UNIVERSITY OF KWAZULU-NATAL

ANALYSIS OF SHEAR-FREE SPHERICALLY SYMMETRIC CHARGED RELATIVISTIC FLUIDS

MANDLENKOSI C. KWEYAMA

Analysis of shear-free spherically symmetric charged relativistic fluids

by

Mandlenkosi Christopher Kweyama

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As the candidate's supervisor I have approved this thesis for submission.

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Abstract

We study the evolution of shear-free spherically symmetric charged fluids in general relativity. This requires the analysis of the coupled Einstein-Maxwell system of equations. Within this framework, the master field equation to be integrated is

$$y_{xx} = f(x)y^2 + g(x)y^3$$

We undertake a comprehensive study of this equation using a variety of approaches. Initially, we find a first integral using elementary techniques (subject to integrability conditions on the arbitrary functions $f(x)$ and $g(x)$). As a result, we are able to generate a class of new solutions containing, as special cases, the models of Maharaj *et al* (1996), Stephani (1983) and Srivastava (1987). The integrability conditions on $f(x)$ and $g(x)$ are investigated in detail for the purposes of reduction to quadratures in terms of elliptic integrals. We also obtain a Noether first integral by performing a Noether symmetry analysis of the master field equation. This provides a partial group theoretic basis for the first integral found earlier. In addition, a comprehensive Lie symmetry analysis is performed on the field equation. Here we show that the first integral approach (and hence the Noether approach) is limited – more general results are possible when the full Lie theory is used. We transform the field equation to an autonomous equation and investigate the conditions for it to be reduced to quadrature. For each case we recover particular results that were found previously for neutral fluids. Finally we show (for the first time) that the pivotal

equation, governing the existence of a Lie symmetry, is actually a fifth order purely differential equation, the solution of which generates solutions to the master field equation.

Dedication

To:

the late, Bezile, my mother,

my wife Nelisiwe,

my daughters Zama and Anelisiwe

and my son Lungelo.

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DECLARATION 1 - PLAGIARISM

I, Mandlenkosi Christopher Kweyama, student number: 851852765, declare that

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DECLARATION 2 - PUBLICATIONS

DETAILS OF CONTRIBUTION TO PUBLICATIONS that form part and/or include research presented in this thesis (include publications in preparation, submitted, in press and published and give details of contributions of each author to experimental work and writing of each publication)

Publication 1

Kweyama MC, Maharaj SD and Govinder KS, First integrals for charged perfect fluid distributions, *Mathematical Methods in the Applied Sciences*, submitted (2010a).

(There were regular meetings between my supervisors and I to discuss research material for publication. The outline of the research paper and discussion of the significance of the results were jointly done. The paper was mainly written by me with some input from my supervisors.)

Publication 2

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(There were regular meetings between my supervisors and I to discuss research material for publication. The outline of the research paper and discussion of the significance of the results were jointly done. The paper was mainly written by me with some input from my supervisors.)

Publication 3

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(There were regular meetings between my supervisors and I to discuss research material for publication. The outline of the research paper and discussion of the significance of the results were jointly done. The paper was mainly written by me with some input from my supervisors.)

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Contents

Chapter 1

Introduction

1.1 Outline

The Einstein field equations describing inhomogeneous processes in gravitational systems, which can be extended to include the electromagnetic field to comprise the Einstein-Maxwell equations, are a system of nonlinear coupled partial differential equations. These systems are difficult to solve in closed form for realistic matter distributions. As our understanding of the gravitational behaviour of these models depends on exact solutions, we need to solve the field equations. The simplest inhomogeneous models have vanishing shear in spherical symmetry with neutral or charged matter. As the case of neutral matter has been extensively studied over the years (Krasinski 1997, Stephani *et al* 2003) we include the effects of the electric field. For this case the integration of the Einstein-Maxwell system reduces to a single master partial differential equation, the condition of pressure isotropy generalised to include the electromagnetic field. The objective of this thesis is to investigate the integrability properties of the governing partial differential equation that contains a term corresponding to charge, for shear-free fluids. This investigation is performed using several approaches.

In this chapter we give relevant background information. This will help to generate solutions of the Einstein field equations which are significant for general relativistic effects. We briefly discuss some of the basic ideas behind the theory of Lie analysis. We also outline the theory of Noether symmetry analysis and invariant solutions.

In chapter 2 we apply an elementary approach suggested by Srivastava (1987). We reduce the Einstein-Maxwell field equations, generalising the transformation due to Faulkes (1969), to a single nonlinear second order partial differential equation that governs the behaviour of charged fluids. As in the case of uncharged fluids, this equation can be treated as an ordinary differential equation. We also derive a first integral of the governing equation by generalising the technique of Srivastava (1987) first used for uncharged fluids. This first integral is subject to two integrability conditions expressed as nonlinear integral equations. We further transform the integrability conditions, into a new system of differential equations which can be integrated in terms of quadratures. We comprehensively investigate the nature of the factors of the quartic arising in the quadrature.

In chapter 3 we study the integrability properties of the underlying partial differential equation for the Einstein-Maxwell system using symmetry methods. Both Noether and Lie point symmetries of the governing equation are considered. Noether symmetries have the interesting property of being associated with physically relevant conservation laws via the well-known Noether theorem in a direct manner. The Lie symmetries are more general, providing a larger set of symmetry generators in general, but do not guarantee integrability and reduction to quadrature in a straight forward manner. We analyse the governing equation for Noether symmetries and this analysis yields a Noether first integral for this equation. We then establish the relationship between the Noether first integral and the first integral obtained using an *ad hoc* approach in chapter 2. We undertake a comprehensive Lie symmetry analysis of the governing equation to investigate the conditions under which it can be reduced to quadratures. In addition we perform a detailed analysis for group invariant solutions.

In chapter 4 we derive a fifth order purely differential equation the solutions of which yield solutions to the master field equation. This is the first time that such an equation, necessary for the existence of the Lie symmetry, has been derived. We then perform a Lie symmetry analysis of this equation. The solution obtained contains the result obtained by Kweyama *et al* (2010b). Furthermore we solve a fourth order integro-differential equation which was deduced by Kweyama *et al* (2010b). Note that the results obtained in this chapter have not been obtained previously.

Finally, in chapter 5, we conclude by summarising the results obtained in this study.

1.2 Lie theory of differential equations

1.2.1 Lie point symmetries of ordinary differential equations

A point symmetry is a symmetry in which the infinitesimals depend only on coordinates (Cantwell 2002). We describe a Lie point symmetry as a point symmetry that depends continuously on at least one parameter, *i.e.* the parameter(s) can vary continuously over a set of scalar nonzero measure. Lie point symmetries of ordinary differential equations are of the form

$$G = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}$$

where the coefficients ξ and η are functions of x and y only.

To be able to apply a point transformation to an nth order ordinary differential equation

$$E\left(x, y, y', y'', ..., y^{(n)}\right) = 0$$

where

$$y' \equiv \frac{dy}{dx}, \quad y'' \equiv \frac{d^2y}{dx^2}, \cdots$$

etc, we need to know how derivatives transform under the infinitesimal transformation

$$\begin{aligned} \bar{x} &= x + \varepsilon\xi(x,y) \\ \bar{y} &= y + \varepsilon\eta(x,y) \end{aligned}$$

which has a generator given by

$$G = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}$$

In terms of the quantities $\bar{x}$ and $\bar{y}$ we have, for the first derivative,

$$\begin{aligned} \frac{d\bar{y}}{d\bar{x}} &= \frac{d(y+\varepsilon\eta)}{d(x+\varepsilon\xi)} \\ &= \frac{\dfrac{dy}{dx} + \varepsilon\dfrac{d\eta}{dx}}{1+\varepsilon\dfrac{d\xi}{dx}} \\ &= (y' + \varepsilon\eta')\left(1 - \varepsilon\xi' + \varepsilon^2\xi'^2 - \cdots\right) \\ &= y' + \varepsilon\left(\eta' - y'\xi'\right) \end{aligned}$$

which we have terminated at $O(\varepsilon^2)$. Note that primes refer to total differentiation with respect to x. For the second derivative we have

$$\begin{aligned} \frac{d^2\bar{y}}{d\bar{x}^2} &= \frac{d}{d\bar{x}}\left(\frac{d\bar{y}}{d\bar{x}}\right) \\ &= \frac{d[y' + \varepsilon\left(\eta' - y'\xi'\right)]}{d\left(x+\varepsilon\xi\right)} \\ &= \frac{\dfrac{dy'}{dx} + \varepsilon\dfrac{d}{dx}\left(\eta' - y'\xi'\right)}{1+\varepsilon\xi'} \\ &= y'' + \varepsilon\left(\eta'' - 2y''\xi' - y'\xi''\right) \end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
\frac{d^3\bar{y}}{d\bar{x}^3} &= y''' + \varepsilon\left(\eta''' - 3y'''\xi' - 3y''\xi'' - y'\xi'''\right) \\
\frac{d^4\bar{y}}{d\bar{x}^4} &= y^{iv} + \varepsilon\left(\eta^{iv} - 4y^{iv}\xi' - 6y'''\xi'' - 4y''\xi''' - y'\xi^{iv}\right)
\end{aligned}$$

and so on. In general we generate the formula (Leach 1995)

$$\frac{d^n\bar{y}}{d\bar{x}^n} = y^{(n)} + \varepsilon\left(\eta^{(n)} - \sum_{i=1}^{n} C_i^n y^{(i+1)}\xi^{(n-i)}\right)$$

where the superscript $^{(i)}$ denotes $\dfrac{d^i}{dx^i}$ and C_i^n is the number of combinations of n objects taken i at a time.

To deal with the infinitesimal transformations of equations and functions involving derivatives, we need the extensions of the generator G. We indicate that a generator G has been extended by writing

$$\begin{aligned}
G^{[1]} &= G + \left(\eta' - y'\xi'\right)\frac{\partial}{\partial y'} \\
G^{[2]} &= G^{[1]} + \left(\eta'' - 2y''\xi' - y'\xi''\right)\frac{\partial}{\partial y''}
\end{aligned}$$

for the first and the second extensions respectively. When generating an extension of G we have to extend G such that all the derivatives appearing in the equation or function are included in the extension. For an nth order differential equation, the nth extension is of the form (Mahomed and Leach 1990)

$$G^{[n]} = G + \sum_{i=1}^{n}\left\{\eta^{(i)} - \sum_{j=1}^{i}\begin{pmatrix} i \\ j \end{pmatrix} y^{(i+1-j)}\xi^{(j)}\right\}\frac{\partial}{\partial y^{(i)}}$$

The generator

$$G = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}$$

is a symmetry of the differential equation

$$E\left(x, y, y', y'', ..., y^{(n)}\right) = 0$$

if and only if

$$G^{[n]}E|_{E=0} = 0$$

which means that the action of the nth extension of G on E is zero when the original equation is satisfied.

1.2.2 Hidden symmetries

Some equations do not admit the required number of point symmetries to enable reduction to quadratures. In an attempt to overcome this limitation, various extensions of the classical Lie approach have been devised. One such extension is due to the observance of the so-called hidden symmetries - point symmetries that arise unexpectedly due to decreasing and/or increasing the order of a differential equation (Edelstein *et al* 2001).

Hidden symmetries have been shown to lead to the solutions of a number of equations that do not possess sufficient Lie point symmetries with the appropriate Lie algebras. Increasing the order of an equation can give rise to Type I hidden symmetries and the reduction of order can give rise to Type II hidden symmetries (Abraham-Shrauner 1993).

1.2.3 Reduction of order

If a differential equation

$$E\left(x, y, ..., y^{(n)}\right) = 0 \tag{1.1}$$

has a symmetry

$$G = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}$$

we can obtain an equation of order $(n-1)$ in a systematic manner. This is achieved by using the zeroth order and first order differential invariants which are two characteristics associated with $G^{[1]}$. The characteristics are obtained by solving the following system of ordinary differential equations

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dy'}{\eta' - y'\xi'}$$

If we integrate the equation involving the first two terms we obtain the characteristic $u = f(x,y)$ and the equation involving the first and the third (equally the second and the third) terms gives the characteristic $v = g(x,y,y')$. Since $Gu = 0$ we call u the zeroth order invariant. Similarly v is called the first order differential invariant since $G^{[1]}v = 0$. A key feature of the Lie method is that all higher derivatives can be expressed in terms of u, v and the derivatives of v with respect to u. As a result equation (1.1) reduces to

$$F\left(u, v, ..., v^{(n-1)}\right) = 0$$

i.e. it reduces to an equation of order one less than the original. If the reduced equation has a symmetry, the order of the equation can be reduced again. The process can be repeated until the original differential equation is reduced to an algebraic equation. This reduction of order reduces an nth order equation to a set of n first order equations *provided* there is a sufficient number of symmetries with the appropriate Lie algebra.

For further information we refer the reader to Bluman and Anco (2002), Bluman and Kemei (1989), Cantwell (2002), Dresner (1999), Hydon (2000) and Olver (1986).

1.3 Noether symmetries and integrals

If a second order ordinary differential equation

$$y'' = N(x, y, y') \tag{1.2}$$

has a Lagrangian $\mathcal{L}(x, y, y')$ then (1.2) is equivalent to the Euler-Lagrange equation

$$\frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial y'}\right) - \frac{\partial \mathcal{L}}{\partial y} = 0$$

The determining equation for a Noether point symmetry

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} \tag{1.3}$$

corresponding to a Lagrangian $\mathcal{L}(x, y, y')$ of (1.2) is

$$X^{[1]}\mathcal{L} + \left(\frac{d\xi}{dx}\right)\mathcal{L} = \frac{dF}{dx}$$

where $F = F(x, y)$ is a gauge function. It is also known that if there is a Noether symmetry corresponding to a Lagrangian of an equation (1.2), then

(1.2) can be reduced to quadratures. This is a critical advantage of Noether point symmetries as emphasised by Wafo Soh and Mahomed (2000). The Noether first integral I_N associated with the Noether point symmetry (1.3) is given by

$$I_N = \xi(x,y)\mathcal{L} + \left(\eta(x,y) - y'\xi(x,y)\right)\mathcal{L}_{y'} - F \tag{1.4}$$

in terms of $\mathcal{L}$ and F.

1.4 Invariant solutions

In general, if an nth order ordinary differential equation

$$F\left(x, y, y', ..., y^{(n)}\right) = 0 \tag{1.5}$$

where

$$y^{(k)} = \frac{d^{(k)}y}{dx^{(k)}}, \qquad k = 1, 2, 3, ..., n$$

admits a one parameter Lie group of point transformations with the infinitesimal generator

$$G = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} \tag{1.6}$$

then the function $y = \phi(x)$ is an invariant solution of (1.5) resulting from its invariance under the point symmetry (1.6) if and only if (Bluman 1990, Bluman and Anco 2002):

1. $y = \phi(x)$ is a solution of the first order ordinary differential equation

$$y' = \frac{\eta(x,y)}{\xi(x,y)}$$

2. $y = \phi(x)$ is a solution of (1.5).

These invariant solutions, where they exist, correspond to singular solutions of the original equation.

