Dynamics and Thermalization of a Fermion in a Fermionic Bath Embedded in a Bosonic Bath

Michael Mwalaba

Submitted in partial fulfillment of the requirements for the degree of
Master of Science in Physics

CENTRE FOR QUANTUM TECHNOLOGY

SCHOOL OF CHEMISTRY AND PHYSICS

UNIVERSITY OF KWAZULU-NATAL

Supervisor: Prof. Francesco Petruccione

Co-supervisor: Dr. Ilya Sinayskiy

Durban, South Africa

December, 2013

As the candidate's Supervisor I have/have not approved this dissertation for submission.

Name: **Prof. Francesco Petruccione**

Signed: ....

Date: ....

As the candidate's Co-supervisor I have/have not approved this dissertation for submission.

Name: **Dr. Ilya Sinayskiy**

Signed: ....

Date: ....

DECLARATION 1 - PLAGIARISM

I, **Mr. Michael Mwalaba**, declare that

1. The research reported in this dissertation, except where otherwise indicated, is my original research.

2. This dissertation has not been submitted for any degree or examination at any other university.

3. This dissertation does not contain other persons' data, pictures, graphs or other information, unless specifically acknowledged as being sourced from other persons.

4. This dissertation does not contain other persons' writing, unless specifically acknowledged as being sourced from other researchers. Where other written sources have been quoted, then: (a) Their words have been re-written but the general information attributed to them has been referenced (b) Where their exact words have been used, then their writing has been placed in italics and inside quotation marks, and referenced.

5. This dissertation does not contain text, graphics or tables copied and pasted from the Internet, unless specifically acknowledged, and the source being detailed in the dissertation and in the References sections.

Signed: ....

DECLARATION 2 - PUBLICATIONS

1. **Michael Mwalaba**, Ilya Sinayskiy and Francesco Petruccione, "Dynamics and Thermalization of a Fermion in a Fermionic Bath Embedded in a Bosonic Bath." - The manuscript for this work is in preparation.

2. **Michael Mwalaba**, Ilya Sinayskiy and Francesco Petruccione, "Dissipative Dynamics of a Spinless Electron Strongly Interacting with an Environment of Spinless Electrons." - This work which is part of this thesis has been submitted for publication in the Proceedings of the 2012 South African Institite of Physics (SAIP) Conference. An oral presentation based on this work at the SAIP 2012 Conference in the master of science category (theoretical track) was awarded the **best prize**.

Abstract

We consider a model consisting of a finite number of quantum dots each of which confines a spinless electron. We zoom in on a single quantum dot containing an electron of interest treating it as being strongly coupled to the surrounding finite bath of electrons. The fermionic bath is embedded in a bosonic Markovian bath. The master equation for the fermion of interest interacting with the fermionic bath is derived. Based on the master equation for this system, the reduced dynamics and thermalization of the spinless electron is studied.

We start with a description of the Hamiltonian of the entire system which we call total Hamiltonian. Because the electron of interest is strongly coupled to the surrounding fermionic bath, we treat the Hamiltonian consisting of the electron of interest, the fermionic bath and the interaction between them as the system Hamiltonian. Then using techniques of linear algebra, we diagolize the system Hamiltonian making it appear in what we are calling quasi-fermionic picture. After this, we take the diagonalized system Hamiltonian back to the total Hamiltonian. We then use this total Hamiltonian to switch to the interaction picture. Since the general expression of the Markovian quantum master equation is in terms of the interaction Hamiltonian, we now substitute our interaction Hamiltonian into it and begin from there to derive the quantum master equation of our system.

In the next step, we solve the derived quantum master equation casting the solution in Kraus representation. Using the explicit form of the Kraus operators and initial conditions, the density matrix of the reduced system is obtained in the quasi-fermionic picture. We then transform to the original fermionic picture and trace out the fermionic bath coming out with the density matrix of the electron of interest.

We then check the normalization of the density matrix of the electron of interest by calculating the trace and then use it to calculate the mean number of fermions. The mean number of fermions is then plotted against time for different coupling strengths and varying numbers of fermions in the fermionic bath to visually check the dynamics and thermalization of the fermion of interest.

A dedication to my hero, Anania Mwalaba, my father.

Acknowledgements

To God Almighty, creator of the heavens, the earth, the seas and all that is in them, God of Abraham, Isaac and Jacob, be all the glory, honor and praise.

This work is based upon research supported by the South African Research Chair Initiative of the Department of Science and Technology and National Research Foundation.

I wish to thank the National Institute for Theoretical Physics (NITHeP) for awarding me the scholarship that supported my research at the University of KwaZulu-Natal (UKZN). I also wish to thank UKZN in general and the School of Chemistry and Physics in particular for providing a conducive environment for study and research.

I am greatly indebted to my supervisor Prof. Francesco Petruccione for accommodating me in his group, the Quantum Research Group, for supervising my research from the beginning all the way to the end without which my work as an MSc student at UKZN would be incomplete. His provision of research facilities cannot be skipped without a word of thanks.

I wish to extend my sincere gratitude to Dr. Ilya Sinayskiy, my co-supervisor, for his consistent effort to ensure that I understood all elements of the project. The exercises he gave me where very nourishing. I also wish to thank members of the Quantum Research Group for academic discussions and great company. A big word of thanks goes to office number 03-102 H-block, where I was stationed. The academic discussions were very helpful. The company and the jokes were a great relief.

Michael Mwalaba

Preface

This document is my thesis submitted in partial fulfillment of the requirements for the degree of Master of Science in Physics at the University of KwaZulu-Natal in Durban, South Africa. It should be seen as a tip of an iceberg in comparison to the entire project from the beginning of my studies for a master of science degree to the end. It is for purposes of brevity that the material presented in this thesis is only that which is crucially relevant to the thesis title. I must say that I remain impressed by the extensive training prior to commencement of the material presented herein; sufficing to say that the foundation for further studies in the field has been thoroughly laid, thanks to my supervisors.

Contents

List of Figures

Chapter 1

Introduction

Recently, there has been a great deal of interest in quantum dots. A quantum dot is a nanocrystal made of semiconductor materials [1]. These semiconductor devices have unique properties resulting into a host of applications. Among others three such areas of application are the following.

1. Fluorescent labelling of cellular components: this is where optical properties of quantum dots are harnessed to label cellular components for tracking purposes in biological research [2]. Electromagnetic radiation (bosonic bath) is used to create excitons in the quantum dot. After some time recombination of the electron and the hole occurs. Upon recombination light of specific wavelength is emitted which is used to label cellular components. Labelling with quantum dots is considered more photostable than with the traditional organic dyes.

2. Temperature sensors: this is an application in microstructures and nanostructures which also makes use of optical properties of quantum dots. It has been studied [3, 4] that spectroscopic properties of quantum dots are temperature dependent. Alterations in the emission spectrum are used as indicators of changes in temperature.

3. Quantum information processing: in this area of application, a quantum dot is a potential candidate for a qubit. Quantum computation at the level of

an individual two-qubit gate has been demonstrated for qubit candidates such as cavity quantum electrodynamics [5–7] and ion-traps [8]. However, it is still uncertain whether such atomic-physics implementations could ever be scaled up to do truly large-scale quantum computation. This has lead to speculation that solid-state devices such as semiconductor quantum dots would be better candidates. Several quantum dots can be entangled offering the potential for scaling to large-scale qubit implementation [9, 10].

In particular, the spin or electric charge of an electron confined in the quantum dot serves as a qubit representation. In the case of spin, the up and down or $|0\rangle$ and $|1\rangle$ states of electron spin serve as the quantum analogue of the classical bit. In the case of charge, the $|0\rangle$ and $|1\rangle$ states correspond to the presence of charge in either of two quantum dots, or two states within a single quantum dot [11, 12]. In both cases, electromagnetic radiation can be used to manipulate the qubit for various quantum information processing tasks [13]. The qubit is an open quantum system and its interaction with the bath of electromagnetic radiation as it thermalizes leads to loss of quantum information, a phenomenon coined decoherence. Decoherence is a well known obstacle to a physical realization of a quantum computer [14–21].

A common feature in all these three examples above is that of dynamics and thermalization of the electron confined in the quantum dot with the electromagnetic radiation as the bosonic bath. Understanding this feature is crucial to modern research and physical implementations of these devices. This is the feature that we intend to investigate in this thesis.

1.1 Model

Our model consists of several single fermion quantum dots coupled to a bosonic bath. The fermions confined in the quantum dots are considered to be strongly coupled to each other since electrons strongly interact with each other. We focus on a single fermion so that the rest of the fermions are treated as a surrounding mesoscopic bath. Overall, our entire system can be seen as a fermion of interest

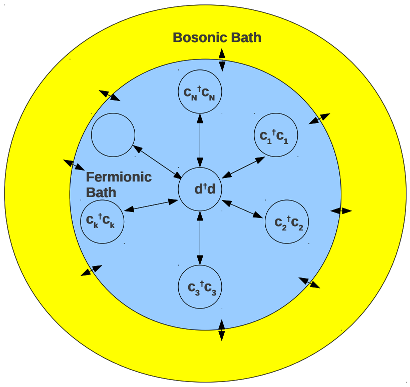

FIGURE 1.1: A fermion of interest ($d^\dagger d$) interacting with a bath of several fermions which is also interacting with a bath of bosons. The operators $d^\dagger, d$ are creation and annihilation operators of the fermion of interest and $c_i^\dagger, c_i$ are creation and annihilation operators of the fermions.

strongly interacting with a surrounding fermionic bath which is also interacting with a Markovian bosonic bath in which it is embedded.

As the title of this thesis depicts, we aim to investigate the dynamics and thermalization of this fermion of interest.

The total Hamiltonian of the entire system reads,

$$H = H_S + H_B + H_{SB} = H_0 + H_{SB}, \tag{1.1}$$

where H_S, referred to from here on simply as the system Hamiltonian, is the Hamiltonian of the fermion of interest interacting with the fermionic bath, i.e.,

$$H_S = \omega d^\dagger d + \sum_{i=1}^{N} \left(\epsilon c_i^\dagger c_i + g d^\dagger c_i + g c_i{}^\dagger d \right), \tag{1.2}$$

where $d^\dagger, d$ are creation and annihilation operators of the fermion of interest and $c_i^\dagger, c_i$ are creation and annihilation operators of the fermions in the mesoscopic bath which are taken to be degenerate with ϵ as the energy of each of the fermions, ω is the energy of the fermion of interest where we have set $\hbar = 1$, and g is the coupling strength for the interaction between the fermion of interest and the mesoscopic bath of fermions. Note that we are treating the Hamiltonian of the fermion of interest together with the fermionic bath and the interaction between them as the system Hamiltonian, H_S, because the interaction is taken to be strong. As an approximation, this interaction is taken to be the same. Since our interest is in the dynamics and thermalization of the electron, we regard all the electrons as spinless in this present work, as the simplest possible case. All the operators $d^\dagger, c_i^\dagger$ satisfy standard anticommutation relations,

$$\{d, d^\dagger\}_+ = 1, \qquad [d, d] = 0 \qquad \text{and} \qquad [d^\dagger, d^\dagger] = 0, \tag{1.3}$$

$$\{c_i, c_j^\dagger\}_+ = \delta_{ij}, \qquad \{c_i, c_j\}_+ = 0 \qquad \text{and} \qquad \{c_i^\dagger, c_j^\dagger\}_+ = 0. \tag{1.4}$$

The Hamiltonian of the bath H_B reads,

$$H_B = \sum_n \omega_n b_n^\dagger b_n. \tag{1.5}$$

The Hamiltonian of interaction of the fermonic bath with the bosonic Markovian environment is denoted by H_{SB} and in the rotating wave approximation, where the intensity of the bosonic bath is low and near resonance with electronic transitions so that strongly oscillating terms in the Hamiltonian are neglected [22], is given by,

$$H_{SB} = \sum_{i=1}^{N} \sum_n g_n b_n c_i^\dagger + g_n^* b_n^\dagger c_i, \tag{1.6}$$

where ω_n is the frequency of the n^{th} mode of the field and $b_n^\dagger, b_n$ are standard bosonic creation and annihilation operators satisfying the commutation relations,

$$[b_n, b_m^\dagger] = \delta_{nm}, \qquad [b_n, b_m] = 0 \qquad \text{and} \qquad [b_n^\dagger, b_m^\dagger] = 0. \tag{1.7}$$

As a single entity, the entire system consisting of the fermion of interest, the fermionic bath, the bosonic bath and interactions between them is a closed quantum system and therefore evolves unitarily. However, the fermion of interest is an open quantum system and its dissipative dynamics undergo non-unitary evolution [23]. To describe this type of dynamics, we will require the general Markovian quantum master equation which within the Born and Markov approximations

[24–27] is given by

$$\frac{d}{dt}\rho_I^S(t) = -\int_0^\infty ds \mathrm{Tr}_B \left\{ [H_I^{SB}(t), [H_I^{SB}(t-s), \rho_I^S(t) \otimes \rho_B]] \right\}, \tag{1.8}$$

where

$$H_I^{SB}(t) = e^{itH_0} H_{SB}(t) e^{-itH_0} \tag{1.9}$$

is the fermion-boson coupling Hamiltonian in the interaction picture,

$$\rho_I^S(t) = e^{itH_0}\rho_S^0(t)e^{-itH_0} = e^{itH_S}\rho_S^0(t)e^{-itH_S} \tag{1.10}$$

is the reduced density matrix in the interaction picture, with $\rho_S^0(t)$ as the reduced density matrix in the Schrödinger picture obtained from the total density matrix $\rho^0(t)$ by taking the partial trace with respect to B, also referred to as tracing out the degrees of freedom of the bath [28, 29],

$$\rho_S^0(t) = \mathrm{Tr}_B\left(\rho^0(t)\right), \tag{1.11}$$

and

$$\rho_B = \frac{e^{-\beta H_B}}{Z} = \frac{e^{-\beta \sum_n \omega_n b_n^\dagger b_n}}{\mathrm{Tr}(e^{-\beta \sum_n \omega_n b_n^\dagger b_n})} = \prod_n \left(1 - e^{-\beta\omega_n}\right) e^{-\beta\omega_n b_n^\dagger b_n}, \tag{1.12}$$

is the density matrix of the bosonic bath, where $\beta = 1/k_B$ is the inverse temperature of the bosonic bath and k_B is the Boltzmann constant. The Born approximation is a weak-coupling assumption in which the state of the bath remains unaffected by the interaction because the bath is assumed to be too large to be affected by the small system. The system S is coupled to the reservoir which causes a damping that destroys the "knowledge" of the past behavior. These considerations lead to the assumption that the system loses all memory of the past making the density matrix local in time. This is known as the Markov approximation [24–27].

From here on, we will be working in the interaction picture. So, for notational simplicity we shall drop the subscript I from ρ_I^S so that in the following pages the general expression of the Markovian quantum master equation will appear as

$$\frac{d}{dt}\rho_S(t) = -\int_0^\infty ds \mathrm{Tr}_B\Bigg\{[H_I^{SB}(t), [H_I^{SB}(t-s), \rho_S(t) \otimes \rho_B]]\Bigg\}. \tag{1.13}$$

The remainder of this thesis involves simplifying, solving and analyzing eqn. (1.13) specific to our model described above and the aim of this thesis. As a starting point, it is convenient that the Hamiltonian H_{SB} appearing on the right hand side of eqn. (1.13) is expressed in the basis set of orthogonal eigenstates of H_S. So, in the next Chapter we are going to diagonalize the system Hamiltonian. Chapter 3 involves simplifying the specific form of eqn. (1.13), the process we are calling derivation of the quantum master equation of the reduced system. Solving the simplied version of this Markovian quantum master equation is the subject of Chapter 4. The results and analysis thereof are presented in Chapter 5, and Chapter 6 concludes.

Chapter 2

Hamiltonian Diagonalization

The Hamiltonian of the system, H_S, as it appears in eqn. (1.2) is not in diagonal form. In order to derive the quantum master equation for the reduced system, it is convenient to diagonalize H_S [27]. In this chapter, we are going to diagonalize H_S using a technique of linear algebra [30, 31].

