UNIVERSITY OF KWAZULU-NATAL

RELATIVISTIC RADIATING STARS WITH GENERALISED ATMOSPHERES

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Relativistic radiating stars with generalised atmospheres

by

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Abstract

In this dissertation we construct radiating models for dense compact stars in relativistic astrophysics. We first utilise the standard Santos (1985) junction condition to model Euclidean stars. By making use of the heuristic Euclidean condition and a linear transformation in the gravitational potentials, we generate a particular exact solution in closed form to the nonlinear stellar boundary condition. Earlier models of spherical nonadiabatic gravitational collapse are then extended by considering the effect of radial perturbations in the matter and metric variables, on the evolution of the stellar fluid and the dynamics of the collapse process. The governing equation describing the temporal behaviour of the model is solved on the stellar surface. The model becomes static in the later stages of collapse. The Santos junction condition is then generalised to describe a radiating star which has a two-fluid atmosphere, consisting of a radiation field and a string fluid. We show that in the appropriate limit when the string energy density goes to zero, the standard result is regained. An exact solution to the generalised boundary condition is found. The generalised boundary condition is extended to hold in the case when the shear is nonvanishing. We demonstrate that our results can be used to model the flow of a string fluid in terms of a diffusion transport process.

To

God and my family,

for their guidance and support.

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DECLARATION 1 - PLAGIARISM

I, Gabriel Govender, student number: 200301326, declare that

1. The research reported in this thesis, except where otherwise indicated, is my original research.
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DECLARATION 2 - PUBLICATIONS

DETAILS OF CONTRIBUTION TO PUBLICATIONS that form part and/or include research presented in this thesis (include publications in preparation, submitted, in press and published and give details of contributions of each author to experimental work and writing of each publication)

Publication 1

Govender G, Govender M and Govinder K S, Thermal behaviour of Euclidean stars, *Int. J. Mod. Phys. D* **19**, 1773-1782 (2010).

(There were regular meetings between myself and my supervisor to discuss research material for publications. The outline of the research papers and discussion of the significance of the results were jointly done. The papers were mainly written by myself with some input from my supervisor.)

Publication 2

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(There were regular meetings between myself and my supervisor to discuss research material for publications. The outline of the research papers and discussion of the significance of the results were jointly done. The papers were mainly written by myself with some input from my supervisor.)

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Contents

Chapter 1

Introduction

Research in the field of relativistic astrophysics is crucial in improving our understanding of the various physical processes and phenomena that drive the dynamics of gravitation. From trying to understand the behaviour of neutron stars, quark stars, black holes, collapse of supermassive black holes and pulsars, to explaining how stars collapse under gravity to form these compact objects, the field of astrophysics has become one of the fundamental cornerstones of modern science. The main focus of this thesis is to study the evolution and dynamics of dense compact stars and stellar configurations, using the framework of general relativity.

Since the inception of the theory of general relativity, a substantial amount of research has been completed involving applications to astrophysics. Four notable examples representing fundamental breakthroughs, in the absence of rotation, in this field are;

- The generation of the first analytical solution to the Einstein field equations describing the exterior gravitational field of a static star. The interior and exterior Schwarzschild solutions provided the first complete relativistic description of the matter content and spacetime geometry for a star (Schwarzschild 1916a, 1916b).

- The derivation of the physical and mathematical conditions governing the dynamics of the gravitational collapse of a star (Oppenheimer and Snyder 1939).

- The discovery of the first radiating solution to the Einstein field equations which describes the radial flow of coherent null radiation in the presence of a spherically symmetric gravitational field (Vaidya 1951).

- More recently, and especially relevant to this thesis, is the construction of the junction conditions relating the matter and thermodynamic variables on the stellar surface. This is crucial in modeling the radiative transfer of heat energy in compact stars (Santos 1985).

These basic results, and subsequent developments, have made it possible to completely model a radiating relativistic compact object in astrophysics. The studies that are presented in this work are separated into two parts. The first part involves the construction of two exact stellar models in the context of the standard Santos formalism for radiating stars. The second part considers the generalisation and extension of the standard Santos (1985) framework. As mentioned earlier, the second part of this study comprises a major theme of this dissertation, and effectively involves extending and reconstructing the Santos junction condition. The purpose is to generate a framework to produce more general and realistic radiating stellar models in future research efforts.

In terms of the standard Santos junction conditions, we first investigate the thermal evolution and stability of a special class of relativistic stars called Euclidean stars. Herrera and Santos (2010) have investigated the general properties of these stars, in both the nonadiabatic and adiabatic limits, and presented the appropriate stellar boundary condition that governs the temporal dynamics of the model. Owing to the high degree of nonlinearity in the boundary condition, no corresponding exact solution was produced in their studies. Consequently the thermal evolution during the latter dissipative phases of collapse was left as an open question. With this application in mind, we extend the existing model for these stars by providing an exact solution to the junction condition; this is then used to complete the description of the thermodynamics. We also consider collapse models with heat flow that were first investigated by Govender *et al* (2003). In their work they studied the effect of radial perturbations in the matter quantities and the gravitational potentials on the collapse process. It was demonstrated that these perturbations allow for the dissipating star to eventually collapse to a static compact state. Their results are useful in constructing models for compact X-ray pulsars, in particular Her X-1 (Sharma and Maharaj 2007). In our contribution to these efforts, we strengthen existing models by mapping out the complete thermodynamic evolution of a radiating star through to the final static state. This enables us to study additional physical features of the model that arise from the perturbations.

In the second part we consider extending the standard Santos formalism to describe more general matter fields. Our intention is to provide a model which is more meaningful physically, and to provide viable mechanisms for the radiative transfer of heat energy in compact relativistic stars. This is effectively done by allowing the mass function in the Vaidya radiating metric to be dependent on both the Eddington retarded time and the comoving radial coordinate. This means that the emission of the null photon radiation across the stellar surface is anisotropic and, significantly, the atmosphere of the star must now be a coupled two-component fluid. We demonstrate in detail how the new generalised junction conditions are derived from first principles. We also show by means of direct application that these generalised conditions have the remarkable consequence that the atmosphere and local interstellar region of such stars exhibit evolutionary and dynamical behaviour that is still yet to be understood. The physics governing these compact stellar systems may be far more complicated than in the standard scenario. This is an area of ongoing research.

This dissertation is organised as follows:

- Chapter 1: Introduction.
- Chapter 2: In this chapter we present a review and background on the fundamental concepts of differential geometry, general relativity and relativistic astrophysics which are essential for constructing the stellar models to be studied. A

number of key definitions and formalisms are highlighted. The Einstein-Maxwell system of field equations are presented for charged fluid distributions as well as those for neutral matter. The Oppenheimer-Volkoff equations for gravitational collapse of stars are introduced and key physical features as well as dynamical quantities are highlighted.

- Chapter 3: We investigate the thermal evolution of radiating Euclidean stars in dissipative collapse. A particular exact solution, to the second order nonlinear boundary condition in two variables, is generated by imposing the Euclidean condition and a linear transformation in the gravitational potentials. This solution is then used, in conjunction with the causal heat transport equation, to construct the complete temperature, relaxation time and proper radius profiles. This work and that of chapter 4 are done in the context of the standard Santos (1985) formalism for radiative transfer in relativistic stars.

- Chapter 4: Here we investigate the gravitational and temporal dynamics of stars that undergo nonadiabatic collapse to eventually reach a static configuration. We consider the effect of radial perturbations in the metric as well as the matter variables, on the evolution of the stellar fluid distribution and the collapse process in terms of the causal and noncausal thermodynamics.

- Chapter 5: This chapter forms a substantial and central part of this study. The Santos (1985) junction conditions for radiating stars are generalised and extended to include the effect of an additional string fluid coupled with the standard null radiation field in the star's atmosphere. This is done by first introducing the generalised Vaidya radiating solution and the stress energy tensor describing a stellar atmosphere consisting of a two-fluid system. The new generalised junction conditions are then derived from first principles by carrying out the smooth matching of a shear-free interior stellar spacetime to the stellar exterior described by the generalised Vaidya metric. This result is then verified by considering the conservation of photon momentum flux across the stellar boundary. The new junction conditions are then utilised to study some of the physics of the two-fluid atmosphere. Profiles for the luminosity and redshift of the emitted radiation are generated, taking into account the effect of the string fluid in the extended condition on the surface of the dissipating star. We also generate an exact solution to the generalised junction condition and show that it describes a model in which a relativistic radiating star is undergoing geodesic heat flow in the presence of a diffusing string atmosphere.

- Chapter 6: In this chapter we extend the results of the previous chapter by considering the role of shearing stresses in the interior stellar fluid on the generalised junction conditions. We verify this result by using an alternative set of geometric conditions that include the relevant spacetime curvatures without following the matching process used in chapter 5. The shearing analogue of the modified conditions are then applied to models describing the evolution of the star's atmosphere including the diffusion of the string fluid component.

- Chapter 7: Conclusion

Chapter 2

Basic theory for relativistic stellar astrophysics

Einstein's theory of general relativity is successful in describing the dynamical behaviour of spherically symmetric matter distributions in strong gravitational fields. A review of the physics of compact objects, black holes and relativistic stellar processes is provided by Shapiro and Teukolsky (1983). For a recent treatment of cosmological models see Gron and Hervik (2007). In this chapter, we present the background theory that enables us to generate a model of a dense compact relativistic star within the context of a localised astrophysical system. We present a brief outline of the relevant differential geometry, the Einstein-Maxwell system of equations for charged matter distributions and the essential physical criteria for a physically acceptable stellar model. For more extensive details on differential manifolds and tensor analysis, and related topics, the reader is referred to Bishop and Goldberg (1968), Misner *et al* (1973) and Wald (1984). In §2.2, the essential components of differential geometry such as the Riemann tensor, the Ricci tensor, the Ricci scalar and the Einstein tensor are introduced. These components are required to generate the Einstein field equations which are the governing equations needed to model a dense gravitating system. We introduce the energy momentum tensor and the special case of a perfect fluid, for modeling astrophysical situations, in §2.3. Then we present a covariant formulation of Maxwell's laws of electromagnetism. This allows us to formulate the Einstein-Maxwell system of equations in which the electromagnetic and matter fields are coupled. In §2.4, the physical conditions necessary for interior solutions for relativistic stellar systems are considered. Finally in §2.5, we briefly discuss the concept of stability in stars and consider the process of nonadiabatic gravitational collapse. We highlight key concepts such as the Oppenheimer-Volkoff equations and the effective adiabatic index.

2.1 Spacetime geometry

In general relativity, we assume that the spacetime $\mathcal{M}$ is a four-dimensional differentiable manifiold endowed with a symmetric, nonsingular metric tensor field $\boldsymbol{g}$. In local regions the manifold has the structure of Euclidean space which implies that it may be covered by overlapping coordinate patches so that special relativity is regained in the relevant limit. The manifold of general relativity, with an indefinite metric tensor field, is called a pseudo-Riemannian manifold. The tensor field $\boldsymbol{g}$ represents the gravitational field and it has signature $(-+++)$. Individual points in the manifold are labelled by the real coordinates $(x^a) = (x^0, x^1, x^2, x^3)$, where $x^0 = ct$ (c is the speed of light in vacuum) is the timelike coordinate and x^1, x^2, x^3 are spacelike coordinates. In this thesis, we use the convention that the speed of light $c = 1$. For more comprehensive treatments of spacetime geometry, the reader is referred to the standard text books in differential geometry such as Bishop and Goldberg (1968), de Felice and Clark (1990), Hawking and Ellis (1973), Misner *et al* (1973) and Wald (1984).

The invariant distance between neighbouring points in $\boldsymbol{M}$ is defined by the line element

$$ds^2 = g_{ab}dx^a dx^b \tag{2.1.1}$$

The metric connection Γ is defined in terms of the metric tensor and its derivatives by

$$\Gamma^a{}_{bc} = \frac{1}{2}g^{ad}(g_{cd,b} + g_{db,c} - g_{bc,d}) \tag{2.1.2}$$

where commas denote partial differentiation. There exists a unique symmetric connection Γ that preserves inner products under parallel transport (do Carmo 1992). The Riemannian (curvature or Riemann-Christoffel) tensor $\boldsymbol{R}$ is given by

$$R^d{}_{abc} = \Gamma^d{}_{ac,b} - \Gamma^d{}_{ab,c} + \Gamma^e{}_{ac}\Gamma^d{}_{eb} - \Gamma^e{}_{ab}\Gamma^d{}_{ec} \tag{2.1.3}$$

On contraction of (2.1.3) we obtain the Ricci tensor

$$\begin{aligned} R_{ab} &= R^c{}_{acb} \\ &= \Gamma^c{}_{ab,c} - \Gamma^c{}_{ac,b} + \Gamma^c{}_{dc}\Gamma^d{}_{ab} - \Gamma^c{}_{db}\Gamma^d{}_{ac} \end{aligned} \tag{2.1.4}$$

which is symmetric. On contracting the Ricci tensor (2.1.4) we obtain

$$\begin{aligned} R &= R^a{}_a \\ &= g^{ab}R_{ab} \end{aligned} \tag{2.1.5}$$

which is the Ricci (or curvature) scalar.

With these definitions it is now possible to construct the Einstein tensor $\boldsymbol{G}$, in terms of the Ricci tensor (2.1.4) and the Ricci scalar (2.1.5), as follows

$$G^{ab} = R^{ab} - \frac{1}{2}Rg^{ab} \tag{2.1.6}$$

Clearly the Einstein tensor $\boldsymbol{G}$ is symmetric. The Einstein tensor has zero divergence so that

$$G^{ab}{}_{;b} = 0 \tag{2.1.7}$$

which follows from the definition of the Einstein tensor (2.1.6). This property is sometimes called the Bianchi identity, and it is a necessary condition to generate the conservation of energy momentum via the Einstein field equations.

2.2 Fluids and electromagnetic fields

For applications in astrophysics the matter distribution is described by a relativistic fluid. The energy momentum tensor for uncharged matter is described by the symmetric tensor $\boldsymbol{T}$ where

$$T^{ab} = (\mu + p)u^a u^b + pg^{ab} + q^a u^b + q^b u^a + \pi^{ab} \tag{2.2.1}$$

where μ is the energy density, p is the isotropic (kinetic) pressure, q^a is the heat flux vector $(q^a u_a) = 0$ and π^{ab} is the anisotropic pressure (stress) tensor $(\pi^{ab}u_a = 0 = \pi^a{}_a)$. These quantities are measured relative to a comoving fluid four-velocity $\boldsymbol{u}$ which is unit and timelike $(u^a u_a = -1)$. In perfect adiabatic fluids there are no heat conduction and stress terms $(q^a = 0, \pi^{ab} = 0)$. For a perfect fluid the energy momentum tensor, equation (2.2.1) becomes

$$T^{ab} = (\mu + p)u^a u^b + pg^{ab} \tag{2.2.2}$$

For many applications in large scale and open astrophysical systems, we require that the matter distribution satisfies a barotropic equation of state

$$p = p(\mu) \tag{2.2.3}$$

on physical grounds. Sometimes the particular equation of state

$$p = (\gamma - 1)\mu$$

where $0 \leq \gamma \leq 1$, is assumed in galaxy and galaxy cluster astrophysics and cosmology to describe matter distributions. This is called the linear γ equation of state. The case $\gamma = 1$ corresponds to pressureless relativistic dust; $\gamma = 2$ gives a stiff equation of state (valid for certain white dwarf stars) in which the speed of sound is equal to the speed of light; $\gamma = 4/3$ corresponds to radiation. Often the particular equation of state

$$p = k\mu^{1+\frac{1}{n}}$$

where k and n are constants, is assumed in relativistic astrophysics. This is called a polytropic equation of state and is fundamental for the realistic description of much stellar matter. Hence, most stars in both the relativistic as well as the Newtonian limits, are modelled as polytropes. Fang and Ruffini (1983) have provided more details on polytropic stars.

The Einstein field equations

$$G^{ab} = T^{ab} \tag{2.2.4}$$

govern the interaction between the curvature of spacetime and the matter content in the absence of electric charge. We have set the coupling constant to be unity in (2.2.4). From (2.1.7) and (2.2.4) we obtain

$$T^{ab}{}_{;b} = 0 \tag{2.2.5}$$

which is the conservation of matter.

We define the electromagnetic field tensor $\boldsymbol{F}$ in terms of the four-potential $\boldsymbol{A}$ by

$$F_{ab} = A_{b;a} - A_{a;b}$$

which is skew-symmetric. The electromagnetic field tensor can be written in terms of the electric field $\boldsymbol{E} = (E^1, E^2, E^3)$ and the magnetic field $\boldsymbol{B} = (B^1, B^2, B^3)$ as follows

$$F^{ab} = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & B^3 & -B^2 \\ -E^2 & -B^3 & 0 & B^1 \\ -E^3 & B^2 & -B^1 & 0 \end{pmatrix} \tag{2.2.6}$$

The electromagnetic contribution $\boldsymbol{E}$ to the total energy momentum is given by the result

$$E_{ab} = F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{cd}F^{cd} \tag{2.2.7}$$

To consider the effect of $\boldsymbol{E}$ on the gravitational field it is necessary to express the fundamental equations of electromagnetism, namely Maxwell's laws, in covariant form. The governing equations are given by

$$F_{ab;c} + F_{bc;a} + F_{ca;b} = 0 \tag{2.2.8a}$$

$$F^{ab}{}_{;b} = J^a \tag{2.2.8b}$$

where $\boldsymbol{J}$ is the four-current density defined by

$$J^a = \sigma u^a \tag{2.2.9}$$

and σ is the proper charge density. For further information on Maxwell's field equations (2.2.8) see Misner *et al* (1973) and Narlikar (2002). Note that the Maxwell equations (2.2.8) are the basic equations that govern the behaviour of the electromagnetic field in a curved background.

We point out that the total energy momentum tensor is the sum of $\boldsymbol{T}$ and $\boldsymbol{E}$. We are now in a position to introduce the Einstein-Maxwell system of equations for a charged fluid in a gravitational field. The interaction between $\boldsymbol{T}$, $\boldsymbol{E}$ and $\boldsymbol{g}$ is governed by the Einstein-Maxwell system of equations

$$G^{ab} = T^{ab} + E^{ab} \tag{2.2.10a}$$

$$F_{ab;c} + F_{bc;a} + F_{ca;b} = 0 \tag{2.2.10b}$$

$$F^{ab}{}_{;b} = J^a \tag{2.2.10c}$$

The system (2.2.10) is a highly nonlinear system of coupled, partial differential equations governing the behaviour of gravitating systems in the presence of an electromagnetic field. In (2.2.10a), we use units in which the coupling constant in the Einstein equations is unity. We need to solve the system (2.2.10) to generate an exact solution;

one approach is to specify a particular form for the matter distribution and electromagnetic field on physical grounds and then integrate the partial differential equations to find the metric tensor field $\boldsymbol{g}$. For uncharged matter, the only equation that has to be satisfied is the Einstein field equation (2.2.10a) with $\boldsymbol{E} = 0$. Note that from (2.1.7) and (2.2.10a) we obtain

$$(T^{ab} + E^{ab})_{;b} = 0 \tag{2.2.11}$$

which is the total conservation of matter and charge which generalises (2.2.5).