Chapter 2

First integrals for charged perfect fluid distributions

2.1 Introduction

Solutions of the Einstein-Maxwell system of equations are important in relativistic astrophysics as they may be used to describe charged compact objects with strong gravitational fields such as dense neutron stars. Several recent treatments, including the works of Ivanov (2002) and Sharma *et al* (2001), demonstrate that the presence of the electromagnetic field affects the values of redshifts, luminosities and maximum mass of a compact relativistic star. The electromagnetic field cannot be ignored when considering the gravitational evolution of stars composed of quark matter as pointed out by Mak and Harko

(2004) and Komathiraj and Maharaj (2007). Therefore exact models describing the formation and evolution of charged stellar objects, within the context of full general relativity, are necessary. Electromagnetic fields play a role in gravitational collapse, the formation of naked singularities, and the collapse of charged shells of matter onto existing black holes (as indicated by Lasky and Lun (2007a, 2007b)). Significant electric fields are also present in phases of intense dynamical activity, in collapsing configurations, with time scales of the order of the hydrostatic time scale for which the usual stable equilibrium configuration assumptions are not reliable (as shown in the treatments of Di Prisco *et al* (2007) and Herrera *et al* (2009)). It is interesting to note that Maxwell's equations play a role in several other scenarios, including the evolution of cosmological models in higher dimensions. De Felice and Ringeval (2009) considered braneworld models, exhibiting Poincare symmetry in extra-dimensions, which admit wormhole configurations.

Spherical symmetry and a shear-free matter distribution are simplifying assumptions usually made when seeking exact solutions to the Einstein field equations with neutral matter. The field equations may then be reduced to a single partial differential equation. What is interesting about this equation is that it can be treated as an ordinary differential equation. A general class of solutions was first found by Kustaanheimo and Qvist (1948). Comprehensive treatments of the uncharged case are provided by Srivastava (1987) and Sussman (1989). The generalisation to include the electromagnetic field is easily performed and is described by the Einstein-Maxwell system. The field equations are again reducible to a single partial differential equation, now containing a term corresponding to charge. A review of known charged solutions, admitting a Friedmann limit, is contained in the treatment of Krasinski (1997). A detailed investigation of the mathematical and physical features of the Einstein-Maxwell system has been performed by Srivastava (1992) and Sussman (1988a, 1988b) respectively.

The objective of this chapter is to investigate the integrability properties of the governing partial differential equation that contains a term corresponding to charge, for shear-free fluids. This investigation is performed using an elementary approach suggested by Srivastava (1987). In §2.2 we reduce the Einstein-Maxwell field equations, generalising the transformation due to Faulkes (1969), to a single nonlinear second order partial differential equation that governs the behaviour of charged fluids. As in the uncharged case, this equation can be treated as an ordinary differential equation. In §2.3 we derive a first integral of the governing equation by generalising the technique of Srivastava (1987) first used for uncharged fluids. This first integral is subject to two integrability conditions expressed as nonlinear integral equations. We transform the integrability conditions, in §2.4, into a new system of differential equations which can be integrated in terms of quadratures. In §2.5 we comprehensively investigate the nature of the factors of the quartic arising in the quadrature. Finally, in §2.6 we discuss the results obtained.

2.2 Field equations

We consider the shear-free motion of a spherically symmetric perfect fluid in the presence of the electromagnetic field. We choose a coordinate system $x^i = (t, r, \theta, \phi)$ which is both comoving and isotropic. In this coordinate system the metric can be written as

$$ds^2 = -e^{2\nu(t,r)}dt^2 + e^{2\lambda(t,r)}\left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]$$

where ν and λ are the gravitational potentials. We are investigating the general case of a self-gravitating fluid in the presence of the electromagnetic field without placing arbitrary restrictions on the potentials. For this model the Einstein equations are supplemented with Maxwell equations. The Einstein field equations for a charged perfect fluid can be written as the system

$$\rho = 3\frac{\lambda_t^2}{e^{2\nu}} - \frac{1}{e^{2\lambda}}\left(2\lambda_{rr} + \lambda_r^2 + \frac{4\lambda_r}{r}\right) - \frac{E^2}{r^4 e^{4\lambda}} \tag{2.1a}$$

$$p = \frac{1}{e^{2\nu}}\left(-3\lambda_t^2 - 2\lambda_{tt} + 2\nu_t\lambda_t\right) + \frac{1}{e^{2\lambda}}\left(\lambda_r^2 + 2\nu_r\lambda_r + \frac{2\nu_r}{r} + \frac{2\lambda_r}{r}\right) + \frac{E^2}{r^4 e^{4\lambda}} \tag{2.1b}$$

$$p = \frac{1}{e^{2\nu}}\left(-3\lambda_t^2 - 2\lambda_{tt} + 2\nu_t\lambda_t\right) + \frac{1}{e^{2\lambda}}\left(\nu_{rr} + \nu_r^2 + \frac{\nu_r}{r} + \frac{\lambda_r}{r} + \lambda_{rr}\right) - \frac{E^2}{r^4 e^{4\lambda}} \tag{2.1c}$$

$$0 = \nu_r\lambda_t - \lambda_{tr} \tag{2.1d}$$

Maxwell's equations yield

$$E = r^2 e^{\lambda-\nu}\Phi_r, \qquad E_r = \sigma r^2 e^{3\lambda} \tag{2.2}$$

In the above ρ is the energy density and p is the isotropic pressure which are measured relative to the four-velocity $u^a = (e^{-\nu}, 0, 0, 0)$. Subscripts refer to partial derivatives with respect to that variable. The quantity $E = E(r)$ is an arbitrary constant of integration and σ is the proper charge density of the fluid. We interpret E as the total charge contained within the sphere of radius r centred around the origin of the coordinate system. Note that $\Phi_r = F_{10}$ is the only nonzero component of the electromagnetic field tensor $F_{ab} = \phi_{b;a} - \phi_{a;b}$ where $\phi_a = (\Phi(t,r), 0, 0, 0)$. The Einstein-Maxwell system (2.1)-(2.2) is a coupled system of equations in the variables ρ, p, E, σ, ν and λ.

The system of partial differential equations (2.1) can be simplified to produce an underlying nonlinear second order equation. Equation (2.1d) can be written as

$$\nu_r = (\ln \lambda_t)_r$$

Then (2.1b) and (2.1c) imply

$$\left[e^{\lambda}\left(\lambda_{rr} - \lambda_r^2 - \frac{\lambda_r}{r}\right) + \frac{2E^2 e^{-\lambda}}{r^4}\right]_t = 0$$

and the potential ν has been eliminated. The Einstein field equations (2.1) can therefore be written in the equivalent form

$$\begin{aligned}
\rho &= 3e^{2h} - e^{-2\lambda}\left(2\lambda_{rr} + \lambda_r^2 + \frac{4\lambda_r}{r}\right) - \frac{E^2}{r^4 e^{4\lambda}} && \text{(2.3a)}\\
p &= \frac{1}{\lambda_t e^{3\lambda}}\left[e^{\lambda}\left(\lambda_r^2 + \frac{2\lambda_r}{r}\right) - e^{3\lambda+2h} - \frac{E^2}{r^4 e^{\lambda}}\right]_t && \text{(2.3b)}\\
e^{\nu} &= \lambda_t e^{-h} && \text{(2.3c)}\\
e^{\lambda}\left(\lambda_{rr} - \lambda_r^2 - \frac{\lambda_r}{r}\right) &= -\tilde{F} - \frac{2E^2}{r^4 e^{\lambda}} && \text{(2.3d)}
\end{aligned}$$

for a charged relativistic fluid. In the above $h = h(t)$ and $\tilde{F} = \tilde{F}(r)$ are arbitrary constants of integration. Equation (2.3d) is the condition of pressure isotropy generalised to include the electric field. To find an exact solution of the field equations, we need to specify the functions h, $\tilde{F}$ and E and solve equation (2.3d) for λ. We can then compute the quantities ρ and p from (2.3a) and (2.3b), and σ follows from (2.2).

It is possible to write (2.3d) in a simpler form by eliminating the exponential factor e^{λ}. We use the transformation, first introduced by Faulkes (1969) for neutral fluids, which has the adapted form

$$\begin{aligned}
x &= r^2\\
y &= e^{-\lambda}\\
f(x) &= \frac{\tilde{F}}{4r^2}\\
g(x) &= \frac{E^2}{2r^6}
\end{aligned}$$

Then (2.3d) becomes

$$y_{xx} = f(x)y^2 + g(x)y^3 \tag{2.4}$$

which is the fundamental equation governing the behaviour of a shear-free charged fluid. Observe that (2.4) is a nonlinear partial differential equation since $y = y(t, x)$. When $g = 0$ then $y_{xx} = f(x)y^2$ for a neutral fluid which has been studied by Maharaj *et al* (1996) and others.

2.3 A charged first integral

It would appear that we need to specify the functions $f(x)$ and $g(x)$ to integrate (2.4). However it is possible *ab inito* to generate a first integral without choosing $f(x)$ and $g(x)$ if we generalise a technique first suggested by Srivastava (1987), and extended by Maharaj *et al* (1996). The first integral generated is subject to a system of integral equations in $f(x)$ and $g(x)$ which can be rewritten as differential equations.

Rather than choose $f(x)$ and $g(x)$ we seek general conditions that reduce the order of (2.4) to produce a first order differential equation. We can formally integrate (2.4) to obtain

$$y_x = \int f(x)y^2 dx + \int g(x)y^3 dx \tag{2.5}$$

We consider $\int fy^2 dx$, the first term of (2.5) and then integrate by parts. We let

$$u = y^2, \qquad dv = f dx$$

then

$$du = 2yy_x dx, \qquad v = \int f dx$$

For convenience we use the notation

$$\int f dx = f_I$$

and hence

$$\int fy^2 dx = f_I y^2 - 2\int f_I y y_x dx \tag{2.6}$$

We also consider $\int gy^3 dx$, the second term of (2.5). Integrating by parts, we let

$$u = y^3, \qquad dv = g dx$$

then

$$du = 3y^2 y_x dx, \qquad v = \int g dx = g_I$$

Hence

$$\int gy^3 dx = g_I y^3 - 3\int g_I y^2 y_x dx \tag{2.7}$$

Using (2.6) and (2.7) in (2.5) yields

$$y_x = f_I y^2 + g_I y^3 - 2\int f_I y y_x dx - 3\int g_I y^2 y_x dx \tag{2.8}$$

We now consider $\int f_I y y_x dx$, the integral in the third term of (2.8). We let

$$u = yy_x, \qquad dv = f_I dx$$

and use (2.4) to obtain

$$du = (y_x y_x + y y_{xx})dx = (y_x^2 + fy^3 + gy^4)dx, \qquad v = \int f_I dx = f_{II}$$

Consequently we get

$$\int f_I y y_x dx = f_{II} y y_x - \int f_{II} y_x^2 dx - \int f f_{II} y^3 dx - \int g f_{II} y^4 dx \qquad (2.9)$$

We then substitute (2.9) in (2.8) to obtain

$$\begin{aligned} y_x &= f_I y^2 + g_I y^3 - 2 f_{II} y y_x + 2\int f_{II} y_x^2 dx + 2\int f f_{II} y^3 dx \\ &\quad +2\int g f_{II} y^4 dx - 3\int g_I y^2 y_x dx \end{aligned} \qquad (2.10)$$

On applying integration by parts on the fourth term of (2.10), we let

$$u = y_x^2, \qquad dv = f_{II} dx$$

then

$$du = 2 y_x y_{xx} dx = \left(2 f y^2 y_x + 2 g y^3 y_x\right) dx, \qquad v = \int f_{II} dx = f_{III}$$

Hence

$$\int f_{II} y_x^2 dx = f_{III} y_x^2 - 2\int f f_{III} y^2 y_x dx - 2\int g f_{III} y^3 y_x dx \qquad (2.11)$$

We now consider $\int f f_{II} y^3 dx$, the integral in the fifth term of (2.10). We let

$$u = y^3, \qquad dv = f f_{II} dx$$

then

$$du = 3 y^2 y_x dx, \qquad v = \int (f f_{II})\, dx = (f f_{II})_I$$

and hence

$$\int f f_{II} y^3 dx = (f f_{II})_I\, y^3 - 3\int (f f_{II})_I\, y^2 y_x dx \qquad (2.12)$$

Similarly the sixth term in (2.10) yields

$$\int gf_{II}y^4dx = (gf_{II})_I\, y^4 - 4\int (gf_{II})_I\, y^3 y_x dx \tag{2.13}$$

On substituting (2.11), (2.12) and (2.13) in (2.10) we obtain

$$\begin{aligned}
y_x &= f_I y^2 + g_I y^3 - 2f_{II}yy_x + 2f_{III}y_x^2 - 4\int ff_{III}y^2y_x dx - 4\int gf_{III}y^3y_x dx \\
&\quad +2\,(ff_{II})_I\, y^3 - 6\int (ff_{II})_I\, y^2 y_x dx + 2\,(gf_{II})_I\, y^4 - 8\int (gf_{II})_I\, y^3 y_x dx \\
&\quad -3\int g_I y^2 y_x dx \\
&= f_I y^2 + g_I y^3 - 2f_{II}yy_x + 2f_{III}y_x^2 + 2\,(ff_{II})_I\, y^3 + 2\,(gf_{II})_I\, y^4 - \int \{[4ff_{III} \\
&\quad +6\,(ff_{II})_I + 3g_I]\, y^2 y_x\}\, dx - \int \{[4gf_{III} + 8\,(gf_{II})_I]\, y^3 y_x\}\, dx \\
&= f_I y^2 + g_I y^3 - 2f_{II}yy_x + 2f_{III}y_x^2 + 2\,(ff_{II})_I\, y^3 + 2\,(gf_{II})_I\, y^4 - \int \{[4ff_{III} \\
&\quad +6\,(ff_{II})_I + 3g_I]\left(\frac{1}{3}\frac{dy^3}{dx}\right)\Bigg\}\, dx - \int \left\{[4gf_{III} + 8\,(gf_{II})_I]\left(\frac{1}{4}\frac{dy^4}{dx}\right)\right\} dx \\
&= f_I y^2 + g_I y^3 - 2f_{II}yy_x + 2f_{III}y_x^2 + 2\,(ff_{II})_I\, y^3 + 2\,(gf_{II})_I\, y^4 - \int \Bigg\{\Bigg[\frac{4}{3}ff_{III} \\
&\quad +2\,(ff_{II})_I + g_I]\left(\frac{dy^3}{dx}\right)\Bigg\}\, dx - \int \left\{[gf_{III} + 2\,(gf_{II})_I]\left(\frac{dy^4}{dx}\right)\right\} dx
\end{aligned} \tag{2.14}$$

For a meaningful result the integrals on the right hand side of (2.14) must be eliminated.

We note that these integrals can be determined if $2ff_{III} + 3\,(ff_{II})_I + \frac{3}{2}g_I$ and $gf_{III} + 2\,(gf_{II})_I$ are constants. This observation yields the following result

$$\begin{aligned}
\tau_0(t) &= -y_x + f_I y^2 + g_I y^3 - 2f_{II}yy_x + 2f_{III}y_x^2 + 2[(ff_{II})_I - \frac{1}{3}K_0]y^3 \\
&\quad +[2\,(gf_{II})_I - K_1]y^4
\end{aligned} \tag{2.15}$$

subject to the integrability conditions

$$K_0 = 2ff_{III} + 3\left(ff_{II}\right)_I + \frac{3}{2}g_I \tag{2.16a}$$

$$K_1 = gf_{III} + 2\left(gf_{II}\right)_I \tag{2.16b}$$

where K_0 and K_1 are constants, and the quantity $\tau_0(t)$ is an arbitrary function of integration. We have therefore established that a first integral of the field equation (2.4) is given by (2.15) subject to conditions (2.16) which are integral equations. We emphasise that we generated the first integral (2.15) without choosing forms for the functions $f(x)$ and $g(x)$. The forms of $f(x)$ and $g(x)$ are constrained by the integrability conditions (2.16).

