Transport on Network Structures

Proscovia Namayanja

Submitted in fulfillment of the academic requirements of the degree
Doctor of Philosophy
in the School of Mathematics, Statistics and Computer Science,
University of KwaZulu-Natal, Durban

March 08, 2013

Abstract

This thesis is dedicated to the study of flows on a network. In the first part of the work, we describe notation and give the necessary results from graph theory and operator theory that will be used in the rest of the thesis. Next, we consider the flow of particles between vertices along an edge, which occurs instantaneously, and this flow is described by a system of first order ordinary differential equations. For this system, we extend the results of Perthame [48] to arbitrary nonnegative off-diagonal matrices (ML matrices). In particular, we show that the results that were obtained in [48] for positive off diagonal matrices hold for irreducible ML matrices. For reducible matrices, the results in [48], presented in the same form are only satisfied in certain invariant subspaces and do not hold for the whole matrix space in general.

Next, we consider a system of transport equations on a network with Kirchoff-type conditions which allow for amplification and/or absorption at the boundary, and extend the results obtained in [33] to connected graphs with no sinks. We prove that the abstract Cauchy problem associated with the flow problem generates a strongly continuous semigroup provided the network has no sinks. We also prove that the acyclic part of the graph will be depleted in finite time, explicitly given by the length of the longest path in the acyclic part.

Preface and Declaration

The work described in this thesis was carried out in the School of Mathematics, Statistics and Computer Sciences, University of KwaZulu-Natal, Durban, from 2010 to December 2012 under the supervision of Professor Jacek Banasiak.

The studies done in this thesis are the original work of the author and have not been submitted in whole or in parts for any degree or diploma in any tertiary institution. Where use of the work of others has been made, proper citation has been done.

Signed: Proscovia Namayanja 08 March 2013

As the candidate's supervisor, I have approved this dissertation for submission.

Signed: Prof. J. Banasiak 08 March 20013

Dedication

Mukujjukira jjajjange Erina Namukasa Nantagya ne kitange Sam Tebukozza.

Declaration 2 - Publications

1. Banasiak, J and Namayanja, P. *Relative Entropy and Discrete Poincaré inequality for Reducible matrices*. Applied Mathematics Letters, 25. 2193-2197. Elsevier Ltd (2012).

2. Banasiak, J and Namayanja, P. *Asymptotic Behaviour of Flows on Reducible Networks.* Journal of Evolution Equations (submitted, under review)

Declaration 3 - Plagiarism

I, Proscovia Namayanja, declare that

1. The research reported in this thesis, except where otherwise indicated, is my original research.
2. This thesis has not been submitted for any degree or examination at any other university.
3. This thesis does not contain other persons data, pictures, graphs or other information, unless specifically acknowledged as being sourced from other persons.
4. This thesis does not contain other persons' writing, unless specifically acknowledged as being sourced from other researchers. Where other written sources have been quoted, then:
   a. Their words have been re-written but the general information attributed to them has been referenced
   b. Where their exact words have been used, then their writing has been placed in italics and inside quotation marks, and referenced.
5. This thesis does not contain text, graphics or tables copied and pasted from the Internet, unless specifically acknowledged, and the source being detailed in the thesis and in the References sections.

Signed

Acknowledgments

I am greatful to my supervisor Prof. Banasiak for all the help he has given me, providing reading material, making time for the many helpful discussions and generally providing a good working atmosphere for me. He has always encouraged me to learn more and has been helpful with many suggestions.

I am thankful to the Organisation for Women in Science for the Developing World (formally TWOWS) for their financial support, without which this research and stay in South Africa would have been hard.

I would like to thank the members of the AGFA team at the University of Tübingen for their hospitality during my short stay there. I especially thank Prof. Dr. Rainer Nagel and Dr. Britta Dorn for the helpful discussions and support, and Retha Heymann for making my stay as comfortable as possible.

Finally, I thank my sister M. Nankinga and brothers D. Ssemwogerere, Elijah Kibirango and J. B Mbogo for those frequent phone calls, emails and sms's, you helped me stay in touch with the rest of the family. I thank the entire family for their encouragement and words of support.

Contents

Chapter 1

Introduction

The study of flows on networks has been carried out for several years and is still an active area of research mainly due to its applications in the applied sciences. From the classical work of L.R. Ford and D. R. Fulkerson [20], much interest has been directed towards network flows. In the earlier works on flows [13], [6], [20], [12], most of the attention was directed towards finding a maximum flow from a source towards a certain destination (sink) subject to certain constraints, along certain links (edges). The constraints constitute what is referred to as the capacity of the edge; that is, the maximum amount that can flow along each edge. In addition to satisfying capacity restrictions, the flow is expected to obey Kirchoff's law at every intermediate vertex. That is; *the total inflow at a vertex v_i must equal the total outflow at v_i if it is an intermediate vertex.* Towards this goal, the famous Max- Flow Min-cut Theorem (or the generalised Max-Flow Min-Cut Theorem) gives the required flow by considering only the flow through the minimum cut. That is, the maximum flow in a network is equal to the capacity of the minimum cut, [13] Chapters 3, 4, or [6], Theorem 3.5.3.

Later, interest shifted from static flows to dynamic processes on networks. Differential operators were then considered on the edges of connected graphs, and such graphs have been called quantum graphs (for instance, see [35]). A quantum graph is a graph consisting of a set of vertices V and a set of edges $E \subseteq V \times V$, where each edge $e_i = (u, v) \in E$ is associated with an interval $[0, l_i]$ and a differential operator acting on the functions of the graph [34]. The number l_i is the length of edge e_i. Quantum graphs are often used in physics and engineering to approximate models of waves in complex structures [35], [37]. The operators considered are of second order (or higher). Examples of such operators include $-\frac{d^2}{dx^2}, -\frac{d^2}{dx^2}+v(x)$ [35], and they are often considered to be self adjoint, which then requires specific boundary/vertex conditions. It is therefore not surprising that a lot of effort has been placed on finding those vertex conditions that allow the operator to be self adjoint. For more on this topic, see [35], [37], [42], [57]. The article [35] provides numerous examples of boundary conditions and characterises those that allow the operator to be self adjoint.

In addition to looking for boundary conditions to ensure that the operator is self adjoint, other problems often discussed in this field include spectral theory (for example see [36], [55], [56], [58]) and inverse spectral problems, [26], [19]. For instance, certain authors have tried to show a relationship between the spectrum of differential operators and the graph itself. For example, the article [37] discusses the spectrum of the Laplace operator on a graph and relates it to certain geometric properties of the underlying graph, and in [38], it was proved that for a Laplace operator on a graph with m edges, the point 0 is an eigenvalue for the Laplacian with algebraic multiplicity $m+1$.

For the inverse spectral problem, one tries to reconstruct the graph from the spectrum of the differential operator defined on it. For example, the authors in [38] showed that a graph can be reconstructed in a unique way from the spectrum of the Laplace operator with free boundary conditions defined at its vertices, provided the lengths of the edges are rationally independent; that is, if there exists no positive number k such that

$$k\left(\frac{1}{l_1}+\frac{1}{l_2}+\cdots+\frac{1}{l_m}\right)\in\mathbb{N},$$

where m is the number of edges in the graph. For more information on wave propagation on networks, see [43], [42], [47], [42], [56].

But while a lot has been done for second order differential operators on graphs, very little attention had been given to first order differential operators. Recently however, various authors [33], [14], [40], [18], have considered the flow problem described by a transport equation on a simple directed graph. To state the problem briefly, suppose that G is a simple, connected, directed graph with a finite number of vertices and edges and flow of particles occurs on the edges, in the direction of the arrows. On edge e_j, particles move with speed c_j and, at position $x\in[0,l_j]$ and time t, the density of these particles is given by $u_j(x,t)$. Since the density depends on both time and space, the resulting flow problem is described by a partial differential equation, in this case a transport equation:

$$\begin{cases} \partial_t u_j(x,t) &= c_j \partial_x u_j(x,t), \\ u_j(x,0) &= f_j(x), \end{cases}$$

accompanied by the Kirchoff law at the vertices as the boundary condition. This describes the interaction of the particles in the vertices before they are redistributed into the outgoing edges at the vertex in question.

For such flows, the focus has been mainly on determining solvability of the problem and its asymptotics [14], [33]. In this regard, semigroup methods have been the primary tool used and we shall follow this trend here. First, one reformulates the problem as an Abstract Cauchy Problem (ACP)

$$ACP \begin{cases} \mathbf{u}'(t) &= A\mathbf{u}(t) \\ \mathbf{u}(0) &= \mathbf{f} \end{cases}$$

for $\mathbf{f}$ in the domain $D(A)$ for which the flow problem on the graph and the abstract problem both make sense, and this domain must be chosen with care so that it captures the boundary/vertex conditions of the flow problem. One then proceeds to investigate the well-posedness of this problem. By this, we mean that one must show that a solution to (ACP) exists and it is unique. In addition, this solution must depend continuously on initial data. If $(A, D(A))$ generates a semigroup, then, by Corollary II.6.9 of [16], it is well-posed and hence the transport problem on the network is solvable and its solution is unique. The asymptotic behaviour of the solution can then be studied through the spectrum of the semigroup, if the explicit formula is available, but since this is rarely the case, we can instead study the spectrum of the operator $(A, D(A))$ and use spectral mapping theorems to understand the spectrum of the semigroup (see Chapter IV of [16], [17], [3] or [46]).

Sikolya [33], Matrai and Sikolya [40], Dorn [14], Engel et al [18], have used a semigroup approach to study transport equations on networks. In most papers, it is assumed that the graph is strongly connected with a Kirchoff type law at the boundary (vertices). Moreover, it is assumed that every vertex has an incoming and outgoing edge and that there is no loss or generation of material in the vertices. In this thesis, we follow the work of these authors but we do not assume strong connectedness of the graph. We also allow for absorption and/or generation to take place in the vertices, which results in a modified Kirchoff law at the boundary. We then investigate the conditions on the graph for the operator $(A, D(A))$ to generate a strongly continuous semigroup, which will imply existence of a unique solution to the transport equation on the network. We then study the asymptotic behaviour of this solution, and then we interpret these results from a graph theoretic point of view.

The first part of this work is a preliminary chapter where we give the notation and terminology that will be used in the thesis. We give some background information from graph theory, with illustrations and examples. In the section for graphs, we give some proofs for certain results that will be used here, some of which are not easily available in literature. We also give a brief overview of Banach lattices and semigroup theory, focusing on those established results which will be used in this thesis. We discuss the Perron-Frobenius theorems for non-negative matrices and ML matrices, which will be used later to study the asymptotic behaviour of the solutions of both the finite dimensional flow problem and the flow problem described by a partial differential equation.

In Chapter 3, we consider a simplified model of the flow of particles from one vertex to another. In this model, the change of state occurs instantaneously and thus, the flow problem is described by a system of ordinary differential equation. We study the asymptotic behaviour of the flow and extend the results obtained by Perthame [48] to arbitrary ML matrices. In particular, we show that the discrete Poincaré lemma, Lemma 6.4 of [48], formulated for positive off diagonal matrices, extends to irreducible matrices, but does not hold in the same way for reducible matrices. Instead, it only holds in certain invariant subspaces of reducible matrices. We use these results together with the help of the relative entropy function to show that there is a norm on $\mathbb{R}^n$ in which the quadratic entropy function is dissipative in the space complementary to that spanned by the Perron eigenvector. This chapter provides a more detailed description of the results in the paper [4].

In Chapter 4, we consider connected finite graphs which are not strongly connected. We improve and extend the results of Sikolya and Kramar [33] to allow for vertices with no incoming flow (sources). We note that the Kirchoff law stated in Equation 3 of [40], [33] and in Equation (2.2) of [14] is incorrect if $c_j \neq 1$ for all j. We rectify this problem and also state a more generalised boundary condition that allows for absorption and/or generation to take place in the vertices. We show that $(A, D(A))$ described in problem (ACP) associated with the flow problem (with absorption and generation at the vertices) generates a strongly continuous semigroup if and only if the outgoing incident matrix is surjective, which is equivalent to saying that every vertex has at least one outgoing edge. We then study the asymptotic behaviour of the semigroup solution using Perron-Frobenius type theorems.

Next, we interpret the results from Chapter 4 from a graph view point in Chapter 5. In particular, we show that asymptotically the mass will collect in the strongly connected components of the graph which have no outgoing flow and the acyclic part of the graph will be depleted in finite time which depends on the length of the longest path (that is, the longest path in the acyclic part of the graph), while the strongly connected components of the graph (with outgoing flow) will be depleted asymptotically.

We conclude the thesis with a brief summary of this research and give other problems related to the networks that could lead to further research in the future.

Chapter 2

Preliminaries

In this chapter, we give some background information necessary to develop the theory of partial differential equations on networks. We start with a few definitions and results in graph theory which will be important in our study of flows on networks.

2.1 Notation

Let A be an $n \times n$ matrix. We write $A \geq 0$ to mean that all the entries in A are non-negative and if the inequality is strict, then all the entries in A are strictly positive. The notation $|A|$ will mean $|A| := (|a_{ij}|)_{1 \leq i,j \leq n}$. Likewise, $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is a non-negative vector if all the components x_i are non-negative and we write $\mathbf{x} \geq \mathbf{0}$ and, if all the components are strictly positive, the inequality is strict. We shall use the notation $|\mathbf{x}|$ to mean $|\mathbf{x}| = (|x_i|)_{1 \leq i \leq n}$. In this thesis, $\mathbb{C}, \mathbb{R}, \mathbb{N}, \mathbb{Q}$ are sets of complex, real, natural, and rational numbers, respective

2.2 Digraphs

Definition 2.2.1. *A graph G is a finite non-empty set V of elements called vertices together with a set of unordered pairs of distinct vertices of G (called edges). A digraph is an ordered pair $G = (V, E)$ containing a finite non-empty set of vertices V and a (possibly empty) set of ordered pairs $(u, v) \in E$; $u, v \in V$ (called the edges, $E \subseteq V \times V$).*

Let u, v be two vertices of a digraph G. If there is an edge with head at v and tail at u, we represent this as $u \longrightarrow v$. If the edges are labelled or if the label on the edge is important, we shall indicate the label on the arrow (for example $u \xrightarrow{e} v$ shows an edge e with tail at u and head at v), otherwise no labels will be provided.

Throughout this thesis, we shall sometimes, where there is no danger of confusion, simply write graph to mean a digraph. Let $G = (V, E)$ be a directed graph with $E \subset V \times V$.

Definition 2.2.2. *Let u, v be vertices of a graph. A $u-v$ walk of graph G is a finite alternating sequence of vertices and edges, beginning at u and ending with vertex v, [10]. The number of edges in a walk is the length of the walk.*

A $u - v$ path is a walk in which no vertex is repeated.

Definition 2.2.3. *G is said to be connected if for every pair u, v of vertices in G, there is either a $u - v$ or a $v - u$ path. It is called strongly connected if for every pair $u, v \in V$, there is a $u - v$ and a $v - u$ path in G.*

A subgraph of G is a graph G' whose vertex set $V(G')$ is a subset of V and $E(G') \subseteq E$ and it is an induced subgraph if two vertices in G' are connected if and only if they are connected in G, [23]. Strongly connected components of the graph G are the maximal induced subgraphs which are strongly connected. If G' is a strongly connected component of G, then its vertex set V_1 is a subset of V and there is a directed path from each vertex in V_1 to every other vertex in V_1. If V_1 and V_2 are vertex sets of strongly connected components of G and $V_1 \neq V_2$, then $V_1 \cap V_2 = \emptyset$, [6] on p. 17, or [7]. In this thesis, when we say that a subgraph is a strongly connected component of a graph, then its vertex set must contain at least two elements.

If a vertex has no outgoing edges, it is called a sink and a source if it has no incoming edges. If it has no outgoing or incoming edges, we say that it is isolated. When we say that a strongly connected component has no outgoing edges (or no outgoing flow), we mean that there exists no vertex $v \in V \setminus V_1$ such that $u \to v$ for any $u \in V_1$, where V_1 is the vertex set of this component. Similarly, when we say that a strongly connected component has no incoming edges, we mean that there is no vertex $v \in V \setminus V_1$ such that $v \to u$ for any $u \in V_1$.

Definition 2.2.4. *Let G be a digraph with vertex set $V = \{v_1, v_2, \cdots v_n\}$. The outgoing Φ^- and incoming Φ^+ incidence matrices of this graph are defined, respectively, as $\Phi^- = (\phi_{ij}^-)$ and*

$\Phi^+ = (\phi^+_{ij})$, *where*

$$\phi^-_{ij} = \begin{cases} 1 & \text{if } v_i \xrightarrow{e_j} \\ 0 & \text{otherwise.} \end{cases} \quad \text{and} \quad \phi^+_{ij} = \begin{cases} 1 & \text{if } \xrightarrow{e_j} v_i \\ 0 & \text{otherwise.} \end{cases}$$

Remark 2.2.5. *Note that both the outgoing and incoming incidence matrices can have at most one non-zero entry in each column, otherwise an edge would have more than one tail or head, respectively.*

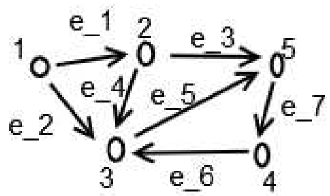
The incidence matrix Φ of the graph is then given by $\Phi = \Phi^+ - \Phi^-$.

Definition 2.2.6. *The adjacency matrix $\tilde{\mathbb{A}}$ of a graph G is defined as $\tilde{\mathbb{A}} = (\tilde{a}_{ij})$, where*

$$\tilde{a}_{ij} = \begin{cases} 1 & \text{if there exists } \; e_k \in E \text{ such that } v_j \xrightarrow{e_k} v_i \\ 0 & \text{otherwise.} \end{cases}$$

Remark 2.2.7. *We note that in most texts on graph theory, the adjacency matrix of a digraph G is the transpose of the matrix defined in Definition 2.2.6. Transposing the adjacency matrix of any directed graph does not change important graph properties like connectedness but simply reverses the direction of the arcs (edges) in the graph. Throughout this thesis, we shall refer to the matrix in Definition 2.2.6 whenever we mention the adjacency matrix of G.*

Example 2.2.8. *Consider the following graph. For this graph, the matrices Φ^-, Φ^+ are given*



below.

$$\Phi^- = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \Phi^+ = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix},$$

while the adjacency matrix is given by

$$\tilde{\mathbb{A}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

A permutation of any two rows of the matrix $\tilde{\mathbb{A}}$ followed by a permutation of the corresponding columns does not change the graph, but will simply change the order of the vertices, [12].

Definition 2.2.9. *The line graph of a digraph G is the graph $L(G) = (V(L), E(L))$, where $V(L) = E(G)$ and*

$$E(L) = \{e_{ij} \text{ if there exist } e_i, e_j \text{ in } G \text{ such that the head of } e_i \text{ coincides with the tail of } e_j\}.$$

In other words, the line graph of a directed graph G is the graph obtained from G by letting the edges in G be the vertices of $L(G)$ (that is, $V(L(G)) = E(G)$). If e_i is an edge in G, we transform this into a vertex in the line graph and label it as v_i' and if v_i' and v_j' are vertices in $L(G)$, then there is an edge from v_i' to v_j' if in G, there is a vertex v such that $\xrightarrow{e_i} v \xrightarrow{e_j}$, otherwise there is no edge from v_i' to v_j'.

Remark 2.2.10. *Any $u - v$ path in G corresponds to a unique path in $L(G)$. To see this, suppose that $u, e_{i_1}, u_1, e_{i_2}, \ldots, e_{i_l}, u_l = v$ is a path; that is,*

$$u \xrightarrow{e_{i_1}} u_{i_1} \xrightarrow{e_{i_2}} \cdots \xrightarrow{e_{i_l}} u_{i_l} = v$$

is a path. Then, from the definition of a line graph, the edges e_{i_h} translate into vertices u'_{i_h}, for all $h = 1, \ldots, l$ in $L(G)$ and there is a path connecting them:

$$u'_{i_1} \xrightarrow{e_{i_1, i_2}} u'_{i_2} \xrightarrow{e_{i_2, i_3}} \cdots \xrightarrow{e_{i_{l-1}, i_l}} u'_{i_l}.$$

Conversely, any $u' - v'$ path in $L(G)$ corresponds to a path in G,

$$u' = u'_0 \xrightarrow{e^1} u'_1 \xrightarrow{e^2} \cdots \xrightarrow{e^k} u'_k = v',$$

Vertices u_i' in $L(G)$ correspond to edges e_{u_i} in G, for all $i = 0, \ldots, k$. For edge e^1 to exist in $L(G)$, there must be a vertex u in G which is the head of e_u and the tail of e_{u_1}. Similarly, *there exists a vertex u_1 in G with the head of e_{u_1} and the tail of e_{u_2}. If we continue with this process, we obtain the following*

$$w \xrightarrow{e_u} u \xrightarrow{e_{e_{u_1}}} u_1 \longrightarrow \cdots \xrightarrow{e_{u_k}} u_k \xrightarrow{e_v} x$$

which is a path in G.

The adjacency matrix $\mathbb{B}$ of the line graph $L(G)$ is the matrix defined as $\tilde{\mathbb{B}} = (\tilde{b}_{ij})$, where

$$\tilde{b}_{ij} = \begin{cases} 1 & \text{if there exists} \quad v_k \in V(G) \text{ such that } \xrightarrow{e_j} v_k \xrightarrow{e_i}, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 2.2.11. *Notice that the definition for the adjacency matrix of the line graph agrees with the general definition of adjacency matrix of a graph given in (2.2.6). If we treat the line graph of G as any graph, we can write down its adjacency matrix using Definition 2.2.6 as*

$$\tilde{b}_{ij} = \begin{cases} 1 & \textit{if } v'_j \longrightarrow v'_i, \\ 0 & \textit{otherwise.} \end{cases}$$

Recalling the relationship between G and $L(G)$, we note that $v'_j \longrightarrow v'_i$ means that there is $u \in V(G)$ such that $\xrightarrow{e_j} u \xrightarrow{e_i}$, which is equivalent to the above definition.

If there are no parallel edges in G and there is more than one outgoing edge at some vertex v_i, we may place weights on these edges. Let the weight on edge e_j (whose tail is at vertex v_i) be w_{ij}. Then the weighted outgoing incidence matrix Φ_w^- is the matrix defined as

$$(\phi_w^-)_{ij} = \begin{cases} w_{ij} & \text{if } \phi_{ij}^- = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

The weighted adjacency matrix for the line graph (denoted here as $\mathbb{B}$) is defined as

$$b_{ij} = \begin{cases} w_{ki} & \text{if } \xrightarrow{e_j} v_k \xrightarrow{e_i}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

Below we provide another representation of matrix $\mathbb{B}$ and the weighted adjacency matrix $\mathbb{A}$.

Lemma 2.2.12. *[33]*

Let Φ^+ be the incoming incidence matrix and Φ_w^- be the weighted outgoing incidence matrix for a directed graph G. Then the following statements hold:

1. *The weighted adjacency matrix of the line graph is given by* $\mathbb{B} = (\Phi_w^-)^T\Phi^+$.

2. *The weighted adjacency matrix* $\mathbb{A}$ *satisfies the equation* $\mathbb{A} = \Phi^+ \left(\Phi_w^-\right)^T$.

3. *If every vertex has an outgoing edge and*

$$\sum_{j\in J_i} w_{ij} = 1, \tag{2.3}$$

where J_i *is the set of indices of the outgoing edges of vertex* v_i*, then* $\mathbb{B}$ *is column stochastic.*

Proof. Let $D = (\Phi_w^-)^T\Phi^+$. Then $d_{ij} = ((\Phi_w^-)^T)_i(\Phi^+)_j^*$, where $((\Phi_w^-)^T)_i$ is the i^{th} row of $(\Phi_w^-)^T$ and $(\Phi^+)_j^*$ is the j^{th} column of Φ^+. Φ_w^- has at most one non zero entry in each column hence $(\Phi_w^-)^T$ has at most one non-zero entry in each row. Let the non-zero entry in $((\Phi_w^-)^T)_i$ be in the k^{th} column. This would imply that the element in the k^{th} row and i^{th} column of Φ_w^- is not zero and $(\phi_w^-)_{ki} = w_{ki}$ and, from (2.1), we have

$$v_k \xrightarrow{e_i}. \tag{2.4}$$

Then $d_{ij} = ((\Phi_w^-)^T)_i(\Phi^+)_j^* = (0,\ldots,w_{ki},0,\ldots,0)(\Phi^+)_j^*$. This product is positive if the non-zero entry in $(\Phi^+)_j^*$ is in the k^{th} row, otherwise $d_{ij} = 0$. But if $\phi_{kj}^+ = 1$, then

$$\xrightarrow{e_j} v_k. \tag{2.5}$$

Combining (2.4) and (2.5) gives $d_{ij} = w_{ki}$ if there is k such that $\xrightarrow{e_j} v_k \xrightarrow{e_i}$, and zero otherwise. Hence $b_{ij} = d_{ij}$ for all i, j.

To prove the second item (2), we show first that $\Phi^+ \left(\Phi^-\right)^T$ is indeed the adjacency matrix described in the Definition 2.2.6. Let $F = \Phi^+ \left(\Phi^-\right)^T$. Then

$$\begin{aligned} f_{ij} &= \left(\phi_{i1}^+,\ldots,\phi_{ik}^+,\ldots,\phi_{im}^+\right)\begin{pmatrix}\phi_{j1}^-\\ \vdots\\ \phi_{jk}^-\\ \vdots\\ \phi_{jm}^-\end{pmatrix}\\ &= \sum_{k=1}^{m}\phi_{ik}^+\phi_{jk}^-. \end{aligned}$$

The sum is positive if there is at least one k such that both ϕ_{ik}^+ and ϕ_{jk}^- are positive. But $\phi_{ik}^+ > 0$ implies $\xrightarrow{e_k} v_i$ and $\phi_{jk}^- > 0$ implies that $v_j \xrightarrow{e_k}$. Hence $v_j \xrightarrow{e_k} v_i$. Now we note that for fixed i and j, there is at most one k such that $v_j \xrightarrow{e_k} v_i$. This is because of the assumption that there are no parallel edges. So f_{ij} is either equal to 0 or 1 and $f_{ij} = 1$ if $v_j \xrightarrow{e_k} v_i$ and 0 otherwise. Hence $F = \mathbb{A}$.

From this we have that the weighted adjacency matrix $\mathbb{A} = \Phi^+ \left(\Phi_w^-\right)^T$ or, explicitly,

$$a_{ij} = \begin{cases} w_{jk} & \text{if } v_j \xrightarrow{e_k} v_i, \\ 0 & \text{otherwise.} \end{cases} \tag{2.6}$$

From Equation (2.2), computing the column sums of $\mathbb{B}$, we obtain

$$\sum_{i=1}^{m} b_{ij} = \sum_{i=1}^{m} w_{ki} = 1$$

for each $j \in \{1, \ldots, m\}$. For the last equality, we have used (2.3). ∎

Recall that for an $n \times n$ matrix A, we say that it is irreducible if there is no permutation that puts it in the form

$$A^* = \begin{pmatrix} A_1 & 0 \\ A_{2,1} & A_2 \end{pmatrix}, \tag{2.7}$$

where A_1 and A_2 are square matrices. Let $A \geq 0$ be an $n \times n$ reducible matrix. Then by permuting its rows, followed by similar permutation of the columns, we can put it in the form (2.7) and, if A_1 or A_2 are still reducible, we repeat the process for these sub matrices until all the square matrices on the main diagonal are either irreducible or 0. We say that the matrix A is in normal form if it is written in the form

$$A = \begin{pmatrix} A_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & A_g & 0 & \cdots & 0 \\ A_{g+1,1} & A_{g+1,2} & \cdots & A_{g+1,g} & A_{g+1} & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ A_{s,1} & A_{s,2} & \cdots & A_{s,g} & A_{s,g+1} & \cdots & A_s \end{pmatrix}, \tag{2.8}$$

where A_i are $n_i \times n_i$ matrices for all $i = 1, 2, \ldots, s$ and they are either irreducible or zero matrices of dimension 1 (see [21], Equation 69).

Definition 2.2.13. *A graph G_1 is isomorphic to a graph G_2 if there exists a one to one mapping f from $V(G_1)$ onto $V(G_2)$ such that $(u, v) \in E(G_1)$ if and only if $(f(u), f(v)) \in E(G_2)$.*

We note that important graph properties like connectedness and orientation are preserved under graph isomorphisms. Let G be a graph with a finite number of vertices and edges. The following result holds:

Lemma 2.2.14. *[44], p.671*

The adjacency matrix $\mathbb{A}$ of a graph G is irreducible if and only if G is strongly connected.

Proof. Suppose that A is reducible, then there is a permutation matrix P that such that

$$\tilde{A} = P^T A P = \begin{pmatrix} A_1 & 0 \\ A_{2,1} & A_2 \end{pmatrix},$$

where A_1 is an $r \times r$ matrix and A_2 is $(n-r) \times (n-r)$ matrix. The zero matrix in $\tilde{A}$ means that the vertices from the set $V_1 = \{v_1, \cdots, v_r\}$ are not accessible from any vertex in the set $V_2 = \{v_{r+1}, \cdots, v_n\}$; that is, if $v_i \in V_1$ and $v_j \in V_2$, then there is no path (of any length) from v_j to v_i. Therefore, the directed graph of $\tilde{A}$ is not strongly connected. Since the graph of A, G_A, is isomorphic to that of $\tilde{A}$, we conclude that the graph of A is not strongly connected.

Now suppose that G_A is not strongly connected. Then there are at least two vertices v_i and v_j such that one is inaccessible from the other. If v_i is inaccessible from v_j, then relabel the vertices such that v_i becomes v_1 and v_j becomes v_n. Any other vertices that are inaccessible from v_j are renamed $v_2, \cdots, v_r$. Therefore the set of vertices that are inaccessible from v_j (relabelled v_n) is $V_1 = \{v_1, \cdots, v_r\}$. All other vertices that are accessible from v_j are relabelled $v_{r+1}, \cdots, v_{n-1}$ and no vertex $v_l \in V_1$ can be accessed from any vertex $v_k \in V_2 = \{v_{r+1}, \cdots, v_n\}$ because if there is a $v_k \in V_2$ such that the edge (v_k, v_l) exists, then the vertex v_l would be accessible from v_n by taking the path $v_n \to v_k \to v_l$ which is not possible.

Let Π be a permutation on the set $\{1, \cdots, n\}$ such that if $i \in \{1, \cdots, n\}$, then Π transforms i into π_i. Then $a_{\pi_i, \pi_j} = 0$ for each $\pi_j \in \{r+1, \cdots, n\}$ and $\pi_i \in \{1, \cdots, r\}$. So, if P is the permutation matrix defined by Π and $\tilde{A} = P^T A P$, then $\tilde{a}_{ij} = a_{\pi_i \pi_j} = 0$ for $\pi_j \in \{r+1, \cdots, n\}$ and $\pi_i \in \{1, \cdots, r\}$. Thus

$$\tilde{A} = \begin{pmatrix} A_1 & 0 \\ A_{2,1} & A_2 \end{pmatrix}.$$

■

We state another important result which will be used in this thesis. Although the result is not new, we could not find its proof in the texts we read. So we provide a proof below.