2.3 Physical conditions

We briefly consider the physical conditions applicable to a relativistic stellar model. For physical viability, any solution applicable to the interior of the stellar body should match smoothly to the appropriate exterior spacetime. The gravitational field outside a static spherically symmetric body, in the absence of charge, is given by

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.3.1}$$

which is the exterior Schwarzschild solution. Here the quantity m is the mass of the stellar body as measured by an observer at infinity. The exterior gravitational field to a static spherically symmetric body, in the presence of charge, has the form

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right) dt^2 + \left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.3.2}$$

In the above q is the constant related to the total charge of the sphere. The line element (2.3.2) is the exterior Reissner-Nordstrom solution. The radial electric field is

$$E = \frac{q}{r^2}$$

and, consequently, the proper charge density is $\sigma = 0$. Therefore, the four current density $\boldsymbol{J} = 0$ which is consistent with an exterior spacetime with no barotropic matter. If $q = 0$ then (2.3.2) reduces to the exterior Schwarzschild line element (2.3.1).

Physical conditions will restrict the solutions of the Einstein-Maxwell system (2.2.10) for a realistic star. It is often assumed by researchers that realistic stellar models for isotropic matter should satisfy the following conditions:

(a) The energy density μ and the pressure p should be positive and finite throughout the interior of the star. The radial pressure should vanish at the boundary $r = b$:

$$0 < \mu < \infty, \ \ 0 < p < \infty, \ \ p(b) = 0$$

(b) The energy density μ and the pressure p should be monotonic decreasing functions from the centre to the boundary:

$$\tfrac{d\mu}{dr} \leq 0, \ \ \tfrac{dp}{dr} \leq 0$$

(c) Causality should be satisfied. The speed of sound should remain less than the speed of light throughout the interior of the star which leads to the condition:

$$0 \leq \tfrac{dp}{d\mu} \leq 1$$

(d) The metric functions $e^{2\nu}$ and $e^{2\lambda}$ and the electric field intensity E should be positive and nonsingular throughout the interior of the star.

(e) At the boundary the interior gravitational potentials should match smoothly to the exterior line elements (2.3.1) and (2.3.2) for neutral and charged matter, respectively. This generates the following conditions on the gravitational potentials:

$$e^{2\nu(b)} = e^{-2\lambda(b)} = 1 - \tfrac{2m}{b}, \ \ (E = 0)$$

$$e^{2\nu(b)} = e^{-2\lambda(b)} = 1 - \tfrac{2m}{b} + \tfrac{q^2}{b^2}, \ \ (E \neq 0)$$

(f) The electric field intensity E should be continuous across the boundary for the case of charged models:

$$E(b) = \tfrac{q}{b^2}$$

(g) The models should be stable with respect to radial perturbations.

It should be observed that not all relativistic stellar models satisfy the full set of the conditions listed above throughout the stellar interior; particular solutions may be valid only in some regions of spacetime. Several examples are listed by Delgaty and Lake (1998) which become singular at the centre. Such solutions need to be treated as an envelope of the star and should be matched to another solution valid for the core. An example of a core-envelope model is provided by Thomas *et al* (2005). Some of the conditions (a)-(g) may be very restrictive. For example, observational evidence suggests that in some stars the energy density μ may be not a strictly decreasing function. However, many researchers, for example Delgaty and Lake (1998), require that an exact solution satisfy these conditions. In addition, it is interesting to study the behaviour of anisotropic matter distributions with radial pressures different from tangential pressures. Such cases were studied by Chaisi and Maharaj (2005), and Dev and Gleiser (2002, 2003) in the case of neutral spheres; Herrera and Ponce de Leon (1985) analysed tangential pressures in the presence of charge. Anisotropic matter and charge distributions may be relevant in the description of quark stars as pointed out by Sharma and Maharaj (2007) and Komathiraj and Maharaj (2007), respectively. Exact solutions to the field equations which do not satisfy all of the conditions (a)-(g) are still of value because they provide useful information which assist in the qualitative analysis of relativistic stars.

2.4 Gravitational collapse of stars

In general a star is in a state of hydrostatic equilibrium if the governing forces that dominate in the stellar matter are balanced. These forces are due to the interior hydrodynamic thermal fluid pressure directed radially outward and the self gravity due to the stars mass which is directed radially inwards. An internal pressure gradient from within is responsible for opposing the inward self gravity, and thus keeping it in hydrostatic equilibrium. For a relativistic star, this pressure gradient forms part of a crucial system of dynamical equations which govern its stability and collapse. These are called the Oppenheimer-Volkoff equations, and they are written for isotropic pressures as

$$m(r) = \frac{1}{2}\int_0^r \mu(x)x^2 dx \tag{2.4.1a}$$

$$\frac{dp}{dr} = -\frac{[p+\mu]\left[m+\frac{1}{2}r^3 p\right]}{r\,[r-2m]} \tag{2.4.1b}$$

where $\mu(r)$ and $p(r)$ are the radial fluid mass density and pressure, respectively. The quantity m is interpreted as the gravitational mass of the star. The above system (2.4.1) describes a star which is static and spherically symmetric (sometimes called Schwarzschild stars). They are crucial in describing stars which are initially in a static state before undergoing dissipative gravitational collapse. The system (2.4.1) becomes important in the early stages of collapse. For more information on (2.4.1) and their role in the stability and collapse dynamics of relativistic stars, the reader is referred to Glendenning (2000). For the general principles underlying relativistic gravitational collapse and the formation of singularities the reader is referred to Penrose (1969).

Once hydrostatic equilibrium is broken, the stellar fluid starts to contract and collapse radially inward and heat energy is released due to the changing gravitational field. This heat energy is used within the core and envelope regions to aid the dissociation of fluid particles and to reionize the neutral fluid. The excess heat energy must then be dissipated across the stellar surface in the form of null radiation by means of radiative transfer (Phillips 1994). In this thesis we focus on the process of heat dissipation in relativistic stars. During the collapsing phases, it is important to be able to describe the temporal evolution of the stellar fluid from within the core region through to the surface across which the radiation is lost. In order to achieve this we have to solve the Maxwell-Cattaneo equation, a causal heat transport equation, and generate the corresponding temperature profiles for particular values of the model parameters and integration constants. Another physical quantity which is of importance in modelling the temperature evolution in a dissipative collapsing stellar fluid is the effective adiabatic index

$$\Gamma_{eff} = \left[\frac{\partial(\ln p)}{\partial(\ln \mu)}\right]_{\Sigma} \tag{2.4.2}$$

at the stellar surface Σ. The effective adiabatic index measures the ability of the stellar matter to resist compression under gravity, and depends on the fluid pressure and energy density profiles which must be obtained by generating an exact solution to the Einstein field equations.

Chapter 3

Thermal behaviour of Euclidean stars

3.1 Introduction

The study of dissipative gravitational collapse achieved prominence with the presentation of the junction conditions by Santos (1985). Earlier work on collapsing stars in general relativity assumed the exterior spacetime to be empty and as a consequence, it was required that the pressure at the boundary vanish. Santos provided the general junction conditions required for the smooth matching of a spherically symmetric, shear-free spacetime to the exterior Vaidya (1951) solution across a timelike hypersurface. An important consequence of the matching conditions is that the pressure on the boundary of the radiating star cannot be zero. It is assumed that the interior of the star is radiating energy in the form of a radial heat flux. The junction conditions due to Santos rejuvenated the study of gravitational collapse and the end states of radiating stars. The simplistic model of Oppenheimer and Snyder (1939) has been generalised to include pressure (Bonnor *et al* 1989), anisotropic stresses (Chan 1997), electromagnetic field (Maharaj and Govender 2000), and the cosmological constant (Govender and Thirukkanesh 2009). These exact models, although simplified, give much insight into the dissipative collapse process as well as physical characteristics of the radiating star such as its temperature and luminosity.

What makes the study of dissipative collapse of stars particularly difficult is the solution to the boundary condition representing the conservation of momentum across the timelike hypersurface. While many exact solutions for shear-free radiating spheres have appeared in the recent literature, there are very few models that include the effects of shear in the interior of the star. One of the first exact models of a shearing radiating star that allowed for an analysis of the gravitational and thermodynamical behaviour of the stellar fluid was found by Naidu *et al* (2006). However, their model was restrictive in the sense that it was acceleration-free, but more importantly, the matter variables such as pressure and density become infinite at the centre of the star. It was pointed out that this model could form part of a core-envelope model of a radiating star. Further exact shearing solutions were obtained by Rajah and Maharaj (2008) in which it was assumed that the particle trajectories within the stellar core were geodesics. An analysis of the temperature profiles for these models reveals unphysical behaviour in that the temperatures closer to the surface of the star become negative. A recent study of shearing, dissipative collapse considered a model of a spherically symmetric matter distribution in which the areal radius is equal to the proper radius throughout the stellar evolution (Herrera and Santos 2010). These Euclidean stars were shown to exhibit very interesting general properties. In this chapter we present an exact solution to the boundary condition that determines the temporal evolution of an Euclidean star. Our solution allows us to study the physical and thermodynamical properties of this class of stars even when the stellar fluid is far from equilibrium. Since Euclidean stars are *not* acceleration-free we are able to draw comparisons with the earlier models of Naidu *et al* (2006) and Rajah and Maharaj (2008).

In this chapter we study the dynamical and thermal evolution of these compact radiating stars and investigate some of the physical features during the collapse process. In §3.2 we provide the details describing the interior stellar fluid distribution and the associated Einstein field equations. The exterior spacetime of the star and the junction conditions that are valid on the stellar surface are defined in §3.3. Section 3.4 is a crucial component of this chapter. Here we introduce the Euclidean condition and generate an exact solution to the boundary condition which will be used to study the thermal evolution of the model. In §3.5 we present a detailed description of the thermodynamics. Profiles for the causal and noncausal temperatures, the relaxation time scale and the proper radius of the collapsing star are generated. Finally some concluding remarks are made in §3.6.

3.2 Shearing spacetimes

The interior spacetime is described by the general spherically symmetric, shearing metric in comoving coordinates

$$ds^2 = -A^2dt^2 + B^2dr^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{3.2.1}$$

where $A = A(t,r)$, $B = B(t,r)$ and $R = R(t,r)$ are metric functions yet to be determined. The matter content for the interior is described by

$$T_{ab} = (\mu + p_T)u_a u_b + p_T g_{ab} + (p_r - p_T)\chi_a\chi_b + q_a u_b + q_b u_a \tag{3.2.2}$$

where μ represents the energy density, p_r the radial pressure, p_T the tangential pressure and q^a the heat flux vector. The fluid four–velocity $\boldsymbol{u}$ is comoving and is given by

$$u^a = \frac{1}{A}\delta^a_0 \tag{3.2.3}$$

The heat flow vector assumes the form

$$q^a = (0, q, 0, 0) \tag{3.2.4}$$

since $q^a u_a = 0$ ensuring radial heat dissipation. We introduce the vector χ^a such that

$$\chi^a\chi_a = 1, \qquad \chi^a u_a = 0 \tag{3.2.5}$$

The expansion scalar and the fluid four acceleration are given by

$$\Theta = u^a{}_{;a}, \qquad a_a = u_{a;b}u^b \tag{3.2.6}$$

where

$$a_a = \left(0, \frac{A'}{A}, 0, 0\right) \tag{3.2.7}$$

and the shear tensor by

$$\sigma_{a_b} = u_{(a;b)} + a_{(a}u_{b)} - \frac{1}{3}\Theta(g_{ab} + u_a u_b) \tag{3.2.8}$$

For the comoving line element (3.2.1) the kinematical quantities take the following form

$$a = \frac{A'}{A} \tag{3.2.9a}$$

$$\Theta = \frac{1}{A}\left(\frac{\dot{B}}{B} + 2\frac{\dot{R}}{R}\right) \tag{3.2.9b}$$

$$\sigma = \frac{1}{A}\left(\frac{\dot{B}}{B} - \frac{\dot{R}}{R}\right) \tag{3.2.9c}$$

where dots and primes denote differentiation with respect to t and r respectively. The nonzero components of the Einstein field equations for the line element (3.2.1) and the energy momentum tensor (3.2.2) are

$$\mu = \frac{1}{A^2}\left(2\frac{\dot{B}}{B}+\frac{\dot{R}}{R}\right)\frac{\dot{R}}{R}-\frac{1}{B^2}\left[2\frac{R''}{R}+\left(\frac{R'}{R}\right)^2-2\frac{B'}{B}\frac{R'}{R}-\left(\frac{B}{R}\right)^2\right] \tag{3.2.10a}$$

$$p_r = -\frac{1}{A^2}\left[2\frac{\ddot{R}}{R}-\left(2\frac{\dot{A}}{A}-\frac{\dot{R}}{R}\right)\frac{\dot{R}}{R}\right]+\frac{1}{B^2}\left(2\frac{A'}{A}+\frac{R'}{R}\right)\frac{R'}{R}-\frac{1}{R^2} \tag{3.2.10b}$$

$$p_T = -\frac{1}{A^2}\left[\frac{\ddot{B}}{B}+\frac{\ddot{R}}{R}-\frac{\dot{A}}{A}\left(\frac{\dot{B}}{B}+\frac{\dot{R}}{R}\right)+\frac{\dot{B}}{B}\frac{\dot{R}}{R}\right]$$

$$+\frac{1}{B^2}\left[\frac{A''}{A}+\frac{R''}{R}-\frac{A'}{A}\frac{B'}{B}+\left(\frac{A'}{A}-\frac{B'}{B}\right)\frac{R'}{R}\right] \tag{3.2.10c}$$

$$q = \frac{2}{AB}\left(\frac{\dot{R}'}{R}-\frac{\dot{B}}{B}\frac{R'}{R}-\frac{\dot{R}}{R}\frac{A'}{A}\right) \tag{3.2.10d}$$

This is an underdetermined system of four highly nonlinear coupled partial differential equations in seven unknowns, viz. A, B, R, μ, p_r, p_T and q.

3.3 Exterior spacetime and junction conditions

The exterior spacetime is taken to be Vaidya's outgoing solution

$$ds^2 = -\left(1-\frac{2m(v)}{R}\right)dv^2 - 2dvdR + R^2\left(d\theta^2+\sin^2\theta d\phi^2\right) \tag{3.3.1}$$

where $m(v)$ represents the gravitational energy contained within the stellar radius and can also be interpreted as the Newtonian mass of the gravitating body as measured by an observer at infinity. The necessary conditions for the smooth matching of the interior spacetime (3.2.1) to the exterior spacetime (3.3.1) have been extensively investigated. We present the main results that are necessary for modelling a radiating star. The continuity of the intrinsic and extrinsic curvature components of the interior and exterior spacetimes across a timelike boundary are

$$m(v)_\Sigma = \left\{ \frac{R}{2}\left[\left(\frac{\dot{R}}{A}\right)^2 - \left(\frac{R'}{B}\right)^2 + 1\right]\right\}_\Sigma \tag{3.3.2a}$$

$$(p_r)_\Sigma = (qB)_\Sigma \tag{3.3.2b}$$

Relation (3.3.2b) determines the temporal evolution of the collapsing star.

3.4 Radiating Euclidean stars

Following Herrera and Santos (2010) we impose the condition that the areal radius of any spherical surface contained within Σ, with centre placed at the origin, is equal to the proper radius from the centre through to $r = b$, the boundary of the star. The physical consequences of this assumption are discussed by Herrera and Santos (2010). This implies that

$$B = R' \tag{3.4.1}$$

The Einstein field equations (3.2.10) reduce to

$$\mu = \frac{1}{A^2}\left(\frac{\dot{R}}{R} + 2\frac{\dot{R}'}{R'}\right)\frac{\dot{R}}{R} \tag{3.4.2a}$$

$$p_r = -\frac{1}{A^2}\left[2\frac{\ddot{R}}{R} - \left(2\frac{\dot{A}}{A} - \frac{\dot{R}}{R}\right)\frac{\dot{R}}{R}\right] + 2\frac{A'}{A}\frac{1}{RR'} \tag{3.4.2b}$$

$$\begin{aligned} p_T = & -\frac{1}{A^2}\left[\frac{\ddot{R}}{R} + \frac{\ddot{R}'}{R'} - \frac{\dot{A}}{A}\frac{\dot{R}}{R} - \left(\frac{\dot{A}}{A} - \frac{\dot{R}}{R}\right)\frac{\dot{R}'}{R'}\right] \\ & + \frac{1}{R'^2}\left[\frac{A''}{A} - \left(\frac{R''}{R'} - \frac{R'}{R}\right)\frac{A'}{A}\right] \end{aligned} \tag{3.4.2c}$$

$$q = -\frac{2}{AR'}\left(\frac{\dot{R}}{R}\frac{A'}{A}\right) \tag{3.4.2d}$$

and the mass function becomes

$$m = \frac{R}{2}\left(\frac{\dot{R}}{A}\right)^2 \tag{3.4.3}$$

The boundary condition (3.3.2b) yields

$$\frac{\ddot{R}}{R} + \frac{1}{2}\left(\frac{\dot{R}}{R}\right)^2 - \frac{\dot{A}}{A}\frac{\dot{R}}{R} - \frac{(A+\dot{R})A'}{RR'} = 0 \tag{3.4.4}$$

valid on $r = b$. We now have a system of six coupled partial differential equations, viz. (3.4.1), (3.4.2) and (3.4.4) in seven unknowns.

We focus on (3.4.4), as a solution of this equation will yield all the relevant kinematical and physical quantities. In doing so, we note that we are requiring (3.4.4) to hold for all r and not just on the boundary $r = b$. Equation (3.4.4) is a nonlinear partial differential equation in the gravitational potentials $A(r,t)$ and $R(r,t)$. We could analyse it as a quasi–linear partial differential equation in $A(r,t)$ only. Then the general solution of (3.4.4) will reduce to a general function of the solutions of two ordinary differential equations. Unfortunately the resulting equations are still difficult to solve. As a consequence, we provide a simple solution to (3.4.4) by imposing the following linear relation

$$R = \varepsilon A \tag{3.4.5}$$

for which ε is an arbitrary constant. (Note that this closes our system of partial differential equations as we now have seven equations in seven unknown functions.) This assumption leads to

$$R(r,t) = \left[C_1(r)e^{\lambda_1 t} + C_2(r)e^{\lambda_2 t}\right]^2 \tag{3.4.6}$$

where

$$\lambda_1 = \frac{1+\sqrt{3}}{2\varepsilon} \qquad \lambda_2 = \frac{1-\sqrt{3}}{2\varepsilon} \tag{3.4.7}$$

which is the general solution to the resulting form of (3.4.4) with the assumption (3.4.5) valid for all r. To prevent divergence of the solution it may be necessary to restrict

(3.4.6) for small time intervals. Utilising the form (3.4.6), the Einstein field equations (3.2.10) yield

$$\mu = \frac{2\varepsilon\dot{z}\left[e^{t/\varepsilon}\left[(3-\sqrt{3})C_2C_1' + (3+\sqrt{3})C_1C_2'\right]\right]}{z'z^6} + \frac{12\varepsilon^2\dot{z}\left(\lambda_1C_1C_1'e^{2\lambda_1 t} + \lambda_2C_2C_2'e^{2\lambda_2 t}\right)}{z'z^6} \tag{3.4.8a}$$

$$p_r = \frac{-2e^{\lambda_1 t}\left(1+\sqrt{3}\right)C_1 - 2e^{\lambda_2 t}\left(1-\sqrt{3}\right)C_2}{z^5} \tag{3.4.8b}$$

$$p_T = -\frac{C_2^2C_2'e^{3\lambda_2 t}\left(3-2\sqrt{3}\right) + C_1^2C_1'e^{3\lambda_1 t}\left(3+2\sqrt{3}\right)}{z'z^6} - \frac{C_1C_2e^{t/\varepsilon}\left[(9+2\sqrt{3})C_1'e^{\lambda_1 t} + (9-2\sqrt{3})C_2'e^{\lambda_2 t}\right]}{z'z^6} \tag{3.4.8c}$$

$$q = -4\varepsilon\frac{\dot{z}}{z^5} \tag{3.4.8d}$$

where we have defined

$$z(r,t) = C_1(r)e^{\lambda_1 t} + C_2(r)e^{\lambda_2 t} \tag{3.4.9}$$

The magnitude of the shear tensor is given by

$$\sigma = \frac{\sqrt{3}e^{(-1+2\sqrt{3})t/\varepsilon}\left(C_2C_1' - C_1C_2'\right)}{z'z^3} \tag{3.4.10}$$

The above relation indicates that the shear vanishes when $C_1(r) \propto C_2(r)$. In the next section we study the thermodynamical properties of our model. In order to ensure that the shear remains finite and nonzero for all time we make the following choice, as a specific example, for our metric function

$$R(r,t) = \left[(a^2+r^2)e^{\lambda_1 t} + (c^2+r^2)e^{\lambda_2 t}\right]^2 \tag{3.4.11}$$

where a and c are constants and λ_1 and λ_2 were defined earlier.

