mathcal{NL} \implies \text{Fix}(\mu) = \{x \in L : \mu(x) = x\} = \mu(L) \in \mathcal{S}'(L)$

The next theorem, give us the following:

Theorem 2.4.10. i) When $S \in \mathcal{S}'(L)$, we have $\mu_S(x) = \bigwedge \{s \in S : x \leq s\} \in S$ i.e. $\mu_S \in \mathcal{NL}$.

ii) $\mathcal{S}'L$ is isomorphism to $(\mathcal{NL})^{op}$ as a complete lattice.

Proof. i) Let us check the following:

- $x \leq \mu_S(x)$ true,

•

$$\begin{aligned}
x \leq y &\implies \{s \in S : y \leq s\} \leq \{s \in S : x \leq s\} \\
&\implies \bigwedge \{s \in S : x \leq s\} \leq \bigwedge \{s \in S : y \leq s\} \\
&\implies \mu_S(x) \leq \mu_S(y)
\end{aligned}$$

- μ_S is idempotent i.e. $\mu_S(x) = \mu_S(\mu_S(x))$.

In fact $\mu_S(\mu_S(x)) = \bigwedge \{s \in S : \mu_S(x) \leq s\} \leq \mu_S(x)$,

and according to the first and the second axioms $x \leq \mu_S(x) \implies \mu_S(x) \leq \mu_S(\mu_S(x))$.

- $\mu_S(x \wedge y) = \mu_S(x) \wedge \mu_S(y)$ because

monotonicity implies $\mu_S(x \wedge y) \leq \mu_S(x) \wedge \mu_S(y)$

and $(x \wedge y) \leq \mu_S(x \wedge y) \iff x \leq (y \rightarrow \mu_S(x \wedge y))$ as $\mu_S(x \wedge y) \in S$ then $y \rightarrow \mu_S(x \wedge y) \in S$

this implies that $\mu_S(x) \leq (y \rightarrow \mu_S(x \wedge y)) \iff \mu_S(x) \wedge y \leq \mu_S(x \wedge y)$.

Let us do the same to y , then we have $\mu_S(x) \wedge \mu_S(y) \leq \mu_S(x \wedge y)$

Thus $\mu_S \in \mathcal{NL}$.

ii) In fact:

$$\mu \rightarrow \mu(L) \rightarrow \mu_{\mu(L)} = \bigwedge \{s \in \mu(L) : x \leq s\} = \bigwedge \{\mu(a) : a \in L; x \leq \mu(a)\} = \mu(x)$$

because $x \leq \mu(a) \implies \mu(x) \leq \mu(\mu(a)) = \mu(a)$; $\mu(x)$ is infimum in $\mu(L)$ of $\mu(a) \geq x$.

$$S \rightarrow \mu_S \rightarrow \mu_S(L); S = \mu_S \text{ because } s \in S \implies s = \bigwedge \{t \in S : s \leq t\} = \mu_S(s).$$

Thus the bijection.

Ordered $S, T \in \mathcal{S}'(L)$

$$S \subseteq T \implies \mu_S(s) = \bigwedge \{s \in S : x \leq s\} \geq \bigwedge \{t \in T : x \leq t\} = \mu_T(x), \text{ then } \mu_S(x) \geq \mu_T(x).$$

□

Proposition 2.4.11. *If $a \in S$ then $(\mu_S(x) \rightarrow a) = (x \rightarrow a)$.*

Proof. In fact:

$$\begin{aligned}
y \leq (\mu_S(x) \rightarrow a) &\iff y \wedge \mu_S(x) \leq a \\
&\implies y \wedge x \leq a \\
&\iff y \leq (x \rightarrow a)
\end{aligned}$$

and

$$\begin{aligned}
y \wedge x \leq a &\iff x \leq (y \rightarrow a) \\
&\implies \mu_S(x) \leq (y \rightarrow a) \\
&\iff y \wedge \mu_S(x) \leq a.
\end{aligned}$$

□

$\mathcal{S}'(L)$ is a complete lattice with meets, intersections and joins being given by the following lemma:

Lemma 2.4.12. *If $S_i; i \in I$ are sublocale sets, then $[S_i; i \in I] = \{\wedge B : B \subseteq \bigcup_{i \in I} S_i\}$ is the smallest sublocale set containing each $S_i; i \in I$.*

Proof. In fact:

- $A \subseteq \{\wedge B : B \subseteq A\}; 1 \in, \wedge A \in \{\wedge B : B \subseteq A\},$
-

$$\begin{aligned}
b_i = \bigwedge_i B_i, B_i \subseteq A, \quad \bigwedge \{b_i : i \in I\} = \bigwedge_i b_i = \bigwedge \left(\bigcup_{i \in I} B_i \right) \\
\text{because } (\bigwedge B) \wedge (\bigwedge C) = \bigwedge (B \wedge C),
\end{aligned}$$

- $p \in L, p \rightarrow \bigwedge B = \bigwedge_{b \in B} (p \rightarrow b)$
 $(p \rightarrow b) \in S_i$ for some i because

$$\begin{aligned}
x \leq (p \rightarrow \bigwedge B) &\iff x \wedge p \leq \bigwedge B \\
&\iff (b \in B, \quad x \wedge p \leq b) \\
&\iff (b \in B \implies x \leq (p \rightarrow b)) \\
&\iff x \leq \bigwedge_{b \in B} (p \rightarrow b).
\end{aligned}$$

□

Theorem 2.4.13. $B \in \mathcal{S}'(L)$, $C_i \in \mathcal{S}'(L); i \in I$

$$\left(\bigcap_{i \in I} C_i\right) \vee B = \bigcap_{i \in I} (C_i \vee B).$$

Before to prove the above theorem, we have to remind that:

Lemma 2.4.14. 1) $a \wedge b = a \wedge c \iff (a \rightarrow b) = (a \rightarrow c)$,

$$2) p \rightarrow (q \rightarrow r) = (p \wedge q) \rightarrow r.$$

Proof. (theorem)

$A \in \mathcal{S}'(L)$, $B_i \in \mathcal{S}'(L); i \in I$, let us show that $A \vee (\bigcap_i B_i) = \bigcap_i (A \vee B_i)$.

In fact:

(\implies)

$$\begin{aligned} \bigcap_{i \in I} B_i \subseteq B_i &\implies A \vee \left(\bigcap_{i \in I} B_i\right) \subseteq A \vee B_i \\ &\implies A \vee \left(\bigcap_{i \in I} B_i\right) \subseteq \bigcap_{i \in I} (A \vee B_i). \end{aligned}$$

(\iff) $x \in \bigcap_{i \in I} (A \vee B_i) \iff \exists x \in A \vee B_i$ for each $i \in I$

for each $i \in I, \exists a_i \in A \quad \exists b_i \in B$ such that $x = a_i \wedge b_i$

$$a = \bigwedge_{i \in I} a_i, \quad x = a_i \wedge b_i \geq a \wedge b_i \geq a \wedge (\bigwedge_i b_i) = x \implies x = a \wedge b_i, \quad i \in I$$

As $(a \rightarrow b) \wedge a = a \wedge b$, then $x = a \wedge b_i, \quad i \in I \implies x = (a \rightarrow b_i) \wedge a$.

And then $i, j \in I$, according to 2.4.14, we have

$$x = (a \rightarrow b_i) \wedge a = (a \rightarrow b_j) \wedge a \iff a \rightarrow (a \rightarrow b_i) = a \rightarrow (a \rightarrow b_j) \iff (a \rightarrow b_i) = (a \rightarrow b_j)$$

$(a \rightarrow b_i)$ does not depend on i , then $(a \rightarrow b_i) \in B_i$, that implies $(a \rightarrow b_i) \in \bigcap_{i \in I} B_i$.

Thus $x = a \wedge (a \rightarrow b_i) \in A \vee (\bigcap_{i \in I} B_i)$. □

Proposition 2.4.15. $\mathcal{S}'(L)$ is the co-frame.

