Fixed Point Theory in Various Generalized Metric-type Spaces

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July 2022

Fixed Point Theory in Various Generalized Metric-type Spaces

by

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Submitted in fulfilment of the

academic requirements for the degree of

Doctor of Philosophy

in the

School of

Mathematics, Statistics and Computer Science

University of KwaZulu-Natal

Durban

July 2022

As the candidate's supervisors, we have approved this thesis for submission.

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Preface

The work described in this dissertation was carried out in the School of Mathematical Sciences, University of KwaZulu-Natal, Durban, from February 2019 to July 2022, under the supervision of Dr V. Singh and co-supervised by Dr P. Singh. This study represents original work by the author and has not otherwise been submitted in any form for any degree or diploma to any other tertiary institution. Where use has been made of the work of others, it is duly acknowledged in the text.

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DECLARATION 1 - PLAGIARISM

I, Thokozani Cyprian Martin Jele, declare that

1. The research reported in this thesis represents my own original research, except where due reference is made.

2. This thesis has not been submitted in the past for any degree at any other institution (academic or otherwise).

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Abstract

In the theory of fixed points, there are numerous articles dealing with generalization of the basic Banach contraction mapping principle. There has been two lines of approach. The first one is concerned with generalizations of the contractive conditions on the mapping space. The other line of investigation deals with various generalizations of the metric spaces and the results that can be obtained in these new frameworks, referred to as metric-type spaces. In this thesis, we elected for the latter approach by providing a more general framework for a b-metric space , G-metric space and S-metric space. In this thesis, we proved that these new metric-type spaces equipped with various contractions type mappings have unique fixed points and provide numerous examples of each metric-type spaced mentioned.

Acknowledgements

Firstly, I would like to thank God for the gift of life and health, also to thank my family for all the support they have been giving me since day one I will forever be grateful for everything. I would also like to thank the University of KwaZulu-Natal for the opportunity to do my studies under their institution from my first year till now.

A special and sincere gratitude to my supervisors Dr Pravin Singh and Dr Virath Singh for allowing to take me as their PhD student after they have also supervised my Masters degree and for all the help, dedication and contribution on the work that was done on this thesis, I would not be where I am today if it was not for them and for that I would forever be grateful.

Not forgetting my partner and my friends for always reminding me that I'm still a student and I need to finish what I have started, that to me came as encouragement not to even think of giving up on my studies, I'm thankful for all that. I would like to dedicate this thesis to my late grandmother Phahlakazi Maria Jele, I still think of you till this day. Ngiyabonga mina.

Notations

$\mathbb{C}$	Set of complex numbers
$\mathbb{R}$	Set of real numbers
$\mathbb{N}$	Set of natural number
(X, d)	Metric space
X	None empty set
$\emptyset$	Empty set
$\{.\}$	Sequence
x^*	Fixed point
$\|\cdot\|_X$	Norm on X
$\subset$	Subset
τ	Metric topology
$T : X \to X$	Contraction mapping on X

Contents

Thesis outline

Chapter 1

In this chapter we introduce the important concepts of metric spaces, their definitions and properties, as well as examine some fixed point theorems.

Chapter 2

The purpose of this chapter is to introduce a new relaxed α, β b-metric type by relaxing the triangle inequality. We investigate the effect that this generalization has on fixed point theorems.

Chapter 3

In this chapter, we generalize Mann's iterative algorithm and prove fixed point results in the framework of $\alpha\beta$-b metric spaces. Firstly a convex structure is imposed on the space and two strong convergence theorems are provided for two different contraction mappings. Also the concept of T-stability is extended to this space.

Chapter 4

In this chapter, we study the existence and uniqueness of fixed point in complex valued b-metric spaces and introduce a new relaxed α, β Complex-valued b-metric type by relaxing the triangle inequality and determine whether the fixed point theorems are applicable in these spaces.

Chapter 5

In this Chapter, we provide a generalization of the G_b-metric and prove some fixed point results of contractive type mappings in this space.

Chapter 6

In this chapter, we provide a generalization of an S-metric space by relaxing the triangle inequality. As applications, we provide some fixed point theorems of mappings with common fixed points in the generalized S-metric space.

Chapter 7

Conclusion.

Chapter 1

Metric spaces and fixed point theorems.

1.1 Metric spaces.

Metric spaces can be thought of as very basic spaces, with only a few axioms, where the ideas of convergence and continuity exist. The distance or a metric is the fundamental property that defines the space and measures how close elements are to each other [22].

Definition 1.1.1. A distance d or a metric on a metric space (X, d) is a function

$$d : X \times X \to \mathbb{R}^+$$

such that the following axioms hold for all $x, y, z \in X$.

(i) $d(x, y) \geq 0$

(ii) $d(x, y) = 0 \iff x = y$

(iii) $d(x, y) = d(y, x)$ (symmetry)

(iv) $d(x,y) \leq d(x,z) + d(z,y)$ (triangular inequality)

Proposition 1.0.1. *The vector space $\mathbb{R}^n$ with the standard Euclidean distance defined by $d_p(x,y) = (\sum_{i=1}^{n} |x_i - y_i|^p)^{\frac{1}{p}}$ where $p \geq 1$ with $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$ in $\mathbb{R}^n$ is a metric space.*

Proof. We shall prove only the triangular inequality $d(x,y) \leq d(x,z) + d(z,y)$ which is non trivial. $d_p(x,y) \leq d_p(x,z) + d_p(z,y)$ where $z = (z_1, z_2, \cdots, z_n)$ in $\mathbb{R}^n$.

$$\begin{aligned} d_p(x,y) &= \left(\sum_{k=1}^{n} |x_k - y_k|^p \right)^{\frac{1}{p}} \\ &= \left(\sum_{k=1}^{n} |(x_k + z_k) + (z_k - y_k)|^p \right)^{\frac{1}{p}} \\ &\leq \left(\sum_{k=1}^{n} |x_k - z_k|^p \right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n} |z_k - y_k|^p \right)^{\frac{1}{p}}. \qquad (1.1) \end{aligned}$$

Equation (1.1) follows from Minkowski's inequality[22]

For the special case when $p = 2$ we have a standard euclidean metric on $\mathbb{R}^n$,

$$d(x,y) = \sqrt{\sum_{k=1}^{n} |x_k - y_k|^2}.$$

It can be easily proved directly that $d(x,y)$ is a metric. Once again it only suffices to prove the triangle inequality.

$$\begin{aligned} d(x,y) &= \left(\sum_{k=1}^{n} |(x_k - z_k) + (z_k - y_k)|^2 \right)^{\frac{1}{2}} \\ &= \left(\sum_{k=1}^{n} |a_k + b_k|^2 \right)^{\frac{1}{2}} \end{aligned}$$

Where $a_k = x_k - z_k$ and $b_k = z_k - y_k$.

Hence

$$d(x,y) = \left(\sum_{k=1}^{n} a_k^2 + 2 \sum_{k=1}^{n} a_k b_k + \sum_{k=1}^{n} b_k^2 \right) \tag{1.2}$$

Define a function

$$f(t) = \sum_{k=1}^{n} (a_k - t b_k)^2 \tag{1.3}$$

Now $f(t)$ is minimum when

$$f'(t) = -2 \sum_{k=1}^{n} (a_k - t b_k)(b_k) = 0$$

which yields

$$t_m = \frac{\sum_{k=1}^{n} a_k b_k}{\sum_{k=1}^{n} b_k^2}$$

With $t = t_m$ in (1.3) we get,

$$\begin{aligned}
f(t_m) &= \sum_{k=1}^{n} \left[a_k - \left(\frac{\sum_{k=1}^{n} a_k b_k}{\sum_{k=1}^{n} b_k^2} \right) b_k \right]^2 \\
&= \sum_{k=1}^{n} \left[a_k^2 - 2 \left(\frac{\sum_{k=1}^{n} a_k b_k}{\sum_{k=1}^{n} b_k^2} \right) a_k b_k + \frac{(\sum_{k=1}^{n} a_k b_k)^2}{(\sum_{k=1}^{n} b_k^2)^2} b_k^2 \right] \\
&= \sum_{k=1}^{n} a_k^2 - 2 \frac{(\sum_{k=1}^{n} a_k b_k)^2}{\sum_{k=1}^{n} b_k^2} + \frac{(\sum_{k=1}^{n} a_k b_k)^2}{(\sum_{k=1}^{n} b_k^2)^2} \sum_{k=1}^{n} b_k^2 \\
&= \sum_{k=1}^{n} a_k^2 - \frac{(\sum_{k=1}^{n} a_k b_k)^2}{\sum_{k=1}^{n} b_k^2}
\end{aligned}$$

Since $f(t_m)$ is positive we have

$$\begin{aligned}
\left(\sum_{k=1}^{n} a_k b_k \right)^2 &\leq \left(\sum_{k=1}^{n} a_k^2 \right) \left(\sum_{k=1}^{n} b_k^2 \right) \\
\sum_{k=1}^{n} a_k b_k &\leq \left(\sum_{k=1}^{n} a_k^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} b_k^2 \right)^{\frac{1}{2}}
\end{aligned}$$

Hence $d(x,y)$ in (1.2) becomes

$$\begin{aligned}
d(x,y) &\le \left(\sum_{k=1}^{n} a_k^2 + 2\left(\sum_{k=1}^{n} a_k^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{n} b_k^2\right)^{\frac{1}{2}} + \sum_{k=1}^{n} b_k^2\right)^{\frac{1}{2}} \\
&= \left(\left[\left(\sum_{k=1}^{n} a_k^2\right)^{\frac{1}{2}} + \left(\sum_{k=1}^{n} b_k^2\right)^{\frac{1}{2}}\right]^2\right)^{\frac{1}{2}} \\
&= \left(\sum_{k=1}^{n} a_k^2\right)^{\frac{1}{2}} + \left(\sum_{k=1}^{n} b_k^2\right)^{\frac{1}{2}} \\
&= \left(\sum_{k=1}^{n} (x_k - z_k)^2\right)^{\frac{1}{2}} + \left(\sum_{k=1}^{n} (z_k - y_k)^2\right)^{\frac{1}{2}} \\
&= d(x,z) + d(z,y)
\end{aligned}$$

$\square$

1.2 Topology of a metric space.

1.2.1 Open spheres

Let d be a metric on set X. For any point $a \in X$ a real number $r > 0$, we denote the set of points within a distance r from a by

$$B_r(a) = \{x \in X : d(x,a) < r\}.$$

It is called the open sphere or ball with centre a and radius r.

One important property of open sphere in a metric space is that if $B_r(x)$ is an open ball with the centre x and radius r then for every $y \in B_r(x)$ there exist an open ball $B_\epsilon(y)$ such that $B_\epsilon(y) \subset B_r(x)$

Definition 1.2.1. Let (X,d) be a metric space. The metric topology τ_d on X is a collection of subsets $U \subset X$ satisfying the property that for each $x \in U$ there exists

$r > 0$ such that $B_r(x) \subset U$.

The collection τ_d is a topology in X if,

(i) $\emptyset, X \in \tau_d$

(ii) Let $\{U_\alpha\}_{\alpha \in A}$ be a collection of τ_d and A an index set. Let $V = \cup_{\alpha \in A} U_\alpha$ and $x \in V$. Then by definition there exist $\alpha \in A$ with $x \in U_\alpha$. By the property satisfied by U_α in τ_d there exist $r > 0$ such that

$$B_r(x) \in U_\alpha$$

since $U_\alpha \subset V$ it follows that $B_r(x) \subset V$ hence $V \in \tau_d$

(iii) Let $\{U_i\}_{i=1,\cdots,n}$ be a subcollection of τ_d. Let $V = \cap_{i=1}^n U_i$ and $x \in V$ for $1 \leq i \leq n$ we get $V \subset U_i$ so $x \in U_i$. By the defining property of τ_d there exist $r_i > 0$ such that

$$B_{r_i}(x) \subset U_i$$

Let $r = \min_{i=1,\cdots,n} r_i$ then

$$B_r(x) \subset B_{r_i}(x) \subset U_i$$

for each $1 \leq i \leq n$ which implies that $B_r(x) \subset V$ hence $V \in \tau_d$.

In general, the intersection of two open spheres need not to be an open sphere. However for every point in the intersection of the open sphere does belong to an open sphere contained in the intersection.

1.2.2 Properties of metric topologies

Theorem 1.1. *The closure $\bar{A}$ of a subset A of a metric space (X, d) is the set of points whose distance from A is zero, i.e $\bar{A} = \{x : d(x, A) = 0\}$.*

Theorem 1.2 (Separation axiom). *Let U, V be closed disjoint susbset of a metric space X. Then there exist disjoint open set U' and V' such that*

$$V \subset V' \text{ and } U \subset U'$$

Definition 1.2.2.

(i) A sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to $x \in X$ if for $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$ for all $n > N$

(ii) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is a Cauchy sequence if and only if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for every $n, m > N$.

Every convergent sequence in a metric space is a Cauchy sequence. The converse is not true.

1.2.3 Complete metric space

Definition 1.2.3. A metric space (X, d) is complete if every Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in X converges to a point $x \in X$.

Theorem 1.3. *The class of open sphere is a set X with metric d is a base for a topology on X.*

Two metrics d and d' on a set X are equivalent if and only if they induce the same topology in X that is if and only if the d-open spheres and the d'-open spheres in X are bases for the same topology on X.

Definition 1.2.4. A metric space (X, d) is isometric to a space (Y, d') if and only if there exists a one - one, onto function $f : X \to Y$ which preserves distances, that is for $x, y \in X$

$$d(x,y) = d'(f(x), f(y))$$

Definition 1.2.5. A metric space (X', d') is a completion of a metric space (X, d) if (X', d') is complete and X is isometric to a dense subset of X'.

Theorem 1.4. *Every metric space (X, d) has a completion and all completions of X are isometric.*

1.3 Fixed Point Theorems

Fixed point theorems concern maps f of a set X into itself that, under certain conditions, admit a fixed point, that is, a point $x \in X$ such that $f(x) = x$. The knowledge of the existence of fixed points has relevant applications in many branches of analysis and topology [26].

Definition 1.3.1. [26]Let (X, d) be a metric space. A mapping $f : X \to X$ is said to be Lipschitz continuous if there exist $\lambda \geq 0$ such that

$$d(f(x_1), f(x_2)) \leq \lambda d(x_1, x_2)$$

for all $x_1, x_2 \in X$. The smallest λ for which the above inequality holds is the Lipschitz constant of f. If $\lambda \leq 1$ f is said to be non-expansive, if $\lambda < 1$ f is said to be a contraction.

Theorem 1.5. *[26] Let (X, d) be a complete metric space and let $f : X \to X$ be a contraction with Lipschitz constant λ. Then f has a unique fixed point $x^* \in X$.*

Proof. Take any point $x_0 \in X$ and define the iterative sequence $x_n = f(x_{n-1})$. We shall first show that $\{x_n\}$ is a Cauchy sequence. It follows that

$$\begin{aligned} d(f(x_n), f(x_{n-1})) &\leq \lambda d(x_n, x_{n-1}) \\ &= \lambda d(f(x_{n-1}), f(x_{n-2})) \\ &\leq \lambda^2 d(x_{n-1}, x_{n-2}) \\ &\leq \lambda^n d(x_1, x_0) \\ &= \lambda^n d(f(x_0), x_0) \end{aligned}$$

For $n \in \mathbb{N}$ and $m \geq 1$

$$\begin{aligned} d(x_{n+m}, x_n) &\leq d(x_{n+m}, x_{n+m-1}) + d(x_{n+m-1}, x_{n+m-2}) + \cdots + d(x_{n+1}, x_n) \\ &= d(f(x_{n+m-1}), f(x_{n+m-2})) + d(f(x_{n+m-2}), f(x_{n+m-3})) + \cdots + d(f(x_n), f(x_{n-1})) \\ &\leq (\lambda^{n+m-1} + \lambda^{n+m-2} + \cdots + \lambda^n) d(f(x_0), x_0) \\ &= \lambda^n \sum_{j=0}^{m-1} \lambda^j d(f(x_0), x_0) \\ &< \lambda^n \left(\sum_{j=0}^{\infty} \lambda^j \right) d(f(x_0), x_0) \\ &= \frac{\lambda^n}{1-\lambda} d(f(x_0), x_0) \end{aligned}$$

Since the $\lim_{n \to \infty} \lambda^n = 0$, hence $\{x_n\}$ is a Cauchy sequence, and admits a limit $x^* \in X$, for X is complete. Furthermore if we use the triangular inequality, we get

$$\begin{aligned} d(x^*, f(x^*)) &\leq d(x^*, x_n) + d(x_n, f(x^*)) \\ &= d(x^*, x_n) + d(f(x_{n-1}), f(x^*)) \\ &\leq d(x^*, x_n) + \lambda d(x_{n-1}, x^*) \\ &\to 0 \text{ as } n \to \infty \end{aligned}$$

Thus we have shown that $d(x^*, f(x^*)) = 0$, hence $f(x^*) = x^*$.

$\square$

Chapter 2

Generalized b-metric and fixed point theorems.

2.1 Introduction

The concept of a b-metric was initiated from the contributions of Bourbaki [7] and Bakhtin [5]. Czerwik [10] gave an axiom which was weaker than the triangular inequality and formally defined a b-metric space with a view of generalizing the Banach contraction mapping theorem. Later on, Fagin et al. [14] discussed some kind of relaxation in the triangular inequality and called this new distance measure a non-linear elastic pattern matching. These applications led us to introduce the concept of a generalized b-metric type and that the results obtained for such spaces become viable in different fields.

Definition 2.1.1. Let X be a non-empty set. A function $d : X \times X \to \mathbb{R}^+$ is a b-metric on X if there exists a real number $\alpha \geq 1$ such that the following conditions hold for all $x, y, z \in X$:

(i) $d(x, y) = 0 \iff x = y$

(ii) $d(x, y) = d(y, x)$

(iii) $d(x, y) \leq \alpha [d(x, z) + d(z, y)]$

The pair (X, d) is a called a b-metric space [41]. A b-metric with $\alpha = 1$ is exactly the usual metric.

Definition 2.1.2. Let X be a non-empty set. A function $\rho : X \times X \to \mathbb{R}^+$ is a generalized α, β b-metric on X if there exists a real numbers $\alpha, \beta \geq 1$ such that the following conditions hold for all $x, y, z \in X$:

(i) $\rho(x, y) = 0 \iff x = y$

(ii) $\rho(x, y) = \rho(y, x)$

(iii) $\rho(x, y) \leq \alpha\rho(x, z) + \beta\rho(z, y)$

We shall refer to (iii) as the α, β relaxed triangle inequality. The pair (X, ρ) is a called a generalized b-metric space. A generalized b-metric with $\alpha = \beta$ is exactly a b-metric. In the special case $\alpha = 1$ we obtain the a strong b-metric, [19, 41]. The following examples justify this generalization found in definition 2.1.2.

