Singularity and Symmetry Analysis of Differential Sequences

ADHIR MAHARAJ

Singularity and Symmetry Analysis of Differential Sequences

ADHIR MAHARAJ BSc (Hons), MSc

Submitted in fulfilment of the academic requirements for the degree of Doctor of Philosophy in the School of Mathematical Sciences, University of KwaZulu-Natal, Private Bag X54001, Durban 4000.

As the candidate's supervisor I have approved this thesis for submission.

Signed:................................

Name:..................................

Date:..................................

Abstract

We introduce the notion of differential sequences generated by generators of sequences. We discuss the Riccati sequence in terms of symmetry analysis, singularity analysis and identification of the complete symmetry group for each member of the sequence. We provide their invariants and first integrals. We propose a generalisation of the Riccati sequence and investigate its properties in terms of singularity analysis. We find that the coefficients of the leading-order terms and the resonances obey certain structural rules. We also demonstrate the uniqueness of the Riccati sequence up to an equivalence class.

We discuss the properties of the differential sequence based upon the equation $ww'' - 2w'^2 = 0$ in terms of symmetry and singularity analyses. The alternate sequence is also discussed. When we analyse the generalised equation $ww'' - (1-c)w'^2 = 0$, we find that the symmetry properties of the generalised sequence are the same as for the original sequence and that the singularity properties are similar. Finally we discuss the Emden-Fowler sequence in terms of its singularity and symmetry properties.

Declaration 1- Plagiarism

I, ADHIR MAHARAJ declare that

1. The research in this thesis, except where otherwise indicated, is my original research.

2. This thesis has not been submitted for any degree or examination at any other university.

3. This thesis does not contain other persons' data, pictures, graphs or other information unless specifically acknowledged as being obtained from other persons.

4. This thesis does not contain other persons' writing unless specifically acknowledged as being obtained from other researchers. Where other written sources have been quoted, then:

Their words have been rewritten, but the general information attributed to them has been cited.

Where their exact words have been used, then their writing has been placed in italics and quotation marks and cited.

5. This thesis does not contain text, graphics or tables copied and pasted from the internet, unless specifically acknowledged, and the source being detailed in the thesis and in the bibliography.

Signed: ....
A Maharaj
September 2009

Declaration 2- Publications

The following articles either appear in journals or are in the preprint stage

1. Andriopoulos K, Leach PGL & Maharaj A (2009) On Differential Sequences *European Journal of Applied Mathematics* (submitted)

2. Leach PGL, Maharaj A & Andriopoulos K (2009) On the generalised Riccati sequence (preprint: School of Mathematical Sciences, University of KwaZulu-Natal, Private Bag X54001, Durban 4000, Republic of South Africa)

3. Maharaj A, Leach PGL & Andriopoulos K (2009) Differential sequences generated by first-order generators of sequences (preprint: School of Mathematical Sciences, University of KwaZulu-Natal, Private Bag X54001, Durban 4000, Republic of South Africa)

4. Maharaj A & Leach PGL (2007) The method of reduction of order and linearisation of the two-dimensional Ermakov system *Mathematical Methods in the Applied Sciences* **30** 2125-2145

5. Leach PGL, Maharaj A & Andriopoulos K (2009) Differential sequences: The Emden-Fowler sequence (preprint: School of Mathematical Sciences, University of KwaZulu-Natal, Private Bag X54001, Durban 4000, Republic of South Africa)

Signed: ....
A Maharaj

Dedication

I dedicate the work compiled in this thesis to my dearest mother, Mrs Saraswathee Maharaj, as a token of my appreciation for all the sacrifices she has made in her life so that I can achieve all the goals that I have set in my life. To my beloved wife, Pratiksha Maharaj, you have been my pillar of strength along this long and at times frustrating journey.

In loving memory of my grandmother
Mrs Chunder Dey Maharaj
10-11-1927 to 11-07-2009

Acknowledgements

I thank my supervisor, Professor PGL Leach, for being ever patient, considerate, understanding and extremely helpful during my graduate studies. He has done more than what is required from a supervisor and a close friend in compiling this thesis. In addition I thank him for introducing me to the interesting field of symmetries and differential equations. I am grateful to him for teaching me the necessary background in Lie theory and symmetries of differential equations and insisting that I produce original mathematics always with his guidance. I also thank Professor Leach for providing me an opportunity to visit Samos where part of this research was compiled.

I thank my co supervisor, Dr Marianna Euler, for her invaluable expertise on recursion operators in assisting me through my doctoral studies and also helping me make sense of the complicated calculations.

I also thank Professor Leach and Dr Euler for the thorough proof-reading of the drafts.

I thank Professor Norbert Euler for assisting me and providing me with the material to compile parts of Chapter Three. I also thank him for his ever willingness to help me to make sense of recursion operators and differential sequences. My gratitude extends to the Department of Mathematics, Luleå University of Technology, for its hospitality in December 2007 and 2008.
To my dear friend, Kostis Andriopoulos, I say thank you, for the hospitality and making my life easy when I was in Greece. Thank you for motivating me to continue with my studies when at times the journey became frustrating. Finally I thank all my friends and family who continuously provided encouragement.

Contents

List of Tables

Chapter 1

Introduction

1.1 Sophus Lie

The concept of a symmetry was formally introduced centuries ago in the sense that, given an object, the object would remain unaltered under a certain action of transformation (for example translation, dilation and reflection). The concept of a group is closely related to that of symmetry of mathematical objects (such as functions and differential equations) and more specifically a symmetry group of an object is defined to be the set of all invertible transformations (symmetries) that leave an object invariant. Galois theory, which was the final outcome of about three centuries of effort by mathematicians to present the solutions of algebraic equations explicitly by radicals, deals with the transformations of a finite group. On the other hand Lie theory deals with symmetries of differential equations using the original idea of Sophus Lie of infinitesimal transformations. Hence the knowledge of symmetry groups can be used to enhance our understanding of natural phenomena, which are mostly formulated in terms of differential equations [113].

While at school Lie decided to take up a military career, but his eyesight was not sufficiently good so he gave up the idea and entered University of Kristiania. There he attended lectures by Sylow in 1862 on Galois theory and by Carl Bjerkneson on Mathematics. Lie certainly had teachers of considerable quality, yet he graduated in 1865 without having shown any great ability in the subject, or any great liking for it. In 1869 Crelle's journal accepted for publication a paper that Lie had written in 1867, which proved vital for he was awarded a scholarship to travel and meet leading mathematicians. In Berlin Lie met Kronecker, Kummer, Weierstrass and Felix Klein, with the last of whom he became a close friend and long-term colleague. In Paris he met Darboux, Chasles and Jordan. While in Paris Lie discovered contact transformations. In 1871 he became an assistant in the University of Kristiania, in the same year submitting his work 'On Complexes, in particular, Line and Spherical Complexes, with Applications to the Theory of Partial Differential Equations' for his doctorate, which was duly awarded in July 1872. This work, essentially based on Plücker's theory of complexes and Monge's geometrical interpretation of partial differential equations, is a splendid blend of Lie's new ideas with results of his contemporaries in projective geometry, contact transformations and the theory of partial differential equations. In 1869-1870 Lie made a crucial discovery, namely that the majority of the older methods of integration of ordinary differential equations, which until then had seemed artificial and not intrinsically related to one another, could be unified by means of group theory. Hence Lie began examining partial differential equations, hoping that he could find a theory which was analogous to that of Galois theory of equations. Lie examined his contact transformations, considering how they affected a process due to Jacobi of generating further solutions of differential equations from a given one. This led to combining transformations in a way which Lie called an infinitesimal group, but which is not a group with our definition, rather what is today called a Lie algebra. During the Winter of 1873 Lie begun to develop systematically what became his theory of continuous transformation groups, later called Lie groups. Although Lie was creating highly innovative mathematics, he became increasingly sad at the lack of recognition he was receiving in the mathematical world. International recognition came late, in 1897, from geometers when he was awarded the first Lobachevskii prize to recognise distinguished works especially on noneuclidean geometry [113].

1.2 Invariance

The symmetry of art is an expression of invariance under some form of translation, rotation or reflection. In Science the concept of symmetry appeals for this leads to a characterisation of the system under consideration. The invariance associated with symmetry produces a simplification which makes understanding easier. To take a simple example the Kepler-Coulomb problem in reduced coordinates is described by the equation of motion

$$\ddot{\mathbf{r}} = -\frac{\mu\hat{\mathbf{r}}}{r^2}, \tag{1.2.1}$$

where μ is a constant, an overdot $(\cdot)$ denotes differentiation with respect to time and an overcaret $(\hat{})$ a unit vector. Equation (1.2.1) is obviously invariant under time translation and rotation. The associated invariants are energy and angular momentum. When the quantum mechanical analogue of (1.2.1) is examined, the spectrum of the energy operator is found to be more degenerate than the conservation of angular momentum would imply. This additional degeneracy was found by Foch [49] to be a consequence of the existence of another conserved quantity, the Laplace-Runge-Lenz vector [69, 112, 83][1][2]. Commonly this additional symmetry is termed an accidental degeneracy which is due to the dynamical symmetry expressed by the Laplace-Runge-Lenz vector. This usage is unfortunate. Equation (1.2.1) is invariant under the similarity transformation

$$\bar{r} = ra^2 \qquad \bar{t} = ta^3$$

where a is the parameter of the transformation, which is no more accidental nor dynamical than the time translation. Indeed to the practised eye all three symmetries are equally obvious.

1.3 Outline of Thesis

In the study and application of differential equations there are certain equations which are prominent by means of their utility of application or their ubiquity of occurrence. In the particular case of ordinary differential equations examples of members of this prominent class are the Riccati equation,

[1] The vector was discovered by Hermann (also Ermanno and Herman) [39] - in a particular case - in 1710 and more generally by Bernoulli [59] in 1711. In 1847 Hamilton reported a third conserved vector [57]. The three vectors, angular momentum, Laplace-Runge-Lenz and Hamilton's, constitute an orthogonal triad which forms a convenient frame of reference for the description of the classical orbits and the representation of the quantum states.

[2] A similar service was performed for the isotropic simple harmonic oscillator by Bargmann [16].

the linear second-order equation, the linear third-order equation of maximal symmetry, the Ermakov-Pinney equation, the Kummer-Schwarz equation and its generalisation. These equations are not independent. As Conte observed [31], the study of any of the Riccati equation, the linear second-order equation and the Kummer-Schwartz equation is equivalent to a study of the other two. More recently the closely connected linear third-order equation of maximal symmetry and the Ermakov-Pinney equation have been added to the list given by Conte. Since these equations are of different orders and have differing point symmetry properties, it is evident that the connections among them are nonlocal.

We are familiar with the use of sequences of numbers and functions in the mathematical and wider scientific literature. The last several decades have seen intensive study of hierarchies based upon nonlinear evolution equations which have their basis in the mathematical modelling of physical phenomena. The extension to ordinary differential equations has only been made recently [42]. We have chosen to term these related equations differential sequences to reflect the dual nature of a definition which combines the idea of an operation which generates the elements of the sequence and also that the elements of the sequence are composed of derivatives. It seems to us that the word sequence is more appropriate in this context than hierarchy and is more in keeping with the tradition of mathematical terminology.

Preliminary theory on symmetry and singularity analyses is presented in Chapters Two and Three. In Chapter Four we discuss some theory of recursion operators for nonlinear partial differential equations and ordinary differential equations. We introduce the concepts of 'generators of sequences' and 'differential sequences'. In the same chapter we illustrate the above theory with some simple examples.

We commence the applications in Chapter Five where, naturally, we begin with the Riccati equation. We obtain a generator of sequences of the Riccati equation and generate its sequence which we term the Riccati sequence. The Riccati sequence is analysed in terms of its symmetry and singularity properties and complete symmetry group.

In Chapter Six we analyse the sequence based upon the Generalised Riccati equation in terms of its singularity properties and conclude that the above mentioned sequence does not possess the Painlevé Property in general.

It is well known as mentioned earlier of the relationship between the Riccati equation and a second-order equation of maximal symmetry which motivated the work conducted in Chapter Seven. We begin with a second-order equation of maximal symmetry and obtain a generator of sequences for the equation. We analyse the resulting sequence and obtain some interesting properties.

Chapter 2

Lie Theory

2.1 Definitions

A group is one of the simplest as well as the most widely studied algebraic structure. We begin this section by defining the notion of a group and a transformation group.

Group: A group, G, is a set of elements with a law of composition, ϕ, between elements satisfying the following axioms [19, p 31]:

(i) Closure property: For any elements a and b of G, $\phi(a, b)$ is an element of G.

(ii) Associative property: For any elements a, b and c of G

$$\phi(a, \phi(b, c)) = \phi(\phi(a, b), c). \tag{2.1.1}$$

(iii) Identity element: There exists an unique element I of G such that for any element a of G

$$\phi(a, I) = \phi(I, a) = a. \tag{2.1.2}$$

(iv) Inverse element: For any element, a, of G there exists an unique inverse element a^{-1} in G such that

$$\phi(a, a^{-1}) = \phi(a^{-1}, a) = I. \tag{2.1.3}$$

Abelian group: A group G is Abelian if $\phi(a, b) = \phi(b, a)$ holds for all elements a and b in G.

Group of transformations: The set of transformations,

$$\bar{\mathbf{x}} = \mathbf{X}(\mathbf{x}; \varepsilon), \tag{2.1.4}$$

defined for each $\mathbf{x}$ in $D \subset R^n$ and depending upon the parameter ε lying in the set $S \subset R$, with $\phi(\varepsilon, \delta)$ defining a composition of parameters ε and δ in S, forms a **group of transformations** on D if

(i) For each parameter ε in S the transformations are one-to-one onto D.

(ii) S with the law of composition ϕ forms a group.

(iii) $\bar{\mathbf{x}} = \mathbf{x}$ when $\varepsilon = I$, *ie,*

$$\mathbf{X}(\mathbf{x}; I) = \mathbf{x}. \tag{2.1.5}$$

(iv) If $\bar{\mathbf{x}} = \mathbf{X}(\mathbf{x},\varepsilon)$ and $\bar{\bar{\mathbf{x}}} = \mathbf{X}(\bar{\mathbf{x}}; \delta)$, then

$$\bar{\bar{\mathbf{x}}} = \mathbf{X}(\mathbf{x}; \phi(\varepsilon; \delta)). \tag{2.1.6}$$

One-parameter Lie group of transformations: A one-parameter Lie group of transformations is a group of transformations which in addition to the above satisfies the following:

(i) ε is a continuous parameter, *ie,* S is an interval in R. (Without loss of generality $\varepsilon = 0$ corresponds to the identity element I.)

(ii) $\mathbf{X}$ is infinitely differentiable with respect to $\mathbf{x}$ in D and an analytic function of ε in S.

(iii) $\phi(\varepsilon, \delta)$ is an analytic function of ε and δ for $\varepsilon \in S$ and $\delta \in S$.

Subgroup: A subgroup of S is a group formed by a subset of elements of G with the same law of composition.

Special linear group: The complex general linear group, $GL(n, C)$, and the real general linear group, $GL(n, R)$, consist of all nonsingular complex and real $n \times n$ matrices respectively [17]. (The latter may be considered as a subgroup of the former.) The complex special linear group, $SL(n, C)$, is the subgroup of $GL(n, C)$ consisting of matrices with determinant one. The real special linear group, $SL(n, R)$, is the intersection of these two subgroups, *ie,*

$$SL(n, R) = SL(n, C) \cap GL(n, R). \tag{2.1.7}$$

Rotation group: The rotation group, *SO(n, R)*, is the special or proper real orthogonal group given by the intersection of the group of orthogonal matrices[1], *O(n, R)*, and the complex special linear group, *ie,*

$$SO(n, R) = O(n, R) \cap SL(n, C). \tag{2.1.8}$$

Lie algebra[2]: A Lie algebra, $\mathcal{L}$, is a vector space together with a product $[x, y]_{LB}$[3] that:
(i) is BILINEAR (*ie,* linear in x and y separately),
(ii) is ANTICOMMUTATIVE (skew-symmetric):

$$[x, y]_{LB} = -[y, x]_{LB}, \tag{2.1.9}$$

(iii) and satisfies the JACOBI IDENTITY

$$[x, [y, z]_{LB}]_{LB} + [y, [z, x]_{LB}]_{LB} + [z, [x, y]_{LB}]_{LB} = 0 \tag{2.1.10}$$

for all vectors x, y, and z in the Lie algebra.

A familiar Lie algebra is the real three-dimensional vector space with the vector cross product as multiplication.

Abelian algebra: A Lie algebra $\mathcal{L}$ is called Abelian (equivalently commutative) if $[x, y]_{LB} = 0 \; \forall \; x, y \in \mathcal{L}$ [105].

Solvable algebra: A Lie algebra, $\mathcal{L}$, is called solvable if the derived series,

$$\mathcal{L} \supseteq \mathcal{L}' = [\mathcal{L}, \mathcal{L}] \tag{2.1.11}$$
$$\supseteq \mathcal{L}'' = [\mathcal{L}', \mathcal{L}'] \tag{2.1.12}$$
$$\supseteq \ldots \tag{2.1.13}$$
$$\supseteq \mathcal{L}^{(k)} = [\mathcal{L}^{(k-1)}, \mathcal{L}^{(k-1)}], \tag{2.1.14}$$

[1]Recall that a matrix is orthogonal if its transpose is its inverse.

[2]The term 'Lie algebra' was coined by H Weyl in 1934. Previously mathematicians simply spoke of "infinitesimal transformations $X_1 f, ..., X_r f$" of the group which Lie and Engel often abbreviated to "the group $X_1 f..., X_r f$" [21].

[3]We call this the Lie Bracket.

terminates with a null ideal, *ie,* $\mathcal{L}^{(k)} = 0, k > 0$ [62]. Note: Any Abelian algebra is solvable and indeed any Lie algebra of dimension < 3 is solvable.

Remark: A Lie algebra is usually defined over the real and complex field. The Jacobi identity plays the same role for Lie algebras that the associative law plays for associative algebras. We define the product associated with the Lie algebras as that of the Lie Bracket, *ie,*

$$[X, Y]_{LB} = XY - YX. \tag{2.1.15}$$

If a differential equation admits the operators X and Y, it also admits their Lie Bracket, $[X, Y]_{LB}$. Lie's main result [116] is the proof that it is always possible to assign to a continuous group (Lie group) a corresponding Lie algebra and vice versa. Thus for the real special linear group, *SL(n, R)*, the corresponding Lie algebra is *sl(n, R)* and for *SO(n, R)*, *so(n, R)*.

2.2 The Lie Analysis

2.2.1 Infinitesimal Transformations: Elementary Considerations

We[4] let (x, y) denote the variables of a two-dimensional space and assume no relationship between x and y. An infinitesimal transformation in this space has the form[5]

$$\bar{x} = x + \varepsilon\xi(x, y) \qquad \bar{y} = y + \varepsilon\eta(x, y) \tag{2.2.1}$$

which can be regarded as generated by the differential operator[6]

$$\Gamma = \xi(x, y)\partial_x + \eta(x, y)\partial_y. \tag{2.2.2}$$

The operator Γ is called the generator of the transformation (2.2.1) since

$$\bar{x} = (1 + \varepsilon\Gamma)x \qquad \bar{y} = (1 + \varepsilon\Gamma)y. \tag{2.2.3}$$

[4]The development of this chapter may be found in [74].

[5]Since there is no implication of dependent and independent variables, (2.2.1) is the most general infinitesimal transformation possible in this instance.

[6]We note ∂_x is the operator $\partial/\partial x$.

In the case of a function $f(x,y)$ the application of the infinitesimal transformation may be written as

$$\begin{aligned} f(\bar{x},\bar{y}) &= f(x+\varepsilon\xi, y+\varepsilon\eta) \\ &= f(x,y)+\varepsilon\left\{\xi\frac{\partial f}{\partial x}+\eta\frac{\partial f}{\partial y}\right\} \\ &= (1+\varepsilon\Gamma)f(x,y), \end{aligned} \tag{2.2.4}$$

where as is usual terms of second or higher order in the infinitesimal parameters are ignored. Γ is a symmetry of $f(x,y)$ if $\Gamma f = 0$, *ie,*

$$\xi\frac{\partial f}{\partial x}+\eta\frac{\partial f}{\partial y}=0. \tag{2.2.5}$$

Equation (2.2.5) has two aspects. Given a function, f, (2.2.5) provides the necessary relationship between ξ and η for Γ to be a symmetry. Clearly there is an infinite number of symmetries for any function $f(x,y)$. Given a symmetry, Γ, with specific coefficient functions, ξ and η, (2.2.5) is a linear partial differential equation from which the class of functions possessing Γ as a symmetry can be found. Suppose

$$\Gamma = y\partial_x - x\partial_y. \tag{2.2.6}$$

Then (2.2.5) becomes

$$y\frac{\partial f}{\partial x}-x\frac{\partial f}{\partial y}=0 \tag{2.2.7}$$

the characteristic of which is found from the solution of the associated Lagrange's system

$$\frac{\mathrm{d}x}{y}=\frac{\mathrm{d}y}{-x} \tag{2.2.8}$$

and is

$$u = x^2+y^2. \tag{2.2.9}$$

Any function of the form $f(x^2+y^2)$ possesses (2.2.6) as a symmetry. Naturally (2.2.6) is just one representative of the class of symmetries of $f(x^2+y^2)$ for, from (2.2.5), we have

$$\xi x+\eta y = 0 \tag{2.2.10}$$

$$\Leftrightarrow \eta = -\frac{x}{y}\xi \tag{2.2.11}$$

so that, given an ξ, we have the required η. The concept is extended easily to any number of variables.

2.2.2 Infinitesimal Transformations: Dependent Variables

In the previous section the variables x and y were independent. Suppose now that x is the independent variable and y is the dependent variable. Under an infinitesimal transformation

$$\bar{x} = x + \varepsilon\xi \qquad \bar{y} = y + \varepsilon\eta \tag{2.2.12}$$

the first derivative transforms according to

$$\frac{\mathrm{d}\bar{y}}{\mathrm{d}\bar{x}} = \frac{\mathrm{d}(y+\varepsilon\eta)}{\mathrm{d}(x+\varepsilon\xi)} \tag{2.2.13}$$

$$= \frac{\dfrac{\mathrm{d}y}{\mathrm{d}x} + \varepsilon\dfrac{\mathrm{d}\eta}{\mathrm{d}x}}{1+\varepsilon\dfrac{\mathrm{d}\xi}{\mathrm{d}x}} \tag{2.2.14}$$

$$= \frac{\mathrm{d}y}{\mathrm{d}x} + \varepsilon\left(\frac{\mathrm{d}\eta}{\mathrm{d}x} - \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}\xi}{\mathrm{d}x}\right). \tag{2.2.15}$$

The differential operator producing the transformation in (2.2.15) is

$$\Gamma_1 = \zeta_1\frac{\partial}{\partial y'}, \tag{2.2.16}$$

where the prime denotes $\mathrm{d}/\mathrm{d}x$ and

$$\zeta_1 = \eta' - y'\xi'. \tag{2.2.17}$$

In a similar fashion we can derive the formula for the transformation of the second derivative. The formula is

$$\frac{\mathrm{d}^2\bar{y}}{\mathrm{d}\bar{x}^2} = \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \varepsilon\left(\frac{\mathrm{d}^2\eta}{\mathrm{d}x^2} - 2\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\frac{\mathrm{d}\xi}{\mathrm{d}x} - \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}^2\xi}{\mathrm{d}x^2}\right) \tag{2.2.18}$$

which is generated by

$$\Gamma_2 = \zeta_2\partial_{y''}, \quad \zeta_2 = \eta'' - 2y''\xi' - y'\xi''. \tag{2.2.19}$$

In general the transformation in the nth derivative is generated by

$$\Gamma_n = \zeta_n\partial_{y^{(n)}}, \quad \zeta_n = \eta^{(n)} - \sum_{j=1}^{n}\binom{n}{j} y^{(n+1-j)}\xi^{(j)}. \tag{2.2.20}$$

In the case of a function, $f(x, y, y', \ldots, y^{(n)})$, the infinitesimal transformation is generated by $\Gamma + \Gamma_1 + \Gamma_2 + \ldots + \Gamma_n$ which we write as $\Gamma^{[n]}$, where [92]

$$\Gamma^{[n]} = \Gamma + \sum_{i=1}^{n} \left\{ \eta^{(i)} - \sum_{j=1}^{i} \begin{pmatrix} i \\ j \end{pmatrix} y^{(i+1-j)} \xi^{(j)} \right\} \partial_{y^{(i)}} \tag{2.2.21}$$

is called the nth extension of Γ.

2.2.3 Symmetries of Differential Equations

We now present a brief introduction to this method of analysing differential equations. For a more detailed approach we refer the reader to [19, 91, 104]. While the subsequent development encompasses only ordinary differential equations, the results apply *mutatis mutandis* to partial differential equations. An ordinary differential equation,

$$F(x, y, y', ..., y^{(n)}) = 0, \tag{2.2.22}$$

admits the one-parameter Lie group of (point) transformations[7]

$$\bar{x} = x + \varepsilon \xi(x, y) \qquad \bar{y} = y + \varepsilon \eta(x, y) \tag{2.2.23}$$

with infinitesimal generator

$$\Gamma = \xi(x, y)\, \partial_x + \eta(x, y)\, \partial_y \tag{2.2.24}$$

if and only if

$$\Gamma^{[n]}\, F(x, y, y', ..., y^{(n)})|_{F\,=\,0} = 0, \tag{2.2.25}$$

where

$$\Gamma^{[n]} = \Gamma^{[n-1]} + \eta^{[n]} \frac{\partial}{\partial y^{(n)}}, \qquad \Gamma^{[0]} = \Gamma, \tag{2.2.26}$$

$$\eta^{[n]} = \frac{d\eta^{[n-1]}}{dx} - y^{(n)} \frac{d\xi}{dx} \tag{2.2.27}$$

[7] We are only concerned here with the study of classical point symmetries. Different types of symmetries, *eg* hidden [4, 5, 6], potential [19, p 301], contact [94] and generalized [8] symmetries, have also been studied. Many, such as hidden and potential, may be interpreted as just variations of point symmetries that appear under different circumstances.

or equivalently

$$\eta^{[n]} = \eta^{(n)} - \sum_{j=0}^{k-1} \left(\begin{array}{c} k \\ j \end{array} \right) y^{(j+1)}\, \xi^{(k-1)} \tag{2.2.28}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x} = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + ... + y^{(n)}\frac{\partial}{\partial y^{(n-1)}}. \tag{2.2.29}$$

Γ, defined by (2.2.24), is called a symmetry of (2.2.22) and (2.2.25) the nth extension[8] or prolongation of Γ. We say that F is invariant under the nth extension of Γ if (2.2.25) holds. The symmetry is usually written in its unextended form for compactness.

For a second-order ordinary differential equation,

$$f(x, y, y', y'') = 0, \tag{2.2.30}$$

we use the second extension (or prolongation) of Γ, *videlicet*

$$\Gamma^{[2]} = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + (\eta' - y'\xi')\frac{\partial}{\partial y'} + (\eta'' - 2y''\xi' - y'\xi'')\frac{\partial}{\partial y''}, \tag{2.2.31}$$

and require that

$$\xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} + (\eta' - y'\xi')\frac{\partial f}{\partial y'} + (\eta'' - 2y''\xi' - y'\xi'')\frac{\partial f}{\partial y''} = 0. \tag{2.2.32}$$

Given ξ and η we can find $f = F(u, v, w)$, where u, v and w are the three characteristics determined from the associated Lagrange's system

$$\frac{\mathrm{d}x}{\xi} = \frac{\mathrm{d}y}{\eta} = \frac{\mathrm{d}y'}{\eta' - y'\xi'} = \frac{\mathrm{d}y''}{\eta'' - 2y''\xi' - y'\xi''}. \tag{2.2.33}$$

Using the first two terms we obtain $u(x, y)$, the zeroth-order invariant, the first three $v(x, y, y')$, the first-order invariant, and all four $w(x, y, y', y'')$, the second-order invariant. However, if f is given, (2.2.32) is an equation for ξ and η and we rewrite it as

$$\begin{aligned} &\xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} + \left(\frac{\partial \eta}{\partial x} + y'\frac{\partial \eta}{\partial y} - y'\frac{\partial \xi}{\partial x} - y'^2\frac{\partial \xi}{\partial y}\right)\frac{\partial f}{\partial y'} \\ &+ \left(\frac{\partial^2 \eta}{\partial x^2} + 2y'\frac{\partial^2 \eta}{\partial x \partial y} + y'^2\frac{\partial^2 \eta}{\partial y^2} + y''\frac{\partial \eta}{\partial y} - 2y''\frac{\partial \xi}{\partial x} - 2y'y''\frac{\partial \xi}{\partial y} - y'\frac{\partial^2 \xi}{\partial x^2}\right. \\ &\qquad \left. -2y'^2\frac{\partial^2 \xi}{\partial x \partial y} - y'^3\frac{\partial^2 \xi}{\partial y^2} - y'y''\frac{\partial \xi}{\partial y}\right)\frac{\partial f}{\partial y''} = 0. \end{aligned} \tag{2.2.34}$$

[8]We need to extend the symmetry to take care of the higher derivatives in the equation.

Since η and ξ are functions of x and y only, we use (2.2.30) to eliminate y'', collect terms by powers of y' and set each coefficient separately equal to zero. The resulting equations form an overdetermined system of partial differential equations which are solved to obtain the explicit forms of ξ and η and thereby Γ. In the solution of (2.2.34) the coefficient functions, ξ and η, contain a number of arbitrary constants of integration say n. Corresponding to each of the constants of integration, or n linearly independent combinations of them, there is a generator of a symmetry, Γ_i, $i = 1, n$. The Lie Brackets of the Γ_i form a Lie algebra.

It is evident that the calculation of ξ and η involves tedious manipulation which is best left to a computer. We have made use of the packages **Program LIE** [58] and SYM [35, 36] to reduce the effort in and increase the accuracy of the calculations.

2.2.4 Types of symmetries

Point symmetry: The coefficient functions depend only upon the independent variable, x, and the dependent variable, y, *ie,*

$$\Gamma = \xi(x,y)\partial_x + \eta(x,y)\partial_y.$$

Contact symmetry: A transformation is contact if ξ and η depend upon x, y and y' in such a way that ζ_1 is independent of y''. Precisely this means

$$\frac{\partial \eta}{\partial y'} = y' \frac{\partial \xi}{\partial y'}.$$

Generalised symmetry: The coefficient functions of a generalised symmetry depend upon derivatives of all admissible orders, *ie,*

$$\Gamma = \xi(x, y', ...)\partial_x + \eta(x, y', ...)\partial_y.$$

The highest order of derivative allowed is limited by the order of the differential equation.

Nonlocal symmetries: The coefficient functions depend upon integrals, in which the integrands are functions of x, y, y', y''... .

2.2.5 Using Symmetries to Solve Differential Equations

Symmetry algebras calculated using Lie's infinitesimal method have numerous applications. These include integration of ordinary differential equations, group invariant solutions of partial differential equations, conservation laws, bifurcation theory etc. We illustrate the usefulness of the knowledge of symmetries in analysing differential equations.

Point transformations

We can relate differential equations of similar order with the same symmetry algebras by a point transformation. The knowledge of the solution of just one equation of a particular class is required to determine the solutions of the remaining equations as they are also related via point transformations. This is illustrated by second-order ordinary differential equations which have the maximal symmetry algebra *sl(3,R)* (with eight symmetries).

Consider the nonlinear second-order ordinary differential equation

$$y'' + 3yy' + y^3 = 0, \tag{2.2.35}$$

which arises in the study of the modified Emden equation and often in practical and impractical applications, and the simple equation

$$Y'' = 0, \tag{2.2.36}$$

both of which have the eight-element Lie algebra, *sl(3,R)* [90]. The solution of (2.2.36) is

$$Y \;=\; A + BX, \tag{2.2.37}$$

while that of (2.2.35) is not exactly obvious. However, one can transform (2.2.35) to (2.2.36). We seek the transformation from (2.2.35) to (2.2.36) which casts $\Gamma_2 = y\partial_x - y^3\partial_y$ into canonical form. Γ_2 assumes canonical form provided

$$\begin{aligned} \xi\frac{\partial Y}{\partial x} + \eta\frac{\partial Y}{\partial y} &= 0 \\ \xi\frac{\partial X}{\partial x} + \eta\frac{\partial X}{\partial y} &= 1, \end{aligned} \tag{2.2.38}$$

where $\xi = y$ and $\eta = -y^3$.