3.5 Thermodynamics

In this section we investigate the evolution of the temperature profile of our model within the context of extended irreversible thermodynamics. The causal transport equation in the absence of rotation and viscous stress is

$$\tau h_a{}^b \dot{q}_b + q_a = -\kappa \left(h_a{}^b \nabla_b T + T \dot{u}_a \right) \tag{3.5.1}$$

where $h_{ab} = g_{ab} + u_a u_b$ projects into the comoving rest space, T is the local equilibrium temperature, κ (≥ 0) is the thermal conductivity, and τ (≥ 0) is the relaxational time-scale which gives rise to the causal and stable behaviour of the theory. To obtain the noncausal Fourier heat transport equation we set $\tau = 0$ in (3.5.1). For the metric (3.2.1), equation (3.5.1) becomes

$$\tau (qB)^{\cdot} + AqB = -\frac{\kappa (AT)'}{B} \tag{3.5.2}$$

In order to obtain a physically reasonable stellar model we will adopt the thermodynamic coefficients for radiative transfer. Hence we are considering the situation where energy is transported away from the stellar interior by massless particles, moving with long mean free path through matter that is effectively in hydrodynamic equilibrium, and that is dynamically dominant. Govender *et al* (1998, 1999) have shown that the choice

$$\kappa = \gamma T^3 \tau_{\mathrm{c}}, \qquad \tau_{\mathrm{c}} = \left(\frac{\alpha}{\gamma} \right) T^{-\omega}, \qquad \tau = \left(\frac{\beta \gamma}{\alpha} \right) \tau_{\mathrm{c}} \tag{3.5.3}$$

is physically reasonable for the thermal conductivity κ, the mean collision time between massive and massless particles τ_c, and the relaxation time τ. The quantities $\alpha \geq 0$, $\beta \geq 0$, $\gamma \geq 0$ and $\omega \geq 0$ are constants. Note that the mean collision time decreases with growing temperature as expected except for the special case $\omega = 0$, when it is constant. With these assumptions the causal heat transport equation (3.5.2) becomes

$$\beta (qB)^{\cdot} T^{-\omega} + A(qB) = -\alpha \frac{T^{3-\omega} (AT)'}{B} \tag{3.5.4}$$

This equation was comprehensively studied in the noncausal ($\beta = 0$) case by Govinder and Govender (2001) as well as in specific causal cases. In the noncausal case, (3.5.4) can be solved to yield

$$(A\tilde{T})^{4-\omega} = \frac{\omega - 4}{\alpha} \int A^{4-\omega} q B^2 dr + F(t), \qquad \omega \neq 0 \qquad (3.5.5a)$$

$$\ln (A\tilde{T}) = -\frac{1}{\alpha} \int q B^2 dr + F(t), \qquad \omega = 4 \qquad (3.5.5b)$$

where $F(t)$ is a function of integration which is fixed by the surface temperature of the star. Note that $\tilde{T}$ corresponds to the noncausal temperature when $\beta = 0$. For a constant mean collision time ($\omega = 0$), (3.5.4) can be integrated to give the causal temperature:

$$(AT)^4 = -\frac{4}{\alpha} \left[\beta \int A^3 B (qB)^{\cdot} dr + \int A^4 q B^2 dr \right] + F(t) \qquad (3.5.6)$$

In (3.5.3) we can think of β as the 'causality' index, measuring the strength of relaxational effects, with $\beta = 0$ giving the noncausal case.

The effective surface temperature of a star is given by

$$(\bar{T}^4)_\Sigma = \left(\frac{1}{r^2 B^2} \right) \left(\frac{L}{4\pi\delta} \right) \qquad (3.5.7)$$

where L is the luminosity at infinity and $\delta (> 0)$ is a constant. The luminosity at infinity can be calculated from

$$L_\infty = -\frac{dm}{dv} \qquad (3.5.8)$$

where $m(v)$ is given in (3.4.3). We are now in a position to analyse the evolution of the temperature in both the causal and noncausal theories.

Figure 3.1 represents the causal temperature (dashed line) and noncausal temperature (solid line) as a function of the radial coordinate. It is clear that the temperature in both the causal and noncausal theories is a maximum at the centre of the star and drops off smoothly as the radial coordinate increases towards the boundary. This trend also indicates that the surface layers of the star are much cooler than the interior regions. As in the acceleration-free case studied by Naidu *et al* (2006) and Rajah and

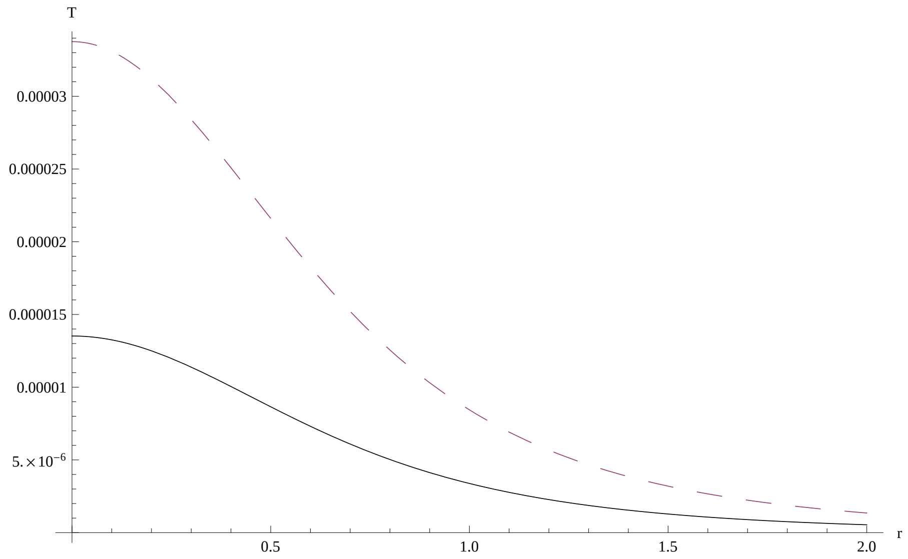

Figure 3.1: Causal (dashed line)and noncausal (solid line) temperature profiles versus radial coordinate.

Maharaj (2008), the causal temperature in Figure 3.1 is everywhere higher than its noncausal counterpart at each interior point of the star. The causal and noncausal temperatures are equal at the boundary of the star. Figure 3.1 also reveals that relaxational effects account for a larger temperature gradient within the stellar core. This is expected at late times during the collapse as the stellar fluid is far from hydrostatic equilibrium.

Figure 3.2 illustrates the trend in the relaxation times for the shear stresses. Following Naidu *et al* (2006) the shear transport equation yields

$$\tau_1 = \frac{-p}{\dot{p} + \frac{8}{15} r_0 \sigma T^4} \tag{3.5.9}$$

where the coefficient of shear viscosity for a radiative fluid

$$\eta = \frac{4}{15} r_0 T^4 \tau_1 \tag{3.5.10}$$

was utilised. In (3.5.9) we have used $p = \frac{1}{3}(p_T - p_r)$ and r_0 is the radiation constant for photons. We have further assumed that $\tau_1 = \beta_1 \tau_c$ where β_1 is a constant. Figure 3.2 also clearly shows that the relaxation time for the shear stresses can vary as much as a factor of $10^2 s$ during the evolution of the collapsing fluid. A similar result was

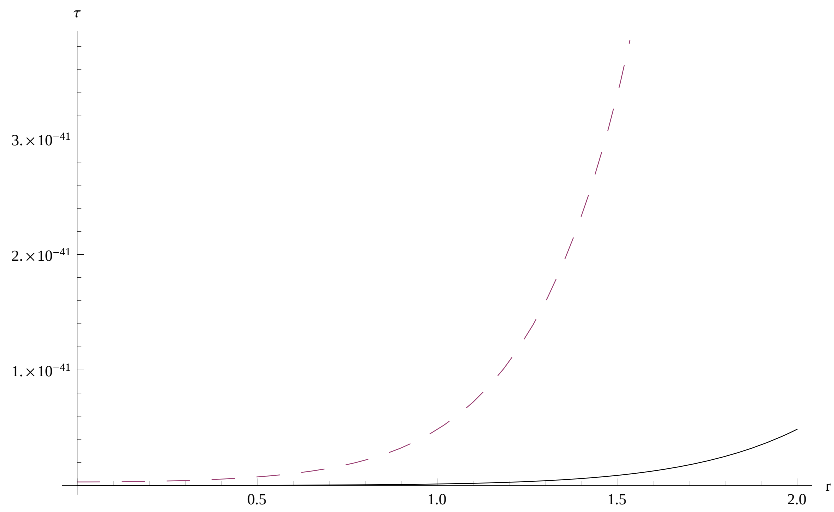

Figure 3.2: Relaxation time for the shear stress (close to equilibrium - dashed line), (far from equilibrium - solid line) versus radial coordinate.

found for the acceleration-free model investigated by Naidu *et al* (2006).

Figure 3.3 shows the proper radius as a function of time. We have followed the conventions of Chan (2003) in generating Figure 3.3. It is a monotonically decreasing function as expected since the star is losing mass in the form of a radial heat flux. It is interesting to note that the formation of the horizon can be avoided in our model even in the presence of shear, by carefully choosing the arbitrary functions $C_1(r)$ and $C_2(r)$. Such a choice would ensure that the mass-to-radius ratio $2m_\Sigma/\bar{r}_\Sigma < 1$ is satisfied and thereby avoids the appearance of the horizon for all time. The horizon-free model of a radiating, shear-free star undergoing collapse was first considered by Banerjee *et al* (2002). The physical viability of this model was studied by Naidu and Govender (2008) where it was shown that the temperature and luminosity profiles were well behaved throughout the stellar interior.

3.6 Discussion

We have presented an exact solution that completely describes the temporal and radial behaviour of a particular class of radiating stars, the Euclidean stars. We have shown

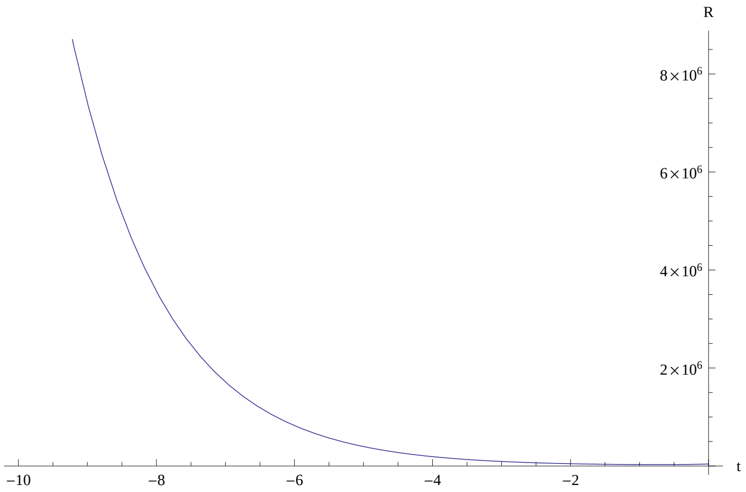

Figure 3.3: Proper radius versus time.

that the model is reasonably well behaved throughout the collapse process, with the physical and thermodynamical variables remaining physically viable. Our model of a radiating star with nonvanishing shear adds to the limited class of such solutions that are currently available in the literature.

Chapter 4

Temperature evolution for a perturbed model

4.1 Introduction

Dissipative processes such as heat generation, shear, particle creation and bulk viscosity during stellar collapse have been extensively studied in the past. It has been shown in numerous models that casual transport equations predict thermodynamical behaviour that are different from their noncausal counterparts. In particular, radiating stellar collapse has been shown to yield causal temperature profiles which are always higher within the stellar core. It is well known that during the latter stages of collapse the core temperature of stars are of the order of 10^9 K. Using a perturbative scheme in which the metric functions and thermodynamical variables are perturbed to first order, Herrera and Santos (1997) have shown that for a temperature range of $10^6 - 10^9$K, the relaxation time may vary from as much as $\tau \approx 10^2$s to as little as $\tau \approx 10^{-4}$ s. Herrera and Santos carried out both Newtonian and post-Newtonian approximations on the causal heat transport equation. They demonstrated that the causal temperature gradient can differ as much as five orders of magnitude from the noncausal temperature gradient.

It is now our aim to calculate and study the complete temperature profiles of a compact star undergoing nonadiabatic gravitational collapse to a final static equilibrium state. In §4.2 we describe the matter distribution and spacetime geometry of the stellar interior. The exterior of the star and junction conditions are defined in §4.3. In §4.4 we construct a radiating model by introducing radial perturbations to first order in the matter and gravitational variables. The thermodynamics of the compact static state is studied in §4.5. We summarise our results in §4.6.

4.2 Interior Spacetime

The interior of the star is described by a spherically symmetric line element with vanishing shear, in comoving and isotropic coordinates, so that

$$ds^2 = -A^2 dt^2 + B^2 \left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{4.2.1}$$

where $A = A(t, r)$ and $B = B(t, r)$ are metric functions. The matter distribution for the stellar interior is represented by the energy momentum tensor of an imperfect fluid

$$T_{ab} = (\mu + p)u_a u_b + p g_{ab} + q_a u_b + q_b u_a \tag{4.2.2}$$

where μ is the energy density, p is the pressure and $q = (q^a q_a)^{\frac{1}{2}}$ is the magnitude of the heat flux. The fluid four-velocity $\boldsymbol{u}$ is comoving and is given by

$$u^a = \frac{1}{A}\delta^a_0 \tag{4.2.3}$$

The heat flow vector takes the form

$$q^a = (0, q, 0, 0) \tag{4.2.4}$$

since $q^a u_a = 0$ and the heat is assumed to flow in the radial direction. The fluid collapse rate $\Theta = u^a{}_{;a}$ of the stellar model is given by

$$\Theta = 3\frac{\dot{B}}{AB} \tag{4.2.5}$$

The Einstein field equations reduce to

$$\mu = 3\frac{1}{A^2}\frac{\dot{B}^2}{B^2} - \frac{1}{B^2}\left(2\frac{B''}{B} - \frac{B'^2}{B^2} + \frac{4}{r}\frac{B'}{B}\right) \tag{4.2.6a}$$

$$p = \frac{1}{A^2}\left(-2\frac{\ddot{B}}{B} - \frac{\dot{B}^2}{B^2} + 2\frac{\dot{A}}{A}\frac{\dot{B}}{B}\right)$$

$$+\frac{1}{B^2}\left(\frac{B'^2}{B^2} + 2\frac{A'}{A}\frac{B'}{B} + \frac{2}{r}\frac{A'}{A} + \frac{2}{r}\frac{B'}{B}\right) \tag{4.2.6b}$$

$$p = -2\frac{1}{A^2}\frac{\ddot{B}}{B} + 2\frac{\dot{A}}{A^3}\frac{\dot{B}}{B} - \frac{1}{A^2}\frac{\dot{B}^2}{B^2} + \frac{1}{r}\frac{A'}{A}\frac{1}{B^2}$$

$$+\frac{1}{r}\frac{B'}{B^3} + \frac{A''}{A}\frac{1}{B^2} - \frac{B'^2}{B^4} + \frac{B''}{B^3} \tag{4.2.6c}$$

$$q = -\frac{2}{AB^2}\left(-\frac{\dot{B}'}{B} + \frac{B'\dot{B}}{B^2} + \frac{A'}{A}\frac{\dot{B}}{B}\right) \tag{4.2.6d}$$

for the line element (4.2.1). In the above system we have used the convention that overhead dots and primes denote derivatives with respect to the comoving and isotropic coordinates t and r respectively.

4.3 Exterior Spacetime

Since the star is radiating energy to the exterior it is natural that the exterior spacetime be described by Vaidya's outgoing solution given by

$$ds^2 = -\left(1 - \frac{2m(v)}{\mathsf{r}}\right)dv^2 - 2dvd\mathsf{r} + \mathsf{r}^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{4.3.1}$$

where $\frac{dm}{dv} < 0$. In order to get a complete description of the radiating star, the interior spacetime is matched to the Vaidya exterior across a timelike hypersurface. These junction conditions were first presented by Santos and are widely utilised to model radiating stars in relativistic astrophysics. Here we present the main results of the matching for easy reference. The continuity of the metric functions and the extrinsic curvature across the boundary Σ yield

$$m(v) = \left(\frac{r^3 B}{2A^2}{B_t}^2 - r^2 B_r - \frac{r^3}{2B}{B_r}^2\right)_\Sigma \quad (4.3.2a)$$

$$p_\Sigma = (qB)_\Sigma \quad (4.3.2b)$$

where $m(v)$ represents the total mass contained within a sphere of radius r in $(4.3.2a)$ and $(4.3.2b)$ ensures the conservation of momentum across the boundary.