Example 1. Let $X = \{1, 2, 3\}$ be a discrete set and let $\rho : X \times X \to \mathbb{R}^+$ be a function defined by

$$\rho(1,1) = \rho(2,2) = \rho(3,3) = 0$$

$$\rho(1,2) = \rho(2,1) = \frac{1}{3}$$

$$\rho(1,3) = \rho(3,1) = 3$$

$$\rho(2,3) = \rho(3,2) = 4$$

From the definition of the b-metric properties (i), (ii) are apparent. For all $x, y, z \in X$ it follows that

$$\rho(x,y) \leq 2\rho(x,z) + 3\rho(z,y).$$

Example 2. Let $X = (1,3)$ and let $\rho : X \times X \to \mathbb{R}^+$ be a function defined by

$$\rho(x,y) = \begin{cases} e^{|x-y|}, & \text{if } x \neq y \\ \\ 0, & \text{iff } x = y. \end{cases}$$

The first two properties of a generalized b-metric are inherent in the definition. We verify the α, β triangle inequality as follows:

For $x \neq y,\ z \in X$ and $\theta \in (0,1)$

$$\begin{aligned} \rho(x,y) &\leq e^{|x-z|+|z-y|} \\ &= e^{\theta|x-z|+(1-\theta)|z-y|} e^{(1-\theta)|x-z|+\theta|z-y|} && (2.1) \\ &\leq \sup_{x,y,z\in X} e^{\theta|x-z|+(1-\theta)|z-y|} \left((1-\theta)e^{|x-z|} + \theta e^{|z-y|}\right) && (2.2) \\ &\leq (1-\theta)e^2 e^{|x-z|} + \theta e^2 e^{|z-y|} \\ &= (1-\theta)e^2\rho(x,z) + \theta e^2 \rho(z,y). \end{aligned}$$

Where from (2.1) to (2.2) we used Jensens inequality [11] on the convex function e^t. For $\theta = \frac{1}{3}$ we have constants $\alpha = \frac{2}{3}e^2$ and $\beta = \frac{1}{3}e^2$.

2.2 Topological properties of the generalized b-metric type.

One introduces a topology on a generalized b-metric space (X, ρ) in the usual way. The open ball $B(x, r)$ with centre $x \in X$ and radius $r > 0$ is given by

$$B(x, r) = \{y \in X : \rho(x, y) < r\}$$

A subset A of X is open if for every $x \in A$ there is a number $r > 0$ such that $B(x, r) \subseteq A$. Denoting by τ_ρ the family of all open subsets of X it follows that τ_ρ satisfies the axioms of a topology.

Let (X, ρ) be a generalized b-metric space. Then the b-metric is continuous if $\rho(x_n, x) \to 0$, $\rho(y_n, y) \to 0$ as $n \to \infty$ implies $\rho(x_n, y_n) \to 0$.

Furthermore, the b-metric is separately continuous if for every $x \in X$, $\rho(y_n, y) \to 0$ as $n \to \infty$ implies $\rho(x, y_n) \to \rho(x, y)$.

The topology τ_ρ generated by a generalized b-metric ρ has a peculiar property in that a ball $B(x, r)$ need not be τ_ρ- open as illustrated by the following example, [25].

Example 3. Let $X = \mathbb{Z}^+ \cup \{0\}$, $\epsilon > 0$ and define $\rho : X \times X \to \mathbb{R}^+$ by

$$\begin{aligned}
\rho(0,1) &= 1 \\
\rho(1,m) &= \frac{1}{m} \\
\rho(0,m) &= 1+\epsilon \ \text{ for } \ m \geq 2 \\
\rho(n,m) &= \frac{1}{n} + \frac{1}{m} \ \text{ for } \ n \geq 2 \\
\rho(n,n) &= 0.
\end{aligned}$$

Then

$$\rho(m,n) \leq \alpha\rho(m,k) + \beta\rho(k,n)$$

for all $m,n,k \in X$. The ball $B(0, 1+\frac{\epsilon}{2}) = \{0,1\}$ and the ball $B(1,r)$ contains an infinite number of terms for every $r > 0$. Now since $1 \in B(0, 1+\frac{\epsilon}{2})$ it follows that $B(1,r) \not\subset B(0,1+\frac{\epsilon}{2})$ for every $r > 0$ illustrating that the ball $B(0,1+\frac{\epsilon}{2})$ is not τ_ρ open.

Let X be a non-empty set and $B : X \times (0,\infty) \to \mathbb{P}(X)$ satisfying

(i) $\bigcap_{r>0} B(x,r) = \{x\}$

(ii) $\bigcup_{r>0} B(x,r) = X$

(iii) $0 < r_1 \leq r_2 \implies B(x,r_1) \subset B(x,r_2)$

(iv) there exists $c \geq 1$ such that $y \in B(x,r) \implies B(x,r) \subset B(y,cr)$ and $B(y,r) \subset B(x,cr)$ for all $x \in X$ and $r > 0$.

A family of subsets satisfying properties (i)-(iv) generates a b-metric on X, [2].

Condition (i)-(iii) are verified easily for an α, β b-metric. We shall show property (iv). If $y \in B(x,r)$ and $z \in B(x,r)$ then $\rho(y,z) \leq \alpha\rho(y,x) + \beta\rho(x,z) < (\alpha+\beta)r$ where $\alpha+\beta \geq 2$ thus $z \in B(y,(\alpha+\beta)r)$ and in a similar manner, if $y \in B(x,r)$ and $z \in B(y,r)$ then $\rho(x,z) \leq \alpha\rho(x,y) + \beta\rho(y,z) < (\alpha+\beta)r$ thus $z \in B(x,(\alpha+\beta)r)$. Thus this family of subsets also generates an α, β b-metric.

2.3 Completeness

Definition 2.3.1. Let (X, ρ) be a generalized b-metric space, and let $\{x_n\}$ be a sequence in X and $x \in X$. Then:

(i) The sequence $\{x_n\}$ converges to x, if $\lim_{n\to\infty} \rho(x_n, x) = 0$.

(ii) The sequence $\{x_n\}$ is a Cauchy in (X, ρ) if $\lim_{n,m\to\infty} \rho(x_n, x_m) = 0$.

(iii) The space (X, ρ) is complete if every Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$ such that $\lim_{n,m\to\infty} \rho(x_n, x_m) = \lim_{n\to\infty} \rho(x_n, x) = 0$.

Definition 2.3.2.

(i) If (X_1, ρ_1) and (X_2, ρ_2) are generalized b-metric spaces then a mapping $i : X_1 \to X_2$ is an isometric embedding if

$$\rho_2(i(x), i(y)) = \rho_1(x, y)$$

for all $x, y \in X_1$.

(ii) A completion of a generalized b-metric space (X, ρ) is a complete b-metric space (Y, ρ) such that there exists an isometric embedding $i : X \to Y$ with $i(X) \subset Y$ dense in Y.

2.4 Fixed point theorem for generalized b-metric spaces

As a consequence of (iii) of Definition 1.2, the α, β triangle inequality, we get for $n, m \in \mathbb{N}$ with $m > n$

$$\begin{aligned}
&\rho(x_n, x_m) \\
&\leq \alpha\rho(x_n, x_{n+1}) + \beta\rho(x_{n+1}, x_m) \\
&\leq \alpha\rho(x_n, x_{n+1}) + \beta[\alpha\rho(x_{n+1}, x_{n+2}) + \beta\rho(x_{n+2}, x_m)] \qquad (2.3)
\end{aligned}$$

Successively applying the α, β triangle inequality, we obtain

$$\rho(x_n, x_m) \leq \alpha \sum_{i=0}^{m-n-2} \beta^i \rho(x_{n+i}, x_{n+i+1}) + \beta^{m-n-1}\rho(x_{m-1}, x_m). \qquad (2.4)$$

In line with the α, β triangle inequality, one may consider the α, β relaxed polygonal inequality given by

$$\rho(x_n, x_m) \leq \frac{(\alpha + \beta)}{2} \sum_{i=0}^{m-n-1} \rho(x_{n+i}, x_{n+i+1}) \qquad (2.5)$$

where $x_i \in X$ for all i.

Definition 2.4.1. Let (X, ρ) be a generalized b-metric space then a mapping $T : X \to X$ is a contraction on X if there is a real number $0 < \lambda < 1$ such that for all $x, y \in X$

$$\rho(Tx, Ty) \leq \lambda\rho(x, y). \qquad (2.6)$$

The Banach fixed point theorem gives a constructive procedure for obtaining approximations to the fixed point called iterations. By the definition, in this method we choose an arbitrary x_0 and calculate recursively a sequence from the relation

$$x_n = T(x_{n-1}) = T^n(x_0). \qquad (2.7)$$

By a repeated use of (2.6) and (2.7) we get

$$
\begin{aligned}
\rho(x_{n+i}, x_{n+i+1}) &= \rho(T^{n+i}(x_0), T^{n+i}(x_1)) \\
&\leq \lambda \rho(T^{n+i-1}(x_0), T^{n+i-1}(x_1)) \\
&\vdots \\
&\leq \lambda^{n+i} \rho(x_0, x_1). \qquad (2.8)
\end{aligned}
$$

Theorem 2.1. *Let (X, ρ) be a complete generalized b-metric space, where ρ satisfies the α, β triangle inequality and $T : X \to X$ a contraction mapping such that $0 < \lambda < \frac{1}{\beta}$. Then T has a unique fixed point $x^* \in X$.*

Proof. We begin by proving that $\{x_n\}$ is a Cauchy sequence. Using (2.4) and (2.8) we get

$$
\begin{aligned}
\rho(x_n, x_{n+k+1}) &\leq \alpha \sum_{i=0}^{k-1} \beta^i \rho(x_{n+i}, x_{n+i+1}) + \beta^k \rho(x_{n+k}, x_{n+k+1}) \\
&\leq \alpha \sum_{i=0}^{k-1} \beta^i \lambda^{n+i} \rho(x_0, x_1) + \beta^k \lambda^{n+k} \rho(x_0, x_1) \\
&= \lambda^n \left[\alpha \sum_{i=0}^{k-1} \beta^i \lambda^i + \beta^k \lambda^k \right] \rho(x_0, x_1) \\
&= \lambda^n \left[\alpha \frac{1 - \beta^k \lambda^k}{1 - \beta\lambda} + \beta^k \lambda^k \right] \rho(x_0, x_1) \\
&= \frac{\lambda^n}{(1 - \beta\lambda)} \left[\alpha - \beta^k \lambda^k (\alpha + \beta\lambda - 1) \right] \rho(x_0, x_1) \\
&< \lambda^n \frac{\alpha}{(1 - \beta\lambda)} \rho(x_0, x_1). \qquad (2.9)
\end{aligned}
$$

Since $\lim_{n\to\infty} \lambda^n = 0$, it follows that $\{x_n\}$ is a Cauchy sequence. By the completeness of (X, ρ) it follows that there exists $x^* \in X$ such that $\lim_{n\to\infty} \rho(x_n, x^*) = 0$. Furthermore, using the α, β triangle inequality, we have

$$\begin{aligned}\rho(x^*, Tx^*) &\leq \alpha\rho(x^*, x_{n+1}) + \beta\rho(x_{n+1}, Tx^*) \\ &\leq \alpha\rho(x^*, x_{n+1}) + \beta\rho(x_{n+1}, x^*) \\ &\to 0 \text{ as } n \to \infty. \end{aligned} \tag{2.10}$$

Hence $\rho(x^*, Tx^*) = 0$ and so $Tx^* = x^*$. Suppose there exists $x^{**}, x^* \in X$ such that $Tx^* = x^*$ and $Tx^{**} = x^{**}$. Then

$$\rho(x^{**}, x^*) = \rho(Tx^{**}, Tx^*) \leq \lambda\rho(x^{**}, x^*), \tag{2.11}$$

which implies that $\rho(x^{**}, x^*) = 0$, i.e., $x^{**} = x^*$. □

Remark 1. Using (2.9) we get

$$\begin{aligned}\rho(x_n, x^*) &\leq \alpha\rho(x_n, x_{n+k+1}) + \beta\rho(x_{n+k+1}, x^*) \\ &\leq \lambda^n \frac{\alpha^2}{(1-\beta\lambda)}\rho(x_0, x_1) + \beta\rho(x_{n+k+1}, x^*). \end{aligned} \tag{2.12}$$

Letting $k \to \infty$ we get the order of convergence:

$$\rho(x_n, x^*) \leq \lambda^n \frac{\alpha^2}{(1-\beta\lambda)}\rho(x_0, x_1) \tag{2.13}$$

which implies at least linear convergence.

Theorem 2.2. *Let (X, ρ) be a complete generalized b-metric space, where ρ satisfies the α, β relaxed polygonal inequality and $T : X \to X$ a contraction mapping such that $0 < \lambda < 1$. Then T has a unique fixed point $x^* \in X$.*

Proof. The proof follows in line with the Theorem 2.1 □

Remark 2. Using (2.5) and (2.8) results in

$$\begin{aligned}
\rho(x_n, x_{n+k}) &\leq \frac{(\alpha+\beta)}{2} \sum_{i=0}^{k-1} \rho(x_{n+i}, x_{n+i+1}) \\
&\leq \frac{(\alpha+\beta)}{2} \left(\lambda^n + \lambda^{n+1} + \cdots + \lambda^{n+k}\right) \rho(x_0, x_1) \\
&= \frac{(\alpha+\beta)}{2} \lambda^n \frac{1-\lambda^{k+1}}{1-\lambda} \rho(x_0, x_1). \qquad (2.14)
\end{aligned}$$

Hence we obtain the priori estimate

$$\rho(x_n, x^*) \leq \frac{(\alpha+\beta)}{2(1-\lambda)} \lambda^n \rho(x_0, x_1). \qquad (2.15)$$

Definition 2.4.2. Let (X, ρ) be a complete b-metric space such that a mapping $T : X \to X$ is a Kannan contraction [17] if there exists $\lambda \in [0, \frac{1}{2})$ such that

$$\rho(Tx, Ty) \leq \lambda \left[\rho(x, Tx) + \rho(y, Ty)\right] \qquad (2.16)$$

for all $x, y \in X$.

Let $x_0 \in X$ be fixed, then for $n \in \mathbb{N}$

$$\begin{aligned}
\rho(T^n x_0, T^{n+1} x_0) &= \rho(T T^{n-1} x_0, T T^n x_0) \\
&\leq \lambda \left[\rho(T^n x_0, T^{n-1} x_0) + \rho(T^{n+1} x_0, T^n x_0)\right], \qquad (2.17)
\end{aligned}$$

then it follows that

$$\rho(T^n x_0, T^{n+1} x_0) - \lambda \rho(T^{n+1} x_0, T^n x_0) \leq \lambda \rho(T^n x_0, T^{n-1} x_0) \qquad (2.18)$$

which implies that

$$\rho(T^n x_0, T^{n+1} x_0) \leq \frac{\lambda}{1-\lambda} \rho(T^n x_0, T^{n-1} x_0). \qquad (2.19)$$

Successively using (2.19) we obtain that

$$\rho(T^n x_0, T^{n+1} x_0) \leq \left(\frac{\lambda}{1-\lambda}\right)^n \rho(x_0, T x_0). \qquad (2.20)$$

Theorem 2.3. *Let (X, ρ) be a complete generalized b-metric space and let $T : X \to X$ be a mapping for which there exists $\lambda \in \left[0, \frac{1}{\beta+1}\right)$ such that*

$$\rho(Tx, Ty) \leq \lambda \left[\rho(x, Tx) + \rho(y, Ty)\right] \tag{2.21}$$

for all $x, y \in X$. Then T has a unique fixed point.

Proof. We begin by showing that for $x_0 \in X$ fixed, and for $n \in \mathbb{N}$

$\{T^n x_0\}$ is a Cauchy sequence in (X, ρ): for $m, n \in X$ with $m > n$, and using inequality (2.4) and (2.20), we get

$$\begin{aligned}
&\rho(T^n x_0, T^m x_0) \\
&\leq \alpha \sum_{i=0}^{m-n-2} \beta^i \rho(T^{n+i} x_0, T^{n+i+1} x_0) + \beta^{m-n-1} \rho(T^{m-1} x_0, T^m x_0) \\
&\leq \alpha \sum_{i=0}^{m-n-2} \beta^i \left(\frac{\lambda}{1-\lambda}\right)^{n+i} \rho(x_0, Tx_0) + \beta^{m-n-1} \left(\frac{\lambda}{1-\lambda}\right)^{m-1} \rho(x_0, Tx_0) \\
&= \left[\alpha \sum_{i=0}^{m-n-2} \beta^i \left(\frac{\lambda}{1-\lambda}\right)^{n+i} + \beta^{m-n-1} \left(\frac{\lambda}{1-\lambda}\right)^{m-1}\right] \rho(x_0, Tx_0) \\
&= \left(\frac{\lambda}{1-\lambda}\right)^n \left[\alpha \sum_{i=0}^{m-n-2} \beta^i \left(\frac{\lambda}{1-\lambda}\right)^{i} + \beta^{m-n-1} \left(\frac{\lambda}{1-\lambda}\right)^{m-n-1}\right] \rho(x_0, Tx_0) \\
&= \left(\frac{\lambda}{1-\lambda}\right)^n \left[\alpha \frac{1 - \left(\frac{\beta\lambda}{1-\lambda}\right)^{m-n-1}}{1 - \left(\frac{\beta\lambda}{1-\lambda}\right)} + \left(\frac{\beta\lambda}{1-\lambda}\right)^{m-n-1}\right] \rho(x_0, Tx_0).
\end{aligned} \tag{2.22}$$

It follows that as $n \to \infty$ that the sequence $\{T^n x_0\}$ is a Cauchy sequence in (X, ρ). Since (X, ρ) is complete there exists a $z_0 \in X$ such that

$$\lim_{n\to\infty} \rho(T^n x_0, z_0) = 0.$$

By the contraction (2.21), we obtain that

$$\rho(T^{n+1} x_0, Tz_0) \leq \lambda \rho(T^{n+1} x_0, T^n x_0) + \lambda \rho(z_0, Tz_0) \tag{2.23}$$

In the limit as $n \to \infty$, we get

$$\rho(z_0, Tz_0) \leq \lambda \rho(z_0, Tz_0). \tag{2.24}$$

Since $\lambda < 1$ we deduce that $\rho(z_0, Tz_0) = 0$. If we assume that $z' \in X$ is any fixed point then we obtain

$$\begin{aligned} \rho(z_0, z') &= \rho(Tz_0, Tz') \\ &\leq \lambda \left[\rho(z_0, Tz_0) + \rho(z', Tz')\right] \\ &= \lambda \left[\rho(z_0, z_0) + \rho(z', z')\right] \\ &= 0 \end{aligned} \tag{2.25}$$

which implies that $z' = z_0$. □

Remark 3. From (2.22) we obtain the priori estimate

$$\rho(T^n(x_0), z_0) \leq \left(\frac{\lambda}{1-\lambda}\right)^n \left(\frac{\alpha}{1 - \frac{\beta\lambda}{1-\lambda}}\right) \rho(x_0, Tx_0). \tag{2.26}$$

Chapter 3

Generalized convex metric and fixed point theorems.

