On Spectral Torsion Theories

by

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To

my wife Aurélie Furaha, daughter Laetitia Ishimwe and

son Adélite Bienfait Iraguha.

Declaration

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This thesis represents original work by the author and has not been submitted in any form for any degree or diploma to another University. Where use was made of the work of others it has been duly acknowledged in the text.

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Introduction

An element x of an abelian group G is said to be *torsion* if there exists a nonzero integer n such that $nx = 0$, i.e., x has a finite order. The collection of all torsion elements forms a subgroup of G denoted by $t(G)$. The abelian group G is said to be *torsion* if $t(G) = G$ and *torsion-free* if $t(G) = 0$. The subject of "Torsion Theory" is a generalization of this notion to modules over arbitrary rings.

A Grothendieck category $\mathcal{C}$ is called *spectral* if every short exact sequence in $\mathcal{C}$ splits or, equivalently, if every object in $\mathcal{C}$ is injective (or projective). If $\mathcal{C}$ is $R - Mod$, the category of unital left modules over a ring R, then a classical theorem in Ring Theory tells us that $\mathcal{C}$ is spectral if and only if every module in $R - Mod$ is semisimple, that is, a direct sum of simple modules, and that this is in the case if and only if R is isomorphic to a finite product of matrix rings over division rings. It is not the case that every object in a spectral category is semisimple, indeed it is possible for a Grothendieck category to have no semisimple objects at all. That this does hold in the case of $R - Mod$ is a consequence of the fact that the category of unital modules over a ring (with identity) is a locally finitely generated category. This is a category all of whose subobject lattices are compactly generated. (The latter property is also referred to as "algebraic".)

It is possible to associate with every module category $R - Mod$ a spectral category called the *spectral category of* $R - Mod$ (abbreviated $R - Spec$). Roughly speaking, the objects of $R - Spec$ are the same as those of $R - Mod$ while the set of morphisms from an object A to an object B in $R - Spec$ is the set of morphisms in $R - Mod$ from A to B modulo an equivalence relation which identifies morphisms $f, g : A \longrightarrow B$ if f and g agree on an essential submodule of A. This means that every essential monomorphism in $R - Mod$ becomes an isomorphism in $R - Spec$ and so every object in $R - Mod$ is identified with its injective hull in $R - Spec$. Thus $R - Spec$ can be viewed as a category whose objects comprise all injective R-modules. Every object in $R - Spec$ is injective and so $R - Spec$ is, indeed, a spectral category. In $R - Spec$ the simple objects are precisely the indecomposable injective modules and an object is semisimple precisely if it is the injective hull of a direct sum of indecomposable injective modules. It is well known that over an arbitrary ring R not every injective module is the injective hull of a direct sum of indecomposable injectives. Viewed in categorical terms, this is because $R - Spec$ is, in general, not locally finitely generated. If, however, the category $R - Mod$ is assumed to be locally noetherian, that is, the subobject lattice of every finitely generated object satisfies the ascending chain condition (and this condition is stronger than finitely locally generated), then it follows from an old theorem of Matlis that every injective module decomposes into a direct sum of indecomposable injective modules. In this instance every object in $R - Spec$ is semisimple. Notice that $R - Mod$ is a locally noetherian category if and only if R is a left noetherian ring. We thus obtain the well known result that over a left noetherian ring, every injective left module decomposes into a direct sum of indecomposable injective modules. If the ring R is commutative and noetherian then there is one-to-one correspondence between the *prime spectrum* of R (this is the set of all prime ideals of R) and the indecomposable injective R-modules, given by $P \longmapsto E(R/P)$. This explains the root of the term "spectral category".

The theory of spectral categories when applied to the spectral category $R - Spec$ can be used to illuminate certain useful properties of injective modules. For example, it is known that the endomorphism ring of every object in a spectral category is Von Neumann regular (VNR) and left self-injective. If E is any injective left R-module and $End_{(R-Spec)}E$ denotes the endomorphism ring of E in $R - Spec$, then $End_{(R-Spec)}E = End_R E/J$ where J is the ideal of $End_R E$ consisting of all R-endomorphisms $f : E \longrightarrow E$ such that $Ker f$ is an essential submodule of E (such endomorphisms clearly need to be identified with the zero map in $R - Spec$). The following classical result now follows: $End_R E/J$ is a VNR left self-injective ring. A variant of Schur's Lemma asserts that the endomorphism ring of any simple object in a spectral (Grothendieck) category is a division ring. In the category $R - Spec$, any indecomposable injective module E is a simple object. Hence $End_{(R-Spec)}E = End_R E/J$ is a division ring where $J = \{f \in End_R E \mid Ker f \text{ is essential in } E\} = \{f \in End_R E \mid Ker f \neq 0\}$. It can be shown that J is the Jacobson radical of $End_R E$. We thus obtain another classical result: $End_R E$ modulo its Jacobson radical is a division ring, so $End_R E$ is a local ring. Our purpose in this thesis is to study spectral categories which arise as the quotient category of $R-Mod$ with respect to certain torsion theories on $R-Mod$. If τ is any torsion theory on $R - Mod$ the class of all τ-torsion-free τ-injective left R-modules constitutes a full subcategory of $R - Mod$ and is denoted by $(R, \tau) - Mod$. It can be shown that $(R, \tau) - Mod$ is a Grothendieck category. For example, if $R = \mathbb{Z}$, the ring of integers, and τ denotes the torsion theory on $\mathbb{Z} - Mod$ corresponding with the classical notion of torsion for abelian groups, then the τ-torsion-free τ-injective modules are precisely the left $\mathbb{Q}$-modules, i.e., the left vector spaces over $\mathbb{Q}$. In this instance the Grothendieck category $(R, \tau) - Mod$ coincides with the full module category $\mathbb{Q} - Mod$. Observe that $\mathbb{Q} - Mod$ is spectral, since $\mathbb{Q}$, being a field, is clearly a semisimple ring. In general, the category $(R,\tau) - Mod$ need not coincide with the full category of modules over some ring. It is also, in general, not spectral. If, for example, τ is the trivial torsion theory then every left R-module is trivially τ-torsion-free and τ-injective so that $(R,\tau) - Mod = R - Mod$. Clearly then $(R,\tau) - Mod = R - Mod$ will be spectral if and only if R is a semisimple ring. The prototype of spectral torsion theory is given by the Goldie torsion theory τ_g. The τ_g-torsion-free τ_g-injective left R-modules are precisely the nonsingular injective left R-modules. Inasmuch as each object in $(R,\tau_g) - Mod$ is injective (in $R - Mod$) it is easy to see that the category $(R,\tau_g) - Mod$ is spectral. We shall see that there are spectral torsion theories which are distinct from the Goldie torsion theory τ_g. Nevertheless, every spectral torsion theory on $R - Mod$ is closely associated with, in a sense which will be be made precise later, the Goldie torsion theory on some factor ring of R.