Remark 2.4.16. $\mathcal{S}'(L), \mathcal{S}(L)$ are co-frames, while $\mathcal{N}L, \mathcal{C}L$ are frames,

then the following picture:

$$\begin{array}{ccc}
 \mathcal{S}(L) & \xrightarrow{\simeq} & \mathcal{S}'(L) \ . \\
 \downarrow \simeq & & \downarrow \simeq \\
 (\mathcal{C}L)^{op} & \xrightarrow{\simeq} & (\mathcal{N}L)^{op}
 \end{array}$$

Proof. see [13]. □

Observation

For a frame L and $a \in L$, let us define the following:

1)

$$\begin{aligned}
 \hat{a} : L &\longrightarrow \downarrow a \\
 x &\longmapsto x \wedge a
 \end{aligned}$$

is the open sublocales, with $\Delta_a = \{(x, y) \in L \times L : x \wedge a = y \wedge a\}$ the associated congruence.

$$\begin{aligned}
 \mu_{\hat{a}} : L &\longrightarrow L \\
 x &\longmapsto \bigvee \{y \in L : x \wedge a = y \wedge a\} = (a \rightarrow x)
 \end{aligned}$$

the corresponding nucleus, with $\mu_{\hat{a}}(L) = \{x \in L : x = (a \rightarrow x)\}$,

2)

$$\begin{aligned}
 \check{a} : L &\longrightarrow \uparrow a \\
 x &\longmapsto x \vee a
 \end{aligned}$$

is the closed sublocales, with $\nabla_a = \{(x, y) \in L \times L : x \vee a = y \vee a\}$ the associated congruence.

$$\begin{aligned}
 \mu_{\check{a}} : L &\longrightarrow L \\
 x &\longmapsto \bigvee \{y \in L : x \vee a = y \vee a\} = \bigvee x / \nabla_a
 \end{aligned}$$

the corresponding nucleus, with $\mu_{\check{a}}(L) = \uparrow a$,

3) In \mathcal{CL} , ∇_0 is the smallest and ∇_1 is the largest,

4) For $a, b \in L$, $\nabla_a \cap \nabla_b = \nabla_{a \cap b}$,

5) $\bigvee_{j \in J} \nabla_{b_j} = \nabla_{\bigvee_{j \in J} b_j}$,

6) The map

$$\begin{aligned} \nabla : L &\longrightarrow \mathcal{CL} \\ a &\longmapsto \nabla_a \end{aligned}$$

is a frame homomorphism,

7) ∇_a and Δ_a are complements in \mathcal{CL} ,

8) For any sublocale $h : L \rightarrow M$ the closed sublocale obtained from the closure of the corresponding sublocale set is the smallest closed sublocale larger than h .

Let

$$\begin{array}{ccc} L & \longrightarrow & M \\ & \searrow \check{s} & \uparrow h' \\ & & \uparrow s \end{array} \quad S = \mu_h(L) = \{\bigvee h^{\leftarrow}(m) : m \in M\}, \quad s = \bigwedge S \quad \text{and} \quad \bar{S} = \uparrow s$$

For each $m \in M$, there exists $U_m = \bigvee h^{\leftarrow}(m) \in S$, and $h(s) \leq h(U_m) = m$

$h(s) \leq 0 \leq h(s) \implies h(s) = 0$ then $h \sqsubseteq \check{s}$.

Same

$$\begin{array}{ccc} L & & \\ & \searrow & \\ & & \uparrow s \longrightarrow M \\ & \searrow & \uparrow h' \\ & & \uparrow p \\ & & \uparrow a \end{array} \quad h \sqsubseteq \check{a}, \quad g(a \vee x) = h(x), \quad g(a) = 0, \quad g(a \vee s) = h(s) = 0 \implies g(s) = 0.$$

$h \sqsubseteq \check{s}$, and $h \sqsubseteq \check{a} \implies \check{s} \sqsubseteq \check{a}$.

Let us define $p(a \vee x) = s \vee x$

$0 = g(a) = g(a \vee a) = h(a) \implies a \leq s, p$.

Proposition 2.4.17. *If $S, T \in \mathcal{S}'(L)$, then we have:*

$\{\bar{1}\} = \{1\}, \overline{\bar{S}} = \bar{S}, \overline{\bar{S} \vee \bar{T}} = \bar{S} \vee \bar{T}$ and $\bar{S} \supseteq S$ imply that $S \mapsto \bar{S}$ is a Kuratowski closure operator on $\mathcal{S}'(L)$.

Coproduct of frames

Given the categories:

- \mathcal{Frm} of frames and frames homomorphisms,
- $\mathcal{BSemLat}$ of (meet) bounded semi-lattices and (meet) semilattice homomorphisms,
- \mathcal{Set} of sets and functions.

The monadicity of \mathcal{Frm} over sets entails: every frame is indeed a quotient frame of a free frame, a powerful tool to describe frames in terms of a set of generators and relations between the terms made from these generators.

We make this more precise: Given any set X let $\mathcal{F}(X)$ denote the free frame on the set X and for any frame let $\mathcal{UF}(X)$ denote the underlying set of $\mathcal{F}(X)$. Now, any subset $R \subseteq \mathcal{UF}(X) \times \mathcal{UF}(X)$ can be thought of as a set of equations between the term obtained in the symbols of X .

We shall now describe the frame freely generated by X in which the equations of R hold good. To do so:

- i) let $\langle R \rangle$ denote the congruence of the form $\mathcal{F}(X)$ generated by R ,
- ii) consider the quotient frame $\mathcal{F}(X)/\langle R \rangle$ and the corresponding quotient homomorphism $\mathcal{S}_R : \mathcal{F}(X) \rightarrow \mathcal{F}(X)/\langle R \rangle$.

We shall now show that $\mathcal{F}(X)/\langle R \rangle$ is the required frame:

- a) surely, if $(x, y) \in R$ then $\mathcal{S}_R(x) = \mathcal{S}_R(y)$;
- b) now, let L be a frame and $f : X \rightarrow \mathcal{U}(L)$ be a function such that: $(x, y) \in R \implies f(x) = f(y)$.

Then by the property of free frames there exists a unique frame homomorphism $f^* : \mathcal{F}(X) \rightarrow L$ such that: $\mathcal{U}(f^*) \circ \eta_X = f$, i.e.

$$\begin{array}{ccc}
 X & \xrightarrow{\eta_X} & \mathcal{U}\mathcal{F}(X) \\
 & \searrow f & \swarrow \mathcal{U}(f^*) \\
 & & \mathcal{U}(L)
 \end{array}$$

Further, since $\langle R \rangle$ is the generated congruence, if $(\alpha, \beta) \in \langle R \rangle$ then both α and β are joins of finite meets of members of R , so that $f^*(x) = f^*(y)$.

Hence there exists a unique $\hat{f} : \mathcal{F}(X)/\langle R \rangle \rightarrow L$ such that $\hat{f}[x] = \hat{f}[y]$.

Hence there exists a unique frame homomorphism $\tilde{f} : \mathcal{F}(X)/\langle R \rangle \rightarrow L$ such that $\tilde{f} = \hat{f} \circ \mathcal{S}_R$, i.e.

$$\begin{array}{ccc}
 \mathcal{F}(X) & \xrightarrow[\text{sur}]{\mathcal{S}_R} & \mathcal{F}(X)/\langle R \rangle \\
 & \searrow \tilde{f} & \swarrow \hat{f} \\
 & & L
 \end{array}$$

This entails that: $\text{Frm}(\mathcal{F}(X)/\langle R \rangle, L) \simeq \{f \in \text{Set}(X, \mathcal{U}(L)); (x, y) \in R \implies f(x) = f(y)\}$

or, in other words, it is enough to provide $f : X \rightarrow \mathcal{U}(L)$ for which the equations of R are preserve by f in L .

So, we shall henceforth provide:

- the generators (the set X),
 - equations between frame terms on these generators (the set R)
- and we get the frame $\mathcal{F}(X)/\langle R \rangle$ as required.

Example 1 (The binary coproduct of frame)

Let L and M be the given frames.

Consider the set of generators $X = \mathcal{U}(L) \times \mathcal{U}(M)$. We shall specify the set R of equations; first we would do very crudely and then later on we would produce very neat looking equations.

Note: $R \subseteq \mathcal{F}(X) \times \mathcal{F}(X) = \mathcal{F}(\mathcal{U}(L), \mathcal{U}(M)) \times \mathcal{F}(\mathcal{U}(L), \mathcal{U}(M))$.

For instance, if $(l, m) \in \mathcal{U}(L) \times \mathcal{U}(M)$, then $\eta_X(L, M) \in \mathcal{F}(X)$.

Towards the description of R :

a) $(\eta_X(1_L, 1_M), 1) \in R$,

b) $(\eta_X(a, 0_M), 0), (0, \eta_X(o_L, b)) \in R$,

c) $(\eta_X(a \wedge a', b \wedge b'), \eta_X(a, b) \wedge \eta_X(a' \wedge b')) \in R$,

d) $(\eta_X(\bigvee S), b), \bigvee_{s \in S} \eta_X(s, b) \in R$ and $(\eta_X(a, \bigvee T), \bigvee_{t \in T} \eta_X(a, \bigvee t)) \in R$.

Some commentary on the set R is required.

Regarding a) We would like to identify the element $\eta_X(1_L, 1_M)$ with the top element 1.

Regarding b) While there is no reason as why should $\eta_X(a, 1_M)$ or $\eta_X(1_L, b)$ be identified with 1 of $\mathcal{F}(X)$, there are enough reasons to expect that the element $\eta_X(a, 0_M)$ or $\eta_X(0_L, b)$ should be identified with 0 of $\mathcal{F}(X)$.