3.1 Introduction

In 1922 Banach [39] proved his famous fixed point theorem that every contraction mapping on a complete metric space has a unique fixed point. Since then there has been numerous extensions to his work, especially in changing the underlying structure of the metric space or introducing new contraction types. Czerwik [10] relaxed the triangular inequality and formally defined a b-metric space. In 1970, Takahasi [40] introduced the concept of convexity in metric spaces and proved fixed point theorems for contraction mappings in such spaces. Chen et. al. [8] discussed fixed point theorems in convex b-metric spaces. Here we discuss such concepts in a convex $\alpha\beta$ b-metric space. Fixed point theory is important in non linear analysis and functional analysis. It finds application in systems of non linear differential, integral and algebraic equations.

3.2 Preliminaries

Definition 3.2.1. [13] Let $I = [0, 1)$. Define $\rho : X \times X \to [0, \infty)$ and a continuous function $\omega : X \times X \times I \to X$. Then ω is said to be a convex structure on X if the following holds.

$$\rho(z, \omega(x, y; \mu)) \leq \mu\rho(z, x) + (1 - \mu)\rho(z, y) \tag{3.1}$$

for each $z \in X$ and $(x, y; \mu) \in X \times X \times I$

Example 4. Let $X = [1, 3]$ and define ρ by

$$\rho(x, y) = \begin{cases} 3^{|x-y|} & x \neq y \\ \\ 0 & x = y \end{cases} \tag{3.2}$$

(X, ρ) is a $\alpha\beta$ b-metric space as

$$\begin{aligned} \rho(x, y) &\leq 3^{|x-z|+|z-y|} \\ &= 3^{\frac{1}{3}|x-z|+\frac{2}{3}|(z-y)|} 3^{\frac{2}{3}|x-z|+\frac{1}{3}|z-y|} \\ &\leq \left(\frac{1}{3}3^{|x-z|} + \frac{2}{3}3^{|z-y|}\right) \sup_X 3^{\frac{2}{3}|x-z|+\frac{1}{3}|z-y|} \\ &= 3\rho(x, z) + 6\rho(z, y) \end{aligned}$$

Since

$$\rho(1, 3) > \rho(1, 2)_+\rho(2, 3) \tag{3.3}$$

(X, ρ) is not a metric space. Define $\omega(x, y; \mu) = \mu x + (1-\mu)y$ for $\mu \in I$, then

$$\begin{aligned}
\rho(z, \omega(x, y; \mu)) &= \rho(z, \mu x + (1-\mu)y) \\
&= 3^{|z-\mu x-(1-\mu)y|} \\
&= 3^{|\mu(z-x)+(1-\mu)(z-y)|} \\
&\leq 3^{\mu|(z-x)|+(1-\mu)|(z-y)|} \\
&\leq \mu 3^{|z-x|} + (1-\mu)3^{|z-y|} \\
&= \mu\rho(z, x) + (1-\mu)\rho(z, y)
\end{aligned}$$

3.3 Main Results

Theorem 3.1. *Let (X, ρ, ω) be a compete convex α, β b-metric space and $T : X \to X$ be a contraction mapping, that is there exists $\lambda \in [0, 1)$ such that $\rho(Tx, Ty) \leq \lambda\rho(x, y), \forall x, y \in X$. Choose $x_0 \in X$ such that $\rho(x_0, Tx_0) < \infty$ and define $x_n = \omega(x_{n-1}, Tx_{n-1}; \mu_{n-1})$, where*

$$0 < \mu_{n-1} < \frac{\frac{1}{\beta^3} - \lambda}{\frac{\alpha}{\beta} - \lambda} \quad , \lambda < \frac{1}{\beta^3}$$

for each $n \in \mathbb{N}$, then T has a unique fixed point in X.

Proof.

$$\rho(x_n, x_{n+1}) = \rho(x_n, \omega(x_n, Tx_n; \mu_n)) \leq (1-\mu_n)\rho(x_n, Tx_n) \tag{3.4}$$

$$\rho(x_n, Tx_n) \leq \alpha\rho(x_n, Tx_{n-1}) + \beta\rho(Tx_{n-1}, Tx_n) \tag{3.5}$$

$$\leq \alpha\rho(\omega(x_{n-1}, Tx_{n-1}; \mu_{n-1}), Tx_{n-1}) + \beta\lambda\rho(x_{n-1}, x_n) \tag{3.6}$$

$$\leq \alpha\mu_{n-1}\rho(x_{n-1}, Tx_{n-1}) + \beta\lambda(1-\mu_{n-1})\rho(x_{n-1}, Tx_{n-1}) \tag{3.7}$$

$$= \left[\alpha\mu_{n-1} + \beta\lambda(1-\mu_{n-1})\right]\rho(x_{n-1}, Tx_{n-1}) \tag{3.8}$$

$$< \frac{1}{\beta^2}\rho(x_{n-1}, Tx_{n-1}) \tag{3.9}$$

$$< \rho(x_{n-1}, Tx_{n-1}) \tag{3.10}$$

Hence $\{\rho(x_n, Tx_n)\}$ is a decreasing sequence of non-negative reals. Hence there exists $\gamma \geq 0$ such that $\lim_{n\to\infty} \rho(x_n, Tx_n) = \gamma$. If $\gamma > 0$ then letting $n \to \infty$ in (3.10) we have $\gamma < \gamma$, a contraction. Hence $\gamma = 0$ and from (3.4) it follows that $\lim_{n\to\infty} \rho(x_n, Tx_n) = 0$. We now show that $\{x_n\}$ is a Cauchy sequence. Let $m > n$ then

$$\rho(x_m, x_n) \leq \alpha\rho(x_n, x_{n+1}) + \beta\rho(x_{n+1}, x_m) \tag{3.11}$$

$$\leq \alpha\rho(x_n, x_{n+1}) + \beta\left[\alpha\rho(x_{n+1}, x_{n+2}) + \beta\rho(x_{n+2}, x_m)\right] \tag{3.12}$$

$$\leq \alpha\rho(x_n, x_{n+1}) + \beta\alpha\rho(x_{n+1}, x_{n+2}) + \beta^2\rho(x_{n+2}, x_m) \tag{3.13}$$

$$\leq \alpha\rho(x_n, x_{n+1}) + \beta\alpha\rho(x_{n+1}, x_{n+2}) + \beta^2\rho(x_{n+2}, x_m) \tag{3.14}$$

$$+ \cdots \beta^{m-n-1}\rho(x_{m-1}, x_m) \tag{3.15}$$

$$< \alpha \sum_{k=0}^{m-n-1} \beta^k \rho(x_{n+k}, x_{n+k+1}) \tag{3.16}$$

Now from (3.4),(3.9) and (3.10) it follows that

$$\rho(x_n, x_{n+1}) = \rho(x_n, \omega(x_n, Tx_n; \mu_n)) \leq (1-\mu_n)\rho(x_n, Tx_n) \tag{3.17}$$

$$< \rho(x_n, Tx_n) \tag{3.18}$$

$$< \frac{1}{\beta^2}\rho(x_{n-1}, Tx_{n-1}) \tag{3.19}$$

$$< \frac{1}{\beta^4}\rho(x_{n-2}, Tx_{n-2}) \tag{3.20}$$

$$< \frac{1}{\beta^{2k}}\rho(x_{n-k}, Tx_{n-k}) \tag{3.21}$$

$$\tag{3.22}$$

Hence

$$\rho(x_{n+k}, x_{n+k+1}) < \frac{1}{\beta^{2k}}\rho(x_n, Tx_n) \tag{3.23}$$

Substituting (3.23) in (3.16) we obtain

$$\rho(x_m, x_n) < \alpha \sum_{k=0}^{m-n-1} \frac{1}{\beta^k}\rho(x_n, Tx_n) \tag{3.24}$$

$$< \alpha\rho(x_n, Tx_n) \sum_{k=0}^{\infty} \left(\frac{1}{\beta}\right)^k \tag{3.25}$$

$$= \frac{\alpha\beta}{\beta-1}\rho(x_n, Tx_n). \tag{3.26}$$

Hence $\lim_{m,n\to\infty} \rho(x_m, x_n) = 0$, which implies that $\{x_n\}$ is Cauchy. By the completeness of X there exists $x^\star \in X$ such that $\lim_{n\to\infty} \rho(x_n, x^\star) = 0$. We now verify that $x^\star$ is a fixed point of T.

$$\rho(x^\star, Tx^\star) \leq \alpha\rho(x^\star, x_n) + \beta\rho(x_n, Tx^\star) \tag{3.27}$$

$$\leq \alpha\rho(x^\star, x_n) + \beta\left[\alpha\rho(x_n, Tx_n) + \beta\rho(Tx_n, Tx^\star)\right] \tag{3.28}$$

$$\leq \alpha\rho(x^\star, x_n) + \beta\alpha\rho(x_n, Tx_n) + \beta^2\lambda\rho(x_n, x^\star) \tag{3.29}$$

$$= \left(\alpha + \beta^2\lambda\right)\rho(x^\star, x_n) + \beta\alpha\rho(x_n, Tx_n) \tag{3.30}$$

Let $n \to \infty$ in (3.30) to conclude that $\rho(x^\star, Tx^\star) \to 0$, hence $Tx^\star = x^\star$. If $x^{\star\star}$ is another fixed point of T then

$$\rho(x^\star, x^{\star\star}) \leq \rho(Tx^\star, Tx^{\star\star}) \leq \lambda\rho(x^\star, x^{\star\star}). \tag{3.31}$$

Hence $\rho(x^\star, x^{\star\star}) = 0$ otherwise $\lambda \geq 1$ is a contradiction and the fixed point is unique.

□

Theorem 3.2. *Let (X, ρ, ω) be a complete convex α, β b-metric space and $T : X \to X$ be defined by $\rho(Tx, Ty) \leq \lambda\left[\rho(x, Tx) + \rho(y, Ty)\right], \forall x, y \in X$ and for $0 < \lambda < \frac{1}{\beta^4}$. Choose $x_0 \in X$ such that $\rho(x_0, Tx_0) < \infty$ and define $x_n = \omega(x_{n-1}, Tx_{n-1}; \mu_{n-1})$, where*

$$0 < \mu_{n-1} < \frac{1}{\alpha}\left(\frac{1}{\beta^2} - \frac{1}{\beta^3} - \frac{1}{\beta^5}\right), \quad 1 + \beta^2 < \beta^3$$

for each $n \in \mathbb{N}$, then T has a unique fixed point in X.

Proof.

$$\rho(x_n, x_{n+1}) = \rho(x_n, \omega(x_n, Tx_n; \mu_n)) \leq (1 - \mu_n)\rho(x_n, Tx_n) \tag{3.32}$$

$$\rho(x_n, Tx_n) \leq \alpha\rho(x_n, Tx_{n-1}) + \beta\rho(Tx_{n-1}, Tx_n) \tag{3.33}$$

$$\leq \alpha\rho(x_n, Tx_{n-1}) + \beta\lambda\left[\rho(x_{n-1}, Tx_{n-1}) + \rho(x_n, Tx_n)\right] \tag{3.34}$$

$$= \alpha\rho(\omega(x_{n-1}, Tx_{n-1}; \mu_{n-1}), Tx_{n-1}) + \beta\lambda\rho(x_{n-1}, Tx_{n-1}) \tag{3.35}$$

$$+ \beta\lambda\rho(x_n, Tx_n) \tag{3.36}$$

$$= \alpha\mu_{n-1}\rho(x_{n-1}, Tx_{n-1}) + \beta\lambda\rho(x_{n-1}, Tx_{n-1}) + \beta\lambda\rho(x_n, Tx_n) \tag{3.37}$$

We observe that $0 < 1 - \beta\lambda$, hence

$$\rho(x_n, Tx_n) \leq \frac{\alpha\mu_{n-1} + \beta\lambda}{1 - \beta\lambda}\rho(x_{n-1}, Tx_{n-1}) \tag{3.38}$$

$$\leq \frac{\rho(x_{n-1}, Tx_{n-1})}{\beta^2} \tag{3.39}$$

as proved in Theorem 3.1. Hence $\rho(x_n, Tx_n)$ is a decreasing sequence that converges to zero and is also Cauchy. If $\lim_{n\to\infty} \rho(x_n, x^\star) = 0$ then

$$\rho(x^\star, Tx^\star) \leq \alpha\rho(x^\star, x_n) + \beta\rho(x_n, Tx^\star) \tag{3.40}$$

$$\leq \alpha\rho(x^\star, x_n) + \beta[\alpha\rho(x_n, Tx_n) + \beta\rho(Tx_n, Tx^\star)] \tag{3.41}$$

$$\leq \alpha\rho(x^\star, x_n) + \alpha\beta\rho(x_n, Tx_n) + \beta^2\lambda[\rho(x_n, Tx_n) + \rho(x^\star, Tx^\star)] \tag{3.42}$$

Then

$$\left(1 - \beta^2\lambda\right)\rho(x^\star, Tx^\star) \leq \alpha\rho(x^\star, x_n) + \left(\alpha\beta^2\lambda\right)\rho(x_n, Tx_n) \tag{3.43}$$

$$\leq \alpha\rho(x^\star, x_n) + \left(\alpha\beta^2\lambda\right)\frac{\rho(x_0, Tx_0)}{\beta^{2n}} \tag{3.44}$$

Letting $n \to \infty$ we obtain $\rho(x^\star, Tx^\star) = 0$, so $x^\star$ is a fixed point of T. If $x^{\star\star}$ is another fixed point of T then

$$\rho(x^\star, x^{\star\star}) = \rho(Tx^\star, Tx^{\star\star}) \leq \lambda[\rho(x^\star, Tx^\star) + \rho(x^{\star\star}, Tx^{\star\star})] = 0 \tag{3.45}$$

proving that the fixed point is unique. □

Lemma 3.3[39] Let $\{y_n\}, \{z_n\}$ be non-negative sequences satisfying $y_{n+1} \leq z_n + hy_n$ for all $n \in \mathbb{N}$, $0 \leq h < 1$, $\lim_{n\to\infty} z_n = 0$, then $\lim_{n\to\infty} y_n = 0$

Definition 3.3.1. Let T be a self map on a compete α, β b-metric space (X, ρ). The iterative procedure $x_{n+1} = f(T, x_n)$ is weakly T-stable if $\{x_n\}$ converges to a fixed point $x^\star$ of T and if $\{y_n\}$ is a sequence in X such that $\lim_{n\to\infty} \rho(y_{n+1}, f(T, y_n)) = 0$ and $\{\rho(y_n, Ty_n)\}$ is bounded then $\lim_{n\to\infty} y_n = x^\star$.

Theorem 3.3. *Under the assumptions of Theorem 3.1, if in addition* $\lim_{n\to\infty} \mu_n = 0$, *then Mann's iteration is weakly T-stable.*

Proof. From Theorem 3.1 $x^\star$ is a fixed point of T in X. If $\{y_n\}$ is a sequence such that $\lim_{n\to\infty} \rho(y_{n+1}, \omega(y_n, Ty_n; \mu_n)) = 0$ and $\{\rho(y_n, Ty_n)\}$ is bounded then

$$\rho(y_{n+1}, x^\star) \leq \alpha\rho(y_{n+1}, \omega(y_n, Ty_n; \mu_n)) + \beta\rho(\omega(y_n, Ty_n; \mu_n), x^\star) \tag{3.46}$$

$$\leq \alpha\rho(y_{n+1}, \omega(y_n, Ty_n; \mu_n)) + \beta[\alpha\rho(\omega(y_n, Ty_n; \mu_n), Ty_n)$$

$$+ \beta\rho(Ty_n, Tx^\star)] \tag{3.47}$$

$$\leq \alpha\rho(y_{n+1}, \omega(y_n, Ty_n; \mu_n)) + \beta\alpha\mu_n\rho(y_n, Ty_n) + \beta^2\lambda\rho(y_n, x^\star)] \tag{3.48}$$

$$= z_n + \beta^2\lambda\rho(y_n, x^\star)]. \tag{3.49}$$

Since $\beta^2\lambda < 1$ and $\{\rho(y_n, Ty_n)\}$ is bounded, $\lim_{n\to\infty} z_n = 0$ and hence by Lemma 3.3 $\lim_{n\to\infty} \rho(y_n, x^\star) = 0$. □

Theorem 3.4. *Under the assumptions of Theorem 3.2, if in addition* $\lim_{n\to\infty} \mu_n = 0$, *and if* α, β, λ *satisfy additionally* $\frac{\alpha\beta^2}{1-\lambda\beta} < 1$ *then Mann's iteration is weakly* T*-stable.*

Proof. From Theorem 3.2 $x^\star$ is a fixed point of T in X. If $\{y_n\}$ is a sequence such that $\lim_{n\to\infty} \rho(y_{n+1}, \omega(y_n, Ty_n; \mu_n)) = 0$ and $\{\rho(y_n, Ty_n)\}$ is bounded then

$$\rho(y_{n+1}, x^\star) \leq \alpha\rho(y_{n+1}, \omega(y_n, Ty_n; \mu_n)) + \beta\rho(\omega(y_n, Ty_n; \mu_n), x^\star) \tag{3.50}$$

$$\leq \alpha\rho(y_{n+1}, \omega(y_n, Ty_n; \mu_n)) + \beta\alpha\rho(\omega(y_n, Ty_n; \mu_n), Ty_n) + \beta^2\rho(Ty_n, x^\star) \tag{3.51}$$

Now

$$\rho(Ty_n, x^\star) = \rho(Ty_n, Tx^\star) \leq \lambda\rho(y_n, Ty_n) \tag{3.52}$$

$$\leq \lambda\alpha\rho(y_n, x^\star) + \lambda\beta\rho(x^\star, Ty_n). \tag{3.53}$$

From which we get

$$\rho(Ty_n, x^\star) \leq \frac{\lambda\alpha}{1-\lambda\beta}\rho(y_n, x^\star). \tag{3.54}$$

$$\rho(y_{n+1}, x^\star) \leq \alpha\rho(y_{n+1}, \omega(y_n, Ty_n; \mu_n)) + \beta\alpha\mu_n\rho(y_n, Ty_n) + \frac{\lambda\alpha\beta^2}{1-\lambda\beta}\rho(y_n, x^\star) \tag{3.55}$$

$$= z_n + \frac{\lambda\alpha\beta^2}{1-\lambda\beta}\rho(y_n, x^\star) \tag{3.56}$$

Since $\frac{\lambda\alpha\beta^2}{1-\lambda\beta} < 1$ and $\{\rho(y_n, Ty_n)\}$ is bounded, $\lim_{n\to\infty} z_n = 0$ and hence by Lemma 3.3 $\lim_{n\to\infty} \rho(y_n, x^\star) = 0$. □

Chapter 4

Complex valued b-metric space with fixed point theorems.