A theme of this thesis is the use of the Goldie torsion theory to illustrate the theory.

Conventions of numbering

In the text, a reference of type Proposition 3.2.6 refers to Proposition 6 in section 2 of Chapter 3. Within Chapter 3 we write Proposition 2.6 and within section 2 we simply use Proposition 6.

Abstract

The purpose of this thesis is to investigate how "spectralness" properties of a torsion theory τ on $R-Mod$ are reflected by properties of the ring R and its ring of quotients R_τ. The development of "spectral" torsion theory owes much to Zelmanowitz [50] and Gomez-Pardo [23]. Gomez-Pardo proved that there exists a bijective correspondence between the set of spectral torsion theories on $R-Mod$ and rings of quotients of R that are Von Neumann regular and left self-injective. Chapter 1 is concerning with the notation used in the thesis and a summary of main results which are needed for understanding the sequel. Chapter 2 is concerned with the construction of a *maximal ring of quotients* of an arbitrary ring R by using the notion of *denseness* and *relative injective hull.* In Chapter 3, we survey the three equivalent ways of formulating Torsion Theory: by means of preradical functors on the category $R-Mod$, pairs of torsion / torsion-free classes and topologizing filters on rings. We shall show that Golan's approach to Torsion Theory via equivalence classes of injectives; and Dickson's one (as presented by Stenström) are equivalent. With a torsion theory τ defined on $R-Mod$ we associate R_τ a ring of quotients of R. The full subcategory $(R,\tau)-Mod$ of $R-Mod$ whose objects are the τ-torsion-free τ-injective left R-modules is a Grothendieck category called the *quotient category* of $R-Mod$ with respect to τ. A left R_τ-module that is τ-torsion-free τ-injective as a left R-module is injective if and only if it is injective as a left R-module (Proposition 3.6.4). Because of its use in the sequel, particular attention is paid to the lattice isomorphism that exists between the lattice of τ-pure submodules of a left R-module M and the lattice of subobjects of the quotient module M_τ in the category $(R,\tau)-Mod$. Chapter 4 introduces the definition of a *spectral torsion theory*: a

torsion theory τ on $R - Mod$ is said to be *spectral* if the Grothendieck category $(R, \tau) - Mod$ is spectral. Using the notion of *relative essential* submodule, one can construct a spectral torsion theory from an arbitrary torsion theory on $R - Mod$. We shall show how an investigation of a general spectral torsion theory on $R - Mod$ reduces to the Goldie torsion theory on $R/t_\tau(R) - Mod$. Moreover, we shall exhibit necessary and sufficient conditions for R_τ to be a *regular left self-injective* ring (Theorem 4.2.10). In Chapter 5, after constructing the torsion functor $Soc_C(-)$ which is associated with the pseudocomplement $\tau^\perp$ of τ in $R - tors$, we show how semiartinian rings can be characterized by means of spectral torsion theories: if a spectral torsion theory τ on $R - Mod$ is generated by the class of τ-torsion simple left R-modules or, equivalently, cogenerated by the class of τ-torsion-free simple left R-modules, then R is a left semiartinian ring (Proposition 5.3.2). Chapter 6 gives Zelmanowitz' important result [50]: R_τ is a semisimple artinian ring if and only if the torsion theory τ is spectral and the associated left Gabriel topology has a basis of finitely generated left ideals. We also exhibit results due to M.J. Arroyo and J. Ríos ([4] and [5]) which illustrate how spectral torsion theories can be used to describe when R_τ is (1) prime regular and left self-injective, (2) a left full linear ring, and (3) a direct product of left full linear rings. We also study the relationship between the flatness of the ring of quotients R_τ and the τ- coherence of the ring R when τ is a spectral torsion theory. It is proved that if τ is a spectral torsion theory on $R - Mod$ then the following conditions are equivalent: (1) R is left τ-coherent; (2) $(R_\tau)_R$ is flat; (3) every right R_τ-module is flat as a right R-module (Proposition 6.3.9). This result is an extension of Cateforis' results.

Partial List of Symbols

$\mathbb{Z}$	ring of integers
$\mathbb{N}$	set of positive integers
$\mathbb{Q}$	field of rational numbers
$\mathbb{R}$	field of real numbers
$\mathbb{Z}_p$	cyclic group $\mathbb{Z}/\mathrm{p}\mathbb{Z}$
$\mathbb{Z}_{(p)}$	localization of $\mathbb{Z}$ at $p\mathbb{Z}$
$\mathbb{Z}_{p^\infty}$	p -primary component of $\mathbb{Q}/\mathbb{Z}$
$B(R)$	set of all central idempotent elements of a ring R
$\emptyset$	empty set
$\subseteq$	inclusion
$\leq$	submodule; partial order
$\leq_e$	essential submodule
$\setminus$	set-theoretic difference
$\in$	element of
$\wedge$	meet
$\vee$	join
$\cap$	intersection
$\Longrightarrow$	implies
$\Longleftrightarrow$	equivalent to
$\cong$	isomorphic to
$Ker\,\alpha$	kernel of α

$Im\,\alpha$	image of α
$[N]\alpha$	$\{(x)\alpha \mid x \in N\}$
$[N]\alpha^{-1}$	$\{x \mid (x)\alpha \in N\}$
1_M	identity homomorphism on R-module M
$\alpha \circ \beta, \alpha\beta$	composition of R-homomorphisms α and β
$A \hookrightarrow B$	injective mapping from A into B
$R - Mod$	category of left R-modules
${}_RM(M_R)$	left (right) R-module
$M \times N$	cartesian product
$M \otimes_R N$	tensor product of M_R and ${}_RN$
$Hom_R(M, N)$	group of R-homomorphisms from M to N
End_RM	ring of R-endomorphisms of M
$R - tors$	lattice of hereditary torsion theories on $R - Mod$
$E(M)$	injective hull of M
$\mathcal{P}_\tau(M)$	lattice of τ-pure submodules of M
$Soc(M)$	socle of M
$Rad(M)$	radical of M
$Z(M)$	singular submodule of M
$(0 :^R M)$	left annihilator of M taken in R
$Q_{max}(R)$	left maximal ring of quotients of R
$Q_{cl}(R)$	left classical ring of quotients of R
R_τ	localization of R with respect to a torsion theory τ

Contents

Chapter 1

Preliminaries

The purpose of this chapter is to provide a brief survey of the background material which is necessary for understanding the sequel. The results are not proved but we indicate to the reader suitable references for more details. The standard texts used in this thesis are Anderson & Fuller [3], Golan [18, 19], Goodearl [24, 25], Passman [34], Stenström [43], van den Berg [45] and Wisbauer [49].

