GROUP ANALYSIS OF EQUATIONS ARISING IN EMBEDDING THEORY

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This dissertation is submitted to the School of Mathematical Sciences, Faculty of Science and Agriculture, University of KwaZulu-Natal, Durban, in fulfillment of the requirements for the degree of Master in Science.

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As the candidate's supervisor, I have approved this dissertation for submission.

Signed: Professor K S Govinder December 2010

Abstract

Embedding theories are concerned with the embedding of a lower dimensional manifold (dim $= n$, say) into a higher dimensional one (usually dim $= n+1$, but not necessarily so). We are concerned with the particular case of embedding $4D$ spherically symmetric equations into $5D$ Einstein spaces. This scenario is of particular relevance to contemporary cosmology and astrophysics.

Essentially, they are $5D$ vacuum field equations with initial data given on a $4D$ spacetime hypersurface. The equations that arise in this framework are highly nonlinear systems of ordinary differential equations and they have been particularly resistant to solution techniques over the past few years. As a matter of fact, to date, despite theoretical results for the existence of solutions for embedding classes of $4D$ spacetimes, no general solutions to the local embedding equations are known.

The Lie theory of extended groups applied to differential equations has proved to be very successful since its inception in the nineteenth century. More recently, it has been successfully utilized in relativity and has provided solutions where none were previously found, as well as explaining the existence of ad hoc methods. In our work, we utilize this method in an attempt to find solutions to the embedding equations. It is hoped that we can place the analysis of these equations onto a firm theoretical basis and thus provide valuable insight into embedding theories.

Declaration

I declare that the contents of this dissertation are original except where due reference has been made. It has not been submitted before for any degree to any other institution.

Michael Okelola

December 2010

Declaration 1 - Plagiarism

I, Michael Okelola, declare that

1. The research reported in this thesis, except where otherwise indicated, is my original research.

2. This thesis has not been submitted for any degree or examination at any other university.

3. This thesis does not contain other persons data, pictures, graphs or other information, unless specifically acknowledged as being sourced from other persons.

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    a. Their words have been re-written but the general information attributed to them has been referenced

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5. This thesis does not contain text, graphics or tables copied and pasted from the Internet, unless specifically acknowledged, and the source being detailed in the thesis and in the References sections.

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- Jothi Moodley for help whenever I got stuck whilst understanding the background of the main problem studied.
- To my family, most especially my parents, for their unwavering support. I cannot quantify your love towards me.
- The National Research Foundation (NRF) of South Africa and the African Institute for Mathematical Sciences (AIMS) for their financial support.

Dedication

I dedicate this work to Him that I call:

Kabiyesi eledunmare, arugbo ojo, oba tose baba fun awon alaini baba, oba tinje emi ni maseberu, oba tinmu ileri se, adagba maparo oye, eleti gbaroye, okan soso ajanaku, alewi lese, alese lewi, alawo tele orun, eru jeje leti okun pupa, oyinkin yinki oba mimo, aterere kari aye, oba lana, loni titi ayeraye, oba tin tele bi eniteni, oba tin gba alai lara, oba toso ti enikan oleso, oba ibere ati opin.

THE MIGHTY GOD.

Contents

Chapter 1

Introduction

Differential equations (DEs) are the connections between calculus and the real world, 'where the rubber meets the road' [29]. From another perspective, DEs are the language in which the laws of nature are expressed. Indeed, the study of DEs began very soon after the invention of the differential and integral calculus, to which it formed a natural sequel. Newton in 1676 solved a differential equation (DE) by the use of an infinite series, but the results were not published until 1693, the same year in which a DE occurred for the first time in the work of Leibniz (whose account of the differential calculus was published in 1684) [42].

Einstein's theory of general relativity uses the language of differential geometry to describe gravity. The resultant Einstein field equations are a set of coupled, highly non-linear partial differential equations that must be solved to yield hopefully physical solutions. This task is non-trivial and frequently requires some subtlety. Indeed, the techniques applied in this dissertation have been successfully used to obtain $4D$ solutions [56, 77]. General relativity is a very successful theory and it has been accurately tested to extreme precision [94]. However it is a theory which attempts to paint a complete picture of the universe. Hence its inconsistency [49, 52] with the theory of quantum mechanics, and the nature of the cosmological dark energy and dark matter, have baffled scientists for a long time. Recently [8, 26, 84], interest has grown in attempting to solve (or soften) these difficulties by considering a higher dimensional bulk into which our $4D$ universe is embedded. This dissertation treats the solution of some DEs arising within this general scenario.

1.1 Differential equations

Any DE expresses a relation between derivatives or between derivatives and given functions of the variables. It thus establishes a relationship between the increments of certain quantities and these quantities themselves. The ancient Greeks established laws of nature in which certain relations between numbers played a privileged role. A law of this type may describe, for example, how a certain state will develop in the immediate future, or the influence of the state of a particle on the particles in the neighborhood. Thus, we have a procedure for the description of a law of nature in terms of infinitesimal differences of time and space. The increments with which the law is conserved appear as derivatives [82]. We can thus define a DE as an equation relating some function f to one or more of its derivatives. By this definition, it is not difficult to see why DEs arise so readily in the sciences. Take for instance, the function

$$y = f(x). \tag{1.1}$$

The derivative, $\frac{df}{dx}$, can be interpreted as the rate of change of f with respect to x. It is this same principle that governs any process of nature, whereby any variables involved are related to their rate of change by the basic scientific principles that govern the process. Indeed, many laws of nature - in chemistry, in biology, in engineering and physics find their most natural expression in the language of DEs. As aptly said earlier on, DEs are the language of nature.

DEs which involve only one independent variable are called ordinary differential equations (ODEs) and those which involve two or more independent variables and partial differential co-coefficients with respect to them are called partial differential equations (PDEs).

Suppose you have an ODE of, say, 2^{nd} order, for example

$$y'' = (x - y)y^3, \tag{1.2}$$

where y' denotes $\dfrac{dy}{dx}$.

The conventional ways of proceeding have been to check whether the DE belongs to a class of equations whose resolution path is already known. If this technique is unsuccessful, the next step might be to look into tables of established solutions to see if the form of the DE is synonymous to the equations with already established solutions. The use of änsatze is thereafter employed as the last option [93].

The Legendre's equation

$$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0, \tag{1.3}$$

the Bessel's equation,

$$x^2y'' + xy' + (x^2 - \alpha^2)y = 0, \tag{1.4}$$

and the Airy's equation,

$$y''' - 4xy' - 2y = 0, \tag{1.5}$$

where $y' = \dfrac{dy}{dx}$, are all classic representations of a process in nature and each has a vast literature surrounding its emergence and subsequent solutions.

1.2 Historical background of the emergence of Lie groups

In this work, we introduce the Lie group theoretic approach to the solution of DEs, particularly in the field of relativity. This we do on the basis of the successes it has been able to achieve over the years, particularly in areas where other techniques have failed.

Ironically, Lie algebras were an area Sophus Lie had little interest in initially pursuing. He instead hoped to develop the equivalent of the Galois theory to DEs. To this end, together with Friedrich Engel, he completed the third and final volume of the massive treatise *Theorie der Transformationsgruppen* [59]. In the late nineteenth century, Lie made the profound and far reaching discovery that all these special methods of solving DEs were in fact special cases of a general integration procedure based on the invariance of the DE under a continuous group of symmetries (Here, a symmetry refers to a group of transformations that transforms the set of all solutions of the differential equation to itself.). By 1884, he had obtained all of his principal results [39].

The applications of Lie groups have now had a profound effect on all areas of mathematics and mathematically-based sciences [80, 90]. As for his original idea of developing the equivalent theory of Galois theory to DEs, one researcher notes that 'the remarkable range of applications of Lie groups to DEs in geometry, in analysis, in physics, and in the engineering over the past 40 years has resurrected Lie's original vision into one of the most active and rewarding fields of contemporary research' [81].

Another scientist who made great advances in the solution of DEs was Emmy Noether, who in 1918 proved two theorems relating symmetry groups of a variational integral to properties of its associated Euler-Lagrange equations. Though this work was neglected for many years, generalised symmetries have now been found to be of importance in the study of nonlinear DEs (which can be viewed as 'completely integrable systems') [80, 90].

1.3 General relativity

The mathematical background for any researcher approaching the realm of general relativity has two cornerstones; differential geometry and differential equations. Until recently, research in general relativity basically attempted to develop and study exact solutions of Einstein's field equations. When things get difficult, the numerical approach was the first way out; exhibiting the differential geometry background and treading the path of differential equations in considerably less detail, if at all [85]. It is precisely this gap that works of this nature seek to fill, by creating a fusion of these two cornerstones.

After an experiment conducted by Michelson in 1891, it was established that light travels at a constant speed through any vacuum, independent of the choice of the reference frame [70]. This result was inconsistent with the Galileo invariant, which assumes that the speed of light is not constant, but time is conserved. This problem was solved using Lorentz transformations instead of Galileo's transformations and the physical consequences of this was pointed out by Einstein in 1905 [61, 62, 63]. The group consisting of the physical consequences of the Lorentian invariance, Einstein's equation of mass-energy equivalence,

$$E = mc^2, \tag{1.6}$$

and the effects of length contraction and time dilatation, became known as special relativity.

Spacetime is a mathematical model that combines space and time into a single continuum. Spacetime is usually interpreted with space being in $3D$ and time playing the role of the fourth dimension. By combining space and time into a single manifold (a mathematical space that on a small enough scale resembles the Euclidean space of the same dimension), physicists have been able to significantly simplify a large number of physical theories, as well succeed in describing in a more uniform manner, the workings of the universe [65].

Special relativity, via Lorentz transformations introduced the notion of spacetime; placing space and time on an equal footing. This concept was extended in general relativity via the inclusion of curvature. Indeed, in relativistic contexts time cannot be separated from the three dimensions of space. This is so because the observed rate at which time passes for an object depends on the object's velocity, relative to the observer and also on the strength of intense gravitational fields (which can slow the passage of time) [62]. The term spacetime has taken on a more generalised meaning beyond treating spacetime events with the normal 3+1 dimensions. Other proposed spacetime theories include those with additional dimensions - normally spatial, but there exist some speculative theories that include additional temporal dimensions - and even some that include dimensions that are neither temporal nor spatial. It is still unclear as to how many dimensions are needed to describe the universe. Speculative theories like the string theory [83] predict 10 or 26 dimensions (with M-theory [71] predicting 11 dimensions; 10 spatial and 1 temporal).

General relativity is very successful in providing a framework for accurate models which describe an impressive array of physical phenomena. With the advent of precision astronomy, GR provides the mathematical foundations for modern (empirically driven) cosmology and astrophysics. Mathematical relativists are still exploring the nature of singularities and the fundamental properties of Einstein's equations [30]. Ever more comprehensive computer simulations of specific spacetimes (such as those describing merging black holes) are still been run [57], and the race for the first direct detection of gravitational waves continues apace [29]. More than ninety years after the theory was first published, research is more active than ever.

On the other hand, there are many interesting open questions, and in particular, the theory as a whole is almost certainly incomplete. In contrast to all other modern theories of fundamental interactions, general relativity is a classical theory - it does not include the effects of quantum physics. The quest for a quantum version of general relativity addresses one of the most fundamental open questions in physics. While there are promising candidates for such a theory of quantum gravity, notably string theory and loop quantum gravity, there is at present no consistent and complete theory. Moreover, testing these theories may well be beyond the scope of terrestial experiments. It has long been hoped that a theory of quantum gravity would also eliminate another problematic feature of general relativity - the presence of spacetime singularities. These singularities are boundaries ('sharp edges') of spacetime at which geometry becomes ill-defined, with the consequence that general relativity itself loses its predictive power. Furthermore, there are singularity theorems which predict that such singularities must exist within the universe if the laws of general relativity were to hold without any quantum modifications. The best-known examples are the singularities associated with the extreme models that describe black holes and the beginning of the universe [35].

It is thus natural that high energy physicists and cosmologists should find common ground in the extreme phenomena of our universe. There we might see signs of the need for new theory and hints for how to build it. In modern cosmological models, most energy in the universe is in forms that have never been detected directly and whose theoretical nature is unclear, namely dark energy and dark matter. Moreover, the standard model requires an early era of inflationary expansion whose nature is poorly constrained by experiment. There have been several controversial proposals to obviate the need for these enigmatic forms of matter and energy, by modifying the laws governing gravity and the dynamics of cosmic expansion, for example the modified Newtonian dynamics [14, 76]. This dissertation is contextualised within another approach: namely to describe these phenomena via the geometric effects of an embedding into a higher dimensional space.

Einstein's theory of general relativity pioneered the idea that gravitation is an effect of the curvature of spacetime. Prior to this, gravity had been viewed as a force from the same perspective that electromagnetism was viewed. Many works have been a sequel to Einstein's theory, describing higher dimensional geometries in an attempt to unify/describe natural forces. Indeed, this theory has been a key element in the understanding of many aspects of cosmology and astrophysics [72].

In their attempts to unify general relativity with electromagnetism, Kaluza and Klein [69] proposed that there exists an extremely compact fifth dimension. This idea of the existence of extra dimensions was abandoned until the early 1960's when string theory was introduced in an attempt to explain strong nuclear forces. The notion that the various string theories represent different limiting perspectives of one 11-dimensional theory (the M-theory) was initiated by the duality transformations of the 1990's. Closer to our time, there has been a large deal of interest in $5D$ brane-world models which was prompted by the Horava-Witten theory [43], in which six of the extra dimensions from the M-theory were compactified, leaving a $5D$ theory. Alternative scenarios include D-branes, which naturally possess odd numbers of spatial dimensions, leading one to consider $6D$ models [88]. Not long after, phenomenological models (models which mathematically express the results of observed phenomena without paying detailed attention to their fundamental significance) such as those of Arkani-Hamed-Dimopoulos-Dvali [7, 8] and Randall-Sundrum [84] followed. These theories all have the potential of eventually explaining the long standing physical problems such as the dark energy and the inflationary field.

All these models require the existence of a $4D$ hypersurface or brane, which is to be embedded into a higher dimensional space, referred to as the bulk, itself satisfying the 5 (or 6)-D Einstein field equations. As a consequence of these, a great deal of interest has arisen in obtaining existence theorems and explicit solutions for such embeddings.

1.4 Outline

In chapter 2 of this work, we will give a broad outline of the method of Lie symmetry analysis. We will thereafter illustrate the technique in the resolution of an example. This example gives a broad and generalised overview of the remarkable strength of the Lie group analysis' approach in the resolution of DEs. Relevant definitions in Lie groups with their corresponding applications will then follow. We proceed with a discussion on obtaining invariant solutions of differential equations via its symmetries and give a practical example on how to implement the method. The chapter will be rounded off by a discussion on the interesting topic of hidden symmetries of DEs. We will conclude by demonstrating the existence of these symmetries by an analysis of a nonlinear DE.

Motivated by various higher dimensional theories in high-energy physics and cosmology, we seek to employ Lie group analysis for the resolution of equations that arise in the embedding of $4D$ spherically symmetric spacetimes into $5D$ Einstein (vacuum) spacetimes. In chapter 3, we provide the background material of the basic problem which we seek to solve in this work. The origin, relevance and ongoing research in this very important field will be highlighted.

Chapter 4 gives a detailed, systematic outline of how we go about obtaining general solutions to the main problem discussed in chapter 3.

In chapter 5, we conclude with the results obtained and their interpretations. Open problems and other possible areas of research close out the chapter.

Chapter 2

An Overview of the Lie group analysis approach

2.1 Introduction

In this chapter, we give a detailed outline of the techniques we shall employ in our study. Section 2.2 is dedicated to giving the general outline of the approach, and we mention the direction (in the form of which particular transformation), our analysis will point. An example of the application of the technique in the resolution of the Emden-Fowler equation will be demonstrated in subsection 2.2.1. Subsection 2.2.2 on the other hand explains the concept behind Lie algebras and we will show its usefulness by analysing the symmetries admitted by the Emden-Fowler equation.