The integral equations (2.16) may be simplified. On setting

$$f_{III} = a \tag{2.17}$$

we can rewrite (2.16b) as

$$ga + 2\left(ga_x\right)_I = K_1$$

which has the solution

$$g = g_0 a^{-3} \tag{2.18}$$

where $a = a(x)$ and g_0 is an arbitrary constant. Equation (2.16a) now becomes

$$\frac{4}{3}a_{xxx}a + 2\left(a_{xxx}a_x\right)_I + \left(g_0 a^{-3}\right)_I = K_0$$

or, if we want a purely differential equation,

$$aa_{xxxx} + \frac{5}{2}a_x a_{xxx} = -\frac{3}{4}g_0 a^{-3} \tag{2.19}$$

We have therefore established that the integrability conditions (2.16) can be transformed into (2.18) and the fourth order ordinary differential equation (2.19). Using (2.16) , (2.17) and (2.18) we rewrite the first integral (2.15) of (2.4) as

$$\tau_0(t) = y_x - a_{xx}y^2 + 2a_x yy_x - 2ay_x^2 + \frac{4}{3}aa_{xxx}y^3 + g_0 a^{-2}y^4 \tag{2.20}$$

subject to condition (2.19). A solution of (2.19) will give a, and then f and g will be found from (2.17) and (2.18) respectively. A full analysis of this case has been reported in Kweyama (2010a).

2.4 Integrability conditions

It is not easy to solve the nonlinear integral equations (2.16). However we can transform these equations into an equivalent system comprising a first order and a fourth order ordinary differential equations which are more convenient to work with.

We let

$$f_{III} = \mathcal{F}$$

so that $f_{II} = \mathcal{F}_x$, $f_I = \mathcal{F}_{xx}$ and $f = \mathcal{F}_{xxx}$. Then it is possible to rewrite (2.16b) as

$$(g\mathcal{F})_x + 2g\mathcal{F}_x = 0 \tag{2.21}$$

Note that the integral equation (2.16b) has been transformed to a first order differential equation in $\mathcal{F}$. Equation (2.21) is integrable and we obtain

$$g = \mathcal{K}_0 \mathcal{F}^{-3} \tag{2.22}$$

where $\mathcal{F} = \mathcal{F}(x)$ and $\mathcal{K}_0$ is an arbitrary constant.

Similarly we can eliminate g in (2.16a), with the help of (2.22), to get the result

$$\mathcal{F}\mathcal{F}_{xxxx} + \frac{5}{2}\mathcal{F}_x\mathcal{F}_{xxx} = -\frac{3}{4}\mathcal{K}_0\mathcal{F}^{-3} \tag{2.23}$$

Therefore the integral equation (2.16a) has been transformed to a fourth order differential equation in $\mathcal{F}$. Equation (2.23) can be integrated to yield

$$\mathcal{F}_{xxx} = -\frac{3}{4}\mathcal{K}_0\mathcal{F}^{-(5/2)}\int \mathcal{F}^{-(3/2)}dx + \mathcal{K}_1\mathcal{F}^{-(5/2)} \tag{2.24}$$

where $\mathcal{K}_1$ is an arbitrary constant of integration. Observe that (2.24) can be written in the form

$$(\mathcal{F}_{xx})_x - \frac{1}{2}\left(\mathcal{F}_x^2\right)_x = \mathcal{K}_1\mathcal{F}^{-(3/2)} - \frac{3}{4}\mathcal{K}_0\mathcal{F}^{-(3/2)}\int \mathcal{F}^{-(3/2)}dx$$

which gives

$$\mathcal{F}\mathcal{F}_{xx} - \frac{1}{2}\mathcal{F}_x^2 = -2\mathcal{K}_2 + \mathcal{K}_1\int \mathcal{F}^{-(3/2)}dx - \frac{3}{8}\mathcal{K}_0\left(\int \mathcal{F}^{-(3/2)}dx\right)^2 \tag{2.25}$$

where $-2\mathcal{K}_2$ is a constant. Equation (2.25) can be written in the form

$$\begin{aligned} 2\left(\mathcal{F}^{1/2}\right)_{xx} &= -2\mathcal{K}_2\mathcal{F}^{-(3/2)} + \mathcal{K}_1\mathcal{F}^{-(3/2)}\int \mathcal{F}^{-(3/2)}dx \\ &\quad -\frac{3}{8}\mathcal{K}_0\mathcal{F}^{-(3/2)}\left(\int \mathcal{F}^{-(3/2)}dx\right)^2 \end{aligned}$$

which is integrated to give

$$\begin{aligned} 2\left(\mathcal{F}^{1/2}\right)_{x} &= -\mathcal{K}_3 - 2\mathcal{K}_2\int \mathcal{F}^{-(3/2)}dx + \frac{1}{2}\mathcal{K}_1\left(\int \mathcal{F}^{-(3/2)}dx\right)^2 \\ &\quad -\frac{1}{8}\mathcal{K}_0\left(\int \mathcal{F}^{-(3/2)}dx\right)^3 \end{aligned} \tag{2.26}$$

where $\mathcal{K}_3$ is a constant. We now rewrite (2.26) as

$$\begin{aligned}\left(\mathcal{F}^{-1}\right)_x &= \mathcal{K}_3\mathcal{F}^{-(3/2)} + 2\mathcal{K}_2\mathcal{F}^{-(3/2)}\int \mathcal{F}^{-(3/2)}dx \\ &\quad -\frac{1}{2}\mathcal{K}_1\mathcal{F}^{-(3/2)}\left(\int \mathcal{F}^{-(3/2)}dx\right)^2 + \frac{1}{8}\mathcal{K}_0\mathcal{F}^{-(3/2)}\left(\int \mathcal{F}^{-(3/2)}dx\right)^3\end{aligned}$$

On integrating this first order differential equation we obtain

$$\begin{aligned}\mathcal{F}^{-1} &= \mathcal{K}_4 + \mathcal{K}_3\int \mathcal{F}^{-(3/2)}dx + \mathcal{K}_2\left(\int \mathcal{F}^{-(3/2)}dx\right)^2 \\ &\quad -\frac{1}{6}\mathcal{K}_1\left(\int \mathcal{F}^{-(3/2)}dx\right)^3 + \frac{1}{32}\mathcal{K}_0\left(\int \mathcal{F}^{-(3/2)}dx\right)^4 \qquad (2.27)\end{aligned}$$

where $\mathcal{K}_4$ is an arbitrary constant.

We can rewrite (2.27) in a simpler form if we let

$$u = \int \mathcal{F}^{-(3/2)}dx \qquad (2.28)$$

so that

$$u_x = \left(\mathcal{F}^{-1}\right)^{3/2}$$

Then we can write (2.27) as

$$u_x = \left(\mathcal{K}_4 + \mathcal{K}_3 u + \mathcal{K}_2 u^2 - \frac{1}{6}\mathcal{K}_1 u^3 + \frac{1}{32}\mathcal{K}_0 u^4\right)^{3/2}$$

which is a first order equation in u. The equivalent integral representation is

$$x - x_0 = \int \frac{du}{\left(\mathcal{K}_4 + \mathcal{K}_3 u + \mathcal{K}_2 u^2 - (1/6)\mathcal{K}_1 u^3 + (1/32)\mathcal{K}_0 u^4\right)^{3/2}} \qquad (2.29)$$

where x_0 is a constant. The quadrature (2.29) can be evaluated in terms of elliptic integrals. We can summarise our result as follows: *the first integral (2.20), with $g = \mathcal{K}_0\mathcal{F}^{-3}$, $f = \mathcal{F}_{xxx}$ and $\mathcal{F}$ given by (2.29) via (2.28), represents a particular class of solutions of (2.4).*

To obtain solutions in closed form, satisfying the integrability conditions (2.16), we need to evaluate the integral (2.29). Particular solutions in terms of elementary functions are admitted. In general the solution will be given in terms of special functions. We can express the solutions to (2.16) in the parametric form as follows

$$f(x) = \mathcal{F}_{xxx} \tag{2.30a}$$

$$g(x) = \mathcal{K}_0 \mathcal{F}^{-3} \tag{2.30b}$$

$$u_x = \mathcal{F}^{-3/2} = [G'(u)]^{-1} \tag{2.30c}$$

$$x - x_0 = G(u) \tag{2.30d}$$

where we have set

$$G(u) = \int \frac{du}{(\mathcal{K}_4 + \mathcal{K}_3 u + \mathcal{K}_2 u^2 - (1/6)\mathcal{K}_1 u^3 + (1/32)\mathcal{K}_0 u^4)^{3/2}} \tag{2.31}$$

If we set $g = 0$ (which forces $\mathcal{K}_0 = 0$) then the charge vanishes and the system (2.30) becomes

$$f(x) = \mathcal{F}_{xxx} \tag{2.32a}$$

$$u_x = \mathcal{F}^{-3/2} = [G'(u)]^{-1} \tag{2.32b}$$

$$x - x_0 = G(u) \tag{2.32c}$$

where

$$G(u) = \int \frac{du}{(\mathcal{K}_4 + \mathcal{K}_3 u + \mathcal{K}_2 u^2 - (1/6)\mathcal{K}_1 u^3)^{3/2}}$$

This corresponds to the results found by Maharaj *et al* (1996) for a neutral shear-free gravitating fluid. Thus their first integral is contained in our class of charged models (2.30)-(2.31).

2.5 Particular solutions

Nine cases arise from the solution (2.30)–(2.31) depending on the nature of the factors of the polynomial $\mathcal{K}_4 + \mathcal{K}_3 u + \mathcal{K}_2 u^2 - (1/6)\mathcal{K}_1 u^3 + (1/32)\mathcal{K}_0 u^4$.

2.5.1 Case I: One order-four linear factor

If $\mathcal{K}_4 + \mathcal{K}_3 u + \mathcal{K}_2 u^2 - (1/6)\mathcal{K}_1 u^3 + (1/32)\mathcal{K}_0 u^4$ has one repeated linear factor then we have

$$\mathcal{K}_4 + \mathcal{K}_3 u + \mathcal{K}_2 u^2 - (1/6)\mathcal{K}_1 u^3 + (1/32)\mathcal{K}_0 u^4 = (a + bu)^4, b \neq 0$$

We evaluate the integral in (2.31) to obtain

$$G(u) = -\frac{1}{5b}(a + bu)^{-5} \tag{2.33a}$$

$$f(x) = \frac{24}{75}(5b)^{4/5}(x - x_0)^{-(11/5)} \tag{2.33b}$$

$$g(x) = \mathcal{K}_0(5b)^{-(12/5)}(x - x_0)^{-(12/5)} \tag{2.33c}$$

In this case it is possible to invert the integral (2.29) and then write $u = u(x)$. The first integral (2.20) has the form

$$\begin{aligned}
\tau_0(t) &= -y_x - \frac{4}{15}(5b)^{4/5}(x-x_0)^{-(6/5)}y^2 - \frac{5}{7}\mathcal{K}_0(5b)^{-(12/5)}(x-x_0)^{-(7/5)}y^3 \\
&\quad -\frac{8}{3}(5b)^{4/5}(x-x_0)^{-(1/5)}yy_x + \frac{10}{3}(5b)^{4/5}(x-x_0)^{4/5}y_x^2 \\
&\quad -2\left[\frac{3856}{10815}(5b)^{8/5}(x-x_0)^{-(7/5)} - \frac{15}{14}\mathcal{K}_0(5b)(x-x_0)^{-(7/5)}\right]y^3 \\
&\quad -\frac{5}{3}\mathcal{K}_0(5b)^{-(8/5)}(x-x_0)^{-(8/5)}y^4 \qquad (2.34)
\end{aligned}$$

where we have used the functional forms in (2.33). The first integral (2.34) corresponds to a shear-free spherically symmetric charged fluid which does not have an uncharged limit since $\mathcal{K}_0 \neq 0$. If $\mathcal{K}_0 = 0$ then the polynomial becomes cubic which is a contradiction. The charged integral (2.34) ($E \neq 0, \mathcal{K}_0 \neq 0, b \neq 0$) is a new solution to the Einstein-Maxwell field equations.

2.5.2 Case II: One order-three linear factor

If $\mathcal{K}_4 + \mathcal{K}_3 u + \mathcal{K}_2 u^2 - (1/6)\mathcal{K}_1 u^3 + (1/32)\mathcal{K}_0 u^4$ has two linear factors, one of which is not repeated, then we have

$$\mathcal{K}_4 + \mathcal{K}_3 u + \mathcal{K}_2 u^2 - (1/6)\mathcal{K}_1 u^3 + (1/32)\mathcal{K}_0 u^4 = (a+bu)(u+c)^3$$

We use the computer package Mathematica (Wolfram 2007) to determine the integral in (2.31) to obtain

$$\begin{aligned}
G(u) &= \frac{2\sqrt{(a+bu)(u+c)}}{35(a-bc)^5}\left[\frac{35b^4}{a+bu} + \frac{93b^3}{u+c} - \frac{29b^2(a-bc)}{(u+c)^2} + \frac{13b(a-bc)^2}{(u+c)^3}\right. \\
&\quad \left. - \frac{5(a-bc)^3}{(u+c)^4}\right] \qquad (2.35)
\end{aligned}$$

expressed completely in terms of elementary functions. In this case, if $g = 0$, $\mathcal{K}_0 = 0$ and $b = 0$, then (2.35) becomes

$$G(u) = a^{-(3/2)}\left(-\frac{2}{7}\right)(u+c)^{-(7/2)}$$

and hence using (2.32) we find

$$f(x) = a^{2/7}\left(\frac{48}{343}\right)\left(-\frac{7}{2}\right)^{6/7}(x-x_0)^{-(15/7)} \tag{2.36}$$

Note that (2.36) is related to the result obtained by Maharaj *et al* (1996).

Again setting $g = 0$, $K_1 = 0$, in (2.15) we get

$$\psi_0(t) = -y_x + f_I y^2 + g_I y^3 - 2f_{II}yy_x + 2f_{III}y_x^2 + 2[(ff_{II})_I - \frac{1}{3}K_0]y^3$$

which was the first integral for uncharged matter found by Maharaj *et al* (1996). Also observe that if $g = 0, K_1 = 0, f(x) = (ax+b)^{-(15/7)}$ then (2.15) yields

$$\begin{aligned}\phi_0(t) &= -6y_x - \frac{21}{4a}(ax+b)^{-8/7}y^2 - \frac{3}{2}\left(\frac{7}{a}\right)^2(ax+b)^{-(1/7)}yy_x \\ &\quad +\frac{1}{4}\left(\frac{7}{a}\right)^3(ax+b)^{6/7}y_x^2 - \frac{1}{6}\left(\frac{7}{a}\right)^3(ax+b)^{-(9/7)}y^3 \end{aligned}\tag{2.37}$$

which was found by Srivastava (1987). Also with $g = 0, K_1 = 0, f(x) = x^{-(15/7)}$ in (2.15) (or if we set $a = 1, b = 0$ in (2.37)) we have

$$\varphi_0(t) = -6y_x - \frac{21}{4}x^{-(8/7)}y^2 - \frac{3}{2}\cdot 7^2x^{-(1/7)}yy_x + \frac{1}{4}\cdot 7^3x^{6/7}y_x^2 - \frac{1}{6}\cdot 7^3x^{-(9/7)}y^3$$

which was established by Stephani (1983). Therefore the first integral (2.20) is a charged generalisation of the particular Maharaj *et al* (1996), Srivastava (1987) and Stephani (1983) neutral models.