We start by expressing H_S in matrix form by expressing it in quadratic form,

$$H_S = \nu^\dagger A \nu, \tag{2.1}$$

where the matrix vector ν and a symmetric coefficient matrix A are to be determined from the given H_S, eqn. (1.2). Matrix A is expected to be diagonal in the basis set of its eigenvectors, i.e. by spectral decomposition,

$$A = \sum_{i=0}^{N} \lambda_i |\lambda_i\rangle\langle\lambda_i|, \qquad \forall i, j \qquad \langle\lambda_i|\lambda_j\rangle = \delta_{ij}. \tag{2.2}$$

In principle, diagonalizing A implies calculating,

$$R^{-1} A R = D, \tag{2.3}$$

where D is the diagonal matrix with the eigenvalues of A on the main diagonal and R is a matrix with an orthonormal basis of eigenvectors of A as column vectors, i.e,

$$R^{-1} = R^{\dagger}. \tag{2.4}$$

Eqn. (2.3) and eqn. (2.4) imply that,

$$A = RDR^{-1} = RDR^{\dagger}. \tag{2.5}$$

Substituting eqn. (2.5) into eqn. (2.1), we get

$$H_S = \nu^{\dagger} RDR^{\dagger}\nu = \xi^{\dagger} D\xi, \tag{2.6}$$

where

$$\xi^{\dagger} = \nu^{\dagger} R, \tag{2.7}$$

is the matrix of eigenvectors of H_S. Thus, the diagonal matrix of eigenvalues of A is the same as the diagonal matrix of eigenvalues of H_S.

Let us find A, D, R and ν. For $N = 1$, eqn. (1.2) can be written as

$$\begin{aligned} H_S &= \omega d^{\dagger} d + \sum_{i=1}^{1} \left(\epsilon c_i^{\dagger} c_i + g d^{\dagger} c_i + g {c_i}^{\dagger} d \right) \\ &= \omega d^{\dagger} d + \epsilon c_1^{\dagger} c_1 + g d^{\dagger} c_1 + g {c_1}^{\dagger} d \\ &= \begin{pmatrix} d^{\dagger} & c_1^{\dagger} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} d \\ c_1 \end{pmatrix} \\ &= a_{11} d^{\dagger} d + a_{22} c_1^{\dagger} c_1 + a_{12} d^{\dagger} c_1 + a_{21} {c_1}^{\dagger} d. \end{aligned} \tag{2.8}$$

Comparison of the second and last line of eqn. (2.8) reveals that

$$a_{11} = \omega, \qquad a_{22} = \epsilon, \qquad \text{and} \qquad a_{12} = a_{21} = g. \tag{2.9}$$

Using eqn. (2.9) in the third line of eqn. (2.8) and comparing with eqn. (2.1), we get

$$A = \begin{pmatrix} \omega & g \\ g & \epsilon \end{pmatrix}, \tag{2.10}$$

and

$$\nu = \begin{pmatrix} d \\ c_1 \end{pmatrix}, \tag{2.11}$$

which is a matrix of operators, d and c's. Diagonalizing A, we obtain the matrix of eigenvalues,

$$D = \left(\frac{\omega+\epsilon}{2} + \frac{\Omega_1}{2}\right)|0\rangle\langle 0| + \left(\frac{\omega+\epsilon}{2} - \frac{\Omega_1}{2}\right)|1\rangle\langle 1|, \tag{2.12}$$

where

$$\Omega_1 = \sqrt{4g^2 + (\omega-\epsilon)^2}, \tag{2.13}$$

and the matrix of eigenvectors,

$$R = [|\lambda_0\rangle, |\lambda_1\rangle], \tag{2.14}$$

where

$$|\lambda_0\rangle = \left(\frac{\omega-\epsilon}{2g} + \frac{\sqrt{4g^2+(\omega-\epsilon)^2}}{2g}\right)|0\rangle + |1\rangle, \tag{2.15}$$

$$|\lambda_1\rangle = \left(\frac{\omega-\epsilon}{2g} - \frac{\sqrt{4g^2+(\omega-\epsilon)^2}}{2g}\right)|0\rangle + |1\rangle. \tag{2.16}$$

In eqns. (2.12), eqn. (2.15) and eqn. (2.16) we have used the orthonormal, $\langle i|j\rangle = \delta_{ij}$, computational basis set in which

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \quad |2\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}, \quad \text{and so on.} \tag{2.17}$$

Repeating the above procedure for $N = 2$, we obtain

$$\nu = d|0\rangle + \sum_{i=1}^{2} c_i|i\rangle, \tag{2.18}$$

the matrix of eigenvalues becomes,

$$D = \left(\frac{\omega+\epsilon}{2} + \frac{\Omega_2}{2}\right)|0\rangle\langle 0| + \left(\frac{\omega+\epsilon}{2} - \frac{\Omega_2}{2}\right)|1\rangle\langle 1| + \epsilon|2\rangle\langle 2|, \tag{2.19}$$

where

$$\Omega_2 = \sqrt{8g^2 + (\omega-\epsilon)^2}, \tag{2.20}$$

and

$$R = [|\lambda_0\rangle, |\lambda_1\rangle, |\lambda_2\rangle], \tag{2.21}$$

where

$$|\lambda_0\rangle = \frac{\omega-\epsilon}{2g} + \frac{\sqrt{8g^2+(\omega-\epsilon)^2}}{2g}|0\rangle + \sum_{k=1}^{2}|k\rangle \tag{2.22}$$

$$|\lambda_1\rangle = \frac{\omega-\epsilon}{2g} - \frac{\sqrt{8g^2+(\omega-\epsilon)^2}}{2g}|0\rangle + \sum_{k=1}^{2}|k\rangle, \tag{2.23}$$

$$|\lambda_2\rangle = |2\rangle - |1\rangle. \tag{2.24}$$

Repeating the above procedure and extending the sum to general N, we obtain

$$\nu = d|0\rangle + \sum_{i=1}^{N} c_i|i\rangle, \tag{2.25}$$

$$D = \left(\frac{\omega+\epsilon}{2} + \frac{\Omega_N}{2}\right)|0\rangle\langle 0| + \left(\frac{\omega+\epsilon}{2} - \frac{\Omega_N}{2}\right)|1\rangle\langle 1| + \sum_{i=2}^{N}\epsilon|i\rangle\langle i|, \tag{2.26}$$

where

$$\Omega_N = \sqrt{4Ng^2 + (\omega-\epsilon)^2}. \tag{2.27}$$

The matrix of eigenvectors becomes

$$R = [|\lambda_0\rangle, |\lambda_1\rangle, |\lambda_2\rangle, \ldots, \lambda_N\rangle], \tag{2.28}$$

where each of the eigenvectors appearing in R are given by

$$|\lambda_0\rangle = \left(\frac{\omega-\epsilon}{2g} + \frac{\Omega_N}{2g}\right)|0\rangle + \sum_{k=1}^{N}|k\rangle \tag{2.29}$$

$$|\lambda_1\rangle = \left(\frac{\omega-\epsilon}{2g} - \frac{\Omega_N}{2g}\right)|0\rangle + \sum_{k=1}^{N}|k\rangle, \tag{2.30}$$

$$|\lambda_i\rangle = |i\rangle - |1\rangle, \qquad i = 2, 3, \cdots, N. \tag{2.31}$$

We are now going to normalize eqn. (2.29), eqn. (2.30) and eqn. (2.31). Before we do that let us for convenience set

$$x = \epsilon - \omega, \tag{2.32}$$

so that from eqn. (2.27), we have

$$\Omega_N^2 = 4Ng^2 + x^2. \tag{2.33}$$

Substituting eqn. (2.32) into eqn. (2.29), we get

$$|\lambda_0\rangle = \frac{\Omega_N - x}{2g}|0\rangle + \sum_{k=1}^{N}|k\rangle. \tag{2.34}$$

Let us now normalize eqn. (2.34); we first calculate the inner product of $|\lambda_0\rangle$,

$$\begin{aligned}\langle\lambda_0|\lambda_0\rangle &= \frac{x^2 - 2x\Omega_N + \Omega_N^2}{4g^2} + N \\ &= \frac{2\Omega_N\left(\Omega_N - x\right)}{4g^2},\end{aligned} \tag{2.35}$$

so that the normalized vector of $|\lambda_0\rangle$ is

$$\begin{aligned}|\bar{\lambda}_0\rangle &= \frac{|\lambda_0\rangle}{\sqrt{\langle\lambda_0|\lambda_0\rangle}} = \frac{\Omega_N - x}{\sqrt{2\Omega_N\left(\Omega_N - x\right)}}|0\rangle + \frac{2g}{\sqrt{2\Omega_N\left(\Omega_N - x\right)}}\sum_{k=1}^{N}|k\rangle \\ &= \frac{2g\sqrt{N}}{\sqrt{2\Omega_N\left(\Omega_N + x\right)}}|0\rangle + \frac{1}{\sqrt{N}}\sqrt{\frac{\Omega_N + x}{2\Omega_N}}\sum_{k=1}^{N}|k\rangle,\end{aligned} \tag{2.36}$$

where in the last line we have used

$$\Omega_N - x = \frac{\left(\Omega_N - x\right)\left(\Omega_N + x\right)}{\Omega_N + x} = \frac{\Omega_N^2 - x^2}{\Omega_N + x} = \frac{4g^2N}{\Omega_N + x} \tag{2.37}$$

and

$$\frac{\Omega_N + x}{\sqrt{2\Omega_N\left(\Omega_N + x\right)}} = \sqrt{\frac{\Omega_N + x}{2\Omega_N}}. \tag{2.38}$$

Similarly, substituting eqn. (2.32) into eqn. (2.30), we get

$$|\lambda_1\rangle = \frac{-\left(\Omega_N + x\right)}{2g}|0\rangle + \sum_{k=1}^{N}|k\rangle, \tag{2.39}$$

whose inner product is given by

$$\begin{aligned}\langle\lambda_1|\lambda_1\rangle &= \frac{\left(x + \Omega_N\right)^2}{4g^2} + N \\ &= \frac{2\Omega_N\left(\Omega_N + x\right)}{4g^2},\end{aligned} \tag{2.40}$$

so that the normalized vector of eqn. (2.39) becomes,

$$\begin{aligned}|\bar{\lambda}_1\rangle &= \frac{|\lambda_1\rangle}{\sqrt{\langle\lambda_1|\lambda_1\rangle}} = \frac{-\left(\Omega_N + x\right)}{\sqrt{2\Omega_N\left(\Omega_N + x\right)}}|0\rangle + \frac{2g}{\sqrt{2\Omega_N\left(\Omega_N + x\right)}}\sum_{k=1}^{N}|k\rangle \\ &= -\sqrt{\frac{\Omega_N + x}{2\Omega_N}}|0\rangle + \frac{2g}{\sqrt{2\Omega_N\left(\Omega_N + x\right)}}\sum_{k=1}^{N}|k\rangle.\end{aligned} \tag{2.41}$$

But it can be seen that

$$\left[\frac{2g\sqrt{N}}{\sqrt{2\Omega_N\left(\Omega_N + x\right)}}\right]^2 + \left[\sqrt{\frac{\Omega_N + x}{2\Omega_N}}\right]^2 = 1, \tag{2.42}$$

we therefore make use of the trigonometric relation: $\sin^2\theta + \cos^2\theta = 1$, by letting

$$\sin\theta = \sqrt{\frac{\Omega_N + x}{2\Omega_N}} \tag{2.43}$$

and

$$\cos\theta = \frac{2g\sqrt{N}}{\sqrt{2\Omega_N\left(\Omega_N + x\right)}}, \tag{2.44}$$

so that eqn. (2.36) and eqn. (2.41) finally become

$$|\bar{\lambda}_0\rangle = \cos\theta|0\rangle + \frac{\sin\theta}{\sqrt{N}}\sum_{k=1}^{N}|k\rangle, \tag{2.45}$$

and

$$|\bar{\lambda}_1\rangle = -\sin\theta|0\rangle + \frac{\cos\theta}{\sqrt{N}}\sum_{k=1}^{N}|k\rangle, \tag{2.46}$$

respectively.

Using eqn. (2.45) and eqn. (2.46) in the following calculations,

$$\begin{aligned}
\langle\bar{\lambda}_0|\bar{\lambda}_0\rangle &= \cos^2\theta + \frac{N\sin^2\theta}{N} = 1 \\
\langle\bar{\lambda}_1|\bar{\lambda}_1\rangle &= \sin^2\theta + \frac{N\cos^2\theta}{N} = 1 \\
\langle\bar{\lambda}_0|\bar{\lambda}_1\rangle &= -\sin\theta\cos\theta + \frac{N\sin\theta\cos\theta}{N} = 0,
\end{aligned} \tag{2.47}$$

confirms that $|\bar{\lambda}_0\rangle$ and $|\bar{\lambda}_1\rangle$ are normal as well as orthogonal, i.e., orthonormal.

Let us now orthonormalize the remaining eigenvectors in R, eqn. (2.31),

$$|\lambda_i\rangle = |i\rangle - |1\rangle, \qquad i = 2, 3, \cdots, N. \tag{2.48}$$

We are going to use the Gram-Schmidt orthonormalization technique [30, 32]. We start by normalizing eqn. (2.48),

$$\begin{aligned}
|\bar{\lambda}_i\rangle &= \frac{|\lambda_i\rangle}{\sqrt{\langle\lambda_i|\lambda_i\rangle}} \\
&= \frac{|i\rangle}{\sqrt{2}} - \frac{|1\rangle}{\sqrt{2}}, \qquad i = 2, 3, \cdots, N.
\end{aligned} \tag{2.49}$$

As a first step of the Gram-Schmidt orthonormalization technique, let us set

$$|\bar{\lambda}'_2\rangle = |\bar{\lambda}_2\rangle \tag{2.50}$$

and

$$|\bar{\lambda}'_3\rangle = c_2|\bar{\lambda}'_2\rangle + c_3|\bar{\lambda}_3\rangle. \tag{2.51}$$

We want to find c_2 and c_3 subject to orthonormalization conditions,

$$\langle\bar{\lambda}'_2|\bar{\lambda}'_3\rangle = 0, \qquad \langle\bar{\lambda}'_3|\bar{\lambda}'_3\rangle = 1. \tag{2.52}$$

Solving eqn. (2.52) simultaneously, we get

$$c_2 = -\frac{1}{\sqrt{3}} \qquad \text{and} \qquad c_3 = \frac{2}{\sqrt{3}}. \tag{2.53}$$

Substituting eqn. (2.53) into eqn. (2.51), we obtain

$$|\bar{\lambda}'_3\rangle = \frac{1}{\sqrt{3\,(2)}}\left(-|1\rangle - |2\rangle + 2|3\rangle\right). \tag{2.54}$$

We repeat the above procedure for $|\bar{\lambda}_4\rangle$, $|\bar{\lambda}_5\rangle$ and $|\bar{\lambda}_6\rangle$ so that overall we get

$$\begin{aligned}
|\bar{\lambda}'_2\rangle &= \frac{1}{\sqrt{2\,(1)}}\left(-|1\rangle + 1|2\rangle\right), \\
|\bar{\lambda}'_3\rangle &= \frac{1}{\sqrt{3\,(2)}}\left(-|1\rangle - |2\rangle + 2|3\rangle\right), \\
|\bar{\lambda}'_4\rangle &= \frac{1}{\sqrt{4\,(3)}}\left(-|1\rangle - |2\rangle - |3\rangle + 3|4\rangle\right), \\
|\bar{\lambda}'_5\rangle &= \frac{1}{\sqrt{5\,(4)}}\left(-|1\rangle - |2\rangle - |3\rangle - |4\rangle + 4|5\rangle\right), \\
|\bar{\lambda}'_6\rangle &= \frac{1}{\sqrt{6\,(5)}}\left(-|1\rangle - |2\rangle - |3\rangle - |4\rangle - |5\rangle + 5|6\rangle\right).
\end{aligned} \tag{2.55}$$

This is clearly a pattern which allows us to generalize,

$$|\bar{\lambda}'_i\rangle = -\frac{1}{\sqrt{i\,(i-1)}}\sum_{k=1}^{i-1}|k\rangle + \sqrt{\frac{i-1}{i}}|i\rangle, \qquad i = 2, 3, \cdots, N. \tag{2.56}$$

Summarizing the above results for orthonormalization of the eigenvectors in R,

$$R = \left[|\lambda_0\rangle|\lambda_1\rangle|\lambda_2\rangle\cdots|\lambda_N\rangle\right], \tag{2.57}$$

we have

$$\begin{aligned}
|\lambda_0\rangle &= \cos\theta|0\rangle + \frac{\sin\theta}{\sqrt{N}}\sum_{k=1}^{N}|k\rangle \\
|\lambda_1\rangle &= -\sin\theta|0\rangle + \frac{\cos\theta}{\sqrt{N}}\sum_{k=1}^{N}|k\rangle \\
|\lambda_i\rangle &= -\frac{1}{\sqrt{i\,(i-1)}}\sum_{k=1}^{i-1}|k\rangle + \sqrt{\frac{i-1}{i}}|i\rangle, \qquad i = 2, 3, \cdots, N,
\end{aligned} \tag{2.58}$$

where we have removed the bar and prime on λ's because all the eigenvectors in R appearing hereafter are taken to be orthonormal.