By applying the method of characteristics for first-order partial differential equations to (2.2.38) we obtain

$$\begin{aligned} \frac{dx}{y} &= \frac{dy}{-y^3} \\ \frac{dx}{y} &= \frac{dy}{-y^3} = \frac{dX}{1} \end{aligned} \tag{2.2.39}$$

for which these solutions [90] are

$$Y = x - \frac{1}{y} \qquad X = -\frac{x^2}{2} + \frac{x}{y}. \tag{2.2.40}$$

Under the transformation (2.2.40), equation (2.2.35) takes the form in (2.2.36). Hence we may apply the transformation (2.2.40) to (2.2.37) to obtain the solution to (2.2.35) which is

$$x - \frac{1}{y} = A + B\left(-\tfrac{1}{2}x^2 + \frac{x}{y}\right),$$

ie

$$y = \frac{Bx + 1}{\frac{1}{2}Bx^2 + x - A}. \tag{2.2.41}$$

Reduction of order

Consider the differential equation

$$E(x, y, ..., y^{(n)}) = 0 \tag{2.2.42}$$

which possesses a symmetry of the general form

$$\Gamma = \xi(x, y)\partial_x + \eta(x, y)\partial_y. \tag{2.2.43}$$

By using the zeroth-order and first-order differential invariants we can obtain an equation of order $(n - 1)$. The zeroth-order and first-order differential invariants are given by the two characteristics associated with $\Gamma^{[1]}$, *ie,*

$$\Gamma^{[1]} = \xi(x, y)\partial_x + \eta(x, y)\partial_y + (\eta' - y'\xi')\partial_{y'},$$

the associated Lagrange's system of which is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dy'}{\eta' - y'\xi'}.$$

In the case of a point symmetry[9] the first two terms give the zeroth-order invariant, $u = f(x, y)$, and together with the third term give the first-order invariant, $v = g(x, y, y')$. All higher derivatives can be expressed in terms of u, v and x such that (2.2.42) reduces to

$$\bar{E}(u, v, ..., v^{(n-1)}) = 0, \qquad (2.2.44)$$

which is an equation of order $(n-1)$. That means we have reduced the order of the original equation by one. If the reduced equation has a symmetry, then the process can be repeated until the solution of the original equation is reduced to quadratures.

Note that, if equation (2.2.42) has two symmetries Γ_1 and Γ_2 with Lie Bracket $[\Gamma_1, \Gamma_2]_{LB} = \lambda\Gamma_1$, where λ is an arbitrary constant, reduction of order by Γ_1 results in the descendant of Γ_2 being a point symmetry of the reduced equation (2.2.44). If a constant λ is not zero, reduction by Γ_2 results in the loss of a descendant of Γ_1 as a point symmetry of the reduced equation [86]. Consequently a point symmetry is not available for a further reduction of order. However, the point symmetry does become an exponential nonlocal symmetry and such a symmetry can be used in reduction of order [52]. If an equation has more than one symmetry, the choice of symmetry for reduction is of great importance.

To summarize, given (2.2.42) with a sufficient number of suitable symmetries, the integration of the equation is achieved by a process of successive reduction of order using the characteristics (invariants) of the symmetries in turn until a first-order ordinary differential equation is obtained. This is integrated to obtain a solution. The next level is obtained by the solution of another first-order ordinary differential equation until the original variables are regained. With each first-order equation solved a constant of integration is obtained so that, after n first-order equations are solved, the solution has n

[9] For a contact symmetry, that is $\xi = \xi(x, y, y')$ and $\eta = \eta(x, y, y')$, both invariants would be expected to be of first-order and the ease of computation for the point symmetries is lost.

constants of integration and represents the general solution of the nth-order differential equation.

The general nth-order ordinary differential equation

$$y^{(n)} + g(x, y, ..., y^{(n-1)}) = 0 \tag{2.2.45}$$

possesses the two symmetries

$$\Gamma_1 = \partial_x \quad \text{and} \quad \Gamma_2 = -qx\partial_x + y\partial_y, \tag{2.2.46}$$

where q is a constant. Since Γ_1 is a symmetry of (2.2.45), g is independent of x. The nth extension of Γ_2 is

$$\Gamma_2^{[n]} = -qx\partial_x + \sum_{i=0}^{i=n}(iq+1)y^{(i)}\partial_{y^{(i)}} \tag{2.2.47}$$

and the invariance of (2.2.45), with g now independent of x, under its action leads to the linear partial differential equation

$$[nq+1]\, y^{(n)} + [(n-1)q+1]\, y^{(n-1)}\frac{\partial g}{\partial_{y^{(n-1)}}} + ... + y\frac{\partial g}{\partial y} = 0 \tag{2.2.48}$$

and its associated Lagrange's system

$$\frac{dg}{(nq+1)g} = \frac{dy^{(n-1)}}{[(n-1)q+1]\, y^{(n-1)}} = ... = \frac{dy}{y}. \tag{2.2.49}$$

Characteristics are obtained by combining the first n terms in turn with the $(n+1)$th and are [74]

$$\xi_i = \frac{y^{(i)}}{y^{iq+1}}, \quad i = 1, n-1, \quad \text{and} \quad \gamma_n = \frac{g}{y^{nq+1}} \tag{2.2.50}$$

so that the general nth-order ordinary differential equation invariant under both Γ_1 and Γ_2 may be written as

$$y^{(n)} + y^{nq+1} f\left(\frac{y^{(n-1)}}{y^{(n-1)q+1}}, ..., \frac{\dot{y}}{y^{q+1}}\right) = 0, \tag{2.2.51}$$

where f is an arbitrary function of its $n-1$ arguments and the overdot indicates differentiation with respect to time. In particular we may write the representative second-order ordinary differential equation as

$$\ddot{y} + y^{2q+1} f(\xi) = 0, \tag{2.2.52}$$

where we have introduced the specialised notations $\xi = \xi_1$ and $\eta = \xi_2$, respectively.
The variables ξ and η are the natural variables for the reduction of order of (2.2.52) using the two symmetries. Thus (2.2.52) becomes the algebraic equation

$$\eta + f(\xi) = 0. \tag{2.2.53}$$

More conventionally one would compute $\dot{\xi}$ as

$$\frac{d\xi}{dx} = y^q \left[\eta - (q+1)\xi^2\right]$$

and using (2.2.53) one would obtain

$$\begin{aligned} f(\xi) + (q+1)\xi^2 + \frac{1}{y^q}\frac{d\xi}{dx} &= 0 \\ \int \frac{\xi d\xi}{f(\xi + (q+1)\xi^2)} + \log y &= constant. \end{aligned}$$

Further progress in the determination of the solutions depends upon the explicit form which f takes.

2.2.6 Complete Symmetry Groups

In 1994 Krause [66, 67] introduced the concept of a complete symmetry group of an ordinary differential equation as being the group associated with the set of symmetries, be they point, contact, generalised or nonlocal, required to specify the equation or system of equations completely. A complete symmetry group realisation in mechanics must possess the following two features [66]:

(a) the group acts freely and transitively on the manifold of all allowed motions of the system;

(b) the given equations of motion are the only ordinary differential equations that remain invariant under the specified action of the group.

Hence every mechanical system may be characterised by the symmetry rules it obeys, *ie,* different mechanical systems cannot have the same symmetry properties. If the systems do have the same symmetry properties, then they must have the same mechanical nature.

The strategy used to find the complete symmetry group of a given nth-order differential equation is that of taking the general nth-order equation, applying the appropriate extension of the symmetries that the equation possesses in turn until one recovers the nth-order differential equation. This is only possible if the given equation or system of equations possesses a sufficient number of point symmetries so as to be completely specified by them. Hence there are three basic categories. The first is the over-symmetric problem for which there exists a sufficient number of point symmetries to find the complete symmetry group. Secondly there is the problem of which one is almost there to find the complete symmetry group. This means that there is an insufficient number of point symmetries so that one needs to find nonlocal symmetries in order to obtain the complete symmetry group. Finally there is the problem were one can do nothing due to the lack of explicit expressions for the symmetries.

To illustrate the procedure in terms of simple calculations we calculate the complete symmetry group of the Kummer-Schwarz equation,

$$2yy''' - 3y''^2 = 0, \tag{2.2.54}$$

which has the point symmetries

$$\begin{aligned} \Gamma_1 &= \partial_x \quad \Gamma_4 = \partial_u \\ \Gamma_2 &= x\partial_x \quad \Gamma_5 = u\partial_u \\ \Gamma_3 &= x^2\partial_x \quad \Gamma_6 = u^2\partial_u. \end{aligned}$$

We consider the general third-order ordinary differential equation

$$y''' = f(x, y, y', y''). \tag{2.2.55}$$

The first symmetry, $\Gamma_1 = \partial_x$, results in (2.2.55) being independent of x, *ie,*

$$y''' = g(y, y', y''). \tag{2.2.56}$$

By the action of

$$\Gamma_2^{[3]} = x\partial_x + 0\partial_y - y'\partial_{y'} - 2y''\partial_{y''} - 3y'''\partial_{y'''} \tag{2.2.57}$$

on (2.2.56) we have

$$-3y''' = -y'\frac{\partial g}{\partial y'} - 2y''\frac{\partial g}{\partial y''}, \tag{2.2.58}$$

which has the associated Lagrange's system

$$\frac{dy'}{y'} = \frac{dy''}{2y''} = \frac{dg}{3g}$$

with characteristics

$$u = \frac{y''}{y'^2}, \quad v = \frac{g}{y'^3}.$$

Hence we have

$$g = y''' = y'^3 h\left(y, \frac{y''}{y'^2}\right). \tag{2.2.59}$$

The action of the third extension of $\Gamma_4 = \partial_y$ on (2.2.59) results in (2.2.59) being independent of y, *ie,*

$$y''' = y'^3 j\left(\frac{y''}{y'^2}\right). \tag{2.2.60}$$

The action of

$$\Gamma_5^{[3]} = 0\partial_x + y\partial_y + y'\partial_{y'} + y''\partial_{y''} + y'''\partial_{y'''} \tag{2.2.61}$$

on (2.2.60) gives

$$\begin{aligned}
y''' &= 3y'^3 j + y'^3\frac{dj}{du}\left(\frac{y''}{y'^2} - \frac{2y''}{y'^3}y'\right)\\
y'^3 j &= 3y'^3 j - y'y''\frac{dj}{du}\\
2j &= u\frac{dj}{du},
\end{aligned}$$

where $u = y''/y'^2$ and the associated Lagrange's system of which is

$$\frac{du}{u} = \frac{dj}{2j}.$$

Hence we have

$$\begin{aligned} k &= u^{-2}j \\ j &= \frac{y''}{y'^2}k. \end{aligned}$$

By substituting $j = \dfrac{y''}{y'^2}k$ into (2.2.60), we may write (2.2.55) as

$$y'y''' = ky''^2, \tag{2.2.62}$$

which is the generalised Kummer-Schwarz equation. We obtain the Kummer-Schwarz equation when we apply the third extension of Γ_3, *ie,*

$$\Gamma_3^{[3]} = x^2\partial_x + 0\partial_y - 2xy'\partial_{y'} - (4xy'' + 2y')\partial_{y''} - (6xy''' + 6y'')\partial_{y'''},$$

the action of which on (2.2.62) gives

$$\begin{aligned} -6xy'y''' - 6y'y'' - 2xy'y''' &= -2ky''(4xy'' + 2y') \\ -8xy'y''' - 6y'y'' &= -(8kxy''^2 + 4ky'y'') \\ k &= 3/2. \end{aligned}$$

The complete symmetry group of (2.2.54) has the algebra $A_1 \oplus sl(2,R)$ and is represented by the symmetries $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_5$.

2.3 Discussion

The application of symmetry groups to the study of differential equations is extended significantly by allowing the infinitesimals that characterise the admitted symmetries to also depend on the derivatives of the dependent variable up to some finite order. Such symmetries are known as generalised symmetries [74]. Lie point symmetries, contact symmetries and generalised symmetries are called local symmetries; they are characterised by infinitesimals of 'local' type. Symmetries that are characterised by infinitesimals that are not of local type are called nonlocal symmetries. Examples include symmetries in which infinitesimals depend on integrals of the dependent variable. In principle a differential equation can admit an infinite number of nonlocal symmetries. The problem, however, is the nonavailablity of a general systematic method for determining these symmetries.

When one can find them, the nonlocal symmetries admitted by a differential equation may provide information on the differential equation not obtainable via local symmetries. A class of nonlocal symmetries called potential symmetries can be computed using Lie's algorithm. Potential symmetries may be put to many uses such as the construction of noninvertible mappings of known solutions to new ones and construction of new invariant solutions *etc* .

In this Thesis we confine our discussion to the application of Lie analysis to scalar equations and sequences of scalar equations. However, Lie analysis may also be applied to systems of equations. For the sake of completeness we have applied Lie analysis to the two-dimensional Ermakov system, we have also presented the complete symmetry group for the system[10].

[10] See Appendix A for details.

Chapter 3

Singularity Analysis

3.1 Introduction

Differential equations arise in almost every area of scientific activity. For over three centuries mathematicians and scientists in general have sought various stratagems to solve them. Most differential equations do not have solutions in terms of known functions, despite many innovative approaches to invent new functions, and indeed are not even integrable even though their solutions exist. The precise meaning of the solution of a system of differential equations can be cast in several ways:

(i) The existence of a sufficient number of functionally independent explicit first integrals;

(ii) the existence of a set of explicit functions describing the variation of the dependent variables with the independent variables;

(iii) the existence of a sufficient number of Lie point symmetries which allows the reduction of a system of differential equations to a system of algebraic equations and

(iv) the possession of the Painlevé Property.

3.2 The ARS Algorithm

We have mentioned above that the precise meaning of the solution of a system of differential equations can be cast in several ways. A feature, central to each of these three equivalent prescriptions of integrability, is the existence of explicit functions, be they solutions, first integrals/invariants or the coefficient functions of Lie symmetries.

There is another approach to the question of integrability which is not concerned with the display of explicit functions but with the demonstration of a specific property. This is the existence of a Laurent series for each of the dependent variables. The series may not be summable to an explicit form, but does represent an analytic function in a punctured disc centred on the singular point about which the expansion is made. The radius of the disc may be infinite. The essential feature of this Laurent series is that it is an expansion about a particular type of movable critical point, a pole. Consequently the existence of these Laurent series is intimately concerned with the singularity analysis of differential equations initiated around the end of the nineteenth century by Kowalevski [64, 65], Painlevé, Gambier and Garnier [61] and continued since by many workers including Chazy [30], Bureau [22, 23, 24, 25], Cosgrove [32] and Cosgrove *et al* [33]. More recently Andriopoulos *et al* [14] demonstrated the existence of nongeneric positive and negative resonances which are found in the performance of the standard singularity analysis. These resonances can lead to the solution of the differential equation under study in an annulus defined by singularities additional to the one about which the analysis is performed.

The connection of this type of singular solution and the solution of partial differential equations by the method of the Inverse Scattering Transform was noticed by Ablowitz *et al* [1, 2, 3] who developed an algorithm, called the ARS algorithm, to test whether the solution of an ordinary differential equation was expressible in terms of a Laurent expansion. If this was the case, the ordinary differential equation was said to pass the Painlevé Test and was conjectured to be integrable. Under more precise conditions Conte [31] has shown when the equation is integrable[1]

[1] The test can also be, *mutatis mutandis*, applied to partial differential equations, but, for the purposes of the present exposition, we confine our attention to ordinary differential equations.

In a review of integrable systems Ramani *et al* [110] provided a listing of the process of the implementation of the ARS algorithm. In the positive sense of the algorithm the polelike nature of the movable singularity is identified, the incidence of arbitrary constants of integration is determined and the consistency of the proposed Laurent expansion with the differential equation established.

Failure of the algorithm at any one of these steps leads to rejection and the equation is deemed to be nonintegrable. We emphasise that this nonintegrability is at the level of a function, represented by a Laurent series, analytic in the complex plane with the exception of singularities which are movable poles.

We detect possible singular behaviour in the solution of a differential equation by means of the leading-order analysis of the equation. Consider the autonomous equation

$$y^{(n)} = E(x, y', y'', \ldots, y^{(n-1)}). \tag{3.2.1}$$

We set $y = \alpha\chi^p$, where $\chi = x - x_0$ and x_0 is the location of the supposed movable singularity, substitute this into (3.2.1) and look for two or more dominant terms. The detection of which terms are dominant is identical to the determination of which terms in an equation are self-similar, *ie* invariant under the action of the symmetry

$$\Gamma_2 = -qx\partial_x + y\partial_y. \tag{3.2.2}$$

If (3.2.1) is invariant under the action of (3.2.2), the exponent in the leading-order term is given by $-1/q$ and the constant α determined by the equation. If (3.2.1) is separately invariant under $x\partial_x$ and $y\partial_y$, *ie*, it possesses two homogeneity symmetries, the value of p is determined by the equation. Otherwise only the self-similar terms contribute to the value of α. The ARS algorithm requires that p be a negative integer. Were this not the case, the algorithm terminates[2]. Once the leading-order term of the solution is obtained, we seek the behaviour of the next-to-leading-order term. In the philosophy of the ARS algorithm we require that this, and all subsequent terms, be compatible with the Laurent series imposed by the analycity criterion. The Laurent

[2] There is an exception for which we see below.

series can be an increasing series of the form

$$y = \sum_{i=0}^{\infty} a_i \chi^{i-n}, \tag{3.2.3}$$

where the singularity at t_0 is a pole of order n, a decreasing series of the form

$$y = \sum_{i=0}^{\infty} a_i \chi^{-(i+n)}, \tag{3.2.4}$$

or a full series. The second and third possibilities were not recognised by Ramani *et al* [110], but are found in [81]. The former series has been called the Right Painlevé Series and the latter the Left Painlevé Series [48]. In terms of the ARS algorithm the next step is to locate the powers at which the arbitrary constants required to make the solution a general solution are introduced. One introduces a single term and insists that its coefficient remain arbitrary, *ie,* one puts

$$y = \alpha\chi^{-n} + \beta\chi^{-n+r}, \tag{3.2.5}$$

where r is call the resonance. The substitution is made into the dominant terms. The solution of $r = -1$ always occurs, the existence of which is related to the one arbitrary constant we have from the beginning of the analysis, namely the location of the x_0 of the singularity. If the other values of r are not integral, the ARS algorithm stops.

The final procedure of the ARS algorithm is to substitute the Laurent series until the resonance farthest from the singularity into the full equation to check that there are no inconsistencies. For a second-order equation with the two symmetries $\Gamma_1 = x\partial_x$ and $\Gamma_2 = -qx\partial_x + y\partial_y$, there can be no inconsistency.

3.3 Example of a scalar equation

Consider the equation

$$y'' + yy' + ky^3 \quad = \quad 0. \tag{3.3.1}$$

To determine the leading-order behaviour we perform the substitution

$$y \quad = \quad \alpha\chi^p, \quad \text{where} \quad \chi = x - x_0,$$

x_0 being the location of the putative movable singularity. Thus equation (3.3.1) takes the form

$$\alpha p(p-1)\chi^{p-2} + \alpha^2 p\chi^{2p-1} + k\alpha^3\chi^{3p} = 0 \qquad (3.3.2)$$

and it is obvious that all terms balance at $p = -1$. This means that all terms are dominant, something that is expected since (3.3.1) possesses the self-similarity symmetry $G = -x\partial_x + y\partial_y$. For this specific value of p one can calculate α using equation (3.3.2), *videlicet*

$$k\alpha^2 - \alpha + 2 = 0. \qquad (3.3.3)$$

Equation (3.3.3) not only specifies the values of α but in this instance introduces the existence of two Laurent series of the form

$$\begin{aligned} y_1 &= \alpha_1\chi^{-1} + ... \\ y_2 &= \alpha_2\chi^{-1} + ..., \end{aligned}$$

where α_1 and α_2 are the two roots of (3.3.3). After the leading-order behaviour has been successfully calculated, the next step is to determine the powers at which the arbitrary constants enter the solution. In other words one takes the series one step further by introducing a term of the form $\mu\chi^{r-1}$ with the aim to calculate r. Making the substitution,

$$y = \alpha\chi^{-1} + \mu\chi^{r-1},$$

into (3.3.1), we obtain the equation

$$\begin{aligned} 2\alpha\chi^{-3} + \mu(r-1)(r-2)\chi^{r-3} + (\alpha\chi^{-1} + \mu\chi^{r-1})(-\alpha\chi^{-2} + \mu(r-1)\chi^{r-2}) \\ k(\alpha\chi^{-1} + \mu\chi^{r-1})^3 = 0. \end{aligned}$$

Separating by powers of χ and specifically for χ^{r-3} , or equally terms linear in μ, we obtain

$$\begin{aligned} (r-1)(r-2) + \alpha(r-1) - \alpha + 3k\alpha^2 &= 0 \\ r^2(\alpha-3)r + (\alpha-4) &= 0 \\ r = -1 \qquad \text{or} \qquad r = 4-\alpha. \end{aligned}$$

In order to have a solution in terms of analytic functions all powers of χ have to be integral. Hence

$$4-\alpha \;=\; n \quad \Leftrightarrow \quad k=\frac{2-n}{(4-n)^2}, \quad \forall n\in \mathcal{Z}-\{4\}\,.$$

Case I: $k=0 \Leftrightarrow n=2,\ \alpha=2,\ r=2$

Equation (3.3.1) now takes the form

$$y''+yy' \;=\; 0. \tag{3.3.4}$$

What remains is the determination of the constants entering the series. We make the substitution of the series, *videlicet*

$$y \;=\; \sum_{i=0}^{\infty}\alpha_i\chi^{i-1}$$

into (3.3.4) to obtain

$$(i-1)(i-2)\alpha_i\chi^{i-3}+(i-1)\alpha_i\alpha_j\chi^{i+j-3} \;=\; 0,$$

where the subscripts $i,j=0,\infty$. Collecting powers of χ we calculate the coefficients of the terms in the series, *ie,*

$$\begin{array}{lccr}
\chi^{-3}: & 2\alpha_0-\alpha_0^2=0 & \overset{\alpha_0\neq 0}{\Rightarrow} & \alpha_0=2\\
\chi^{-2}: & -\alpha_0\alpha_1=0 & \Rightarrow & \alpha_1=0\\
\chi^{-1}: & \alpha_2\alpha_0-\alpha_2\alpha_0=0 & \Rightarrow & \alpha_2 \text{ is arbitrary}\\
\chi^{0}: & 2\alpha_3+2\alpha_3\alpha_0+\alpha_2\alpha_1-\alpha_0\alpha_3=0 & \Rightarrow & \alpha_3=0\\
\chi^{1}: & 6\alpha_4+3\alpha_4\alpha_0+2\alpha_3\alpha_1+\alpha_2^2-\alpha_0\alpha_4 & \Rightarrow & \alpha_4=-\dfrac{\alpha_2^2}{10}.
\end{array}$$

Hence we have constructed the general solution of the particular case of (3.3.1) when $k=0$, *ie,*

$$y \;=\; 2\chi^{-1}+\alpha_2\chi-\frac{1}{10}\alpha_2^2\chi^3+...,$$

or equally

$$y = c\left\{\frac{2}{c(x-x_0)} + c(x-x_0) - \frac{1}{10}(c(x-x_0))^3 + ...\right\}.$$

Case II: $k = 1/9 \Leftrightarrow n = 1, \alpha = 3$ and $\alpha = 6$ and $r = 1$ and $r = -2$ respectively

In this case we have the interesting situation in which there exists two Laurent series, oftentimes called the Right Painlevé Series (RPS) and Left Painlevé Series (LPS), because they ascend and descend respectively. They obviously are valid in different domains. The general solution is found to be

$$y = \left\{\begin{array}{c} 3\chi^{-1} + \mu_1\chi^0 + ...(RPS) \\ 6\chi^{-1} + \mu_2\chi^{-3} + ...(LPS) \end{array}\right\}.$$

For the RPS we substitute

$$y_1 = \sum_{i=1}^{\infty} \alpha_i \chi^{i-1}$$

into (3.3.1) with $k = 1/9$ and obtain

$$(i-1)(i-2)\alpha_i\chi^{i-3} + (i-1)\alpha_i\alpha_j\chi^{i+j-3} + \frac{1}{9}\alpha_i\alpha_j\alpha_k\chi^{i+j+k-3} = 0,$$

where $i, j, k = 0, \infty$. Collecting powers of χ we calculate the coefficients of the terms in the series, *ie,*

$$\begin{array}{lccr} \chi^{-3}: & 2\alpha_0 - \alpha_0^2 + \frac{1}{9}\alpha_0^3 = 0 & \overset{\alpha_0\neq 0}{\Rightarrow} & \alpha_0 = 3 \text{ or } \alpha_0 = 6 \\ \chi^{-2}: & -\alpha_0\alpha_1 + \frac{1}{3}\alpha_1\alpha_0^2 = 0 & \Rightarrow & \alpha_1 \text{ is arbitrary or } \alpha_1 = 0 \\ \chi^{-1}: & \frac{1}{3}\alpha_2\alpha_0^2 + \frac{1}{3}\alpha_1^2\alpha_0 = 0 & \Rightarrow & \alpha_2 = -\frac{1}{3}\alpha_1^2 \text{ or } \alpha_2 = 0 \\ \chi^{0}: & 2\alpha_3 + \alpha_3\alpha_0 + \alpha_2\alpha_1 + \frac{1}{3}\alpha_3\alpha_0^2 & & \\ & \frac{2}{3}\alpha_2\alpha_1\alpha_0 + \frac{1}{9}\alpha_1^3 = 0 & \Rightarrow & \alpha_3 = \frac{1}{10}\alpha_1^3 \text{ or } \alpha_3 = 0. \end{array}$$

On the other hand for the LPS we substitute

$$y_2 = \sum_{i=1}^{\infty} \alpha_i \chi^{-i-1}$$

into equation (3.3.1) with $k = 1$ and obtain

$$(i+1)(i+2)\alpha_i\chi^{-i-3} - (i+1)\alpha_i\alpha_j\chi^{-i-j-3} + \frac{1}{9}\alpha_i\alpha_j\alpha_k\chi^{-i-j-k-3} = 0,$$

where $i, j = 0, \infty$. Collecting powers of χ we calculate the coefficients of the terms in the series, *ie,*

$$\begin{array}{llcr}
\chi^{-3}: & 2\alpha_0 - \alpha_0^2 + \frac{1}{9}\alpha_0^3 = 0 & \overset{\alpha_0\neq 0}{\Rightarrow} & \alpha_0 = 6 \text{ or } \alpha_0 = 3 \\
\chi^{-4}: & 6\alpha_1 - 3\alpha_1\alpha_0 + \frac{1}{3}\alpha_1\alpha_0^2 = 0 & \Rightarrow & \alpha_1 \text{ is arbitrary} \\
\chi^{-5}: & 12\alpha_2 - 4\alpha_2\alpha_0 - 2\alpha_1^2 & & \\
 & +\frac{1}{9}(3\alpha_2\alpha_0^2 + 3\alpha_1^2\alpha_0) = 0 & \Rightarrow & \alpha_2 \text{ is arbitrary} \\
\chi^{-6}: & 20\alpha_3 - 5\alpha_3\alpha_0 - 5\alpha_2\alpha_1 & & \\
 & +\frac{1}{9}(3\alpha_3\alpha_0^2 + 6\alpha_2\alpha_1\alpha_0 + \alpha_1^3) = 0 & \Rightarrow & \alpha_3 = \frac{1}{2}(\alpha_1\alpha_2 - \frac{1}{9}\alpha_1^3) \\
\chi^{-7}: & 30\alpha_4 - 6\alpha_4\alpha_0 - 6\alpha_3\alpha_1 - 3\alpha_2^2 & & \\
 & +\frac{1}{9}(3\alpha_4\alpha_0^2 + 6\alpha_3\alpha_1\alpha_0 + 3\alpha_2\alpha_1^2 & & \\
 & +3\alpha_2^2\alpha_0) = 0 & \Rightarrow & \alpha_4 = \frac{1}{6}\{\alpha_1^2(\frac{2}{3}\alpha_2 - \frac{2}{9}\alpha_1^2) + \alpha_2^2\}.
\end{array}$$

For $\alpha_0 = 3$ all other coefficients are zero since this gives the RPS.

The general solution is given by

$$y = \left\{ \begin{array}{c} 3\chi^{-1} + \alpha_{11} - \frac{1}{3}\alpha_{11}^2\chi + \frac{1}{10}\alpha_{11}^3\chi^2 + ...(RPS) \\ 6\chi^{-1} + \alpha_{21}\chi^{-2} + \alpha_{22}\chi^{-3} + \frac{1}{2}\alpha_{21}(\alpha_{22} - \frac{1}{9}\alpha_{21}^2)\chi^{-4} \\ +\frac{1}{6}\{\alpha_{11}^2(\frac{2}{3}\alpha_{22} - \frac{1}{9}\alpha_{11}^2) + \alpha_{22}^2\}\chi^{-5} + ...(LPS) \end{array} \right\}.$$

or, written in a slightly neater form (and by setting $\alpha_{21} = 0$, which is equivalent to putting the singularity at the origin)

$$y = \left\{ \begin{array}{l} \alpha_{11}\left\{\dfrac{3}{\alpha_{11}(x-x_0)} + 1 - \dfrac{1}{3}\alpha_{11}(x-x_0) + \dfrac{1}{10}(\alpha_{11}(x-x_0))^2 + ...(RPS)\right\} \\ \frac{6}{x-x_0}\left(1 + \dfrac{\alpha_{22}}{6(x-x_0)^2} + \dfrac{\alpha_{22}^2}{36(x-x_0)^4}...\right)(LPS) \end{array} \right\}.$$

3.4 Discussion

We recognise that the subject of singularity analysis of ordinary differential equations provides for different philosophies. These different philosophies are also connected to the understanding of what integrability is. There are those who require that the solution of an ordinary differential equation be an analytic function. As soon as the ARS algorithm throws up a nonintegral power, the algorithm is terminated. A concession, implied by the qualifier 'weak', is to require that the solution be analytic in part of the complex plane, *ie,* rational powers, either in the singularity or in the series expansion, be admitted. In stating that an equation passes the weak Painlevé Test a certain amount of common sense must be used to interpret the utility of the result. The rational exponent involved cannot be permitted to have too large a denominator or else the complex plane is divided into unworkable parts, particularly in the neighbourhood of the singularity. In a realistic scheme of things, where in the end solutions must be computed, there is no numerical difference between an irrational exponent and a rational exponent with a large denominator [53].

In the initial development of the singularity analysis the solution of a differential equation was considered as a Laurent expansion convergent in a punctured disc centered upon the movable singularity, and the more recently introduced Left Painlevé Series [48] which is an asymptotic solution and may be regarded as a Laurent expansion outside a disc centred on the movable singularity [53] and the full series which may be regarded as a Laurent expansion in an annulus about the singularity [14].

Chapter 4

Methodologies for Generating Differential Sequences

4.1 Introduction

It is known that one can construct integrable partial differential equations (or systems of partial differential equations) by the use of a so-called recursion operator, $R[u]$, which generates an infinite number of Lie-Bäcklund symmetries [103, 46, 47]. These equations are usually described as being symmetry integrable. The main problem is to find the recursion operator for a given system or to show that an infinite number of Lie-Bäcklund symmetries exists or that it does not exist (the latter being the more demanding task). Since the recursion operator can in general contain nonlocal variables even for equations linearisable by a (nonlocal) transformation [107], the procedure is not an easy one especially for higher-order equations and for systems.

Consider a general partial differential equation of order n described by

$$F(x, t, y, y_t, y_x, ..., y_{nx}) = 0, \tag{4.1.1}$$

where $y_x = \dfrac{\partial y}{\partial x}$, $y_{xx} = \dfrac{\partial^2 y}{\partial x^2}$..., $y_{nx} = \dfrac{\partial^n y}{\partial x^n}$.

Equation (4.1.1) admits a Lie-Bäcklund symmetry,

$$Z = \eta(x, t, y, y_x, ..., y_{sx})\partial_y, \tag{4.1.2}$$

if and only if the invariance condition [43]

$$Z^{(n)}F = 0 \tag{4.1.3}$$

is satisfied under the condition $F = 0$ and its differential consequences. A Lie-Bäcklund symmetry is said to be nontrivial if $s > n$. Note that condition (4.1.3) is a linear partial differential equation in the variable η, *ie,*

$$Z^{(n)}F = L[y]\eta = 0 \tag{4.1.4}$$

in which $L[y]$ is the linear operator

$$L[y] = \frac{\partial F}{\partial y} + \frac{\partial F}{\partial y_t}D_t + \frac{\partial F}{\partial y_x}D_x + ... + \frac{\partial F}{\partial y_{nx}}D_x^n, \tag{4.1.5}$$

where D_x and D_t are total derivatives with respect to x and t, respectively. By definition the linearised equation for (4.1.1) is given by

$$L[y]v = 0, \tag{4.1.6}$$

which provides a necessary condition for the infinitesimal transformation

$$\varphi = y + \epsilon v \tag{4.1.7}$$

to satisfy (leave invariant) (4.1.1). Therefore

$$v = \eta(x, t, y, y_x, ..., y_{sx}) \tag{4.1.8}$$

is a special solution of the linearised equation, (4.1.6), for every Lie-Bäcklund symmetry of (4.1.1).