Proposition 2.2.15. *Let G be a connected digraph. Then $L(G)$ is strongly connected if and only if G is strongly connected.*

Proof. Suppose that G is strongly connected. Take an arbitrary pair of vertices u', v' in $L(G)$. A vertex in $L(G)$ is obtained from an edge in G. Let the corresponding edges in G be e_u and e_v, respectively. Let u be the head of e_u and v be the tail of e_v. Since G is strongly connected, there is a path from u to v and from v to u. Let the path be

$$u_0 \xrightarrow{e_u} u \xrightarrow{e_1} u_{i_1} \xrightarrow{e_{i_2}} \cdots \xrightarrow{e_{i_l}} v \xrightarrow{e_v} v_0.$$

But each edge on the $u - v$ path in G translates into a vertex in $L(G)$ and these vertices are connected. Hence there is a $u' - v'$ path in $L(G)$ (see Remark 2.2.10). Since u', v' were chosen arbitrarily, we conclude that $L(G)$ is strongly connected.

Conversely, suppose that $L(G)$ is strongly connected. Then every vertex in $L(G)$ has an incoming and outgoing edge. Let u be a vertex in G. Since G is connected, then for u there is either an incoming or outgoing edge. Assume that we have an outgoing edge but no incoming edge. This edge will become a vertex in $L(G)$, say u'. But since there is no incoming edge at u, u' in $L(G)$ will have no incoming edge (and therefore its a source), implying that $L(G)$ is not strongly connected, which is a contradiction. So u must have an incoming edge. If we assume that u has an incoming edge and no outgoing edge, then this edge will become a vertex, v' in $L(G)$ and since u has no outgoing edge, v' will have no outgoing edges, which again implies that $L(G)$ is not strongly connected, a contradiction to the hypothesis that $L(G)$ is strongly connected. Therefore, if $L(G)$ is strongly connected and G is connected, then every vertex of G must have incoming and outgoing edges. Pick two vertices u, v in G. Let e_{i_1} be an outgoing edge of u and let e_{i_l} be an incoming edge of v ($u \neq v$). If there is no $u - v$ path in G, then by Remark 2.2.10, there is no path from vertex u'_{i_1} to vertex u'_{i_l} in $L(G)$, which implies that $L(G)$ is not strongly connected. This is a contradiction since $L(G)$ is strongly connected. Therefore, there exists a $u - v$ path in G for every ordered pair (u, v). ■

From the preceding result, we see that for a connected graph G, $\mathbb{B}$ is irreducible if and only if G is strongly connected. For more interesting results on line digraphs, see Section 4.5 of [6] on p. 182; [10] or [23].

Remark 2.2.16. *In general, it is not true that if $L(G)$ is strongly connected then G is strongly connected as well. For example, the graph G below is not connected, but its line graph is*

Figure 2.1: Graph G

strongly connected.

Figure 2.2: Its line graph L(G)

In this thesis, we use the term "reducible network" to refer to a simple graph (a graph with no self loops and no parallel edges) which is not strongly connected. The reducible part of this term comes from the fact that its adjacency matrix is a reducible matrix. From now onwards, we assume that the graph is simple.

Lemma 2.2.17. *Let G be a digraph. Let m be the number of edges and n the number of vertices ($m \geq n$). If there is at least one incoming (outgoing) edge at every vertex of G, then the matrix Φ^+, (respectively, Φ^-) is surjective. Moreover, $\Phi^-(\Phi^-_w)^T = I_n$.*

Proof. We show this for Φ^- only. Since each vertex has an outgoing edge, there is a non-zero entry in each row in Φ^- and exactly one non-zero entry in each column. Therefore, by construction of this matrix, all the rows are linearly independent, implying that it has full row rank, n. Hence, Φ^- must have a right inverse. If $n = m$, then Φ^- is invertible.

Moreover, Equation (2.3) implies that $(\Phi^-_w)^T$ is column stochastic, and since the non-zero terms in Φ^- coincide with those in Φ^-_w, the $(i,j)^{th}$ entry in $\Phi^-(\Phi^-_w)^T$ is given by

$$(\phi^-_{i1}, \ldots, \phi^-_{ik}, \ldots, \phi^-_{im}) \begin{pmatrix} (\phi^-_{j1})_w \\ \vdots \\ (\phi^-_{jk})_w \\ \vdots \\ (\phi^-_{jm})_w \end{pmatrix} = \sum_{l=1}^{m} \phi^-_{il} (\phi^-_{jl})_w = \sum_{l=1}^{m} \phi^-_{il} w_{jl}.$$

If $i = j$, then $w_{jl} > 0$ if and only if $\phi_{jl}^- = 1$. By Equation (2.3), the sum becomes

$$\sum_{l=1}^{m} w_{jl} = 1.$$

If $i \neq j$, then $\phi_{il}^- = 1$ implies that $\phi_{jl}^- = 0$ (and hence $(\phi_{jl}^-)_w = 0$). This is because the edge e_l cannot have two tails. Similarly, $(\phi_{jl}^-)_w = 1$ implies $\phi_{il}^- = 0$. Hence

$$\sum_{l=1}^{m} \phi_{il}^-(\phi_{jl}^-)_w = 0, \quad i \neq j.$$

Therefore,

$$\Phi^-(\Phi_w^-)^T = I_n. \tag{2.9}$$

■

In general, Φ^- is not invertible even when G is strongly connected. But as a result of (2.9), we have

$$\left((\Phi_w^-)^T\Phi^-\right)^2 = (\Phi_w^-)^T\Phi^-(\Phi_w^-)^T\Phi^- = (\Phi_w^-)^T\Phi^- \tag{2.10}$$

and

$$\begin{aligned}\left(I - (\Phi_w^-)^T\Phi^-\right)^2 &= (I - (\Phi_w^-)^T\Phi^-)(I - (\Phi_w^-)^T\Phi^-)\\ &= (I - (\Phi_w^-)^T\Phi^-) - (\Phi_w^-)^T\Phi^-(I - (\Phi_w^-)^T\Phi^-)\\ &= I - (\Phi_w^-)^T\Phi^- - (\Phi_w^-)^T\Phi^- + (\Phi_w^-)^T\Phi^-\\ &= I - (\Phi_w^-)^T\Phi^-.\end{aligned}$$

Hence $(\Phi_w^-)^T\Phi^-$ is a projection.

Using Lemma 2.2.12, we can see that $\mathbb{B}(\Phi_w^-)^T = (\Phi_w^-)^T\Phi^+(\Phi_w^-)^T = (\Phi_w^-)^T\mathbb{A}$ and

$$\begin{aligned}\mathbb{B}^2(\Phi_w^-)^T &= (\Phi_w^-)^T\Phi^+(\Phi_w^-)^T\Phi^+(\Phi_w^-)^T\\ &= (\Phi_w^-)^T\mathbb{A}^2.\end{aligned}$$

By induction, we obtain $\mathbb{B}^n(\Phi_w^-)^T = (\Phi_w^-)^T\mathbb{A}^n$ for all $n \in \mathbb{N}_0$.

In the following, we state the relationship between the spectrum of $\mathbb{B}$ to that of $\mathbb{A}$. We found this to be interesting although we did not find it in literature.

Lemma 2.2.18. *Let G be a digraph whose every vertex has an outgoing edge. Then the matrices $\mathbb{A}$ and $\mathbb{B}$ have the same eigenvalues.*

Proof. Let $\mathbb{B}\mathbf{x} = \lambda\mathbf{x}$ for some $\lambda \in \mathbb{C}$ and $\mathbf{x} \neq \mathbf{0}$. Then using Lemma 2.2.12, we have $\lambda\Phi^+\mathbf{x} = \Phi^+\mathbb{B}\mathbf{x} = \Phi^+(\Phi_w^-)^T\Phi^+\mathbf{x} = \mathbb{A}\Phi^+\mathbf{x}$. Since every vertex has an outgoing edge, every column has exactly one non-zero entry (by Remark 2.2.5). Hence $\mathbf{x} \neq \mathbf{0} \Rightarrow \Phi^+\mathbf{x} \neq \mathbf{0}$, implying that $(\lambda, \Phi^+\mathbf{x})$ is an eigenpair of $\mathbb{A}$.

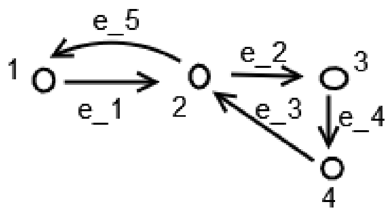
Conversely, if α is an eigenvalue of $\mathbb{A}$, then $\alpha\mathbf{y} = \mathbb{A}\mathbf{y}$ for some $\mathbf{y} \neq 0$. Then

$$\begin{aligned}\alpha(\Phi_w^-)^T\mathbf{y} = (\Phi_w^-)^T\mathbb{A}\mathbf{y} &= (\Phi_w^-)^T\Phi^+(\Phi_w^-)^T\mathbf{y} \\ &= \mathbb{B}(\Phi_w^-)^T\mathbf{y}.\end{aligned}$$

The matrix Φ^- has no zero rows since every vertex has an outgoing edge, and by Remark 2.2.5, there is exactly one non-zero entry in each column of Φ^-, hence there is exactly one non-zero entry in each row of $(\Phi_w^-)^T$ and every column of $(\Phi_w^-)^T$ has at least one positive entry. Therefore, $\mathbf{y} \neq \mathbf{0}$ implies that $(\Phi_w^-)^T\mathbf{y} \neq \mathbf{0}$, hence $(\alpha, (\Phi_w^-)^T\mathbf{y})$ is an eigenpair of $\mathbb{B}$. ∎

Remark 2.2.19. *Since $\mathbb{A}$ is an $n \times n$ matrix and $\mathbb{B}$ an $m \times m$ matrix with $m \geq n$, it follows that some eigenvalues of $\mathbb{B}$ are repeated whenever $m > n$. Since there are at most n linearly independent eigenvectors of $\mathbb{A}$, it follows that $\mathbb{B}$ is a defective matrix since it can only have n linearly independent eigenvectors ($< m$). That is, $\mathbb{B}$ is singular whenever $m > n$. If $\mathbb{A}$ is defective, then $\mathbb{B}$ must also be defective.*

Example 2.2.20. *Consider the graph below. We will place weights such that*



$$\Phi_w^- = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

We compute the weighted adjacency matrix $\mathbb{A} = \Phi^+ \left(\Phi_w^-\right)^T$ and the weighted adjacency matrix *for the line graph* $\mathbb{B} = \left(\Phi_w^-\right)^T \Phi^+$ *to get*

$$\mathbb{A} = \begin{pmatrix} 0 & \frac{2}{3} & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}; \quad \mathbb{B} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \frac{2}{3} & 0 & \frac{2}{3} & 0 & 0 \end{pmatrix}.$$

The eigenvalues of $\mathbb{A}$ *are*

$$\left\{0, 1, \frac{-3 \pm \sqrt{3}\imath}{6}\right\}.$$

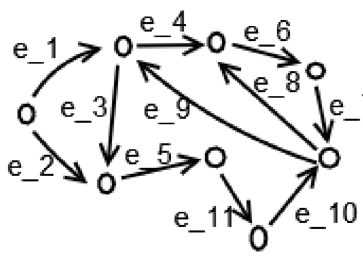
Note that $\mathbb{B}$ *has the same eigenvalues, with* $\lambda = 0$ *being a repeated eigenvalue of algebraic multiplicity* 2.

Definition 2.2.21. *[13], p. 121.*
A cut set of a connected graph G *is a set* S *of edges such that both of the following conditions hold:*

- *the removal of all the edges in* S *disconnects* G,
- *the removal of some but not all of the edges in* S *does not disconnect* G.

A directed graph can have more than one cut set, but any cut set divides the set of vertices into two disjoint sets.

Example 2.2.22. *For the graph below, the following are cut sets of this graph:*



$$C_1 = \{e_4, e_5, e_9\}; \quad C_2 = \{e_1, e_2\}; \quad C_3 = \{e_4, e_9, e_{10}\};$$

$$C_4 = \{e_6, e_8, e_9, e_5\}; \quad C_5 = \{e_6, e_8, e_9, e_{10}\}.$$

On the other hand, the set $\{e_8, e_9\}$ *is not a cut set of* G.

2.3 Perron-Frobenius Theorems

As we have seen in the previous section, graphs and non-negative matrices are closely related. Due to the relationship between irreducibility of the adjacency matrix $\mathbb{A}$ and the structure of the graph (Lemma 2.2.14), it is sometimes easier to study the graph via its adjacency matrix. Since adjacency matrices are non-negative, their spectral properties fall into the scope of the Perron-Frobenius Theory. Here we provide a brief overview of the theory. The presentation is based on [51], [44] and [21].

Definition 2.3.1. *For an $n \times n$ matrix A, the spectrum of A (denoted $\sigma(A)$) is the set of eigenvalues and its spectral radius $r(A)$ is the number*

$$r(A) = \max_{\lambda \in \sigma(A)} |\lambda|.$$

Theorem 2.3.2. *(Perron-Frobenius theorem for positive matrices)*
Let $A > 0$ be an $n \times n$ matrix. Then $r(A) > 0$ has an associated positive eigenvector $\mathbf{x}$. Moreover, $r(A) > |\lambda|$ for any other eigenvalue λ of A and its algebraic multiplicity is 1.

Proof. This theorem and its proof appear in [44], Section 8.2. ■

Lemma 2.3.3. *(Perron-Frobenius Theorem for irreducible matrices)*
Let $A \geq 0$ be an irreducible matrix. Then there exists an eigenvalue r such that

1. *r is real and $r > 0$,*
2. *there exists strictly positive left and right eigenvectors associated with the eigenvalue r,*
3. *r is a simple root of the characteristic polynomial of A,*
4. *the eigenvectors associated with r are unique up to constant multiples.*
5. *$r = r(A)$, the spectral radius of A.*

The proof of this result can be found in [44], Section 8.3. Notice that when A is not strictly positive, the eigenvalue $r = r(A)$ is no longer dominant but satisfies the inequality $r(A) \geq |\lambda|$ for any other eigenvalue λ of A.

When the matrix A is reducible, $r(A)$ is still an eigenvalue but the associated eigenvector $\mathbf{x}$ may not be strictly positive as in the case of positive and irreducible matrices.

Theorem 2.3.4. *[21]*
To the maximal eigenvalue $r(A)$ of a reducible matrix $A \geq 0$, there corresponds a positive eigenvector if and only if each A_i for $i = 1, \ldots, g$ in the normal form of A (2.8) has eigenvalue r satisfying $r \notin \sigma(A_j)$ for any $j = g+1, \ldots, s$.

Even when the right eigenvector $\mathbf{x}$ is positive, typically the left eigenvector will not be strictly positive, or vice versa. Indeed, we have the following theorem regarding existence of positive left and right eigenvectors.

Theorem 2.3.5. *[21]*
Let $A \geq 0$ be a reducible matrix and $r(A)$ be its spectral radius. Both A and A^T have positive eigenvectors corresponding to $r(A)$ if and only if A is block diagonal and $r(A) \in \sigma(A_i)$ for all $i = 1, \ldots, s$.

Definition 2.3.6. *A non-negative off-diagonal matrix $\tilde{A}$ is called an ML-matrix. It is called irreducible if there exists a non-negative irreducible matrix A and $\eta \in \mathbb{R}$ such that $\tilde{A} = A - \eta I$, otherwise we say that it is reducible.*

ML matrices are of interest because they are some of the simplest generators of positive semigroups in finite dimensional spaces, [17]. They are commonly used in modelling basic phenomena in the natural sciences in continuous time, like birth and death problems, migration between different patches of land or transition from one age group to another. Due to their obvious relationship with non-negative matrices, we can formulate a Perron-Frobenius type theorem for ML matrices through the associated non-negative matrix.

Theorem 2.3.7. *Theorem 2.6, [51]*
Let A be an irreducible ML matrix. Then there exists an eigenvalue τ such that

1. *τ is real*
2. *τ is associated with strictly positive right and left eigenvectors which are unique up to constant multiples.*
3. *$\tau > \Re(\lambda)$ for any other eigenvalue $\lambda \neq \tau$ of A*

For reducible ML matrices, we have the following result.

Theorem 2.3.8. *[39], p.205*

Let A be an ML matrix. Then there exists a real number τ and a non-negative vector $\mathbf{x}$ such that

- $A\mathbf{x} = \tau\mathbf{x}$;
- *if $\lambda \neq \tau$ is any other eigenvalue of A, then $\Re\lambda < \tau$.*

Remark 2.3.9. *Note that for an ML matrix A, the dominant eigenvalue is the number τ satisfying $\tau > \Re\lambda$, for any other eigenvalue λ of A. This property of τ is very important in the study of long term behaviour of systems of ordinary differential equations, where the coefficient matrix is an ML matrix (see Chapter 3). In contrast, for difference equations, we consider the dominant eigenvalue to be the number r with maximum modulus.*

In this thesis, we study flows in networks using semigroup theory. In particular, we investigate existence of positive semigroups for (ACP) associated with the network flow problem. In order to discuss positive semigroups, we need some introduction and results from the theory of Banach lattices.

2.4 Banach Lattices

An ordered set is a set endowed with the binary relation $\leq$ which is transitive, reflexive and antisymmetric. If $(Y, \leq)$ is an ordered set, then $y \geq x$ means that $x \leq y$, $x < y$ means that $x \leq y, x \neq y$. If Y is a subset of $\mathbb{R}^n$, then $x \leq y$ means that $x_i \leq y_i$ for each $i \in \{1, \ldots, n\}$ and $x, y \in Y$.

Definition 2.4.1. *An ordered vector space is a vector space X equipped with a partial order which is compatible with the vector space structure*

- *$x \geq y$ implies that $x + z \geq y + z$ for all $x, y, z \in X$;*
- *$x \geq y$ implies that $\alpha x \geq \alpha y$ for all $x, y \in X$ and $\alpha > 0$.*

Definition 2.4.2. *Let X be an ordered set and S be a non empty subset of X. An element $b \in X$ is said to be an upper bound of S if $x \leq b$ for any $x \in S$. The least upper bound* *(supremum) is the element $l \in X$ which satisfies $l \leq b$ for any upper bound b of S. We call the element a a lower bound of S if $x \geq a$ for all $x \in S$. It is called the greatest lower bound (or infimum) if it greater or equal to any other lower bound of S.*

Definition 2.4.3. *An ordered set $(X, \leq)$ is called a lattice if for each pair $(x, y) \in X \times X$, the supremum and infimum of x, y denoted $x \vee y := \sup\{x, y\}$ and $x \wedge y := \inf\{x, y\}$, both exist in X.*

So a lattice vector space X is an ordered vector space such that the infimum and supremum of any pair of elements in X is also contained in X. For an element x in a lattice vector space, the positive and negative parts of x, denoted, respectively as x^+ and x^-, are defined as

$$x^+ = x \vee 0; \quad x^- = -x \vee 0$$

and the absolute value of x denoted $|x|$, is the element

$$|x| = x \vee (-x).$$

For example, if we consider the vector lattice $\mathbb{R}^4$ and $x = (1, -2, 3, -4)^T$, then $x^+ = (1, 0, 3, 0)^T$, $x^- = (0, 2, 0, 4)^T$ and $|x| = (1, 2, 3, 4)$. If X is an ordered vector space, then $X^+ = \{x \in X : x \geq 0\}$ is called the positive cone of X.

Examples of vector lattices include the usual sequence spaces c_0, ℓ^1, ℓ^∞. Other examples include the function spaces $L_p(\Omega)$. Let $f, g \in L_p(\Omega)$, then we say that $f \leq g$ if $f(x) \leq g(x)$ for almost all $x \in \Omega$. Equipped with such an order, $L_p(\Omega)$ becomes a vector lattice, [3], p.43.

Definition 2.4.4. *Let X be a vector lattice. A subset Y of X is called a solid if $|x| \leq |y|$ and $y \in Y$ both imply that $x \in Y$. A solid subspace of X is called an ideal.*

For example, the sequence space ℓ^1 is an ideal in c_0. We note first that $\ell^1 \subset c_0$. To see that ℓ^1 is an ideal of c_0, suppose that $y = (y_1, y_2, \ldots) \in \ell^1$. Then since

$$\sum_{i \in \mathbb{N}} |y_i| < \infty,$$

$y_i \to 0$ as $i \to \infty$. Hence $y \in c_0$. Let $|x| \leq |y|$. That is, $|x_i| \leq |y_i|$ for all $i \in \mathbb{N}$. Then

$$\sum_{i=1}^{\infty} |x_i| \leq \sum_{i=1}^{\infty} |y_i| < \infty.$$

Therefore, $x \in \ell^1$, implying that ℓ^1 is an ideal of c_0.

Example 2.4.5. *Let $X = \mathbb{R}^n$. Define*

$$A = \textit{Span}\{e_1, \ldots, e_m; \quad m < n\}; \text{ where } e_i = (0, \ldots, 0, 1, 0, \ldots, 0),$$

where 1 is in the i^{th} position. Then A is an ideal of X.

Definition 2.4.6. *An ordered set E is called directed if any pair of elements in the set has an upper bound. Then, by a net we refer to a function $(x_\alpha)_{\alpha \in E}$ which maps elements of E into X [3], p. 55.*

A net $(x_\alpha)_{\alpha \in E}$ in an ordered set X is said to be decreasing if $x_{\alpha_1} \geq x_{\alpha_2}$ for any $\alpha_1, \alpha_2 \in E$ with $\alpha_1 < \alpha_2$, and it said to be increasing if $x_{\alpha_2} \geq x_{\alpha_1}$, [3].

If (x_α) is decreasing and x is the infimum of x_α for all $\alpha \in E$, we write this as $x_\alpha \downarrow x$ and conversely if (x_α) is increasing.

Definition 2.4.7. *A net (x_n) of arbitrary elements of X is said to be order convergent to x if there exist nets $(y_\alpha)_{\alpha \in B_1}, (z_\beta)_{\beta \in B_2}$ such that $y_\alpha \uparrow x$, $z_\beta \downarrow x$ and, for any α, β, there is N such that $y_\alpha \leq x_n \leq z_\beta$ for all $n \geq N$.*

If x_n is order convergent to x, we write this as $x_n \xrightarrow{o} x$.

Definition 2.4.8. *A subset A of a partially ordered set is said to be order closed if $(x_n)_{n \in \mathbb{N}} \in A$ and $x_n \rightarrow x$ both imply that $x \in A$. An order closed ideal is called a band.*

For example, the set A defined in Example 2.4.5 is a band in X: Choose a sequence $0 \leq (x_k)_{k \in \mathbb{N}} \in A$ and let $x_k \xrightarrow{o} x$. But $(x_k)_{k \in \mathbb{N}} \in A$ implies that $x_k^{m+1} = x_k^{m+2} = \cdots = x_k^n = 0$. If this sequence converges, the pointwise limit must equal to the limit. Hence $x_k^i = x^i \Rightarrow x^i = 0$ for all $i = m+1, \ldots, n$, implying that $x \in A$.

The space ℓ^1 is an ideal of ℓ^∞ but it is not a band: consider the sequence x_n whose terms are given by

$$\begin{aligned} x_1 &= (1, 0, 0, 0, \ldots), \\ x_2 &= (1, 1, 0, 0, \ldots), \\ x_3 &= (1, 1, 1, 0, \ldots) \\ \vdots &= \quad \vdots \end{aligned}$$

then $x_n \xrightarrow{o} (1, 1, 1, \ldots, 1, \ldots) \notin \ell^1$.

Definition 2.4.9. *A norm on a vector lattice is called a lattice norm if $|x| \leq |y|$ implies that $\|x\| \leq \|y\|$.*

If X is an ordered vector lattice and $\|\cdot\|$ is a lattice norm on X, then $(X, \|\cdot\|)$ is a normed vector lattice and it is called a Banach lattice if it is complete [53]. Examples of Banach lattices include $X = L_1([0,1])^m$, with $\|\cdot\|$ defined as

$$\|f\|_X = \sum_{i=1}^{m} \int_0^1 |f_i(x)|dx.$$

To see that this is indeed a lattice norm, let $|f| \leq |g|$. Then this implies that $|f_i(x)| \leq |g_i(x)|$ almost everywhere. Then

$$\|f\|_X = \sum_{i=1}^{m} \int_0^1 |f_i(x)|dx \leq \sum_{i=1}^{m} \int_0^1 |g_i(x)|dx = \|g\|.$$

Definition 2.4.10. *Let X, Z be two ordered vector spaces. A linear operator $T : X \to Z$ is called positive if $x \geq 0$ implies $Tx \geq 0$, for $x \in X$. It is called strictly positive if $x > 0$ implies $Tx > 0$ for all $x \in X$.*

Let $T : X \to Z$ be positive and $x \in X$. From the definition of absolute value $|x|$, we have the following relations: $x \leq |x|$ and $-|x| \leq x$. Hence, $|x| - x \geq 0$ and $|x| + x \geq 0$. Since T is positive, we have $T(|x| - x) \geq 0$ and $T(|x| + x) \geq 0$. Since T is linear, we have $T|x| - Tx \geq 0$ and $T(|x| + x) = T|x| + Tx \geq 0$. Hence, $-T|x| \leq Tx \leq T|x|$. This implies that $|Tx| \leq T|x|$, for all $x \in X$.

If X is a normed space with lattice structure, and T is a positive operator on X, then the behaviour of T on X can be determined by studying its behaviour on the positive cone X^+, [3], p. 53.

Definition 2.4.11. *Let X, Y be two Banach lattices and $A, B : X \to Y$. Then $A \leq B$ if for every $x \in X$, $Ax \leq Bx$.*

Proposition 2.4.12. *[3], Proposition 2.67*
If A is a positive operator, then its norm is given by

$$\|A\|_* = \sup_{x \geq 0, \|x\| \leq 1} \|Ax\|.$$

If $0 \leq Ax \leq Bx$, then from the definition of the lattice norm, we must have $\|Ax\| \leq \|Bx\|$. Using this and Proposition 2.4.12, we see that if A and B are positive operators satisfying $A \leq B$, then

$$\|A\|_* = \sup_{x\geq 0, \|x\|\leq 1} \|Ax\| \leq \sup_{x\geq 0, \|x\|\leq 1} \|Bx\| = \|B\|_*,$$

therefore, if $\|\cdot\|$ is a lattice norm, then

$$\|A\|_* \leq \|B\|_*, \tag{2.11}$$

where $\|\cdot\|_*$ is the operator norm.

2.5 Positive Semigroups

Let X be a non trivial Banach space. Consider the abstract Cauchy problem

$$\begin{cases} u'(t) & = Au, \\ u(0) & = u_0, \end{cases} \tag{2.12}$$

where A is a linear operator acting on a subspace $D(A)$ of X. That is, $A : D(A) \to X$, where $D(A)$ is a subspace of X where Au makes sense. First, we assume the solution to the abstract Cauchy problem is unique. Let $u(t, u_0)$ be the solution subject to the initial condition $u(0) = u_0$ and let $T(t)u_0 = u(t, u_0)$; with $T(0)u_0 = u_0$. For $u(\cdot, u_0)$ to be a solution to (2.12), it must be continuously differentiable, hence $T(t)u_0$ is also continuously differentiable. Let $v_0 \in D(A)$ be another initial condition and $\alpha, \beta \in \mathbb{R}$. Then $\alpha u_0 + \beta v_0 \in D(A)$ and since A is assumed to be a linear operator, we have

$$\begin{aligned}
(\alpha u(t, u_0) + \beta u(t, v_0))' &= \alpha u'(t, u_0) + \beta u'(t, v_0) \\
&= \alpha A u(t, u_0) + \beta A u(t, v_0) \\
&= A(\alpha u(t, u_0) + \beta u(t, v_0)) \\
&= \alpha \frac{d}{dt}(T(t)u_0) + \beta \frac{d}{dt} T(t) v_0.
\end{aligned}$$

Hence $\alpha u(t, u_0) + \beta u(t, v_0)$ solves the abstract Cauchy problem with initial condition $u(0) = \alpha u_0 + \beta v_0$. That is, $\alpha T(t)u_0 + \beta T(t)v_0$ is a solution. By definition, $u(t, \alpha u_0 + \beta v_0)$ solves the problem (2.12) with initial condition $\alpha u_0 + \beta v_0$. Hence, by uniqueness,

$$u(t, \alpha u_0 + \beta v_0) = T(t)(\alpha u_0 + \beta v_0) = \alpha u(t, u_0) + \beta u(t, v_0) = \alpha T(t)u_0 + \beta T(t)v_0.$$

Therefore, $T(t)$ is a linear map.

If $t, s \geq 0$, then if $u(t, u(s))$ and $u(t+s, u_o)$ are solutions to (ACP), we have $u(t, u(s)) = T(t)u(s)$ and $u(t+s, u_o) = T(t+s)u_o$. By uniqueness of the solution, we have $u(t+s, u_o) = u(t, u(s))$, hence $T(t+s)u_o = T(t)u(s) = T(t)T(s)u_o$, hence $T(t+s) = T(t)T(s)$.

Below, we formally define families of operators $T(t)$ with the above properties below.

Definition 2.5.1. *A family of linear bounded operators $(T(t))_{t\geq 0}$ which satisfies the conditions:*

- $T(0)x = x$ *for all* $x \in D(A)$,
- $T(t+s) = T(t)T(s)$,

is called a one parameter semigroup of bounded operators.

The semigroup $(T(t))_{t\geq 0}$ is called a strongly continuous semigroup if, in addition, the maps $t \mapsto T(t)x$ are continuous from $\mathbb{R}^+$ to X, for every $x \in X$. Strongly continuous semigroups are also referred to as C_0 semigroups. As seen above, semigroups arise naturally in dynamical systems, [3], Chapters 1, 3. However, (2.12) does not always have a solution on the whole space X, but only if restricted to some subspace of X. This motivates the following definitions. The operator $(A, D(A))$, where $D(A)$ is the space

$$D(A) := \left\{ x \in X : \lim_{t \to 0^+} \frac{T(t)x - x}{t} \text{ exists in } X \right\}$$

and

$$Ax := \lim_{t \to 0^+} \frac{T(t)x - x}{t}, \quad x \in D(A),$$

is called the generator of the semigroup $(T(t))_{t\geq 0}$. Typically, we expect that this operator A should coincide with the operator A in (2.12), but this is not always the case. So the operator $(A, D(A))$ defined above is the realisation of A in (2.12). If $(A, D(A))$ generates a semigroup $(T(t))_{t\geq 0}$, then solutions to (2.12) are then given by $u(t, u_0) = T(t)u_0$. We will now list a few useful facts about strongly continuous semigroups that will be used in this thesis.

Theorem 2.5.2. *Let $(A, D(A))$ be a generator of a strongly continuous semigroup. Then $(A, D(A))$ is linear, closed and densely defined.*

Proof. Linearity of the generator follows from the definition of $(A, D(A))$. The rest of the proof can be found in [16], Chapter II, Theorem 1.4. ■

Lemma 2.5.3. *[3], p.70*
Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup. Then there exist constants ω and $M > 0$ such that $\|T(t)\| \leq Me^{\omega t}$.

Recall that the spectrum of an operator is the set $\sigma(A) = \{\lambda \in \mathbb{C} : \lambda I - A$ is not bijective$\}$. The complement of this set is called the resolvent set $\rho(A)$ and the operator $R(\lambda, A) := (\lambda I - A)^{-1}, \lambda \in \rho(A)$, is called the resolvent operator.