1.1 Basic concepts

Throughout this thesis, some familiarity with the fundamental *algebraic structures: group, abelian group, ring, division ring, field, modules* is assumed. In category theory we shall assume that the reader is familiar with the notions of *covariant (contravariant) functor, natural transformations and full subcategory.* Unless otherwise stated, the symbol R shall denote an associative ring with identity 1 and all modules are unital. We use the symbol ${}_RM$ to indicate that M is a left R-module. If M is a left R-module, $N \leq M$ means N is a submodule of M. The homomorphisms between modules are usually written (and composed) on the opposite side of the scalars but other functions are written on the left of the argument. We shall denote by $R-Mod$ (resp., $Mod-R$) the category of left (resp., right) R-modules, $Hom_R(M,N)$ the abelian group of all left R-homomorphisms from M to N and End_RM the ring of all left endomorphisms of M.

For convenience we discuss below some homological results.

A diagram

$$\begin{tikzcd}
M \arrow[r, "\alpha"] \arrow[d, "\beta"'] & N \arrow[d, "\gamma"] \\
P \arrow[r, "\eta"'] & Q
\end{tikzcd}$$

of R-homomorphisms and left R-modules is said to *commute* or to be *commutative* if $\alpha\gamma = \beta\eta$.

Let $f : M \to N$ be an R-homomorphism. We shall write $Imf = [M]f = \{(m)f \mid m \in M\}$ for the image of f, $Kerf = \{m \in M \mid (m)f = 0\}$ for the kernel of f and $[N]f^{-1} = \{m \in M \mid (m)f \in N\}$.

The sequence (finite or infinite)

$$\dots \longrightarrow M_k \xrightarrow{\alpha_k} M_{k+1} \xrightarrow{\alpha_{k+1}} M_{k+2} \longrightarrow \dots.$$

of R-homomorphisms is said to be *exact* if $Im\,\alpha_k = Ker\,\alpha_{k+1}$ for each successive pair $\alpha_k,\ \alpha_{k+1}$.

An exact sequence of the form $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ ($\star$) is called a *short exact sequence* or an *extension* of C by A.

The short exact sequence ($\star$) is said to *split* if there exists an R-homomorphism $\varphi : C \longrightarrow B$ with $\varphi\beta = 1_C$ where 1_C denotes the identity map on C, or equivalently, if there exists an R-homomorphism $\psi : B \longrightarrow A$ with $\alpha\psi = 1_A$. In that case $B \cong A \oplus C$.

Suppose that the following diagram of left R-modules and R-homomorphisms is commutative and has exact rows:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

If β is an isomorphism then α is an epimorphism if and only if γ is a monomorphism [3, page 52]. Let R and S be rings. A covariant functor $F : R-Mod \longrightarrow S-Mod$ is called *left (resp., right) exact* if given a short exact sequence $0 \to A \to B \to C \to 0$ in $R-Mod$ then $0 \to F(A) \to F(B) \to F(C)$ (resp., $F(A) \to F(B) \to F(C) \to 0$) is exact in $S - Mod$. The functor F is exact if it is both left and right exact. If E is an arbitrary left R-module, the covariant functor $Hom_R(E, -)$ and the contravariant functor $Hom_R(-, E)$ from $R - Mod$ to $\mathbb{Z} - Mod$ are left exact [3, Proposition 16.6].

If $\{M_i \mid i \in I\}$ is a family of left R-modules, we write $\prod_{i\in I} M_i$ for the *direct product* and $\bigoplus_{i\in I} M_i$ for the *direct sum*. In cases where the index set is understood we shall drop the I and simply write $\prod M_i, \bigoplus M_i$. If $M_i = M$ for all $i \in I$ we write $\prod_I M$ ($\bigoplus_I M$) for the direct product (direct sum). If, in addition, the index set I is finite, of order $n \in \mathbb{N}$, say, we write $\bigoplus_I M = \prod_I M = M^n$. We have the known results: $Hom_R(\bigoplus_i M_i, N) \cong \prod_i Hom_R(M_i, N)$ and $Hom_R(N, \prod M_i) \cong \prod Hom_R(N, M_i)$ for $N \in R - Mod$, $\{M_i \mid i \in I\}$ a family of left R-modules [3, page 182].

If $\{N_i \mid i \in \mathbb{N}\}$ is a family of R-submodules of M then $\sum N_i = \bigoplus N_i$ if and only if $N_{n+1} \cap (\sum_{i=1}^{n} N_i) = 0$ for all n [24, page 5].