We define another direction in our work by investigating the group invariant solutions of a DE. These are basically singular solutions of a DE – solutions which cannot be obtained from the general solution. Via a theorem by Bluman [11], in section 2.3, we give a detailed picture of this approach. We will illustrate this method by again considering the Emden-Fowler equation.

We also discuss the concept of hidden symmetries in section 2.4 as they arise in our analysis. We conclude the chapter in section 2.5.

2.2 Lie analysis

In this section, we briefly outline the technique of Lie symmetry analysis. We will employ this method in the resolution of ODEs that arise from relativistic models. The strength of this approach lies mainly in the ability of the technique to solve ODEs by using their symmetries. By a symmetry, we mean the generator of a transformation which leaves the form of the DE invariant. The main application of Lie point symmetries is searching for exact solutions by the reduction of an n^{th} order differential equation through its symmetries to an $(n-1)^{th}$ order differential equation, with the hope that the reduced equation will then be solvable.

One of the ways of solving DEs is via transforming the dependent or independent variable. This makes the resultant DE a simpler equation on substitution of these new variables. When the transformation depends on the variables alone, it is called a point transformation. This is the transformation we shall concern ourselves with here, though other forms of transformations exists (e.g contact transformations [45]).

Let us consider an invertible one-parameter group of transformations

$$\tilde{x} = \tilde{x}(x, y; \varepsilon) \qquad\qquad \tilde{y} = \tilde{y}(x, y; \varepsilon) \tag{2.1}$$

of the (x, y) plane. These transformations depend on the real parameter ε and have the conditions

$$\tilde{x}|_{\varepsilon=0} = x \qquad\qquad \tilde{x}|_{\varepsilon=0} = y,$$

imposed on them. Transformations of the form of equation (2.1) are called point transformations (unlike contact transformations where the transformed values also depend on the derivative y'). The one-parameter group of transformations (2.1) is thus called a group of point transformations.

The infinitesimal transformations of functions $\tilde{x}$ and $\tilde{y}$ can be approximately estimated, via Taylor series expansion, as

$$\tilde{x} \approx x + \varepsilon\xi(x, y),$$

$$\tilde{y} \approx y + \varepsilon\eta(x, y).$$

The expansion was done with respect to the parameter ε in the neighbourhood $\varepsilon = 0$, where

$$\xi(x,y) = \left.\frac{\mathrm{d}\tilde{x}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0},$$
$$\eta(x,y) = \left.\frac{\mathrm{d}\tilde{y}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0}.$$

An infinitesimal operator G is then written in terms of the 1^{st} order differential operator

$$G = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}.$$

Theorem 2.2.1. *The function F(x,y) = 0 is an invariant of a group of point transformations with the infinitesimal operator G if and only if it satisfies the condition [86]*

$$GF \equiv 0.$$

□

From Theorem 2.2.1, we can, in like manner, easily show that an n^{th} order ODE

$$E(x, y, y', ..., y^{(n)}) = 0, \tag{2.2}$$

is invariant under the generator

$$G = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y},$$

provided

$$G^{[n]}E|_{E=0} = 0, \tag{2.3}$$

holds, where

$$G^{[n]} = G + \sum_{i=1}^{n}\left(\eta^{(i)} - \sum_{j=0}^{i-1}\binom{i}{j}y^{(i+1)}\xi^{(i-j)}\right).$$

This means that the action of the n^{th} extension of G on E is zero when the original equation is satisfied.

Note that

$$\xi' = \frac{\partial \xi}{\partial x} + y'\frac{\partial \xi}{\partial y} \tag{2.4}$$

for the first derivative,

$$\xi'' = \frac{\partial^2 \xi}{\partial x^2} + 2y' \frac{\partial^2 \xi}{\partial x \partial y} + y'^2 \frac{\partial^2 \xi}{\partial y^2} + y'' \frac{\partial \xi}{\partial y} \tag{2.5}$$

for the second derivative and so on (similar expressions apply to η). As a result, condition (2.3) is an identity in the powers of y'. Equating coefficients of the different powers of y' to zero results in a system of linear partial differential equations in η and ξ. Solving this system explicitly for η and ξ yields the symmetry G. This analysis can be automated. We use a combination of program LIE [40] and the SYM package [25] to determine the symmetries of the equations we study.

Once the symmetries are known explicitly, they can be used to reduce the order of the DE. In order to reduce equation (2.2), we obtain reduction variables via $G^{[1]}z = 0$, where $z = z(x, y, y')$ is an arbitrary function of its arguments. This results in the equation

$$\xi \frac{\partial Z}{\partial x} + \eta \frac{\partial Z}{\partial y} + (\eta' - y'\xi') \frac{\partial Z}{\partial y'} = 0 \tag{2.6}$$

which has the associated Lagrange's system

$$\frac{\mathrm{d}x}{\xi} = \frac{\mathrm{d}y}{\eta} = \frac{\mathrm{d}y'}{\eta' - y'\xi'}.$$

Solving the 1^{st} and 2^{nd} terms in the system gives the zeroth order differential invariant, while the 2^{nd} and 3^{rd} terms gives the first order differential invariant. These are the new variables that make the n^{th} order DE become an $(n-1)^{th}$ order DE.

2.2.1 Special case of the generalised Emden-Fowler equation

The generalised Emden-Fowler equation

$$y'' = f(x) y^n \tag{2.7}$$

is of great importance in the analysis of the gravitational behaviour of many cosmological and astrophysical models [60, 98]. We will look at a particular case of this problem in order to highlight the ability of the Lie group analysis approach in the resolution of a problem.

Statement of the problem

We consider a particular case of equation (2.7) with $f(x) = 1$ and $n = 2$, i.e.

$$y'' = y^2. \tag{2.8}$$

We wish to find a solution to this equation using the Lie group approach.

Solution of the problem by the Lie group approach

In determining the symmetries of equation (2.8), we require condition (2.3) to hold for (2.8) i.e.

$$\begin{aligned}\left(\frac{\partial^2\eta}{\partial x^2} + 2y'\frac{\partial^2\eta}{\partial x\partial y} + y'^2\frac{\partial^2\eta}{\partial y^2} + y^2\frac{\partial\eta}{\partial y}\right) - 2y^2\left(\frac{\partial\xi}{\partial x} + y'\frac{\partial\eta}{\partial y}\right) \\ -y'\left(\frac{\partial^2\xi}{\partial x^2} + 2y'\frac{\partial^2\xi}{\partial x\partial y} + y'^2\frac{\partial^2\xi}{\partial y^2} + y^2\frac{\partial\xi}{\partial y}\right) = 2\eta y.\end{aligned} \tag{2.9}$$

Observe that, while ξ and η do not depend on derivatives of y, these derivatives appear in (2.9). This allows us to equate different powers of y to zero to obtain an over-determined system of linear PDEs [45]. In the case of (2.9), the system is

$$y'^3: \qquad \frac{\partial^2\xi}{\partial y^2} = 0 \tag{2.10}$$

$$y'^2: \qquad \frac{\partial^2\eta}{\partial y^2} - 2\frac{\partial^2\xi}{\partial x\partial y} = 0 \tag{2.11}$$

$$y'^1: \qquad 2\frac{\partial^2\eta}{\partial x\partial y} - \frac{\partial^2\xi}{\partial x^2} - 3y^2\frac{\partial\eta}{\partial y} = 0 \tag{2.12}$$

$$y'^0: \qquad \frac{\partial^2\eta}{\partial x^2} - 2y^2\frac{\partial\xi}{\partial x} + y^2\frac{\partial\eta}{\partial y} = 2\eta y \tag{2.13}$$

The solution of (2.10)–(2.13) yields

$$\eta = A_0 + xA_1, \tag{2.14}$$

$$\xi = -2yA_1, \tag{2.15}$$

which gives the two symmetries of equation (2.8) as

$$G_1 = \frac{\partial}{\partial x}, \tag{2.16}$$

$$G_2 = \frac{1}{2}x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}. \tag{2.17}$$

Due to the admittance of these two symmetries, equation (2.8) can now be reduced to quadratures. We shall now proceed to reduce equation (2.8) via G_1. The new variables for transformation will then be

$$u = y \qquad\qquad V(u) = y'. \tag{2.18}$$

Under this transformation, equation (2.8) reduces to

$$VV' = u^2, \tag{2.19}$$

a 1^{st} order DE. This implies that

$$V = \sqrt{\frac{2}{3}(u^3 + 3\beta)}, \tag{2.20}$$

where β is a constant of integration.

To obtain a solution to equation (2.8), the technique requires that we invert the solution in equation (2.20) through the transformations hitherto made.

Using (2.18), we will have that

$$y' = \sqrt{\frac{2}{3}(y^3 + 3\beta)}. \tag{2.21}$$

We solve this to obtain

$$x - \alpha = F\left[\sin^{-1}\left\{\frac{19}{25}\sqrt{\left(\frac{13}{15} - \frac{i}{2}\right) + \left(\frac{3}{5} - \frac{8i}{23}\right)\frac{y}{\beta^{\frac{1}{3}}}}\right\}, \left(\frac{1}{2} + \frac{13i}{15}\right)\right] \left(\frac{17}{9} + \frac{11i}{10}\right)\sqrt{\frac{12 - 7i}{42\beta^{\frac{1}{3}}}} \tag{2.22}$$

where α is an arbitrary constant and the function F is the incomplete elliptic integral of the first kind.

2.2.2 Lie algebras

Definition 2.2.2. A Lie algebra $\mathcal{L}$ is a vector space over a field $\mathcal{F}$ augmented by a bilinear composition law [,], known as the Lie bracket for $\mathcal{L}$, such that the following properties hold:

(1) Bilinearity:

$$[\alpha v_1 + \beta v_2, v_3] = \alpha[v_1, v_3] + \beta[v_2, v_3],$$
$$[v_1, \alpha v_2 + \beta v_3] = \alpha[v_1, v_2] + \beta[v_1, v_3].$$

(2) Anti-commutativity:

$$[v_1, v_2] = -[v_2, v_1]$$

(3) The Jacobi identity

$$[v_1, [v_2, v_3]] + [v_2, [v_3, v_1]] + [v_3, [v_1, v_2]] = 0$$

$\forall$ vectors $v_i, i = 1, ..., 3 \in \mathcal{L}$ and constants α, β [86]. $\square$

Let $G_i; i = 1, ..., m$ be linearly independent infinitesimal generators associated with an m-dimensional Lie invariant transformation group. On introduction of the Lie bracket

$$[G_1, G_2] = G_1G_2 - G_2G_1,$$

it is established that $\{G_i\}$ generate an m-dimensional Lie algebra. Furthermore it can be shown that if

$$[G_1, G_2] = \lambda G_1,$$

where λ is a non-zero constant, then reduction via G_1 will result in $G_2^{[1]}$ being a point symmetry of the reduced equation. On the other hand, reduction via G_2 will result in $G_1^{[1]}$ not being a point symmetry for the reduced equation [45]. In the case of $\lambda = 0$, symmetries G_1 and G_2 will commute i.e. reduction via either of them will result in the other being a point symmetry of the reduced equation.

We shall extensively employ the above technique in the choice of which symmetries will be most appropriate for the reduction of the equations we studied.

Recall that we reduced the order of equation (2.8) via G_1. This was due to the fact that the Lie bracket analysis of the symmetries,

$$[G_1, G_2] = \frac{1}{2}G_1. \tag{2.23}$$

We then inferred that only reduction via G_1 will result in $G_2^{[1]}$ being a point symmetry of the resultant equation (and not vice-versa). As a result of this route of reduction, we were able to solve the reduced equation (2.19).

To confirm our assertion, we will attempt to reduce via G_2. This results in the 1^{st} order equation

$$V'(3V - V'u) - V' = 1. \tag{2.24}$$

As expected, this equation cannot be solved directly as opposed to (2.19) (we note that a solution has been obtained via the nonlocal symmetries [3]).

2.3 Group invariant solutions

In addition to the above method to find the general solution of an ODE via symmetries, we can also use symmetries to find singular solutions.

Let Δ be a group of DEs defined over an open subset $M \subset X \text{x} Y \simeq \mathbb{R}^p \text{x} \mathbb{R}^q$; where X, Y are the spaces of the independent and dependent variables respectively. Let G be a local group of transformations acting on M. A solution $y = f(x)$ of Δ is said to be G-invariant if it remains unchanged by all the group transformations in G [80].

In particular, if an ODE admits a one-parameter Lie group of transformations, then special cases called invariant solutions can be constructed without knowledge of the general solution of the ODE [11]. In this section, we shall give an outline of the theorem by Bluman [11], on how group invariant solutions are obtained.

Theorem 2.3.1. *Suppose*

$$F(x, y, y', ..., y^n) = 0 \tag{2.25}$$

admits the one-parameter Lie group of transformations

$$x^* = X(x, y; \epsilon) = x + \epsilon\xi(x, y) + O(\epsilon^2),$$
$$y^* = Y(x, y; \epsilon) = y + \epsilon\eta(x, y) + O(\epsilon^2),$$

with infinitesimal generator

$$G = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}, \tag{2.26}$$

in domain $D \subset R^2$.

Without loss of generality, assume that $\xi(x, y) \neq 0$ *in D. Let*

$$\psi(x, y) = \frac{\eta(x, y)}{\xi(x, y)}, \qquad Y = \frac{\partial}{\partial x} + \psi(x, y)\frac{\partial}{\partial y} = \frac{1}{\xi(x, y)}X$$

and

$$y_k = Y^{k-1}\psi, \qquad k = 1, 2, ...n.$$

Then the general solution of

$$Q(x, y) = F(x, y, \psi, Y\psi, ..., Y^{n-1}\psi) = 0,$$

yields an invariant solution $\phi(x, y) = 0$, *of the DE (2.25).*

To practically typify how to obtain invariant solutions, we shall again consider the Emden-Fowler equation (2.8). Recall that equation (2.8) admits the symmetry

$$G_2 = \frac{1}{2}x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}. \tag{2.27}$$

Following Theorem 2.3.1, we now define

$$\psi = \frac{\eta}{\xi} = \frac{-2y}{x} \tag{2.28}$$

and

$$Y = \frac{\partial}{\partial x} - \frac{2y}{x}\frac{\partial}{\partial y}. \tag{2.29}$$

We will now have that

$$y' = \psi = \frac{\eta}{\xi} = \frac{-2y}{x}, \tag{2.30}$$

$$y'' = Y\psi = \left[\frac{\partial}{\partial x} - \frac{2y}{x}\frac{\partial}{\partial y}\right]\left[\frac{-2y}{x}\right] = \frac{6y}{x^2}. \tag{2.31}$$

The new DE,

$$Q(x, y) = \frac{6y}{x^2} - y^2 = 0, \tag{2.32}$$

has general solutions

$$y = 0 \qquad\qquad y = \frac{6}{x^2}. \tag{2.33}$$

These are two invariant solutions of the Emden-Fowler equation (2.8).

Note that we cannot obtain these solutions by setting the constants in (2.22) to special values.

2.4 Hidden symmetries

Research has now shown that under special cases, DEs on reduction, admit symmetries that were not evident when they were in their original state. This breakthrough in research has enabled scientists solve problems that were hitherto abandoned due to its initial admittance of an inadequate number of symmetries [90].

There are two classes of hidden symmetries: Type I hidden symmetries, which occur when one or more Lie symmetries are lost during the decrease in order of a DE, and Type II symmetries which are lost during the increase in the order of a DE [2, 90].

A good example to show the existence of hidden symmetries is the equation

$$2FF^{(iv)} + 4F'F''' = 0, \tag{2.34}$$

which arise in the study of the Emden-Fowler equation (2.7) when $n = 2$ [68].