Definition 2.5.4. *Let X be a Banach lattice. The semigroup $(T(t))_{t\geq 0}$ is said to be positive if, for all $x \in X^+$ and $t \geq 0$, one has $T(t)x \geq 0$. The operator $(A, D(A))$ is called resolvent positive if there exists ω such that $(\omega, \infty) \in \rho(A)$ and $R(\lambda, A) \geq 0$ for all $\lambda > \omega$.*

Remark 2.5.5. *Let X be a Banach lattice and A be a positive operator on X. If $|\lambda| > r(A)$, then the resolvent of A, $R(\lambda, A)$ is also positive, [1], p.254.*

We often have information about the operator $(A, D(A))$ but have no explicit formula for the semigroup or its resolvent. The following result due to Hille and Yosida enables us to draw certain conclusions about the semigroup from the structure of its generator.

Theorem 2.5.6. *Hille-Yosida, [3], Theorem 3.5*
Let $(A, D(A))$ be a linear operator on a Banach space X and let $M \geq 1; \omega \in \mathbb{R}$ be constants. Then the following assertions are equivalent

1. *The C_0 semigroup $(T(t))_{t\geq 0}$ generated by $(A, D(A))$ satisfies $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$ and $(\omega, \infty) \in \rho(A)$.*
2. *$(A, D(A))$ is closed, densely defined and for every $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega$, we have $\lambda \in \rho(A)$ and*
$$\|R(\lambda, A)^n\| \leq \frac{M}{(\Re\lambda - \omega)^n}, \quad n \geq 1, \Re\lambda > \omega.$$

Theorem 2.5.7. *Theorem II. 1.10, [17]*
Let $(T(t))_{t\geq 0}$ be a C_0 semigroup on the Banach space X and let $\omega \in \mathbb{R}$, $M \geq 1$ be constants such that $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$. Then for $(A, D(A))$, the generator of $(T(t))_{t\geq 0}$, the following properties hold.

1. *If $R(\lambda)(x) := \int_0^\infty e^{-\lambda t}T(t)xdt$ exists for some $\lambda \in \mathbb{C}$ and for all $x \in X$, then $\lambda \in \rho(A)$ and $R(\lambda)x = R(\lambda, A)x$.*

2. *If $\Re\lambda > \omega$, then $\lambda \in \rho(A)$, and the resolvent is given by the integral given in item (1) above.*

The proof can be found in [16], p.55.

Definition 2.5.8. *A semigroup $(T(t))_{t\geq 0}$ is called a contraction semigroup if $\|T(t)\| \leq 1$.*

Corollary 2.5.9. *A linear, closed and densely defined operator $(A, D(A))$ generates a semigroup of contractions if and only if $(0, \infty) \in \rho(A)$ and*

$$\|R(\lambda, A)x\| \leq \frac{1}{\lambda}\|x\|, \quad \forall\lambda > 0. \tag{2.13}$$

Sometimes, it is easier to find the resolvent of the operator $(A, D(A))$ rather than the semigroup itself. Therefore, we need to establish a relationship between resolvents and semigroups. We note that if X is a Banach lattice and the resolvent $R(\lambda, A)$ is known to be a positive operator for $\lambda > \omega$, for some $\omega \in \mathbb{R}$, then each $T(t), t \geq 0$ will be a positive operator as well (see [3], p. 97).

Definition 2.5.10. *Let A be a linear operator on a Banach space X. The spectral bound of A is the number*

$$s(A) = \sup\{\Re\lambda : \lambda \in \sigma(A)\}.$$

Theorem 2.5.11. *[3], Theorem 3.39*
Let $(A, D(A))$ be a densely defined linear operator with positive resolvent. If there exists $\lambda_o > s(A)$ and $c > 0$ such that for all $x \geq 0$, $\|R(\lambda_o, A)x\|_X \geq c\|x\|_X$, then $(A, D(A))$ is the generator of a positive semigroup on X.

Definition 2.5.12. *A positive semigroup $(T(t))_{t\geq 0}$ with generator $(A, D(A))$ on a Banach lattice X is called irreducible if it has no non trivial closed invariant ideals. That is; if I is an ideal of X and $T(t)I \in I$, then either $I = \{0\}$ or $I = X$.*

Chapter 3

Finite Dimensional Flows

3.1 Introduction

In this chapter, we employ techniques related to Perron-Frobenius theorems to investigate long time behaviour of flow on networks. To prepare the ground, we begin with some results pertaining to simpler problems, such as finite dimensional flows where the Perron-Frobenius structure of the governing matrix plays an essential role. The results of this chapter not only allow us to further develop and explore a number of necessary techniques but also the obtained results are of independent interest. First, however, we introduce the models discussed in this chapter.

3.1.1 Direct modelling with ML matrices

ML matrices have been introduced in Definition 2.3.6. Here, we present a typical way they appear in applications and explain how they are related to flows on networks which is the main topic of this thesis.

Consider a population that is divided into n subgroups according to a certain criteria (like age, sex, size, geographical location, etc). Let $u_i(t)$ be the size of the population in the i^{th} state at time t. At any time t, individuals in any state can die, migrate to other states or give birth to other individuals. Let b_i, d_i and a_{ji} be the birth rate, death rate and the rate at which individuals in state i migrate to state j, respectively. Then the change in the number of individuals in state

i in a small interval of time Δt can be expressed as

$$u_i(t+\Delta t) = u_i(t) + b_i u_i(t)\Delta t - d_i u_i(t)\Delta t + \sum_{j=1, j\neq i}^{n} a_{ji} u_j(t)\Delta t. \tag{3.1}$$

Note that in the above equation, d_i also includes the rate at which individuals migrate from state i to other states. Assuming $u_i(t)$ is differentiable, we rearrange the terms in (3.1) and take limits as $\Delta t \to 0$ to get

$$\lim_{\Delta t \to 0} \frac{u_i(t+\Delta t) - u_i(t)}{\Delta t} = u_i'(t) = (b_i - d_i)u_i(t) + \sum_{j=1, j\neq i}^{n} a_{ij} u_j(t).$$

Therefore, the dynamics of the whole population can be summarised in the following problem

$$\begin{cases} \mathbf{u}'(t) &= A\mathbf{u}(t), \\ \mathbf{u}(0) &= \mathbf{u}_0. \end{cases} \tag{3.2}$$

The matrix A is an ML matrix and so if $\mathbf{u}_0 \geq 0$, the solution is non-negative. The total population will then be given by the sum of the individuals in all the states. From Theorem 2.3.8, A has a dominant eigenvalue τ which determines the long term behaviour of the population. Note that

$$\sum_{j=1, j\neq i}^{n} a_{ji} u_i(t)$$

represents the total number of individuals leaving state i through migration at time t. In particular, if in the described model the number of births and the total number of individuals leaving (due to migration and death) each state are equal, the diagonal coefficients are sums of the other terms in the respective columns taken with negative sign and thus, the sum of each column is zero. Such matrices are called Kolmogorov matrices and it is known that they have dominant eigenvalue 0, and they describe conservative processes. That is, processes in which the total number of individuals in the population remains constant, [4].

Any problem of the form (3.2) with ML matrix A can be reduced to an equivalent problem with matrix $\tilde{A}$ having 0 as the dominant eigenvalue, although not a Kolmogorov matrix. Indeed, let r be the dominant eigenvalue of A and consider the following. Let $\mathbf{v} = e^{-rt}\mathbf{u}$. Then

$$\begin{cases} \mathbf{v}'(t) &= (A - rI)\mathbf{v} = \tilde{A}\mathbf{v}; \\ \mathbf{v}(0) &= \mathbf{u}(0), \end{cases} \tag{3.3}$$

and since $\sigma(\tilde{A}) = \sigma(A) - r$, 0 is the dominant eigenvalue of $\tilde{A}$. The solutions $\mathbf{v}(t)$ are non-negative if $\mathbf{u}(0)$ is non-negative.

Remark 3.1.1. *We observe that a Kolmogorov model in (3.2) can be considered as a simplified network transport problem in which the states form vertices and the transport between them occurs instantaneously with the rates given by the coefficients of the matrix A. A more detailed justification is given in Remark 4.1.6.*

The main aim of this chapter is to extend the results in [48]. In [48], Perthame showed decay of the general relative entropy function for (3.2) in the case of strictly positive off-diagonal matrices A. Under this assumption on A, he also proved a discrete Poincaré inequality which allowed him to establish that, under suitable renorming of the state space, the solutions to (3.2) decay exponentially (strictly) in the space complementary to that spanned by the Perron vector of A. He also noted that 'interesting things occur for non-negative matrices'. In this chapter, we explore this statement. In particular, we extend the result in Lemma $6.3.1$ of [48] to general irreducible matrices (see Lemma 3.2.3) and also show that this result does not hold for reducible matrices in general, but only holds in certain invariant subspaces of the reducible matrix A.

More precisely, we will work with the problem (3.2), rescaled as in (3.3) for arbitrary irreducible matrix A. Instead of writing $\tilde{A}$, we shall simply write A and A has dominant eigenvalue 0. Let $\mathbf{N}$ and $\mathbf{v}$ be the positive right and left eigenvectors of A, respectively. Then the relative entropy function,

$$\sum_{i=1}^{n} v_i N_i H\left(\frac{u_i(t)}{N_i}\right),$$

where $H(\cdot)$ is any convex function is non increasing. This allows us to prove a number of classical estimates for the solution of (3.3) in a unified way. Moreover, using $H(u) = u^2$ and extending the Poincaré inequality, we will show that the semigroup generated by A is strictly contractive in the subspace orthogonal to $\mathbf{N}$ with respect to the scalar product,

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} \frac{v_i}{N_i} x_i y_i$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. In the second part of this chapter (Section 3.3), we reformulate these results in a more restricted case so that they can be applied to reducible matrices.

3.2 Irreducible matrices

Let A be an ML matrix with dominant eigenvalue 0. If A is irreducible, then strictly positive right and left Perron eigenvectors $\mathbf{N}$ and $\mathbf{v}$, respectively, exist as seen in Section 2.3. We fix a unique right eigenvector $\mathbf{N}$ by normalising it in the following way:

$$\sum_{i=1}^{n} N_i = 1, \quad \sum_{i=1}^{n} N_i v_i = 1. \tag{3.4}$$

In the next result, we show that Proposition 6.6 in [48] for solutions to the system (3.3), where A has strictly positive off-diagonal entries still holds when A is irreducible but not necessarily strictly positive off-diagonal.

Theorem 3.2.1. *Let A be an ML matrix with $s(A) = 0$ and corresponding right and left eigenvectors $\mathbf{N}$ and $\mathbf{v}$, respectively. Let $H(\cdot)$ be a differentiable convex function on $\mathbb{R}$. Then the solution to the initial value problem (3.3) satisfies*

$$\frac{d}{dt}\sum_{i=1}^{n} v_i N_i H\left(\frac{u_i(t)}{N_i}\right) \le 0.$$

Proof.

$$\begin{aligned}
\frac{d}{dt}\sum_{i=1}^{n} v_i N_i H\left(\frac{u_i(t)}{N_i}\right) &= \sum_{i=1}^{n} v_i N_i \frac{d}{dt} H\left(\frac{u_i(t)}{N_i}\right) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} v_i N_i a_{ij} \frac{u_j(t)}{N_i} H'\left(\frac{u_i(t)}{N_i}\right) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} v_i a_{ij} u_j H'\left(\frac{u_i(t)}{N_i}\right) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} v_i a_{ij} N_j H'\left(\frac{u_i(t)}{N_i}\right)\left[\frac{u_j(t)}{N_j} - \frac{u_i(t)}{N_i}\right].
\end{aligned}$$

The last equation is due to the fact that $A\mathbf{N} = \mathbf{0}$, hence

$$\sum_{i=1}^{n}\sum_{j=1}^{n} v_i a_{ij} N_j H'\left(\frac{u_i(t)}{N_i}\right)\frac{u_i(t)}{N_i} = \sum_{i=1}^{n} v_i H'\left(\frac{u_i(t)}{N_i}\right)\frac{u_i(t)}{N_i}\left(\sum_{j=1}^{n} a_{ij} N_j\right) = 0.$$

Since $H(.)$ is convex, we have

$$H'\left(\frac{u_i(t)}{N_i}\right)\left[\frac{u_j(t)}{N_j} - \frac{u_i(t)}{N_i}\right] \le H\left(\frac{u_j(t)}{N_j}\right) - H\left(\frac{u_i(t)}{N_i}\right)$$

hence

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{n} v_i a_{ij} N_j H'\left(\frac{u_i(t)}{N_i}\right)\left[\frac{u_j(t)}{N_j} - \frac{u_i(t)}{N_i}\right] &\le \sum_{i=1}^{n}\sum_{j=1}^{n} v_i a_{ij} N_j \left[H\left(\frac{u_j(t)}{N_j}\right) - H\left(\frac{u_i(t)}{N_i}\right)\right] \\
&= 0.
\end{aligned}$$

This completes the proof. ■

Remark 3.2.2. *In some applications of this result, we use convex functions like $|\cdot|$ or the maximum which are not differentiable. However, every convex function on $\mathbb{R}$ is absolutely continuous ([50], Chapter 5.5, Proposition 17) and therefore differentiable almost everywhere. The composition of an absolutely convex continuous function and a differentiable function is differentiable almost everywhere. We also use the result that if an absolutely continuous function f satisfies $f' \leq 0$, then it is non increasing.*

Let $\mathbf{x} \in \mathbb{R}^n$, then the function $\|\cdot\|$ defined as

$$\|\mathbf{x}\| = \left(\sum_{i=1}^{n} \frac{v_i}{N_i} x_i^2 \right)^{\frac{1}{2}} \tag{3.5}$$

is a norm on $\mathbb{R}^n$ if $\mathbf{N} > \mathbf{0}$ and $\mathbf{v} > \mathbf{0}$, which is the case for irreducible matrices. However, if A is reducible and $\mathbf{N} > 0$, then $\mathbf{v}$ may not be strictly positive, hence $\|\cdot\|$ is just a seminorm on $\mathbb{R}^n$. The result below is the discrete version of Poincaré's lemma for irreducible matrices.

Lemma 3.2.3. *Let $(a_{ij})_{1\leq i,j\leq n}$ be an irreducible matrix with Perron eigenvectors $\mathbf{v}, \mathbf{N} > \mathbf{0}$. Then there is a constant $\alpha > 0$ such that for any vector $\mathbf{m}$ satisfying*

$$\sum_{i=1}^{n} v_i m_i = 0, \tag{3.6}$$

the inequality

$$\sum_{i=1}^{n} \sum_{j=1}^{n} v_i a_{ij} N_j \left(\frac{m_j}{N_j} - \frac{m_i}{N_i} \right)^2 \geq \alpha \sum_{i=1}^{n} \frac{v_i}{N_i} m_i^2 \tag{3.7}$$

holds.

Proof. Let us introduce a new inner product between two vectors $\mathbf{x}$ and $\mathbf{y}$ defined as

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} \frac{v_i}{N_i} x_i y_i \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n. \tag{3.8}$$

This inner product defines a norm (defined in (3.5)) on $\mathbb{R}^n$ which is equivalent to any other norm on $\mathbb{R}^n$, by completeness. Let $\mathbf{m} \neq \mathbf{0}$ be a vector satisfying (3.6). We shall normalise $\mathbf{m}$ and call this normalised vector $\overline{\mathbf{m}}$ so that

$$\sum_{i=1}^{n} \frac{v_i}{N_i} \overline{m}_i^2 = 1.$$

We notice that $\overline{\mathbf{m}}$ still satisfies

$$\sum_{i=1}^{n} v_i \overline{m}_i = 0. \tag{3.9}$$

Dividing Equation (3.7) with $\|\mathbf{m}\|^2$, where $\|\mathbf{m}\|$ is given by (3.5), gives

$$\sum_{i=1}^{n}\sum_{j=1}^{n} v_i a_{ij} N_j \left(\frac{\overline{m}_j}{N_j} - \frac{\overline{m}_i}{N_i}\right)^2 \geq \alpha > 0. \tag{3.10}$$

Now suppose that there is no α satisfying (3.10). This means that for each k there exists a vector $(\overline{\mathbf{m}}^k)_{k\geq 1}$, satisfying

$$\sum_{i=1}^{n} v_i \overline{m}_i^k = 0, \quad \sum_{i=1}^{n} \frac{v_i}{N_i}(\overline{m}_i^k)^2 = 1,$$

and

$$\sum_{i=1}^{n}\sum_{j=1}^{n} v_i a_{ij} N_j \left(\frac{\overline{m}_j^k}{N_j} - \frac{\overline{m}_i^k}{N_i}\right)^2 \leq \frac{1}{k}. \tag{3.11}$$

The sequence $(\overline{\mathbf{m}}^k)_{k\geq 1}$ is bounded and its terms are on the $n-$sphere of radius 1. This sphere is compact, so by the Bolzano-Weierstrass theorem, there exists a subsequence of $(\overline{\mathbf{m}}_k)_{k\geq 1}$ that converges to a vector $\overline{\overline{\mathbf{m}}}$ which is also on the $n-$sphere. Taking limits on both sides of inequality (3.11), we find that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} v_i a_{ij} N_j \left(\frac{\overline{\overline{m}}_j}{N_j} - \frac{\overline{\overline{m}}_i}{N_i}\right)^2 = 0. \tag{3.12}$$

A is irreducible, so for every pair i and j ($i \neq j$), there exists a sequence of indices $j, i_r, i_{r-1}, \cdots, i_1, i$ such that $a_{i,i_1} a_{i_1,i_2} \cdots a_{i_{r-1},i_r} a_{i_r,j} > 0$ (see [44], p. 671). This means that $a_{i,i_1} > 0$, which implies that for that particular pair i, i_1, Equation (3.12) holds if and only if

$$\frac{\overline{\overline{m}}_i}{N_i} = \frac{\overline{\overline{m}}_{i_1}}{N_{i_1}};$$

$a_{i_1,i_2} > 0$ which implies that Equation (3.12) holds if and only if

$$\frac{\overline{\overline{m}}_{i_1}}{N_{i_1}} = \frac{\overline{\overline{m}}_{i_2}}{N_{i_2}} = \frac{\overline{\overline{m}}_i}{N_i}.$$

If we continue with the same reasoning for all the terms in the product, we find that

$$\frac{\overline{\overline{m}}_{i_{r-1}}}{N_{i_{r-1}}} = \frac{\overline{\overline{m}}_{i_r}}{N_{i_r}} = \frac{\overline{\overline{m}}_{i_j}}{N_{i_j}},$$

hence

$$\frac{\overline{\overline{m}}_i}{N_i} = \frac{\overline{\overline{m}}_{i_1}}{N_{i_1}} = \frac{\overline{\overline{m}}_{i_2}}{N_{i_2}} = \cdots = \frac{\overline{\overline{m}}_{i_{r-1}}}{N_{i_{r-1}}} = \frac{\overline{\overline{m}}_{i_r}}{N_{i_r}} = \frac{\overline{\overline{m}}_j}{N_j}.$$

By the process we have just described, it follows that for every pair, i, j, $\overline{\overline{m}}_i/N_i = \overline{\overline{m}}_j/N_j$. Therefore, $\overline{\overline{m}}_i = \nu N_i$ for some constant ν. Then, using the assumption in (3.6)

$$0 = \sum_{i=1}^{n} \overline{\overline{m}}_i v_i = \sum_{i=1}^{n} \nu N_i v_i = \nu.$$

But if $\nu = 0$, then $\overline{\overline{m}}_i = 0$ for all $1 \le i \le n$. This implies that the sequence of vectors $(\overline{\mathbf{m}}^k)_{k\ge 1}$ converges to a zero vector, a contradiction since the zero vector is not contained on the unit sphere. Therefore, $\alpha > 0$ satisfying (3.10) does exist. Hence (3.7) holds. ■

Remark 3.2.4. *The above result holds for any irreducible matrix A, regardless of its dominant eigenvalue. That is, Lemma 3.2.3 holds even when $s(A) \neq 0$. In particular, it holds for non-negative irreducible matrices as well.*

As a consequence of Theorem 3.2.1 together with Lemma 3.2.3, we have the following theorem regarding the solution of the problem (3.3) (an extension of Proposition 6.5 in [48] to irreducible matrices).

Theorem 3.2.5. *Let A be an irreducible ML matrix with $s(A) = 0$. Then for any solution $\mathbf{u}(t)$ satisfying (3.3), the following is true:*

1. $$\rho := \sum_{i=1}^{n} v_i u_i(t) = \sum_{i=1}^{n} v_i u_i(0), \tag{3.13}$$

2. $$\sum_{i=1}^{n} v_i |u_i(t)| \le \sum_{i=1}^{n} v_i |u_i(0)|, \tag{3.14}$$

3. *there exist $\alpha > 0$ such that*
$$\sum_{i=1}^{n} v_i N_i \left(\frac{u_i(t) - \rho N_i}{N_i}\right)^2 \le e^{-\alpha t} \sum_{i=1}^{n} v_i N_i \left(\frac{u_i(0) - \rho N_i}{N_i}\right)^2, \tag{3.15}$$

4. *if there exist constants C_1, C_2 such that $C_1 N_i \le u_i(0) \le C_2 N_i$, then*
$$C_1 N_i \le u_i(t) \le C_2 N_i, \quad t \ge 0. \tag{3.16}$$

Proof. We pick $H(u) = u$ and use Lemma 3.2.1.

$$\begin{aligned} \frac{d}{dt}\sum_{i=1}^{n} v_i N_i H\left(\frac{u_i(t)}{N_i}\right) &= \frac{d}{dt}\sum_{i=1}^{n} v_i N_i \left(\frac{u_i(t)}{N_i}\right) \\ &= \frac{d}{dt}\sum_{i=1}^{n} v_i u_i(t) \le 0. \end{aligned}$$

From this, we conclude that
$$\sum_{i=1}^{n} v_i u_i(t) \le \sum_{i=1}^{n} v_i u_i(0). \tag{3.17}$$

The function $H(u) = -u$ is also convex and, using this function in Lemma 3.2.1, yields

$$\sum_{i=1}^{n} v_i u_i(t) \geq \sum_{i=1}^{n} v_i u_i(0). \tag{3.18}$$

From these equations ((3.17) and (3.18)), we can conclude that (3.13) holds.

To prove the inequality (3.14), we use the function $H(u) = |u|$, which is convex. Then

$$\begin{aligned}
\frac{d}{dt}\sum_{i=1}^{n} v_i N_i H\left(\frac{u_i(t)}{N_i}\right) &= \frac{d}{dt}\sum_{i=1}^{n} v_i N_i \left|\frac{u_i(t)}{N_i}\right| \\
&= \frac{d}{dt}\sum_{i=1}^{n} v_i \left|u_i(t)\right| \\
&\leq 0 \text{ a.e.}
\end{aligned}$$

Thus, from Remark 3.2.2,

$$\sum_{i=1}^{n} v_i |u_i(t)| \leq \sum_{i=1}^{n} v_i |u_i(0)|.$$

To prove the third item, let $H(u) = u^2$ and $\mathbf{h}(t) = \mathbf{u}(t) - \rho\mathbf{N}$. Then

$$\begin{aligned}
\frac{d}{dt}\sum_{i=1}^{n} v_i N_i H\left(\frac{h_i(t)}{N_i}\right) &= \sum_{i=1}^{n} v_i N_i \frac{d}{dt} H\left(\frac{u_i(t)-\rho N_i}{N_i}\right) \\
&= \sum_{i=1}^{n} v_i N_i \frac{d}{dt}\left(\frac{u_i(t)-\rho N_i}{N_i}\right)^2 \\
&= 2\sum_{i=1}^{n}\sum_{j=1}^{n} v_i a_{ij} u_j(t)\left(\frac{u_i(t)-\rho N_i}{N_i}\right) \\
&= 2\sum_{i=1}^{n}\sum_{j=1}^{n} v_i a_{ij} N_j\left(\frac{u_j(t)-\rho N_j}{N_j}\right)\left(\frac{u_i(t)-\rho N_i}{N_i}\right) \\
&= -\sum_{i=1}^{n}\sum_{j=1}^{n} v_i a_{ij} N_j\left(\frac{u_j(t)-\rho N_j}{N_j} - \frac{u_i(t)-\rho N_i}{N_i}\right)^2.
\end{aligned}$$

Using the result in equation (3.13) together with the normalising conditions in (3.4), we see that the vector $\mathbf{m} = \mathbf{u}(t) - \rho\mathbf{N}$ satisfies the conditions of Lemma 3.2.3 above, hence, we use the mentioned lemma to get

$$-\sum_{i=1}^{n}\sum_{j=1}^{n} v_i a_{ij} N_j\left(\frac{u_j(t)-\rho N_j}{N_j} - \frac{u_i(t)-\rho N_i}{N_i}\right)^2 \leq -\alpha\sum_{i=1}^{n}\frac{v_i}{N_i}\left(u_i(t)-\rho N_i\right)^2,$$

hence

$$\frac{d}{dt}\sum_{i=1}^{n} v_i N_i\left(\frac{u_i(t)-\rho N_i}{N_i}\right)^2 \leq -\alpha\sum_{i=1}^{n}\frac{v_i}{N_i}\left(u_i(t)-\rho N_i\right)^2,$$

Hence,

$$\frac{d}{dt}\sum_{i=1}^{n}\frac{v_i}{N_i}(u_i(t)-\rho N_i)^2+\alpha\sum_{i=1}^{n}\frac{v_i}{N_i}(u_i(t)-\rho N_i)^2\le 0$$

implies

$$\frac{d}{dt}\left(e^{\alpha t}\sum_{i=1}^{n}\frac{v_i}{N_i}(u_i(t)-\rho N_i)^2\right)\le 0.$$

Thus

$$e^{\alpha t}\sum_{i=1}^{n}\frac{v_i}{N_i}(u_i(t)-\rho N_i)^2-\sum_{i=1}^{n}\frac{v_i}{N_i}(u_i(0)-\rho N_i)^2\le 0$$

and from this, we get (3.15).

Finally, for the upper bound, we use the function $H(u)=[(u-C_2)_+]^2$, where $(u-C_2)_+$ refers to the positive part of $u(t)-C_2$. For this function, we write $(u-C_2)_+=0$ if $u-C_2\le 0$ on some interval. Note that although the function $y=(u-C_2)_+$ is not differentiable in general, its square, $[(u-C_2)_+]^2$ is a differentiable function. If the assumptions are satisfied, then $u_i(0)/N_i\le C_2$, thus

$$\sum_{i=1}^{n}v_iN_iH\left(\frac{u_i(0)}{N_i}\right)=\sum_{i=1}^{n}v_iN_i\left[\left(\frac{u_i(0)}{N_i}-C_2\right)_+\right]^2=0.$$

Hence by Lemma 3.2.1, we have

$$\frac{d}{dt}\sum_{i=1}^{n}v_iN_iH\left(\frac{u_i(t)}{N_i}\right)=\frac{d}{dt}\sum_{i=1}^{n}v_iN_i\left[\left(\frac{u_i(t)}{N_i}-C_2\right)_+\right]^2\le 0;$$

that is

$$\sum_{i=1}^{n}v_iN_i\left[\left(\frac{u_i(t)}{N_i}-C_2\right)_+\right]^2\le\sum_{i=1}^{n}v_iN_i\left[\left(\frac{u_i(0)}{N_i}-C_2\right)_+\right]^2=0,$$

which implies that $u_i(t)\le C_2N_i$.

For the lower bound, we choose $H(u)=[(u-C_1)_-]^2$, where $(u-C_1)_-$ is the negative part of $u(t)-C_1$. Note that $u(t)-C_1\ge 0$ on some interval of time if and only if $(u(t)-C_1)_-=0$ on that interval. Then if the assumptions are satisfied, that is, $C_1N_i\le u_i(0)$, then $u_i(0)/N_i-C_1\ge 0$. Hence

$$\sum_{i=1}^{n}v_iN_iH\left(\frac{u_i(0)}{N_i}\right)=\sum_{i=1}^{n}v_iN_i\left[\left(\frac{u_i(0)}{N_i}-C_1\right)_-\right]^2=0.$$

By Lemma 3.2.1,

$$\frac{d}{dt}\sum_{i=1}^{n}v_iN_iH\left(\frac{u_i(t)}{N_i}\right)=\frac{d}{dt}\sum_{i=1}^{n}v_iN_i\left[\left(\frac{u_i(t)}{N_i}-C_1\right)_-\right]^2\le 0;$$

$$\Rightarrow\sum_{i=1}^{n}v_iN_i\left[\left(\frac{u_i(t)}{N_i}-C_1\right)_-\right]^2\le\sum_{i=1}^{n}v_iN_i\left[\left(\frac{u_i(0)}{N_i}-C_1\right)_-\right]^2=0.$$

Since all the terms of the sum are positive, we must have

$$\sum_{i=1}^{n} v_i N_i \left[\left(\frac{u_i(t)}{N_i} - C_1 \right)_- \right]^2 = 0,$$

and hence,

$$\left[\left(\frac{u_i(t)}{N_i} - C_1 \right)_- \right]^2 = 0 \Rightarrow \frac{u_i(t)}{N_i} - C_1 \geq 0$$

for each $i \in \{1, \ldots, n\}$. Therefore, $u_i(t) \geq C_1 N_i$. ■

3.3 Reducible matrices

If A is a reducible ML matrix, then Lemma 3.2.3 does not hold in general. To illustrate this, consider the following examples.

Example 3.3.1. *Let*

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0.5 \end{pmatrix}.$$

A is reducible with $r = 1$, one of the right (and respectively, left) eigenvectors is $\mathbf{N} = 0.25(1, 1, 2)^T$ (respectively, $\mathbf{v} = 2(1, 1, 0)$). Pick

$$\mathbf{m} = \left(\pm\frac{1}{4}, \mp\frac{1}{4}, m \right), \quad m \in \mathbb{R}.$$

This vector satisfies the requirements in Lemma 3.2.3 but the lemma does not hold since

$$\sum_{i=1}^{3} \sum_{j=1}^{3} v_i a_{ij} N_j \left(\frac{m_j}{N_j} - \frac{m_i}{N_i} \right)^2 = 0;$$

while

$$\sum_{i=1}^{3} \frac{v_i}{N_i} m_i^2 = \frac{2}{1/4} \left(\frac{1}{4} \right)^2 + \frac{2}{1/4} \left(\frac{-1}{4} \right)^2 = 1.$$

Clearly, there is no positive number α for which (3.7) holds.

If A is reducible, write A in the normal form given by (2.8). Then $\mathbf{N}$ is non-negative but may not be strictly positive and the eigenvalue 0 may not be simple, [21]. It is important to note that even when a positive right eigenvector $\mathbf{N}$ exists, it is not always unique (up to constant multiples). It is possible to have two vectors $\mathbf{N}_1, \mathbf{N}_2 > 0$ both satisfying $A\mathbf{N}_1 = \mathbf{0}$, $A\mathbf{N}_2 = \mathbf{0}$, but $\mathbf{N}_1 \neq \beta\mathbf{N}_2$ for any $\beta \in \mathbb{N}$.

To be able to extend the results from the previous section to reducible matrices, we need to be able to divide by N_i. But this, in general, is not possible. However, further developments (as will be seen below) allow for the vector $\mathbf{N}$ to have zeros.