1.2 Lattices

A *lattice* L is a partially ordered set in which every pair of elements x, y has a least upper bound called the *join* of x and y, written $x \vee y$ and a greatest lower bound called the *meet* of x and y written $x \wedge y$. It follows then by induction that every nonempty finite set of elements has a join and a meet. The operations $\vee$ and $\wedge$ are commutative and associative in L. Furthermore, for any $a \leq b$ where $\leq$ is the partial order on L, we have that $(c \wedge b) \vee a \leq (c \vee a) \wedge b$ for all $c \in L$. If the inverse inequality holds, i.e., $(c \wedge b) \vee a = (c \vee a) \wedge b$, then the lattice L is said to be *modular*. A lattice L is said to be *distributive* if $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for all $a, b, c \in L$. Note that any distributive lattice is modular. A lattice L is said to be *complete* if every subset S of L has a least upper bound written $\bigvee S$ and called the *join* of S and a greatest lower bound denoted by $\bigwedge S$ and called the *meet* of S. For example if R is an arbitrary ring and M a left R-module, then the submodules of M form a complete modular lattice denoted by $L(M)$. If S is a nonempty subset of $L(M)$, then $\bigcap S$ and $\sum S$ are its meet and join respectively. The submodules 0 and M are the unique smallest and largest elements of $L(M)$. We have a well known result: if L is a partially ordered set and every subset of L has a least upper bound in L, then L is a complete lattice [43, Proposition III.1.2]. Let L be a complete lattice with smallest element 0 and greatest element 1. An element $a \in L$ is said to be *essential* if $a \wedge c = 0$ implies $c = 0$ whenever $c \in L$. A *complement* of $a \in L$ is an element $c \in L$ such that $a \wedge c = 0$ and $a \vee c = 1$. If every element of the lattice L has a complement in L, then L is said to be *complemented*. A complemented distributive lattice is called a *boolean lattice*. Let R be an arbitrary ring. The elements $r \in R$ such that $rx = xr$ for all $x \in R$ form a subring of R called the *center* of R. A *central idempotent* element of R is an element e belonging to the center of R such that $e^2 = e$. The central idempotent elements of a ring R form a boolean lattice denoted by $B(R)$ [43, page 71]. We define a *pseudocomplement* of $a \in L$ to be the an element $c \in L$ such that $a \wedge c = 0$ and c is maximal with this property. A nonzero element $a \in L$ is said to be an *atom* (resp., *coatom*) if $b < a$ (resp., $b > a$) implies $b = 0$ (resp., $b = 1$). The lattice L is said to be *atomic* if for every nonzero $b \in L$ there exists an atom a such that $a \leq b$.

An element c of a complete lattice L is *compact* if whenever $D \subseteq L$ and $c \leq \bigvee D$ there exists a finite subset D' of D such that $c \leq \bigvee D'$. The lattice L is said to be *compact* if its greatest element is compact and is said to be *compactly generated* if every element of L is a join of compact elements. Note that the lattice $L(M)$ is compact if and only if M is finitely generated [43, page 74].

Let L and L' be partially ordered by $\leq$. A map $f : L \longrightarrow L'$ is *order preserving* if whenever $a \leq b$ in L, then $f(a) \leq f(b)$ in L'. If L and L' are lattices, then f is a *lattice homomorphism* if $f(a \vee b) = f(a) \vee f(b)$ and $f(a \wedge b) = f(a) \wedge f(b)$ for all $a, b \in L$. Note that if f is bijective with inverse $f^{-1} : L' \longrightarrow L$, then f is a lattice isomorphism if and only if both f and f^{-1} are order preserving.

1.3 Projective and injective modules

A left R-module P is said to be *projective* if given any diagram

$$\begin{tikzcd}
N \arrow[r] & M \arrow[r] & 0 \\
P \arrow[u, dotted, "\beta"] \arrow[ur, "\alpha"'] & &
\end{tikzcd}$$

with exact row, there exists an R-homomorphism $\beta : P \longrightarrow N$ that makes the diagram commute.

Proposition 1.3.1 [34, Theorem 2.8 & Lemma 21.2] *The following conditions are equivalent for* $P \in R - Mod$*:*

(i) P *is projective;*

(ii) $Hom_R(P, -)$ *is an exact functor;*

(iii) every short exact sequence of the form $0 \longrightarrow A \longrightarrow B \longrightarrow P \longrightarrow 0$ *splits;*

(iv) P *is isomorphic to a direct summand of a free left* R*-module.* □

Proposition 1.3.2 [27, Proposition 2.5] *A direct sum* $\bigoplus P_i$ *of left* R*-modules is a projective module if and only if each* P_i *is projective.* □

A left R-module Q is said to be *injective* if given any diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[dr] & M \arrow[d, dotted, "h"] \\
 & & Q
\end{tikzcd}$$

with exact row, there exists an R-homomorphism $h : M \longrightarrow Q$ which makes the diagram commute. It is well known that for any ring R, the module ${}_RR$ is projective but not necessary injective. A ring R is said to be *left self-injective* if ${}_RR$ is injective. We have the following analogue of Proposition 1.

Proposition 1.3.3 [34, pages 206-210] *The following conditions are equivalent for a left R-module Q:*

(i) Q is injective;

(ii) $Hom_R(-, Q)$ is exact;

(iii) every short exact sequence of the form $0 \longrightarrow Q \longrightarrow L \longrightarrow M \longrightarrow 0$ splits;

(iv) Q is a direct summand of every left R-module which contains it. $\square$

Theorem (Baer's Criterion) 1.3.4 [34, Lemma 21.3] *The left R-module Q is injective if and only if given any left ideal I of R and any R-homomorphism $\sigma : I \longrightarrow Q$ there exists an R-homomorphism $\hat{\sigma} : R \longrightarrow Q$ that extends σ.* $\square$

Theorem (Baer's Theorem) 1.3.5 [34, Theorem 21.6] *Every left R-module is contained in an injective left R-module.* $\square$

Dually to Proposition 2, a direct product $\prod Q_i$ of left R-modules is injective if and only if each Q_i is injective [34, Lemma 21.7]. A direct sum of injective left R-modules need not, in general, be injective. In fact, we have

Theorem 1.3.6 [27, Theorem 3.46] *A ring R is left noetherian if and only if every direct sum of injective left R-modules is injective.* $\square$

Proposition 1.3.7 [34, Lemma 21.7] *A direct summand of an injective left R-module is injective.* $\square$

1.4 Essential extensions and injective hulls

Let V be a left R-module and $W \leq V$. We say that V is an *essential extension* of W or W is an *essential submodule* of V if W has a nonzero intersection with every nonzero submodule of V. In this situation we shall write $W \leq_e V$. Golan [18, page 7] uses the term *large* instead of *essential.* Note that every module M has at least one essential submodule, namely M itself and that $0 \leq_e M$ only if $M = 0$. A left R-module M is said to be *essentially finitely generated* if M contains a finitely generated essential submodule. M is called *essentially finitely presented* if there exists a short exact sequence $0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$ with F finitely generated free and K essentially finitely generated. An R-monomorphism $f : V \longrightarrow W$ is said to be *essential* if $[V]f$ is an essential submodule of W. Using Zorn's Lemma, one can show that for any submodule N of a left R-module M, there exists a submodule N' of M which is maximal with respect to the condition that $N \cap N' = 0$. Such a submodule N' (not necessary unique) is called an *orthogonal complement* of N. In this situation, $N \oplus N'$ is an essential submodule of M.