2.5.3 Case III: One order-two linear factor; one order-one quadratic factor

If $\mathcal{K}_4 + \mathcal{K}_3 u + \mathcal{K}_2 u^2 - (1/6)\mathcal{K}_1 u^3 + (1/32)\mathcal{K}_0 u^4$ has two factors, one linear and repeated and the other is irreducible to linear factors, then we have

$$\mathcal{K}_4 + \mathcal{K}_3 u + \mathcal{K}_2 u^2 - (1/6)\mathcal{K}_1 u^3 + (1/32)\mathcal{K}_0 u^4 = \left(a + bu + cu^2\right)(u+d)^2,$$
$$b^2 - 4ac < 0$$

The function (2.31) is integrated to obtain

$$\begin{aligned} G(u) \;=\; & -\left\{ \frac{1}{(a-bd+cd^2)\,u^2} + \frac{5(b-2cd)}{2(a-bd+cd^2)u} \right. \\ & -\frac{15(b-2cd)^4 - 62c(b-2cd)^2(a-bd+cd^2) + 24c^2(a-bd+cd^2)^2}{2(a-bd+cd^2)\left[4c(a-bd+cd^2)-(b-2cd)^2\right]} \\ & \left. -\frac{c(b-2cd)\left[15(b-2cd)^2 - 52c(a-bd+cd^2)\right]u}{2(a-bd+cd^2)\Delta} \right\} \times \\ & \frac{1}{2\sqrt{(a-bd+cd^2)+(b-2cd)u+cu^2}} \\ & + \frac{15(b-2cd)^2 - 12c(a-bd+cd^2)}{8(a-bd+cd^2)^3} \times \\ & \int \frac{du}{u\sqrt{a-bd+cd^2+(b-2cd)u+cu^2}} \end{aligned}$$

where $\Delta = 4(a-bd+cd^2)c - (b-2cd)^2$ and the integral on the right hand side can be expressed in terms of elementary functions. The exact form of the integral depends on the signs of $a - bd + cd^2$ and Δ (see Gradshteyn and Ryzhik (1980), equations 2.266 and 2.269.6).

2.5.4 Case IV: One order-two linear factor; two order-one linear factors

With one repeated and two non-repeated linear factors we have

$$\mathcal{K}_4 + \mathcal{K}_3 u + \mathcal{K}_2 u^2 - (1/6)\mathcal{K}_1 u^3 + (1/32)\mathcal{K}_0 u^4 = (a + bu)(cu + d)(u + e)^2$$

In this case the expression for the integral in (2.31) can be evaluated with the help of the computer package Mathematica (Wolfram 2007). The resulting expression is expressible in terms of only elementary functions. This expression is very lengthy and not illuminating, and is therefore not included in this work.

2.5.5 Case V: Two order-two linear factors

If $\mathcal{K}_4 + \mathcal{K}_3 u + \mathcal{K}_2 u^2 - (1/6)\mathcal{K}_1 u^3 + (1/32)\mathcal{K}_0 u^4$ has two linear factors each of which is repeated, then we have

$$\mathcal{K}_4 + \mathcal{K}_3 u + \mathcal{K}_2 u^2 - (1/6)\mathcal{K}_1 u^3 + (1/32)\mathcal{K}_0 u^4 = (a + bu)^2(u + c)^2$$

The integral in (2.31) may be easily determined so that

$$\begin{aligned} G(u) &= \frac{1}{(a-bc)^5}\left[6b^2 \ln\frac{u+c}{a+bu} + \frac{3b^2(a-bc)}{a+bu} + \frac{b^2(a-bc)^2}{2(a+bu)^2} + \frac{3b(a-bc)}{u+c}\right. \\ &\quad \left. - \frac{(a-bc)^2}{2(u+c)^2}\right] \end{aligned}$$

Thus for the case of two order-two linear factors the integral can be expressed completely in terms of elementary functions.

2.5.6 Case VI: No repeated linear factors

If $\mathcal{K}_4 + \mathcal{K}_3 u + \mathcal{K}_2 u^2 - (1/6)\mathcal{K}_1 u^3 + (1/32)\mathcal{K}_0 u^4$ has no repeated linear factors, then we have

$$\mathcal{K}_4 + \mathcal{K}_3 u + \mathcal{K}_2 u^2 - (1/6)\mathcal{K}_1 u^3 + (1/32)\mathcal{K}_0 u^4 = e(a+u)(b+u)(c+u)(d+u), e \neq 0$$

In this case we obtain, in terms of elementary functions and elliptic integrals, the result (Dieckmann 2010)

$$\begin{aligned} G(u) \quad = \quad & \frac{2e^{-(3/2)}}{(a-b)\sqrt{(a+u)(b+u)(c+u)(d+u)}} \left[\frac{(a+u)(b+u)}{(b-c)(a-d)} \left[\frac{2}{(b-d)^2} \right.\right. \\ & \left. \frac{1}{(b-d)(c-d)} + \frac{1}{(a-c)(c-d)+} \right] + \frac{b+u}{a-c} \left[\frac{2(d+u)}{(a-b)(a-d)^2} \right. \\ & \left.\left. \frac{1}{(b-c)(b-d)} - \frac{1}{(a-d)(b-d)} \right] - \frac{1}{(b-c)(b-d)} \right] - \frac{4e^{-(3/2)}}{(a-b)\sqrt{b-d}} \times \\ & \left[\frac{1}{(a-d)^2(c-d)\sqrt{a-c}} + \frac{\sqrt{a-c}}{(a-b)(b-c)^2(b-d)} \right. \\ & \left. + \frac{a-b-c+d}{(c-d)^2(a-c)^{3/2}(b-c)} \right] E(\alpha, p) \\ & + \frac{2e^{-(3/2)}}{(a-c)^{3/2}(b-d)^{3/2}(b-c)(a-d)} \times \\ & \left[\frac{2(a+b-c-d)^2}{(b-c)(a-d)} + \frac{(a-b-c+d)^2}{(a-b)(c-d)} \right] F(\alpha, p), \\ & (0 < d < c < b < a) \end{aligned} \tag{2.38}$$

where we have let

$$\alpha = \arcsin \sqrt{\frac{(a-c)(d+u)}{(a-d)(c+u)}}, \qquad p = \frac{(b-c)(a-d)}{(a-c)(b-d)}$$

In (2.38), $F(\alpha, p)$ is the elliptic integral of the first kind and $E(\alpha, p)$ is the elliptic integral of the second kind. This result is similar to one of the results obtained by Maharaj *et al* (1996). However their uncharged model is not regainable from the expression above as the polynomial here is necessarily quartic.

2.5.7 Case VII: One order-two quadratic factor

If $\mathcal{K}_4 + \mathcal{K}_3 u + \mathcal{K}_2 u^2 - (1/6)\mathcal{K}_1 u^3 + (1/32)\mathcal{K}_0 u^4$ has one repeated quadratic irreducible factor, then we have

$$\mathcal{K}_4 + \mathcal{K}_3 u + \mathcal{K}_2 u^2 - (1/6)\mathcal{K}_1 u^3 + (1/32)\mathcal{K}_0 u^4 = (a + bu + cu^2)^2$$

In this case we obtain

$$\begin{aligned} G(u) &= \frac{b + 2cu}{4ac - b^2}\left[\frac{1}{2(a + bu + cu^2)^2} + \frac{3c}{(4ac - b^2)(a + bu + cu^2)}\right] \\ &\quad + \frac{6c^2}{(4ac - b^{2)^2}}\int \frac{du}{a + bu + cu^2} \end{aligned}$$

which can be expressed in terms of only elementary functions. The exact form of the integral depends on the sign of $4ac - b^2$ (see Gradshteyn and Ryzhik (1980), equations 2.172 and 2.173.2).

2.5.8 Case VIII: Two order-one quadratic factors

With two non-repeated quadratic factors we have

$$\mathcal{K}_4 + \mathcal{K}_3 u + \mathcal{K}_2 u^2 - (1/6)\mathcal{K}_1 u^3 + (1/32)\mathcal{K}_0 u^4 = (a + bu + cu^2)(d + eu + u^2)$$

In this case the expression for the integral in (2.31), using the computer package Mathematica (Wolfram 2007), is obtainable but is not included in this work as it is very lengthy. It may be expressed in terms of elementary functions and elliptic integrals.

2.5.9 Case IX: One order-one cubic factor

If $\mathcal{K}_4 + \mathcal{K}_3 u + \mathcal{K}_2 u^2 - (1/6)\mathcal{K}_1 u^3 + (1/32)\mathcal{K}_0 u^4$ has one irreducible cubic factor, then we have

$$\mathcal{K}_4 + \mathcal{K}_3 u + \mathcal{K}_2 u^2 - (1/6)\mathcal{K}_1 u^3 + (1/32)\mathcal{K}_0 u^4 = (a + bu + cu^2 + du^3)(e + u)$$

The integral in (2.31) can again be found with the help of the computer package Mathematica (Wolfram 2007). It is given in terms of elementary functions, elliptic integrals and special functions. However it is so lengthy that it is also not included in this work.

2.6 Discussion

In this chapter we have modelled the behaviour of shear-free charged fluids, and reduced the solution of the Einstein-Maxwell system of field equations to a single nonlinear partial differential equation. By treating this equation as an ordinary differential equation, a first integral was found using elementary methods. It is remarkable to note that the first integral is obtainable without specifying the arbitrary functions contained in the governing equation. The first integral is subject to a system of two integral equations which were replaced by a system of two differential equations which can be integrated up to a quadrature. Consequently we have found a new class of parametric solutions to the Einstein-Maxwell system for a charged gravitating shear-free fluid. The new solution is given by the parametric equations (2.30)-(2.31).

A detailed analysis of the factors of the quartic arising in the quadrature was performed. Two cases of interest arise. Firstly we are in a position to explicitly invert the quadrature when there is one repeated linear factor and explicitly write the first integral. Then the model has to be necessarily charged. We believe that this is a new result. Secondly we can explicitly invert the quadrature when there is one order-three linear factor. This case contains that of vanishing charge and we regain the results of Maharaj *et al* (1996), Srivastava (1987) and Stephani (1983).

A comprehensive mathematical analysis of the integrability properties of (2.4) using the symmetry properties of the equation may provide further solutions and insights. For example the treatment of Halburd (1999), for the uncharged shear-free case, established an equivalence with the generalised Chazy equation and provided a new class of integrable equations. The symmetry analysis is presented in the next chapter.

Chapter 3

Noether and Lie symmetries for charged perfect fluids

3.1 Introduction

The Einstein-Maxwell system of equations plays a central role in relativistic astrophysics when describing spherically symmetric gravitational fields in static manifolds. In these situations we are modelling charged compact objects with strong gravitational fields such as dense relativistic stars. Recent investigations indicate that the electromagnetic field significantly affects physical quantities in relativistic stellar systems: equations of state, redshifts, luminosities, stability and maximum masses of compact relativistic stars. The presence of electric charge is a necessary ingredient in the structure and gravitational evolution of stars composed of quark matter. Other applications include the role of electromagnetic fields in gravitational collapse, formation of black holes and the existence of naked singularities. Electric fields cannot be ignored in spherical gravitational collapse with phases of intense dynamical activity and particle interaction. Maxwell's equations also play an important role in cosmological models in higher dimensions, brane world models and wormhole configurations. For a sample of these applications the reader is referred to Sharma *et al* (2001), Ivanov (2002), Mak and Harko (2004), Lasky and Lun (2007a), Thirukkanesh and Maharaj (2008), Herrera *et al* (2009) and De Felice and Ringeval (2009).

When solving the Einstein field equations with neutral matter distributions, we often make the assumption that the spacetime is shear-free and spherically symmetric. Kustaanheimo and Qvist (1948) were the first to present a general class of solutions. The generalisation to include the presence of the electromagnetic field may be easily achieved. The field equations are reducible to a single partial differential equation. A review of known charged solutions, with a Friedmann limit, is given by Krasinski (1997). Srivastava (1987) and Sussmann (1988a, 1988b) undertook a detailed study of the mathematical and physical features of the Einstein-Maxwell system in spherical symmetry. Wafo Soh and Mahomed (2000) used symmetry methods to systematically study the underlying partial differential equation. They showed that all previously known solutions can be related to a Noether point symmetry

The main aim of this chapter is to study the integrability properties of the underlying partial differential equation for the Einstein-Maxwell system using symmetry methods. Both Noether and Lie point symmetries of the governing equation are considered. Noether symmetries have the interesting property of being associated with physically relevant conservation laws in a direct manner via the well-known Noether theorem. The Lie symmetries are more general, providing a larger set of symmetry generators in general, but do not guarantee integrability and reduction to quadratures in a straight forward manner. In §3.2, we give the single nonlinear second order partial differential equation that governs the behaviour of charged fluids and the first integral of the governing equation obtained earlier. This first integral is subject to two integrability conditions expressed as nonlinear integral equations which, in chapter 2, were transformed into a fourth order differential equation. In §3.3, we analyse the governing equation for Noether symmetries via its Lagrangian and this analysis yields a general Noether first integral for this equation. We then establish the relationship between the Noether first integral and the first integral obtained earlier using an *ad hoc* approach. In §3.4, we undertake a comprehensive Lie symmetry analysis of the governing equation to investigate the conditions under which it can be reduced to quadratures. We show how the Noether results are a subset of the Lie analysis results. Lastly, in §3.5, we discuss the results obtained and relate some invariant solutions to known results.

3.2 Background

We briefly summarise the relevant equations required for this chapter without rederiving the results in chapter 2. We are analysing the shear-free motion of a fluid distribution in the presence of an electric field. It is possible to choose coordinates $x^i = (t, r, \theta, \phi)$ such that the line element can be written in the form

$$ds^2 = -e^{2\nu(t,r)}dt^2 + e^{2\lambda(t,r)} \left[dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]$$

which is simultaneously comoving and isotropic. Since an electromagnetic field is present, the Einstein field equations are supplemented with Maxwell's equations to describe a self-gravitating charged fluid.

After considerable simplification the Einstein-Maxwell system produces the differential equation

$$y'' = f(x)y^2 + g(x)y^3 \tag{3.1}$$

where primes denote differentiation with respect to the variable x and $y = y(t, x)$. Equation (3.1) is the fundamental nonlinear partial differential equation which determines the behaviour of the self-gravitating charged fluid in general relativity. However, we can treat it as an ordinary differential equation as only derivatives with respect to x appear. If $g = 0$ then we regain a neutral fluid which has been studied by Maharaj *et al* (1996) and Wafo Soh and Mahomed (1999), amongst others.