Using eqn. (2.2) and eqn. (2.25) into eqn. (2.1), we get

$$\begin{aligned}
H_S &= \nu^\dagger A \nu \\
&= \left(d^\dagger\langle 0| + \sum_{i=1}^{N} c_i^\dagger\langle i|\right)\left(\sum_{i=0}^{N}\lambda_i|\lambda_i\rangle\langle\lambda_i|\right)\left(d|0\rangle + \sum_{k=1}^{N} c_k|k\rangle\right) \\
&= \sum_{i=0}^{N}\lambda_i\xi_i^\dagger\xi_i,
\end{aligned} \tag{2.59}$$

where in moving from the second line to the third line, we have set

$$\xi_i = d\langle\lambda_i|0\rangle + \sum_{k=1}^{N} c_k\langle\lambda_i|k\rangle. \tag{2.60}$$

Upon substitution of eqn. (2.58) into eqn. (2.60), we obtain the eigenvectors

$$\begin{aligned}
\xi_0 &= \cos\theta d + \frac{\sin\theta}{\sqrt{N}}\sum_{i=1}^{N} c_i, \quad \lambda_0 = \frac{\omega+\epsilon}{2} + \frac{\Omega_N}{2}, \\
\xi_1 &= -\sin\theta d + \frac{\cos\theta}{\sqrt{N}}\sum_{i=1}^{N} c_i, \quad \lambda_1 = \frac{\omega+\epsilon}{2} - \frac{\Omega_N}{2}, \\
\xi_i &= -\frac{1}{\sqrt{i(i-1)}}\sum_{k=1}^{i-1} c_k + \sqrt{\frac{i-1}{i}}c_i, \quad \lambda_i = \epsilon, \quad i = 2, 3\cdots N,
\end{aligned} \tag{2.61}$$

written with the corresponding eigenvalues as earlier obtained in eqn. (2.26). The coefficients Ω_N, $\cos\theta$ and $\sin\theta$ read,

$$\begin{aligned}\Omega_N &= \sqrt{4g^2N + (\epsilon-\omega)^2},\\ \cos\theta &= \frac{2g\sqrt{N}}{\sqrt{2\Omega_N\left(\Omega_N + (\epsilon-\omega)\right)}},\\ \sin\theta &= \sqrt{\frac{\Omega_N + (\epsilon-\omega)}{2\Omega_N}}.\end{aligned} \tag{2.62}$$

We note that the fact that the linear combinations of c_i appear in eqn. (2.61) is a consequence of choosing all the c-levels degenerate in eqn. (1.2).

Having casted H_S in the form of eqn. (2.59), we can conclude the chapter by stating that we have successfully diagonalized the Hamiltonian of the system.

Chapter 3

Quantum Master Equation

In this chapter, we are going to derive the Markovian quantum master equation. But before we do that it is imperative that we bring the results of Chapter 2 into perspective. In Chapter 2, we diagonalized the Hamiltonian of the system H_S, eqn. (2.59). In this form, we introduced new operators, ξ_i, whose properties we do not know yet. Therefore, the entire first section of this chapter is devoted towards investigating properties of these new operators. We are also going to cast the interaction Hamiltonian, H_{SB}, in terms of ξ_i's. In Section 3.2, we will convert the total Hamiltonian to the interaction picture followed by the derivation of the Markovian quantum master equation in section 3.3.

3.1 Quasi-Fermionic Picture

The fact that the original system Hamiltonian, H_S, eqn. (1.2), consists entirely of fermions, gives us a sound starting point for checking properties of the new operators, ξ_i, into which H_S has been expressed. Fermions are subject to the standard anticommutation relations. So, we are going to subject ξ_i to the anticommutator and observe what happens.

For $i = 0$, the anticommutator, $\{\xi_0, \xi_0^\dagger\}_+$ gives

$$\begin{aligned}\{\xi_0, \xi_0^\dagger\}_+ &= \{\cos\theta d + \frac{\sin\theta}{\sqrt{N}}\sum_{i=1}^{N} c_i, \cos\theta d^\dagger + \frac{\sin\theta}{\sqrt{N}}\sum_{i=1}^{N} c_i^\dagger\}_+ \\ &= \cos^2\theta + \frac{N\sin^2\theta}{N} \\ &= 1. \end{aligned} \tag{3.1}$$

Similarly, for $i = 1$,

$$\begin{aligned}\{\xi_1, \xi_1^\dagger\}_+ &= \{-\sin\theta d + \frac{\cos\theta}{\sqrt{N}}\sum_{i=1}^{N} c_i, -\sin\theta d^\dagger + \frac{\cos\theta}{\sqrt{N}}\sum_{i=1}^{N} c_i^\dagger\}_+ \\ &= \sin^2\theta + \frac{N\cos^2\theta}{N} \\ &= 1. \end{aligned} \tag{3.2}$$

Combining ξ_0 and $\xi_1^\dagger$ in the anticommutator, we obtain

$$\begin{aligned}\{\xi_0, \xi_1^\dagger\}_+ &= \{\cos\theta d + \frac{\sin\theta}{\sqrt{N}}\sum_{i=1}^{N} c_i, -\sin\theta d^\dagger + \frac{\cos\theta}{\sqrt{N}}\sum_{i=1}^{N} c_i^\dagger\}_+ \\ &= -\sin\theta\cos\theta + \sin\theta\cos\theta \\ &= 0. \end{aligned} \tag{3.3}$$

In a similar manner, we obtain the following results,

$$\{\xi_0, \xi_i^\dagger\}_+ = 0, \qquad \{\xi_1, \xi_i^\dagger\}_+ = 0, \qquad \{\xi_i, \xi_i^\dagger\}_+ = 1 \qquad \text{and} \qquad \{\xi_k, \xi_i^\dagger\}_+ = 0, \tag{3.4}$$

where $i \neq k$. Eqn. (3.1) through to eqn. (3.4) indicates that $\xi_i^\dagger$ and ξ_i are creation and annihilation operators satisfying anticommutation relations,

$$\{\xi_i, \xi_j^\dagger\}_+ = \delta_{ij}, \qquad \{\xi_i, \xi_j\}_+ = 0 \qquad \text{and} \qquad \{\xi_i^\dagger, \xi_j^\dagger\}_+ = 0, \tag{3.5}$$

which look exactly like the standard anticommutation relations that fermions obey. Therefore, the new operators are fermion-like, allowing us to call them quasi-fermions. So, the diagonalization process of Chapter 2, transformed the system

Hamiltonian, H_S, from the original fermionic picture to the new quasi-fermionic picture.

Let us now transform the interaction Hamiltonian, $H_{SB} = \sum_{i=1}^{N} \sum_n g_n c_i^\dagger b_n + g_n^* c_i b_n^\dagger$, into this new quasi-fermionic picture.

Multiplying eqn. (2.60) by $\sum_{i=0}^{N} |\lambda_i\rangle$ to the right, we get

$$\begin{aligned}\sum_{i=0}^{N} \xi_i |\lambda_i\rangle &= d \sum_{i=0}^{N} |\lambda_i\rangle\langle\lambda_i|0\rangle + \sum_{k=1}^{N} c_k \sum_{i=0}^{N} |\lambda_i\rangle\langle\lambda_i|k\rangle \\ &= d|0\rangle + \sum_{k=1}^{N} c_k |k\rangle. \end{aligned} \tag{3.6}$$

When we multiply eqn. (3.6) by $\langle 0|$ to the left, we obtain

$$d\langle 0|0\rangle + \sum_{k=1}^{N} c_k \langle 0|k\rangle = \sum_{i=0}^{N} \xi_i \langle 0|\lambda_i\rangle = d. \tag{3.7}$$

Upon substitution of eqn. (2.58) into eqn. (3.7), we obtain the expression for d in terms of ξ_0 and ξ_1,

$$d = \sum_{i=0}^{N} \xi_i \langle 0|\lambda_i\rangle = \cos\theta \xi_0 - \sin\theta \xi_1. \tag{3.8}$$

Multiplying the left handside of eqn. (3.6) by $\langle i|$, for $i = 1, 2, \cdots, N$, and simplifying, we get

$$\sum_{i=1}^{N} c_i = \sum_{i=1}^{N} \sum_{k=0}^{N} \langle i|\lambda_k\rangle \xi_k = \sum_{k=0}^{N} \alpha_k \xi_k, \tag{3.9}$$

where

$$\alpha_k = \sum_{i=1}^{N} \langle i|\lambda_k\rangle. \tag{3.10}$$

If we set $k = 0$ in eqn. (3.10) and use $|\lambda_0\rangle$ from eqn. (2.58), we obtain

$$\alpha_0 = \sum_{i=1}^{N} \langle i|\lambda_0\rangle = \sum_{i=1}^{N} \langle i| \left(\cos\theta |0\rangle + \frac{\sin\theta}{\sqrt{N}} \sum_{k=1}^{N} |k\rangle \right) = \sqrt{N} \sin\theta. \tag{3.11}$$

Similarly,

$$\alpha_1 = \sum_{i=1}^{N} \langle i|\lambda_1\rangle = \sum_{i=1}^{N} \langle i| \left(-\sin\theta|0\rangle + \frac{\cos\theta}{\sqrt{N}} \sum_{k=1}^{N} |k\rangle \right) = \sqrt{N}\cos\theta, \tag{3.12}$$

and for $k = 2, 3 \cdots N$, we have

$$\alpha_k = \sum_{i=1}^{N} \langle i|\lambda_k\rangle = \sum_{i=1}^{N} \langle i| \left(-\frac{1}{\sqrt{k\,(k-1)}} \sum_{j=1}^{k-1} |j\rangle + \sqrt{\frac{k-1}{k}} |k\rangle \right) = 0. \tag{3.13}$$

Using eqn. (3.11), eqn. (3.12) and eqn. (3.13) in eqn. (3.9), we obtain

$$\begin{aligned} \sum_{i=1}^{N} c_i &= \sum_{k=0}^{N} \alpha_k \xi_k = \alpha_0 \xi_0 + \alpha_1 \xi_1 + \sum_{k=2}^{N} \alpha_k \xi_k \\ &= \sqrt{N} \left(\sin\theta \xi_0 + \cos\theta \xi_1 \right). \end{aligned} \tag{3.14}$$

To obtain the interaction Hamiltonian, eqn. (1.6), in terms of the new operators, ξ_i's, we substitute eqn. (3.14) into eqn. (1.6). This substitution gives

$$\begin{aligned} H_{SB} &= \sum_{i=1}^{N} \sum_{n} g_n c_i^{\dagger} b_n + g_n^* c_i b_n^{\dagger} \\ &= \sqrt{N} \sum_{n} \left(g_n \left(\sin\theta \xi_0^{\dagger} + \cos\theta \xi_1^{\dagger} \right) b_n + g_n^* \left(\sin\theta \xi_0 + \cos\theta \xi_1 \right) b_n^{\dagger} \right). \end{aligned} \tag{3.15}$$

The total Hamiltonian, eqn. (1.1), therefore becomes

$$\begin{aligned} H &= H_S + H_B + H_{SB} \\ &= \sum_{i=0}^{N} \lambda_i \xi_i^{\dagger} \xi_i + \sum_{n} \omega_n b_n^{\dagger} b_n \\ &\quad + \sqrt{N} \sum_{n} \left(g_n \left(\sin\theta \xi_0^{\dagger} + \cos\theta \xi_1^{\dagger} \right) b_n + g_n^* \left(\sin\theta \xi_0 + \cos\theta \xi_1 \right) b_n^{\dagger} \right). \end{aligned} \tag{3.16}$$

The fact that the bosonic bath couples to only two of the quasi-fermion states is a big simplification, resulting from the fact that the c-operators in eqn. (1.2) are all degenerate.

Thus, we have expressed the total Hamiltonian in terms of the new operators, ξ_i, a transformation into the new quasi-fermionic picture.

3.2 Transition to the Interaction Picture

The starting point for the derivation of the master equation is the general expression for the Markovian quantum master equation (1.13). The Hamiltonian appearing in this expression is in the interaction picture. So, we are now going to switch to the interaction picture within the quasi-fermionic picture.

Expressing the interaction term in eqn. (3.16) as

$$H_{SB} = H_{SB1} + H_{SB2}, \tag{3.17}$$

where

$$H_{SB1} = \sum_n \sqrt{N} g_n^* \left(\sin\theta \xi_0 + \cos\theta \xi_1\right) b_n^\dagger, \tag{3.18}$$

and its complex conjugate

$$H_{SB2} = \sum_n \sqrt{N} g_n \left(\sin\theta \xi_0^\dagger + \cos\theta \xi_1^\dagger\right) b_n, \tag{3.19}$$

and converting it to the interaction picture, we have

$$H_I^{SB}(t) = e^{it(H_S+H_B)} H_{SB} e^{-it(H_S+H_B)} = H_I^{SB1} + H_I^{SB2}. \tag{3.20}$$

where we have used eqn. (3.17), eqn. (3.18) and eqn. (3.19) so that

$$\begin{aligned} H_I^{SB1} &= e^{it(H_S+H_B)} H_{SB1} e^{-it(H_S+H_B)} \\ &= \sum_n \sqrt{N} g_n^* \left(\sin\theta e^{it(H_S+H_B)} \xi_0 b_n^\dagger e^{-it(H_S+H_B)} + \cos\theta e^{it(H_S+H_B)} \xi_1 b_n^\dagger \right) \\ &= \sum_n \sqrt{N} g_n^* \sin\theta e^{itH_S} \xi_0 e^{-itH_S} e^{-itH_B} b_n^\dagger e^{-itH_B} \\ &\quad + \sum_n \sqrt{N} g_n^* \cos\theta e^{itH_S} \xi_1 e^{-itH_S} e^{-itH_B} b_n^\dagger e^{-itH_B}, \end{aligned} \tag{3.21}$$

and its complex conjugate

$$\begin{aligned} H_I^{SB2} &= e^{it(H_S+H_B)} H_{SB2} e^{-it(H_S+H_B)} \\ &= \sum_n \sqrt{N} g_n \sin\theta e^{itH_S} \xi_0^\dagger e^{-itH_S} e^{-itH_B} b_n e^{-itH_B} \\ &\quad + \sum_n \sqrt{N} g_n \cos\theta e^{itH_S} \xi_1^\dagger e^{-itH_S} e^{-itH_B} b_n e^{-itH_B}. \end{aligned} \tag{3.22}$$

Note that in eqn. (3.21) and eqn. (3.22) we have used the fact that H_S and H_B commute, $[H_S, H_B] = 0$. Using the Baker–Campbell–Hausdorf formula [33],

$$e^{\alpha A} B e^{-\alpha A} = B + \alpha[A,B] + \frac{\alpha^2}{2!}[A,[A,B]] + \frac{\alpha^3}{3!}[A,[A,[A,B]]] + \cdots, \tag{3.23}$$

we have

$$e^{itH_S} \xi_0 e^{-itH_S} = \xi_0 + it[H_S, \xi_0] + \frac{(it)^2}{2!}[H_S,[H_S,\xi_0]] + \cdots \tag{3.24}$$

We first solve for the commutator appearing in eqn. (3.24), which gives us

$$[H_S, \xi_0] = \sum_{i=0}^N \lambda_i [\xi_i^\dagger \xi_i, \xi_0] = -\sum_{i=0}^N \lambda_i \{\xi_i^\dagger, \xi_0\}_+ \xi_i = -\sum_{i=0}^N \lambda_i \xi_i \delta_{i0} = -\lambda_0 \xi_0. \tag{3.25}$$

Substituting eqn. (3.25) into eqn. (3.24), we obtain

$$e^{itH_S}\xi_0 e^{-itH_S} = \xi_0 + (-it\lambda_0)\,\xi_0 + \frac{(it)^2\,(-\lambda_0)^2}{2!}\xi_0 + \frac{(it)^3\,(-\lambda_0)^3}{3!}\xi_0 + \cdots$$
$$= \xi_0\left\{1 + (-it\lambda_0) + \frac{(-it\lambda_0)^2}{2!} + \cdots\right\} = \xi_0 e^{-it\lambda_0}. \qquad (3.26)$$

Again using eqn. (3.23), we have

$$e^{itH_B}b_n^\dagger e^{-itH_B} = b_n^\dagger + it[H_B, b_n^\dagger] + \frac{(it)^2}{2!}[H_B,[H_B,b_n^\dagger]] + \cdots \qquad (3.27)$$