Theorem 1.4.1 [34, Lemma 23.1] *A left R-module Q is injective if and only if it has no proper essential extension.* □

Before stating the next result, we need to introduce some notation. Let X and Y be nonempty subsets of a left R-module M. We define

$$(X :^R Y) = \{r \in R \mid ry \in X \text{ for all } y \in Y\}.$$

If there is no ambiguity about the ring, we drop R and write $(X : Y)$. If X is a submodule of M, then $(X : Y)$ is a left ideal of R. In particular if $X = 0$, then the left ideal $(0 : Y)$ of R is called the *left annihilator* of Y. A left R-module M is said to be *faithful* if the left annihilator of M in R is zero. If X is a submodule of M and S a left ideal of R, $(X :^M S) = \{x \in M \mid sx \in X \text{ for all } s \in S\}$ is a submodule of M.

Propostion 1.4.2 [45, Proposition 4.7] & [24, page 16] *Let A, B and C be left R-modules. Then:*

(i) if $A \leq B \leq C$, then $A \leq_e C$ if and only if $A \leq_e B \leq_e C$;

(ii) if $A \leq_e B \leq C$ and $E \leq_e F \leq C$, then $A \cap E \leq_e B \cap F$;

(iii) if $f : B \longrightarrow C$ is an R-homomorphism and $A \leq_e C$, then $[A]f^{-1} \leq_e B$;

(iv) if $L \leq_e M$, then for $x \in M$, $(L : x) \leq_e {}_RR$. □

Theorem 1.4.3 [34, Theorem 23.2] *Let W be a left R-module. There then exists an injective left R-module V containing W with $W \leq_e V$. Furthermore, V is uniquely determined up to isomorphism.* □

For a left R-module W, we call the uniquely determined (up to isomorphism) injective module V with $W \leq_e V$, the *injective hull* of W and denote it by $E(W)$. Stenström [43, page 118] uses the term *envelope* instead of *hull.* For example, if R is a commutative integral domain with field of fractions F, then $E({}_RR) = {}_RF$.

Proposition 1.4.4 [34, Lemma 23.4] *Let W be an essential submodule of a left R-module V. Then:*

(i) $E(W) \cong E(V)$. Furthermore, any essential extension of W is contained up to isomorphism in $E(W)$;

(ii) If $\{W_i \mid 1 \leq i \leq n\}$ is a finite family of left R-modules and $W = \bigoplus_{i=1}^{n} W_i$, then $E(V) = \bigoplus_{i=1}^{n} E(W_i)$. □

1.5 The socle and radical of a module

A left R-module M is said to be *simple* if it has no other submodule than 0 and M. A left R-module M is said to be *semisimple* if every submodule of M is a direct summand of M. The lattice $L(M)$ of submodules of M is complemented if and only if M is a semisimple module [43, page 66]. We call a ring R *left semisimple* if R is semisimple as left module over itself. A characterization of left semisimple ring is given by

Proposititon 1.5.1 [49, 20.7] *For a ring R, the following conditions are equivalent:*

(i) R is left semisimple;

(ii) every left ideal of R is a direct summand in R;

(iii) R is isomorphic to a finite direct product of finite matrix rings over division rings;

(iv) every (finitely generated) left R-module is projective;

(v) every left R-module is injective;

(vi) every short exact sequence in $R-Mod$ splits;

(vii) every simple left R-module is projective. □

Condition (iii) of the proposition shows that the "semisimplicity" of a ring is left-right symmetric, so we may omit the prefix in "left semisimple" and speak simply of a "semisimple" ring. The rings described in the proposition are also called *semisimple artinian* .

The *socle* of a left R-module M is defined to be the sum of all simple submodules of M. We write $Soc(M)$ for the socle of M and if there is no nonzero simple submodule of M then $Soc(M) = 0$.

Proposition 1.5.2 [34, Proposition 23.5] & [27, Page 242] *Let M, N be left R-modules. The following statements are true:*

(i) for any $f \in Hom_R(M, N)$, $[Soc(M)]f \subseteq Soc(N)$;

(ii) for any submodule K *of* M, $Soc(K) = K \cap Soc(M)$;

(iii) $Soc(\bigoplus M_i) = \bigoplus Soc(M_i)$;

(iv) $Soc(M)$ *is the intersection of all essential submodules of* M;

(v) $Soc({}_R R)$ *is an ideal of* R. ☐

Dual to the socle of M, the *Jacobson radical* of M is defined to be the intersection of all maximal proper submodules of M. It is denoted by $Rad(M)$. If M has no maximal submodule then $Rad(M) = M$. It is well known that the Jacobson radical of R as left R-module coincides with the Jacobson radical of R as right R-module [43, page 179]. We may thus write $Rad(R)$ without ambiguity.

Proposition 1.5.3 [49, 21.6] *Let* M *and* N *be left* R*-modules.*

(i) For any $f \in Hom_R(M, N)$, $[Rad(M)]f \subseteq Rad(N)$.

(ii) $Rad(M/Rad(M)) = 0$.

(iii) $Rad(R)$ *is an ideal of* R. ☐

Note that for a submodule K of M, it is not true, in general, that $Rad(K) = K \cap Rad(M)$. For example, taking $\mathbb{Z}$ as a submodule of ${}_{\mathbb{Z}}\mathbb{Q}$ we have that $Rad({}_{\mathbb{Z}}\mathbb{Z}) = 0$ while $Rad({}_{\mathbb{Z}}\mathbb{Q}) = \mathbb{Q}$ since $\mathbb{Q}$ has no maximal $\mathbb{Z}$-submodule.

1.6 Regular self-injective rings

By a *regular* ring we mean a *Von Neumann regular* ring. This is a ring R such that for every $r \in R$ there exists $r' \in R$ such that $rr'r = r$. We denote by $B(R)$ the set of all central idempotent elements of R. If $B(R)$ contains only 0 and 1 then R is said to be *indecomposable*.