4.1 Introduction

The concept of a b-metric was initiated from the contributions of Bourbaki [7] and Bakhtin [5]. Czerwik [10] gave an axiom which was weaker than the triangular inequality and formally defined a b-metric space with a view of generalizing the Banach contraction mapping theorem. Later on, Fagin et al. [14] discussed some kind of relaxation in the triangular inequality and called this new distance measure a non-linear elastic pattern matching. In 2011, A. Azzam, B. Fisher and M. Khan introduced the notion of a complex valued metric space and called the complex-valued metric space as an extension of the classical metric space and proved some common fixed point theorems, [3]. In a similar way various authors have studied and proved the fixed point results for mappings satisfying different types of contractive conditions in the framework of complex-valued metric spaces,[6]. In 2013, Rao, et al. introduced the concept of a complex-valued b-metric space which is a generalization of the concept of a complex-valued metric space, [27] and subsequent to that A.A Mukheimer obtained common fixed point results, [21]. In this thesis, we generalize the concept of a complex-valued b-metric and prove the common fixed results satisfying certain expressions in this new space.

4.2 Preliminaries

Let $\mathbb{C}$ be the set of complex numbers and if $z_1, z_2 \in \mathbb{C}$ then define a partial ordering $\precsim$ on $\mathbb{C}$ as follows:

$z_1 \precsim z_2 \iff \text{Re}(z_1) \leq \text{Re}(z_2)$ and $\text{Im}(z_1) \leq \text{Im}(z_2)$

Futhermore, if $z_3 \in \mathbb{C}$, we obtain that following:

(i) If $0 \precsim z_1 \precsim z_2$ then $|z_1| < |z_2|$

(ii)If $z_1 \precsim z_2$ and $z_2 \precsim z_3$ then $z_1 \precsim z_3$

(iii) If $a, b \in \mathbb{R}$ and $a \leq b$ then $az \precsim bz$ for all $z \in \mathbb{C}$

Definition 4.2.1. Let X be a non-empty set. A function $d : X \times X \to \mathbb{C}$ is a complex-valued b-metric on X, [27], if there exists a real number $\alpha \geq 1$ such that the following conditions hold for all $x, y, z \in X$:

(i) $0 \precsim d(x, y)$ and $d(x, y) = 0 \iff x = y$

(ii) $d(x, y) = d(y, x)$

(iii) $d(x, y) \precsim \alpha\,[d(x, z) + d(z, y)]$

The pair (X, d) is a called a complex-valued b-metric space.

Definition 4.2.2. Let X be a non-empty set. A function $\rho : X \times X \to \mathbb{C}$ is a generalized α, β complex-valued b-metric on X if there exists real numbers $\alpha, \beta \geq 1$ such that the following conditions hold for all $x, y, z \in X$:

(i) $0 \precsim \rho(x, y)$ and $\rho(x, y) = 0 \iff x = y$

(ii) $\rho(x, y) = \rho(y, x)$

(iii) $\rho(x, y) \precsim \alpha\rho(x, z) + \beta\rho(z, y)$

The pair (X, ρ) is a called a α, β complex-valued b-metric space.

The following example justifies the generalization found in the definition.

Example 5. Let $X = (1, 3)$ and let $\rho : X \times X \to \mathbb{C}$ be a function defined by

$$\rho(x, y) = \begin{cases} e^{|x-y|} + ie^{|x-y|}, & \text{if } x \neq y \\ \\ 0, & \text{iff } x = y. \end{cases}$$

To show that the example is a generalized α, β complex-valued b-metric, we only need to verify the α, β triangle inequality:

For $x \neq y,\ z \in X$ and $\theta \in (0, 1)$

$$\begin{aligned}
\rho(x, y) &\precsim (1 + i)e^{|x-z|+|z-y|} \\
&= (1 + i)e^{\theta|x-z|+(1-\theta)|z-y|}e^{(1-\theta)|x-z|+\theta|z-y|} \\
&\precsim \sup_{x,y,z\in X} e^{\theta|x-z|+(1-\theta)|z-y|} \left((1 - \theta)(1 + i)e^{|x-z|} + \theta(1 + i)e^{|z-y|}\right) \\
&\precsim (1 - \theta)e^2(1 + i)e^{|x-z|} + \theta(1 + i)e^2e^{|z-y|} \\
&= (1 - \theta)e^2\rho(x, z) + \theta e^2\rho(z, y).
\end{aligned}$$

For $\theta = \frac{1}{3}$ we have constants $\alpha = \frac{2}{3}e^2$ and $\beta = \frac{1}{3}e^2$.

One introduces a topology on a generalized α, β b-metric space (X, ρ) in the usual way. The open ball $B(x, \epsilon)$ with centre $x \in X$ and radius $0 \prec \epsilon \in \mathbb{C}$ is given by

$$B(x, \epsilon) = \{y \in X : \rho(x, y) \prec \epsilon\}$$

A subset A of X is open if for every $x \in A$ there is a number $0 \prec \epsilon \in \mathbb{C}$ such that $B(x, \epsilon) \subseteq A$.

Definition 4.2.3. Let (X, ρ) be a generalized α, β complex-valued b-metric space, and let $\{x_n\}$ be a sequence in X and $x \in X$. Then:

(i) The sequence $\{x_n\}$ converges to $x \in X$, if for every $0 \prec \epsilon \in \mathbb{C}$ then there is $N \in \mathbb{N}$ such that $\rho(x_n, x) \prec \epsilon$. The sequence $\{x_n\}$ converges to $x \in X \iff |\rho(x_n, x)| \to 0$ as $n \to \infty$, [27].

(ii) The sequence $\{x_n\}$ is a Cauchy in (X, ρ) if for every $\epsilon \in \mathbb{C}$ there is $N \in \mathbb{N}$ such that $\rho(x_n, x_{n+m}) \prec \epsilon$, where $m \in \mathbb{N}$. The sequence $\{x_n\}$ is a Cauchy sequence in $(X, \rho) \iff |\rho(x_n, x_{n+m})| \to 0$ as $n \to \infty$, where $m \in \mathbb{N}$, [27].

(iii) The space (X, ρ) is complete if every Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$.

4.3 Fixed point theorem for generalized α, β complex-valued b-metric spaces

As a consequence of (iii) of Definition 4.2.2 , the α, β triangle inequality, we get for $n, m \in \mathbb{N}$ with $m > n$

$$\begin{aligned}
&\rho(x_n, x_m) \\
&\precsim \alpha\rho(x_n, x_{n+1}) + \beta\rho(x_{n+1}, x_m) \\
&\precsim \alpha\rho(x_n, x_{n+1}) + \beta[\alpha\rho(x_{n+1}, x_{n+2}) + \beta\rho(x_{n+2}, x_m)] \qquad (4.1)
\end{aligned}$$

Successively applying the α, β triangle inequality, we obtain

$$\rho(x_n, x_m) \precsim \alpha \sum_{i=0}^{m-n-2} \beta^i \rho(x_{n+i}, x_{n+i+1}) + \beta^{m-n-1}\rho(x_{m-1}, x_m). \qquad (4.2)$$

This theorem is a generalization of the fixed point theorem studied by Mishra, et al., in [20].

Theorem 4.1. *Let (X, ρ) be a complete generalized α, β complex-valued b-metric space and $T : X \to X$ a mapping such that*

$$\rho(Tx, Ty) \precsim a\rho(x, Tx) + b\rho(y, Ty) + c\rho(x, y)$$

for all $x, y \in X$, where $a, b,$ and c are non-negative real numbers satisfying $a+\beta(b+c) < 1$, then T has a unique fixed point.

Proof. We begin by proving that for $x_0 \in X$, the sequence $\{x_n\}$ generated by the recussive formula $x_n = Tx_{n-1} = T^n x_0$ is a Cauchy sequence in X. For $n \in \mathbb{N}$ we obtain

$$\rho(x_{n+2}, x_{n+1}) = \rho(Tx_{n+1}, Tx_n) \tag{4.3}$$

$$\preccurlyeq a\rho(x_{n+1}, Tx_{n+1}) + b\rho(x_n, Tx_n) + c\rho(x_{n+1}, x_n)$$

$$= a\rho(x_{n+2}, x_{n+1}) + b\rho(x_{n+1}, x_n) + c\rho(x_{n+1}, x_n)$$

$$= a\rho(x_{n+2}, x_{n+1}) + (b+c)\rho(x_{n+1}, x_n) \tag{4.4}$$

$$(1-a)\rho(x_{n+2}, x_{n+1}) \preccurlyeq (b+c)\rho(x_{n+1}, x_n)$$

$$\rho(x_{n+2}, x_{n+1}) \preccurlyeq \left(\frac{b+c}{1-a}\right)\rho(x_{n+1}, x_n) \tag{4.5}$$

If we let $\gamma = \left(\frac{b+c}{1-a}\right)$ then by repeated use of (4.5) we get

$$\rho(x_{n+2}, x_{n+1}) \preccurlyeq \gamma^{n+1}\rho(x_1, x_0). \tag{4.6}$$

Using (4.2) and (4.6) for $k \in \mathbb{N}$, we get

$$\begin{aligned}
\rho(x_n, x_{n+k+1}) &\preccurlyeq \alpha \sum_{i=0}^{k-1} \beta^i \rho(x_{n+i}, x_{n+i+1}) + \beta^k \rho(x_{n+k}, x_{n+k+1}) \\
&\preccurlyeq \alpha \sum_{i=0}^{k-1} \beta^i \gamma^{n+i} \rho(x_0, x_1) + \beta^k \gamma^{n+k} \rho(x_0, x_1) \\
&= \gamma^n \left[\alpha \sum_{i=0}^{k-1} \beta^i \gamma^i + \beta^k \gamma^k\right] \rho(x_0, x_1) \\
&= \gamma^n \left[\alpha \frac{1-\beta^k\gamma^k}{1-\beta\gamma} + \beta^k \gamma^k\right] \rho(x_0, x_1) \\
&= \frac{\gamma^n}{(1-\beta\gamma)} \left[\alpha - \beta^k\gamma^k(\alpha + \beta\gamma - 1)\right] \rho(x_0, x_1) \\
&\prec \gamma^n \frac{\alpha}{(1-\beta\gamma)} \rho(x_0, x_1). \qquad (4.7)
\end{aligned}$$

Now,

$$|\rho(x_n, x_{n+k+1})| \leq \gamma^n \frac{\alpha}{(1-\beta\gamma)} |\rho(x_0, x_1)| \tag{4.8}$$

Since $a + \beta(b + c) < 1$ for $\beta \geq 1$ then $\beta\gamma < 1$ and $\gamma < 1$. Taking the limit $n \to \infty$ we get $\gamma^n \to 0$, which implies that $|\rho(x_n, x_{n+k+1})| \to 0$ as $n \to \infty$ thus the sequence $\{x_n\}$ is a Cauchy sequence. Since X is a complete α, β complex-valued b-metric space the sequence converges to $x^* \in X$. We show that x^* is a fixed point of T. Using the α, β triangle inequality, we have

$$\begin{aligned}
\rho(x^*, Tx^*) &\precsim \alpha\rho(x^*, x_{n+1}) + \beta\rho(x_{n+1}, Tx^*) \\
&\precsim \alpha\rho(x^*, x_{n+1}) + \beta\rho(Tx_n, Tx^*) \\
&\precsim \alpha\rho(x^*, x_{n+1}) + \beta[a\rho(x_n, Tx_n) + b\rho(x^*, Tx^*) + c\rho(x_n, x^*)] \qquad (4.9)
\end{aligned}$$

$$\begin{aligned}
(1 - b\beta)\rho(x^*, Tx^*) &\precsim \alpha\rho(x^*, x_{n+1}) + a\beta\rho(x_n, Tx_n) + c\beta\rho(x_n, x^*) \\
\rho(x^*, Tx^*) &\precsim \frac{1}{(1-b\beta)} [\alpha\rho(x^*, x_{n+1}) + a\beta\rho(x_n, x_{n+1}) + c\beta\rho(x_n, x^*)] \qquad (4.10)
\end{aligned}$$

Taking the absolute value of both sides, we get

$$\begin{aligned}
|\rho(x^*, Tx^*)| &\leq \frac{1}{(1-b\beta)} |[\alpha\rho(x^*, x_{n+1}) + a\beta\rho(x_n, x_{n+1}) + c\beta\rho(x_n, x^*)]| \\
&\leq \frac{1}{(1-b\beta)} |\alpha\rho(x^*, x_{n+1})| + a\beta\gamma^n |\rho(x_0, x_1)| + c\beta |\rho(x_n, x^*)| \qquad (4.11)
\end{aligned}$$

Since x_n converges to x^* , taking limit $n \to \infty$ implies that $|\rho(x^*, Tx^*)| \to 0$ which yields $x^* = Tx^*$. To show uniqueness of the fixed point. Assume that there exists $x^{**} \in X$ such that $Tx^{**} = x^{**}$. Then

$$\rho(x^{**}, x^*) = \rho(Tx^{**}, Tx^*) \precsim a\rho(x^{**}, Tx^{**}) + b\rho(x^*, Tx^*) + c\rho(x^*, x^{**}) \tag{4.12}$$

which implies that $\rho(x^{**}, x^*) \precsim c\rho(x^*, x^{**})$. Taking the absolute value of both sides, we get $|\rho(x^{**}, x^*)| \leq c\,|\rho(x^*, x^{**})|$. This implies that $\rho(x^*, x^{**}) = 0$. Thus $x^* = x^{**}$. $\square$

Theorem 4.2. *Let (X, ρ) be a complete generalized complex-valued α, β b-metric space and let $T : X \to X$ be a mapping such that,*

$$\rho(Tx, Ty) \precsim a\rho(x, Ty) + b\rho(y, Tx) \tag{4.13}$$

Which implies that for every $x, y \in X$, where a, b are non-negative constants with $\beta b < \frac{1}{1+\alpha}$. Then T has a fixed point in X and has a unique fixed point if $a + b < 1$.

Proof. Let $x_0 \in X$ be fixed then consider the sequence generated by the formula $x_n = Tx_{n-1} = T^n x_0$. Let $n \in \mathbb{N}$ then we get

$$\begin{aligned}
\rho(x_{n+2}, x_{n+1}) &= \rho(Tx_{n+1}, Tx_n) \\
&\precsim a\rho(x_{n+1}, Tx_n) + b\rho(x_n, Tx_{n+1}) \\
&= a\rho(x_{n+1}, x_{n+1}) + b\rho(x_n, x_{n+2}) \\
&= b\rho(x_n, x_{n+2}) \\
&\precsim b\alpha\rho(x_n, x_{n+1}) + b\beta\rho(x_{n+1}, x_{n+2}) \\
(1 - b\beta)\rho(x_{n+2}, x_{n+1}) &\precsim b\alpha\rho(x_n, x_{n+1}) \\
\rho(x_{n+2}, x_{n+1}) &\precsim \tfrac{b\alpha}{1-b\beta}\rho(x_n, x_{n+1}) \qquad (4.14)
\end{aligned}$$

Letting $\gamma = \frac{b\alpha}{1-b\beta}$ and repeated use of (4.14) yields

$$\begin{aligned}
\rho(x_{n+1}, x_{n+2}) &\precsim \gamma\rho(x_n, x_{n+1}) \\
&\precsim \gamma^2\rho(x_{n-1}, x_n) \\
&\vdots \\
&\precsim \gamma^{n+1}\rho(x_0, x_1) \qquad (4.15)
\end{aligned}$$

Let $m \in \mathbb{N}$ then

$$\begin{aligned}
&\rho(x_n, x_{n+m}) \\
&\precsim \alpha\rho(x_n, x_{n+1}) + \beta\rho(x_{n+1}, x_{n+m}) \\
&\precsim \alpha\rho(x_n, x_{n+1}) + \beta\left[\alpha\rho(x_{n+1}, x_{n+2}) + \beta\rho(x_{n+2}, x_{n+m})\right] \\
&\precsim \alpha\rho(x_n, x_{n+1}) + \alpha\beta\rho(x_{n+1}, x_{n+2}) + \cdots + \alpha\beta^{m-2}\rho(x_{n+m-2}, x_{n+m-1}) + \beta^{m-1}\rho(x_{n+m-1}, x_{n+m}) \\
&\precsim \alpha\gamma^n\rho(x_0, x_1) + \alpha\beta\gamma^{n+1}\rho(x_0, x_1) + \cdots + \alpha\gamma^{n+m-2}\beta^{m-2}\rho(x_0, x_1) + \beta^{m-1}\gamma^{n+m-1}\rho(x_0, x_1) \\
&\precsim \alpha\gamma^n\rho(x_0, x_1)\left[1 + \beta\gamma + \cdots + \gamma^{m-2}\beta^{m-2}\right] + \beta^{m-1}\gamma^{n+m-1}\rho(x_0, x_1) \\
&\precsim \alpha\gamma^n\rho(x_0, x_1)\left[\tfrac{1-(\beta\gamma)^{m-1}}{1-\beta\gamma}\right] + \beta^{m-1}\gamma^{n+m-1}\rho(x_0, x_1) \\
&\precsim \alpha\tfrac{\gamma^n}{1-\beta\gamma}\rho(x_0, x_1)[\alpha - (\beta\gamma)^{m-1}(\alpha - 1 + \beta\gamma)] \\
&\prec \alpha\tfrac{\gamma^n}{1-\beta\gamma}\rho(x_0, x_1)
\end{aligned}$$

Since $\beta b < \frac{1}{1+\alpha}$ then $0 < \gamma < \frac{1}{\beta}$ and $\beta\gamma < 1$. Taking the limit $n \to \infty$, we get $\gamma^n \to 0$. This implies that $|\rho(x_n, x_{n+m})| \to 0$ as $n \to \infty$. The sequence $\{x_n\}$ is a Cauchy sequence in X. Since (X, ρ) is a complete complex-valued b-metric space then $\{x_n\}$ converges to $x^* \in X$.