Solving the commutator appearing in eqn. (3.27), we get

$$[H_B, b_n^\dagger] = [\sum_m \omega_m b_m^\dagger b_m, b_n^\dagger] = \sum_m \omega_m (b_m^\dagger b_n^\dagger b_m - b_n^\dagger b_m^\dagger b_m + b_m^\dagger \delta_{mn}) = \omega_n b_n^\dagger, \quad (3.28)$$

where we have made use of eqn. (1.5). Substituting eqn. (3.28) into eqn. (3.27), we have

$$e^{itH_B}b_n^\dagger e^{-itH_B} = b_n^\dagger\left\{1 + (it\omega_n) + \frac{(it\omega_n)^2}{2!} + \frac{(it\omega_n)^3}{3!} + \cdots\right\} = b_n^\dagger e^{it\omega_n}. \qquad (3.29)$$

Similarly,

$$e^{itH_S}\xi_1 e^{-itH_S} = \xi_1 + it[H_S,\xi_1] + \frac{(it)^2}{2!}[H_S,[H_S,\xi_]] + \cdots \qquad (3.30)$$

The commutator appearing in eqn. (3.30) can be shown to simplify as follows,

$$[H_S,\xi_1] = \sum_{i=0}^{N}\lambda_i[\xi_i^\dagger\xi_i,\xi_1] = -\sum_{i=0}^{N}\lambda_i\{\xi_i^\dagger,\xi_1\}_+\xi_i = -\sum_{i=0}^{N}\lambda_i\xi_i\delta_{i1} = -\lambda_1\xi_1. \quad (3.31)$$

Substituting eqn. (3.31) into eqn. (3.30), we obtain

$$e^{itH_S}\xi_1 e^{-itH_S} = \xi_1\left\{1 + (-it\lambda_1) + \frac{(-it\lambda_1)^2}{2!} + \frac{(-it\lambda_1)^3}{3!} + \cdots\right\} = \xi_1 e^{-it\lambda_1}. \quad (3.32)$$

When we use eqn. (3.32) and eqn. (3.29) in eqn. (3.21), H_I^{SB1} becomes

$$\begin{aligned} H_I^{SB1} &= \sum_n \sqrt{N} g_n^* \sin\theta e^{itH_S} \xi_0 e^{-itH_S} e^{-itH_B} b_n^\dagger e^{-itH_B} \\ &\quad + \sum_n \sqrt{N} g_n^* \cos\theta e^{itH_S} \xi_1 e^{-itH_S} e^{-itH_B} b_n^\dagger e^{-itH_B}. \\ &= \sum_n \sqrt{N} g_n^* \sin\theta e^{it(\omega_n - \lambda_0)} \xi_0 b_n^\dagger \\ &\quad + \sum_n \sqrt{N} g_n^* \cos\theta e^{it(\omega_n - \lambda_1)} b_n^\dagger. \end{aligned} \tag{3.33}$$

Let us set the two terms appearing in eqn. (3.33) as

$$\begin{aligned} B_{0t}^\dagger &= \sum_n \sqrt{N} g_n^* \sin\theta e^{it(\omega_n - \lambda_0)} b_n^\dagger, \\ B_{1t}^\dagger &= \sum_n \sqrt{N} g_n^* \cos\theta e^{it(\omega_n - \lambda_1)} b_n^\dagger, \end{aligned} \tag{3.34}$$

so that

$$H_I^{SB1} = \xi_0 B_{0t}^\dagger + \xi_1 B_{1t}^\dagger, \tag{3.35}$$

and its complex conjugate given by

$$H_I^{SB2} = \xi_0^\dagger B_{0t} + \xi_1^\dagger B_{1t}. \tag{3.36}$$

Therefore, if we substitute eqn. (3.35) and eqn. (3.36) into eqn. (3.20), the total Hamiltonian in the interaction picture becomes

$$\begin{aligned} H_I^{SB} &= H_I^{SB1} + H_I^{SB2} \\ &= \xi_0 B_{0t}^\dagger + \xi_1 B_{1t}^\dagger + \xi_0^\dagger B_{0t} + \xi_1^\dagger B_{1t}. \end{aligned} \tag{3.37}$$

This brings us to the conclusion of this section. We have transformed the total Hamiltonian from the original fermionic picture to the new quasi-fermionic picture. We further switched the total Hamiltonian to the interaction picture. Looking back at eqn. (1.13), the general expression of the Markovian quantum master equation, we now have everything we need from our model to start the derivation of the Markovian quantum master equation for our reduced system.

3.3 Derivation of Quantum Master Equation

We are now set to derive the Markovian quantum master equation from the general expression, eqn. (1.13),

$$\dot{\rho}_S(t) = -\int_0^\infty ds \mathrm{Tr}_B[H_I^{SB}(t), [H_I^{SB}(t-s), \rho_S(t) \otimes \rho_B(0)]]. \tag{3.38}$$

Note that eqn. (3.38) has a commutator in another commutator in the integrand on the right handside. This entails that it will diverge when expanded. So, we will handle it in steps. We will first work with the integrand until it is in a convenient form to allow us to carry out the integration.

Substituting eqn. (3.37) into the commutator in the integrand of eqn. (3.38) and taking the trace over the bath's degrees of freedom, we obtain

$$
\begin{aligned}
&\mathrm{Tr}_B\left\{[H_I^{SB}(t),[H_I^{SB}(t-s),\rho_S(t)\otimes\rho_B]]\right\}\\
&=\xi_0\xi_0\rho_S\mathrm{Tr}_B(B_{0t}^\dagger B_{0t-s}^\dagger\rho_B)-\xi_0\rho_S\xi_0\mathrm{Tr}_B(B_{0t}^\dagger B_{0t-s}^\dagger\rho_B)\\
&-\xi_0\rho_S\xi_0\mathrm{Tr}_B(B_{0t-s}^\dagger B_{0t}^\dagger\rho_B)+\rho_S\xi_0\xi_0\mathrm{Tr}_B(B_{0t-s}^\dagger B_{0t}^\dagger\rho_B)\\
&+\xi_1\xi_0\rho_S\mathrm{Tr}_B(B_{1t}^\dagger B_{0t-s}^\dagger\rho_B)-\xi_0\rho_S\xi_1\mathrm{Tr}_B(B_{1t}^\dagger B_{0t-s}^\dagger\rho_B)\\
&-\xi_1\rho_S\xi_0\mathrm{Tr}_B(B_{0t-s}^\dagger B_{1t}^\dagger\rho_B)+\rho_S\xi_0\xi_1\mathrm{Tr}_B(B_{0t-s}^\dagger B_{1t}^\dagger\rho_B)\\
&+\xi_0^\dagger\xi_0\rho_S\mathrm{Tr}_B(B_{0t}B_{0t-s}^\dagger\rho_B)-\xi_0\rho_S\xi_0^\dagger\mathrm{Tr}_B(B_{0t}B_{0t-s}^\dagger\rho_B)\\
&-\xi_0^\dagger\rho_S\xi_0\mathrm{Tr}_B(B_{0t-s}^\dagger B_{0t}\rho_B)+\rho_S\xi_0\xi_0^\dagger\mathrm{Tr}_B(B_{0t-s}^\dagger B_{0t}\rho_B)\\
&+\xi_1^\dagger\xi_0\rho_S\mathrm{Tr}_B(B_{1t}B_{0t-s}^\dagger\rho_B)-\xi_0\rho_S\xi_1^\dagger\mathrm{Tr}_B(B_{1t}B_{0t-s}^\dagger\rho_B)\\
&-\xi_1^\dagger\rho_S\xi_0\mathrm{Tr}_B(B_{0t-s}^\dagger B_{1t}\rho_B)+\rho_S\xi_0\xi_1^\dagger\mathrm{Tr}_B(B_{0t-s}^\dagger B_{1t}\rho_B)\\
&+\xi_0\xi_1\rho_S\mathrm{Tr}_B(B_{0t}^\dagger B_{1t-s}^\dagger\rho_B)-\xi_1\rho_S\xi_0\mathrm{Tr}_B(B_{0t}^\dagger B_{1t-s}^\dagger\rho_B)\\
&-\xi_0\rho_S\xi_1\mathrm{Tr}_B(B_{1t-s}^\dagger B_{0t}^\dagger\rho_B)+\rho_S\xi_1\xi_0\mathrm{Tr}_B(B_{1t-s}^\dagger B_{0t}^\dagger\rho_B)\\
&+\xi_1\xi_1\rho_S\mathrm{Tr}_B(B_{1t}^\dagger B_{1t-s}^\dagger\rho_B)-\xi_1\rho_S\xi_1\mathrm{Tr}_B(B_{1t}^\dagger B_{1t-s}^\dagger\rho_B)\\
&-\xi_1\rho_S\xi_1\mathrm{Tr}_B(B_{1t-s}^\dagger B_{1t}^\dagger\rho_B)+\rho_S\xi_1\xi_1\mathrm{Tr}_B(B_{1t-s}^\dagger B_{1t}^\dagger\rho_B)\\
&+\xi_0^\dagger\xi_1\rho_S\mathrm{Tr}_B(B_{0t}B_{1t-s}^\dagger\rho_B)-\xi_1\rho_S\xi_0^\dagger\mathrm{Tr}_B(B_{0t}B_{1t-s}^\dagger\rho_B)\\
&-\xi_0^\dagger\rho_S\xi_1\mathrm{Tr}_B(B_{1t-s}^\dagger B_{0t}\rho_B)+\rho_S\xi_1\xi_0^\dagger\mathrm{Tr}_B(B_{1t-s}^\dagger B_{0t}\rho_B)\\
&+\xi_1^\dagger\xi_1\rho_S\mathrm{Tr}_B(B_{1t}B_{1t-s}^\dagger\rho_B)-\xi_1\rho_S\xi_1^\dagger\mathrm{Tr}_B(B_{1t}B_{1t-s}^\dagger\rho_B)\\
&-\xi_1^\dagger\rho_S\xi_1\mathrm{Tr}_B(B_{1t-s}^\dagger B_{1t}\rho_B)+\rho_S\xi_1\xi_1^\dagger\mathrm{Tr}_B(B_{1t-s}^\dagger B_{1t}\rho_B)\\
&+\xi_0\xi_0^\dagger\rho_S\mathrm{Tr}_B(B_{0t}^\dagger B_{0t-s}\rho_B)-\xi_0^\dagger\rho_S\xi_0\mathrm{Tr}_B(B_{0t}^\dagger B_{0t-s}\rho_B)\\
&-\xi_0\rho_S\xi_0^\dagger\mathrm{Tr}_B(B_{0t-s}B_{0t}^\dagger\rho_B)+\rho_S\xi_0^\dagger\xi_0\mathrm{Tr}_B(B_{0t-s}B_{0t}^\dagger\rho_B)\\
&+\xi_1\xi_0^\dagger\rho_S\mathrm{Tr}_B(B_{1t}^\dagger B_{0t-s}\rho_B)-\xi_0^\dagger\rho_S\xi_1\mathrm{Tr}_B(B_{1t}^\dagger B_{0t-s}\rho_B)\\
&-\xi_1\rho_S\xi_0^\dagger\mathrm{Tr}_B(B_{0t-s}B_{1t}^\dagger\rho_B)+\rho_S\xi_0^\dagger\xi_1\mathrm{Tr}_B(B_{0t-s}B_{1t}^\dagger\rho_B)\\
&+\xi_0^\dagger\xi_0^\dagger\rho_S\mathrm{Tr}_B(B_{0t}B_{0t-s}\rho_B)-\xi_0^\dagger\rho_S\xi_0^\dagger\mathrm{Tr}_B(B_{0t}B_{0t-s}\rho_B)\\
&-\xi_0^\dagger\rho_S\xi_0^\dagger\mathrm{Tr}_B(B_{0t-s}B_{0t}\rho_B)+\rho_S\xi_0^\dagger\xi_0^\dagger\mathrm{Tr}_B(B_{0t-s}B_{0t}\rho_B)\\
&+\xi_1^\dagger\xi_0^\dagger\rho_S\mathrm{Tr}_B(B_{1t}B_{0t-s}\rho_B)-\xi_0^\dagger\rho_S\xi_1^\dagger\mathrm{Tr}_B(B_{1t}B_{0t-s}\rho_B)\\
&-\xi_1^\dagger\rho_S\xi_0^\dagger\mathrm{Tr}_B(B_{0t-s}B_{1t}\rho_B)+\rho_S\xi_0^\dagger\xi_1^\dagger\mathrm{Tr}_B(B_{0t-s}B_{1t}\rho_B)
\end{aligned}
$$

$$
\begin{aligned}
&+ \xi_0\xi_1^\dagger\rho_S\mathrm{Tr}_B(B_{0t}^\dagger B_{1t-s}\rho_B) - \xi_1^\dagger\rho_S\xi_0\mathrm{Tr}_B(B_{0t}^\dagger B_{1t-s}\rho_B)\\
&- \xi_0\rho_S\xi_1^\dagger\mathrm{Tr}_B(B_{1t-s}B_{0t}^\dagger\rho_B) + \rho_S\xi_1^\dagger\xi_0\mathrm{Tr}_B(B_{1t-s}B_{0t}^\dagger\rho_B)\\
&+ \xi_1\xi_1^\dagger\rho_S\mathrm{Tr}_B(B_{1t}^\dagger B_{1t-s}\rho_B) - \xi_1^\dagger\rho_S\xi_1\mathrm{Tr}_B(B_{1t}^\dagger B_{1t-s}\rho_B)\\
&- \xi_1\rho_S\xi_1^\dagger\mathrm{Tr}_B(B_{1t-s}B_{1t}^\dagger\rho_B) + \rho_S\xi_1^\dagger\xi_1\mathrm{Tr}_B(B_{1t-s}B_{1t}^\dagger\rho_B)\\
&+ \xi_0^\dagger\xi_1^\dagger\rho_S\mathrm{Tr}_B(B_{0t}B_{1t-s}\rho_B) - \xi_1^\dagger\rho_S\xi_0^\dagger\mathrm{Tr}_B(B_{0t}B_{1t-s}\rho_B)\\
&- \xi_0^\dagger\rho_S\xi_1^\dagger\mathrm{Tr}_B(B_{1t-s}B_{0t}\rho_B) + \rho_S\xi_1^\dagger\xi_0^\dagger\mathrm{Tr}_B(B_{1t-s}B_{0t}\rho_B)\\
&+ \xi_1^\dagger\xi_1^\dagger\rho_S\mathrm{Tr}_B(B_{1t}B_{1t-s}\rho_B) - \xi_1^\dagger\rho_S\xi_1^\dagger\mathrm{Tr}_B(B_{1t}B_{1t-s}\rho_B)\\
&- \xi_1^\dagger\rho_S\xi_1^\dagger\mathrm{Tr}_B(B_{1t-s}B_{1t}\rho_B) + \rho_S\xi_1^\dagger\xi_1^\dagger\mathrm{Tr}_B(B_{1t-s}B_{1t}\rho_B),
\end{aligned}
\tag{3.39}
$$

where on the right handside we have deliberately suppressed the time dependence of ρ_S for notational simplicity.