As seen in Example 3.3.1, Lemma 3.2.3 does not hold for reducible matrices, even when a positive right eigenvector exists. This is because the eigenvalue 0 may have algebraic multiplicity greater than one and $\mathbf{v}$ may have zeros. Indeed if $\mathbf{N} > \mathbf{0}$, then the right eigenspace corresponding to 0 are spanned by the vectors

$$\mathbf{N}_i = (\mathbf{0}, \mathbf{N}^i, \mathbf{0}, \mathbf{y}_i^{g+1}, \cdots, \mathbf{y}_i^s); i = 1, \ldots, g; \tag{3.19}$$

where $\mathbf{N}^i > \mathbf{0}$ is a normalised right eigenvector for A_i corresponding to 0 and

$$\begin{aligned} \mathbf{y}_i^{g+1} &= (-A_{g+1})^{-1} A_{g+1,i} \mathbf{N}^i \\ \mathbf{y}_i^j &= (-A_j)^{-1} \left(A_{j,i} \mathbf{N}^i + \sum_{h=g+1}^{j-1} A_{j,h} \mathbf{y}_i^h \right). \end{aligned}$$

The vectors $\mathbf{y}_i^h$ are well defined since $s(A_h) < 0$ for all $h = g+1, \ldots, s$. The matrices A_j and $A_{h,j}$ described here are the matrices in the normal form (2.8). Since $A_h, h = g+1, \ldots, s$ are all irreducible, Theorem 2.1 of [51] guarantees that the matrices $(-A_h)^{-1}$ are all strictly positive, hence $\mathbf{y}_i^h \geq 0$. The left eigenspace is then spanned by the vectors

$$\mathbf{v}_i = (\mathbf{0}, \mathbf{v}^i, \mathbf{0}) \quad \forall i = 1, \ldots, g \quad \mathbf{v}^i A_i = \mathbf{0}. \tag{3.20}$$

In order to state analogous results (to Lemma 3.2.3 and Theorem 3.2.5) for reducible matrices, regardless of whether a positive eigenvector exists or not, we need the following definitions.

We partition the set $\{1, \ldots, n\}$ into sets $I_k = \{n_1 + \cdots + n_{k-1} + 1, \ldots, n_1 + \cdots + n_k\}$ for $1 \leq k \leq s$ corresponding to matrices A_k that have r as an eigenvalue. Define

$$I^\Theta = I_{k_1} \cup \cdots \cup I_{k_l}, \tag{3.21}$$

where $\Theta = \{k_1, \ldots, k_l\} \subset \{1, \ldots, s\}$ is arbitrary. Further, we define

$$X^\Theta = \mathsf{Span}\{\mathbf{e}_i\}_{i \in I^\Theta}, \tag{3.22}$$

where $\mathbf{e}_i = (\delta_{ik})_{1 \leq k \leq n}$ and δ_{ik} is the Kronecker delta. Let $X_k = X^{\{k\}}$ so that for general Θ defined above, $X^\Theta = X_{k_1} \oplus \cdots \oplus X_{k_l}$. Let $\mathbf{N} = (\mathbf{N}^1, \cdots, \mathbf{N}^s)$ be an eigenvector of A. For a non-negative left eigenvector of A, we shall write $\mathbf{v} = (\mathbf{v}^1, \cdots, \mathbf{v}^s)$. Then using the normal form of A in (2.8), we see that $\mathbf{v}^i = \mathbf{0}$ if $s(A_i) < 0$.

To make this indexing clearer, suppose in the normal form of A in (2.8), and the matrices A_1, A_g and A_{s-1} all have $r = 0$ as an eigenvalue (and $s(A_i) < 0$ for $i \neq 1, g, s-1$). Then $I_1 = \{1, \ldots, n_1\}, I_g = \{n_1 + \cdots + n_{g-1} + 1, n_1 + \cdots + n_{g-1} + 2, \ldots, n_1 + \cdots + n_g\}$ and $I_{s-1} = \{n_1 + \cdots + n_{s-2} + 1, n_1 + \cdots + n_{s-2} + 2, \ldots, n_1 + \cdots + n_{s-1}\}$. $I^\Theta = I_1 \cup I_g \cup I_{s-1}$, where $\Theta = \{r_1, r_2, r_3\} = \{1, g, s-1\} \subset \{1, \ldots, s\}$.

Consider an arbitrary reducible ML matrix A with dominant eigenvalue $s(A) = 0$. The following result holds for this matrix.

Lemma 3.3.2. *Let Θ be an arbitrary set of indices from the set $\{1, \ldots, s\}$ such that X^Θ is invariant under A. Let $A^\Theta = A|_{X^\Theta} : X^\Theta \to X^\Theta$, $\mathbf{N}^\Theta = (\mathbf{N}^k)_{k\in\Theta}$ and $\mathbf{v}^\Theta = (\mathbf{v}^k)_{k\in\Theta}$. Then*

1. *$\mathbf{v}^\Theta A^\Theta = \mathbf{0}$.*

2. *If $A_{ij} = \mathbf{0}$ whenever $i \in \Theta$ and $j \notin \Theta$, then $A^\Theta \mathbf{N}^\Theta = \mathbf{0}$.*

3. *If $A^\Theta \mathbf{N}^\Theta = \mathbf{0}$ and $A_k \mathbf{N}^k = \mathbf{0}$ for some $k \in \Theta$, then $A_{k,l} = \mathbf{0}$ for all $k > l \in \Theta$. Hence if $A_k \mathbf{N}^k = \mathbf{0}$ for all $k \in \Theta$, then $\mathbf{v}^l A_l = \mathbf{0}$ for all $l \in \Theta$.*

Proof. For X^Θ to be invariant, $A_{ij} = \mathbf{0}$ for $i \notin \Theta$ and $j \in \Theta$. Since $\mathbf{v}A = \mathbf{0}$, then if $k \in \Theta$,

$$\mathbf{v}^k A_k + \sum_{l=k+1}^{s} \mathbf{v}^l A_{l,k} = \mathbf{0}.$$

This together with the invariance of X^Θ under A implies that

$$\mathbf{v}^k A_k + \sum_{l=k+1, l\in\Theta}^{s} \mathbf{v}^l A_{l,k} = \mathbf{0} \quad \forall k \in \Theta,$$

hence $\mathbf{v}^\Theta A^\Theta = \mathbf{0}$.

To prove item (2), pick $k \in \Theta$. Then

$$\mathbf{0} = A_k \mathbf{N}^k + \sum_{l=1}^{k-1} A_{k,l} \mathbf{N}^l = A_k \mathbf{N}^k + \sum_{l=1, l\in\Theta}^{k-1} A_{k,l} \mathbf{N}^l = A^\Theta \mathbf{N}^\Theta.$$

Finally, for $k \in \Theta$,

$$(A^\Theta \mathbf{N}^\Theta)_k = \sum_{l=1}^{k-1} A\mathbf{N}^l + A\mathbf{N}^k = \sum_{l=1}^{k-1} A\mathbf{N}^l = \mathbf{0}.$$

If $l \in \Theta$, then $\mathbf{N}^l > \mathbf{0}$ and $A_{k,l} \geq \mathbf{0}$ hence $A_{k,l} = \mathbf{0}$. Therefore,

$$\mathbf{0} = \mathbf{v}^l A_l + \sum_{k=l+1}^{s} \mathbf{v}^k A_{k,l} = \mathbf{v}^l A_l.$$

This gives the required result. ∎

Lemma 3.3.3. *Let I^Θ be an arbitrary set of indices defined by (3.21) and $\mathbf{N} = (N_1, \ldots, N_n)$ be any right eigenvector of A with $N_i > 0$ for $i \in I^\Theta$ and $A_r \mathbf{N}^r = \mathbf{0}$ for any $r \in \Theta$. Then there is a positive constant α such that for any non zero vector $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{R}^n$ satisfying*

$$\sum_{i \in I_k} v_i m_i = 0 \tag{3.23}$$

for each $k \in \Theta$, the following inequality holds:

$$\sum_{i \in I^\Theta} \sum_{j \in I^\Theta} v_i a_{ij} N_j \left(\frac{m_j}{N_j} - \frac{m_i}{N_i} \right)^2 > \alpha \sum_{i \in I^\Theta} \frac{v_i}{N_i} m_i^2. \tag{3.24}$$

Proof. By Lemma 3.3.2, $\mathbf{v}^k > \mathbf{0}$ for $k \in \Theta$. Thus

$$\|\mathbf{m}\| = \sqrt{\sum_{i \in \Theta} \frac{v_i}{N_i} m_i^2}$$

is a norm on the space $Y = \{\mathbf{e}_k\}_{k \in I^\Theta}$. If $\mathbf{m} = \mathbf{0}$, the result holds trivially, so we assume that $\mathbf{m} \neq \mathbf{0}$. We now divide both sides of Equation 3.24 by $\|\mathbf{m}\|^2$ to get

$$\sum_{i \in I^\Theta} \sum_{j \in I^\Theta} v_i a_{ij} N_j \left(\frac{\tilde{m}_j}{N_j} - \frac{\tilde{m}_i}{N_i} \right)^2 > \alpha; \tag{3.25}$$

where $\tilde{\mathbf{m}} = \mathbf{m}/\|\mathbf{m}$. Note that the vector $\tilde{\mathbf{m}}$ satisfies the assumptions of the lemma. Now suppose that (3.3.3) is false. Then there exists a sequence $(\tilde{\mathbf{m}}^l)_{l \geq 0}$ on the unit sphere in Y satisfying the relation

$$\sum_{i \in I^\Theta} \sum_{j \in I^\Theta} v_i a_{ij} N_j \left(\frac{\tilde{m}_j^l}{N_j} - \frac{\tilde{m}_i^l}{N_i} \right)^2 \leq \frac{1}{l}. \tag{3.26}$$

Since the sequence is on the unit sphere in Y, it contains a convergent subsequence, whose limit is $\tilde{\tilde{\mathbf{m}}} \in Y$. This limit also exists on the unit sphere (Bolzano-Weierstrass) and it satisfies the assumptions of the lemma. Taking limits on (3.26), we get

$$\sum_{i \in I^\Theta} \sum_{j \in I^\Theta} v_i a_{ij} N_j \left(\frac{\tilde{\tilde{m}}_j}{N_j} - \frac{\tilde{\tilde{m}}_i}{N_i} \right)^2 = 0. \tag{3.27}$$

For $r_i, \ldots, r_k \in I^{\Theta}$, equation (3.27) is equivalent to the equation below:

$$\sum_{i\in I_{r_1}}\sum_{j\in I_{r_1}} v_i a_{ij} N_j \left(\frac{\tilde{\tilde{m}}_j}{N_j} - \frac{\tilde{\tilde{m}}_i}{N_i}\right)^2 + \cdots + \sum_{i\in I_{r_k}}\sum_{j\in I_{r_k}} v_i a_{ij} N_j \left(\frac{\tilde{\tilde{m}}_j}{N_j} - \frac{\tilde{\tilde{m}}_i}{N_i}\right)^2 = 0. \tag{3.28}$$

For $i = j$ the terms of the term are all equal to zero. For $i \neq j$, $a_{ij} \geq 0$ and $v_k > 0, N_k > 0$ provided $k \in I^{\Theta}$. So equation (3.27) holds if and only if each term on the left hand side of the equation is zero. Consider the term corresponding to A_{r_i}, for $1 \leq i \leq k$. Since A_{r_i} is irreducible, for every pair $k, j \in I_{r_i}$, there exists a sequence of indices $j, k_r, k_{r-1}, \cdots, k_1, k$ such that $a_{k,k_1} a_{k_1,k_2} \cdots a_{k_{r-1},k_r} a_{k_r,j} > 0$. Therefore, Equation 3.28 holds if and only if

$$\frac{\tilde{\tilde{m}}_j}{N_j} = \frac{\tilde{\tilde{m}}_{k_r}}{N_{k_r}} = \cdots = \frac{\tilde{\tilde{m}}_{k_1}}{N_{k_1}} = \frac{\tilde{\tilde{m}}_k}{N_k}.$$

Therefore, $\tilde{\tilde{m}}_k = \nu^i N_k$ for some constant ν^i, hence from (3.23), we obtain

$$0 = \sum_{k\in I_{r_i}} v_k \tilde{\tilde{m}}_k = \nu^i \sum_{k\in I_{r_i}} N_k v_k$$

and since $\mathbf{N}^{r_i}, \mathbf{v}^{r_i} > \mathbf{0}$, we have $\nu^i = 0$. But this also means that $\tilde{\tilde{m}}_k = 0$ for all $k \in I_{r_i}$. If we repeat this procedure for all $k \in \Theta$, we obtain $\tilde{\tilde{\mathbf{m}}} = \mathbf{0}$ in Y, a contradiction. ■

Remark 3.3.4. *If A has a positive right eigenvector, then any vector $\mathbf{m} \in \mathbb{R}^n$ which satisfies (3.23) also satisfies*

$$\sum_{i=1}^{n} v_i m_i = 0$$

trivially.

Example 3.3.5. *Let*

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 18 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0.5 \end{pmatrix}.$$

For this matrix, $r = 3$, and if we pick $\mathbf{N} = (5/61)(1, 1, 1, 6, 16/5)^T$ and $\mathbf{v} = (61/70)(1, 1, 6, 1, 0)$, we see that $A\mathbf{N} = 3\mathbf{N}$ and $\mathbf{v}A = 3\mathbf{v}$. We select a vector $\mathbf{m}$ such that

$$\sum_{i=1}^{5} v_i m_i = 0.$$

Let $\mathbf{m} = (-k, k, k_1, -6k_1, m)^T$*, where* $m, k, k_1 \in \mathbb{R}$ *are arbitrary. This vector satisfies the requirements of Lemma 3.2.3.*

$$\sum_{i=1}^{5}\sum_{j=1}^{5} v_i a_{ij} N_j \left(\frac{m_j}{N_j} - \frac{m_i}{N_i}\right)^2 = \sum_{i=1}^{5} v_i \left[\frac{5}{61} a_{i1}\left((-k)\frac{61}{5} - \frac{m_i}{N_i}\right)^2 + a_{i2}\frac{5}{61}\left(k\frac{61}{5} - \frac{m_i}{N_i}\right)^2 \right.$$

$$\left. + a_{i3}\frac{5}{61}\left((k_1)\frac{61}{5} - \frac{m_i}{N_i}\right)^2 + a_{i4}\frac{30}{61}\left((-6k_1)\frac{61}{30} - \frac{m_i}{N_i}\right)^2 + a_{i5}\frac{16}{61}\left((m)\frac{61}{16} - \frac{m_i}{N_i}\right)^2\right]$$

$$= \frac{122^2}{25(7)}(k^2 + 9k_1^2) = \frac{122^2}{175}(k^2 + 9k_1^2).$$

and the right hand side of the inequality is

$$\begin{aligned}\sum_{i=1}^{5} \frac{v_i}{N_i} m_i^2 &= \frac{61}{70} \times \frac{61}{5}\left[(-k)^2 + k^2 + 6k_1^2 + \frac{1}{6}.(-6k_1)^2\right] \\ &= \frac{61^2}{175}\left(k^2 + 6k_1^2\right) \\ &< \sum_{i=1}^{5}\sum_{j=1}^{5} v_i a_{ij} N_j \left(\frac{m_j}{N_j} - \frac{m_i}{N_i}\right)^2.\end{aligned}$$

Therefore, for some $\alpha > 0$*, the result holds.*

From Example 3.3.5 above, we observe that $I_1 = \{1, 2\}; I_2 = \{3, 4\}$, $I^\Theta = \{1, 2, 3, 4\}$ and $\Theta = \{1, 2\}$. In the first case when $\mathbf{m} = (-k, k, k_1, -6k_1, m)^T$, we observe that

$$\sum_{i=1}^{2} v_i m_i = 0 = \sum_{i=3}^{4} v_i m_i,$$

hence the assumption in (3.23) is satisfied, and hence the result holds. In general, any vector $\mathbf{m}$ of the form $\mathbf{m} = (-k, k, k_1, -6k_1, m)^T, m \in \mathbb{R}$ yields a positive result for this matrix in Example 3.3.5, regardless of the vector $\mathbf{v}$ used.

Remark 3.3.6. *If* A *is a* 3×3 *block triangular matrix in normal form with* $g = 2$*,* A_1 *and* A_2 *being* 1×1 *blocks (i.e scalars) and* $\mathbf{N} > 0$*,* $\mathbf{v} \geq 0$*, then the only vectors* $\mathbf{m}$ *for which equation 3.23 holds are vectors of the form* $(0, 0, \mathbf{m}^3)^T$*, where* $\mathbf{m}^3$ *is a vector whose dimension depends on the dimension of* A_3*. This is because if* $\mathbf{m} = (m_1, m_2, \mathbf{m}^3)^T$*, then (3.23) holds if and only if* $m_1 = m_2 = 0$*. But for such a vector, the left hand side of the inequalities (3.7) and (3.24) are always* 0*. Therefore, there is no vector* $\mathbf{m}$ *for which the lemma holds. This explains why Example 3.3.1 gives a negative answer.*

In general, if $\mathbf{N} > \mathbf{0}$ *and* $2 \leq g \leq s$ *and* A_i *are one dimensional, with* $r(A_i) = r(A), i = 1, \ldots, g$*, then there is no non-trivial vector* $\mathbf{m}$ *that satisfies the result.*

3.4 Relative Entropy Inequality for Reducible Matrices

We now consider the differential equation in (3.3), where A is of the form (2.8) and $s(A) = 0$. We still use the indexing given in the previous section to state the following result.

Theorem 3.4.1. *Let Θ be an arbitrary set of indices from the set $\{1, \ldots, s\}$ such that X^Θ is invariant under A, $A^\Theta \mathbf{N}^\Theta = \mathbf{0}$ and $\mathbf{N}^k > \mathbf{0}$ for $k \in \Theta$. Let H be a convex function on $\mathbb{R}$. Then any solution $\mathbf{u}$ to the initial value problem in (3.3), with $\mathbf{u}_0 \in X^\Theta$, satisfies*

$$\frac{d}{dt} \sum_{i \in I^\Theta} v_i N_i H\left(\frac{u_i(t)}{N_i}\right) \leq 0.$$

Hence, for all $t \geq 0$,

1.
$$\sum_{i \in I^\Theta} v_i u_i(t) = \sum_{i \in I^\Theta} v_i u_i(0); \tag{3.29}$$
$$\sum_{i \in I^\Theta} v_i |u_i(t)| \leq \sum_{i \in I^\Theta} v_i |u_i(0)|. \tag{3.30}$$

2. *If there exists constants C_1, C_2 such that $C_1 N_i \leq u_i(0) \leq C_2 N_i$, then $C_1 N_i \leq u_i(t) \leq C_2 N_i$ for any $i \in I^\Theta$, such that $\mathbf{v}^i > \mathbf{0}$;*

3. *If $A_k \mathbf{N}^k = \mathbf{0}$ for any $k \in \Theta$, then there is a constant $\alpha > 0$ such that*
$$\sum_{i \in I^\Theta} \frac{v_i}{N_i} \left(u_i(t) - \rho^i N_i\right)^2 \leq e^{-\alpha t} \sum_{i \in I^\Theta} \frac{v_i}{N_i} \left(u_i(0) - \rho^i N_i\right)^2; \tag{3.31}$$
where
$$\rho^i = \sum_{k \in I_r} v_k u_k(0) \text{ for } i \in I_r.$$

Proof.

$$\begin{aligned} \frac{d}{dt} \sum_{i \in I^\Theta} v_i N_i H\left(\frac{u_(t)}{N_i}\right) &= \sum_{i \in I^\Theta} \sum_{j \in I^\Theta} v_i a_{ij} u_j(t) H'\left(\frac{u_i(t)}{N_i}\right) \\ &= \sum_{i \in I^\Theta} \sum_{j \in I^\Theta} v_i a_{ij} N_j H'\left(\frac{u_i(t)}{N_i}\right) \left[\frac{u_j(t)}{N_j} - \frac{u_i(t)}{N_i}\right]. \end{aligned}$$

Using the convexity of the function H, we obtain

$$\sum_{i \in I^\Theta} \sum_{j \in I^\Theta} v_i a_{ij} N_j H'\left(\frac{u_i(t)}{N_i}\right) \left[\frac{u_j(t)}{N_j} - \frac{u_i(t)}{N_i}\right] \leq \sum_{i \in I^\Theta} \sum_{j \in I^\Theta} v_i a_{ij} N_j \left[H\left(\frac{u_j(t)}{N_j}\right) - H\left(\frac{u_i(t)}{N_i}\right)\right].$$

Since $A^{\Theta}\mathbf{N}^{\Theta} = \mathbf{0}, \mathbf{v}^{\Theta}A^{\Theta} = \mathbf{0}$, we have

$$\sum_{j\in I^{\Theta}} a_{ij}N_j = 0, \quad \sum_{i\in I^{\Theta}} v_i a_{ij} = 0$$

for each $i \in I^{\Theta}$, hence

$$\sum_{i\in I^{\Theta}}\sum_{j\in I^{\Theta}} v_i a_{ij} N_j H\left(\frac{u_j(t)}{N_j}\right) = \left(\sum_{i\in I^{\Theta}} v_i a_{ij}\right) \sum_{j\in I^{\Theta}} N_j H\left(\frac{u_j(t)}{N_j}\right) = 0,$$

$$\sum_{i\in I^{\Theta}}\sum_{j\in I^{\Theta}} v_i a_{ij} N_j H\left(\frac{u_i(t)}{N_i}\right) = \sum_{i\in I^{\Theta}} v_i H\left(\frac{u_i(t)}{N_i}\right)\left(\sum_{j\in I^{\Theta}} a_{ij}N_j\right) = 0.$$

Therefore,

$$\sum_{i\in I^{\Theta}}\sum_{j\in I^{\Theta}} v_i a_{ij} N_j \left[H\left(\frac{u_j(t)}{N_j}\right) - H\left(\frac{u_i(t)}{N_i}\right)\right] = 0.$$

The proof of items 1, 2 and 3 is analogous to that shown in Theorem 3.2.5. ■

3.5 Positive left eigenvector

Suppose the matrix A is an ML matrix with a positive left eigenvector, but the right eigenvector is only non-negative. In this section, we show that we can still use the results obtained in the preceding sections in this chapter to study the long term behaviour of the solution to (3.3).

Suppose A is in normal form (2.8). If we transpose this matrix A, and note that A^T has the same eigenvalues as A, but its right eigenvector $\mathbf{N}$ will now be strictly positive while its left vector $\mathbf{v}$ will be non-negative. By doing this, we have a problem similar to the one in the preceding sections. If we permute this transposed matrix and put it in the normal form (2.8), this will reorder the diagonal blocks in the sense that $(A^T)_i = A^T_{s-i+1}$, for all $i = 1, \ldots, s$, where $(A^T)_i$ is the i^{th} diagonal block in A^T .

By Theorem 2.3.4, all blocks $(A^T)_i$ with $s((A^T)_i) = s(A_{s+1-i}) = 0$, for $1 \le i \le 1 + s - g'$, must be isolated and the others must satisfy $s(A_{s+1-i}) < 0$, for $1 + s - g' \le i \le s$. Therefore, in the normal form of A, we must have $A_{ik} = 0$ for $g' \le i \le s$ and $g' \le k \le i - 1$.

In this case, we have $\mathbf{N}^i > \mathbf{0}$ for $g' \le i \le s$. Furthermore,

$$X^{\Theta} = \text{Span}\left\{\begin{pmatrix}\mathbf{0}\\ \mathbf{N}^{g'}\\ \mathbf{0}\end{pmatrix}, \ldots, \begin{pmatrix}\mathbf{0}\\ \mathbf{0}\\ \mathbf{N}^{s}\end{pmatrix}\right\}$$

is invariant under A, and $A_i\mathbf{N}^i = \mathbf{0}$ as $A_{i,k} = 0$ for $g' \leq k \leq i-1, g' \leq i \leq s$ and $\mathbf{N}^i = \mathbf{0}$ for $i < g'$. Hence, Theorem 3.4.1 can now be used on this matrix, with $\Theta = \{g', \ldots, s\}$.

Chapter 4

Reducible Networks

4.1 Introduction

In this section, we present a survey of some of the results about the transport equation on network structures obtained in [14], [33], [40] and [15]. Consider a simple, strongly connected digraph G with a finite number of vertices (n of them) and m edges. The edges are all assumed to be of unit length, hence the space variable $x \in [0, 1]$. The edges are parameterised contrary to the direction of the flow. That is, the tail of each edge is assumed to be at position 1 while the head is at position 0. The flow of particles along edge e_j is then described by the transport equation

$$\begin{cases} \partial_t u_j(x,t) &= c_j \partial_x u_j(x,t), \quad \forall j = 1, \ldots, m, \\ u_j(x,0) &= f_j(x), \end{cases} \tag{4.1}$$

where c_j is the speed of the particles along the edge and $u_j(x,t)$ is the density of particles on edge e_j at position x and time t. If there is no absorption or generation of material at any vertex, then we have the Kirchoff law at the vertices, and this gives us the boundary condition

$$\phi^-_{ij} c_j u_j(1,t) = w_{i,j} \left[\sum_{k=1}^{m} \phi^+_{i,k} (c_k u_k(0,t)) \right], \quad \forall i \in \{1, \ldots, n\}. \tag{4.2}$$

Equations (4.1) and (4.2) define the transport problem on a strongly connected graph. This problem has been studied using semigroup methods by several authors [33], [15], [40] on finite graphs and were extended to infinite strongly connected graphs in [14].

Remark 4.1.1. *If we model a mass conserving transport on the network, then the conditions at* *the nodes must be given by the Kirchoff law which balances the flow in each node. That is, the total mass flowing in a node must equal to the total mass flowing out. From this point of view, boundary condition stated in [33], [15], [40] is incorrect as it balances densities and not flow.*

First, we write (4.1) together with the boundary condition (4.2) as an abstract Cauchy problem in the state space $X = L^1([0,1])^m$

$$\begin{cases} \frac{d}{dt}\mathbf{u} & = A_0\mathbf{u}(t) \\ \mathbf{u}(0) & = (f_j)_{j=1,\ldots,m}. \end{cases} \tag{4.3}$$

where A_0 is the realisation of the expression $A = diag(c_j\partial_x)_{1\leq j\leq m}$, on the domain

$$D(A_0) = \left\{\mathbf{u} \in W_1^1([0,1])^m; \mathbf{u} \text{ satisfies } (4.2)\right\}. \tag{4.4}$$

Following the proof given in Section 2.3 of [14], we can show that the domain $D(A_0)$ can also be written in the following way

$$D(A_0) = \left\{\mathbf{u} \in W_1^1([0,1])^m : \mathbf{u}(1) = C^{-1}\mathbb{B}C\mathbf{u}(0)\right\}, \tag{4.5}$$

where $C = diag(c_1, \ldots, c_m)$ and $\mathbb{B}$ is the adjacency matrix defined in Chapter 2. Below, we list some of their results.

Proposition 4.1.2. *[33], Proposition 2.5*
The operator $(A_0, D(A_0))$ is a generator of a positive bounded semigroup and hence the flow problem (4.1) with a Kirchoff law (4.2) is well-posed.

Proposition 4.1.3. *[15]*
For a graph G and weighted adjacency matrices $\mathbb{A}$, $\mathbb{B}$ defined in Definition 2.2.6 and equation (2.2), respectively, and the corresponding flow semigroup $(T(t))_{t\geq 0}$, the following statements are equivalent.

1. *G is strongly connected*
2. *$\mathbb{A}$ is irreducible*
3. *$\mathbb{B}$ is irreducible*
4. *$(T(t))_{t\geq 0}$ is irreducible.*

Note that the above result is correct if we add that the graph is connected. As seen in Remark 2.2.16, the implication $3 \Rightarrow 1$ may fail to hold. That is, by Lemma 2.2.14, $\mathbb{B}$ is irreducible if and only if $L(G)$ is strongly connected. But the example in the mentioned remark shows that it is possible for $L(G)$ to be strongly connected while G is not. This only occurs in the presence of isolated vertices in G. Hence, without the assumption that G is connected, $3 \Rightarrow 1$ fails.

Corollary 4.1.4. *[14]*
If $c_j = 1$ for all j, then the semigroup $(T(t))_{t\geq 0}$ is given by

$$(T(t)f)(s) = \mathbb{B}^n f(t+s-n), \quad n \leq t+s \leq n+1, n \in \mathbb{N}_0. \tag{4.6}$$

In [33], it was proved that the spectrum of $(T(t))_{t\geq 0}$ depends not only on the structure of the cycles in G but also on the rational dependency of the flow velocities on the edges that form a cycle. First, suppose that the edges $e_{i_1}, e_{i_2}, \dots, e_{i_l}$ form a cycle in G, and $c_{i_1}, c_{i_2}, \dots, c_{i_l}$ are the speeds of the particles flowing along these edges. Then if there is a $k > 0$ such that

$$k\left(\frac{1}{c_{i_1}} + \frac{1}{c_{i_2}} + \cdots + \frac{1}{c_{i_l}}\right) \in \mathbb{N}, \tag{4.7}$$

then these speeds are said to be rationally dependent. The following result is due to Kramar and Sikolya [33] (Theorem 4.5).

Theorem 4.1.5. *Let G be strongly connected and (4.7) holds. Then there is a decomposition $X = X_1 \oplus X_2$ such that*

- *The semigroup is uniformly stable on the space X_1*
- *The semigroup $(T(t)|_{X_2})_{t\geq 0}$ is periodic with period τ given by*

$$\tau = \frac{1}{k}\textit{gcd}\left\{k\left(\frac{1}{c_{i_1}} + \frac{1}{c_{i_2}} + \cdots + \frac{1}{c_{i_l}}\right): \quad e_{i_1}, \dots, e_{i_l} \text{ form a cycle}\right\},$$

where $k > 0$ is a number satisfying (4.7).

Remark 4.1.6. *Having introduced the necessary notation, we provide a formal justification of the claim that the flow in the network described in (4.1) is related to the finite dimensional system in (3.2). For simplicity, assume that $c_j = 1$ for $j = 1, \dots, m$ and integrate (4.1) with respect to x over $[0,1]$.*

$$\int_0^1 \partial_t u_j(\tilde{x}, t)dx = \int_0^1 \partial_x u_j(x,t)dx$$

If $\mathbf{u}(0) \in D(A_0)$, we can interchange the derivative and integral to get

$$\frac{d}{dt}\int_0^1 u_j(x,t)dx = u_j(1) - u_j(0);$$

hence

$$\frac{d}{dt}\int_0^1 \mathbf{u}(x,t)dx = \mathbf{u}(1) - \mathbf{u}(0) = (\mathbb{B} - I)\mathbf{u}(0). \tag{4.8}$$

Note that to obtain the last equation (4.8), we used the definition of the domain in (4.5). Clearly the above system is not closed. To close it (approximately), we introduce the quantity

$$v_i(t) = \int_0^1 u_i(x,t)dx,$$

which is the total mass (also in this case, it is the average density) concentrated on edge e_i. If we assume that the density is almost homogeneous on each particular edge, then $\mathbf{v} \approx \mathbf{u}(0,t)$. Formally, the approximate closure of (4.8) is stated below

$$\begin{cases} \dfrac{d\mathbf{v}}{dt} & = (\mathbb{B} - I)\mathbf{v} \\ \mathbf{v}(0) & = \int_0^1 \mathbf{f}(x)dx, \end{cases} \tag{4.9}$$

where $\mathbf{f}$ is the vector function defined in (4.3). That is, the closed form of (4.8) is the system of equations (3.2) with $A = \mathbb{B} - I$ (note that here we refer to the matrix A in (3.2) and not the differential expression introduced above). In particular, we note that since $\mathbb{B}$ is column stochastic, A is a Kolmogorov matrix.

There are indications that (4.9) can be obtained from (4.8) as some asymptotic limit but so far, we have not been able to provide conclusive results in this direction.

