Categorical Systems Biology: An Appreciation of Categorical Arguments in Cellular Modelling.

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This dissertation is submitted to the School of Mathematical Sciences, Faculty of Science and Agriculture, University of KwaZulu-Natal, Durban, in fulfillment of the requirements for the degree of Master in Science.

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As the candidate's supervisor, I have approved this dissertation for submission.

Signed: Professor Jacek Banasiak June 2012

As the candidate's co-supervisor, I have approved this dissertation for submission.

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Abstract

With big science projects like the human genome project, [2], and preliminary attempts to seriously study brain activity, e.g. [9], mathematical biology has come of age, employing formalisms and tools from most branches of mathematics.

Recent results, [51] and [53], have extended the relational (or categorical) approach of Rosen [44], to demonstrate that (in a very general class of systems) cellular self-organization/self-replication is implicit in metabolism and repair/stability. This is a powerful philosophical statement and removes the need of teleological argument. However, the result carries a technical limitation to Cartesian closed categories, which excludes many mathematical languages.

We review the relevant literature on metabolic-repair pathways, category theory and systems theory, before performing a critique of this work. We find that the restriction to Cartesian closed categories is purely for simplicity, and describe how equivalent arguments may be built for monoidal closed categories. Moreover, any symmetric monoidal category may be "embedded" in a closed one. We discuss how these constructions/techniques provide the formal structure to treat self-organization/self-replication in most contemporary mathematical (modelling) languages. These results significantly soften the impact on current modelling paradigms while extending the philosophical implications.

Declaration 1

I declare that the contents of this dissertation are original except where due reference has been made. It has not been submitted before for any degree to any other institution.

Maurine Atieno Songa
June 2012

Declaration 2 - Publication List

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Dedication

For my grandmother Leonora, and my son Kellan, with love.

Contents

Chapter 1

Introduction

Systems biology provides a conceptual basis and working methodologies for the scientific explanation of biological phenomenon, [51]. Such biological phenomenon are modelled as formal systems using mathematical representations which then may be analysed. Of particular interest is the process of modelling in (or translating existing models into) very abstract mathematical languages, e.g. category theory. Mathematical modelling at this level helps biologists to decide on structural elements, (e.g. which variables to consider), and allows for meta-statements or predictions to be made, which may be proven using the machinery of the formal language. Thus, the conceptual framework may be used to explain hitherto unknown relationships, as well as to make predictions about the future behaviour of a system, [50].

Mathematical modelling is of great significance when analysing the organisation and control of genetic pathways, [50]. A pathway is a sub-system whose behaviour is determined by a finite set of variables, [51]. Such molecular processes are complex and are difficult to measure or quantify. Mathematical models can therefore be used to provide accurate numerical predictions. Applications of such analysis arise in the areas of metabolic engineering and control, [25, 50].

It is possible for a system consisting of individual simple components to exhibit an overall complex behaviour. Cellular automata are an example of such a system. Stephen Wolfram, [49], discusses the complex nature of cellular automata and their application to a variety of biological systems. In reference [39], recent work on applying genetic algorithms to the design of cellular automata that perform computations which require global coordination is reviewed. This is an evolutionary process in which the functioning of simple components of the system give rise to coordinated global information processing, [39]. Such computations have been known to occur in biological systems such as insect colonies, the immune system and the brain. Computer programmes are often used to understand the behaviour of systems. However, they do not necessarily capture the structural organization of systems or higher level processes, [9]. Brain activity is an example of a system which deals with structural information. Brown and Porter, [9], find it reasonable that category theory and higher dimensional category theory could be necessary for modelling the kind of behaviour displayed by brain activity.

These observations are paralleled by computer scientists seeking to build analogous relational formal systems, also using category theory. While there are obvious implications for artificial intelligence, more modest goals include: minimizing the amount of debugging required in software designed for sensitive contexts (e.g. medical diagnostic tools) which cannot be tested safely, easily or cheaply; database design; and learning systems (e.g. voice transcription), [42].

From a more philosophical perspective, the notion of causality arises as an explanation of change when studying systems. Causation is the process by which we explain change. Aristotle introduced four types of causal entailment: the material cause; the formal cause; the efficient cause; and the final cause. The material cause refers to that with which something is made; the formal cause is the idea after which something is made; the efficient cause is the mechanism of change in a system; while the final cause is the objective that something is meant to achieve, [51].

Robert Rosen, [43, 44], on whose work our research is largely based, claimed that the defining nature of living systems is that they are closed to efficient causation (i.e. in this context, that self-replication/self-organization is manifested intrinsically). Closure to efficient causation removes the need for a final cause, and so eliminates the need for teleological argument. As we shall see in Chapter 5, many systems, in many languages, may display efficient causation so that this is almost certainly not "the", but rather "a" defining characteristic of life. Alternatively, we may need to embrace a more broad definition of what is meant by "living."

Rosen's definition of life in terms of closure to efficient causation does not stand unchallenged, [43, 44]. For example, Schrödinger, [45], characterized life in terms of open systems maintaining their own low-entropy order by exploiting the energy difference between their low-entropy food and their high entropy by-products. Some times these challenges may be easily reconciled. In the above example, Schrödinger's thermodynamical condition is an example of material causation, and need not be in conflict with the organizational requirement of closure with respect to efficient causation.

However, there does exist much controversy over the possible role closure to efficient causation should play in defining life. Firstly, close analysis of Rosen-style arguments, [12], reveals that they prove only that certain mathematical structures (Cartesian closed categories) are sufficient to yield closure to efficient causation. This should be interpreted in light of Rosen's understanding of simulation (a Turing machine having predictive power) versus modelling (in which the structure of the phenomenon is understood). This idea of closure to efficient causation is an attempt to model life, and he argued that life (so defined) precludes the possibility of simulation. His is controversial because computation is central to the study of living systems, and many modern mathematical languages underlying these computations are not Cartesian closed categories.

Many researchers have addressed these issues, phrasing their arguments in different ways. We now provide a brief summary of this work, describing its relationship to M-R systems and summarising its relevance to the question of computability. For further details, the reader is referred to Cárdenas et al., [12], for a review.

Organizational closure is central to both M-R systems and autopoiesis, [34]. They both describe efficient closure in terms of catalytic closure. Every catalyst must be a product of metabolism via Rosen-style arguments. (These do not preclude openness with respect to material cause). Autopoiesis also includes a physical separation between individuals (e.g. cell wall, skin etc.). On the other hand, a focus on organizational closure is absent in Gánti's [17] chemoton which focuses instead on processes occurring in a particular order, despite this being equivalent to a particular implementation of an autopoietic system, [53].

Kauffman, [21, 22], discusses the topic in terms of autocatalytic sets and the idea of catalytic closure. This broadly agrees with definitions in M-R systems [43, 44, 51] and autopoiesis [34, 53], the principal difference being that autocatalytic sets tend to be very large in order to possess the statistical properties needed for closure to be assured. M-R systems, being model based, would be as simple as possible whilst still encapsulating the key behaviour under study.

Chemero and Turvey, [14, 15], claim that catalytic closure does not mean closure to efficient cause. They base their argument on hypersets (sets in which the restriction that sets cannot be members of themselves is relaxed), and show how simple graphs of hypersets allow one to recognise if a system embodies circular definitions. However, there are several controversies.

Mossio et al., [40] have technical issues with regard to the meaning of impredictability and computability, and Cárdenas et al., [12], note that their definition of catalytic closure is flawed.

Recent results in the theory of computer programming [40] conclude that a system closed to efficient causation can have computable models. Similar arguments have been made for autopoiesis, [36]. These results in turn are contested on technical grounds, [12]. Nevertheless, particular M-R system models exist that appear to be computable, or at least not obviously noncomputable, [27, 28].

From the above, it should be clear that there is a distinct lack of consensus about the definitions to be used when discussing living systems. (For a more complete review see reference [12].) Moreover, the issue has not been approached from the perspective of category theory.

In this dissertation we consider how to "fill the hole" in the Rosen-style argument. We focus on the key miss-assumption that Cartesian closure is required for closure to efficient causation and demonstrate that such mathematical structures can, for many practical purposes, always be constructed. Moreover, using the language of monoidal categories, it is possible to consider "nested" closed and/or open categories. This allows one to naturally embed open/closed categories inside other open/closed categories, key for dealing with complex systems. The richness of this language holds the possibility of both reconciling practice and theory, and of potentially unifying the various approaches.

Rosen, [44], first applied the tools of category theory to study efficient causation in metabolic-repair systems in 1958. Using a language sufficiently rich in entailment, he demonstrated that the existence of metabolism and a repair map implies the existence of a replication map. Casti, [13], extended this discussion to the general case where metabolism is linear, avoiding the one-input/one-output simplicity of Rosen's initial study. He also studied the effect of environmental fluctuations on Lamarckian inheritance and mutation (see Chapter 3). (Note that by Lamarckian we mean only the possibility of environmental influences "permanently" affecting repair and replication). His work can be naturally extended to non-linear systems as the complexity of biological properties of a system increases. This is possible provided that the degree of non-linearity, (the extent to which it differs from the linear structure), can be precisely specified, [13].

The self-replication arguments mentioned above have analogous counterparts in the discussion of self-organization; the two discourses are formally similar, [51]. Recent studies by Wolkenhauer and Hofmeyr have demonstrated how category theory can be used to explain the self-organizing principle in a very broad class of systems, including cell function in living systems, [51]. Motivated by Rosen's work on metabolic-repair systems [50], and phrasing the abstract cell description using the systems theory of Mesarovic and Takahara, [38], they show that: given a basic cellular process and cell function, the coordination of cell function is autonomously realised from within the cell. The existence of the coordination process ensures that the cell is closed to efficient causation and that every process is entailed from within the cell, [51]. In this argument, the living cell is represented in terms of a Cartesian closed category (see Chapter 2). Several common modelling methods are, however, not Cartesian closed. Several categories, such as the category of Abelian groups, the category of vector spaces over a fixed field, the category of posets and the category of topological spaces, are not Cartesian closed (see Chapter 2). These spaces may provide the basis for models and simulations, and the lack of the exponential object and evaluation map associated with Cartesian closure would imply that it is impossible to achieve efficient causation. Wolkenhauer and Hofmeyr's conclusion is, therefore, that these models and computer simulations fail to capture the autonomous self-organization of living things. On the other hand, we note that they actually only show that Cartesian closure is sufficient for efficient causation, and not that it is necessary. Their argument requires only closure under exponentiation.

In Mesarovic and Takahara's formalization of abstract systems [38], the Cartesian product is applied only for simplicity. We describe the extension to the case in which the tensor product is considered instead, leading to the notion of monoidal closure. Moreover, we outline a technique whereby any symmetric monoidal category can be embedded in a closed one. These extensions to more general descriptions of closure to efficient causation allow Rosen-style arguments to be constructed much more generally. This dissertation thus provides an analysis and extension of earlier results, softening the impact on current modelling paradigms while strengthening the philosophical implications and hinting at a more unified picture.

We now describe and motivate, in more detail, the structure of the work contained in this dissertation.

Organisms, cells, genes, proteins are complex structures whose relationships and properties are largely determined by their function as a whole, [51]. Mathematical modelling seeks to reduce these complex realities into simpler notions that can be more easily studied, [25]. However, there is a need to study only those models which provide relevant biological meaning. Rosen argued that the focus of attention should lie in the principles that lead to a certain phenomenon rather than the phenomenon itself, [50]. Newtonian mechanics describes cellular systems using energy and matter and represents them using states and dynamical laws. This, in itself, may be insufficient to fully capture the relationships between various components. Since our objective is to describe the functioning of a system as a whole rather than its individual components, relational language is introduced. A system is then described in terms of its components, their function and how they contribute to the entire organisation of the system, [50]. Rosen introduced relational biology which described entailment without bringing states into use and applied it to metabolism-repair systems. Contemporary theory, [13, 51], uses both techniques; particularly to describe time evolution within a relational structure.

In order to explain cellular activity, we use the notion of a relation among objects. A relation between systems is manifested either through their different structures or the functioning of each component of a system. A system can be identified in terms of the complexity of its attributes, or by the objective that it is set to achieve in the end. The representation of every system depends on the knowledge about the system. Each system, no matter how complex, can be thought of as a situation in which inputs are mapped to outputs through some kind of transformation, [38]. This can then be modified to include additional structure.

Our interest lies not in the individual functioning of systems. We are instead interested in studying the classes of things that exhibit similar characteristics and their underlying structures. It is for this reason that the language of category theory is introduced. In Chapter 2, we introduce the basic concepts of category theory. Closure to efficient causation was described by Rosen in terms of Cartesian closed categories. Therefore, the mathematical requirements leading to Cartesian closure are described in this chapter.

In Chapter 3, we review Rosen's work on metabolic-repair systems, [43]. Rosen argued that if the metabolic and repair components of a cell exist, then necessarily the replication component also exists, [44]. In other words, replication is implicitly defined given metabolism and repair. These arguments were based on efficient causation for cellular activity. In order to guarantee efficient causation, a Cartesian closed category is introduced. They also restrict the system to a one-input/one-output type system. The latter limitation is averted by Casti [13] in his study of linear metabolism-repair systems by introducing the inputs and outputs of a given system as a sequence of vectors. Under this construction, any finitely realisable metabolism can be a metabolism-repair system. The former is addressed in Chapter 5.

Chapter 4 introduces Mesarovic and Takahara's approach to tackling systems. In this chapter, the structural organization of a system is introduced. This approach, unlike other modelling methods like differential equations, begins by introducing the system concepts using minimal mathematical observations. Additional mathematical structure is only added to suit the characterizations of the real life system as required. Here, the inputs and outputs of a given system are not seen merely as elements but are described as signals. The functioning of a system, as much as it depends on the inputs and outputs, also depends on the systems' past history. The intrinsic description of a system is captured by the introduction of state objects and state space. The state objects and state space capture the systems past information and allow for predictability of the future behaviour of a system, [38].

In Chapter 5, we discuss Wolkenhauer and Hofmeyr's [51] general argument for self-organization, within the Cartesian paradigm. It is shown that, given the metabolism and repair components in any cell function, the coordination principle is realised autonomously from within the cell. We note that this argument rests on the existence of exponentiation; Cartesian closure is sufficient, but not necessary. Due to Mesarovic and Takahara's approach to dealing with systems, we find that a system is only as rich as the mathematical structure that has been added to it. We observe that the use of Cartesian closure to depict closure to efficient causation is merely for simplicity. In the case where we describe a system as a relation among objects and translate this mathematically using tensor products, we arrive at the notion of monoidal closure. We find that, although some modelling formalisms are not Cartesian closed, they may be monoidal closed and hence capable of describing closure to efficient causation; for example, the category **Ab** of Abelian groups and the category $\mathbf{Vect}_{\mathbf{K}}$ of vector spaces over a fixed field **K**.

By adding more structure to monoidal categories, we introduce the notion of enriched categories and note that any symmetric monoidal category can be embedded in a closed one. This allows for the extension of Rosen-style arguments to categories such as the category of topological spaces. Moreover, similar arguments point to the existence of closed subcategories of, for instance, **Top**. This is significant because actual models may well "live" in subcategories of the more well known categories.

Chapter 6 provides a summary of the work presented in this dissertation, and identifies some potentially fruitful areas of further research.

Chapter 2

Category Theory

In the study of category theory, we are interested not merely in individual things but in the general functioning a group of things can realize. Faced with the problem of categorization of different entities that display certain similar features, the language of category theory becomes the most useful solution. In this chapter, we largely rely on MacLane's book, "Categories for the Working Mathematician," [32]. Our main goal is to understand the properties of a Cartesian closed category as this is the basis for most of the arguments developed in the literature. Beginning with the basic definitions and examples of categories (§2.1 and §2.2), we then study functors (§2.3), natural transformations (§2.4), constructions on categories (§2.5), universal arrows (§2.6), limits and colimits (§2.7), and adjoints (§2.8). These are critical to the study of exponentiation which is a necessary requirement for the existence of closure (§2.9). Analogous constructions apply to the case of monoidal closure (see §5.3).

2.1 Introduction To Category Theory

Category theory is a branch of Mathematics in which mathematical systems are represented by the use of diagrams consisting of objects and arrows. An arrow $f : X \to Y$ represents a function, where X and Y are sets. A rule $x \to fx$ assigns to each $x \in X,\ fx \in Y$.

There are axioms for categories. We first begin by describing categories using axioms without including any set theory. This leads us to the notion of a meta-category.

We begin with a simpler notion of a meta-graph. This consists of objects $a, b, c \ldots$, arrows $f, g, h \ldots$, and two operations. These operations are the domain which assigns to each arrow an object a, and the codomain which assigns to each arrow an object b.

A meta-category is a meta-graph with additional operations. These are:

- Identity: To each object a, we have an arrow $id_a = 1_a : a \to a$.
- Composition: For any two arrows $f : a \to b,\ g : b \to c$, there exists an arrow
  $h = g \bullet f : a \to c$ known as the composition of f and g.

These two operations are subject to two axioms, namely:

- Associativity: For objects a, b, c, d and arrows $f : a \to b,\ g : b \to c$, $k : c \to d, k \bullet (g \bullet f) = (k \bullet g) \bullet f$.
- Unit law: For all arrows $f : a \to b,\ g : b \to c,\ 1_b \bullet f = f$ and $g \bullet 1_b = g$.

The interpretation of these category axioms within set theory is known as category theory. A category is any such particular interpretation and consists of objects and arrows. For each arrow there are objects known as the domain and codomain of the arrow. For any object, there exists an identity arrow from the object into the same object. Any two arrows satisfy the composition axiom. The identity and the composition operations are subject to the unit and associativity laws.

Categories can also be interpreted in terms of hom-sets. In this case, we deal with arrows only rather than objects. For objects $a, b \in \mathbf{C}$, the hom-set is the set of all arrows from a to b. It is written as:

$$\mathbf{C}(a,b) = \hom(a,b) = \{f | \ f : a \to b\}.$$

A category can now be described using the arrows only approach as in reference [32]:

For objects $a, b, c \in \mathbf{C}$, there exists a function assigning to each ordered pair $\langle a, b\rangle$ of objects a set $\hom(a,b) = \{f|f : a \to b\}$ such that $a \in \mathrm{dom}(f)$, $b \in \mathrm{cod}(f)$.

The identity and composition operations need to be defined as:

- To each object b, there exists an arrow $I_b \in \hom(b,b)$ known as the identity of b.
- For an ordered triple of objects $\langle a, b, c\rangle$, the composition function $\langle g, f\rangle \mapsto g \bullet f$ for $g \in \hom(b,c), f \in \hom(a,b)$ is given as: $\hom(b,c) \times \hom(a,b) \to \hom(a,c)$.

Just as is the case in the objects and arrows approach, these operations are subject to the associativity and unit axioms. For arrows $a \to b, \ b \to c, \ c \to d$,

$$\begin{aligned} \hom(a,d) &= \hom(c,d) \times \hom(b,c) \times \hom(a,b) \\ &= \hom(b,d) \times \hom(a,b) \\ &= \hom(c,d) \times \hom(a,c). \end{aligned}$$

In order for each arrow to have a distinct domain and codomain, the disjointness axiom; if $\langle a, b\rangle \neq \left\langle a', b'\right\rangle$ then $\hom(a,b) \cap \hom(a',b') = \emptyset$, should also be satisfied.

Suppose X and Y are sets. We can form the set:

$X \times Y = \{(x, y) \mid x \in X,\ y \in Y\}$.

This set is known as the Cartesian product of X and Y. This notion coincides with that of products in the category **Set** of small sets and will be discussed in §2.7.

2.2 Small and Large Categories

The theory of large and small categories is based upon the fact that there exists a set of all sets with certain similarity in structure. For example, we consider a group as a set with certain added structure. Given a property $f(x)$ of sets x, we form the set $\{x|x \in f(x)\}$, which is a collection of all those sets x with the structure of $f(x)$. Such a construction may, however, lead to the existence of paradoxical sets such as the set of all sets which are not members of themselves, [32].

In order to overcome this obstacle, the notion of a universe is introduced. A universe is defined to be U with the following properties:

- Suppose we have a set x and a collection of sets $u = \{x|x \in u\} \in U$, then $x \in U$.
- If $u \in U$ and $v \in U$ are a collection of sets each with similarity in structure, then we have the set with exactly u, v as elements $\{u, v\} \in U$, the ordered pair $\langle u, v \rangle \in U$ and the Cartesian product $u \times v \in U$.
- If $x \in U$, then the power set $\rho(x) \in U$ and the union $\cup x \in U$.
- The set of all finite ordinals $\omega \in U$.
- If $f : a \to b$ is a surjective function with $a \in U$ and $b \subset U$, then $b \in U$.

We will now base our subsequent operations on the existence of the universe U. If we hold U fixed, and call a set $u \in U$ a small set, then U is the set of all small sets. A function $f : u \to v$ is small if u, v are small. We can then form a category with objects: the set of all small sets; and (small) functions as arrows between them. This category is known as **Set**. This category is in itself not small since U is not small as this would imply $U \in U$.

Similarly, we can form the categories **Grp**, **Cat**, **Top** of all small groups, all small categories and all small topological spaces respectively.

Definition 2.2.1. A category in which both the collection of objects and the collection of arrows are (small) sets is known as a small category.

For an explanation of large categories, we define a class $C \subset U$ of the universe. The first condition for U to be a universe states that $x \in u \in U \Rightarrow x \in U$. This means that every element of U is also a subset of U. Every small set is then also a class. However, some classes are not small sets. Examples of classes which are not small include the categories **Set**, **Grp**, **Cat**. The introduction of classes allows us to sensibly define categories such as **Set**, **Grp**, **Cat** even though they are not small.

Definition 2.2.2. A large category is one in which both the collection of objects and the collection of arrows are classes.

Examples of Categories

- **Set**: The category of all small sets. Its objects are all small sets and the arrows are functions between them.