Regarding c) Take $a, a' \in L, b, b' \in M$ and we get the element $(a, b), (a', b')$ and $(a \wedge a', b \wedge b')$ of X ; while the insertion produces:

$\eta_X(a, b), \eta_X(a', b')$ and $\eta_X(a \wedge a', b \wedge b')$ in $\mathcal{F}(X)$ and $\eta_X(a, b) \wedge \eta_X(a', b')$ with operation of meet in $\mathcal{F}(X)$; we identify then the meet $\eta_X(a, b) \wedge \eta_X(a', b')$ in $\mathcal{F}(X)$ with $\eta_X(a \wedge a', b \wedge b')$.

Regarding d) similar to c).

This view point enable us writing R as a set of equations. To drive the home further, let us write $x \oplus y$ for the term $\eta_X(x, y)$ of $\mathcal{F}(X)$. Then, the describing equations for R are:

a) $1_L \oplus 1_M = 1$,

b) $a \wedge 0_M = 0 = 0_L \wedge b$,

c) $(a \oplus b) \wedge (a' \oplus b') = (a \wedge a') \oplus (b \wedge b')$,

d) $(a \oplus \bigvee T) = \bigvee_{t \in T} (a \oplus t), (\bigvee S \oplus b) = \bigvee_{s \in S} (s \oplus b)$.

Intended the coproduct, now written as $L \oplus M$, is then generated by the elements of the form $l \oplus m, l \in L, m \in M$, subject to the equations (a)-(d) above.

Evidently

$$\begin{aligned}\tau_0 : L &\longrightarrow L \oplus M \\ l &\longmapsto l \oplus 1_M\end{aligned}$$

and

$$\begin{aligned}\tau_1 : M &\longrightarrow L \oplus M \\ l &\longmapsto 1_L \oplus m\end{aligned}$$

are frames homomorphism by virtue of (c), (d) and if $L \xrightarrow{f} A \xleftarrow{g} M$ are frame homomorphisms then the function:

$$\begin{aligned}h : X &\longrightarrow \mathcal{U}(A) \\ (l, m) &\longmapsto f(l) \wedge g(m)\end{aligned}$$

satisfies the equations:

- $h(1_L, 1_M) = 1$,
- $h(0_L, b) = 0 = h(a, 0_M)$,
- $h(l \wedge l', m \wedge m') = f(l \wedge l') \wedge g(m \wedge m') = h(l, m) \wedge h(l', m')$,
- $h(l, \bigvee T) = f(l) \wedge g(\bigvee T) = f(l) \wedge \bigvee_{t \in T} g(t) = \bigvee_{t \in T} f(l) \wedge g(t) = \bigvee_{t \in T} h(l, t)$,

and similarly the other one.

Hence, there exists a unique frame homomorphism $\hat{h} : L \oplus M \longrightarrow A$ such that $\hat{h}(l \oplus m) = h(l, m)$.

In particular:

$$\hat{h}(1_L \oplus m) = h(1_L, m) = g(m) \quad \text{and} \quad \hat{h}(l \oplus 1_M) = h(l, 1_M) = f(l) \implies \hat{h} \circ \tau_0 = f, \hat{h} \circ \tau_1 = g$$

Thus $L \oplus M$ is indeed the coproduct of L and M .

Remark 2.4.18. See Note on Kuratowski Mrowka theorems in point free context [14].

If $\Phi_i : L_i \rightarrow M_i, (i = 1, 2)$ are homomorphisms with $L \xrightarrow{i_L} L \oplus M \xleftarrow{i_M} M$, we write

$$\Phi_1 \oplus \Phi_2 : L_1 \oplus L_2 \longrightarrow M_1 \oplus M_2,$$

for the homomorphism given by $\Phi_1 \oplus \Phi_2 \circ i_{L_1} = i_{M_1} \circ \Phi_1$. Obviously,

$$(\Phi_1 \oplus \Phi_2)(a_1 \oplus a_2) = \Phi_1(a_1) \oplus \Phi_2(a_2).$$

The coproduct $A \oplus 2$ can be identified with A (then $a \oplus 1$ is a and $a \oplus 0 = 0$).

Example 2 (The frame of pointfree reals):

generator = $\mathbb{Q} \times \mathbb{Q}$, i.e pairs of rationales.

Convention For $p, q \in \mathbb{Q}$

- $(p, q) = \{x \in \mathbb{R} : p < x < q\}$,
- $\langle p, q \rangle =$ the ordered pair of p and q ,
- $\ll p, q \gg =$ term from $\langle p, q \rangle$.

Equations

- a) $\ll p, q \gg \wedge \ll r, s \gg = \ll p \vee r, q \wedge s \gg$,
- b) $p \leq r \leq q \leq s \Rightarrow \ll p, q \gg \vee \ll r, s \gg = \ll p, s \gg$,
- c) $\ll p, q \gg = \bigvee_{p < x < y < q} \ll x, y \gg$,
- d) $\bigvee_{p, q \in \mathbb{Q}} \ll p, q \gg = 1$.

Frame of Reals

The description of pointfree topology, can be used to introduce the reals in pointfree way, by defining a suitable frame, independent of any notion of real numbers. We refer [3] for more information.

Definition 2.4.19. The frame of reals, denoted by $\mathcal{L}(\mathbb{R})$ is generated by $\{\langle p, q \rangle : p, q \in \mathbb{Q}\}$ such that:

$$(R_1) \quad \langle p, q \rangle \wedge \langle r, s \rangle = \langle p \vee r, q \wedge s \rangle.$$

$$(R_2) \quad \langle p, q \rangle \vee \langle r, s \rangle = \langle p \wedge r, q \vee s \rangle \text{ if } p \leq r \leq q \leq s.$$

$$(R_3) \quad \langle p, q \rangle = \bigvee \{ \langle r, s \rangle, p < r < s < q \}.$$

$$(R_4) \quad \top = \bigvee \{ \langle p, q \rangle : p, q \in \mathbb{Q} \}.$$

Note that, the condition $\langle p, q \rangle = \perp$ whenever $p \geq q$ is a consequence of (R_3) .

Proposition 2.4.20. 1) $p \geq q \implies \langle p, q \rangle = \perp$.

2) If $p < q$ and $r < s$ then $\langle p, q \rangle \leq \langle r, s \rangle \iff (p, q) \subseteq (r, s)$.

3) The process of making open sets out of generators is a frame homomorphism

$$(-, -) : \mathcal{L}(\mathbb{R}) \rightarrow \mathcal{O}\mathbb{Q}.$$

Definition 2.4.21. It will be useful here to adopt the equivalent description of $\mathcal{L}(\mathbb{R})$

$$1) \quad \langle p, - \rangle = \bigvee_{p \in \mathbb{Q}} \langle p, q \rangle,$$

$$2) \quad \langle -, q \rangle = \bigvee_{p \in \mathbb{Q}} \langle p, q \rangle.$$

Lemma 2.4.22. If $p < q$, then we have:

$$\begin{cases} \langle -, p \rangle \leq \langle -, q \rangle \\ \langle p, - \rangle \geq \langle q, - \rangle \\ \langle p, - \rangle \vee \langle -, q \rangle = 1. \end{cases}$$

Proof. •

$$\langle -, p \rangle \wedge \langle -, q \rangle = \left(\bigvee_{t < p} \langle t, p \rangle \right) \wedge \langle -, q \rangle = \bigvee_{t < p} (\langle t, p \rangle \wedge \langle -, q \rangle) \quad (2.4)$$

and

$$\langle t, p \rangle \wedge \langle -, q \rangle = \langle t, p \rangle \wedge \left(\bigvee_{u < q} \langle u, q \rangle \right) = \bigvee_{u < q} (\langle t, p \rangle \wedge \langle u, q \rangle) \quad (2.5)$$

The equation (2.4) in (2.5), we then have:

$$\langle -, p \rangle \wedge \langle -, q \rangle = \bigvee_{t < p; u < q} (\langle t, p \rangle \wedge \langle u, q \rangle).$$

The necessary condition is $t, u < p$, because if $t < p < u < q$ then $\langle t, p \rangle \wedge \langle u, q \rangle = 0$

So $\langle -, p \rangle \wedge \langle -, q \rangle = \bigvee_{t, u < p} (\langle t, p \rangle \wedge \langle u, q \rangle) = \bigvee_{t < p} \langle t \wedge u, p \rangle = \bigvee_{t < p} \langle t, p \rangle = \langle -, p \rangle$.

Therefore $\langle -, p \rangle \leq \langle -, q \rangle$.