Proposition 1.6.1 [25, Lemma 9.5] *Let J be a two-sided ideal of a regular left self-injective ring R. Then there exists a unique $e \in B(R)$ for which ${}_RJ \leq_e {}_RRe$.*$\square$

Recall that a ring R is said to be *prime* if for nonzero ideals I, J of R we have $IJ \neq 0$. An ideal P of R is called a *prime ideal* of R if $P \supseteq I$ or $P \supseteq J$ whenever I and J are ideals of R such that $P \supseteq IJ$. This is equivalent to R/P being a prime ring.

Proposition 1.6.2 [25, Proposition 9.6] *The following statements are equivalent for a regular left self-injective ring R:*

(i) R is indecomposable;

(ii) R is prime. $\square$

Proposition 1.6.3 [25, Corollary 9.11] *The following conditions are equivalent for a regular left self-injective ring R:*

(i) $B(R)$ is atomic;

(ii) R is isomorphic to a direct product of prime rings. $\square$

A *left full linear* ring is the ring of all linear transformations (written on the right) of any vector space over a division ring. A left full linear ring is an example of a regular left self-injective ring [24, Proposition 2.23].

Proposition 1.6.4 [25, Theorem 9.12] *The following statements are equivalent for a ring R:*

(i) R is isomorphic to a left full linear ring;

(ii) R is a prime regular left self-injective ring and $Soc({}_RR) \neq 0$. $\square$

Proposition 1.6.5 [25, Theorem 9.13] *The following statements are equivalent for a ring R:*

(i) R is isomorphic to a direct product of left full linear rings;

(ii) R is a regular left self-injective ring and $Soc({}_RR) \leq_e {}_RR$. $\square$

1.7 Commutative localization

Let R be a commutative ring. We call S a multiplicatively closed subset of R provided that $1 \in S$ and for every pair $s, t \in S$, $st \in S$. A *localization of R with respect to S* is defined as a ring T together with a ring homomorphism $\varphi : R \longrightarrow T$ satisfying the following three conditions:

L1 $\varphi(s)$ is a unit in T for all $s \in S$;

L2 $Ker\varphi = \{r \in R \mid tr = 0 \text{ for some } t \in S\}$;

L3 every element in T can be expressed in the form $\varphi(t)^{-1}\varphi(r)$ with $r \in R$ and $t \in S$.

When the ring T exists, it satisfies the following universal property:

Proposition 1.7.1 [8, Proposition 3.1] *Suppose $\varphi : R \longrightarrow T$ is a localization of R with respect to multiplicatively closed subset S. For every ring T' and ring homomorphism $\varphi' : R \longrightarrow T'$ satisfying* L1 *and* L2*, there exists a unique ring homomorphism $\theta : T \longrightarrow T'$ such that the diagram*

$$\begin{tikzcd} R \arrow[rr, "\varphi"] \arrow[dr, "\varphi'"'] & & T \arrow[dl, "\theta"] \\ & T' & \end{tikzcd}$$

commutes. Moreover, if $\varphi' : R \longrightarrow T'$ is also a localization of R with respect to S, then θ is a ring isomorphism. $\square$

Thus if T exists, then it is unique up to isomorphism. The existence of T is proved by defining an equivalence relation $\sim$ on the set $R \times S$ by $(r, s) \sim (r', s')$ if and only if there exists $t \in S$ such that $t(s'r - sr') = 0$. We write $S^{-1}R$ for the set of corresponding equivalence classes and denote by $\frac{r}{s}$ the equivalence class of (r, s) under $\sim$. Addition and multiplication operations in $S^{-1}R$ are defined by $\frac{r}{s} + \frac{r'}{s'} = \frac{s'r+sr'}{ss'}$ and $\frac{r}{s} \cdot \frac{r'}{s'} = \frac{rr'}{ss'}$. This makes $S^{-1}R$ into a ring with the identity $\frac{1}{1}$. Defining $\varphi : R \longrightarrow S^{-1}R$ by $\varphi(r) = \frac{r}{1}$, we see that φ is a homomorphism of rings satisfying L1, L2 and L3. By L2, φ is injective if and only if S does not contain any zero divisor of R. In particular the set of all non-zero divisors of R is a multiplicatively closed subset and in this case the ring $S^{-1}R$ is called the *total ring of quotients* of R. If R is an integral domain, then its total ring of quotients is its field of fractions.

If P is a prime ideal of an arbitrary commutative ring R, then $S = R \setminus P = \{r \in R \mid r \notin P\}$ is a multiplicatively closed subset of R. We write R_P in place of $S^{-1}R$. The localization R_P has a unique maximal ideal $PR_P = \{\frac{p}{s} \mid p \in P, s \in R \setminus P\}$. A ring with this property is called a *local* ring.

The construction of the ring of quotients $S^{-1}R$ can be extended to a module of quotients $S^{-1}M$ of a left R-module M. The localization of M with respect to S is defined as $S^{-1}M = \{\frac{m}{s} \mid m \in M,\ s \in S\}$ and $\frac{m}{s} = \frac{m'}{s'}$ if and only if $t(s'm - sm') = 0$ for some $t \in S$. Define addition in $S^{-1}M$ and scalar multiplication by elements of $S^{-1}R$ by $\frac{m}{s} + \frac{m'}{s'} = \frac{s'm+sm'}{ss'}$ and $\frac{r}{s} \cdot \frac{m}{s'} = \frac{rm}{ss'}$. Then $S^{-1}M$ becomes a left $S^{-1}R$-module. There is a canonical R-homomorphism $\psi_M : M \longrightarrow S^{-1}M$ given by

$m \longmapsto \frac{m}{1}$. We call $t(M) = Ker\psi_M = \{m \in M \mid sm = 0 \text{ for some } s \in S\}$ the *S-torsion submodule* of M. A left R-module M is said to be *S-torsion* if $t(M) = M$ and *S-torsion-free* if $t(M) = 0$.

Proposition 1.7.2 [43] *Let R be a commutative ring. Then for every left R-module M, $t(M/t(M)) = 0$.*

Proof: If $x \in M$ and $\bar{x} = x + t(M) \in t(M/t(M))$, then there exists $s \in S$ such that $s\bar{x} = 0$. Thus $sx \in t(M)$ from which it follows that $tsx = 0$ for some $t \in S$. Since S is a multiplicatively closed set, $ts \in S$ and so $x \in t(M)$. Therefore $\bar{x} = 0$. ☐

Let $M \otimes_R N$ denote the tensor product of $M \in Mod - R$ by $N \in R - Mod$.