Definition: [43] *An operator-valued function $R[y]$ (differential operator, integrodifferential operator or nonlocal operator) is called a recursion operator for the equation (4.1.1) if it generates an infinite set of distinct nontrivial Lie-Bäcklund symmetries as follows:*

$$\eta_{i+1} = R[y]\eta_i \equiv R^i[y]\eta_1 \quad \forall\, i \in N, \tag{4.1.9}$$

where $L[y]\eta = 0$ for every η generated.

A recursion operator can be realised by the use of the linearised equation (4.1.6) which is associated with the (in general) nonlinear equation (4.1.1).

This leads to the determining condition for a recursion operator, namely [19, p 288]

$$L[y]R[y]v = 0, \tag{4.1.10}$$

for any y that satisfies (4.1.1) and any v that satisfies the associated linearised equation $L[y]v = 0$ [19]. It can be shown that condition (4.1.10) is equivalent to the commutator condition

$$[L[y], R[y]]_{LB}\,\varphi = 0, \tag{4.1.11}$$

where y and φ are any solutions to (4.1.1).

We take the Lie bracket of (4.1.11), *ie,*

$$[L[y]R[y] - R[y]L[y]]\,\varphi = 0 \tag{4.1.12}$$

where $\varphi = y + \epsilon v$ and using $L[y]v = 0$ we have (4.1.10) to be equivalent to (4.1.11).

For autonomous evolution equations we have

$$y_t = E(y, y_x, y_{xx}, ..., y_{nx}) \tag{4.1.13}$$

and it is convenient to introduce the linear operator,

$$\begin{aligned} L &= L[y] - \frac{\partial E}{\partial y_t}D_t \\ &= \frac{\partial E}{\partial y} + \frac{\partial E}{\partial y_x}D_x + \frac{\partial E}{\partial y_{xx}}D_x^2 + ... - \frac{\partial E}{\partial y_t}D_t, \end{aligned} \tag{4.1.14}$$

where $L[y]v = 0$ is given by (4.1.5). The determining condition for a recursion operator of an evolution equation (4.1.1) takes the form

$$[L[y], R[y]]_{LB}\,\varphi = (D_t R[y])\varphi, \tag{4.1.15}$$

where y and φ satisfy (4.1.1). We may define the recursion operator as

$$R[y] = \sum_{j=0}^{l} G_j(x, y, y_x, ...)D_x^j + \sum_{j=1}^{s} Q_j(x, y, y_x, ...)D_x^{(-1)} \circ J(x, y, y_x, ...).$$

We further observe that the recursion operator of an evolution equation generates an hierarchy of higher-order integrable evolution equations as follows

$$y_t = R[y]^i E(y, y_x, y_{xx}, ..., y_{nx}), \quad i \in N. \tag{4.1.16}$$

Every equation in this hierarchy can be considered integrable when every equation admits an infinite number of Lie-Bäcklund symmetries. This happens when the recursion operator is hereditary, *ie,* the Lie symmetry algebra generated by $R[y]$ is Abelian [50].

4.2 Proper Differential Sequences

We consider a kth-order integrodifferential operator of the form

$$R^{[k]}[y] = G_k D^k + G_{k-1} D^{(k-1)} + ... + G_0 + JD^{-1} \circ Q \tag{4.2.1}$$

which operates on

$$F(y, y', y'', ..., y^{(n)}) = 0 \tag{4.2.2}$$

to generate a differential sequence of the following form,

$$\begin{aligned}
E_1 &:= F(y, y', y'', ..., y^{(n)}) = 0 && (4.2.3)\\
E_2 &:= R^{[k]}[y]F(y, y', y'', ..., y^{(n)}) = 0 && (4.2.4)\\
E_3 &:= (R^{[k]})^2[y]F(y, y', y'', ..., y^{(n)}) = 0 && (4.2.5)\\
E_4 &:= (R^{[k]})^3[y]F(y, y', y'', ..., y^{(n)}) = 0 &&\\
&\vdots &&\\
E_m &:= (R^{[k]})^{(m-1)}[y]F(y, y', y'', ..., y^{(n)}) = 0. && (4.2.6)
\end{aligned}$$

The adjoint of (4.2.1) is

$$(R^{[k]})^*[y] = \sum_{i=0}^{k} (-1)^i D^i \circ G_i - JD^{(-1)} \circ Q. \tag{4.2.7}$$

Euler *et al* [45] termed (4.2.3) the *seed equation* of the differential sequence (4.2.6) and they also note that (4.2.6) is of order $n + (m-1)k$.
Let

$$L[y] = \frac{\partial E_i}{\partial y} + \frac{\partial E_i}{\partial y'} D + \frac{\partial E_i}{\partial y''} D^2 + ... + \frac{\partial E_i}{\partial y^{(n)}} D^n \tag{4.2.8}$$

denote the linear operator and

$$L^*[y] = \frac{\partial E_i}{\partial y} - D \circ \frac{\partial E_i}{\partial y'} + D^2 \circ \frac{\partial E_i}{\partial y''} + ... + (-1)D^n \circ \frac{\partial E_i}{\partial y^{(n)}} \qquad (4.2.9)$$

denote the adjoint linear operator.

We define the symmetry generator for the sequence (4.2.6) as

$$Z^i(E_i) \quad = \quad Q(x, y, y', y'', ..., y^{(j)})\partial_y$$

under the condition

$$L_{[E_i]}Q_{|_{E_i = 0}} = 0.$$

4.2.1 Recursion Operators for Ordinary Differential Equations

The following definitions and propositions are summarised from Euler *et al* [45]

Definition 1: [45] *The sequence (4.2.6) admits a p-dimensional Lie symmetry algebra, Ω, spanned by the linearly independent symmetry generators,*

$$Z_1^i(E_i), Z_2^i(E_i), ..., Z_n^i(E_i),$$

if each equation in the sequence (4.2.6) admits a p-dimensional Lie symmetry algebra, Ω', isomorphic to Ω.

Definition 2: [45] $J = J(x, y, y', y'', ...)$ *is an integrating factor for the differential sequence (4.2.6) if J is an integrating factor for each equation in the sequence.*

Definition 3: [45] *The operator $R^{[k]}[y]$ of the form (4.2.1) is defined as a kth-order recursion operator of the differential sequence (4.2.6) under the following conditions:*

$$\left[L[y], R^{[k]}[y]\right]_{LB} \quad = \quad 0, \quad i = 1, 2, ..., m, \qquad (4.2.10)$$

$$\left(R^{[k]}\right)^* [y] J_k \quad = \quad \alpha J_l \quad \forall \quad k, l = 1, 2, ..., n, \qquad (4.2.11)$$

where α is a nonzero constant, $i = 1, 2, ..., m$, and p is the total number of integrating factors, J_l, valid for all members of the sequence. J_l may be zero for some values of l.

Example 1: We consider the equation

$$y'' + y'^2 + y' = 0 \tag{4.2.12}$$

with the linearised operator being

$$L[y] \;=\; D^2 + 2y'D + D. \tag{4.2.13}$$

We assume that (4.2.12) admits a generator of the form

$$R \;=\; D + A. \tag{4.2.14}$$

Using (4.1.11) to obtain the determining equations we have

$$\begin{aligned}
[L[y], R[y]]_{LB}\,\varphi &= \Big[D^2 + 2y'D + D, D + A\Big]_{LB}\,\varphi \\
&= \Big[D^2[D + A] + 2y'D[D + A] + D[D + A]\Big]\,\varphi \\
&\quad - \Big[D[D^2 + 2y'D + D] + A[D^2 + 2y'D + D]\Big]\,\varphi \\
&= [A'' + 2A'D + 2y'A' + A' - 2y''D]\,\varphi \\
&= A''\varphi + 2A'\varphi' + 2y'A'\varphi + A'\varphi - 2y''\varphi'. \qquad (4.2.15)
\end{aligned}$$

Hence we have the following determining equations

$$\varphi \;:\; A'' + 2y'A' + A' = 0 \tag{4.2.16}$$
$$\varphi' \;:\; 2A' - 2y'' = 0 \Rightarrow A = y' + constant \tag{4.2.17}$$

where (4.2.16) is just the differential consequence of (4.2.12) and (4.2.12) possesses the recursion operator $R = D + y'$ when we set $constant = 0$.

Definition 4: [45] *A proper differential sequence of ordinary differential equations is a differential sequence which admits at least one recursion operator of the form (4.2.1).*

Definition 5: [45] *An integrable differential sequence is defined as a proper differential sequence of ordinary differential equations for which each equation in the sequence is integrable.*

Proposition 1: [45] *J_s is an integrating factor for the sequence (4.2.6) if and only if the following conditions are satisfied:*

$$L^*_{E_i[y]} J_s(x, y, y_x, ...) \mid_{E_i=0} = 0, \quad i = 1, 2, ..., m, \tag{4.2.18}$$

$$\frac{\partial J_s}{\partial y_{(q-2r)x}} + \sum_{j=1}^{2r-1} (-1)^{j-1} \frac{\partial}{\partial y_{(q-2r)x}} \left[D^{j-1} \left(\frac{\partial f_i}{\partial y_{(j+q-2r)x}} J_s \right) \right]$$
$$+ \frac{\partial}{\partial y_{(q-1)x}} \left(D^{2r-1}_{E_i} \right) = 0, \quad s = 1, 2, ..., p, \quad r = 1, 2, ..., \left[\frac{q}{2} \right]. \tag{4.2.19}$$

Here $[\frac{p}{2}]$ is the largest natural number less than or equal to the number $\frac{q}{2}$, $i = 1, 2, ..., m$, and p is the total number of integrating factors, J_s, valid for all members of the sequence, ie, $s = 1, 2, ..., p$.

Proposition One is the natural extension of the derivation of the necessary and sufficient conditions for integrating factors of scalar ordinary differential equations which appears in [20].

4.2.2 An Alternate sequence

We construct a sequence,

$$\bar{E_1}, \bar{E_2}, ..., \bar{E_m}, \tag{4.2.20}$$

alternate to (4.2.6), where the order of the differential equations in the alternate sequence, (4.2.20), does not increase, but stays fixed by the seed equation, $\bar{E_1}$. If the two sequences share the same seed equation that is, $E_1 = \bar{E_1}$, then we may show that the two sequences (4.2.6) and (4.2.20) are either compatible or completely compatible by applying the following definitions and Proposition 2.

The following definitions and propositions are summarised from Euler *et al* [45]

Definition 6: [45] *Two equations, E_j and $\bar{E_j}$ from the sequences (4.2.6) and (4.2.20) respectively, are called compatible if the equations admit at least one common solution. The two equations are called completely compatible if the general solution of $\bar{E_j}$ gives the general solution for E_j. Two sequences of* *m equations, $E_1, E_2, ..., E_m$ and $\bar{E_1}, \bar{E_2}, ..., \bar{E_m}$ with the same seed equation $E_1 = \bar{E_1}$, are called compatible if each equation in the sequence admits at least one common solution between the corresponding members in the two sequences. The sequences are completely compatible if the general solution of $\bar{E_j}$ provides the general solution for E_j for all members of the sequence, ie for all $j = 1, 2, ...m$. The sequence (4.2.20) is termed an alternate sequence to (4.2.6) if the two sequences are at least compatible.*

Proposition 2: [45] *Consider a proper differential sequence, $E_1, E_2, ..., E_m$, with recursion operator $R^{[k]}[y]$. An alternate sequence, $\bar{E_1}, \bar{E_2}, ..., \bar{E_m}$, of the form*

$$\begin{aligned} \bar{E_1} &:= F(y, y', y'', ..., y^{(n)}) = 0 \qquad (4.2.21) \\ \bar{E_{j+1}} &:= F(y, y', y', ..., y^{(n)}) = Q_j(x, y, y', ..., \omega^1, \omega^2, ..., \omega^l; c_1, c_2, ..., c_s), \\ j &= 1, 2, ..., m-1, \end{aligned}$$

is compatible with the proper differential sequence, $E_1, E_2, ..., E_m$, with $E_1 = \bar{E_1}$ if

$$R^{[k]}Q_1 = 0 \qquad (4.2.22)$$

$$R^{[k]}Q_i = Q_{i-1}, \quad i = 1, 2, ...m. \qquad (4.2.23)$$

Here $\omega^1, \omega^2, ..., \omega^l$ are nonlocal coordinates defined by

$$\frac{d\omega^1}{dx} = g_1(y), \qquad (4.2.24)$$

$$\frac{d\omega^2}{dx} = g_2(\omega^1), \quad \frac{d\omega^3}{dx} = g_3(\omega^3) \quad , ..., \quad \frac{d\omega^l}{dx} = g_l(\omega^{l-1}) \qquad (4.2.25)$$

for some differential functions g_k.

Example 2: When $R[y]$ operates upon (4.2.12) it generates the following proper differential sequence with (4.2.12) being the seed equation,

$$\begin{aligned} E_1 &:= y'' + y'^2 + y' = 0 \qquad (4.2.26) \\ E_2 &:= y''' + 3y'y'' + y'' + y'^2 + y'^3 = 0 \qquad (4.2.27) \\ E_3 &:= y'''' + 3y'y''' + y''' + 3y''^2 + 3y'^2y'' + 6y'y''y'^3 + y'^4 = 0 \\ &\vdots \\ E_m &:= (R[y])^{m-1}(y'' + y'^2 + y') = 0. \qquad (4.2.28) \end{aligned}$$

We now apply Proposition Two in order to obtain the sequence alternate to (4.2.28) which has the same seed equation (4.2.12). To obtain Q_1 we have

$$\begin{aligned} R[y]Q_1 &= 0 \\ D(Q_1) + y'Q_1 &= 0 \Rightarrow Q_1(y, c_1) = c_1 e^{-y}, \end{aligned}$$

where c_1 is an arbitrary constant of integration. Hence the second member of the alternate sequence is

$$y'' + y'^2 + y' = c_1 e^{-y}. \tag{4.2.29}$$

To obtain the third member of the alternate sequence we repeat the above procedure as follows

$$\begin{aligned} R[y]Q_2 &= Q_1 \\ D(Q_2) + y'Q_2 &= Q_1, \end{aligned}$$

which possesses the general solution

$$Q_2(x, y, c_1, c_2) = e^{-y}(c_1 x + c_2), \tag{4.2.30}$$

where c_1 and c_2 are arbitrary constants of integration. Hence the third member of the alternate sequence is

$$y'' + y'^2 + y' = e^{-y}(c_1 x + c_2). \tag{4.2.31}$$

The fourth member of the sequence may be written as

$$\begin{aligned} y'' + y'^2 + y' &= e^{-y}\left(\tfrac{1}{2}c_1 x^2 + c_2 x + c_3\right) \qquad (4.2.32) \\ &= e^{-y} D^{-2} c_1. \end{aligned}$$

The alternate sequence to the proper differential sequence (4.2.28) may be written as

$$\begin{aligned} \bar{E}_1 &:= y'' + y'^2 + y' = 0 \\ \bar{E}_j &:= y'' + y'^2 + y' = e^{-y} D^{-(j-2)} c_1, \quad j = 2, 3, ..., m. \end{aligned}$$

The alternate sequence in expanded form is

$$\bar{E}_1 := y'' + y'^2 + y' = 0$$

$$
\begin{aligned}
\bar{E}_2 &:= y'' + y'^2 + y' = c_1 e^{-y} \\
\bar{E}_3 &:= y'' + y'^2 + y' = e^{-y}(c_1 x + c_2) \\
\bar{E}_4 &:= y'' + y'^2 + y' = e^{-y}\left(\frac{1}{2}c_1 x^2 + c_2 x + c_3\right) \\
&\vdots \\
\bar{E}_m &:= y'' + y'^2 + y' = e^{-y}\left(\sum_{j=1}^{m-1} \frac{c_j}{(m-j-1)!} x^{m-j-1}\right). \quad (4.2.33)
\end{aligned}
$$

We now check whether the two sequences, (4.2.28) and (4.2.33), are compatible or completely compatible with Definition Six in mind.

When we compare E_2 and $\bar{E}_2$, we note that a first integral for E_2 is given by $\bar{E}_2$ in the form

$$
\bar{E}_2 := c_1 = e^y(y'' + y'^2 + y'). \quad (4.2.34)
$$

We can conclude that the general solution of $\bar{E}_2$ provides the general solution of E_2, where c_1 is one of the constants of integration for E_2. Therefore according to Definition 6 $\bar{E}_2$ and E_2 are completely compatible.

We now compare $\bar{E}_3$ and E_3 and note that $\bar{E}_3$ is a second integral of E_3 in the form

$$
\bar{E}_3 := c_1 x + c_2 = e^y(y'' + y'^2 + y'). \quad (4.2.35)
$$

We may conclude that the general solution of $\bar{E}_3$ provides the general solution of E_3, where c_1 and c_2 are the two constants of integration for E_3. The two equations $\bar{E}_3$ and E_3 are completely compatible. A similar argument may be used for all members of the proper differential sequence (4.2.28). We may conclude that sequences (4.2.28) and (4.2.33) are completely compatible.

In general we may write

$$
\bar{E}_m := \left(\sum_{j=1}^{m-1} \frac{c_j}{(m-j-1)!} x^{m-j-1}\right) = e^y(y'' + y'^2 + y'). \quad (4.2.36)
$$

4.3 Generators of sequences

We consider a kth-order integrodifferential operator of the form

$$G^{[k]}[y] = A_k D^k + A_{k-1} D^{(k-1)} + ... + A_1 + A_0 D^{-1} \circ Q \tag{4.3.1}$$

which operates on

$$F(y, y', y'', ..., y^{(n)}) = 0 \tag{4.3.2}$$

to generate a differential sequence of the following form,

$$\begin{aligned}
E_1 &:= F(y, y', y'', ..., y^{(n)}) = 0 && (4.3.3)\\
E_2 &:= G^{[k]}[y] F(y, y', y'', ..., y^{(n)}) = 0 && (4.3.4)\\
E_3 &:= (G^{[k]})^2[y] F(y, y', y'', ..., y^{(n)}) = 0 && (4.3.5)\\
E_4 &:= (G^{[k]})^3[y] F(y, y', y'', ..., y^{(n)}) = 0 &&\\
&\vdots &&\\
E_m &:= (G^{[k]})^{(m-1)}[y] F(y, y', y'', ..., y^{(n)}) = 0. && (4.3.6)
\end{aligned}$$

We assume that (4.3.1) no longer satisfies the two conditions of Definition Three. Therefore to differentiate between a recursion operator and (4.3.1) we term (4.3.1) a generator of sequences. The adjoint of (4.3.1) is

$$(G^{[k]})^*[y] = \sum_{i=1}^{k} (-1)^i D^i \circ A_i - A_0 D^{(-1)} \circ Q \tag{4.3.7}$$

and the linearising operator for each member of (4.3.6) is

$$L_{[E_i]}[y] = \frac{\partial E_i}{\partial y} + \frac{\partial E_i}{\partial y'} D + \frac{\partial E_i}{\partial y''} D^2 + ... + \frac{\partial E_i}{\partial y^{(n)}} D^n \tag{4.3.8}$$

and the adjoint is

$$L^*_{[E_i]}[y] = \frac{\partial E_i}{\partial y} - D \circ \frac{\partial E_i}{\partial y'} + D^2 \circ \frac{\partial E_i}{\partial y''} + ... + (-1)^n D^n \circ \frac{\partial E_i}{\partial y^{(n)}} \tag{4.3.9}$$

Example 3: Consider the equation

$$u'' + u'^2 = 0 \tag{4.3.10}$$

with recursion operator

$$R \;=\; D + u'. \tag{4.3.11}$$

When we apply the transformation $y = u'$, (4.3.10) and (4.3.11) may be written as

$$y' + y^2 = 0, \quad \text{and} \quad G = D + y,$$

where we note that the first-order equation is of Riccati type. When we take the Lie Bracket, we have

$$\begin{aligned}
[L[y], R[y]]_{LB} &= [D + 2y, D + y]_{LB} \\
&= D^2 + y' + yD + 2yD + 2y^2 - D^2 - 2y' - 2yD - yD - 2y^2 \\
&= -y' \qquad (4.3.12)
\end{aligned}$$

which is clearly not zero and does not satisfy the first condition of Definition Three.
The resulting differential sequence is

$$\begin{aligned}
E_1 &:= y' + y^2 = 0 \\
E_2 &:= y'' + 3yy' + y^3 = 0 \\
E_3 &:= y''' + 4yy'' + 3y'^2 + 6y^2y' + y^4 = 0 \\
&\vdots \\
E_m &:= (D + y)^{(m-1)}(y' + y^2) = 0. \qquad (4.3.13)
\end{aligned}$$

By applying Proposition One we are able to generate the alternate sequence with the same seed equation. The sequence alternate to the differential sequence (4.3.13) has the following form

$$\begin{aligned}
\bar{E_1} &:= y' + y^2 = 0 \\
\bar{E_2} &:= y' + y^2 = c_1 \exp\left[-\int y dx\right] \\
\bar{E_3} &:= y' + y^2 = \exp\left[-\int y dx\right](c_1 x + c_2) \\
&\vdots \\
\bar{E_m} &:= y' + y^2 = \exp\left[-\int y dx\right]\left(\sum_{j=1}^{m-1} \frac{c_j}{(m-j-1)!} x^{m-j-1}\right) \qquad (4.3.14)
\end{aligned}$$

We now check whether the two sequences (4.3.13) and (4.3.14) are compatible or completely compatible with Definition Six in mind.

When we compare E_2 and $\bar{E}_2$, we note that a first integral for E_2 is given by $\bar{E}_2$ in the form

$$c_1 \quad = \quad \exp\left[\int y\mathrm{d}x\right](y' + y^2). \tag{4.3.15}$$

We can conclude that the general solution of $\bar{E}_2$ provides the general solution of E_2, where c_1 is one of the constants of integration for E_2. Therefore according to Definition Six, $\bar{E}_2$ and E_2 are completely compatible.

We now compare $\bar{E}_3$ and E_3 and note that $\bar{E}_3$ is a second integral of E_3 in the form

$$c_1 x + c_2 \quad = \quad \exp\left[\int y\mathrm{d}x\right](y' + y^2). \tag{4.3.16}$$

We may conclude that the general solution of $\bar{E}_3$ provides the general solution of E_3, where c_1 and c_2 are the two constants of integration for E_3. The two equations $\bar{E}_3$ and E_3 are completely compatible. A similar argument may be used for all members of the differential sequence (4.3.13).
In general we may write

$$\bar{E_m} \quad := \quad \left(\sum_{j=1}^{m-1} \frac{c_j}{(m-j-1)!} x^{m-j-1}\right) = \exp\left[\int y\mathrm{d}x\right](y' + y^2). \tag{4.3.17}$$

We may conclude that sequences (4.3.13) and (4.3.14) are completely compatible.

4.4 Discussion

We have shown that, even though the operator does not satisfy the two conditions of Definition Three, we are still able to generate an alternate sequence for the differential sequence (4.3.13). Moreover the two sequences are also completely compatible. In Chapter Five we note that the differential sequence (4.3.13) generated by $G = D + y$ possess interesting symmetry and singularity properties.

Chapter 5

Riccati sequence

5.1 Introduction

A considerable part[1] of the motivation[2] for this study is found in the works of Peterssen *et al* [107] and Euler *et al* [41] in which the authors obtain recursion operators for linearisable $1+1$ evolution equations. The specific equation of relevance to this work is the eighth of their classification, *videlicet*

$$u_t = u_{xx} + \lambda_8 u_x + h_8 u_x^2,$$

where λ_8 is an arbitrary constant and h_8 is an arbitrary function of the dependent variable u. The corresponding recursion operator is

$$\textbf{VIII} \qquad R_8[u] = D_x + h_8 u_x.$$

The usual area of application of recursion operators has been that of partial differential equations and in particular evolution equations of which the equation above is a sample. Euler *et al* [42] applied the idea of a recursion operator to ordinary differential equations. They did not present a treatment of all possible classes, but concentrated upon two representative equations, *videlicet* the Riccati equation and the Ermakov-Pinney equation.

[1] The essential content of this Chapter may be found in [15].

[2] The content of the work done by Peterssen *et al* [107] applied to partial differential equations which motivated us to investigate recursion operators/generators of sequences in the case of ordinary differential equations and more specifically their eight class of the classification since it may be transformed into the Riccati equation.

The generator of sequences which generates the Riccati sequence is

$$\mathrm{G} = D + y, \tag{5.1.1}$$

where D denotes total differentiation with respect to x. The generator of sequence essentially originates from Case VIII, $R_8[u]$, for partial differential equations when the dependence on t is removed, *ie* $u_t = 0$, and one writes $u_x = y$. We note that in applications to ordinary differential equations the notation usually used in the context of partial differential equations can be and is simplified.

When (5.1.1) acts upon[3] y (which can be denoted as G_0), it generates, in succession, all the members of the sequence which we denote by G_n, *videlicet*

$$\begin{aligned}
G_1 &:= y' + y^2 = 0 && (5.1.2)\\
G_2 &:= y'' + 3yy' + y^3 = 0 && (5.1.3)\\
G_3 &:= y''' + 4yy'' + 3y'^2 + 6y^2y' + y^4 = 0 && (5.1.4)\\
G_4 &:= y'''' + 5yy''' + 10y'y'' + 10y^2y'' + 15yy'^2 + 10y^3y' + y^5 = 0\\
&\vdots\\
G_n &:= (D + y)^n y = 0. && (5.1.5)
\end{aligned}$$

As an aid in our discussion we distinguish between an element of the sequence, denoted as indicated above by G_n, and its left side by writing the latter as $\tilde{G}_n$.

In addition to the formal definition given in (5.1.5) there is a differential recurrence relation.

Lemma: *The members of the Riccati sequence satisfy the differential recurrence relation*

$$\frac{\partial}{\partial y}\left(\tilde{G}_{m+1}\right) = (m + 2)\tilde{G}_m. \tag{5.1.6}$$

Proof: By inspection of G_1, G_2 and G_3 it is evident that (5.1.6) is true for the initial members of the sequence. We assume that

$$\frac{\partial}{\partial y}\left(\tilde{G}_{k+1}\right) = (k + 2)\tilde{G}_k \tag{5.1.7}$$

[3]This is a matter of notation. In this Thesis we have chosen the simplest. In another context, especially when one wishes to be 'physically' correct, one would choose $\exp[\int y \mathrm{d}x]$ since, if y is replaced by $-x$, one obtains the generating function for the solution of the time-independent Schrödinger equation for the simple harmonic oscillator [42].

for some $k \geq 2$. Then

$$\begin{aligned}
\frac{\partial}{\partial y}\left(\tilde{G}_{k+2}\right) &= \frac{\partial}{\partial y}\left[(D+y)\tilde{G}_{k+1}\right] \\
&= \left[(D+y)\partial_y + 1\right]\tilde{G}_{k+1} \\
&= (D+y)(k+2)\tilde{G}_k + \tilde{G}_{k+1} \\
&= (k+3)\tilde{G}_{k+1}
\end{aligned}$$

as required.
Remark: This recurrence relation is a little intriguing bearing in mind the connection to the generating function for the Hermite polynomials. One recalls the differential recurrence relation for the Hermite polynomials, *videlicet*

$$\frac{\mathrm{d}H_n(x)}{\mathrm{d}x} = 2nH_{n-1}(x),$$

and notes the similarity of structure to that given in (5.1.6).

The Riccati sequence contains as its two lower members the Riccati equation and the Painlevé-Ince equation. The Riccati equation [111] has an history now of almost three centuries and is one of the few examples known for which there exists a nonlinear superposition principle. Its close relationship to the linear second-order differential equation and the Kummer-Schwartz equation has already been noted [31]. The Painlevé-Ince equation [106, 61] is a notable example of a nonlinear second-order differential equation of maximal symmetry possessing both Left and Right Painlevé Series [48] and arises in a remarkable number of applications[4].

In this Chapter we report the notable properties of the members of the Riccati sequence. We concentrate upon the number of Lie point symmetries, singularity properties, first integrals, explicit integrability and complete symmetry groups. In Section 5.2 we present the symmetry analysis of the members of the sequence. In terms of the explicit integrability of the members of the sequence the lack of point symmetry in the higher-order members is remarkable and indicates the necessity for considering nonlocal symmetries. We note that this was the case in the treatment of a pair of equations of Ermakov-Pinney

[4]For a listing see [90, 48, 29, 14]. The authors of the third paper place (5.1.3) in a broader class of equations which they describe as being of modified Emden type.

type [78]. In Section 5.3 we elaborate upon the results of the singularity analysis of the lower members of the Riccati sequence presented by Euler *et al* [42] by considering the pattern of the values of the resonances for the general member of the sequence. Section 5.4 is devoted to a consideration of the invariants and first integrals of the members of the sequence. A short discussion on the symmetries of the integrals is presented. In Section 5.5 we present the complete symmetry group of the general member of the sequence after firstly considering the results for G_4 to give a concrete basis for the theoretical discussion. In Section 5.6 we recall the solution of G_n and extend our discussion to an equation containing combinations of the G_i. In the case of the second-order equation so formed the properties have been known for a long time [106, 61, 63]. Among our concluding remarks in Section 5.6 we summarise the remarkable properties found for the Riccati sequence which is based upon a rather elementary generator of sequence. We indicate that the route to complexification presented in Section 5.6 broadens the class of differential equations which may be subsumed into a more general concept of a Riccati sequence.

5.2 Symmetry Analysis

As a first-order ordinary differential equation (5.1.2) possesses an infinite number of Lie point symmetries. Equation (5.1.3) was examined in [90] for its Lie point symmetries which were found to be the following eight, here written in a more elegant way to match the results of the succeeding members of the sequence, indeed the maximal number for a second-order ordinary differential equation,

$$\begin{aligned}
\Gamma_1 &= x^2 y\partial_x - y[2 + xy(xy-2)]\partial_y \\
\Gamma_2 &= y\partial_x - y^3\partial_y \\
\Gamma_3 &= xy\partial_x + y^2(1-xy)\partial_y \\
\Gamma_4 &= x\partial_x - y\partial_y \\
\Gamma_5 &= x^3(xy-2)\partial_x - x(xy-2)[2 + xy(xy-2)]\partial_y \\
\Gamma_6 &= -x^2(xy-2)\partial_x + xy(xy-2)(xy-1)\partial_y \\
\Gamma_7 &= x^2\partial_x + (2-2xy)\partial_y \\
\Gamma_8 &= \partial_x
\end{aligned}$$

with the algebra $sl(3,R)$.

We proceed with G_3 to find an unexpected three, *ie*

$$\begin{aligned}\Gamma_1 &= \partial_x\\ \Gamma_2 &= x\partial_x - y\partial_y\\ \Gamma_3 &= x^2\partial_x + (3-2xy)\partial_y\end{aligned}$$

with the algebra $sl(2,R)$.

Proposition: *The general member of the Riccati sequence possesses the symmetries*

$$\begin{aligned}\Gamma_1 &= \partial_x\\ \Gamma_2 &= x\partial_x - y\partial_y\\ \Gamma_3 &= x^2\partial_x + (n-2xy)\partial_y\end{aligned}$$

with the algebra $sl(2,R)$.

Remark: For G_2 there are an additional five symmetries and for G_1 an additional infinity.

Proof: The possession of Γ_1 is straightforward since (5.1.5) is autonomous. The result for Γ_2 holds by closure of the Lie Algebra. We proceed with Γ_3.

It is easy to show that

$$\begin{aligned}\Gamma_3^{[1]}\tilde{G}_1 &= -2x(1+1)\tilde{G}_1\\ \Gamma_3^{[2]}\tilde{G}_2 &= -2x(2+1)\tilde{G}_2\\ \Gamma_3^{[3]}\tilde{G}_3 &= -2x(3+1)\tilde{G}_3\end{aligned}$$

and one can easily assume that

$$\Gamma_3^{[m]}\tilde{G}_m = -2x(m+1)\tilde{G}_m. \qquad (5.2.1)$$

The task is now to demonstrate that from this property it follows that

$$\Gamma_3^{[m+1]}\tilde{G}_{m+1} = -2x(m+2)\tilde{G}_{m+1}. \qquad (5.2.2)$$

We commence with the definition and so we have

$$\begin{aligned}\Gamma_3^{[m+1]}\tilde{G}_{m+1} &= \Gamma_3^{[m+1]}(D+y)\tilde{G}_m\\ &= \left[\Gamma_3^{[m+1]},(D+y)\right]_{LB}\tilde{G}_m+(D+y)\Gamma_3^{[m+1]}\tilde{G}_m\\ &= \left[\Gamma_3^{[m+1]},(D+y)\right]_{LB}\tilde{G}_m+(D+y)\left(\Gamma_3^{[m]}+\partial_y\right)\tilde{G}_m,\end{aligned}$$

where the last line is a consequence of remembering the definition of Γ_3 and that the $(m+1)$th derivative in $\Gamma_3^{[m+1]}$ does not act on $\tilde{G}_m$.