4.4 A radiating model

Following Govender *et al* (2003) we consider a model in which the star undergoes dissipative collapse and evolves to a stable equilibrium state. In order to obtain an analytical model we impose the following conditions on our metric functions and thermodynamical variables

$$A(r,t) = A_0(r) + \epsilon a(r)H(t) \quad (4.4.1a)$$

$$B(r,t) = B_0(r) + \epsilon b(r)H(t) \quad (4.4.1b)$$

$$\mu(r,t) = \mu_0(r) + \epsilon\bar{\mu}(r,t) \quad (4.4.1c)$$

$$p(r,t) = p_0(r) + \epsilon\bar{p}(r,t) \quad (4.4.1d)$$

where the heat flux is of the order of ϵ $(0 < \epsilon \ll 1)$ and (A_0, B_0) represent the final static configuration. For the static end state we have

$$\mu_0 = -\frac{1}{B_0^2}\left[2\frac{B_0''}{B_0} - \left(\frac{B_0'}{B_0}\right)^2 + \frac{4}{r}\frac{B_0'}{B_0}\right] \tag{4.4.2a}$$

$$p_0 = \frac{1}{B_0^2}\left[\left(\frac{B_0'}{B_0}\right)^2 + \frac{2}{r}\frac{B_0'}{B_0} + \frac{2}{r}\frac{A_0'}{A_0} + 2\frac{A_0'}{A_0}\frac{B_0'}{B_0}\right] \tag{4.4.2b}$$

The pressure isotropy equation for the static configuration is

$$\left(\frac{A_0'}{A_0} + \frac{B_0'}{B_0}\right)' - \left(\frac{A_0'}{A_0} + \frac{B_0'}{B_0}\right)^2 - \frac{1}{r}\left(\frac{A_0'}{A_0} + \frac{B_0'}{B_0}\right) + 2\left(\frac{A_0'}{A_0}\right)^2 = 0 \tag{4.4.3}$$

and the perturbed quantities up to first order in ϵ are

$$\bar{\mu} = -3\mu_0\frac{b}{B_0}H + \frac{1}{B_0^3}\left[-\left(\frac{B_0'}{B_0}\right)^2 b + 2\left(\frac{B_0'}{B_0} - \frac{2}{r}\right)b' - 2b''\right]H \tag{4.4.4a}$$

$$\begin{aligned}\bar{p} &= -2p_0\frac{b}{B_0}H - 2\frac{b}{A_0^2 B_0}\ddot{H} \\ &\quad + \frac{2}{B_0^2}\left[\left(\frac{B_0'}{B_0} + \frac{1}{r} + \frac{A_0'}{A_0}\right)\left(\frac{b}{B_0}\right)' + \left(\frac{B_0'}{B_0} + \frac{1}{r}\right)\left(\frac{a}{A_0}\right)'\right]H\end{aligned} \tag{4.4.4b}$$

$$\bar{q} = \frac{2\epsilon}{B_0^2}\left(\frac{b}{A_0 B_0}\right)'\dot{H} \tag{4.4.4c}$$

The condition of pressure isotropy for the perturbed matter distribution yields

$$\begin{aligned}&\left[\left(\frac{a}{A_0}\right)' + \left(\frac{b}{B_0}\right)'\right]' - 2\left[\left(\frac{a}{A_0}\right)' + \left(\frac{b}{B_0}\right)'\right]\left(\frac{A_0'}{A_0} + \frac{B_0'}{B_0}\right) \\ &\quad -\frac{1}{r}\left[\left(\frac{a}{A_0}\right)' + \left(\frac{b}{B_0}\right)'\right] + 4\frac{A_0'}{A_0}\left(\frac{a}{A_0}\right)' = 0\end{aligned} \tag{4.4.5}$$

Introducing the following parameters

$$\alpha = \frac{A_0^2}{bB_0}\left[\left(\frac{B_0'}{B_0} + \frac{1}{r} + \frac{A_0'}{A_0}\right)\left(\frac{b}{B_0}\right)' + \left(\frac{B_0'}{B_0} + \frac{1}{r}\right)\left(\frac{a}{A_0}\right)'\right] \tag{4.4.6a}$$

$$\beta = \frac{A_0^2}{2b}\left(\frac{b}{A_0 B_0}\right)' \tag{4.4.6b}$$

allows us to write (4.4.4b) and (4.4.4c) as

$$\bar{p} = -2p_0\frac{b}{B_0}H + 2\frac{\alpha b}{A_0^2 B_0}H - 2\frac{b}{A_0^2 B_0}\ddot{H} \tag{4.4.7a}$$

$$\bar{q} = \frac{4\epsilon b}{A_0^2 B_0^2}\beta\dot{H} \tag{4.4.7b}$$

Substituting (4.4.7a) and (4.4.7b) into (4.3.2b) we obtain the temporal evolution equation

$$\ddot{H} + 2\beta\dot{H} - \alpha H = 0 \tag{4.4.8}$$

where we have taken $(p_0)_\Sigma = 0$ which is necessary for the static end state. Bearing in mind that we are investigating a collapse scenario, more specifically the evolution leading to a final equilibrium configuration, we take

$$H(t) = H_0 e^{-(\beta_\Sigma + \sqrt{\alpha_\Sigma + \beta_\Sigma^2})t} \tag{4.4.9}$$

which is obtained by direct integration of (4.4.8) and obeys the following set of boundary conditions

$$H(t)|_{t=\infty} = 0, \qquad H(t)|_{t=0} = H_0$$

where H_0 is a constant. For the proper description of a physically reasonable stellar situation we require $H(t)$ to decrease with time, so we must have $\alpha_\Sigma > 0$.

4.5 Thermodynamics

Our primary interest is to investigate the physical viability of a collapsing star evolving into a final static configuration. To this end, we seek to obtain the temperature profile of our model within the context of extended irreversible thermodynamics. The role of relaxational effects during dissipative gravitational collapse have been highlighted in many studies. Previous works have shown that the inclusion of relaxation effects, especially during the late stages of collapse (when the stellar fluid is far from hydrostatic equilibrium) lead to higher temperatures within the stellar core. The causal transport equation in the absence of rotation and viscous stress is

$$\tau h_a{}^b \dot{q}_b + q_a = -\kappa \left(h_a{}^b \nabla_b T + T\dot{u}_a \right) \tag{4.5.1}$$

where $h_{ab} = g_{ab} + u_a u_b$ projects into the comoving rest space, T is the local equilibrium temperature, κ (≥ 0) is the thermal conductivity, and τ (≥ 0) is the relaxational time-scale which gives rise to the causal and stable behaviour of the theory. To obtain the noncausal Fourier heat transport equation we set $\tau = 0$ in (4.5.1). For the metric (4.2.1), equation (4.5.1) becomes

$$\tau (qB)^{\dot{}} + AqB = -\frac{\kappa (AT)'}{B} \tag{4.5.2}$$

In order to obtain a physically reasonable stellar model we will adopt the thermodynamic coefficients for radiative transfer. Hence we are considering the situation where energy is transported away from the stellar interior by massless particles (photons), moving with a long mean free path through matter that is effectively in hydrodynamic equilibrium, and that is dynamically dominant. Govender *et al* (1998, 1999) have shown that the choice

$$\kappa = \gamma T^3 \tau_{\rm c}, \qquad \tau_{\rm c} = \left(\frac{\alpha}{\gamma}\right) T^{-\omega}, \qquad \tau = \left(\frac{\beta\gamma}{\alpha}\right) \tau_{\rm c} \tag{4.5.3}$$

is physically reasonable for the thermal conductivity κ, the mean collision time between massive and massless particles τ_c and the relaxation time τ. The quantities $\alpha \geq 0$, $\beta \geq 0$, $\gamma \geq 0$ and $\omega \geq 0$ are constants. Note that the mean collision time decreases with growing temperature as expected except for the special case $\omega = 0$, when it is constant.

With these assumptions the causal heat transport equation (4.5.2) becomes

$$\beta (qB)^{\dot{}} T^{-\omega} + A(qB) = -\alpha \frac{T^{3-\omega} (AT)'}{B} \tag{4.5.4}$$

In (4.5.3) we can think of β as the 'causality' index measuring the strength of relaxational effects, with $\beta = 0$ giving the noncausal case. For our perturbative model, we write

$$T = \mathcal{T}_0 + \epsilon\bar{\mathcal{T}}S(t) \tag{4.5.5}$$

where $\mathcal{T}_0$ represents the equilibrium temperature and $S(t)$ is an arbitrary function. Utilising (4.5.5) in (4.5.4) we obtain

$$\bar{\mathcal{T}}(r) = \frac{-2}{\alpha}\left[\frac{\beta}{A_0}\frac{\ddot{S}}{S}\int\left(\frac{b}{A_0B_0}\right)'\mathcal{T}_0^{-3}dr + \frac{1}{A_0}\frac{\dot{S}}{S}\int A_0\mathcal{T}_0^{\omega-3}\left(\frac{b}{A_0B_0}\right)'dr\right]$$

$$-\frac{a\mathcal{T}_0}{A_0} + \frac{C_1}{A_0} \tag{4.5.6}$$

where C_1 is an integration constant and

$$(A_0\mathcal{T}_0)' = 0 \tag{4.5.7}$$

Relation (4.5.7) leads to

$$\mathcal{T}_0 = \frac{C_0}{A_0} \tag{4.5.8}$$

where $C_0 > 0$ is a constant. As pointed out by Herrera and Santos (1997), this is a well known result first obtained by Tolman which ensures the existence of a temperature gradient that prevents heat flux from regions of higher to regions of lower gravitational field intensity during thermal equilibrium. This result follows since A_0 is an increasing factor of r. In order to investigate the evolution of the temperature we assume that the end state of collapse is described by the static Schwarzschild interior solution in isotropic coordinates

$$A_0 = \zeta_1 - \zeta_2\frac{1-r^2}{1+r^2} \tag{4.5.9a}$$

$$B_0 = \frac{2R}{1+r^2} \tag{4.5.9b}$$

where ζ_1, ζ_2 and R are constants. We can easily calculate the energy density and pressure for the static configuration as

$$\mu_0 = \frac{3}{R^2} \tag{4.5.10a}$$

$$p_0 = \frac{1}{R^2}\left(-1 + \frac{2\zeta_2(1-r^2)}{\zeta_1(1+r^2) - \zeta_2(1-r^2)}\right) \tag{4.5.10b}$$

The vanishing of the pressure at the boundary and the continuity of the metric functions across Σ leads to

$$\frac{\zeta_1}{\zeta_2} = 3\frac{(1-r_\Sigma^2)}{(1+r_\Sigma^2)}$$

and

$$\zeta_2 = \frac{1}{2}$$

As demonstrated by Bonnor *et al* (1989), the physical requirements $\mu_0 > 0$, $p_0 > 0$, $p_0 < \mu_0$ and $0 \leq r \leq r_\Sigma$ are satisfied provided that

$$r_\Sigma^2 < \frac{1}{3} \tag{4.5.11a}$$

$$\frac{2m_0}{r_\Sigma} = \frac{4r_\Sigma^2}{(1+r_\Sigma^2)^2} < \frac{3}{4} \tag{4.5.11b}$$

where

$$m_0 = \frac{4Rr_\Sigma^3}{(1+r_\Sigma^2)^3}$$

represents the total mass within the static sphere up to the boundary Σ. Furthermore, the pressure isotropy condition for the nonstatic configuration is ensured by choosing

$$\frac{a}{A_0} = \frac{b}{B_0} = \frac{k_1}{2}\int rB_0^2 dr + k_2 \tag{4.5.12}$$

where k_1 and k_2 are constants of integration.

It is clear from Figure 4.1 that the causal temperature is greater than the noncausal temperature everywhere within the stellar interior. We must point out that the contributions from $\bar{\mathcal{T}}$ in (4.5.5) to the overall temperature profile $\mathcal{T}$ are due to the positive contribution of $\bar{\mathcal{T}}$ and the relaxational effects in $\bar{\mathcal{T}}$. Plots of $\bar{\mathcal{T}}$ in both the casual and

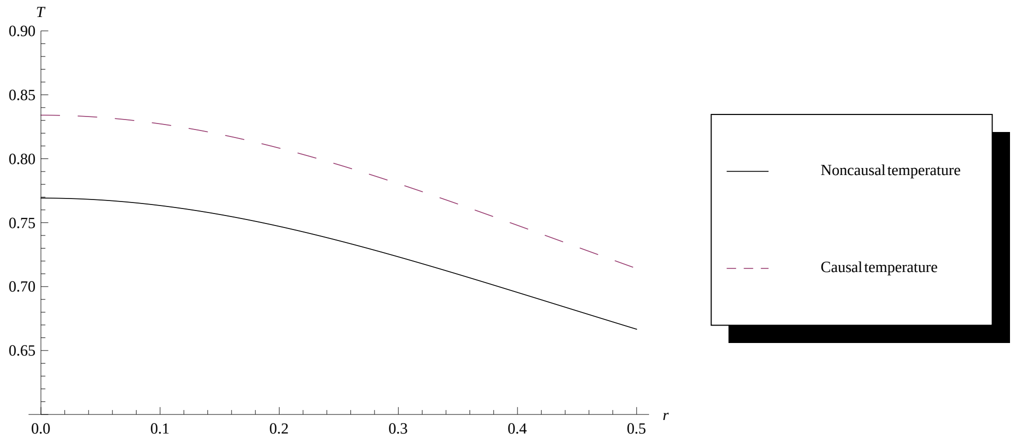

Figure 4.1: Causal (dashed line)and noncausal (solid line) temperature profiles versus radial coordinate.

noncausal cases indicate that $\bar{\mathcal{T}}$ is a positive decreasing function from the centre of the star through to the stellar surface. This perturbative contribution is greatly enhanced by relaxational effects. Figure 4.1 clearly indicates that the causal temperature gradient is steeper than its noncausal counterpart closer to the core with the difference dropping off as one gets to the stellar surface.

4.6 Discussion

We have provided a complete description of a radiating star, undergoing dissipative gravitational collapse in the form of radial heat emission, with the final end state being static. The temperature profile is obtained for both the casual and noncausal regimes. It is clear that the temperature induced by perturbations leads to higher core temperatures. This effect is enhanced by relaxational effects which is noticeable at late stages of collapse (when the system is far from hydrostatic equilibrium). Our results confirm earlier findings by Herrera and Santos (1997) in which it was shown that the causal temperature gradient can be as high as five orders of magnitude greater than its noncausal counterpart. Our results are also in keeping with the perturbative results of Govender *et al* (1999) in which it was shown that for a sphere collapsing from an initial static configuration, relaxational effects become dominant during the latter stages of collapse.

Chapter 5

Radiating stars with generalised Vaidya atmospheres

5.1 Introduction

The study of radiating stars in the context of general relativity has generated much interest in researchers because of the variety of applications in relativistic astrophysics. These studies are important as they enable us to investigate physical features such as surface luminosity, dynamical stability, particle production at the stellar surface, relaxation effects, causal temperature gradients and other thermodynamical processes. Some relevant references investigating these issues are given by Di Prisco *et al* (2007), Govender *et al* (1998), Herrera *et al* (2009) and Pinheiro and Chan (2008). Relativistic radiating stars are also important in the process of gravitational collapse, describing the final state of stars, formation of singularities and black hole physics, in four and higher dimensions. Recent investigations in this regard are contained in the works of Goswami and Joshi (2007), Joshi (2007) and Madhav *et al* (2005). In particular, the validity of the cosmic censorship conjecture can be tested in this physical scenario.

The model of a relativistic radiating star undergoing dissipation was completed by Santos (1985) by analysing the junction conditions at the stellar surface. By matching a shear-free interior spacetime to the radiating Vaidya exterior spacetime, he showed that at the surface the pressure is nonvanishing and proportional to the magnitude of the heat flux. Subsequently several explicit relativistic radiating stellar models have been found by investigating the appropriate boundary condition. Kramer (1992) and Maharaj and Govender (1997) generated nonstatic radiating spheres from a static model by allowing certain parameters to become functions of time. Kolassis *et al* (1988) and Thirukkanesh and Maharaj (2009) assumed geodesic fluid trajectories to produce new radiating models. In the approach of De Oliviera *et al* (1985) and Nogueira and Chan (2004) the model has an initial static configuration before the radiating sphere starts gradually to collapse. Exact solutions for shear-free interiors which are conformally flat generate radiating stellar models as shown by Herrera *et al* (2004), Herrera *et al* (2006), Maharaj and Govender (2005) and Misthry *et al* (2008). Stellar models which are radiating with nonzero shear are difficult to analyse because of the complexity of the boundary condition. However even in this case there have been advances in obtaining exact solutions. Particular exact models have been found by Naidu *et al* (2006) and Rajah and Maharaj (2008).

We now seek to generalise the Santos junction conditions by matching a shear-free interior spacetime to the generalised Vaidya exterior spacetime (Husain 1996). The energy momentum tensor of the generalised Vaidya spacetime may be interpreted as a superposition of two fluids, a pressureless null dust and a null string fluid. The physical properties of the generalised Vaidya spacetime have been discussed by Husain (1996) and Wang and Wu (1999). Glass and Krisch (1998) have interpreted the exterior spacetime as a superposition of two fluids outside a relativistic star, the original Vaidya null fluid and a new null fluid composed of strings. By assuming diffusive transport for the string fluid Glass and Krisch (1999) found new solutions to Einstein's equations with transverse stresses. Physically reasonable energy transport mechanisms have been generated by Krisch and Glass (2005) in the stellar interior with the generalised Vaidya metric as the exterior spacetime. These investigations, and other treatments, have largely focussed on physical processes in the *exterior* of the stellar model with a generalised Vaidya atmosphere. To fully describe a radiating stellar model requires generation of the junction conditions at the stellar surface. This is now our aim.

We follow the convention that the coupling constant $\frac{8\pi G}{c^4}$ and the speed of light c are unity; the metric has signature $(-+++)$. In §5.2 we introduce the relevant definitions and theory describing the junction conditions, and we present the junction conditions in their most general form. We discuss in §5.3 the defining geometries for the interior and exterior spacetimes, and the respective energy momentum tensors; the Einstein field equations are presented in full. In §5.4 we perform the matching of the interior and exterior spacetimes across the stellar surface in detail. The new set of junction conditions are derived for the generalised Vaidya spacetime. In §5.5 we indicate how the new junction conditions generalise the junction conditions previously derived by Santos (1985). The physical significance of our new result is highlighted in terms of a string fluid. We consider the new junction condition in the context of conservation of momentum flux across the stellar boundary in §5.6. In §5.7 we discuss the luminosity and redshift. An exact solution to the generalised boundary condition is given in §5.8. We discuss the significance of our results in §5.9.

5.2 Junction conditions

Spacetime needs to be divided into two distinct regions, the interior spacetime $\mathcal{M}^-$ and the exterior spacetime $\mathcal{M}^+$ for a stellar model. The boundary of the star Σ serves as the matching surface for $\mathcal{M}^-$ and $\mathcal{M}^+$. The boundary or stellar surface is a timelike three-dimensional hypersurface. We assume that Σ is endowed with an intrinsic metric $g_{\alpha\beta}$ so that

$$ds^2_\Sigma = g_{\alpha\beta} d\xi^\alpha d\xi^\beta$$

The intrinsic coordinates on Σ are given by ξ^α where $\alpha = 1, 2, 3$. The line elements in the exterior and interior spacetimes, respectively, have the form

$$ds^2_\pm = g_{ab} d\chi^a_\pm d\chi^b_\pm$$

The coordinates in the exterior and interior spacetimes, respectively, are $\chi^a_\pm$ where $a = 0, 1, 2, 3$. For consistency we require that

$$(ds^2_+)_\Sigma = (ds^2_-)_\Sigma = ds^2_\Sigma \tag{5.2.1}$$

so that the line elements match on the boundary Σ. This implies that the coordinates of Σ in $\mathcal{M}^\pm$ are $\chi^a_\pm = \chi^a_\pm(\xi^\alpha)$. It is clear that the first junction condition (5.2.1) is generated by the continuity of the metric across Σ.