We show that x^* is a fixed point of T.

$$\begin{aligned}
\rho(x^*, Tx^*) &\precsim \alpha\rho(x^*, x_n) + \beta\rho(x_n, Tx^*) \\
&= \alpha\rho(x^*, x_n) + \beta\rho(Tx_{n-1}, Tx^*) \\
&\precsim \alpha\rho(x^*, x_n) + \beta\left[a\rho(x_{n-1}, Tx^*) + b\rho(Tx_{n-1}, Tx^*)\right] \\
&= \alpha\rho(x^*, x_n) + \beta a\rho(x_{n-1}, Tx^*) + \beta b\rho(x_n, x^*) \\
&\precsim (\alpha + \beta b)\rho(x_n, x^*) + \beta a\alpha\rho(x_{n-1}, x^*) + a\beta^2\rho(x^*, Tx^*) \\
[1 - a\beta^2]\rho(x^*, Tx^*) &\precsim (\alpha + \beta b)\rho(x_n, x^*) + a\beta\alpha\rho(x_{n-1}, x^*). \qquad (4.16)
\end{aligned}$$

Since $\{x_n\}$ converges to x^* we get $|\rho(x^*, x_n)| \to 0$ as $n \to \infty$, and taking the absolute value of both sides of (4.16) we obtain $|\rho(x^*, Tx^*)| \leq 0$ thus $\rho(x^*, Tx^*) = 0$, which implies that $Tx^* = x^*$. To prove the uniqueness of the fixed point we assume that there $x^*, x^{**} \in X$ such that $Tx^* = x^*$ and $Tx^{**} = x^{**}$. Now

$$\begin{aligned}
\rho(x^*, x^{**}) &= \rho(Tx^*, Tx^{**}) && (4.17) \\
&\precsim a\rho(x^*, Tx^{**}) + b\rho(x^{**}, Tx^*) && (4.18) \\
&= a\rho(x^*, x^{**}) + b\rho(x^{**}, x^*) && (4.19) \\
&= (a + b)\rho(x^*, x^{**}) && (4.20)
\end{aligned}$$

Thus we get $|\rho(x^*, x^{**})| \leq |(a + b)|\,|\rho(x^*, x^{**})|$. Since $a + b < 1$ we get $|\rho(x^*, x^{**})| = 0$ thus $x^* = x^{**}$.

□

Kir et al. studied the following fixed point theorem in b-metric spaces and we generalized the result into a α, β complex-valued b-metric spaces, [19].

Theorem 4.3. *Let (X, ρ) be a α, β complex-valued b-metric space and let $T : X \to X$*

be a mapping such that

$$\rho(Tx, Ty) \preccurlyeq \lambda[\rho(x, Tx) + \rho(y, Ty)],$$

where $\lambda \in [0, \frac{1}{2})$, for all $x, y \in X$. Then T has a unique fixed point.

Proof. We begin by showing that for $x_0 \in X$ fixed, the sequence $\{x_n\}$ where $x_n = Tx_{n-1} = T^n x_0$ is a Cauchy sequence in (X, ρ).

For $n \in \mathbb{N}$, we have

$$\begin{aligned}
\rho(x_{n+2}, x_{n+1}) &= \rho(Tx_{n+1}, Tx_n) \\
&\preccurlyeq \lambda[\rho(x_{n+1}, Tx_{n+1}) + \rho(x_n, Tx_n)] \\
&= \lambda[\rho(x_{n+1}, x_{n+2}) + \rho(x_n, Tx_n)] \\
(1-\lambda)\rho(x_{n+2}, x_{n+1}) &\preccurlyeq \lambda\rho(x_{n+1}, x_n) \\
\rho(x_{n+2}, x_{n+1}) &\preccurlyeq \tfrac{\lambda}{(1-\lambda)}\rho(x_{n+1}, x_n) \qquad (4.21)
\end{aligned}$$

Repeated use of (4.21), for $n \in \mathbb{N}$, we get

$$\rho(x_{n+2}, x_{n+1}) \preccurlyeq \left(\tfrac{\lambda}{1-\lambda}\right)^{n+1} \rho(x_1, x_0).$$

For $m, n \in X$

$$\rho(x_n, x_{n+m})$$

$$\preccurlyeq \alpha\rho(x_n, x_{n+1}) + \beta\rho(x_{n+1}, x_{n+m})$$

$$\preccurlyeq \alpha\rho(x_n, x_{n+1}) + \beta\alpha\rho(x_{n+1}, x_{n+2}) + \beta^2\rho(x_{n+2}, x_{n+m})$$

$$\preccurlyeq \alpha\rho(x_n, x_{n+1}) + \alpha\beta\rho(x_{n+1}, x_{n+2}) + \cdots + \alpha\beta^{m-2}\rho(x_{n+m-2}, x_{n+m-1})$$

$$+ \beta^{m-1}\rho(x_{n+m-1}, x_{n+m})$$

$$\preccurlyeq \alpha\left(\tfrac{\lambda}{1-\lambda}\right)^n \rho(x_0, x_1) + \alpha\beta\left(\tfrac{\lambda}{1-\lambda}\right)^{n+1} \rho(x_0, x_1) + \cdots + \alpha\beta^{m-2}\left(\tfrac{\lambda}{1-\lambda}\right)^{n+m-2} \rho(x_0, x_1)$$

$$+ \beta^{m-1}\left(\tfrac{\lambda}{1-\lambda}\right)^{n+m-1} \rho(x_0, x_1)$$

$$= \alpha\left(\tfrac{\lambda}{1-\lambda}\right)^n \rho(x_0, x_1)\left[1 + \beta\left(\tfrac{\lambda}{1-\lambda}\right) + \cdots + \beta^{m-2}\left(\tfrac{\lambda}{1-\lambda}\right)^{m-2}\right] + \beta^{m-1}\left(\tfrac{\lambda}{1-\lambda}\right)^{n+m-1} \rho(x_0, x_1)$$

Since $\lambda \in [0, \frac{1}{2})$ implies that $0 \leq \frac{\lambda}{1-\lambda} < 1$. Taking the absolute value of both sides we get $|\rho(x_n, x_{n+m})| \to 0$ as $n \to \infty$. It follows that that the sequence $\{x_n\}$ is a Cauchy sequence in (X, ρ). Since (X, ρ) is complete there exists a $x^* \in X$ such that

$$\lim_{n\to\infty} \rho(x_n, x^*) = 0.$$

We now show that x^* is a fixed point of the mapping T.

$$\begin{aligned}
\rho(x^*, Tx^*) &\preccurlyeq \alpha\rho(x^*, x_n) + \beta\rho(x_n, Tx^*) \\
&= \alpha\rho(x^*, x_n) + \beta\rho(Tx_{n-1}, Tx^*) \\
&\preccurlyeq \alpha\rho(x^*, x_n) + \beta\lambda\rho(x_{n-1}, Tx_{n-1}) + \beta\lambda\rho(x^*, Tx^*) \\
[1 - \beta\lambda]\,\rho(x^*, Tx^*) &\preccurlyeq \alpha\rho(x^*, x_n) + \beta\lambda\rho(x_{n-1}, x_n) \qquad (4.22)
\end{aligned}$$

Taking the absolute value of both sides of (4.22) and taking $n \to \infty$, we obtain $\rho(x^*, Tx^*) = 0$. This implies that x^* is a fixed point of T.

To prove the uniqueness of the fixed point, we assume that there are $x^*, x^{**} \in X$ such that $Tx^* = x^*$ and $Tx^{**} = x^{**}$. Now

$$\begin{aligned}\rho(x^*, x^{**}) &= \rho(Tx^*, Tx^{**}) \\ &\preccurlyeq \lambda\left[\rho(x^*, Tx^*) + \rho(x^{**}, Tx^{**})\right] \\ &= 0\end{aligned}$$

Thus we get $|\rho(x^*, x^{**})| \leq 0$, which implies that $x^* = x^{**}$. Hence x^* is a unique fixed point.

$\square$

Chapter 5

Generalized G_b metric

5.1 Introduction

Mustafa et al.[24] introduced a new structure of generalized metric spaces which they called G-metric spaces as a generalization of metric spaces, to develop and introduce a new fixed point theory for various mappings in this new structure,[23]. Various authors have proved some fixed point theorems in these spaces,[9, 24, 35].

Recently, Sedghi et al.[30] have introduced D^*-metric spaces which is a modification of the definition of D-metric spaces introduced by Dhage, [12] and proved some basic properties in D^*-metric spaces, [31]. Furthmore, they introduced the concept of S-metric spaces and presented some properties for common fixed point theorem for a self-mapping on complete S-metric spaces, [33].

Using the concepts of G-metric and b-metric, Ahhajani et al.[1] define a new type of metric which they called a G_b-metric. They studied some basic properties of such a metric and proved common fixed point theorem for mappings satisfying weakly compatible condition in complete partially ordered G_b-metric spaces and presented a nontrivial example to verify their effectiveness and applicability, [1].

Definition 5.1.1. Let X be a nonempty set. A generalization of a metric or G-metric is a function $G : X \times X \times X \to [0, \infty)$ satisfying the following properties,[23]:

(i) for all $x, y, z \in X$ $G(x, y, z) = 0 \iff x = y = z$

(ii) for all $x, y \in X$, $x \neq y$, $0 < G(x, x, y)$

(iii) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$, $z \neq y$

(iv) $G(x, y, z) = G(x, z, y) = G(y, x, z) = \cdots$ symmetry in all variables

(v) for all $x, y, z, w \in X$, $G(x, y, z) \leq G(x, w, w) + G(w, y, z)$

The pair (X, G) is a G-metric space.

Definition 5.1.2. Let X be a nonempty set and $s \geq 1$ be a real number. A generalization of a G-metric is a function $G_b : X \times X \times X \to [0, \infty)$ satisfying the following properties,[1]:

(i) for all $x, y, z \in X$ $G_b(x, y, z) = 0 \iff x = y = z$

(ii) for all $x, y \in X$, $x \neq y$, $0 < G_b(x, x, y)$

(iii) for all $x, y, z \in X$, $z \neq y$ $G_b(x, x, y) \leq G_b(x, y, z)$

(iv) $G_b(x, y, z) = G_b(x, z, y) = G_b(y, x, z) = \cdots$ symmetry in all variables

(v) for all $x, y, z, w \in X$, $G_b(x, y, z) \leq s\,[G_b(x, w, w) + G_b(w, y, z)]$

The pair (X, G_b) is a generalized b-metric space or G_b-metric space.

Example 6. Let $X = \mathbb{R}$ then define a mapping $G_b : X \times X \times X \to [0, \infty)$ by $G_b(x, y, z) = \frac{1}{9}\left(|x - y| + |y - z| + |x - z|\right)^2$. Then (X, G_b) is a G_b-metric space[1].

Definition 5.1.3. Let X be a nonempty set and $\alpha, \beta \geq 1$ are real numbers. A generalization of a G_b-metric is a function $G_b^{\alpha\beta} : X \times X \times X \to [0, \infty)$ satisfying the following properties:

(i) for all $x, y, z \in X$ $G_b^{\alpha\beta}(x, y, z) = 0 \iff x = y = z$

(ii) for all $x, y \in X$, $x \neq y$, $0 < G_b^{\alpha\beta}(x, x, y)$

(iii) for all $x, y, z \in X$, $z \neq y$, $G_b^{\alpha\beta}(x, x, y) \leq G_b^{\alpha\beta}(x, y, z)$

(iv) $G_b^{\alpha\beta}(x, y, z) = G_b^{\alpha\beta}(x, z, y) = G_b^{\alpha\beta}(y, x, z) = \cdots$ symmetry in all variables

(v) for all $x, y, z, w \in X$, $G_b^{\alpha\beta}(x, y, z) \leq \alpha G_b^{\alpha\beta}(x, w, w) + \beta G_b^{\alpha\beta}(w, y, z)$

The pair $(X, G_b^{\alpha\beta})$ is an α, β generalized b-metric space or $G_b^{\alpha\beta}$-metric space. If $\alpha = \beta = 1$ then $G^{\alpha\beta} = G$. If $\alpha = \beta = s$ then $G_b^{\alpha\beta} = G_b$. For every generalized b-metric G_b it is not always possible to find an $\alpha, \beta \geq 1$ such that $1 \leq \alpha, \beta < s$ satisfying the property (v) of definition 5.1.3.

Example 7. Let $X = (1, 3)$ then define $G_b^{\alpha\beta} : X \times X \times X \to [0, \infty)$ by

$$G_b^{\alpha\beta}(x, y, z) = \begin{cases} e^{|x-y|+|y-z|+|z-x|}, & x \neq y \neq z \\ 0 & x = y = z, \end{cases}$$

To show that $G_b^{\alpha\beta}(x, y, z)$ is a $G_b^{\alpha\beta}$-metric we verify properties (i)-(v) of definition 5.1.3. Properties (i)-(iv) are easily verified. We only verify property (v) of definition 5.1.3.

Let $x, y, z \in X$ such that $x \neq y \neq z$ then

$$
\begin{aligned}
& G_b^{\alpha\beta}(x,y,z) \\
& \leq e^{|x-w|+|y-w|+|y-z|+|z-w|+|x-w|} \\
& \leq e^{2|x-w|+|y-w|+|y-z|+|z-w|} \\
& \leq \sup_{x,y,z\in X} e^{\frac{2}{3}|2(x-w)|+\frac{1}{3}[|y-w|+|y-z|+|z-w|]} e^{\frac{1}{3}|2(x-w)|+\frac{2}{3}[|y-w|+|y-z|+|z-w|]} \\
& \leq \frac{e^{\frac{14}{3}}}{3} e^{|2(x-w)|} + \frac{2e^{\frac{14}{3}}}{3} e^{[|y-w|+|y-z|+|z-w|]} \\
& = \frac{e^{\frac{14}{3}}}{3} G_b^{\alpha\beta}(x,w,w) + \frac{2e^{\frac{14}{3}}}{3} G_b^{\alpha\beta}(w,y,z)
\end{aligned}
$$

with $\alpha = \frac{e^{\frac{14}{3}}}{3} \geq 1$ and $\beta = \frac{2e^{\frac{14}{3}}}{3} \geq 1$, $\alpha \neq \beta$ and $\alpha < \beta$.

5.2 Properties

The following properties can be deduced from the Definition 5.1.3.

Proposition 5.0.1. *Let $(X, G_b^{\alpha\beta})$ be a $G_b^{\alpha\beta}$-metric space. For all $x, y, z \in X$*

(i) $G_b^{\alpha\beta}(x,y,z) \leq \alpha G_b^{\alpha\beta}(y,x,x) + \beta G_b^{\alpha\beta}(x,x,z)$;

(ii) $G_b^{\alpha\beta}(x,y,y) \leq (\alpha+\beta) G_b^{\alpha\beta}(y,x,x)$.

Definition 5.2.1. A $G_b^{\alpha\beta}$-metric is symmetric if $G_b^{\alpha\beta}(x,x,y) = G_b^{\alpha\beta}(x,y,y)$ for all $x, y \in X$.

Definition 5.2.2. Let $(X, G_b^{\alpha\beta})$ be a $G_b^{\alpha\beta}$-metric space then for $x_0 \in X$, $\epsilon > 0$, a $G_b^{\alpha\beta}$ ball with center x_0 and radius ϵ is

$$B_{G_b^{\alpha\beta}}(x_0, \epsilon) = \left\{ y \in X; G_b^{\alpha\beta}(x_0, y, y) < \epsilon \right\}$$

Proposition 5.0.2. *Let $x_0 \in X$, $\epsilon > 0$ and if $y \in B_{G_b^{\alpha\beta}}(x_0, \epsilon)$ then there exists $\delta > 0$ and $c \geq 1$ such that $B_{G_b^{\alpha\beta}}(y, \delta) \subset B_{G_b^{\alpha\beta}}(x_0, c\epsilon)$.*

Proof. Let $y \in B_{G_b^{\alpha\beta}}(x_0, \epsilon)$ then $G_b^{\alpha\beta}(x_0, y, y) < \epsilon$ and taking $\delta = \epsilon - G_b^{\alpha\beta}(x_0, y, y) > 0$. Now, let $w \in B_{G_b^{\alpha\beta}}(y, \delta)$ then $G_b^{\alpha\beta}(y, w, w) < \delta$. Then it follows that

$$\begin{aligned} G_b^{\alpha\beta}(x_0, w, w) &\leq \alpha G_b^{\alpha\beta}(x_0, y, y) + \beta G_b^{\alpha\beta}(y, w, w) \\ &\leq \alpha G_b^{\alpha\beta}(x_0, y, y) + \beta(\epsilon - G_b^{\alpha\beta}(x_0, y, y)) \\ &\leq \alpha G_b^{\alpha\beta}(x_0, y, y) + \beta\epsilon \\ &\leq (\alpha + \beta)\epsilon \end{aligned}$$

Thus $w \in B_{G_b^{\alpha\beta}}(x_0, (\alpha+\beta)\epsilon)$. Since $\alpha, \beta \geq 1$ and taking $c = \alpha+\beta \geq 2$ we conclude. $\square$

Definition 5.2.3. Let $(X, G_b^{\alpha\beta})$ be a $G_b^{\alpha\beta}$-space and $\{x_n\}$ a sequence in X :

(i) The sequence $\{x_n\}$ is a $G_b^{\alpha\beta}$-Cauchy sequence if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $G_b^{\alpha\beta}(x_n, x_m, x_k) < \epsilon$ for all $n, m, k \geq N$.

(ii) The sequence $\{x_n\}$ is a $G_b^{\alpha\beta}$-convergent sequence if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ and $x \in X$ such that $G_b^{\alpha\beta}(x_n, x_m, x) < \epsilon$ for all $n, m \geq N$.

Proposition 5.0.3. *Let $(X, G_b^{\alpha\beta})$ be a $G_b^{\alpha\beta}$-metric space.*
The sequence $\{x_n\}$ is $G_b^{\alpha\beta}$-Cauchy $\iff$ for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $G_b^{\alpha\beta}(x_n, x_m, x_m) < \epsilon$ for all $n, m \geq N$.

Proof. $\implies$: From Definition 5.2.3 it follows easily if we take $k = m$.
$\impliedby$: If $\epsilon_{\alpha\beta} = \frac{\epsilon}{2\max\{\alpha,\beta\}}$ then $\epsilon_{\alpha\beta} > 0$ for any $\epsilon > 0$ then there exists $N \in \mathbb{N}$ such that $G_b^{\alpha\beta}(x_n, x_m, x_m) < \epsilon_{\alpha\beta}$ for all $n, m \geq N$. From definition 5.1.3, property (iv) and (v),

we get

$$
\begin{aligned}
G_b^{\alpha\beta}(x_n, x_m, x_k) &\leq \alpha G_b^{\alpha\beta}(x_n, x_m, x_m) + \beta G_b^{\alpha\beta}(x_m, x_m, x_k) \\
&< \alpha \left(\frac{\epsilon}{2 \max \{\alpha, \beta\}} \right) + \beta \left(\frac{\epsilon}{2 \max \{\alpha, \beta\}} \right) \\
&= \epsilon
\end{aligned}
$$

for all $n, m, k \geq N$. $\square$

Proposition 5.0.4. *[1] Let $(X, G_b^{\alpha\beta})$ be a $G_b^{\alpha\beta}$-metric space. The following statements are equivalent:*

(a) If $\{x_n\}$ is a $G_b^{\alpha\beta}$-convergent sequence.

(b) for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $G_b^{\alpha\beta}(x_n, x_n, x) < \epsilon$ for all $n \geq N$.

(c) for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $G_b^{\alpha\beta}(x_n, x, x) < \epsilon$ for all $n \geq N$.