Equation (2.34) admits symmetries

$$X_1 = \frac{\partial}{\partial x}, \tag{2.35}$$

$$X_2 = x\frac{\partial}{\partial x}, \tag{2.36}$$

$$X_3 = F\frac{\partial}{\partial F}. \tag{2.37}$$

By the Lie bracket analysis of the symmetries, we proceed to reduce equation (2.34) via X_1. The differential invariants for this symmetry are

$$u = F \qquad\qquad V(u) = \dot{F}. \tag{2.38}$$

Subsequent substitution of these new variables into equation (2.34) results in the 3^{rd} order equation

$$V\left[2uV'^3 + VV'(5V' + 8uV'') + V^2(5V'' + 2uV''')\right] = 0. \tag{2.39}$$

Equation (2.39) now admits symmetries

$$Y_1 = u\frac{\partial}{\partial u}, \tag{2.40}$$

$$Y_2 = V\frac{\partial}{\partial V}, \tag{2.41}$$

$$Y_3 = 2u^2\frac{\partial}{\partial u} + uV\frac{\partial}{\partial V}. \tag{2.42}$$

Observe that Y_3 is a new symmetry whose existence was unknown during the admittance of symmetries by the initial DE. This is a Type II hidden symmetry and can be used for further reduction of the DE [2, 90].

2.5 Remarks and synopsis

In this chapter, we gave a detailed outline of the Lie group analysis approach. This method was then illustrated via examples. We indicated how to reduce the order of an equation, find its group invariant solutions and discussed the notion of hidden symmetries.

We discuss the origin of the problem we will study in the next chapter.

Chapter 3

Background to the main problem

3.1 Introduction

This chapter will be dedicated to providing sufficient general relativistic (GR) background to the problem which we seek to solve. We shall employ differential geometry as a tool in this construction, before seeking to solve the resultant system of DEs in the next chapter. Section 3.2 will highlight basic definitions, culminating in the definition of isometric embeddings (both local and global). That section will also discuss the Gauss, Codazzi and Ricci equations, which provide the necessary platform for embedding one spacetime into another. Section 3.3 covers the vast amount of work that has being done in previous years, leading up to the contemporary formulation of the embedding equations. We shall outline these equations, in the context of spherical symmetry, in the next section - section 3.4 - making the following chapter a natural sequel (where we look at obtaining solutions to these equations). Section 3.5 consists of remarks and a brief synopsis of the chapter.

We shall consistently denote quantities concerning the embedded space with a tilde while an overbar will be used to denote quantities obtained from the n-dimensional component of the higher-dimensional metric. Also note that we shall denote differentiation with respect to y and r by dots and primes respectively. We follow the

Einstein summation convention for repeated indices.

3.2 Differential Geometry

Differential geometry is a mathematical discipline that utilises the tools of differential and integral calculus, coupled with (multi)-linear algebra in the study of geometry. Over the years, it has grown into a field now majorly concerned with the geometric structures of manifolds. A systematic approach will be utilised in the introduction of this field - by giving basic definitions of the important concepts and a discussion on the structure of these concepts.

These materials are drawn from references [10, 32, 55, 92, 99]. For further information, the reader is referred to these sources.

Suppose f is a function between two sets M and N. Then f is a 'homeomorphism' if

(1) f is continuous,

(2) its inverse f^{-1} is continuous, and

(3) f is bijective i.e. is 1-1 and onto.

A manifold M is a space that looks locally like the Euclidean space but may have a different structure globally. Thus a line and a circle are one-dimensional manifolds, a plane and sphere (the surface of a ball) are two-dimensional manifolds, and so on. A manifold need not be equipped with any system of measurement. For example a plane, considered purely as a manifold, is like the Euclidean plane stripped of the notions of length and angle. More formally, every point of an n-dimensional manifold has a neighborhood homeomorphic to a neighborhood of the n-dimensional space $\mathbb{R}^n$.

For most applications, a special kind of topological manifold called the differential manifold is always used. To accurately define this concept, we need to understand what an atlas is.

An n-dimensional chart at $p \in M$ is a pair (λ, μ), where λ is an open set of M

containing p. The function μ is a homeomorphism of λ onto an open set of $\mathbb{R}^n$. Additional features of this pair is that λ is a coordinate neighborhood while μ is a coordinate map. If ψ^i are the coordinate functions on $\mu(\lambda)$, then

$$\mu^i = \psi^i \circ \mu \tag{3.1}$$

are local coordinates on λ. The set of all n-dimensional charts $(\lambda_\alpha, \mu_\alpha)$ such that $\{\lambda_\alpha\}$ covers M, is an atlas.

Definition 3.2.1. The set M is a differentiable manifold if and only if it comes equipped with a countable atlas, and satisfies the Hausdorff property. □

The Hausdorff property is defined such that, for any two points $x \neq y$ in M, there are disjoint open sets A and B, with $x \in A$ and $y \in B$. All metric spaces are in fact Hausdorff spaces.

In order to measure the distances and angles on a manifold, the manifold must be (pseudo)-Riemannian. A (pseudo)-Riemannian manifold is simply a differentiable manifold in which each tangent space is equipped with an inner product $\langle\cdot,\cdot\rangle$ in a manner that varies smoothly from point to point. Not every differentiable manifold can be given a (pseudo)-Riemannian structure though - there are topological restrictions on doing so.

The Euclidean space itself carries a natural structure of the (pseudo)-Riemannian manifold (i.e. the tangent spaces are naturally identified with the Euclidean space itself and carry the standard scalar product of the space). Many familiar curves and surfaces, including for example all n-spheres, are specified as subspaces of a Euclidean space and inherit a metric from their embedding in it.

Now, let us suppose that M is a C^k manifold ($k \geq 1$) and p is a point in M. The tangent space of M at p, denoted by T_pM is the set of all the tangent vectors to M at p. Let $f : M \to N$ be a smooth map between two smooth manifolds. At $p \in M$, f induces a differential map $f_* : T_pM \to T_{f(p)}N$ given by

$$(f_*X)(\alpha) \equiv X(\alpha \circ f),$$

where $X \in T_pM$ and α is a real-valued function defined in a neighborhood of $f(p)$ [10].

An m-dimensional submanifold of M is characterised by $x^a = x^a(u^1, u^2, ..., u^n)$; for $a = 1, 2, ..., m$, where n is the dimension of the manifold and is strictly less than m $(n < m)$. An $(n-1)$-dimensional submanifold (for $n \geq 3$), parametrically characterised by

$$h(x^1, x^2, ..., x^n) = 0,$$

is called a hypersurface.

The relation

$$ds^2 = f(x^a, dx^a),$$

which is the measure of the infinitesimal distance between two points, is referred to as the metric. The quantity dx^a is the difference between the coordinates of the points in the manifold. We are particularly concerned with (pseudo)-Riemannian metric spaces, whose metric is of the form

$$ds^2 = g_{ab}(x^c)dx^a dx^b.$$

The forms dx^ν are the one-form gradients of the scalar coordinate fields x^ν while the symmetric coefficients g_{ab} are a set of real-valued functions (since g is a tensor field defined at all points of a spacetime manifold). The interval is timelike when $ds^2 < 0$, lightlike when $ds^2 = 0$ and spacelike when $ds^2 > 0$ (the metric is called Riemannian if this is always true).

The fundamental theorem of Riemannian geometry states that, given a metric on any (pseudo)-Riemannian manifold, there is a unique symmetric connection which preserves the scalar product under parallel transport. This unique connection is characterised by the Christoffel symbol

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad}(g_{cd,b} + g_{db,c} - g_{bc,d}). \tag{3.2}$$

This is referred to as the Christoffel symbol of the second kind. On the other hand, the relation

$$\Gamma_{abc} = \frac{1}{2}(g_{ac,b} + g_{ba,c} - g_{bc,a}) \tag{3.3}$$

is referred to as the Christoffel symbol of the first kind. Note that commas denote partial differentiation.

We shall employ the Christoffel symbol of the second kind in the definition of the covariant derivative of a manifold. The covariant derivative ∇ is a smooth map between tensors of ranks (p, q) and $(p, q+1)$, having components

$$\begin{aligned}
\nabla_d V^{a_1,\ldots a_p}{}_{b_1,\ldots,b_q} = \quad & V^{a_1,\ldots a_p}{}_{b_1,\ldots,b_q;d} \\
= \quad & V^{a_1,\ldots a_p}{}_{b_1,\ldots,b_q,d} \\
& + \Gamma^{a_1}{}_{cd} V^{ca_2,\ldots a_p}{}_{b_1,\ldots,b_q} + \ldots + \Gamma^{a_p}{}_{cd} V^{a_1,\ldots c}{}_{b_1,\ldots,b_q} \\
& - \Gamma^{c}{}_{b_1 d} V^{a_1,\ldots a_p}{}_{cb_2,\ldots,b_q} - \ldots - \Gamma^{c}{}_{b_q d} V^{a_1,\ldots a_p}{}_{b_1,\ldots,c}.
\end{aligned}$$

3.2.1 Curvature and the field equations

Geometric investigations of manifolds are done via two perspectives - intrinsic and extrinsic. The extrinsic curvature of a manifold depends on how the manifold is embedded into a higher dimensional space, providing a geometrical relationship between the embedded and embedding spaces. The intrinsic curvature on the other hand is confined to the manifold, assuming no knowledge of what happens outside the manifold.

The intrinsic curvature of a manifold (due to the non-commutativity of covariant differentiation) is characterised by

$$\begin{aligned}
\mathbf{J}^a{}_{;cd} - \mathbf{J}^a{}_{;dc} &= R^a{}_{bcd} \mathbf{J}^b \\
\mathbf{J}_{b;dc} - \mathbf{J}_{b;cd} &= R^a{}_{bcd} \mathbf{J}_a,
\end{aligned}$$

where $\mathbf{J}$ is some vector. The quantity

$$R^a{}_{bcd} = \Gamma^a{}_{bc,d} - \Gamma^a{}_{bd,c} + \Gamma^a{}_{ed} \Gamma^e{}_{bc} - \Gamma^a{}_{ec} \Gamma^e{}_{bd}$$

is called the Riemann tensor or curvature tensor of the second kind. The quantity

$$R_{abcd} = g_{ae} R^e{}_{bcd} = \Gamma_{abc,d} - \Gamma_{abd,c} + \Gamma_{afd} \Gamma^f{}_{bc} - \Gamma_{afc} \Gamma^f{}_{bd}$$

is referred to as the Riemann tensor of the first kind.

These Riemann tensors satisfy the properties of:

(1) Anti-symmetry on its first and second pairs i.e. $R_{abcd} = -R_{bacd} = -R_{abdc}$

(2) Symmetry on pair exchange i.e. $R_{abcd} = R_{cdab}$

(3) Cyclic identity i.e. $R_{abcd} + R_{adbc} + R_{acdb} = 0$

The relation

$$R_{ab} = \Gamma^{d}{}_{ad,b} - \Gamma^{d}{}_{ab,d} + \Gamma^{d}{}_{eb}\Gamma^{e}{}_{ad} - \Gamma^{d}{}_{ed}\Gamma^{e}{}_{ab}$$

is the expression for the symmetric Ricci tensor, obtained from the contraction $R_{ab} = R^{d}_{abd}$. Its scalar equivalent is the Ricci scalar R, given by

$$R = R^{b}{}_{b} = g^{ab}R_{ab}.$$

When a manifold has constant curvature, the Riemann tensor can be written as

$$R_{amsq} = \frac{1}{K}\left(g_{as}g_{mq} + g_{aq}g_{ms}\right),$$

where the constant K is referred to as the radius of curvature of the manifold.

The Riemann curvature described above provides a measure of the intrinsic curvature - it makes no assumptions about a higher dimensional embedding space. The latter perspective leads to the notion of extrinsic curvature. An example of the extrinsic curvature of a $3D$ spacetime in a $4D$ spacetime with metric

$$ds^2 = g_{ab}dx^a dx^b = g_{AB}dx^A dx^B - \phi dt^2,$$

is given by

$$\Omega_{ab} = -\frac{1}{2\phi}\frac{\partial g_{ab}}{\partial t},$$

where $\phi = \phi(x, y, z, t)$ [94]. This notion is developed further in the next section.

Before outlining the details of embedding theory, we need to describe the embedding space. This is done via the Einstein field equations (EFE):

$$G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R = KT_{ab} + g_{ab}\Lambda,$$

where G_{ab} is the Einstein tensor, Λ is the cosmological constant, T_{ab} is the matter tensor and K is a constant $\left(\frac{8\pi G}{c^4}\right)$ comprised of Newton gravitational constant (G) and the speed of light (c). The constant K is usually set to 1 for convenience. The EFE can be interpreted as a set of equations showing how the curvature of spacetime is related to the matter/energy content of the universe.

The form of the Ricci tensor (R_{ab}) in an $(n+1)$-dimensional Einstein (vacuum) spacetime is

$$R_{ab} = \frac{2\Lambda}{1-n} g_{ab}.$$

The Ricci scalar (R), is of the form

$$R = \frac{2n\Lambda}{1-n}.$$

When $\Lambda = 0$; $R_{ab} = 0 = R$, implying a Ricci-flat space. We shall employ Einstein spaces as our embedding spaces - motivated by a programme of 'geometrizing' the physics, we seek embedding spaces that are devoid of matter. Later on in this section, we will highlight Eisenhart's [28] alternative form of the EFE in a bid to embed an n-dimensional space V_n into an m-dimensional space V_m.

3.2.2 Embedding theory

As aforementioned in the introductory chapter, a great deal of interest arose in embeddings in a quest, via the phenomenological models [7, 8, 84], to eventually explain long standing physical problems such as dark energy, dark matter and inflationary fields. Indeed, all these higher dimensional theories of gravity require a sound knowledge of differential geometry to understand how to embed one manifold into another. The Gauss, Codazzi and Ricci equations provide us with these tools by relating the twisting of the manifolds relative to another, and by relating the intrinsic and extrinsic curvatures.

An embedding f is essentially an homeomorphism onto itself. In other words, a mapping $f : M \to N$ between manifolds M and N is an embedding if f yields a homeomorphism between M and $f(M)$. This implies that $f(M)$ is then a submanifold. Embeddings could either be local or global.

Definition 3.2.2. Suppose M^n is an n-dimensional analytic manifold with metric g_{ij} and N^{n+k} is an $(n+k)$-dimensional analytic manifold with metric $\tilde{g}_{\mu\nu}$. Then $f : M^n \to N^{n+k}$ is a global isometric embedding if:

(1) f is a homeomorphism onto its image,

(2) $f_* : T_pM^n \to T_{f(p)}N^{n+k}$ is injective $\forall p \in M^n$, and

(3) $g_p(R,S) = \tilde{g}_{f(p)}\left(f_*(R), f_*(S)\right) \quad \forall\, R,S \in T_pM^n, \quad \forall\, p \in M^n$ [32]. □

The function f is a local isometric embedding if and only if given a subset $U \subset M^n$ (where U is an open coordinated neighborhood of the point p), the three conditions above hold for $f : U \to V, \quad V \subset N^{n+k}$.

The Gauss, Codazzi and Ricci (GCR) equations

In a bid to locally embed an n-dimensional space V_n into an m-dimensional space V_m as a hypersurface (where $m > n$), Eisenhart [28] produced three equations

$$R_{hijk} = \sum_{\sigma} e_{(\sigma)}(\Omega^{(\sigma)}_{hj}\Omega^{(\sigma)}_{ik} - \Omega^{(\sigma)}_{hk}\Omega^{(\sigma)}_{ij}) + \bar{R}_{\alpha\beta\gamma\delta}y^{\alpha}{}_{,h}y^{\beta}{}_{,i}y^{\gamma}{}_{,j}y^{\delta}{}_{,k}, \tag{3.4}$$

$$\Omega^{(\sigma)}_{ij,k} - \Omega^{(\sigma)}_{ik,j} = \sum_{\sigma} e_{(\tau)}(t^{(\tau\sigma)}_{k}\Omega^{(\tau)}_{ij} - t^{(\tau\sigma)}_{j}\Omega^{(\tau)}_{ik}) + \bar{R}_{\alpha\beta\gamma\delta}y^{\alpha}{}_{,i}y^{\gamma}{}_{,j}y^{\delta}{}_{,k}n^{\beta(\sigma)}, \tag{3.5}$$

$$\begin{aligned} t^{(\tau\sigma)}_{j,k} - t^{(\tau\sigma)}_{k,j} = \sum_{\varrho} e_{(\varrho)}(t^{(\varrho\tau)}_{j}t^{(\varrho\tau)}_{k} - t^{(\varrho\tau)}_{k}t^{(\varrho\tau)}_{j}) + g^{lh}(\Omega^{(\tau)}_{lk}\Omega^{(\sigma)}_{hj} - \Omega^{(\tau)}_{lj}\Omega^{(\sigma)}_{hk}) \\ -\bar{R}_{\alpha\lambda\mu\nu}y^{\mu}{}_{,j}y^{\nu}{}_{,k}n^{\lambda(\sigma)}n^{\alpha(\tau)}, \end{aligned} \tag{3.6}$$

known as the Gauss, Codazzi and Ricci equations respectively, where $e_{\sigma} = \pm 1$ and $t^{\tau\sigma}_{j}$ describes the twisting of the $n^{\alpha(\sigma)}$ vectors in relation to one another, for $\sigma, \tau = n, ..., m-1$ and $\sigma \neq \tau$.