•

$$\begin{aligned} \langle p, - \rangle \vee \langle q, - \rangle &= \left(\bigvee_{p < t} \langle t, p \rangle \right) \vee \left(\bigvee_{q < s} \langle q, s \rangle \right) \\ &= \bigvee_{p < t, q < s} (\langle p, t \rangle \vee \langle q, s \rangle) \\ &= \left(\bigvee_{p < t < q < s} \langle p, t \rangle \vee \langle q, s \rangle \right) \vee \left(\bigvee_{p < q < t < s} \langle p, t \rangle \vee \langle q, s \rangle \right) \\ &\quad \vee \left(\bigvee_{p < q < s < t} \langle p, t \rangle \vee \langle q, s \rangle \right) \\ &= \left(\bigvee_{p < t < q < s} \langle p, s \rangle \right) \vee \left(\bigvee_{p < q < t < s} \langle p, s \rangle \right) \vee \left(\bigvee_{p < q < s < t} \langle p, t \rangle \right) \\ &= \left(\bigvee_{p < q < s < t} \langle p, s \rangle \right) \vee \left(\bigvee_{p < t < q < s} \langle p, t \rangle \right) \\ &= \bigvee_{p < q < s < t; p < t < q < s} \langle p, s \vee t \rangle = \bigvee_{p < s, t} \langle p, s \vee t \rangle = \bigvee_{p < t} \langle p, t \rangle \\ &= \langle p, - \rangle \end{aligned}$$

Then $\langle p, - \rangle \geq \langle q, - \rangle$.

•

$$\begin{aligned}
\langle p, - \rangle \vee \langle -, q \rangle &= \left(\bigvee_{p < s} \langle p, s \rangle \right) \vee \left(\bigvee_{t < q} \langle t, q \rangle \right) \\
&= \bigvee_{p < s; t < q} (\langle p, s \rangle \vee \langle t, q \rangle) \\
&= \left(\bigvee_{p < t < s < q} \langle p, s \rangle \vee \langle t, q \rangle \right) \vee \left(\bigvee_{p < s < t < q} \langle p, s \rangle \vee \langle t, q \rangle \right) \\
&\quad \vee \left(\bigvee_{t < p < s < q} \langle p, s \rangle \vee \langle t, q \rangle \right) \vee \left(\bigvee_{t < p < q < s} \langle p, s \rangle \vee \langle t, q \rangle \right) \\
\langle p, - \rangle \vee \langle -, q \rangle &= \langle p, q \rangle \vee \langle p, q \rangle \vee \langle t, q \rangle \vee \langle t, s \rangle,
\end{aligned}$$

$\langle t, s \rangle = 1$ then second member is equal to 1. Thus $\langle p, - \rangle \vee \langle -, q \rangle = 1$.

□

We refer the ready in [3] for the proof of the following proposition

Proposition 2.4.23. *a) $p < q \implies \langle p, - \rangle \vee \langle -, q \rangle = 1$.*

$$b) \langle p, q \rangle^* = \langle q, - \rangle \vee \langle -, p \rangle.$$

In what follows, we observe that $\mathcal{L}(\mathbb{R})$ has enough points, because for a, b, c and d in L , $\langle a, b \rangle < \langle c, d \rangle \implies \exists p : \mathbb{Q} \times \mathbb{Q} \rightarrow 2$ such that $(R_1), (R_2), (R_3)$ and R_4 are satisfied and $p(\langle a, b \rangle) = 0 < 1 = p(\langle c, d \rangle)$,

- 1) $p(\langle x \vee u, y \wedge v \rangle) = p(\langle x, y \rangle) \wedge p(\langle u, v \rangle)$.
- 2) $p(\langle x \wedge u, y \vee v \rangle) = p(\langle x, y \rangle) \vee p(\langle u, v \rangle)$ for $x < u < y < v$.
- 3) $p(\langle x, y \rangle) = \bigvee \{ p(u, v) : x < u < v < y \}$.
- 4) $\bigvee p(x, y) = 1$.

We define $p\langle x, y \rangle = 0$ if and only if $(x, y) \subseteq (a, b)$.

Proposition 2.4.24. *$\theta : \mathbb{R} \rightarrow \Sigma\mathcal{L}(\mathbb{R})$ is a homeomorphism if and only if its adjunction correspondent is an isomorphism.*

There exists an epimorphism $\mathcal{L}(\mathbb{R}) \rightarrow \mathbb{R}$ induced by the following function:

$$\begin{aligned} (-, -) : \mathbb{Q} \times \mathbb{Q} &\longrightarrow \mathcal{O}\mathbb{R} \\ \langle p, q \rangle &\longmapsto (p, q) \end{aligned}$$

satisfy the following condition:

- 1) $(p \vee r, q \wedge s) = (p, q) \cap (r, s)$.
- 2) $(p \wedge r, q \vee s) = (p, q) \cup (r, s)$ if $p < r < q < s$.
- 3) $(p, q) = \bigcup \{(r, s) : p < r < s < q\}$.
- 4) $\mathbb{R} = \bigcup_{p, q \in \mathbb{R}} (p, q)$.

The following diagram let us know the existence of the unique τ surjective:

$$\begin{array}{ccccc} \mathbb{Q} \times \mathbb{Q} & \xrightarrow{i} & \mathcal{F}(\mathbb{Q} \times \mathbb{Q}) & \xrightarrow{\xi} & \mathcal{L}(\mathbb{R}) \\ & \searrow^{(-, -)} & \downarrow \beta & \nearrow_{\tau} & \\ & & \mathcal{O}\mathbb{R} & & \end{array}$$

Because:

Let $\mathcal{U} \in \mathcal{O}\mathbb{R}$, $u \in \mathcal{U}$ implies there exist $p_u, q_u \in \mathbb{Q}$ such that $p_u < u < q_u$

$$u = \bigcup_{u \in \mathbb{Q}} (p_u, q_u) = \bigcup_{u \in \mathbb{Q}} (-, -)\langle p_u, q_u \rangle = \bigcup_{u \in \mathbb{Q}} \tau(\xi i\langle p_u, q_u \rangle) = \tau(\bigvee_{u \in \mathbb{Q}} \xi i\langle p_u, q_u \rangle) = \tau\xi(\bigvee_{u \in \mathbb{Q}} i\langle p_u, q_u \rangle)$$

then that implies τ is surjective.

In particular, \mathbb{R} is a subspace of $\Sigma\mathcal{L}(\mathbb{R})$.

Combining the natural isomorphism

$$\mathcal{T}op(X, \Sigma\mathcal{L}(\mathbb{R})) \simeq \mathcal{F}rm(\mathcal{L}(\mathbb{R}), \mathcal{O}X)$$

for $\mathcal{L}(\mathbb{R})$, with the homeomorphism $\mathbb{R} \simeq \Sigma\mathcal{L}(\mathbb{R})$, one obtains

$$\mathcal{T}op(X, \mathbb{R}) \simeq \mathcal{F}rm(\mathcal{L}(\mathbb{R}), \mathcal{O}X).$$

2.5 Completely Regular Frames, Normal Frames, Compact Frames and the Stone-Ćech of Frames

Completely Regular Frames, Normal Frames

Definition 2.5.1. Let L be a frame and a, b be elements of a frame L .

We say a is rather below b in L and denoted it by

$$a \prec b \quad \text{if} \quad a^* \vee b = 1.$$

A frame L is said to be regular if for each $a \in L$

$$a = \bigvee \{x : x \prec a\}.$$

We shall give some properties about the "rather below" relation.

Lemma 2.5.2. 1) $a \prec b \implies a \leq b$ for any $0 \prec a \prec 1$.

$$2) \quad a \prec b \implies a \leq b.$$

$$3) \quad a \prec b \implies b^* \prec a^*.$$

$$4) \quad a \prec b \text{ and } a \prec b \implies a \prec (b \wedge c).$$

$$5) \quad a \prec c \text{ and } b \prec c \implies (a \vee b) \prec c.$$

$$6) \quad a \prec b \prec c \implies a \prec c.$$

$$7) \quad a \prec a \iff a \text{ is complemented.}$$

$$8) \quad \text{For any frame homomorphism } f : L \longrightarrow M,$$

$$x \prec y \implies f(x) \prec f(y).$$

Definition 2.5.3. Let L be a frame and a, b be elements of a frame L .

We say a is completely below b in L and denoted it by

$$a \prec\prec b$$

if there exists a sequence $(c_{n,k})_{0 \leq k \leq 2^n}$ of elements of L such that:

- i) $a = c_{n,0}$ and $b = c_{n,2^n}$,
- ii) $0 \leq k < 2^n \implies c_{n,k} \prec c_{n,k+1}$,
- iii) $0 \leq k \leq 2^n \implies c_{n+1,2k} = c_{n,k}$.