We now use eqn. (1.12) and eqn. (3.34) to solve the trace terms appearing on the right handside of eqn. (3.39). Tracing over the bosonic bath's degrees of freedom, we obtain vanishing terms:

$$
\begin{aligned}
\mathrm{Tr}_B(B_{0t}^\dagger B_{0t-s}^\dagger\rho_B) &= \mathrm{Tr}_B(B_{1t}^\dagger B_{0t-s}^\dagger\rho_B) = \mathrm{Tr}_B(B_{0t-s}^\dagger B_{1t}^\dagger\rho_B) = \mathrm{Tr}_B(B_{1t-s}^\dagger B_{0t}^\dagger\rho_B)\\
&= \mathrm{Tr}_B(B_{1t-s}^\dagger B_{1t}^\dagger\rho_B) = \mathrm{Tr}_B(B_{1t}^\dagger B_{1t-s}^\dagger\rho_B) = \mathrm{Tr}_B(B_{0t-s}B_{0t}\rho_B)\\
&= \mathrm{Tr}_B(B_{0t-s}B_{1t}\rho_B) = \mathrm{Tr}_B(B_{1t}B_{0t-s}\rho_B) = \mathrm{Tr}_B(B_{1t}B_{1t-s}\rho_B)\\
&= \mathrm{Tr}_B(B_{0t-s}^\dagger B_{0t}^\dagger\rho_B) = \mathrm{Tr}_B(B_{0t}B_{0t-s}\rho_B) = \mathrm{Tr}_B(B_{1t-s}B_{1t}\rho_B)\\
&= 0,
\end{aligned}
\tag{3.40}
$$

and non-vanishing terms:

$$
\mathrm{Tr}_B(B_{0t}B_{0t-s}^\dagger\rho_B) = N\sin^2\theta\sum_m |g_m|^2 e^{-is(\omega_m-\lambda_0)}(n+1) = \alpha_1(s), \tag{3.41}
$$

$$
\mathrm{Tr}_B(B_{0t-s}^\dagger B_{0t}\rho_B) = N\sin^2\sum_m |g_m|^2 e^{-is(\omega_m-\lambda_0)}n = \beta_1(s), \tag{3.42}
$$

$$\mathrm{Tr}_B(B_{0t}B^{\dagger}_{1t-s}\rho_B) = N\sin\theta\cos\theta\sum_m |g_m|^2 e^{it(\lambda_0-\lambda_1)}e^{-is(\omega_m-\lambda_1)}(n+1) = \alpha_2(t,s), \tag{3.43}$$

$$\mathrm{Tr}_B(B^{\dagger}_{1t-s}B_{0t}\rho_B) = N\sin\theta\cos\theta\sum_m |g_m|^2 e^{it(\lambda_0-\lambda_1)}e^{-is(\omega_m-\lambda_1)}n = \beta_2(t,s), \tag{3.44}$$

$$\mathrm{Tr}_B(B_{1t}B^{\dagger}_{1t-s}\rho_B) = N\cos^2\theta\sum_m |g_m|^2 e^{-is(\omega_m-\lambda_1)}(n+1) = \alpha_3(s), \tag{3.45}$$

$$\mathrm{Tr}_B(B^{\dagger}_{1t-s}B_{1t}\rho_B) = N\cos^2\sum_m |g_m|^2 e^{-is(\omega_m-\lambda_1)}n = \beta_3(s), \tag{3.46}$$

$$\mathrm{Tr}_B(B^{\dagger}_{1t}B_{0t-s}\rho_B) = N\sin\theta\cos\theta\sum_m |g_m|^2 e^{it(\lambda_0-\lambda_1)}e^{is(\omega_m-\lambda_0)}n = \beta_4(t,s), \tag{3.47}$$

and

$$\mathrm{Tr}_B(B_{0t-s}B^{\dagger}_{1t}\rho_B) = N\sin\theta\cos\theta\sum_m |g_m|^2 e^{it(\lambda_0-\lambda_1)}e^{is(\omega_m-\lambda_0)}(n+1) = \alpha_4(t,s), \tag{3.48}$$

where we have employed the Bose-Einstein statistics [28] which gives the average occupation number as

$$n = \frac{1}{(e^{\beta\omega_n}-1)}. \tag{3.49}$$

In the limit of a large number of bosons, we transform the summation appearing in eqns. (3.41) through to (3.48) to the continuum [27],

$$\sum_m |g_m|^2 \longrightarrow \int d\omega_m J(\omega_m), \tag{3.50}$$

where $J(\omega_m)$ is the spectral density. We also note that

$$\mathrm{Tr}_B(B_{0t-s}B^{\dagger}_{0t}\rho_B) = \alpha_1^*(s), \tag{3.51}$$

$$\mathrm{Tr}_B(B^{\dagger}_{0t}B_{0t-s}\rho_B) = \beta_1^*(s) \tag{3.52}$$

$$\mathrm{Tr}_B(B_{1t-s}B^{\dagger}_{0t}\rho_B) = \alpha_2^*(t,s), \tag{3.53}$$

$$\mathrm{Tr}_B(B_{0t}^\dagger B_{1t-s}\rho_B) = \beta_2^*(t,s), \tag{3.54}$$

$$\mathrm{Tr}_B(B_{1t-s}B_{1t}^\dagger\rho_B) = \alpha_3^*(s), \tag{3.55}$$

$$\mathrm{Tr}_B(B_{1t}^\dagger B_{1t-s}\rho_B) = \beta_3^*(s), \tag{3.56}$$

$$\mathrm{Tr}_B(B_{0t-s}^\dagger B_{1t}\rho_B) = \beta_4^*(t,s), \tag{3.57}$$

and

$$\mathrm{Tr}_B(B_{1t}B_{0t-s}^\dagger\rho_B) = \alpha_4^*(t,s). \tag{3.58}$$

Substituting eqns. (3.40) through to (3.48) and eqns. (3.51) through to (3.58) into eqn. (3.39), we get

$$\begin{aligned}
&\mathrm{Tr}_B\left\{[H_I^{SB}(t),[H_I^{SB}(t-s),\rho_S(t)\otimes\rho_B]]\right\} = \\
&+\xi_0^\dagger\xi_0\rho_S\alpha_1 - \xi_0\rho_S\xi_0^\dagger(\alpha_1+\alpha_1^*) - \xi_0^\dagger\rho_S\xi_0(\beta_1+\beta_1^*) \\
&-\xi_1\rho_S\xi_1^\dagger(\alpha_3+\alpha_3^*) - \xi_1^\dagger\rho_S\xi_1(\beta_3+\beta_3^*) + \rho_S\xi_0\xi_0^\dagger\beta_1 \\
&+\xi_1^\dagger\xi_0\rho_S\alpha_4^* - \xi_0\rho_S\xi_1^\dagger\alpha_4^* - \xi_1^\dagger\rho_S\xi_0\beta_4^* + \rho_S\xi_0\xi_1^\dagger\beta_4^* \\
&+\xi_0^\dagger\xi_1\rho_S\alpha_2 - \xi_1\rho_S\xi_0^\dagger\alpha_2 - \xi_0^\dagger\rho_S\xi_1\beta_2 + \rho_S\xi_1\xi_0^\dagger\beta_2 \\
&+\xi_1^\dagger\xi_1\rho_S\alpha_3 + \rho_S\xi_1\xi_1^\dagger\beta_3 + \xi_0\xi_0^\dagger\rho_S\beta_1^* + \rho_S\xi_0^\dagger\xi_0\alpha_1^* \\
&+\xi_1\xi_0^\dagger\rho_S\beta_4 - \xi_0^\dagger\rho_S\xi_1\beta_4 - \xi_1\rho_S\xi_0^\dagger\alpha_4 + \rho_S\xi_0^\dagger\xi_1\alpha_4 \\
&+\xi_0\xi_1^\dagger\rho_S\beta_2^* - \xi_1^\dagger\rho_S\xi_0\beta_2^* - \xi_0\rho_S\xi_1^\dagger\alpha_2^* + \rho_S\xi_1^\dagger\xi_0\alpha_2^* \\
&+\xi_1\xi_1^\dagger\rho_S\beta_3^* + \rho_S\xi_1^\dagger\xi_1\alpha_3^*.
\end{aligned} \tag{3.59}$$

To further simplify eqn. (3.59), we will need the following relations,

$$\begin{aligned} \alpha_1 + \alpha_1^* &= 2\mathrm{Re}\alpha_1, \\ \alpha_1^* &= \mathrm{Re}\alpha_1 - i\mathrm{Im}\alpha_1, \\ \alpha_1 &= \mathrm{Re}\alpha_1 + i\mathrm{Im}\alpha_1, \\ \mathrm{Im}\alpha_1 &= \frac{\alpha_1 - \alpha_1^*}{2i}, \end{aligned} \tag{3.60}$$

$$\begin{aligned} \alpha_2 + \alpha_2^* &= 2\mathrm{Re}\alpha_2, \\ \alpha_2^* &= \mathrm{Re}\alpha_2 - i\mathrm{Im}\alpha_2, \\ \alpha_2 &= \mathrm{Re}\alpha_2 + i\mathrm{Im}\alpha_2, \\ \mathrm{Im}\alpha_2 &= \frac{\alpha_2 - \alpha_2^*}{2i}, \end{aligned} \tag{3.61}$$

$$\begin{aligned} \beta_1 + \beta_1^* &= 2\mathrm{Re}\beta_1, \\ \beta_1^* &= \mathrm{Re}\beta_1 - i\mathrm{Im}\beta_1, \\ \beta_1 &= \mathrm{Re}\beta_1 + i\mathrm{Im}\beta_1, \\ \mathrm{Im}\beta_1 &= \frac{\beta_1 - \beta_1^*}{2i}, \end{aligned} \tag{3.62}$$

$$\begin{aligned} \beta_2 + \beta_2^* &= 2\mathrm{Re}\beta_2, \\ \beta_2^* &= \mathrm{Re}\beta_2 - i\mathrm{Im}\beta_2, \\ \beta_2 &= \mathrm{Re}\beta_2 + i\mathrm{Im}\beta_2, \\ \mathrm{Im}\beta_2 &= \frac{\beta_2 - \beta_2^*}{2i}, \end{aligned} \tag{3.63}$$

where we have invoked the notation of complex numbers with Reα_1, Reα_2, Reβ_1 and Reβ_2 corresponding to the real parts of α_1, α_2, β_1 and β_2, and Imα_1, Imα_2, Imβ_1 and Imβ_2 corresponding to the imaginary parts of α_1, α_2, β_1 and β_2, respectively. Similar expressions can be obtained for $\alpha_3, \alpha_4, \beta_3$ and β_4 and when used together in eqn. (3.59), we obtain

$$
\begin{aligned}
&\mathrm{Tr}_B\left\{[H_I^{SB}(t),[H_I^{SB}(t-s),\rho_S(t)\otimes\rho_B]]\right\} = \\
&-2\mathrm{Re}\alpha_1\left(\xi_0\rho_S\xi_0^\dagger - \frac{1}{2}\{\xi_0^\dagger\xi_0,\rho_S\}_+\right) \\
&-2\mathrm{Re}\beta_1\left(\xi_0^\dagger\rho_S\xi_0 - \frac{1}{2}\{\xi_0\xi_0^\dagger,\rho_S\}_+\right) \\
&-2\mathrm{Re}\alpha_3\left(\xi_1\rho_S\xi_1^\dagger - \frac{1}{2}\{\xi_1^\dagger\xi_1,\rho_S\}_+\right) \\
&-2\mathrm{Re}\beta_3\left(\xi_1^\dagger\rho_S\xi_1 - \frac{1}{2}\{\xi_1\xi_1^\dagger,\rho_S\}_+\right) \\
&+i[\left(\mathrm{Im}\alpha_1\xi_0^\dagger\xi_0 - \mathrm{Im}\beta_1\xi_0\xi_0^\dagger + \mathrm{Im}\alpha_3\xi_1^\dagger\xi_1 - \mathrm{Im}\beta_3\xi_1\xi_1^\dagger\right),\rho_S] \\
&+\mathrm{Re}\alpha_2\left(\xi_1^\dagger\xi_0\rho_S + \rho_S\xi_1^\dagger\xi_0 - \xi_1\rho_S\xi_0^\dagger - \xi_0\rho_S\xi_1^\dagger\right) \\
&+\mathrm{Re}\beta_2\left(\xi_0\xi_1^\dagger\rho_S + \rho_S\xi_1\xi_0^\dagger - \xi_0^\dagger\rho_S\xi_1 - \xi_1^\dagger\rho_S\xi_0\right) \\
&+\mathrm{Re}\alpha_4\left(\xi_1^\dagger\xi_0\rho_S + \rho_S\xi_0^\dagger\xi_1 - \xi_1\rho_S\xi_0^\dagger - \xi_0\rho_S\xi_1^\dagger\right) \\
&+\mathrm{Re}\beta_4\left(\xi_1\xi_0^\dagger\rho_S + \rho_S\xi_0\xi_1^\dagger - \xi_0^\dagger\rho_S\xi_1 - \xi_1^\dagger\rho_S\xi_0\right) \\
&+i\mathrm{Im}\alpha_2\left(\xi_1^\dagger\xi_0\rho_S - \rho_S\xi_1^\dagger\xi_0 + \xi_0\rho_S\xi_1^\dagger - \xi_1\rho_S\xi_0^\dagger\right) \\
&+i\mathrm{Im}\beta_2\left(\xi_1^\dagger\rho_S\xi_0 - \xi_0\xi_1^\dagger\rho_S + \rho_S\xi_1\xi_0^\dagger - \xi_0^\dagger\rho_S\xi_1\right) \\
&+i\mathrm{Im}\alpha_4\left(\rho_S\xi_0^\dagger\xi_1 - \xi_1\rho_S\xi_0^\dagger + \xi_0\rho_S\xi_1^\dagger - \xi_1^\dagger\xi_0\rho_S\right) \\
&+i\mathrm{Im}\beta_4\left(\xi_1\xi_0^\dagger\rho_S - \xi_0^\dagger\rho_S\xi_1 + \xi_1^\dagger\rho_S\xi_0 - \rho_S\xi_0\xi_1^\dagger\right).
\end{aligned}
\tag{3.64}
$$

Substituting eqn. (3.64) into the general expression for Markovian quantum master equation (1.13), we have

$$
\begin{aligned}
\dot{\rho}_S = \quad & -\int_0^\infty ds \mathrm{Tr}_B[H_I^{SB}(t),[H_I^{SB}(t-s),\rho_S(t)\otimes\rho_B(0)]] \\
= \quad & 2\int_0^\infty ds \mathrm{Re}\alpha_1 \left(\xi_0\rho_S\xi_0^\dagger - \frac{1}{2}\{\xi_0^\dagger\xi_0,\rho_S\}_+\right) \\
& + 2\int_0^\infty ds \mathrm{Re}\beta_1 \left(\xi_0^\dagger\rho_S\xi_0 - \frac{1}{2}\{\xi_0\xi_0^\dagger,\rho_S\}_+\right) \\
& + 2\int_0^\infty ds \mathrm{Re}\alpha_3 \left(\xi_1\rho_S\xi_1^\dagger - \frac{1}{2}\{\xi_1^\dagger\xi_1,\rho_S\}_+\right) \\
& + 2\int_0^\infty ds \mathrm{Re}\beta_3 \left(\xi_1^\dagger\rho_S\xi_1 - \frac{1}{2}\{\xi_1\xi_1^\dagger,\rho_S\}_+\right) \\
& - i[\int_0^\infty ds \left(\mathrm{Im}\alpha_1\xi_0^\dagger\xi_0 - \mathrm{Im}\beta_1\xi_0\xi_0^\dagger + \mathrm{Im}\alpha_3\xi_1^\dagger\xi_1 - \mathrm{Im}\beta_3\xi_1\xi_1^\dagger\right), \rho_S] \\
& - \int_0^\infty ds \mathrm{Re}\alpha_2 \left(\xi_1^\dagger\xi_0\rho_S + \rho_S\xi_1^\dagger\xi_0 - \xi_1\rho_S\xi_0^\dagger - \xi_0\rho_S\xi_1^\dagger\right) \\
& - \int_0^\infty ds \mathrm{Re}\beta_2 \left(\xi_0\xi_1^\dagger\rho_S + \rho_S\xi_1\xi_0^\dagger - \xi_0^\dagger\rho_S\xi_1 - \xi_1^\dagger\rho_S\xi_0\right) \\
& - \int_0^\infty ds \mathrm{Re}\alpha_4 \left(\xi_1^\dagger\xi_0\rho_S + \rho_S\xi_0^\dagger\xi_1 - \xi_1\rho_S\xi_0^\dagger - \xi_0\rho_S\xi_1^\dagger\right) \\
& - \int_0^\infty ds \mathrm{Re}\beta_4 \left(\xi_1\xi_0^\dagger\rho_S + \rho_S\xi_0\xi_1^\dagger - \xi_0^\dagger\rho_S\xi_1 - \xi_1^\dagger\rho_S\xi_0\right) \\
& - i\int_0^\infty ds \mathrm{Im}\alpha_2 \left(\xi_1^\dagger\xi_0\rho_S - \rho_S\xi_1^\dagger\xi_0 + \xi_0\rho_S\xi_1^\dagger - \xi_1\rho_S\xi_0^\dagger\right) \\
& - i\int_0^\infty ds \mathrm{Im}\beta_2 \left(\xi_1^\dagger\rho_S\xi_0 - \xi_0\xi_1^\dagger\rho_S + \rho_S\xi_1\xi_0^\dagger - \xi_0^\dagger\rho_S\xi_1\right) \\
& - i\int_0^\infty ds \mathrm{Im}\alpha_4 \left(\rho_S\xi_0^\dagger\xi_1 - \xi_1\rho_S\xi_0^\dagger + \xi_0\rho_S\xi_1^\dagger - \xi_1^\dagger\xi_0\rho_S\right) \\
& - i\int_0^\infty ds \mathrm{Im}\beta_4 \left(\xi_1\xi_0^\dagger\rho_S - \xi_0^\dagger\rho_S\xi_1 + \xi_1^\dagger\rho_S\xi_0 - \rho_S\xi_0\xi_1^\dagger\right),
\end{aligned}
\tag{3.65}
$$

where we have again deliberately suppressed the time dependence of ρ_S for notational simplicity.