The first task is to compute the Lie Bracket. We have

$$\begin{aligned}&\Bigg[\partial_x+\dot{\sum_{k=0}}y^{(k+1)}\partial_{y^{(k)}}+y,x^2\partial_x+(m+1-2xy)\partial_y-2x\sum_{j=1}^{m+1}(j+1)y^{(j)}\partial_{y^{(j)}}\\ &\quad-\sum_{j=1}^{m+1}j(j+1)y^{(j-1)}\partial_{y^{(j)}}\Bigg]_{LB}\\ &\quad= 2x\partial_x-2y\partial_y-2\sum_{j=1}^{m+1}(j+1)y^{(j)}\partial_{y^{(j)}}-2xy'\partial_y-(m+1-2xy)\\ &\qquad-2x\sum_{j=1}^{m+1}(j+1)\dot{\sum_{k=0}}\left[y^{(k+1)}\partial_{y^{(k)}},\,y^{(j)}\partial_{y^{(j)}}\right]_{LB}\\ &\qquad-\sum_{j=1}^{m+1}j(j+1)\dot{\sum_{k=0}}\left[y^{(k+1)}\partial_{y^{(k)}},y^{(j-1)}\partial_{y^{(j)}}\right]_{LB},\end{aligned}$$

where the overdot on the summation means that the sum is taken to whatever order of derivative is required.

We compute the two subsidiary Lie Brackets separately.

$$\begin{aligned}\mathrm{LB}_1 &= -2x\sum_{j=1}^{m+1}(j+1)\dot{\sum_{k=0}}\left\{y^{(k+1)}\delta_{k,j}\partial_{y^{(j)}}-y^{(j)}\delta_{j,k+1}\partial_{y^{(k)}}\right\}\\ &= -2x\sum_{j=1}^{m+1}(j+1)\left\{y^{(j+1)}\partial_{y^{(j)}}-y^{(j)}\partial_{y^{(j-1)}}\right\}\\ &= 2x\left\{2y'\partial_y+\sum_{j=1}^{m}y^{(j+1)}\partial_{y^{(j)}}-(m+2)y^{(m+1)}\partial_{y^{(m+1)}}\right\}.\end{aligned}$$

In a similar way we find that

$$\mathrm{LB}_2 = 2\sum_{j=1}^{m}(j+1)y^{(j)}\partial_{y^{(j)}} - (m+1)(m+2)y^{(m+1)}\partial_{y^{(m+1)}} + 2y\partial_y.$$

We return to the main calculation and after some simplification of terms which cancel we continue as below.

$$\begin{aligned}
\Gamma_3^{[m+1]}\tilde{G}_{m+1} &= \Bigg\{-2x\left(\partial_x + y'\partial_y + \sum_{j=1}^{m+1} y^{(j)+1}\partial_{y^{(j)}}\right) \\
&\quad +2(m+2)y^{(m+1)}\partial_{y^{(m+1)}} + m + 1 - 2xy - 2xy^{(m+2)}\partial_{y^{(m+1)}} \\
&\quad +2x(m+2)y^{(m+2)}\partial_{y^{(m+1)}} + (m+1)(m+2)y^{(m+1)}\partial_{y^{(m+1)}} \\
&\qquad + (D+y)\left(-2x(m+1)+\partial_y\right)\Bigg\}\tilde{G}_m \\
&= -2x(m+2)(D+y)\tilde{G}_m + \left[(D+y)\partial_y - (m+1)\right]\tilde{G}_m \\
&= -2x(m+2)\tilde{G}_{m+1}
\end{aligned}$$

since

$$\left[(D+y)\partial_y - (m+1)\right]\tilde{G}_m = 0.$$

This last statement is equivalent to the result of the Lemma, *ie*

$$\frac{\partial}{\partial y}\left(\tilde{G}_{m+1}\right) = (m+2)\tilde{G}_m,$$

and so the result is proven.

Note that in the proof we have made use of the fact that $\tilde{G}_m$ contains derivatives only up to $y^{(m)}$.

Remark: Note that in the proof above we have again made use of the fact that $\tilde{G}_m$ contains derivatives only up to $y^{(m)}$.

5.3 Singularity Analysis

We use singularity analysis as a tool to determine whether a given differential equation is integrable in terms of functions almost everywhere analytic. For equations which pass the Painlevé Test we are then encouraged to seek closed-form solutions. We find that all members of the Riccati sequence possess the Painlevé Property. As we see below, explicit integrability follows. Independently of this integrability the results of the singularity analysis provide some very interesting patterns in terms of the parameters, that is the p, α and r of the standard analysis. Similarly interesting patterns have been reported in [79].

Euler *et al* [42] presented the singularity analysis of the Riccati sequence. For all elements the leading-order behaviour is $\alpha\chi^{-1}$ with the possible values of α being listed in the Table 5.1 and Table 5.2. We summarise their results in the table below.

In Table 5.2 we advance from [42] and present the properties in terms of the singularity analysis for the general member of the Riccati sequence. In that way we are able to comment upon the integrability or otherwise of all members of the sequence in the sense of Painlevé.

The pattern of the resonances is as follows: The set of resonances for $\alpha = j$ is obtained from the set for $\alpha = j-1$ by subtracting the number $n+1$ from the largest positive resonance of the latter set.

It follows from Tables 5.1 and 5.2 that all members of the Riccati sequence pass the Painlevé Test and each is integrable in terms of analytic functions [31].

When we apply the Riccati transformation

$$y = \alpha\frac{w'}{w} \tag{5.3.1}$$

to the nth member of the Riccati sequence, we observe that the most simplified equation, *ie* $w^{(n+1)} = 0$, arises when we chose $\alpha = 1$ which is a consequence of the singularity analysis itself [13]. Therefore (5.3.1) may be written as

$$x = x, \quad w = \exp\left[\int y\mathrm{d}x\right].$$

It is a matter of simple calculation to verify the following proposition.

Table 5.1: Singularity analysis for the first four members of the Riccati sequence.

Member	Leading-order coefficients	Resonances
G_1	$\alpha = 1$	$r = -1$
G_2	$\alpha = 1$ $\alpha = 2$	$r = -1, 1$ $r = -1, -2$
G_3	$\alpha = 1$ $\alpha = 2$ $\alpha = 3$	$r = -1, 1, 2$ $r = -1, 1, -2$ $r = -1, -2, -3$
G_4	$\alpha = 1$ $\alpha = 2$ $\alpha = 3$ $\alpha = 4$	$r = -1, 1, 2, 3$ $r = -1, 1, 2, -2$ $r = -1, 1, -2, -3$ $r = -1, -2, -3, -4$

Proposition: *The general solution of the nth member of the Riccati sequence, $n \geq 1$, is given by*

$$y_n = \frac{\left(\sum_{i=0}^{n} A_i x^i\right)'}{\sum_{i=0}^{n} A_i x^i}, \tag{5.3.2}$$

where the A_i, $i = 0, n$, are constants of integration.

Table 5.2: Singularity analysis for the general member of the Riccati sequence.

Member	Leading-order coefficients	Resonances
G_n	$\alpha = 1$ $\alpha = 2$ $\vdots$ $\alpha = n$	$r = -1, 1, 2, \ldots, n-1$ $r = -1, 1, \ldots, n-2, -2$ $\vdots$ $r = -1, -2, \ldots, -n$

5.4 Invariants and First Integrals

For the purposes of this section we commence with some definitions.

Definition 1: An invariant I of an ordinary differential equation is said to be any nontrivial function which satisfies

$$\frac{\mathrm{d}I}{\mathrm{d}x} = 0 \tag{5.4.1}$$

on solution curves of the differential equation where $I = I(x, y, y', ..., y^{(n-1)})$.

Definition 2: A first integral of an nth-order ordinary differential equation is any function $I = I(x, y, y', \ldots, y^{(n-1)})$ which satisfies (5.4.1).

Note that we do depart from some standard definitions for reasons which become evident below. There are various conventions concerning the meanings of the expressions 'first integral', 'invariant' and 'conserved quantity'. We are not concerned by the third in this Chapter since we do not use that expression. Some writers use the first two expressions interchangeably. Others prefer to distinguish between the two by insisting that the former be autonomous whereas the latter is allowed to depend upon the independent variable. This is a sensible distinction under appropriate circumstances.

Indeed in terms of the integration of an ordinary differential equation the distinction can be quite critical. In this Section we have varied the definitions to suit the very precise purpose of distinguishing between two classes of function both of which have the property of having a zero total derivative on solution curves of the differential equation.

We wish to identify all invariants and first integrals, as defined above, of each member of the Riccati sequence. In order to do so we take the nth member of that sequence, perform an increase of order by using the Riccati transformation to obtain $w^{(n+1)} = 0$. We compute the fundamental integrals and invariants of that equation and by reverting the transformation one can deduce the whole set of invariants for the nth member of the Riccati sequence, *videlicet*

$$I_j = \left(\sum_{i=1}^{j} \frac{(-1)^{i+1}}{(j-i)!} x^{j-i} \tilde{G}_{(n-i)} \right) \exp\left[\int y \mathrm{d}x \right], \quad j = 1, n+1. \tag{5.4.2}$$

Note that for the invariants of $w^{(n)} = 0$ we would have the above invariants (5.4.2) except that instead of $\tilde{G}_{(n-i)}$ we would have $w^{(n+1-i)}$ and there would be no exponential term.

We now turn our attention to first integrals. One takes the ratio of two separate invariants to obtain a first integral. An independent set for G_n can be variously defined. The set $\{\mathcal{F}_{ij}\}$, defined by

$$\mathcal{F}_{ij} = \frac{I_j}{I_i}, \quad j = 1, i-1, i+1, n, \tag{5.4.3}$$

is an independent set of first integrals for G_n. Such a simple formula is not available if one wishes to describe a set of autonomous first integrals.

An interesting aspect arises in the symmetry properties of the first integrals which we briefly note. If one computes the symmetries of all first integrals of (5.1.3), one finds that they all share the algebra $A_1 \oplus_s A_2$. By a curious misfortune this feature does not persist and therefore one is confronted with an absolute zero for contact (not to mention point) symmetries for the first integrals of (5.1.4)[5].

[5]When equation (5.1.4) is related to $w^{(iv)} = 0$ by an increase of order, the symme-

5.5 Complete symmetry groups

We start with G_4 to give a flavour of the procedure and then we prove the general result for any member of the Riccati sequence.

Proposition: *The complete symmetry group of G_4 is given by the symmetries*

$$\begin{aligned}
\Delta_1 &= -\exp\left[-\int y\mathrm{d}x\right]\{y\}\partial_y \\
\Delta_2 &= -\exp\left[-\int y\mathrm{d}x\right]\{xy-1\}\partial_y \\
\Delta_3 &= -\exp\left[-\int y\mathrm{d}x\right]\{x^2y-2x\}\partial_y \\
\Delta_4 &= -\exp\left[-\int y\mathrm{d}x\right]\{x^3y-3x^2\}\partial_y \\
\Delta_5 &= -\exp\left[-\int y\mathrm{d}x\right]\{x^4y-4x^3\}\partial_y.
\end{aligned}$$

Proof: We calculate the fourth extensions of $\Delta_1 - \Delta_5$.

$$\begin{aligned}
\Delta_1^{[4]} &= -\exp\left[-\int y\mathrm{d}x\right]\{y\partial_y + (y'-y^2)\partial_{y'} + (y''-3yy'+y^3)\partial_{y''} \\
&\quad +(y'''-4yy''-3y'^2+6y^2y'-y^4)\partial_{y'''} \\
&\quad +(y^{(iv)}-5yy'''-10y'y''+10y^2y''+15yy'^2-10y^3y'+y^5)\partial_{y^{(iv)}}\} \\
\Delta_2^{[4]} &= -\exp\left[-\int y\mathrm{d}x\right]\{-\partial_y + 2y\partial_{y'} + 3(y'-y^2)\partial_{y''} + 4(y''-3yy'+y^3)\partial_{y'''} \\
&\quad +5(y'''-4yy''-3y'^2+6y^2y'-y^4)\partial_{y^{(iv)}}\} + x\Delta_1^{[4]} \\
\Delta_3^{[4]} &= -2\exp\left[-\int y\mathrm{d}x\right]\{-\partial_{y'} + 3y\partial_{y''} + 6(y'-y^2)\partial_{y'''} + 10(y''-3yy'+y^3)\partial_{y^{(iv)}}\} \\
&\quad +2x\Delta_{2eff}^{[4]} + x^2\Delta_1^{[4]} \\
\Delta_4^{[4]} &= -6\exp\left[-\int y\mathrm{d}x\right]\{-\partial_{y''} + 4y\partial_{y'''} + 10(y'-y^2)\partial_{y^{(iv)}}\} \\
&\quad +3x\Delta_{3eff}^{[4]} + 3x^2\Delta_{2eff}^{[4]} + x^3\Delta_1^{[4]} \\
\Delta_5^{[4]} &= -24\exp\left[-\int y\mathrm{d}x\right]\{-\partial_{y'''} + 5y\partial_{y^{(iv)}}\}
\end{aligned}$$

---

tries of the first integrals of the latter become, by inverting the transformation, nonlocal symmetries for the ones of the former.

$$+4x\Delta^{[4]}_{4eff} + 6x^2\Delta^{[4]}_{3eff} + 4x^3\Delta^{[4]}_{2eff} + x^4\Delta^{[4]}_1,$$

where the subscript *eff* stands for the effective[6] part of that symmetry. When we act all the above extensions on the general fourth-order ordinary differential equation, *videlicet*

$$y^{(iv)} = f(x, y, y', y'', y'''),$$

we obtain the system of five equations

$$\begin{aligned}
&\frac{\partial f}{y'''} = -5y \\
&4y\frac{\partial f}{y'''} - \frac{\partial f}{y''} = 10(y' - y^2) \\
&6(y' - y^2)\frac{\partial f}{y'''} + 3y\frac{\partial f}{y''} - \frac{\partial f}{y'} = 10(y'' - 3yy' + y^3) \\
&4(y'' - 3yy' + y^3)\frac{\partial f}{y'''} + 3(y' - y^2)\frac{\partial f}{y''} + 2y\frac{\partial f}{y'} \\
&-\frac{\partial f}{y} = 5(y''' - 4yy'' - 3y'^2 + 6y^2y' - y^4) \\
&(y''' - 4yy'' - 3y'^2 + 6y^2y' - y^4)\frac{\partial f}{y'''} + (y'' - 3yy' + y^3)\frac{\partial f}{y''} + (y' - y^2)\frac{\partial f}{y'} \\
&+y\frac{\partial f}{y} = y^{(iv)} - 5yy''' - 10y'y'' + 10y^2y'' + 15yy'^2 - 10y^3y' + y^5 \qquad (5.5.1)
\end{aligned}$$

which can be solved to give the following expressions for the derivatives of all arguments in f

$$\begin{aligned}
\frac{\partial f}{\partial y'''} &= -5y \\
\frac{\partial f}{\partial y''} &= -10(y' + y^2) \\
\frac{\partial f}{\partial y'} &= -10(y'' + 3yy' + y^3) \\
\frac{\partial f}{\partial y} &= -5(y''' + 4yy'' + 3y'^2 + 6y^2y' + y^4).
\end{aligned}$$

[6]This means that whatever we omit from the expression would not play any role in the current calculation. It is therefore not necessary to include as it is not used.

When these expressions are substituted into (5.5.1), we recover G_4.

Proposition: *The complete symmetry group of G_n is given by the $(n+1)$ symmetries*[7]

$$\Delta_i = -\exp\left[-\int y\mathrm{d}x\right]\{x^{i-1}y-(i-1)x^{i-2}\}\partial_y,\ i=1,\,n+1.$$

Proof: We recall that G_n stands for the nth member of the Riccati sequence and $\tilde{G}_n$ for the left hand side of G_n. It is essential for the notation required in this proof to introduce the adjoint generator of sequence, $G.S.^{\alpha} = D - y$, which generates the adjoint Riccati sequence, *videlicet*

$$\begin{aligned}
(G_1^{\alpha}) &:= y' - y^2 = 0\\
(G_2^{\alpha}) &:= y'' - 3yy' + y^3 = 0\\
(G_3^{\alpha}) &:= y''' - 4yy'' - 3y'^2 + 6y^2y' - y^4 = 0\\
(G_4^{\alpha}) &:= y^{(iv)} - 5yy''' - 10y'y'' + 10y^2y'' + 15yy'^2 - 10y^3y' + y^5 = 0\\
(G_5^{\alpha}) &:= y^{(v)} - 6yy^{(iv)} - 15y'y''' + 15y^2y''' + 60yy'y'' - 10y''^2 - 20y^3y''\\
&\quad +15y'^3 - 45y^2y'^2 + 15y^4y' - y^6 = 0 \qquad (5.5.2)\\
&\ \vdots\\
(G_n^{\alpha}) &:= (D-y)^n y = 0. \qquad (5.5.3)
\end{aligned}$$

The left hand side of the above members is denoted by $\tilde{G}_n^{\alpha}$.

Remark: We note that there is a reflection here between the Riccati sequence and its adjoint in that the formulæ for the sequence are reflected in the formulæ for the adjoint sequence.

Lemma: *The general members of the Riccati and the adjoint Riccati sequences satisfy the recurrence relation*

$$\tilde{G}_n = \tilde{G}_n^{\alpha} + \sum_{i=1}^{n}\binom{n+1}{i}\tilde{G}_{i-1}^{\alpha}\tilde{G}_{n-i}. \qquad (5.5.4)$$

[7]The correct number of symmetries required to specify an equation completely has been discussed in [9] and [10].

Proof: It is easy to show that (5.5.4) holds for $n = 2$ and $n = 3$. We assume that (5.5.4) is true for $n = k$ and we prove that it is true for $n = k + 1$.

$$\begin{aligned}
\tilde{G}_{k+1} &= (D+y)\tilde{G}_k = (D+y)\left[\tilde{G}_k^\alpha + \sum_{i=1}^{k}\binom{k+1}{i}\tilde{G}_{i-1}^\alpha \tilde{G}_{k-i}\right] \\
&= \tilde{G}_{k+1}^\alpha + 2y\tilde{G}_k^\alpha + \sum_{i=1}^{k}\binom{k+1}{i}\left\{D\left[\tilde{G}_{i-1}^\alpha \tilde{G}_{k-i}\right] + y\tilde{G}_{i-1}^\alpha \tilde{G}_{k-i}\right\} \\
&= \tilde{G}_{k+1}^\alpha + 2y\tilde{G}_k^\alpha + \sum_{i=1}^{k}\binom{k+1}{i}\left(\tilde{G}_i^\alpha + y\tilde{G}_{i-1}^\alpha\right)\tilde{G}_{k-i} \\
&\quad + \sum_{i=1}^{k}\binom{k+1}{i}\tilde{G}_{i-1}^\alpha\left(\tilde{G}_{k-i+1} - y\tilde{G}_{k-i}\right) + y\sum_{i=1}^{k}\binom{k+1}{i}\tilde{G}_{i-1}^\alpha \tilde{G}_{k-i} \\
&\overset{(n=k)}{=} \tilde{G}_{k+1}^\alpha + 2y\tilde{G}_k^\alpha + \sum_{i=1}^{k}\binom{k+1}{i}\tilde{G}_i^\alpha \tilde{G}_{k-i} + \sum_{i=1}^{k}\binom{k+1}{i}\tilde{G}_{i-1}^\alpha \tilde{G}_{k-i+1} \\
&\quad + y(\tilde{G}_k - \tilde{G}_k^\alpha) \\
&= \tilde{G}_{k+1}^\alpha + y\tilde{G}_k^\alpha + y\tilde{G}_k + \sum_{i=2}^{k+1}\binom{k+1}{i-1}\tilde{G}_{i-1}^\alpha \tilde{G}_{k-i+1} \\
&\quad + \sum_{i=1}^{k}\binom{k+1}{i}\tilde{G}_{i-1}^\alpha \tilde{G}_{k-i+1} \\
&= \tilde{G}_{k+1}^\alpha + y\tilde{G}_k^\alpha + y\tilde{G}_k + \sum_{i=1}^{k+1}\left\{\binom{k+1}{i-1} + \binom{k+1}{i}\right\}\tilde{G}_{i-1}^\alpha \tilde{G}_{k-i+1} \\
&\quad - \tilde{G}_k^\alpha \tilde{G}_0 - \tilde{G}_0^\alpha \tilde{G}_k \\
&= \tilde{G}_{k+1}^\alpha + \sum_{i=1}^{k+1}\binom{k+2}{i}\tilde{G}_{i-1}^\alpha \tilde{G}_{k-i+1},
\end{aligned}$$

which proves the result.

The nth extension of Δ_i is

$$\begin{aligned}
\Delta_i^{[n]} &= -\exp\left[-\int y\mathrm{d}x\right]\left\{-(i-1)!\partial_{y^{(i-2)}} + \frac{i!}{1!}y\partial_{y^{(i-1)}} + \frac{(i+1)!}{2!}\tilde{G}_1^\alpha \partial_{y^{(i)}}\right. \\
&\quad \left. + \frac{(i+2)!}{3!}\tilde{G}_2^\alpha \partial_{y^{(i+1)}} + \ldots + \frac{(n+1)!}{(n-i+2)!}\tilde{G}_{n-i+1}^\alpha \partial_{y^{(n)}}\right\} \\
&\quad + \sum_{k=0}^{i-2}\frac{1}{k!}\frac{\mathrm{d}^k}{\mathrm{d}x^k}(x^{i-1})\Delta_{(k+1)eff}^{[n]}, \quad i = 1, n+1,
\end{aligned}$$

where

$$\partial_{y^{(-1)}} = 0,\ \partial_{y^{(0)}} = \partial_y,\ \partial_{y^{(1)}} = \partial_{y'} \text{ etc},\ \frac{\mathrm{d}^0}{\mathrm{d}x^0} = 1,\ \frac{\mathrm{d}^1}{\mathrm{d}x^1} = \frac{\mathrm{d}}{\mathrm{d}x} \text{ etc.}$$

The action of $\Delta_i^{[n]}$ on the general nth-order ordinary differential equation, *videlicet*

$$y^{(n)} = f(x, y, y', y'', y''', \ldots, y^{(n-1)}),$$

gives

$$\begin{aligned}\binom{n+1}{n-i+2}\tilde{G}^{\alpha}_{n-i+1} &= -\frac{\partial f}{\partial y^{(i-2)}} + \binom{i}{1} y \frac{\partial f}{\partial y^{(i-1)}} + \binom{i+1}{2}\tilde{G}^{\alpha}_1 \frac{\partial f}{\partial y^{(i)}} \\ &+ \binom{i+2}{3}\tilde{G}^{\alpha}_2 \frac{\partial f}{\partial y^{(i+1)}} + \ldots + \binom{n}{n-i+1}\tilde{G}^{\alpha}_{n-i+1}\frac{\partial f}{\partial y^{(n-1)}},\ i = 1,\ n+1.\end{aligned}$$

For $i = 1$ this is

$$\tilde{G}^{\alpha}_n = \sum_{j=0}^{n-1} \tilde{G}^{\alpha}_j \frac{\partial f}{\partial y^{(j)}}. \tag{5.5.5}$$

The remaining n may be conveniently written as the system of equations

$$\mathbf{L}^{\alpha}_n = \mathbf{Q}^{\alpha}_n \mathbf{F}_n, \tag{5.5.6}$$

where

$$\begin{aligned}(\mathbf{L}^{\alpha}_n)_i &= \binom{n+1}{i}\tilde{G}^{\alpha}_{n-i}, \qquad i = 1, n, \\ (\mathbf{F}_n)_i &= \frac{\partial f y^{(i-1)},}{\partial \quad i = 1, n,} \\ (\mathbf{Q}^{\alpha}_n)_{ij} &= \begin{cases} -1, & i = j, \\ 0, & i > j, \\ \binom{j}{i}\tilde{G}^{\alpha}_{j-i-1}, & i < j. \end{cases}\end{aligned} \tag{5.5.7}$$

We claim that the solution of (5.5.6) is

$$\mathbf{F}_n = \mathbf{Q}_n \mathbf{L}^{\alpha}_n = -\mathbf{L}_n, \tag{5.5.8}$$

where

$$(\mathbf{Q}_n)_{ij} = \begin{cases} -1, & i = j, \\ 0, & i > j, \\ -\binom{j}{i}\tilde{G}_{j-i-1}, & i < j, \end{cases} \tag{5.5.9}$$

and

$$(\mathbf{L}_n)_i = \binom{n+1}{i}\tilde{G}_{n-i}, \qquad i = 1, n.$$

Given (5.5.6), the first part of (5.5.8) follows if the Lemma below is proven.

Lemma: *The matrices* $\mathbf{Q}_n^\alpha$ *and* $\mathbf{Q}_n$ *defined in (5.5.7) and (5.5.9) are the inverses of each other, ie, they satisfy*

$$\mathbf{Q}_n^\alpha \mathbf{Q}_n = \mathbf{I}_n. \tag{5.5.10}$$

Proof: That (5.5.10) is true for $n = 1$ is obvious. We assume that it is true for $n = k$, *ie,*

$$\mathbf{Q}_k^\alpha \mathbf{Q}_k = \mathbf{I}_k. \tag{5.5.11}$$

Then

$$\begin{aligned}
\mathbf{Q}_{k+1}^\alpha \mathbf{Q}_{k+1} &= \begin{bmatrix} \mathbf{Q}_k^\alpha & \mathbf{L}_k^\alpha \\ \mathbf{O}_k^T & -1 \end{bmatrix} \begin{bmatrix} \mathbf{Q}_k & -\mathbf{L}_k \\ \mathbf{O}_k^T & -1 \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{Q}_k^\alpha \mathbf{Q}_k, & -\mathbf{Q}_k^\alpha \mathbf{L}_k - \mathbf{L}_k^\alpha \\ \mathbf{O}_k^T & 1 \end{bmatrix} \\
&\overset{(5.5.11)}{=} \mathbf{I}_{k+1}
\end{aligned}$$

provided $\mathbf{Q}_k^\alpha \mathbf{L}_k + \mathbf{L}_k^\alpha = 0$. We consider the ith element of this term.

$$\begin{aligned}
(\mathbf{Q}_k^\alpha \mathbf{L}_k - \mathbf{L}_k^\alpha)_i &= \sum_j (\mathbf{Q}_k^\alpha)_{ij} (\mathbf{L}_k)_j + (\mathbf{L}_k^\alpha)_i \\
&= -(\mathbf{L}_k)_i + \sum_{j=i+1}^{k} (\mathbf{Q}_k^\alpha)_{ij} (\mathbf{L}_k)_j + (\mathbf{L}_k^\alpha)_i \\
&= -\binom{k+1}{i}\tilde{G}_{k-i} + \sum_{j=i+1}^{k} \binom{j}{i} \tilde{G}_{j-i-1}^\alpha \binom{k+1}{j} \tilde{G}_{k-j} \\
&\qquad + \binom{k+1}{i}\tilde{G}_{k-i}^\alpha \\
&= -\binom{k+1}{i}\left\{ \tilde{G}_{k-i} - \tilde{G}_{k-i}^\alpha - \sum_{j=i+1}^{k} \binom{k+1-i}{j-i} \tilde{G}_{j-i-1}^\alpha \tilde{G}_{k-j} \right\} \\
&\overset{(5.5.4)}{=} 0.
\end{aligned}$$

We now proceed to prove the second part of (5.5.8).

Lemma: *If* $\mathbf{L}_n^\alpha = \mathbf{Q}_n^\alpha \mathbf{F}_n$, *then* $\mathbf{Q}_n \mathbf{L}_n^\alpha = -\mathbf{L}_n$.

Proof: Consider an element of the product $\mathbf{Q}_n\mathbf{L}_n^\alpha$. This is

$$\begin{aligned}
(\mathbf{Q}_n)_{ij}(\mathbf{L}_n^\alpha)_j &= -\binom{n+1}{i}\tilde{G}_{n-i}^\alpha + \sum_{j=i+1}^{n}\left[-\binom{j}{i}\tilde{G}_{j-i-1}\right] \\
&\quad \times \binom{n+1}{j}\tilde{G}_{n-j}^\alpha \\
&= -\binom{n+1}{i}\left\{\tilde{G}_{n-i}^\alpha + \sum_{j=i+1}^{n}\binom{j}{i}\binom{n+1}{j}/\binom{n+1}{i}\right. \\
&\quad \left. \times\, \tilde{G}_{n-j}^\alpha \tilde{G}_{j-i-1}\right\} \\
&\overset{(5.5.4)}{=} -\binom{n+1}{i}\tilde{G}_{n-i} \\
&= -(\mathbf{L}_n)_i.
\end{aligned}$$

Therefore (5.5.6) is written as

$$\begin{aligned}
\mathbf{F}_n &= (\mathbf{Q}_n^\alpha)^{-1}\mathbf{L}_n^\alpha \\
&= \mathbf{Q}_n\mathbf{L}_n^\alpha \\
&= -\mathbf{L}_n. \qquad (5.5.12)
\end{aligned}$$

Equation (5.5.5) is equally written as

$$\tilde{G}_n^\alpha = \sum_{i=1}^{n}\tilde{G}_{i-1}^\alpha \frac{\partial f}{\partial y^{(i-1)}},$$

which, when (5.5.12) and (5.5.4) are used, results to G_n and the proposition is proven.

5.6 Discussion

The Riccati sequence, which has been the subject of the present study, illustrates with a degree of excellence, which one hopes can be surpassed, the type of mathematical properties which are likely to make such sequences an object of fond study [42]. The Riccati equation has a long and distinguished history in both the theory of differential equations and the application to divers phenomena. The Riccati sequence and its adjoint are based on recursion operators closely related in form to the Dirac operators [37] of the quantum mechanical simple harmonic oscillator which in themselves are simply autonomous versions of two of the Lie point symmetries of the classical simple harmonic oscillator. With such a lavish heritage one should not be too surprised that the Riccati sequence[8] exhibits such properties in terms of symmetry and integrability.

Each element of the sequence can be linearised by means of the nonlocal transformation – often called a Riccati transformation – and so is trivially integrable. Although G_1 and G_2 – the Riccati and Painlevé-Ince equations – display exceptional symmetry in the sense of Lie point symmetries, the remaining elements of the sequence possess just the three-element algebra $sl(2,R)$ of Lie point symmetries. The distinguishing feature of the Lie symmetries of the elements of the sequence is the possession of $n+1$ (in the case of G_n) exponential nonlocal symmetries which completely specify the elements. The algebra of these symmetries is $(n+1)A_1$.

As a consequence of the ability to linearise the equations of the Riccati sequence each element possesses an invariant derived from the so-called fundamental integrals of the parent linear equation[9]. These invariants are not integrals in the conventional sense as they contain the integral of the dependent variable. However, the integral of the dependent variable appears as an exponential term and so functions of these invariants which are homogeneous of degree zero in the dependent variable are first integrals. It was for this reason that we introduced specialised meanings for the two expressions, 'invariant' and 'first integral', for the purposes of this Chapter. Since there are $(n+1)$ linearly independent (exponential nonlocal) invariants, there are n functions of the independent integrals which reflect most adequately the

[8]Equally the adjoint sequence. In the following discussion the properties of the adjoint sequence may be inferred *mutatis mutandis* from those of the Riccati sequence.

[9]These integrals have been termed fundamental since an nth-order linear ordinary differential equation possesses n linearly independent integrals which are linear in the dependent variable and its first $(n-1)$th derivatives. The integrals are known for their interesting algebraic properties.

integrability of each member of the Riccati sequence.

In terms of singularity analysis the Riccati sequence possesses properties which may even be regarded to outshine the symmetry properties. Not only is each element of the sequence explicitly integrable in terms of analytic functions apart from isolated polelike singularities – a property which has been used to illustrate certain subtle and not widely appreciated implications of singularity analysis [14] – but also there are patterns to the possible values of the coefficients of the leading-order terms and the values of the resonances for each of the principal branches. In passing one recalls that each branch is principal and the need to be concerned with the implications of branches with unfortunate properties is obviated. The patterns of the resonances deserve further, separate treatment, particularly in respect of other possible sequences.

The Riccati sequence has provided an excellent vehicle for the introduction of the notion of differential sequences of ordinary differential equations. Its generator of sequence is simple. Its generating function is elementary. Two of the elements, G_1 and G_2, are well-known in the literature as well as in applications. Already Euler *et al* [42] have made a brief excursion into the properties of the Ermakov-Pinney sequence and, as the name suggests, this is based upon the Ermakov-Pinney equation [38, 109] which has been found in many varied applications.

In the Introduction of this Thesis we recalled the classical association of the Riccati equation, the linear second-order ordinary differential equation and the third-order Kummer-Schwarz equation. Since these equations are related by means of nonlocal transformations, there is no reason to exclude from this select group other equations which are similarly related[10]. One looks forward to further revelations of fascinating properties of other differential sequences.