It is possible to find a first integral of (3.1) without choosing explicit forms of the functions $f(x)$ and $g(x)$. We use integration by parts, an approach adopted by Maharaj *et al* (1996) in investigating the integrability properties of the field equation $y'' = f(x)y^2$ of a neutral spherically symmetric shear-free fluid. We can integrate (3.1) by parts to obtain

$$\begin{aligned} y' &= f_I y^2 + g_I y^3 - 2f_{II} y y' + 2f_{III} y'^2 + 2\left(f f_{II}\right)_I y^3 + 2\left(g f_{II}\right)_I y^4 \\ &\quad - K_0 y^3 - K_1 y^4 + \tau_0(t) \end{aligned} \tag{3.2}$$

The result (3.2) is subject to the integrability conditions

$$K_0 = \frac{4}{3} f f_{III} + 2\left(f f_{II}\right)_I + g_I \tag{3.3a}$$

$$K_1 = g f_{III} + 2\left(g f_{II}\right)_I \tag{3.3b}$$

where K_0 and K_1 are constants and $\tau_0(t)$ is an arbitrary function of integration. On setting $f_{III} = a$ we can solve (3.3b) to get

$$g = g_0 a^{-3} \tag{3.4}$$

where $a = a(x)$ and g_0 is an arbitrary constant. Equation (3.3a) can now be written as a purely differential equation,

$$a a^{(iv)} + \frac{5}{2} a' a''' = -\frac{3}{4} g_0 a^{-3} \tag{3.5}$$

We have therefore established that the integrability conditions (3.3) can be transformed into (3.4) and the fourth order ordinary differential equation (3.5). We rewrite the first integral (3.2) of (3.1) as

$$\tau_0(t) = y' - a'' y^2 + 2a' y y' - 2a y'^2 + \frac{4}{3} a a''' y^3 + g_0 a^{-2} y^4 \tag{3.6}$$

subject to condition (3.5). A solution of (3.5) will give a, and then f and g follow.

3.3 Noether symmetries and integration

Given that Noether's theorem generates first integrals in a direct manner, we now investigate (3.1) for Noether symmetries in order to generate first integrals. For equation (3.1) a Lagrangian is

$$\mathcal{L} = \frac{1}{2}y'^2 + \frac{1}{3}f(x)y^3 + \frac{1}{4}g(x)y^4 \tag{3.7}$$

The Lagrangian $\mathcal{L}$ admits the Noether point symmetry

$$G = a\frac{\partial}{\partial x} + (by + c)\frac{\partial}{\partial y}$$

provided

$$b = \frac{1}{2}a' \tag{3.8a}$$

$$g = g_1 a^{-3} \tag{3.8b}$$

$$f = a^{-5/2}\left(f_1 - 3g_1\int ca^{-3/2}dx\right) \tag{3.8c}$$

$$a''' = 4a^{-5/2}c\left(f_1 - 3g_1\int ca^{-3/2}dx\right) \tag{3.8d}$$

$$c = C_0 + C_1 x \tag{3.8e}$$

$$F = \frac{1}{4}a''y^2 + c'y \tag{3.8f}$$

where $a = a(x)$, $b = b(x)$, $c = c(x)$ and f_1 and g_1 are constants. From (3.8c) and (3.8d) we have

$$f = \frac{a'''}{4c}$$

When we differentiate (3.8d) once with respect to x we obtain

$$caa^{(iv)} + \frac{5}{2}ca'a''' - ac'a''' = -12g_1c^3a^{-3} \tag{3.9}$$

Applying the transformation

$$X = \frac{C_1}{C_0 + C_1 x}, \qquad A = \frac{aC_1^2}{(C_0 + C_1 x)^2} \tag{3.10}$$

reduces (3.9) to

$$AA^{(iv)} + \frac{5}{2}A'A''' = -12g_1 C_1^2 A^{-3}$$

which is in the form of (3.5).

From (1.4), given by

$$I_N = \xi(x,y)\mathcal{L} + (\eta(x,y) - y'\xi(x,y))\,\mathcal{L}_{y'} - F$$

we have that (3.1) admits the Noether first integral

$$\begin{aligned} I_N &= a\left[\frac{1}{2}y'^2 + \frac{1}{3}y^3 a^{-(5/2)}\left(f_1 - 3g_1\int ca^{-(3/2)}dx\right) + \frac{1}{4}g_1 a^{-3}y^4\right] \\ &\quad + \left(\frac{1}{2}a'y + c - ay'\right)y' - \frac{1}{4}a''y^2 - c'y \end{aligned} \tag{3.11}$$

Substituting (3.8d) in (3.11) yields

$$I_N = cy' - \frac{1}{4}a''y^2 + \frac{1}{2}a'yy' - \frac{1}{2}ay'^2 + \frac{aa'''}{12c}y^3 + \frac{1}{4}g_1 a^{-2}y^4 - c'y \tag{3.12}$$

On comparing our first integral (3.6) and the Noether first integral (3.12) we observe that, with $c = 1/4$, $g_0 = g_1$ and primes denoting derivatives with respect to x,

$$\tau_0(t) = 4I_N$$

As a result, our earlier *ad hoc* approach yielded a first integral which is a special case of that obtained via Noether's theorem. This further implies that (3.6) admits the Noether symmetry

$$Y = 4a\frac{\partial}{\partial x} + (2a'y + 1)\frac{\partial}{\partial y}$$

thus supporting the results of our approach, and the fact that (3.1) could be reduced to quadratures.

3.4 Lie analysis

As a final attempt at solving (3.1) we undertake a Lie symmetry analysis (Olver 1986). Lie symmetries are usually a larger set of symmetries for a problem as opposed to the set of Noether symmetries. However, the disadvantage is that no simple formula exists to find first integrals associated with Lie symmetries – direct integration of (often difficult) equations is usually needed. However, as we show here, this approach allows for a larger class of solutions than the Noether approach.

It is a simple matter to verify that

$$G = a\frac{\partial}{\partial x} + (by + c)\frac{\partial}{\partial y} \tag{3.13}$$

is a symmetry of (3.1) and the relationship among the functions $a(x)$, $b(x)$, $c(x)$, $f(x)$ and $g(x)$ is given by the following system of ordinary differential equations

$$\begin{aligned} a'' &= 2b' && (3.14a) \\ b'' &= 2fc && (3.14b) \\ c'' &= 0 && (3.14c) \\ af' + (2a' + b)f &= -3cg && (3.14d) \\ ag' + (2a' + 2b)g &= 0 && (3.14e) \end{aligned}$$

From (3.14a)

$$2b = a' + \alpha \tag{3.15}$$

where α is an arbitrary constant, from (3.14a) and (3.14b)

$$f = \frac{a'''}{4c} \tag{3.16}$$

from (3.14c)

$$c = C_0 + C_1 x$$

where C_0 and C_1 are arbitrary constants, and finally from (3.14e)

$$g = g_2 a^{-3} \exp\left(-\int \frac{\alpha dx}{a}\right) \tag{3.17}$$

where g_2 is an arbitrary constant. By integrating (3.14d) and using (3.17) we obtain an alternative form of (3.16), and it is

$$f = a^{-5/2} \exp\left(-\int \frac{\alpha dx}{2a}\right) \left[f_2 - 3g_2 \int ca^{-(3/2)} \exp\left(\int \frac{\alpha dx}{a}\right) dx\right] \tag{3.18}$$

where f_2 is an arbitrary constant of integration. From (3.14d), (3.16) and (3.17) we observe that a is a solution of the equation

$$caa^{(iv)} + \left[c\left(\frac{5a'}{2} + \frac{\alpha}{2}\right) - c'a\right] a''' = -12 g_2 c^3 a^{-3} \exp\left(-\int \frac{\alpha dx}{a}\right) \tag{3.19}$$

Alternatively (3.19) may be obtained by equating equations (3.16) and (3.18) and differentiating once with respect to x.

Observe the similarity between the results obtained here in the case of Lie analysis and those obtained in the previous section on Noether symmetries. The main difference is the occurrence of the parameter α in the Lie analysis. Thus, we have a more general result (as expected) in the Lie analysis as compared to the Noether analysis. In addition, this result is even more general than that obtained in Kweyama *et al* (2010a) as our $c(x)$ is nonconstant.

The transformation which converts the symmetry (3.13) to $\dfrac{\partial}{\partial X}$ makes (3.1) autonomous. The simplest expression of this transformation is

$$X = \int \frac{dx}{a} \tag{3.20a}$$

$$Y = y \exp\left(-\int \frac{bdx}{a}\right) - \int \frac{c}{a} \exp\left(-\int \frac{bdx}{a}\right) dx \tag{3.20b}$$

From (3.20) we have

$$Y'' = \exp\left(-\int \frac{bdx}{a}\right) \left(a^2 y'' + aa'y' - 2aby' + b^2 y - ab'y + bc - ac'\right) \tag{3.21}$$

We also have

$$y = a^{1/2} \exp\left(\int \frac{\alpha dx}{2a}\right) (Y + I) \tag{3.22}$$

so that

$$y' = a^{1/2} \exp\left(\int \frac{\alpha dx}{2a}\right) \left(\frac{Y'}{a} + \frac{bY}{a} + \frac{bI}{a} + ca^{-(3/2)} \exp\left(-\int \frac{\alpha dx}{2a}\right)\right) \tag{3.23}$$

where we let

$$I = \int ca^{-(3/2)} \exp\left(-\int \frac{\alpha dx}{2a}\right) dx$$

We substitute (3.17), (3.18), (3.22) and (3.23) in (3.1) and we obtain

$$\begin{aligned} y'' &= a^{-(3/2)} \exp\left(\int \frac{\alpha dx}{2a}\right) \left(f_2 Y^2 + 2f_2 Y I + f_2 I^2 - 3g_2 Y I^2 \right. \\ & \left. -2g_2 I^3 + g_2 Y^3\right) \end{aligned} \tag{3.24}$$

Now using (3.8a),(3.22), (3.23) and (3.24) in (3.21) yields

$$\begin{aligned} Y'' &= f_2 Y^2 + g_2 Y^3 - \alpha Y' - \left(\frac{1}{2} a a'' - \frac{a'^2}{4} - 2f_2 I + 3g_2 I^2\right) Y - \frac{\alpha^2}{4} Y \\ & - \left(\frac{1}{2} a a'' - \frac{a'^2}{4}\right) I + f_2 I^2 - 2g_2 I^3 \\ & - a^{-(1/2)} \exp\left(-\int \frac{\alpha dx}{2a}\right) \left(ac' - \frac{a'c}{2} + \frac{\alpha c}{2}\right) - \frac{\alpha^2}{4} I \end{aligned} \tag{3.25}$$

We substitute (3.18) in (3.16) and we get

$$\begin{aligned} \frac{1}{2} a''' &= 2ca^{-(5/2)} \exp\left(-\int \frac{\alpha dx}{2a}\right) \left[f_2 - 3g_2 \int ca^{-(3/2)} \times \right. \\ & \left. \exp\left(-\int \frac{\alpha dx}{2a}\right) dx\right] \end{aligned} \tag{3.26}$$

Note that (3.26) can be obtained by integrating (3.19). On multiplying (3.26) by a and then integrating we get

$$\begin{aligned} \frac{1}{2} \int a a''' dx &= 2f_2 I - 6g_2 \int \left[ca^{-(3/2)} \exp\left(-\int \frac{\alpha dx}{2a}\right) \int ca^{-(3/2)} \times \right. \\ & \left. \exp\left(-\int \frac{\alpha dx}{2a}\right) dx\right] dx + M \end{aligned} \tag{3.27}$$

From (3.27) we have

$$M = \frac{1}{2} a a'' - \frac{1}{4} a'^2 - 2f_2 I + 3g_2 I^2 \tag{3.28}$$

Again we multiply (3.26) by aI and then integrate to obtain

$$\begin{aligned} \frac{1}{2} \int a a''' I dx &= 2f_2 \int ca^{-(3/2)} exp\left(-\int \frac{\alpha dx}{2a}\right) I dx \\ & -6g_2 \int ca^{-(3/2)} exp\left(-\int \frac{\alpha dx}{2a}\right) I^2 dx - N \end{aligned}$$

and therefore

$$\begin{aligned} N &= -a^{-(1/2)} \exp\left(-\int \frac{\alpha dx}{2a}\right)\left(ac' - \frac{1}{2}a'c + \frac{1}{2}\alpha c\right) \\ &\quad - \left(\frac{1}{2}aa'' - \frac{1}{4}a'^2 + \frac{\alpha^2}{4}\right) I + f_2 I^2 - 2g_2 I^3 \end{aligned} \tag{3.29}$$

Now substituting (3.28) and (3.29) in equation (3.25) yields

$$Y'' + \alpha Y' + \left(M + \frac{\alpha^2}{4}\right) Y = f_2 Y^2 + g_2 Y^3 + N \tag{3.30}$$

where f_2 and g_2 are arbitrary constants introduced in (3.17) and (3.18) respectively. The quantities M and N are arbitrary constants that arise in integrations of (3.19).

Note that in the neutral perfect fluid case we must have $g_2 = N = 0$ and (3.30) reduces to that of Maharaj *et al* (1996). However, it is difficult to make direct comparison to the results therein as the full equations are not always given, and the equation referencing is not always clear.

To proceed further, we need to analyse equation (3.19). We note that the form of (3.19) was obtained under the assumption that $c \neq 0$. We consider both nonzero c and vanishing c in turn in our subsequent analysis.

3.4.1 Case I: $c \neq 0$

When $C_0 \neq 0$ and $C_1 = 0$, (3.19) may be written as

$$aa^{(iv)} + \frac{1}{2}\left(5a' + \alpha\right) a''' = -12 g_2 a^{-3} \exp\left(-\int \frac{\alpha dx}{a}\right) \tag{3.31}$$

Rescaling a and x in (3.31) yields

$$aa^{(iv)} + \frac{1}{2}\left(5a' + \alpha\right)a''' = a^{-3}\exp\left(-\int \frac{\alpha dx}{a}\right) \tag{3.32}$$

when $\alpha \neq 0$ and

$$aa^{(iv)} + \frac{5}{2}a'a''' = a^{-3} \tag{3.33}$$

when $\alpha = 0$.

When $C_0 \neq 0$ and $C_1 \neq 0$, we apply transformation (3.10) to (3.19) and we obtain (again with the rescaling of A and X)

$$AA^{(iv)} + \frac{1}{2}\left(5A' - \alpha\right)A''' = A^{-3}\exp\left(\int \frac{\alpha dX}{A}\right) \tag{3.34}$$

when $\alpha \neq 0$ and

$$AA^{(iv)} + \frac{5}{2}A'A''' = A^{-3}$$

when $\alpha = 0$. Changing the sign of α in (3.34) brings it to the form of (3.32), and so the critical equations are (3.32) and (3.33).