Proposition 5.0.5. *[1] Let $(X, G_b^{\alpha\beta})$ be a $G_b^{\alpha\beta}$-metric space. Then the following statements are equivalent:*

(i) $\{x_n\}$ is $G_b^{\alpha\beta}$-convergent to $x \in X$;

(ii) $G_b^{\alpha\beta}(x_n, x_n, x) \to 0$ as $n \to \infty$;

(iii) $G_b^{\alpha\beta}(x_n, x, x) \to 0$ as $n \to \infty$.

Definition 5.2.4. A $G^{\alpha\beta}$-metric space $(X, G_b^{\alpha\beta})$ is $G_b^{\alpha\beta}$ complete if every $G_b^{\alpha\beta}$- Cauchy sequence is $G^{\alpha\beta}$-convergent.

5.3 Fixed Point Theorems

Theorem 5.1. *Let $(X, G_b^{\alpha\beta})$ be a $G_b^{\alpha\beta}$-complete metric space. If a mapping $T : X \to X$ satisfies the following*

$$G_b^{\alpha\beta}(Tx, Ty, Tz) \leq \lambda G_b^{\alpha\beta}(x, y, z) \tag{5.1}$$

for all $x, y, z \in X$, where $0 \leq \lambda < \frac{1}{\beta}$. Then T has a unique fixed point.

Proof. Let $x_0 \in X$ be arbitrary and define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for $n \in \mathbb{N}$. Then it follows from inequality (5.1), for the sequence $\{x_n\}$ we get

$$G_b^{\alpha\beta}(x_n, x_{n+1}, x_{n+1}) = G_b^{\alpha\beta}(Tx_{n-1}, Tx_n, Tx_n) \leq \lambda G_b^{\alpha\beta}(x_{n-1}, x_n, x_n) \tag{5.2}$$

Repeatedly applying inequality (5.2), we get

$$G_b^{\alpha\beta}(x_n, x_{n+1}, x_{n+1}) \leq \lambda^n G^{\alpha\beta}(x_0, x_1, x_1) \tag{5.3}$$

For $n, m \in \mathbb{N}$ and proposition 5.0.3, we get

$$\begin{aligned}
&G_b^{\alpha\beta}(x_n, x_{n+m}, x_{n+m}) \\
&\leq \alpha G_b^{\alpha\beta}(x_n, x_{n+1}, x_{n+1}) + \alpha\beta G_b^{\alpha\beta}(x_{n+1}, x_{n+2}, x_{n+2}) \\
&+ \alpha\beta^2 G_b^{\alpha\beta}(x_{n+2}, x_{n+3}, x_{n+3}) + \cdots + \beta^{m-1} G_b^{\alpha\beta}(x_{n+m-1}, x_{n+m}, x_{n+m}) \\
&\leq \alpha G_b^{\alpha\beta}(x_n, x_{n+1}, x_{n+1}) + \alpha\beta G_b^{\alpha\beta}(x_{n+1}, x_{n+2}, x_{n+2}) \\
&+ \alpha\beta^2 G_b^{\alpha\beta}(x_{n+2}, x_{n+3}, x_{n+3}) + \cdots + \alpha\beta^{m-1} G_b^{\alpha\beta}(x_{n+m-1}, x_{n+m}, x_{n+m}) \\
&\leq \alpha \left(\lambda^n + \lambda^{n+1}\beta + \cdots + \lambda^{n+m-1}\beta^{m-1}\right) G_b^{\alpha\beta}(x_0, x_1, x_1) \\
&\leq \alpha\lambda^n \frac{1}{1 - \beta\lambda} G_b^{\alpha\beta}(x_0, x_1, x_1)
\end{aligned}$$

Since $\lambda < \frac{1}{\beta}$ we conclude that for every $\epsilon > 0$ there exsits $N \in \mathbb{N}$ such that $G_b^{\alpha\beta}(x_n, x_{n+m}, x_{n+m}) < \epsilon$ for $n \geq N$, thus $\{x_n\}$ is a $G_b^{\alpha\beta}$-Cauchy sequence in $(X, G_b^{\alpha\beta})$.

Since $(X, G_b^{\alpha\beta})$ is a complete-$G_b^{\alpha\beta}$ metric space there exists x^* and $N_1 \in \mathbb{N}$ such that $G_b^{\alpha\beta}(x_n, x_n, x^*) < \epsilon$ for $n \geq N_1$.

To show that x^* is a fixed point of T. Using the contraction condition we get

$$\begin{aligned} G_b^{\alpha\beta}(x_{n+1}, Tx^*, Tx^*) &= G_b^{\alpha\beta}(Tx_n, Tx^*, Tx^*) \\ &\leq \lambda G_b^{\alpha\beta}(x_n, x^*, x^*) \end{aligned}$$

Taking the limit as $n \to \infty$, we get $G_b^{\alpha\beta}(x^*, Tx^*, Tx^*) = 0$. Thus $Tx^* = x^*$.

To prove uniqueness we assume that T has fixed points x^* and x^{**}. Then it follows that

$$G_b^{\alpha\beta}(x^{**}, x^*, x^*) = G_b^{\alpha\beta}(Tx^{**}, Tx^*, Tx^*) \leq \lambda G_b^{\alpha\beta}(x^{**}, x^*, x^*)$$

Since $0 \leq \lambda < 1$, we get $x^{**} = x^*$. □

Theorem 5.2. *Let $X, G_b^{\alpha\beta}$ be a $G_b^{\alpha\beta}$-complete metric space and a mapping $T : X \to X$ such that*

$$\begin{aligned} &G_b^{\alpha\beta}(Tx, Ty, Tz) \\ &\leq \lambda \max\big\{G^{\alpha\beta}(x, y, z), G^{\alpha\beta}(x, Tx, Tx), G^{\alpha\beta}(y, Ty, Ty), \\ &G^{\alpha\beta}(z, Tz, Tz), G^{\alpha\beta}(x, Ty, Ty), G^{\alpha\beta}(y, Tz, Tz), G^{\alpha\beta}(z, Tx, Tx)\big\} \end{aligned} \tag{5.4}$$

for all $x, y, z \in X$ with $0 \leq \lambda < \frac{1}{\alpha+\beta}$. Then T has a unique fixed point.

Proof. Let $x_0 \in X$ define a sequence $x_{n+1} = Tx_n$ then for sequence $\{x_n\}$, we get from inequality (5.4),

$$
\begin{aligned}
&G_b^{\alpha\beta}(x_n, x_{n+1}, x_{n+1}) \\
&\leq \lambda \max\Big\{G_b^{\alpha\beta}(x_{n-1}, x_n, x_n), G_b^{\alpha\beta}(x_n, x_{n+1}, x_{n+1}), G_b^{\alpha\beta}(x_{n-1}, x_{n+1}, x_{n+1}), \\
&\qquad\qquad G_b^{\alpha\beta}(x_n, x_n, x_n)\Big\} \\
&\leq \lambda \max\Big\{G_b^{\alpha\beta}(x_{n-1}, x_n, x_n), G_b^{\alpha\beta}(x_n, x_{n+1}, x_{n+1}), \alpha G_b^{\alpha\beta}(x_{n-1}, x_n, x_n) \\
&\qquad\qquad +\beta G_b^{\alpha\beta}(x_n, x_{n+1}, x_{n+1})\Big\} \\
&= \lambda\alpha G_b^{\alpha\beta}(x_{n-1}, x_n, x_n) + \lambda\beta G_b^{\alpha\beta}(x_n, x_{n+1}, x_{n+1}) \qquad (5.5)
\end{aligned}
$$

From inequality (5.5), and that $\lambda < \frac{1}{\alpha+\beta}$, we get

$$
G_b^{\alpha\beta}(x_n, x_{n+1}, x_{n+1}) \leq \frac{\lambda\alpha}{1-\lambda\beta} G^{\alpha\beta}(x_{n-1}, x_n, x_n). \qquad (5.6)
$$

For $n, m \in \mathbb{N}$, recursively applying inequality (5.6), we get

$$
\begin{aligned}
&G_b^{\alpha\beta}(x_n, x_{n+m}, x_{n+m}) \\
&\leq \alpha G_b^{\alpha\beta}(x_n, x_{n+1}, x_{n+1}) + \alpha\beta G_b^{\alpha\beta}(x_{n+1}, x_{n+2}, x_{n+2}) \\
&+ \alpha\beta^2 G_b^{\alpha\beta}(x_{n+2}, x_{n+3}, x_{n+3}) + \cdots + \beta^{m-1} G_b^{\alpha\beta}(x_{n+m-1}, x_{n+m}, x_{n+m}) \\
&\leq \alpha G_b^{\alpha\beta}(x_n, x_{n+1}, x_{n+1}) + \alpha\beta G_b^{\alpha\beta}(x_{n+1}, x_{n+2}, x_{n+2}) \\
&+ \alpha\beta^2 G_b^{\alpha\beta}(x_{n+2}, x_{n+3}, x_{n+3}) + \cdots + \alpha\beta^{m-1} G_b^{\alpha\beta}(x_{n+m-1}, x_{n+m}, x_{n+m}) \\
&\leq \alpha \left(\left(\frac{\lambda\alpha}{1-\lambda\beta}\right)^n + \left(\frac{\lambda\alpha}{1-\lambda\beta}\right)^{n+1} \beta + \cdots + \left(\frac{\lambda\alpha}{1-\lambda\beta}\right)^{n+m-1} \beta^{m-1} \right) \\
&\quad G_b^{\alpha\beta}(x_0, x_1, x_1) \\
&\leq \alpha \left(\frac{\lambda\alpha}{1-\lambda\beta}\right)^n \frac{1-\beta\lambda}{1-\beta\lambda(1+\alpha)} G_b^{\alpha\beta}(x_0, x_1, x_1)
\end{aligned}
$$

Since $\lambda < \frac{1}{\alpha+\beta}$ we conclude that for every $\epsilon > 0$ there exsits $N \in \mathbb{N}$ such that $G_b^{\alpha\beta}(x_n, x_{n+m}, x_{n+m}) < \epsilon$ for $n \geq N$, thus $\{x_n\}$ is a $G_b^{\alpha\beta}$-Cauchy sequence in $(X, G_b^{\alpha\beta})$.

Since $(X, G_b^{\alpha\beta})$ is a complete-$G_b^{\alpha\beta}$ metric space there exists x^* and $N_1 \in \mathbb{N}$ such that $G_b^{\alpha\beta}(x_n, x_n, x^*) < \epsilon$ for $n \geq N_1$.

To show that x^* is a fixed point of T. For the $G_b^{\alpha\beta}$-convergent sequence $\{x_n\}$, we get

$$\begin{aligned}
&G_b^{\alpha\beta}(x_n, Tx^*, Tx^*)\\
&\leq \lambda \max\Big\{G_b^{\alpha\beta}(x_{n-1}, x^*, x^*), G_b^{\alpha\beta}(x_{n-1}, Tx^*, Tx^*),\\
&G_b^{\alpha\beta}(x^*, Tx^*, Tx^*), G_b^{\alpha\beta}(x_{n-1}, x_n, x_n), G_b^{\alpha\beta}(x^*, x_n, x_n)\Big\} \qquad (5.7)
\end{aligned}$$

Letting $n \to \infty$ in inequality (5.7), we get $G_b^{\alpha\beta}(x^*, Tx^*, Tx^*) \leq \lambda G_b^{\alpha\beta}(x^*, Tx^*, Tx^*)$. Since $\lambda < \frac{1}{\alpha+\beta} < \frac{1}{2}$ the inequality is only valid if $G_b^{\alpha\beta}(x^*, Tx^*, Tx^*) = 0$ which implies $Tx^* = x^*$. For the uniqueness of the fixed point we assume that $x^{**} \in X$ is a fixed point of T. Then from inequality (5.4), we get

$$\begin{aligned}
&G_b^{\alpha\beta}(Tx^*, Tx^*, Tx^{**})\\
&\leq \lambda \max\{G^{\alpha\beta}(x^*, x^*, x^{**}), G^{\alpha\beta}(x^*, Tx^*, Tx^*), G^{\alpha\beta}(x^{**}, Tx^{**}, Tx^{**}),\\
&\qquad G^{\alpha\beta}(x^{**}, Tx^{**}, Tx^{**}), G^{\alpha\beta}(x^*, Tx^{**}, Tx^{**}), G^{\alpha\beta}(x^*, Tx^{**}, Tx^{**}),\\
&\qquad G^{\alpha\beta}(x^{**}, Tx^*, Tx^*)\} \qquad (5.8)
\end{aligned}$$

Thus, we obtain from Proposition 5.0.1,

$$\begin{aligned}
G_b^{\alpha\beta}(x^*, x^*, x^{**}) &\leq \lambda \max\Big\{G_b^{\alpha\beta}(x^*, x^*, x^{**}), G_b^{\alpha\beta}(x^*, x^{**}, x^{**})\Big\}\\
&\leq \lambda(\alpha+\beta)G^{\alpha\beta}(x^*, x^*, x^{**}).
\end{aligned}$$

It follows that

$$[1 - \lambda(\alpha+\beta)]\, G_b^{\alpha\beta}(x^*, x^*, x^{**}) \leq 0$$

Since $1 - \lambda(\alpha+\beta) > 0$, we conclude that $G_b^{\alpha\beta}(x^*, x^*, x^{**}) = 0$ thus $x^* = x^{**}$. $\square$

Theorem 5.3. *Let $X, G_b^{\alpha\beta}$ be a $G_b^{\alpha\beta}$-complete metric space and a mapping $T : X \to X$ such that*

$$
\begin{aligned}
G_b^{\alpha\beta}(Tx, Ty, Tz) \leq \lambda_1 G^{\alpha\beta}(x, y, z) &+ \lambda_2 G^{\alpha\beta}(x, Tx, Tx) + \lambda_3 G^{\alpha\beta}(y, Ty, Ty) \\
&+ \lambda_4 G^{\alpha\beta}(z, Tz, Tz) + \lambda_5 G^{\alpha\beta}(x, Ty, Ty) + \lambda_6 G^{\alpha\beta}(y, Tz, Tz) \\
&+ \lambda_7 G^{\alpha\beta}(z, Tx, Tx) \qquad (5.9)
\end{aligned}
$$

for all $x, y, z \in X$ with $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + (\alpha + \beta)\lambda_5 + \lambda_6 + \lambda_7 < 1$. Then T has a unique fixed point.

Proof. Let $x_0 \in X$ be arbitrary and define a sequence $\{x_n\}$, where $x_{n+1} = Tx_n$. For the sequence $\{x_n\}$ and from inequality (5.9), we get

$$
\begin{aligned}
&G_b^{\alpha\beta}(x_n, x_{n+1}, x_{n+1}) \\
&\leq \lambda_1 G^{\alpha\beta}(x_{n-1}, x_n, x_n) + \lambda_2 G^{\alpha\beta}(x_{n-1}, x_n, x_n) + \lambda_3 G^{\alpha\beta}(x_n, x_{n+1}, x_{n+1}) \\
&+ \lambda_4 G^{\alpha\beta}(x_n, x_{n+1}, x_{n+1}) + \lambda_5 G^{\alpha\beta}(x_{n-1}, x_{n+1}, x_{n+1}) + \lambda_6 G^{\alpha\beta}(x_n, x_{n+1}, x_{n+1}) \\
&+ \lambda_7 G^{\alpha\beta}(x_n, x_n, x_n) \\
&\leq (\lambda_1 + \lambda_2) G_b^{\alpha\beta}(x_{n-1}, x_n, x_n) + (\lambda_3 + \lambda_4 + \lambda_6) G_b^{\alpha\beta}(x_n, x_{n+1}, x_{n+1}) \\
&+ \lambda_5 \left(\alpha G_b^{\alpha\beta}(x_{n-1}, x_n, x_n) + \beta G_b^{\alpha\beta}(x_n, x_{n+1}, x_{n+1} \right) \qquad (5.10)
\end{aligned}
$$

From inequality (5.10), we get

$$
\begin{aligned}
&[1 - (\lambda_3 + \lambda_4 + \lambda_6) - \lambda_5 \beta]\, G_b^{\alpha\beta}(x_n, x_{n+1}, x_{n+1}) \\
&\leq (\lambda_1 + \lambda_2 + \lambda_5 \alpha) G_b^{\alpha\beta}(x_{n-1}, x_n, x_n)
\end{aligned}
$$

Since $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + (\alpha + \beta)\lambda_5 + \lambda_6 < 1$, we get

$$
G_b^{\alpha\beta}(x_n, x_{n+1}, x_{n+1}) \leq \left(\frac{\lambda_1 + \lambda_2 + \lambda_5 \alpha}{1 - (\lambda_3 + \lambda_4 + \lambda_6) - \lambda_5 \beta} \right) G_b^{\alpha\beta}(x_{n-1}, x_n, x_n)
$$

Following an argument as in theorem 5.1, we can conclude that the sequence $\{x_n\}$ is a $G_b^{\alpha\beta}$-Cauchy sequence in X. Since $(X, G^{\alpha\beta})$ is complete it follows that the sequence $\{x_n\}$ is $G^{\alpha\beta}$-convergent to $x^* \in X$. To show that x^* is a fixed point for T. From inequality (5.9), we have

$$\begin{aligned}
&G_b^{\alpha\beta}(x_n, Tx^*, Tx^*)\\
&\leq \lambda_1 G^{\alpha\beta}(x_{n-1}, x^*, x^*) + \lambda_2 G^{\alpha\beta}(x_{n-1}, x_n, x_n) + \lambda_3 G^{\alpha\beta}(x^*, Tx^*, Tx^*)\\
&+ \lambda_4 G^{\alpha\beta}(x^*, Tx^*, Tx^*) + \lambda_5 G^{\alpha\beta}(x_{n-1}, Tx^*, Tx^*) + \lambda_6 G^{\alpha\beta}(x^*, Tx^*, Tx^*)\\
&+ \lambda_7 G^{\alpha\beta}(x^*, x_n, x_n)
\end{aligned}$$

Taking the limit $n \to \infty$ in the above inequality, we get

$$(1 - \lambda_3 - \lambda_4 - \lambda_5 - \lambda_6) G_b^{\alpha\beta}(x^*, Tx^*, Tx^*) \leq 0$$

It follows that $Tx^* = x^*$. To prove uniqueness we assume that x^{**} is a fixed point for T. Then from inequality (5.9), we get

$$\begin{aligned}
&G_b^{\alpha\beta}(x^*, x^{**}, x^*)\\
&\leq \lambda_1 G^{\alpha\beta}(x^*, x^{**}, x^*) + \lambda_2 G^{\alpha\beta}(x^*, Tx^*, Tx^*) + \lambda_3 G^{\alpha\beta}(x^{**}, Tx^{**}, Tx^{**})\\
&+ \lambda_4 G^{\alpha\beta}(x^*, Tx^*, Tx^*) + \lambda_5 G^{\alpha\beta}(x^*, Tx^{**}, Tx^{**}) + \lambda_6 G^{\alpha\beta}(x^{**}, Tx^*, Tx^*)\\
&+ \lambda_7 G^{\alpha\beta}(x^*, Tx^*, Tx^*)
\end{aligned}$$

It follows that

$$\begin{aligned}
G_b^{\alpha\beta}(x^*, x^{**}, x^*) \leq \lambda_1 G_b^{\alpha\beta}(x^*, x^{**}, x^*) &+ \lambda_5 G_b^{\alpha\beta}(x^*, x^{**}, x^{**})\\
&+ \lambda_6 G_b^{\alpha\beta}(x^{**}, x^*, x^*)
\end{aligned}$$

Using Proposition 5.0.1, we get

$$(1 - \lambda_1 - \lambda_6 - \lambda_5(\alpha + \beta)) G_b^{\alpha\beta}(x^*, x^{**}, x^*) \leq 0. \tag{5.11}$$

Thus, we get $G_b^{\alpha\beta}(x^*, x^{**}, x^*) = 0$ which implies $x^{**} = x^*$. $\square$

Chapter 6

Generalized S-metric

6.1 Introduction

The concept of metric spaces is a very important in Mathematics with a wide range of applicability in many fields in applied sciences. Many authors have given generalizations of metric spaces in several ways. G*ä*hler, introduced the concept of 2-metric spaces,[16] and Dhage, [12] introduced the concepts of D-metric spaces. Mustafa al et.[24] introduced a new structure of a generalized metric space which they called G-metric spaces as a generalization of metric spaces, [23]. They developed and introduced new fixed point theory for various mappings in this new space.