As a final remark we recall that the evolution equations and associated Recursion Operators [107] which are the source material for this Chapter contain arbitrary parameters and unspecified functions. In this Chapter we have

[10]One is well aware that all properties may not travel well through the process of nonlocal transformation, but that obstacle is already encountered under far less exotic transformations.

deliberately kept to the minimum required to make a sensible sequence. It is a little like treating the autonomous linear oscillator rather than a time-dependent, damped and forced oscillator. The latter can be transformed to the former by means of well-defined transformations and a study of the former is easier due to the simplicity possible in the presentation. Despite our personal preference for simplicity we do nevertheless accept that there are those who wish to see the treatment of the unsimplified systems.

Consider

$$E_n = \sum_{i=0}^{n} f_i(t) G_i = 0. \tag{5.6.1}$$

Equation (5.6.1) is linearisable by means of the Riccati transformation (5.3.1), with $\alpha = 1$, to

$$\sum_{i=0}^{n} f_i(t) w^{(i+1)} = 0. \tag{5.6.2}$$

Equation (5.6.2) possesses $(n+1)$ solution symmetries and the homogeneity symmetry.

Whilst there can be no dispute that the solution of (5.6.2) is more difficult than that of $w^{(n+1)} = 0$, for our purposes there is no difference. Subject to some mild conditions on the functions $f_i(t)$ – basically that they be continuous [61, p 72] – the solutions of (5.6.2) exist and that is all that is required for inclusion into the general framework of the treatment above. An nth-order linear ordinary differential equation can have $n+1$, $n+2$ or $n+4$ Lie point symmetries which represent the n solution symmetries, linearity, autonomy (in the right variables) and the general $sl(2,R)$ symmetry of equations of maximal symmetry [93]. There is no gainsaying that the additional symmetries make the process of solution in closed form just that much easier. However, all we need is the existence of the n solution symmetries to provide the symmetries necessary for our purposes. Consequently we have taken the clearer part so that the ideas and results contained in this Chapter be as evident as can be possible in such matters.

Chapter 6

Generalised Riccati sequence

6.1 Introduction

In[1] the third volume of his collected works [111] we are presented with the first-known study of the equation now called the Riccati equation from some forty years earlier. The equation proved to have some very interesting properties so that it has been a standard part of the repertoire for some centuries and probably the bane of most students during a first course on differential equations due to the rather intricate description of its general solution. In recent times the Riccati equation, its extension, the matrix Riccati equation, and their discretised forms have attracted considerable attention in applications[2]. For over a century the Riccati equation has been known to be the sole first-order equation of the form $y' = f(x, y)$ to possess the Painlevé Property [106, 61].

Another equation of not quite so much prominence but nevertheless well-known is the Painlevé-Ince equation [106, 61] which has received attention due to its occurrence in divers areas of application, in theoretical studies of univalent functions and its symmetry and singularity properties. Most recently it has received attention for its place in differential sequences [41, 42, 15] on the one hand and as a special case of the integrable class of modified Painlevé-Ince equations [29]. It is to the former characteristic that we wish to address ourselves in this Chapter.

[1]The essential content of this Chapter may be found in [80].

[2]For a contemporary summary of this aspect of the equation see Cariñena *et al* [28].

Euler *et al* [42] presented the Riccati sequence as based on the action of a recursion operator on a generating function. Depending upon where one wishes to start the generating function can be either $\exp[\int y\mathrm{d}x]$ or y. It can even be the 'left side' of the Riccati equation, *videlicet* $y' + y^2$. Successive members of the sequence are generated by the repeated action of $G = D + y$, where D denotes total differentiation with respect to the independent variable, x. Euler *et al* [42] were more concerned with the mode of construction of the sequence and its integrability in the sense of Painlevé. Aspects of the latter feature have been treated in Chapter Five more with a view to the patterns displayed by the coefficients of the leading-order terms and the resonances. This Chapter also contained its treatment of the complete symmetry groups of the elements of the sequence since the representation of the group was completely in terms of nonlocal symmetries even though each element of the sequence is readily integrated and this very ease of integration been used recently to illustrate the different types of Laurent expansion which can arise from the singularity analysis of a nonlinear differential equation [14]. A contemporary study of a sequence based upon the Emden-Fowler equation of index[3] $(0, 2)$ by Leach *et al* [79] found that such patterns were not peculiar to the Riccati sequence and did not need to have a specific form.

In this Chapter we look to a generalisation of the Riccati sequence. This generalisation is very simply constructed. Indeed we wish to emphasise from the very beginning of this Chapter that we are concerned with the simplest representations of generators of sequences and so the most elementary expressions for the members of a sequence. For those who wish to consult a more general treatment of recursion operators and sequences we refer to the paper by Petersson *et al* [107]. The reader who is more familiar with sequences of evolution partial differential equations is reminded that in the case of ordinary differential equations there is a greater latitude to manipulate the equation and/or the operator than in the case of the former equations. This is simply because there is a zero on the other side of the equals sign rather than a derivative with respect to time.

The basic form of the Painlevé-Ince equation is

$$y'' + 3yy' + y^3 = 0.$$

[3]The equation $y'' = kx^m y^n$ is called an Emden-Fowler equation of index (m, n).

It is characterised by the possession of eight Lie point symmetries [90]. This is in contrast to the modified Painlevé-Ince equation which is often written as

$$y'' + yy' + ky^3 = 0,$$

where $k \neq 1/9$. Otherwise the two equations are the same up to a rescaling of the dependent variable. The latter equation possesses only two Lie point symmetries. Its integrability has received and continues to receive attention [81, 29]. The Painlevé-Ince equation is the second member, after the Riccati equation, of the Riccati sequence. A natural question is whether the modified Painlevé-Ince equation is also a member of a differential sequence and, if this be the case, does that differential sequence have anything like the interesting and curious properties displayed by the Riccati sequence itself and the Emden-Fowler sequence. In particular we are interested in the existence of patterns in the values of the coefficients of leading-order terms and the resonances. Indeed the motivation for the present work is a result of the existence of a modified Painlevé-Ince equation as the result of the reduction of a fourth-order equation using two Lie point symmetries.

In the next section we demonstrate the construction of a differential sequence which includes the modified Painlevé-Ince equation as its second element. In Section 6.2 we determine the coefficients of the leading-order terms and the values of the resonances for each of these coefficients. We consider the results of the singularity analysis in terms of the integrability of the equations of this differential sequence. As a byproduct of these considerations we see that the elements of the Riccati sequence occupy a distinguished position.

The equation which motivated this work is

$$y'' + 10yy' + 12y^3 = 0 \tag{6.1.1}$$

which arose in the reduction of a fourth-order equation using two of its Lie point symmetries. One recalls that in the generation of the Riccati sequence we had the operation $(D + y)(y' + y^2)$ which leads one to the Painlevé-Ince equation. What similar operation leads to (6.1.1)? If one assumes an operation of the form $(D + by)(y' + cy^2)$, one finds that the permissible pairs of the parameters are $(4, 3)$ and $(6, 2)$. With either pair one can create a differential sequence.

We look at the implications of the first pair. The second, third and fourth members of this sequence are (6.1.1) and

$$
\begin{aligned}
y^{(3)} + 14yy'' + 10y'^2 + 76y^2y' + 48y^4 &= 0 \quad (6.1.2) \\
y^{(4)} + 18yy^{(3)} + 34y'y'' + 132y^2y'' + 192yy'^2 + 496y^3y' & \\
+192y^5 &= 0. \quad (6.1.3)
\end{aligned}
$$

It is obvious that the exponent of the leading-order term is -1. The results of a singularity analysis performed in the usual way of determining the coefficient of the leading-order term and then the resonances of these three equations are presented in Table 6.1 below.

In each case there is a coefficient of the leading-order term which allows for a Right Painlevé Series [48] in fractional powers. However, the other coefficients imply 'No Painlevé Property', be it the standard or the weak property.

We consider whether the other pair of constants which leads to (6.1.1) does anything different. By construction we obtain (6.1.1) after the first operation, $(D + 6y)(y' + 2y^2)$. The following two equations are

$$
\begin{aligned}
y^{(3)} + 16yy'' + 10y'^2 + 96y^2y' + 72y^4 &= 0 \quad (6.1.4) \\
y^{(4)} + 22yy^{(3)} + 36y'y'' + 192y^2y'' & \\
+252yy'^2 + 864y^3y' + 432y^5 &= 0. \quad (6.1.5)
\end{aligned}
$$

The singularity analyses of these equations give the results in Table 6.2. We see that (6.1.4) and (6.1.5) possess a similar pattern of behaviour to (6.1.2) and (6.1.3).

Given a generator of sequences, $D + by$, and an initial Riccati function, $y' + cy^2$, we seek values of the parameters b and c for which it is possible to obtain a sequence of equations which have interesting singularity properties, other than the Riccati sequence.

Table 6.1: Coefficients of the leading-order term and corresponding resonances for (6.1.1), (6.1.2) and (6.1.3)

Equation	Values of α	Resonances
(6.1.1)	$\frac{1}{3}$	$-1,\ \frac{2}{3}$
	$\frac{1}{2}$	$-1,\ -1$
(6.1.2)	$\frac{1}{3}$	$-1,\ \frac{2}{3},\ \frac{5}{3}$
	$\frac{1}{2}$	$-1,\ -1,\ 1$
	$\frac{3}{4}$	$-1,\ \frac{1}{4}\left(-7-i\sqrt{11}\right),\ \frac{1}{4}\left(-7+i\sqrt{11}\right)$
(6.1.3)	$\frac{1}{3}$	$-1,\ \frac{2}{3},\ \frac{5}{3},\ \frac{8}{3}$
	$\frac{1}{2}$	$-1,\ -1,\ 1,\ 2$
	$\frac{3}{4}$	$-1,\ 1,\ \frac{1}{4}\left(-7-i\sqrt{11}\right),\ \frac{1}{4}\left(-7+i\sqrt{11}\right)$
	1	$-2,\ -1,\ \frac{1}{2}\left(-5-i\sqrt{7}\right),\ \frac{1}{2}\left(-5+i\sqrt{7}\right)$

We construct a few of the earlier members of the putative sequence to see if there is anything of interest. The Generalised Riccati sequence is defined in terms of the generator of sequences, $D+by$, and the initial Riccati function, $y'+cy^2$, as

$$(D+by)^n\left(y'+cy^2\right)=0, \quad n=0,1,2,\ldots. \tag{6.1.6}$$

The differential sequence after the initial Riccati equation is

$$\begin{aligned}
GR_1 &:= y''+2cyy'+byy'+bcy^3=0 \qquad (6.1.7)\\
GR_2 &:= y^{(3)}+2cyy''+2byy''+2cy'^2+by'^2+5bcy^2y'+b^2y^2y'+b^2cy^4=0\\
GR_3 &:= y^{(4)}+2cyy^{(3)}+3byy^{(3)}+6cy'y''+4by'y''+7bcy^2y''+3b^2y^2y''
\end{aligned}$$

Table 6.2: Coefficients of the leading-order term and corresponding resonances for (6.1.4) and (6.1.5)

Equation	Values of α	Resonances
(6.1.4)	$\frac{1}{3}$	$-1,\ \frac{2}{3},\ 1$
	$\frac{1}{2}$(bis)	$-1,\ -1,\ 0$
(6.1.5)	$\frac{1}{3}$	$-1,\ \frac{2}{3},\ 1,\ 2$
	$\frac{1}{2}$(bis)	$-1,\ -1,\ 0,\ 1$
	$\frac{2}{3}$	$-2,\ -1,\ \frac{1}{6}\left(-5-i\sqrt{23}\right),\ \frac{1}{6}\left(-5+i\sqrt{23}\right).$

$$
\begin{aligned}
& +12bcyy'^2+3b^2yy'^2+9b^2cy^3y'+b^3y^3+b^3cy^5=0 && (6.1.8)\\
GR_4 &:= y^{(5)}+2cyy^{(4)}+4byy^{(4)}+8cy'y^{(3)}+7by'y^{(3)}+9bcy^2y^{(3)} \\
& +6b^2y^2y^{(3)}+6cy''^2+4by''^2+44bcyy'y'' \\
& +16b^2yy'y''+16b^2cy^3y''+4b^3y^3y''+12bcy'^3+3b^2y'^3+39b^2cy^2y'^2 \\
& +6b^3y^2y'^2+14b^3cy^4+b^4y^4y'+b^4cy^6=0. && (6.1.9)\\
& \vdots \\
GR_n &:= (D+by)^n\left(y'+cy^2\right)=0. && (6.1.10)
\end{aligned}
$$

When one sees such a mess of symbology, the expectation of being able to do anything – perhaps apart from the first equation which we know leads to a quadratic equation to be solved – can reasonably be estimated to be approximately zero.

Note in table 6.2 the expression ‘bis’ means twice.

6.2 Singularity Analysis

We summarise the results of the coefficients of the leading-order term and corresponding resonances in Table 6.3 and 6.4 below.

Table 6.3: Coefficients of the leading-order term and corresponding resonances

Equation	Values of α	Resonances
$(GR1)$	$\frac{2}{b}$	$-1,\ \frac{2(b-2c)}{b}$
	$\frac{1}{c}$	$-1,\ \frac{-b+2c}{c}$
$(GR2)$	$\frac{2}{b}$	$-1,\ 1,\ \frac{2(b-2c)}{b}$
	$\frac{3}{b}$	$-1,\ \frac{b-6c\pm\sqrt{13b^2-48bc+36c^2}}{2b}$
	$\frac{1}{c}$	$-1,\ \frac{-b+2c}{c},\ \frac{-b+3c}{c}$
$(GR3)$	$\frac{2}{b}$	$-1,\ 1,\ 2,\ \frac{2(b-2c)}{b}$
	$\frac{3}{b}$	$-1,\ 1,\ \frac{b-6c\pm\sqrt{13b^2-48bc+36c^2}}{2b}$
	$\frac{4}{b}$	$-2,\ -1,\ \frac{b-8c\pm\sqrt{17b^2-80bc+64c^2}}{2b}$
	$\frac{1}{c}$	$-1,\ \frac{-b+2c}{c},\ \frac{-b+3c}{c},\ \frac{-b+4c}{c}$

The summary of the results obtained by means of the standard[4] singularity analysis of equations (6.1.7-6.1.10) is presented in Table B.1. Although we can write down an expression for the nth element of the sequence as

$$(D+by)^{n-1}\left(y'+cy^2\right)=0, \quad n=1,2,\ldots, \tag{6.2.1}$$

[4]By 'standard' we mean the process of determining the exponent of the leading-order term, the possible coefficients of the leading-order term and the resonances in the spirit of the ARS algorithm [1, 2, 3].

Table 6.4: Coefficients of the leading-order term and corresponding resonances

Equation	Values of α	Resonances
$(GR4)$	$\frac{2}{b}$	$-1,\ 1,\ 2,\ 3,\ \frac{2(b-2c)}{b}$
	$\frac{3}{b}$	$-1,\ 1,\ 2,\ \frac{b-6c\pm\sqrt{13b^2-48bc+36c^2}}{2b}$
	$\frac{4}{b}$	$-2,\ -1,\ 1,\ \frac{b-8c\pm\sqrt{17b^2-80bc+64c^2}}{2b}$
	$\frac{5}{b}$	$-3,\ -2,\ -1,\ \frac{b-10c\pm\sqrt{21b^2-120bc+100c^2}}{2b}$
	$\frac{1}{c}$	$-1,\ \frac{-b+2c}{c},\ \frac{-b+3c}{c},\ \frac{-b+4c}{c},\ \frac{-b+5c}{c}$
$(GR5)$	$\frac{2}{b}$	$-1,\ 1,\ 2,\ 3,\ 4,\ \frac{2(b-2c)}{b}$
	$\frac{3}{b}$	$-1,\ 1,\ 2,\ 3,\ \frac{b-6c\pm\sqrt{13b^2-48bc+36c^2}}{2b}$
	$\frac{4}{b}$	$-2,\ -1,\ 1,\ 2,\ \frac{b-8c\pm\sqrt{17b^2-80bc+64c^2}}{2b}$
	$\frac{5}{b}$	$-3,\ -2,\ -1,\ 1,\ \frac{b-10c\pm\sqrt{21b^2-120bc+100c^2}}{2b}$
	$\frac{6}{b}$	$-4,\ -3,\ -2,\ -1,\ \frac{b-12c\pm\sqrt{25b^2-168bc+144c^2}}{2b}$
	$\frac{1}{c}$	$-1,\ \frac{-b+2c}{c},\ \frac{-b+3c}{c},\ \frac{-b+4c}{c},\ \frac{-b+5c}{c},\ \frac{-b+6c}{c}.$

a neat and explicit formula is not at hand. However, it is an easy matter to infer the coefficients and resonances for the nth member of the sequence.

The exponent of the leading-order term is always -1. The possible coefficients of the leading-order term of the nth element of the sequence are

$$\frac{2}{b},\ \frac{3}{b},\ \frac{4}{b},\ \ldots,\ \frac{n}{b},\ \frac{1}{c}, \tag{6.2.2}$$

ie, there are n possible coefficients. One is related to the parameter c whereas the remaining $n-1$ are related to the parameter b. In the possible coefficients of the leading-order term there is no mixing of the parameters, b and c. Including the generic resonance, -1, there are n resonances for each of the possible coefficients of the leading-order term. Corresponding to the first and last coefficients as listed in (6.2.2) the expressions for the resonances are rational functions of the parameters. For the remaining $n-2$ resonances the former include algebraic functions of the parameters. We observe that both rational and algebraic functions are homogeneous of equal degree in b and c. It is convenient to introduce the parameter $\sigma := c/b$ as the critical determinant of the singularity properties of the elements of the sequence. For an element of the sequence to possess the Painlevé Property or its weak form σ must be rational. There is no need for b and c to be rational. All that is required of them is that they be rational modulo some number which may itself be irrational or transcendental.

As the resonances corresponding to the final coefficient in (6.2.2) have a quite distinct pattern, we deal with them firstly for the nth element of the sequence. They are given by

$$-1,\ \frac{-b+2c}{c},\ \frac{-b+3c}{c},\ \ldots,\ \frac{-b+nc}{c}$$

or in terms of the parameter σ as

$$-1,\ 2-\frac{1}{\sigma},\ 3-\frac{1}{\sigma},\ \ldots,\ n-\frac{1}{\sigma}$$

from which it is apparent that integral resonances, as required for the possession of the Painlevé Property, occur only if σ is the inverse of an integer.

The other coefficient with resonances expressed as rational functions corresponds to the first coefficient in (6.2.2). In terms of σ the resonances are

$$-1,\ 1,\ 2,\ \ldots,\ n,\ 2(1-2\sigma).$$

Except for the final value all of these are integral. For the final value to be integral σ is constrained to be an integer, an half integer or a quarter integer. In combination with the requirement for the coefficient, $1/\sigma$, an element of the sequence possesses the Painlevé Property for at most of the values, $\sigma = 1$, $\frac{1}{2}$ and $\frac{1}{4}$.

We now turn our attention to the resonances which contain algebraic functions. It is evident that the integral resonances behave according to the tableau below.

Table 6.5: Variation of the values of the integral resonances with the coefficient of the leading-order term for general n

Values of α	Integral		Resonances			
$\frac{3}{b}$	-1	1	2	3	$\ldots$	$n-3$
$\frac{4}{b}$	-2	-1	1	2	$\ldots$	$n-4$
$\frac{5}{b}$	-3	-2	-1	1	$\ldots$	$n-5$
$\frac{6}{b}$	-4	-3	-2	-1	$\ldots$	$n-6$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\frac{n}{b}$	$-(n-2)$	$-(n-3)$	$-(n-4)$	$-(n-5)$	$\ldots$	-1

The pattern is quite obvious. It is remarkable how so many of the resonances are integral without any constraint on the values of the two parameters, b and c.

The two resonances which are expressed as algebraic functions of the parameters b and c contain quadratics in σ which appear in Table 6.6.
In passing we note that the entry for $2/b$ could be included in this tabulation. If we put $n = 1$ in the general formula, we obtain $9 - 24\sigma + 16\sigma^2 = (3 - 4\sigma)^2$ and this accounts for the first, -1, and last resonances.

Table 6.6: The quadratic expressions in σ contained in the square roots for general n

Values of α	Integral Resonances
$\frac{3}{b}$	$13 - 48\sigma + 36\sigma^2$
$\frac{4}{b}$	$17 - 80\sigma + 64\sigma^2$
$\frac{5}{b}$	$21 - 120\sigma + 100\sigma^2$
$\frac{6}{b}$	$25 - 168\sigma + 144\sigma^2$
$\vdots$	$\vdots$
$\frac{n}{b}$	$(4n+1) - 4n(n+1)\sigma + 4n^2\sigma^2$

To avoid complex resonances, which are anathema to the success of the singularity analysis, for any member of the sequence we require that

$$(4n+1) - 4n(n+1)\sigma + 4n^2\sigma^2 \geq 0 \quad \forall \quad n. \tag{6.2.3}$$

The roots of the quadratic are given by

$$\sigma = \frac{n+1 \pm \sqrt{n^2 - 2n}}{2n}$$

so that

$$\begin{aligned} \sigma &\leq \frac{n+1 - \sqrt{n^2-2n}}{2n} \quad \text{or} \\ \sigma &\geq \frac{n+1 + \sqrt{n^2-2n}}{2n} \quad \forall n. \end{aligned}$$

In the limit as $n \longrightarrow \infty$ this means that the interval $\sigma \in (0, 1)$ is excluded.

In combination with the constraints on the value of σ obtained above it means that only the value $\sigma = 1$ remains. When we substitute this value for σ into the left side of (6.2.3), we obtain $(4n+1) - 4n(n+1) + 4n^2 = 1\ \forall\, n$. When the coefficient of the leading-order term is n/b, the algebraic resonance is given by

$$r = -\tfrac{1}{2}\left[2n\sigma - 1 \pm \sqrt{(4n+1) - 4n(n+1)\sigma + 4n^2\sigma^2}\right].$$

We substitute the value $\sigma = 1$ to obtain

$$r = -n,\ -(n+1)$$

so that all resonances are integral.

We conclude that the only possibility for a Generalised Riccati sequence to possess the Painlevé Property is in the case that $\sigma = 1$, *ie,* $b = c$. In the case of *the* Riccati sequence as it is usually presented [41, 42, 15] $b = 1 = c$. However, the elements of the sequence may always have the dependent variable rescaled according to $u \longrightarrow by$ without loss of the Painlevé Property. The effect of this is to change the definition of the $(n+1)$th element from $(D+y)^n(y'+y^2)$ to $(D+by)^n(y'+by^2)$ after a superfluous b has been removed.

6.3 Discussion

In this Chapter we have presented the Generalised Riccati sequence and determined the coefficients of the leading-order term and the values of the resonances for each of those coefficients. It is remarkable that closed-form expressions could be found for both coefficients and resonances. Moreover the general forms of these expressions are obvious by a little inspection. Although we introduced two parameters into the definition of the sequence, the values of the resonances were expressed in terms of their ratio, σ. We saw that the possession of integral resonances was possible only in the case that $\sigma = 1$ which is the Riccati sequence itself, subject to the rescaling mentioned above. The patterns to be found for the coefficients of the leading-order terms and the resonances of the elements of the Riccati sequence are, in themselves, quite fascinating. In the case of the Generalised Riccati sequence we see that there is an even richer complexity in the structure of the resonances since the n resonances may be divided into two groups. The smaller group, based upon the parameter c, comprises just a single series of resonances. The larger group, based upon the parameter b, includes the remaining $n-1$ sets of resonances. In both groups the gross behaviour of the elements of the sequence is affected by the value of the ratio σ. For $\sigma \in (0,1)$ complex resonances must necessarily enter into the higher elements of the sequence. In fact we saw this already at $n=2$ in the specific example of the modified Painlevé-Ince equation which prompted our investigation.

Chapter 7

Differential Sequences Generated by First-Order Generators of Sequences

7.1 Introduction

Hierarchies, or differential sequences[1], generated by means of recursion operators acting upon nonlinear evolution partial differential equations are now well-known and have been studied for some decades. Leach *et al* [79] provided a generalisation of the Riccati sequence and investigated its properties in terms of singularity analysis and found that the coefficients of the leading-order terms and the resonances obey certain structural rules. Leach *et al* [80] also demonstrated the uniqueness of *the* Riccati sequence up to an equivalence class.

In the initial studies of the Riccati and Ermakov-Pinney sequences the recursion operators were adapted from the operators for the corresponding evolution partial differential equations as listed in Euler *et al* [41]. A more direct method is to adapt the general procedure for the construction of recursion operators described in some detail in the text of Bluman and Kumei [19, p 283] to the particular instance of ordinary differential equations.

In this Chapter we construct a differential sequence based upon the second-

[1]The essential content of this Chapter may be found in [89].

order equation

$$uu'' - 2u'^2 = 0 \tag{7.1.1}$$

and the generator of sequences

$$G = uD - u'. \tag{7.1.2}$$

The construction of our starting point is as follows. When we apply the Riccati transformation

$$y = \alpha\frac{u'}{u}, \quad \alpha = -1 \tag{7.1.3}$$

to the Riccati equation

$$y' + y^2 = 0 \tag{7.1.4}$$

and corresponding generator of sequences

$$G = D + y, \tag{7.1.5}$$

we obtain the recursion operator (7.1.2) and the nonlinear second-order ordinary differential equation, (7.1.1), which is an equation of maximal symmetry, after multiplying (7.1.1) and (7.1.2) by u^2 and u respectively.

Our interest in the elements of the resulting differential sequence is in the patterns found when the equations are examined for their singularity properties. We demonstrate that the leading-order behaviour and the resonances display regular patterns as we consider higher members of the sequence. Furthermore we find that an elaboration of the generator of sequences leads to a sequence of essentially the same properties. A natural feature of the members of both sequences is that after the first element in each, which possesses the eight-element algebra, $sl(3, R)$, all possess the three element algebra $A_2 \oplus A_1$ in the notation of the Mubarakzyanov [95, 96, 97] classification scheme.

7.2 The Elements of the Differential sequence

For ease of reference to the different elements of the differential sequence we label the equations as S_i, $i = 1, \ldots$. When we have occasion to refer to only the left side of the equation, we use the notation $\tilde{S}_i$. When (7.1.2) operates on

(7.1.1) (which we denote as S_1), it generates, in succession, all the members of the sequence, *videlicet*

$$\begin{aligned}
S_1 &:= uu'' - 2u'^2 = 0 \\
S_2 &:= u^2u''' - 4uu'u'' + 2u'^3 = 0 \\
S_3 &:= u^3u'''' - 3u^2u'u''' - 4u^2u''^2 + 6uu'^2u'' - 2u'^4 = 0 \\
S_4 &:= u^4u''''' - u^3u'u'''' - 11u^3u''u''' + 3u^2u'^2u''' + 8u^2u'u''^2 \\
&\quad -8uu'^3u'' + 2u'^5 = 0 \\
&\vdots \\
S_m &=: (uD - u')^{m-1}(uu'' - 2u'^2) = 0,
\end{aligned}$$

where we note that, even with a simple initial equation and a generator of sequences which is about as elementary as one could imagine, the equations quickly become quite complicated. In this set of equations we have a sufficient number to enable us to infer general properties of the members of this differential sequence.

7.3 Exponents for the Leading-order Terms

Since all of the elements of the differential sequence are homogeneous in both the independent and the dependent variables, all terms in each equation are dominant, the coefficient of the leading-order term is arbitrary and to determine the exponents one must solve the algebraic equation given by the coefficient of the powers. In order for one to obtain a singularity, the exponent in the leading-order term must either be negative or rational. For S_1, the exponents of the leading-order term are $p = -1$ and $p = 0$. The only acceptable value is $p = -1$. It is quite evident that α is arbitrary. In the case for which the exponent of the leading-order term is a positive integer we make the substitution $w = 1/z$ and repeat the above analysis. The results are displayed in Table 7.2. When the exponent p is zero, we substitute a Maclaurin expansion

$$u = \Sigma_{i=0}^{\infty} a_i \chi^i \qquad (7.3.1)$$

into each member of the sequence and find that the first $n + 1$ coefficients are arbitrary and subsequent coefficients are expressed in terms of these constants. We repeat the process for each of the S_i. The results are recorded in the Table 7.1.

Table 7.1: The exponents of the leading-order term for S_1 to S_9

Equation	Exponents
S_1	$-1,\ 0$
S_2	$-1,\ 0,\ 2$
S_3	$-1,\ 0,\ \frac{3}{2},\ 2$
S_4	$-1,\ 0,\ \frac{4}{3},\ \frac{3}{2},\ 2$
S_5	$-1,\ 0,\ \frac{5}{4},\ \frac{4}{3},\ \frac{3}{2},\ 2$
S_6	$-1,\ 0,\ \frac{6}{5},\ \frac{5}{4},\ \frac{4}{3},\ \frac{3}{2},\ 2$
S_7	$-1,\ 0,\ \frac{7}{6},\ \frac{6}{5},\ \frac{5}{4},\ \frac{4}{3},\ \frac{3}{2},\ 2$
S_8	$-1,\ 0,\ \frac{8}{7},\ \frac{7}{6},\ \frac{6}{5},\ \frac{5}{4},\ \frac{4}{3},\ \frac{3}{2},\ 2$
S_9	$-1,\ 0,\ \frac{9}{8},\ \frac{8}{7},\ \frac{7}{6},\ \frac{6}{5},\ \frac{5}{4},\ \frac{4}{3},\ \frac{3}{2},\ 2.$

The pattern is well established. In all cases there are the values -1 and 0. The former is a simple pole and acceptable. In terms of singularity analysis the latter is unacceptable. In addition we find fractional powers given by $i/(i-1)$, $i = 2, 9$. The first fraction, being the integer 2, is not acceptable for the singularity analysis. However, the other fractional powers provide candidates for the possibility of possession of the weak Painlevé Property.

Table 7.2: The resonances for the leading-order behaviour $p = 2$ of S_1 to S_n. Recall that in the transformed sequence there is a double pole corresponding to this $p = 2$ exponent.

Equation	Resonances
S_1	$0,\ -1$
S_2	$-1,\ 0,\ 6$
S_3	$-1,\ -1,\ 0,\ 6$
S_4	$-2,\ -1,\ -1,\ 0,\ 6$
S_5	$-3,\ -2,\ -1,\ -1,\ 0,\ 6$
S_6	$-4,\ -3,\ -2,\ -1,\ -1,\ 0,\ 6$
S_7	$-5,\ -4,\ -3,\ -2,\ -1,\ -1,\ 0,\ 6$
S_8	$-6,\ -5,\ -4,\ -3,\ -2,\ -1,\ -1,\ 0,\ 6$
S_9	$-7,\ -6,\ -5,\ -4,\ -3,\ -2,\ -1,\ -1,\ 0,\ 6.$
S_n	$2-n\ldots,\ -3,\ -2,\ -1,\ -1,\ 0,\ 6.$

7.4 Resonances

To determine the resonances we write $w = \alpha\chi^p + \mu\chi^{p+r}$, where the value of p is an acceptable value given in Table 7.1. In the case of S_1 we obtain the resonances $r = -1,\ 0$. This was to be expected as there is only one resonance apart from the generic -1 and we already knew that the coefficient of the leading-order term was arbitrary.

Since for the higher members of the sequence we have multiple exponents for the leading-order term, we present the results of the calculations in two parts. In Table 7.3 we give the resonances for the polelike singularity. We follow with the values of the resonances for the fractional exponents in Tables 7.4 and 7.5.

Table 7.3: The resonances for the polelike singularity of S_1 to S_n

Equation	Resonances
S_1	-1, 0
S_2	-1, 0, 3
S_3	-1, 0, 3, 5
S_4	-1, 0, 3, 5, 7
S_5	-1, 0, 3, 5, 7, 9
S_6	-1, 0, 3, 5, 7, 9, 11
S_7	-1, 0, 3, 5, 7, 9, 11, 13
S_8	-1, 0, 3, 5, 7, 9, 11, 13, 15
S_9	-1, 0, 3, 5, 7, 9, 11, 13, 15, 17
S_n	-1, 0, 3, ..., $2n-1$.

In the case of the polelike singularity the resonances are all nonnegative integers and indicate the existence of a Right Painlevé Series as the solution of each of the members of the differential sequence which is analytic in a punctured disc centred on the movable singularity.

The resonances for the fractional exponents of the leading-order term tell an entirely different story. Firstly there are two irrational resonances and this immediately removes the possibility of the possession of the weak Painlevé Property. Secondly the resonances for the same fractional exponent are identical for each element of the differential sequence with the exception of one additional resonance at each increment in the sequence. Thirdly for the highest fractional exponent all of the resonances are nonnegative except for the generic -1. For the other fractional exponents there is a mixture of signs for the resonances. Andriopoulos *et al* [13] have explicitly demonstrated the interpretation of resonances of mixed sign. Fourthly there is the occurrence of a double -1 resonance which signals the necessity of the introduction of a logarithmic term into the expansion. This in combination with the irrational resonances indicates that the fractional leading-order exponents cannot be considered as cases for which the members of the differential sequence possess the weak Painlevé Property. However, one could consider the possibility of the existence of a subsidiary solution [29] if the constants entering the expansion at the unacceptable values of the resonances are set equal to zero.