Case I (a): $\alpha = 0$

If $\alpha = 0$ then (3.30) becomes

$$Y'' + MY = f_2Y^2 + g_2Y^3 + N \tag{3.35}$$

To solve equation (3.35) we firstly multiply by Y' to obtain

$$\frac{d}{dX}\left(\frac{1}{2}Y'^2 + \frac{1}{2}MY^2\right) = \frac{d}{dX}\left(\frac{1}{3}f_2Y^3 + \frac{1}{4}g_2Y^4 + NY\right) \tag{3.36}$$

When we integrate (3.36) we obtain

$$Y'^2 = \frac{1}{2}g_2Y^4 + \frac{2}{3}f_2Y^3 - MY^2 + 2NY + 2L \tag{3.37}$$

Again we integrate (3.37) to obtain the solution of (3.35) expressed as the quadrature

$$X - X_0 = \int \frac{dY}{\sqrt{\frac{1}{2}g_2Y^4 + \frac{2}{3}f_2Y^3 - MY^2 + 2NY + 2L}}$$

where L is an arbitrary constant introduced in the first integration of (3.35). A full discussion of the evaluation of this quadrature can be found in Kweyama *et al* (2010a).

Note that when $\alpha = 0$ in the transformation (3.20) we have

$$y = a^{1/2}(Y + I) \tag{3.38}$$

We substitute (3.38) in the Noether first integral (3.12) and we obtain

$$\begin{aligned} I_N &= \frac{1}{4}g_1Y^4 + \left(g_1I + \frac{a^{5/2}a'''}{12c}\right)Y^3 + \left[\left(\frac{3}{2}g_1I + \frac{a^{5/2}a'''}{4c}\right)I + \frac{1}{8}a'^2 \right. \\ &\left. -\frac{1}{4}aa''\right]Y^2 + \left[\left(g_1I + \frac{a^{5/2}a'''}{4c}\right)I^2 + \left(\frac{1}{4}a'^2 - \frac{1}{2}aa'''\right)I \right. \\ &\left. +a^{-(1/2)}\left(\frac{a'c}{2} - ac'\right)\right]Y - \frac{1}{2}Y'^2 + \left(\frac{1}{4}g_1I + \frac{a^{5/2}a'''}{12c}\right)I^3 \\ &+ \left(\frac{1}{8}a'^2 - \frac{1}{4}aa''\right)I^2 + a^{-(1/2)}\left(\frac{a'c}{2} - ac'\right)I + \frac{c^2}{2a} \end{aligned} \tag{3.39}$$

where

$$I = \int ca^{-(3/2)}dx$$

We substitute (3.8d) given by ($\alpha = 0$)

$$a''' = 4ca^{-(5/2)}(f_1 - 3g_1I) \tag{3.40}$$

in (3.39) and we obtain

$$
\begin{aligned}
I_N &= \frac{1}{4}g_1 Y^4 + \frac{1}{3}f_1 Y^3 - \frac{1}{2}\left(\frac{1}{2}aa'' - \frac{1}{4}a'^2 - 2f_1 I + 3g_1 I^2\right) Y^2 \\
&\quad + \left[a^{-(1/2)}\left(\frac{a'c}{2} - ac'\right) - \left(\frac{1}{2}aa'' - \frac{1}{4}a'^2\right) I + f_1 I^2 - 2g_1 I^3\right] Y \\
&\quad - \frac{1}{2}Y'^2 - \frac{1}{2}\left(\frac{1}{2}aa'' - \frac{1}{4}a'^2\right) I^2 + \frac{1}{3}f_1 I^3 - \frac{3}{4}g_1 I^4 \\
&\quad + a^{-(1/2)}\left(\frac{a'c}{2} - ac'\right) I + \frac{c^2}{2a} \\
&= \frac{1}{4}g_1 Y^4 + \frac{1}{3}f_1 Y^3 - \frac{1}{2}MY^2 + NY + \frac{1}{2}Y'^2 - \frac{1}{2}\left(\frac{1}{2}aa'' - \frac{1}{4}a'^2\right) I^2 \\
&\quad + \frac{1}{3}f_1 I^3 - \frac{3}{4}g_1 I^4 + a^{-(1/2)}\left(\frac{a'c}{2} - ac'\right) I + \frac{c^2}{2a} \qquad (3.41)
\end{aligned}
$$

where (with $\alpha = 0$)

$$M = \frac{1}{2}aa'' - \frac{a'^2}{4} - 2f_1 I + 3g_1 I^2$$

and

$$N = -a^{-(1/2)}\left(ac' - \frac{a'c}{2}\right) - \left(\frac{1}{2}aa'' - \frac{a'^2}{4}\right) I + f_1 I^2 - 2g_1 I^3$$

From (3.40) we have

$$\frac{1}{2}a''' = 2ca^{-(5/2)}\left(f_1 - 3g_1 I\right) \qquad (3.42)$$

We multiply (3.42) by aI^2 and then integrate to get

$$\int \frac{1}{2}aa''' I^2 dx = 2f_1 \int ca^{-(3/2)} I^2 dx - 6g_1 \int ca^{-(3/2)} I^3 dx - 2L \qquad (3.43)$$

We determine the integrals in (3.43) using integration by parts and we obtain

$$
\begin{aligned}
L &= -\frac{1}{2}\left(\frac{1}{2}aa'' - \frac{1}{4}a'^2\right) I^2 + \frac{1}{3}f_1 I^3 - \frac{3}{4}g_1 I^4 + \int \left[\left(\frac{1}{2}aa'' - \frac{1}{4}a'^2\right) \times \right. \\
&\quad \left. ca^{-(3/2)} \int ca^{-(3/2)} dx\right] dx) \qquad (3.44)
\end{aligned}
$$

If we differentiate the last two terms in (3.41) we obtain

$$\left(\frac{1}{2}aa'' - \frac{1}{4}a'^2\right) ca^{-(3/2)} \int ca^{-(3/2)} dx$$

Hence

$$\begin{aligned} &\int \left[\left(\frac{1}{2}aa'' - \frac{1}{4}a'^2\right) ca^{-(3/2)} \int ca^{-(3/2)} dx\right] dx \\ = &\ a^{-(1/2)} \left(\frac{a'c}{2} - ac'\right) I + \frac{c^2}{2a} \end{aligned} \tag{3.45}$$

We substitute (3.45) in (3.44) to obtain

$$\begin{aligned} L \ = &\ -\frac{1}{2}\left(\frac{1}{2}aa'' - \frac{1}{4}a'^2\right) I^2 + \frac{1}{3}f_1 I^3 - \frac{3}{4}g_1 I^4 \\ &+ a^{-(1/2)} \left(\frac{a'c}{2} - ac'\right) I + \frac{c^2}{2a} \end{aligned} \tag{3.46}$$

and on substituting (3.46) in (3.41) we get

$$I_N = \frac{1}{4}g_1 Y^4 + \frac{1}{3}f_1 Y^3 - \frac{1}{2}MY^2 + NY + \frac{1}{2}Y'^2 + L$$

Hence we have

$$-L = \frac{1}{4}g_1 Y^4 + \frac{1}{3}f_1 Y^3 - \frac{1}{2}MY^2 + NY - \frac{1}{2}Y'^2$$

which is the result of the first integration (refer to (3.37)) in reducing (3.35) to quadratures. This again indicates that the Noether results are a subset of the Lie results.

We now have to determine $a(x)$. We make the observation that if $\alpha = 0$ in (3.31) and $g_0 = 16g_2$ and $g_1 = g_2/C_1^2$, (3.5) and (**??**) become (3.31) which reduces to (3.33). We therefore consider (3.33) which is given by

$$aa^{(iv)} + \frac{5}{2}a'a''' = a^{-3} \tag{3.47}$$

for further analysis and reduction to quadratures. In carrying out the Lie analysis of (3.47), using PROGRAM LIE, we find that it has two Lie point symmetries, namely

$$G_1 = \frac{\partial}{\partial x} \tag{3.48a}$$

$$G_2 = x\frac{\partial}{\partial x} + \frac{4}{5}a\frac{\partial}{\partial a} \tag{3.48b}$$

Usually, when an nth order equation admits an $m < n$ dimensional Lie algebra of symmetries, there is little hope for the solution of the equation via those symmetries. However, in this case we are able to reduce the equation due to the presence of hidden symmetries (Abraham-Shrauner 1992).

The symmetry G_1 determines the variables for reduction

$$u = a, \qquad v = a'$$

and the reduced equation is

$$u^4v^3v''' + 4u^4v^2v'v'' + \frac{5}{2}u^3v^3v'' + u^4vv'^3 + \frac{5}{2}u^3v^2v'^2 - 1 = 0$$

This equation admits the following two symmetries

$$U_1 = u\frac{\partial}{\partial u} - \frac{1}{4}v\frac{\partial}{\partial v}$$

$$U_2 = 2u^2\frac{\partial}{\partial u} + uv\frac{\partial}{\partial v}$$

The variables for reduction via U_2 are

$$r = u^{-(1/2)}v, \qquad s = u^{3/2}v' - \frac{1}{2}u^{1/2}v$$

and the reduced equation is

$$r^3s^2s'' + r^3ss'^2 + 4r^2s^2s' + rs^3 - 1 = 0 \tag{3.49}$$

The Lie symmetry analysis of (3.49) yields the following two symmetries

$$X_1 = r\frac{\partial}{\partial r} - \frac{s}{3}\frac{\partial}{\partial s} \tag{3.50a}$$

$$X_2 = \frac{\partial}{\partial r} - \frac{s}{r}\frac{\partial}{\partial s} \tag{3.50b}$$

The reduction variables generated by X_2 are

$$p = rs, \qquad q = rs' + s$$

and the reduced equation is

$$p^2qq' + pq^2 - 1 = 0$$

with solution

$$q^2 = \frac{2}{p} + \frac{q_0}{p^2}$$

where q_0 is a constant. We can now invert these transformations to find the solution of (3.33). Alternatively, we can integrate (3.33) directly and write down the solution as

$$u_x = a^{-(3/2)} = [G'(u)]^{-1}$$

$$x - x_0 = G(u)$$

where we have set

$$G(u) = \int \frac{du}{(\mathcal{K}_4 + \mathcal{K}_3u + \mathcal{K}_2u^2 - (1/6)\mathcal{K}_1u^3 + (1/32)\mathcal{K}_0u^4)^{3/2}}$$

and the $\mathcal{K}_i, i = 0, \ldots, 3$ are constants of integration related to M, N, f_2 and g_2 and $\mathcal{K}_4$ is arbitrary. As pointed out earlier, this result was obtained in Kweyama *et al* (2010a), but for constant $c(x)$. In the case of nonconstant $c(x)$, the solution is the same, except that we replace a and x in this solution with A and X respectively. To obtain the solution to (3.19) (with $\alpha = 0$) we need to apply the inverse of (3.10).

Case I (b): $\alpha \neq 0$

When $\alpha \neq 0$, we cannot directly reduce (3.30) to quadratures. We need to investigate the constraints under which it possesses a second point symmetry. We find that if $f_2 \neq 0$, $g_2 \neq 0$ then (3.30) has the following two symmetries

$$G_1 = \frac{\partial}{\partial X} \tag{3.51a}$$

$$G_2 = e^{(\alpha/3)X}\frac{\partial}{\partial X} - e^{(\alpha/3)X}\left(\frac{\alpha}{3}Y + \frac{\alpha f_2}{9g_2}\right)\frac{\partial}{\partial Y} \tag{3.51b}$$

provided the following conditions are satisfied

$$M = -\frac{f_2^2}{3g_2} - \frac{\alpha^2}{36}, \qquad N = \frac{f_2^3}{27g_2^2} - \frac{2\alpha^2 f_2}{27g_2}$$

Utilising (3.51b) we obtain the transformation

$$\mathcal{X} = -\frac{3}{\alpha}e^{-(\alpha/3)X}, \qquad \mathcal{Y} = e^{(\alpha/3)X}\left(Y + \frac{f_2}{3g_2}\right)$$

and equation (3.30) becomes

$$\mathcal{Y}'' = g_2\mathcal{Y}^3$$

with solution

$$\mathcal{X} - \mathcal{X}_0 = \int \frac{d\mathcal{Y}}{\sqrt{\frac{g_2}{2}\mathcal{Y}^4 + \mathcal{C}}}$$

When $c \neq 0,\ g_2 = 0,\ f_2 \neq 0$ we find that $g \equiv 0$ (and so we are in the neutral perfect fluid realm). Now (3.30) has the following two symmetries

$$Y_1 = \frac{\partial}{\partial X} \tag{3.52a}$$

$$Y_2 = e^{(\alpha/5)X}\frac{\partial}{\partial X} + \left(\frac{\alpha^3}{500 f_2} + \frac{\alpha M}{5 f_2} - \frac{2\alpha}{5}Y\right) e^{(\alpha/5)X}\frac{\partial}{\partial Y} \tag{3.52b}$$

provided the following condition is satisfied

$$\begin{aligned} N &= \frac{M^2}{4 f_2} + \frac{\alpha^2 M}{8 f_2} + \frac{49\alpha^4}{40000 f_2} \\ &= \frac{1}{4 f_2}\left(M + \frac{\alpha^2}{4}\right)^2 - \frac{36\alpha^4}{2500 f_2} \end{aligned} \tag{3.53}$$

This condition is equivalent to the one obtained by Mellin *et al* (1994) for the case where $n = 2$ in their analysis of the generalised Emden-Fowler equation. We use (3.52b) to obtain the following transformation

$$\mathcal{X} = -\frac{5}{\alpha}e^{(-\alpha/5)X}, \qquad \mathcal{Y} = \left(Y - \frac{M}{2 f_2} - \frac{\alpha^2}{200 f_2}\right) e^{(2\alpha/5)X}$$

which, together with (3.53) reduces equation (3.30) to

$$\mathcal{Y}'' = f_2 \mathcal{Y}^2$$

with solution

$$\mathcal{X} - \mathcal{X}_0 = \int \frac{d\mathcal{Y}}{\sqrt{\frac{2 f_2}{3}\mathcal{Y}^3 + \mathcal{C}_1}}$$

We take this opportunity to make two minor corrections to the work of Mellin *et al* (1994): While the expression for N given by their equation (7.11) is correct, it is obtained by multiplying their equation (7.9) by

$$a\int da^{-3/2}\exp\left[\frac{1}{2}\int(p-2C_0/a)\,dx\right]dx$$

and then integrating, not multiplying by $a\int da^{-3/2}dx$ as indicated in their paper. Also, the coefficient of C_0 should be 2 in their equation (7.8).