- **Cat**: The category of all categories whose objects are small categories and the arrows are functors between them.

- **Mon**: The category of monoids. Its objects are all small monoids and the arrows are all morphisms of monoids. A monoid is a semi-group with a unity element.

- **Grp**: The category of groups. Its objects are all small groups and the arrows are all morphisms of groups.
- **Ab**: The category of Abelian groups. Its objects are all small Abelian groups and the arrows are all morphisms of Abelian groups.
- **Rng**: The category of rings. Its objects are all small rings and the arrows are the ring morphisms.
- **Top**: The category of topological spaces. Its objects are small topological spaces and the arrows are continuous maps.
- $\mathbf{Vect_K}$: The category of vector spaces over a fixed field K. Its objects are all small vector spaces and the arrows are the linear transformations.
- **Grph**: The category of graphs. Its objects are small graphs and its arrows are the morphisms between the graphs.
- **Pos**: The category of posets. Its objects are partially ordered sets and its arrows are monotone mappings.
- **Toph**: The homotopy category of topological spaces. Its objects are topological spaces while the arrows are homotopy classes of continuous maps.

For details on the definitions required to construct the above categories, the reader is referred to standard texts such as MacLane's book "Categories for the Working Mathematician," [32], Borceux's "Handbook of Categorical Algebra 1: Basic Category Theory," [8], and Steve Awodey's "Category Theory," [4]. For monoids, the reader is referred to Chapter 5 which is largely based on G. M. Kelly's "Basic Concepts of Enriched Category Theory" [24].

2.3 Functors

We have already put together certain things that display similarities. In cases where different categories have been formed within a certain universe, we may find that the objects and arrows of a certain category are related to those of another category. A functor takes the objects of one category to the objects of another category. It also maps the arrows of one category to another category. We may have a functor $F :$ **Grp** $\rightarrow$ **Grp**. In this case we are dealing only with the categories of groups. We may also have functors such as $G :$ **Grp** $\rightarrow$ **Set** which include two different categories. A functor is therefore a morphism of categories. The study of functors becomes very useful when we study exponentiation, a very significant notion required for the existence of Cartesian and monoidally closed categories. This is studied later in §2.9 and §5.3 respectively.

Definition 2.3.1. For categories $\mathbf{C}$ and $\mathbf{B}$, a functor $T : \mathbf{C} \rightarrow \mathbf{B}$ consists of two functions:

(a) Object function T which assigns to each object $c \in \mathbf{C}$ an object $Tc \in \mathbf{B}$.

(b) Arrow function T which assigns to each arrow $f : c \rightarrow c^{'}$ of $\mathbf{C}$ an arrow $Tf : Tc \rightarrow Tc^{'}$ of $\mathbf{B}$ such that $T(1_c) = 1_{Tc}$ and $T(g \bullet f) = Tg \bullet Tf$.

Functors can be composed. Given two functors $T : \mathbf{C} \rightarrow \mathbf{B}$ and $S : \mathbf{B} \rightarrow \mathbf{A}$ there exists an arrow $S \bullet T : \mathbf{C} \rightarrow \mathbf{A}$. Composition of functors is associative.

A functor $T : \mathbf{C} \rightarrow \mathbf{B}$ is an isomorphism iff there exists a functor $S : \mathbf{B} \rightarrow \mathbf{C}$ for which both composites $S \bullet T$ and $T \bullet S$ are identity functors. A functor is then an isomorphism if it is invertible. In §2.8, we shall see that the definition of adjoints depends on two functors being invertible. Adjoints arise widely in the study of Cartesian and monoidally closed categories.

A functor $T : \mathbf{C} \rightarrow \mathbf{B}$ is full when to every pair $c, c^{'}$ of objects of $\mathbf{C}$ and to every arrow $g : Tc \rightarrow Tc^{'}$ of $\mathbf{B}$ there is an arrow $f : c \rightarrow c^{'}$ of $\mathbf{C}$ with $g = Tf$.

A functor is therefore said to be full iff it is surjective. This is useful when studying limits and colimits in §2.7, one of the requirements for the existence of a Cartesian or monoidally closed category. We shall see that the surjective nature of a functor leads to the existence of equalizers in §2.7.3.

A functor $T : \mathbf{C} \to \mathbf{B}$ is faithful when to every pair $c, c^{'}$ of objects of $\mathbf{C}$ and to every pair $f_1, f_2 : c \to c^{'}$ of (parallel) arrows of $\mathbf{C}$, $Tf_1 = Tf_2 : Tc \to Tc^{'}$ implies $f_1 = f_2$. For a functor to be considered faithful, it has to be injective. This is a property that arises when studying coequalizers in §2.7.3. Equalizers and coequalizers are examples of limits and colimits.

Another role of faithful functors arises in the description of concrete categories. Suppose we have a category $\mathbf{C}$, and a faithful functor $g : \mathbf{C} \to \mathbf{Set}$, then the pair $\langle \mathbf{C}, g \rangle$ is a concrete category. Examples of concrete categories include **Grp**, the category of groups, and **Top**, the category of topological spaces. The description of a concrete category makes it possible for us to think of the objects of a category simply as sets with additional structure, and the arrows as the morphisms which preserve that structure.

A functor which forgets some or all of the structure of an algebraic object is known as a forgetful functor. The functor $U : \mathbf{Grp} \to \mathbf{Set}$ is an example of a forgetful functor. To each group G, it assigns the set UG of its elements. We notice that the set UG forgets multiplication and hence group structure, [32]. The functor also assigns to each arrow $f : G \to G^{'}$, the same function f regarded as a function between sets.

Suppose $\mathbf{C}$ is a category. We can form another category $\mathbf{S}$ which constitutes a collection of some of the objects and some of the arrows of $\mathbf{C}$. The category $\mathbf{S}$ is referred to as the subcategory of $\mathbf{C}$. To ensure that a subcategory $\mathbf{S}$ is indeed a category, it is required to include with each arrow f, both the codomain and domain of f, with each object s, its identity arrow I_s, and with each pair of composable arrows $s \to s^{'} \to s^{''}$, their composite, [32].

Since the injection map from a subcategory **S** to a category **C** sends each object and arrow of **S** into itself in **C**, it is known as the inclusion functor. This functor is faithful, [32]. Whenever the inclusion functor is full, **S** is known as a full subcategory of **C**.

2.4 Natural Transformations

We have seen the relationship between two categories and the formation of functors. Functors are in themselves categories. The objects of one functor can be mapped to the objects of another functor. Similarly, arrows from a functor can be mapped to those of another functor. We then end up with a morphism between functors. This is known as a natural transformation. Natural transformations are useful while studying adjoints. The relationship between natural transformations and adjoints will be studied in §2.8. In particular, if every arrow in a natural transformation is invertible, we obtain a natural isomorphism.

Definition 2.4.1. A natural transformation $\tau : S \to T$ between two functors $S, T : \mathbf{C} \to \mathbf{B}$ is a function which assigns to each object $c \in \mathbf{C}$ an arrow $\tau_c : Sc \to Tc$ of $\mathbf{B}$ such that every arrow $f : c \to c^{'}$ in $\mathbf{C}$ makes the following square diagram commute:

$$
\begin{array}{ccc}
Sc & \xrightarrow{\tau_c} & Tc \\
{\scriptstyle Sf}\downarrow & & \downarrow{\scriptstyle Tf} \\
Sc^{'} & \xrightarrow[\tau_{c^{'}}]{} & Tc^{'}
\end{array}
$$

Suppose $1_T : T \to T$ is the identity natural transformation, then $1_T(c) = 1_{T_c}$.

2.5 Constructions on Categories

Different categories can have relationships between their respective objects and arrows. In order to study the functionality of a universe as a whole, bigger categories involving the already existing ones can be formed. Certain structures may also be changed. For example, opposites of categories discussed below. In this section we look at categories which can be formed from the already existing ones. We begin with products and opposites of categories. We also study comma categories. This is useful in the study of universal arrows, [48].

For a category to be Cartesian closed, it should have all its finite limits. One of the requirements is the existence of products. The opposites of categories provide us with duality properties for the already existing category. Suppose $\mathbf{C} \rightarrow \mathbf{B}$ is a functor in **Set**, then the opposite functor $\mathbf{B} \rightarrow \mathbf{C}$ is known as the contravariant functor and belongs to the category $\mathbf{Set^{op}}$. Certain functor categories aid us in the study of exponentials. Some constructions of categories are given below.

(1) Products of Categories. Suppose that we are given two categories $\mathbf{B}$ and $\mathbf{C}$. We can construct a new category $\mathbf{B} \times \mathbf{C}$ known as the product of the two given categories. An object of this new category is a pair $\langle b, c \rangle$ of objects $b \in \mathbf{B}$ and $c \in \mathbf{C}$. An arrow $\langle b, c \rangle \rightarrow \left\langle b^{'}, c^{'} \right\rangle$ of $\mathbf{B} \times \mathbf{C}$ is a pair $\langle f, g \rangle$ of arrows $f : b \rightarrow b^{'}$ and $g : c \rightarrow c^{'}$. The composite $\left(f^{'}, g^{'}\right) \bullet (f, g) = \left\langle f^{'} \bullet f, g^{'} \bullet g \right\rangle$.

This gives us two projections

$$\mathbf{B} \xleftarrow{P} \mathbf{B} \times \mathbf{C} \xrightarrow{Q} \mathbf{C} \tag{2.1}$$

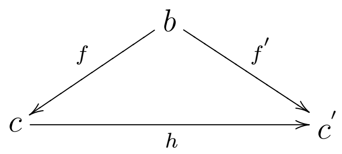
defined as $P \langle f, g \rangle = f$ and $Q \langle f, g \rangle = g$.

(2) Opposites of Categories. Given any category $\mathbf{A}$, one can form a category $\mathbf{A}^{op}$ known as the opposite category. The objects of $\mathbf{A}^{op}$ are the

same as the objects of $\mathbf{A}$. However, an arrow $f : a \to b$ in $\mathbf{A}$ is the arrow $f^{op} : b \to a$ in $\mathbf{A}^{op}$. Suppose $\mathbf{F}, \mathbf{G}$ are categories and $\alpha : \mathbf{F} \to \mathbf{G}$, the opposite category $\alpha^{op} : \mathbf{G}^{op} \to \mathbf{F}^{op}$ reverses the arrows in $\mathbf{F}$ and $\mathbf{G}$. However, since several categories and functors are involved, the duality statement does not reverse the functors, [32].

(3) Comma Categories. Suppose b is an object in the category of $\mathbf{C}$, we can construct the comma category $(b \downarrow \mathbf{C})$. The objects of this category are all pairs $\langle f, c \rangle, c \in \mathbf{C}, f : b \to c\ ;\ f \in \mathbf{C}$.

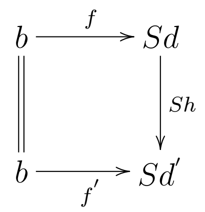
The arrows $h : \langle f, c \rangle \to \left\langle f', c' \right\rangle$ are arrows $h : c \to c'$ for which $h \bullet f = f'$. This can be illustrated by the following commutative triangle:



The comma category $(\mathbf{C} \downarrow b)$ has objects as arrows from c to a, and arrows such that $f = f' \bullet h$. The above descriptions define comma categories in terms of objects and categories.

Comma categories can also be described in terms of objects and functors, [32]. Suppose $\mathbf{C}, \mathbf{D}$ are categories, $b \in \mathrm{Obj}(\mathbf{C})$, and $S : \mathbf{D} \to \mathbf{C}$ is a functor, we can form a comma category $(b \downarrow S)$. Its objects are all pairs $\langle f, d \rangle$ where $d \in \mathrm{Obj}(\mathbf{D})$, $f : b \to Sd$. The arrows of this category are those of the form $f : b \to Sd$, $f' : b \to Sd'$ such that for $h : d \to d'$, $f' = Sh \bullet f$.

This construction is shown in the figure below:



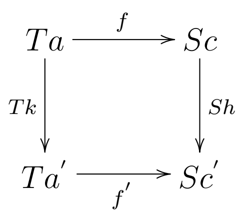
The category $(S \downarrow b)$ has as objects all pairs $\langle f, d \rangle$, $d \in$ Obj($\mathbf{D}$), $f : Sd \to b$. Its arrows are those of the form $f : Sd \to b$, $f^{'} : Sd^{'} \to b$ such that for $h : d \to d^{'}$, $f = f^{'} \bullet Sh$.

In general, comma categories can be described in terms of categories and functors. Consider categories $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and functors $T : \mathbf{A} \to \mathbf{B}$, $S : \mathbf{C} \to \mathbf{B}$ between them. The comma category $\langle T \downarrow S \rangle$ has as objects the triple

$$\{\langle a, c, f \rangle : a \in \text{Obj}(\mathbf{A}), c \in \text{Obj}(\mathbf{C}), f : Ta \to Sc\}.$$

Its arrows are those of the form $f : Ta \to Sc$, $f^{'} : Ta^{'} \to Sc^{'}$ such that for $k : a \to a^{'}, h : c \to c^{'}$, $f^{'} \bullet Tk = Sh \bullet f$.

This construction is shown in the commutative figure below:



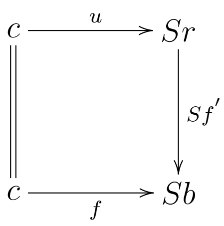
(4) Functor Categories. Given two categories $\mathbf{B}$ and $\mathbf{C}$, we can form a functor category $\mathbf{B}^{\mathbf{C}}$. The objects of this category are its respective functors and the arrows are the natural transformations between the functors. Note that this unfortunate historical notation for a functor category may be confused with that for the exponential (see §2.9). For the category **Cat** of all small categories, these two notions coincide.

In the case where **B** and **C** are both sets, then the functor category $\mathbf{B}^{\mathbf{C}}$ is also a set consisting of all functions between **C** and **B**.

2.6 Universal Arrows

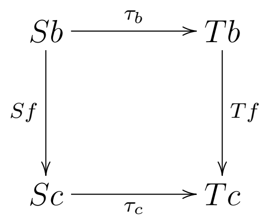
Universal arrows provide a way to define limits and colimits, which are studied in §2.7. There we also provide an alternative definition of universal arrows using comma categories and initial objects: Daniele Turi [48] defines a universal arrow from an object $c \in \mathbf{C}$ to a functor $F : \mathbf{B} \to \mathbf{C}$ as consisting of an initial object (see Definition 2.7.2) in the comma category $(c \downarrow F)$. We find that the importance of universal arrows extends to the existence of adjunctions. The presence of adjunctions implies that a category is Cartesian closed.

Definition 2.6.1. Given a functor $S : \mathbf{B} \to \mathbf{C}$ and c an object of $\mathbf{C}$, a universal arrow from c to S is a pair $\langle r, u \rangle$ where r is an object of $\mathbf{B}$, u is an arrow $u : c \to Sr$ of $\mathbf{C}$ such that to every pair $\langle b, f \rangle$ with b an object of $\mathbf{B}$ and $f : c \to Sb$ an arrow of $\mathbf{C}$, there exists a unique arrow $f' : r \to b$ of $\mathbf{B}$ such that $Sf' \bullet u = f$. This can be represented by the following commutative diagram:



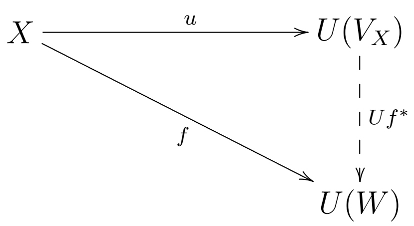
Natural transformations can also be defined in terms of the comma categories of two given functors. Suppose $S, T : \mathbf{B} \to \mathbf{C}$ are two parallel functors (having the same domain and codomain), which implies that the given functors $S = T$. A natural transformation from S to T is a functor $\tau : \mathbf{B} \to (S \downarrow T)$ such that $\delta_0 \bullet \tau = \delta_1 \bullet \tau = id_{\mathbf{B}}$. Here, δ_1 and δ_2 are projection functors, [48].

This implies that to each $b \in \text{obj}(\mathbf{B})$, an arrow $\tau_b : Sb \to Tb$ of $\mathbf{C}$ is mapped such that for every $f : b \to c \in \mathbf{C}$, the following square commutes:



Examples

(1) Let $U : \mathbf{Vect_K} \to \mathbf{Set}$ be a functor between the category of all vector spaces over a fixed field K and the category of sets. Let X be any set. Our goal is to find a universal arrow from X to U. For any set X, we can construct a vector space V_X with basis X. This V_X is an object of the category $\mathbf{Vect_K}$. We now need to find an arrow in the category $\mathbf{Set}$. We have an arrow $u : X \to U(V_X)$. For any other vector space W and $f : X \to U(W)$ an arrow of $\mathbf{Set}$, there exists a unique arrow $f^* : V_X \to W$ with $Uf^* \bullet u = f$. A universal arrow from X to U is therefore a pair $\langle V_X, u \rangle$ such that the diagram below commutes:



(2) Let $U : \mathbf{Cat} \to \mathbf{Grph}$ be a functor and G be a small graph. A universal arrow from G to U is a pair consisting of $\mathbf{C}$ an object in $\mathbf{Cat}$, and $u : G \to U\mathbf{C}$ an arrow in $\mathbf{Grph}$, such that for any other category B and an arrow $f : G \to UB$, there exists a unique arrow $f^* : \mathbf{C} \to B$ with $Uf^* \bullet u = f$.

The commutative diagram for example two is similar to that given for the first example.

Properties of Universality

There are equivalent notions of universal arrows. Here, we look at the relationship between universal arrows and universal elements. An equivalent notion of representable functors is also introduced.

(1) Let $S : \mathbf{D} \to \mathbf{C}$ be a functor. Suppose $\mathbf{D}(r, d) \cong \mathbf{C}(c, Sd)$ is a natural isomorphism:

   There exists $r \in \mathbf{D}$, $u : c \to Sr$ such that $\langle r, u \rangle$ is universal from c to S iff the function sending $f^* : r \to d$ into $Sf^* \bullet u : c \to Sd$ is a bijection of hom-sets $\mathbf{D}(r, d) \cong \mathbf{C}(c, Sd)$. The bijection is natural in d.

   Given r, c, any given natural isomorphism is determined by a unique arrow $u : c \to Sr$ such that $\langle r, u \rangle$ is universal from c to S.

(2) Suppose $\mathbf{D}$ has small hom-sets and $K : \mathbf{D} \to \mathbf{Set}$ is a functor. The following are equivalent:

   (a) K is representable. This implies that there exists a pair $\langle r, \alpha \rangle$, with r an object of $\mathbf{D}$ and $\alpha : \mathbf{D}(r, -) \cong K$ a natural isomorphism.

   (b) K has a universal element. A universal element of the functor K is a pair $\langle r, e \rangle$ with r an object of $\mathbf{D}$, and e an element such that for every pair $\langle d, x \rangle$ with $x \in Kr$, there exists a unique arrow $f : r \to d$ of $\mathbf{D}$ with $(Kf)e = x$.

   (c) There exists a universal arrow from the one point set to K.

These properties of universality show us that the notions of universal arrows, universal elements and representable functors are equivalent. A universal arrow from c to S leads us to a natural isomorphism $\mathbf{D}(r, d) \cong \mathbf{C}(c, Sd)$ which in turn leads us to a representation of the functor $K : \mathbf{D} \to \mathbf{Set}$.

These properties amount to the following Yoneda Lemma, [52].

Lemma 2.6.2. *Yoneda Lemma [52]. Suppose* $K : \mathbf{D} \to \mathbf{Set}$ *is a functor and ensure that* $\mathbf{D}$ *is a category with small hom-sets. If* r *is an object in* $\mathbf{D}$*, then there exist bijections* $y : \mathbf{Set}^{\mathbf{D}}(\mathbf{D}(r, -), K) \cong Kr$ *natural in both* K *and* r*.*

The Yoneda Lemma can also be stated in form of the following corollaries:

- Suppose $K : \mathbf{D}^{op} \to \mathbf{Set}$, then there exist bijections:

  $$y' : \mathbf{Set}^{\mathbf{D}^{op}}(\mathbf{D}(-, r), K) \cong Kr$$

  natural in r and K. This is known as the dual of the Yoneda Lemma.

- For objects $r, s \in \mathbf{D}$, $\mathbf{Set}^{\mathbf{D}}\left[\mathbf{D}(r, -), \mathbf{D}(s, -)\right] \cong \mathbf{D}(h, -)$ where h is a unique arrow $r \to s$.

- Suppose $Y : \mathbf{D}^{op} \to \mathbf{Set}^{\mathbf{D}}$, then Y is a full and faithful functor. It's dual also yields a similar functor.

The Yoneda Lemma [52] further intensifies the universality property. As shown in [48], it can be used to prove that a category is Cartesian closed. The dual of the lemma leads to the proposition that for every small category $\mathbf{C}$, $\mathbf{Set}^{\mathbf{C}^{op}}$ is Cartesian closed. From the lemma, the category $\mathbf{Set}^{\mathbf{C}^{op}}$ has all its finite limits. In cases where the smaller category $\mathbf{D}$ lacks some of its limits, the situation is amended by considering it as part of the functor category $\mathbf{Set}^{\mathbf{C}^{op}}$, [48]. Similar constructions are used in §5.3.

2.7 Limits and Colimits in Categories

Limits and colimits are an important step towards defining and understanding the notion of Cartesian closure. They ensure the existence of finite limits. Limits and colimits are also of importance in our discussion of adjoints and are therefore useful in finding exponentials.

We begin by defining some special cases: (Coproducts and products; coequalizers and equalizers; pushouts and pullbacks).

Definition 2.7.1. Terminal Object: An object t is terminal in a category $\mathbf{C}$ if to each object a in $\mathbf{C}$, there exists exactly one arrow $a \to t$. If t is terminal, the only arrow $t \to t$ is the identity. Any two terminal objects in $\mathbf{C}$ are isomorphic in $\mathbf{C}$.