These equations are essentially the field equations for V_m given initial data V_n. They are the machinery to be used in describing the twisting of the manifolds relative to another, and relating the intrinsic and extrinsic curvatures. The Gauss and Codazzi equations must be solved on V_n (i.e. on the hypersurface) while the Ricci equation should be solved on V_m (off the hypersurface). To date, there do not exist any known general solution to these equations. Several works have been built upon these equations and these are what we shall highlight in the next section.

Note that for embeddings with codimension one, there is no twisting (i.e. $t_j^{\tau\sigma} = 0$), so that the Ricci equation disappears. The components of the Ricci tensor for V_m are subsequently formulated into a propagation equation [22].

3.3 Existence theorems

Schläfli [89] was the first scientist to consider the problem of embedding an n-dimensional (pseudo)-Riemannian manifold locally into an Euclidean manifold by suggesting that the dimension of the embedding space should be $\frac{n(n+1)}{2}$. This work was motivated by a desire to understand (pseudo)-Riemannian spaces in terms of the more familiar Euclidean manifolds.

Janet [48] and Cartan [16] proved the Schläfli theorem on the dimension of the embedding space before Friedman [31] treated the indefinite case. Since then, several theorems have been propounded, building upon this foundation.

One of the prominent contributors to this research was the American mathematician, John Forbes Nash [38]. He was able to show, by two theorems, that every (pseudo)-Riemannian manifold can be isometrically embedded (i.e. preserving the length of every path in the embedding) into some Euclidean space.

The 1^{st} theorem (Nash-Kuiper theorem) deals with continuously differentiable (C^1) embeddings.

Theorem 3.3.1. *Let (M, g) be a Riemannian manifold and $f : M^n \to \mathbb{R}^m$ a C^∞-embedding into the Euclidean space, where $m \geq n + 1$. Then for arbitrary $\epsilon > 0$, there is an embedding $f_\epsilon : M^n \to \mathbb{R}^m$ which is* *(i) in class* C^1*,*

(ii) isometric for any two vectors $v, w \in T_p(M)$ *in the tangent space at* $p \in M$*:*

$$g(v, w) = \langle df_\epsilon(v), df_\epsilon(w) \rangle,$$

(iii) ϵ*-close to* f*:*

$$|f(p) - f_\epsilon(p)| < \epsilon \qquad\qquad \forall p \in M. \tag{3.7}$$

$\square$

The theorem was originally proved by John Nash with the condition that $m \geq n+2$. It was later generalised by Nicholas Kuiper to the case of $m \geq n+1$ [54].

The 2^{nd} theorem deals with analytic embeddings that are smooth of class C^k, $3 \leq k \leq \infty$.

Theorem 3.3.2. *Given an* n*-dimensional Riemannian manifold* M*, which is either analytic or of order* $C^k, 3 \leq k \leq \infty$*. Then there exists a number* m *(where* $m = n^2 + 5n + 3$*) and an injective map* $f : M \to \mathbb{R}^m$ *(which is also analytic or of order* C^k*); such that for every* $p \in M$*, the derivative* df_p *is a linear map from the tangent space* T_pM *to* $\mathbb{R}^m$*. This mapping is compatible with the given inner product on* T_pM *and the standard dot product of* $\mathbb{R}^m$ *in the sense that*

$$\langle u, v \rangle = df_p(u) \cdot df_p(v),$$

$\forall$ *vectors* $u, v \in T_pM$*. This is an undetermined system of PDEs.* $\square$

Nash's embedding theorems are global in the sense that the whole manifold is embedded into $\mathbb{R}^m$.

Nash [78] also showed that every n-dimensional Riemannian manifold is embeddable in $\mathbb{R}^m$. He established that this was possible when $m = \frac{n}{2}(3n + 11)$ for compact M and $m = \frac{n}{2}(n + 1)(3n + 11)$ for non-compact M, where the embeddings are C^k isometric for $k \geq 3$ (indicating that one needs a higher number of dimensions for a smoother embedding).

Clarke [19] and Greene [36] succeeded in providing extensions to the indefinite case. From Clarke's proof, it was found that the application of his results to a non-compact Riemannian manifold will lead to a lower dimension of the Euclidean embedding space, than that obtained by Nash. Greene on his part, succeeded in demonstrating that the embedding can be made C^∞ isometric with $m = n(n+5)$ for compact M and $m = 4(2n+1)(n+3)$ for non-compact M. In more recent times, further results have built upon this foundation. Subsequent subsections are dedicated to describing these results, which are the basis for our work.

3.3.1 The Dahia-Romero (DR) theorem

Despite the historical bias towards Euclidean embedding spaces, there is no real reason to restrict one's attention in this manner. Indeed, Rund [87] dealt with spacetimes of constant curvature. Dahia and Romero [21, 22], on their part, were able to propound theorems on the local embedding of an n-dimensional (pseudo)-Riemannian manifold into both Einstein spaces and more general pseudo-Riemannian manifolds. This they did by building upon an earlier result by Campbell and Magaard [15, 66].

The Campbell-Magaard (CM) theorem

Subsequent to the work described above, several other studies [32, 87, 95, 96] followed, showing embeddings into particular Euclidean embedding spaces and Riemann manifolds. We are interested in the Campbell-Magaard theorem (stated by Campbell [15] but proved by Magaard [66]), giving a local existence theorem for embeddings into Ricci-flat (pseudo)-Riemannian spaces.

Theorem 3.3.3. *An n-dimensional (pseudo)-Riemannian manifold with analytic metric can be locally, analytically and isometrically embedded into an $(n+1)$-dimensional Ricci-flat ($\tilde{R}_{\alpha\beta} = 0$) manifold [15, 66].* □

Progression of the CM theorem to the DR theorem

Anderson and Lidsey [5] considered the case when the embedded manifold is an Einstein space. They also presented results which showed the criteria for embedding a given manifold in a spacetime that represents the solution to Einstein's equations sourced by massless scalar fields.

Further results were obtained by a group consisting of Anderson, Dahia, Lidsey and Romero [6]. They showed that Einstein and Ricci-flat spacetimes may be embedded into spacetimes sourced by self-interacting scalar fields. Just before this result, Dahia and Romero [21, 22] had already extended the Campbell-Magaard theorem to Einstein embedding spaces.

Theorem 3.3.4. *An n-dimensional (pseudo)-Riemannian manifold can be locally, analytically and isometrically embedded in an $(n + 1)$-dimensional Einstein manifold [22].* □

They improved on this result by extending it to arbitrary non-degenerate pseudo-Riemannian manifolds.

Theorem 3.3.5. *An n-dimensional pseudo-Riemannian manifold can be locally, analytically and isometrically embedded in an $(n + 1)$-dimensional pseudo-Riemannian manifold with a non-degenerate Ricci tensor which is equal, up to a local analytic diffeomorphism, to the Ricci tensor of an arbitrarily given pseudo-Riemannian manifold [21].* □

There are conditions to be satisfied for the existence of a local isometric embedding of M^n (an n-dimensional analytic manifold with metric g_{ij}) into N^{n+1}, which is an $(n + 1)$-dimensional manifold with metric $\tilde{g}_{\mu\nu}$. The following theorem gives these conditions and is used as a lemma in the proof of Theorems 3.3.4 and 3.3.5. It sets out the formalism used in this dissertation.

Theorem 3.3.6. *There exists a local isometric analytic embedding of a pseudo-Riemannian manifold (M^n, g) at $p \in U \subset M^n$ into a pseudo-Riemannian manifold $(N^{n+1}, \tilde{g})$ if and only if there exist analytic functions $\bar{g}_{ik}(x^1, ..., x^n, y)$ and*

$\bar{\phi}(x^1, ..., x^n, y)$ in a neighborhood of $(x^1_p, ..., x^n_p, 0)$ with $\bar{\phi} \neq 0$, $\bar{g}_{ik} = \bar{g}_{ki}$, $|\bar{g}_{ik}| \neq 0$ and $\bar{g}_{ik}(x^1, ..., x^n, 0) = g_{ik}(x^1, ..., x^n)$, and such that the metric for some $V \subseteq N^{n+1}$ is

$$ds^2 = \tilde{g}_{\alpha\beta} dy^\alpha dy^\beta \tag{3.8}$$

$$= \bar{g}_{ik} dx^i dx^k + \epsilon\bar{\phi}^2 (dy)^2, \tag{3.9}$$

where $\epsilon^2 = 1$. □

Since the metric is expressed in Gaussian form, we can set $\phi = 1$ without any loss of generality. It suffices to show that the embedding functions exist for the case of embedding M^n into N^{n+1}. To prove the theorem, it then suffices to show that the conditions for embedding - into Einstein and pseudo-Riemannian spaces - holds. An elaborate and well detailed form of the proof was given by Moodley [72].

The following forms

$$\begin{aligned}\tilde{G}^{n+1}{}_{n+1} &= -\frac{1}{2}\bar{g}^{ik}\bar{g}^{jm}(\bar{R}_{ijkm} + \epsilon(\bar{\Omega}_{ik}\bar{\Omega}_{jm} - \bar{\Omega}_{jk}\bar{\Omega}_{im})), \\ \tilde{R}_{i(n+1)} &= \bar{\phi}\bar{g}^{gk}(\bar{\nabla}_j\bar{\Omega}_{ik} - \bar{\nabla}_i\bar{\Omega}_{jk}),\end{aligned} \tag{3.10}$$

were then obtained for the Gauss and Codazzi equations respectively. As stated in the conclusion of the CM theorem, the codimension now reduces to one and there's no twisting in only one extra dimension. The Ricci equation is thus replaced by the new equation

$$\tilde{R}_{ik} = \bar{R}_{ik} + \epsilon\bar{g}^{jm}(\bar{\Omega}_{ik}\bar{\Omega}_{jm} - 2\bar{\Omega}_{jk}\bar{\Omega}_{im}) - \frac{\epsilon}{\bar{\phi}}\frac{\partial\bar{\Omega}_{ik}}{\partial y} + \frac{1}{\bar{\phi}}\bar{\nabla}_i\bar{\nabla}_k\bar{\phi}, \tag{3.11}$$

called the propagation equation (because it is used to propagate off the hypersurface in the specification of the rest of the bulk). The existence of solutions to the Gauss and Codazzi equations (on the hypersurface), and to the propagation equation (in the bulk) is guaranteed by an application of the Cauchy-Kowalewski theorem for PDEs [17, 53].

By using local Sobolev spaces, Dahia and Romero [23, 24] have also constructed an alternative approach to the embedding problem. They did this by asserting that, for any $4D$ spacetime, there exists initial data sets whose Cauchy development for the Einstein vacuum equations is a $5D$ vacuum space. The $4D$ spacetime can then be locally, analytically and isometrically embedded into the $5D$ vacuum space. This result eliminated potential concerns raised by Anderson [4] about causality in the embedding construction.

3.3.2 The Moodley-Amery(MA) theorem

Though the embedding results obtained by Dahia and Romero charted a new era in embedding theory, they were nonetheless local existences results, relevant only for local embeddings, with no mention of an application globally.

Global embedding theory has been found to be useful in the quest for new solutions in GR [95, 96]. Global embeddings provide insight into the global properties of a manifold and are the context for many physical applications, such as phenomenological higher dimensional cosmology. Due to the importance of the global embeddings, a natural sequel to the DR theorem will be to embed a (pseudo)-Riemannian space globally into an Einstein space. This is precisely what Katzourakis [50, 51] attempted to do.

The Katzourakis theorem

Katzourakis claimed to have provided the globalisation theorem for the Campbell-Magaard-Dahia-Romero theorem. His claim was on the basis of successfully proving that there exists a global isometric embedding of an arbitrary n-dimensional pseudo-Riemannian space M into an $(n+1)$-dimensional Einstein space $\mathcal{E} := M \times Y$, where Y is a 1-dimensional analytic manifold.

Theorem 3.3.7. *Any n-dimensional real analytic pseudo-Riemannian manifold $(M; \nabla^M, g_M)$ has a global isometric embedding into an $(n+1)$-dimensional Einstein manifold $(\mathcal{E}; \nabla^{\mathcal{E}}, g_{\mathcal{E}})$ satisfying:*

$$Ric^{\nabla^{\mathcal{E}}}(g\mathcal{E}) = \frac{2\Lambda}{n-1} g_{\mathcal{E}}$$

for $\Lambda \in \mathbb{R}$. $\square$

As a remark to the theorem, he posits that repeated applications of Theorem 3.3.7 would show that M can also be embedded into a space with codimension greater than 1. A corollary to the theorem is that any analytic manifold of the form

$$\mathcal{E}^{n+d} \cong M^{(n)} \times Y^{(d)}, \quad d \geq 1$$

admits an Einstein metric, making it an Einstein space.

Progression of the Katzourakis result to the Moodley-Amery result

After making a thorough analysis of Katzourakis assertions, Moodley and Amery [75] came to the conclusion that Katzourakis' theorem rests upon the assumption that the local Einstein embedding has the form $M \times Y$ for any embedded space M.

Dahia and Romero [22] had already shown that any n-dimensional pseudo-Riemannian space M can be locally embedded into an $(n+1)$-dimensional Einstein space equipped with metric

$$\mathrm{diag}[\bar{g}_{ik}(x^i, y), \epsilon\phi^2(x^i, y)].$$

Here, $\bar{g}_{ik}$ depends on the $(n+1)$th coordinate y, reducing to the metric for M only along the hypersurface $y = c$. Based on this, Moodley and Amery claimed that it is not true that the form of the local embedding constructed by Katzourakis is $M \times Y$ for any M. This was highlighted as the major limitation of his theorem. In addition, they also pointed out that the corollary to his theorem is similarly limited.

They went further to show by a counter example that Katzourakis' assertions were wrong. Recall the Gauss, Codazzi and propagation equations (3.10)–(3.11). Now taking N^{n+1} to be an Einstein space, where

$$\tilde{R}_{\alpha\beta} = \frac{2\Lambda}{1-n}\bar{g}_{\alpha\beta} \qquad\qquad \tilde{G}_{\alpha\beta} = \Lambda\tilde{g}_{\alpha\beta},$$

the system of equations transforms into

$$\Lambda = -\frac{1}{2}\bar{g}^{ik}\bar{g}^{jm}(\bar{R}_{ijkm} + \epsilon(\bar{\Omega}_{ik}\bar{\Omega}_{jm} - \bar{\Omega}_{jk}\bar{\Omega}_{im})), \tag{3.12}$$

$$0 = \bar{\phi}\bar{g}^{jk}(\bar{\nabla}_j\bar{\Omega}_{ik} - \bar{\nabla}_i\bar{\Omega}_{jk}), \tag{3.13}$$

$$\frac{2\Lambda}{1-n}\bar{g}_{ik} = \bar{R}_{ik} + \epsilon\bar{g}^{jm}(\bar{\Omega}_{ik}\bar{\Omega}_{jm} - 2\bar{\Omega}_{jk}\bar{\Omega}_{im}) - \frac{\epsilon}{\bar{\phi}}\frac{\partial\bar{\Omega}_{ik}}{\partial y} + \frac{1}{\bar{\phi}}\bar{\nabla}_i\bar{\nabla}_k\bar{\phi}. \tag{3.14}$$

Moodley and Amery looked at the case of a static spherical spacetime where the Ricci scalar is a function of r, for the embedding equations into a product space: $M \times Y$. Since $\bar{g}_{ik} = g_{ik}$ and does not depend on y, Ω_{ik} vanishes, and the Codazzi equation does too by implication. Equations (3.12) and (3.14) (i.e. the Gauss and propagation equations), now become

$$R_{ik} = \frac{2\Lambda}{1-n}g_{ik}, \tag{3.15}$$

$$R = -2\Lambda, \tag{3.16}$$

where Λ is the cosmological constant.