4.2 Disconnected graphs

Suppose that G is a digraph with a finite number of vertices n and a finite number of edges m. Suppose that there is material flowing along the edges of this graph into the vertices. We assume that no material is absorbed in each of the vertices. Let us parameterise the edges such that the length of each edge is 1 and that the head of each edge is located at position 0 and its tail is at 1. Let $u_j(x,t)$ denote the amount of substance flowing in the j^{th} edge at time t and position $x \in (0,1)$ with speed c_j. We start with the disconnected graph where every vertex has an incoming and outgoing edge. Suppose G is the graph shown in Figure 4.1 below. Since

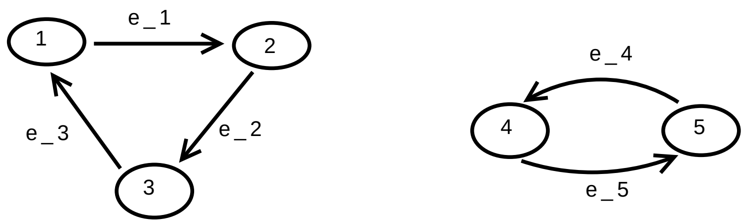


Figure 4.1: A disconnected network

the network consists of two completely disconnected graphs, the flow problem on this network can be divided into two different flow problems corresponding to the two irreducible graphs and the problem is then reduced to that solved by Sikolya in [33] for the finite dimensional case (or that by Dorn [14] for the infinite dimensional case).

In general, if a network consists of a finite number of disjoint strongly connected graphs, we solve the flow problem on each subgraph separately using the methods developed in [33] or in [14]. If G is a collection of g disjoint strongly connected graphs, $(A_0, D(A_0))$ generates a C_0 semigroup if and only if $(A_0^i, D(A_0^i))$ generate C_0 semigroups for every $i = 1, \ldots, g$, where $(A_0, D(A_0))$ is the operator defined in (4.3) and (4.4), defined on the whole graph G, $(A_0^i, D(A_0^i))$ is the realisation of the same expression ∂_x on the i^{th} maximal strongly connected subgraph of G, and the solution to the entire network problem is the direct sum of the solutions to the flow problem on the separate components.

4.3 Connected graphs

From this section onwards, we consider the graph G to be connected but not strongly connected. Suppose that on each edge e_j, the flow of particles is described by the equation

$$F1 \begin{cases} \partial_t u_j(x,t) & = c_j \partial_x u_j(x,t) + g_j(x,t,u_j(x,t)), \quad x \in (0,1), \quad t \geq 0 \\ u_j(x,0) & = f_j(x) \\ \phi_{ij}^- \alpha_j c_j u_j(1,t) & = w_{i,j} \left[\sum_{k=1}^m \phi_{i,k}^+ (\gamma_k c_k u_k(0,t)) + h_i(t) \right], \quad \forall i \quad (BC). \end{cases} \tag{4.10}$$

If $h_i(t) > 0$, then there is an input term at vertex v_i and if $h_i(t) < 0$, then the vertex is losing material. The above equation (4.10) is a more general problem which will not be discussed here. We will only consider the homogeneous problem with $h_i(t) = 0$ for all i and $g_j(x,t,u_j(x,t)) = 0$

for all $j = 1, \dots, m$. That is;

$$F2 \begin{cases} \partial_t u_j(x,t) & = c_j \partial_x u_j(x,t), \quad x \in (0,1), \quad t \geq 0 \\ u_j(x,0) & = f_j(x) \\ \phi^-_{ij} \alpha_j c_j u_j(1,t) & = w_{i,j} \sum_{k=1}^m \phi^+_{i,k} \gamma_k c_k u_k(0,t), \quad \forall i \quad (BC) \\ 0 & = \sum_{k=1}^m \phi^+_{i,k} (\gamma_k c_k u_k(0,t)), \text{ if } v_i \text{ is a sink} \quad (BC2), \end{cases} \tag{4.11}$$

where α_j, γ_j are absorption or generation coefficients at the head of the edge e_j and are all bounded above by a finite positive number α.

4.4 Existence and uniqueness

Let $X := L^1([0,1])^m$ with norm

$$\|\mathbf{f}\|_X = \sum_{i=1}^m \int_0^1 |f_i(x)| dx, \quad f \in X.$$

Let us denote $C = diag(c_j)_{1 \leq j \leq m}, \mathbb{G} = diag(\alpha_j)_{1 \leq j \leq m}$ and $\mathbb{E} = diag(\gamma_j)_{1 \leq j \leq m}$. Then the flow problem (4.11) can be written in an abstract way (ACP) in X

$$\begin{cases} \frac{d}{dt}\mathbf{u} & = A_0 \mathbf{u}(t) \\ \mathbf{u}(0) & = \mathbf{u}_0 = (f_j)_{j=1,\dots,m}. \end{cases} \tag{4.12}$$

where A_0 is the realisation of the expression $A = diag(c_j \partial_x)_{1 \leq j \leq m}$, with domain

$$D(A_0) = \left\{ \mathbf{u} \in W_1^1([0,1])^m ; \mathbf{u} \text{ satisfies the boundary conditions in } (4.11) \right\}. \tag{4.13}$$

We state the following result on the existence of a C_0 semigroup.

Theorem 4.4.1. *For the flow problem in (4.11) the following statements are equivalent.*

1. *The operator $(A_0, D(A_0))$ generates a C_0 semigroup.*
2. *The matrix Φ^- is surjective.*
3. *Every vertex in the graph has an outgoing edge.*

Proof. $2 \Leftrightarrow 3$: If v_i has no outgoing edge, then the i^{th} row of Φ^-, $(\Phi^-)_i$, is a row of zeros, hence Φ^- cannot be a full row rank matrix, and thus Φ^- is not surjective.

Conversely, if Φ^- is not surjective, then at least one row, say, row k is a linear combination of some other rows, i.e,

$$(\Phi^-)_k = \sum_{j=i}^{n'} \beta_j (\Phi^-)_j$$

for some β_j not all equal to 0. In particular if $\phi^-_{kl} > 0$, then $\phi^-_{jl} > 0$ for some j. But this is not possible since each column must have at most one non zero entry (see Remark 2.2.5). So if Φ^- is not surjective, then the only possibility is that it has a row of zeros, implying that at least one of the vertices in the graph has no outgoing edges.

$1 \Rightarrow 3$: Let $T(t)$ be the semigroup generated by the realisation of the operator in (4.11) and consider $\mathbf{u}(t) = T(t)\mathbf{f}$, $\mathbf{f} \in D(A_0)$. If v_i has no outgoing edge, then, from the boundary conditions, we have

$$0 = \sum_{k=1}^{m} \phi^+_{i,k}(\gamma_k c_k u_k(0,t)), \quad t > 0$$

Particularly, $u_k(x,t) = f_k(x + c_k t), 0 \le x + c_k t \le 1 \Rightarrow u_k(0,t) = f(c_k t)$. So

$$0 = \sum_{k=1}^{m} \phi^+_{i,k}(\gamma_k c_k f_k(c_k t)), \quad 0 \le t \le \frac{1}{c_k}.$$

Define $c^{-1} := (\max\{c_k\})^{-1}$, then

$$0 = \sum_{k=1}^{m} \phi^+_{i,k}(\gamma_k c_k f_k(c_k t)), \quad 0 \le t \le c^{-1}.$$

There is a sequence $(\mathbf{f}^r)_{r\in\mathbb{N}}$ in $D(A_0)$ which approximates $\mathbf{1} = (1,\dots,1)$ in X. For this sequence, we have

$$\begin{aligned}
0 \le \|\mathbf{1}|_{(0,c^{-1})} - (f^r_k(c_k\cdot))_{1\le k\le m}\|_X &= \sum_{k=1}^{m} \int_0^{c^{-1}} |1 - f^r_k(c_k t)| dt \\
&= \sum_{k=1}^{m} \frac{1}{c_k} \int_0^{c_k c^{-1}} |1 - f^r_k(z)| dz \\
&\le \sum_{k=1}^{m} \frac{1}{c_k} \int_0^{1} |1 - f^r_k(z)| dz \overset{r\to\infty}{\longrightarrow} 0.
\end{aligned}$$

Since convergence in X implies convergence almost everywhere of a subsequence of $(\mathbf{f}^r)_{r\in\mathbb{N}}$, we have

$$0 = \sum_{k=1}^{m} \phi^+_{ik}\gamma_k c_k$$

almost everywhere on $(0, c^{-1})$, and thus everywhere. Since the graph is connected and we have assumed that there is no outgoing edge at v_i, then there must be an incoming edge, hence, at least one term in the sum is positive and some are negative. This implies that the set of initial conditions satisfying the boundary conditions BC is not dense in X. Hence $(A_0, D(A_0))$ cannot generate a semigroup.

$2 \Rightarrow 1$ Conversely, suppose that Φ^- is surjective. Then there is at least one non-zero entry in each of its rows. Since these rows are linearly independent, it is a full row rank matrix. So the system of equations $\Phi^- x = y$ is consistent for any vector $y \in \mathbb{C}^m$. In particular, the system of equations in

$$\Phi^- \mathbb{G} C \mathbf{u}(1,t) = \Phi^+ \mathbb{E} C \mathbf{u}(0,t)$$

is consistent. Note that

$$D(A_0) = \left\{ \mathbf{u} \in W^{1,1}([0,1])^m | \mathbf{u}(1) = C^{-1}\mathbb{G}^{-1}\mathbb{B}\mathbb{E}C\mathbf{u}(0) = \mathbb{P}u(0) \right\}, \tag{4.14}$$

where $\mathbb{B}$ is the adjacency matrix of the line graph defined in (2.2). Below, we show that (4.14) is equivalent to (4.13): From the boundary conditions in (4.11), we have

$$\phi_{ij}^- \alpha_j c_j u_j(1,t) = w_{ij} \sum_{k=1}^{m} \phi_{ik}^+ \gamma_k c_k u_k(0,t)$$

for all $i = 1, \ldots, n$. We want to show that this condition is equivalent to $\mathbf{u}(1,t) = \mathbb{P}\mathbf{u}(0,t) = C^{-1}\mathbb{G}^{-1}\mathbb{B}\mathbb{E}C\mathbf{u}(0,t)$. Fix j. Then there is exactly one i for which $\phi_{ij}^- = 1$ and the rest is 0. For this i,

$$\begin{aligned}\phi_{ij}^- c_j \alpha_j u_j(1) = \alpha_j c_j u_j(1) &= w_{ij} \sum_{k=1}^{m} \phi_{ik}^+ c_k \gamma_k u_k(0) \\ &= \sum_{k=1}^{m} w_{ij} \phi_{ik}^+ c_k \gamma_k u_k(0).\end{aligned}$$

But $w_{ij}\phi_{ik}^+ > 0$ if and only if $w_{ij} > 0$ and $\phi_{ik}^+ = 1$, which implies that $\xrightarrow{e_k} v_i \xrightarrow{e_j}$. Since there is only one such v_i for fixed j and k, it follows that

$$w_{ij}\phi_{ik}^+ = \sum_{l=1}^{n} w_{lj}\phi_{lk}^+ = \mathbb{B}_{jk}.$$

So

$$\begin{aligned}\alpha_j c_j u_j(1) &= \sum_{k=1}^{m} \mathbb{B}_{jk} \gamma_k c_j u_k(0) \\ &= \left(\mathbb{B}\mathbb{E}C\mathbf{u}(0)\right)_j,\end{aligned}$$

where $\left(\mathbb{B}\mathbb{E}C\mathbf{u}(0)\right)_j$ is the j^{th} row of $\mathbb{B}\mathbb{E}C\mathbf{u}(0)$. But $\alpha_j c_j u_j(1,t) = \left(\mathbb{G}C\mathbf{u}(1)\right)_j$, hence $\mathbb{G}C\mathbf{u}(1,t) = \mathbb{B}\mathbb{E}C\mathbf{u}(0,t) \Rightarrow \mathbf{u}(1,t) = \mathbb{P}\mathbf{u}(0,t)$.

Conversely, suppose that $\mathbf{u}(1,t) = \mathbb{P}\mathbf{u}(0,t)$. Then the j^{th} component of $\mathbf{u}(1,t)$ satisfies

$$u_j(1,t) = c_j^{-1}\alpha_j^{-1}\left(\mathbb{B}\mathbb{E}C\mathbf{u}(0,t)\right)_j,$$

where $\left(\mathbb{B}\mathbb{E}C\mathbf{u}(0,t)\right)_j$ is the j^{th} entry of $\mathbb{B}\mathbb{E}C\mathbf{u}(0,t)$, hence

$$\begin{aligned}
\alpha_j c_j u_j(1,t) &= \left(\mathbb{B}\mathbb{E}C\mathbf{u}(0,t)\right)_j \\
&= \left(\left(\Phi_w^-\right)^T\right)_j \Phi^+\mathbb{E}C\mathbf{u}(0,t) \\
&= w_{ij}\sum_{k=1}^{m}\phi_{ik}^+\gamma_k c_k u_k(0,t).
\end{aligned}$$

We have used $\left(\left(\Phi_w^-\right)^T\right)_j$ to refer to the j^{th} row of (Φ_w^-) and in each row, there is only one non zero entry which is w_{ij}, in the i^{th} column. Since we have assumed that there is an outgoing edge at every vertex and $w_{ij} \neq 0$ if and only if $\phi_{ij}^- = 1$, we have

$$\phi_{ij}^-\alpha_j c_j u_j(1,t) = w_{ij}\sum_{k=1}^{m}\phi_{ik}^+\gamma_k c_k u_k(0,t)$$

which is our boundary condition.

We claim that $(A_0, D(A_0))$ is linear, closed and densely defined. It is easy to see that $D(A_0)$ is a linear space and A_0 is linear operator. To show that it is closed, suppose that $\mathbf{u}_n = (u_n^1, \ldots, u_n^m) \in D(A_0)$ and there is $\mathbf{u} \in X$ such that $\mathbf{u}_n \to \mathbf{u}$. Let $\mathbf{y} \in X$ be a vector such that $A_0\mathbf{u}_n \to \mathbf{y}$. Then $\mathbf{u}_n(1) = \mathbb{P}\mathbf{u}_n(0)$. Since $\mathbf{u}_n \in D(A_0)$, it follows that $\mathbf{u}_n \in L^1([0,1])^m$ and that it has a generalised derivative in $L^1([0,1])^m$ (by definition of $(W^{1,1}([0,1])^m)$). Let the generalised derivative be $\mathbf{v}_n$. Then

$$-\int_0^1 \mathbf{u}_n(x)\zeta'(x)dx = \int_0^1 \mathbf{v}_n(x)\zeta(x)dx, \quad \forall\zeta \in C_0^\infty([0,1]).$$

Now $\mathbf{u}_n \to \mathbf{u}$ implies

$$\int_0^1 \mathbf{v}_n(x)\zeta(x)dx = -\int_0^1 \mathbf{u}_n(x)\zeta'(x)dx \to -\int_0^1 \mathbf{u}(x)\zeta'(x)dx.$$

But $\mathbf{v}_n$ being a genaralised derivative of $\mathbf{u}_n$ means that $\mathbf{u}_n' = \mathbf{v}_n$, so $C\mathbf{u}_n' = C\mathbf{v}_n \to \mathbf{y}$. Hence

$$\int_0^1 C\mathbf{v}_n\zeta dx \to \int_0^1 \mathbf{y}\zeta dx$$

implying that

$$\begin{array}{rcl}
\int_0^1 C\mathbf{v}_n(x)\zeta dx & = & -\int_0^1 C\mathbf{u}_n(x)\zeta' dx \\
\downarrow & & \downarrow \\
\int_0^1 \mathbf{y}(x)\zeta(x)dx & = & -\int_0^1 C\mathbf{u}(x)\zeta'(x)dx.
\end{array}$$

Therefore, $\mathbf{y}$ is the generalised derivative of $C\mathbf{u}$, i.e; $A\mathbf{u} = \mathbf{y}$.

To show that $\mathbf{u} \in D(A_0)$, we note that $\mathbf{u}_n \in W^{1,1}([0,1])^m$ implies that $u_n^i(x) \in C([0,1])$ for each $i = 1, \ldots, m$ by Lemma 8.2 of [8]. Therefore, $u_n^i \to u^i$ implies $u_n(1) \to u(1)$ and $u_n(0) \to u(0)$. Now $u_n^i(1) = (\mathbb{P}\mathbf{u}_n(0))^i$. Since the operator $\mathbb{P}$ is linear on $D(A_0)$ and norm bounded, it is continuous. So we have that

$$\mathbb{P}\mathbf{u}_n(0) \to \mathbb{P}\mathbf{u}(0) \Rightarrow (\mathbb{P}\mathbf{u}_n(0))^i \to (\mathbb{P}\mathbf{u}(0))^i.$$

Hence $u_n^i(1) = (\mathbb{P}\mathbf{u}_n(0))^i \to (\mathbb{P}\mathbf{u}(0))^i$, implying that

$$\begin{array}{lcl} u_n^i(1) & = & (\mathbb{P}\mathbf{u}_n(0))^i \\ \downarrow & & \downarrow \\ u^i(1) & = & (\mathbb{P}\mathbf{u}(0))^i, \quad \forall i. \end{array}$$

So $\mathbf{u}(1) = \mathbb{P}\mathbf{u}(0)$, hence $\mathbf{u} \in D(A_0)$.

To show that $D(A_0)$ is dense in X, note that $(C_0^\infty([0,1]))^m \subset D(A_0)$. By Corollary 1.14, [49], $C_0^\infty(\Omega)$ is dense in $L^p(\Omega)$ for $1 \leq p < \infty$; that is, $\overline{C_0^\infty(\Omega)} = L^p(\Omega)$. Therefore, $(C_0^\infty([0,1]))^m \subset D(A_0)$ and $\overline{C_0^\infty(\Omega)} = L^p(\Omega)$ both imply that $\overline{D(A_0)} = (L^p([0,1]))^m$.

We show existence of a C_0 semigroup by analysing the resolvent. We solve a resolvent equation $\lambda\mathbf{f} - C\mathbf{f}' = \mathbf{g}$, where $\mathbf{f} \in D(A_0)$ and $\mathbf{g} \in X$ using the method of variation of parameters to obtain $(R(\lambda, A)\mathbf{g})(x)$. This is equivalent to $\mathbf{f}' - \lambda C^{-1}\mathbf{f} = -C^{-1}\mathbf{g}$. This equation implies that

$$f_j'(x) - \frac{\lambda}{c_j} f_j(x) = -\frac{1}{c_j} g_j(x) \quad \forall j = 1, \ldots, m. \tag{4.15}$$

Solving the homogeneous part of this equation gives

$$f_j(x) = e^{\frac{\lambda}{c_j}x} \nu_j, \tag{4.16}$$

where ν_j is an arbitrary constant. Let $f_{j,p}(x) = e^{\frac{\lambda}{c_j}x} D_j(x)$, where $D_j(x)$ is an unknown function to be determined. Then

$$f_{j,p}'(x) = \frac{\lambda}{c_j} e^{\frac{\lambda}{c_j}x} D_j(x) + e^{\frac{\lambda}{c_j}x} D_j'(x).$$

Substituting in (4.15), we get the simple ODE, which we solve to get

$$D_j(x) = \frac{1}{c_j} \int_x^1 e^{-\frac{\lambda}{c_j}s} g_j(s) ds,$$

so that $f_{j,p}(x)$ is given by

$$f_{j,p}(x) = \frac{1}{c_j} e^{\frac{\lambda}{c_j}x} \int_x^1 e^{-\frac{\lambda}{c_j}s} g_j(s) ds.$$

Hence the general solution to the entire problem is given by

$$\begin{aligned} f_j(x) &= e^{\frac{\lambda}{c_j}x}\nu_j + e^{\frac{\lambda}{c_j}x}D_j(x) \\ &= e^{\frac{\lambda}{c_j}x}\nu_j + \frac{1}{c_j}\int_x^1 e^{\frac{\lambda}{c_j}(x-s)}g_j(s)ds. \end{aligned}$$

We rewrite in the form

$$c_jf_j(x) = c_je^{\frac{\lambda}{c_j}x}\nu_j + \int_x^1 e^{\frac{\lambda}{c_j}(x-s)}g_j(s)ds, \tag{4.17}$$

therefore

$$C\mathbf{f} = Ce^{\lambda xC^{-1}}\nu + \int_x^1 e^{\lambda(x-s)C^{-1}}\mathbf{g}(s)ds.$$

Since $\mathbf{f} \in D(A_0)$, $\mathbb{G}C\mathbf{f}(1) = \mathbb{G}Ce^{\lambda C^{-1}}\nu = \mathbb{BE}C\mathbf{f}(0)$. But

$$C\mathbf{f}(0) = C\nu + \int_0^1 e^{\lambda(-s)C^{-1}}\mathbf{g}(s)ds,$$

hence

$$\begin{aligned} \mathbb{G}Ce^{\lambda C^{-1}}\nu &= \mathbb{BE}C\nu + \mathbb{BE}\int_0^1 e^{\lambda(-s)C^{-1}}\mathbf{g}(s)ds \\ \Rightarrow \left(\mathbb{G}Ce^{\lambda C^{-1}} - \mathbb{BE}C\right)\nu &= \mathbb{BE}\int_0^1 e^{\lambda(-s)C^{-1}}\mathbf{g}(s)ds \\ \Rightarrow \left(I - \mathbb{G}^{-1}C^{-1}e^{-\lambda C^{-1}}\mathbb{BE}C\right)\nu &= C^{-1}\mathbb{G}^{-1}e^{-\lambda C^{-1}}\mathbb{BE}\int_0^1 e^{\lambda(-s)C^{-1}}\mathbf{g}(s)ds \end{aligned}$$

Therefore,

$$\nu = \left(I - \mathbb{G}^{-1}C^{-1}e^{-\lambda C^{-1}}\mathbb{BE}C\right)^{-1}C^{-1}\mathbb{G}^{-1}e^{-\lambda C^{-1}}\mathbb{BE}\int_0^1 e^{\lambda(-s)C^{-1}}\mathbf{g}(s)ds. \tag{4.18}$$

Since the norm of $e^{-\lambda C^{-1}}$ can be made as small as one wishes by taking large λ, we see that ν in (4.18) is uniquely defined by the Neumann series provided λ is sufficiently large and hence the resolvent of A_0 exists. We find an estimate for the resolvent by noting, first, that the Neumann series expansion ensures that A_0 is a resolvent positive operator and hence the norm estimates can be obtained for non-negative entries. Next, we recall that $\mathbb{B}$ is column stochastic, hence each column sums to 1. Adding the rows of the system

$$\mathbb{G}Ce^{\lambda C^{-1}}\nu = \mathbb{BE}C\nu + \mathbb{BE}\int_0^1 e^{\lambda(-s)C^{-1}}\mathbf{g}(s)ds$$

gives

$$\sum_{j=1}^m \alpha_jc_je^{\frac{\lambda}{c_j}}\nu_j = \sum_{j=1}^m \gamma_jc_j\nu_j + \sum_{j=1}^m \gamma_j\int_0^1 e^{\frac{-\lambda}{c_j}s}g_j(s)ds. \tag{4.19}$$

We integrate Equation (4.17) to get

$$\begin{aligned}\int_0^1 f_j(x)dx &= \nu_j \int_0^1 e^{\frac{\lambda}{c_j}x}dx + \frac{1}{c_j}\int_0^1\int_x^1 e^{-\frac{\lambda}{c_j}(x-s)}g_j(s)dsdx\\ &= \nu_j\frac{c_j}{\lambda}\left(e^{\frac{\lambda}{c_j}}-1\right)+\frac{1}{\lambda}\int_0^1\left(e^{\frac{\lambda}{c_j}s}-1\right)e^{-\frac{\lambda}{c_j}s}g_j(s)ds\\ &= \nu_j\frac{c_j}{\lambda}\left(e^{\frac{\lambda}{c_j}}-1\right)+\frac{1}{\lambda}\int_0^1\left(1-e^{-\frac{\lambda}{c_j}s}\right)g_j(s)ds.\end{aligned}$$

Introducing a weighted space $\mathbb{X}$ with norm

$$\|\mathbf{f}\|_{\mathbb{X}} = \sum_{j=1}^m \alpha_j \|f_j\|_{L_1([0,1])}$$

and considering $\mathbf{g} \geq 0$, we get

$$\|\mathbf{f}\|_{\mathbb{X}} = \sum_{j=1}^m \alpha_j \int_0^1 f_j(x)dx = \frac{1}{\lambda}\sum_{j=1}^m \nu_j c_j \alpha_j\left(e^{\frac{\lambda}{c_j}}-1\right)+\frac{1}{\lambda}\sum_{j=1}^m \alpha_j\int_0^1\left(1-e^{-\frac{\lambda}{c_j}s}\right)g_j(s)ds$$

and using (4.19), we obtain

$$\|\mathbf{f}\|_{\mathbb{X}} = \frac{1}{\lambda}\sum_{j=1}^m \nu_j c_j(\gamma_j-\alpha_j)+\frac{1}{\lambda}\sum_{j=1}^m(\gamma_j-\alpha_j)\int_0^1 e^{-\frac{\lambda}{c_j}s}g_j(s)ds+\frac{1}{\lambda}\sum_{j=1}^m \alpha_j\int_0^1 g_j(x)dx. \quad (4.20)$$

There are three cases to consider

1. $\gamma_j \leq \alpha_j$ for all $j = 1, \ldots, m$: Since $\mathbb{G}, \mathbb{E}, C$ are diagonal matrices, they commute, so we have

$$e^{-\lambda C^{-1}}C^{-1}\mathbb{G}^{-1}\mathbb{B}\mathbb{E}C \leq (C\mathbb{G})^{-1}e^{-\lambda C^{-1}}\mathbb{B}C\mathbb{G}$$

   Since $(C\mathbb{G})^{-1}e^{-\lambda C^{-1}}\mathbb{B}C\mathbb{G}$ is similar to $e^{-\lambda C^{-1}}\mathbb{B}$ and $r\left(e^{-\lambda C^{-1}}\mathbb{B}\right) = r\left(e^{-\lambda C^{-1}}\right)r(\mathbb{B}) \leq r(\mathbb{B}) = 1$, we have

$$r\left(e^{-\lambda C^{-1}}C^{-1}\mathbb{G}^{-1}\mathbb{B}\mathbb{E}C\right) \leq 1$$

   for any $\lambda > 0$. Therefore, $R(\lambda, A_0)$ is well defined for $\lambda > 0$. Using $\gamma_j \leq \alpha_j$ and equation (4.20) we get

$$\|R(\lambda, A)\mathbf{g}\|_{\mathbb{X}} = \|\mathbf{f}\|_{\mathbb{X}} \leq \frac{1}{\lambda}\sum_{j=1}^m \alpha_j\int_0^1 g_j(x)dx = \frac{1}{\lambda}\|\mathbf{g}\|_{\mathbb{X}}, \quad \lambda > 0.$$

   Since $(A_0, D(A_0))$ is dense in $\mathbb{X}$, it generates a C_0 semigroup of contraction.

2. If $\gamma_j \geq \alpha_j$ for all $j = 1, \ldots, m$, then from Equation (4.20),

$$\|R(\lambda, A)\mathbf{g}\|_{\mathbb{X}} \geq \frac{1}{\lambda}\|\mathbf{g}\|_{\mathbb{X}}.$$

Since $(A_0, D(A_0))$ is dense, by definition, we use Theorem 2.5.11 to conclude that $(A_0, D(A_0))$ is a generator of a positive semigroup in $\mathbb{X}$, and hence in X since $\|.\|_{\mathbb{X}}$ is equivalent to $\|.\|_X$.

3. $\gamma_j < \alpha_j$ for some $j \in I_1$ and $\gamma_j \geq \alpha_j$ for some $j \in I_2$, where $I_1 \cap I_2 = \emptyset$ and $I_1 \cup I_2 = \{1, 2, \ldots, m\}$. Let $\mathbb{D} = \mathrm{diag}(l_j)$, where $l_j = \alpha_j$ for $j \in I_1$ and $l_j = \gamma_j$ for $j \in I_2$. Then
$$e^{-\lambda C^{-1}} C^{-1} \mathbb{G}^{-1} \mathbb{B} \mathbb{E} C \leq (C\mathbb{G})^{-1} e^{-\lambda C^{-1}} \mathbb{B}(C\mathbb{D}).$$

   Let $A_{\mathbb{D}}$ be the restriction of A_0 to the domain
$$D(A_{\mathbb{D}}) = \left\{\mathbf{u} \in \left(W_1^1([0,1])\right)^m : \mathbf{u}(1) = \mathbb{G}^{-1} C^{-1} \mathbb{B} \mathbb{D} C \mathbf{u}(0)\right\},$$

   then clearly, if $\mathbf{g} \geq 0$, the resolvent $R(\lambda, A_{\mathbb{D}})$ is also positive. The resolvent of A_0 and $A_{\mathbb{D}}$ also satisfy $0 \leq R(\lambda, A_0) \leq R(\lambda, A_{\mathbb{D}})$ for any λ for which $R(\lambda, A_{\mathbb{D}})$ exists. Using the previous case, we see that $A_{\mathbb{D}}$ generates a positive semigroup. We also have the inequality $R(\lambda, A)^k \leq R(\lambda, A_{\mathbb{D}})^k$ for any $k \in \mathbb{N}$ and for some $\omega > 0, M \geq 1$. Therefore, using inequality 2.11, together with Theorem 2.5.6 (Hille-Yosida) we have
$$\|R(\lambda, A_0)^k\| \leq \|R(\lambda, A_{\mathbb{D}})^k\| \leq M\,(\lambda - \omega)^{-k}, \quad \lambda > \omega.$$

Therefore, $(A_0, D(A_0))$ is a generator of a positive semigroup as well. ■

Remark 4.4.2. *If $\alpha_j = \gamma_j$, then from Equation (4.20), we have*
$$\|f\|_{\mathbb{X}} = \frac{1}{\lambda}\sum_{j=1}^{m} \alpha_j \int_0^1 g_j(x)dx \Rightarrow \|R(\lambda, A_0)\mathbf{g}\|_{\mathbb{X}} = \frac{1}{\lambda}\|\mathbf{g}\|_{\mathbb{X}}.$$

Thus the semigroup is conservative in $\mathbb{X}$.

From Corollary 2.6 of [33], the semigroup $(T(t))_{t\geq 0}$ is contractive only when $c_j = c$ for all $j = 1, \ldots, m$, but as seen in the above calculations this is not necessary. The semigroup was not contractive because the Kirchoff law (boundary condition) stated in Equation (3) of [33] was incorrect (also see Remark 4.1.1).

4.5 Spectral properties and asymptotic behaviour

4.5.1 Same speed

In [14], the author showed that if $c_j = 1$ and $\alpha_j, \gamma_j = 1$ for all $j = 1, \ldots, m$ and the graph is strongly connected, then the semigroup is given by

$$(T(t)\mathbf{f})(x) = \mathbb{B}^n \mathbf{f}(t + x - n) \quad 0 \leq t + x - n \leq 1, n \in \mathbb{N}_0.$$

This formula still holds for a finite, connected graph (not strongly connected), with an outgoing edge in every vertex, even when absorption and/or generation is allowed.