Definition 2.7.2. Initial Object: An object s is initial in $\mathbf{C}$ if to each object a there exists exactly one arrow $s \to a$.

Definition 2.7.3. Null object: A null object z in $\mathbf{C}$ is an object which is both terminal and initial; for any two objects a, b of $\mathbf{C}$, there is a unique arrow $a \to z \to b$ known as the zero arrow.

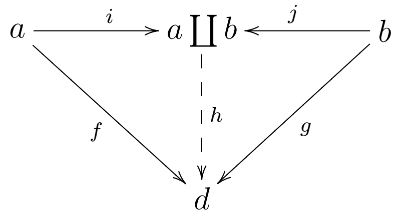
Examples

- In the category **Set** of sets, the empty set is an initial object while the one element set $\{x\}$ is a terminal object.
- In the category **Grp** of groups, the one element group is a terminal object while any terminal object is also initial. Here, the one element group is therefore a null object.
- In the category **Ab** of Abelian groups, the zero group is both initial and terminal. Here, the zero group is therefore a null object.
- In the category **Top** of topological spaces, the one point space is terminal while the empty space is initial.

Coproducts and Products

(a) Coproducts: Suppose $a, b \in \mathbf{C}$, the coproduct of a and b consists of an object $c = a \coprod b$ along with two projections $i : a \to a \coprod b,\ j : a \coprod b \leftarrow b$

such that for any pair $f : a \to d,\ g : b \to d,\ \exists\,!\ h : c \to d; f = h \bullet i$, $g = h \bullet j$. This yields the following commutative diagram:



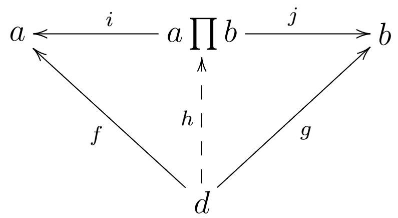
The coproduct of a and b if it exists, is unique upto isomorphism.

The coproduct of a and b exists iff the functor $\mathbf{C}(a \coprod b, -) : \mathbf{C} \to \mathbf{Set}$ is representable and is represented by the coproduct of a and b, which is $\mathbf{C}(a \coprod b, -)$.

The bijection $\mathbf{C}(d, a \coprod b) \cong \mathbf{C}(d, a) \times \mathbf{C}(d, b)$ is natural in d.

The properties of coproduct include:
$a \coprod b \simeq b \coprod a,\ (a \coprod b) \coprod c \simeq a \coprod (b \coprod c)$, and $a \coprod 0 \simeq a$.

(b) The notion of the product is the dual of that of the coproduct. Suppose $a, b \in \mathbf{C}$ the product of a and b consists of an object $c = a \prod b$ and two projections $i : a \prod b \to a$ and $j : a \prod b \to b$ such that for any pair $f : d \to a,\ g : d \to b$, there exists a unique arrow $h : d \to a \prod b$. Here the universality properties imply that the following diagram commutes:



The product of a and b if it exists, is unique upto isomorphism.

The product of a and b exists iff the functor $\mathbf{C}(a \prod b, -) : \mathbf{C}^{op} \to \mathbf{Set}$ is representable and is represented by the product of a and b, which is $\mathbf{C}(a \prod b, -)$.

The bijection $\mathbf{C}(d, a \prod b) \cong \mathbf{C}(d, a) \times \mathbf{C}(d, b)$ is natural in d.

The properties of products include:

$a \prod b \simeq b \prod a$, $(a \prod b) \prod c \simeq a \prod (b \prod c)$ and $a \prod 1 \simeq a$.

Examples

In the category **Set**, the coproduct of a and b is the disjoint union while the product is the usual Cartesian product.

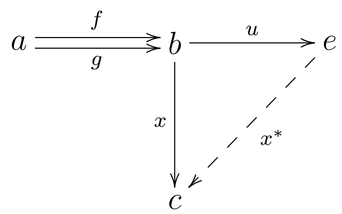
Suppose **Pos** is the category of posets. Then the coproduct of a and b is the supremum $sup\{a, b\}$ while the product is the infimum $\inf\{a, b\}$.

In **Top**, the coproduct of a and b is the topological sum while the product is the product of spaces.

In **Grp**, the free product is the coproduct of a and b while the product is the direct product.

Coequalizers and Equalizers

(a) Coequalizers. Suppose **C** is a category in which $f, g : a \to b$ are two parallel arrows. A coequalizer of $\langle f, g \rangle$ is an arrow $u : b \to e$ such that (i) $u \bullet f = u \bullet g$; (ii) If $x : b \to c$ such that $x \bullet f = x \bullet g$, then there exists a unique arrow x^* such that $x = x^* \bullet u$.

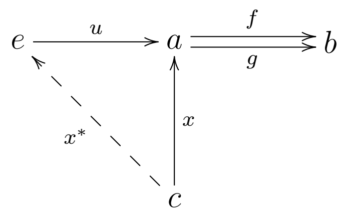


The coequalizer of f and g, if it exists, is unique upto isomorphism.

There exists a bijection $\mathbf{C}(e, c) \simeq \{x \in \mathbf{C}(b, c) \ ; \ x \bullet f = x \bullet g\}$.

The coequalizer is a generalisation of a quotient by an equivalence relation.

(b) Equalizers. Suppose $f, g : a \to b$ is a pair of parallel arrows. An equalizer of f and g is an arrow $u : e \to a$ such that (i) $f \bullet u = g \bullet u$ (ii) For any arrow $x : c \to a$ with $f \bullet x = g \bullet x$, there exists a unique arrow $x^* : c \to e$ such that $x = u \bullet x^*$.



The equalizer of f and g, if it exists, is unique upto isomorphism.

There exists a bijection $\mathbf{C}(c, e) \simeq \{x \in \mathbf{C}(c, a)\ ;\ f \bullet x = g \bullet x\}$.

Examples

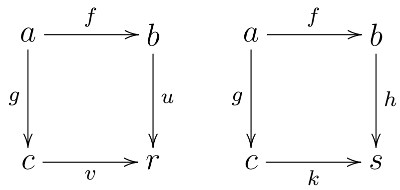
In **Set**, take ρ to be the smallest equivalence relation containing $\{(f(x), g(x))\,, x \in X\}$. We then take $e = c/\rho$. The coequalizer is therefore given by $u : b \to c/\rho$. The equalizer of a and b is obtained by taking $e = \{x \in a : f(x) = g(x)\}$. Here, a, c, b, e are as in the above diagrams of the coequalizers and equalizers.

In **Top**, we obtain the coequalizer by applying a similar construction as in **Set**. However, we now use the quotient topology. The equalizer is obtained by using the subspace topology.

In **Grp**, the coequalizer of a and b is obtained by defining a congruence relation on a. The congruence relation is the smallest equivalence relation, [8], generated by all the pairs $(f(a), g(a))$ for all $a \in \mathbf{A}$. The equalizer is defined as in **Set**.

Pushouts and Pullbacks

(a) Pushouts. Suppose **C** is a category which contains two arrows $f : a \to b,\ g : a \to c$ such that for any other two arrows $u : b \to r$, $v : c \to r,\ u \bullet f = v \bullet g$. The pushout of f and g is a commutative square, such as that on the left below, such that to every other square such as the one on the right below with $h : b \to s,\ k : c \to s\ ;\ h \bullet f = k \bullet g$, there exists a unique arrow $t : r \to s$ with $t \bullet u = h$ and $t \bullet v = k$.

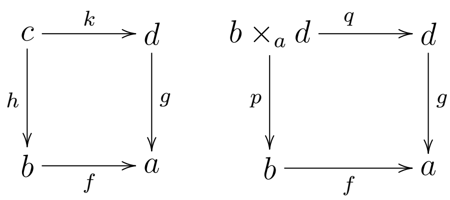


In this case, the square on the left is known as the pushout square of f and g.

When $f = g$, the pushout of f and f is known as the cokernel pair of f.

(b) Pullbacks. Suppose **C** is a category which contains a pair of arrows $f : b \to a,\ g : d \to a$ such that for any other pair of arrows $k : c \to d,\ h : c \to b,\ g \bullet k = f \bullet h$. The pullback of f and g is a universal arrow consisting of a pair of arrows $q : b \times_a d \to d,\ p : b \times_a d \to b$ such that $g \bullet q = f \bullet p$. For every commutative square, there exists a unique arrow $r : c \to b \times_a d$ with $k = q \bullet r,\ h = p \bullet r$. Here, $b \times_a d$ is known as the fibred product or the product over a.

The pullback square is shown on the right as follows:



When $f = g$, the pullback of f and f is known as the kernel pair of f. A pullback is the dual of a pushout.

Example

In **Set**, the pushout of f and g is the disjoint union $b \amalg c$ with the elements fx and gx identified for each $x \in a$. The pullback is given by:
$\{\langle x, y\rangle \mid x \in X,\ y \in Y, fx = gy\}$.

Having dealt with some special cases, we now define the more general concepts of limits and colimits.

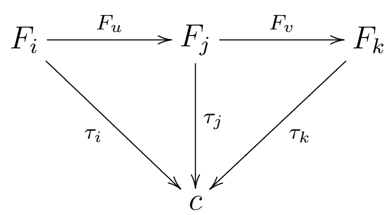
Colimits and Limits

(a) Colimits. Suppose we have a functor $F : \mathbf{J} \to \mathbf{C}$ where $\mathbf{J}$ is an index category. Construct the diagonal functor $\lhd : \mathbf{C} \to \mathbf{C}^{\mathbf{J}}$. This sends each object c to the functor $\lhd c$. Given $f : c \to c'$ we have the natural transformation $\lhd f : \lhd c \to \lhd c'$.

A universal arrow $\langle r, u\rangle$ from F to $\lhd$ is called the colimit of F.

(i) r is an object of $\mathbf{C}$, written as $r = ColimF$.

(ii) $u : F \to \lhd r$ is a natural transformation which is universal among natural transformations $\tau : F \to \lhd c$. For each object $i \in \mathbf{J}$, τ consists of arrows $\tau_i : F_i \to c$ such that for each $u : i \to j$, $\tau_j \bullet Fu = \tau_i$.

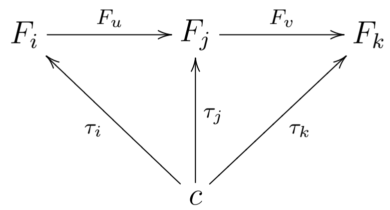


(b) Limits. Limits are dual to the colimits. Suppose we have a functor $F : \mathbf{J} \to \mathbf{C}$.

A universal arrow $\langle r, v \rangle$ from $\lhd$ to F is called the limit of F.

(i) r is an object of $\mathbf{C}$ written as $r = LimF$.

(ii) $v : \lhd r \to F$ is a natural transformation which is universal among natural transformations $\tau : \lhd c \to F$. For each object $i \in \mathbf{J}$, τ consists of arrows $\tau_i : c \to F_i$ such that for every arrow $u : i \to j,\ \tau_j = Fu \bullet \tau_i$.



2.8 Adjoints

Adjoints bring natural transformations into use. They are also identified by the presence of limits and colimits. The right adjoint is an indication of the presence of exponentials. Adjoints can therefore be used to check whether a category is Cartesian closed.

For a fixed field K we consider the functors

$$\mathbf{Set} \underset{\mathbf{U}}{\overset{\mathbf{V}}{\rightleftarrows}} \mathbf{Vect_K} \tag{2.2}$$

We begin with the universal mapping property. Consider the construction of a vector space V_X with basis X. There exists a function $u : X \to U(V_X)$. The universal mapping property states that for every vector space W and every function $u : X \to U(W)$ there exists a unique homomorphism $f^* : V_X \to W$ such that $U(f^*) \bullet u$. We now consider the map

$$\phi : \mathbf{Vect_K}(V(X), W) \cong \mathbf{Set}(X, U(W)).$$

This is a bijection due to the universal property mapping. The above construction provides us with a simple example of an adjunction.

Definition 2.8.1. An adjunction between categories X, A consists of functors F, G as shown:

$$X \underset{G}{\overset{F}{\rightleftarrows}} A$$

and:

(i) A natural transformation $\eta : 1_X \to GF$ with the universal mapping property:

Suppose $\eta_X : x \to GFx$. For any $a \in A,\ x \in X$ and $f : x \to Ga$, there exists a unique $g : Fx \to a$ such that $f = Gg \bullet \eta_X$.

(ii) A natural transformation $\varepsilon : FG \to 1_A$ such that given $\varepsilon_a : FGa \to a$, for any $a \in A,\ x \in X$ and $\varphi^{-1}g : Fx \to a$, there exists a unique $g : x \to Ga$ such that $\varphi^{-1}g = \varepsilon_a \bullet Fg$.

F is the left adjoint for G, G is the right adjoint for F, while η and ε are known as the unit and the counit of the adjunction respectively.

There are several equivalent definitions for adjunctions. Adjoints can be defined in terms of hom-sets as:

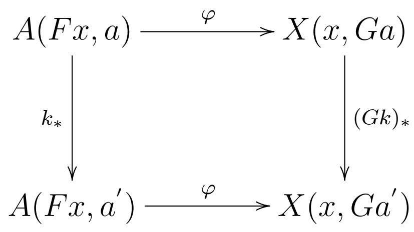
Definition 2.8.2. Given two functors $F : X \to A$ and $G : A \to X$, then F is a left adjoint for G or G is a right adjoint for F iff for each $x \in X,\ a \in A$, there exist bijections

$$\varphi : \ A(Fx, a) \cong X(x, Ga) \tag{2.3}$$

natural in x and a.

Naturality in a

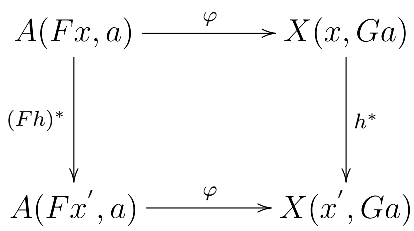
For naturality in a, we fix x and take $k : a \to a'$. We end up with the following diagram which should commute due to the naturality of the bijection (2.3).



In the diagram, k_* is an abbreviation for $A(Fx, k)$ while $(Gk)_*$ stands for $X(x, Gk)$.

Naturality in x

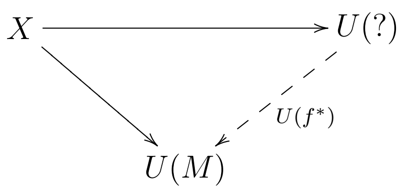
For naturality in x, we fix a and take $h : x^{'} \to x$. The following commutative diagram is the result of naturality in x.



In the diagram, h^* is an abbreviation for $X(h, Ga)$ while $(Fh)^*$ stands for $A(Fh, a)$.

2.8.1 Examples of Adjoints

(1) A monoid is a category with one object. Consider the functor from the category of monoids to the category of small sets given by U : **Mon** $\to$ **Set**. Suppose X is a set. Our objective is to find universal arrows from X to U. We are faced with the construction:



For M, a monoid and given the above construction, there exists a unique $f^* : ? \to M$. Our left adjoint is given by the unknown value of ?. Considering the above case therefore, the left adjoint will be given by the free monoid, say $F(X)$, on X.

(2) Suppose $U : \mathbf{Ab} \to \mathbf{Grp}$ is a functor from the category of Abelian groups to the category of groups. For any group G, the commutator subgroup $[G, G]$ is a normal subgroup of G. The factor commutator functor $\mathbf{Grp} \to \mathbf{Ab}$ is defined by $G \to G/[G, G]$. The factor commutator group $G/[G, G]$ is the left adjoint.

(3) Suppose we have the functor $F : \mathbf{C} \to \mathbf{1}$ where $\mathbf{1}$ is the category that contains one object and one identity arrow. A right adjoint would be an object say $U : \mathbf{1} \to \mathbf{C}$ such that the bijection $(FC, -) \cong (C, U(-))$ exists. In order for the functor to have a right adjoint, the object U has to be a terminal object in $\mathbf{C}$.

2.9 Cartesian Closed Categories

In the study of categories, one begins with a simple structure and goes on to add more structure. One of the additional structures of a category is closure. A more special case of this additional structure is Cartesian closure. This helps us look into the internal compositions of categories. It brings into use the ideas of universal arrows, adjoints, and the various limits and colimits. We begin by introducing the idea of exponentiation.

Definition 2.9.1. Let $\mathbf{C}$ be a category with binary products and a terminal object. Take A to be an object in $\mathbf{C}$. The functor $- \times A : \mathbf{C} \to \mathbf{C}$ is represented by the following diagram:

$$
\begin{array}{ccc}
X & \longrightarrow & X \times A \\
{\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h \times id_A} \\
Y & \longrightarrow & Y \times A
\end{array}
$$

Suppose **B** is a category and take B, an object in **B**. In order to find a universal arrow from $- \times A$ to $B \in \mathbf{B}$, we need:

(a) $\hat{B}$ an object in **C**.

(b) An arrow $\hat{B} \times A \xrightarrow{\gamma} B$.

(c) Whenever for any $X \in \mathbf{C}$, $f : X \times A \longrightarrow B$, there exists a unique $f^* : X \longrightarrow \hat{B}$ such that $\gamma \bullet (f^* \times id_A) = f$.

These requirements are summarised in Figure (2.4):

$$
\begin{array}{ccc}
\hat{B} \times A & \xrightarrow{\gamma} & B \\
{\scriptstyle f^* \times id_A}\nwarrow & & \nearrow{\scriptstyle f} \\
 & X \times A &
\end{array}
\tag{2.4}
$$

Such a $\hat{B}$, if it exists, is unique upto isomorphism. It is called the exponential object of B to A and is denoted by B^A. The evaluation map is given by $\gamma : \hat{B} \times A \longrightarrow B$.

Proposition 2.9.2. *A category* **C** *has exponential objects iff every object A, $(- \times A) : \mathbf{C} \to \mathbf{C}$ has a right adjoint $(-)^A : \mathbf{C} \to \mathbf{C}$.*

This construction of exponentials enables us to define Cartesian closed categories.

Proposition 2.9.3. *It can be shown that the following requirements are equivalent conditions needed for a category to be Cartesian closed:*

- *A category* **C** *with binary products, exponentials, and terminal object is Cartesian closed. This implies that each of the functors* $\mathbf{C} \to \mathbf{1}$, $\mathbf{C} \to \mathbf{C} \times \mathbf{C}$, $\mathbf{C} \to \mathbf{C}$ *must have a specified right adjoint.*

- *A category* **C** *with coproducts, exponentials, and initial object is Cartesian closed.*

- *The two requirements above imply that a category* **C** *with a null object, products, coproducts and exponentials is Cartesian closed.*

2.9.1 Examples

(1) The category **Set** is Cartesian closed.

The disjoint union $X + Y \cong X \cup Y = \{(0,x)|x \in X\} \cup \{(1,y)|y \in Y\}$ of two sets X and Y is the coproduct of X and Y. This is because the left injection $X \to X + Y$ maps x to $< 0, x >$. The initial object of **Set** is the empty set. The category of small sets therefore has coproducts as the disjoint union of sets and initial object as the empty set.

In the category **Set** of small sets, $\hat{B} = B^A$ is the set of all functions from A to B.

The existence of coproducts, initial objects and exponentials implies that the category **Set** is Cartesian closed.

(2) The category **Cat** is Cartesian closed.

The initial object in the category of all small categories **Cat** is the empty category, the terminal object is the category **1** with one object and one arrow (identity).

Suppose **A** and **B** are two categories in **Cat**. The exponential B^A is the functor category. This is a category whose objects are the respective functors and the arrows are the natural transformations between them.

The existence of exponentials as the functor category in **Cat**, the initial object and the coproduct leads us to the conclusion that **Cat** is Cartesian closed.

(3) The category **Grp** is not Cartesian closed.

Suppose $A, B \in$ **Grp**. A natural transformation between $f, g : A \to B$ is an element $b \in B$ such that for every $a \in A$ we have $f(a) \bullet b = b \bullet g(a)$. Therefore $b^{-1} \bullet f(a) \bullet b = g(a)$. The natural transformation is therefore an inner automorphism $y \longmapsto b^{-1}yb$. Every such arrow $f \mapsto g$ has an inverse.

For exponentiation, we require that $B^A \in$ **Grp**. For B^A to be a group, it has to have one object in which every arrow has a two sided inverse under composition, [4]. Since there may be more than a single homomorphism $A \to B$, B^A is therefore not always a group. The first requirement of existence of exponentials is not satisfied.

We remember that a group is a category in which every arrow has a (two sided) inverse under composition, [32]. In reference [4], it is recommended that the category of groups be enlarged to include the categories with more than one object, but still having inverses for all arrows. Such a category is known as the category of groupoids. A groupoid is a category in which every arrow is invertible. Since the inverse of any path is the same path drawn in the opposite direction, for any pair of categories $A, B \in$ **Grpd**, $B^A \in$ **Grpd**. The category **Grpd** is therefore Cartesian closed.

(4) The category of Abelian groups.

The category **Ab** has as objects all small additive Abelian groups and arrows the homomorphisms between them. In **Ab**, the zero group is both initial and terminal object. It is therefore known as a null object. The coproduct in **Ab** is the tensor product of $A \otimes B$ of A, B in **Ab**.

We now study the natural transformations and consider whether there exists a unique natural isomorphism between two Abelian groups. This example is in reference to the natural transformations described by Saunders MacLane in reference [32].

A homomorphism between two Abelian groups G and H is a map $f : G \to H$ such that $f(g \bullet h) = f(g) \bullet g(h)$. An example of such a homomorphism is the character group $D(g)$ of an Abelian group. This is the set of all functions from a group G to the additive group of real numbers modulo 1 :

$$t : G \to \mathbf{R}/\mathbf{Z}.$$

Each homomorphism $f : G \to H \in \mathbf{Ab}$ determines a character homomorphism $Df : DH \to DG$ in $\mathbf{Ab}$. Note that D is an arrow in the opposite direction. The character group is therefore not a functor but a contravariant functor from $\mathbf{Ab}$ to $\mathbf{Ab}$. For this reason, f is an isomorphism but is not natural.