Theorem 1.7.3 [32] *Let R be a commutative ring with multiplicatively closed subset S and M a left R-module. Then $S^{-1}M \cong S^{-1}R \otimes_R M$.*

Proof: The map $S^{-1}R \times M \longrightarrow S^{-1}M$ defined by $(\frac{a}{s}, m) \longmapsto \frac{am}{s}$ is an R-bilinear map, so that there exists a linear map $\alpha : S^{-1}R \otimes_R M \longrightarrow S^{-1}M$ such that $\alpha(\frac{a}{s} \otimes m) = \frac{am}{s}$. On the other hand, the map $\beta : S^{-1}M \longrightarrow S^{-1}R \otimes_R M$ defined by $\beta(\frac{m}{s}) = \frac{1}{s} \otimes m$ is well-defined, for if $\frac{m}{s} = \frac{m'}{s'}$ then $ts'm = tsm'$ for some $t \in S$ and so $\frac{1}{s} \otimes m = \frac{ts'}{tss'} \otimes m = \frac{1}{tss'} \otimes ts'm = \frac{1}{tss'} \otimes tsm' = \frac{1}{s'} \otimes m'$.

It is easily checked that α and β are mutually inverse $S^{-1}R$-homomorphisms. ☐

Proposition 1.7.4 [45, Proposition 8.3] *Let R be a commutative ring with multiplicatively closed subset S. Suppose $M, N \in R - Mod$ and $\alpha \in Hom_R(M, N)$. The mapping $S^{-1}\alpha : S^{-1}M \longrightarrow S^{-1}N$ defined by $S^{-1}\alpha(\frac{m}{s}) = \frac{\alpha(m)}{s}$ for $\frac{m}{s} \in S^{-1}M$ is an $S^{-1}R$-module homomorphism which makes the diagram*

$$\begin{tikzcd} M \arrow[r, "\alpha"] \arrow[d, "\psi_M"'] & N \arrow[d, "\psi_N"] \\ S^{-1}M \arrow[r, "S^{-1}\alpha"] & S^{-1}N \end{tikzcd}$$

commute. □

In categorical terms, $S^{-1}(-)$ can be viewed as a covariant functor from $R - Mod$ to $S^{-1}R - Mod$. This functor is called the *localization functor of R with respect to S*. By Theorem 3, we have that $S^{-1}(-)$ and $S^{-1}R \otimes_R (-)$ are naturally equivalent.

Now let S be a multiplicatively closed subset of an arbitrary ring R (not necessary commutative). Then a *classical left quotient ring* of R with respect to S is defined to be the ring $S^{-1}R = Q_{cl}(R)$ together with a ring homomorphism $\varphi : R \longrightarrow S^{-1}R$ such that conditions L1, L2, L3 are satisfied.

Contrary to the situation for commutative rings, $S^{-1}R$ need not exist for every multiplicatively closed subset S. The ring of quotients $S^{-1}R$ exists if and only if S satisfies the following two conditions:

(i) if $s \in S$ and $a \in R$ there exists $t \in S$ and $b \in R$ with $ta = bs$; and

(ii) if $as = 0$ for $s \in S$ and $a \in R$ then there exists $t \in S$ such that $ta = 0$. In that case S is called a *left denominator set* and $S^{-1}R$ is unique up to isomorphism. We refer the reader to [34, Chapter 25] for further details.

Chapter 2

The Maximal Ring of Quotients

2.1 Introduction

Let $\mathbb{Z}$ be the ring of integers and $\mathbb{Q}$ the field of rationals. We know of course that $\mathbb{Q}$ is the injective hull of $\mathbb{Z}$ as a left module over $\mathbb{Z}$, i.e., $E({}_{\mathbb{Z}}\mathbb{Z}) = {}_{\mathbb{Z}}\mathbb{Q}$. Forget momentarily the usual multiplication in $\mathbb{Q}$ and look at $\mathbb{Q}$ as a left $\mathbb{Z}$-module. The intention is to give $\mathbb{Q}$ a ring structure which agrees with its $\mathbb{Z}$-module structure.

Proposition 2.1.1 *Let $x \in {}_{\mathbb{Z}}\mathbb{Q}$ and define a $\mathbb{Z}$-homomorphism $q_x : {}_{\mathbb{Z}}\mathbb{Z} \longrightarrow {}_{\mathbb{Z}}\mathbb{Q}$ by $(z)q_x = zx$. Then there exists a unique $\mathbb{Z}$-homomorphism $\hat{q}_x : {}_{\mathbb{Z}}\mathbb{Q} \longrightarrow {}_{\mathbb{Z}}\mathbb{Q}$ that extends q_x.*

Proof: The existence of such a $\mathbb{Z}$-homomorphism $\hat{q}_x$ is ensured by the injectivity of ${}_{\mathbb{Z}}\mathbb{Q}$. Assume that there exist f and g which extend q_x. Since $Ker(f-g) \supseteq \mathbb{Z}$ and $\mathbb{Q}/\mathbb{Z}$ is torsion, $Im(f-g) \cong \mathbb{Q}/Ker(f-g)$ is torsion. On the other hand, $Im(f-g)$ is contained in $\mathbb{Q}$ and $\mathbb{Q}$ is torsion-free. It follows that $Im(f-g) = 0$ from which we infer $f = g$. □

Define a multiplication operation $\star$ on $\mathbb{Q}$ as follows: for $p, r \in \mathbb{Q}$ put $p \star r = (1)(\hat{q}_p \circ \hat{q}_r)$.