Substitution of equation (3.16) into (3.15) now yields

$$R_{ik} = \frac{Rg_{ik}}{n-1}.$$

This then implies that

$$R^e_k = \frac{R\delta^e_k}{n-1}, \tag{3.17}$$

$$R = \frac{Rn}{n-1}. \tag{3.18}$$

We can deduce from equation (3.18) that $R = 0$, and that $R_{ik} = 0$ by implication, i.e. the embedded space must be Ricci flat for the local result to have form $M \times Y$. This shows that Katzourakis' global embedding $M \times Y$ fails since there does not exist a local embedding of form $M \times Y$ for any non Ricci-flat space M. In addition, $R = 0 \implies \Lambda = 0$, showing that the global embedding space $M \times Y$ must be

Ricci-flat. This conclusively makes the Katzourakis theorem only a globalisation for Ricci-flat embedded spaces, but not for non-Ricci-flat spaces [72].

At this point, Moodley and Amery corrected Katzourakis' result and extended it to any (pseudo)-Riemannian embedded space.

Theorem 3.3.8. *Any n-dimensional real analytic pseudo-Riemannian manifold* (M, g_M) *has a global isometric analytic embedding into an (n+1)-dimensional Einstein manifold* $(\mathcal{E}, \tilde{g}_{\mathcal{E}})$*, where*

$$\tilde{R}_{\mu\nu} = \frac{2\Lambda}{1-n}\tilde{g}_{\mu\nu} \tag{3.19}$$

for $\Lambda \in \mathbb{R}$. $\square$

A brief summary of the proof of this theorem begins by assuming a global embedding space of $\bar{\mathcal{E}}$ of same Einsteinian metric structure as that of the specified local embedding space. This specified local embedding space consists of the $(n+1)$-dimensional patches into which the n-dimensional patches of the embedded space are embedded as a hypersurface.

These patches are then 'manually' inserted into the global embedding manifold: the $y = 0$ hypersurface, $\sum$, of $\bar{\mathcal{E}}$ is excised and the resultant cover for $\bar{\mathcal{E}} - \sum$ is combined with the local embedding patches. Then the paracompactness of metric spaces is used to ensure that the metrics can be made to match across the manifold. Moodley and Amery thus obtained a construction which guaranteed the global embedding space possessing the specified Ricci curvature, valid for any analytic embedded spacetime.

The principal differences between the proof by Katzourakis and the correction given by Moodley-Amery lies in the specification of the bulk cover and the counting arguments used to manifest the existence of the global metric.

By making more general definitions for $\bar{\mathcal{E}}$ to be any arbitrarily defined metric space, they were able to extend this result to a more general theorem:

Theorem 3.3.9. *If any n-dimensional real analytic metric space has a local isometric analytic embedding into some specified m-dimensional metric space* $(m \geq n+1)$, *then there exists a global isometric analytic embedding into that space.* □

3.4 The embedding equations

3.4.1 Embedding SS spacetimes

We consider $4D$ spacetimes that are spherically symmetric (SS). The term 'spherically symmetric' implies that it is invariant under rotations [41]. The property of being spherically symmetric can now be summed up as a metric which appears the same in all directions, i.e. isotropic.

In order to embed SS spacetimes M, with a metric of the form

$$ds^2 = -e^{2\nu(t,r)}dt^2 + e^{2\lambda(t,r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{3.20}$$

where ν and λ are functions of r and t, equations (3.10)–(3.11) now transform into [60, 72]

$$\frac{\partial^2 \bar{g}_{ik}}{\partial^2 y} = \frac{-4\epsilon\Lambda\bar{g}_{ik}}{3} - \frac{\bar{g}^{jm}}{2}\left(\frac{\partial\bar{g}_{ik}}{\partial y}\frac{\partial\bar{g}_{jm}}{\partial y} - 2\frac{\partial\bar{g}_{im}}{\partial y}\frac{\partial\bar{g}_{ik}}{\partial y}\right) - 2\epsilon\bar{R}_{ik}, \tag{3.21}$$

$$0 = g^{jk}(\nabla_j\Omega_{ik} - \bar{\nabla}_i\bar{\Omega}_{jk}), \tag{3.22}$$

$$-2\Lambda = R + g^{ik}g^{jm}\epsilon(\Omega_{ik}\Omega_{jm} - 2\Omega_{jk}\Omega_{im}), \tag{3.23}$$

where $\Omega_{ik}(x^j, 0) = \Omega_{ik}$ and the form of the Ricci tensor is given by

$$\bar{R}_{ab} = \bar{\Gamma}^d_{ad,b} - \bar{\Gamma}^d_{ab,d} + \bar{\Gamma}^d_{eb}\bar{\Gamma}^d_{ad} - \bar{\Gamma}^d_{ed}\bar{\Gamma}^d_{ab}.$$

This system of equations has the initial conditions

$$\bar{g}_{ik}(t, r, \theta, \phi, 0) = g_{ik}(t, r, \theta, \phi), \tag{3.24}$$

$$\frac{\partial\bar{g}_{ik}(t, r, \theta, \phi, 0)}{\partial y} = -2\Omega_{ik}(t, r, \theta, \phi). \tag{3.25}$$

Note that N^{n+1} is an Einstein space, where $\tilde{R}_{\alpha\beta} = \frac{2\Lambda}{1-n}\tilde{g}_{\alpha\beta}$ and $\tilde{G}_{\alpha\beta} = \Lambda\tilde{g}_{\alpha\beta}$.

Dahia and Romero [21] assert by their theorem that a solution to the system exists. The general form of the SS spacetime has Ricci tensors

$$
\begin{aligned}
R_{00} &= e^{2(\nu-\lambda)}(-\nu'^2 - \nu'' + \nu'\lambda' - \frac{2}{r}\nu') + \lambda_{tt} + \lambda_t^2 - \nu_t\lambda_t, \\
R_{01} &= -\frac{2}{r}\lambda_t, \\
R_{11} &= \nu'^2 + \nu'' - \nu'\lambda' - \frac{2}{r}\lambda' - e^{2(\nu-\lambda)}(\lambda_{tt} + \lambda_t^2 - \nu_t\lambda_t), \\
R_{22} &= -1 + e^{-2\lambda} + re^{-2\lambda}\nu' - re^{-2\lambda}\lambda', \\
R_{33} &= R_{22}\sin^2\theta,
\end{aligned}
$$

with coordinates (t, r, θ, ϕ).

3.4.2 Solving the Gauss, Codazzi and the propagation equations

In what follows, the goal shall is to discover which SS spacetimes may be embedded into a specified Einstein space. Accordingly, we assume

$$\Omega_{ik} = \mathrm{diag}[a(r)g_{00}, b(r)g_{11}, c(r)g_{22}, c(r)g_{33}], \tag{3.26}$$

so that the Gauss and Codazzi equations (3.22)–(3.23) transforms into [6]

$$(b-a)\nu' - a' - 2c' + \frac{2b}{r} - \frac{2c}{r} = 0, \tag{3.27}$$

$$-2\epsilon\Lambda - \epsilon R = 2c^2 + 2ab + 4ac + 4bc. \tag{3.28}$$

The Gauss and Codazzi equations in (3.27)–(3.28) have been solved [60, 72] for the general case of $R(r)$ as

$$
\begin{aligned}
c &= 2d + \frac{2\Lambda + R}{6\epsilon d}, \\
d &= \frac{18\epsilon\int e^{\nu} + I}{24\epsilon e^{\nu}} \pm \sqrt{\left[\frac{18\epsilon\int e^{\nu} + I}{24\epsilon e^{\nu}}\right]^2 - \frac{2\Lambda + R}{12\epsilon}},
\end{aligned} \tag{3.29}
$$

where I is a constant of integration, $b(r) = c(r)$ and $d(r) = a(r) + c(r)$.

By setting $a' = b' = c' = 0$ when R is a constant, solutions also exists for

$$a = b = c = \frac{f}{2} \qquad \text{and} \qquad f = \pm\sqrt{\frac{-2\epsilon\Lambda - \epsilon R}{3}},$$

when $\nu' \neq 0$ and

$$c = b \qquad \text{and} \qquad b = -\frac{a}{2} \pm \frac{1}{2}\left(a^2 - \frac{4\epsilon\Lambda}{3} - \frac{2\epsilon R}{3}\right)^{\frac{1}{2}},$$

when $\nu' = 0$.

At this juncture, Moodley and Amery [72] made an assumption for the bulk metric:

$$\bar{g}_{ik}(y, r) = \text{diag}[A(y, r)g_{00}, B(y, r)g_{11}, C(y, r)g_{22}, C(y, r)g_{33}].$$

Based on this, they constructed [75] new forms of the propagation equations:

$$\ddot{A} + \frac{\dot{A}}{2}\left[-\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} + \frac{\dot{D}}{D}\right] + \frac{4\epsilon\Lambda}{3}A = -2\epsilon\frac{A}{B}R_{00}g^{00}, \tag{3.30}$$

$$\ddot{B} + \frac{\dot{B}}{2}\left[\frac{\dot{A}}{A} - \frac{\dot{B}}{B} + \frac{\dot{C}}{C} + \frac{\dot{D}}{D}\right] + \frac{4\epsilon\Lambda}{3}B = -2\epsilon R_{11}g^{11}, \tag{3.31}$$

$$\ddot{C} + \frac{\dot{C}}{2}\left[\frac{\dot{A}}{A} + \frac{\dot{B}}{B} - \frac{\dot{C}}{C} + \frac{\dot{D}}{D}\right] + \frac{4\epsilon\Lambda}{3}C = -2\epsilon\frac{C}{B}R_{22}g^{22} - \frac{2\epsilon}{r^2}\left[\frac{C}{B} - 1\right], \tag{3.32}$$

$$\ddot{D} + \frac{\dot{D}}{2}\left[\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} - \frac{\dot{D}}{D}\right] + \frac{4\epsilon\Lambda}{3}D = -2\epsilon\frac{D}{B}R_{33}g^{33} - \frac{2\epsilon}{r^2}\left[\frac{D}{B} - \frac{D}{C}\right], \tag{3.33}$$

with initial conditions

$$\begin{array}{llll} A(0) = 1 & B(0) = 1 & C(0) = 1 & D(0) = 1, \\ \dot{A}(0) = -2\Omega_{00}g^{00} & \dot{B}(0) = -2\Omega_{11}g^{11} & \dot{C}(0) = -2\Omega_{22}g^{22} & \dot{D}(0) = -2\Omega_{33}g^{33}. \end{array}$$

From the metric $\bar{g}_{ik}$ of the bulk, we have

$$R_{00}g^{00} = e^{-2\lambda}\left[\nu'' + \nu'^2 - \nu'\lambda' + \frac{2}{r}\nu'\right] = \alpha_1, \tag{3.34}$$

$$R_{11}g^{11} = e^{-2\lambda}\left[\nu'' + \nu'^2 - \nu'\lambda' + \frac{2}{r}\nu'\right] = \alpha_2, \tag{3.35}$$

$$\frac{C}{B}R_{22}g^{22} + \left[\frac{C}{B} - 1\right]\frac{1}{r^2} = \frac{C}{B}e^{-2\lambda}\left[\frac{\nu'}{r} - \frac{\lambda'}{r} + \frac{1}{r^2}\right] - \frac{1}{r^2} = \alpha_3(y), \tag{3.36}$$

$$\frac{D}{B}R_{33}g^{33} + \left[\frac{D}{B} - \frac{D}{C}\right]\frac{1}{r^2} = \frac{D}{B}e^{-2\lambda}\left[\frac{\nu'}{r} - \frac{\lambda'}{r} + \frac{1}{r^2}\right] - \frac{D}{C}\frac{1}{r^2} = \alpha_4(y). \tag{3.37}$$

Setting $\frac{D}{C}\alpha_3 = \alpha_4$ will imply an equivalence of equations (3.36) and (3.37). Solutions to these equations are not immediately clear, necessitating further simplifying assumptions.

Moodley and Amery [75] now made an assumption of $B(y) = C(y)$ where the r-dependence is now absorbed into g_{00} and g_{CD}. This new construction has a metric of the form

$$\bar{g}_{ik}(y, r) = \mathrm{diag}[A(y)g_{00}, F(y)g_{CD}],$$

where $F(y)$ is the new form of $B(y) = C(y)$.

SS spacetimes that are embedded in this way may be shown to satisfy [73]

$$R_{00}g^{00} = \alpha_1 \qquad\qquad R_{11}g^{11} = R_{22}g^{22} = R_{33}g^{33} = \alpha_2,$$

where α_1 and α_2 are constants such that $\alpha_1 = \alpha_2$ or $\alpha_1 = 0$. These constraints arise from the solution to the Gauss and Codazzi equations.

When $\alpha_1 = \alpha_2$;

$$\lambda(r) = -\frac{1}{2}\ln\left[\frac{\alpha_1}{3}r^2 + \frac{B}{r} + 1\right], \qquad\qquad \nu(r) = -\lambda(r) + g(t),$$

where $B, C \in \mathbb{R}$. These are the generalised Schwarzschild-de Sitter spacetimes.

On the other hand, when $\alpha_1 \neq \alpha_2$ and $\alpha_1 = 0$;

$$\lambda(r) = -\frac{1}{2}\ln\left[\frac{\alpha_2}{2}r^2 + 1\right], \qquad\qquad \nu(r) = g(t),$$

where $C \in \mathbb{R}$. This is the static Einstein universe.

These spacetimes have a Ricci scalar of $R = \alpha_1 + 3\alpha_2$.

The propagation equations now become

$$\ddot{A} + \frac{\dot{A}}{2}\left[\frac{3\dot{F}}{F} - \frac{\dot{A}}{A}\right] + \frac{4\epsilon\Lambda}{3}A = -2\epsilon\frac{A}{F}\alpha_1, \tag{3.38}$$

$$\ddot{F} + \frac{\dot{F}}{2}\left[\frac{\dot{A}}{A} + \frac{\dot{F}}{F}\right] + \frac{4\epsilon\Lambda}{3}F = -2\epsilon\alpha_2, \tag{3.39}$$

with initial conditions in each case:

(i) $\alpha_1 = \alpha_2 : A(0) = F(0) = 1, \dot{A}(0) = \dot{F}(0) = \pm\sqrt{\frac{-2\epsilon\Lambda - \epsilon R}{3}}$.

(ii) $\alpha_1 \neq \alpha_2$ and $\alpha_1 = 0$: $A(0) = F(0) = 1, \dot{A}(0) = -2a, \dot{F}(0) = a \pm (a^2 - \frac{4\epsilon\Lambda}{3} - \frac{2\epsilon R}{3})^{\frac{1}{2}}$;

where a is an unspecified constant.

The following solutions have been previously obtained [5, 75]:

When $\alpha_1 = \alpha_2$:

$$A(y) = F(y) = \begin{cases} \left[\cosh\left(\frac{1}{2}\sqrt{\frac{-2\epsilon\Lambda}{3}}y\right) + \sqrt{1 + \frac{R}{2\Lambda}}\sinh\left(\frac{1}{2}\sqrt{\frac{-2\epsilon\Lambda}{3}}y\right)\right]^2 & \text{for} \quad \Lambda \neq 0, \\ \\ \left[1 + \frac{1}{2}\sqrt{\frac{-\epsilon R}{3}}y\right]^2 & \text{for} \quad \Lambda = 0. \end{cases}$$

The $4D$ embedded spacetimes are the generalised Schwarzschild-de Sitter spacetimes with $R = 4\alpha_1$ (where $-\alpha_1$ is the $4D$ cosmological constant).