One way of making D into a covariant functor is by letting G be a finite Abelian group. We now consider isomorphisms between the categories of two finite Abelian groups. The isomorphism $f : G \to DG$ depends on a representation of G as a direct product of cyclic groups. It is therefore not natural.

Although every Abelian group G is isomorphic to its character group DG, the isomorphism is not natural since the natural transformation depends on artificial choices of generators. Without a unique natural isomorphism, the existence of a right adjoint is impossible. This implies that in the category $\mathbf{Ab}$, the exponential $\{G^H : G, H \in \mathbf{Ab}\}$ is not defined. The category of Abelian groups is therefore not Cartesian closed.

(5) The category of vector spaces over a fixed field.

A case similar to the category of Abelian groups is the category of vector spaces over a fixed field.

Consider $\mathbf{Vect_K}$, the category of all vector spaces over a fixed field K. Each vector space V has a dual space V^*. Every linear transformation $f : V \to W$ gives rise to the dual linear transformation $f^* : W^* \to V^*$ in $\mathbf{Vect_K}$. Here f^* is a contravariant functor.

Every vector space is isomorphic to its dual. In the case when the vector is not finite dimensional, we end up with a contravariant functor, making it impossible for the exponential to be defined. When the vector space is finite dimensional, we are able to make this into a covariant functor. However, this isomorphism depends on a given choice of bases and is therefore not natural. We are then unable to find a right adjoint of the functor $\mathbf{Vect_K} \to \mathbf{Vect_K}$. The absence of a right adjoint implies that the exponential is not defined and that the category of vector spaces over a fixed field is not Cartesian closed.

(6) The category of posets. The arrows in a category **Pos** of posets are the monotone functions. Suppose $(a, b) \in$ **Pos** of posets, the greatest lower bound of a and b is the initial object while the lowest upper bound of a and b is the terminal object. The infimum, $inf\{a, b\}$, of objects is the product while the supremum, $sup\{a, b\}$, is the coproduct.

Suppose we take the exponential B^A to be the set of monotone functions $B^A = \{f : A \to B \mid f \text{ is monotone}\}$ ordered pointwise [4] such that $f \leq g$ iff $fa \leq ga$ for all $a \in A$.

We first aim to show that the evaluation $\gamma : B^A \times A \to B$ is monotone. Given $(f, a) \leq (f', a')$ in $B^A \times B$, we have

$$\begin{aligned} \gamma(f, a) &= f(a) \\ &\leq f(a') \\ &\leq f'(a') = \gamma(f', a') \end{aligned}$$

This implies that the evaluation map γ is monotone.

Next, we take $f^* : X \to B^A, f : X \times A \to B$ and aim to show that f^* is monotone if f is monotone.

Suppose f is monotone. Let $x \leq x^{'}$. We need to show that:
$f^*(x) \leq f^*(x^{'})$. Now,

$$\begin{aligned} f^*(x)(a) &= f(x, a) \\ &\leq f(x^{'}, a) \\ &\leq f^{'}(x^{'}, a) = f^*(x^{'}(a)) \end{aligned}$$

We have that f^* is monotone if f is monotone.

The category **Pos** of posets is Cartesian closed only if the function space is ordered pointwise i.e for $B^A = \{f : A \to B \mid f \text{ is monotone}\}$, $f \leq g$ if and only if $fa \leq ga$ for all $a \in A$. Otherwise the category **Pos** is not Cartesian closed.

(7) The category of topological spaces.

The category of topological spaces is not Cartesian closed. The coproduct in **Top** is the disjoint union of spaces [32] while the product is the direct product. The functor $A \times - :$ **Top** $\to$ **Top** does not preserve quotients and hence does not preserve coequalizers, [1]. For further information on the category **Top** of topological spaces the reader is referred to Chapter 5, §5.3.

2.10 Summary

Categories are common in mathematics and related fields such as computer science, [1]. Sets, vector spaces, groups, topological spaces, Banach spaces, manifolds, ordered sets, and automata, all naturally give rise to categories, [1]. In this chapter, basic categorical language was introduced. A category consists of objects and arrows. The relationship between different categories leads to the study of functors. We examined functors such as products, co-products, pullbacks, pushouts, equalizers, coequalizers, limits and colimits. The existence of these is an essential requirement for defining the closure of Cartesian categories. We also introduced adjoints and studied the relationship between the right adjoint and the existence of exponentials. For a category to be Cartesian closed, finite limits and exponentials have to exist. We concluded this chapter by giving examples of Cartesian closed and non-Cartesian closed categories.

Chapter 3

M-R Systems

In this chapter, we shall review Rosen's work on metabolic-repair systems, [44]. This is abbreviated as M-R systems. In his work, Rosen argued that the existence of the metabolism and repair components implies that the replication component also exists. In other words, replication is implicitly defined given metabolism and repair. We shall first study Rosen's discussion on causality and anticipatory behaviour as described in reference [50]. However, we note that Rosen's method of studying M-R processes is limited to a certain class of systems and we discuss some extensions of these techniques, [13]. In Chapter 5, we shall additionally note that Rosen's (and subsequent) arguments are unnecessarily restricted to Cartesian closed categories.

In order to include anticipatory behaviour in living systems, Rosen applied relational biology. We no longer consider the material properties of the cell. Instead we study the different components of the cell, their particular functions, and their contribution to the functioning of the system as a whole, [50].

We realize that Rosen's approach to relational biology only describes the functional characteristics of living systems; the structural characteristics are largely ignored in this framework, [13]. This is alleviated by Mesarovic and

Takahara's [38] systems theory (see Chapter 4) as applied by Wolkenhauer and Hofmeyr (see §5.1 and §5.2). Rosen's argument is also restricted to one-input/one-output systems. We will then introduce the work done by Casti in his paper "Linear Metabolic-Repair Systems," [13]. At first, a simple (one-input/one-output) M-R system is considered and different cases regarding metabolic and repair changes to environmental fluctuations are studied. We describe the mathematical description of metabolism (§3.1) and repair (§3.2) before using Cartesian closure to construct a replication map via the exponential (§3.3). We also discuss Lamarckian inheritance and mutation. This analysis is then generalised to include many inputs/outputs using vectors (see §3.4). The linear analysis can also be extended to non-linear cases provided that the non-linearity is adequately quantified. Metabolic Repair systems may also form networks. The mathematical questions surrounding the behaviour of such networks are studied by Casti, [13]. In §5.1, we summarise Wolkenhauer and Hofmeyr's generalisation (to Mesarovic and Takahara's system theory) of the arguments presented in this chapter.

3.1 Metabolism

Suppose we consider a collection of cells which (collectively) accepts inputs and produces outputs. Suppose also that at least one cell accepts inputs and produces outputs. The inputs can be from the environment or from the output of another cell. Each cell's output is either released into the environment or utilised as an input by another cell. Such a network is known as a metabolic network.

We begin by considering a simple metabolic process with only a single cell. The functioning of a cell is influenced both by external and internal causes. The external cause can also be referred to as the environmental state. Let A represent the set of environmental inputs to the cell. Let B represent the set of the outputs the cell is able to produce. Suppose that $a \in A$ and $b \in B$. The metabolic map is given by:

$$f: \quad A \longrightarrow B$$
$$a \mapsto b$$

The output $b = f(a)$.

Biologically, as in reference [50], we may think of the map f as an abstract enzyme which converts a substrate $a \in A$ into a product $b \in B$. Supposing that the set $F(A, B)$ represents all the realisable metabolisms between A and B, we have that $f \in F(A, B)$. This is the set of all mappings from A to B.

Various constraints determine the set of realisable mechanisms. In most instances, the map f uses classical Newtonian machinery to capture the cell's metabolic activity. In Newtonian mechanics, cellular systems are described in terms of energy and masses with forces acting on them. For a system to be rendered closed in this case, the environmental influences are internalised by adding state variables and more parameters to the system, [50].

There have been certain deficiencies pointed out both by Casti [13] and Rosen [44] that limit the use of the classical Newtonian paradigm in the description of living systems. These limitations include:

- While using the classical Newtonian machinery, anticipatory behaviour is not defined.

  Anticipatory behaviour includes predictions and expectations at a future time that are derived given current knowledge of a system. It is also known as intrinsic control, [50].

- It is impossible to explain biological processes of repair and replication in a natural way.

The Newtonian paradigm therefore does not capture biological, social and behavioural phenomena as well as we would like them to. Similar comments may be made with regard to self-organizational phenomena such as brain activity, [9]. Modern theory typically uses both relational and state space techniques, the former to address the above intuitions and the latter to address, for instance, time evolution, [13, 51].

3.2 Repair Mechanism

In a metabolic network, we have that at least one cell accepts inputs from the environment and produces outputs to the environment. Biologically, a cell stops functioning after some time and can therefore not contribute to the metabolic activity. In certain cases there exists such a cell whose death can cause failure to the entire system. Each cell should therefore be associated with a repair component that produces a new copy of the failed cell. This process is known as the process of repair.

Metabolic activity is influenced both by the environment and the internal functions. We require a process which can be used to stabilise the disturbances caused by the environment and the internal metabolism of a cell. This is the significance of the repair mechanism. The repair process has to receive as its input at least one environmental output, and all its internal inputs. Suppose a repair mechanism receives as its input the output from a failed metabolic activity. Then that repair component cannot be re-introduced back into the network. Just like the metabolic components, the repair components can also die. We would then require another process to create new copies of the repair components. Biologically, we may assume that the repair map represents, for example, a systems' genetic component.

We first introduce the notion of causation. Rosen's arguments, [44], were largely based on efficient causation. For each process we have a cause, and an effect. There are four different kinds of causal entailment. These causes originate from Aristotle's descriptions of causality. Two of these are related to the structural organisation of a system while the other two explain the functional organisation. The different causations are listed below:

- Material cause: This is the structure/matter underlying a certain object.
- Formal cause: This is the form of an object and is subject to alterations.
- Efficient cause: This is the source/agency that brings about change within the state of an object.
- Final cause: This is the purpose for which something exists.

In the metabolic map, given an input a, we end up with $b = f(a)$ as the output. We say that b is the image of a. Alternatively, we say that b is entailed by a. This is equivalent to saying that a and f entails $f(a)$. The question why $f(a)$ is answered by: "because a and because f". Here, a is the material cause while f is the efficient cause.

In order to study efficient causation, the metabolism map is expanded to include another map. We shall now consider the map

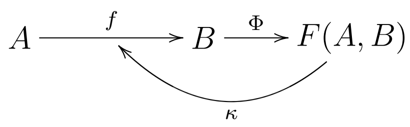
$$A \xrightarrow{f} B \xrightarrow{g} C \ .$$

Here, a entails $f(a)$ and $f(a) = b$. Formally, this is written as
$f \Rightarrow (a \Rightarrow f(a))$. We also have that b entails $g(b)$. This implies that b entails $g(f(a))$ and is written as $g \Rightarrow (b \Rightarrow g(b))$.

For f to have an efficient cause, we let C be the collection of all mappings from A to B. Then $C = F(A, B)$. The map g generates a new f for any $b \in B$. Thus $g(b)$ is in itself a mapping such that g entails f. This implies that $f = g(f(a))$.

The question "why f," is then answered by because b as the material cause, because b as the final cause, and because g as the efficient cause. We now have g as the efficient cause of f. To ensure that g has an efficient cause, we shall need to introduce another map (h, say) and so on. However, we will stop at the map g.

The process of repair becomes:



(3.1)

where Φ is the repair map.

Conclusion:

Suppose $f : A \to B$ denotes the metabolic activity of any cellular process. Given an environmental input $a \in A$,

(1) In the absence of any disturbances, f produces the cellular output $b \in B$. This means that in the case when there is no change either from f itself or from the environment, the repair map Φ produces f. Thus

$$\Phi_f(f(a)) = f. \tag{3.2}$$

If we have an input $a \in A$ that satisfies the condition (3.2), then the cell is stable.

(2) In the case of disturbances either from the external environment or the internal functions, the cell stabilises the fluctuations by producing a new $f^* \neq f$ for any $b \in B$. Here, the cell repairs itself and is stabilised when $\Phi_{f^*}(b) = f^*$.

Due to environmental changes from b to $\bar{b}$, $b \neq \bar{b}$, stability is only achievable either when

$$f(b) = f(\bar{b}) \text{ or } \Phi_f(f(\bar{b})) = f.$$

3.3 Replication

The question of replication is similar to asking ourselves how we would generate an efficient cause for Φ. One way to do this would be to add another map such that

$$A \xrightarrow{f} B \xrightarrow{\Phi} F(A,B) \xrightarrow{\beta} F(B, F(A,B)) \ . \tag{3.3}$$

This method of simply adding a new map to the already existing maps would lead to an infinite chain since the last map is always open to efficient causation.

3.3.1 How To Create A Replication Map

We need to create a replication map from the metabolism and the repair map. Let **C** be a category. Referring to the previous chapter on category theory, we find the definition of a category as a collection of objects and their associated arrows. Let the (A, B, f, Φ) be a simple M-R system on the category **C**.

Suppose that **C** is a concrete category. As previously defined in §2.3, this is a category equipped with a faithful functor to the category **Set** of sets. Such categories enable us to define the objects of a category simply as sets with an additional structure. The arrows of a category are the functions that preserve that given structure. Examples of concrete categories include: **Grp**, **Set** and **Top**.

Suppose also that **C** is closed under Cartesian products such that for $A, B, C, \in Obj(\mathbf{C})$, $f \in F(A, B)$ and $g \in F(A, B)$,

- $A \times C,\ B \times C\ \in \mathbf{C}$.
- $f \times g \in F(A \times C, B \times C)$.

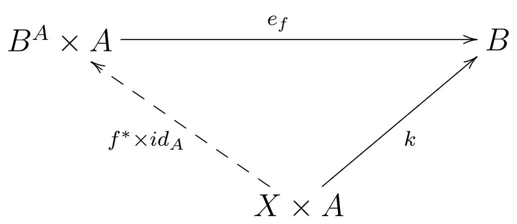
If **C** is closed under Cartesian products, then for $F(A, B) \in \mathbf{C}$, there exists an exponential, which is the collection of all functions from A to B denoted by:

$$B^A = \{f | f : A \to B\}. \tag{3.4}$$

The exponential B^A in Equation (3.4) satisfies $B^A \in \mathbf{C}$. This means that as previously discussed in §2.9, there exists an evaluation map:

$$e_f : B^A \times A \to B. \tag{3.5}$$

For any $k : X \times A \to B$, there exists a unique $f^* : X \to B^A$ such that $k = e_f \bullet (f^* \times id_A)$. This is shown in the triangle below:



This concludes the description of the metabolism process. We next study the evaluation map of the repair process.

Remembering that $F(A, B) = B^A$, the process of repair becomes:

$$A \xrightarrow{f} B \xrightarrow{\Phi} B^A .$$

The map Φ belongs to the collection of mappings from B to B^A.

Defining the evaluation map for Φ therefore leads us to

$$e_\Phi : ((B^A)^B \times B) \to B^A, \tag{3.6}$$

where $(B^A)^B \in \mathbf{C}$. This evaluation map can be expressed as a bijection between functions in two variables and in one variable.

The definition of an evaluation map for Φ implies that for $k : B^A \times B \to B^A$, there exists a unique map $f^* : B^A \to (B^A)^B$ such that $k = e_\Phi \bullet (f^* \times id_B)$.

This process can be graphically seen in Figure (3.7):

$$\begin{array}{ccc} (B^A)^B \times B & \xrightarrow{\quad e_\Phi \quad} & B^A \\ & {\scriptstyle f^* \times id_B} \nwarrow \quad \nearrow {\scriptstyle k} & \\ & B^A \times B & \end{array} \tag{3.7}$$

Our objective was to find a way in which the repair map Φ could be closed to efficient causation. We have achieved this by assuming that the category $\mathbf{C}$ is Cartesian closed and defining the evaluation maps both for the metabolism and repair maps following Rosen's set-up as reviewed by Wolkenhauer in reference [50].

Conclusion:

- For the repair map Φ to entail $f : A \to B$, the exponential B^A should exist. Furthermore, having defined these objects in the Cartesian closed category $\mathbf{C}$, we must have $B^A \in \mathbf{C}$.
- For the repair map Φ to be entailed by something, the exponential $(B^A)^B$ must exist in the Cartesian closed category.
- The efficient closure of the repair map using its evaluation map e_Φ leads us to the existence of a unique replication map $f^* : B^A \to (B^A)^B$.

Therefore, given a metabolic function $f : A \to B$ and a repair map
$\Phi : B \to F(A, B)$, replication is implicitly defined. This is made possible by

considering a Cartesian closed category and studying the evaluation maps for metabolism and repair. This leads to the existence of an exponential which in categorical terms implies the existence of a right adjoint. Note that in the situation when there are no changes in f itself or from the environment, $f^*(f) = \Phi(f)$.

This concludes our discussion of Rosen's method of creating a replication map while avoiding infinite regress. We now turn to the questions of Lamarckian inheritance and mutation.

3.3.2 Comments

The concept of replication raises two further questions of interest. These two were described by John Casti in his study of Linear M-R systems, [13]. They are:

1. Lamarckian changes.

   This addresses the possibility of environmental changes leading to changes in the replication map. Consider the collection of cellular processes from Equation (3.3). Stability in the repair and replication maps is achievable when

   $$\Phi_f(f(a)) = f = [\beta_f(f)(f(a))].$$

   Suppose we have an environmental change from a to $a^{'}$, then the output changes from $f(a) = b$ to $f(a^{'}) = b^{'}$. Assuming that the repair mechanism can correct the disturbances caused by the change in environment such that $\Phi_f(b) = \Phi_f(b^{'}) = f$, then $(\beta_f \bullet \Phi_f) \bullet (b) = (\beta_f \bullet \Phi_f)(b^{'}) = \Phi_f$. Thus replication remains unaffected by the changes in environment. From the above formulation, Casti [13] concluded that whenever we have fluctuations in the environment that can be corrected through the repair mechanism, it is impossible for Lamarckian changes to occur.

However, Rosen's formulation of M-R systems provided a stronger conclusion. Rosen argued that no environmental change of any kind can lead to Lamarckian changes in the replication map. The stability condition for the replication map suggests that $\beta_f(f) = \Phi_f$. If we let A and B be arbitrary sets, there exists a map $\hat{e} : B^A \to (B^A)^B \times B$. If one assumes that $\hat{e}$ is one to one, then there exists a map $e : (B^A)^B \times B \to B^A$. This is the evaluation map as discussed in Equation (3.6).

Since $B^A = F(A, B)$, the evaluation map gives us the replication map.

Rosen's argument therefore heavily relies on the fact that the map $\hat{e}$ should be one to one. This, in Casti's explanation, [13], translates to the restriction that there exist only a single input and a single output. It is this severe restriction that led him to the idea of linear metabolic repair systems, a formalization which works for all finitely realisable metabolisms (see §3.4).

2. Mutation.

   Suppose external disturbances can modify the replication map. The process in which such changes in the replication map result in changes in the metabolic map is known as mutation. Mutation is the change in the genetic material of an organism. Casti's aim is to study the kind of changes in the metabolic map which can result in this situation, [13]. In other words, given the metabolic map f, what are the circumstances which can cause a change in the replication map such that we have $\beta_f \to \beta_{f'}$?

   This proves to be an enormous task since we have that the replication map β_f does not act directly upon the metabolism map f. The replication map only acts directly on the repair map Φ_f.

   In nature, it is usually assumed that mutations arise from random events acting upon a system from outside, [13]. The M-R system set up does not however allow for this phenomenon. In the case of metabolic-repair systems, we therefore need to consider directed mutations arising from within the cell itself, or imposed on the system by an outside controller, [13].

If the mutations arise from within the cell, we are faced with the problem of incorporating the relevant feedback loops into mathematical representation, [13]. On the other hand, mutations arising due to an outside controller are equivalent to naturally induced mutations which aim to determine direct paths between the replication map and the metabolic map.

In order to overcome the problem of an infinite regress, Rosen used the concepts of category theory, [43]. Category theory provides a conceptual framework rich enough in entailment [50] to allow us achieve closure to efficient causation without running into the infinite chain problems encountered when the method of invoking additional maps is used.

3.4 Linear M-R Systems

This section is largely based on the work by Casti in his paper "Linear Metabolism-Repair Systems," [13]. In his attempt to address the limitations of Rosen's framework, he introduces a construction that works for all finitely realizable metabolisms. He goes on to construct a canonical state space for a linear dynamical system. Various application areas of M-R systems, together with their extension to complex systems, are also discussed.

In the map $f : A \to B$, we consider $a \in A$ as an input time series leading to an output $b \in B$ [50]. The elements of A are sequences in $\mathbb{R}^m$ while those of B are sequences of vectors in $\mathbb{R}^p$. Elements $a \in A$ and $b \in B$ are thus of the form

$$
\begin{aligned}
a &= \{a_0, a_1, a_2, \ldots\} \ \ a_i \in \mathbb{R}^m, \\
b &= \{b_1, b_2, b_3, \ldots\} \ \ b_i \in \mathbb{R}^p.
\end{aligned}
$$

We notice that the output begins one time step after the first input. This is to ensure that the idea of causation is satisfied: effects have to follow causes. The time is also taken as discrete. Suppose that the metabolic map f is linear, then there exists a sequence of matrices $B = \{A_1, A_2, A_3, \ldots\}, \quad A_i \in \mathbb{R}^{p\times m}$ known as the behaviour sequence such that

$$
b_t = \sum_{i=0}^{t-1} A_{t-i} a_i, \quad t = 1, 2, \ldots
$$

The relation b_t is written in form of the Toepliz matrix F [18] as:

$$
\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} A_1 & 0 & 0 & \cdots \\ A_2 & A_1 & 0 & \cdots \\ A_3 & A_2 & A_1 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \end{bmatrix}.
$$

The behaviour sequence takes the block Hankel form, [29]

$$
H = \begin{bmatrix} A_1 & A_2 & A_3 & \cdots \\ A_2 & A_3 & A_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}.
$$

Suppose F, G, H are matrices such that $F \in \mathbb{R}^{n\times n}, \ G \in \mathbb{R}^{n\times m}, \ H \in \mathbb{R}^{p\times n}$, then the system (F, G, H) can be expressed dynamically as:

$$
\begin{aligned}
x_{t+1} &= F x_t + G a_t \ ; \ x_0 = 0, \ x_t \in \mathbb{R}^n \\
b_t &= H x_t \ ; \ t = 0, 1, 2 \ldots
\end{aligned}
$$

where n is an integer. The precise meanings of X, F and G are given below, for the finite dimensional case.