A frame L is said completely regular if for each $a \in L$

$$a = \bigvee \{x : x \prec\prec a\}.$$

From the lemma 2.5.2, we immediately have:

Lemma 2.5.4. 1) $a \prec\prec b \implies a \leq b$.

$$2) a \prec\prec b \implies a \prec b.$$

$$3) a \prec\prec b \implies b^* \prec\prec a^*.$$

$$4) a \prec\prec b \text{ and } a \prec\prec c \implies a \prec\prec (b \wedge c).$$

$$5) a \prec\prec c \text{ and } b \prec\prec c \implies (a \vee b) \prec\prec c.$$

$$6) a \prec\prec b \prec\prec c \implies a \prec\prec c.$$

$$7) a \prec\prec a \iff a \text{ is complemented.}$$

$$8) \text{ For any frame homomorphism } f : L \longrightarrow M,$$

$$x \prec\prec y \implies f(x) \prec\prec f(y).$$

Definition 2.5.5. A frame L is normal if whenever $a \vee b = 1$ for $a, b \in L$ implies there exist $u, v \in L$ such that

$$a \vee u = 1 = b \vee v \quad \text{and} \quad u \wedge v = 0.$$

Definition 2.5.6. A relation R is said to be interpolative if

$$aRb \implies \exists c : aRcRb$$

Lemma 2.5.7. ([13]) *In a normal frame the rather below \prec is interpolative.*

Proof. Let $a \prec b$ in a normal frame L . Then there are u, v such that $u \leq v^*, u \vee b = 1$ and $a^* \vee v = 1$. Thus, $a \prec v$ and $v^* \vee b \geq u \vee b = 1$ so that also $v \prec b$. \square

In particular, in a normal frame $x \prec y \implies x \prec\prec y$.

Definition 2.5.8. A frame homomorphism $f : L \rightarrow M$ is said to be:

- a) Dense, if and only if $f(a) = 0 \implies a = 0$.
- b) Co-dense, if and only if $f(a) = 1 \implies a = 1$.

Remark 2.5.9. For any continuous map $f : X \rightarrow Y$,

- a) f^{\leftarrow} is dense if and only if $f(X)$ is dense in Y .
- b) f^{\leftarrow} is co-dense if and only if every non-empty proper closed subset of Y intersect $f(X)$.

Corollary 2.5.10. *If $p < r < s < q \implies \langle r, s \rangle \prec\prec \langle p, q \rangle$.*

Proof. $p < r < s < q$ let us prove that $\langle r, s \rangle \prec \langle p, q \rangle$.

In fact for using the above proposition we have:

$$\begin{aligned}
 \langle r, s \rangle^* \vee \langle p, q \rangle &= (\langle -, r \rangle \vee \langle s, - \rangle) \vee \langle p, q \rangle \\
 &= (\langle -, r \rangle \vee \langle p, q \rangle) \vee (\langle s, - \rangle \vee \langle p, q \rangle) \\
 &= \langle -, q \rangle \vee \langle p, - \rangle \\
 &= 1,
 \end{aligned}$$

from it then follows that also

$$\langle r, s \rangle \prec \langle \frac{1}{2}(p+r), \frac{1}{2}(s+q) \rangle \prec \langle p, q \rangle.$$

Providing an explicit interpolation for the relation \prec which can be continued indefinitely, proving that $\langle r, s \rangle \prec\prec \langle p, q \rangle$. \square

Theorem 2.5.11. $\mathcal{L}(\mathbb{R})$ is a completely regular frame.

Definition 2.5.12. 1) A trial in L is a function $t : \mathbb{Q} \rightarrow L$ such that:

i) $r < s \implies t(r) \prec t(s) \implies t(r)^* \vee t(s) = 1,$

ii) $\bigvee_{r \in \mathbb{Q}} t(r) = 1 = \bigvee_{r \in \mathbb{Q}} t(r)^*.$

2) A descending trial in L is a function $s : \mathbb{Q} \rightarrow L$ such that:

i) $r < s \implies t(s) \prec t(r) \implies t(s)^* \vee t(r) = 1,$

ii) $\bigvee_{r \in \mathbb{Q}} t(r) = 1 = \bigvee_{r \in \mathbb{Q}} t(r)^*.$

Lemma 2.5.13. For any trial $t : \mathbb{Q} \rightarrow L$ and any descending trial $s : \mathbb{Q} \rightarrow L$, the function

a) $\varphi_t \langle p, q \rangle = \bigvee \{t(r)^* \wedge t(s) : p < r < s < q\},$

b) $\varphi_s \langle p, q \rangle = \bigvee \{t(r) \wedge t(s)^* : p < r < s < q\},$

determine frame homomorphism φ_t, φ_s on $\mathcal{L}(\mathbb{R}) \rightarrow L$.

Proof. To check that, it is enough to verify that φ_t, φ_s on $\mathcal{L}(\mathbb{R}) \rightarrow L$ satisfy the four axioms of definition 2.4.19. □

In what follows, we consider $\mathcal{R}(L) = \mathcal{Frm}(\mathcal{L}(\mathbb{R}), L)$, the set of all continuous real functions on L .

Proposition 2.5.14. A frame L is normal if and only if for any $a, b \in L$ such that $a \vee b = 1$ there exists $\varphi : \mathcal{L}(\mathbb{R}) \rightarrow L$ such that:

- $\varphi(\langle -, 0 \rangle \vee \langle 1, - \rangle) = 0.$
- $\varphi(0, -) \leq a.$
- $\varphi(-, 1) \leq b.$

Proof. Given the three conditions let us show that L is normal.

Let $\langle -, \frac{1}{2} \rangle, \langle \frac{1}{2}, - \rangle \in \mathcal{L}(\mathbb{R})$ with $\langle -, \frac{1}{2} \rangle \wedge \langle \frac{1}{2}, - \rangle = 0$

$\implies \varphi \langle -, \frac{1}{2} \rangle \wedge \varphi \langle \frac{1}{2}, - \rangle = 0$

$a \vee \varphi \langle -, \frac{1}{2} \rangle \geq \varphi \langle 0, - \rangle \vee \varphi \langle -, \frac{1}{2} \rangle \geq \varphi \langle 0, - \rangle \vee \varphi \langle -, 0 \rangle = 1.$

Then $a \vee \varphi \langle -, \frac{1}{2} \rangle = 1$

$b \vee \langle \frac{1}{2}, - \rangle \geq \langle -, 1 \rangle \vee \langle \frac{1}{2}, - \rangle \geq \langle -, 1 \rangle \vee \langle 1, - \rangle = 1.$

Then $b \vee \langle \frac{1}{2}, - \rangle = 1.$

Therefore the frame L is normal.

Now given L is a normal frame, $a \vee b = 1$ there exist $u, v \in L, u \wedge v = 0, a \vee u = 1 = b \vee v$

So $v \leq u^* \implies b \vee u^* = 1 \implies u \prec b \implies u \prec \prec b$

there exists $(w_{n,k})_{0 \leq k \leq 2^n}$ such that

$$w_{n,0} = u, w_{n,2^n} = b$$

$$w_{n,k} \prec w_{n,k+1}; k = 0, 1, \dots, 2^n - 1$$

$$w_{n+1,2k} = w_{n,k}; k = 0, 1, \dots, 2^n$$

Define

$$t(r) = \begin{cases} 0 & \text{if } r < 0 \\ \bigvee_{\frac{k}{2^n} \leq r} w_{n,k} & \text{if } 0 \leq r \leq 1 \\ 1 & \text{if } r > 1 \end{cases} \quad (2.6)$$

- $t(0) = \bigvee_{\frac{k}{2^n} \leq 0} w_{n,k} = \bigvee_n w_{n,0} = u,$
- $t(1) = \bigvee_{\frac{k}{2^n} \leq 1} w_{n,k} = \bigvee_n w_{n,2^n} = b,$
- $0 \leq p \leq q \leq 1 \implies \exists r, R, p < \frac{R}{2^n} < \frac{R+1}{2^n} < q \implies t(p) \leq w_{n,k} \prec w_{n,k+1} \leq t(q) \implies t(p) \prec t(q),$
- $\bigvee_p t(p) = \bigvee_p t(p)^* = 1$

t is trial, according to the lemma 2.5.13 there exists the frame homomorphism $\varphi_t : \mathbb{Q} \times \mathbb{Q} \longrightarrow L$ such that:

- $\varphi(\langle -, 0 \rangle \vee \langle 1, - \rangle) = 0$,
- $\varphi\langle 0, - \rangle \leq a$,
- $\varphi\langle -, 1 \rangle \leq b$.

□

Proposition 2.5.15. *In any frame L , $a \prec\prec b$ if and only if there exists $\varphi : \mathcal{L}(\mathbb{R}) \rightarrow L$ such that:*

- $a \leq \varphi\langle -, \frac{1}{2} \rangle$,
- $\varphi\langle -, 1 \rangle \leq b$.

Proof. (\Leftarrow) Let us show that $a \prec\prec b$.

$\langle -, \frac{1}{2} \rangle \prec\prec \langle -, 1 \rangle$ is already true.

Let $p^* = \bigvee\{q : p \wedge q = 0\}$, $\varphi(p^*) = \bigvee\{\varphi(q) : p \wedge q = 0\} \leq \bigvee\{t : \varphi(p) \wedge t = 0\} = \varphi(p)^*$.