When $\alpha_1 \neq \alpha_2$, $\alpha_1 = 0$ and $\Lambda = 0$ [75, 100]:

$$\begin{cases} A(y) = 1, \\ F(y) = \left[1 + \frac{1}{2}\sqrt{-2\epsilon\alpha_2}y\right]^2. \end{cases} \tag{3.40}$$

These solutions clearly do not exhaust the solution space, since they specify particular values for the α's and Λ. In the next chapter, we shall provide further solutions.

3.5 Remarks and synopsis

The relevance of the study of global embedding theories cannot be over-estimated. They have been found to be useful in the quest for new solutions in GR [95, 96] and they provide insight into the global properties of a manifold (causality, for example). Indeed, most of the phenomenological higher dimensional cosmological models are posited using a global language.

In section 3.2 we highlighted fundamental definitions relevant to a sound understanding of embedding theory. We initiated this by stating some basic definitions in differential geometry, leading to more specific definitions for local and global embeddings. We concluded the section by highlighting the GCR equations produced by Eisenhart [28]. These are the tools for embedding manifolds into each other.

Section 3.3 dealt with existence theorems for embeddings. The works done by Schläfli [89], Nash [38] and Greene *et al* [36] were briefly discussed. A detailed study was then performed on more recent works - from the Dahia-Romero theorems [21, 22] to the Moodley-Amery theorem [72].

Section 3.4 was dedicated to looking at the embedding equations that arise in a bid to embed spacetimes, particularly the SS spacetimes (this is part of ongoing doctoral work by Moodley [73]). These culminate in an initial value problem which we shall solve in the next chapter.

Solutions to the highly nonlinear ODEs that arise from these embedding equations are not easily obtained. To date, there have been no known solutions to these ODEs (for the general case of $\Lambda \neq 0$ under the second set of initial conditions), necessitating works of this nature. We attempt to seek a solution to the initial value problem (3.38)–(3.39) in the next chapter, hoping that we will be able to place investigations in this sophisticated field upon a firm basis.

Chapter 4

Application of Lie group analysis

4.1 Introduction

In this chapter, we will attempt to employ the techniques of group analysis discussed in chapter 2. Our goal is to solve the initial value problem (IVP) presented at the end of chapter 3.

The chapter is structured as follows: Section 4.2 is a statement of the problem we intend solving. This gives a proper perspective to our objective. Section 4.3 will address the problem using the reduction-of-order technique via Lie group analysis. This will be done via the symmetries admitted by the system of equations. Section 4.4 is dedicated to obtaining the group invariant solutions to the IVP. Section 4.5 is a synopsis of results obtained and brief remarks thereon.

4.2 Statement of the problem

We wish to find solutions to the IVP from the previous chapter given by

$$\ddot{A} + \frac{\dot{A}}{2}\left[\frac{3\dot{F}}{F} - \frac{\dot{A}}{A}\right] + \frac{4\epsilon\Lambda}{3}A = -2\epsilon\frac{A}{F}\alpha_1, \tag{4.1}$$

$$\ddot{F}+\frac{\dot{F}}{2}\left[\frac{\dot{A}}{A}+\frac{\dot{F}}{F}\right]+\frac{4\epsilon\Lambda}{3}F=-2\epsilon\alpha_2, \tag{4.2}$$

subject to

(i) $\alpha_1=\alpha_2: A(0)=F(0)=1, \dot{A}(0)=\dot{F}(0)=\pm\sqrt{\frac{-2\epsilon\Lambda-\epsilon R}{3}}$.

(ii) $\alpha_1\neq\alpha_2$ and $\alpha_1=0$: $A(0)=F(0)=1, \dot{A}(0)=-2a, \dot{F}(0)=a\pm(a^2-\frac{4\epsilon\Lambda}{3}-\frac{2\epsilon R}{3})^{\frac{1}{2}}$; where a is an unspecified constant.

4.3 Symmetry analysis

Using the SYM [25] package and Program LIE [40], we find that equations (4.1) and (4.2) admit symmetries:

$$G_1=\frac{\partial}{\partial y}, \tag{4.3}$$

$$G_2=A\frac{\partial}{\partial A}. \tag{4.4}$$

By imposing restrictions on the system, we can obtain the following additional symmetries:

i) $\Lambda=0$:

$$\hat{G}_3=2F\frac{\partial}{\partial F}+y\frac{\partial}{\partial y}. \tag{4.5}$$

ii) $\Lambda\neq 0$ and $\alpha_1=\alpha_2=0$:

$$\tilde{G}_3=F\frac{\partial}{\partial F}, \tag{4.6}$$

$$\tilde{G}_4=(3A\log(A)-3A\log(F))\frac{\partial}{\partial A}+(-F\log(A)+F\log(F))\frac{\partial}{\partial F}. \tag{4.7}$$

Since symmetries G_1 and G_2 commute, we shall now proceed to reduce equations (4.1) and (4.2) by each of them separately. We shall begin with G_2, and then see whether reduction by G_1 will make a difference to the results already obtained in the former case.

4.3.1 Reduction via G_2

The characteristics for a DE invariant under $G_2^{[1]}$ come from the solution of

$$\frac{\mathrm{d}y}{0} = \frac{\mathrm{d}A}{A} = \frac{\mathrm{d}\dot{A}}{\dot{A}} = \frac{\mathrm{d}F}{0} = \frac{\mathrm{d}\dot{F}}{0}.$$

The new variable (in this case the 1^{st} order differential invariant) for the reduction of the order of equations (4.1)–(4.2) will then be

$$B = \frac{\dot{A}}{A}. \tag{4.8}$$

Equation (4.1) now reduces to

$$\dot{B} + \frac{\dot{B}}{2}\left[\frac{3\dot{F}}{F} + B\right] + \frac{4\epsilon\Lambda}{3} = \frac{-2\epsilon\alpha_1}{F}. \tag{4.9}$$

The knowledge of a group of symmetries of a system of differential equations has much the same consequences as knowledge of a similar group of symmetries of a higher order equation [80]. With this in mind, we shall now solve for B in (4.2), differentiate once, and substitute into (4.9). The resultant equation is

$$\begin{aligned}-\frac{14\epsilon\Lambda}{3} + \frac{2\alpha_1\epsilon}{F} - \frac{2\alpha_2\epsilon}{F} + \frac{8\alpha_2^2\epsilon^2}{\dot{F}^2} + \frac{16\alpha_2\epsilon^2\Lambda F}{\dot{F}^2} + \frac{8\epsilon^2\Lambda^2F^2}{\dot{F}^2} + \frac{12\alpha_2\epsilon\ddot{F}}{\dot{F}^2}&\\ + \frac{12\epsilon\Lambda F\ddot{F}}{\dot{F}^2} + \frac{4\ddot{F}^2}{\dot{F}^2} + \frac{2\dddot{F}}{\dot{F}} = 0,&\end{aligned} \tag{4.10}$$

a 3^{rd} order equation in F. This admits the sole symmetry

$$G_1 = \frac{\partial}{\partial y}.$$

We also find additional symmetries as follows:

i) $\Lambda = 0$:

$$\hat{G}_3 = 2F\frac{\partial}{\partial F} + y\frac{\partial}{\partial y},$$

ii) $\Lambda = 0$ and $\alpha_1 = \alpha_2$:

$$\begin{aligned}\hat{G}_3 &= 2F\frac{\partial}{\partial F} + y\frac{\partial}{\partial y},\\ \hat{G}_4 &= \frac{\partial}{\partial F},\end{aligned}$$

iii) $\Lambda \neq 0$ and $\alpha_1 = \alpha_2$:

$$\tilde{G}_3 = \left[F + \frac{\alpha_2}{\Lambda}\right] \frac{\partial}{\partial F}.$$

We shall now proceed to reduce equation (4.10) via G_1.

The differential invariants for this symmetry are

$$u = F \qquad\qquad V(u) = \dot{F}. \tag{4.11}$$

Substitution of these new variables in (4.10) results in the 2^{nd} order equation

$$-\frac{14\epsilon\Lambda}{3} + \frac{2\alpha_1\epsilon}{u} - \frac{2\alpha_2\epsilon}{u} + \frac{8\alpha_2^2\epsilon^2}{V^2} + \frac{16\alpha_2\epsilon^2\Lambda u}{V^2} + \frac{8\epsilon^2\Lambda^2u^2}{V^2} + \frac{12\alpha_2\epsilon\dot{V}}{V}$$
$$+\frac{12\epsilon\Lambda u\dot{V}}{V} + 6\dot{V}^2 + 2V\ddot{V} = 0. \tag{4.12}$$

This equation does not admit any symmetries, but we can obtain symmetries when we impose restrictions on it:

i) $\Lambda = 0$ and $\alpha_1 = \alpha_2$:

$$\begin{aligned}
\hat{G}_3 &= 2u\frac{\partial}{\partial u} + V\frac{\partial}{\partial V}, \\
\hat{G}_4 &= \frac{\partial}{\partial u}, \\
\hat{G}_5 &= \left[16u^2 - \frac{V^4}{\alpha_2^2\epsilon^2}\right]\frac{\partial}{\partial u} + \left[16uV + \frac{4V^3}{\alpha_2\epsilon}\right]\frac{\partial}{\partial V}.
\end{aligned}$$

Observe that $\hat{G}_5$ is an example of a hidden symmetry discussed in chapter 2.

On computation of the Lie bracket of these symmetries, we have that

$$[\hat{G}_3, \hat{G}_4] = 2\hat{G}_3, \tag{4.13}$$

$$[\hat{G}_4, \hat{G}_5] = 16\hat{G}_3 \tag{4.14}$$

and

$$[\hat{G}_3, \hat{G}_5] = 2\hat{G}_5. \tag{4.15}$$

Based on this analysis, reduction of equation (4.12) will be by $\hat{G}_4$ and $\hat{G}_5$, though under the restrictions of $\Lambda = 0$ and $\alpha_1 = \alpha_2$.

Reduction via $\hat{G}_4$ has new variables

$$m = V \qquad\qquad N(m) = \dot{V}, \tag{4.16}$$

and this results in a Riccati equation

$$\frac{8\alpha_2^2\epsilon^2}{m^2} + \frac{12\alpha_2\epsilon N}{m} + 6N^2 + 2m\dot{N} = 0. \tag{4.17}$$

The Riccati equation is solved to obtain

$$N(m) = \left[\left(1 - \frac{\Psi_1 U\left(\frac{3}{2} + \frac{i\sqrt{3}}{2}, 2, \frac{2i\sqrt{3}\alpha_2\epsilon}{m}\right) -_1 F_1\left(\frac{3}{2} + \frac{i\sqrt{3}}{2}, 2, \frac{2i\sqrt{3}\alpha_2\epsilon}{m}\right)}{{}_1F_1\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}, 1, \frac{2i\sqrt{3}\alpha_2\epsilon}{m}\right) + \Psi_1 U\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}, 1, \frac{2i\sqrt{3}\alpha_2\epsilon}{m}\right)}\right)\right.$$
$$\left.\left(\frac{-3\alpha_2\epsilon + i\sqrt{3}\alpha_2\epsilon}{3m}\right)\right], \tag{4.18}$$

where Ψ_1 is an arbitrary constant. The function ${}_1F_1$ is the Kummer (confluent hypergeometric) function of the 1^{st} kind and the function U is the Kummer (confluent hypergeometric) function of the 2^{nd} kind [64].

Obtaining the general solution to the system of equations (4.1) and (4.2) requires that we invert the solution in (4.18) through the various transformations we have previously made [80, 93]. Unfortunately this is not possible due to the divergent nature of these particular Kummer series. Furthermore, an attempt to carry out a phase plane analysis of (4.18) (i.e. trying to observe whether it converges under special conditions or values of the variables) was also unsuccessful. No other progress in this case was possible. See later for group invariant solutions though.

ii) $\Lambda = 0$ and $\alpha_2 = 0$:

$$\hat{G}_3 = 2u\frac{\partial}{\partial u} + V\frac{\partial}{\partial V}$$

iii) $\Lambda \neq 0$ and $\alpha_1 = \alpha_2$:

$$\tilde{G}_3 = \left[u + \frac{\alpha_2}{\Lambda}\right]\frac{\partial}{\partial u} + V\frac{\partial}{\partial V}$$

Reduction via $\hat{G}_4$: $\Lambda = 0$ and $\alpha_1 = \alpha_2$

Based on the Lie bracket analysis, we can also reduce equation (4.10) by $\hat{G}_4$, though under the restrictions of $\Lambda = 0$ and $\alpha_1 = \alpha_2$.

The characteristics for the reduced equation will be obtained from

$$\frac{\mathrm{d}y}{0} = \frac{\mathrm{d}F}{1} = \frac{\mathrm{d}\dot{F}}{0}.$$

This has new variables

$$u = y \qquad\qquad V(u) = \dot{F}. \tag{4.19}$$

Substitution of these new variables into (4.10) results in the 2^{nd} order equation,

$$\frac{4\alpha_2^2\epsilon^2}{V} + \frac{6\alpha_2\epsilon\dot{V}}{V} + 2\dot{V}^2 + \ddot{V} = 0. \tag{4.20}$$

Equation (4.20) admits symmetries:

$$\hat{G}_3 = 2V\frac{\partial}{\partial V} + u\frac{\partial}{\partial u},$$
$$\hat{G}_6 = \frac{\partial}{\partial u}.$$

Further reduction under these restrictions is via $\hat{G}_6$ (from the Lie bracket analysis), with new variables

$$m = V \qquad\qquad N(m) = \dot{V}. \tag{4.21}$$

This yields a Riccati equation

$$\dot{N} + \frac{6\alpha_2\epsilon N}{m} + 2N^2 + \frac{4\alpha_2^2\epsilon^2}{m} = 0, \tag{4.22}$$

which is solved to obtain

$$N(m) = \frac{\sqrt{2}\alpha_2\epsilon\left(-J_{6\alpha_2\epsilon}[4\alpha_2\epsilon\sqrt{2m}]\,\Psi_2\ \Gamma[6\alpha_2\epsilon] + J_{-6\alpha_2\epsilon}[4\alpha_2\epsilon\sqrt{2m}]\,\Gamma[2-6\alpha_2\epsilon]\right)}{\sqrt{m}\left(J_{-1+6\alpha_2\epsilon}[4\alpha_2\epsilon\sqrt{2m}]\,\Psi_2\,\Gamma[6\alpha_2\epsilon] + J_{1-6\alpha_2\epsilon}[4\alpha_2\epsilon\sqrt{2m}]\,\Gamma[2-6\alpha_2\epsilon]\right)}, \tag{4.23}$$

where Ψ_2 is an arbitrary constant. The function J is the Bessel function of the 1^{st} kind and Γ is the gamma function. Inversion of this result is again not possible due to the divergence of these Bessel series.

It is essential to point out that the Lie bracket analysis asserts that reduction of equation (4.10) can only be via G_1 and $\hat{G}_4$. This will have a consequence of making the other symmetries point-symmetries of the reduced equation. Nonetheless, we attempted to reduce equation (4.10) via $\hat{G}_3$ and $\tilde{G}_3$ to confirm this assertion. As expected, the resultant DEs were not solvable.