Table 7.4: The resonances for the fractional exponents of the leading-order term for S_3 to S_7

Equation	Exponents	Resonances
S_3	$\frac{3}{2}$	$-1,\ 0,\ \frac{1}{4}\left(11-\sqrt{61}\right),\ \frac{1}{4}\left(11+\sqrt{61}\right)$
S_4	$\frac{3}{2}$ $\frac{4}{3}$	$-1,\ -\frac{1}{2},\ 0,\ \frac{1}{4}\left(11-\sqrt{61}\right),\ \frac{1}{4}\left(11+\sqrt{61}\right)$ $-1,\ 0,\ \frac{2}{3},\ \frac{1}{6}\left(15-\sqrt{113}\right),\ \frac{1}{6}\left(15+\sqrt{113}\right)$
S_5	$\frac{3}{2}$ $\frac{4}{3}$ $\frac{5}{4}$	$-1,\ -1,\ -\frac{1}{2},\ 0,\ \frac{1}{4}\left(11-\sqrt{61}\right),\ \frac{1}{4}\left(11+\sqrt{61}\right)$ $-1,\ -\frac{1}{3},\ 0,\ \frac{2}{3},\ \frac{1}{6}\left(15-\sqrt{113}\right),\ \frac{1}{6}\left(15+\sqrt{113}\right)$ $-1,\ 0,\ \frac{1}{2},\ \frac{3}{4},\ \frac{1}{8}\left(19-\sqrt{181}\right),\ \frac{1}{8}\left(19+\sqrt{181}\right)$
S_6	$\frac{3}{2}$ $\frac{4}{3}$ $\frac{5}{4}$ $\frac{6}{5}$	$-\frac{3}{2},\ -1,\ -1,\ -\frac{1}{2},\ 0,\ \frac{1}{4}\left(11-\sqrt{61}\right),\ \frac{1}{4}\left(11+\sqrt{61}\right)$ $-1,\ -\frac{2}{3},\ -\frac{1}{3},\ 0,\ \frac{2}{3},\ \frac{1}{6}\left(15-\sqrt{113}\right),\ \frac{1}{6}\left(15+\sqrt{113}\right)$ $-1,\ -\frac{1}{4},\ 0,\ \frac{1}{2},\ \frac{3}{4},\ \frac{1}{8}\left(19-\sqrt{181}\right),\ \frac{1}{8}\left(19+\sqrt{181}\right)$ $-1,\ 0,\ \frac{2}{5},\ \frac{3}{5},\ \frac{4}{5},\ \frac{1}{10}\left(23-\sqrt{265}\right),\ \frac{1}{10}\left(23+\sqrt{265}\right)$
S_7	$\frac{3}{2}$ $\frac{4}{3}$ $\frac{5}{4}$ $\frac{6}{5}$ $\frac{7}{6}$	$-2,\ -\frac{3}{2},\ -1,\ -1,\ -\frac{1}{2},\ 0,\ \frac{1}{4}\left(11-\sqrt{61}\right),\ \frac{1}{4}\left(11+\sqrt{61}\right)$ $-1,\ -1,\ -\frac{2}{3},\ -\frac{1}{3},\ 0,\ \frac{2}{3},\ \frac{1}{6}\left(15-\sqrt{113}\right),\ \frac{1}{6}\left(15+\sqrt{113}\right)$ $-1,\ -\frac{1}{2},\ -\frac{1}{4},\ 0,\ \frac{1}{2},\ \frac{3}{4},\ \frac{1}{8}\left(19-\sqrt{181}\right),\ \frac{1}{8}\left(19+\sqrt{181}\right)$ $-1,\ -\frac{1}{5},\ 0,\ \frac{2}{5},\ \frac{3}{5},\ \frac{4}{5},\ \frac{1}{10}\left(23-\sqrt{265}\right),\ \frac{1}{10}\left(23+\sqrt{265}\right)$ $-1,\ 0,\ \frac{1}{3},\ \frac{1}{2},\ \frac{2}{3},\ \frac{5}{6},\ \frac{1}{12}\left(27-\sqrt{365}\right),\ \frac{1}{12}\left(27+\sqrt{365}\right)$

Table 7.5: The resonances for the fractional exponents of the leading-order term for S_8 and S_9

Equation	Exponents	Resonances
S_8	$\frac{3}{2}$	$-\frac{5}{2},\ -2,\ -\frac{3}{2},\ -1,\ -1,\ -\frac{1}{2},\ 0,\ \frac{1}{4}\left(11-\sqrt{61}\right),\ \frac{1}{4}\left(11+\sqrt{61}\right)$
	$\frac{4}{3}$	$-\frac{4}{3},\ -1,\ -1,\ -\frac{2}{3},\ -\frac{1}{3},\ 0,\ \frac{2}{3},\ \frac{1}{6}\left(15-\sqrt{113}\right),\ \frac{1}{6}\left(15+\sqrt{113}\right)$
	$\frac{5}{4}$	$-1,\ -\frac{3}{4},\ -\frac{1}{2},\ -\frac{1}{4},\ 0,\ \frac{1}{2},\ \frac{3}{4},\ \frac{1}{8}\left(19-\sqrt{181}\right),\ \frac{1}{8}\left(19+\sqrt{181}\right)$
	$\frac{6}{5}$	$-1,\ -\frac{2}{5},\ -\frac{1}{5},\ 0,\ \frac{2}{5},\ \frac{3}{5},\ \frac{4}{5},\ \frac{1}{10}\left(23-\sqrt{265}\right),\ \frac{1}{10}\left(23+\sqrt{265}\right)$
	$\frac{7}{6}$	$-1,\ -\frac{1}{6},\ 0,\ \frac{1}{3},\ \frac{1}{2},\ \frac{2}{3},\ \frac{5}{6},\ \frac{1}{12}\left(27-\sqrt{365}\right),\ \frac{1}{12}\left(27+\sqrt{365}\right)$
	$\frac{8}{7}$	$-1,\ 0,\ \frac{2}{7},\ \frac{3}{7},\ \frac{4}{7},\ \frac{5}{7},\ \frac{6}{7},\ \frac{1}{14}\left(31-\sqrt{481}\right),\ \frac{1}{14}\left(31+\sqrt{481}\right)$
S_9	$\frac{3}{2}$	$-3,\ -\frac{5}{2},\ -2,\ -\frac{3}{2},\ -1,\ -1,\ -\frac{1}{2},\ 0,\ \frac{1}{4}\left(11-\sqrt{61}\right),\ \frac{1}{4}\left(11+\sqrt{61}\right)$
	$\frac{4}{3}$	$-\frac{5}{3},\ -\frac{4}{3},\ -1,\ -1,\ -\frac{2}{3},\ -\frac{1}{3},\ 0,\ \frac{2}{3},\ \frac{1}{6}\left(15-\sqrt{113}\right),\ \frac{1}{6}\left(15+\sqrt{113}\right)$
	$\frac{5}{4}$	$-1,\ -1,\ -\frac{3}{4},\ -\frac{1}{2},\ -\frac{1}{4},\ 0,\ \frac{1}{2},\ \frac{3}{4},\ \frac{1}{8}\left(19-\sqrt{181}\right),\ \frac{1}{8}\left(19+\sqrt{181}\right)$
	$\frac{6}{5}$	$-1,\ -\frac{3}{5},\ -\frac{2}{5},\ -\frac{1}{5},\ 0,\ \frac{2}{5},\ \frac{3}{5},\ \frac{4}{5},\ \frac{1}{10}\left(23-\sqrt{265}\right),\ \frac{1}{10}\left(23+\sqrt{265}\right)$
	$\frac{7}{6}$	$-1,\ -\frac{1}{3},\ -\frac{1}{6},\ 0,\ \frac{1}{3},\ \frac{1}{2},\ \frac{2}{3},\ \frac{5}{6},\ \frac{1}{12}\left(27-\sqrt{365}\right),\ \frac{1}{12}\left(27+\sqrt{365}\right)$
	$\frac{8}{7}$	$-1,\ -\frac{1}{7},\ 0,\ \frac{2}{7},\ \frac{3}{7},\ \frac{4}{7},\ \frac{5}{7},\ \frac{6}{7},\ \frac{1}{14}\left(31-\sqrt{481}\right),\ \frac{1}{14}\left(31+\sqrt{481}\right)$
	$\frac{9}{8}$	$-1,\ 0,\ \frac{1}{4},\ \frac{3}{8},\ \frac{1}{2},\ \frac{5}{8},\ \frac{3}{4},\ \frac{7}{8},\ \frac{1}{16}\left(35-\sqrt{613}\right),\ \frac{1}{16}\left(35+\sqrt{613}\right)$

7.5 Alternate sequence

The alternate sequence[2] for the differential sequence,

$$
\begin{aligned}
S_1 &:= uu'' - 2u'^2 = 0\\
S_2 &:= u^2u''' - 4uu'u'' + 2u'^3 = 0\\
S_3 &:= u^3u'''' - 3u^2u'u''' - 4u^2u''^2 + 6uu'^2u'' - 2u'^4 = 0\\
S_4 &:= u^4u''''' - u^3u'u'''' - 11u^3u''u''' + 3u^2u'^2u''' + 8u^2u'u''^2\\
&\quad -8uu'^3u'' + 2u'^5 = 0\\
&\quad \vdots\\
S_m &\quad (uD - u')^{m-1}(uu'' - 2u'^2) = 0,
\end{aligned}
$$

may be written as

$$
\begin{aligned}
\bar{S}_1 &:= uu'' - 2u'^2 = 0\\
\bar{S}_2 &:= uu'' - 2u'^2 = c_1u\\
\bar{S}_3 &:= uu'' - 2u'^2 = c_1u\int \frac{\mathrm{d}x}{u} + c_2u\\
\bar{S}_4 &:= uu'' - 2u'^2 = c_1u\int \left(\int \frac{\mathrm{d}x}{u}\right)\frac{\mathrm{d}x}{u} + c_2u\\
&\quad \vdots
\end{aligned}
$$

where $c_1, c_2, ...$ are arbitrary constants of integration. Evidently one could continue in like fashion. However, we now have an alternate sequence of integrodifferential equations. When we compare S_2 and $\bar{S}_2$, we note that a first integral for S_2 is given by $\bar{S}_2$ in the form

$$c_1 = \tfrac{1}{u}(uu'' - 2u'^2). \tag{7.5.1}$$

When we compare S_3 and $\bar{S}_3$, we note that $\bar{S}_3$ is a second integral of S_3 in the form

$$c_1\int \tfrac{1}{u}\mathrm{d}x + c_2 = \tfrac{1}{u}(uu'' - 2u'^2), \tag{7.5.2}$$

where we may substitute (7.5.1) in (7.5.2). A similar argument may be used for all the members of the above differential sequence. Thus we may conclude that the above two sequences are completely compatible.

[2]The alternate sequence is constructed as in Chapter Four.

7.6 Discussion

Equation (7.1.1) possesses eight Lie point symmetries which constitute a representation of the algebra, $sl(3, R)$. All other elements of this differential sequence possess just three Lie point symmetries. In all cases the symmetries are

$$\begin{aligned} \Gamma_1 &= \partial_x \\ \Gamma_2 &= x\partial_x \\ \Gamma_2 &= u\partial_u, \end{aligned} \qquad (7.6.1)$$

where Γ_1 represents the autonomy of each member of the differential sequence and Γ_2 and Γ_3 represent the homogeneity of each equation with respect to the independent and dependent variables, respectively.

This dramatic loss of symmetry as one moves from the second-order representative of the sequence to all higher-order representatives is not without precedent. Another example is the Riccati sequence [15, 42] for which the second-order representative is the well-known Painlevé-Ince equation (We treated this equation in Chapter Five.),

$$y'' + 3yy' + y^3 = 0.$$

This equation possesses eight Lie point symmetries [90] and yet all higher members of the sequence have just three Lie point symmetries.

It is possible to obtain a generalisation of the differential sequence based upon equation (7.1.1) using the recursion operator (7.1.2). The method of construction is very similar. One simply introduces the Riccati transformation

$$y = c\frac{u'}{u},$$

where c is a constant. After multiplying by u the generator of sequences is then

$$R = uD + cu'$$

and the first few members of the generalised differential sequence are

$$GS_1 \;:=\; uu'' - (1-c)u'^2 = 0$$

$$
\begin{aligned}
GS_2 &:= u^2u^{(3)} + 3cuu'u'' - uu'u'' + c^2u'^3 - cu'^3 = 0 \\
GS_3 &:= u^3u^{(4)} + 4cu^2u'u^{(3)} + u^2u'u^{(3)} + 3cu^2u''^2 \\
&\quad -u^2u''^2 + 6c^2uu'^2u'' - cuu'^2u'' - uu'^2u'' + c^3u'^4 - c^2u'^4 = 0 \\
&\vdots \\
GS_m : &\quad (uD + cu')^{(m-1)}(uu'' - (1-c)u'^2) = 0.
\end{aligned}
$$

We find that the symmetry properties of the generalised sequence are the same as for the original sequence. The equation GS_1 possesses eight Lie point symmetries and the subsequent members of the sequence just the obvious three representing autonomy and homogeneity in the two variables.

The exponents of the leading-order terms are summarised in Table 7.6. The pattern is quite clear. A polelike singularity can exist if $c = -1/n$, where n is a positive integer. If c is a negative integer, apart from -1, the corresponding singularity ceases to exist. Thus for example in the case of GS_2 the possible values of the exponent when $c = -2$ are just 0 and $-1/2$. For rational values of c there is the possibility of the possession of the Weak Painlevé Property. We summarise the possible values of the resonances in Table 7.7. Apart from the generic -1 and the expected zero the formulæ for the other resonances involve the parameter c. Even for rational values of c the resonances containing square roots are generally irrational and there is a considerable interval for positive values of c for which they are complex. Only the first two exponents for the leading-order behaviour lead to principal branches for rational c.

Table 7.6: The exponents of the leading-order term for GS_1 to GS_9

Equation	Exponents
GS_1	$0,\ \frac{1}{c}$
GS_2	$0,\ \frac{1}{c},\ \frac{2}{2+c}$
GS_3	$0,\ \frac{1}{c},\ \frac{2}{2+c},\ \frac{3}{3+c}$
GS_4	$0,\ \frac{1}{c},\ \frac{2}{2+c},\ \frac{3}{3+c},\ \frac{4}{4+c}$
GS_5	$0,\ \frac{1}{c},\ \frac{2}{2+c},\ \frac{3}{3+c},\ \frac{4}{4+c},\ \frac{5}{5+c}$
GS_6	$0,\ \frac{1}{c},\ \frac{2}{2+c},\ \frac{3}{3+c},\ \frac{4}{4+c},\ \frac{5}{5+c},\ \frac{6}{6+c}$
GS_7	$0,\ \frac{1}{c},\ \frac{2}{2+c},\ \frac{3}{3+c},\ \frac{4}{4+c},\ \frac{5}{5+c},\ \frac{6}{6+c},\ \frac{7}{7+c}$
GS_8	$0,\ \frac{1}{c},\ \frac{2}{2+c},\ \frac{3}{3+c},\ \frac{4}{4+c},\ \frac{5}{5+c},\ \frac{6}{6+c},\ \frac{7}{7+c},\ \frac{8}{8+c}$
GS_9	$0,\ \frac{1}{c},\ \frac{2}{2+c},\ \frac{3}{3+c},\ \frac{4}{4+c},\ \frac{5}{5+c},\ \frac{6}{6+c},\ \frac{7}{7+c},\ \frac{8}{8+c},\ \frac{9}{9+c}$
GS_n	$0,\ \frac{1}{c},\ \frac{2}{2+c},\ \frac{3}{3+c},\ \ldots,\ \frac{n}{n+c}$

Table 7.7: The resonances for the nonzero exponents of the leading-order term for GS_1 to GS_5

Equation	Exponents	Resonances
GS_1	$\frac{1}{c}$	$-1,\ 0$
GS_2	$\frac{1}{c}$	$-1,\ 0,\ \frac{c-2}{c}$
	$\frac{2}{2+c}$	$-1,\ 0,\ \frac{2(2-c)}{2+c}$
GS_3	$\frac{1}{c}$	$-1,\ 0,\ \frac{c-2}{c},\ \frac{2c-3}{c}$
	$\frac{2}{2+c}$	$-1,\ 0,\ \frac{2(2-c)}{2+c},\ \frac{c}{2+c}$
	$\frac{3}{3+c}$	$-1,\ 0,\ -\frac{5c-6-\sqrt{c^2-24c+36}}{2(c+3)},\ -\frac{5c-6+\sqrt{c^2-24c+36}}{2(c+3)}$
GS_4	$\frac{1}{c}$	$-1,\ 0,\ \frac{c-2}{c},\ \frac{2c-3}{c},\ \frac{3c-4}{c}$
	$\frac{2}{2+c}$	$-1,\ 0,\ \frac{2(2-c)}{2+c},\ \frac{c}{2+c},\ \frac{2c}{2+c}$
	$\frac{3}{3+c}$	$-1,\ 0,\ \frac{c}{3+c},\ -\frac{5c-6-\sqrt{c^2-24c+36}}{2(c+3)},\ -\frac{5c-6+\sqrt{c^2-24c+36}}{2(c+3)}$
	$\frac{4}{4+c}$	$-1,\ 0,\ -\frac{2c}{4+c},\ -\frac{7c-8-\sqrt{c^2-48c+64}}{2(4+c)},\ -\frac{7c-8+\sqrt{c^2-48c+64}}{2(4+c)}$
GS_5	$\frac{1}{c}$	$-1,\ 0,\ \frac{c-2}{c},\ \frac{2c-3}{c},\ \frac{3c-4}{c},\ \frac{4c-5}{c}$
	$\frac{2}{2+c}$	$-1,\ 0,\ \frac{2(2-c)}{2+c},\ \frac{c}{2+c},\ \frac{2c}{2+c},\ \frac{3c}{2+c}$
	$\frac{3}{3+c}$	$-1,\ 0,\ \frac{c}{3+c},\ \frac{2c}{3+c},\ -\frac{5c-6-\sqrt{c^2-24c+36}}{2(c+3)},\ -\frac{5c-6+\sqrt{c^2-24c+36}}{2(c+3)}$
	$\frac{4}{4+c}$	$-1,\ 0,\ -\frac{2c}{4+c},\ \frac{c}{4+c},\ -\frac{7c-8-\sqrt{c^2-48c+64}}{2(4+c)},\ -\frac{7c-8+\sqrt{c^2-48c+64}}{2(4+c)}$
	$\frac{5}{5+c}$	$-1,\ 0,\ -\frac{3c}{5+c},\ -\frac{2c}{5+c},\ -\frac{9c-10-\sqrt{c^2-80c+100}}{2(5+c)},\ -\frac{9c-10+\sqrt{c^2-80c+100}}{2(4+c)}$

Chapter 8

Conclusion

8.1 Summary

There are four standard approaches to the analysis of ordinary and partial differential equations. The approaches are singularity analysis, symmetry analysis, dynamical systems and numerical methods. In this Thesis we have focused on the symmetry and singularity analyses of sequences of ordinary differential equations.

In Chapters Two and Three we provided a description of Lie analysis and singularity analysis as the differential sequences in this Thesis were analysed using the above two methods of analysis. We note that not only can we apply the above analyses to scalar equations and systems of equations but also to sequences of ordinary differential equations. It is really remarkable to analyse the nth member of a sequence without even knowing the structure of the equation.

We have studied the singularity and symmetry properties of differential sequences. In our analysis we examined the Riccati sequence which has a first-order ordinary differential equation as the seed equation. As a first-order ordinary differential equation (5.1.2) possesses an infinite number of Lie point symmetries. The second member of the sequence is the Painlevé-Ince equation which is a second-order ordinary differential equation of maximal symmetry [90]. As we proceed to the higher-order members we note that we lose symmetries. We conclude that the general member of the Riccati sequence possesses only three symmetries for $n > 2$. However, all is not lost as all the members of the Riccati sequence pass the Painlevé Test. This encourages one to seek closed-form solutions. As we have noted earlier each member of the Riccati sequence is integrable by applying the Riccati transformation (5.3.1) with $\alpha = 1$.

In Chapter Six we presented the Generalised Riccati sequence which was generated by a first-order generator of sequence and determined the coefficients of the leading-order term and the values of the resonances for each of the coefficients. We saw that possession of integral resonances was possible only in the case that $\sigma = 1$ which is the Riccati sequence itself, subject to the rescaling mentioned in Chapter Six.

In Chapter Seven we analysed the Differential sequence with a second-order ordinary differential equation of maximal symmetry as its seed equation. Once again we note the considerable loss of symmetry as we progress to the higher-order members.

8.2 Future work

In this Thesis we have only analysed differential sequences generated by first-order generators of sequences. However, one may construct sequences by using higher-order generators. A well-known paradigm of such a sequence is the Emden-Fowler sequence with a generator of sequence of the form[1]

$$G \; = \; \left(D^2 + 2y - D^{-1}y'\right)^n$$

The Emden-Fowler Differential sequence is

$$\begin{aligned} EF_1 &:= y'' + \frac{3}{2}y^2 = 0 \\ EF_2 &:= y^{(iv)} + 5yy'' + \frac{5}{2}y'^2 + \frac{5}{2}y^3 = 0 \end{aligned}$$

[1]The details for the derivation of the generator of sequence for the Emden-Fowler equation may be found in the Appendix B. We apply reduction of order to a fifth-order ordinary differential equation to obtain the Emden-Fowler equation. Details may also be found in Appendix B.

$$
\begin{aligned}
EF_3 &:= y^{(vi)} + 7yy^{(iv)} + 14y'y''' + \frac{21}{2}y''^2 + \frac{35}{2}y^2y'' + \frac{35}{2}yy'^2 + \frac{35}{8}y^4 = 0. \\
EF_4 &:= y^{(viii)} + 9yy^{(vi)} + 27y'y^{(v)} + 57y''y^{(iv)} + \frac{69}{2}y'''^2 + \frac{63}{2}y^2y^{(iv)} \\
&\quad +126yy'y''' + \frac{189}{2}yy''^2 + \frac{231}{2}y'^2y'' + \frac{105}{2}y^3y'' + \frac{315}{4}y^2y'^2 \\
&\quad +\frac{63}{8}y^5 = 0 \\
&\vdots \\
EF_n &:= \left(D^2 + 2y - D^{-1}y'\right)^n y = 0 \qquad (8.2.1)
\end{aligned}
$$

We use singularity analysis as a tool to determine whether a given differential equation is integrable in terms of functions almost everywhere analytic. We find that all members of the Emden-Fowler sequence possess the Painlevé Property. The results of the singularity analysis provide some very interesting patterns in terms of the parameters, that is the p, α and r of the standard analysis.

The Emden-Fowler equation,(8.2.1), passes the Painlevé Test with $p = -2, \alpha = -4$ and $r = -1, 6$. For all elements of the sequence the leading-order behaviour is $\alpha\chi^{-2}$ with possible values being listed in Table 8.1. The singularity analysis for the first five members of the sequence is shown in Table 8.1 in which the resonance -1 has been omitted.

Proposition I: The nth member of the Emden-Fowler sequence as defined above has the exponent of the leading-order term as $p = -2$ and the possible coefficients of the leading-order term as

$$\alpha_j = -2j(j+1), \qquad j = 1, n,$$

where n is the ranking of the equation in the sequence.

Proposition II: For the n possible leading-order behaviour of the nth element of the Emden-Fowler sequence the sum of the resonances is determined by the highest derivative in the equation, *ie,* $y^{(2n)}$, expanded as

$$y^{(2n)} = \alpha(\chi^{-2})^{(2n)} + \mu(\chi^{r-2})^{(2n)},$$

since all other terms are of lower degree in r. Thus one has

$$\sum_{i=1}^{n} r_i = -2n(n+1), \; n = 1, 2, \ldots .$$

Note that the number of terms involved in the determination of the value of α, the coefficient of the leading order term, is limited in a sense as one observes that the additional value coming from an increase in order is just that. The coefficients of earlier members are replicated and so their sum in the next equation immediately gives the next value of α.

Table 8.1: Singularity analysis of the first three members of the Emden-Fowler sequence.

Member	Leading-order coefficients	Resonances
G_1	$\alpha = -4$	$r = -1, 6$
G_2	$\alpha = -4$ $\alpha = -12$	$r = 2, 8, 5$ $r = -3, 8, 10$
G_3	$\alpha = -4$ $\alpha = -12$ $\alpha = -24$	$r = 2, 4, 5, 7, 10$ $r = -3, 2, 7, 10, 12$ $r = -5, -3, 10, 12, 14$
G_4	$\alpha = -4$ $\alpha = -12$ $\alpha = -24$ $\alpha = -40$	$r = -1, 2, 4, 5, 6, 7, 9, 12$ $r = -3, -1, 2, 4, 7, 9, 12, 14$ $r = -5, -3, -1, 2, 9, 12, 14, 16$ $r = -7, -5, -3, -1, 12, 14, 16, 18$
G_5	$\alpha = -4$ $\alpha = -12$ $\alpha = -24$ $\alpha = -40$ $\alpha = -60$	$r = -1, 2, 4, 5, 6, 7, 9, 11, 14$ $r = -3, -1, 2, 4, 6, 7, 9, 11, 14, 16$ $r = -5, -3, -1, 2, 4, 9, 11, 14, 16, 18$ $r = -7, -5, -3, -1, 2, 11, 14, 16, 18, 20$ $r = -9, -7, -5, -3, -1, 14, 16, 18, 20, 22$

Appendix A

An application of Lie analysis to the two-dimensional Ermakov system

A.1 Introduction

The elementary[1] two-dimensional Ermakov system[2], *videlicet*

$$\begin{aligned} \ddot{x} &= \frac{1}{x^3} \\ \ddot{y} &= \frac{1}{y^3}, \end{aligned} \tag{A.1.1}$$

possesses the Lie point symmetries

$$\begin{aligned} \Gamma_1 &= \partial_t \\ \Gamma_2 &= t\partial_t + \tfrac{1}{2}(x\partial_x + y\partial_y) \\ \Gamma_3 &= t^2\partial_t + t(x\partial_x + y\partial_y). \end{aligned} \tag{A.1.2}$$

We note that a general structure for the Lie point symmetries above is

$$\Gamma_i = f_i(t)\partial_t + \tfrac{1}{2}\dot{f}_i(t)(x\partial_x + y\partial_y), \quad i = 1, 3, \tag{A.1.3}$$

[1]The reduction from the more general system including $\omega^2(t)x$ and $\omega^2(t)y$ is readily achieved [71].

[2]The essential content of this section is found in [88].

where the overdot represents differentiation with respect to time, t, and f depends upon time only.
If we take a general second-order system of equations, *videlicet*

$$\begin{aligned} \ddot{x} &= F(t,x,y,\dot{x},\dot{y}) \\ \ddot{y} &= G(t,x,y,\dot{x},\dot{y}), \end{aligned} \tag{A.1.4}$$

and the second extension of (A.1.3) with $i=1$,

$$\begin{aligned} \Gamma_1^{[2]} &= f_1\partial_t + \tfrac{1}{2}\dot{f}_1\left(x\partial_x + y\partial_y\right) + \tfrac{1}{2}\left(\ddot{f}_1 x - \dot{f}_1\dot{x}\right)\partial_{\dot{x}} + \tfrac{1}{2}\left(\ddot{f}_1 y - \dot{f}\dot{y}\right)\partial_{\dot{y}} \\ &\quad + \tfrac{1}{2}\left(\dddot{f}_1 x - 3\dot{f}_1\ddot{x}\right)\partial_{\ddot{x}} + \tfrac{1}{2}\left(\dddot{f}_1 y - 3\dot{f}_1\ddot{y}\right)\partial_{\ddot{y}}, \end{aligned} \tag{A.1.5}$$

the action of (A.1.5) on equations (A.1.4a) and (A.1.4b) is

$$\begin{aligned} & f_1\frac{\partial F}{\partial t} + \tfrac{1}{2}\dot{f}_1\left(x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y}\right) + \tfrac{1}{2}\left(\ddot{f}_1 x - \dot{f}_1\dot{x}\right)\frac{\partial F}{\partial \dot{x}} + \tfrac{1}{2}\left(\ddot{f}_1 y - \dot{f}\dot{y}\right)\frac{\partial F}{\partial \dot{y}} \\ &= \tfrac{1}{2}\left(\dddot{f}_1 x - 3\dot{f}_1 F\right) \end{aligned} \tag{A.1.6}$$

and

$$\begin{aligned} & f_1\frac{\partial G}{\partial t} + \tfrac{1}{2}\dot{f}_1\left(x\frac{\partial G}{\partial x} + y\frac{\partial G}{\partial y}\right) + \tfrac{1}{2}\left(\ddot{f}_1 x - \dot{f}_1\dot{x}\right)\frac{\partial G}{\partial \dot{x}} + \tfrac{1}{2}\left(\ddot{f}_1 y - \dot{f}\dot{y}\right)\frac{\partial G}{\partial \dot{y}} \\ &= \tfrac{1}{2}\left(\dddot{f}_1 y - 3\dot{f}_1 G\right), \end{aligned} \tag{A.1.7}$$

respectively. The associated Lagrange's system for (A.1.6) is

$$\frac{\mathrm{d}t}{f_1} = \frac{\mathrm{d}x}{\frac{1}{2}\dot{f}_1 x} = \frac{\mathrm{d}y}{\frac{1}{2}\dot{f}_1 y} = \frac{\mathrm{d}\dot{x}}{\frac{1}{2}(\ddot{f}_1 x - \dot{f}_1\dot{x})} = \frac{\mathrm{d}\dot{y}}{\frac{1}{2}(\ddot{f}_1 y - \dot{f}_1\dot{y})} = \frac{\mathrm{d}F}{\frac{1}{2}(\dddot{f}_1 x - 3\dot{f}_1 F)} \tag{A.1.8}$$

when equations (A.1.4) are taken into account.
The invariants arising from combination of the first with the second, third, fourth, fifth and sixth elements of (1.1.99), respectively, are

$$\begin{aligned} & u_1 = xf_1^{-1/2}, \quad u_2 = yf_1^{-1/2}, \quad u_3 = \dot{x}f_1^{1/2} - x(f_1^{1/2})^{\dot{}}, \\ & u_4 = \dot{y}f_1^{1/2} - y(f_1^{1/2})^{\dot{}} \quad \text{and} \quad \mathcal{F} = Ff_1^{3/2} - x(f_1^{1/2})^{\ddot{}}f_1. \end{aligned}$$

It is evident with these five invariants that we may write

$$F = \frac{(f_1^{1/2})\ddot{} x}{f_1^{1/2}} + f_1^{-3/2}\mathcal{F}\left(\frac{x}{f_1^{1/2}}, \frac{y}{f_1^{1/2}}, f_1^{1/2}\dot{x} - (f_1^{1/2})\dot{} x, f_1^{1/2}y - (f_1^{1/2})\dot{} y\right). \tag{A.1.9}$$

A considerable simplification in appearance is achieved if we make the replacement $f_1 = \rho^2$ for then the system (A.1.4) may be written as

$$\begin{aligned} \ddot{x} &= \frac{\ddot{\rho}x}{\rho} + \frac{1}{\rho^3}\mathcal{F}\left(\frac{x}{\rho}, \frac{y}{\rho}, \rho\dot{x} - \dot{\rho}x, \rho\dot{y} - \dot{\rho}y\right) \\ \ddot{y} &= \frac{\ddot{\rho}x}{\rho} + \frac{1}{\rho^3}\mathcal{G}\left(\frac{x}{\rho}, \frac{y}{\rho}, \rho\dot{x} - \dot{\rho}x, \rho\dot{y} - \dot{\rho}y\right). \end{aligned} \tag{A.1.10}$$

The generalised canonical transformation [26, 27], equally a Kummer-Liouville transformation [68, 87],

$$T = \int \rho^{-2}, \qquad X = \frac{x}{\rho}, \qquad \dot{X} = \rho\dot{x} - \dot{\rho}x, \tag{A.1.11}$$

which has found much application in both Hamiltonian and Quantum Mechanics [70, 84, 108], reduces system (A.1.10) to

$$\begin{aligned} X'' &= \mathcal{F}(X, Y, X', Y') \\ Y'' &= \mathcal{G}(X, Y, X', Y') \end{aligned} \tag{A.1.12}$$

which is independent of the variable t so that system (A.1.12) is autonomous. The symmetry, Γ_1, of (A.1.3) with $i = 1$ has been reduced to ∂_T. We now revert to lowercase variables and invoke the structure of $sl(2, R)$ in the standard form

$$[\Gamma_1, \Gamma_2]_{LB} = \Gamma_1, \quad [\Gamma_1, \Gamma_3]_{LB} = 2\Gamma_2 \quad \text{and} \quad [\Gamma_2, \Gamma_3]_{LB} = \Gamma_3.$$