The realization problem of Casti [13] is formulated in a way such that the behaviour of the system (F, G, H) is in agreement with that of the behaviour sequence. This is because the aim is to come up with the best representation that captures the properties of a system. Casti also ensures that the system (F, G, H) is the simplest possible linear system whose properties are in accordance with the behaviour sequence. In the paper [13], the integer n is assumed to be finite and known; hence invoking the use of Ho's algorithm [54] in the construction of the system (F, G, H).

There are several steps involved in the construction of a state space using Ho's algorithm, [54]. These include:

- Consider the behaviour sequence H in its Hankel array form, [29].
- Find non-singular matrices P and Q such that $PHQ = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} = N$ where N is the normal form of the behaviour sequence H and n is the rank of H.
- Partition P and Q into $P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$ and $Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}$ where P has n rows and Q has n columns.
- Calculate the following matrices

$$D = P \begin{bmatrix} H_2 & H_3 & \cdots \\ H_3 & \cdots & \cdots \\ \vdots & \vdots & \vdots \end{bmatrix} Q \ ; \quad E = P \begin{bmatrix} H_1 \\ H_2 \\ \vdots \end{bmatrix} \ ; \ V = \begin{bmatrix} H_1 & H_2 & \cdots \end{bmatrix}.$$

- Let D denote the infinite array obtained by shifting each row of H to the left.
- Let R_n, R_p, C^m and C^n be the matrices obtained by retaining the first n rows of H, retaining the first p rows of H, retaining the first m rows of H, and retaining the first n columns of H, respectively.

- Letting (F, G, H) be given by,
  $(R_n DC^n,\ R_n EC^m,\ R_p VC^n)$, gives a canonical minimal realization of the behaviour sequence H, [13].

Similar constructions are applied to the repair and replication maps. These maps are assumed linear and the inputs and outputs are once again written as sequences of vectors. Casti showed that, having assumed a finite dimensional realization for the metabolic component, we also end up with finite dimensional repair replication components, [13]. If complexity is measured using state space, these dimensions for the repair and replication are smaller than those of the metabolic component. The canonical realizations for metabolism, repair and replication can all be computed using Ho's algorithm.

Casti [13] goes on to study how the complexity of a system increases as the metabolism component gains additional structure. He shows that as long as the dimension represented system is finite, the repair mechanism is always at least as complex as the metabolic process. This claim appears to contradict the earlier statement that the dimensions of the repair and replication systems are smaller than that of metabolism. However, in the latter case, a different notion of complexity involving the objects of the behavioural sequence of metabolism, and the linear maps of the repair network, is applied. Unless $p = 1$, the collection of repair maps always contains more objects than the behaviour sequence of metabolism, [13]. By this measure, the repair system is always at least as complex as the metabolism.

Casti's construction works for all finitely realizable metabolisms, unlike Rosen's introduction of the evaluation map which is restricted to a one-input/one-output type of system. Several questions arise from Casti's formulation of a simple M-R system. The question as to whether environmental fluctuations of a system can lead to changes in the replication map is linked to the

Lamarckian inheritance. Rosen reasoned that such changes are not possible under any circumstances. However, his argument required that the replication map be invertible. Casti's set up of linear M-R systems shows that as long as the repair process is able to rectify the environmental changes, then no Lamarckian changes can occur in the replication mechanism, [13].

Suppose that it is possible for environmental disturbances to change the replication mechanism of an M-R system. How can such changes in the replication map modify the metabolic map? This process corresponds to mutation and is complicated due to the absence of a direct relationship between the replication and the metabolic functioning of a system. Moreover, mutation would require the generation of random environmental fluctuations, a process which for the time being does not fit within our representation of a metabolic-repair system.

In the conclusion of his paper, Casti [13] outlines various application areas of M-R systems. He notes that, besides biological motivations such as genetics and DNA analysis, metabolic-repair systems are highly applicable to technological developments within a network of industrial firms; economic processes; and evolutionary view of social organisations.

Casti focuses on linear metabolism, repair and replication. He observes that Ho's algorithm can be extended to calculate the canonical state representation of a larger class of input/output systems which are non-linear. However, the type of non-linearity, and the extent to which it differs from the linear case, needs to be precisely specified.

3.5 Summary

In this chapter, we have reviewed Rosen's work on metabolism-repair systems. Using his relational language, we see how, given metabolism and repair, the replication of cell function is automatically realised from within the cell. Cartesian closed categories are used to achieve closure to efficient causation. The limitation of Rosen's approach is that it is applicable only when considering a one-input/one-output system. This is amended by Casti's introduction of inputs and outputs as a sequence of vectors, a method which works for all metabolisms provided that they are finitely realizable. We reviewed Casti's representation of M-R systems, applications and extensions of his work to linear M-R systems. In Chapters 4 and 5, we generalise the metabolism-repair system and describe it within the set-theoretic framework of Mesarovic and Takahara [38] by introducing state objects and state space. In Chapter 5, we further extend the categorical context of describing such systems to include monoidal closed and enriched categories.

Chapter 4

Systems Theory

The formalization approach to systems theory as developed by Mesarovic and Takahara in [38] aims to take real life observations and transform them into mathematical representations that exhibit the properties and behaviours of the system. This differs philosophically from other approaches such as using differential equations and automata, whose starting point is not always the real life concepts. In this paradigm, a notion such as that of a dynamical system is not just a mathematical observation, but is supposed to represent, as precisely and economically as possible, a definition of a real life phenomena. While there is necessarily a process of abstraction involved in any modelling, the approach of Mesarovic and Takahara described here has at its heart a degree of phenomenology not always present in other languages or approaches. It is also a sufficiently general language as to include, as special cases, most contemporary modelling languages, [51]. Philosophical issues aside, its generality is appealing for our purposes. In Chapter 5 we discuss results relating to efficient causation, using this language to describe our cellular system (see §5.1).

The procedures of the formalization approach as used in [38] that need to be followed are:

(1) Taking a basic real life concept and expressing it using simple mathematical structure.

(2) Adding simple mathematical structure to condition (1) such that various properties of a system are defined.

(3) Validating of already known results for a particular system. Here, properties such as causality are investigated.

(4) Determination of new results by generalising the results found after following procedures (1) – (3). This leads to a broader formulation of a particular problem.

In Rosen's arguments, we realise that the structure is not included. Furthermore, the elements of the cell are defined as material objects. In order to amend this limitation, those elements are now identified as signals and cellular activity is represented as a dynamical system. The system theoretic set-up of Mesarovic and Takahara in [38] is described in this chapter and applied to a cellular model in §5.1.

We shall follow the formalization approach described above. From the definition of a system as a relation among objects, we obtain the most basic representation of a system (see §4.1). From then on, various mathematical structures are added to the basic model keeping in mind the properties we would like to investigate (see §4.2). We introduce notions of time (§4.2) and state (§4.3 and §4.4), and describe how one constructs a state space (§4.5). These notions are used to define causality, and the causal nature of M-R systems is analysed (see §4.6).

We should remember that the goal of systems theory is not to study the properties of a specific system. The main objective is be to investigate the properties of a class of systems that display similar properties, [38]. We therefore look for a mathematical approach which classifies objects of the same kind and investigates underlying structures. As a consequence of its inherently relational nature, the language of category theory proves to be the most important mathematical approach that we can apply. In Chapter 5 we use categorical arguments to investigate efficient causation in systems described using the machinery of this chapter.

4.1 Formalization of an Abstract System

We shall begin with the first formalization procedure. The real life definition of a system is a relation among objects. For the simplest mathematical representation of a relation, we shall use the set theoretic definition of a system as a proper relation on sets, [38]. (From a categorical perspective this definition restricts our attention to concrete categories.) This is formulated as follows:

A system is a relation $S \subset V_1 \times \cdots \times V_n$ over the sets $V_1, \cdots, V_n$. The components $V_1, \cdots, V_n$ represent the objects of a system. The set V_i is a collection of objects. An object is specifically identified by the given properties which it contains. A given element $v_i \in V_i$ represents one way in which a given property can be identified, [38]. Studying the structure and behaviour of objects within a given system yields the relationship between those objects. We could also identify objects by their specific functions within the system i.e one can build an object by listing elements, or by deciding membership in the object defined by functionality.

Let $\{V_i | i \in I\}$ be a family of sets, with I, an index set. We define the Cartesian product of this family as $\underline{V} = \times\{V_i | i \in I\}$. The Cartesian product of all system objects is referred to as the context of the system. Here, S is a proper subset of $\underline{V}$. When using abstract methods to represent a system, it is required that not all elements of the Cartesian product belong to the system. Otherwise, one cannot identify a system within a context.

We note that the Cartesian product of all system objects is taken to be the context of the system only because the first procedure in the formalization approach requires a minimal mathematical representation, and it is a ubiquitous mathematical construct. One could define the context of the system using other products such as tensor products. This will emerge as a crucial element of our analysis in Chapter 5.

The second procedure in the formalization approach of Mesarovic and Takahara [38] requires us to enrich our basic mathematical description with additional structure. There are several ways to enrich the mathematical representation of a system as a proper relation on sets. These include:

- Adding more structure to the system objects either by introducing mappings within the system as algebras, or by defining elements of the objects as functions.
- Combining systems to form complex systems. objects consist of sets of functions/specified functions.
- Specifying the internal functionality of the system. These are referred to as goal-seeking systems.

4.2 System Structures

Structure refers to the interrelationships between different aspects of a system. The structural description of a system refers to the processes occurring within a system such as the physical, chemical and biochemical aspects, [38]. The functional description, however, refers to the intrinsic roles achieved by different parts of a system. It is in this regard that a system is viewed in different ways:

1. Taking into account the behaviour of a system including its physical

conditions, a system can be said to be a mapping of inputs into outputs. This is referred to as an input/output representation of a system. An input/output system can also be referred to as a terminal system. Since a system is a relation between objects, every system can be considered terminal. (Note that this does not necessarily mean the possession of a "terminal object" in the sense of Definition 2.7.1).

2. Suppose a system contains a certain activity which achieves certain objectives such that, given a certain stimuli, it determines an appropriate response, then the representation of the system is referred to as goal-seeking.

   Whenever the functioning of a system is noticeable from outside, it is referred to as a goal-seeking system [38]. A goal-seeking system is therefore an input/output system enriched with more structure, including additional environmental conditions and internal goals.

3. Another possible type of system is known as the complex system. This is a system of systems. The complex system therefore is a universe which contains different subsystems. This enables us to relate the systems themselves to other systems.

   This above structural description leads us to four classes of systems. These include:

- Input/output systems: Here, the system is defined as a function between the influence of the environment on the system (input) and the influence of the system on the environment (output).

- Complex terminal systems: This is a system of input/output systems.

- Goal-seeking systems: This is a system which is described in terms of the objective it aims to achieve and the processes it undergoes towards attaining that goal.

- Complex goal-seeking systems: This is a (possibly goal-seeking) system of goal-seeking systems.

Further knowledge of a system then helps to us determine other properties of that system. We are able to discern whether a system is goal-seeking or complex by investigating the structure of a system, [38].

Referring to Rosen's work [44], the simplest metabolic repair system can be classified as terminal. We now study the M-R systems while adding the concepts of time and state. The addition of state and time using the systems theoretic framework enables us to look into causation as a relation between the changes of states of things, [51]. By doing so, one overcomes Rosen's limitation of having considered the objects of a system merely as material things.

4.3 Introducing Time Into Terminal Systems

In §4.1, we studied ways in which the second procedure in the formalization approach can be achieved. We are required to enrich the notion of a system as a proper relation on sets with additional mathematical structure. Choosing the simplest mathematical structure, we shall enrich the proper relation on sets by defining the elements of objects $v_i \in V_i$ as functions in themselves. This leads us to the notion of a time system.

We consider the fact that the first output is realised one time step after the first input. The system context $\underline{V}$ is now partitioned into two classes, namely:

- Objects representing the influence of the environment on the system. These are called inputs or stimuli and are given by

$$\underline{\Omega} = \{V_1, \cdots V_m\}. \quad (4.1)$$

- Objects representing the influence of the system on the environment. These are called outputs or responses and are represented by

$$\underline{\Gamma} = \{V_{m+1}, \cdots V_n\}. \tag{4.2}$$

We then have the following definition:

Definition 4.3.1. Given a system $S \subset \times\{V_i | i \in I\}$. Let $I_x \subset I$ and $I_y \subset I$ be a partition of I such that $I_x \cap I_y = \emptyset$ and $I_x \cup I_y = I$.

The set $\Omega = \times\{V_i | i \in I_x\}$ is called the input object, and $\Gamma = \times\{V_i | i \in I_y\}$ is known as the output object.

The input/output system is thus given by $S \subseteq \Omega \times \Gamma$. The relation $S \subseteq \Omega \times \Gamma$ is called a binary relation over Ω and Γ.

We define $\underline{\mathrm{D}}(\mathrm{S}) = \{\omega | (\exists y)((\omega, y) \in S)\}$ as the domain of the relation, while $\underline{\mathrm{R}}(\mathrm{S}) = \{y | (\exists \omega)((\omega, y) \in S)\}$ is the range of the relation.

In §4.1, we found that the three ways in which additional mathematical structures can be included are not mutually exclusive. One could combine two or more of the conditions depending on the properties of the system that we are interested in. In our case, our main objective is to achieve closure to efficient causation. The example of the M-R system as studied in [51] and [50] is geared towards proving that, given the metabolism and repair of cell function, organization/replication is realised automatically from within the cell. Our interest therefore lies in the internal functioning of a system.

In order to represent the internal structure of a system, a time set is introduced in the form of a linearly ordered set. We let the time set I, with minimal element o be given by $I = \{t : t \geq o\}$. Appropriate restrictions are included $\forall\ t,\ t^{'} > t$ as follows:

$$\begin{aligned} I_t &= \{t^{'} | t^{'} \geq t\},\ I^t = \{t^{'} | t^{'} < t\},\ I_{tt^{'}} = \{t^* | t \leq t^* < t^{'}\}, \\ \overline{I}_{tt^{'}} &= I_{tt^{'}} \cup \{t^{'}\},\ \overline{I}^t = I^t \cup \{t\}. \end{aligned}$$

Definition 4.3.2. Suppose U and Y are arbitrary sets, I a time set, U^I and Y^I, the sets of all maps of I into U and Y respectively, $\Omega \subseteq U^I$ and $\Gamma \subseteq Y^I$. An abstract time system S on Ω and Γ is a relation

$$S \subseteq \Omega \times \Gamma.$$

The sets U and Y are called alphabets of the input set Ω and the output set Γ respectively. The maps Ω and Γ are time objects while the elements $\omega : I \to U$ and $y : I \to Y$ are abstract time functions. The values of ω and y are denoted by $\omega(t)$ and $y(t)$ respectively.

Restrictions on $\omega \in \Omega$ are given by:

$$\omega_t = \omega|I_t,\ \omega^t = \omega|I^t,\ \omega_{tt'} = \omega|I_{tt'},\ \overline{\omega}^t = \omega|\overline{I}^t,\ \overline{\omega}_{tt'} = \omega|\overline{I}_{tt'}.$$

Similar restrictions are defined for $y \in \Gamma$.

The concatenation of two signals, denoted $\bullet$ is also defined. Let $\omega \in U^I$ and $\omega^* \in U^I$. Then for any t, we can define another element $\omega' \in U^I$ as

$$\omega'(\tau) = \begin{cases} \omega(\tau) \text{ if } \ \tau < t \\ \\ \omega^*(\tau) \text{ if } \ \tau > t \end{cases}, \tag{4.3}$$

where ω' is represented by $\omega' = \omega^t \bullet \omega^*_t$.

4.4 Introducing State Into Terminal Systems

The internal functioning of a time system can also be described by the notion of a state. We realize that we need to enrich the proper relation on sets such that certain properties of a system are satisfied. One of these properties of interest is causality. The idea of a state aids us in the following ways:

- Enabling the predictability of an output given the input at a certain time.

- The behaviour of a system is not only determined by the stimulus only. The systems past history should also be considered. By using the notion of a state, a system's past history is encapsulated.

Suppose C denotes a state object. If $c \in C$ is the input at time $t \in I$, then the pair (c, t) is known as an event. For a time system to be a function, the initial state and the input have to be known.

Given the state c_t, an output γ_t can be specified. Given also an input $\omega_{tt'}$ which changes the state of the system from a time t to $t^{'}$, a new state at time $t^{'}$ can be specified.

The state enables us to represent a system in terms of functions. We shall study some of the functions that aid us in this type of representation. We begin by defining the following functions:

Definition 4.4.1. Let $S \subseteq \Omega \times \Gamma$ be a time system. Let C_0 be an arbitrary set and $\rho_0 : C_0 \times \Omega \to \Gamma$ be a function. Then if ρ_0 satisfies the relation

$$(\omega, y) \in S \Leftrightarrow (\exists c)(y = \rho_0(c, \omega)),$$

ρ_0 and C_0 are called an initial response function of S, and an initial state object of S, respectively.

Definition 4.4.2. Let S be a time system and $S_t \subset \Omega_t \times \Gamma_t$ be its restriction from t. Let C_t be the state object, while c_t is the state time at t. A response function $\rho_t : C_t \times \Omega_t \to \Gamma_t$ is a function which satisfies

$$(\omega_t, y_t) \in S_t \Rightarrow (\exists c_t)(y_t = \rho_t(c_t, \omega_t)).$$

A family of response functions $\forall t \in I$ for a given system S is given by:

$$\rho = \{\rho_t \mid \rho_t : C_t \times \Omega_t \to \Gamma_t \text{ where } \quad t \in I\}. \tag{4.4}$$

We have seen that given an initial state, a future output can be determined. We have also seen how knowledge about an input which contributes to change of states can determine the future state of a system. We shall now describe this using a dynamical system representation using a pair of families of mappings, namely; the family of response functions and the family of state transition maps.

The family of response functions is given by Equation (4.4). We will now define the family of state transition maps:

Definition 4.4.3. Let $S \subseteq \Omega \times \Gamma$ be a time system. Let C_t be a state object and $\phi_{tt'} : C_t \times \Omega_{tt'} \to C_{t'}$ be a function. The function $\phi_{tt'}$ is known as a state transition function if it satisfies the following three conditions:

(a) The consistency property:
For $\omega_t = \omega_{tt'} \bullet \omega_{t'}$, $\rho_t(c_t,\ \omega_t)|I_{t'} = \rho_{t'}\left(\phi_{tt'}(c_t,\ \omega_{tt'}),\ \omega_{t'}\right)$.

(b) The composition property:
For $\omega_{tt''} = \omega_{tt'} \bullet \omega_{t't''}, \phi_{tt''}(c_t, \omega_{tt''}) = \phi_{t't''}(\phi_{tt'}(c_t,\ \omega_{tt'}), \omega_{t't''})$.

(c) The unit property: $\phi_{tt}(c_t,\ \omega_{tt}) = c_t$.

A family of state transition maps is given by:

$$\phi = \{\phi_{tt'} : C_t \times \Omega_{tt'} \to C_{t'}\}. \tag{4.5}$$

The pair (ρ, ϕ) as defined in Equation (4.4) and Equation (4.5) is known as a pre-dynamical representation of a system.

Suppose that for every $t \in I,\ C_t = C$. Then the pair (ρ, ϕ) is a dynamical system representation of S and C becomes the state space.

4.5 Construction of the State Space

We have now defined a system in terms of a state. Generally, it is assumed that a system does not exist unless a state space is given, [38]. In the formalization approach described in §4.1, we are required to introduce mathematical structures consistent with the properties of a system. So far, we have studied the state objects of dynamical systems and how a system evolves at different states. However, there may be no relation between two different states. This presents us with the following limitations while using only the notion of state time:

- The impact of changing states over time becomes of less importance.
- Since states at different times may not be related, we cannot therefore compare them.

The remedy to these limitations is the natural introduction of a state space. We do so by beginning with a family of state objects. Our main concern now becomes the construction of a state space beginning with the simpler notion of the input/output pairs as given by Mesarovic and Takahara in reference [38].

In order to make sure that there is a relationship between two different states, we assume that two states are equal if they lead to the same future behaviour of a system, [51].

Consider the response function of a dynamical system $\rho_t : C_t \times \Omega_t \to \Gamma_t$.