4.3.2 Reduction via G_1

We shall now consider the case of the symmetry G_1, which was admitted by the system of equations (4.1) and (4.2). The characteristics of the new DE are obtained from

$$\frac{\mathrm{d}y}{1} = \frac{\mathrm{d}A}{0} = \frac{\mathrm{d}\dot{A}}{0} = \frac{\mathrm{d}F}{0} = \frac{\mathrm{d}\dot{F}}{0}.$$

From the system above, the new variables for the reduction of order will be

$$p = F \qquad Q(p) = \dot{F} \qquad R(y) = A. \tag{4.24}$$

We will now have that

$$Q' = \frac{\mathrm{d}Q}{\mathrm{d}p} = \frac{\mathrm{d}Q}{\mathrm{d}y}.\frac{\mathrm{d}y}{\mathrm{d}p} = \frac{\ddot{F}}{\dot{F}} \qquad \Longrightarrow \ddot{F} = QQ', \tag{4.25}$$

$$\dot{A} = \frac{\mathrm{d}R}{\mathrm{d}y} = \frac{\mathrm{d}R}{\mathrm{d}F}.\frac{\mathrm{d}F}{\mathrm{d}y} = \frac{\mathrm{d}R}{\mathrm{d}p}.\frac{\mathrm{d}F}{\mathrm{d}y} = R'\dot{F} \quad \Longrightarrow \dot{A} = R'Q, \tag{4.26}$$

and

$$\ddot{A} = \frac{\mathrm{d}^2R}{\mathrm{d}y^2} = R''Q^2 + R'QQ'. \tag{4.27}$$

Upon substitution into equations (4.1) and (4.2), we have that

$$(R''Q^2 + R'QQ') + \frac{R'Q}{2}\left[\frac{3Q}{p} - \frac{R'Q}{R}\right] + \frac{4\epsilon\Lambda}{3}R = -2\epsilon\alpha_1\frac{R}{p}, \tag{4.28}$$

$$QQ' + \frac{Q}{2}\left[\frac{R'Q}{R} + \frac{Q}{p}\right] + \frac{4\epsilon\Lambda}{3}p = -2\epsilon\alpha_2. \tag{4.29}$$

Now solving for R in equation (4.29), and substituting into equation (4.28), we obtain

$$\frac{4\epsilon^2}{Q}\left[9\alpha_2^2 + 12\alpha_2\Lambda p + 4\Lambda^2p^2\right] + 9Q'\left[6\alpha_2 + 4\epsilon\Lambda p + 9QQ'\right]$$
$$+\frac{9\epsilon Q}{p}[\alpha_1 - \alpha_2] - 9Q^2\left[Q'' + \frac{Q'}{p}\right] - 12\epsilon\Lambda Q = 0. \tag{4.30}$$

This 2^{nd} order nonlinear equation admits no symmetries except under the following restrictions:

i) $\Lambda = 0$:

$$\bar{G}_1 = 2p\frac{\partial}{\partial p} + Q\frac{\partial}{\partial Q}.$$

ii)$\Lambda \neq 0$ and $\alpha_1 = \alpha_2 = 0$:

$$\bar{\bar{G}}_1 = p\frac{\partial}{\partial p} + Q\frac{\partial}{\partial Q}.$$

Equation (4.30) can not be reduced to quadratures due to the inadequate number of symmetries it admits. Unfortunately no hidden symmetries were found in any of the reductions.

We shall now consider subcases of symmetries admitted by the system of equations (4.1) and (4.2) to see whether this will yield solutions.

4.3.3 Reduction via $\tilde{G}_3$

Considering equations (4.1) and (4.2) again, observe that they admit the symmetry $\tilde{G}_3$, though under the restrictions of $\alpha_1 = \alpha_2 = 0$. We will attempt to find a general solution, under this special case, to these equations.

The new invariant

$$B = \frac{\dot{F}}{F} \tag{4.31}$$

obtained from the associated Lagrange system will now be used to reduce the system of equations.

Reducing equation (4.2) with this invariant and subsequently substituting equation (4.1) into it results in

$$-\frac{8\epsilon\Lambda}{9} + \frac{32\epsilon^2\Lambda^2A^2}{27\dot{A}^2} + \frac{8\epsilon\Lambda A\ddot{A}}{3\dot{A}^2} - \frac{2\ddot{A}}{3A} + \frac{4\ddot{A}^2}{3\dot{A}^2} - \frac{2\dddot{A}}{3\dot{A}} = 0. \tag{4.32}$$

This 3^{rd} order equation in A admits

$$G_1 = \frac{\partial}{\partial y},$$
$$G_2 = A\frac{\partial}{\partial A}.$$

Under the restrictions of $\Lambda = 0$, it admits, in addition,

$$\overrightarrow{G}_3 = y\frac{\partial}{\partial y},$$
$$\overrightarrow{G}_4 = A\log(A)\frac{\partial}{\partial A}.$$

We will now proceed to reduce equation (4.32) via G_1.

Reduction via G_1 with new variables

$$u = A \qquad\qquad V(u) = \dot{A}, \tag{4.33}$$

results in the 2^{nd} order equation

$$-\frac{4\epsilon\Lambda}{3} + \frac{16\epsilon^2\Lambda^2u^2}{9V^2} + \frac{4\epsilon\Lambda u\dot{V}}{V} - \frac{V\dot{V}}{u} + \dot{V}^2 - V\ddot{V} = 0. \tag{4.34}$$

This has the symmetry

$$G_2 = u\frac{\partial}{\partial u} + V\frac{\partial}{\partial V}.$$

When $\Lambda = 0$, it also yields additional symmetries:

$$\begin{aligned}
\overrightarrow{G}_5 &= [V\log(V)]\frac{\partial}{\partial V},\\
\overrightarrow{G}_6 &= [V\log(u)]\frac{\partial}{\partial u},\\
\overrightarrow{G}_7 &= [u\log(V)]\frac{\partial}{\partial u},\\
\overrightarrow{G}_9 &= [u\log(u)]\frac{\partial}{\partial u},\\
\overrightarrow{G}_{10} &= [u\log(u)\log(V)]\frac{\partial}{\partial u} + \left[V\log(V)^2\right]\frac{\partial}{\partial V},\\
\overrightarrow{G}_{11} &= \left[u\log(u)^2\right]\frac{\partial}{\partial u} + [V\log(u)\log(V)]\frac{\partial}{\partial V},\\
\overrightarrow{G}_{12} &= u\frac{\partial}{\partial u}.
\end{aligned}$$

Further reduction of equation (4.34) via G_2 with new variables

$$m = \frac{V}{u} \qquad\qquad N(m) = \dot{V}, \tag{4.35}$$

results in a Riccati equation

$$-\frac{4\epsilon\Lambda}{3} + \frac{16\epsilon^2\Lambda^2}{9m^2} + \frac{4\epsilon\Lambda N}{m} + N(N-m) - m\dot{N}(N-m) = 0, \tag{4.36}$$

with solution

$$N(m) = \frac{4\epsilon\Lambda}{3m} + \frac{m(12+\Psi_3) \pm 4\sqrt{-24\epsilon\Lambda + \Psi_3\left(2\epsilon\Lambda + 3m^2\right)}}{-12+\Psi_3}, \tag{4.37}$$

where Ψ_3 is an arbitrary constant.

We now attempt to invert the solution in (4.37) through the various transformations. From (4.35), we have that

$$\dot{V}(u) = \frac{4u\epsilon\Lambda}{3V} + \frac{(\frac{V}{u})(12+\Psi_3) \pm 4\sqrt{-24\epsilon\Lambda + \Psi_3\left(2\epsilon\Lambda + 3(\frac{V}{u})^2\right)}}{-12+\Psi_3}. \tag{4.38}$$

Solving this for $V(u)$, and substituting for (4.33), we obtain

$$\frac{e^{-6\Psi_4(-12+\Psi_3)}[-3A(-12+\Psi_3)]^4\left[1+\frac{6\dot{A}}{A\sqrt{2\epsilon\Lambda(-12+\Psi_3)+\frac{3\Psi_3\dot{A}^2}{A^2}}}\right]\left[2\epsilon\Lambda+\frac{3\dot{A}^2}{A^2}\right]}{\left[1-\frac{6\dot{A}}{A\sqrt{2\epsilon\Lambda(-12+\Psi_3)+\frac{3\Psi_3\dot{A}^2}{A^2}}}\right]\left[\frac{6\Psi_3\dot{A}}{A}+2\sqrt{3\Psi_3}\sqrt{2\epsilon\Lambda(-12+\Psi_3)+\frac{3\Psi_3\dot{A}^2}{A^2}}\right]^{\pm 4\sqrt{\frac{3}{\Psi_3}}}} = 1, \tag{4.39}$$

where Ψ_3, Ψ_4 are arbitrary constants. Unfortunately, further inversion is not possible, nor is the satisfaction of the initial conditions.

4.3.4 General symmetry reduction

The previous attempts only used individual symmetries. We will now take a general linear combination of the two general symmetries,

$$\vartheta\frac{\partial}{\partial y} + \varpi A\frac{\partial}{\partial A},$$

where ϑ and ϖ are constants.

However, noting that we still have to take the boundary conditions into account, we check if there are any conditions on this linear combination before applying the reductions to the equations. This leads to the condition

$$\vartheta + \varpi \frac{A}{\dot{A}} = 0. \tag{4.40}$$

Instead of taking this to be a restriction on ϑ and ϖ (which would require $\vartheta = \varpi = 0$), we take it to constrain $A(y)$ as:

$$A(y) = \exp\left(-\frac{\varpi}{\vartheta} y\right) \Psi_5, \tag{4.41}$$

where Ψ_5 is an arbitrary constant of integration.

Now, solving the system of equations (4.1)–(4.2) for $A(y)$ and $F(y)$, we obtain

$$\begin{cases} A(y) = \exp\left(-2ay\right), \\ \\ F(y) = 1, \end{cases} \tag{4.42}$$

where $a = \frac{\varpi}{2\vartheta} = \sqrt{\epsilon\alpha_2}$. These satisfy the initial conditions under restrictions

$$\Lambda \neq 0, \qquad \alpha_1 = 0, \qquad \Lambda = -\frac{3}{2}\alpha_2.$$

4.4 Group invariant solutions

As a final approach, we will now try to solve the system (4.1)–(4.2) by seeking group invariant solutions.

4.4.1 General symmetry for (4.1)–(4.2)

We take

$$X_1 = \zeta \frac{\partial}{\partial y} + A \frac{\partial}{\partial A}, \tag{4.43}$$

where ζ is a constant, as our general symmetry without loss of generality. We shall now attempt to obtain an invariant solution via this symmetry.

Following the formalism by Bluman [11], we have that

$$\psi = \frac{\eta}{\xi} = \frac{A}{\zeta},$$

and

$$Y = \frac{\partial}{\partial y} + \psi \frac{\partial}{\partial A} = \frac{\partial}{\partial y} + \left(\frac{A}{\zeta}\right) \frac{\partial}{\partial A}. \tag{4.44}$$

Since

$$y_k = y^{(k)} = Y^{k-1}\psi, \qquad\qquad k = 1, 2, ..., n, \tag{4.45}$$

we will have that

$$\begin{aligned}
\dot{A} &= \psi = \frac{\eta}{\xi} = \frac{A}{\zeta}, \\
\ddot{A} &= Y\psi = \left[\frac{\partial}{\partial y} + \frac{A}{\zeta}\frac{\partial}{\partial A}\right]\left[\frac{A}{\zeta}\right] = \frac{\dot{A}}{\zeta} + \frac{A}{\zeta^2}.
\end{aligned}$$

Substituting these into equation (4.1) and solving the resultant DE for F, we obtain

$$F = \exp\left(\frac{-(3 + 4\epsilon\zeta^2\Lambda)y}{9\zeta}\right)\Psi_6 - \frac{6\alpha_1\epsilon\zeta^2}{3 + 4\epsilon\zeta^2\Lambda}, \tag{4.46}$$

where Ψ_6 is a constant.

Invoking the initial conditions, we find that

$$F = \left[1 + \frac{6\alpha_1\epsilon\zeta^2}{3 + 4\epsilon\zeta^2\Lambda}\right] \exp\left(\frac{-(3 + 4\epsilon\zeta^2\Lambda)y}{9\zeta}\right) - \frac{6\alpha_1\epsilon\zeta^2}{3 + 4\epsilon\zeta^2\Lambda}. \tag{4.47}$$

where, when $\alpha_1 = \alpha_2$:

$$\Lambda = \frac{3}{16\zeta^2}\left[\sqrt{3\left(51 - 112\alpha_1\epsilon\zeta^2\right)} + \epsilon\left(13 - 8\alpha_1\epsilon\zeta^2\right)\right],$$

and for $\alpha_1 \neq \alpha_2, \quad \alpha_1 = 0$:

$$\Lambda = \frac{-3}{8\zeta^2}\left[3\sqrt{13 + 4\zeta\left(3a + a^2\zeta - 2\alpha_2\epsilon\zeta\right)} + \epsilon\left(11 + 6a\zeta^2\right)\right].$$

Unfortunately, the resulting equation in A could not be solved.

4.4.2 Solution for (4.10) via $\hat{G}_3$: $\Lambda = 0$

By Theorem 2.3.1, we have that

$$\psi = \frac{\eta}{\xi} = \frac{2F}{y},$$

and

$$Y = \frac{\partial}{\partial y} + \psi \frac{\partial}{\partial F} = \frac{\partial}{\partial y} + \left(\frac{2F}{y}\right)\frac{\partial}{\partial F}. \tag{4.48}$$

Since

$$y_k = y^{(k)} = Y^{k-1}\psi, \qquad k = 1, 2, ..., n, \tag{4.49}$$

we will have that

$$\begin{aligned}
\dot{F} &= \psi = \frac{\eta}{\xi} = \frac{2F}{y},\\
\ddot{F} &= Y\psi = \left[\frac{\partial}{\partial y} + \frac{2F}{y}\frac{\partial}{\partial F}\right]\left[\frac{2F}{y}\right] = 0,\\
\dddot{F} &= Y^2\psi = \left[\frac{\partial}{\partial y} + \frac{2F}{y}\frac{\partial}{\partial F}\right]^2\left[\frac{2F}{y}\right] = 0.
\end{aligned}$$

Substituting these into equation (4.10) (with $\Lambda = 0$), we obtain the invariant solution

$$F = \frac{\alpha_2^2 \epsilon y^2}{\alpha_2 - \alpha_1}. \tag{4.50}$$

However, it does not satisfy the initial condition $F(0) = 1$, and is discarded.

4.4.3 Solution for (4.10) via $G_1 + \hat{G}_3$: $\Lambda \neq 0$, $\alpha_1 = \alpha_2$

Since

$$\psi = \frac{\eta}{\xi} = F + \frac{\alpha_2}{\Lambda},$$

and

$$Y = \frac{\partial}{\partial y} + \left[F + \frac{\alpha_2}{\Lambda}\right]\frac{\partial}{\partial F};$$

we will now have that

$$\begin{aligned}
\dot{F} &= \psi = \frac{\eta}{\xi} = F + \frac{\alpha_2}{\Lambda},\\
\ddot{F} &= Y\psi = \left[\frac{\partial}{\partial y} + \left(F + \frac{\alpha_2}{\Lambda}\right)\frac{\partial}{\partial F}\right]\left[F + \frac{\alpha_2}{\Lambda}\right] = F + \frac{\alpha_2}{\Lambda},\\
\dddot{F} &= Y^2\psi = \left[\frac{\partial}{\partial y} + \left(F + \frac{\alpha_2}{\Lambda}\right)\frac{\partial}{\partial F}\right]^2\left[F + \frac{\alpha_2}{\Lambda}\right] = F + \frac{\alpha_2}{\Lambda}.
\end{aligned}$$

Substituting these into equation (4.10), gives the solution

$$F(y) = \frac{-\alpha_2}{\Lambda} \pm 2\epsilon\sqrt{\frac{3\alpha_2(\alpha_2 - 1)}{9 + \epsilon\Lambda(11 + 12\epsilon\Lambda)}}, \tag{4.51}$$

which is a constant.

$F(y)$ will only satisfy the system of equations (4.1) and (4.2), with the initial conditions if we set $F(y) = 1$. In fact, if we start with $F(y) = 1$ and do not assume $\alpha_1 = \alpha_2$, we can also obtain a solution. Substitution of this into equation (4.1), and solving the 2^{nd} order DE gives the general result

$$A(y) = \left[\left(\frac{1}{2} - \frac{a}{2\sqrt{\frac{-2\epsilon\Lambda}{3}}}\right)\exp\left(2\sqrt{\frac{-\epsilon\Lambda}{6}}y\right) + \left(\frac{1}{2} + \frac{a}{2\sqrt{\frac{-2\epsilon\Lambda}{3}}}\right)\exp\left(-2\sqrt{\frac{-\epsilon\Lambda}{6}}y\right)\right]^2 \tag{4.52}$$

under the constraints that $\alpha_1 = 0$ and $\Lambda = -\frac{3}{2}\alpha_2$.