Sam al et.[29] established some useful propositions to show that many fixed point theorems on (non-symmetric) G-metric spaces follow directly from results on metric spaces.

Sedghi et al. introduced D^*-metric spaces,[32] which are modifications of the definition of D-metric spaces introduced by Dhage, [12]. These authors, further introduced the concept of S-metric space and gave some properties with applications as common fixed point theorems for self mappings on complete S-metric spaces [30].

Definition 6.1.1. Let X be a nonempty set. A function $S : X \times X \times X \to [0, \infty)$ is a S-metric on X if for all $x, y, z, w \in X$:

(i) $S(x, y, z) = 0 \iff x = y = z$

(ii) $S(x, y, z) \leq S(x, x, w) + S(y, y, w) + S(z, z, w)$

The pair (X, S) is called an S-metric space, [30].

Example 8. Let $X = \mathbb{R}^n$ and $\|\cdot\|_X$ be a norm on X, then the function S defined by

$$S(x, y, z) = \|y + z - 2x\|_X + \|y - z\|_X$$

is an S- metric on X.

Definition 6.1.2. Let X be a nonempty set and assume that there exists a real number $\alpha \geq 1$. A function $S_b : X \times X \times X \to [0, \infty)$ is an S_b-metric on X if for all $x, y, z, w \in X$:

(i) $S_b(x, y, z) = 0 \iff x = y = z$;

(ii) $S_b(x, x, y) = S_b(y, y, x)$ for all $x, y \in X$;

(iii) $S_b(x, y, z) \leq \alpha\, [S_b(x, x, w) + S_b(y, y, w) + S_b(z, z, w)]$.

The pair (X, S_b) is called an S_b-metric space [38],[34]. If $\alpha = 1$, we have that the S_b-metric is equivalent to the S-metric. It should be noted that the symmetry property follows from the triangle property with $\alpha = 1$.

Definition 6.1.3. Let X be a nonempty set and assume that there exists real numbers $\alpha, \beta, \gamma \geq 1$. A function $S_{\alpha\beta\gamma} : X \times X \times X \to [0, \infty)$ is an $S_{\alpha\beta\gamma}$-metric on X if for all $x, y, z, w \in X$:

(i) $S_{\alpha\beta\gamma}(x,y,z)=0 \iff x=y=z$;

(ii) $S_{\alpha\beta\gamma}(x,y,z)\leq \alpha S_{\alpha\beta\gamma}(x,x,w)+\beta S_{\alpha\beta\gamma}(y,y,w)+\gamma S_{\alpha\beta\gamma}(z,z,w)$.

The pair $(X,S_{\alpha\beta\gamma})$ is called an $S_{\alpha\beta\gamma}$-metric space. If $\alpha=\beta=\gamma=1$, we obtain that $S=S_{\alpha\beta\gamma}$. If $\alpha=\beta=\gamma$ then we obtain that $S_{\alpha\beta\gamma}=S_b$. Furthermore, if $\alpha,\beta\geq 1$ and $\gamma=1$ then we have the symmetry property, $S_{\alpha\beta\gamma}(x,x,y)=S_{\alpha\beta\gamma}(y,y,x)$ for all $x,y\in X$. The following example justifies the weakening in the triangle inequality found in Definition 6.1.3.

Example 9. Let $X=(1,2)$ and define $S_{\alpha\beta\gamma}(x,y,z)$ by

$$S_{\alpha\beta\gamma}(x,y,z)=\begin{cases}2^{|x-y|+|y-z|+|z-x|} & x\neq y\neq z\\ \\ 0 \iff & x=y=z.\end{cases}\tag{6.1}$$

It suffices to verify property (iii) of definition 6.1.3. For $x\neq y\neq z$ we have

$$\begin{aligned}
&S_{\alpha\beta\gamma}(x,y,z) &(6.2)\\
&=2^{|x-y|+|y-z|+|z-x|}\\
&\leq 2^{|x-w|+|w-y|+|y-w|+|w-z|+|z-w|+|w-x|}\\
&=2^{2|x-w|+2|y-w|+2|z-w|}\\
&=2^{\frac{1}{2}(2|x-w|)+\frac{3}{8}(2|y-w|)+\frac{1}{8}(2|z-w|)}2^{|x-w|+\frac{5}{4}|y-w|+\frac{7}{4}|z-w|} &(6.3)\\
&\leq\left[\frac{1}{2}\left(2^{2|x-w|}\right)+\frac{3}{8}\left(2^{2|y-w|}\right)+\frac{1}{8}\left(2^{2|z-w|}\right)\right]\sup_{(1,2)}2^{|x-w|+\frac{5}{4}|y-w|+\frac{7}{4}|z-w|} &(6.4)\\
&=8S_{\alpha\beta\gamma}(x,x,w)+6S_{\alpha\beta\gamma}(y,y,w)+2S_{\alpha\beta\gamma}(z,z,w), &(6.5)
\end{aligned}$$

where we have obtained (6.4) from (6.3) by using Jensen's inequality, [28].

Definition 6.1.4. Let $(X,S_{\alpha\beta\gamma})$ a $S_{\alpha\beta\gamma}$-metric space. For $\epsilon>0$ and $x\in X$, we define the open ball $B_{S_{\alpha\beta\gamma}}(x,\epsilon)=\{y\in X; S_{\alpha\beta\gamma}(y,y,x)<\epsilon\}$.

Definition 6.1.5. Let $(X, S_{\alpha\beta\gamma})$ be a $S_{\alpha\beta\gamma}$-metric space and $A \subset X$:

(i) If for every $x \in A$ there exists $\epsilon > 0$ such that $B_{S_{\alpha\beta\gamma}}(x, \epsilon) \subset A$, then the subset A is open.

(ii) Subset A is bounded if there exists $\epsilon > 0$ such that $S_{\alpha\beta\gamma}(x, x, y) < \epsilon$ for all $x, y \in A$.

(iii) A sequence $\{x_n\}$ in X converges to $x \in X \iff$ for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $S_{\alpha\beta\gamma}(x_n, x_n, x) < \epsilon$ for all $n \geq N$.

(iv) A sequence $\{x_n\}$ is a Cauchy sequence if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $S_{\alpha\beta\gamma}(x_n, x_n, x_m) < \epsilon$ for all $n, m \geq N$.

(v) The $S_{\alpha\beta\gamma}$-metric space $(X, S_{\alpha\beta\gamma})$ is complete if every Cauchy sequence in X is convergent.

Lemma 6.1. *Let $(X, S_{\alpha\beta\gamma})$ be an $S_{\alpha\beta\gamma}$-metric space. If a sequence in X is convergent then the limit point is unique.*

Proof. Let $\{x_n\}$ be a convergent sequence in X. Then for every $\epsilon > 0$ there exists $x \in X$ and $N_1 \in \mathbb{N}$ such that $S_{\alpha\beta\gamma}(x_n, x_n, x) < \frac{\epsilon}{2(\alpha+\beta)\gamma}$ for all $n \geq N_1$. Assume that there exist $y \in X$ and $N_2 \in \mathbb{N}$ such that $S_{\alpha\beta\gamma}(x_n, x_n, y) < \frac{\epsilon}{2\gamma^2}$ for all $n \geq N_2$. From definition 6.1.3, property (ii) it follows that

$$\begin{aligned}
S_{\alpha\beta\gamma}(x, x, y) &\leq \alpha S_{\alpha\beta\gamma}(x, x, x_n) + \beta S_{\alpha\beta\gamma}(x, x, x_n) + \gamma S_{\alpha\beta\gamma}(y, y, x_n) \\
&= (\alpha + \beta) S_{\alpha\beta\gamma}(x, x, x_n) + \gamma S_{\alpha\beta\gamma}(y, y, x_n) \\
&\leq (\alpha + \beta)\gamma S_{\alpha\beta\gamma}(x_n, x_n, x) + \gamma^2 S_{\alpha\beta\gamma}(x_n, x_n, y) \\
&< \epsilon
\end{aligned}$$

for all $n \geq \max\{N_1, N_2\}$. It follows that $S_{\alpha\beta\gamma}(x, x, y) = 0$ thus we get $x = y$. $\square$

6.2 Some fixed point results

Definition 6.2.1. Let $(X, S_{\alpha\beta\gamma})$ be a $S_{\alpha\beta\gamma}$-metric space. A mapping $T : X \to X$ is a contraction if there exists a constant $0 \leq \lambda < 1$ such that

$$S_{\alpha\beta\gamma}(Tx, Tx, Ty) \leq \lambda S_{\alpha\beta\gamma}(x, x, y)$$

for all $x, y \in X$.

Theorem 6.2. *Let $(X, S_{\alpha\beta\gamma})$ be a complete $S_{\alpha\beta\gamma}$-metric space and $T : X \to X$ be a contraction with $0 \leq \lambda < \frac{1}{\gamma^2}$. Then T has a unique fixed point $x \in X$.*

Proof. To show uniqueness, we assume that there exists $x, y \in X$ with $Tx = x$ and $Ty = y$. Then

$$\begin{aligned} S_{\alpha\beta\gamma}(x, x, y) &= S_{\alpha\beta\gamma}(Tx, Tx, Ty) && (6.6) \\ &\leq \lambda S_{\alpha\beta\gamma}(x, x, y) && (6.7) \end{aligned}$$

since $\lambda < 1$, we conclude $S_{\alpha\beta\gamma}(x, x, y) = 0$ thus we get $x = y$. To show existence we show that for $x \in X$ that $\{T^n x\}$ is a Cauchy sequence in X. For $n \in \mathbb{N}$, we recursively obtain that

$$\begin{aligned} S_{\alpha\beta\gamma}(T^n x, T^n x, T^{n+1} x) &\leq \lambda S_{\alpha\beta\gamma}(T^{n-1} x, T^{n-1} x, T^n x) \\ &\vdots \\ &\leq \lambda^n S_{\alpha\beta\gamma}(x, x, Tx) && (6.8) \end{aligned}$$

For $n, m \in \mathbb{N}$, and from inequality (6.8), we get

$$
\begin{aligned}
& S_{\alpha\beta\gamma}(T^n x, T^n x, T^{n+m} x) \\
& \leq (\alpha+\beta) S_{\alpha\beta\gamma}(T^n x, T^n, T^{n+1} x) + (\alpha+\beta)\gamma^2 S_{\alpha\beta\gamma}(T^{n+1} x, T^{n+1}, T^{n+2} x) \\
& + \cdots + (\alpha+\beta)\gamma^{2(m-2)} S_{\alpha\beta\gamma}(T^{n+m-2} x, T^{n+m-2}, T^{n+m-1} x) \\
& + (\gamma)^{2(m-1)} S_{\alpha\beta\gamma}(T^{n+m-1} x, T^{n+m-1}, T^{n+m} x) \\
& \leq (\alpha+\beta) \sum_{i=0}^{m-1} \gamma^{2i} S_{\alpha\beta\gamma}(T^{n+i} x, T^{n+i} x, T^{n+i+1} x) \\
& \leq (\alpha+\beta) \sum_{i=0}^{m-1} \gamma^{2i} \lambda^{n+i} S_{\alpha\beta\gamma}(x, x, Tx) \\
& \leq (\alpha+\beta) \lambda^n S_{\alpha\beta\gamma}(x, x, Tx) \frac{1}{1-(\lambda\gamma^2)}.
\end{aligned}
$$

It follows that $\{T^n x\}$ is a Cauchy sequence and since X is complete there exists $x_0 \in X$ such that $\lim_{n\to\infty} T^n x = x_0$. Since T is continuous it follows that $x_0 = \lim_{n\to\infty} T^{n+1} x = \lim_{n\to\infty} T T^n x = T(\lim_{n\to\infty} T^n x) = T x_0$. Therefore x_0 is a fixed point of T. Taking $m \to \infty$, we get

$$
S(T^n x, T^n x, x_0) \leq (\alpha+\beta) \lambda^n S_{\alpha\beta\gamma}(x, x, Tx) \frac{1}{1-(\lambda\gamma^2)}.
$$

□

Example 10. Let $X = [0, 1]$ and define $S_{\alpha\beta\gamma}(x, y, z)$ by

$$
S_{\alpha\beta\gamma}(x, y, z) = \left(\frac{1}{4}|x-y| + \frac{1}{4}|y-z| + \frac{1}{2}|z-x| \right)^2. \tag{6.9}
$$

Then, we have that

$$
S_{\alpha\beta\gamma}(x, x, w) = \frac{9}{16}|x-w|^2 \tag{6.10}
$$

$$
S_{\alpha\beta\gamma}(y, y, w) = \frac{9}{16}|y-w|^2 \tag{6.11}
$$

$$
S_{\alpha\beta\gamma}(z, z, w) = \frac{9}{16}|z-w|^2 \tag{6.12}
$$

and by Jensen's inequality [28], it follows that

$$S_{\alpha\beta\gamma}(x,y,z) \leq \frac{1}{4}|x-y|^2 + \frac{1}{4}|y-z|^2 + \frac{1}{2}|z-x|^2. \tag{6.13}$$

As

$$\begin{aligned} |x-y|^2 &\leq (|x-w|+|w-y|)^2 \\ &= |x-w|^2 + |w-y|^2 + 2|x-w||y-w| \\ &\leq 2|x-w|^2 + 2|w-y|^2 \end{aligned} \tag{6.14}$$

and similar relations hold for $|y-z|^2$ and $|z-x|^2$ we can simplify (6.13) as follows

$$S_{\alpha\beta\gamma}(x,y,z) \leq \frac{3}{2}|x-w|^2 + |y-w|^2 + \frac{3}{2}|z-w|^2. \tag{6.15}$$

Finally using (6.10)-(6.12) we conclude that

$$S_{\alpha\beta\gamma}(x,y,z) \leq \frac{24}{9}S_{\alpha\beta\gamma}(x,x,w) + \frac{16}{9}S_{\alpha\beta\gamma}(y,y,w) + \frac{24}{9}S_{\alpha\beta\gamma}(z,z,w) \tag{6.16}$$

It follows that $(X, S_{\alpha\beta\gamma})$ is a complete $S_{\alpha\beta\gamma}$-metric space. Let $T : X \to X$ defined by

$$Tx = \frac{1}{x+2},$$

then T is a contraction on X as shown below.

$$\begin{aligned} S_{\alpha\beta\gamma}(Tx,Tx,Ty) &= \frac{9}{16}\left|Tx-Ty\right|^2 \\ &= \frac{9}{16}\left|\frac{1}{x+2} - \frac{1}{y+2}\right|^2 \\ &= \frac{9}{16}\frac{|x-y|^2}{|x+2|^2|y+2|^2} \\ &\leq \frac{1}{16}S_{\alpha\beta\gamma}(x,x,y) \end{aligned} \tag{6.17}$$

where $\frac{1}{16} = \lambda \leq \frac{1}{\gamma^2} = \left(\frac{9}{24}\right)^2$. Thus by theorem 6.2, T has a fixed point $x^\star = \sqrt{2}-1 \in X$.

6.3 Some common fixed points results of mappings

Lemma 6.3. *Let $(X, S_{\alpha\beta\gamma})$ be an $S_{\alpha\beta\gamma}$-metric space and assume that there exists a sequence $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n\to\infty} S_{\alpha\beta\gamma}(x_n, x_n, y_n) = 0$ whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} S_{\alpha\beta\gamma}(x, x, x_n) = 0$ for some $x \in X$ then $\lim_{n\to\infty} y_n = x$.*

Proof. From Definition 6.1.3, property (ii) we get

$$\begin{aligned} S_{\alpha\beta\gamma}(y_n, y_n, x) &\leq (\alpha+\beta)S_{\alpha\beta\gamma}(y_n, y_n, x_n) + \gamma S_{\alpha\beta\gamma}(x, x, x_n) \\ &\leq (\alpha+\beta)\gamma S_{\alpha\beta\gamma}(x_n, x_n, y_n) + \gamma S_{\alpha\beta\gamma}(x, x, x_n) \end{aligned}$$

It follows that $\lim_{n\to\infty} \sup\{(\alpha+\beta)S_{\alpha\beta\gamma}(x_n, x_n, y_n) + \gamma S_{\alpha\beta\gamma}(x, x, x_n)\} = 0$. Since $S_{\alpha\beta\gamma}(\cdot,\cdot,\cdot) \geq 0$, we get

$$0 \leq \lim_{n\to\infty} \inf\{(\alpha+\beta)S_{\alpha\beta\gamma}(x_n, x_n, y_n) + \gamma S_{\alpha\beta\gamma}(x, x, x_n)\} \tag{6.18}$$

$$\leq \lim_{n\to\infty} \sup\{(\alpha+\beta)S_{\alpha\beta\gamma}(x_n, x_n, y_n) + \gamma S_{\alpha\beta\gamma}(x, x, x_n)\} = 0. \tag{6.19}$$

Hence, we get $\lim_{n\to\infty} S_{\alpha\beta\gamma}(y_n, y_n, x) = 0$. Thus we obtain $\lim_{n\to\infty} y_n = x$. □

Definition 6.3.1. Let $(X, S_{\alpha\beta\gamma})$ be a $S_{\alpha\beta\gamma}$-metric space. A pair of mappings $\{f, g\}$ are compatible iff $\lim_{n\to\infty} S_{\alpha\beta\gamma}(fgx_n, fgx_n, gfx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x$ for some $x \in X$.