Let us now set

$$\begin{aligned}
A &= 2\int_0^\infty ds \mathrm{Re}\alpha_1(s), \qquad B = 2\int_0^\infty ds \mathrm{Re}\beta_1(s), \qquad C = 2\int_0^\infty ds \mathrm{Re}\alpha_3(s) \\
D &= 2\int_0^\infty ds \mathrm{Re}\beta_3(s), \qquad E = \int_0^\infty ds \mathrm{Re}\alpha_2(t,s), \qquad F = \int_0^\infty ds \mathrm{Re}\beta_2(t,s), \\
G &= \int_0^\infty ds \mathrm{Re}\alpha_4(t,s), \qquad H = \int_0^\infty ds \mathrm{Re}\beta_4(t,s), \qquad I = \int_0^\infty ds \mathrm{Im}\alpha_2(t,s), \\
H_{LS} &= \int_0^\infty ds \left(\mathrm{Im}\alpha_1 \xi_0^\dagger \xi_0 - \mathrm{Im}\beta_1 \xi_0 \xi_0^\dagger + \mathrm{Im}\alpha_3 \xi_1^\dagger \xi_1 - \mathrm{Im}\beta_3 \xi_1 \xi_1^\dagger\right) \\
J &= \int_0^\infty ds \mathrm{Im}\beta_2(t,s), \qquad K = \int_0^\infty ds \mathrm{Im}\alpha_4(t,s), \qquad L = \int_0^\infty ds \mathrm{Im}\beta_4(t,s),
\end{aligned} \tag{3.66}$$

so that eqn. (3.65) becomes

$$\begin{aligned}
\dot{\rho}_S = \quad & A\left(\xi_0 \rho_S \xi_0^\dagger - \frac{1}{2}\{\xi_0^\dagger \xi_0, \rho_S\}_+\right) + B\left(\xi_0^\dagger \rho_S \xi_0 - \frac{1}{2}\{\xi_0 \xi_0^\dagger, \rho_S\}_+\right) \\
&+ C\left(\xi_1 \rho_S \xi_1^\dagger - \frac{1}{2}\{\xi_1^\dagger \xi_1, \rho_S\}_+\right) + D\left(\xi_1^\dagger \rho_S \xi_1 - \frac{1}{2}\{\xi_1 \xi_1^\dagger, \rho_S\}_+\right) \\
&- i[H_{LS}, \rho_S] \\
&- E\left(\xi_1^\dagger \xi_0 \rho_S + \rho_S \xi_1^\dagger \xi_0 - \xi_1 \rho_S \xi_0^\dagger - \xi_0 \rho_S \xi_1^\dagger\right) \\
&- F\left(\xi_0 \xi_1^\dagger \rho_S + \rho_S \xi_1 \xi_0^\dagger - \xi_0^\dagger \rho_S \xi_1 - \xi_1^\dagger \rho_S \xi_0\right) \\
&- G\left(\xi_1^\dagger \xi_0 \rho_S + \rho_S \xi_0^\dagger \xi_1 - \xi_1 \rho_S \xi_0^\dagger - \xi_0 \rho_S \xi_1^\dagger\right) \\
&- H\left(\xi_1 \xi_0^\dagger \rho_S + \rho_S \xi_0 \xi_1^\dagger - \xi_0^\dagger \rho_S \xi_1 - \xi_1^\dagger \rho_S \xi_0\right) \\
&- iI\left(\xi_1^\dagger \xi_0 \rho_S - \rho_S \xi_1^\dagger \xi_0 + \xi_0 \rho_S \xi_1^\dagger - \xi_1 \rho_S \xi_0^\dagger\right) \\
&- iJ\left(\xi_1^\dagger \rho_S \xi_0 - \xi_0 \xi_1^\dagger \rho_S + \rho_S \xi_1 \xi_0^\dagger - \xi_0^\dagger \rho_S \xi_1\right) \\
&- iK\left(\rho_S \xi_0^\dagger \xi_1 - \xi_1 \rho_S \xi_0^\dagger + \xi_0 \rho_S \xi_1^\dagger - \xi_1^\dagger \xi_0 \rho_S\right) \\
&- iL\left(\xi_1 \xi_0^\dagger \rho_S - \xi_0^\dagger \rho_S \xi_1 + \xi_1^\dagger \rho_S \xi_0 - \rho_S \xi_0 \xi_1^\dagger\right).
\end{aligned} \tag{3.67}$$

We note that in the fifth term of eqn. (3.65) and eqn. (3.67), we have identified the first term in the commutator with the Lamb-Shift Hamiltonian, H_{LS}, by comparison with standard quantum master equations in the theory of open quantum systems [27].

Solving for A,

$$\begin{aligned}
A &= 2\int_0^\infty ds Re\alpha_1(s) \\
&= \int_0^\infty ds(\alpha_1(s) + \alpha_1(s)^*) \\
&= \int_0^\infty ds \int d\omega_m J(\omega_m) N \sin^2\theta e^{-is(\omega_m-\lambda_0)}(n(\omega_m)+1) \\
&\quad + \int_0^\infty ds \int d\omega_m J(\omega_m) N \sin^2\theta e^{is(\omega_m-\lambda_0)}(n(\omega_m)+1) \\
&= 2\pi N \sin^2\theta J(\lambda_0)(n(\lambda_0)+1),
\end{aligned} \tag{3.68}$$

where we have used the following property of the Dirac delta function [28],

$$\int_{-\infty}^{+\infty} ds e^{is(\omega_m-\lambda_0)} = 2\pi\delta(\omega_m - \lambda_0). \tag{3.69}$$

We will make use of the cotangent hyperbolic function,

$$\coth\left(\frac{x}{2}\right) = \frac{e^{\frac{x}{2}} + e^{-\frac{x}{2}}}{e^{\frac{x}{2}} - e^{-\frac{x}{2}}} = 2\left(\frac{1}{e^x - 1}\right) + 1 = 2n(x) + 1. \tag{3.70}$$

From eqn. (3.70), we obtain two equations,

$$\begin{aligned}
n(\lambda_0) &= \frac{1}{2}\left[\coth\left(\frac{\lambda_0}{2}\right) - 1\right] \\
n(\lambda_0) + 1 &= \frac{1}{2}\left[\coth\left(\frac{\lambda_0}{2}\right) + 1\right].
\end{aligned} \tag{3.71}$$

Substituting eqn. (3.71) into eqn. (3.68), we obtain

$$A = 2\pi N \sin^2\theta J(\lambda_0)(n(\lambda_0)+1) = \pi N \sin^2\theta J(\lambda_0)\left[\coth\left(\frac{\lambda_0}{2}\right) + 1\right]. \tag{3.72}$$

Repeating the same procedure for B, C and D, we get

$$B = 2\pi N \sin^2\theta J(\lambda_0) n(\lambda_0) = \pi N \sin^2\theta J(\lambda_0)\left[\coth\left(\frac{\lambda_0}{2}\right) - 1\right],$$
$$C = 2\pi N \cos^2\theta J(\lambda_1)(n(\lambda_1)+1) = \pi N \cos^2\theta J(\lambda_1)\left[\coth\left(\frac{\lambda_1}{2}\right) + 1\right],$$
$$D = 2\pi N \cos^2\theta J(\lambda_1) n(\lambda_1) = \pi N \cos^2\theta J(\lambda_1)\left[\coth\left(\frac{\lambda_1}{2}\right) - 1\right]. \tag{3.73}$$

To get E, we carry out the following integration,

$$\begin{aligned}
E &= \int_0^\infty ds Re\alpha_2(t,s) \\
&= \frac{1}{2}\int_0^\infty ds(\alpha_2(t,s) + \alpha_2(t,s)^*) \qquad (3.74)\\
&= \frac{1}{2}\int_0^\infty ds \int d\omega_m J(\omega_m) N \sin\theta\cos\theta e^{it(\lambda_0-\lambda_1)} e^{-is(\omega_m-\lambda_1)}(n(\omega_m)+1) \\
&\quad + \frac{1}{2}\int_0^\infty ds \int d\omega_m J(\omega_m) N \sin\theta\cos\theta e^{-it(\lambda_0-\lambda_1)} e^{is(\omega_m-\lambda_1)}(n(\omega_m)+1) \\
&= \frac{\pi N \sin\theta\cos\theta J(\lambda_1)}{2}(n(\lambda_1)+1)\left(e^{it(\lambda_0-\lambda_1)} + e^{-it(\lambda_0-\lambda_1)}\right) \\
&\quad + \frac{iPN\sin\theta\cos\theta}{2}\int d\omega_m \frac{J(\omega_m)(n(\omega_m)+1)}{\omega_m - \lambda_1}\left(e^{-it(\lambda_0-\lambda_1)} - e^{it(\lambda_0-\lambda_1)}\right),
\end{aligned}$$

where we have used the following property of the Dirac delta function [28, 31],

$$\int_0^\infty dk e^{\pm ikx} = \pi\delta(x) \pm iP\frac{1}{x}, \tag{3.75}$$

with P being the principal value.

For $|\lambda_0 - \lambda_1| \gg 1$ which holds for $N \gg 1$, E vanishes and similarly F, G, H, I, J, K and L. The Lamb-Shift Hamiltonian, H_{LS}, will only shift the energy levels of the quasi-fermions by a value much smaller than any value in the system Hamiltonian, H_S, and does not affect dissipative dynamics [27]. Since we are only interested in the dissipative dynamics, we will ignore the term with the Lamb-Shift Hamiltonian, so that eqn. (3.67) now reads

$$\begin{aligned}\dot{\rho}_S = \quad & A\left(\xi_0\rho_S\xi_0^\dagger - \frac{1}{2}\{\xi_0^\dagger\xi_0, \rho_S\}_+\right) + B\left(\xi_0^\dagger\rho_S\xi_0 - \frac{1}{2}\{\xi_0\xi_0^\dagger, \rho_S\}_+\right) \\ & + C\left(\xi_1\rho_S\xi_1^\dagger - \frac{1}{2}\{\xi_1^\dagger\xi_1, \rho_S\}_+\right) + D\left(\xi_1^\dagger\rho_S\xi_1 - \frac{1}{2}\{\xi_1\xi_1^\dagger, \rho_S\}_+\right). \end{aligned} \tag{3.76}$$

Substituting expressions for A, B, C, and D, eqn. (3.72) and eqns. (3.73) into eqn. (3.76), we obtain the following quantum master equation,

$$\dot{\rho}_S = \sum_{i=0}^{1} \gamma_i^+ \left(\xi_i\rho_S\xi_i^\dagger - \frac{1}{2}\{\xi_i^\dagger\xi_i, \rho_S\}_+\right) + \gamma_i^- \left(\xi_i^\dagger\rho_S\xi_i - \frac{1}{2}\{\xi_i\xi_i^\dagger, \rho_S\}_+\right), \tag{3.77}$$

where the damping rates $\gamma_i^\pm$ are given by,

$$\gamma_0^\pm = \pi N \sin^2\theta J(\lambda_0)\left(\coth\frac{\beta\lambda_0}{2} \pm 1\right), \tag{3.78}$$

$$\gamma_1^\pm = \pi N \cos^2\theta J(\lambda_1)\left(\coth\frac{\beta\lambda_1}{2} \pm 1\right), \tag{3.79}$$

where $J(\lambda_i)$ is the spectral density and β is the inverse temperature of the bosonic Markovian bath.

We conclude this chapter by noting that eqn. (3.77) is the quantum master equation for our reduced system with the assumption that the number of fermions in the fermionic bath is much larger than one. We note that this master equation is in the quasi-fermionic picture with the summation running over two quasi-fermions corresponding to ξ_0 and ξ_1. The objective of this chapter of deriving the quantum master equation for our reduced system has been accomplished. In the next chapter, we move on to solve this equation.

Chapter 4

Solution of the Quantum Master Equation

In the previous chapter, we obtained the Markovian quantum master equation for the reduced system, eqn. (3.77), in quasi-fermionic picture with the summation running over two quasi-fermions corresponding to ξ_0 and ξ_1. In this chapter, we present the solution of this quantum master equation. From the outset, it is emphasized that the approach we are going to take to obtain the solution is based on a feature that is particular to our quantum master equation. This feature is that we can rearrange the quantum master equation into two terms that commute with each other. These two terms correspond to the two quasi-fermions, ξ_0 and ξ_1.

4.1 Solution

The quantum master equation, eqn. (3.77), can be written as

$$\begin{aligned}\dot{\rho}_S &= \sum_{i=0}^{1} \gamma_i^+ \left(\xi_i \rho_S \xi_i^\dagger - \frac{1}{2}\{\xi_i^\dagger \xi_i, \rho_S\}_+ \right) + \gamma_i^- \left(\xi_i^\dagger \rho_S \xi_i - \frac{1}{2}\{\xi_i \xi_i^\dagger, \rho_S\}_+ \right) \\ &= \sum_{i=0}^{1} \alpha_i(\rho_S), \end{aligned} \tag{4.1}$$

where

$$\alpha_i(\rho_S) = \gamma_i^+ \left(\xi_i \rho_S \xi_i^\dagger - \frac{1}{2}\{\xi_i^\dagger \xi_i, \rho_S\}_+ \right) + \gamma_i^- \left(\xi_i^\dagger \rho_S \xi_i - \frac{1}{2}\{\xi_i \xi_i^\dagger, \rho_S\}_+ \right) \tag{4.2}$$

is a superoperator α_i acting on ρ_S. Substituting explicit expressions for α_0 and α_1 into the commutator gives

$$[\alpha_0, \alpha_1]\rho_S = 0, \tag{4.3}$$

which means that α_0 and α_1 commute. α_0 and α_1 are independent, indeed they act on different Hilbert spaces and as a result we can deduce from eqn. (4.1) the single quasi-fermion master equation

$$\dot{\rho}_{Si}(t) = \alpha_i \rho_{Si}(t), \tag{4.4}$$

where i is either 0 or 1 corresponding to quasi-fermion ξ_0 or ξ_1, respectively.

4.2 Solution in Kraus Representation

In this section, we are going to present the solution of eqn. (4.1) in the Kraus representation [11, 34] which allows us to express the single quasi-fermion solution as

$$\rho_{Si}(t) = \sum_{k=0}^{M} E_{ki}(t)\rho_{Si}(0)E_{ki}^\dagger(t), \tag{4.5}$$

where $E_{ki}(t)$ are the corresponding Kraus operators subject to the normalization condition

$$\sum_{k=0}^{M} E_{ki}^\dagger(t)E_{ki}(t) = I, \tag{4.6}$$

and the total solution as

$$\rho_S(t) = \sum_{j,k=0}^{M} E_{j0}(t)E_{k1}(t)\rho_S(0)E_{k1}^\dagger(t)E_{j0}^\dagger(t) \tag{4.7}$$

subject to the normalization condition,

$$\sum_{j,k=0}^{M} E_{k1}^{\dagger}(t)E_{j0}^{\dagger}(t)E_{j0}(t)E_{k1}(t) = I, \tag{4.8}$$

which should hold if the first normalization condition, eqn. (4.6), holds, i.e.

$$\begin{aligned}\sum_{j,k=0}^{M} E_{k1}^{\dagger}(t)E_{j0}^{\dagger}(t)E_{j0}(t)E_{k1}(t) &= \sum_{k=0}^{M} E_{k1}^{\dagger}(t)\left(\sum_{j=0}^{M} E_{j0}^{\dagger}(t)E_{j0}(t)\right)E_{k1}(t) \\ &= \sum_{k=0}^{M} E_{k1}^{\dagger}(t)\,(I)\,E_{k1}(t) \\ &= \sum_{k=0}^{M} E_{k1}^{\dagger}(t)E_{k1}(t) \\ &= I. \end{aligned} \tag{4.9}$$

The implication of this is that we are going to obtain an explicit expression for the total solution, eqn. (4.7), by obtaining explicit expressions for the Kraus operators corresponding to a single quasi-fermion solution, eqn. (4.5).

Now recall that we have been working in the interaction picture. We will need the solution to be in the Schrödinger picture. Switching from the interaction picture to the Schrödinger picture using eqn. (1.10), the single quasi-fermion solution, eqn. (4.5), becomes,

$$\begin{aligned}\rho_{Si}^{0}(t) &= e^{-iH_S t}\rho_{Si}(t)e^{iH_S t} \\ &= \sum_{k=0}^{M} E_{ki}^{0}(t)\rho_{Si}^{0}(0)E_{ki}^{0\dagger}(t),\end{aligned} \tag{4.10}$$

and the total solution, eqn. (4.7), becomes,

$$\begin{aligned}\rho_{S}^{0}(t) &= e^{-iH_S t}\rho_{S}(t)e^{iH_S t} \\ &= \sum_{j,k=0}^{M} E_{j0}^{0}(t)E_{k1}^{0}(t)\rho_{S}^{0}(0)E_{k1}^{0\dagger}(t)E_{j0}^{0\dagger}(t),\end{aligned} \tag{4.11}$$

where the superscript 0 corresponds to the Schrödinger picture.