The second junction condition is generated by the continuity of the extrinsic curvature of Σ across the boundary. The extrinsic curvature of Σ is defined by

$$K^\pm_{\alpha\beta} \equiv -n^\pm_a \frac{\partial^2 \chi^a_\pm}{\partial\xi^\alpha \partial\xi^\beta} - n^\pm_a \Gamma^a{}_{bc} \frac{\partial\chi^b_\pm}{\partial\xi^\alpha} \frac{\partial\chi^c_\pm}{\partial\xi^\beta} \tag{5.2.2}$$

In the above $n^\pm_a(\chi^b_\pm)$ are the components of the vector normal to Σ. The second junction condition is then given by

$$(K^+_{\alpha\beta})_\Sigma = (K^-_{\alpha\beta})_\Sigma \tag{5.2.3}$$

Note that the junction conditions (5.2.1) and (5.2.3) are equivalent to the Lichnerowicz (1955) and O' Brien and Synge (1952) junction conditions.

5.3 Interior and exterior spacetimes

The line element for the interior manifold $\mathcal{M}^-$ is given by

$$ds^2 = -A^2(t, r)dt^2 + B^2(t, r)[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)] \tag{5.3.1}$$

in comoving and isotropic coordinates. The interior spacetime is expanding and accelerating but is shear-free. The following Einstein tensor components

$$G^-_{00} = 3\frac{\dot{B}^2}{B^2} - \frac{A^2}{B^2}\left(2\frac{B''}{B} - \frac{B'^2}{B^2} + \frac{4}{r}\frac{B'}{B}\right) \tag{5.3.2a}$$

$$G^-_{01} = -\frac{2}{B^2}\left(B\dot{B}' - B'\dot{B} - B\dot{B}\frac{A'}{A}\right) \tag{5.3.2b}$$

$$\begin{aligned} G^-_{11} = & \frac{1}{A^2}\left(-2B\ddot{B} - \dot{B}^2 + 2B\dot{B}\frac{\dot{A}}{A}\right) + \\ & \frac{1}{B^2}\left(B'^2 + 2BB'\frac{A'}{A} + \frac{2}{r}B^2\frac{A'}{A} + \frac{2}{r}BB'\right) \end{aligned} \tag{5.3.2c}$$

$$\begin{aligned} G^-_{22} = & -2r^2\frac{B\ddot{B}}{A^2} + 2r^2B\dot{B}\frac{\dot{A}}{A^3} - r^2\frac{\dot{B}^2}{A^2} + \\ & r\frac{A'}{A} + r\frac{B'}{B} + r^2\frac{A''}{A} - r^2\frac{B'^2}{B^2} + r^2\frac{B''}{B} \end{aligned} \tag{5.3.2d}$$

$$G^-_{33} = \sin^2\theta G^-_{22} \tag{5.3.2e}$$

are nonvanishing for the shear-free metric (5.3.1). In the above dots and primes denote differentiation with respect to the coordinates t and r respectively. A physically relevant interior matter distribution that is consistent with (5.3.1) and (5.3.2) is given by

$$T^-_{ab} = (\mu + p)u_a u_b + p g_{ab} + q_a u_b + q_b u_a \tag{5.3.3}$$

where μ is the energy density, p is the isotropic pressure, q_a is the radial heat flux vector and $u^a = \frac{1}{A}\delta^a_0$ is the comoving fluid four-velocity. The Einstein field equations

$G^-_{ab} = T^-_{ab}$ for the interior manifold $\mathcal{M}^-$ are given by

$$\mu = 3\frac{\dot{B}^2}{A^2B^2} - \frac{1}{B^2}\left(2\frac{B''}{B} - \frac{B'^2}{B^2} + \frac{4}{r}\frac{B'}{B}\right) \tag{5.3.4a}$$

$$p = \frac{1}{A^2}\left(-2\frac{\ddot{B}}{B} - \frac{\dot{B}^2}{B^2} + 2\frac{\dot{A}}{A}\frac{\dot{B}}{B}\right)$$

$$+\frac{1}{B^2}\left(\frac{B'^2}{B^2} + 2\frac{A'}{A}\frac{B'}{B} + \frac{2}{r}\frac{A'}{A} + \frac{2}{r}\frac{B'}{B}\right) \tag{5.3.4b}$$

$$p = -2\frac{\ddot{B}}{A^2B} + 2\frac{\dot{A}}{A^3}\frac{\dot{B}}{B} - \frac{\dot{B}^2}{A^2B^2} + \frac{1}{r}\frac{A'}{AB^2}$$

$$+\frac{1}{r}\frac{B'}{B^3} + \frac{A''}{AB^2} - \frac{B'^2}{B^4} + \frac{B''}{B^3} \tag{5.3.4c}$$

$$q = -\frac{2}{AB^2}\left(-\frac{\dot{B}'}{B} + \frac{B'\dot{B}}{B^2} + \frac{A'}{A}\frac{\dot{B}}{B}\right) \tag{5.3.4d}$$

where we have used (5.3.2) and (5.3.3).

The line element for the exterior manifold $\mathcal{M}^+$ is taken to be

$$ds^2 = -\left(1 - 2\frac{m(v,\mathsf{r})}{\mathsf{r}}\right)dv^2 - 2dvd\mathsf{r} + \mathsf{r}^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{5.3.5}$$

where $m(v,\mathsf{r})$ is the mass function, and is related to the gravitational energy within a given radius r (Lake and Zannias 1991, Poisson and Israel 1990). This metric is often called the generalised Vaidya spacetime since it reduces to the Vaidya spacetime when $m = m(v)$ which is the mass of the star as measured by an observer at infinity. The following nonzero Einstein tensor components

$$G^{+}_{00} = -\frac{2}{\mathsf{r}^2} m_v \tilde{v}^2 + 2\frac{(\mathsf{r} - 2m)}{\mathsf{r}^3} m_{\mathsf{r}} \quad (5.3.6a)$$

$$G^{+}_{01} = \frac{2}{\mathsf{r}^2} m_{\mathsf{r}} \quad (5.3.6b)$$

$$G^{+}_{22} = -\mathsf{r} m_{\mathsf{rr}} \quad (5.3.6c)$$

$$G^{+}_{33} = \sin^2\theta G^{+}_{22} \quad (5.3.6d)$$

are all defined in terms of $m = m(v, \mathsf{r})$ and we have used the notation

$$m_v = \frac{\partial m}{\partial v}, \qquad m_{\mathsf{r}} = \frac{\partial m}{\partial \mathsf{r}}, \qquad \tilde{v} = \frac{dv}{d\tau}$$

where τ is a timelike coordinate on the hypersurface.

It has been demonstrated by Husain (1996) and Wang and Wu (1999) that an energy momentum tensor consistent with (5.3.5) and (5.3.6) is

$$T^{+}_{ab} = T^{(n)}_{ab} + T^{(m)}_{ab} \quad (5.3.7a)$$

$$T^{(n)}_{ab} = \tilde{\mu} l_a l_b \quad (5.3.7b)$$

$$T^{(m)}_{ab} = (\rho + P)(l_a n_b + l_b n_a) + P g_{ab} \quad (5.3.7c)$$

which represents a superposition of a pressureless null dust and a null string fluid. In general T^{+}_{ab} represents a Type II fluid as defined by Hawking and Ellis (1973). The null vector l^a is a double null eigenvector of the energy momentum tensor T^{+}_{ab}. The weak and strong energy conditions, and the dominant energy conditions are satisfied for proper choices of the mass function $m(v, \mathsf{r})$.

In (5.3.7) we have introduced the two null vectors

$$l_a = \delta_a^0 \tag{5.3.8a}$$

$$n_a = \frac{1}{2}\left[1 - 2\frac{m(v,\mathsf{r})}{\mathsf{r}}\right]\delta_a^0 + \delta_a^1 \tag{5.3.8b}$$

where $l_a l^a = n_a n^a = 0$ and $l_a n^a = -1$. The Einstein field equations $G^+_{ab} = T^+_{ab}$ for the exterior manifold $\mathcal{M}^+$ are then given by

$$\tilde{\mu} = -2\frac{m_v}{\mathsf{r}^2}\tilde{v}^2 \tag{5.3.9a}$$

$$\rho = 2\frac{m_{\mathsf{r}}}{\mathsf{r}^2} \tag{5.3.9b}$$

$$P = -\frac{m_{\mathsf{rr}}}{\mathsf{r}} \tag{5.3.9c}$$

where we have utilised (5.3.6) and (5.3.7). We interpret $\tilde{\mu}$ as the energy density of the null dust radiation; ρ and P are the null string energy density and null string pressure, respectively.

5.4 Matching

The intrinsic metric to the hypersurface Σ is defined by

$$ds^2_\Sigma = -d\tau^2 + \mathcal{Y}^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{5.4.1}$$

with coordinates $\xi^\alpha = (\tau, \theta, \phi)$ and $\mathcal{Y} = \mathcal{Y}(\tau)$. The timelike coordinate τ is defined only on Σ and the coordinates are comoving. In the interior manifold $\mathcal{M}^-$, the equation of the hypersurface Σ is defined by

$$f(t, r) = r - r_\Sigma = 0$$

where r_Σ is a constant. This implies that the vector $\partial f/\partial \chi^a_-$ is orthogonal to Σ. Therefore the unit normal vector to Σ is

$$n_a^- = [0, B(r_\Sigma, t), 0, 0] \tag{5.4.2}$$

On the hypersurface Σ we must set $dr = 0$ in (5.3.1) and when comparing with (5.4.1) we find

$$A(r_\Sigma, t)dt \quad = \quad d\tau \tag{5.4.3a}$$

$$r_\Sigma B(r_\Sigma, t) \quad = \quad \mathcal{Y}(\tau) \tag{5.4.3b}$$

for the first junction condition (5.2.1).

The extrinsic curvature $K^-_{\alpha\beta}$ can be evaluated with the quantities (5.2.2), (5.3.1) and (5.4.2). The surviving nonzero components are

$$K^-_{11} \quad = \quad \left(-\frac{1}{B}\frac{A'}{A}\right)_\Sigma \tag{5.4.4a}$$

$$K^-_{22} \quad = \quad [r(rB)']_\Sigma \tag{5.4.4b}$$

$$K^-_{33} \quad = \quad \sin^2\theta K^-_{22} \tag{5.4.4c}$$

which are valid on the stellar surface Σ. In the exterior region $\mathcal{M}^+$ the stellar surface is defined by the equation

$$f(\mathsf{r}, v) = \mathsf{r} - \mathsf{r}_\Sigma(v) = 0$$

Consequently the vector orthogonal to the stellar surface Σ is $\frac{\partial f}{\partial \chi^a_+} = \left(-\frac{d\mathsf{r}_\Sigma}{dv}, 1, 0, 0\right)$. Then the unit vector normal to the hypersurface Σ can be written as

$$n_a^+ = \left(1 - \frac{2m}{\mathsf{r}_\Sigma} + 2\frac{d\mathsf{r}_\Sigma}{dv}\right)^{-1/2}\left(-\frac{d\mathsf{r}_\Sigma}{dv}, 1, 0, 0\right) \tag{5.4.5}$$

For the exterior region $\mathcal{M}^+$ the first junction condition (5.2.1) yields the results

$$\mathsf{r}_\Sigma(v) = \mathcal{Y}(\tau) \tag{5.4.6a}$$

$$\left(1 - \frac{2m}{\mathsf{r}} + 2\frac{d\mathsf{r}}{dv}\right)_\Sigma = \left(\frac{dv}{d\tau}\right)^{-2}_\Sigma = \left(\frac{1}{\tilde{v}^2}\right)_\Sigma \tag{5.4.6b}$$

when comparing the line elements (5.3.5) and (5.4.1). Using equation (5.4.6*b*) the unit normal vector (5.4.5) can be written as

$$n_a^+ = \left(-\frac{d\mathsf{r}}{d\tau}, \frac{dv}{d\tau}, 0, 0\right) = (-\tilde{\mathsf{r}}, \tilde{v}, 0, 0) \tag{5.4.7}$$

In the above we have utilised the notation

$$\tilde{\mathsf{r}} = \frac{d\mathsf{r}}{d\tau}, \qquad \tilde{v} = \frac{dv}{d\tau}$$

The extrinsic curvature $K^+_{\alpha\beta}$ may now be evaluated from the quantities (5.2.2), (5.3.5), (5.4.6*b*) and (5.4.7). The nonvanishing components of the extrinsic curvature tensor are calculated and are given as follows

$$K^+_{11} = \left[\frac{\tilde{\tilde{v}}}{\tilde{v}} - \tilde{v}\frac{m}{\mathsf{r}^2} + \tilde{v}\frac{m_\mathsf{r}}{\mathsf{r}}\right]_\Sigma \tag{5.4.8a}$$

$$K^+_{22} = \left[\tilde{v}\left(1 - \frac{2m}{\mathsf{r}}\right)\mathsf{r} + \mathsf{r}\tilde{\mathsf{r}}\right]_\Sigma \tag{5.4.8b}$$

$$K^+_{33} = \sin^2\theta K^+_{22} \tag{5.4.8c}$$

which are valid only on the stellar surface Σ. Observe the appearance of the term containing m_r in K^+_{11} which does not exist in the treatment of Santos (1985). As we shall see later this has a profound effect on the physics of the model.

Equations (5.4.3) and (5.4.6) correspond to the first junction condition (5.2.1). Observe that the quantity τ was defined on the surface Σ as an intermediate variable. On eliminating τ we find that

$$A(r_\Sigma, t)dt = \left(1 - \frac{2m}{\mathsf{r}_\Sigma} + 2\frac{d\mathsf{r}_\Sigma}{dv}\right)^{1/2} dv \tag{5.4.9a}$$

$$\mathsf{r}_\Sigma(v) = rB(\mathsf{r}_\Sigma, t) \tag{5.4.9b}$$

Equations (5.4.9) are the necessary and sufficient conditions for the first junction condition (5.2.1) to be valid.

By equating the extrinsic curvature components (5.4.4) and (5.4.8) we generate the second set of junction conditions (5.2.3). These are given by

$$\left(-\frac{1}{B}\frac{A'}{A}\right)_\Sigma = \left[\frac{\tilde{\tilde{v}}}{\tilde{v}} - \tilde{v}\frac{m}{\mathsf{r}^2} + \tilde{v}\frac{m_\mathsf{r}}{\mathsf{r}}\right]_\Sigma \tag{5.4.10a}$$

$$(r(rB)')_\Sigma = \left[\tilde{v}\left(1 - \frac{2m}{\mathsf{r}}\right)\mathsf{r} + \mathsf{r}\tilde{\mathsf{r}}\right]_\Sigma \tag{5.4.10b}$$

The mass profile in terms of the metric functions can be generated by eliminating r, $\tilde{\mathsf{r}}$ and $\tilde{v}$ from (5.4.10b) with the help of (5.4.3) and (5.4.6). We find that

$$m(v, \mathsf{r}) = \left[\frac{rB}{2}\left(1 + r^2\frac{\dot{B}^2}{A^2} - \frac{1}{B^2}(B + rB')^2\right)\right]_\Sigma \tag{5.4.11}$$

which is the total gravitational energy contained within the stellar surface Σ. From (5.4.3a) and (5.4.6a) we can produce the relationship

$$\tilde{\mathsf{r}}_\Sigma = \left(r\frac{\dot{B}}{A}\right)_\Sigma$$

Using this expression for $\tilde{\mathsf{r}}_\Sigma$ and substituting equation (5.4.11) into the junction condition (5.4.10b) we obtain the following expression

$$\tilde{v}_\Sigma = \left(r\frac{\dot{B}}{A} + \frac{(rB)'}{B}\right)^{-1}_\Sigma \tag{5.4.12}$$

Now differentiating the above expression with respect to τ, and using the restriction $(5.4.3a)$ on the surface, we get

$$\tilde{\tilde{v}} \;=\; \left[\frac{1}{A}\left(1 + r\frac{B'}{B} + r\frac{\dot{B}}{A}\right)^{-2}\left(r\dot{B}\frac{\dot{A}}{A^2} + r\frac{B'\dot{B}}{B^2} - r\frac{\dot{B}'}{B} - r\frac{\ddot{B}}{A}\right)\right]_{\Sigma} \tag{5.4.13}$$

Then substituting (5.4.12) and (5.4.13) into the junction condition $(5.4.10a)$ and using the restrictions $(5.4.3b)$ and $(5.4.6a)$ on the surface, we get

$$\left(-\frac{1}{B}\frac{A'}{A}\right) \;=\; \left(1 + r\frac{B'}{B} + r\frac{\dot{B}}{A}\right)^{-1}$$
$$\times\left[\frac{m_{\mathsf{r}}}{\mathsf{r}} + \frac{B'}{B^2} + \frac{r}{2}\frac{B'^2}{B^3} - \frac{r}{2}\frac{\dot{B}^2}{BA^2} - \frac{1}{A}\left(r\frac{\dot{B}'}{B} + r\frac{\ddot{B}}{A} - r\frac{B'\dot{B}}{B^2} - r\dot{B}\frac{\dot{A}}{A^2}\right)\right]_{\Sigma}$$

This expression may be simplified further: multiply with $1 + r\frac{B'}{B} + r\frac{\dot{B}}{A}$ and utilise $(5.3.4b)$ and $(5.3.4d)$. We then arrive at the result

$$p = \left(qB - 2\frac{m_{\mathsf{r}}}{r^2B^2}\right)_{\Sigma}$$

which generalises the junction condition of Santos (1985).

Hence we have demonstrated that equations (5.4.10) are equivalent to

$$m(v,\mathsf{r}) \;=\; \left[\frac{rB}{2}\left(1 + r^2\frac{\dot{B}^2}{A^2} - \frac{1}{B^2}(B + rB')^2\right)\right]_{\Sigma} \tag{5.4.14a}$$

$$p \;=\; \left(qB - 2\frac{m_{\mathsf{r}}}{\mathsf{r}^2}\right)_{\Sigma} \tag{5.4.14b}$$

We have shown that (5.4.14) are the necessary and sufficient conditions for the second junction condition (5.2.3) to be valid.

5.5 Santos conditions generalised

We have generated the relationships (5.4.9) and (5.4.14) so that the junction conditions (5.2.1) and (5.2.3) are satisfied for the shear-free interior spacetime (5.3.1) and the generalised Vaidya exterior spacetime (5.3.5) across the hypersurface Σ. This generalises the Santos (1985) result for a relativistic radiating star when $m = m(v)$. Observe that when m depends on the coordinate v only then (5.4.14b) becomes

$$p = qB \tag{5.5.1}$$

at the boundary Σ, which is the earlier Santos junction condition. When (5.5.1) is valid then the pressure p on the boundary depends only on the heat flux q. We have shown here that if $m = m(v, \mathsf{r})$ then (5.4.14b) is valid, and the pressure p on the boundary depends on the heat flux q and the gradient $m_{\mathsf{r}}(v, \mathsf{r})$.