Theorem 6.4. *Assume that f, g, F, G are self maps of a complete $S_{\alpha\beta\gamma}$-metric space $(X, S_{\alpha\beta\gamma})$ with $f(X) \subset F(X)$, $g(X) \subset G(X)$ and the pairs $\{f, G\}$, $\{g, F\}$ are compatible. If*

$$\begin{aligned} &S_{\alpha\beta\gamma}(fx, fy, gz) \\ &\leq \lambda \max\{S_{\alpha\beta\gamma}(Gx, Gy, Fz), S_{\alpha\beta\gamma}(fx, fx, Gx), \\ &\qquad S_{\alpha\beta\gamma}(gz, gz, Fz), S_{\alpha\beta\gamma}(fy, fy, gz)\} \end{aligned} \tag{6.20}$$

for $x, y, z \in X$ with $0 < \lambda < \frac{1}{\gamma^4}$. Then mappings f, g, F, G have a unique common fixed point in X provided that F, G are continuous.

Proof. Let $x_0 \in X$ then $fx_0 = Fx_1$ for some $x_1 \in X$ since $f(X) \subset F(X)$ and $gx_1 = Gx_2$ for some $x_2 \in X$ since $g(X) \subset G(X)$. In general, we get $y_{2n} = fx_{2n} = Fx_{2n+1}$ for some $x_{2n+1} \in X$ and $y_{2n+1} = gx_{2n+1} = Gx_{2n+2}$ for some $x_{2n+2} \in X$. We shall show that the sequence $\{y_n\}$ is a Cauchy sequence in X. For the sequence $\{y_n\}$ using the inequality (6.20), we get

$$\begin{aligned}
&S_{\alpha\beta\gamma}(y_{2n}, y_{2n}, y_{2n+1}) \\
&= S_{\alpha\beta\gamma}(fx_{2n}, fx_{2n}, gx_{2n+1}) \\
&\leq \lambda \max\{S_{\alpha\beta\gamma}(Gx_{2n}, Gx_{2n}, Fx_{2n+1}), S_{\alpha\beta\gamma}(fx_{2n}, fx_{2n}, Gx_{2n}), \\
&S_{\alpha\beta\gamma}(gx_{2n+1}, gx_{2n+1}, Fx_{2n+1}), S_{\alpha\beta\gamma}(fx_{2n}, fx_{2n}, gx_{2n+1})\} \\
&\leq \lambda \max\{S_{\alpha\beta\gamma}(y_{2n-1}, y_{2n-1}, y_{2n}), S_{\alpha\beta\gamma}(y_{2n}, y_{2n}, y_{2n-1}), \\
&S_{\alpha\beta\gamma}(y_{2n+1}, y_{2n+1}, y_{2n}), S_{\alpha\beta\gamma}(y_{2n}, y_{2n}, y_{2n+1})\} \\
&\leq \lambda\gamma \max\{S_{\alpha\beta\gamma}(y_{2n-1}, y_{2n-1}, y_{2n}), S_{\alpha\beta\gamma}(y_{2n}, y_{2n}, y_{2n+1})\} \qquad (6.21)
\end{aligned}$$

If $S_{\alpha\beta\gamma}(y_{2n}, y_{2n}, y_{2n+1}) > S_{\alpha\beta\gamma}(y_{2n-1}, y_{2n-1}, y_{2n})$ then from inequality (6.21) we get

$$\begin{aligned}
S_{\alpha\beta\gamma}(y_{2n}, y_{2n}, y_{2n+1}) &\leq \lambda\gamma \max\{S_{\alpha\beta\gamma}(y_{2n}, y_{2n}, y_{2n+1})\} \\
&< S_{\alpha\beta\gamma}(y_{2n}, y_{2n}, y_{2n+1})
\end{aligned}$$

is a contradiction. Hence, $S_{\alpha\beta\gamma}(y_{2n}, y_{2n}, y_{2n+1}) \leq S_{\alpha\beta\gamma}(y_{2n-1}, y_{2n-1}, y_{2n})$ therefore

$$\begin{aligned}
S_{\alpha\beta\gamma}(y_{2n}, y_{2n}, y_{2n+1}) &\leq \lambda\gamma S_{\alpha\beta\gamma}(y_{2n-1}, y_{2n-1}, y_{2n}) \\
&\leq \lambda\gamma^2 S_{\alpha\beta\gamma}(y_{2n}, y_{2n}, y_{2n-1}) \qquad (6.22)
\end{aligned}$$

In a similar manner, have that

$$
\begin{aligned}
&S_{\alpha\beta\gamma}(y_{2n-1}, y_{2n-1}, y_{2n})\\
&\leq \gamma S_{\alpha\beta\gamma}(y_{2n}, y_{2n}, y_{2n-1})\\
&= \gamma S_{\alpha\beta\gamma}(fx_{2n}, fx_{2n}, gx_{2n-1})\\
&\leq \lambda\gamma \max\{S_{\alpha\beta\gamma}(Gx_{2n}, G_{2n}, Fx_{2n-1}), S_{\alpha\beta\gamma}(fx_{2n}, f_{2n}, Gx_{2n})\ ,\\
&S_{\alpha\beta\gamma}(gx_{2n-1}, gx_{2n-1}, Fx_{2n-1}), S_{\alpha\beta\gamma}(fx_{2n}, f_{2n}, gx_{2n-1})\}\\
&= \gamma\lambda \max\{S_{\alpha\beta\gamma}(y_{2n-1}, y_{2n-1}, y_{2n-2}), S_{\alpha\beta\gamma}(y_{2n}, y_{2n}, y_{2n-1})\ ,\\
&S_{\alpha\beta\gamma}(y_{2n-1}, y_{2n-1}, y_{2n-2}), S_{\alpha\beta\gamma}(y_{2n}, y_{2n}, y_{2n-1})\}\\
&= \gamma\lambda \max\{S_{\alpha\beta\gamma}(y_{2n}, y_{2n}, y_{2n-1}), S_{\alpha\beta\gamma}(y_{2n-1}, y_{2n-1}, y_{2n-2})\}
\end{aligned}
$$

If $S_{\alpha\beta\gamma}(y_{2n}, y_{2n}, y_{2n-1}) > S_{\alpha\beta\gamma}(y_{2n-1}, y_{2n-1}, y_{2n-2})$ then it follows that

$$
\begin{aligned}
S_{\alpha\beta\gamma}(y_{2n-1}, y_{2n-1}, y_{2n}) &\leq \lambda\gamma S_{\alpha\beta\gamma}(y_{2n}, y_{2n}, y_{2n-1})\\
&\leq \lambda\gamma^2 S_{\alpha\beta\gamma}(y_{2n-1}, y_{2n-1}, y_{2n}) \qquad (6.23)
\end{aligned}
$$

which is a contradiction. Hence

$$
\begin{aligned}
S_{\alpha\beta\gamma}(y_{2n-1}, y_{2n-1}, y_{2n}) &\leq \lambda\gamma S_{\alpha\beta\gamma}(y_{2n-1}, y_{2n-1}, y_{2n-2})\\
&\leq \lambda\gamma^2 S_{\alpha\beta\gamma}(y_{2n-2}, y_{2n-2}, y_{2n-1}) \qquad (6.24)
\end{aligned}
$$

Thus, from inequality (6.22) and (6.24) we obtain

$$
S_{\alpha\beta\gamma}(y_n, y_n, y_{n-1}) \leq \lambda\gamma^2 S_{\alpha\beta\gamma}(y_{n-1}, y_{n-1}, y_{n-2}) \qquad (6.25)
$$

where $\lambda\gamma^2 < 1$ and $n \geq 2$. It follows by repeated application of inequality (6.25) that,

$$\begin{aligned}
S_{\alpha\beta\gamma}(y_n, y_n, y_{n-1}) &\leq \lambda\gamma^2 S_{\alpha\beta\gamma}(y_{n-1}, y_{n-1}, y_{n-2}) \\
&\vdots \\
&\leq (\lambda\gamma^2)^{n-1} S_{\alpha\beta\gamma}(y_1, y_1, y_0)
\end{aligned} \tag{6.26}$$

It follows from (6.26) that

$$S_{\alpha\beta\gamma}(y_n, y_n, y_{n+1}) \leq \gamma S_{\alpha\beta\gamma}(y_{n+1}, y_{n+1}, y_n) \leq \gamma(\lambda\gamma^2)^n S_{\alpha\beta\gamma}(y_1, y_1, y_0) \tag{6.27}$$

For $n, m \in \mathbb{N}$ we get

$$\begin{aligned}
&S_{\alpha\beta\gamma}(y_n, y_n, y_{n+m}) \\
&\leq (\alpha+\beta) S_{\alpha\beta\gamma}(y_n, y_n, y_{n+1}) + (\alpha+\beta)\gamma^2 S_{\alpha\beta\gamma}(y_{n+1}, y_{n+1}, y_{n+2}) + \cdots \\
&+ (\alpha+\beta)(\gamma^2)^{m-2} S_{\alpha\beta\gamma}(y_{n+m-2}, y_{n+m-2}, y_{n+m-1}) \\
&+ (\gamma^2)^{m-1} S_{\alpha\beta\gamma}(y_{n+m-1}, y_{n+m-1}, y_{n+m}) \\
&\leq (\alpha+\beta) \sum_{i=0}^{m-1} (\gamma)^{2i} S_{\alpha\beta\gamma}(y_{n+i}, y_{n+i}, y_{n+i+1}) \\
&\leq (\alpha+\beta)\gamma(\lambda\gamma^2)^n \sum_{i=0}^{m-1} \left((\gamma)^4\lambda\right)^i S_{\alpha\beta\gamma}(y_1, y_1, y_0) \\
&< (\alpha+\beta)\gamma(\lambda\gamma^2)^n \frac{1}{1-\gamma^4\lambda} S_{\alpha\beta\gamma}(y_1, y_1, y_0)
\end{aligned}$$

since $\lambda\gamma^2 < 1$, it follows that $\{y_n\}$ is a Cauchy sequence in a complete $S_{\alpha\beta\gamma}$-metric space, thus there exists $y \in X$ such that $\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} fx_{2n} = \lim_{n\to\infty} Fx_{2n+1} = y = \lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} gx_{2n+1} = \lim_{n\to\infty} Gx_{2n+2}$ We shall now show that y is a common fixed point for mappings f, g, F, G. Since G is continuous, we get $\lim_{n\to\infty} G(Gx_{2n+2}) = Gy$ and $\lim_{n\to\infty} Gfx_{2n} = Gy$ since f, G are compatible $\lim_{n\to\infty} S_{\alpha\beta\gamma}(fGx_{2n}, fGx_{2n}, Gfx_{2n}) =$

0 so by lemma 6.3 it follows that $\lim_{n\to\infty} fGx_{2n} = Gy$. It follows from inequality (6.20),

$$
\begin{aligned}
&S_{\alpha\beta\gamma}(fGx_{2n}, fGx_{2n}, gx_{2n+1})\\
&\leq \lambda \max\{S_{\alpha\beta\gamma}(GGx_{2n}, GGx_{2n}, Fx_{2n+1}), S_{\alpha\beta\gamma}(fGx_{2n}, fGx_{2n}, GGx_{2n}),\\
&S_{\alpha\beta\gamma}(gx_{2n+1}, gx_{2n+1}, Fx_{2n+1}), S_{\alpha\beta\gamma}(fGx_{2n}, fGx_{2n}, gx_{2n+1})\}
\end{aligned}
$$

Taking the limit $n \to \infty$, we get

$$
\begin{aligned}
&S_{\alpha\beta\gamma}(Gy, Gy, y)\\
&\leq \lambda \max\{S_{\alpha\beta\gamma}(Gy, Gy, y), S_{\alpha\beta\gamma}(Gy, Gy, Gy), S_{\alpha\beta\gamma}(y, y, y), S_{\alpha\beta\gamma}(Gy, Gy, y)\}\\
&= \lambda S_{\alpha\beta\gamma}(Gy, Gy, y)
\end{aligned}
$$

since $\lambda < 1$, we get $S_{\alpha\beta\gamma}(Gy, Gy, y) = 0$ thus $Gy = y$. In a similar manner, since F is continuous we get, $\lim_{n\to\infty} FFx_{2n+1} = Fy$, $\lim_{n\to\infty} Fgx_{2n+1} = Fy$ since g and F are compatible, $\lim_{n\to\infty} S_{\alpha\beta\gamma}(gFx_{2n+1}, gFx_{2n+1}, Fgx_{2n+1}) = 0$ and it follows that $\lim_{n\to\infty} gFx_{2n+1} = Fy$. From inequality (6.20),

$$
\begin{aligned}
&S_{\alpha\beta\gamma}(fx_{2n}, fx_{2n}, gFx_{2n+1})\\
&\leq \lambda \max\{S_{\alpha\beta\gamma}(Gx_{2n}, Gx_{2n}, FFx_{2n+1}), S_{\alpha\beta\gamma}(fx_{2n}, fx_{2n}, Gx_{2n}),\\
&S_{\alpha\beta\gamma}(gFx_{2n+1}, gFx_{2n+1}, FFx_{2n+1}), S_{\alpha\beta\gamma}(fx_{2n}, fx_{2n}, gFx_{2n+1})\}
\end{aligned}
$$

Taking the limit $n \to \infty$, we get

$$
\begin{aligned}
&S_{\alpha\beta\gamma}(y, y, Fy)\\
&\leq \lambda \max\{S_{\alpha\beta\gamma}(y, y, Fy), S_{\alpha\beta\gamma}(y, y, y),\\
&S_{\alpha\beta\gamma}(Fy, Fy, Fy), S_{\alpha\beta\gamma}(y, y, Fy)\}\\
&\leq \lambda S_{\alpha\beta\gamma}(y, y, Fy)
\end{aligned}
$$

since $\lambda < 1$, it follows that $Fy = y$. Furthermore, we obtain that

$$\begin{aligned}
&S_{\alpha\beta\gamma}(fy, fy, gx_{2n+1}) \\
&\leq \lambda \{S_{\alpha\beta\gamma}(Gy, Gy, Fx_{2n+1}), S_{\alpha\beta\gamma}(fy, fy, Gy), S_{\alpha\beta\gamma}(gx_{2n+1}, gx_{2n+1}, Fx_{2n+1}), \\
&S_{\alpha\beta\gamma}(fy, fy, gx_{2n+1})\}
\end{aligned}$$

Taking the limit $n \to \infty$, and $Gy = Fy = y$ we have

$$\begin{aligned}
&S_{\alpha\beta\gamma}(fy, fy, y) \\
&\leq \lambda \max \{S_{\alpha\beta\gamma}(Gy, Gy, y), S_{\alpha\beta\gamma}(fy, fy, y), \\
&S_{\alpha\beta\gamma}(y, y, y), S_{\alpha\beta\gamma}(fy, fy, y)\} \\
&= \lambda S_{\alpha\beta\gamma}(fy, fy, y)
\end{aligned}$$

since $\lambda < 1$, $fy = y$. Finally, we have $Gy = Fy = fy = y$ and

$$\begin{aligned}
&S_{\alpha\beta\gamma}(y, y, gy) = S_{\alpha\beta\gamma}(fy, fy, gy) \\
&\leq \lambda \max \{S_{\alpha\beta\gamma}(Gy, Gy, Fy), S_{\alpha\beta\gamma}(fy, fy, Gy), S_{\alpha\beta\gamma}(gy, gy, Fy), S_{\alpha\beta\gamma}(fy, fy, gy)\} \\
&= \lambda S_{\alpha\beta\gamma}(y, y, gy)
\end{aligned}$$

It follows that $gy = y$. Thus we get $Fy = Gy = gy = fy = y$. It remains to show that the common fixed point is unique. Assume that there exists $x \in X$ such that $Fx = Gx = gx = fx = x$ then

$$\begin{aligned}
&S_{\alpha\beta\gamma}(x, x, y) = S_{\alpha\beta\gamma}(fx, fx, gy) \\
&\leq \lambda \max \{S_{\alpha\beta\gamma}(Gx, Gx, Fy), S_{\alpha\beta\gamma}(fx, fx, Gy), S_{\alpha\beta\gamma}(gy, gy, Fy), S_{\alpha\beta\gamma}(fx, fx, gy)\} \\
&= \lambda \max \{S_{\alpha\beta\gamma}(x, x, y), S_{\alpha\beta\gamma}(x, x, x), S_{\alpha\beta\gamma}(x, x, y)\} \\
&= \lambda S_{\alpha\beta\gamma}(x, x, y)
\end{aligned}$$

which implies that $S_{\alpha\beta\gamma}(x, x, y) = 0$ thus $x = y$. □

Corollary 6.4.1. *Let $(X, S_{\alpha\beta\gamma})$ be a complete $S_{\alpha\beta\gamma}$- metric space and let $f, g : X \to X$ be mappings such that*

$$\begin{aligned}
&S_{\alpha\beta\gamma}(fx, fy, gz) \\
&\le \lambda \max\{S_{\alpha\beta\gamma}(x, y, z), S_{\alpha\beta\gamma}(fx, fx, x), \\
&\qquad S_{\alpha\beta\gamma}(gz, gz, z), S_{\alpha\beta\gamma}(fy, fy, gz)\}
\end{aligned}$$

for all $x, y, z \in X$ with $0 \le \lambda < 1$ then there exists a unique fixed point for mappings f and g.

Proof. The proof follows in a similar manner as in Theorem 6.4, by taking mappings F and G as identity mappings. $\square$

Chapter 7

Conclusion

In Chapter 1, we presented the concept of a metric space and showed that a contraction mapping on a complete metric space has a unique fixed point which set the tone and style for sections to follow. We further presented definitions of concepts that were used in the thesis.

In Chapter 2, we presented the concept of a b-metric and showed by relaxing the s-triangle inequality, we can formulate the concept of a generalized b-metric type. We proved that a contraction type mapping on a complete generalized b-metric space has a unique fixed point. We proved that if one replaces the s-inequality of a b-metric by a relaxed polygonal inequality, one can formulate the concept of generalized b- metric space type and can prove that a contraction mapping on a complete generalized b-metric space type has a unique fixed point. We further proved that a Kannan contraction on a complete generalized b-metric space type has a unique fixed point.

In Chapter 3, we imposed a convex structure on a generalized b-metric space type to formulated the concept of a generalized convex metric type space. We proved that a contraction mapping in a complete generalized convex metric type space has a unique fixed point.

In Chapter 4, we imposed a partial ordering on the set of complex numbers and extended the concept of a generalized b-metric type space to a generalized complex valued b-metric type space. We proved that a Reich contraction type mapping on a complete generalized complex valued b-metric space type has a fixed point.

In Chapter 5, we relaxed the rectangle inequality and formulated the concept of a generalized G_b- metric space. We further presented some properties of the generalized G_b metric type. We proved that in a complete generalized G_b metric space type with a contraction mapping a fixed point exists.

In Chapter 6, we formulated the concept of a generalized S_b-metric space and proved that a pair of compatible mappings satisfying a contraction condition on a complete generalized S_b-metric space type has common fixed points.

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