3.4.2 Case II: $c=0$

From (3.14a), (3.14b), (3.14d) and (3.14e) we have

$$b = \frac{1}{2}(a'+\alpha) \tag{3.54a}$$

$$a = a_0+a_1x+a_2x^2 \tag{3.54b}$$

$$f = f_2a^{-5/2}\exp\left(-\int\frac{\alpha dx}{2a}\right) \tag{3.54c}$$

$$g = g_2a^{-3}\exp\left(-\int\frac{\alpha dx}{a}\right) \tag{3.54d}$$

The symmetry (3.13) now takes the form

$$G=a\frac{\partial}{\partial x}+\frac{1}{2}(a'+\alpha)\,y\frac{\partial}{\partial y}$$

Using the transformation

$$X = \int\frac{dx}{a},\qquad Y=ya^{-1/2}\exp\left(-\int\frac{\alpha dx}{2a}\right)$$

equation (3.1) is transformed into the autonomous equation

$$Y''+\alpha Y'+\beta Y=f_2Y^2+g_2Y^3 \tag{3.55}$$

where

$$\beta = \frac{1}{4}\left(\alpha^2-\Delta\right),\qquad \Delta=a_1^2-4a_0a_2$$

In carrying out the standard Lie point symmetry analysis on (3.55) we have the following cases:

Case II (a):

If $f_2 \neq 0$, $g_2 \neq 0$, (3.55) has the following two symmetries

$$G_1 = \frac{\partial}{\partial X} \tag{3.56a}$$

$$G_2 = e^{(\alpha/3)X}\frac{\partial}{\partial X} - e^{(\alpha/3)X}\left(\frac{\alpha}{3}Y + \frac{2\alpha^3}{9f_2}\right)\frac{\partial}{\partial Y} \tag{3.56b}$$

provided the following conditions apply

$$\beta = -\frac{4\alpha^2}{9}, \qquad g_2 = \frac{f_2^2}{2\alpha^2} \tag{3.57}$$

We use (3.56b) to obtain the following transformation

$$\mathcal{X} = -\frac{3}{\alpha}e^{-(\alpha/3)X}, \qquad \mathcal{Y} = e^{-(\alpha/3)X}\left(Y + \frac{2\alpha^2}{3f_2}\right) \tag{3.58}$$

Using (3.57) and (3.58) the equation (3.55) becomes

$$\mathcal{Y}'' = \frac{f_2^2}{2\alpha^2}\mathcal{Y}^3 \tag{3.59}$$

and the solution of (3.59) is

$$\mathcal{X} - \mathcal{X}_0 = \int \frac{d\mathcal{Y}}{\sqrt{\frac{f_2^2}{4\alpha^2}\mathcal{Y}^4 + \mathcal{C}}}$$

Case II (b):

If $f_2 = 0$, $g_2 \neq 0$ (which implies that $f = 0$), then (3.55) has the following two symmetries

$$G_1 = \frac{\partial}{\partial X} \tag{3.60a}$$

$$G_2 = e^{(\alpha/3)X}\frac{\partial}{\partial X} - \frac{\alpha}{3}e^{(\alpha/3)X}Y\frac{\partial}{\partial Y} \tag{3.60b}$$

subject to the following condition

$$\beta = \frac{2\alpha^2}{9} \tag{3.61}$$

Using (3.60b) we obtain the following transformation

$$\mathcal{X} = -\frac{3}{\alpha}e^{-(\alpha/3)X}, \qquad \mathcal{Y} = e^{(\alpha/3)X}Y \tag{3.62}$$

Using (3.61) and (3.62) equation (3.55) is transformed to

$$\mathcal{Y}'' = g_2\mathcal{Y}^3 \tag{3.63}$$

and the solution of (3.63) is

$$\mathcal{X} - \mathcal{X}_0 = \int \frac{d\mathcal{Y}}{\sqrt{\frac{g_2}{2}\mathcal{Y}^4 + \mathcal{C}}}$$

This is an intrinsically charged result - there is no uncharged analogue.

Case II (c):

If $f_2 \neq 0$, $g_2 = 0$ (which implies $g = 0$), then (3.55) has two symmetries provided

$$\beta = \pm\frac{6\alpha^2}{25} \tag{3.64}$$

and can be transformed to

$$\mathcal{Y}'' = f_2\mathcal{Y}^2$$

which has the solution

$$\mathcal{X} - \mathcal{X}_0 = \int \frac{d\mathcal{Y}}{\sqrt{\frac{2f_2}{3}\mathcal{Y}^3 + \mathcal{C}}}$$

This result was previously obtained in the neutral case by Maharaj *et al* (1996). A *consequence* of (3.64) is that $\Delta > 0$ (Note that this is not imposed on (3.55) as was done in Maharaj *et al* (1996).) and hence a in (3.54b) has real roots.

3.5 Discussion

We have undertaken a comprehensive analysis of (3.1) in order to determine which forms of the functions f and g would lead to first integrals and/or solutions of the equation. We reviewed our previous *ad hoc* approach, and showed that those results were contained in the results obtained via Noether's theorem. This occurred when the funciton c obtained in the Noether analysis was set to $\frac{1}{4}$. These latter results were then shown to be further contained with those obtain via the Lie analysis. This occurs when $\alpha = 0$. In this case (3.17) and (3.18) take the forms of (3.8b) and (3.8c) respectively, and so (3.8b) and (3.8c) are special cases of (3.17) and (3.18) respectively.

Usefully, the first two approaches, namely the *ad hoc* approach and the Noether symmetry analysis, yielded first integrals directly. The final (Lie) approach needed two stages of analysis in order to reduce the equation to quadratures.

We were able to completely analyse (3.1) for Lie point symmetries in an exhaustive analysis. All possible cases for the functions f and g were analysed. We showed that these results reduced to the neutral case results ($g = 0$) of Maharaj *et al* (1996) in most cases. However, we were also able to find an inherently charged case in §3.4.2 that has no uncharged analogue.

While a complete analysis was produced for the case of $\alpha = 0$, only a partial analysis could be performed when $\alpha \neq 0$. Nonetheless, we were still able to provide conditions under which (3.1) could be reduced to quadratures. It still remains to solve (3.32). The main difficulty is that this equation is an integro-differential equation and such equations are notoriously difficult to solve. However, we can still reduce (3.32) to a first order equation by eliminating a'' from (3.28) and (3.29). Further work in this direction is ongoing.

Note that, in the case that $c = 0$, we were able to find constraints under which we could reduce (3.1) to quadratures with f and g given explicitly and $\alpha \neq 0$. It is interesting to observe that the constraints we found in all subcases of §3.4.2 forced the quadratic a in (3.54b) to have real roots.

To complete our analysis, we analyse equation (3.33) for invariant solutions. We take the symmetries calculated in §3.4.1 and investigate the possibility of group invariant solutions of (3.33). The only two results of significance arise in the following cases: Firstly, we use the symmetry (3.48b) of the equation (3.33). This yields the invariant solution

$$a = -\left(\frac{625}{24}\right)^{1/5} x^{4/5} \tag{3.65}$$

of (3.33). Secondly, we use the symmetry (3.50a) of the equation (3.49). We obtain

$$x - x_0 = \pm \int \frac{da}{\left[-\left(\frac{32}{3}\right)^{1/3} a^{-(1/3)} + Ka^{2/3}\right]^{3/4}} \tag{3.66}$$

as the invariant solution of (3.33). The second solution can only be given implicitly and so is not of much practical use. However, if we let $c = \frac{1}{4}$ in (3.16), $\alpha = 0$ in (3.17) and $K = 0$ in (3.66) and then substitute each solution into (3.16) and (3.17) we obtain the explicit forms of f and g for which we can solve (3.1). Substituting (3.66) into (3.16) and (3.17) yields

$$\begin{aligned} f(x) &= \pm\left(\frac{20000}{768}\right)^{1/5}\left(\frac{24}{75}\right)(x - x_0)^{-(11/5)} \\ g(x) &= \pm\left(\frac{32}{3}\right)^{-(3/5)}\left(\frac{5}{4}\right)^{-(12/5)} g_2\,(x - x_0)^{-(12/5)} \end{aligned}$$

and if we substitute (3.65) into 3.16) and (3.17) we obtain

$$\begin{aligned} f(x) &= -\left(\frac{625}{24}\right)^{1/5}\left(\frac{24}{75}\right) x^{-(11/5)} \\ g(x) &= -\left(\frac{625}{24}\right)^{-(3/5)} g_2 x^{-(12/5)} \end{aligned}$$

Note that in both cases we obtain the forms

$$f(x) \propto (x - x_0)^{-(11/5)}, \qquad g(x) \propto (x - x_0)^{-(12/5)}$$

These forms for f and g were earlier obtained by Kweyama (2010a) using an *ad hoc* approach which yielded a new charged first integral to the Einstein-

Maxwell field equations. In this case, (3.1) admits two Lie point symmetries and can be reduced to quadratures.

We can also reuse (3.65) by invoking (3.10) to obtain

$$a = \kappa \left(\frac{c'}{c} \right)^{-(6/5)}$$

which is a solution to (3.19) provided

$$g_2 = -\frac{24\kappa^5}{625C_1^2}$$

This corresponds to the choice

$$f(x) \propto (x - x_0)^{-(14/5)}\,, \qquad g(x) \propto (x - x_0)^{-(18/5)}$$

for which (3.1) again has two Lie point symmetries and can be reduced to quadratures.

In summary, we have given a complete Noether and Lie point symmetry analysis of (3.1). For the Lagrangian (3.7) we were able to provide the most general Noether point symmetry, the most general first integral associated with this symmetry and indicated that this integral was equivalent to that found via an *ad hoc* approach, *i.e* (3.2). Finally we determined the most general Lie point symmetry admitted by (3.1) and gave conditions under which the equation could be reduced either to a first order equation, or to quadratures.

Chapter 4

A fifth order differential equation for charged perfect fluids

4.1 Introduction

We have demonstrated in previous chapters the value of elementary integration techniques, Noether symmetries and Lie point symmetries in providing first integrals and solutions to the governing dynamical equation in charged gravitating fluids. In this chapter we consider a different approach which we believe is unique and has not been considered before for the governing equation in shear-free spherically symmetric spacetimes. For the existence of a Lie point symmetry we derive, in §4.2, a fifth order purely differential equation that must be satisfied. A detailed analysis of this equation is performed. We solve, for the first time, the relevant integro-differential equation arising from the integration of the fifth order differential equation. A brief discussion of the results is given in §4.3.

4.2 Lie analysis

We can verify that

$$G = a(x)\frac{\partial}{\partial x} + (b(x)y + c(x))\frac{\partial}{\partial y}$$

is a symmetry of

$$y'' = f(x)y^2 + g(x)y^3 \tag{4.1}$$

The relationship among the functions $a(x)$, $b(x)$, $c(x)$, $f(x)$ and $g(x)$ is given by the following system of ordinary differential equations

$$a'' = 2b' \tag{4.2a}$$

$$b'' = 2fc \tag{4.2b}$$

$$c'' = 0 \tag{4.2c}$$

$$af' + (2a' + b)f = -3cg \tag{4.2d}$$

$$ag' + (2a' + 2b)g = 0 \tag{4.2e}$$

We now combine equations (4.2a)-(4.2e) into one fifth order ordinary differential equation. From (4.2a) we have

$$2b = a' + \alpha \tag{4.3}$$

where α is an arbitrary constant, from (4.2b) we have

$$f = \frac{a'''}{4c} \tag{4.4}$$

and from (4.2c) we have

$$c = C_0 + C_1 x \tag{4.5}$$

Previous analyses (Maharaj *et al* (1996) and Kweyama *et al* (2010b) have then solved (4.2d) and (4.2e) for f and g, respectively, and taking (4.4) into account, obtained the fourth order integro–differential equation for a:

$$caa^{(iv)} + \left[c\left(\frac{5a'}{2} + \frac{\alpha}{2}\right) - c'a\right] a''' = -12g_2c^3a^{-3}\exp\left(-\int \frac{\alpha dx}{a}\right) \tag{4.6}$$

The nature of this particular equation has made it difficult to deduce much information in general. Here we focus on obtaining a fifth order purely *differential* equation.

We substitute (4.3) and (4.4) into (4.2d) to obtain

$$g = -\frac{aa^{(iv)}}{12c^2} + \frac{aa'''c'}{12c^3} - \frac{5a'a'''}{24c^2} - \frac{\alpha a'''}{24c^2} \tag{4.7}$$

We then use (4.3) and (4.7) in (4.2e) to obtain the fifth order differential equation

$$-\alpha^2c^2a''' - 8\alpha c^2a'a''' - 15c^2a'^2a''' + 4\alpha acc'a''' + 18aca'c'a'''$$
$$-6a^2c'^2a''' - 5ac^2a''a''' - 3\alpha ac^2a^{(iv)} - 13ac^2a'a^{(iv)}$$
$$+6a^2cc'a^{(iv)} - 2a^2c^2a^{(v)} = 0 \tag{4.8}$$

where $c(x)$ is given by (4.5). This is the first time that the symmetry analysis of (4.1) has reduced to solving a fifth order differential equation. Given that the equation is purely differential, one can make recourse to numerical techniques if analytic solutions are elusive.

The fifth order equation (4.8) can be transformed into autonomous form via the transformation

$$X = \frac{C_1}{C_0 + C_1x}, \qquad A = \frac{aC_1^2}{(C_0 + C_1x)^2} \tag{4.9}$$

We end up with the following equation:

$$2A^2A^{(v)} + 13AA'A^{(iv)} - 3\alpha AA^{(iv)} + 5AA''A''' + 15A'^2A'''$$
$$-8\alpha A'A''' + \alpha^2A''' = 0 \tag{4.10}$$

Once we obtain solutions to this equation, we can find f and g through direct substitution into (4.4) and (4.7) respectively, after inverting (4.9).

We thus have the result that the equation

$$y'' = \frac{a'''}{4c}y^2 + \left(-\frac{aa^{(iv)}}{12c^2} + \frac{aa'''c'}{12c^3} - \frac{5a'a'''}{24c^2} - \frac{\alpha a'''}{24c^2}\right)y^3$$

admits the Lie point symmetry

$$G = a(x)\frac{\partial}{\partial x} + ((a'(x) + \alpha)y/2 + c(x))\frac{\partial}{\partial y}$$

where

$$c(x) = C_0 + C_1 x$$

and $a(x)$ satisfies the fifth order ordinary differential equation

$$\begin{aligned} & 2A^2A^{(v)} + 13AA'A^{(iv)} - 3\alpha AA^{(iv)} + 5AA''A''' + 15A'^2A''' \\ & -8\alpha A'A''' + \alpha^2 A''' = 0 \end{aligned} \tag{4.11}$$

with

$$X = \frac{C_1}{C_0 + C_1 x}, \qquad A = \frac{aC_1^2}{(C_0 + C_1 x)^2}$$

As observed previously (Kweyama *et al*, 2010b), this is a general result incorporating the *ad hoc* approach of Kweyama *et al* (2010a) and Wafo Soh and Mahomed (2000). However, note that the fifth order equation (4.10) has not been derived previously.