Let the equivalence relation $E_t \subset C_t \times C_t$ be such that:

$$(c_t, c_{t'}) \in E_t \Leftrightarrow (\forall \omega_t)\ \rho_t(c_t, \omega_t) = \rho_t(c_{t'}, \omega_t).$$

Mathematically, the assumption that two states are equal if they lead to the same future behaviour is translated into the following equivalent conditions, [37]:

- In terms of the state transition map, the state at time t equals the state at time $t^{'}$ if, given an input ω_t, the output at time t equals that at time $t^{'}$. This is given as

$$[c_t] = [c_{t'}] \Rightarrow [\phi_{tt'}(c_t, \omega_{tt'})] = [\phi_{tt'}(c_{t'}, \omega_{tt'})] \text{ for every } \omega_{tt'} \in \Omega_{tt'}.$$

- In terms of the family of response maps,

$$[c_t] = [c_{t'}] \Rightarrow (\forall \omega_t)(\rho_t(c_t, \omega_t) = \rho_t(c_{t'}, \omega_t)).$$

The equivalence between the description in terms of a state transition map and the family of response functions can be proven by showing that the right hand side of the two equations are equivalent. We have:

$$\begin{aligned} & (\forall \omega_t)(\rho_t(c_t, \omega_t) = \rho_t(c_{t'}, \omega_t)), \\ \Leftrightarrow\ & \forall(\omega_{tt'})(\forall \omega_{t'})(\rho_{t'}(\phi_{tt'}(c_t, \omega_{tt'}), \omega_t^{'}) = \rho_{t'}(\phi_{tt'}(c_{t'}, \omega_{tt'}), \omega_{t'}), \\ \Leftrightarrow\ & \forall(\omega_{tt'})[\phi_{tt'}(c_t, \omega_{tt'})] = [\phi_{tt'}(c_{t'}, \omega_{tt'})]. \end{aligned}$$

The state object C_t might contain redundant states, [38]. In order to get rid of these states, a new state set is constructed as a quotient of the equivalence relation. We end up with a new state space $\hat{C}_t = C_t/E_t$.

The reduced state space leads us to a new state transition map

$$\phi_{tt'} : (C_t/E_t) \times \Omega_{tt'} \to (C_{t'}/E_{t'}).$$

The notion of a state brings into perspective the history of a system. Suppose c_0 is the initial state time and $\omega^t = \omega | I^t$ is as given by the restrictions in §4.3. The initial response map $\rho_0 : C_0 \times \Omega \to \Gamma$ lets us study the present and future behaviour of a system. Due to the assumption that any two states are equal if the future behaviour of a system is the same, any two elements of $C_0 \times \Omega^t$ are equal if they both lead to the same future behaviour of a system.

This leads to the Nerode realization [41] which we give as defined in reference [38]. The Nerode equivalence is given by $E^t \subset (C_0 \times \Omega^t) \times (C_0 \times \Omega^t)$ where C_0 is the initial state object. The state transition map $\phi_{tt'}$ and the response function ρ_t are defined as:

$$\begin{aligned} \text{State transition function} &: \quad \phi_{tt'}((c_0, \omega^t), \omega_{tt'}^{'}) = (c_0, \omega^t \bullet \omega_{tt'}^{'}), \\ \text{Response function} &: \quad \rho_t((c_0, \omega^t), \omega^t) = \rho_0(c_0, \omega^t \bullet \omega_{t'})|I_t. \end{aligned}$$

To achieve the minimal mathematical representation for a dynamical system, we must find ways in which we can represent the system so that it has the smallest number of state variables, [38]. Having introduced a class of systems, we establish an equivalence relation $E^t : t \in I$.

The state object is given by $C_t = (C_0 \times \Omega^t)/E^t$.

The response map is then given by $\rho_t : (C_0 \times \Omega^t)/E^t \times \Omega_t \to \Gamma_t$, with $c_t = [c_0, \omega^t]$.

To construct a state space, we select for each $t \in I$ an arbitrary but fixed element $c_t^* \in C_t$.

$$\text{Let } C = \bigcup_{t \in I} C_t.$$

Let the state transition map

$$\phi_{tt'} \quad : \quad C \times \Omega_{tt'} \to C \text{ be such that:}$$

$$\phi_{tt'}(c, \omega_{tt'}) = \begin{cases} \phi'_{tt'}(c, \omega_{tt'}) \text{ if } \ c \in C_t \\ \\ \phi'_{tt'}(c_t^*, \omega_{tt'}) \text{ otherwise} \end{cases},$$

An equivalence relation $E_c \subset C \times C$ is defined and then the quotient set $\hat{C} = C/E_c$ is taken as the state space.

We have managed to introduce one basic attribute that is commonly used to classify different systems. The concept of a state and the state space allows us to study the internal organization of a system. Beginning with a basic input/output system, the future behaviour of a system is captured by introducing a family of response functions and a family of state transition maps. In order to get rid of redundant state objects during the construction of the state space, equivalence relations are introduced. A new state object and state space are then defined as quotient sets of equivalence sets.

4.6 Causality

The theory of causality was first described by Aristotle who defined four different types of causes as discussed earlier in §3.2. Causality is the process by which every event has a set of causes that uniquely determine it. Generally, the effects are meant to occur after the causes. Supposing the causes of a certain event are known, its events can be uniquely determined. Causality need not be necessarily linked to time, [38]. The concept of causality can be introduced in two ways as stated in reference [38]:

(1) Clearly establishing the difference between the inputs and the outputs.

(2) Defining the value of the output such that it depends only on the past and present. It should not depend on the future.

There are several definitions of causality when the output depends solely on the present and past. These include non-anticipation and past-determinacy. Past-determinacy is a special case of non-anticipation. In [38] therefore, non-anticipation is referred to as causality while past-determinacy is strong causality. For the output of a time system S to be uniquely determined by the initial state and past input, the two notions of causality and strong causality are defined as follows:

Definition 4.6.1. Consider an initial response function $\rho_0 : C \times \Omega \rightarrow \Gamma$ for a time system $S \subset \Omega \times \Gamma$, ρ_0 is causal iff

$$(\forall\ c)(\forall\ t)(\forall\ \omega, \omega^{'}) \in \Omega,\ \bar{\omega}^t = \bar{\omega}'^t \Rightarrow \rho_0(c, \omega)|\bar{I}^t = \rho_0(c, \omega^{'})|\bar{I}^t.$$

If $\omega^t = \omega'^t \Rightarrow \rho_0(c, \omega)|\bar{I}^t = \rho_0(c, \omega^{'})|\bar{I}^t$. then ρ_0 is strongly causal.

The above Definition 4.6.1 gives a weaker notion of causality. A stronger notion of causality is defined as follows:

Definition 4.6.2. A time system $S \subset \Omega \times \Gamma$ is past-determined from τ if and only if the following conditions are satisfied.

(1) For any $(\omega, y), (\omega^{'}, y^{'}) \in S$ and for any $t \geq \tau$, $(\omega, y)|I^{\tau} = (\omega^{'}, y^{'})|I^{\tau}$ and $\bar{\omega}^{t} = \bar{\omega}^{'t} \Rightarrow \bar{y}^{t} = \bar{y}^{'t}$.

(2) For any $(\omega^{\tau}, y^{\tau}) \in S^{\tau}$ and for any $\omega^{'}_{\tau}$ there exists $y^{'}_{\tau} \in \Gamma_{\tau}$ such that $(\omega^{\tau} \bullet \omega^{'}_{\tau}, y^{\tau} \bullet y^{'}_{\tau}) \in S$.

In the case when condition (1) is replaced with: if for any $(\omega, y), (\omega^{'}, y^{'}) \in S$ and $t \geq \tau$, $(\omega, y)|I^{\tau} = (\omega^{'}, y^{'})|I^{\tau}$ and $\omega^{t} = \omega^{'t} \Rightarrow \bar{y}^{t} = \bar{y}^{'t}$, S is said to be strongly past-determined.

Example 4.6.3. In §3.4, we discussed Casti's work on the linear representation of M-R systems, [13]. Dynamically, the system was expressed as:

$$\begin{aligned} x_{t+1} &= Fx_t + Ga_t \ ; \ x_0 = 0, \\ b_t &= Hx_t \ ; \ t = 0, 1, 2 \ldots \end{aligned} \tag{4.6}$$

where n is an integer and F, G, H are matrices.

The system (F, G, H) described by Equation (4.6) is strongly past-determined from n.

Proof. Suppose that n is the dimension of x.

Then, $b_k = HF^k x_0 + H \sum_{i=0}^{k-1} F^{k-1-i} G a_i$.

Let $b^{'}_k = HF^k x^{'}_0 + H \sum_{i=0}^{k-1} F^{k-1-i} G a^{'}_i$.

Suppose also that $(a^{\tau}, b^{\tau}) = (a^{'\tau}, b^{'\tau})$ for $\tau = n$, then we have that:
$Hx_0 = Hx^{'}_0, \ HFx_0 = HFx^{'}_0, \ldots, HF^{n-1}x_0 = HF^{n-1}x^{'}_0$.

By the Cayley Hamilton theorem which states that every square matrix satisfies its own characteristic equation [6], $HF^k x_0 = HF^k x^{'}_0$.

Thus for any $t \geq \tau = n$, $a^{t} = a^{'t} \Rightarrow \bar{b}^{t} = \bar{b}^{'t}$, [13]. □

The dynamical system described by Equation (4.6) is therefore strongly past-determined. We bear in mind that the above system in Equation (4.6) is a linear representation of a simple M-R system. This leads us to conclude that a simple M-R system with a finite amount of inputs and outputs displays its causal nature as being strongly past-determined.

4.7 Summary

The Rosen formulation of metabolic-repair systems in Chapter 3 was based on the material organization of cell function. This restriction proves limiting. In order to bring into use the structural organization of cell function, the notion of dynamical systems within the set theoretic framework of Mesarovic and Takahara in reference [38] is introduced. Modelling methods such as differential equations and automata are special cases of the general time system formalism of Mesarovic and Takahara, [51]. It begins by taking real life observations and transforming them into minimal mathematical representations that display equivalent behaviour, [38]. Basic mathematical structure is then enriched to depict the functioning of the natural system. The notion of a time system is introduced since the functioning of a system does not merely depend on the input but also on the systems past history. The internal functionality of a system is captured by the construction of states in a state space. Causality may then be described in terms of relationships between changes of state in a system, [51]. In Chapter 5 we describe our abstract cellular model in terms of the systems theory of this chapter.

Chapter 5

Self-organization and efficient closure

Wolkenhauer and Hofmeyr introduce an abstract cell model that describes the self-organizing principle of cell function (or, equivalently, self-replication) in living systems in the paper [51]. Motivated by Rosen's notion of closure to efficient causation, they argue that the category associated with the self-organization of cell function has to be Cartesian closed. The abstract cell model is defined using state objects. A map of basic cellular processes is defined. (This corresponds to the metabolic map studied in Chapter 3.) Since the characterisation of a living cell depends on what it achieves, a cell function map is introduced. (This corresponds to the repair map in Chapter 3.) Wolkenhauer and Hofmeyr then characterize the functioning of a cell with three dynamic principles: control; regulation; and adaptation, [51]. These three principles correspond to the effects of environmental disturbances on M-R systems studied in Chapter 3, and enable us to relate the basic cellular process and the cell function.

In order for a cell to function as a whole, different cell functions are required. This leads to the introduction of a coordination map which can be compared to the replication map in Chapter 3. Wolkenhauer and Hofmeyr then proceed to use Cartesian closure to show that, given a basic cellular process and a cell function, the coordination map is intrinsically defined. This implies that the coordination of cell function is a self-organizing principle. Closure to efficient causation is once again introduced using the context of a Cartesian closed category. It is argued that, in order to ensure the existence of a self-organized coordination map, the category in which it is defined has to be Cartesian closed. However, several modelling formalisms such as: the category of Abelian groups; the category of topological spaces; the category of posets; and the category of all vector spaces over some fixed field, are generally not Cartesian closed (see Chapter 2). Wolkenhauer and Hofmeyr therefore argue that, because these non-Cartesian closed methods are used in most models and codes, computer simulations based on, for example, differential equations, do not capture essential behaviours of living systems such as autonomous self-organisation.

In §5.1, we describe an abstract cell model using the systems theoretic language of Mesarovic and Takahara [38], described in Chapter 4. Section 5.2 deals with the construction of a coordination map such that it is closed to efficient causation. We briefly outline the argument of Wolkenhauer and Hofmeyr [51], in which Cartesian closure is shown to be sufficient for exponentiation, and hence closure to efficient causation. (This is a generalisation of Rosen's [44] arguments to a very broad setting.) These authors further claim that Cartesian closure is necessary, and, on these grounds, dismiss many contemporary mathematical languages as unsuitable for modelling self-organization/self-replication phenomena. However, we observe that this is not, in fact, the case. Indeed, their argument proves only that exponentiation is sufficient for closure to efficient causation; and Cartesian closure is sufficient for exponentiation. To this end we consider alternative constructions which also yield exponentiation.

In §5.3.1, we extend the discussion to include monoidal closed categories. In Chapter 4, we found that Mesarovic and Takahara's formalisation of a system introduced Cartesian products only for simplicity. One can equivalently use tensor products; this leads to the notion of monoidal closure. Of the modelling approaches listed by Wolkenhauer and Hofmeyr in [51] as not Cartesian closed, some are monoidally closed. Monoidal closure, just like Cartesian closure, involves the existence of an exponential object and an evaluation map. It is therefore an argument that can be used to obtain closure to efficient causation in a wider class of languages. For example, the category **Ab** of Abelian groups and the category $\mathbf{Vect_K}$ of vector spaces over a fixed field K.

In §5.3.2 we discuss categories enriched with additional structure, [9]. Enriched categories have been applied in hierarchical models for cell systems. Brown, Paton and Porter [10] examine how network models for complex systems can be enriched using ordinary categorical notions. In reference [10], enriched categories are applied to models of the internal actions of certain major cell types in the liver. In particular, we observe that any symmetric monoidal category can be embedded in a closed one. For example, the category **Top** of topological spaces may be embedded in the category **Pstop** of pseudotopological spaces.

These results extend the philosophical impact of Wolkenhauer and Hofmeyr's results [51], while providing reassurance for models constructed in many non-Cartesian closed languages.

5.1 An abstract cell model

Representing a terminal system in the form of state spaces, a formal model for the M-R system is proposed by Wolkenhauer and Hofmeyr in reference [51]. This is based on the fact that, given a state c_t at time t, we are able to determine what the new state will be at time $t^{'}$. This new state is given by $\phi_{tt'}(c_t, \omega_{tt'})$. If C denotes the state object, then metabolism can now be represented by the following map:

$$\begin{aligned} \sigma : C_t &\longrightarrow C_{t'} \\ c_t &\mapsto \phi_{tt'}(c_t, \omega_{tt'}) = c_{t'} = \sigma(c_t). \end{aligned}$$

The state c_t at time t entails the state $c_{t'}$ at time $t^{'}$, and the input $\omega_{tt'}$ after the change of state from c_t to $c_{t'}$.

We now require the basic cellular process to be closed to efficient causation. Suppose we have a change of state objects with variation in times
$C_t \to C_{t'} \to C_{t''}$.

Let $H(C_{t'}, C_{t''})$ be the collection of mappings from $C_{t'}$ to $C_{t''}$.

The cell function ψ, which is a way of repairing/stabilizing the basic cellular process, is given by:

$$\begin{aligned} \psi : C_{t'} &\rightarrow H(C_{t'}, C_{t't''}) \\ c_{t'} &\mapsto \psi(c_{t'}) = \phi_{t't''}(c_{t'}, \omega_{t't''}). \end{aligned}$$

Remark 5.1.1. There are several possible cases to consider with respect to stability:

1. Given a state c_t at t and an input ω_t, stability of the cell function for the metabolism σ is given by $\psi_\sigma(c) = \sigma$.

2. Suppose we have a state transition from time t to time $t^{'}$ such that the environment changes from ω_t to ${\omega^*}_{tt'}$. For stability to occur, we need to have that $\sigma(c) = \sigma^*(c^*)$. If $\sigma(c) \neq \sigma^*(c^*)$, then stability is not achieved and we need to move on to the repair mechanism. We have three different instances involving repair:

   – We can achieve stability by letting $\psi_{\sigma^*}(c^*) = \sigma^*$. Here, σ^* duplicates σ and the entire metabolic activity is altered.

- Let $\psi_{\sigma^*}(c^*) = \sigma$, then the metabolic activity undergoes periodic changes in time.
- It is also possible to have the situation where stability of the cell function for metabolism σ^* is given by $\psi_{\sigma^*}(c^*) = \sigma^+ \neq \sigma,\ \sigma^*$. The input/output behaviour changes in a sequence, [13]. This sequence stops either if
  (a) $\exists\ N$ such that $\psi_\sigma(\sigma^N(c^*)) = \sigma^N$. The cell is stable in c^*.
  (b) $\exists\ N$ such that $\psi_\sigma(\sigma^N(c)) = \sigma^{N-k}$; $k = 1, 2, \ldots N-1$. The cells metabolic structure undergoes periodic changes.

In order for a cellular structure to function as a whole, a collection of different cell functions are required, [51]. Denote $H(C_{t'}, C_{t''})$, the collection of all mappings from $C_{t'}$ to $C_{t''}$, by Σ. Our aim is to find a way in which the repair map $\psi : C_{t'} \rightarrow H(C_{t'}, C_{t't''})$ can be closed to efficient causation. Suppose we introduce a cell function map ψ. The map from the collection of all cell function maps Σ to the collection of all maps such as ψ is known as the coordination map. This is given by:

$$\begin{aligned} \Upsilon : \Sigma &\rightarrow H(C_{t'}, \Sigma) \\ \sigma &\mapsto \psi. \end{aligned}$$

This is equivalent to the replication principle according to Rosen as discussed in §3.3.

As is the case for the cell function map, we have several different cases for stability:

1. Given an input ω, stability occurs when $\Upsilon_\sigma(\sigma) = \psi_\sigma$.
2. Suppose we have a state transition from time t to time $t^{'}$ such that the metabolic cellular process shifts from ω to ω^*, then stability occurs if $\psi_\omega = \psi^*_\omega$. If this is not the case, then we require to apply the coordination principle.

If $\psi_\omega \neq \psi_\omega^*$, then we have the three cases that can lead us back to a stable process. These are:

- $\Upsilon_{\sigma^*}(\sigma^*) = \psi_\sigma$. The cell stabilises itself by going back to its original basic process σ.
- $\Upsilon_{\sigma^*}(\sigma^*) = \psi_{\sigma^*}$. Here σ^* duplicates σ.
- $\Upsilon_{\sigma^*}(\sigma^*) = \psi_{\sigma^+},\ \sigma^+ \neq \sigma, \sigma^*$. This leads to a sequence of processes.

5.2 How To Create A Coordination Map

In this section, our objective is to create a coordination map such that it is closed to efficient causation. The method in which another map is added to the already existing ones would lead to an infinite regress. We will therefore apply categorical arguments to ensure that the coordination map is closed to efficient causation. This will lead us to show that given a basic cellular process and a cell function, the coordination of the cell is automatically achievable from within the cell as concluded in references [50] and [51]. These processes can be summarised by the mapping:

$$C_t \xrightarrow{\sigma} C_{t'} \xrightarrow{\psi} \Sigma \xrightarrow{\Upsilon} H(C_{t'}, \Sigma)\ . \tag{5.1}$$

Closure to efficient causation means that the codomain of any cellular process should also act as the domain of another cellular process, [51]. The overall functioning of a cell lies not in the way in which one particular cell functions but in the general functioning of different classes of cells. We therefore use the language of category theory. We consider the class of cells which realize a particular function. This will form a concrete category known as **Cell**, [51]. Using the definition of a category as studied in Chapter 2, we have that **Cell** may be defined as containing the following:

1. Objects such as $c_t \in C_t,\ c_{t'} \in C_{t'}$ where C_t and $C_{t'}$ are both collections

of objects. Here, c_t is the state at time t, while C_t is the state object at t.

2. A collection of arrows, an example of which is $\sigma : C_t \to C_{t'}$, such that for each arrow, say σ, there exist objects $dom(\sigma)$ and $cod(\sigma)$ known as the domain and the codomain of σ respectively. In the case of the mapping σ, we have $dom(\sigma) = C_t$ and $cod(\sigma) = C_{t'}$.

3. Given arrows $\sigma : C_t \to C_{t'},\ \psi : C_{t'} \to \Sigma$, there exists an arrow $\psi \bullet \sigma : C_t \to \Sigma$, known as the composition of ψ and σ.

4. For each state object C_t there exists an arrow $I_{C_t} : C_t \to C_t$ which is the identity arrow.

   These properties should satisfy the associativity principle:

$$\Upsilon \bullet (\psi \bullet \sigma) = (\Upsilon \bullet \psi) \bullet \sigma,$$

   for all the maps as given.

   The unit principle $\sigma \bullet I_{C_t} = \sigma = I_{C_{t'}}\ \forall\ \sigma : C_t \to C_{t'}$ should also be satisfied.

In order to achieve closure to efficient causation, **Cell** may be taken as Cartesian closed. This implies that **Cell** has a terminal object, products and exponentials as studied in §2.9. According to Proposition 2.9.2 for the existence of exponentials we have:

Definition 5.2.1. For **Cell**, a category and $C_t, C_{t'} \in$ **Cell**, there exists an object ${C_{t'}}^{C_t} \in$ **Cell** and an arrow $\varepsilon : {C_{t'}}^{C_t} \times C_t \to C_{t'}$ such that for any $X \in$ **Cell**, $\tilde{\sigma} : X \times C_t \to C_{t'}$, there exists a unique $\hat{\sigma} : X \to {C_{t'}}^{C_t}$ such that $\varepsilon \bullet (\hat{\sigma} \times id_{C_t}) = \tilde{\sigma}$.

This definition is summarised below in Figure (5.2) and is similar to Figure (2.4):

$$\begin{array}{ccc} {C_{t'}}^{C_t} \times C_t & \xrightarrow{\quad \varepsilon \quad} & C_{t'} \\ & {\scriptstyle \hat{\sigma} \times id_{C_t}} \nwarrow \quad \nearrow {\scriptstyle \tilde{\sigma}} & \\ & X \times C_t & \end{array} \tag{5.2}$$

5.2.1 Basic Definitions

Similarity between systems can be captured in various ways. Similarity in structure is investigated by defining different important modelling relations relating different systems.

We shall use some of the following definitions, (see Mesarovic and Takahara, [38], and Steve Awodey, [4]).