Proposition 4.5.1. *Let G be a connected graph where every vertex has an outgoing edge and $c_j = 1$ for all $j = 1, \ldots, m$. Then the semigroup $T(t)$ generated by $(A_0, D(A_0))$ on G is given by*

$$T(t)\mathbf{f}(x) = \mathbb{P}^n \mathbf{f}(t + x - n); \quad n \in \mathbb{N}_0, 0 \leq t + x - n \leq 1, \tag{4.21}$$

where $\mathbb{P} = \mathbb{G}^{-1}\mathbb{B}\mathbb{E}$.

Proof. Let us denote by $\mathcal{T}(t)\mathbf{f}$ the formula of the operator given by the right hand side of (4.21). First, we observe that $\mathcal{T}(t)\mathbf{f}$ is strongly continuous. This can be shown in the same way we show that translation semigroups are strongly continuous. Computing the Laplace transform of $\mathcal{T}(t)$,

$$\begin{aligned} R(\lambda)\mathbf{g}(s) &= \int_0^\infty e^{-\lambda t} \mathcal{T}(t)\mathbf{g}(s)dt \\ &= \int_0^{1-s} e^{-\lambda t}\mathbf{g}(t+s)dt + \sum_{n=1}^{\infty} \int_{n-s}^{1+n-s} e^{-\lambda t}\mathbb{P}^n \mathbf{g}(t+s-n)dt \\ &= \int_s^1 e^{-\lambda(\tau - s)}\mathbf{g}(\tau)d\tau + \sum_{n=1}^{\infty} \int_0^1 e^{-\lambda(n-s+\tau)}\mathbb{P}^n \mathbf{g}(\tau)d\tau. \end{aligned}$$

Recall, from previous calculations, that $R(\lambda, A_0)$ is given by

$$R(\lambda, A_0)\mathbf{g}(x) = e^{\lambda x}\nu + \int_x^1 e^{\lambda(x-s)}\mathbf{g}(s)ds,$$

where ν is given by (4.18). We expand the expression for ν using power series to obtain

$$\begin{aligned}\nu &= \sum_{n=0}^{\infty} \left(\mathbb{G}^{-1}e^{-\lambda}\mathbb{B}\mathbb{E}\right)^n \mathbb{G}^{-1}e^{-\lambda}\mathbb{B}\mathbb{E}\int_0^1 e^{-\lambda s}\mathbf{g}(s)ds \\ &= \mathbb{G}^{-1}e^{-\lambda}\mathbb{B}\mathbb{E}\int_0^1 e^{-\lambda s}\mathbf{g}(s)ds + \sum_{n=1}^{\infty} \left(\mathbb{G}^{-1}e^{-\lambda}\mathbb{B}\mathbb{E}\right)^n \mathbb{G}^{-1}e^{-\lambda}\mathbb{B}\mathbb{E}\int_0^1 e^{-\lambda s}\mathbf{g}(s)ds \\ &= e^{-\lambda}\int_0^1 e^{-\lambda s}\mathbb{P}\mathbf{g}(s)ds + \sum_{n=1}^{\infty} e^{-n\lambda}\mathbb{P}^{n+1}e^{-\lambda}\int_0^1 e^{-\lambda s}\mathbf{g}(s)ds.\end{aligned}$$

Thus, multiplying by $e^{\lambda x}$ with sufficiently large λ, we get

$$\begin{aligned}e^{\lambda x}\nu &= \int_0^1 e^{-\lambda(1-x+s)}\mathbb{P}\mathbf{g}(s)ds + \sum_{n=1}^{\infty}\mathbb{P}^{n+1}\int_0^1 e^{-\lambda(n-x+1+s)}\mathbf{g}(s)ds \\ &= \sum_{n=0}^{\infty}\int_0^1 e^{-\lambda(n-x+1+s)}\mathbb{P}^{n+1}\mathbf{g}(s)ds.\end{aligned}$$

Therefore, the resolvent $R(\lambda, A_0)$ is given by

$$\begin{aligned}R(\lambda, A_0)\mathbf{g}(x) &= \sum_{n=0}^{\infty}\int_0^1 e^{-\lambda(n-x+1+s)}\mathbb{P}^{n+1}\mathbf{g}(s)ds + \int_x^1 e^{\lambda(x-s)}\mathbf{g}(s)ds \\ &= \sum_{n'=1}^{\infty}\int_0^1 e^{-\lambda(n'-x+s)}\mathbb{P}^{n'}\mathbf{g}(s)ds + \int_x^1 e^{\lambda(x-s)}\mathbf{g}(s)ds;\end{aligned}$$

where $n' = n+1$, and the equation is true for all $x \in [0,1]$ and $\mathbf{g}(x) \in L^1([0,1])^m$. Therefore, $R(\lambda, A_0) = R(\lambda)$ and from uniqueness of the Laplace transform ([2], Theorem 1.7.3), $(\mathcal{T}(t))_{t\geq 0} = (T(t))_{t\geq 0}$. ■

Proposition 4.5.2. *If $\gamma_j = \alpha_j = 1$ for all $j = 1, \dots, m$ and $\lambda \in \sigma_p(A_0)$, then $(e^{\lambda}, \mathbf{f}(0))$ is an eigenpair of $\mathbb{B}$.*

Proof. Solving the eigenvalue equation $A\mathbf{f} = \lambda\mathbf{f}$ gives $\mathbf{f}(x) = e^{\lambda x}\mathbf{f}(0)$. This is an eigenvector of A only if $\mathbf{f}(x) \in D(A_0)$. This means that $\mathbf{f}(1) = e^{\lambda}\mathbf{f}(0) = \mathbb{B}\mathbf{f}(0)$. This implies that $(e^{\lambda}, \mathbf{f}(0))$ is an eigenpair of $\mathbb{B}$ for any eigenvalue λ of A_0. ■

Since $\mathbb{B}$ is column stochastic, 1 is an eigenvalue of $\mathbb{B}$, implying that some eigenvalues of A_0 are of the form $2\imath k\pi$. Note also that $\mathbb{B}$, being column stochastic, implies that $\mathbb{B}^n$ is also column stochastic for $n \in \mathbb{N}$, hence $\|\mathbb{B}^n\|_1 = 1$. From Proposition 4.5.2, we see that $\lambda \in \sigma_p(A)$ implies $e^{\lambda} \in \sigma(\mathbb{B}) = \sigma(T(1)) \Rightarrow e^{\sigma_p(A)} \subset \sigma(T(1)) = \sigma(\mathbb{B})$. Hence $e^{t\sigma_p(A_0)} \subset \sigma(T(t))$ for all $t > 0$ (by the spectral mapping theorem [16], Theorem IV.3.6).

Lemma 4.5.3. *The spectrum of $\mathbb{P}$ is the same as the spectrum of $T(1)$.*

Proof.

$$
\begin{aligned}
(T(t)f)(x) &= \mathbb{P}^n f(t+x-n) \\
\Rightarrow (T(1)f)(x) &= \mathbb{P}^n f(1+x-n) = \mathbb{P}\mathbb{P}^{n-1} f(0+x-(n-1)) \\
&= \mathbb{P}(T(0)f)(x) = \mathbb{P}f(x)
\end{aligned}
$$

So $T(1) = \mathbb{P}$, hence $\sigma(\mathbb{P}) = \sigma(T(1))$ ■

Since $T(t)$ is a strongly continuous semigroup, by Theorem V.2.6 of [17], we have $e^{\sigma(A)} = \sigma(T(1)) \setminus \{0\}$.

4.5.2 Same speed, with $\gamma_j = \alpha_j$

In this section, we assume that $\alpha_j = \gamma_j$ for all $j = 1, 2, \ldots, m$, and $c_j = 1$ for all j. Then $\mathbb{P} = \mathbb{G}^{-1}\mathbb{B}\mathbb{G}$ and the two matrices $\mathbb{P}$ and $\mathbb{B}$ have the same set of eigenvalues. Suppose that $\mathbf{v}$ is an eigenvector of $\mathbb{P}$. Then there is $\lambda \in \mathbb{C}$ such that $\mathbb{P}\mathbf{v} = \lambda\mathbf{v} \Rightarrow \mathbb{B}\mathbb{G}C\mathbf{v} = \lambda\mathbb{G}C\mathbf{v}$. So $(\lambda, \mathbb{G}C\mathbf{v})$ is an eigenpair of $\mathbb{B}$.

Conversely, if α is an eigenvalue of $\mathbb{B}$, then $\mathbb{B}\mathbf{y} = \alpha\mathbf{y} \Rightarrow C^{-1}\mathbb{G}^{-1}\mathbb{B}\mathbf{y} = \alpha C^{-1}\mathbb{G}^{-1}\mathbf{y}$. But $C^{-1}\mathbb{G}^{-1}\mathbb{B} = \mathbb{P}C^{-1}\mathbb{G}^{-1}$, hence $\mathbb{P}C^{-1}\mathbb{G}^{-1}\mathbf{y} = \alpha C^{-1}\mathbb{G}^{-1}\mathbf{y}$. Thus $(\alpha, (\mathbb{G}C)^{-1}\mathbf{y})$ is an eigenpair of $\mathbb{P}$. Since $\mathbb{B}$ is column stochastic, 1 is an eigenvalue of both $\mathbb{P}$ and $\mathbb{B}$.

Similarly, $\mathbf{v}\mathbb{P} = \lambda\mathbf{v} \Rightarrow \mathbf{v}(\mathbb{G}C)^{-1}\mathbb{B} = \lambda\mathbf{v}(\mathbb{G}C)^{-1}$ and $\mathbf{u}\mathbb{B} = \alpha\mathbf{u} \Rightarrow \mathbf{u}\mathbb{G}C\mathbb{P} = \alpha\mathbf{u}\mathbb{G}C$. Since $\mathbb{B}$ is column stochastic, $(1, \mathbf{1})$ is its left eigenpair, hence the left eigenvector of $\mathbb{P}$ corresponding to eigenvalue 1 is $\mathbf{v} = \mathbf{1}\mathbb{G}C = (\alpha_1 c, \alpha_2 c, \ldots, \alpha_m c) = (c_1\alpha_1, \ldots, c_m\alpha_m)$. If $c_j = 1$ for all j, then $\mathbf{v} = \mathbf{1}\mathbb{G} = (\alpha_1, \alpha_2, \ldots, \alpha_m)$.

We have seen that the spectrum of the semigroup $T(t)$ is related to the spectrum of $\mathbb{P}$ and hence that of $\mathbb{B}$, so we shall study the long term behaviour of the semigroup via matrix $\mathbb{B}$.

Since $\mathbb{B}$ is reducible, we can write it in normal form (2.8), from which we see that B_s is column stochastic, so $1 \in \sigma(B_s)$. If there is another matrix B_k with 1 as an eigenvalue, then $B_{h,k} = 0$

for all $h > k$. Let $B_k, B_{k+1}, \dots, B_s$ all have 1 as an eigenvalues. Then

$$\mathbb{B} = \begin{pmatrix} B_1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots & \vdots & \cdots & \vdots & & \cdots & \vdots \\ 0 & 0 & \cdots & B_g & 0 & \cdots & 0 & 0 & \cdots & 0 \\ B_{g+1,1} & B_{g+1,2} & \cdots & B_{g+1,g} & B_{g+1} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots & & \cdots & \vdots \\ B_{k,1} & B_{k,2} & \cdots & B_{k,g} & B_{k,g+1} & \cdots & B_k & 0 & \cdots & 0 \\ B_{k+1,1} & B_{k+1,2} & \cdots & B_{k+1,g} & B_{k+1,g+1} & \cdots & 0 & B_{k+1} & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & & \ddots & \vdots \\ B_{s,1} & B_{s,2} & \cdots & B_{s,g} & B_{s,g+1} & \cdots & 0 & 0 & \cdots & B_s \end{pmatrix}, \tag{4.22}$$

From the matrix representation (4.22) we see that $B_k, \cdots B_s$ are all column stochastic, hence $1 \in \sigma(B_i)$ for each $i = k, \dots, s$ and $1 \notin \sigma(B_j)$ for $j = 1, \dots, k-1$. The right eigenvectors of $\mathbb{B}$ corresponding to $r(\mathbb{B}) = 1$ are given by

$$\mathbf{N}_i = (0, \dots, 0, \mathbf{N}^i, 0, \dots, 0)^T, \text{ where } \quad B_i \mathbf{N}^i = \mathbf{N}^i.$$

That is, $\mathbf{N}^i > 0$ are the Perron eigenvectors of B_i for all $i = k, \dots, s$. The left eigenvectors of $\mathbb{B}$ corresponding to 1 are $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ where

$$\begin{aligned} \mathbf{v}_k &= (\mathbf{y}_1^k, \mathbf{y}_2^k, \dots, \mathbf{y}_{k-1}^k, \mathbf{v}^k, 0, \dots, 0) \\ \mathbf{v}_{k+1} &= (\mathbf{y}_1^{k+1}, \mathbf{y}_2^{k+1}, \dots, \mathbf{y}_{k-1}^{k+1}, 0, \mathbf{v}^{k+1}, 0, \dots, 0) \\ \vdots \quad & \vdots \\ \mathbf{v}_s &= (\mathbf{y}_1^s, \mathbf{y}_2^s, \dots, \mathbf{y}_{k-1}^s, 0, \dots, 0, \mathbf{v}^s) \end{aligned}$$

where $\mathbf{v}^i \mathbb{B} = \mathbf{v}^i = \mathbf{1}_{n_i}$ for each $i = k, k+1, \dots, s$. We have used the notation $\mathbf{1}_{n_i}$ to mean

$$\mathbf{1}_{n_i} = (\underbrace{1, 1, \dots, 1}_{n_i \text{ times}}).$$

By solving the system $\mathbf{v}_i \mathbb{B} = \mathbf{v}_i$, we get

$$\begin{aligned} \mathbf{y}_{k-1}^i &= \mathbf{v}^i B_{i,k-1} \left(I - B_{k-1}\right)^{-1}, \\ \mathbf{y}_h^i &= \left[\sum_{j=h+1}^{k-1} \mathbf{y}_j^i B_{j,h} + \mathbf{v}^i B_{i,h} \right] \left(I - B_h\right)^{-1}, \end{aligned} \tag{4.23}$$

for all $1 \leq h \leq k-2$ and $i = 1, \ldots, m$. The vectors in (4.23) are all positive, by Theorem 2.1 of Seneta, [51].

Lemma 4.5.4. *The positive left eigenvector* $\mathbf{1}$ *is a sum of the linearly independent eigenvectors* $\mathbf{v}_k, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_s$.

Proof.

$$\begin{aligned}\sum_{i=k}^{m} \mathbf{y}_{k-1}^{i} = {} & \mathbf{1}_{n_k} B_{k,k-1}(I - B_{k-1})^{-1} + \mathbf{1}_{n_{k+1}} B_{k+1,k-1}(I - B_{k-1})^{-1} + \\ & \cdots + \mathbf{1}_{n_s} B_{s,k-1}(I - B_{k-1})^{-1} \\ & = (\mathbf{1}_{n_k} B_{k,k-1} + \mathbf{1}_{n_{k+1}} B_{k+1,k-1} + \cdots + \mathbf{1}_{n_s} B_{s,k-1})(I - B_{k-1})^{-1}.\end{aligned}$$

Since $\mathbb{B}$ is column stochastic,

$$\mathbf{1}_{n_{k-1}} B_{k-1} + \mathbf{1}_{n_k} B_{k,k-1} + \mathbf{1}_{n_{k+1}} B_{k+1,k-1} + \cdots + \mathbf{1}_{n_s} B_{s,k-1} = \mathbf{1}_{n_{k-1}}.$$

Therefore,

$$\begin{aligned}\mathbf{1}_{n_k} B_{k,k-1} + \mathbf{1}_{n_{k+1}} B_{k+1,k-1} + \cdots + \mathbf{1}_{n_s} B_{s,k-1} &= \mathbf{1}_{n_{k-1}} - \mathbf{1}_{n_{k-1}} B_{k-1} \\ &= \mathbf{1}_{n_{k-1}}(I - B_{k-1}).\end{aligned}$$

Therefore,

$$\sum_{i=k}^{s} \mathbf{y}_{k-1}^{i} = \mathbf{1}_{n_{k-1}}. \tag{4.24}$$

Using this equation (4.24) and the fact that $\mathbb{B}$ is column stochastic, we obtain

$$\begin{aligned}\mathbf{y}_{k-2}^{i} &= \left[\mathbf{1}_{n_i} B_{i,k-2} + \mathbf{y}_{k-11}^{i} B_{k-1,k-2}\right](I - B_{k-2})^{-1} \\ \sum_{i=k}^{s} \mathbf{y}_{k-2}^{i} &= \left[\sum_{i=k}^{s} \mathbf{1}_{n_i} B_{i,k-2} + \sum_{i=k}^{s} \mathbf{y}_{k-1}^{i} B_{k-1,k-2}\right](I - B_{k-2})^{-1} \\ &= \left[\left(\mathbf{1}_{n_{k-2}} - \mathbf{1}_{n_{k-1}} B_{k-1,k-2} - \mathbf{1}_{n_{k-2}} B_{k-2}\right) + \mathbf{1}_{n_{k-1}} B_{k-1,k-2}\right](I - B_{k-2})^{-1} \\ &= \mathbf{1}_{n_{k-2}}.\end{aligned}$$

Indeed, similar calculations show that for any $1 \leq h \leq k-2$,

$$\sum_{i=1}^{m} \mathbf{y}_{h}^{i} = \mathbf{1}_{n_h}.$$

■

Let the number of distinct eigenvalues of $\mathbb{B}$ be ν and k_i be the algebraic multiplicity of λ_i for every $\lambda_i \in \sigma(\mathbb{B})$. Let the number of distinct eigenvalues on the spectral circle be r and let $\lambda_1 = 1$, then $k_1 = s - k + 1$, $|\lambda_2| = \cdots = |\lambda_r| = 1$. We assume that $\mathbf{N}_i$ are normalised so that

$$\mathbf{v}_i.\mathbf{N}_i = 1, \quad i = k, \ldots, s. \tag{4.25}$$

Denote by d_i the index of imprimitivity of the matrix B_i, that is, d_i is the number of distinct eigenvalues of B_i of modulus 1. Then these eigenvalues $\lambda_i \in \sigma(B_i)$ can be written in the form $\lambda_i^l = e^{\frac{2l\imath\pi}{d_i}}$ for $l = 0, 1, \ldots, d_i - 1$ for each $i = k, \ldots, s$ and each of them is simple (see [44], p. 676). Further, denote

$$Z = \{\lambda \in \sigma(B_i) : |\lambda| < 1\}.$$

Theorem 4.5.5. *There is a decomposition of the space $X = X_k \oplus \cdots \oplus X_s \oplus Y$ such that*

1. *The spaces $X_l, l = k, \ldots, s$; and Y are invariant under $(T(t))_{t\geq 0}$;*
2. *$(T(t)|_{X_l})_{t\geq 0}$ is periodic with period $d_l, l = k, \ldots, s$;*
3. *$(T(t)|_Y)_{t\geq 0}$ is exponentially stable of the type $0 > \omega > \max\{\ln|\lambda|; \lambda \in Z\}$.*

Proof. Let P_λ be the spectral projection onto the generalised eigenspace corresponding to λ. Then, for $\mathbf{u} \in \mathbb{R}^m$,

$$\mathbb{P}^n\mathbf{u} = \mathbb{G}^{-1}\mathbb{B}^n\mathbb{G}\mathbf{u} = \mathbb{G}^{-1}\sum_{i=k}^{s}\sum_{l=0}^{d_i-1}\lambda_i^{ln}P_{\lambda_i}\mathbb{G}\mathbf{u} + \mathbb{G}^{-1}\sum_{\lambda_i\in Z}\lambda_i^n p_{\lambda_i}(n)P_\lambda\mathbb{G}\mathbf{u}, \tag{4.26}$$

where $p_\lambda(n)$ is a matrix valued polynomial in n of order less or equal to k_i. Since $\mathbb{B}\mathbf{N} = \mathbf{N}$, $\mathbb{P}\mathbb{G}^{-1}\mathbf{N} = \mathbb{G}^{-1}\mathbf{N}$, the spectral projections $\mathcal{P}_{\lambda_i^0}$ corresponding to $1 = \lambda_i^0 \in \sigma(B_i)$, for $i = k, \ldots, s$ is given by

$$\mathcal{P}_{\lambda_i^0}\mathbf{u} = (\mathbb{G}\mathbf{v}_i.\mathbf{u})\mathbb{G}^{-1}\mathbf{N}_i.$$

Similarly, the spectral projections $\mathcal{P}_{\lambda_i^l}$ have analogous form,

$$\mathcal{P}_{\lambda_i^l}\mathbf{u} = \left(\mathbb{G}\mathbf{e}^*_{\lambda_i^l}.\mathbf{u}\right), \mathbb{G}^{-1}\mathbf{e}_{\lambda_i^l}; \tag{4.27}$$

where $\mathbf{e}_{\lambda_i^l}$ is the right eigenvector of $\mathbb{B}$ corresponding to λ_i^l and $\mathbf{e}^*_{\lambda_i^l}$ is the associated left eigenvector, normalised so that $\mathbf{e}^*_{\lambda_i^l}.\mathbf{e}_{\lambda_i^l} = 1$. Then, recalling Proposition 4.5.1 and equation

(4.26), the semigroup generated by $(A_0, D(A_0))$ with $c_j \equiv 1$ is given by

$$\begin{aligned} T(t)\mathbf{u}(x) &= \left[\mathbb{G}^{-1}\mathbb{B}^n\mathbb{G}\mathbf{u}\right](t+x-n) = \mathbb{G}^{-1}\sum_{i=k}^{s}\sum_{l=0}^{d_i-1}\lambda_i^{ln}\left[\mathcal{P}_{\lambda_i^l}\mathbb{G}\mathbf{u}\right](t+x-n) \\ &\quad + \mathbb{G}^{-1}\sum_{\lambda_i\in Z}\lambda_i^n\left[p_{\lambda_i}(n)\mathcal{P}_\lambda\mathbb{G}\mathbf{u}\right](t+x-n); n\in\mathbb{N}_0, \quad 0\le t+x-n\le 1. \end{aligned} \tag{4.28}$$

Define X_i and Y to be the spaces

$$X_i = \bigoplus_{l=0}^{d_i-1}\mathcal{P}_{\lambda_i^l}X, \quad i=k,\dots,s,$$

$$Y = \bigoplus_{\lambda\in Z}P_\lambda X.$$

The only element common in all these spaces is the vector $\mathbf{0}$.

To prove item (2), let $\mathbf{u}\in X_i = L_1([0,1],\mathbb{X}_i)$, where $\mathbb{X}_i = \text{Span}\{\mathbb{G}^{-1}e_{\lambda_i^l}\}_{l=0,\dots,d_i-1}$ and $i=k,\dots,s$. Consider the equation

$$\left[T(t)|_{X_i}\mathbf{u}\right](x) := [T_i(t)\mathbf{u}](x) = \sum_{l=0}^{d_i-1}\lambda_i^{ln}[\mathcal{P}_{\lambda_i^l}\mathbf{u}](t+x-n), \quad 0\le t+x-n\le 1 \tag{4.29}$$

for $i=k,\dots,s$. Then $[T_i(t)\mathbf{u}]$ extends to a periodic group in X_i with period d_i, the index of imprimitivity of the matrix B_i. To verify this, we evaluate $T_i(t+d_i)$. First, we take n' satisfying $0\le(t+d_i)+x-n'\le 1$, that is $n'=n+d_i$. Then

$$\begin{aligned} (T_i(t+d_i)\mathbf{u})(x) &= \sum_{l=0}^{d_i-1}\lambda_i^{ln'}[P_{\lambda_i^l}\mathbf{u}](t+d_i+x-n'), \quad 0\le t+x-n'\le 1 \\ &= \sum_{l=0}^{d_i-1}e^{\frac{2\imath l\pi(n+d_i)}{d_i}}[P_{\lambda_i^l}\mathbf{u}](t+x-n), \quad 0\le t+x-n\le 1 \\ &= \sum_{l=0}^{d_i-1}e^{\frac{2\imath nl\pi}{d_i}}e^{2\imath\pi l}[P_{\lambda_i^l}\mathbf{u}](t+x-n), \quad 0\le t+x-n\le 1 \\ &= [T_i(t)\mathbf{u}](x), \quad 0\le t+x-n\le 1, \quad n\in\mathbb{N}_0. \end{aligned}$$

Hence, the period of $(T_i(t))_{t\ge 0}$ is $\tau_i\le d_i$. We can show that the period is indeed equal to d_i by using a similar argument to that used in Theorem 4.5, [33] or Theorem 24, [14], but we give a more elementary proof here by using the structure of the matrix given in (4.22). Taking the

Laplace transform of $(T_i(t))_{t\geq 0}$ with $\mathbf{u}_l = P_{\lambda_i^l}\mathbf{u}, \mathbf{u} \in X_i$ gives

$$\begin{aligned}
\int_0^\infty e^{-t\lambda}[T_i(t)\mathbf{u}](x)dt &= \int_0^\infty e^{-t\lambda}\sum_{l=0}^{d_i-1}\lambda_i^{ln}[P_{\lambda_i^l}\mathbf{u}](t+x-n)dt \\
&= \sum_{l=0}^{d_i-1}\left(\int_0^\infty e^{-\lambda t}\mathbf{u}_l(t+x)dt + \sum_{n=1}^{\infty} e^{\frac{2\imath\pi ln}{d_i}}\int_0^\infty e^{-\lambda t}\mathbf{u}_l(t+x-n)dt\right) \\
&= \sum_{l=0}^{d_i-1}\left(\int_x^1 e^{-\lambda(s-x)}\mathbf{u}_l(s)ds + \sum_{n=1}^{\infty} e^{\frac{2\imath\pi ln}{d_i}}\int_0^1 e^{-\lambda(s+n-x)}\mathbf{u}_l(s)ds\right) \\
&= \sum_{l=0}^{d_i-1}\left(\int_x^1 e^{-\lambda(s-x)}\mathbf{u}_l(s)ds + \sum_{n=1}^{\infty} e^{\left(\frac{2\imath\pi l}{d_i}-\lambda\right)n}\int_0^1 e^{-\lambda(s-x)}\mathbf{u}_l(s)ds\right).
\end{aligned}$$

From this, we see that the resolvent of the generator of $(T_i(t))_{t\geq 0}$ has poles at $\lambda \in \mathbb{C}$ which satisfy $e^{\lambda} = e^{\frac{2\imath\pi l}{d_i}}, l = 0, 1, \ldots d_i - 1$. From Lemma IV. 2.25 of [16], if τ is the period of a semigroup, then the resolvent of the generator has poles at $2\pi\imath.\mathbb{Z}/\tau$. By comparing, we conclude that τ must be a multiple of d_i, hence $\tau_i = d_i$ is the period of the semigroup $(T_i(t))_{t\geq 0}$.

To prove the last item, let $\tau_n = t - n$ with $0 \leq \tau_n < 1$. Using boundedness of P_λ, we have

$$\begin{aligned}
\|\lambda^n[\mathbf{p}_\lambda(n)P_\lambda\mathbf{u}](t+.-n)\|_X &\leq c(|\lambda|^n|\mathbf{p}_\lambda(n)|\int_{\tau_n}^1|\mathbf{u}(s)|ds + |\lambda|^{n+1}|\mathbf{p}_\lambda(n+1)|\int_0^{\tau_n}|\mathbf{u}(s)|ds) \\
c'\overline{\lambda}^n\|\mathbf{u}\|_X &\leq c''\|\mathbf{u}\|_X e^{t\ln\overline{\lambda}}
\end{aligned}$$

for some positive constants c, c', c'' and $|\lambda| < \overline{\lambda} < 1 \in \mathbb{Z}$. ■

4.5.3 The primitive case

If $d_i = 1$ for some $i \in \{k, \cdots, s\}$, then B_i is primitive, and thus X_i is spanned by the Perron eigenvector $\mathbf{N}^i$ extended by zero. That is, X_i is spanned by the vector $\mathbf{N}_i = (\mathbf{0}, \mathbf{N}^i, \mathbf{0})$, where $B_i\mathbf{N}^i = \mathbf{N}^i$. Thus for $\mathbf{f} \in D(A_0)$, the solution to the flow problem restricted to the space X_i is given by

$$[T_i(t)\mathbf{f}](x) = (\mathbb{G}\mathbf{v}_i.\mathbf{f}(t+x-n))\mathbb{G}^{-1}\mathbf{N}_i, n \in \mathbb{N}_0, 0 \leq t+x-n \leq 1. \tag{4.30}$$

If there is only one block in the normal form of $\mathbb{B}$ (B_s) with eigenvalue 1, then this eigenvalue is simple (because of irreducibility of B_s). If P_s is primitive, then any other eigenvalue of $\mathbb{B}$ is of magnitude less than 1. Then $\mathbf{N} = (0, \ldots, 0, \mathbf{N}^s)^T$ and $\mathbf{v} = (1, \ldots, 1)$, hence X is decomposed into two $(T(t))_{t\geq 0}$ invariant subspaces, $X = X_1 \oplus Y$, where $T(t)$ restricted to X_1 is given by

$$\begin{aligned} T(t)|_{X_1} \mathbf{f}(x) &= (\mathbb{G}(1,\dots,1)\mathbf{f}(t+x-n))\mathbb{G}^{-1}\mathbf{N} \\ &= \sum_{i=1}^{m} \alpha_i f_i(t+x-n)\mathbb{G}^{-1}\mathbf{N} = f(t+x-n)\mathbb{G}^{-1}\mathbf{N}, \end{aligned}$$

where $f(t+x-n) \in L^1([0,1])$. This semigroup $T(t)|_{X_1}$ is periodic with period 1.

If there is more than one block with eigenvalue 1 and each of them is not cyclic, then the semigroups $(T(t)|_{X_l})_{t\geq 0}$ defined in Theorem 4.5.5 are periodic with period 1.

4.5.4 Same speed, $\gamma_j \neq \alpha_j$

If there is at least one j such that $\gamma_j \neq \alpha_j$, then $\mathbb{P} = \mathbb{G}^{-1}\mathbb{B}\mathbb{E}$ and $\mathbb{B}$ are no longer similar matrices, so the asymptotic behaviour of $T(t)$ cannot be studied through the spectral properties of $\mathbb{B}$. Further, 1 may not be the maximum eigenvalue of $\mathbb{P}$. But we can define a new matrix $\tilde{\mathbb{P}} := (1/r(\mathbb{P}))\mathbb{P}$. This new matrix $\tilde{\mathbb{P}}$ has spectral radius $r(\tilde{\mathbb{P}}) = 1$. We then proceed as shown in the preceding section using the matrix $\tilde{\mathbb{P}}$.

Using the Jordan decomposition, we can write the powers of $\mathbb{P}$ in terms of its eigenvectors and generalised eigenvectors as follows.

$$\mathbb{P}^n = r^n \mathbf{N}_k \mathbf{v}_k + \cdots + r^n \mathbf{N}_s \mathbf{v}_s + \sum_{i=2}^{\nu} \sum_{j=1}^{k_i} \sum_{l=0}^{j-1} \binom{n}{l} \lambda_i^{n-l} \mathbf{x}_{i,j-l} \mathbf{v}_j^i,$$

where $\mathbf{v}_j^i$ are left eigenvectors and generalised left eigenvectors corresponding to eigenvalue λ_i, whose algebraic multiplicity is k_i, for all $i = 2, \dots, \nu$. Similarly, $\mathbf{x}_{i,j-l}$ are right eigenvectors and generalised right eigenvectors corresponding to λ_i. Then

$$\tilde{\mathbb{P}}^n = \frac{\mathbb{P}^n}{r^n} = \mathbf{N}_k \mathbf{v}_k + \cdots + \mathbf{N}_s \mathbf{v}_s + \sum_{i=2}^{\nu} \sum_{j=1}^{k_i} \sum_{l=0}^{j-1} \binom{n}{l} \frac{\lambda_i^{n-l}}{r^n} \mathbf{x}_{i,j-l} \mathbf{v}_j^i.$$

The semigroup $\tilde{T}(t)\mathbf{f}(x) = \tilde{\mathbb{P}}^n \mathbf{f}(t+x-n)$ then satisfies Theorem 4.5.5. Since

$$\tilde{T}(t)\mathbf{f}(x) = \tilde{\mathbb{P}}^n \mathbf{f}(t+x-n) = \frac{1}{r^n}\mathbb{P}^n f(t+x-n),$$

we have $r^n \tilde{T}(t)\mathbf{f}(x) = T(t)\mathbf{f}(x)$ for all $n \in \mathbb{N}_0$ such that $0 \leq t+x-n \leq 1$.