The Lie Bracket of Γ_1 with Γ_2 as given in (A.1.3) with $i = 2$ implies that $\dot{f}_2$ is a constant. Thus we have

$$\Gamma_2 = t\partial_t + \tfrac{1}{2}\left(x\partial_x + y\partial_y\right).$$

In a similar fashion with Γ_1 and Γ_3 we obtain that $f_3 = t^2 + constant$ so that

$$\Gamma_3 = \left(t^2 + K\right)\partial_t + t\left(x\partial_x + y\partial_y\right),$$

where K is some constant. The final bracket, that of Γ_2 and Γ_3, requires that K be zero. Hence the canonical representation of $sl(2,R)$, *videlicet*

$$\begin{aligned} \Gamma_1 &= \partial_t \\ \Gamma_2 &= t\partial_t + (x\partial_x + y\partial_y) \\ \Gamma_3 &= t^2\partial_t + t\,(x\partial_x + y\partial_y)\,, \end{aligned} \tag{A.1.13}$$

is the appropriate one to use once we choose Γ_1 to be ∂_t. We note that the representation (A.1.2) for (A.1.1) is exactly the same as this representation for the more general system.
The action of the second extension of Γ_2, *videlicet*

$$\Gamma_2^{[2]} = t\partial_t + \tfrac{1}{2}(x\partial_x + y\partial_y) - \tfrac{1}{2}\dot{x}\partial_{\dot{x}} - \tfrac{1}{2}\dot{y}\partial_{\dot{y}} - \tfrac{3}{2}\ddot{x}\partial_{\ddot{x}} - \tfrac{3}{2}\ddot{y}\partial_{\ddot{y}}, \tag{A.1.14}$$

on the first member of the autonomous system (A.1.12) leads to the associated Lagrange's system,

$$\frac{\mathrm{d}x}{x} = \frac{\mathrm{d}y}{y} = \frac{\mathrm{d}\dot{x}}{-\dot{x}} = \frac{\mathrm{d}\dot{y}}{-\dot{y}} = \frac{\mathrm{d}F}{-3F}, \tag{A.1.15}$$

and three of the characteristics are easily found to be $v_1 = x/y$, $v_2 = x\dot{x}$ and $v_3 = Fx^3$. The determination of v_4 is slightly complicated. The combination $\dot{y}\times(1.1.106a) - \dot{x}\times(1.1.106b) - y\times(1.1.106c) - x\times(1.1.106d)$ gives $v_4 = x\dot{y} - \dot{x}y$. A similar calculation applies for the second of (A.1.12). Now the system (A.1.4) has the form

$$\begin{aligned} \ddot{x} &= \frac{1}{x^3}F\left(\frac{x}{y}, x\dot{x}, x\dot{y} - \dot{x}y\right) \\ \ddot{y} &= \frac{1}{y^3}G\left(\frac{x}{y}, x\dot{x}, x\dot{y} - \dot{x}y\right). \end{aligned} \tag{A.1.16}$$

The second extension of Γ_3 is

$$\begin{aligned} \Gamma_3^{[2]} &= t^2\partial_t + t(x\partial_x + y\partial_y) + (x - t\dot{x})\partial_{\dot{x}} + (y - t\dot{y})\partial_{\dot{y}} - 3t(\ddot{x}\partial_{\ddot{x}} + \ddot{y}\partial_{\ddot{y}}) \\ &= t^2\partial_t + 2t\left[t\partial_t + \tfrac{1}{2}(x\partial_x + y\partial_y) - \tfrac{1}{2}(\dot{x}\partial_x + \dot{y}\partial_y) - \tfrac{3}{2}(\ddot{x}\partial_{\ddot{x}} + \ddot{y}\partial_{\ddot{y}})\right] + x\partial_{\dot{x}} + y\partial_{\dot{y}} \\ &= t^2\Gamma_1^{[2]} + 2t\Gamma_2^{[2]} + x\partial_{\dot{x}} + y\partial_{\dot{y}} \end{aligned} \tag{A.1.17}$$

which we may write as the effective operator

$$\Gamma_{3eff}^{[2]} = x\partial_{\dot{x}} + y\partial_{\dot{y}}. \tag{A.1.18}$$

The action of (A.1.18) on system (A.1.16) provides the general system without loss of generality as

$$\begin{aligned}\ddot{x} &= \frac{1}{x^3}F\left(\frac{x}{y}, x\dot{y}-\dot{x}y\right)\\ \ddot{y} &= \frac{1}{y^3}G\left(\frac{x}{y}, x\dot{y}-\dot{x}y\right).\end{aligned}\tag{A.1.19}$$

The structure and nature of Ermakov systems are best appreciated in plane polar coordinates in which the equations in (A.1.19) become

$$\begin{aligned}\ddot{r}-r\dot{\theta}^2 &= \frac{1}{r^3}F(\theta, r^2\dot{\theta})\\ r\ddot{\theta}+2\dot{r}\dot{\theta} &= \frac{1}{r^3}G(\theta, r^2\dot{\theta}),\end{aligned}\tag{A.1.20}$$

where the functions in (A.1.19) are related to those in (A.1.20) according to

$$\begin{aligned}F\left(\theta, r^2\dot{\theta}\right) &= \frac{F}{\cos^2\theta}+\frac{G}{\sin^2\theta}\\ G\left(\theta, r^2\dot{\theta}\right) &= -\frac{F\sin\theta}{\cos^3\theta}+\frac{G\cos\theta}{\sin^3\theta}.\end{aligned}$$

A.2 Implementation of the method of reduction of order

The implementation of the method of reduction of order [99, 100] consists of two parts. The system of differential equations is written in terms of first-order equations and ignorable variables may be eliminated to reduce the order of the system. To enable an algorithmic calculation of symmetries the reduced system of first-order equations is rewritten to include at least one second-order equation. There is always a question of an appropriate choice of variables.
We set

$$w_1 = r, \quad w_2 = \theta, \quad w_3 = \dot{r} \quad \text{and} \quad w_4 = \dot{\theta}.$$

Then system (A.1.20) becomes a system of four first-order ordinary differential equations, *videlicet*

$$\begin{aligned} \dot{w}_1 &= w_3 & \dot{w}_3 &= \frac{1}{w_1^3} F(w_2, w_1^2 w_4) + w_1 w_4^2 \\ \dot{w}_2 &= w_4 & \dot{w}_4 &= \frac{1}{w_1^4} G(w_2, w_1^2 w_4) - 2\frac{w_3 w_4}{w_1}. \end{aligned} \tag{A.2.1}$$

We choose θ as the new independent variable. The system (A.2.1) may be written as

$$w_1' = \frac{w_3}{w_4} \tag{A.2.2}$$

$$w_3' = \frac{1}{w_1^3 w_4} F(\theta, w_1^2 w_4) + w_1 w_4 \tag{A.2.3}$$

$$w_4' = \frac{1}{w_1^4 w_4} G(\theta, w_1^2 w_4) - 2\frac{w_3}{w_1}, \tag{A.2.4}$$

where the prime denotes differentiation with respect to θ. We eliminate w_3 through the relation $w_3 = w_1' w_4$. Then (A.2.3) and (A.2.4) become

$$w_1'' w_4 + w_1' w_4' = \frac{1}{w_1^3 w_4} F(\theta, w_1^2 w_4) + w_1 w_4 \tag{A.2.5}$$

$$(w_1^2 w_4)' = \frac{1}{w_1^2 w_4} G(\theta, w_1^2 w_4), \tag{A.2.6}$$

respectively. When we let $z = w_1^2 w_4$, (A.2.6) may be written as a first-order equation for z, *ie*

$$z' = \frac{1}{z} G(\theta, z). \tag{A.2.7}$$

On substitution of (A.2.4) into (A.2.5) and division by $w_1^2 w_4$ we have

$$\frac{w_1''}{w_1^2} - 2\frac{(w_1')^2}{w_1^3} + \frac{w_1'}{w_1^6 w_4^2} G(\theta, w_1^2 w_4) - \frac{1}{w_1} - \frac{1}{w_1^5 w_4^2} F(\theta, w_1^2 w_4) = 0 \tag{A.2.8}$$

which may be written as

$$\left(\frac{1}{w_1}\right)'' + \left(\frac{1}{w_1}\right)' \frac{G(\theta, z)}{z^2} + \frac{1}{w_1} + \frac{F(\theta, z)}{w_1 z^2} = 0. \tag{A.2.9}$$

When we set $u_1 = w_1^{-1}$, (A.2.9) becomes

$$u_1'' + u_1' \frac{G(\theta, z)}{z^2} + \left(1 + \frac{F(\theta, z)}{z}\right) u_1 = 0. \tag{A.2.10}$$

Hence the system (A.1.20) reduces to a linear second-order differential equation plus a first-order differential equation, *ie*

$$u_1'' + u_1'\frac{G(\theta,z)}{z^2} + \left(1+\frac{F(\theta,z)}{z}\right)u_1 = 0 \qquad \text{(A.2.11)}$$

$$z' = \frac{1}{z}G(\theta,z). \qquad \text{(A.2.12)}$$

Suppose that the solution of the z equation can be written as $I = u_2(\theta,z)$. Then the change of variable to u_2 in place of z provides the system

$$u_1'' + a(\theta,u_2)u_1' + b(\theta,u_2)u_1 = 0 \qquad \text{(A.2.13)}$$

$$u_2' = 0. \qquad \text{(A.2.14)}$$

The transformation to normal form is

$$u_1(\theta) = U_1(\theta)\exp\left[-\tfrac{1}{2}\int a(\theta,u_2(\theta))\mathrm{d}\theta\right]$$

which allows us to write (A.2.13) as

$$u_1'' + \omega^2 u_1 = 0,$$

where

$$\omega^2 = b - \tfrac{1}{2}\frac{\mathrm{d}^2 a}{\mathrm{d}\theta^2} + \tfrac{1}{4}\left(\frac{\mathrm{d}a}{\mathrm{d}\theta}\right)^2.$$

We may write (A.2.13/A.2.14) as

$$u_1'' + \omega^2 u_1 = 0 \qquad \text{(A.2.15)}$$

$$u_2' = 0 \qquad \text{(A.2.16)}$$

respectively. The change of independent and dependent variables by the transformation [26, 27],

$$\Theta = \int \rho^{-2}(\theta)\mathrm{d}\theta, \qquad U_1 = \frac{u_1}{\rho(\theta)}, \qquad U_2 = u_2, \qquad \text{(A.2.17)}$$

with $\rho(\theta)$ being the solution of

$$\rho'' + \omega^2\rho = \frac{1}{\rho^3}$$

allows the reduction of (A.2.13/A.2.14) to the structure of the reduced Kepler Problem, *ie* a linear oscillator plus a conservation law, *videlicet*

$$\begin{aligned} \frac{\mathrm{d}^2U_1}{\mathrm{d}\Theta^2} + U_1 &= 0 \\ \frac{\mathrm{d}U_2}{\mathrm{d}\Theta} &= 0. \end{aligned} \tag{A.2.18}$$

However, there is a difference. We note that the independent variable is no longer time t, but the coordinate Θ. We now turn our attention to the periodicity in Θ. We have

$$\begin{aligned} \Theta(\theta) &= \int_{\theta_0}^{\theta} \rho^{-2}(\eta)\mathrm{d}\eta \\ \Theta(\theta + 2\pi) &= \int_{\theta_0}^{\theta+2\pi} \rho^{-2}(\eta)\mathrm{d}\eta \\ \Theta(\theta + 2\pi) - \Theta(\theta) &= \int_{\theta}^{\theta+2\pi} \rho^{-2}(\eta)\mathrm{d}\eta \\ &\neq 2\pi \end{aligned} \tag{A.2.19}$$

in general. From (A.2.19) we deduce that the system (A.2.18) is no longer naturally periodic and that identification of (A.2.13/A.2.14) with the reduced form of the Kepler Problem is in general not possible. To maintain periodicity the angular variable cannot be changed so that the only possibility for (A.2.13/A.2.14) to be identified with the reduced form of the Kepler Problem is if (A.2.13) can be written in the normal form $U_1'' + U_1 = 0$ by means of a transformation of u_1 only and the required normal form demands that

$$b - \tfrac{1}{2}a' + \tfrac{1}{4}a^2 = 1. \tag{A.2.20}$$

One can view this condition as a differential equation to be satisfied by a given b. We write equation (A.2.20) as

$$\psi'' + (b-1)\psi = 0$$

by introducing the Riccati transformation $a = -2\psi'/\psi$. Alternatively the condition may be interpreted as defining b given a. In terms of the functions of the original system we have

$$F(\theta, z) = \frac{1}{4z^3}\left\{2z^2\frac{\partial G}{\partial \theta} + 2zG\frac{\partial G}{\partial z} - 4zG - G^2\right\} \tag{A.2.21}$$

in which form we observe that the identification of the two reduced forms is a matter of the balance – a not completely intuitive balance – between F and G in system (A.1.20).

A.3 Complete Symmetry Group of the Two-dimensional Ermakov System

Although we have noted the physical constraint on the reduction of the two-dimensional Ermakov system to a combination of a simple harmonic oscillator and a conservation law, this does not inhibit us from using the latter combination as a basis for the determination of the complete symmetry group of the two-dimensional Ermakov system. The basic theory for our treatment is to be found in a number of papers of Andriopoulos *et al* [9, 10, 75]. The Lie point symmetries of the system

$$u_1'' + u_1 = 0 \qquad \text{(A.3.1)}$$
$$u_2' = 0 \qquad \text{(A.3.2)}$$

are

$$\begin{aligned} \Lambda_{1\pm} &= \exp[\pm i\theta]\partial_{u_1} \\ \Lambda_2 &= u_1\partial_{u_1} \\ \Lambda_{3\pm} &= \exp[\pm 2i\theta]\left(\partial_\theta \pm iu_1\partial_{u_1}\right) \\ \Lambda_4 &= \partial_\theta \\ \Lambda_{5\pm} &= \exp[\pm i\theta]u_1\left(\partial_\theta \pm iu_1\partial_{u_1}\right) \\ \Lambda_6 &= \partial_{u_2}. \end{aligned}$$

We note that Λ_6 can have an arbitrary function of u_2 as coefficient since it is a constant[3] and we make use of this to determine the complete symmetry group.

Proposition: The complete symmetry group of system (A.3.1/A.3.2) has the algebra $A_1 \oplus_s 3A_1$ and is represented by the operators

$$\begin{aligned} K_{1\pm} &= \exp[\pm i\theta]\partial_{u_1} \\ K_2 &= u_1\partial_{u_1} + u_2\partial_{u_2} \\ K_3 &= \partial_{u_2}. \end{aligned}$$

[3]One finds the same phenomenon occurring in the properties of the contact symmetries of second-order ordinary differential equations [94].

Remark: In general the dimension of the algebra equals the sum of the number of dependent variables plus one, but systems such as this have one dimension fewer [10].
Proof: We commence with the general forms

$$u_1'' = f(\theta, u_1, u_2, u_1') \tag{A.3.3}$$
$$u_2' = g(\theta, u_1, u_2, u_1') \tag{A.3.4}$$

and apply the second extensions of $K_{\pm 1}$, K_2 and K_3 in turn. We obtain, respectively,

$$\begin{array}{ll} f = u_1 + F(\theta, u_2) & g = G(\theta, u_2), \\ F = A(\theta)u_2 & G = B(\theta)u_2 \quad \text{and} \\ A = 0 & B = 0 \end{array} \tag{A.3.5}$$

so that indeed the system (A.3.1/A.3.2) is recovered.
Before we treat the general case we take a particular example for which it is possible to do all of the calculations in explicit form to provide a precise guide to the flavour of the exercise. As our example we consider the system

$$\ddot{r} - r\dot{\theta}^2 = \frac{1}{r^3} \tag{A.3.6}$$
$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \tag{A.3.7}$$

which represents the two-dimensional motion of a particle in a Newton-Cotes potential [115] [p 83]. We summarise the first part of the transformation resulting from the reduction of order and introduction of a new independent variable as

$$u = \frac{1}{r} \quad \text{and} \quad z = r^2\dot{\theta} \tag{A.3.8}$$

with the new independent variable being θ. System (A.3.6/A.3.7) becomes

$$\begin{aligned} u'' + \left(1 + \frac{1}{z^2}\right)u &= 0 \\ z' &= 0. \end{aligned}$$

We express this in the more compact form

$$u'' + w^2 u = 0 \tag{A.3.9}$$
$$w' = 0, \tag{A.3.10}$$

where $w^2 := 1 + z^{-2}$. We obtain the formal form of system (A.3.1/A.3.2) by means an adaptation of the Kummer-Liouville transformation usually used for the time-dependent oscillator. Thus we write

$$u_1 = \frac{u}{\rho}, \qquad u_2 = \frac{w}{\rho} \qquad \text{and} \qquad \Theta = \int \frac{\mathrm{d}\theta}{\rho^2}, \tag{A.3.11}$$

where $\rho^2 = w^{-1}$.

It is a trivial matter to show that the relevant K symmetries of system (A.3.1/A.3.2) become

$$\begin{aligned} K_{1\pm} &= \exp[\pm iw\theta]\partial_u \\ K_2 &= u\partial_u + w\partial_w \\ K_3 &= \partial_w \end{aligned} \tag{A.3.12}$$

up to a common constant multiplier which we omit.

A symmetry of the form

$$\Gamma = \tau\partial_t + \eta\partial_r + \zeta\partial_\theta + \left(\dot{\zeta} - \dot{\theta}\dot{\tau}\right)\partial_{\dot{\theta}}, \tag{A.3.13}$$

which we have extended once in the θ variable to be able to make the transition to the w variable, becomes

$$\Gamma = \zeta\partial_\theta - \frac{\eta}{r^2}\partial_u + \left\{-\frac{4r\dot{\theta}\eta}{z^3} + \left(\dot{\zeta} - \dot{\theta}\dot{\tau}\right)\left(-\frac{2r^2}{z^3}\right)\right\}\partial_w \tag{A.3.14}$$

in terms of the differential operators of (A.3.12). We commence with K_3 for which it is evident by comparison with (A.3.14) that

$$\zeta = 0, \quad \eta = 0 \quad \text{and} \quad \frac{2r^2\dot{\theta}\dot{\tau}}{z}^3 = 1.$$

Hence we have the symmetry

$$\Lambda_3 = \left(\int \tfrac{1}{2}z^2\mathrm{d}t\right)\partial_t. \tag{A.3.15}$$

Since z is a constant, it can be removed from the integral and indeed from the symmetry so far as its role in the construction of the extension of the symmetry is concerned. However, we keep it for algebraic purposes.

When we look at K_2, we see that

$$\zeta = 0, \quad \eta = -r^2 u = -r \quad \text{and} \quad \dot{\tau} = \tfrac{1}{2}\left(1 - 3z^2\right)$$

so that the symmetry is

$$\Lambda_2 = \left(\int \tfrac{1}{2}\left(1 - 3z^2\right)\mathrm{d}t\right)\partial_t - r\partial_r. \tag{A.3.16}$$

In the case of $K_{1\pm}$ we have

$$\zeta = 0, \quad \eta = -r^2 \exp[\pm iw\theta] \quad \text{and} \quad \dot{\tau} = -2$$

so that the final symmetries are

$$\Lambda_{1\pm} = \left(\int 2r \exp[\pm iw\theta]\mathrm{d}t\right)\partial_t + r^2 \exp[\pm iw\theta]\partial_r. \tag{A.3.17}$$

In addition we have the symmetry $\Lambda_4 = \partial_t$ which was the symmetry which enabled us to reduce the order in the first place.

To verify that the symmetries are in fact a representation of the complete symmetry group system (A.3.6/A.3.7) we apply them in turn to the general system

$$\begin{aligned} \ddot{r} &= f(r, \theta, \dot{r}, \dot{\theta}) \\ \ddot{\theta} &= g(r, \theta, \dot{r}, \dot{\theta}) \end{aligned}$$

in which the effect of Λ_4 has already been invoked. The second extensions of the symmetries are

$$\begin{aligned} \Lambda_{1\pm}^{[2]} &= \left(\int 2r \exp[\pm iw\theta]\mathrm{d}t\right)\partial_t + r^2 \exp[\pm iw\theta]\partial_r \pm ir^2 w \exp[\pm iw\theta]\partial_{\dot{r}} && \text{(A.3.18)} \\ &\quad -2r\dot{\theta}\exp[\pm iw\theta]\partial_{\dot{\theta}} + \left(\pm 2ir\dot{r}w\dot{\theta} \pm ir^2 w\ddot{\theta} - r^2 w^2 \dot{\theta}^2 - 2r\ddot{r}\right)\exp[\pm iw\theta]\partial_{\ddot{\theta}} \\ \Lambda_2^{[2]} &= \left(\int \tfrac{1}{2}\left(1 - 3z^2\right)\mathrm{d}t\right)\partial_t - r\partial_r - \tfrac{3}{2}\dot{r}\left(1 - z^2\right)\partial_{\dot{r}} && \text{(A.3.19)} \\ &\quad -\tfrac{1}{2}\dot{\theta}\left(1 - 3z^2\right)I\partial_{\dot{\theta}} - \ddot{r}\partial_{\ddot{r}} - \ddot{\theta}\left(1 - 3z^2\right)\partial_{\ddot{\theta}} \\ \Lambda_3^{[2]} &= \left(\int \tfrac{1}{2}z^2\mathrm{d}t\right)\partial_t - \tfrac{1}{2}\dot{r}z^2\partial_{\dot{r}} - \tfrac{1}{2}\dot{\theta}z^2\partial_{\dot{\theta}} - \ddot{r}z^2\partial_{\ddot{r}} - \ddot{\theta}z^2\partial_{\ddot{\theta}}. && \text{(A.3.20)} \end{aligned}$$

We commence with the angular equation. The effect of the action of $\Lambda^{[2]}_{1\pm}$ is to force g to take the form

$$g = -\frac{2\dot{r}\dot{\theta}}{r} + \frac{1}{r^4}G(\theta, r^2\dot{\theta}). \tag{A.3.21}$$

When we apply $\Lambda^{[2]}_2$, we find that

$$G = \left(r^2\dot{\theta}\right)^n \mathcal{G}(\theta), \tag{A.3.22}$$

where the exponent is given by $n = 2\left(5 - 3z^2\right)/3\left(1 - z^2\right)$. Not surprisingly the final symmetry, $\Lambda^{[2]}_3$, forces $\mathcal{G}$ to be zero and we recover the angular equation, (A.3.7).

Proceeding in the same fashion with the radial equation we find successively

$$f = w^2 r\dot{\theta}^2 + \frac{1}{r^2}F(\theta, r^2\dot{\theta}) \tag{A.3.23}$$

$$F = \mathcal{F}(\theta)\left(r^2\dot{\theta}\right)^{-m}, \quad m = 2/3\left(1 + z^2\right), \tag{A.3.24}$$

$$\mathcal{F} = 0. \tag{A.3.25}$$

The radial equation is

$$\begin{aligned} \ddot{r} &= w^2 r\dot{\theta}^2 = \left(1 + \frac{1}{z^2}\right) r\dot{\theta}^2 \\ &= r\dot{\theta}^2 + \frac{r\dot{\theta}^2}{r^4\dot{\theta}^2} \end{aligned}$$

and (A.3.6) is recovered.

We have spent some time on this simple example because every calculation can be performed explicitly and so the computation is transparent. We turn now to the general case which we must treat somewhat differently since not all of the transformations are explicit. The general symmetry (A.3.13) becomes

$$\Gamma = \zeta\partial_\theta - \frac{\eta}{r^2}\partial_u + \left[2r\dot{\theta}\eta + \left(\dot{\zeta} - \dot{\theta}\dot{\tau}\right)r^2\right]\partial_z, \tag{A.3.26}$$

where $u = r^{-1}$ and $z = r^2\dot{\theta}$. (Were we considering Γ as an algebraic operator, it would be necessary to replace $\dot{\theta}$ by zr^{-2}.) We recall that the angular equation is

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{1}{r^3}G(\theta, r^2\dot{\theta}).$$

In terms of the new variables this is

$$z' = \frac{G(\theta, z)}{z}$$

for which the existence of a solution is guaranteed provided that the right hand side is a continuous function of its variables [61, p 72]. We write the integral as

$$I = u_2(\theta, z)$$

which we can write as

$$z = H(\theta, u_2) \qquad u_2 = J(\theta, z). \tag{A.3.27}$$

Consequently we can write the system (A.1.20) as

$$\begin{aligned} u'' + \frac{G}{z^2}u' + \left(1 + \frac{F}{z^2}\right)u &= 0 && \text{(A.3.28)} \\ u_2' &= 0. && \text{(A.3.29)} \end{aligned}$$

It is convenient to put (A.3.28) into normal form by means of the transformation

$$u = v \exp\left[-\tfrac{1}{2}\int \frac{G}{z^2}\mathrm{d}\theta\right] \qquad v = u \exp\left[\tfrac{1}{2}\int \frac{G}{z^2}\mathrm{d}\theta\right].$$

The normal form of (A.3.28) is

$$v'' + \omega^2 v = 0, \tag{A.3.30}$$

where

$$\omega^2 = 1 + \frac{F}{z^2} - \left(\frac{G}{2z^2}\right)' - \left(\frac{G}{2z^2}\right)^2.$$

The differential operator (A.3.26) is now

$$\begin{aligned} \Gamma &= \zeta\partial_\theta + \left[\zeta u \frac{G}{2z^2}\exp\left(\int \frac{G}{2z^2}\mathrm{d}\theta\right) - \frac{\eta}{r^2}\exp\left(\int \frac{G}{2z^2}\mathrm{d}\theta\right)\right]\partial_v \\ &\quad + \left\{\zeta\frac{\partial J}{\partial\theta} + \left[2r\dot{\zeta}\eta + \left(\dot{\zeta} - \dot{\theta}\dot{\tau}\right)r^2\right]\frac{\partial J}{\partial z}\right\}\partial_{u_2}. && \text{(A.3.31)} \end{aligned}$$

The final transformation to render the nonautonomous oscillator equation, (A.3.30), autonomous is

$$V = \frac{v}{\rho} \qquad \psi = \int \frac{\mathrm{d}\theta}{\rho^2},$$

where $\rho(\theta)$ is a solution of the Pinney equation

$$\rho'' + \omega^2\rho = \frac{1}{\rho^3}.$$

The ultimate form of the general differential operator is then

$$\begin{aligned}\Gamma &= \frac{\zeta}{\rho^2}\partial_\psi + \left[-\frac{\zeta u\rho'}{\rho^t wo} + \zeta u\left(\frac{G}{2z^2}\right) - \frac{\eta}{r^2\rho}\right]\exp\left[\int\frac{G}{2z^2}\mathrm{d}\theta\right]\partial_V \\ &\quad + \left\{\zeta\frac{\partial J}{\partial\theta} + \left[2r\dot{\zeta}\eta + \left(\dot{\zeta} - \dot{\theta}\dot{\tau}\right)r^2\right]\frac{\partial J}{\partial z}\right\}\partial_{u_2}. \qquad \text{(A.3.32)}\end{aligned}$$

In terms of the current variables a representation of the complete symmetry group of the autonomous system is given by

$$K_{1\pm} = \exp\left[\pm i\psi\right]\partial_V \qquad \text{(A.3.33)}$$

$$K_2 = V\partial_V + u_2\partial_{u_2} \qquad \text{(A.3.34)}$$

$$K_3 = \partial_{u_2}. \qquad \text{(A.3.35)}$$

To determine the coefficient functions in Γ for each of these symmetries we make the identifications by comparing the form of (A.3.32) with each of (A.3.33), (A.3.34) and (A.3.35) in turn. Commencing with K_3 we see that $\zeta = 0$, $\eta = 0$ and $-r^2\dot{\theta}\dot{\tau}\partial J/\partial z = 1$. We obtain

$$\Gamma_3 = \left(\int P\mathrm{d}t\right)\partial_t, \qquad \text{(A.3.36)}$$

where we have made use of the constancy of u_2, *ie* J, to introduce the latter into the integral and define

$$P = \frac{J}{z\dfrac{\partial J}{\partial z}} \qquad \text{(A.3.37)}$$

for a reason which becomes apparent below.
In the case of K_2 we have

$$\zeta = 0, \quad -\eta\exp\left[\int\frac{G}{2z^2}\mathrm{d}\theta\right]/r^2\rho = V,$$

ie $-\eta = r$ and consequently $\dot{\tau} = -\int(2+P)$. Part of the symmetry is already included in Γ_3 and so we write

$$\Gamma_{2eff} = 2\partial_t + r\partial_r. \qquad \text{(A.3.38)}$$

Finally for $K_{1\pm}$ we have that

$$\zeta = 0, \quad \eta = -r^2\rho\exp\left[\int\frac{G}{2z^2}\mathrm{d}\theta \pm i\int\frac{\mathrm{d}\theta}{\rho^2}\right]$$

and

$$\dot{\tau} = -2r\rho\exp\left[\int\frac{G}{2z^2}\mathrm{d}\theta \pm i\int\frac{\mathrm{d}\theta}{\rho^2}\right]$$

so that

$$\Gamma_{1\pm} = \int\left\{2r\rho\exp\left[\int\frac{G}{2z^2}\mathrm{d}\theta \pm i\int\frac{\mathrm{d}\theta}{\rho^2}\right]\right\}\mathrm{d}t\partial_t + r^2\rho\exp\left[\int\frac{G}{2z^2}\mathrm{d}\theta \pm i\int\frac{\mathrm{d}\theta}{\rho^2}\right]\partial_r. \tag{A.3.39}$$

A.4 Complete Symmetry Group of the General Two-dimensional Ermakov System

Our task is now to apply these four symmetries to the general system

$$\begin{aligned} \ddot{r} &= f(r,\theta,\dot{r},\dot{\theta}) && \text{(A.4.1)} \\ \ddot{\theta} &= g(r,\theta,\dot{r},\dot{\theta}), && \text{(A.4.2)} \end{aligned}$$

where f and g are initially arbitrary functions of their indicated arguments, to validate that these four symmetries plus ∂_t specify the generalised two-dimensional Ermakov system completely. For the sake of future reference we write the second extensions of these four operators. Recall that the overdot represents differentiation with respect to time and the prime presents differentiation with respect to θ. In the expressions for the second extensions we have eliminated the ∂_t component since this is of no relevance to the autonomous system, (1.1.172/A.4.2). In the case of $\Gamma_{1\pm}$ we have also removed a common factor, $\exp\left[\int\frac{G}{2z^2}\mathrm{d}\theta \pm i\int\frac{\mathrm{d}\theta}{\rho^2}\right]$, made use of the definition of z and the differential equation governing ρ. The second extensions are

$$\begin{aligned} \Gamma_{1\pm(eff)} &= r^2\rho'\partial_r + \left\{r^2\rho\dot{\theta} + r^2\rho\dot{\theta}\left[\pm\frac{i}{\rho^2} - \frac{G}{2z^2}\right]\right\}\partial_{\dot{r}} \\ &\quad -2r\rho\dot{\theta}\partial_{\dot{\theta}} + \left\{\left(r\rho' \pm i\frac{r}{\rho}\right)\left[r\ddot{\theta} + 2\dot{r}\dot{\theta} - \frac{r\dot{\theta}^2G}{z^2}\right]\right. \end{aligned}$$

$$
\begin{aligned}
& -\frac{r^2\dot\theta^2}{\rho}\left[1+\frac{F}{z^2}-\left(\frac{G}{2z^2}\right)^2\right]-r\rho\left(r\ddot\theta+2\dot r\dot\theta\right)\frac{G}{2z^2}+r^2\rho\dot\theta^2\left(\frac{G}{2z^2}\right)^2-2r\ddot r\rho\Bigg\}\partial_{\ddot r}\\
& +\left\{-\rho\left[4r\ddot\theta+2r\dot r-\frac{2G}{r^3}\right]\pm\frac{2ir\dot\theta^2}{\rho}\right\}\partial_{\ddot\theta}
\end{aligned}
$$

$$
\begin{aligned}
\Gamma_{2eff} &= r\partial_r-\dot r\partial_{\dot r}-2\dot\theta\partial_{\dot\theta}-3\ddot r\partial_{\ddot r}-4\ddot\theta\partial_{\ddot\theta}\\
\Gamma_3 &= \dot rP\partial_{\dot r}+\dot\theta P\partial_{\dot\theta}+\left[2\ddot rP+\dot r\dot P\right]\partial_{\ddot r}+\left[2\ddot\theta P+\dot\theta\dot P\right]\partial_{\ddot\theta}.
\end{aligned}
$$

We commence with the angular equation, (A.4.2), and the action of the $\pm$ part of $\Gamma_{1\pm(eff)}$. We obtain

$$\frac{\partial g}{\partial\dot r}=-2\frac{\dot\theta}{r}$$

whence (A.4.2) may be written as

$$r\ddot\theta+2\dot r\dot\theta=A(r,\theta,\dot\theta). \tag{A.4.3}$$

We now apply the rest of $\Gamma_{1\pm(eff)}$ to obtain

$$r\frac{\partial A}{\partial r}-2\dot\theta\frac{\partial A}{\partial\dot\theta}=-3A$$

so that (A.4.3) is now

$$r\ddot\theta+2\dot r\dot\theta=\frac{B(\theta,r^2\dot\theta)}{r^3}. \tag{A.4.4}$$

When we apply $\Gamma_{2(eff)}$ to (A.4.4), the equation is restored identically, *ie* there is no further restriction on the structure of the equation. The same is true in the case of Γ_3. However, the argument is more than a little subtler and we provide some detail. After we apply the second extension of the symmetry and substitute the expression for P, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial J}{\partial z}\right]=\left[\frac{\dot z}{z}-\frac{1}{r^2}\right]\frac{\partial J}{\partial z}.$$

We make use of the definition $u_2=J(\theta,z)$ and (A.3.29) to manipulate the left side of this equation as follows.

$$
\begin{aligned}
LHS &= \dot\theta\frac{\partial J}{\partial z\partial\theta}+\dot z\,\frac{\partial^2 J}{\partial z^2}\\
&= \frac{z}{r^2}\frac{\partial^2 J}{\partial z\partial\theta}+\frac{B}{r^2}\frac{\partial^2 J}{\partial z^2}
\end{aligned}
$$

$$= \frac{1}{r^2}\left\{\frac{\partial}{\partial z}\left(z\frac{\partial J}{\partial \theta}+B\frac{\partial J}{\partial z}\right)-\frac{\partial J}{\partial \theta}-\frac{\partial B}{\partial z}\frac{\partial J}{\partial z}\right\}$$
$$= -\frac{1}{r^2}\left\{\frac{\partial J}{\partial \theta}+\frac{\partial B}{\partial z}\frac{\partial J}{\partial z}\right\}$$

so that the total equation reduces to the identity

$$0 = \frac{1}{r^2}\frac{\partial J}{\partial \theta}+\frac{\dot{z}}{z}\frac{\partial J}{\partial z}$$

as a consequence of the conservation of u_2.