4.3 Kraus operators

This section is aimed at obtaining the explicit expressions for the Kraus operators in the Schrödinger picture. We do have the freedom of choice for the Kraus operators. However, the normalization condition,

$$\sum_{k=0}^{M} E_{ki}^{0\dagger}(t)E_{ki}^{0}(t) = I, \tag{4.12}$$

has to be satisfied. Because the dynamics of a fermion can be mapped to a two-level system, we will use as parameterization of our Kraus operators the Kraus representation of amplitude channel for the two-level system [11], i.e.

$$\begin{aligned} E_{0i}^{0}(t) &= \omega\left(\alpha\xi_i^\dagger\xi_i + \beta\xi_i\xi_i^\dagger\right), \\ E_{1i}^{0}(t) &= \omega\delta\xi_i^\dagger, \\ E_{2i}^{0}(t) &= \Omega\left(\beta\xi_i^\dagger\xi_i + \alpha\xi_i\xi_i^\dagger\right), \\ E_{3i}^{0}(t) &= \Omega\delta\xi_i, \end{aligned} \tag{4.13}$$

where $\alpha, \beta, \delta, \omega$ and Ω are to be determined. Checking our choice of the Kraus operators for the normalization condition,

$$\begin{aligned} \sum_{k=0}^{3} E_{ki}^{0\dagger}(t)E_{ki}^{0}(t) &= E_{0i}^{0\dagger}(t)E_{0i}^{0}(t) + E_{1i}^{0\dagger}(t)E_{1i}^{0}(t) + E_{2i}^{0\dagger}(t)E_{2i}^{0}(t) + E_{3i}^{0\dagger}(t)E_{3i}^{0}(t) \\ &= \omega^2\left(\alpha\xi_i^\dagger\xi_i + \beta\xi_i\xi_i^\dagger\right)^2 + \omega^2\delta^2\xi_i\xi_i^\dagger + \Omega^2\left(\beta\xi_i^\dagger\xi_i + \alpha\xi_i\xi_i^\dagger\right)^2 \\ &\quad +\Omega^2\delta^2\xi_i^\dagger\xi_i \\ &= 1, \end{aligned} \tag{4.14}$$

we observe that it is satisfied. So, the four chosen Kraus operators fix M in eqn. (4.10) which can now be written as

$$\rho^0_{Si}(t) = \sum_{k=0}^{3} E^0_{ki}(t)\rho^0_{Si}(0)E^{0\dagger}_{ki}(t). \tag{4.15}$$

All we need to do now is to determine the coefficients $\alpha, \beta, \delta, \omega$ and Ω. To do that we are going to use the quantum master equation for a single quasi-fermion in its original form, eqn. (3.77),

$$\dot{\rho}_{Si} = \gamma_i^+ \left(\xi_i \rho_S \xi_i^\dagger - \frac{1}{2}\{\xi_i^\dagger \xi_i, \rho_S\}_+ \right) + \gamma_i^- \left(\xi_i^\dagger \rho_S \xi_i - \frac{1}{2}\{\xi_i \xi_i^\dagger, \rho_S\}_+ \right). \tag{4.16}$$

We expect the solution of this master equation to be of the form [27],

$$\begin{aligned} \rho_{Si}(t) &= \sum_{j,k=0}^{1} c_{jk}(t)(\xi_i^\dagger)^j |0\rangle\langle 0| \xi_i^k \\ &= c_{00}(t)|0\rangle\langle 0| + c_{01}(t)|0\rangle\langle 1| + c_{10}(t)|1\rangle\langle 0| + c_{11}(t)|1\rangle\langle 1|, \end{aligned} \tag{4.17}$$

where the coefficients $c_{jk}(t)$ are to be determined. Substituting eqn. (4.17) on both sides of eqn. (4.16) and equating coefficients, we get

$$\begin{aligned} \dot{c}_{00}(t) &= \gamma_i^+ c_{11}(t) - \gamma_i^- c_{00}(t), \\ \dot{c}_{01}(t) &= -\frac{\gamma_i^\beta}{2} c_{01}(t), \\ \dot{c}_{10}(t) &= -\frac{\gamma_i^\beta}{2} c_{10}(t), \\ \dot{c}_{11}(t) &= -\gamma_i^+ c_{11}(t) + \gamma_i^- c_{00}(t), \end{aligned} \tag{4.18}$$

which upon solving yields,

$$
\begin{aligned}
c_{00}(t) &= \frac{\gamma_i^+ + \gamma_i^- e^{-t\gamma_i^\beta}}{\gamma_i^\beta} c_{00}(0) + \frac{\gamma_i^+}{\gamma_i^\beta}\left(1 - e^{-t\gamma_i^\beta}\right) c_{11}(0), \\
c_{01}(t) &= c_{01}(0) e^{-t\frac{\gamma_i^\beta}{2}}, \\
c_{10}(t) &= c_{10}(0) e^{-t\frac{\gamma_i^\beta}{2}}, \\
c_{11}(t) &= \frac{\gamma_i^-}{\gamma_i^\beta}\left(1 - e^{-t\gamma_i^\beta}\right) c_{00}(0) + \frac{\gamma_i^- + \gamma_i^+ e^{-t\gamma_i^\beta}}{\gamma_i^\beta} c_{11}(0), \qquad (4.19)
\end{aligned}
$$

where $\gamma_i^\beta = \gamma_i^+ + \gamma_i^-$. Switching from the interaction picture to the Schrödinger picture using eqn. (2.59), eqn. (4.17) and eqn. (4.19) we have

$$
\begin{aligned}
\rho_{Si}^0(t) &= e^{-iH_S t} \rho_{Si}(t) e^{iH_S t} \\
&= c_{00}^0(t)|0\rangle\langle 0| + c_{01}^0(t)|0\rangle\langle 1| + c_{10}^0(t)|1\rangle\langle 0| + c_{11}^0(t)|1\rangle\langle 1|, \qquad (4.20)
\end{aligned}
$$

where the coefficients are

$$
\begin{aligned}
c_{00}^0(t) &= \frac{\gamma_i^+ + \gamma_i^- e^{-t\gamma_i^\beta}}{\gamma_i^\beta} c_{00}(0) + \frac{\gamma_i^+}{\gamma_i^\beta}\left(1 - e^{-t\gamma_i^\beta}\right) c_{11}(0), \\
c_{01}^0(t) &= c_{01}(0) e^{-t\frac{\gamma_i^\beta}{2} + i\lambda_i t}, \\
c_{10}^0(t) &= c_{10}(0) e^{-t\frac{\gamma_i^\beta}{2} - i\lambda_i t}, \\
c_{11}^0(t) &= \frac{\gamma_i^-}{\gamma_i^\beta}\left(1 - e^{-t\gamma_i^\beta}\right) c_{00}(0) + \frac{\gamma_i^- + \gamma_i^+ e^{-t\gamma_i^\beta}}{\gamma_i^\beta} c_{11}(0). \qquad (4.21)
\end{aligned}
$$

Let us now use the Kraus operators, eqn. (4.13), in the single fermion solution, eqn. (4.15). Substituting eqn. (4.20) on both sides of eqn. (4.15) and equating the coefficients, we obtain

$$\begin{aligned}
c_{00}^{0}(t) &= \left(\omega^2\beta^2 + \Omega^2\alpha^2\right) c_{00}^{0}(0) + \Omega^2\delta^2 c_{11}^{0}(0), \\
c_{01}^{0}(t) &= \left(\omega^2 + \Omega^2\right) \alpha\beta c_{01}^{0}(0), \\
c_{10}^{0}(t) &= \left(\omega^2 + \Omega^2\right) \alpha\beta c_{10}^{0}(0), \\
c_{11}^{0}(t) &= \omega^2\delta^2 c_{00}^{0}(0) + \left(\omega^2\alpha^2 + \Omega^2\beta^2\right) c_{11}^{0}(0).
\end{aligned} \tag{4.22}$$

A comparison of eqn. (4.22) with eqn. (4.21) reveals that

$$\begin{array}{cccccc}
\Omega^2 & = & \frac{\gamma_i^+}{\gamma_i^\beta}, & \omega^2 & = & \frac{\gamma_i^-}{\gamma_i^\beta}, \\
\alpha^2 & = & 1, & \alpha & = & 1, \\
\beta & = & e^{-\frac{t}{2}\gamma_i^\beta - i\lambda_i t}, & \delta & = & \sqrt{1 - e^{-t\gamma_i^\beta}}.
\end{array} \tag{4.23}$$

When we substitute eqn. (4.23) into eqn. (4.13), we obtain

$$\begin{aligned}
E_{0i}^{0}(t) &= \cos\alpha_i \left(\xi_i^\dagger \xi_i + f_i(t)\xi_i\xi_i^\dagger\right), \\
E_{1i}^{0}(t) &= \cos\alpha_i g_i(t)\xi_i^\dagger, \\
E_{2i}^{0}(t) &= \sin\alpha_i \left(\xi_i\xi_i^\dagger + f_i^*(t)\xi_i^\dagger\xi_i\right), \\
E_{3i}^{0}(t) &= \sin\alpha_i g_i(t)\xi_i,
\end{aligned} \tag{4.24}$$

as the Kraus operators, where

$$\cos\alpha_i = \sqrt{\frac{\gamma_i^-}{\gamma_i^\beta}} = \sqrt{\frac{1}{1+e^{\beta\lambda_i}}} = \sqrt{p_i}, \tag{4.25}$$

$$\sin\alpha_i = \sqrt{\frac{\gamma_i^+}{\gamma_i^\beta}} = \sqrt{1 - \frac{1}{1+e^{\beta\lambda_i}}} = \sqrt{1-p_i}, \tag{4.26}$$

$$f_i(t) = \exp\left(-\frac{\gamma_i^\beta}{2}t - i\lambda_i t\right) \quad \text{and} \quad g_i(t) = \sqrt{1-|f_i(t)|^2}. \tag{4.27}$$

In the remainder of this thesis, we will be working in the Schrödinger picture. Therefore, for notational simplicity, we are going to suppress the superscript 0 corresponding to the Schrödinger picture so that the total solution of the quantum master equation, eqn. (4.11), will appear as

$$\rho_S(t) = \sum_{j,k=0}^{3} E_{j0}(t)E_{k1}(t)\rho_S(0)E_{k1}^\dagger(t)E_{j0}^\dagger(t), \tag{4.28}$$

where the Kraus operators are given as

$$\begin{aligned} E_{0i}(t) &= \cos\alpha_i \left(\xi_i^\dagger\xi_i + f_i(t)\xi_i\xi_i^\dagger\right), \\ E_{1i}(t) &= \cos\alpha_i g_i(t)\xi_i^\dagger, \\ E_{2i}(t) &= \sin\alpha_i \left(\xi_i\xi_i^\dagger + f_i^*(t)\xi_i^\dagger\xi_i\right), \\ E_{3i}(t) &= \sin\alpha_i g_i(t)\xi_i. \end{aligned} \tag{4.29}$$

To conclude this chapter, we state that we have solved the quantum master equation for the reduced sytem which we derived in Chapter 3. We switched from the interaction picture to the Schrödinger picture expressing the solution in the Kraus representation. We showed that the Kraus operators obey the normalization condition. This implies that the trace [11] of the solution is 1; which must be the case for any correct probability preserving density matrix [11, 27]. Setting $t = 0$ in the solution reduces it to $\rho_S(0)$ as expected. We can therefore be confident that eqn. (4.28) is the correct solution.

Chapter 5

Results and Discussion

In the previous chapter, we solved the derived quantum master equation for the reduced system consisting of the fermion of interest, the fermionic bath and the interaction between them. However, the subject of this thesis is the fermion of interest. Therefore, the first section of this chapter is devoted to a calculation of the density matrix of the fermion of interest from the solution, eqn. (4.28), which we obtained in the previous chapter. We will then use this calculated density matrix of the fermion of interest to calculate the mean number of fermions. This will be followed by a plot of the mean number of fermions against time to observe the dynamics of the fermion of interest. We will finally investigate thermalization in Section 5.2.

5.1 Dynamics of the Fermion of Interest

In this section we endeavor to obtain the reduced density matrix of the fermion of interest in the original fermionic picture. We consider that initially we only have a single fermion of interest and no fermions in the mesoscopic bath i.e.,

$$\rho_S(0) = |\psi_d\rangle\langle\psi_d| \otimes |\psi_c\rangle\langle\psi_c| = d^\dagger|0\rangle\langle 0|d, \tag{5.1}$$

where we have used the state vector $|\psi_d\rangle = d^\dagger|0\rangle$ for the electron of interest and $|\psi_c\rangle = |0\rangle$ for the mesoscopic bath.

Substituting eqn. (5.1) into eqn. (4.28), we have

$$\rho_S(t) = \sum_{j,k=0}^{3} E_{j0}(t)E_{k1}(t)d^\dagger|0\rangle\langle 0|dE^\dagger_{k1}(t)E^\dagger_{j0}(t). \tag{5.2}$$

Next, we transform from the quasi-fermionic picture to the original fermionic picture by substituting into the Kraus operators, eqn. (4.29), the transformation equations

$$\begin{aligned} \xi_0 &= \cos\theta d + \frac{\sin\theta}{\sqrt{N}}\sum_{i=1}^{N} c_i, \\ \xi_1 &= -\sin\theta d + \frac{\cos\theta}{\sqrt{N}}\sum_{i=1}^{N} c_i, \end{aligned} \tag{5.3}$$

which we obtained earlier on in Chapter 2, eqn. (2.61). We then trace out the degrees of freedom of the mesoscopic bath, c's, from the reduced density matrix, eqn. (5.2). Thus,

$$\begin{aligned} \rho_e(t) &= \mathrm{Tr}_c[\rho_S(t)] \\ &= \sum_{i_1,i_2,\cdots,i_N=0}^{1} \langle 0|(c_N)^{i_N}\cdots(c_2)^{i_2}(c_1)^{i_1}\rho_S(t)(c_1^\dagger)^{i_1}(c_2^\dagger)^{i_2}\cdots(c_N^\dagger)^{i_N}|0\rangle \\ &= \kappa(t)d^\dagger|0\rangle\langle 0|d + (1-\kappa(t))|0\rangle\langle 0|, \end{aligned} \tag{5.4}$$

where

$$\begin{aligned} \kappa(t) &= p_0\cos^2\theta + p_1\sin^2\theta + w(t) \\ &\quad +|f_0(t)|^2\cos^2\theta\left(\cos^2\theta\,(1-p_0p_1) - p_0\right) \\ &\quad +|f_1(t)|^2\sin^2\theta\left(\sin^2\theta\,(1-p_0p_1) - p_1\right), \end{aligned} \tag{5.5}$$

and

$$w(t) = 2Re(f_0(t)f_1^*(t))\sin^2\theta\cos^2\theta. \tag{5.6}$$

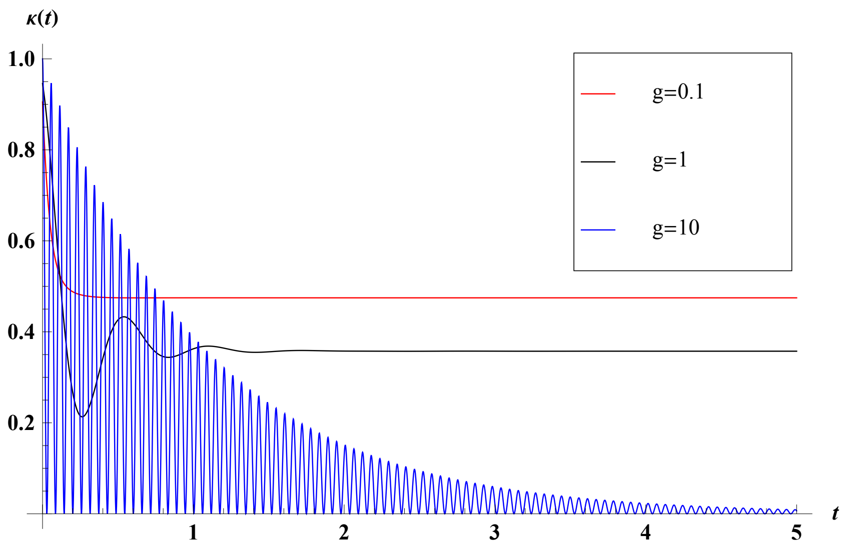

FIGURE 5.1: Time dependence of the mean number of the fermion of interest, $\kappa(t)$, as a function of time, t, for different values of the coupling strength, g, to the mesoscopic bath. The rest of the parameters are chosen to be the same for all three curves: $\epsilon = 1$, $\omega = 1.3$, $N = 30$, $J(\lambda_0)$=$J(\lambda_1)$=0.01 and $\beta = 0.1$.