If we let $\alpha = 0$ in (4.10) we obtain

$$2A^2A^{(v)} + 13AA'A^{(iv)} + 5AA''A''' + 15A'^2A''' = 0 \tag{4.12}$$

The Lie analysis of (4.12), using PROGRAM LIE, gives the following three symmetries

$$\begin{aligned} V_1 &= \frac{\partial}{\partial X} \\ V_2 &= X\frac{\partial}{\partial X} \\ V_3 &= A\frac{\partial}{\partial A} \end{aligned}$$

The Lie bracket relationships are

$$[V_1, V_2] = V_1, \qquad [V_1, V_3] = 0, \qquad [V_2, V_3] = 0$$

We therefore use V_1 to reduce the order of (4.12). The variables for reduction are

$$u = A, \qquad v = A'$$

and the reduced equation is

$$2u^2v^4v^{(iv)} + 14u^2v^3v'v''' + 13uv^4v''' + 8u^2v^3v''^2 + 22u^2v^2v'^2v'' + 57uv^3v'v''$$
$$+15v^4v'' + 2u^2vv'^4 + 18uv^2v'^3 + 15v^3v'^2 = 0 \tag{4.13}$$

We analyse (4.13) for symmetries, and obtain the following three symmetries using PROGRAM LIE:

$$X_1 = u\frac{\partial}{\partial u}$$
$$X_2 = v\frac{\partial}{\partial v}$$
$$X_3 = 2u^2\frac{\partial}{\partial u} + uv\frac{\partial}{\partial v}$$

We determine the Lie bracket relationships of the symmetries and we obtain

$$[X_1, X_2] = 0, \qquad [X_1, X_3] = X_3, \qquad [X_2, X_3] = 0$$

We then use X_3 to reduce the order of (4.13). The variables for reduction are

$$r = u^{-(1/2)}v, \qquad s = u^{3/2}v' - \frac{1}{2}u^{1/2}v$$

and the reduced equation in terms of new variables r and s is

$$2r^4s^3s''' + 8r^4s^2s's'' + 14r^3s^3s'' + 2r^4ss'^3 + 22r^3s^2s'^2 + 22r^2s^3s' + 2rs^4 = 0 \tag{4.14}$$

The Lie symmetry analysis of (4.14) gives three symmetries, namely

$$\begin{aligned} U_1 &= r\frac{\partial}{\partial r} \\ U_2 &= s\frac{\partial}{\partial s} \\ U_3 &= \frac{\partial}{\partial r} - \frac{s}{r}\frac{\partial}{\partial s} \end{aligned}$$

The Lie bracket relationships of the symmetries are

$$[U_1, U_2] = 0, \qquad [U_1, U_3] = U_3, \qquad [U_2, U_3] = 0$$

We therefore use U_3 to reduce the order of (4.14). The variables for reduction are

$$p = rs, \qquad q = rs' + s$$

and the reduced equation is

$$2p^3q^2q'' + 2p^3qq'^2 + 8p^2q^2q' + 2pq^3 = 0$$

which simplifies to

$$p^2qq'' + p^2q'^2 + 4pqq' + q^2 = 0 \tag{4.15}$$

Note that equation (4.15) is in fact

$$\left(p^2q^2\right)'' = 0 \tag{4.16}$$

and the solution of (4.16) is

$$q^2 = \frac{C_2}{p} + \frac{C_3}{p^2} \tag{4.17}$$

Note that this result contains the result obtained by Kweyama *et al* (2010b) in §3.4.1.

We now return to (4.8) in the event that $\alpha \neq 0$. Integrating the equation (4.8) once yields the fourth order integro-differential equation

$$caa^{(iv)} + \left[c\left(\frac{5a'}{2} + \frac{\alpha}{2}\right) - c'a\right]a''' = -12g_2c^3a^{-3}\exp\left(-\int\frac{\alpha dx}{a}\right) \qquad (4.18)$$

In chapter 3 we observed that (4.18) can be obtained by differentiating the following equation once with respect to x

$$\begin{aligned}\frac{1}{2}a''' &= 2ca^{-(5/2)}\exp\left(-\int\frac{\alpha dx}{2a}\right)\left[f_2 - 3g_2\int ca^{-(3/2)}\times\right.\\ &\left.\exp\left(-\int\frac{\alpha dx}{2a}\right)dx\right] \qquad (4.19)\end{aligned}$$

We therefore deduce that integrating (4.18) yields (4.19). We multiply (4.19) by a and then integrate to get

$$\begin{aligned}\frac{1}{2}\int aa'''dx &= 2f_2I - 6g_2\int\left[ca^{-(3/2)}\exp\left(-\int\frac{\alpha dx}{2a}\right)\times\right.\\ &\left.\int ca^{-(3/2)}\exp\left(-\int\frac{\alpha dx}{2a}\right)dx\right]dx + M \qquad (4.20)\end{aligned}$$

where M is an arbitrary constant of integration. From (4.20) we have

$$M = \frac{1}{2}aa'' - \frac{1}{4}a'^2 - 2f_2I + 3g_2I^2 \qquad (4.21)$$

Again we multiply (4.19) by aI and then integrate to obtain

$$\begin{aligned}\frac{1}{2}\int aa'''Idx &= 2f_2\int ca^{-(3/2)}\exp\left(-\int\frac{\alpha dx}{2a}\right)Idx\\ &-6g_2\int ca^{-(3/2)}\exp\left(-\int\frac{\alpha dx}{2a}\right)I^2dx - N \qquad (4.22)\end{aligned}$$

where N is an arbitrary constant of integration. After performing the integrals in (4.22) we find that

$$\begin{aligned}N &= -a^{-(1/2)}\exp\left(-\int\frac{\alpha dx}{2a}\right)\left(ac' - \frac{1}{2}a'c + \frac{1}{2}\alpha c\right)\\ &-\left(\frac{1}{2}aa'' - \frac{1}{4}a'^2 + \frac{\alpha^2}{4}\right)I + f_2I^2 - 2g_2I^3 \qquad (4.23)\end{aligned}$$

From (4.21) we have

$$\frac{1}{2}aa'' - \frac{1}{4}a'^2 = M + 2f_2 I - 3g_2 I^2 \tag{4.24}$$

When substituting (4.24) in (4.23) we obtain the following equation

$$\begin{aligned} N &= -a^{-(1/2)} \exp\left(-\int \frac{\alpha dx}{2a}\right)\left(ac' - \frac{1}{2}a'c + \frac{1}{2}\alpha\ c\right) - MI - f_2 I^2 \\ &\quad + g_2 I^3 - \frac{\alpha^2}{4} I \end{aligned} \tag{4.25}$$

We thus have the result that the equation

$$y'' = \frac{a'''}{4c}y^2 + \left(-\frac{aa^{(iv)}}{12c^2} + \frac{aa'''c'}{12c^3} - \frac{5a'a'''}{24c^2} - \frac{\alpha a'''}{24c^2}\right)y^3$$

admits the Lie point symmetry

$$G = a(x)\frac{\partial}{\partial x} + ((a'(x) + \alpha)y/2 + c(x))\frac{\partial}{\partial y} \tag{4.26}$$

where

$$c(x) = C_0 + C_1 x$$

and $a(x)$ is constrained by the first order integro–differential equation

$$\begin{aligned} N &= -a^{-(1/2)} \exp\left(-\int \frac{\alpha dx}{2a}\right)\left(ac' - \frac{1}{2}a'c + \frac{1}{2}\alpha\ c\right) - MI - f_2 I^2 \\ &\quad + g_2 I^3 - \frac{\alpha^2}{4} I \end{aligned}$$

and M, N, f_2 and g_2 are all arbitrary constants of integration. This is the first time that the admittance of symmetry by (4.1) has been reduced to solving essentially a first order equation. This is a general result incorporating both the *ad hoc* approach of Kweyama *et al* (2010a) and the Noether symmetry results of Wafo Soh and Mahomed (2000).

In the event that $\alpha = 0$, we have that (4.25) reduces to

$$N = -a^{-(1/2)}\left(ac' - \frac{1}{2}a'c\right) - MI - f_2 I^2 + g_2 I^3 \tag{4.27}$$

this time with

$$I = \int c a^{-(3/2)} dx$$

We can now integrate (4.27) and obtain the same result as in (Wafo Soh and Mahomed 2000). See Kweyama *et al* (2010b) for a full discussion of this case.

It remains to solve (4.25) when $\alpha \neq 0$. If we set

$$X = \frac{C_1}{C_0 + C_1 x}, \qquad A = \frac{aC_1^2}{(C_0 + C_1 x)^2} \tag{4.28}$$

then (4.25) reduces to

$$\begin{aligned} N &= C_1\left[-A^{-(1/2)}\exp\left(\int \frac{\alpha dX}{2A}\right)\left(\frac{A'}{2} + \frac{\alpha}{2}\right)\right] + C_1\left(M + \frac{\alpha^2}{4}\right) I \\ &\quad - C_1^2 f_2 I^2 - C_1^3 g_2 I^3 \end{aligned}$$

where

$$I = \int A^{-(3/2)} \exp\left(\int \frac{\alpha dX}{2A}\right) dX$$

Hence

$$\begin{aligned} \frac{N}{C_1} &= -A^{-(1/2)}\exp\left(\int \frac{\alpha dX}{2A}\right)\left(\frac{A'}{2} + \frac{\alpha}{2}\right) + \left(M + \frac{\alpha^2}{4}\right) I \\ &\quad - C_1 f_2 I^2 - C_1^2 g_2 I^3 \end{aligned}$$

We therefore have the equation

$$\begin{aligned} A^{-(1/2)}\exp\left(\int \frac{\alpha dX}{2A}\right)\left(\frac{A'}{2} + \frac{\alpha}{2}\right) &= -\frac{N}{C_1} + \left(M + \frac{\alpha^2}{4}\right) I \\ &\quad - C_1 f_2 I^2 - C_1^2 g_2 I^3 \end{aligned} \tag{4.29}$$

We can now integrate (4.29) further to obtain

$$
\begin{aligned}
A^{1/2}\exp\left(\int\frac{\alpha dX}{2A}\right) &= \int\Big[-\frac{N}{C_1}+\left(M+\frac{\alpha^2}{4}\right)I-C_1f_2I^2 \\
&\quad -C_1^2g_2I^3\Big]\,dX+P \qquad (4.30)
\end{aligned}
$$

which is an implicit solution for (4.8) with $\alpha\neq 0$. We note that no reductions of (4.25) (or its fourth order counterpart (4.18)) have been previously found for nonzero α.

Having found a via (4.30) and (4.28) we now focus on (4.1). Using (4.26) we can transform (4.1) into the autonomous form

$$
Y''+\alpha Y'+\left(M+\frac{\alpha^2}{4}\right)Y=f_2Y^2+g_2Y^3+N \qquad (4.31)
$$

via the transformation

$$
X = \int\frac{dx}{a},\qquad Y=y\exp\left(-\int\frac{bdx}{a}\right)-\int\frac{c}{a}\exp\left(-\int\frac{bdx}{a}\right)dx
$$

The form (4.31) can be more directly analysed. However, when $\alpha\neq 0$, we find that we cannot directly reduce (4.31) to quadratures. Further (Lie) analysis yields that, when $f_2\neq 0$, $g_2\neq 0$ then (4.31) has the following two symmetries

$$
\begin{aligned}
G_1 &= \frac{\partial}{\partial X} \\
G_2 &= e^{(\alpha/3)X}\frac{\partial}{\partial X}-e^{(\alpha/3)X}\left(\frac{\alpha}{3}Y+\frac{\alpha f_2}{9g_2}\right)\frac{\partial}{\partial Y}
\end{aligned}
$$

provided the following conditions are satisfied

$$
M=-\frac{f_2^2}{3g_2}-\frac{\alpha^2}{36},\qquad N=\frac{f_2^3}{27g_2^2}-\frac{2\alpha^2 f_2}{27g_2} \qquad (4.32)
$$

Here, G_2 is a new symmetry – G_1 is just the transformed form of (4.26). If we further transform (4.31) using G_2 we obtain

$$\mathcal{Y}'' = g_2\mathcal{Y}^3$$

with solution

$$\mathcal{X} - \mathcal{X}_0 = \int \frac{d\mathcal{Y}}{\sqrt{\frac{g_2}{2}\mathcal{Y}^4 + \mathcal{C}}}$$

where

$$\mathcal{X} = -\frac{3}{\alpha}e^{-(\alpha/3)X}, \qquad \mathcal{Y} = e^{(\alpha/3)X}\left(Y + \frac{f_2}{3g_2}\right)$$

As noted previously (Kweyama *et al*(2010b)), the values in (4.32) correspond directly to a simplification of the eigenvalue problem associated with a dynamical systems analysis of (4.31).

4.3 Discussion

We derived (for the first time) a fifth order nonlinear differential equation, the solutions of which generate the solutions to the governing Einstein-Maxwell field equation. This equation was obtained by undertaking a Lie analysis of the master equation (4.1). In the case of $\alpha = 0$ we performed a Lie symmetry analysis of the fifth order differential equation. This allowed us to solve the equation. On letting $C_2 = 2$ in (4.17) we regain the result obtained by Kweyama *et al* (2010b). For the $\alpha \neq 0$ case, we were not able to make progress via a Lie symmetry analysis. However, we were able to integrate the fifth order differential equation directly. We obtained an implicit solution for this case - a solution that has not been found previously. We also looked at the implications of this reduction for the original equation.

Chapter 5

Conclusion

The focus of this thesis was to perform an investigation of the integrability conditions of the second order partial differential equation

$$y_{xx} = f(x)y^2 + g(x)y^3$$

governing the evolution of shear-free spherically symmetric charged fluids in general relativity. We presented an extensive analysis of this equation using different approaches.

In chapter 1 we provided some relevant background information. A discussion of some concepts behind the theory of Lie analysis was given with the aim of clarifying the approach. We also briefly outlined the theory of Noether symmetry analysis and invariant solutions.

In chapter 2 we reduced the Einstein-Maxwell field equations to a single nonlinear second order partial differential equation. This was achieved by generalising the transformation due to Faulkes (1969). The second order differential equation is the master equation governing the behaviour of shear-free spherically symmetric charged fluids. We also derived a first integral of the governing equation. This was achieved by generalising the technique first used by Srivastava (1987) for uncharged fluids. The first integral is subject to two conditions in the form of integral equations. We transformed the integral equations into a system of purely differential equations which could be integrated up to quadratures. This yielded a new class of solutions to the Einstein-Maxwell system for a charged shear-free fluid. This new solution was given in the parametric form. A detailed analysis was performed based on the nature of the factors of the quartic arising in the quadrature. In the case of one repeated linear factor we were able to invert the quadrature and wrote the first integral explicitly. As far as we know this is a new result. In the case of one order-three linear factor we could also invert the quadrature and regain the results of Maharaj *et al* (1996), Srivastava (1987) and Stephani (1983) for uncharged fluids.

In chapter 3 we undertook a comprehensive analysis of the master equation using symmetry methods. We analysed the equation for Noether symmetries and this analysis yielded a Noether first integral for the equation. By selecting a particular value of a parameter, we showed that the first integral obtained in chapter 2 using an *ad hoc* approach was a special case of the Noether first integral. We also analysed the governing equation for Lie point symmetries. The results obtained using the Noether analysis were further shown to be contained in those obtained via the Lie analysis. We considered all the possible cases for the functions f and g which are in the master equation (5). We were able to show that most of the results obtained in this analysis reduced to the results obtained by Maharaj *et al* (1996) for the uncharged case ($g = 0$). It was interesting to note that we were also able to find an inherently charged case that has no uncharged analogue. For the case where $\alpha \neq 0$ we only performed a partial analysis. The complete analysis of the integro-differential equation arising in this case was performed in chapter 4. In addition we investigated the possibility of group invariant solutions of the master equation. We were able to find conditions under which we could reduce the equation to quadratures with f and g given explicitly.

In chapter 4 we derived a fifth order differential equation, and this equation had not been obtained before for the charged case. The solutions of this equation can generate the solutions of the master equation. For $\alpha = 0$ we obtained a solution which is a generalisation of the result obtained in chapter 3. We also integrated the equation completely when $\alpha \neq 0$ and $c \neq 0$. Again this solution had not been obtained before. This is a remarkable result and should lead to new solutions which is the object of ongoing investigations.

Exact solutions of the Einstein-Maxwell system are important for applications in general relativity theory but not many classes of solutions are known. Our new classes of solutions presented in this thesis may be useful in this context, and could provide a deeper insight into the behaviour of the gravitational field. We have demonstrated in this thesis that applying the method of Lie symmetries provides new insights into the Einstein-Maxwell system of equations.

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