The generalised Vaidya spacetime has physical significance and contains many known solutions of the Einstein field equations with spherical symmetry. It contains the monopole solution, the de Sitter and Anti-de Sitter solutions, the charged Vaidya solution and the radiating dyon solution. The physical features and the energy momentum complexes, that provide acceptable energy momentum distributions for these systems, have been studied by Barriola and Vilenkin (1989), Chamorro and Virbhadra (1995), Virbhadra (1990a, 1990b,1999) and Yang (2007). Glass and Krisch (1998,1999) and Krisch and Glass (2005) have interpreted the generalised Vaidya spacetime to represent a superposition of an atmosphere composed of two fluids: a string fluid and a pressureless null dust fluid. This atmosphere may model several physical situations at different distance scales, eg. the exterior regions of black holes (distance scale of multiples of the Schwarzschild radius) and globular clusters containing a component of dark matter (distance scale of the order of parsecs). The additional term $2\frac{m_{\mathsf{r}}}{\mathsf{r}^2}$ in the boundary condition (5.4.14b) arises from the matching at the surface Σ. This quantity has physical significance and can be interpreted as a particular contribution from the energy momentum tensor. We observe that the term $2\frac{m_{\mathsf{r}}}{\mathsf{r}^2}$ in (5.4.14b) is the same quantity as that in (5.3.9b). Therefore we may interpret the quantity $2\frac{m_{\mathsf{r}}}{\mathsf{r}^2}$ as the string density ρ.

We can therefore write (5.4.14b) in the more transparent form

$$p = [qB - \rho]_{\Sigma} \tag{5.5.2}$$

at the boundary Σ. Consequently for a radiating star with outgoing dissipation in the form of radial heat flow, with the generalised Vaidya spacetime as the exterior, the pressure on the surface depends on the interior heat flux q and the exterior string density ρ. The appearance of the quantity ρ in (5.5.2) allows for more general behaviour that was the case in the Santos (1985) treatment. From (5.5.1) we observe that $q = 0$ implies that $p = 0$ on Σ and the exterior manifold $\mathcal{M}^+$ must be the Schwarzschild exterior metric with m being constant. In (5.5.2) we note that we obtain the Schwarzschild exterior geometry when $q = 0 = \rho$. However it is clear from (5.5.2) that when $qB = \rho$ then $p = 0$ on Σ and the exterior spacetime remains the generalised Vaidya spacetime with $m = m(v, \mathsf{r})$. In addition, when $q = 0$ then $p = -\rho$ on Σ and the interior is not radiating.

5.6 Momentum Flux

It is possible to provide a physical interpretation of our result by consideration of the momentum flux across the boundary Σ. Since the quantity (5.4.14a) represents the total gravitational energy for a sphere of radius r within Σ we can write $m(v, \mathsf{r}) = m(t, r)$. Partially differentiate (5.4.14a) with respect to t to give

$$\left(\frac{\partial m}{\partial t}\right)_\Sigma = \left(\frac{r^3}{2}\frac{\dot{B}^3}{A^2} + \frac{r^3 B\dot{B}\ddot{B}}{A^2} - \frac{r^3 B\dot{B}^2\dot{A}}{A^3} - r^2\dot{B}' - \frac{r^3 B'\dot{B}'}{B} + \frac{r^3}{2}\frac{B'^2\dot{B}}{B^2}\right)_\Sigma \tag{5.6.1}$$

Then using the interior field equations (5.3.4b) and (5.3.4d) we can write (5.6.1) as

$$\left(\frac{\partial m}{\partial t}\right)_\Sigma = -\frac{r^3}{2}\dot{B}B^2 p - \frac{r^2}{2}AB^2(B + rB')q \tag{5.6.2}$$

Now taking note of (5.4.14b) and simplifying (5.6.2) we obtain

$$\left(\frac{\partial m}{\partial t}\right)_\Sigma = -\frac{r^2}{2}\frac{pA}{\dot{v}} - \left(\frac{2m_{\mathsf{r}}}{\mathsf{r}^2}\right)\frac{r^2}{2}\frac{A}{\dot{v}} + \left(\frac{2m_{\mathsf{r}}}{\mathsf{r}^2}\right)\frac{\dot{r}r^2 A}{2} \tag{5.6.3}$$

where we have used (5.4.3a). Using the standard property of partial differentiation $dm = m_v dv + m_{\mathsf{r}} d\mathsf{r}$ we have

$$\left(\frac{\partial m}{\partial t}\right)_\Sigma = \frac{\tilde{v}}{\tilde{t}} m_v + \frac{\tilde{\mathsf{r}}}{\tilde{t}} m_{\mathsf{r}} \tag{5.6.4}$$

Finally (5.6.3) and (5.6.4) yield

$$p_\Sigma = -\frac{2}{r^2}\tilde{v}^2 m_v - \frac{2}{r^2} m_{\mathsf{r}} \tag{5.6.5}$$

which reduces to the corresponding Santos (1985) equation when $m = m(v)$.

The radial flux of momentum across the hypersurface Σ is defined by

$$F^\pm = e_0^{\pm a} n^{\pm b} T^\pm_{ab}$$

where $e_0^{\pm a}$ and $n^{\pm b}$ are vectors which are respectively tangent and normal to Σ. For conservation of momentum flux across Σ we must have

$$F^+ = F^- \tag{5.6.6}$$

In the interior manifold $\mathcal{M}^-$ we have the forms

$$e_0^{-a} = \frac{1}{A}\delta_0^a$$

$$n^{-b} = \frac{1}{B}\delta_1^b$$

and T^-_{ab} is given by (5.3.3). Then we can generate the quantity

$$F^- = -qB \tag{5.6.7}$$

In the exterior manifold $\mathcal{M}^+$ we have the forms

$$e_0^{+a} = \left(1 - \frac{2}{\mathsf{r}}m + 2\frac{d\mathsf{r}}{dv}\right)^{-1/2}\left(\delta_0^a + \frac{d\mathsf{r}}{dv}\delta_1^a\right)$$

$$n^{+b} = -\tilde{v}\delta_0^b + \left[\tilde{\mathsf{r}} + \tilde{v}\left(1 - \frac{2}{\mathsf{r}}m\right)\right]\delta_1^b$$

and T^+_{ab} is given by (6.2.11). This produces the quantity

$$F^+ = \frac{2}{r^2}\tilde{v}^2 m_v \tag{5.6.8}$$

Then (5.6.5)-(5.6.8) yields the result

$$p_\Sigma = \left(qB - \frac{2}{r^2}\frac{m_{\mathsf{r}}}{B^2}\right)_\Sigma$$

which is the same as (5.4.14b). Therefore the junction condition (5.4.14b) corresponds to the conservation of the radial flux of momentum across the hypersurface Σ. It represents the local conservation of momentum.

5.7 Surface redshift and luminosity

The junction condition given by (5.5.2) effectively relates the isotropic fluid pressure on the stellar surface to the interior heat flux and the exterior string fluid density. This relationship should have an effect on observable quantities measured on the surface of the star as well as in the surrounding region. Investigations carried out by Chan (1997, 2003) show that the surface redshift and luminosity and the asymptotic luminosity play a crucial role in understanding the formation of astrophysical black holes during the radiative gravitational collapse of a dense star. In general, the redshift of the emitted photon radiation observed on the stellar boundary is given by the equation

$$z_\Sigma = \sqrt{\frac{L_\Sigma}{L_\infty}} - 1 \tag{5.7.1}$$

where L_Σ is the luminosity of the radiation on the surface of the star and is given by

$$\begin{aligned} L_\Sigma &= -\left(\frac{dv}{d\tau}\right)^2 \frac{dm}{dv} \\ &= -\left(\frac{dv}{d\tau}\right) \frac{\partial m}{\partial t}\frac{dt}{d\tau} \end{aligned} \tag{5.7.2}$$

where $\frac{dv}{d\tau}$ is given in terms of the interior gravitational potentials by

$$\frac{dv}{d\tau} = \left(r\frac{\dot{B}}{A} + \frac{(rB)'}{B}\right)^{-1} \tag{5.7.3}$$

$\frac{\partial m}{\partial t}$ is given in terms of the isotropic pressure p and the magnitude of the heat flux q by

$$\frac{\partial m}{\partial t} = -\frac{1}{2}\left(pr^3B^2\dot{B} + qr^2AB^2(B + rB')\right) \tag{5.7.4}$$

and

$$\frac{dt}{d\tau} = \frac{1}{A} \tag{5.7.5}$$

L_∞ represents the luminosity of the radiation as measured by a stationary observer at an asymptotic distance ($r \longrightarrow \infty$) away from the radiating star and has the form

$$\begin{aligned} L_\infty &= -\frac{dm}{dv} \\ &= -\frac{\partial m}{\partial t}\frac{dt}{d\tau}\left(\frac{dv}{d\tau}\right)^{-1} \end{aligned} \tag{5.7.6}$$

With the new generalised junction condition (5.5.2), equation (5.7.4) can be rewritten in the form

$$\frac{\partial m}{\partial t} = -\frac{1}{2}pr^2B^2\left(r\dot{B} + A + r\frac{AB'}{B}\right) + \frac{1}{2}\rho r^2 AB^2\left(1 + r\frac{B'}{B}\right) \tag{5.7.7}$$

and using (5.7.7), (5.7.5) and (5.7.3) in (5.7.6) and (5.7.2) we are able to construct the general forms for the surface and asymptotic luminosities as follows

$$L_\Sigma = \frac{1}{2}r^2B^2\left[p\left(r\frac{\dot{B}}{A} + \frac{(rB)'}{B}\right) - \rho\left(1 + r\frac{B'}{B}\right)\right]\left(r\frac{\dot{B}}{A} + \frac{(rB)'}{B}\right)^{-1} \tag{5.7.8a}$$

$$L_\infty = \frac{1}{2}r^2B^2\left[p\left(r\frac{\dot{B}}{A} + \frac{(rB)'}{B}\right) - \rho\left(1 + r\frac{B'}{B}\right)\right]\left(r\frac{\dot{B}}{A} + \frac{(rB)'}{B}\right) \tag{5.7.8b}$$

It is interesting to note that in the above system L_∞ and L_Σ differ by the quantity $\left(r\frac{\dot{B}}{A} + \frac{(rB)'}{B}\right)^2$. When $\rho = 0$ we have that

$$L_\Sigma = \frac{1}{2}r^2B^2p, \qquad L_\infty = \frac{1}{2}r^2B^2p\left(r\frac{\dot{B}}{A} + \frac{(rB)'}{B}\right)^2 \tag{5.7.9}$$

which are the standard results so that $\rho \neq 0$ for the generalised Vaidya metric generates different values for the surface and asymptotic luminosities. Remarkably the form of the ratio $\frac{L_\Sigma}{L_\infty}$, remains the same as in the standard scenario, and can be written as

$$\frac{L_\Sigma}{L_\infty} = \left(\frac{dv}{d\tau}\right)^{-2} \tag{5.7.10}$$

Now with the system (5.7.8) and equation (5.7.1) we can generate the surface redshift as follows

$$z_\Sigma = \left(r\frac{\dot{B}}{A} + \frac{(rB)'}{B}\right)^{-1} - 1 \tag{5.7.11}$$

Observe that the structural form of the surface redshift (5.7.11) does not change from the standard form; however the gravitational potentials A and B in (5.7.11) are different from the interior potentials as they now satisfy the Einstein field equations as well as a *new* junction condition, namely equation (5.5.2). In view of this fact it would be appropriate to now relabel the new set of metric potentials in the following way

$$A \longrightarrow X, \qquad\qquad B \longrightarrow Y$$

and with the above, equation (5.7.11) may be recast as

$$z_\Sigma = \left(r\frac{\dot{Y}}{X} + \frac{(rY)'}{Y} \right)^{-1} - 1 \tag{5.7.12}$$

This distinguishes the surface redshifts for the standard Vaidya and generalised Vaidya metrics. We can consider (5.7.11) as a special case of (5.7.12).

We can summarise our results as follows:

- In the limit when $\rho = 0$ (when $p = qB$), we regain the standard forms for the surface and asymptotic luminosities.
- A stationary observer, located at some asymptotic distance ($r \longrightarrow \infty$) away from the star, observes a much weaker luminosity signal when the atmosphere of the star contains the string fluid (when compared with the standard Vaidya exterior).
- The surface redshift for the generalised Vaidya metric contains the standard result even though the structural form for the formula remains the same in both cases.

5.8 An exact solution

We now turn our attention to relativistic stellar models in which the null fluid particles move along geodesics from the core and up through to the stellar surface where the radiation is lost to the exterior. Such models were investigated by Govender and Thirukkanesh (2009), Rajah and Maharaj (2008) and Thirukkanesh and Maharaj

(2009). In particular the Govender and Thirukkanesh model studied a radiating star undergoing geodesic heat flow in the presence of the cosmological constant. This situation may be interpreted as a star dissipating energy while immersed in a spacetime background filled with a fluid having negative pressure. These works were carried out using the standard Santos junction condition $p = qB$. We would now like to extend these models by using the new generalised junction condition (5.5.2), and making use of the fact that the stellar atmosphere is now a well defined two-fluid system. For our investigations, we consider the string fluid density ρ to be constant on the stellar surface and as we will see in the next chapter that this is not physically unreasonable as this situation corresponds to the diffusion of the string fluid.

For geodesic motion of fluid particles in heat dissipation, the gravitational potential $A = 1$. The fluid pressure and radial heat flux are given by

$$p = \left(-2\frac{\ddot{B}}{B} - \frac{\dot{B}^2}{B^2}\right) + \frac{1}{B^2}\left(\frac{B'^2}{B^2} + \frac{2}{r}\frac{B'}{B}\right) \tag{5.8.1a}$$

$$q = -\frac{2}{B^2}\left(-\frac{\dot{B}'}{B} + \frac{\dot{B}B'}{B^2}\right) \tag{5.8.1b}$$

from (5.3.4b) and (5.3.4d) respectively. The condition of pressure isotropy admits the following analytical form

$$B(r,t) = -\frac{d}{c_1(t)r^2 - c_2(t)} \tag{5.8.2}$$

for the gravitational potential B. Here d is an arbitrary constant and $c_1(t)$ and $c_2(t)$ are functions of integration which have to be determined in order to complete the exact solution. With the above system (5.8.1) and the form (5.8.2) the generalised junction condition given by (5.5.2) may be written as

$$-4db(\dot{c_1}c_2 - c_1\dot{c_2})(c_1b^2 - c_2) - 4c_1c_2(c_1b^2 - c_2)^2 - 2d^2(\ddot{c_1}b^2 - \ddot{c_2})(c_1b^2 - c_2)$$
$$+5d^2(\dot{c_1}b^2 - \dot{c_2})^2 - \rho d^2(c_1b^2 - c_2)^2 = 0 \tag{5.8.3}$$

where we have taken $r = r_\Sigma = b =$ constant, on the stellar surface. It is important to note the presence of the additional term that arises due to the presence of the string fluid density $\rho \neq 0$. The density ρ is taken to be constant on the stellar boundary and this corresponds to the diffusion of the string fluid in the exterior of the star. In the limit when the string density goes to zero, we regain the earlier equation obtained by Thirukkanesh and Maharaj (2009). To integrate (5.8.3) we make use of the new transformation

$$u(t) = c_1 b^2 - c_2 \tag{5.8.4}$$

Then equation (5.8.3) can be rewritten as

$$4bdu^2\dot{c_1} + 4(u^2 - bd\dot{u})uc_1 - 4b^2u^2c_1^2 = d^2\left[(2u\ddot{u} - 5\dot{u}^2) + \rho u^2\right] \tag{5.8.5}$$

Equation (5.8.5) is a Riccati equation in c_1 but is still difficult to solve in general. If we let $u = \alpha$ (constant) then (5.8.5) becomes

$$\dot{c_1} + \frac{\alpha}{bd}c_1 - \frac{b}{d}c_1^2 = \frac{d}{4b}\rho \tag{5.8.6}$$

We now make use of the transformation

$$c_1 = -\frac{d}{b}\frac{\dot{w}}{w} \tag{5.8.7}$$

where $w(t)$ is an arbitrary function. Then equation (5.8.6) becomes

$$\ddot{w} + \frac{\alpha}{bd}\dot{w} + \frac{\rho}{4}w = 0 \tag{5.8.8}$$

which is a second order linear ordinary differential equation with constant coefficients and can be easily integrated. The general solution to equation (5.8.8) is given by

$$w(t) = g_1(t)\exp\left[\lambda_1 t\right] + g_2(t)\exp\left[\lambda_2 t\right] \tag{5.8.9}$$

In the above solution $g_1(t)$ and $g_2(t)$ are functions of integration and

$$\lambda_1 = \frac{1}{2}\left(\sqrt{\frac{\alpha^2}{b^2d^2} - \rho} - \frac{\alpha}{bd}\right), \qquad \lambda_2 = -\frac{1}{2}\left(\sqrt{\frac{\alpha^2}{b^2d^2} - \rho} + \frac{\alpha}{bd}\right)$$

Then the functions $c_1(t)$ and $c_2(t)$ become

$$c_1(t) = -\frac{d}{2b}\left[\frac{g_1(t)\left(\eta - \frac{\alpha}{bd}\right) - g_2(t)\left(\frac{\alpha}{bd} + \eta\right)\exp(-\eta t)}{g_1(t) + g_2(t)\exp(-\eta t)}\right] \tag{5.8.10a}$$

$$c_2(t) = -\frac{bd}{2}\left[\frac{g_1(t)\left(\eta - \frac{\alpha}{bd}\right) - g_2(t)\left(\frac{\alpha}{bd} + \eta\right)\exp(-\eta t)}{g_1(t) + g_2(t)\exp(-\eta t)}\right] - \alpha \tag{5.8.10b}$$

Consequently the gravitational potential B has the form

$$B(r,t) = \frac{-2db}{d\left[\frac{g_1(t)\left(\eta - \frac{\alpha}{bd}\right) - g_2(t)\left(\frac{\alpha}{bd} + \eta\right)\exp(-\eta t)}{g_1(t) + g_2(t)\exp(-\eta t)}\right](b^2 - r^2) + 2b\alpha} \tag{5.8.11}$$

and the metric has the form

$$ds^2 = -dt^2 + \frac{4d^2b^2}{\left[\Omega(t)(b^2 - r^2)d + 2b\alpha\right]^2}\left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] \tag{5.8.12}$$

where $\Omega(t)$ is an arbitrary function given by

$$\Omega(t) = \frac{g_1(t)\left(\eta - \frac{\alpha}{bd}\right) - g_2(t)\left(\frac{\alpha}{bd} + \eta\right)\exp(-\eta t)}{g_1(t) + g_2(t)\exp(-\eta t)} \tag{5.8.13}$$

in terms of $g_1(t)$ and $g_2(t)$.

Our new exact solution (5.8.12) is similar, in structure, to the solution found by Govender and Thirukkanesh (2009). However our model results from a different physical scenario since the atmosphere of our star does not contain the cosmological constant Λ, but a two-fluid system in which one of the components is a string fluid. This model corresponds to geodesic heat dissipation in a relativistic star when the string fluid in the stellar atmosphere is undergoing diffusion. The solution (5.8.12) can now be used in the framework of irreversible causal and noncausal thermodynamics, as in the previous chapters, to study the temperature evolution of the radiating star. This is ongoing research.