In conclusion, we have now obtained solutions

$$\begin{cases} A(y) = \left[\left(\frac{1}{2} - \frac{a}{2\sqrt{\frac{-2\epsilon\Lambda}{3}}}\right)\exp\left(2\sqrt{\frac{-\epsilon\Lambda}{6}}y\right) + \left(\frac{1}{2} + \frac{a}{2\sqrt{\frac{-2\epsilon\Lambda}{3}}}\right)\exp\left(-2\sqrt{\frac{-\epsilon\Lambda}{6}}y\right)\right]^2, \\ \\ F(y) = 1, \end{cases} \tag{4.53}$$

which satisfies the initial conditions under the restrictions that $\Lambda = -\frac{3}{2}\alpha_2$ and $\alpha_1 = 0$. Observe that the solution obtained in equation (4.42) is actually a subset of equation (4.53) at the point $a = \sqrt{\epsilon\alpha_2}$.

4.4.4 Solution for (4.12) via $\hat{G}_5$: $\Lambda = 0,\ \alpha_1 = \alpha_2$

The only other symmetry which we shall investigate is

$$\hat{G}_5 = \left[16u^2 - \frac{V^4}{\alpha_2^2\epsilon^2}\right]\frac{\partial}{\partial u} + \left[16uV + \frac{4V^3}{\alpha_2\epsilon}\right]\frac{\partial}{\partial V}. \tag{4.54}$$

The reason for this is because the remaining symmetries are either the same (sub-) cases or prolongations of the symmetries already investigated above. This symmetry is special, in that it is a hidden symmetry that emerged during reduction.

Using the invariant solutions approach, we now have that

$$\psi = \frac{\eta}{\xi} = \frac{4V\alpha_2\epsilon}{4u\alpha_2\epsilon + V^2},$$

and

$$Y = \frac{\partial}{\partial u} + \left(\frac{4V\alpha_2\epsilon}{4u\alpha_2\epsilon + V^2}\right)\frac{\partial}{\partial V}.$$

Then,

$$\begin{aligned}
\dot{V} &= \psi = \frac{4V\alpha_2\epsilon}{4u\alpha_2\epsilon + V^2} \\
\ddot{V} &= Y\psi = \left[\frac{\partial}{\partial u} + \left(\frac{4V\alpha_2\epsilon}{4u\alpha_2\epsilon + V^2}\right)\frac{\partial}{\partial V}\right]\left[\frac{4V\alpha_2\epsilon}{4u\alpha_2\epsilon + V^2}\right] = \frac{-32\alpha_2^2\epsilon^2V^3}{(4u\alpha_2\epsilon + V^2)^3}.
\end{aligned}$$

Substituting these into equation (4.12), we will now obtain invariant solutions

$$V = \pm 2.89i\sqrt{\alpha_2\epsilon u}, \tag{4.55}$$

$$V = \pm 0.6i\sqrt{(2.05 + 1.05i)\alpha_2\epsilon u}, \tag{4.56}$$

$$V = \pm 0.6i\sqrt{(2.05 - 1.05i)\alpha_2\epsilon u}. \tag{4.57}$$

Recall that $V(u) = \dot{F}(y)$ and $u = F(y)$, so equations (4.55)–(4.57) become

$$\dot{F}(y) = \pm 2.89i\sqrt{\alpha_2\epsilon F(y)}, \tag{4.58}$$

$$\dot{F}(y) = \pm 0.6i\sqrt{(2.05 + 1.05i)\alpha_2\epsilon F(y)}, \tag{4.59}$$

$$\dot{F}(y) = \pm 0.6i\sqrt{(2.05 - 1.05i)\alpha_2\epsilon F(y)}. \tag{4.60}$$

Solving equation (4.58) for $F(y)$, we will obtain

$$F(y) = -2.08\alpha_2\epsilon y^2 + 1.44i\sqrt{\alpha_2\epsilon}\,y\Psi_7 + 0.25\Psi_7, \tag{4.61}$$

where Ψ_7 is an arbitrary constant. The initial conditions reduce this solution to

$$F(y) = 1, \tag{4.62}$$

which was already considered in section 4.4.3. This also occurs for equations (4.59) and (4.60).

4.5 Remarks and synopsis

In summary, we have been able to obtain a solution through group analysis of the system of equations (4.1)–(4.2). We obtained this solution via both reduction of order and the group invariant approach. This solution, to the best of our knowledge, is new.

Chapter 5

Conclusion

5.1 Summary

In this work, we have successfully employed both the tools of differential geometry and differential equations. These were used in the understanding of the embedding theories and resolution of the DEs that come up in the process of embedding.

The 1^{st} chapter of this work discussed the historical background of DEs and our proposed method (the Lie group analysis approach) of solving DEs - in particular the nonlinear ones. We also discussed the emergence of general relativity and the inconsistencies with other fundamental theories. The extension of GR to higher dimensions was motivated in terms of its potentials to resolve these difficulties.

Chapter 2 gave a detailed outline of the techniques we employed in the resolution of the IVP. The concepts behind the Lie group approach and Lie algebras was followed by a section on obtaining invariant solutions to DEs. The phenomena of hidden symmetries rounded up the chapter. After each concept, we illustrated these techniques with examples.

In chapter 3, we gave the background to the IVP (which was the motivation for this work). We employed the tools of differential geometry in the construction of this background. From the basic concepts in differential geometry, to the theories underlying embeddings; we charted a path leading to the system of highly nonlinear DEs which we analysed in chapter 4.

Chapter 4 was a group analysis of the IVP (4.1)–(4.2). We employed the tools of reduction-of-order and group invariance and succeeded in obtaining the following solutions:

i) Using reduction of order techniques, we obtained the solution:

$$N(m) = \left[\left(1 - \frac{\Psi_1 U\left(\frac{3}{2} + \frac{i\sqrt{3}}{2}, 2, \frac{2i\sqrt{3}\alpha_2\epsilon}{m}\right) -_1 F_1\left(\frac{3}{2} + \frac{i\sqrt{3}}{2}, 2, \frac{2i\sqrt{3}\alpha_2\epsilon}{m}\right)}{{}_1F_1\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}, 1, \frac{2i\sqrt{3}\alpha_2\epsilon}{m}\right) + \Psi_1 U\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}, 1, \frac{2i\sqrt{3}\alpha_2\epsilon}{m}\right)} \right) \left(\frac{-3\alpha_2\epsilon + i\sqrt{3}\alpha_2\epsilon}{3m} \right) \right], \tag{5.1}$$

where Ψ_1 is an arbitrary constant. We were unable to invert this solution due to the divergence of these particular Kummer series.

In addition, we obtained

$$N(m) = \frac{\sqrt{2}\alpha_2\epsilon\left(-J_{6\alpha_2\epsilon}[4\alpha_2\epsilon\sqrt{2m}]\,\Psi_2\;\Gamma[6\alpha_2\epsilon] + J_{-6\alpha_2\epsilon}[4\alpha_2\epsilon\sqrt{2m}]\,\Gamma[2-6\alpha_2\epsilon]\right)}{\sqrt{m}\left(J_{-1+6\alpha_2\epsilon}[4\alpha_2\epsilon\sqrt{2m}]\,\Psi_2\,\Gamma[6\alpha_2\epsilon] + J_{1-6\alpha_2\epsilon}[4\alpha_2\epsilon\sqrt{2m}]\,\Gamma[2-6\alpha_2\epsilon]\right)}, \tag{5.2}$$

where Ψ_2 is an arbitrary constant. Inversion of this result was again not possible due to the divergence of these Bessel series.

We also obtained an implicit solution in A,

$$\frac{e^{-6\Psi_4(-12+\Psi_3)}[-3A(-12+\Psi_3)]^4\left[1 + \frac{6\dot{A}}{A\sqrt{2\epsilon\Lambda(-12+\Psi_3)+\frac{3\Psi_3\dot{A}^2}{A^2}}}\right]\left[2\epsilon\Lambda + \frac{3\dot{A}^2}{A^2}\right]}{\left[1 - \frac{6\dot{A}}{A\sqrt{2\epsilon\Lambda(-12+\Psi_3)+\frac{3\Psi_3\dot{A}^2}{A^2}}}\right]\left[\frac{6\Psi_3\dot{A}}{A} + 2\sqrt{3\Psi_3}\sqrt{2\epsilon\Lambda(-12+\Psi_3)+\frac{3\Psi_3\dot{A}^2}{A^2}}\right]^{\pm 4\sqrt{\frac{3}{\Psi_3}}}} = 1. \tag{5.3}$$

This solution did not satisfy the initial conditions.

Lastly, using this technique, we obtained the solution

$$\begin{cases} A(y) = \exp(-2ay), \\ \\ F(y) = 1. \end{cases} \tag{5.4}$$

This solution successfully satisfied the IVP under the restrictions that

$$a = \sqrt{\epsilon\alpha_2}, \quad \alpha_2 = -\frac{3}{2}\Lambda, \quad \alpha_1 = 0, \quad \Lambda \neq 0.$$

Note that this solution is a subset of the solution in equation (4.53).

ii) Using group invariant techniques, we were able to obtain the solution

$$\begin{cases} A(y) = \left[\left(\frac{1}{2} - \frac{a}{2\sqrt{\frac{-2\epsilon\Lambda}{3}}}\right)\exp\left(2\sqrt{\frac{-\epsilon\Lambda}{6}}y\right) + \left(\frac{1}{2} + \frac{a}{2\sqrt{\frac{-2\epsilon\Lambda}{3}}}\right)\exp\left(-2\sqrt{\frac{-\epsilon\Lambda}{6}}y\right)\right]^2 \\ \\ F(y) = 1, \end{cases} \tag{5.5}$$

under the constraints that $\alpha_1 = 0$ and $\Lambda = -\frac{3}{2}\alpha_2$. This solution satisfied the initial conditions of the system and is accepted as a solution to the IVP.

To the best of our knowledge, solution (5.5) is a new solution to this system of embedding equations.

5.2 Significance of the result

The new result obtained in this work will further aid the understanding of embedding theories. It is of particular significance in the following ways:

- Particular solutions for SS spacetimes have previously only existed for embeddings with Ricci tensor $R_{ij} = 0$ of the $4D$ spacetimes, together with Einstein spaces. The new solution for the propagation equations obtained in this work (together with those obtained by Anderson et al [5, 58]) have successfully completed the embedding of the general Schwarzschild-de Sitter metric and the Einstein static universe into both the $\Lambda = 0$ and $\Lambda \neq 0$ $5D$ vacuum spacetimes.

- Wesson [100] embedded Friedman-Lemaitre-Robertson-Walker (FLRW) into $5D$ Minkowski in the context of the space-time matter theory. In addition, within the brane-world paradigm, both the FLRW and the Bianchi Type I spacetimes are embedded as singular branes in $5D$ anti de-Sitter space (ADS) [46, 67]. The static universe we embedded is a special case of the FLRW metric. Unlike the FLRW results, our embeddings are manifestly not $\mathbb{Z}_2$-symmetric and possess non-singular energy momentum, albeit only for the static case.

- Apart from not being $\mathbb{Z}_2$-symmetric, they are also not of the form of the simplest Randall-Sundrum scenarios [84] which agrees with results obtained by Londal [60] (which severely constrain astrophysical solutions).

5.3 Open problems

Solutions to the embedding equations have always been very difficult to find. This has in most instances necessitated strong simplifying assumptions.

Recall that the original assumption made by Moodley and Amery [75] was

$$\bar{g}_{ik}(y, r) = \mathrm{diag}[A(y, r)g_{00}, B(y, r)g_{11}, C(y, r)g_{22}, C(y, r)g_{33}].$$

Based on this assumption, the propagation equations became

$$\ddot{A} + \frac{\dot{A}}{2}\left[-\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} + \frac{\dot{D}}{D}\right] + \frac{4\epsilon\Lambda}{3}A = -2\epsilon\frac{A}{B}R_{00}g^{00}, \tag{5.6}$$

$$\ddot{B} + \frac{\dot{B}}{2}\left[\frac{\dot{A}}{A} - \frac{\dot{B}}{B} + \frac{\dot{C}}{C} + \frac{\dot{D}}{D}\right] + \frac{4\epsilon\Lambda}{3}B = -2\epsilon R_{11}g^{11}, \tag{5.7}$$

$$\ddot{C} + \frac{\dot{C}}{2}\left[\frac{\dot{A}}{A} + \frac{\dot{B}}{B} - \frac{\dot{C}}{C} + \frac{\dot{D}}{D}\right] + \frac{4\epsilon\Lambda}{3}C = -2\epsilon\frac{C}{B}R_{22}g^{22} - \frac{2\epsilon}{r^2}\left[\frac{C}{B} - 1\right], \tag{5.8}$$

$$\ddot{D} + \frac{\dot{D}}{2}\left[\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} - \frac{\dot{D}}{D}\right] + \frac{4\epsilon\Lambda}{3}D = -2\epsilon\frac{D}{B}R_{33}g^{33} - \frac{2\epsilon}{r^2}\left[\frac{D}{B} - \frac{D}{C}\right], \tag{5.9}$$

Since the solution to this system was not immediately clear, strong simplifying assumptions were made. This resulted in the system which we analysed in this work. Now that we have successfully obtained a solution to the system (4.1)–(4.2), the next step will be to employ these techniques in the resolution of system (5.6)–(5.9).

Moodley and Amery [73] also made an equivalent assumption of a metric of the form

$$\bar{g}_{ik}(y,r) = \mathrm{diag}[-e^{A(y,r)}, e^{B(y,r)}, (C(y,r))^2, (C(y,r))^2 \sin^2\theta]. \tag{5.10}$$

The propagation equations that were constructed based on this assumptions are:

$$\ddot{A} + \frac{\dot{A}^2}{2} + \frac{\dot{A}\dot{B}}{2} + \frac{2\dot{A}\dot{C}}{C} + \frac{4\epsilon\Lambda}{3} = -2\epsilon e^{-B}\left(\frac{A''}{2} + \frac{A'^2}{4} - \frac{A'B'}{4} + \frac{A'C'}{C}\right)$$

$$\ddot{B} + \frac{\dot{B}^2}{2} + \frac{\dot{A}\dot{B}}{2} + \frac{2\dot{B}\dot{C}}{C} + \frac{4\epsilon\Lambda}{3} = -2\epsilon e^{-B}\left(\frac{A''}{2} + \frac{A'^2}{4} - \frac{A'B'}{4} - \frac{B'C'}{C} + \frac{2C''}{C}\right)$$

$$2C\ddot{C} + 2\dot{C}^2 + C\dot{C}\dot{A} + C\dot{C}\dot{B} + \frac{4\epsilon\Lambda}{3}C^2 = -2\epsilon e^{-B}\left(-e^{-B} + CC'' + C'^2 - \frac{B'CC'}{2} + \frac{A'CC'}{2}\right).$$

This form of the propagation equations is particularly well suited to the study of $4D$ spacetimes like the global monopole and the Reissner-Nordström spacetimes. The former has been successfully embedded in $5D$ Minkowski space [75], but the $\Lambda \neq 0$ case and the Reissner-Nordström solution remain open problems. These spacetimes are of particular physical interest because they arise as the near and far field limits of particular solutions in Gauss-Bonnett gravity [69]. We again believe that the knowledge garnered from solving system (4.1)–(4.2) can also be applied here.

Finally we note that time dependence may be introduced into the assumptions made for the extrinsic curvature; both on and off the initial hypersurface. The resultant equations will undoubtedly be much more complicated, but it is quite possible that the propagation equations solved in this dissertation could be appropriate for classes of solutions more general than the Schwarzschild-de Sitter and the static Einstein universe. This issue is currently being investigated further.

Hopefully, this work will give useful insights into solving these highly nonlinear DEs which have eluded solution to date. A further implication of these new insights is that they may aid attempts to build more realistic higher dimensional models in cosmology and astrophysics.

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