If all the blocks P_i are primitive, then there are no other eigenvalues of modulus r, hence taking limits as $n \to \infty$, we get

$$\tilde{\mathbb{P}}^n \to \mathbf{N}_k \mathbf{v}_k + \cdots + \mathbf{N}_s \mathbf{v}_s,$$

hence $\tilde{T}(t)f(x) = \tilde{\mathbb{P}}^n f(t+x-n)$ has asymptotic behaviour

$$\tilde{T}(t)\mathbf{f}(x) \approx (\mathbf{v}_k\mathbf{f}(t+x-n))\mathbf{N}_k + \cdots + (\mathbf{v}_s\mathbf{f}(t+x-n))\mathbf{N}_s.$$

The asymptotic behaviour of the new matrix gives the asynchronous growth of the powers of $\mathbb{P}$, and the qualitative behaviour of the semigroup $(\tilde{T}(t))_{t\geq 0}$ defined by $\tilde{T}(t)\mathbf{f}(x) = \tilde{\mathbb{P}}^n f(t+x-n)$ is the same behaviour of the original semigroup $(T(t))_{t\geq 0}$.

4.6 Different speed along the vertices

Until now, we have considered flow problems with $c_j = 1$ for all j. In this section, we show that the flow problem can still be solved using the results we have obtained already by converting it to a problem with $c_j \equiv 1$. In this section, we adopt the assumption from [33] that the speeds are linearly dependent over the field of rational numbers $\mathbb{Q}$. That is, for all $j = 1, \ldots, m$,

$$\frac{N}{c_j} \in \mathbb{N} \text{ for some } N \in \mathbb{N}. \tag{4.31}$$

If c_j can be written in the form $c_j = \frac{1}{l_j}$, where $l_j \in \mathbb{N}$ for all j, we make the transformation $\tilde{x} = (1/c_j)x$. Then, the flow problem becomes:

$$F3 \begin{cases} \partial_t u_j(\tilde{x}, t) & = \partial_{\tilde{x}} u_j(\tilde{x}, t), \quad \tilde{x} \in [0, \frac{1}{c_j}], \quad t \geq 0, \\ u_j(x, 0) & = f_j(x), \\ \phi_{ij}^- c_j u_j(\frac{1}{c_j}, t) & = w_{i,j} \sum_{k=1}^m \phi_{i,k}^+ (c_k u_k(0,t)), \quad \forall i. \end{cases} \tag{4.32}$$

If the speed c_j can be written in the form $c_j = N/l_j$ for some $N, l_j \in \mathbb{N}$ and $j \in \{1, \ldots, m\}$, then we first rescale time using the transformation $\tau = Nt$. This will put the flow problem into the form where $0 < \tilde{c}_j < 1$, considered above,

$$\begin{cases} \partial_\tau u_j(x, \tau) & = \frac{1}{l_j} \partial_x u_j(x, \tau), \quad x \in (0,1), \quad \tau \geq 0, \\ u_j(x, 0) & = f_j(x), \\ \phi_{ij}^- c_j u_j(1, \tau) & = w_{i,j} \sum_{k=1}^m \phi_{i,k}^+ (c_k u_k(0,\tau)), \quad \forall i. \end{cases} \tag{4.33}$$

Notice that in this, case, $\tilde{c}_j = 1/l_j$, which is the first case we considered. Then we can apply the transformation $\tilde{x} = x l_j = (1/\tilde{c}_j)x$ and this will put the problem into the form given in (4.32). From now on, we assume that the flow problem is in the form (4.32).

We then divide the interval $[0, \frac{1}{c_j}]$ into unit intervals by creating artificial vertices along the edge e_j. The number of artificial vertices created along e_j will be $(\frac{1}{c_j} - 1)$. We label the edges joining these vertices as $e_{j_1}, e_{j_2}, \ldots, e_{j_{1/c_j}}$. In this way, we have created a larger network with n' vertices and m' edges and each edge is of unit length; that is $s \in [0,1]$ with

$$m' = \sum_{j=1}^{m} \frac{1}{c_j}; \quad n' = n + \sum_{j=1}^{m} \left(\frac{1}{c_j} - 1 \right) = n + m' - m.$$

The density of particles on edge e_{j_i} will be denoted by u_{j_i} for all $j = 1, \ldots, m$ and $i = 1, 2, \ldots, \frac{1}{c_j}$.

For each $i = 1, \ldots, \frac{1}{c_j}$, $i - 1 \leq \tilde{x} \leq i$, we introduce a new variable s, which is defined on $[0,1]$ and depends on $\tilde{x}$ through the following equation:

$$s = \tilde{x} - i + 1, \tag{4.34}$$

for each i and $\tilde{x}$ defined above.

Remark 4.6.1. *On the larger network, we shall again represent the outgoing incidence matrix as Φ^- and the incoming incidence matrix as Φ^+. These matrices are $n^{'} \times m^{'}$ matrices. Every artificial vertex has exactly one incoming and outgoing edge, and the weight on the outgoing edge of an artificial vertex is 1. If v_i is not an artificial vertex and e_j is an outgoing edge of v_i, then the weight on e_j, w_{ij}, is the same as in the original network.*

The flow problem on this larger network is simply the flow problem in (4.10) with a few modifications:

$$\begin{cases} \partial_t u_{j_i}(s,t) & = \partial_s u_{j_i}(s,t), \quad s \in [0,1], \quad t \geq 0, \\ u_{j_i}(s,0) & = \varphi_{j_i}(s), \\ \phi^-_{lj_i} c_{j_i} u_{j_i}(1,t) & = w_{l,j_i} \sum_{k=1}^{m} \phi^+_{l,k_{\frac{1}{c_j}}} c_{k_{\frac{1}{c_j}}} u_{k_{\frac{1}{c_j}}}(0,t), \quad \forall l = 1, \ldots, n'. \end{cases} \tag{4.35}$$

where

$$\varphi_{j_i}(s) = \begin{cases} f_j(\tilde{x}) & \text{if } 0 \leq \tilde{x} \leq 1; \quad i = 1, \\ f_j(\tilde{x}) & \text{if } 1 \leq \tilde{x} \leq 2; \quad i = 2, \\ \vdots & \quad \vdots \\ f_j(\tilde{x}) & \text{if } \frac{1}{c_j} - 1 \leq \tilde{x} \leq \frac{1}{c_j}; \quad i = \frac{1}{c_j} \end{cases}$$

Since each artificial vertex has exactly one incoming and one outgoing edge and the speed on all the artificial edges is the same, the boundary condition on each of these artificial vertices is

$$\phi^-_{lj_i} c_{j_i} u_{j_i}(1,t) = c_j u_{j_i}(1,t) = \phi^+_{lj_{i-1}} c_{j_{i-1}} u_{j_{i-1}}(0,t) = c_j u_{j_{i-1}}(0,t) \tag{4.36}$$

for all $i = 2, \dots, \frac{1}{c_j}$ and $j = 1, 2, \dots, m$. If we define the operator $(\mathcal{A}, D(\mathcal{A}))$ where

$$D(\mathcal{A}) = \{\mathbf{v} \in W_1^1([0,1])^{m'} : \mathbf{v}(1) = C^{-1}\mathcal{B}C\mathbf{v}(0)\}$$

and $\mathcal{A}\mathbf{v} = \mathbf{v}'$, then the flow problem on the new graph is equivalent to the abstract Cauchy problem

$$\begin{cases} \mathbf{v}'(t) & = \mathcal{A}\mathbf{v}(t), \\ \mathbf{v}(0) & = \varphi. \end{cases}$$

Here $\mathcal{B}$ is the adjacency matrix for the line graph of the new expanded network.

Define a function $\mathbf{v}(s) = (v_1(s), \dots, v_m(s))$, where $v_j = (v_{j_1}, \dots, v_{j_{\frac{1}{c_j}}})$ is defined as

$$v_j = v_{j_i}(s), s = \tilde{x} - i + 1, 1 \le i \le c_j^{-1}, \quad 1 \le j \le m. \tag{4.37}$$

Then we define $\mathbf{u}(x)$ by

$$u_j(x) = v_j(c_j\tilde{x}) \tag{4.38}$$

Theorem 4.6.2. *Let $\mathbb{S} : L_1([0,1])^{m'} \to L_1([0,1])^m$ be the transformation $\mathbf{u} = \mathbb{S}\mathbf{v}$ defined by (4.38). Then $\mathbb{S}$ is an isomorphism.*

Proof. We note first that $\mathbb{S}$ is invertible, since

$$v_j(s) = \begin{cases} u_j(\tilde{x}) & \text{if } 0 \le \tilde{x} \le 1; \quad i = 1 \\ u_j(\tilde{x}) & \text{if } 1 \le \tilde{x} \le 2; \quad i = 2 \\ \vdots & \quad \vdots \\ u_j(\tilde{x}) & \text{if } \frac{1}{c_j} - 1 \le \tilde{x} \le \frac{1}{c_j}; \quad i = \frac{1}{c_j}, \end{cases} \tag{4.39}$$

where s is as defined in (4.34) for all $j = 1, \dots, m$, defines the inverse of $\mathbb{S}$. Then

$$\begin{aligned} \|\mathbb{S}\mathbf{v}\| &= \sum_{j=1}^{m} \int_0^1 |u_j(x)| dx \\ &= \sum_{j=1}^{m} c_j \int_0^{\frac{1}{c_j}} |v_j(\tilde{x})| d\tilde{x} \\ &= \sum_{j=1}^{m} c_j \sum_{i=1}^{\frac{1}{c_j}} \int_{i-1}^{i} |v_{j_i}(\tilde{x})| d\tilde{x} \\ &= \sum_{j=1}^{m} c_j \sum_{i=1}^{\frac{1}{c_j}} \int_0^1 |v_{j_i}(s)| ds \\ &\le \max\{c_j\}\|\mathbf{v}\|. \end{aligned}$$

On the other hand, $\|\mathbb{S}^{-1}\mathbf{u}\| = \|\mathbf{u}\|$

$$\begin{aligned}
\|\mathbb{S}^{-1}\mathbf{u}\| &= \sum_{j=1}^{m}\left[\int_0^1 |v_{j_1}(s)|ds + \int_0^1 |v_{j_2}(s)|ds + \cdots + \int_0^1 |v_{j_{\frac{1}{c_j}}}(s)|ds\right] \\
&= \sum_{j=1}^{m}\sum_{i=1}^{\frac{1}{c_j}} \int_0^1 |v_{j_i}(s)|ds \\
&= \sum_{j=1}^{m}\sum_{i=1}^{\frac{1}{c_j}} \int_{i-1}^{i} |v_j(\tilde{x})|d\tilde{x} \\
&= \sum_{j=1}^{m}\int_0^{\frac{1}{c_j}} |v_j(\tilde{x})|d\tilde{x} = \sum_{j=1}^{m}\frac{1}{c_j}\int_0^1 |u_j(x)|dx \\
&\le \max_j\left(\frac{1}{c_j}\right)\|\mathbf{u}\|.
\end{aligned}$$

We conclude that $\mathbb{S}$ is an isomorphism. ■

Lemma 4.6.3. *The function $\mathbf{v} \in D(\mathcal{A})$ if and only if $\mathbf{u} = \mathbb{S}\mathbf{v} \in D(A)$.*

Proof. From the preceding results (in particular Theorem 4.4.1), we know that (4.35) admits semigroup solutions. Suppose that

$$\mathbf{v}(s) = \big(v_{1,1}(s), \ldots, v_{1,1/c_1}(s), \ldots, v_{m,1}(s), v_{m,2}(s), \ldots, v_{m,1/c_m}\big) \in D(\mathcal{A}).$$

Then Equation (4.36) gives us continuity of the flow at the artificial vertices. That is, $v_{j_i}(1) = v_{j_{i-1}}(0)$ for all $i = 2, \ldots, \frac{1}{c_j}$. So the function $v_j(s)$, defined in equation (4.37), is continuous on $[0, \frac{1}{c_j}]$ for all $j = 1, \ldots, m$ and

$$\begin{aligned}
\sum_{j=1}^{m}\int_0^{\frac{1}{c_j}} |v_j(\tilde{x})|d\tilde{x} &= \sum_{j=1}^{m}\left[\int_0^1 |v_j(\tilde{x})|d\tilde{x} + \int_1^2 |v_j(\tilde{x})|d\tilde{x} + \cdots + \int_{\frac{1}{c_j}-1}^{\frac{1}{c_j}} |v_j(\tilde{x})|d\tilde{x}\right] \\
&= \sum_{j=1}^{m}\frac{1}{c_j}\left[\int_0^1 |v_{j_1}(x)|dx + \int_0^1 |v_{j_2}(x)|dx + \cdots + \int_0^1 |v_{j_{\frac{1}{c_j}}}(x)|dx\right] < \infty
\end{aligned}$$

since each $v_{j_i}(x) \in L_1[0,1]$. So $v_j \in L_1([0, \frac{1}{c_j}])$ for any $j = 1, \ldots, m$. Next, we show that $v_j \in W_1^1([0, \frac{1}{c_j}])$, $j = 1, \ldots, m$. Since each $v_{j_i} \in W_1^1([0,1])$, there exists $g_{j_i} \in L_1([0,1])$ such that

$$\int_0^1 v_{j_i}\phi' ds = -\int_0^1 g_{j_i}\phi ds, \quad \forall \phi \in C_0^\infty([0,1]),$$

for each $j = 1, \ldots, m$ and $i = 1, \ldots, \frac{1}{c_j}$. On the other hand, let $\xi \in C_0^\infty([0, 1/c_j])$. Then

$$\int_0^{\frac{1}{c_j}} v_j\xi' d\tilde{x} = \int_0^1 v_j\xi' d\tilde{x} + \int_1^2 v_j\xi' d\tilde{x} + \cdots + \int_{\frac{1}{c_j}-1}^{\frac{1}{c_j}} v_j\xi' d\tilde{x}$$

From this equation, using Corollary 8.10 of [8], we get

$$\begin{aligned}
\int_0^{\frac{1}{c_j}} v_j \xi' d\tilde{x} &= -\int_0^1 \hat{g}_{j_1}\xi d\tilde{x} + (v_{j_1}\xi)|_0^1 - \int_1^2 \hat{g}_{j_2}\xi d\tilde{x} + (v_{j_2}\xi)|_0^1 - \cdots - \int_{\frac{1}{c_j}-1}^{\frac{1}{c_j}} \hat{g}_{j_{\frac{1}{c_j}}}\xi d\tilde{x} \\
&\quad + (v_{j_{\frac{1}{c_j}}}\xi)\Big|_{\frac{1}{c_j}-1}^{\frac{1}{c_j}} \\
&= \left(-\int_0^1 \hat{g}_{j_1}\xi d\tilde{x} + v_{j_1}(1)\xi(1)\right) + \left(-\int_1^2 \hat{g}_{j_2}\xi d\tilde{x} + v_{j_2}(2)\xi(2) - v_{j_2}(1)\xi(1)\right) + \cdots \\
&\quad + \left(-\int_{\frac{1}{c_j}-1}^{\frac{1}{c_j}} \hat{g}_{j_{\frac{1}{c_j}}}\xi d\tilde{x} - v_{j_{\frac{1}{c_j}}}\left(\frac{1}{c_j}-1\right)\xi\left(\frac{1}{c_j}-1\right)\right),
\end{aligned}$$

where $\hat{g}_{j_i}(\tilde{x}) = g_{j_i}(s)$ with $s = \tilde{x} - i + 1$ for $j = 1, \dots, m$ and $i = 1, \dots, 1/c_j$. By the Kirchoff law at the artificial vertices and continuity of the test functions ξ on $[0, \frac{1}{c_j}]$, all the boundary terms in the integral cancel out. That is,

$$v_{j_1}(1) = v_{j_2}(1), v_{j_2}(2) = v_{j_3}(2), \cdots, v_{j_{\frac{1}{c_j}}}\left(\frac{1}{c_j}-1\right) = v_{j_{\frac{1}{c_j}-1}}\left(\frac{1}{c_j}-1\right).$$

Hence

$$\begin{aligned}
\int_0^{\frac{1}{c_j}} v_j \xi' d\tilde{x} &= -\int_0^1 \hat{g}_{j_1}\xi d\tilde{x} - \int_1^2 \hat{g}_{j_2}\xi d\tilde{x} - \cdots - \int_{\frac{1}{c_j}-1}^{\frac{1}{c_j}} \hat{g}_{j_{\frac{1}{c_j}}}\xi d\tilde{x} \\
&= -\int_0^{\frac{1}{c_j}} g_j v_j d\tilde{x};
\end{aligned}$$

where

$$g_j(\tilde{x}) = \begin{cases} \hat{g}_{j_1}(\tilde{x}), & \text{if } 0 \le \tilde{x} \le 1, \\ \hat{g}_{j_2}(\tilde{x}), & \text{if } 1 \le \tilde{x} \le 2, \\ \vdots & \quad \vdots \\ \hat{g}_{j_{\frac{1}{c_j}}}(\tilde{x}), & \text{if } \frac{1}{c_j} - 1 \le \tilde{x} \le \frac{1}{c_j}. \end{cases}$$

Since $g_j(\tilde{x}) \in L_1([0, \frac{1}{c_j}])$, $v_j(\tilde{x}, t) \in W_1^1([0, \frac{1}{c_j}])$ for each $j = 1, \dots, m$, hence $v_j \in W_1^1([0, \frac{1}{c_j}])$, $j = 1, \dots, m$. We have $\mathbf{u} \in W_1^1([0,1])^m$, where $\mathbf{u}$ is defined by (4.38). Since the Kirchoff laws at the original vertices have not been changed, we must have $\mathbf{u} \in D(A_0)$. ■

Thus, the solution to (4.35) is given by

$$(\mathcal{T}(t)f)(s) = C^{-1}\mathcal{B}^n C\psi(t+s-n), n \in \mathbb{N}_0, \quad 0 \le t+s-n < 1\, s \in [0,1],$$

where

$$\psi(s) = \begin{pmatrix} \varphi_{1,1}(s) \\ \vdots \\ \varphi_{1,\frac{1}{c_1}}(s) \\ \vdots \\ \varphi_{m,1}(s) \\ \vdots \\ \varphi_{m,\frac{1}{c_m}}(s) \end{pmatrix}$$

and from this, we obtain the solution to the original problem through equation (4.38) above. Using the transformation in Theorem 4.6.2, we have

$$T(t)\mathbf{u} = \mathbb{S}^{-1}\mathcal{T}(Nt)\mathbb{S}\mathbf{u}. \tag{4.40}$$

That is, in order to get $u_1(x,t)$, we take the submatrix of $\mathcal{B}$ which contains only the first $\frac{1}{c_1}$ rows of $\mathcal{B}$ and multiply it with φ. To get $u_2(x,t)$, we take the next $\frac{1}{c_2}$ rows by starting from the $(\frac{1}{c_1}+1)^{th}$ row up to the $(\frac{1}{c_1}+\frac{1}{c_2})^{th}$ row and so on.

Chapter 5

A Graph Theoretic Point of View

5.1 Introduction

In the preceding chapter, we proved the existence of semigroup solutions to the flow problem on the graph with no sinks. In this chapter, we extend the results of Chapter 4 to graphs with non trivial acyclic part. We will show that asymptotically, the flow will remain in certain parts of the graph with cycles and that these subgraphs where the flow remains asymptotically are those cycles with no outgoing flow. We also show that the flow on the edges in the acyclic part of the graph will be depleted in finite time while the flow in the cyclic parts of the graph with both incoming and outgoing flow will be depleted asymptotically. We start with certain useful graph descriptions which will enable asymptotic description of the flow to be easier.

5.2 Graph Descriptions

Let $G = (V(G), E(G))$ be a digraph with n vertices and m edges and let $v \in V(G)$. The out-degree (denoted $d^-(v)$) of v is the number of outgoing edges of v and the in-degree of v (denoted $d^+(v)$) is the number of incoming edges of v. We will assume that $G(V, E)$ is connected but not necessarily strongly connected. We assume that each vertex has at least one outgoing edge ($d^-(v) > 0$ for every $v \in V$). Let V_o be the set of vertices which are sources in the graph.

Let $Q = (V(Q), E(Q)) = (E(G), E(Q))$ be the line graph of G. As seen in the first chapter,

$e'_i = (v'_i v'_j)$ is an edge of Q if there exists a vertex $v \in V(G)$ such that $\xrightarrow{e_i} v \xrightarrow{e_j}$. Let us put this in a formal definition:

Definition 5.2.1. *A vertex $v \in G$ is said to generate an edge e_{ij} in Q if there exist edges e_i and e_j in G such that*

$$\xrightarrow{e_i} v \xrightarrow{e_j} .$$

We introduce the map

$$\Phi_G : E(G) \to V(Q) \tag{5.1}$$

by defining $\Phi_G(e) = \bar{e}$, where $\bar{e}$ is the vertex in Q corresponding to the edge e in G. Therefore, it is a one-to-one function. We also define Ψ_G to be the multifunction

$$\Psi_G : V(G) \to 2^{E(Q)}. \tag{5.2}$$

This function assigns to a vertex $v \in V(G)$ a set of edges generated by v. If $u \in V_o$, then it does not generate any edges in Q. Every vertex in $V(G) \setminus V_o$ generates at least one edge since there are no sinks.

More precisely, if v is a vertex in $V(G) \setminus V_o$, then it generates $d^-(v)d^+(v)$ edges. To verify this, notice that if v has out degree $d^-(v)$, then every edge that is incoming at v is adjacent to all the $d^-(v)$ edges going out of v, hence there are $d^-(v)$ edges generated in Q for every incoming edge at v, and since there are $d^+(v)$ incoming edges at v, there is a total of $d^-(v)d^+(v)$ edges in Q generated by v.

On the other hand, the inverse of Ψ_G is a map, that is, if $\Psi_G(v) \cap \Psi_G(v') \neq \emptyset$, then $v = v'$. Indeed if ϵ is generated by a vertex $v \in V(G)$, then there is an incoming edge u and an outgoing edge w such that the vertices $\bar{u}$ and $\bar{w}$ are incident to ϵ. Since an edge has only one tail and one head, if ϵ was generated by another vertex $v' \in G$, then v' would be the head of u and the tail of w, which is impossible.

Since the graph G has no sinks, it cannot be acyclic (see Proposition 1.4.2, [6]), hence the line graph Q is not acyclic as well. However, if G has sources, then there exists an acyclic subgraph of G (and hence, Q has an acyclic subgraph as well). We shall see how to obtain the maximal acyclic subgraph of Q below. Let $V_o(Q)$ be the set of sources in Q. Then $V_o(Q) = \Phi_G(e_j)$ where e_j are edges in G with their tails in $V_o(G)$. Let Q_2 be the subgraph of Q which consists of all cycles and all paths joining the cycles and Q_1 be the graph obtained by deleting Q_2

from Q. Then $V_o(Q) \subset V(Q_1)$. Let C_Q be the cut set separating Q_1 and Q_2. Then the set $E(Q) = C_Q \cup E(Q_1) \cup E(Q_2)$.

We can easily determine all the vertices and edges of Q_1 by using a topological sorting algorithm, [12]:

1. Pick a vertex $v \in V_o(Q)$ and label it v_1.
2. Delete v_1 from Q and call the resulting graph $\tilde{Q}$.
3. Pick another vertex $v \in V_o(\tilde{Q})$ and label it v_2.
4. Delete v_2 from $\tilde{Q}$ and call the resulting graph $\tilde{Q}$ (again).
5. Repeat the above procedure until we get a graph $\tilde{Q} = Q_2$ such that $V_o(\tilde{Q_2}) = \emptyset$ or until all the vertices in the graph are labelled.

Lemma 5.2.2. *If Q' is the subgraph of Q which was left after the execution of the topological sorting algorithm, then it must contain a cycle.*

Proof. If $v \in V(Q')$, then v is the head of some edge in Q'. Since by assumption every vertex has an outgoing edge, it follows that every vertex in Q' has an incoming and outgoing edge, there are no sources and no sinks, so Q' must have a cycle, by Proposition 1.4.2, [6]. ■

Let Q_1 be the graph whose vertex set contains the vertices $\{v_1, v_2, \ldots\}$ which were labelled by the topological sorting algorithm. That is, $V(Q_1) = V(Q) \setminus V(Q_2)$. Then we note that $\tilde{Q}$ is a subgraph of Q and $V_o(Q) \subseteq V(Q_1)$. Clearly, if $v \in V(Q_2)$, then $v \notin V(Q_1)$. This is because for $v \in V(Q_2)$, it must either be on a cycle or an a path between two cycles. In either case $v \in V(Q_1)$ would mean that there is a cycle contained in Q_1, a contradiction.

Remark 5.2.3. *A graph Q (or G) may have more than one cut set, but the cut set separating the acyclic part of Q (that is Q_1) from the cyclic part Q_2 is unique. To verify this, Let C_1 and C_2 be two cuts separating Q_1 and Q_2. Then note that $E(Q) = E(Q_1) \cup E(Q_2) \cup C_1 = E(Q_1) \cup E(Q_2) \cup C_2$. Since $E(Q_1) \cap E(Q_2) = \emptyset$, it follows that $C_1 = C_2$.*

We can also use the topological sorting algorithm to find the maximal acyclic subgraph of G, and we shall call this graph G_1 (G_2 will then be the graph obtained by deleting G_1 from G).

Lemma 5.2.4. *The function Φ_G maps the edges of the graph G which form cycles to the vertices of the graph Q which are on cycles or between cycles; i.e*

$$\Phi_G(E(G_2)) = V(Q_2); \quad \Phi_G(E(G_1)) \cup C_G = V(Q_1) \tag{5.3}$$

and

$$\Psi_G(V(G_2)) = E(Q_2) \cup C_Q; \quad \Psi_G(V(G_1)) = E(Q_1). \tag{5.4}$$

Proof. Let $u_{i_1}, e_{i_1}, u_{i_2}, \ldots, e_{i_{k-1}}, u_{i_k}$ be a cycle in G_2. Since e_{j-1} and e_{i_j} connect through u_{i_j}, there is an edge in $\Psi_G(u_{i_j})$ with head at $\Phi_G(e_{i_j})$ and an edge in $\Psi_G(u_{j+1})$ with tail at $\Phi_G(e_{i_j})$ so that $\Psi_G(u_{i_1})$ contains an edge connecting $\Phi_G(e_{i_{k-1}})$ and $\Phi_G(e_{i_1})$. Therefore, if there is a cycle in G, there is a corresponding cycle in Q. On the other hand, if $v_{i_1}, e_{i_1}, v_{i_2}, e_{i_2}, \ldots, e_{i_{l-1}}, v_{i_l} = v_{i_1}$ is a cycle in Q_2, then

$$\Phi_G^{-1}(v_{i_1}), \Psi_G^{-1}(e_{i_1}), \ldots, \Psi_G^{-1}(e_{i_{l-1}}), \Phi_G^{-1}(v_{i_l})$$

is a cycle in G. Hence there is a cycle in G_2 if and only if there is a corresponding cycle in Q_2.

Let $e \in E(G_2)$, then $\Phi_G(e)$ is a vertex in Q. If e is on a cycle, then v is also on a cycle, hence $v \in V(Q_2)$. If e is on a path joining two cycles, then there is a v_{i_1} and v_{i_k} both on cycles such that $v_{i_1}, e_{i_1}, \ldots, e_{i_j} = e, v_{i_{j+1}}, e_{i_{j+1}}, \ldots, e_{i_{k-1}}, v_{i_k}$. Let v' be a vertex on a cycle, adjacent to the vertex v_{i_1} (that is; $v'v_{i_1}$ is an edge on the cycle). Then $\Phi_G(v'v_{i_1})$ is a vertex in a corresponding cycle in Q_2. Now since v_{i_1} has at least two outgoing edges (one edge is on the cycle and the other is on the path leading to $\Phi_G(e)$, which is not on the cycle), $\Psi_G(v_{i_1})$ consists of at least two edges. Also, v_{i_k} has at least two incoming edges, one edge is on the cycle and the other is on the path which contains e. Since v_{i_k} is on a cycle, there is at least one outgoing edge on the cycle. Hence $\Phi_G(e)$ is on a path joining two cycles and therefore belongs to Q_2. This shows that $\Phi_G(E(G_2)) \subseteq V(Q_2)$.

Let u be on a path joining two cycles in Q_2. Then there are vertices u_{i_1} and u_{i_k}, both on cycles, such that $u_{i_1}, e_{i_1}, \ldots, e_{j-1}, u_{i_j} = u, e_{i_j}, \ldots, u_{i_k}$ is a path between cycles. u_{i_1} must have at least two outgoing edges; e_{i_1} (leading to vertex u) and the other (call it e) on a cycle, and at least one incoming edge. So there is some vertex $v \in G_2$ which generates at least two edges. In particular, $e, e_{i_1} \in \Psi_{G_2}(v)$. Indeed if this was not the case, then we see that if e is generated by u and e_{i_1} by v ($u \neq v$), then there would be edges $\bar{e}_i$, $\bar{e}_j$, $\bar{e}_l$ and $\bar{e}_m$ such that

$$\xrightarrow{\bar{e}_i} u \xrightarrow{\bar{e}_j}; \qquad \xrightarrow{\bar{e}_l} v \xrightarrow{\bar{e}_m},$$

where $e = \bar{e}_{ij}$ and $e_{i_1} = \bar{e}_{lm}$. Since e and e_{i_1} are incident from the same vertex in Q, it follows that $\bar{e}_i = \bar{e}_l$, which gives the implication that $\bar{e}_i$ has two different heads, which is impossible. So indeed e and e_{i_1} are generated by the same vertex in G_2. On the other hand, u_{i_k} has at least two incoming edges; one on the cycle (call it e') and the other is the edges $e_{i_{k-1}}$ and at least one outgoing edge (on the cycle). So $u_{i_k} = \Phi_G^{-1}(e')$ with e' on a cycle in G and the two incoming edges of u_{i_k} are generated by the same vertex v' in G_2, with v' on a cycle. Indeed to see that $e_{i_{k-1}}, e' \in \Psi_G(v')$, suppose first that they are generated by two different vertices v_1 and v'. Then there are edges $\bar{e}_{l_1}, \bar{e}_{l_2}, \bar{e}_{k_1}, \bar{e}_{k_2}$ in G such that $e' = \bar{e}_{k_1k_2}, e_{i_{k-1}} = \bar{e}_{l_1l_2}$ and

$$\xrightarrow{\bar{e}_{l_1}} v_1 \xrightarrow{\bar{e}_{l_2}}; \qquad \xrightarrow{\bar{e}_{k_1}} v' \xrightarrow{\bar{e}_{k_2}}.$$

Since e' and $e_{i_{k-1}}$ are incident to the same vertex (u_{i_k}) in Q, it follows that $\bar{e}_{l_2} = \bar{e}_{k_2}$, which gives the implication that $\bar{e}_{k_2}$ has two different tails, which is impossible. Thus the path starting at $\Phi_G^{-1}(u_{i_1})$ and ending at $\Phi_G^{-1}(u_{i_k})$ is a path joining two different cycles in G_2. Hence $\Phi_G(E(G_2)) \supseteq V(Q_2)$ and this gives the equality in (5.3).