Then $\varphi(p^*) \leq \varphi(p)^*$,

$p \prec q \implies p^* \vee q = 1 \implies \varphi(p^*) \vee \varphi(q) = 1 \leq \varphi(p)^* \vee \varphi(q)$

So $\varphi(p)^* \vee \varphi(q) = 1$, then $\varphi(p) \prec \varphi(q)$. Therefore if $p \prec q \implies \varphi(p) \prec \varphi(q)$

$p \prec\prec q$, by definition there are $p_r \in \mathcal{L}(\mathbb{R})$ for r dyadic rational in $[0, 1]$ with $p_r \prec p_s$ for $r < s$, there are $\varphi(p_r) \in L$ for r dyadic rational in $[0, 1]$ with $\varphi(p_r) \prec \varphi(p_s)$ for $r < s$. Therefore $\varphi(p) \prec\prec \varphi(q)$.

Now since $\langle -, \frac{1}{2} \rangle \prec\prec \langle -, 1 \rangle \implies \varphi\langle -, \frac{1}{2} \rangle \prec\prec \varphi\langle -, 1 \rangle$. As $a \leq \varphi\langle -, \frac{1}{2} \rangle \prec\prec \varphi\langle -, 1 \rangle \leq b$, then $a \prec\prec b$.

(\Leftarrow) Given $a \prec\prec b$

- $c_{n,0} = a$, $c_{n,2^n} = b$,
- $c_{n+1,2k} = c_{n,k}$,
- $c_{n,k} \prec c_{n,k+1}$.

According to equation (2.6) with $t(r) = \begin{cases} 0 & \text{if } r < 0 \\ \bigvee_{\frac{k}{2^n} \leq r} w_{n,k} & \text{if } 0 \leq r \leq 1 \\ 1 & \text{if } r > 1 \end{cases}$

we say t is trial, there exists the frame homomorphism $\varphi_t : \mathcal{L}(\mathbb{R}) \longrightarrow L$ such that:

- $\varphi\langle -, \frac{1}{2} \rangle = \bigvee_{p < \frac{1}{2}} \varphi\langle p, \frac{1}{2} \rangle = \bigvee_{p < r < s < \frac{1}{2}} t_r^* \wedge t_s \geq a$,
- and $\varphi\langle -, 1 \rangle = \bigvee_{p < 1} \varphi\langle p, 1 \rangle = \bigvee_{p < r < s < 1} t_r^* \wedge t_s \leq b$.

□

Corollary 2.5.16. *A frame L is completely regular if and only if it is generated by the images of the homomorphisms $\varphi : \mathcal{L}(\mathbb{R}) \longrightarrow L$.*

Proof. (\implies) L is completely regular frame, if for each $a \in L$, $a = \bigvee_{x \prec a} x = \bigvee_{x \prec a} \varphi_x\langle -, \frac{1}{2} \rangle$.

(\impliedby) Now L is generated by the images of φ .

- i) If $\varphi : \mathcal{L}(\mathbb{R}) \longrightarrow L$,
then $A = \varphi(\mathcal{L}(\mathbb{R}))$ is a completely regular subframe of L .
- ii) The frame generated by completely regular subframes is completely regular
 $A_i \subseteq L, i \in I$ subframe.

□

Compact Frames and the Stone-Ćech Compactification of Frames

Definition 2.5.17. 1) A cover of a frame L is a subset $A \subseteq L$ such that $\bigvee A = 1$, and a subcover B of A is a subset $B \subseteq A$ which is still a cover.

2) A frame L is said to be compact if each cover A of L has a finite subcover.

Remark 2.5.18. a) Each subframe of a compact frame is compact.

b) Each closed sublocale $\uparrow c$ of a compact frame is compact.

Proposition 2.5.19. 1) *Let L be regular and let M be compact. Then each dense $h : L \longrightarrow M$ is one to one.*

2) *Let L be regular and $h : L \longrightarrow M$ a sublocale with compact M . Then h is closed.*

Definition 2.5.20. An ideal in a frame L is a non-void subset $J \subseteq L$ such that:

(i) $b \leq a \in J \implies b \in J$,

(ii) $a, b \in J \implies a \vee b \in J$.

We shall denote by $\mathcal{J}L$ the set of all ideals in a frame L ordered by inclusion.

Remark 2.5.21. It was proved in [13] that $\mathcal{J}L$ is a compact frame; and a mapping

$$\begin{aligned} \nu_L : \mathcal{J}L &\longrightarrow L \\ J &\longmapsto \bigvee J \end{aligned}$$

is a dense sublocale homomorphism.

For a frame homomorphism $h : L \longrightarrow M$ let us define

$$\begin{aligned} \mathcal{J}h : \mathcal{J}L &\longrightarrow \mathcal{J}M \\ L &\longmapsto \downarrow h[J]. \end{aligned}$$

Proposition 2.5.22. \mathcal{J} is a functor $\mathcal{F}rm \longrightarrow \mathcal{F}rm$ and $\nu = (\nu_L)_L$ is a natural transformation.

Compactification

For a frame L , a compactification of L is defined to be a compact regular frame M together with a dense onto frame $h : M \longrightarrow L$.

Definition 2.5.23. A ideal I of L is called completely regular if for any $a \in I$ there is some $b \in I$ with $a \prec\prec b$.

We shall denote by $\mathcal{R}L$ the set of all regular ideals in L .

Remark 2.5.24. For any frame L , $\mathcal{R}L$ is a subframe of $\mathcal{J}L$. In particular, it is compact.

For an element a of a frame L set

$$\sigma(a) = \{x : x \prec\prec a\}.$$

Proposition 2.5.25. $\mathcal{R}L$ is a completely regular frame.

Stone-Čech Compactification

Define $\mathcal{R}h = \mathcal{J}h$ for homomorphism $h : L \rightarrow M$ and $\nu_L : \mathcal{R}L \rightarrow L$. These formulas yield a functor $\mathcal{R} : CRegFrm \rightarrow CRegFrm$ and a natural transformation $\nu_L : \mathcal{R} \rightarrow Id$ such that:

- 1) each $\mathcal{R}L$ is regular compact,
- 2) each ν_L is a dense sublocale homomorphism,
- 3) ν_L is an isomorphism if and only if L is compact.

2.6 The Frame of non-negative reals and point free Semirings of localic continuous functions

Definition 2.6.1. The frame of non-negative reals is one of the frame derived from $\mathcal{L}(\mathbb{R})$ and it is denoted by $\mathcal{L}[0, -) = \uparrow(-, 0)$.

Furthermore, we establish a result concerning $\mathcal{L}(\mathbb{R})$ in the coproduct of frames. We instead proceed as follows: taking $\mathcal{L}(\mathbb{R})$ instead M and L in the coproduct of frames, we shall prove the following lemma:

Lemma 2.6.2. The map from frame $\mathcal{L}(\mathbb{R})$ to the coproduct of frames $\mathcal{L}(\mathbb{R}) \oplus \mathcal{L}(\mathbb{R})$ given by:

$$\begin{aligned} \alpha : \mathcal{L}(\mathbb{R}) &\longrightarrow \mathcal{L}(\mathbb{R}) \oplus \mathcal{L}(\mathbb{R}) \\ (p, q) &\longmapsto \bigvee \{(r, s) \oplus (u, v) : \langle r, s \rangle + \langle u, v \rangle \subseteq \langle p, q \rangle\} \end{aligned}$$

is the frame morphism, i.e the function $f : \mathbb{Q} \times \mathbb{Q} \longrightarrow \mathcal{L}(\mathbb{R}) \oplus \mathcal{L}(\mathbb{R})$ defined by $f(a, b) = \bigvee \{(r, s) \oplus (u, v) : \langle r, s \rangle + \langle u, v \rangle \subseteq \langle a, b \rangle\}$ for $a, b \in \mathbb{Q}$, satisfies the conditions of definition of $\mathcal{L}(\mathbb{R})$.

Proof. Let us prove the four conditions:

a) In fact:

$$f(p, q) \wedge f(p', q') = \bigvee \{(r, s) \oplus (u, v) : \langle r, s \rangle + \langle u, v \rangle \subseteq \langle p, q \rangle\} \wedge \bigvee \{(r', s') \oplus (u', v') : \langle r', s' \rangle + \langle u', v' \rangle \subseteq \langle p', q' \rangle\}$$

$$f(p, q) \wedge f(p', q') = \bigvee \{[(r, s) \oplus (u, v)] \wedge [(r', s') \oplus (u', v')] : \langle r, s \rangle + \langle u, v \rangle \subseteq \langle p, q \rangle \text{ and } \langle r', s' \rangle + \langle u', v' \rangle \subseteq \langle p', q' \rangle\}$$

$$f(p, q) \wedge f(p', q') = \bigvee \{[(r, s) \wedge (r', s')] \oplus [(u, v) \wedge (u', v')] : \langle r, s \rangle + \langle u, v \rangle \subseteq \langle p, q \rangle \text{ and } \langle r', s' \rangle + \langle u', v' \rangle \subseteq \langle p', q' \rangle\}$$

$$f(p, q) \wedge f(p', q') = \bigvee \{(r \vee r', s \wedge s') \oplus (u \vee u', v \wedge v') : \langle r, s \rangle + \langle u, v \rangle \subseteq \langle p, q \rangle \text{ and } \langle r', s' \rangle + \langle u', v' \rangle \subseteq \langle p', q' \rangle\},$$

where the condition for the join implies:

$$\langle r \vee r', s \wedge s' \rangle + \langle u \vee u', v \wedge v' \rangle \subseteq \langle p \vee p', q \wedge q' \rangle;$$

while on the another hand we have:

$$f(p \vee p', q \wedge q') = \bigvee \{(r \vee r', s \wedge s') \oplus (u \vee u', v \wedge v') : \langle r \vee r', s \wedge s' \rangle + \langle u \vee u', v \wedge v' \rangle \subseteq \langle p \vee p', q \wedge q' \rangle\}.$$

Hence every term in the first join is equal to the term in the second join,

then $f(p, q) \wedge f(p', q') = f(p \vee p', q \wedge q')$.

b) Let $p \leq p' < q' \leq q$ in \mathbb{Q} . Then $f(p, q) \vee f(p', q') \leq f(p, q)$ is trivial. For the reverse inequality.