Given $h : F \longrightarrow G$, we say that

- h is injective, if $h(f_1) = h(f_2)$ implies $f_1 = f_2 \ \forall \ f_1, f_2 \in F$.

  Here, $h(F) = G$.

- h is surjective, if $\forall \ g_1 \in G$ there exists some $f_1 \in F$ with $h(f_1) = g_1$. This means that $h(F) \subseteq G$.

- h is invertible, if there exists $k : G \to F$ such that $k \bullet h : F \to F$ is an identity. If $k \bullet h : F \to F = \ \mathrm{Id}_F$, then h is a right inverse. This is also known as a section.

- h is a left inverse, if there exists $k : G \to F$ such that $h \bullet k : G \to G = \mathrm{Id}_G$. This is also known as a retraction.

- h is an epimorphism if for any H and $r : G \to H, \ s : G \to H, \ r \bullet h = s \bullet h$ implies $r = s$. This is shown below.

$$F \xrightarrow{\ h \ } G \underset{s}{\overset{r}{\rightrightarrows}} H$$

- h is a monomorphism if for any H and (r, s), $r : G \to F$, $s : G \to F$, $h \bullet r = h \bullet s$ implies $r = s$. This is shown below.

$$H \underset{s}{\overset{r}{\rightrightarrows}} F \xrightarrow{h} G$$

5.2.2 The Self-Organizing Coordination Principle

We have given the basic definitions required to highlight the properties of a system. Our goal is to show that given metabolism and repair/stability, replication/organization is autonomously realised from within the cell. Wolkenhauer and Hofmeyr in [51] devised a proposition within Cartesian closed categories detailing the conditions required in order for a living cell to be closed to efficient causation. This proposition provides a sufficient condition for a living system to be closed under efficient causation having been described within the context of a Cartesian closed category.

Proposition 5.2.2. *For a Cartesian closed category, in order to ensure that a living cell is closed to efficient causation, it is sufficient that the map $\hat{\sigma}$, of basic cellular processes in the exponential object ${C_{t'}}^{C_t}$, has a retraction.*

Proof. Consider the basic cellular map $\hat{\sigma} : X \to {C_{t'}}^{C_t}$ and let ${C_{t'}}^{C_t} = \Sigma$.

Suppose $\hat{\sigma}$ has at least one retraction, then from the basic definitions in Subsection 5.2.1, there exists $\sigma^* : \Sigma \to X$ such that $\sigma^* \bullet \hat{\sigma} = \mathrm{id}_X$.

If $\hat{\sigma}$ has at least one retraction, then for all σ^*, and $\hat{\sigma} = (\hat{\sigma_1},\ \hat{\sigma_2})$,
$\sigma^* \bullet \hat{\sigma_1} = \sigma^* \bullet \hat{\sigma_2}$. This implies that $\hat{\sigma_1} = \hat{\sigma_2}$ and hence the basic cellular map is therefore injective.

Suppose $\sigma^* = (\sigma_1^*, \sigma_2^*)$ and the basic cellular map $\hat{\sigma} = (\sigma_1 \hat{,} \hat{\sigma_2})$ is injective such that $\sigma_1^* \bullet \hat{\sigma_1} = \sigma_2^* \bullet \hat{\sigma_2}$.

For any map $\hat{\sigma}^{-1} = (\hat{\sigma}_1^{-1}, \hat{\sigma}_2^{-1}) : \Sigma \to X$ we have $(\sigma_1^* \bullet \hat{\sigma_1}) \bullet \hat{\sigma}^{-1} = (\sigma_2^* \bullet \hat{\sigma_2}) \bullet \hat{\sigma}^{-1}$.

Applying the laws of associativity gives us: $\sigma_1^* \bullet (\hat{\sigma_1} \bullet \hat{\sigma}^{-1}) = \sigma_2^* \bullet (\hat{\sigma_2} \bullet \hat{\sigma}^{-1})$.

This implies that $\sigma_1^* = \sigma_2^*$ and thus the basic cellular map is a monomapping according to the basic definitions in Subsection 5.2.1.

The map σ^* is known as a retraction for the basic cellular map $\hat{\sigma}$.

The definition of an exponential (in Proposition 5.2.1 and in §2.9) states that X can be any object. Let $X = H(C_t^{'}, \Sigma)$ and suppose that there exists a retraction $\sigma^* : \Sigma \to X$. The map given by the retraction σ^* is exactly the coordination map $\Upsilon : \Sigma \to H(C_t^{'}, \Sigma)$ as in Equation (5.1).

According to the universal mapping property of exponentials [4], there exists an isomorphism of hom-sets

$$H(X \times C_t, C_{t'}) \cong H(X, \Sigma).$$

This can be viewed as a bijective correspondence between functions of the form $X \times C_t \to C_{t'}$ and those of the form $X \to \Sigma$ and can be written as

$$\frac{X \times C_t \to C_{t'}}{X \to \Sigma}$$

The bijection is mediated by an evaluation map $\varepsilon : C_{t'}^{C_t} \times C_t \to C_{t'}$, [4].

The object $C_{t'}^{C_t}$ is known as the exponential, [51]. □

Remark 5.2.3.

- Given the basic cellular process map $\hat{\sigma}$, the presence of a retraction ensures that the map $\hat{\sigma}$ is injective. This implies that every cell function ψ is associated with at least one basic cellular process, [51].

- Since the basic cellular map $\hat{\sigma}$ is injective, the retraction σ^* is surjective. Taking X to be any object, the retraction σ^* becomes the coordination map. The surjective nature of the coordination map implies that every cell function is entailed by at least one cellular process, [51].

The conclusion relies heavily on the assumption that there exists a retraction of the map $\hat{\sigma}$. In order for a retraction to exist, the map $\hat{\sigma}$ should be one-to-one. During our study of Rosen's work in §3.3.1, we ran into this similar requirement of injectivity that limits the system of study to a one-input/one-output type of system. However, Casti [13] amends this limitation by describing an M-R system using a sequence of vectors, a method which works for all finitely realizable metabolisms, including the case at hand.

Thus, Wolkenhauer and Hofmeyr in [51] have shown that, given the existence of a retraction σ^*, the coordination of cell function Υ is realised autonomously from within the cell in the Cartesian closed category framework. This is amenable to extensions like those of Casti [13] studied in Chapter 3.

Our major concern is that closure to efficient causation is assumed to be based on the existence of a Cartesian closed category. In Chapter 2, we gave examples of Cartesian and non-Cartesian closed categories. The category **Set** of small sets and the category **Cat** of all small categories are Cartesian closed. However, we found that the categories **Ab** of all small Abelian groups, $\mathbf{Vect_K}$ of all vector spaces over a fixed field K, and the category **Top** of all small topological groups are not Cartesian closed. The category **Pos** of posets is also not Cartesian closed unless the function space is ordered pointwise. These categories, though not Cartesian closed, are widely used in mathematical modelling. Their importance, and the significance of the results achieved while using such modelling techniques, is therefore brought into question since they appear unable to capture the autonomous self-organization of cell function. Should we then dismiss such techniques of mathematical modelling as mimicking the behaviour of living systems rather than representing the real life situations of living systems? We note that Proposition 5.2.2 uses Cartesian closure only as a sufficient condition, and that no proof that it is necessary is supplied. In fact, given Casti's extension to many-input/many-output systems, the real meaning of Proposition 5.2.2

is that exponentiation is sufficient to obtain a Rosen-style argument for efficient causation. In the next section, we therefore consider other possible constructions which yield exponentials and evaluation maps.

5.3 Monoidal and Enriched Categories

Closure to efficient causation has been described using the context of Cartesian closure. However, replacing the usual Cartesian product in categories with the tensor product leads to the study of monoidal categories. These too, may be closed, posses evaluation maps and exponential objects, and be used as a requirement for efficient closure. Adding structure to pre-existing categories leads to the notion of enriched categories. Using these notions we observe that any symmetric monoidal category (e.g. **Top**) may be embedded in a closed one. This additional structure thus enables us to apply to biological systems, those modelling formalisms, which would otherwise be considered neither Cartesian nor monoidally closed.

5.3.1 Monoidal Categories

In Chapter 2, we introduced various axioms in category theory. Any such interpretation that consists of objects and arrows under the unity, composition and associativity restrictions was then defined as a category. Chapter 2 provided an extensive discussion of Cartesian categories. This was due to the fact that both Robert Rosen [44] and Olaf Wolkenhauer [51] describe the achievement of efficient causation in the context of a Cartesian closed category. However, most conventional modelling methods in mathematics such as topological spaces, vector spaces and Abelian groups are not Cartesian closed. In Mesarovic and Takahara's modelling approach [38], the use of Cartesian products is merely used to achieve the simplest mathematical structure. It may be replaced by the tensor product, [24]. We therefore examine the use of tensor products instead of the Cartesian product. This eventually leads to the notion of monoidal closed categories which are also be a condition sufficient for exponentiation, and hence efficient causation in the sense of Chapter 3. We also aim to show that some of these non-Cartesian closed conventional modelling methods are monoidal closed, thereby extending the scope of the arguments presented in §5.1 and §5.2, as well as providing reassurance that these languages are appropriate to the discussion of self-organization/self-replication.

This description of monoidal categories is largely based on G. M. Kelly's work in reference [24]. Some essential definitions are listed below:

Definition 5.3.1. A category $\mathbf{V}$ is said to be monoidal if it consists of:

- A category $\mathbf{V}_0 \in \mathbf{V}$, an object I of $\mathbf{V}_0$, an arrow $\otimes : \mathbf{V}_0 \times \mathbf{V}_0 \to \mathbf{V}_0$.
- Natural isomorphisms $a_{XYZ} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$,
  $l_X : I \otimes X \to X$, $r_X : X \otimes I \to X$.

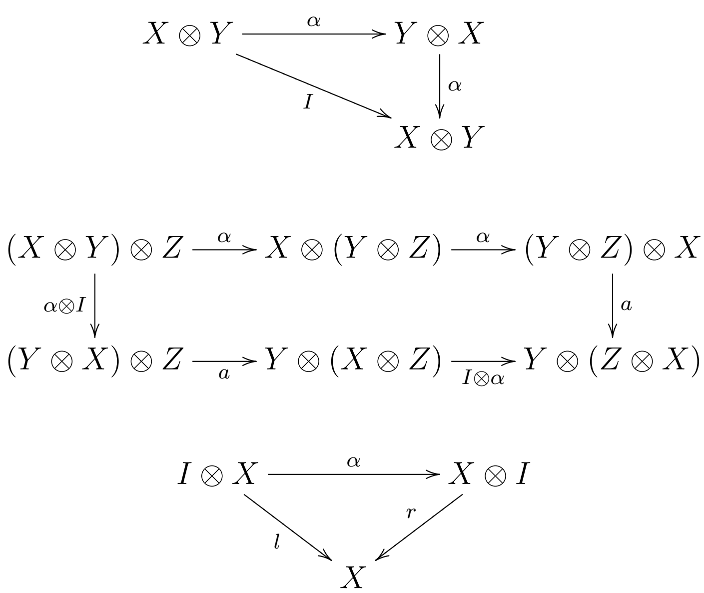
Just as in the definition of a category (see Chapter 2), the above conditions are subject to the identity and associativity axioms. In monoidal categories, the functor $\otimes$ is the tensor product, while the object I is the identity.

By replacing the tensor product with the usual Cartesian product, we end up with a Cartesian category. These two notions can also be combined to form a Cartesian monoidal category. This is achieved by letting $\mathbf{V}_0$ be any category with finite products, replacing the tensor product with the usual Cartesian product, and taking the identity object I be the terminal object. Examples of such categories as given in [24] include: the category **Set** of all small sets; the category **Cat** of all small categories; the category **Top** of topological spaces; the category **Grpd** of groupoids; the category **Ord** of ordered sets; the category **CGTop** of compactly generated topological spaces; the category

HCGTop of Hausdorff compactly generated topological spaces; the category **QTop** of quasitopological spaces; and the category **Shv(S)** of sheaves for a given site S. (Note that a site is a category endowed with a (Grothendieck) topology, [33]).

Objects of one monoidal category can be mapped to the objects of another monoidal category to form functor categories. The descriptions of functors and natural transformations follow as in Chapter 2. However, the Cartesian product is replaced by the tensor product.

Definition 5.3.2. A monoidal category **V** is said to be symmetric if there exists a natural isomorphism $\alpha_{XY} : X \otimes Y \to Y \otimes X$ that satisfies the coherence axioms (unit, associativity, commutativity) such that the following diagrams commute:



Symmetric monoidal categories are essential in our study of enriched categories which shall be discussed in the next subsection.

In Chapter 2, we defined adjunctions in terms of Cartesian closure. Definition 2.8.1 brings into use two natural transformations. These are known as the unit and counit of an adjunction. In the same way, adjunctions may be defined when dealing with a monoidal category. This leads us to a condition necessary for monoidal categories with the additional structure known as closure, [24].

Remark 5.3.3. The monoidal category $\mathbf{V}$ is said to be closed if each functor $- \otimes Y : \mathbf{V}_0 \to \mathbf{V}_0$ has a right adjoint $[Y, -]$. In terms of hom-sets, the adjunction is given by a bijection

$$\pi : \mathbf{V}_0(X \otimes Y, Z) \cong \mathbf{V}_0(X, [Y, Z]) \tag{5.3}$$

This bijection is natural in X and Y. The notation $[Y, Z]$ can also be written as $Y \to Z$.

The two natural transformations known as the unit and counit are given by $\eta : X \to [Y, X \otimes Y]$ and $\epsilon : [Y, Z] \otimes Y \to Z$ respectively.

Similarly to Definition 2.8.1, adjunctions can be constructed whenever we have a universal arrow from or to every object of a given category. The presence of a universal arrow gives rise to the functor $\mathbf{V}_0 \otimes \mathbf{V}_0 \to \mathbf{V}_0$ which is a right adjoint to the diagonal functor.

Any category $\mathbf{A}$ possesses an external hom-functor $\mathrm{Hom}_{\mathbf{A}} : \mathbf{A}^{op} \otimes \mathbf{A} \to \mathbf{Set}$, [46]. If $\mathbf{A} = \mathbf{Set}$, this functor is internal.

The existence of a right adjoint is equivalent to the existence of exponentials. The exponential is defined similarly to Definition 2.9.1. In [24], Kelly states that in the sense of the commutativity axiom in Figure (5.5) in the next subsection, all the given examples of Cartesian monoidal categories (in the list preceding Definition 5.3.1) are closed except the category **Top** of topological spaces.

Examples

- The category **Ab** of Abelian groups is monoidal closed.

  Consider the internal hom-functor $\mathbf{Ab}^{\mathrm{op}} \otimes \mathbf{Ab} \to \mathbf{Ab}$. The set $\mathbf{Ab}(X, Y)$ is given by $\mathbf{Ab}(X, Y) : X \to Y$. This set, taken as the exponential, has the structure of an Abelian group. Considering the category of Abelian groups as the category of $\mathbb{Z} - \mathbb{Z}$ bimodules $X \to Y$ satisfying the condition $zfz^{'}(x) = f(xz)z^{'}$, ensures that each functor has a right adjoint, [46]. Hence the category **Ab** is monoidal closed.

- The category $\mathbf{Vect_K}$ of vector spaces over a fixed field K is monoidal closed.

  The proof that the category $\mathbf{Vect_K}$ is monoidal closed is similar to that of **Ab**.

- The category **Cat** of all small categories is monoidal closed.

  Let **A** and **B** be two categories in **Cat**. The exponential $\mathbf{B^A}$ is given by $\text{Hom}(\mathbf{A}, \mathbf{B})$. This is the category whose objects are all functors from **A** to **B**, and whose arrows are the natural transformations between such functors. This forms the functor category. The categories **A** and **B** are small; hence the functor category $\mathbf{B^A}$ is also small. The category formed by the exponential of two small categories therefore also belongs to the category **Cat** of all small categories. The exponential asserts the presence of right adjoints; hence **Cat** is monoidal closed.

- The category **Set** of all small sets is monoidal closed.

  Let A, B, C be sets [46]. There exists natural bijections $A^{B \otimes C} \cong (A^B)^C$ which associate with each function $f : B \otimes C \to A$ in two variables, the function $\bar{f} : C \to A^B$ in one variable whose values at $x \in C$ are the functions $\bar{f}(x) : B \to A$ defined by the equations $\bar{f}(x)(y) = f(x, y)$. This implies the existence of exponentials in the category **Set** and hence **Set** is monoidal closed.

- The category **Top** of topological spaces is not monoidal closed.

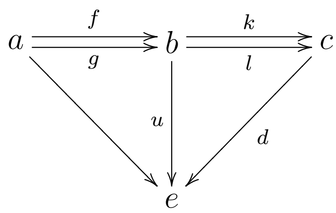
  In the category of topological spaces, the epimorphism is given by the quotient map. This enables us to form epimorphisms which are useful in the construction of a coequalizer (see Chapter 2, Section 2.7). It is therefore essential that any monoidal closed category preserves limits and colimits. One of the colimits is the coequalizer whose presence is determined by the preservation of regular epimorphisms.

In [23], Kelly defines a morphism $f : A \to B$ in a category $\mathbf{A}$ to be a regular epimorphism if any $g : A \to C$ satisfying $gx = gy$ whenever $fx = fy$ is of the form $g = hf$ for a unique h.

Now consider the category **Top** and let $A, B, C \in \mathbf{Top}$. This gives us the exponential category $- \otimes \mathbf{A} : \mathbf{Top} \to \mathbf{Top}$.

Suppose for $a \in A,\ b \in B,\ c \in C$, we can form the given epimorphisms as required for the existence of a coequalizer such that we have $a \underset{g}{\overset{f}{\rightrightarrows}} b \xrightarrow{u} e$ with $u \bullet f = u \bullet g$ and $b \underset{l}{\overset{k}{\rightrightarrows}} c \xrightarrow{d} e$ with $d \bullet k = d \bullet l$ where e is the quotient map in **Top**.

The composition from a to c is given by the following diagram:



(5.4)

Our aim is to define the composition $[a, b] \otimes [b, c] = [a, c]$ and to determine whether it yields regular epimorphisms.

From Diagram (5.4), we realise that the composition $[a, b] \otimes [b, c]$ is given by $(k \bullet d)(u \bullet f)$.

The direct composition $[a, c]$ is given by $d \bullet (k \bullet f)$.

Taking $u = (k \bullet d)$, we have that $u(u \bullet f) = d \bullet (k \bullet f)$.

Applying the associativity law, we have:

$u(u \bullet f) = (d \bullet k) \bullet f \Rightarrow u(u \bullet f) = (u \bullet f)$.

We are dealing with the functor $- \otimes \mathbf{A} : \mathbf{Top} \to \mathbf{Top}$. From the calculation we have that e is a quotient map in $[a, b]$ and $[b, c]$. However, it is not a quotient map in $[a, c]$. Regular epimorphisms are therefore not preserved in **Top** and as such coequalizers are not preserved. Hence **Top** is not monoidally closed.

A monoidally closed category, is amenable to precisely the same argument as given by Wolkenhauer and Hoymeyr, [51] (and outlined in §5.1 and §5.2). This is because they posses an exponential and an evaluation map. Canonical examples of categories that are monoidally, (but not Cartesian) closed include **Ab** and $\mathbf{Vect}_{\mathbf{K}}$.

5.3.2 Enriched Categories

In the previous section, we discussed how some categories, although not Cartesian closed, are monoidally closed. This is a very significant fact considering that Rosen style arguments require only closure under exponentiation to achieve closure to efficient causation, which is indeed the case for monoidally closed categories such as **Ab** and $\mathbf{Vect}_{\mathbf{K}}$. Several potentially useful categories are still excluded: **Top** and **Pos** (in general) are not monoidally closed. However, we can extend this result yet further: any symmetric monoidal category (e.g. **Top**) may be embedded in a closed one, [24].

In order to construct such an embedding, we employ the notion of an enriched category. Kelly [24] introduces the concept of enriched categories in terms of monoidal categories. An enriched category is one which contains additional structure over and above sets of morphisms. There are two ways of thinking about enriched categories. We may either have additional structure on a pre-assigned ordinary category; or we may construct an ordinary category from an enriched category. In both cases, the ordinary category constructed should be isomorphic to the one we started out with. In order to examine the enriched case, Kelly begins by introducing the concept of a symmetric monoidal category $\mathbf{V}$ as defined in §5.3.1. The definition of a category enriched over a symmetric monoidal category is as follows:

Definition 5.3.4. Let $\mathbf{V}$ be a symmetric monoidal category and $\mathbf{V}_0 \in \mathbf{V}$, then a

$\mathbf{V}$-category $\mathbf{A}$ or a category $\mathbf{A}$ enriched over $\mathbf{V}$ consists of:

- A set $Ob(\mathbf{A})$ of objects such as (A, B, C).
- A hom-object $\mathbf{A}(A, B) \in \mathbf{V}_0$ for each pair of objects in $\mathbf{A}$.
- The composition morphism $\mathbf{A}(B, C) \otimes \mathbf{A}(A, B) \to \mathbf{A}(A, C)$ for each triple of objects (A, B, C).
- Identity element $I_A : I \to \mathbf{A}(A, A)$ for each object.