From the equations $E(G) = E(G_1) \cup E(G_2) \cup C_G$ and $V(Q) = V(Q_1) \cup V(Q_2)$, we have

$$\begin{aligned} V(Q_1) &= V(Q) \setminus V(Q_2) \\ &= \Phi_G(E(G)) \setminus \Phi_G(E(G_2)) \\ &= \Phi_G(E(G_1) \cup E(G_2) \cup C_G) \setminus \Phi_G(E(G_2)) \\ &= \Phi_G(E(G_1) \cup C_G). \end{aligned}$$

To prove that $\Psi_G(V(G_1)) = E(Q_1)$, suppose that $v \in V(G_1)$ and let $e_{ij} \in \Psi_G(v)$. Then there exist edges e_i, e_j in $E(G)$ such that $\xrightarrow{e_i} v \xrightarrow{e_j}$. We want to show that $e_{ij} \in E(Q_1)$. If $v \in V(G_1)$, then there is no closed path involving it or involving the edges e_i and e_j. In particular, $\Phi_G(e_i)$ is a source in Q ($\Phi_G(e_i) \in V(Q_1)$) as indeed is any of the $d^+(v)$ edges incoming at v. That is, $\Phi_G(e_k) \in V(Q_1)$ whenever e_k is an incoming edge of v, for all $v \in V(G_1)$. Similarly, $\Phi_G(e_j) \in V(Q_1)$ as well (see from proof of (5.3)) and since e_{ij} is an edge from $\Phi_G(e_i)$ to $\Phi_G(e_j)$, it follows that $e_{ij} \in E(Q_1)$. Hence, $e_{ij} \in \Psi_G(V(G_1)) \Rightarrow e_{ij} \in E(Q_1)$, implying that $\Psi_G(V(G_1)) \subseteq E(Q_1)$.

Now suppose that $e = e_{ij} \in E(Q_1)$. We want to show that this edge is generated by a vertex v in G_1. $e_{ij} \in E(Q_1)$ implies that $\Phi_G(e_i), \Phi_G(e_j) \in V(Q_1)$, with $\Phi_G(e_i) \xrightarrow{e_{ij}} \Phi_G(e_j)$. Using the equation in (5.3), we have

$$\Phi_G^{-1}(e_i), \Phi_G^{-1}(e_j) \in E(G_1) \cup C_G.$$

This implies that either $\Phi_G^{-1}(e_i) \in E(G_1)$ and $\Phi_G^{-1}(e_j) \in C_G$ or $\Phi_G^{-1}(e_i), \Phi_G^{-1}(e_j) \in E(G_1)$. Since

$$\xrightarrow{\Phi_G^{-1}(e_i)} v \xrightarrow{\Phi_G^{-1}(e_j)},$$

both cannot be in the cut set of G because then e_{ij} would not be in $E(Q_1)$. $\Phi_G^{-1}(e_i)$ cannot be in the cut set of G either because then $\Phi_G^{-1}(e_j) \in E(G_1)$ and this would contradict the second equation of (5.3). In either case ($\Phi_G^{-1}(e_i) \in E(G_1)$ and $\Phi_G^{-1}(e_j) \in C_G$ or $\Phi_G^{-1}(e_i), \Phi_G^{-1}(e_j) \in E(G_1)$), $v \in V(Q_1)$, hence $E(Q_1) \subseteq \Psi_G(V(G_1))$. Thus $E(Q_1) = \Psi_G(V(G_1))$.

Using this result, we also have that

$$\begin{aligned}
\Psi_G(V(G)) &= E(Q) = C_Q \cup E(Q_1) \cup E(Q_2) \\
&= C_Q \cup E(Q_2) \cup \Psi_G(V(G_1)) \\
\Rightarrow \Psi_G(V(G) \setminus V(G_1)) &= C_Q \cup E(Q_2),
\end{aligned}$$

hence $\Psi_G(V(G_2)) = C_Q \cup E(Q_2)$ which is the first equality in (5.4). ■

Remark 5.2.5. *In most texts on graph theory, the topological sorting algorithm is performed on a directed acyclic graph. We have adapted this algorithm on a graph which may contain cycles to obtain two subgraphs, one of which is acyclic, and the other containing cycles. Most of the results obtained in this section can not be found easily in literature. However, some partial results do exist and have been expanded in this thesis. For example, Theorem 14.4 of [12] states that the vertices of a directed graph can be arranged in a topological order if and only if the graph is acyclic. This can be related to Lemma 5.2.2 in this section in the sense that the vertices of Q_2, which contains cycles, can not be arranged by a topological sorting algorithm.*

5.3 Asymptotic behaviour

If now we reorder the vertices in Q such that those in Q_1 are given smaller numbers and thus appear first, we can then write the adjacency matrix $\mathbb{B}$ in the form

$$\mathbb{B} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbb{B}_{2,1} & \mathbb{B}_2 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbb{B}_3 & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \mathbb{B}_{g-1,1} & \mathbb{B}_{g-1,2} & \mathbb{B}_{g-1,3} \cdots & \mathbb{B}_{g-1,g-2} & \mathbb{B}_{g-1} & \mathbf{0} & \mathbf{0} & \cdots & & \\ \mathbb{B}_{g,1} & \mathbb{B}_{g,2} & \mathbb{B}_{g,3} & \cdots & \mathbb{B}_{g,g-2} & \mathbb{B}_{g,g-1} & \mathbb{B}_g & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbb{B}_{g+1,1} & \mathbb{B}_{g+1,2} & \mathbb{B}_{g+1,3} & \cdots & \mathbb{B}_{g+1,g-2} & \mathbb{B}_{g+1,g-1} & \mathbf{0} & \mathbb{B}_{g+1} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbb{B}_{s,1} & \mathbb{B}_{s,2} & \mathbb{B}_{s,3} & \cdots & \mathbb{B}_{s,g-2} & \mathbb{B}_{s,g-1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbb{B}_s \end{pmatrix},$$

where the first zero matrix $\mathbf{0}$ is an $n_1 \times n_1$ matrix corresponding to the sources in Q. The matrix $\mathbb{B}_2$ corresponds to the other vertices in Q_1 (the cardinality of $V(Q_1)$ is $n_1 + n_2$, i.e $|V(Q_1)| = n_1 + n_2$) and the non zero entries in this matrix are below the main diagonal. The matrix $\mathbb{B}_3$ corresponds to strong components of the graph which act as sources (subgraph of Q where there is no flow from other parts of the graph into this subgraph, but there is outflow). The other matrices that follow ($\mathbb{B}_4, \ldots, \mathbb{B}_{g-1}$) represent cyclic subgraphs with flow to other parts of the graph. The matrices in the last section ($\mathbb{B}_g, \ldots, \mathbb{B}_s$) are adjacency matrices for strong components of Q with no outflow (which is why $\mathbb{B}_{h,j} = \mathbf{0}$ for all $h = g, \ldots, s$ and $j = h+1, \ldots, s$). Let Z be the set of eigenvalues below:

$$Z = \{\lambda \in \sigma(\mathbb{B}_i); 3 \leq i \leq s : |\lambda| < 1\}. \tag{5.5}$$

Then

$$\mathbb{B}^n \mathbf{u} = \sum_{i=k}^{s} \sum_{l=0}^{d_i - 1} \lambda_i^{ln} P_{\lambda_i} \mathbf{u} + \sum_{\lambda_i \in Z} \lambda_i^n p_{\lambda_i}(n) P_{\lambda} \mathbf{u} + \mathbb{B}_0^n \mathbf{u}, \tag{5.6}$$

where

$$\mathbb{B}_0 = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbb{B}_{2,1} & \mathbb{B}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}. \tag{5.7}$$

This matrix is the adjacency matrix of the subgraph Q_1. The first row of zeros (in bold) represents the n_1 rows of zeros which are obtained from the n_1 sources. This matrix $\mathbb{B}_0$ is idempotent with index not exceeding n_2 and its only eigenvalue is 0. That is, $\mathbb{B}_0^n = \mathbf{0}, n \geq n_2$. Note that n_2 is the number of vertices in Q_1 which are not sources or, if we use the original graph G, then n_2 is the number of edges in G_1 plus the cut set C_G minus the number of edges with tails in the set $V_0(G)$ (see Equation (5.3)). Then $\mathbb{R}^m$ can be decomposed as $\mathbb{R}^m = \mathbb{R}^{n_1+n_2} \oplus \mathbb{R}^{m'}$ where $m' = m - (n_1 + n_2)$. Note that the spectral projections onto the fixed space are given by

$$P_i \mathbf{u} = (\mathbf{v}_i.\mathbf{u})\, \mathbf{N}_i; \quad \mathbf{u} \in \mathbb{R}^m, \tag{5.8}$$

for all $i = g, g+1, \ldots, s$. Similarly, the spectral projections onto the cyclic space have the form

$$P_{\lambda_i^l} \mathbf{u} = \left(\mathbf{e}^*_{\lambda_i^l}.\mathbf{u}\right) \mathbf{e}_{\lambda_i^l}, \tag{5.9}$$

where $\mathbf{e}_{\lambda_i^l}$ is the right eigenvector of $\mathbb{B}$ corresponding to λ_i^l and $\mathbf{e}^*_{\lambda_i^l}$ is the associated left eigenvector; normalised so that

$$\mathbf{e}^*_{\lambda_i^l}.\mathbf{e}_{\lambda_i^l} = 1 \tag{5.10}$$

Then, if $c_j = 1$ for all j and $\alpha_j = \gamma_j$, the semigroup generated by $(A_0, D(A_0))$ is now given by

$$\begin{aligned} T(t)\mathbf{u}(x) &= [\mathbb{P}^n \mathbf{u}]\,(t + x - n) \\ &= \mathbb{G}^{-1} \sum_{i=k}^{s} \sum_{l=0}^{d_i - 1} \lambda_i^{ln}\, [P_{\lambda_i} \mathbb{G}\mathbf{u}](t + x - n) + \mathbb{G}^{-1} \sum_{\lambda_i \in Z} \lambda_i^n\, [p_{\lambda_i}(n) P_{\lambda} \mathbb{G}\mathbf{u}]\,(t + x - n) \\ &+ \mathbb{G}^{-1}\, [\mathbb{B}_0^n \mathbb{G}\mathbf{u}]\,(t + x - n). \end{aligned} \tag{5.11}$$

Theorem 5.3.1. *For any $\mathbf{u} \in X$, $[T(t)\mathbf{u}]\,|_{E(G_1)\cup C_G} = 0$ for $t \geq n_2$.*

Proof. The matrix $\mathbb{B}_0$ is an adjacency matrix of an acyclic graph with $n_1 + n_2$ vertices. Since $\mathbb{B}_0$ is lower triangular and $\mathbb{B}_2^{n_2} = \mathbf{0}$, hence $\mathbb{B}_0^k = \mathbf{0}$ for all $k \geq n_2$. Since $\mathbb{B}_0$ is the restriction of $\mathbb{B}$ on $E(G_1) \cup C_G$, we conclude that $[T(t)\mathbf{u}]\,|_{E(G_1)\cup C_G} = 0$ for $t \geq n_2$. ■

This result tells us that, irrespective of the initial distribution of mass, after $t = n_2$, all the edges in the acyclic part of the graph will be depleted. In fact we can improve this result by noting that if b_{ij}^r is the $(i,j)^{th}$ entry in $\mathbb{B}_0^r$ then, by Theorem 2.2, [10], b_{ij}^r gives the number of $v_i - v_j$ paths of length r. So $[T(t)\mathbf{u}]\,|_{E(G_1)\cup C_G} = 0$ for $t \geq k + 1$, where k is the length of the longest path in Q_1. The longest path in Q_1 is of length at most $n_1 + n_2 - 1$ and this number $n_1 + n_2 - 1$ is maximum when there is only one source, $(n_1 = 1)$ hence $k \geq n_2$.

Remark 5.3.2. *Using the argument in Theorem 4.5 of [33], we see that the period of* $(T(t)_l)_{t\geq 0}$ *equals the greatest common divisor of the lengths of the cycles composed of edges in* G *which are among the states* $n_1 + \cdots + n_{l-1} + 1, \ldots, n_1 + \cdots + n_l$.

Remark 5.3.3. *Using the argument in [33], Theorem 4.5 together with Lemma 5.2.4, to ascertain that the cycles in* G *correspond to cycles in* Q *in a one-to-one way, we see that the period of the semigroups* $(T_i(t))_{t\geq 0}$*, described in the previous chapter, is equal to the greatest common divisor of the cycle lengths of edges in* G *which are among the states* $n_1 + \cdots + n_{i-1} + 1, \ldots, n_1 + \cdots + n_i$*. In other words, those states corresponding to the subgraph whose adjacency matrix is* $\mathbb{B}_i$*, for* $i = k, \ldots, s$.

5.4 Different speeds

The results of the preceding sections in this chapter were obtained under the assumption that the speed of particles along every edge is the same ($c_j = 1$). Now we revisit Section 4.6 and give a graphical picture of the asymptotic behaviour of the flow problem. In Section 4.6, we showed that we can transfer the problem into a flow problem with same speed on all the vertices by expanding the network into a larger networks with more edges. We also showed that the abstract Cauchy problem on the larger network generates a C_0 semigroup $\mathcal{T}(t)$.

Consider the diagonal block $B_i, g \leq i \leq s$ in the matrix $\mathbb{B}$ shown above. Let Q_i be the digraph whose adjacency matrix is B_i. Since B_i is irreducible, Q_i must be strongly connected (by Lemma 2.2.14). Moreover, Q_i is an invariant strongly connected component of Q. That is, any path that originates at some vertex v of Q_i remains entirely in Q_i. Clearly then, there exists a subgraph G_i of G whose line graph is Q_i. Indeed, by definition, the set of edges of G_i corresponds to the set of vertices of Q_i and each edge in Q_i joins two vertices u', v' in Q_i, and thus there is a corresponding vertex in G_i which is the head of $\Phi_G^{-1}(u')$ and the tail of $\Phi_G^{-1}(v')$. Moreover, G_i is a strongly connected, invariant component of G. Indeed, if there was an outgoing edge of G_i with tail at $v \in V(G_i)$, then this vertex would also generate an outgoing edge in Q_i. That is, there would be an edge with tail in Q_i and head in $Q - Q_i$, meaning that Q_i would not be invariant, a contradiction.

We now state the following result.

Theorem 5.4.1. *Under the assumptions stated in (4.31), there is a decomposition*

$$X = X_g \oplus \cdots \oplus X_s \oplus Y_e \oplus Y_l$$

such that

1. *the spaces* $X_i, Y_e, Y_l, i = g, \ldots, s$ *are invariant under* $(T(t))_{t\geq 0}$*;*

2. $(T(t)|_{X_i})_{t\geq 0}$ *is periodic with period*

$$\tau_i = \frac{1}{N}\mathit{gcd}\left\{N\left(\frac{1}{c_{i_1}} + \cdots + \frac{1}{c_{i_k}}\right); e_{i_1}, \ldots, e_{i_k} \text{ form a cycle}\right\} \tag{5.12}$$

   for $i = g, \ldots, s$*;*

3. $(T(t)|_{Y_e})_{t\geq 0}$ *is exponentially stable of the type* $0 > \omega > max\{N \ln|\lambda|; \lambda \in Z\}$*; where* Z *is as defined in (5.5)*

4. $(T(t)|_{Y_l})_{t\geq 0}$ *is nilpotent and*

$$T(t)|_{Y_l} = 0; \quad t \geq \kappa, \tag{5.13}$$

   where Y_l *can be identified with* $E(G_1) \cup C_G$ *and*

$$\kappa = \max\left\{\frac{1}{c_{i_1}} + \cdots + \frac{1}{c_{i_k}}; e_{i_1}, \ldots, e_{i_k} \text{ form a path in } G_1 \cup C_G\right\}. \tag{5.14}$$

Proof. The transformation of the graph described in Section 4.6 does not change the number of cycles in the graph and does not affect the split between the acyclic part and the cyclic part of the graph. However, it increases the lengths of the cycles (and the length of any $u-v$ path, for any $u, v \in V(G)$) and it rescales time by N. Therefore, from representation (4.40), Theorem 4.5.5 holds with the same number of periodic semigroups. However, their periods and the time it takes to deplete the acyclic part of the graph changes. To make this precise, note that there is a one-to-one correspondence between cycles in Q_i and cycles in G_i, and this correspondence extends to the cycle lengths (see Remark 5.3.3). Then, using the argument of [33], Theorem 4.5, we see that if $e_{i_1}, \ldots, e_{i_k}$ form a cycle in G_i, then the length of the corresponding cycle in Q is given by

$$N\left(\frac{1}{c_{i_1}} + \cdots + \frac{1}{c_{i_k}}\right).$$

Thus, the period of the semigroups $(T_i(t)|_{X_i})_{t\geq 0}$ is the greatest common divisor of all such numbers for which $e_{i_1}, \ldots, e_{i_k}$ form cycles in G_i. Therefore, by (4.40), the period of $(T_i(t)|_{X_i})_{t\geq 0}$ is given by (5.12) for each $i = g, \ldots, s$. ■

Chapter 6

Conclusion

The original part of this thesis commences in Chapter 3. In this chapter, we considered a simple model of transport between states described by a system of ODEs. Such a model can be considered as a simplified model of transport on a network where exchange between states takes place instantaneously. The main objective of Chapter 3 is to study the asymptotic behaviour of these systems when the coefficient matrix is an ML matrix using the so called relative entropy function and the discrete Poincaré lemma. In particular, we extended the results of Perthame [48] to arbitrary irreducible ML matrices. In this regard, we showed that there is a norm in $\mathbb{R}^n$ for which the quadratic entropy function decays exponentially in the subspace orthogonal to the Perron eigenvector $\mathbf{N}$ if the matrix is irreducible. We also extended the results to reducible ML matrices.

Next, in Chapter 4, we provided a more general proof for the generation of positive semigroups on networks. In particular, we proved that there is no semigroup if the graph has a sink. In other words, the operator $(A_0, D(A_0))$ associated with the abstract Cauchy problem for the system of transport equations on the network generates a semigroup if every vertex in the graph has an outgoing edge. We then provided a representation theorem for the flow in the case where the speeds along the edges are linearly dependent over $\mathbb{Q}$.

Using the representation theorem we provided a more explicit description of the long term behaviour of the flow on reducible networks where every vertex has an outgoing edge and an incoming edge, thus improving the results in [33]. That is, we considered graphs of the form shown in Figure 6.1. We showed that for such graphs, the flow collects in those strongly

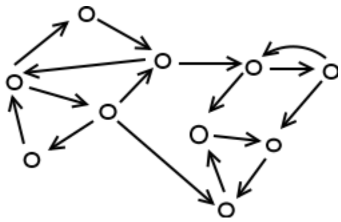


Figure 6.1: Connected graph with no sources

connected components with no outgoing flow. That is, sink components (every such strongly connected component must have at least two vertices!), and the period of the semigroup $T(t)$, restricted to these strong components, is given by the greatest common divisor of the cycle lengths in these components. Further, we showed that the flow in the strongly connected source components of the graph is depleted as $t \to \infty$.

We extended the above results to graphs with non-trivial acyclic parts, such as that shown in Figure 6.2. In particular, we proved that the flow in the acyclic part of the graph is depleted in finite time and that, if $c_j = 1$ for all $j = 1, \ldots, m$, this time of depletion does not exceed the length of the longest path in the acyclic part of the graph.

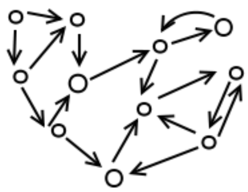


Figure 6.2: A graph with non-trivial acyclic part

6.1 Open Problems and Further Research

One of the open problems is to show that the approximation of solutions to the full transport equation on the network by a system of ordinary differential equations suggested in Remark 4.1.6 can be proved. We hope to show that if the length of each edge is taken to be arbitrarily small ($l_j \to 0$), then the solution to (4.11) (with no sinks) can be approximated by solutions of the system of ordinary differential equations in (4.9).

We showed that the abstract $(A_0, D(A_0))$ in the Cauchy problem in (4.12) associated with the transport equation on a network where some vertices have no outgoing edges does not generate a semigroup. The problem is that for a first order partial differential equation to have a unique solution, the boundary condition must be set at the start of the flow and not at the end, which is what condition (BC2) does in (4.11). In order to rectify this, we modify the boundary conditions at the sink. For example, if we introduce output functions at the sinks, and require this output to depend on the total inflow into that sink, then the boundary condition (BC2) in 4.11 will be given

$$h_i(\mathbf{u}(t)) = \xi_i(t) \sum_{i=1}^{m} \phi_{ik}^{+} \gamma_k c_k u_k(0,t)$$

if v_i is a sink, where h_i is the output at v_i. If, in addition we need the flow to be conservative, then it is possible to have a unique solution for the flow problem. However, our investigations have not yet been complete or exhaustive.

The study of flows on infinite networks has already been done by Dorn [14] where the graph was assumed to be strongly connected. However, strong connectedness of the graph depends on how one defines a path. For instance, in Chapter 2, we defined a strongly connected graph as one where for every pair of vertices u, v, there is a $u - v$ and $v - u$ path. For infinite graphs, there is a possibility that this path is infinite; that is, it contains an infinite number of vertices and edges. How one defines strong connectedness has an impact on the reducibility of the matrices $\mathbb{A}$ and $\mathbb{B}$. Most authors consider an infinite digraph to be strongly connected if the $u - v$ and $v - u$ paths are finite for every $u, v \in V(G)$. Using this definition, $\mathbb{A}$ and $\mathbb{B}$ and the corresponding semigroup $(T(t))_{t\geq 0}$ for the flow problem in (4.11) are irreducible if and only if G is strongly connected (see Proposition 36, [14] or [31]). We feel that there is still more that can be done regarding infinite networks.

To our knowledge, a systematic study of adjoint operators for first order differential operators on graphs have not yet been done on $L^1(\Omega)$, where Ω is a closed interval in $\mathbb{R}$. We note that studying these operators is a delicate process and their behaviour is intertwined with vertex conditions. We would like to know whether the results obtained in [48], Chapter 3, for the scalar transport equation

$$\begin{cases} \partial_t u(x,t) & = \partial_x u(x,t), \\ u(x,0) & = f(x), \end{cases}$$

still hold on a network.

Bibliography

[1] Abramovich, A A & Aliprantis, C D. An Invitation to Operator Theory. American Mathematical Society, USA (2002).

[2] Arendt, W, Batty, C J K, Heiber, M & Neubrander, F. Vector-valued Laplace Transforms and Cauchy Problems. Birkhauser Verlag Basel, Switzerland (2001).

[3] Arlotti, L & Banasiak, J. Perturbation of Positive Semigroups with Applications. Springer Monographs in Mathematics. Springer Verlag, London (2006).

[4] Banasiak, J & Namayanja, P. *Relative Entropy and Discrete Poincarê inequality for Reducible matrices*. App. Math. Lett, 25, 2193-2197. (2012).

[5] Banasiak, J. Modelling with Difference and Differential Equations. Cambridge University Press (2009).

[6] Bang-Jensen, J & Gutin, G. Digraphs: Theory, Algorithms and Applications, 2nd ed. Springer Verlag, London (2009).

[7] Bender, M A & Ron, D. *Testing Properties of Directed graphs: Acyclicity and Connectivity*. Journal of Random structures and Algorithms, vol 20, Issue 2, 184-205 (2002).

[8] Brezis, H. Functional Analysis, Sobolev Spaces and Partial Differential Equations. Springer, New York (2010).

[9] Casarino, V, Engel, K-J, Nagel, R & Nickel, G. *A semigroup Approach to Boundary Feedback Systems*. Journal of Integral Equations and Operator Theory, vol. 47, no. 3, 289-306 (2003).

[10] Chartrand, G & Lesniak, L. Graphs and Digraphs. Second Edition. Wadsworth, Belmont, California 94002 (1986).

[11] Cull, P, Flahive, M & Robson, R. Difference Equations: From Rabbits to Chaos. Springer Science and Business Media (2005).

[12] Deo, N. Graph Theory with Applications to Engineering and Computer Science. Prentice Hall, Englewood Cliffs, N.Y. (1974).

[13] Dolan, A & Aldous, J. Networks and Algorithms: An Introductory Approach. John Wiley & Sons Ltd, Chichester, England (1993).

[14] Dorn, B. Flows in infinite networks- a semigroup approach. Verlag Dr. Hut, München (2009).

[15] Dorn, B, Kramar Fijavž, M, Nagel, R & Radl, A. *The Semigroup Approach to Transport Processes in Networks*. Physica, D, 239, 1416-1421 (2010).

[16] Engel, K J & Nagel, R. One Parameter Semigroups for Linear Evolution Equations. Graduate Texts in Mathematics, vol 194. Springer-Verlag, New York (2000).

[17] Engel, K J & Nagel, R. A Short Course on Operator Semigroups. Springer Science and Business Media, LLC (2006).

[18] Engel, K J, Kramar Fijavž, M. K, Nagel, R & Sikolya, E. *Vertex Control of Flows in Networks*. Networks and Heterogeneous Media 3, 709-722 (2008).

[19] Freiling, G, Ignatiev, M & Yurko, V. *An Inverse Problem for Sturm-Liouville Operators with Singular Potentials on Star-type graphs* (1991).

[20] Ford, L R & Fulkerson, D R. Flows in Networks. Princeton University Press, Princeton (1962).

[21] Gantmacher, F R. Applications of the Theory of Matrices. Interscience Publishers, New York (1959).

[22] Greiner, G. *Perturbing the Boundary Conditions of a Generator*. Houston J. Math. 13, no. 2 (1987)

[23] Godsil, C & Royle, G. Algebraic Graph Theory. Graduate Texts in Mathematics. Springer-Verlag, New York (2001).

[24] Hante, F M; Leugering, G & Seidman, T I. *Modeling and Analysis of Modal Switching in Networked Transport Systems*. Appl. Math. Opt., vol 59, no. 2, 275-292 (2009).

[25] Higdon, R L. Initial Boundary Value Problems for Linear Hyperbolic Systems. SIAM (1986).

[26] Isakov, V. Inverse Problems for Partial Differential Equations. Applied Mathematical Sciences. Springer-Verlag, New York, vol 127, 2nd ed, (1998).

[27] Israel, A B & Greville, N E. Generalized Inverses: Theory and Applications. 2nd Ed. Springer-Verlag, New York (2003).

[28] Jameson, G J O. Topology and Normed Spaces. Chapman and Hall Ltd, London (1974).

[29] Joshi, M S & Wassermann A J. Lecture Notes on Partial Differential Equations (2000).

[30] Kato, T. Perturbation Theory for Linear Operators. Springer-Verlag, Berlin (1966).

[31] Keicher, V. *Almost Periodicity of Stochastic Operators on* $\ell^1(\mathbb{N})$. Tbilisi Mathematical Journal 1, 105-131 (2008).

[32] Keicher, V. *Convergence of Positive Co-Semigroups to Rotation Groups*. PhD Dissertation, Universität Tübingen (2008).

[33] Kramar, M and Sikolya, E. *Spectral Properties and Asymptotic Periodicity of Flows in Networks*. Math. Z. 249, 139-162 (2005).

[34] Kuchment, P. *Quantum Graphs: An Introduction and a Brief Survey*. Analysis on Graphs and its Applications. Proc. Symp. Pure Math. AMS, 291-314 (2008).

[35] Kuchment, P. *Quantum Graphs I: Some basic Structures*. Waves in Random Media 14, no.1, 107-128 (2004).

[36] Kuchment, P. *Quantum Graphs II: Some Spectral Properties of Quantum and Combinatorial graphs*. J. Phys. A: Math. Gen. 38, no. 22, 4887 (2005).

[37] Kurasov, P & Nowaczyk, M. *Geometric Properties of Quantum Graphs and Vertex Scattering Matrices*. Opuscula Mathematica, vol 30, no. 3, 295-309 (2010).

[38] Kurasov, P & Nowaczyk, M. *Inverse Spectral Problems for Quantum Graphs*. J. Phys. A: Math. Gen. 38, 4901 - 4915. (2005).

[39] Luenberger, D G. Introduction to Dynamic Systems: Theory, Models and Applications. John Wiley & Sons, USA (1979).

[40] Mátrai, T & Sikolya, E. *Asymptotic Behaviour of Flows in Networks*. Forum Math. 19, 429-461 (2007).

[41] McKibben, M A. Discovering Evolution Equations with Applications, Volume 1- Deterministic Equations. Chapman and Hall/CRC, Boca Raton, FL (2011).

[42] Mehmeti, A, von Below, J & Nicaise, S (Eds). Partial Differential Equations On Multistructures. CRC Press (2001).

[43] Mehmeti, A. Nonlinear Waves in Networks. Mathematical Research 80. Akademie Verlag, Berlin (1994).

[44] Meyer, C D. Matrix Analysis and Applied Linear Algebra. SIAM (2000).

[45] Nagel, R (Ed). One-parameter Semigroups of Positive Operators. Springer-Verlag, Berlin (1986).

[46] van Neerven, J. The Asymptotic Behaviour of Semigroups of Linear Operators. Birkhäuser Verlag, Basel (1996).

[47] Nicaise, S. *Some Results on Spectral Theory over Networks, Applied to Nerve Transmission*. Lecture Notes in Mathematics. Springer, vol. 1171, 532-541 (1985).

[48] Perthame, B. Transport Equations in Biology. Birkhäuser Verlag, Basel, Switzerland (2007).

[49] Robinson, J C. Infinite Dimensional Dynamical Systems: An Introduction to Dissipative Parabolic PDEs and the Theory of Global Attractors. Cambridge University Press (2001).

[50] Royden, H L. Real Analysis. Third Edition. Macmillan Publishing Company, New York (1988).

[51] Seneta, E. *Nonnegative Matrices and Markov Chains*. Springer Science+Business Media (2006).

[52] Sikolya, E. *Semigroups for Flows in Networks*. PhD Dissertation, Universität Tübingen (2004).

[53] Schaeffer, H H. Banach Lattices and Positive Operators. Springer-Verlag, Berlin (1974)

[54] van Steen, M. Graph Theory and Complex Networks: An Introduction. Self published (2010). Also available as http://www.di.unipi.it/ ricci/book-watermarked.pdf.

[55] von Below, J. *A Characteristic Equation associated with an Eigenvalue Problem on C^2 Networks*. Linear Algebra and Applications, 71, 309-325 (1985).

[56] von Below, J. *Can One hear the Shape of a Network?* Lecture Notes in Pure and Applied Mathematics 219, Marcel Dekker, New York, 19-36 (2001).

[57] von Below, J. *Kirchoff laws and Diffusion on Networks*. Linear Algebra and Applications. 121, 692-697 (1989).

[58] von Below, J. *Sturm-Liouville Eigenvalue Problems on Networks*. Math. Meth. Appl. Sc 10, 383-395 (1988).