The symmetries ∂_t and $\Gamma_{1\pm}$ are sufficient to specify the structure of the angular equation, but we note that this is in terms of an arbitrary function $B(\theta, z)$ rather than the unspecified function $G(\theta, z)$ to be found in the differential operator.

We now turn to the radial equation. The action of the $\pm$ part of the second extension of $\Gamma_{1\pm}$ reduces to

$$\frac{\partial f}{\partial \dot{r}} = \frac{B-G}{zr}$$

when (A.4.4) is taken into account. Hence

$$f = \dot{r}\frac{B-G}{zr} + C(r, \theta, \dot{\theta}).$$

The application of Γ_{2eff} immediately gives

$$C = \frac{1}{r^3}D(\theta, z)$$

so that the radial equation now has the form

$$\ddot{r} = \dot{r}\frac{B-G}{zr} + \frac{1}{r^3}D(\theta, z). \tag{A.4.5}$$

We revert to the other part of the second extension of $\Gamma_{1\pm}$.
With all the results thus far accumulated the relevant remaining part of the second extension of $\Gamma_{1\pm}$ is

$$r^2\rho\partial_r + \left[r^2\rho'\dot{\theta} - r^2\rho\dot{\theta}\left(\frac{G}{2z^2}\right)\right]\partial_{\dot{r}} - 2r\rho\dot{\theta}\partial_{\dot{\theta}}$$
$$+\left\{\frac{\rho'}{r^2}(B-G) - r^2\rho\dot{\theta}^2\left(1+\frac{F}{z^2}+2r^2\rho\dot{\theta}^2\left(\frac{G}{2z^2}\right)^2-\rho\frac{BG}{2r^2z^2}\right)-2r\ddot{r}\rho\right\}\partial_{\ddot{r}}$$

and its action on (A.4.5) gives

$$
\begin{aligned}
&r^2\dot{\theta}^2\left[\frac{1}{\rho^3}-\rho\left(1+\frac{F}{z^2}-\left(\frac{G}{2z^2}\right)^2\right)\right]+\frac{\rho' B}{r^2}\\
&-r^2\rho'\dot{\theta}^2\left(\frac{G}{2z^2}\right)+\frac{\rho B}{r^2}\left(\frac{G}{2z^2}\right)-2\dot{r}\rho\frac{B-G}{z}-\frac{2\rho D}{r^2}\\
&=\left[r^2\rho'\dot{\theta}-r^2\rho\dot{\theta}\left(\frac{G}{2z^2}\right)\right]\frac{B-G}{zr}-\frac{\rho\dot{r}}{z}(B-G)-\frac{3\rho D}{r^2}.
\end{aligned}
$$

From the coefficient of $\dot{r}$ it is apparent that $B = G$ so that now we have recovered the angular equation. The equation now simplifies considerably to give

$$D = r^4\dot{\theta}^2\left(1+\frac{F}{z^2}\right),$$

ie, the radial equation now has the form

$$\ddot{r} - r\dot{\theta}^2 = \frac{F}{r^3} \tag{A.4.6}$$

which is the desired form.

Appendix B

Derivation of the Emden-Fowler sequence

B.1 Reduction of fifth-order ordinary differential equation to Emden-Fowler equation

The Kummer-Schwarz sequence is defined in terms of

$$R^n v = 0, \tag{B.1.1}$$

where

$$v = u''' - \frac{3u''^2}{2u'} \tag{B.1.2}$$

and the recursion operator [44] is

$$R = D^2 - 2\frac{u''}{u'}D + \frac{u'''}{u'} - \frac{u''^2}{u'^2} - u'D^{-1}\left(\frac{u^{(iv)}}{u'^2} - 4\frac{u'''u''}{u'^3} + \frac{3u''^3}{u'^4}\right). \tag{B.1.3}$$

The first two members of the sequence are

$$u''' - \frac{3u''^2}{2u'} = 0, \tag{B.1.4}$$

which is the Kummer-Schwarz equation and possesses the algebra $sl(2,R) \oplus sl(2,R)$ with the representation, $\partial_x, x\partial_x, x^2\partial_x, \partial_u, u\partial_u, u^2\partial_u$, and

$$u^{(v)} - 5\frac{u''u^{(iv)}}{u'} - \frac{5u'''^2}{2u'} + \frac{25u'''u''^2}{2u'^2} - \frac{45u''^4}{8u'^3} = 0, \tag{B.1.5}$$

which has the five-element algebra $sl(2,R)\oplus A_2$, represented by $\partial_x, x\partial_x, \partial_u, u\partial_u$ and $u^2\partial_u$.

The reduction of the Kummer-Schwarz equation is well known and we commence our analysis with the fifth-order equation (B.1.5). We reduce using the $sl(2,R)$ subalgebra.

We commence with ∂_u for which the invariants are evidently x and $w = u'$. The fourth-order equation is

$$w^{(iv)} - 5\frac{w'''w'}{w} - \frac{5w''^2}{2w} + \frac{25w''w'^2}{2w^2} - \frac{45w'^4}{8w^4} = 0 \qquad \text{(B.1.6)}$$

and the remaining symmetries are $\partial_x, x\partial_x - w\partial_w, w\partial_w, (2w\int w\mathrm{d}x)\partial_w$ in which the third element of the $sl(2,R)$ subalgebra has become nonlocal due to the Lie Bracket $[\partial_u, u^2\partial_u]_{LB} = 2u\partial_u$.

We now reduce by $w\partial_w$ for which the invariants are x and $z = w'/w$. The reduced equation is

$$z''' - zz'' + \tfrac{1}{2}z'^2 - \tfrac{3}{2}z'z^2 + \tfrac{3}{8}z^4 = 0 \qquad \text{(B.1.7)}$$

and the remaining symmetries are $\partial_x, x\partial_x - z\partial_z$ and $2\exp[\int z\mathrm{d}x]\partial_z$. We observe that the nonlocal symmetry has now become an exponential nonlocal symmetry and so is suitable to be used for reduction of order. The invariants are x and $y = z' - \frac{1}{2}z^2$. The second-order equation is

$$y'' + \frac{3}{2}y^2 = 0 \qquad \text{(B.1.8)}$$

which we recognize as an Emden-Fowler equation of index $(0,2)$. Two point symmetries, ∂_x and $x\partial_x - 2y\partial_y$, persist for (B.1.8) which is integrable in terms of elliptic functions. It also possesses the Painlevé Property with $p = -2, \alpha = -4$ and $r = -1, 6$.

The reduction has been solely in terms of the dependent variable and so it is an easy matter to reconcile the w of (B.1.8) with the u of (B.1.5). The sequence of transformations is

$$w = u' \;\rightarrow\; z = \frac{w'}{w} \;\rightarrow y = z' - \tfrac{1}{2}z^2 \qquad \text{(B.1.9)}$$

which can be conflated to

$$y = \frac{u'''}{u'} - \frac{3u''^2}{2u'^2}, \tag{B.1.10}$$

ie, y is the Schwarzian and $y = 0$ is essentially the first element of the Kummer-Schwarz sequence. Given the solution, $w(x)$, of (B.1.8) we may use (B.1.10) to solve (B.1.5). Equation (B.1.10) may be written as

$$\left(u'^{-\frac{1}{2}}\right)'' + \tfrac{1}{2}\left(u'^{-\frac{1}{2}}\right) y = 0 \tag{B.1.11}$$

which has the form of the equation of a time-dependent linear oscillator and in a formal sense we may write the solution as

$$u(x) = \int (A\xi \sin x + B\xi \cos x)^{-2} \mathrm{d}x, \tag{B.1.12}$$

where ξ is any solution of the Ermakov-Pinney equation

$$\xi'' + \tfrac{1}{2} y\xi = \frac{1}{\xi^3} \tag{B.1.13}$$

and

$$X = \int \xi^{-2} \mathrm{d}x. \tag{B.1.14}$$

Curiously a more explicit form of the solution can be obtained since the integral in (B.1.12) can be evaluated to give

$$u(x) = C - \frac{1}{A}\frac{1}{A\tan X(x) + B}, \tag{B.1.15}$$

where A, B and C are the constants of integration arising from the integration of the third-order equation (B.1.10). The function $X(x)$ contains the two constants of integration arising from (B.1.8).

Note that one can only regard the integration of (B.1.5) presented above as formal since the likelihood of being able to present an explicit solution of (B.1.11) (regarded as a second-order equation) when y is given in terms of elliptic functions is slight. Since an elliptic function is doubly periodic, (B.1.11) has the form of an Hill's equation.

Equation (B.1.5) is homogeneous in both x and u. Hence the exponent of the leading-order term is the solution of an algebraic equation of degree eight.

B.2 Derivation of the generator of sequence of the Emden-Fowler equation

The reduction of the fifth-order Kummer-Schwarz equation, (B.1.5), to a second-order equation in the Schwarzian prompts one to look at the generator of sequence, (B.1.3), in terms of the Schwarzian.
Firstly we observe that (B.1.4) is $u'y = 0$ so that the next member of the sequence is

$$\left\{D^2 - 2\frac{u''}{u'}D + \frac{u'''}{u'} - \frac{u''^2}{u'^2} - u'D^{-1}\left(\frac{u^{(iv)}}{u'^2} - 4\frac{u'''u''}{u'^3} + \frac{3u''^3}{u'^4}\right)\right\}u'y = 0 \tag{B.2.1}$$

and we make a further examination of (B.2.1). Noting that

$$y' = \frac{u^{(iv)}}{u'} - 4\frac{u'''u''}{u'^2} + \frac{3u''^3}{u'^3} \tag{B.2.2}$$

we observe that the integral part of (B.2.1) is

$$-u'D^{-1}\left(\frac{y'}{u'}\right)(u'y) = -\tfrac{1}{2}u'y^2. \tag{B.2.3}$$

We now look at the rest of (B.2.1).

$$\left(D^2 - 2\frac{u''}{u'}D + \frac{u'''}{u'} - \frac{u''^2}{u'^2}\right)u'y = u'(y'' + 2y^2) \tag{B.2.4}$$

and so (B.2.1) may be written as

$$u'\left(D^2 + 2y - D^{-1}y'\right)y = 0 \tag{B.2.5}$$

or

$$u'(y'' + \frac{3}{2}y^2) = 0 \tag{B.2.6}$$

and in (B.2.6) we recover (B.1.8).

We introduce the Emden-Fowler sequence

$$\left(D^2 + 2y - D^{-1}y'\right)^n y = 0,\ n = 1, 2, \dots, \tag{B.2.7}$$

where we omit $n = 0$ as a too trivial result.

Bibliography

[1] Ablowitz MJ, Ramani A & Segur H (1978) Nonlinear evolution equations and ordinary differential equations of Painlevé Type *Lettere al Nuovo Cimento* **23** 333-337

[2] Ablowitz MJ, Ramani A & Segur H (1980) A connection between nonlinear evolution equations and ordinary differential equations of P type I *Journal of Mathematical Physics* **21** 715-721

[3] Ablowitz MJ, Ramani A & Segur H (1980) A connection between nonlinear evolution equations and ordinary differential equations of P type II *Journal of Mathematical Physics* **21** 1006-1015

[4] Abraham-Shrauner B & Guo A (1992) Hidden symmetries associated with the projective group of nonlinear first-order ordinary differential equations *Journal of Physics A: Mathematical and General* **25** 5597-5608

[5] Abraham-Shrauner B & Leach PGL (1993) Hidden symmetries of nonlinear ordinary differential equations in *Exploiting Symmetry in Applied and Numerical Analysis* E Allgower, K Georg and R Miranda edd (Lectures in Applied Mathematics **29**, AMS, Providence) 1-10

[6] Abraham-Shrauner B (1993) Hidden symmetries and linearization of the modified Painlevé-Ince equation *Journal of Mathematical Physics* **34** 4809-4816

[7] Abraham-Shrauner B, Govinder KS & Leach PGL (1995) Integration of second order equations not possessing point symmetries *Physics Letters A* **203** 169-174

[8] Anderson IM, Kamran N & Olver PJ (1993) Internal, external and generalised symmetries *Advances in Mathematics* **100** 53-100

[9] Andriopoulos K, Leach PGL & Flessas GP (2001) Complete symmetry groups of ordinary differential equations and their integrals: some basic considerations *Journal of Mathematical Analysis and Applications* **262** 256-273

[10] Andriopoulos K & Leach PGL (2002) The economy of complete symmetry groups for linear higher dimensional systems *Journal of Nonlinear Mathematical Physics* **9 Second Supplement** 10-23

[11] Andriopoulos K & Leach PGL (2005) Some group theoretical aspects of nonlinear quantal oscillators *Journal of Nonlinear Mathematical Physics* **12** 32-43

[12] Andriopoulos K & Leach PGL (2005) Wave-functions for the time-dependent linear oscillator and Lie point symmetries *Journal of Physics A: Mathematical and General* **38** 4365-4374

[13] Andriopoulos K & Leach PGL (2006) Autonomous self-similar ordinary differential equations and the Painlevé connection *Journal Mathematical Analysis and Applications* **328** 625- 639

[14] Andriopoulos K & Leach PGL (2006) An interpretation of the presence of both positive and negative nongeneric resonances in the singularity analysis *Physics Letters A* **359** 199-203

[15] Andriopoulos K, Leach PGL & Maharaj A (2009) On Differential Sequences *European Journal of Applied Mathematics* (submitted)

[16] Bargmann V (1936) Zür Theorie des Wasserstoffatoms *Zeitschrift für Physik* **99** 576-582

[17] Belinfante JGK & Kolman BA (1972) *A Survey of Lie Groups and Lie Algebras with Applications and Computational Methods* (SIAM, Philadelphia)

[18] Bluman GW & Cole JD (1974) *Similarity Methods for Differential Equations* (Springer-Verlag, New York)

[19] Bluman GW & Kumei S (1989) *Symmetries and Differential Equations* (Applied Mathematical Sciences **81**, Springer-Verlag, New York)

[20] Bluman GW & Anco SC (2002) *Symmetry and Integration Methods for Differential Equations* (Springer-Verlag, New York)

[21] Bourbaki N (1975) *Elements of Mechanics: Lie Groups and Lie Algebras* (Addison-Wesley, Reading, Massachusetts)

[22] Bureau FJ (1964) Differential equations with fixed critical points *Annali di Matematica pura ed applicata* **LXIV** 229-364

[23] Bureau FJ (1964) Differential equations with fixed critical points *Annali di Matematica pura ed applicata* **LXVI** 1-116

[24] Bureau FJ (1972) Équations différentielles du second ordre en Y et du second degré en $\ddot{Y}$ dont l'intégrale générale est á points critiques fixes *Annali di Matematica pura ed applicata* **XCI** 163-281

[25] Bureau FJ (1972) Integration of some nonlinear systems of ordinary differential equations *Annali di Matematica pura ed applicata* **XCVI** 345-359

[26] Burgan J-R (1978) *Sur les groupes de transformation en physique mathématique. Application aux fluides de l'éspace des phases, et à la mécanique quantique* (thèsé, Université d'Orléans, Orléans)

[27] Burgan J-R, Feix MR, Fijalkow E, Gutierrez J and Munier A (1978) Utilisation des groupes de transformation pour la resolution des équations aux derivées partielles in *Applied Inverse Problems* Sabatier PC ed (Cahiers Mathématiques, Montpellier) 67-76

[28] Cariñena José, Guha Partha & Leach Peter (2007) Geometrical Aspects of The Riccati Equation (preprint: Departamento de Fisica Teórica, Facultad de Ciencias, Universidad de Zaragoza, 50009 Zaragoza, Spain)

[29] Chandrasekar VK, Senthilvelan M & Lakshmanan M (2007) On the general solution for the modified Emden type equation $\ddot{x}+\alpha x\dot{x}+\beta x^3=0$ *Journal of Physics A: Mathematical and Theoretical* **40** 4717-4727

[30] Chazy J (1911) Sur les équations différentielles du troisième ordre et d'ordre supérieur dont l'intégrale générale a ses points critiques fixes *Acta Mathematica* **34** 317-385

[31] Conte R (1994) Singularities of differential equations and integrability in *Introduction to Methods of Complex Analysis and Geometry for Classical Mechanics and Nonlinear Waves* Benest D and Frœschlé C edd (Éditions Frontières, Gif-sur-Yvette) 49-143

[32] Cosgrove CM (1993) All binomial-type Painlevé equations of the second-order and degree three or higher *Studies in Applied Mathematics* **90** 119-187

[33] Cosgrove C M & Scoufis G (1993) Painlevé classification of a class of differential equations of the second order and second degree *Studies in Applied Mathematics* **88** 25-87

[34] Davis Harold T (1962) *Introduction to Nonlinear Differential and Integral Equations* (Dover, New York)

[35] Dimas S & Tsoubelis D (2005) SYM: A new symmetry-finding package for Mathematica *Group Analysis of Differential Equations* Ibragimov NH, Sophocleous C & Damianou PA edd (University of Cyprus, Nicosia) 64-70

[36] Dimas S & Tsoubelis D (2006) A new Mathematica-based programme for solving overdetermined systems of PDE *8th International Mathematica Symposium* (Avignon, France)

[37] Dirac PAM (1932) *Principles of Quantum Mechanics* (Oxford, at the Clarendon Press)

[38] Ermakov V (1880) Second-order differential equations. Conditions of complete integrability *Universita Izvestia Kiev Series III* **9** 1-25, (2008) *Applicable Analysis and Discrete Mathematics* **2** 123-145 (trans AO Harin, redactor PGL Leach)

[39] Ermanno GJ (1710) Metodo d'investigare l'Orbite de'Pianeti, nell'ipotesi che le forze centrali o pure le gravita delgi stessi Pianeti sono in ragione reciproca de'quadrati delle distanze, che i medesimi tengonzo dal centro, a cuisi dirigono le forze stesse *Giornale de Letterati D'Italia* **2** 447-467

[40] Ervin VJ, Ames WF & Adams E (1984) *Nonlinear waves in pellet fusion* in Wave Phenomena: Modern Theory and Applications Rodgers C & Moodie TB edd (North Holland, Amsterdam)

[41] Euler M, Euler N & Petersson N (2003) Linearizable hierarchies of evolution equations in $(1+1)$ dimensions *Studies in Applied Mathematics* **111** 315-337

[42] Euler M, Euler N & Leach PGL (2006) The Riccati and Ermakov-Pinney Hierarchies *Journal of Nonlinear Mathematical Physics* **14** 290-310

[43] Euler N (2007) Notes on Integration Methods for Nonlinear Differential Equations (*Lecture notes*) (Department of Mathematics, Luleå Tekniska Universitet, Sweden)

[44] Euler M & Euler N (2008) Kummer-Schwarz sequence (private communication)

[45] Euler N & Leach PGL (2008) Aspects of proper differential sequences of ordinary differential equations *Journal of Theoretical and Mathematical Physics* **159** 474-487

[46] Fokas AS (1980) A symmetry approach to exactly solvable evolution equations *Journal of Mathematical Physics* **21** 1318-1325

[47] Fokas AS (1987) Symmetries and Integrability *Studies in Applied Mathematics* **77** 253-299

[48] Feix MR, Géronimi C, Cairó, Leach PGL, Lemmer RL & Bouquet SÉ (1997) On the singularity analysis of ordinary differential equations invariant under time translation and rescaling *Journal of Physics A: Mathematical and General* **30** 7437-7461

[49] Foch V (1935) Zür Theories des Wasserstoffatoms *Zeitschrift für Physik* **98** 145-154

[50] Fuchssteiner B (1979) Application of hereditary symmetries to nonlinear evolution equations *Nonlinear Analysis: Theory, Methods and Application* **3** 849-862

[51] Gazizov RK & Ibragimov NH (1998) Lie symmetry analysis of differential equations in finance *Nonlinear Dynamics* **17** 387-407

[52] Géronimi C, Feix MR & Leach PGL (2001) Exponential nonlocal symmetries and nonnormal reduction of order *Journal of Physics A: Mathematical and General* **34** 10109-10117

[53] Géronimi C, Leach PGL & Feix MR (2002) Singularity analysis and a function unifying the Painlevé and Ψ series *Journal of Nonlinear Mathematical Physics* 9 **Second Supplement** 36-48

[54] Golubev VV (1950) *Lectures on Analytical Theory of Differential Equations* (Gosteekhizdat, Moscow-Leningrad)

[55] González-Gascón F & González-López A (1988) Newtonian systems of differential equations integrable via quadratures, with trivial group of point symmetries *Physics Letters A* **129** 153-156

[56] Govinder KS & Leach PGL (2003) On the equivalence of linear third order differential equations under nonlocal transformations *Journal of Mathematical Analysis and Applications* **287** 399-404

[57] Hamilton WR (1847) On the application of the method of quaternions to some dynamical questions *Proceedings of the Royal Irish Academy* **3** appendix III xxxvi-l 645-658

[58] Head AK (1993) LIE, a PC program for Lie analysis of differential equations *Computer Physics Communications* **77** 241-248

[59] Herman J (1710) Extrait d'une Lettre de M Herman à M Bernoulli, dat'ee de Padoüe le 12 juillet 1710 *Mémoires de l'Academie Royale des Scienies* 519-521

[60] Hua DD, Cairó L, Feix MR, Govinder KS & Leach PGL (1996) Connection between the existence of first integrals and the Painlevé Property in Lotka-Volterra and Quadratic Systems *Proceedings of the Royal Society of London* **452** 859-880

[61] Ince EL (1927) *Integration of Ordinary Differential Equations* (Longmans, Green & Co, London)

[62] Jacobson N (1979) *Lie Algebras* (Dover, New York)

[63] Kamke E (1983) *Differentialgleichungen Lösungsmethoden und Lösungen* (BG Teubner, Stuttgart)

[64] Kowalevski S (1889) Sur le problème de la rotation d'un corps solide autour d'un point fixe *Acta Mathematica* **12** 177-232

[65] Kowalevski S (1889) Sur une propriété du système d'équations différentielles qui définit la rotation d'un corps solide autour d'un point fixe *Acta Mathematica* **14** 89-93

[66] Krause J (1994) On the complete symmetry group of the classical Kepler system *Journal of Mathematical Physics* **35** 5734-5748

[67] Krause J (1995) On the complete symmetry group of the Kepler problem in Arima A ed *Proceedings of the XXth International Colloquium on Group Theoretical Methods in Physics* (World Scientific, Singapore) 286-290

[68] Kummer EE (1887) De generali quadam æquatione differentiali tertii ordinis *Journal für Reine und Angewandte Mathematik* **100** 1-9 (reprinted from the Programm des evangelischen Königlischen und Stadtgymnasiums in Liegnitz for the year 1834)

[69] Laplace PS (1798) *Traité de Méchanique Céleste* (Gauthiers-Villars, Paris) Tome **I** 165

[70] Leach PGL (1977) On a direct method for the determination of an exact invariant for the time-dependent harmonic oscillator *Journal of the Australian Mathematical Society, Series B* **20** 97-105

[71] Leach PGL (1977) On the theory of time-dependent linear canonical transformations as applied to Hamiltonians of harmonic oscillator type *Journal of Mathematical Physics* **18** 1608-1611

[72] Leach PGL (1985) First integrals for the modified Emden equation $\ddot{q}+\alpha(t)\dot{q}+q^n=0$ *Journal of Mathematical Physics* **26** 2510-2514

[73] Leach PGL, Feix MR & Bouquet S (1988) Analysis and solution of a nonlinear second order differential equation through rescaling and through a dynamical point of view *Journal of Mathematical Physics* **29** 2563-2569

[74] Leach PGL (2000) *An Introduction to Symmetry, Singularity and Integrability of Nonlinear Ordinary Differential Equations* (School of Mathematical Sciences, University of KwaZulu-Natal, South Africa)

[75] Leach PGL, Andriopoulos K & Nucci MC (2003) The Ermanno-Bernoulli constants and representations of the complete symmetry group of the Kepler Problem *Journal of Mathematical Physics* **44** 4090-4106

[76] Leach PGL & Nucci MC (2004) Reduction of the Classical MICZ-Kepler Problem to a two-dimensional simple harmonic oscillator *Journal of Mathematical Physics* **45** 3590-3604

[77] Leach PGL & Andriopoulos K (2005) Newtonian economics in *Group Analysis of Differential Equations*, Ibragimov NH, Sophocleous C & Damianou PA edd (University of Cyprus, Nicosia) 134-142

[78] Leach PGL, Karasu (Kalkani) A, Nucci MC & Andriopoulos K (2005) Ermakov's superintegrable toy and nonlocal symmetries *SIGMA:* **1** Paper 018, 15 pages

[79] Leach PGL, Maharaj A & Andriopoulos K (2009) Differential sequences: The Emden-Fowler sequence (preprint: School of Mathematical Sciences, University of KwaZulu-Natal, Durban 4041, Republic of South Africa)

[80] Leach PGL, Maharaj A & Andriopoulos K (2009) On the generalised Riccati sequence (preprint: School of Mathematical Sciences, University of KwaZulu-Natal, Durban 4041, Republic of South Africa)

[81] Lemmer RL & Leach PGL (1993) The Painlevé test, hidden symmetries and the equation $y'' + yy' + ky^3 = 0$ *Journal of Physics A: Mathematical and General* **26** 5017-5024

[82] Lemmer RL & Leach PGL (1999/1420) A classical viewpoint on quantum chaos *Arab Journal of Mathematical Sciences* **5** 1-17

[83] Lenz W (1924) Über den Bewegungsverlauf und die Quantenzustande der gestörten Keplerbewegung *Zeitschrift für Physik* **24** 197-207

[84] Lewis H Ralph Jr & Riesenfeld WB (1969) An exact quantum theory of the time-dependent harmonic oscillator and of a charged particle in a time-dependent electromagnetic field *Journal of Mathematical Physics* **10** 1458-1473

[85] Lewis HR & Leach PGL (1982) A direct approach for determining exact invariants for one-dimensional time-dependent classical systems *Journal of Mathematical Physics* **25** 2371-2374

[86] Lie Sophus (1967) *Differentialgleichungen* (Chelsea, New York)

[87] Liouville J (1837) Sur le développement des fonctions ou parties de fonctions en séries dont les divers termes sont assujettis à satisfaire à une même équatione différentielle du second ordre contenent un paramètre variable *Journal de Mathématiques* **II** 16-35

[88] Maharaj A & Leach PGL (2007) The method of reduction of order and linearization of the two-dimensional Ermakov system *Mathematical Methods in the Applied Sciences* **30** 2125-2145

[89] Maharaj A, Leach PGL & Andriopoulos K (2009) Differential sequences generated by first-order generators of sequences (preprint: School of Mathematical Sciences, University of KwaZulu-Natal, Durban 4041, Republic of South Africa)

[90] Mahomed FM & Leach PGL (1985) The linear symmetries of a nonlinear differential equation *Quæstiones Mathematicæ* **8** 241-274

[91] Mahomed FM (1986) *Classification of Nonlinear Ordinary Differential Equations According to their Symmetries* (dissertation, University of the Witwatersrand, Johannesburg)

[92] Mahomed FM (1989) *Symmetry Lie algebras of nth-order ordinary differential equations* (thesis, University of Witwatersrand, Johannesburg)

[93] Mahomed FM & Leach PGL (1990) Symmetry Lie algebras of nth-order ordinary differential equations *Journal of Mathematical Analysis and Applications* **151** 80-107

[94] Mahomed FM & Leach PGL (1991) Contact symmetries of second order differential equations *Journal of Mathematical Physics* **32** 2051-2055

[95] Mubarakzyanov GM (1963) On solvable Lie algebras *Izvestia Vysshikh Uchebn Zavendeniǐ Matematika* **32** 114-123

[96] Mubarakzyanov GM (1963) Classification of real structures of five-dimensional Lie algebras *Izvestia Vysshikh Uchebn Zavendeniǐ Matematika* **34** 99-106

[97] Mubarakzyanov GM (1963) Classification of solvable six-dimensional Lie algebras with one nilpotent base element *Izvestia Vysshikh Uchebn Zavendeniǐ Matematika* **35** 104-111

[98] Naicker V, Andriopoulos K & Leach PGL (2005) Symmetry reduction of some nonlinear Hamilton-Jacobi-Bellman equations arising in the mathematics of finance *Journal of Nonlinear Mathematical Physics* **12** 264-274

[99] Nucci MC (1996) The complete Kepler group can be derived by Lie group analysis *Journal of Mathematical Physics* **37** 1772-1775

[100] Nucci MC & Leach PGL (2000) The determination of nonlocal symmetries by the method of reduction of order *Journal of Mathematical Analysis and Application* **251** 871-884

[101] Nucci MC & Leach (2004) Jacobi's last multiplier and symmetries for the Kepler Problem plus a lineal story *Journal of Physics A: Mathematical and General* **37** 7743-7753

[102] Nucci MC & Leach PGL (2005) Jacobi's last multiplier and the complete symmetry group of the Ermakov-Pinney equation *Journal of Nonlinear Mathematical Physics* **12** 305-320

[103] Olver Peter J (1977) Evolution equations possessing infinitely many symmetries *Journal of Mathematical Physics* **18** 1212-1215

[104] Olver PJ (1986) *Applications of Lie Groups to Differential Equations* (Graduate Texts in Mathematics **107** Springer-Verlag, Berlin)

[105] Ovsiannikov LV (1982) *Group Analysis of Differential Equations* (Academic Press, New York)

[106] Painlevé P (1902) Sur les équations différentielles du second ordre et d'ordre supérieur dont l'intégrale générale est uniforme *Acta Mathematica* **25** 1-86

[107] Petersson N, Euler N & Euler M (2004) Recursion operators for a class of integrable third-order evolution equations *Studies in Applied Mathematics* **112** 201-225

[108] Pillay T & Leach PGL (2000) Chaos, integrability and symmetry *South African Journal of Science* **96** 371-376

[109] Pinney E (1950) The nonlinear differential equation $y''(x)+p(x)y+cy^{-3}=0$ *Proceedings of the American Mathematical Society* **1** 681

[110] Ramani A, Grammaticos B & Bountis T (1989) The Painlevé property and singularity analysis of integrable and nonintegrable systems *Physics Reports* **180** 159-245

[111] Riccati Jacopo (1764) *Opere* (Jacopo Giusti, Lucca) 83-96

[112] Runge CDT (1963) *Vector Analysis* (trans H Levy, Methuen, London) Vol I

[113] Stubhaug A (2000) *The Mathematician Sophus Lie* (trans RH Daly) (Springer-Verlag, Berlin)

[114] Vawda F E (1994) *An Application of the Lie Analysis to Classical Mechanics* (dissertation, University of the Witwatersrand, Johannesburg)

[115] Whittaker ET (1944) *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies* (Dover, New York)

[116] Yaglom IM (1988) *Felix Klein and Sophus Lie* (Birkhäuser, Boston)