These are subject to:

(1) Associativity axiom such that the following diagram commutes:

$$\begin{array}{ccc}
\mathbf{A}(C,D)\otimes\mathbf{A}(B,C)\otimes\mathbf{A}(A,B) & \longrightarrow & \mathbf{A}(C,D)\otimes\mathbf{A}(A,C) \\
\downarrow & & \downarrow \\
\mathbf{A}(B,D)\otimes\mathbf{A}(A,B) & \longrightarrow & \mathbf{A}(A,D)
\end{array}$$

(2) Unit axiom such that the following diagram commutes:

$$\begin{array}{ccccc}
I\otimes\mathbf{A}(A,B) & \longleftarrow & \mathbf{A}(A,B) & \longrightarrow & \mathbf{A}(A,B)\otimes I \\
\downarrow & & \| & & \downarrow \\
\mathbf{A}(B,B)\otimes\mathbf{A}(A,B) & \longrightarrow & \mathbf{A}(A,B) & \longleftarrow & \mathbf{A}(A,B)\otimes\mathbf{A}(A,A)
\end{array}$$

Taking $\mathbf{V} = \mathbf{Set}$, gives us a (locally small) ordinary category, $\mathbf{V} = \mathbf{Cat}$ gives us a 2-category, $\mathbf{V} = \mathbf{Ab}$ gives us an Ab-category, while $\mathbf{V} = \mathbf{Top}$ gives us a topological category. In Chapter 2, we observed that the category

Ab of Abelian groups is not Cartesian closed. However, when this category **Ab** is equipped with a multiplication $\otimes$ and a unit $\mathbb{Z}$, then it is monoidally closed. This is an example of a category enriched in **Ab**. The discussion on the category $\mathbf{Vect_K}$ is similar to that on the category **Ab** of Abelian groups.

Objects and arrows of one enriched category can be mapped onto those of another enriched category to form functors between them.

Definition 5.3.5. Suppose $\mathbf{A}$, $\mathbf{B}$ are categories enriched over $\mathbf{V}$. We say that $T : \mathbf{A} \to \mathbf{B}$ is a functor if it consists of two functions:

- An object function T which assigns to each object $A \in \mathbf{A}$ an object $TA \in \mathbf{B}$.
- An arrow function T which assigns to every $f : A \to B$ of $\mathbf{A}$, an arrow $Tf : TA \to TB$ of $\mathbf{B}$. These are subject to the commutativity and identity axioms.

In terms of Hom-sets, we require that the following diagram commutes:

$$\begin{array}{ccc} \mathbf{A}(B,C) \otimes \mathbf{A}(A,B) & \longrightarrow & \mathbf{A}(A,C) \\ {\scriptstyle T\otimes T}\downarrow & & \downarrow{\scriptstyle T} \\ \mathbf{B}(TB,TC) \otimes \mathbf{B}(TA,TB) & \longrightarrow & \mathbf{B}(TA,TC) \end{array} \tag{5.5}$$

Enriched functors can also be composed. A natural transformation is defined as:

Definition 5.3.6. Consider two enriched functors $T, S : \mathbf{A} \to \mathbf{B}$. A monoidal natural transformation $\alpha : T \to S$ is a function which assigns to each $A \in \mathbf{A}$ an arrow $\alpha_A : TA \to SB$ of $\mathbf{B}$ such that for every $f : A \to B$, there exist natural bijections

$$\mathbf{B}(TB, SB) \otimes \mathbf{B}(TA, TB) \cong \mathbf{B}(SA, SB) \otimes \mathbf{B}(TA, SA) \cong \mathbf{B}(TA, SB).$$

Indexed Limits and Colimits

For **V**-categories, we require the notion of indexed limits. These still have the properties of limits as discussed in Chapter 2. There are two ways of thinking of a **V**-functor. We may consider a **V**-functor $F : \mathbf{K} \to \mathbf{V}$ as an indexing type. We may also consider a **V**-functor $G : \mathbf{K} \to \mathbf{B}$ as a diagram in **B** of type F, [24].

For each $B \in \mathbf{B}$, the **V**-functor $\mathbf{B}(B, G-) : \mathbf{K} \to \mathbf{V}$ naturally arises.

Consider the existence of the object $[\mathbf{K}, \mathbf{V}](F(\mathbf{B}(B, G-)))$ in **V**. If such an object exists for all B, then it is the value of a **V**-functor $\mathbf{B}^{op} \to \mathbf{V}$, [24].

If in addition such a functor admits a representation

$$\mathbf{B}(B, \{F, G\} \cong [\mathbf{K}, \mathbf{V}], (F, \mathbf{B}(B, G-))) \tag{5.6}$$

with counit such as $\mu : F \to \mathbf{B}(\{F, G\}, G-)$, then $(\{F, G\}, \mu)$ is the limit of G indexed by F.

Notice that at times the indexed limit is abbreviated as $\{F, G\}$.

The notion of a colimit is dual to that of a limit. An indexed colimit in **B** is an indexed limit in $\mathbf{B}^{op}$. We replace **K** by $\mathbf{K}^{op}$ in the definition of an indexed limit. We therefore have $G : \mathbf{K} \to \mathbf{B}$, $F : \mathbf{K}^{op} \to \mathbf{V}$.

If the **V**-functor exists and admits representation

$$\mathbf{B}(F * G, B) \cong [\mathbf{K}^{op}, \mathbf{V}](F, \mathbf{B}(G-, B)) \tag{5.7}$$

with a unit such as $\upsilon : F \to \mathbf{B}(G-, F * G)$, then $(F * G, \upsilon)$ is the colimit of G indexed by F.

Notice that at times the indexed colimit is abbreviated as $F * G$.

There are two notions which arise from those of indexed limits and colimits. These are the ends and coends in a **V**-category and are defined below:

Definition 5.3.7. The end $\int_{A \in \mathbf{A}} G(A, A)$ of $G : \mathbf{A}^{op} \times \mathbf{A} \to \mathbf{B}$ is defined as:

$$\int_{A \in \mathbf{A}} G(A, A) = \{Hom_{\mathbf{A}}, G\},$$

where $hom_{\mathbf{A}} : \mathbf{A}^{op} \otimes \mathbf{A} \to \mathbf{V}$ for some $\mathbf{V}$-category $\mathbf{A}$.

It is determined by $\mathbf{B}(B, \int_A G(A, A)) \cong \int_A \mathbf{B}(B, G(A, A))$.

Definition 5.3.8. The coend $\int^{A \in \mathbf{A}} G(A, A)$ of $G : \mathbf{A}^{op} \times \mathbf{A} \to \mathbf{B}$ is defined as:

$$\int^{A \in \mathbf{A}} G(A, A) = Hom_{\mathbf{A}} * G.$$

It is determined by $\mathbf{B}(\int^A G(A, A), B) \cong \int_A \mathbf{B}(G(A, A), B)$.

In Chapter 2 we stated the Yoneda Lemma which is very significant in the study of category theory. It also exists in enriched categories. We have a weaker notion of this lemma and a stronger version of the lemma where we no longer use bijection of sets, [24].

Below, we give the strong Yoneda lemma.

Lemma 5.3.9. *The Yoneda Embedding* $\mathbf{A} \to [\mathbf{A}^{op}, \mathbf{V}]$.

Given a $\mathbf{V}$*-functor* $F : \mathbf{A} \to \mathbf{V}$*, and an object* K *of* $\mathbf{A}$*, we have the map,*

$$F_{KA} : \mathbf{A}(K, A) \to [FK, FA],$$

which is $\mathbf{V}$*-natural in* A*.*

The transform $\alpha_A : FK \to [\mathbf{A}(K, A), FA]$ *of* F_{KA} *under the adjunction* $\mathbf{V}_0(X, [Y, Z]) \cong \mathbf{V}_0(Y, [X, Z])$ *is* $\mathbf{V}$*-natural in* A*.*

The stronger Yoneda lemma ensures that FK is expressed as the end $\int_A [\mathbf{A}(K, A), FA]$, so that we have an isomorphism;

$$\alpha : FK \cong [\mathbf{A}, \mathbf{V}](\mathbf{A}(K, -), F).$$

For more details about the Yoneda lemma and the proof, the reader is referred to G. M. Kelly's "Basic Concepts of Enriched Category Theory", [24].

Proposition 5.3.10. *Any symmetric monoidal category can be embedded in a closed one.*

Our objective is to show that any symmetric monoidal category $\mathbf{V}$ can be extended to a higher universe $\mathbf{V}'$.

The category $\mathbf{V}'$ is to be closed.

Let $\mathbf{A}$ be a category enriched over $\mathbf{V}$ and $\{\mathbf{A}\}$ be the set of such categories.

Let $\mathbf{Set}'$ be the category of sets in some universe containing as elements, $Ob(\mathbf{Set}), Ob(\mathbf{V})$ and $Ob(\mathbf{A})$ for each $\mathbf{A} \in \{\mathbf{A}\}$. Take the functor category ${\mathbf{V}_0}' = [\mathbf{V}_0^{op},\ \mathbf{Set}']$ to be enriched in $\mathbf{Set}'$ such that it preserves all limits and colimits. This is because the category $\mathbf{Set}'$ is closed.

Let the Yoneda full embedding sending X to $\mathbf{V}_0(-, X)$ be denoted by $y : \mathbf{V}_0 \to \mathbf{V}'_0$. Then there exists a unique monoidal closed structure $\mathbf{V}' = (\mathbf{V}'_0, \otimes', I', [\ ,\ ]')$ on $\mathbf{V}'_0$ extending that on $\mathbf{V}_0$.

Proof. The category $\mathbf{V}'$ is closed therefore the left adjoint functors $F\otimes' -$ and $- \otimes' G$ must preserve all colimits. Since $\mathbf{V}_0^{op}$ is small, we have the pointwise colimits $F * Y \cong F$ and $Y * G \cong G$.

This implies that:

$F \otimes' G \cong (F * y) \otimes' G \cong F * (y \otimes' G) \cong F * (G * (y \otimes' y)) \cong (F \otimes G) * (y \otimes y)$

We require the functor $\otimes'$ to extend to $\otimes$. We therefore need to have:

$$yX \otimes' yY = \mathbf{V}_0(-, X) \otimes' \mathbf{V}_0(-, Y) \cong \mathbf{V}_0(-, X \otimes Y) = y(X \otimes Y) \quad (5.8)$$

The colimit of G indexed by F given by $F\otimes' G$ is given by $(F - \otimes G?) * y(- \otimes ?)$ where the notations $-$ and ? are used to represent unexpressed variables.

This can be written in the coend form $F\otimes G = \int^{X,Y} FX \times GY \times \mathbf{V}_0(-, X\otimes Y)$.

Next we apply representation (5.7) to Equation (5.8).

We have that $\mathbf{B} = \mathbf{V}'_0, B = H, V = \mathbf{Set}', F = F - \times G?$ and $G = y(-\otimes ?)$.

We end up with

$$\mathbf{V}_0'(F \otimes' G, H) \cong [\mathbf{V}_0^{op} \times \mathbf{V}_0^{op}, \mathbf{Set}'](F - \times G?), \mathbf{V}_0'(y - \otimes ?), H)$$

When $\mathbf{V} = \mathbf{B}$, the expression on the right hand side of (5.7) is equivalent to $\{F, \mathbf{B}(G-, B)\}$.

By the Yoneda Lemma, we have that $\mathbf{V}_0'(y(-\otimes ?), H) \cong H(-\otimes ?)$.

Therefore we have:

$$\begin{aligned} \mathbf{V}_0'(F \otimes' G, H) &\cong [\mathbf{V}_0^{op} \times \mathbf{V}_0^{op}, \mathbf{Set}'](F - \times G?), H(-\otimes ?) \\ &\cong \mathbf{V}_0'(F, \mathbf{V}_0'(G?, H(-\otimes ?))) \end{aligned}$$

This implies that the functor $- \otimes' G$ has a desired right adjoint [24]

$$[\ ,\ ]' : \mathbf{V}_0'(F \otimes' G, H) \to \mathbf{V}_0'(F, [G, H]') \tag{5.9}$$

Here, $[G, H]' = \mathbf{V}_0'(G?, H(-\otimes ?))$.

This can be written in the end-form as $\int_Y [GY, H(- \otimes Y)]$.

By the Yoneda embedding studied earlier in the section, we have $[\mathbf{V}_0(-, X), \mathbf{V}_0(-, Z)]' = \int_Y [\mathbf{V}_0(Y, X), \mathbf{V}_0(- \otimes Y, Z)]$ gives the adjunction $\mathbf{V}_0(- \otimes X, Z) \cong \mathbf{V}_0(-\ , [X, Z])$.

Therefore $[yX, yZ]' \cong y[X, Z]$. This implies that the functor $[\ ,\ ]'$ is an extension of the $\mathbf{V}$-functor $[\ ,\ ]$.

We have seen that the functor $- \otimes' G$ has a desired right adjoint. We also have that $\otimes'$ is symmetric. From this, we have that $(F \otimes' G) \otimes' H$ preserves colimits in each variable separately, [24].

Since $F \cong F * y$ we have that $(F \otimes' G) \otimes' H = ((F \times G) \times H) * (y \otimes' y)$.

We require $\otimes'$ to extend $\otimes$ hence we require

$$\begin{aligned} (yX \otimes' yY) \otimes' yZ &= ((\mathbf{V}_0(-, X) \otimes' \mathbf{V}_0(-, Y))) \otimes' \mathbf{V}_0(-, Z) \\ &\cong \mathbf{V}_0(-, ((X \otimes Y) \otimes Z)) \\ &= y((X \otimes Y) \otimes Z) \end{aligned}$$

Thus we can define $(F \otimes' G) \otimes' H$ as $((F - \times G?) \times H!) * y(- \otimes ?)$ which can be written in the coend form as:

$$(F \otimes' G) \otimes' H \cong \int^{X,Y,Z} FX \times GY \times HZ \times \mathbf{V}_0(-, (X \otimes Y) \otimes Z)$$

We also have a similar expression for $F \otimes' (G \otimes' H)$. This shows us that the associativity of $\otimes$ implies associativity of $\otimes'$. This is also true for commutativity. The derivation of the desired right adjoint $[\ ,\ ]'$ in Equation (5.9) makes no use of any closed structure of the symmetric monoidal **V**. Thus any symmetric monoidal category can be embedded in a closed one, [24]. $\square$

In [1] there is a discussion on sub-categories and super-categories. We can think of these as the two perspectives of enriched categories mentioned earlier. The category **Top** of all topological spaces has Cartesian closed super-categories such as **Pstop**; the category of pseudotopological spaces where the power objects carry the structure of continuous convergence. The category **Top** also has closed sub-categories such as **KTop** of compact Hausdorff generated topological spaces where the power objects are formed by the K-Top coreflexions of the compact open topologies, [1]. Some categories, like the category of topological spaces, are neither Cartesian closed (see Chapter 2) nor monoidal closed (see §5.3.1). It therefore seems as if such categories do not well describe the behaviour of natural systems, despite having been previously used to model biological systems. It is in this regard that we introduce the notion of enriched categories. Kelly in [24], shows that any symmetric monoidal category (e.g. **Top**) can, by passing to a higher universe, be embedded in a closed one. This means that even though we have certain categories which are not closed, they are still useful and can be described in terms of a closed one, although it should be remembered that there do exist categories that are not symmetric monoidal.

Moreover, the fact that many categories of interest may be embedded in closed categories presents us with mathematical structure rich enough to contemplate two avenues of potentially fruitful research. Firstly it would be interesting to reconsider the actual mathematical (categorical) underpinnings of the various "computable" approaches (e.g. autopoiesis, automata, etc.) to see if the apparent disagreement with Rosen's conclusions may be alleviated and the disparate approaches somewhat unified. Here we need to note that many models do not necessarily require all of a particular language – they may naturally "live" in a "subspace." Secondly, the natural embedding of efficiently closed systems within larger closed systems suggests a technique that may assist in discussions of complexity. In particular, one would like to consider exponentiation and closure more generally. These suggest that categorical techniques are worth investigating.

5.4 Summary

In this chapter, we have reviewed Wolkenhauer and Hofmeyr's, [51], arguments on the self-organizing principles of cell function. These arguments are inspired by Rosen's work on metabolic-repair systems (see Chapter 3), and describe an abstract cell model using Mesarovic and Takahara's approach, [38]. The unifying factor in these Rosen-style arguments is that, for a living cell to be closed to efficient causation, the category within which it is defined is taken to be Cartesian closed. We note, however, that Cartesian products are used only for simplicity and for the reason that one wishes to begin with a minimal mathematical representation. Morever, all that is required is closure under exponentiation. We have explored the case for monoidal closure by replacing the Cartesian product by the tensor product. Monoidally closed categories also posses an evaluation map and an exponential object, which is the condition required for a living cell/self-organizing system to be closed to efficient causation.

The notion of symmetric monoidal categories lead to an interesting study of enriched categories. We find that any symmetric monoidal category can be embedded in a closed one. This eases our concern about the categories which are used to model systems despite their being neither Cartesian nor monoidally closed. We conclude that categories such as **Top** can still capture the behaviour of a living cell/self-organizing system by being described within a closed category.

These observations extend the philosophical impact of the Wolkenhauer and Hofmeyr result, while simultaneously providing a protective result for (and constructive techniques to study) self-organization/self-replication in non-Cartesian closed categories. The obvious next step is to consider the impact of these results on particular models.

Chapter 6

Conclusion

Systems biology seeks to provide scientific explanations for biological phenomena. Biological systems are, however, complex in nature and one requires basic simulations in order to understand the functioning of the systems as a whole. Similar observations could be made of fields such as computer science and engineering, [10]. One such system is the metabolic-repair system which was studied by Rosen [44] and provides the basis for arguments by Wolkenhauer and Hofmeyr in reference [51] and Casti in reference [13]. Rosen argued that in order for a living system to be an organism, it has to be closed to efficient causation. Adding maps to already existing ones such that everything within the system is entailed by something leads to an infinite regress. It is in this regard that the language of category theory is introduced. These arguments then require that, for a living cell to be an organism, it has to be described within the setting of a Cartesian closed category so as to exploit the existence of an exponential to ensure efficient causation. We, however, note that Cartesian closure is sufficient, but not necessary for exponentiation. Moreover, we observe that the Cartesian structure is introduced only because the language of Mesarovic and Takahara [38], requires us to begin with a minimal mathematical structure and to add more structure only as required by the behaviour of the system. It quite naturally extends to, for example, the tensor product and monoidal categories.

We have demonstrated that closure to efficient causation can also be described within the context of monoidally closed categories. This is because such categories contain an evaluation map and an exponential object. This leads to similar arguments for the self-organizing principle of cell function. Given the representation of a living cell in terms of a monoidally closed category, the living system is closed to efficient causation whenever the basic cellular process map has a retraction. This extends the Wolkenhauer and Hofmeyr result to categories such as **Ab** and $\mathbf{Vect_K}$.

We have further managed to extend these results to include categories which are neither Cartesian nor monoidally closed. The category of topological spaces, **Top**, is neither Cartesian closed nor monoidally closed. However, since it is symmetric monoidal, it can be embedded in a closed one. We note that the ubiquity of closed structures and exponentiation lends credence to the idea that closure to efficient causation is not necessarily "the," but rather "a" defining characteristic of life, [3]. The work presented in this dissertation is summarised below:

- Chapter 1: Various mathematical modelling approaches and the application areas of categorical systems biology are discussed and motivated. We outline the structure of the dissertation.

- Chapter 2: Basic concepts of category theory leading to Cartesian closure are introduced.

- Chapter 3: We review Rosen's work on metabolism-repair systems and extend this to Casti's study of linear metabolism-repair systems.

- Chapter 4: Mesarovic and Takahara's systems theoretic approach which involves state time and state objects is reviewed. This allows us to study the evolution of a system over time. Real life observations are transformed into mathematical systems beginning with minimal structure.

- Chapter 5: We review Wolkenhauer and Hofmeyr's treatment of self-organization within the context of Mesarovic and Takahara's systems theory and Cartesian closure. We observe that we may extend these results to monoidally closed and symmetric monoidal categories. The extension to monoidal and enriched categories provides language rich enough to enable us to extend Rosen-style arguments to include most mathematical modelling languages.

- Chapter 6: We present future work, application areas and a summary of this dissertation.

Future work includes such applications as:

- Mathematical investigations of the impact and utility of monoidally closed and/or enriched categorical techniques within particular modelling scenarios. The natural starting point would be to consider various M-R models [12], and to identify the appropriate categorical description applicable to each. This is complicated by the fact that the model may well be posed in a very different language, but would assist in identifying whether and how the model is closed to efficient causation via Rosen-style arguments, and, if not, whether one can "close" the model via appropriate embeddings. In addition, we isolate the following issues as being of particular interest.

  - The colimit notion has been used in the quest to understand how the brain works to relay information, [9]. This is particularly interesting given the nascent field of quantum biology (e.g. [19, 25, 26, 35, 47]), particularly as applied to brain function.

– Brown, Porter and Paton [9] observe that categories that vary over time can be introduced to deal with structures which vary over time. This seems interesting and could help in solving the dilemma we had in Chapter 3 when studying the effect of environmental changes on the replication map (our discussion on mutation.) Mutation required us to mathematically formulate a phenomenon in which random events acted on a system over time. This can naturally be extended to a complex system consisting of many such mutating systems, with obvious implications for evolution and/or learning systems.

– It would also be interesting to consider applications outside of biology, e.g. unification theory and cosmology, as these are contexts in which one might expect closure to efficient causation. Categorical techniques are used to incorporate the principal of general covariance in the algebraic approach to quantum field theory on curved backgrounds, [11], and are likely to feature in a full theory of quantum gravity.

  Partial orders are used in causal set cosmology [7, 16], which attempts to build a discrete picture of space-time consistent (at large scales and low energies) with that of Einstein. Possible applications of category theory techniques have been considered by among others, Isham, [20]. It would be interesting to investigate whether arguments such as those presented in this dissertation can be carried over to a cosmological context. Within the standard model, there already exist inflationary scenarios which embody such patterns: e.g. Linde's chaotic eternal inflation, [30, 31].

- Investigations generalizing the results described in this dissertation.

  – Clearly, identifying exponentiation in other contexts would be nice.

- It would also be helpful to know whether the structure of causal efficiency can be made more explicit via, for example, a closed subcategory of the full category applicable to some given model.
- The notion of closure appears ubiquitously in the study of algebraic logic, [5]. It would be interesting to consider whether, and how, modelling lessons could be learned by moving the discussion to this level.

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