5.9 Discussion

In this chapter we have produced a general model of a relativistic radiating star by performing the smooth matching of a shear-free interior spacetime to the generalised Vaidya exterior spacetime, across a timelike spatial hypersurface. We have demonstrated that with the generalised Vaidya radiating metric, the junction conditions on the stellar surface change substantially, and consequently represents a more general atmosphere surrounding the star. The atmosphere is a superposition of the pressureless null dust and a string fluid. We find that the density of the string affects the fluid pressure at the stellar boundary. We have shown explicitly that

$$p = qB - \rho_{string}$$

at the stellar surface. If the weak and strong energy conditions or the dominant energy conditions are satisfied then $\rho_{string} \geq 0 \quad (\mu \neq 0)$ and $\rho_{string} \geq P_{string} \geq 0 \quad (\mu \neq 0)$ respectively. This indicates that for outgoing heat flux in gravitational collapse, the string density reduces the pressure on the stellar boundary. It is interesting to note that we have shown using a geometric approach that the derivative of the mass function with respect to the exterior radial coordinate is related to the string density.

We have also demonstrated the importance and impact that the generalised junction condition has in constructing physically viable star models, as well as its crucial role in describing the physics of relativistic stellar atmospheres. Our new junction condition has been directly applied in the construction of the luminosity and redshift profiles on the stellar surface as well as in the local atmosphere of the star. We have shown explicitly, that these luminosities have a different form from the standard results. It was also found that the surface redshift of the emitted null radiation is affected by the new junction condition since the metric potentials are different. The metric potentials have to satisfy the new generalised condition on the stellar boundary.

The generalised junction condition has also been used to extend the models of Govender and Thirukkanesh (2009) and Thirukkanesh and Maharaj (2009). We have extended the model of a radiating star undergoing geodesic heat flow in the presence of a generalised atmosphere. An exact solution to the generalised junction condition (5.5.2) was found in terms of elementary functions. Even though the form of the solution is the same as Govender and Thirukkanesh (2009) we are in a position to model a two-fluid atmosphere with a constant string energy density. In future work we intend to relate our results to astrophysical objects.

Chapter 6

Generalised junction conditions with shear

6.1 Introduction

Exact solutions of the Einstein field equations describing stellar configurations have formed an active area of research within the framework of relativistic astrophysics. The discovery of the Schwarzschild metrics (1916a, 1916b) have enabled researchers to model various static stars with a wide spectrum of interior matter distributions including perfect fluid sources, charged interiors and anisotropic matter. However, these static models may represent only a small part of a star's evolution. Observations indicate that stars in general are continuously radiating energy to the exterior spacetime while undergoing gravitational collapse. With the discovery of the exterior Vaidya radiating solution (1951), it became possible to model a dynamically unstable, relativistic star emitting null radiation across the stellar surface. The Vaidya solution is a unique spherically symmetric solution of the Einstein field equations which describes a pure radiation atmosphere. The junction conditions required for the complete description of a spherically symmetric, shear-free stellar core undergoing nonadiabatic collapse with a radiation atmosphere was first provided by Santos (1985). In this scenario, an imperfect contracting sphere with radial heat flow is matched to Vaidya's outgoing metric across a timelike hypersurface. The main feature of the junction conditions require that the pressure at the boundary of the star be nonzero which differed from static interiors matched to the exterior Schwarzschild solution. In the latter, the pressure at the boundary was required to vanish as there was no heat flux across the surface of the star.

The Santos junction conditions have subsequently been generalised to include shear, electromagnetic field, bulk viscosity and nonsphericity. These models have produced a rich vein of physically tractable stellar models including acceleration-free collapse, collapse from an initial static configuration, gamma-ray bursts, expansion free collapse and Euclidean stars. The physical viability of these models have been extensively studied within the framework of extended irreversible thermodynamics. Relaxational effects have been highlighted within the stellar interior, particularly in the late stages of collapse.

In the previous chapter the Santos junction conditions were generalised to describe stars that have a two-fluid atmosphere. The exterior of such a star is described by the generalised Vaidya solution in which the dynamical mass is a function of both the temporal and radial coordinate. The generalised Vaidya solution has been widely employed in the study of the end state of gravitational collapse. It was shown that the energy momentum tensor that is consistent with the generalised Vaidya solution consists of a two-component fluid of strings and null radiation. In the previous chapter we have shown that the matching of a general spherically symmetric, shear-free radiating stellar interior to the exterior generalised Vaidya spacetime requires that the pressure on the boundary be nonvanishing, similar to the Santos junction conditions. The main difference is that the string density makes a contribution at the boundary. This is an important result which highlights the role of the string component of the atmosphere on the internal dynamics. We now aim to extend the results of the previous chapter to include the effects of shear in the interior of the radiating model.

In §6.2 we present the generalised junction condition for a radiating star with non-vanishing shearing stresses. The Lichnerowiscz and O' Brien and Synge conditions are studied in detail in §6.3. The junction conditions are derived using a geometric argument. In §6.4 we study some physical features of the generalised boundary condition and investigate models of isotropic strings and string fluid diffusion. Finally, in §6.5 we discuss our results.

6.2 Generalised junction conditions

The stellar interior is taken to be the most general relativistic fluid having nonzero shear, expansion and acceleration and is described by the line element

$$ds^2 = -A^2dt^2 + B^2dr^2 + R^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{6.2.1}$$

Here A, B and R are the metric potentials describing the gravitational field inside the star and are functions of the interior comoving coordinates t and r. The interior stellar fluid is described by the following energy momentum tensor

$$T^{-}_{ab} = (\mu + p)u_a u_b + p g_{ab} + \pi_{ab} + q_a u_b + q_b u_a \tag{6.2.2}$$

where μ and p are the energy density and isotropic pressure respectively, u_a and q_a are the fluid four-velocity and the heat flux vectors respectively, and π_{ab} is the anisotropic stress (pressure) tensor. In general the anisotropic pressure tensor has the following form

$$\pi_{ab} = (p_r - p_T)\left(n_a n_b - \frac{1}{3}h_{ab}\right) \tag{6.2.3}$$

Here p_r is the radial component of the interior pressure, p_T is the tangential pressure component, and $\boldsymbol{n}$ is a unit radial vector given by

$$n^a = \frac{1}{B}\delta^a_1 \tag{6.2.4}$$

The interior isotropic fluid pressure p is related to the radial and tangential components of pressure by the equation

$$p = \frac{1}{3}\left[p_r + 2p_T\right]$$

The kinematical quantities associated with the metric (6.2.1) may be given as

$$\sigma = \frac{1}{A}\left(\frac{\dot{B}}{B} - \frac{\dot{R}}{R}\right) \tag{6.2.5a}$$

$$\Theta = \frac{1}{A}\left(\frac{\dot{B}}{B} + 2\frac{\dot{R}}{R}\right) \tag{6.2.5b}$$

$$a = \frac{A'}{A} \tag{6.2.5c}$$

where σ is the shear, Θ is the scalar expansion and a is the fluid acceleration. The energy momentum tensor (6.2.2) admits the following nonvanishing components

$$T^{-}_{00} = \mu A^2 \tag{6.2.6a}$$

$$T^{-}_{01} = -qAB \tag{6.2.6b}$$

$$T^{-}_{11} = p_r B^2 \tag{6.2.6c}$$

$$T^{-}_{22} = p_T R^2 \tag{6.2.6d}$$

$$T^{-}_{33} = \sin^2\theta T^{-}_{22} \tag{6.2.6e}$$

and we may write the nonzero components of the Einstein curvature tensor as

$$G^{-}_{00} = 2\frac{\dot{B}}{B}\frac{\dot{R}}{R} + \frac{\dot{R}^2}{R^2} - \frac{A^2}{B^2}\left[2\frac{R''}{R} + \frac{R'^2}{R^2} - 2\frac{B'}{B}\frac{R'}{R} - \frac{B^2}{R^2}\right] \tag{6.2.7a}$$

$$G^{-}_{01} = -2\left(\frac{\dot{R}'}{R} - \frac{\dot{B}}{B}\frac{R'}{R} - \frac{\dot{R}}{R}\frac{A'}{A}\right) \tag{6.2.7b}$$

$$G^{-}_{11} = -\frac{B^2}{A^2}\left[2\frac{\ddot{R}}{R} - \left(2\frac{\dot{A}}{A} - \frac{\dot{R}}{R}\right)\frac{\dot{R}}{R}\right] + \left(2\frac{A'}{A} + \frac{R'}{R}\right)\frac{R'}{R} - \frac{B^2}{R^2} \tag{6.2.7c}$$

$$\begin{aligned} G^{-}_{22} &= -\frac{R^2}{A^2}\left[\frac{\ddot{B}}{B} + \frac{\ddot{R}}{R} - \frac{\dot{A}}{A}\left(\frac{\dot{B}}{B} + \frac{\dot{R}}{R}\right) + \frac{\dot{B}}{B}\frac{\dot{R}}{R}\right] \\ &\quad + \frac{R^2}{B^2}\left[\frac{A''}{A} + \frac{R''}{R} - \frac{A'}{A}\frac{B'}{B} + \left(\frac{A'}{A} - \frac{B'}{B}\right)\frac{R'}{R}\right] \end{aligned} \tag{6.2.7d}$$

$$G^{-}_{33} = \sin^2\theta G^{-}_{22} \tag{6.2.7e}$$

where dots and primes represent derivatives with respect to the coordinates t and r respectively. With the systems (6.2.6) and (6.2.7) the Einstein field equations $G^{-}_{ab} = T^{-}_{ab}$ for the stellar interior become

$$\mu = \frac{2}{A^2}\frac{\dot{B}}{B}\frac{\dot{R}}{R} + \frac{1}{R^2} + \frac{1}{A^2}\frac{\dot{R}^2}{R^2} - \frac{1}{B^2}\left(2\frac{R''}{R} + \frac{R'^2}{R^2} - 2\frac{B'}{B}\frac{R'}{R}\right) \tag{6.2.8a}$$

$$p_r = -\frac{1}{A^2}\left[2\frac{\ddot{R}}{R} - 2\frac{\dot{A}}{A}\frac{\dot{R}}{R} + \frac{\dot{R}^2}{R^2}\right] + \frac{1}{B^2}\left[2\frac{A'}{A}\frac{R'}{R} + \frac{R'^2}{R^2}\right] - \frac{1}{R^2} \tag{6.2.8b}$$

$$\begin{aligned} p_T &= -\frac{1}{A^2}\left[\frac{\ddot{B}}{B} + \frac{\ddot{R}}{R} - \frac{\dot{A}}{A}\frac{\dot{B}}{B} - \frac{\dot{A}}{A}\frac{\dot{R}}{R} + \frac{\dot{B}}{B}\frac{\dot{R}}{R}\right] \\ &\quad + \frac{1}{B^2}\left[\frac{A''}{A} + \frac{R''}{R} - \frac{A'}{A}\frac{B'}{B} + \frac{A'}{A}\frac{R'}{R} - \frac{B'}{B}\frac{R'}{R}\right] \end{aligned} \tag{6.2.8c}$$

$$q = -\frac{2}{AB^2}\left(-\frac{\dot{R}'}{R} + \frac{\dot{B}}{B}\frac{R'}{R} + \frac{\dot{R}}{R}\frac{A'}{A}\right) \tag{6.2.8d}$$

Note that using the system (6.2.5), the heat flux (6.2.8d) may be rewritten in terms of the dynamical quantities as follows

$$qB = \frac{1}{3}(\Theta - \sigma)' - \sigma\frac{R'}{R} \tag{6.2.9}$$

We observe that in the above equation (6.2.9), if the interior stellar fluid is not shearing and expanding ($\sigma = 0, \Theta = 0$), then the heat flux q must vanish and the star is not radiating. The stellar interior may be nonradiating ($q = 0$) but can still be shearing and expanding.

The exterior spacetime of the star is described by the generalised Vaidya line element

$$ds^2 = -\left(1 - 2\frac{m(v, \mathsf{r})}{\mathsf{r}}\right) dv^2 - 2dvd\mathsf{r} + \mathsf{r}^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{6.2.10}$$

where $m(v, \mathsf{r})$ is the mass of the star as observed at infinity. The atmosphere of the star is considered to be a two-fluid system consisting of a pressureless null dust and a null string fluid and is defined by the following energy momentum tensor

$$T^{(+)}_{ab} = \tilde{\mu} l_a l_b + (\rho + P)(l_a n_b + l_b n_a) + P g_{ab} \tag{6.2.11}$$

where $\tilde{\mu}$ is the energy density of the null dust and ρ and P are the energy density and pressure of the null string fluid respectively. Wang and Wu (1999) have provided a detailed description and analysis of the above energy momentum tensor (6.2.11), for relativistic stars having an atmosphere that is described by the generalised Vaidya radiating metric. The nonvanishing components of the Einstein curvature tensor for the metric (6.2.10) are given as follows

$$G^{+}_{00} = -\frac{2}{\mathsf{r}^2} m_v \tilde{v}^2 + 2\frac{(\mathsf{r} - 2m)}{\mathsf{r}^3} m_{\mathsf{r}} \tag{6.2.12a}$$

$$G^{+}_{01} = \frac{2}{\mathsf{r}^2} m_{\mathsf{r}} \tag{6.2.12b}$$

$$G^{+}_{22} = -\mathsf{r} m_{\mathsf{rr}} \tag{6.2.12c}$$

$$G^{+}_{33} = \sin^2\theta G^{+}_{22} \tag{6.2.12d}$$

and the relevant components of the energy momentum tensor for (6.2.10) are given by

$$T^+_{00} = \tilde{\mu} + \rho - \frac{2m}{\mathsf{r}}\rho \tag{6.2.13a}$$

$$T^+_{01} = \rho \tag{6.2.13b}$$

$$T^+_{22} = \mathsf{r}^2 P \tag{6.2.13c}$$

Using the systems (6.2.12) and (6.2.13) the Einstein field equations $G^+_{ab} = T^+_{ab}$ for the exterior manifold $\mathcal{M}^+$ may be written as

$$\tilde{\mu} = -2\frac{m_v}{\mathsf{r}^2}\tilde{v}^2 \tag{6.2.14a}$$

$$\rho = 2\frac{m_{\mathsf{r}}}{\mathsf{r}^2} \tag{6.2.14b}$$

$$P = -\frac{m_{\mathsf{rr}}}{\mathsf{r}} \tag{6.2.14c}$$

which is of the same form (5.3.9).

In general, the junction conditions can be derived by carrying out the smooth matching of the interior and exterior spacetime geometries across the stellar surface Σ following the procedure in chapter 5. This process requires the use of the first and second fundamental forms. The first fundamental form is written as

$$(ds^2_+)_\Sigma = (ds^2_-)_\Sigma = ds^2_\Sigma \tag{6.2.15}$$

and relates the interior and exterior spacetime metrics on the stellar surface Σ. We can show that condition (6.2.15) yields the following equations that hold on the stellar surface

$$A(r_\Sigma, t)dt = \left(1 - \frac{2m}{\mathsf{r}_\Sigma} + 2\frac{d\mathsf{r}_\Sigma}{dv}\right)^{1/2} dv \tag{6.2.16a}$$

$$\mathsf{r}_\Sigma(v) = rB(\mathsf{r}_\Sigma, t) \tag{6.2.16b}$$

The extrinsic curvature for the interior and exterior spacetimes are given by the second fundamental form

$$(K^+_{\alpha\beta})_\Sigma = (K^-_{\alpha\beta})_\Sigma \tag{6.2.17}$$

where

$$K^\pm_{\alpha\beta} \equiv -n^\pm_a \frac{\partial^2 \chi^a_\pm}{\partial\xi^\alpha \partial\xi^\beta} - n^\pm_a \Gamma^a{}_{bc} \frac{\partial\chi^b_\pm}{\partial\xi^\alpha} \frac{\partial\chi^c_\pm}{\partial\xi^\beta} \tag{6.2.18}$$

Making use of the first and second fundamental forms (6.2.15) and (6.2.17) and performing lengthy calculations as in chapter 5, we are able to generate the junction conditions which are valid on the stellar surface Σ. They are written as follows

$$m(r,t) = \frac{R}{2}\left[1 + \left(\frac{\dot{R}}{A}\right)^2 - \left(\frac{R'}{B}\right)^2\right] \tag{6.2.19a}$$

$$p_r = qB - 2\frac{m_\mathsf{r}}{\mathsf{r}^2} \tag{6.2.19b}$$

We can also generate these junction conditions using the Lichnerowiscz (1955) and O' Brien and Synge (1952) conditions.

6.3 Derivation of the master equation

6.3.1 The Lichnerowiscz junction condition

The Lichnerowiscz (1955) approach is the first part of a geometric method that can be used to derive the junction condition that relates the matter variables on the stellar surface. A comprehensive discussion of this technique, for the modelling of a radiating star, is provided by de Oliveira *et al* (1987). The Lichnerowiscz junction condition may be written in general as

$$\left[G_{\alpha\beta}\eta^{\alpha}\eta^{\beta}\right] = 0$$

This can be written more precisely as

$$G^{-}_{\alpha\beta}\eta^{\alpha}_{-}\eta^{\beta}_{-} = G^{+}_{\alpha\beta}\eta^{\alpha}_{+}\eta^{\beta}_{+} \tag{6.3.1}$$

The left hand side of equation (6.3.1) represents the relationship between the geometry of the interior manifold and the three dimensional spatial hypersurface and the right hand side represents the relationship between that of the exterior manifold and the hypersurface. $G^{\pm}_{\alpha\beta}$ are the intrinsic and extrinsic components of the Einstein curvature tensor and $\eta^{\alpha}_{\pm}$ are intrinsic and extrinsic vectors that are tangent to the stellar surface.

In the stellar interior the components of the Einstein curvature tensor are given by the system (6.2.7) and the interior tangent vector has the form

$$\eta^{\alpha}_{-} = \left(0, \frac{1}{B}, 0, 0\right) \tag{6.3.2}$$

The left hand side of the Lichnerowiscz condition in equation (6.3.1) may be written as

$$\begin{aligned} G^{-}_{\alpha\beta}\eta^{\alpha}_{-}\eta^{\beta}_{-} &= G^{-}_{11}\eta^{1}_{-}\eta^{1}_{-} \\ &= p_r \end{aligned} \tag{6.3.3}$$

For the exterior spacetime $\mathcal{M}^{+}$ the Einstein curvature tensor components are given by the system (6.2.12) and the exterior tangent vector has the form

$$\eta^{\alpha}_{+} = \left(-\tilde{v}, \tilde{\mathsf{r}} + \left(1 - 2\frac{m}{\mathsf{r}}\right)\tilde{v}, 0, 0\right) \tag{6.3.4}$$

In the above we have utilised the notation

$$\tilde{\mathsf{r}} = \frac{d\mathsf{r}}{d\tau}, \qquad \tilde{v} = \frac{dv}{d\tau}$$

where τ is a timelike coordinate on the spatial hypersurface. The exterior Lichnerowiscz condition on the right hand side of equation (6.3.1) may be written as

$$
\begin{aligned}
G^{+}_{\alpha\beta}\eta^{\alpha}_{+}\eta^{\beta}_{+} &= G^{+}_{00}\eta^{0}_{+}\eta^{0}_{+} + G^{+}_{01}\eta^{0}_{+}\eta^{1}_{+} + G^{+}_{10}\eta^{1}_{+}\eta^{0}_{+} + G^{+}_{11}\eta^{1}_{+}\eta^{1}_{+} \\
&= -\frac{2}{\mathsf{r}^2}\frac{dm}{dv}\tilde{v}^2 - 2\frac{m_{\mathsf{r}}}{\mathsf{r}^2} \qquad (6.3.5)
\end{aligned}
$$

With (6.3.3) and (6.3.5) the Lichnerowiscz condition given by equation (6.3.1) can be written as

$$
p_r = -\frac{2}{\mathsf{r}^2}\frac{dm}{dv}\tilde{v}^2 - 2\frac{m_{\mathsf{r}}}{\mathsf{r}^2} \qquad (6.3.6)
$$

which defines the radial fluid pressure.