Calculating the trace of $\rho_e(t)$, we obtain

$$\mathrm{Tr}[\rho_e(t)] = \kappa(t)\langle 0|dd^\dagger|0\rangle + (1-\kappa(t))\langle 0|0\rangle = 1, \tag{5.7}$$

which confirms that $\rho_e(t)$ preserves probability as expected for a correct density matrix [27].

To investigate the dynamics of the fermion of interest, let us calculate the mean number of the fermion of interest. Using eqn. (5.4), the mean number of the fermion of interest is obtained to be

$$\begin{aligned}\langle d^\dagger d\rangle &= \mathrm{Tr}[d^\dagger d\rho_e(t)] \\ &= \kappa(t)\mathrm{Tr}[d^\dagger d\left(d^\dagger|0\rangle\langle 0|d\right)] + (1-\kappa(t))\mathrm{Tr}[d^\dagger d\left(|0\rangle\langle 0|\right)] \\ &= \kappa(t). \end{aligned}\tag{5.8}$$

Fig. 5.1 and Fig. 5.2 show our investigation of the behaviour of the mean number of the fermion of interest, $\kappa(t)$, with time, t, for different values of the number of fermions, N, in the mesoscopic bath and for different values of the coupling strength, g, to the mesoscopic bath.

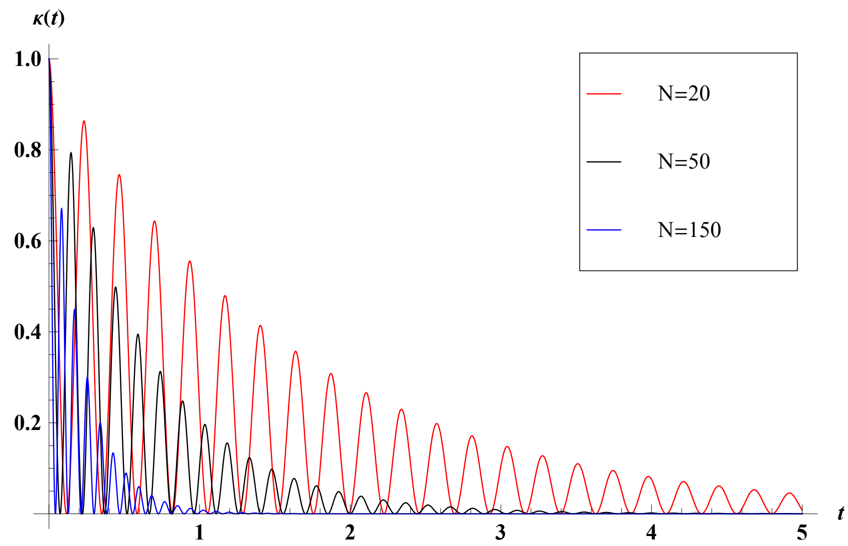

FIGURE 5.2: Time dependence of the mean number of the fermion of interest, $\kappa(t)$, as a function of time, t, for different values of the number of fermions, N, in the mesoscopic bath. The rest of the parameters are chosen to be the same for all three curves: $\epsilon = 1$, $\omega = 1.3$, $g = 3$, $J(\lambda_0)$=$J(\lambda_1)$=0.01 and $\beta = 1$.

In Fig. 5.1 and Fig. 5.2 the dynamics of the mean number of the fermion of interest decay exponentially with time and approaches equilibrium as time goes to infinity. This indicates that thermalization is achieved as time goes to infinity in agreement with literature [35]. In Fig. 5.1 we analyze different regimes of the interaction between the fermion of interest and meso-reservoir of fermions. It is clear that in the weak coupling case (Fig. 5.1: red curve) Markovian dissipation [27] is observed. However, increasing the interaction strength (g), the process of thermalization shows strong oscillations, a sign of strong interactions which is typical of non-Markovian behaviour [25] (Fig. 5.1: black and blue curves). We note that the original master equation, eqn. (1.13), for the total system was derived by assuming Markovian dynamics, but we see emergence of this non-Markovian behaviour for the reduced system consisting of the fermion of interest and the mesocopic bath. In Fig. 5.2 we analyze the influence of the number of fermions in the mesoscopic bath on the dynamics of the fermion. It is clear from Fig. 5.2 that increasing the number of fermions strongly influences the frequency of oscillations: as the number of fermions in the fermionic bath increases, interactions increase which in turn increases the frequency of oscillations.

5.2 Thermalization of the Fermion of Interest

Both Fig. 5.1 and Fig. 5.2 show signs of thermalization as time approaches infinity. In this section, we are going to show that thermalization is indeed achieved as time approaches infinity.

Using the well known thermal equilibrium state [11, 27],

$$\rho_{TS} = \frac{e^{-\beta H_S}}{Z}, \tag{5.9}$$

where $Z = \mathrm{Tr}[e^{-\beta H_S}]$ is the normalizing partition function, the mean number of the fermion of interest in thermal equilibrium is

$$\kappa_{TS} = \langle d^\dagger d \rangle = \mathrm{Tr}[d^\dagger d \rho_{TS}] = \frac{\mathrm{Tr}[d^\dagger d e^{-\beta H_S}]}{\mathrm{Tr}[e^{-\beta H_S}]}. \tag{5.10}$$

Substituting the system Hamiltonian, eqn. (2.59), into eqn. (5.10), we obtain

$$\begin{aligned} \kappa_{TS} &= \frac{\mathrm{Tr}\big[d^\dagger d e^{-\beta(\lambda_0 \xi_0^\dagger \xi_0 + \lambda_1 \xi_1^\dagger \xi_1)}\big]\mathrm{Tr}\big[e^{-\beta \sum_{i=2}^N \lambda_i \xi_i^\dagger \xi_i}\big]}{\mathrm{Tr}\big[e^{-\beta(\lambda_0 \xi_0^\dagger \xi_0 + \lambda_1 \xi_1^\dagger \xi_1)}\big]\mathrm{Tr}\big[e^{-\beta \sum_{i=2}^N \lambda_i \xi_i^\dagger \xi_i}\big]} \\ &= \frac{\mathrm{Tr}\big[d^\dagger d e^{-\beta(\lambda_0 \xi_0^\dagger \xi_0 + \lambda_1 \xi_1^\dagger \xi_1)}\big]}{\mathrm{Tr}\big[e^{-\beta(\lambda_0 \xi_0^\dagger \xi_0 + \lambda_1 \xi_1^\dagger \xi_1)}\big]}, \end{aligned} \tag{5.11}$$

so that upon using

$$d = \cos\theta \xi_0 - \sin\theta \xi_1$$

from eqn. (3.8) and simplifying, we get

$$\kappa_{TS} = \frac{e^{-\beta\lambda_0}\cos^2\theta + e^{-\beta(\lambda_0+\lambda_1)} + e^{-\beta\lambda_1}\sin^2\theta}{(1+e^{-\beta\lambda_0})(1+e^{-\beta\lambda_1})}. \tag{5.12}$$

Multiplying both the numerator and denominator of eqn. (5.12) by $e^{\beta(\lambda_0+\lambda_1)}$, we get

$$\kappa_{TS} = \frac{e^{\beta\lambda_1}\cos^2\theta + 1 + e^{\beta\lambda_0}\sin^2\theta}{(1+e^{\beta\lambda_0})(1+e^{\beta\lambda_1})}. \tag{5.13}$$

When we replace 1 on the numerator in eqn. (5.13) by $\cos^2\theta + \sin^2\theta$ and simplify, we obtain

$$\kappa_{TS} = p_0 \cos^2\theta + p_1 \sin^2\theta, \tag{5.14}$$

where p_0 and p_1 are given by eqn. (4.25). Eqn. (5.14) is the expected mean number of the fermion of interest in thermal equilibrium.

Let us now see if thermalization is achieved as time approaches infinity from the reduced dynamics which we have derived in this present work. We set $t = \infty$ in eqn. (5.8) and eqn. (5.5). This gives us,

$$\kappa(\infty) = p_0 \cos^2\theta + p_1 \sin^2\theta, \tag{5.15}$$

which is exactly the same as the actual thermal equilibrium mean number, eqn. (5.14). This indicates that thermalization is indeed achieved as time approaches infinity. The equality of the derived thermal equilibrium mean number and the actual thermal equilibrium mean number also gives us confidence in the detailed derivations conducted in this thesis.

Chapter 6

Conclusion

In conclusion, we derived and solved analytically the quantum master equation for the spinless electron interacting with a mesosopic bath of spinless electrons with restrictions on the system-bath interaction. We considered the fermion of interest to be strongly coupled to the surrounding mesosopic bath of electrons which is weakly coupled to the Markovian bosonic bath. The coupling strength between the electron of interest and the rest of the electrons in the fermionic bath was taken to be the same. By tracing out the degrees of freedom of the mesosopic bath of fermions from the solution of the quantum master equation, we calculated the density matrix of the fermion of interest. This density matrix of the fermion of interest was then used to calculate the mean number of the fermion of interest which we used in our analysis of the dynamics and thermalization of the fermion of interest.

We plotted graphs of the mean number of the fermion of interest against time for different values of the number of fermions in the mesoscopic bath and for different values of the coupling strength to the mesoscopic bath.

In the weak coupling case Markovian dissipation was observed and in the strong coupling regime non-Markovian behaviour was observed [25, 27]. Increasing the number of fermions strongly influenced the frequency of oscillations: as the number of fermions in the fermionic bath increased, interactions increased which in turn increased the frequency of oscillations.

We observed that as time approached infinity the fermion of interest was thermalized by the Markovian bosonic bath through its weak interaction with the fermionic bath as expected from literature [11, 27, 35]. To verify thermalization, we conducted two calculations. Firstly, we calculated the mean number of the fermion of interest in thermal equilibrium state from the derived density matrix of the fermion of interest. Secondly, we calculated the mean number of the fermion of interest in thermal equilibrium state from the well known [11, 27] expression for the density matrix of the thermal equilibrium state. A comparison of these two results revealed an exact match confirming that thermalization was indeed achieved as time approached infinity.

In the future, we plan to take into account spin-spin interactions and consider more general initial conditions and more general fermion-fermion interactions.

Bibliography

[1] C. Kittel and P. McEuen. *Introduction to Solid State Physics*, volume 7. Wiley New York, 1996.

[2] T. Jamieson, R. Bakhshi, D. Petrova, R. Pocock, M. Imani, and A. M. Seifalian. Biological Applications of Quantum Dots. *Biomaterials*, 28(31):4717–4732, 2007.

[3] L. Sha, Z. Kai, Y. Jui-Ming, L. Liwei, and Y. Haw. Single Quantum Dots as Temperature Markers. *Nano Letters*, 7(10):3102–3105, 2007.

[4] R. Bakalova, H. Ohba, Z. Zhelev, M. Ishikawa, and Y. Baba. Quantum Dots as Photosensitizers? *Nature Biotechnology*, 22(11):1360–1361, 2004.

[5] H. Walther, B. T. H. Varcoe, B. Englert, and T. Becker. Cavity Quantum Electrodynamics. *Reports on Progress in Physics*, 69(5):1325, 2006.

[6] T. Niemczyk, F. Deppe, H. Huebl, E. P. Menzel, F. Hocke, M. J. Schwarz, J. J. García-Ripoll, D. Zueco, T. Hümmer, E. Solano, et al. Circuit Quantum Electrodynamics in the Ultrastrong-Coupling Regime. *Nature Physics*, 6(10):772–776, 2010.

[7] A. Blais, R. Huang, A. Wallraff, S. M. Girvin, and R. J. Schoelkopf. Cavity Quantum Electrodynamics for Superconducting Electrical Circuits: An Architecture for Quantum Computation. *Physical Review A*, 69(6):062320, 2004.

[8] A. Steane. The Ion Trap Quantum Information Processor. *Applied Physics B: Lasers and Optics*, 64(6):623–643, 1997.

[9] D. Loss and D. P. DiVincenzo. Quantum Computation with Quantum Dots. *Physical Review A*, 57(1):120, 1998.

[10] J. Gemmer and G. Mahler. Entanglement and the Factorization-Approximation. *The European Physical Journal D-Atomic, Molecular, Optical and Plasma Physics*, 17(3):385–393, 2001.

[11] M. A. Nielsen and I. L. Chuang. *Quantum Computation and Quantum Information*. Cambridge University Press, 2010.

[12] D. S. Kosov, T. Prosen, and B. Žunkovič. A Markovian Kinetic Equation Approach to Electron Transport through a Quantum Dot Coupled to Superconducting Leads. *Journal of Physics: Condensed Matter*, 25(7):075702, 2013.

[13] C. H. Bennett and D. P. DiVincenzo. Quantum Information and Computation. *Nature*, 404(6775):247–255, 2000.

[14] Z. H. Wojciech. Decoherence, Einselection, and the Quantum Origins of the Classical. *Review of Modern Physics*, 75:715–775, 2003.

[15] H. Sommer. Emergence of Classical Reality from a Quantum Mechanical Background. *Chaos, Solitons & Fractals*, 39(3):1027–1036, 2009.

[16] H. Tasaki. From Quantum Dynamics to the Canonical Distribution: General Picture and a Rigorous Example. *Physical Review Letters*, 80(7):1373, 1998.

[17] M. Michel, M. Hartmann, J. Gemmer, and G. Mahler. Fourier's Law Confirmed for a Class of Small Quantum Systems. *The European Physical Journal B-Condensed Matter and Complex Systems*, 34(3):325–330, 2003.

[18] J. Gemmer, A. Otte, and G. Mahler. Quantum Approach to a Derivation of the Second Law of Thermodynamics. *Physical Review Letters*, 86(10):1927, 2001.

[19] T. Endo. Quantum Theory of Measurement. *Journal of the Physical Society of Japan*, 57(1):71–74, 1988.

[20] H. Grabert, P. Schramm, and G. Ingold. Quantum Brownian Motion: the Functional Integral Approach. *Physics Reports*, 168(3):115–207, 1988.

[21] H. Grabert, P. Schramm, and G. Ingold. Localization and Anomalous Diffusion of a Damped Quantum Particle. *Physical Review Letters*, 58(13):1285, 1987.

[22] B. Thimmel, O. Terzidis, and P. Nalbach. Rotating Wave Approximation. *The European Physical Journal B-Condensed Matter*, 1998.

[23] S. Ajisaka, F. Barra, C. Mejía-Monasterio, and T. Prosen. Nonequlibrium Particle and Energy Currents in Quantum Chains Connected to Mesoscopic Fermi Reservoirs. *Physical Review B*, 86(12):125111, 2012.

[24] G. M. Moy, J. J. Hope, and C. M. Savage. Born and Markov Approximations for Atom Lasers. *Physical Review A*, 59(1):667, 1999.

[25] P. Stenius and A. Imamoglu. Stochastic Wavefunction Methods Beyond the Born-Markov and Rotating-Wave Approximations. *Quantum and Semiclassical Optics: Journal of the European Optical Society Part B*, 8(1):283, 1996.

[26] M. Nakatani and T. Ogawa. Quantum Master Equations for Composite Systems: Is Born-Markov Approximation Really Valid? *Journal of the Physical Society of Japan*, 79(8), 2010.

[27] H-P. Breuer and F. Petruccione. *The Theory of Open Quantum Systems*. Oxford University Press, 2002.

[28] B. H. Bransden. *Quantum Mechanics, 2/E*. Pearson Education India, 2000.

[29] J. J. Sakurai and S. F. Tuan. *Modern Quantum Mechanics, Revised Edition*. Addison-Wesley Publishing Company, Inc., 1994.

[30] W. Greub. *Linear Algebra*. Springer, 1975.

[31] E. Kreyszig. *Advanced Engineering Mathematics*. John Wiley & Sons,Inc, 2006.

[32] A. Ruhe. Numerical Aspects of Gram-Schmidt Orthogonalization of Vectors. *Linear Algebra and its Applications*, 52:591–601, 1983.

[33] R. S. Strichartz. The Campbell-Baker-Hausdorff-Dynkin Formula and Solutions of Differential Equations. *Journal of Functional Analysis*, 72(2):320–345, 1987.

[34] K. Kraus. *States, Effects and Operations*. Springer Verlag, 1983.

[35] S. Camalet. Thermalisation by a Boson Bath in a Pure State. *The European Physical Journal B*, 61(2):193–199, 2008.