Consider the set $\{(r, s) \oplus (u, v) : \langle r, s \rangle + \langle u, v \rangle \subseteq \langle p, q \rangle\}$ in the join defining the element on the right. Then, take

$$r = r_0 < r_1 < \cdots < r_n = s, \quad u = u_0 < u_1 < \cdots < u_n = v$$

such that each $\langle r_i, r_{i+2} \rangle + \langle u_k, u_{k+2} \rangle$ fits into an interval of length less than $q' - p'$. Now, whenever $\langle r_i, r_{i+2} \rangle + \langle u_k, u_{k+2} \rangle$ is not contained in $\langle p', q' \rangle$, and therefore contains some

$x \geq q'$, it cannot contain any $y \leq p'$ because $x - y > q' - p'$, and hence it is contained in $\langle p, q \rangle$.

It follows that

$$(r_i, r_{i+2}) \oplus (u_k, u_{k+2}) \leq f(p, q) \vee f(p', q')$$

and since

$$(r, s) \oplus (u, v) = \bigvee \{(r_i, r_{i+2}) \oplus (u_k, u_{k+2}) : i, k = i, \dots, n\}.$$

By repeated application of b) this proves the inequality in equation.

c) Let us show that $f(p, q) = \bigvee \{f(p', q') : p < p' < q' < q\}$.

In fact:

$$\begin{aligned} f(p, q) &= \bigvee \{(r, s) \oplus (u, v) : \langle r, s \rangle + \langle u, v \rangle \subseteq \langle p, q \rangle\} \\ &= \bigvee \left\{ \left(\bigvee_{r < r' < s' < s} (r', s') \right) \oplus (u, v) : \langle r, s \rangle + \langle u, v \rangle \subseteq \langle p, q \rangle \right\} \\ &= \bigvee \left\{ \bigvee ((r', s') \oplus (u, v)) : r < r' < s' < s \text{ and } \langle r, s \rangle + \langle u, v \rangle \subseteq \langle p, q \rangle \right\} \\ &= \bigvee \left\{ (r', s') \oplus \left(\bigvee_{u < u' < v' < v} (u', v') \right) : r < r' < s' < s \text{ and } \langle r, s \rangle + \langle u, v \rangle \subseteq \langle p, q \rangle \right\} \\ &= \bigvee \left\{ \bigvee (r', s') \oplus (u', v') : r < r' < s' < s, u < u' < v' < v \text{ and } \langle r, s \rangle + \langle u, v \rangle \subseteq \langle p, q \rangle \right\} \\ &= \bigvee \left\{ (r', s') \oplus (u', v') : r < r' < s' < s, u < u' < v' < v \text{ and } \langle r, s \rangle + \langle u, v \rangle \subseteq \langle p, q \rangle \right\} \\ &= \bigvee \left\{ (r', s') \oplus (u', v') : \exists p', q'; p < p' < q' < q, \langle r', s' \rangle + \langle u', v' \rangle \subseteq \langle p', q' \rangle \right\} \\ &= \bigvee \{f(p', q') : p < p' < q' < q\}. \end{aligned}$$

d) $\bigvee_{p, q \in \mathbb{Q}} f(p, q) = 1$ is obvious.

□

In order to state the next step, we introduce the following auxiliary morphisms:

$\tau : 2 \rightarrow \mathcal{L}(\mathbb{R})$ the unique morphism,

the twist isomorphisms is defined by:

$$\begin{aligned} tw : \mathcal{L}(\mathbb{R}) \oplus \mathcal{L}(\mathbb{R}) &\longrightarrow \mathcal{L}(\mathbb{R}) \oplus \mathcal{L}(\mathbb{R}) \\ (u, v) \oplus (x, y) &\longmapsto (x, y) \oplus (u, v) \end{aligned}$$

the codiagonal is defined by

$$\begin{aligned}\nabla : \mathcal{L}(\mathbb{R}) \oplus \mathcal{L}(\mathbb{R}) &\longrightarrow \mathcal{L}(\mathbb{R}) \\ (u, v) \oplus (x, y) &\longmapsto (u, v) \wedge (x, y).\end{aligned}$$

Proposition 2.6.3. *The morphisms α, β satisfy the following laws making $\mathcal{L}(\mathbb{R})$ into the coring in frames with addition α , and multiplication β :*

- i) $(\alpha \oplus id) \circ \alpha = (id \oplus \alpha) \circ \alpha$ (Associativity),
- ii) $(e \oplus id) \circ \alpha = id = (id \oplus e) \circ \alpha$ (Identity),
- iii) $tw \circ \alpha = \alpha$ (Commutativity),
- iv) $\nabla \circ (\eta \oplus id) \circ \alpha = \tau \circ e$ (inverses)
- v) $\beta \circ (\beta \oplus id) = \beta \circ (id \oplus \beta)$ (Associativity),
- vi) $\beta \circ tw = \beta$ (Commutativity),
- vii) $(id \oplus \beta) \circ \alpha = (\nabla \oplus id \oplus id) \circ (id \oplus tw \oplus id) \circ (\alpha \oplus \alpha) \circ \beta$ (distributivity).

Proof. Let us show that $\mathcal{L}(\mathbb{R})$ equipped with addition law is a group, i.e. associativity, identity, inverses commutativity.

- i) $(\alpha \oplus id) \circ \alpha = (id \oplus \alpha) \circ \alpha$ (Associativity) i.e the following diagram commutes:

$$\begin{array}{ccc}\mathcal{L}(\mathbb{R}) & \xrightarrow{\alpha} & \mathcal{L}(\mathbb{R}) \oplus \mathcal{L}(\mathbb{R}) \\ \alpha \downarrow & & \downarrow id \oplus \alpha \\ \mathcal{L}(\mathbb{R}) \oplus \mathcal{L}(\mathbb{R}) & \xrightarrow{\alpha \oplus id} & \mathcal{L}(\mathbb{R}) \oplus \mathcal{L}(\mathbb{R}) \oplus \mathcal{L}(\mathbb{R})\end{array}$$

For $(p, q) \in \mathcal{L}(\mathbb{R})$, $(\alpha \oplus id) \circ \alpha(p, q) = (\alpha \oplus id)(\alpha(p, q)) = (\alpha \oplus id)(\bigvee\{(r, s) \oplus (u, v) : (r, s) + (u, v) \subseteq (p, q)\}) = \bigvee\{\alpha(r, s) \oplus (u, v) : (r, s) + (u, v) \subseteq (p, q)\} = \bigvee\{\bigvee\{(x, y) \oplus (z, t) :$

$$(x, y) + (z, t) \subseteq (r, s)\} \oplus (u, v) : (r, s) + (u, v) \subseteq (p, q)\} = \bigvee\{(x, y) \oplus (z, t) \oplus (u, v) : (x, y) + (z, t) + (u, v) \subseteq (p, q)\} \quad (1)$$

$$\text{and another side } (id \oplus \alpha) \circ \alpha(p, q) = (id \oplus \alpha)(\alpha(p, q)) = (id \oplus \alpha)(\bigvee\{(r, s) \oplus (u, v) : (r, s) + (u, v) \subseteq (p, q)\}) = \bigvee\{(r, s) \oplus \bigvee\{(x, y) \oplus (z, t) : (x, y) + (z, t) \subseteq (u, v)\} : (r, s) + (u, v) \subseteq (p, q)\} = \bigvee\{(r, s) \oplus (x, y) \oplus (z, t) : (r, s) + (x, y) + (z, t) \subseteq (p, q)\} \quad (2)$$

$$(1)=(2) \text{ implies } (\alpha \oplus id) \circ \alpha = (id \oplus \alpha) \circ \alpha.$$

ii) $(e \oplus id) \circ \alpha = id = (id \oplus e) \circ \alpha$ (Identity) i.e the following diagram commutes:

$$\begin{array}{ccc} \mathcal{L}(\mathbb{R}) & \xrightarrow{\alpha} & \mathcal{L}(\mathbb{R}) \oplus \mathcal{L}(\mathbb{R}) \\ & \searrow id & \downarrow e \oplus id \\ & & \mathcal{L}(\mathbb{R}) \end{array} \quad \begin{array}{ccc} \mathcal{L}(\mathbb{R}) & \xrightarrow{\alpha} & \mathcal{L}(\mathbb{R}) \oplus \mathcal{L}(\mathbb{R}) \\ & \searrow id & \downarrow id \oplus e \\ & & \mathcal{L}(\mathbb{R}) \end{array}$$

$$\text{For } (p, q) \in \mathcal{L}(\mathbb{R}), (e \oplus id) \circ \alpha(p, q) = (e \oplus id)(\alpha(p, q)) = (e \oplus id)(\bigvee\{(r, s) \oplus (u, v) : (r, s) + (u, v) \subseteq (p, q)\}) = \bigvee\{e(r, s) \oplus (u, v) : (r, s) + (u, v) \subseteq (p, q)\} = \bigvee\{1 \oplus (u, v) : (r, s) + (u, v) \subseteq (p, q); r < 0 < s\} = (p, q) = id(p, q) \quad (1)$$

$$\text{and another side } \alpha = (id \oplus e) \circ \alpha(p, q) = (id \oplus e)(\alpha(p, q)) = (id \oplus e)(\bigvee\{(r, s) \oplus (u, v) : (r, s) + (u, v) \subseteq (p, q)\}) = \bigvee\{(r, s) \oplus e(u, v) : (r, s) + (u, v) \subseteq (p, q)\} = \bigvee\{(r, s) \oplus 1 : (r, s) + (u, v) \subseteq (p, q); u < 0 < v\} = (p, q) = id(p, q) \quad (2)$$

$$\text{Then } (1)=(2) \text{ implies } (e \oplus id) \circ \alpha = id = (id \oplus e) \circ \alpha.$$

iii) $tw \circ \alpha = \alpha$ (Commutativity) i.e the following diagram commutes:

$$\begin{array}{ccc} \mathcal{L}(\mathbb{R}) & \xrightarrow{\alpha} & \mathcal{L}(\mathbb{R}) \oplus \mathcal{L}(\mathbb{R}) \\ \parallel & & \downarrow tw \\ \mathcal{L}(\mathbb{R}) & \xrightarrow{\alpha} & \mathcal{L}(\mathbb{R}) \oplus \mathcal{L}(\mathbb{R}) \end{array}$$

$$\text{For } (p, q) \in \mathcal{L}(\mathbb{R}), tw \circ \alpha(p, q) = tw(\alpha(p, q)) = tw(\bigvee\{(r, s) \oplus (u, v) : (r, s) + (u, v) \subseteq (p, q)\}) = \bigvee\{(u, v) \oplus (r, s) : (u, v) + (r, s) \subseteq (p, q)\} = \alpha(p, q).$$

v) After writing out the values of the right and left sides of this equations at a generator $(a, b) \in \mathcal{L}(\mathbb{R})$, one finds that establishing equality depends on showing that certain pairs of sets have the same supremum in $\mathcal{L}(\mathbb{R}) \oplus \mathcal{L}(\mathbb{R}) \oplus \mathcal{L}(\mathbb{R})$

Since $i), ii), iii)$ and $v)$ are done then $vi)$ and $vii)$ are automatically solved.

□

Proposition 2.6.4. *Given the following morphisms*

$\alpha : \mathcal{L}(\mathbb{R}) \longrightarrow \mathcal{L}(\mathbb{R}) \oplus \mathcal{L}(\mathbb{R}), f : \mathcal{L}(\mathbb{R}) \longrightarrow \uparrow(-, 0), \quad \xi : \mathcal{L}(\mathbb{R}) \longrightarrow 2$ and $f \oplus f : \mathcal{L}(\mathbb{R}) \oplus \mathcal{L}(\mathbb{R}) \longrightarrow \uparrow(-, 0) \oplus \uparrow(-, 0)$ such that:

$$\begin{array}{ccc}
 \mathcal{L}(\mathbb{R}) & \xrightarrow{f} & \uparrow(-, 0) & \mathcal{L}(\mathbb{R}) & \xrightarrow{\alpha} & \mathcal{L}(\mathbb{R}) \oplus \mathcal{L}(\mathbb{R}) \\
 \downarrow \xi & \nearrow \text{---} & & \downarrow f & & \downarrow f \oplus f \\
 2 & & & \uparrow(-, 0) & \text{---} & \uparrow(-, 0) \oplus \uparrow(-, 0)
 \end{array}$$

there exist $\alpha' : \uparrow(-, 0) \longrightarrow \uparrow(-, 0) \oplus \uparrow(-, 0)$ and $\xi' : \uparrow(-, 0) \longrightarrow 2$ such that the above diagrams commute i.e $f \oplus f \circ \alpha = \alpha' \circ f$ and $\xi' \circ f = \xi$.

Proof. Let us show that $\text{cong}(f) \subseteq \text{cong}((f \oplus f) \circ \alpha)$ and $\text{cong}(\xi) \subseteq \text{cong}(f)$ then there exist respectively α' and ξ' .

In fact let $(p, q), (r, s) \in \text{cong}(f)$ then $f(p, q) = f(r, s)$ that means

$$(p, q) \vee (-, 0) = (r, s) \vee (-, 0).$$

$$\text{In general } (p, q) \vee (-, 0) = \begin{cases} (-, q) & \text{if } p < 0 \leq q \\ (-, 0) & \text{if } q < 0 \end{cases}$$

$$\text{and } (p, q) \vee (-, 0) = (r, s) \vee (-, 0) \iff \begin{cases} \text{either } p = r, & q = s \\ \text{or } p, r < 0 < q = r \\ \text{or } q \leq 0, & s < 0 \\ \text{or } s \leq 0, & q < 0 \end{cases}$$

Let us study

$$\begin{aligned}
(f \oplus f) \circ \alpha(p, q) &= (f \oplus f) \left(\bigvee_{(u,v)+(w,x) \subseteq (p,q)} \langle u, v \rangle \oplus \langle w, x \rangle \right) \\
&= (f \oplus f) \left(\bigvee_{p \leq u+w < v+x \leq q} \langle u, v \rangle \oplus \langle w, x \rangle \right) \\
&= (f \oplus f) \left(\bigvee_{v \in \mathbb{R}} \bigvee_{v-(q+p) < u < v} \langle u, v \rangle \oplus \langle p-u, q-v \rangle \right) \\
&= \bigvee_{v \in \mathbb{R}} \bigvee_{v-(q+p) < u < v} (\langle u, v \rangle \vee \langle -, 0 \rangle) \oplus (\langle p-u, q-v \rangle \vee \langle -, 0 \rangle)
\end{aligned}$$

Let us take $T_{u,v} = (\langle u, v \rangle \vee \langle -, 0 \rangle) \oplus (\langle p-u, q-v \rangle \vee \langle -, 0 \rangle)$

If: $v \leq 0$ then $\langle u, v \rangle \vee \langle -, 0 \rangle = \langle -, 0 \rangle$ this implies that $T_{u,v} = \langle -, 0 \rangle \oplus 1$,

else if: $u \leq 0 < v$ and $q \leq v$ then $\langle p-u, q-v \rangle \vee \langle -, 0 \rangle = \langle -, 0 \rangle$ this implies that $T_{u,v} \leq 1 \oplus \langle -, 0 \rangle$,

else if: $p, u \leq 0 < v < q$ then $T_{u,v} \subseteq \langle -, v \rangle \oplus \langle -, q-v \rangle$,

and thus

a) if $q \leq 0$ then $(f \oplus f) \circ \alpha(p, q) = (1 \oplus \langle -, 0 \rangle) \vee (\langle -, 0 \rangle \oplus 1)$,

b) if $p \leq 0 \leq q$ then

$$(f \oplus f) \circ \alpha(p, q) = (1 \oplus \langle -, 0 \rangle) \vee (\langle -, 0 \rangle \oplus 1) \vee \bigvee_{v \in \mathbb{R}} \langle -, v \rangle \oplus \langle -, q-v \rangle.$$

Hence $\text{cong}(f) \subseteq \text{cong}((f \oplus f) \circ \alpha)$. □

For the next step, we set up the morphism $\alpha' : \uparrow(-, 0) \longrightarrow \uparrow(-, 0) \oplus \uparrow(-, 0)$ and $\beta' : \uparrow(-, 0) \oplus \uparrow(-, 0) \longrightarrow \uparrow(-, 0)$, it is obvious that these morphisms satisfy the above proposition make $\uparrow(-, 0)$ into the coring frame with addition α' and multiplication β' .

Corollary 2.6.5. *The collection $\mathcal{Frm}(\uparrow(-, 0), L)$ of frame homomorphisms from the frame of non-negative reals to a frame is a semiring.*

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