6.3.2 The O' Brien and Synge junction condition

The O' Brien and Synge (1952) approach is the second part of the geometric method that can be used to derive the general junction condition that is valid on the stellar surface. In general the O' Brien and Synge junction condition can be written as

$$
\left[G_{\alpha\beta}l^{\alpha}\eta^{\beta}\right] = 0
$$

and this may be written more precisely as

$$
G^{-}_{\alpha\beta}l^{\alpha}_{-}\eta^{\beta}_{-} = G^{+}_{\alpha\beta}l^{\alpha}_{+}\eta^{\beta}_{+} \qquad (6.3.7)
$$

The O' Brien and Synge condition relates the geometry of the interior and exterior manifolds to the spatial hypersurface and is defined in terms of two distinct intrinsic and extrinsic tangent vectors $l^{\alpha}_{\pm}$ and $\eta^{\beta}_{\pm}$. For the interior manifold we make use of the tangent vector given by (6.3.2) and the new vector l^{α}_{-}:

$$
\eta^{\alpha}_{-} = \left(0, \frac{1}{B}, 0, 0\right), \qquad l^{\alpha}_{-} = \left(\tilde{t}, 0, 0, 0\right) \qquad (6.3.8)
$$

where

$$
\tilde{t} = \frac{dt}{d\tau}
$$

We may write the O' Brien and Synge condition for the stellar interior in the following way

$$\begin{aligned} G^{-}_{\alpha\beta} l^{\alpha}_{-} \eta^{\beta}_{-} &= G^{-}_{00} l^{0}_{-} \eta^{0}_{-} + G^{-}_{01} l^{0}_{-} \eta^{1}_{-} + G^{-}_{10} l^{1}_{-} \eta^{0}_{-} + G^{-}_{11} l^{1}_{-} \eta^{1}_{-} \\ &= -qB \end{aligned} \tag{6.3.9}$$

In a similar manner the right hand side of equation (6.3.7) for the exterior O' Brien and Synge condition may be written in terms of the exterior tangent vectors given by (6.3.4) and l^{α}_{+}:

$$\eta^{\alpha}_{+} = \left(-\tilde{v}, \mathsf{r} + \left(1 - 2\frac{m}{\mathsf{r}}\right)\tilde{v}, 0, 0\right), \qquad l^{\alpha}_{+} = \left(\tilde{v}, \frac{1}{\tilde{v}} - \mathsf{r} - \tilde{v}\left(1 - 2\frac{m}{\mathsf{r}}\right), 0, 0\right) \tag{6.3.10}$$

The exterior O' Brien and Synge condition can now be written in expanded form as

$$\begin{aligned} G^{+}_{\alpha\beta} l^{\alpha}_{+} \eta^{\beta}_{+} &= G^{+}_{00} l^{0}_{+} \eta^{0}_{+} + G_{01} l^{0}_{+} \eta^{1}_{+} + G^{+}_{10} l^{1}_{+} \eta^{0}_{+} \\ &= \frac{2}{\mathsf{r}^2}\frac{dm}{dv}\tilde{v}^2 \end{aligned} \tag{6.3.11}$$

We can now write down the form of the O' Brien and Synge condition by utilising equations (6.3.9) and (6.3.11) which yields

$$-qB = \frac{2}{\mathsf{r}^2}\frac{dm}{dv}\tilde{v}^2 \tag{6.3.12}$$

which defines the heat flux.

6.3.3 The general junction condition

The Lichnerowiscz and O' Brien and Synge conditions can now be combined to generate the general junction condition on the stellar surface. With the Lichnerowiscz condition we obtained the equation

$$p_r = -\frac{2}{\mathsf{r}^2}\frac{dm}{dv}\tilde{v}^2 - 2\frac{m_{\mathsf{r}}}{\mathsf{r}^2} \tag{6.3.13}$$

It is clear that the Lichnerowiscz condition relates the radial pressure in the stellar interior to the energy densities of the null dust and null string fluid in the stellar exterior. The interior radial pressure is balanced by the exterior energy densities on the boundary of the star. The O' Brien and Synge condition generated the equation

$$-qB = \frac{2}{\mathsf{r}^2}\frac{dm}{dv}\tilde{v}^2 \tag{6.3.14}$$

It can be seen that on the stellar boundary the interior heat flux is equal to the exterior null dust energy density. This confirms the fact that the null dust that is present in the atmosphere of a radiating star is due only to the presence of a nonzero interior heat source, namely the radial heat flux. Relating (6.3.13) and (6.3.14) we can generate the junction condition

$$p_r = qB - 2\frac{m_{\mathsf{r}}}{\mathsf{r}^2} \tag{6.3.15}$$

Note that this is precisely the result (6.2.19*b*) that we have obtained by performing the formal matching of the interior and exterior spacetimes across the stellar surface obtained by following the general procedure as in chapter 5. Observe that the second term on the right hand side of equation (6.3.15) corresponds to the string density ρ in the exterior Einstein field equation (6.2.14*b*) hence equation (6.3.15) may be written in the more compact form

$$p_r = qB - \rho \tag{6.3.16}$$

This result has the same form as in chapter 5 but in this case the shear is nonzero, and consequently (6.3.16) is a *new* differential equation.

6.4 Some qualitative features

The junction condition (6.3.16) is crucial in modelling a relativistic radiating star, as it explicitly relates the interior matter variables to the exterior matter variables on the stellar boundary. For an interior stellar fluid that has nonzero shear, the radial pressure is balanced by the interior heat flux as well as the energy density of the exterior string fluid. It is possible that for a shearing anisotropic stellar fluid, the radial component of pressure may be zero ($p_r = 0$). In such a case the above junction condition demands that the interior heat flux is balanced by the exterior string density

$$qB = \rho \tag{6.4.1}$$

In this case, although the pressure on the surface is zero the interior of the star is still radiating and the stellar atmosphere is still a two-fluid system. When the stellar interior is not radiating the radial heat flux must vanish ($q = 0$) and this results in the radial pressure balancing the string density on the surface,

$$p_r = -\rho \tag{6.4.2}$$

Here the pressure must decrease towards the boundary and is possibly suppressed by the density of the exterior string fluid.

6.4.1 Two-fluid models

A number of radiating stellar models for which the interior is shearing and anisotropic and the atmosphere is a two-fluid system have been investigated. Glass and Krisch (1998, 1999) and Krisch and Glass (2005) have extensively studied various physical situations for the dynamics of the atmosphere which is described by the generalised Vaidya radiating metric and the resulting null dust and string fluid system. These treatments were carried out for general spherically symmetric spacetimes; Ghosh and Deshkar (2010) have subsequently extended and generalised these models to include plane symmetric as well as cylindrically symmetric spacetime geometries. In these treatments the authors concentrated only on the physics of the interior *or* the exterior of the radiating star. We have shown that the junction conditions are satisfied if (6.3.16) holds, and now the model can be treated as a single system.

6.4.2 Isotropic string fluid

An isotropic string fluid is described as a cloud of strings for which the string pressure is isotropic, i.e. $P_\mathsf{r} = P_\perp$. This condition of pressure isotropy results in the following differential equation for the string fluid mass

$$\frac{m_{\mathsf{rr}}}{2\mathsf{r}} = \frac{m_\mathsf{r}}{\mathsf{r}^2}$$

which upon integration admits the radial mass profile

$$m(v, \mathsf{r}) = \mathsf{r}^3 c_1(v) + c_2(v) \tag{6.4.3}$$

Using the above mass (6.4.3) the junction condition (6.2.19b) in terms of the exterior radial mass gradient m_r becomes

$$p_r = qB - 6c_1(v) \tag{6.4.4}$$

This is the governing equation that has to be solved in order to provide a more complete model of an isotropic string atmosphere.

6.4.3 Diffusive transport

It is possible to model the flow of a string fluid in terms of a diffusive transport process. Diffusion has been used in the description of cosmic strings by Vilenkin (1981). It is also largely understood that diffusive processes may play a pivotal role in understanding quantum gravitational fluctuations (Percival 1995, Percival and Strunz 1997), particularly in the very early stages of the universe. For diffusion of the exterior string fluid, the string number density n and the string fluid density ρ are related by the following equation

$$\rho = M_s n$$

where M_s is the constant string element mass. The governing diffusion equation for the string fluid is written as

$$\dot{m} = 4\pi\mathcal{D}\mathsf{r}^2 \frac{\partial \rho}{\partial \mathsf{r}} \tag{6.4.5}$$

Here $\mathcal{D}$ is the positive coefficient of self diffusion, which is taken to be constant. It has been shown by Glass and Krisch (1999) that the above diffusion equation can be solved for ρ and the resulting solutions then integrated to generate forms for m. These analytical forms for ρ and m are exact solutions to the exterior Einstein field equations in the region outside the star and may be interpreted as either anisotropic fluids or diffusing string fluids. The above mentioned solutions are given below

$$\rho = \rho_0 + k_1/\mathsf{r} \tag{6.4.6a}$$

$$\rho = \rho_0 + (k_2/6)\mathsf{r}^2 + k_2\mathcal{D}v \tag{6.4.6b}$$

$$\rho = \rho_0 + k_3(\mathcal{D}v)^{-3/2}\exp[-\mathsf{r}^2/(4\mathcal{D}v)] \tag{6.4.6c}$$

$$\rho = \rho_0 + (k_6/\mathsf{r})\exp(-k_4^2\mathcal{D}v)[\sin(k_4\mathsf{r}) + k_5\cos(k_4\mathsf{r})] \tag{6.4.6d}$$

where ρ_0 is a static or constant string fluid density. The explicit spatial dependance of the above density profiles (6.4.6) now allow us to investigate their asymptotic behaviour as well as their effect on the junction condition (6.3.16).

It is clear that the density solutions given by (6.4.6a), (6.4.6c) and (6.4.6d) represent spatially decaying behaviours, i.e. the string fluid density decays as the distance from the stellar surface and the stars atmosphere increases. These solutions also show that in regions very far away from the stellar atmosphere ($\mathsf{r} \to \infty$) the string fluid density must become constant ($\rho \to \rho_0$). The evolutionary behaviour exhibited in these profiles are consistent with diffusion of the string fluid. With the density solutions (6.4.6a), (6.4.6c)

and (6.4.6d) the junction condition (6.3.16) takes the following forms respectively

$$p_r = qB - \rho_0 - k_1/\mathsf{r} \quad (6.4.7a)$$

$$p_r = qB - \rho_0 - k_3(\mathcal{D}v)^{-3/2}\exp[-\mathsf{r}^2/(4\mathcal{D}v)] \quad (6.4.7b)$$

$$p_r = qB - \rho_0 - (k_6/\mathsf{r})\exp(-k_4^2\mathcal{D}v)[\sin(k_4\mathsf{r}) + k_5\cos(k_4\mathsf{r})] \quad (6.4.7c)$$

It is important to note that in the above system (6.4.7), the radial pressure at the stellar surface is dependent on the static string fluid density ρ_0. If the constants ρ_0, k_1, k_3 and k_6 are all strictly positive then the radial pressure remains reduced and is consistent with the outflow of null radiation. If these constants are strictly negative then the radial pressure at the surface of the star is increased and this does not depict behaviour that is physically reasonable. The junction condition (6.3.16) thus places the restriction on the density solutions (6.4.6a), (6.4.6c) and (6.4.6d) that

$$\rho_0 > 0, \quad k_1 > 0, \quad k_3 > 0, \quad k_6 > 0 \quad (6.4.8)$$

for the acceptable description of a radiating stellar system with a two-fluid atmosphere.

Glass and Krisch (1999) have shown that the density solution given by (6.4.6b) has two distinct types of behaviour. When $k_2 > 0$, the string fluid density grows with increasing radius and is not realistic, and when $k_2 < 0$, the density decays with radius indicating that the string atmosphere could be bounded. With the density profile (6.4.6b) the junction condition (6.3.16) becomes

$$p_r = qB - \rho_0 - (k_2/6)\mathsf{r}^2 - k_2\mathcal{D}v \quad (6.4.9)$$

For the above form of the junction condition (6.4.9) we have the restriction that $\rho_0 > 0, k_2 > 0$ in order to have an acceptable radial pressure at the surface of the star.

6.5 Discussion

The general junction condition on the surface of a relativistic radiating star having an interior stellar fluid with nonzero shear has been presented. The matching of the interior spacetime geometry to that of the exterior shows that on the stellar boundary the interior radial pressure is related to the interior heat flux as well as the exterior string fluid energy density as follows

$$p_r = qB - \rho$$

We have also demonstrated that the above junction condition can be derived alternatively by using a systematic geometric approach which involves the Lichnerowiscz and the O' Brien and Synge conditions. In this treatment the junction conditions on the stellar boundary have been obtained purely from the geometry of the spacetime manifolds, and consequently it is the geometry that prescribes the way in which the interior and exterior matter variables of a relativistic radiating star are related on the spatial hypersurface. Physically reasonable stellar situations have been discussed and the associated governing equations are highlighted. These equations are consistency equations and must be solved as differential equations on the surface in order to yield exact radiating models. In future we intend to use our results to construct more complete models of astrophysical objects.

Chapter 7

Conclusion

The major theme of this dissertation has been to generalise the stellar junction conditions that are necessary for the well defined modelling of the radiative transfer of heat energy in dense compact relativistic stars. The extended junction conditions may be used to model a radiating star with a two-fluid atmosphere. In addition to this, it has been our aim to construct new models for radiating relativistic stars, both in the standard as well as the new generalised formalism. Within the standard framework of a radiating star, we have studied the exact thermal behaviour of a special class of relativistic stars by making use of the 'Euclidean condition' which allows one of the gravitational potentials to be transformed away. These Euclidean stars were modelled as having interior gravitating fluids with nonvanishing shear and undergoing nonadiabatic spherical gravitational collpase with a radial heat flux. Furthermore, we have also investigated models in which a compact star is evolving under the action of radial perturbations in the metric as well as matter variables. It has been demonstrated that these perturbations play a pivotal role in the various stages of collapsing stellar material and that the late phase thermodynamics and matter behaviour are substantially affected.

We now provide an overview of the main results obtained during the course of our investigations:

- In Chapter 2, we provided the basic theory that is essential for the construction and study of stellar and other localised astrophysical systems within the context of general relativity. A concrete formalism for the geometry and matter distribution in the presence of strong spherically symmetrical gravitational fields on curved spacetime backgrounds was provided. Stability criteria and physical conditions for the general dynamics as well as gravitational collapse of stars were briefly discussed.

- Chapter 3 focussed on a study involving a special class of radiating stars which satisfy a transformation property called the Euclidean condition. In these Euclidean stars the areal radius, which is the radius measured according to a changing area, is equal to the proper radius of the dissipating star, as measured from the central core region to the outer surface layer. The governing second order nonlinear ordinary differential equation

$$\frac{\ddot{R}}{R} + \frac{1}{2}\left(\frac{\dot{R}}{R}\right)^2 - \frac{\dot{A}}{A}\frac{\dot{R}}{R} - \frac{(A+\dot{R})A'}{RR'} = 0$$

  was examined on the stellar boundary. A particular analytical solution which is regular, well behaved and without any singularities at the centre of the star was found. The new radiating solution

$$R(r,t) = \left[C_1(r)e^{\lambda_1 t} + C_2(r)e^{\lambda_2 t}\right]^2$$

  enabled us to study the causal and noncausal thermal evolution of the collapsing stellar matter from the central core region up to the stellar surface.

- In Chapter 4 we investigated the effects of radial perturbations in the gravitational as well as the matter variables for a radiating star. A perturbative construction was adopted to allow for a relativistic star to undergo nonadiabatic spherical collapse which eventually leads to a static compact stellar configuration. These perturbations affect the dynamics of the dissipation as well as the collapse and they have a marked impact on the thermodynamics of the stellar model. The boundary condition has the exact solution

$$T(t) = T_0 e^{-(\beta_\Sigma + \sqrt{\alpha_\Sigma + \beta_\Sigma^2})t}$$

which ensures that the model becomes static in the later stages of evolution. An expression for the perturbed temperature can be found explicitly, and this plays a major role in understanding the thermal evolution of the model in the final stages just before reaching equilibrium.

- Chapter 5 extends the formalism developed by Santos in 1985. The Santos junction condition indicates that for a dense star undergoing nonadiabatic gravitational collapse, the interior fluid pressure is proportional to the magnitude of the heat flux across the star's surface. The exterior of the star is described by the conventional Vaidya solution with outgoing null radiation that is radially isotropic with mass function depending only on the retarded time. We know that the Santos junction condition can be generalised and extended by allowing the mass function to be dependant on both the retarded time and comoving radial coordinates. The extrinsic curvature of the star's interior and exterior were matched and we were able to arrive at the following new generalised junction condition

  $$p = qB - \rho_{string}$$

  which shows that the interior isotropic fluid pressure now has an additional dependance on the exterior string fluid energy density. The effect of the additional string fluid, on the atmosphere and local exterior region of the star was investigated by generating the surface and asymptotic luminosity profiles as well as the surface redshift and showing that they are reduced by the string energy density. We also generated a new exact solution to the expanded form of the generalised junction condition given by

$$ds^2 = -dt^2 + \frac{4d^2b^2}{[\Omega(t)(b^2 - r^2)d + 2b\alpha]^2}\left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]$$

This solution provides a physically meaningful interpretation of the idea of a dissipating star in the presence of a two-fluid atmosphere.

- In Chapter 6 we extended the results of chapter 5 for a fluid having nonvanishing shear and showed that the same junction condition holds and that the radial component of the pressure now depends on the heat flux as well as the string density. An alternate geometric approach to derive the generalised junction condition was also provided utilising the Lichnerowiscz and O' Brien and Synge conditions. The physical impact and importance of the new junction condition were further studied by applying it to the two-fluid star models proposed by Glass and Krisch (1998, 1999) and Krisch and Glass (2005).

It has been clearly shown in this dissertation that the junction conditions on the stellar boundary are crucial for the construction of physically reasonable and acceptable models of dissipating stars in relativistic astrophysics. In view of the fact that our central result for the generalised Vaidya metric is new, it will have far reaching consequences for the framework and formalism for radiating stars. It is our aim, in future work, to apply our result to previous stellar models constructed and to provide more concrete solutions to existing problems. The results presented in this thesis will be directly applied to the following stellar astrophysical problems:

- The geodesic motion of fluid particles in radiating stars.
- The horizon-free gravitational collapse of stars.
- Collapse of dense stars from an initial static configuration.
- Radiating stars with conformal flatness.

- Radiating stars having electromagnetic fields.
- Stellar models with polytropic equations of state.

amongst many others. This will form the basis for work to be carried out in the future.

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