Proposition 2.1.2 *The usual* $+$ *and* $\star$ *defined above give* $\mathbb{Q}$ *a ring structure which agrees with its left* $\mathbb{Z}$*-module structure.*

Proof: To show the associativity of $\star$ we need the observation: $\hat{q}_{p\star r} = \hat{q}_p \circ \hat{q}_r$. Let $z \in \mathbb{Z}$. Then $(z)(\hat{q}_p \circ \hat{q}_r) = ((z)\hat{q}_p)\hat{q}_r = (zp)\hat{q}_r = z((p)\hat{q}_r) = z(((1)q_p)\hat{q}_r) = z((1)q_p)\hat{q}_r = z(1)(\hat{q}_p \circ \hat{q}_r) = z(p \star r)$. Thus $\hat{q}_p \circ \hat{q}_r$ and $\hat{q}_{p\star r}$ both extend $q_{p\star r}$; so they must be equal. Now let p, t and r be any elements of $\mathbb{Q}$, then we have $(p \star t) \star r = (1)(\hat{q}_{p\star t} \circ \hat{q}_r) = (1)((\hat{q}_p \circ \hat{q}_t) \circ \hat{q}_r) = (1)(\hat{q}_p \circ (\hat{q}_t \circ \hat{q}_r)) = p \star (t \star r)$.
The other ring properties are easily checked.
It remains to show that the ring structure agrees with the left $\mathbb{Z}$-module structure in the sense that the $\mathbb{Z}$-module embedding $\wr : {}_{\mathbb{Z}}\mathbb{Z} \hookrightarrow {}_{\mathbb{Z}}\mathbb{Q}$ becomes a ring monomorphism, or equivalently, if $r \in \mathbb{Q}$ and $n \in \mathbb{Z}$, then $nr = (n)\wr \star r$. We have $(n)\wr \star r = (1)(\hat{q}_{(n)\wr} \circ \hat{q}_r) = ((1)\hat{q}_{(n)\wr})\hat{q}_r = (1(n)\wr)\hat{q}_r = (n)\hat{q}_r = nr$. □

The purpose of the following section is to generalize the above ideas to an arbitrary ring R and its injective hull $E({}_R R)$. The easiest way is to require ${}_R R$ to be "torsion-free" in some sense which implies that $E({}_R R)$ is also "torsion-free". This requirement should be seen as analogous to $\mathbb{Z}$ and $\mathbb{Q}$ being torsion-free abelian groups in the "classical" sense. We need to introduce the notions of *"singularity"* and *"denseness"*.

2.2 The maximal ring of quotients of a ring

Let M be a left R-module. The *singular* submodule of M denoted by $Z(M)$ is defined as follows:

$$Z(M) = \{x \in M \mid (0 : x) \text{ is an essential left ideal in } R\}.$$

M is said to be *nonsingular* if $Z(M) = 0$ and *singular* if $Z(M) = M$. It can be shown that a left R-module C is singular if and only if there exists a short exact sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ of left R-modules and R-homomorphisms such that $[A]f$ is essential in B and that C is nonsingular if and only if $Hom_R(A, C)$ $= 0$ for every singular left R-module A [24, Proposition 1.20]. A ring R is said to be *left nonsingular* if ${}_RR$ is nonsingular. For example, semisimple rings and regular rings are nonsingular [27, Page 249]. Notice that for a regular ring R, all finitely generated nonsingular left R-modules are projective (injective) if and only if R is left self-injective [25, Theorem 9.2]. Furthermore, if $\{M_i \mid 1 \leq i \leq n\}$ is a family of direct summands of a nonsingular injective left R-module M then $M_1 + M_2 + \ldots + M_n$ is also a direct summand of M [25, Proposition 9.1]. Some properties of nonsingular and singular modules are exhibited in the next result.

Proposition 2.2.1 [24] *The class of all singular left R-modules is closed under taking submodules, direct sums and homomorphic images. The class of nonsingular left R-modules is closed under taking submodules, direct products and essential extensions.*

Proof: Let $N \leq M$ be left R-modules. Note that $Z(N) = N \cap Z(M)$. To see this take $x \in N$ with $(0 : x)$ essential in ${}_RR$. Whence $x \in Z(M)$. Therefore

$x \in N \cap Z(M)$ whenever $x \in Z(N)$. Conversely, if $x \in N \cap Z(M)$, x belongs to N and to $Z(M)$. Thus x is contained in N and $(0 : x)$ is essential in ${}_RR$, so $x \in Z(N)$ from which we conclude that $Z(N) = N \cap Z(M)$. If now $Z(M) = M$ then $Z(N) = N \cap M = N$ and if $Z(M) = 0$ then $Z(N) = N \cap 0 = 0$. Thus the classes of singular modules and nonsingular modules are closed under taking submodules. Furthermore, if $N \leq_e M$ and $Z(N) = 0$ we have that $0 = N \cap Z(M)$ implies $Z(M) = 0$ meaning that the class of nonsingular modules is closed under taking essential extensions.

Let $\{C_i \,|\, i \in I\}$ be a collection of singular left R-modules. For each C_i there exists a short exact sequence $0 \longrightarrow A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i \longrightarrow 0$ with essential monomorphism f_i. This gives rise to an exact sequence $0 \longrightarrow \bigoplus A_i \xrightarrow{\oplus f_i} \bigoplus B \xrightarrow{\oplus g_i} \bigoplus C_i \longrightarrow 0$. Using the fact that each A_i is essential in B_i, we conclude that $\bigoplus A_i$ is essential in $\bigoplus B_i$.

Let M, N be left R-modules and $f : M \longrightarrow N$ an R-homomorphism. Let $m \in Z(M)$ and $r \in (0 : m)$. Since $rm = 0$, $(rm)f = r[(m)f] = 0$. Therefore $r \in (0 : (m)f)$ and $(0 : m) \subseteq (0 : (m)f)$. Thus $(0 : (m)f)$ is essential in ${}_RR$ because $(0 : m)$ is essential in ${}_RR$, so $[Z(M)]f \subseteq Z(N)$.

Let $\{C_i \mid i \in I\}$ be a family of nonsingular left R-modules and A an arbitrary singular left R-module. Then $Hom_R(A, \prod C_i) \cong \prod Hom_R(A, C_i) = 0$. It follows that $\prod C_i$ is nonsingular. □

If A is essential in B then certainly B/A is singular. The converse holds if B is nonsingular for suppose $A' \cap A = 0$ for some submodule A' of B. Then $A' \cong (A \oplus A')/A \leq B/A$ and hence A' is singular. But being a submodule of a nonsingular module B, A' must be nonsingular. Thus $A' = 0$, whence A is an essential submodule of B. We have thus proved:

Proposition 2.2.2 [24] *Let B be a nonsingular left R-module. Then A is essential in B if and only if B/A is singular.* $\square$

In the particular case of $R = \mathbb{Z}$; every nonzero left ideal of $\mathbb{Z}$ is an essential left ideal. Hence if M is a $\mathbb{Z}$-module we have $Z(M) = \{x \in M \mid (0 : x) \neq 0\}$ which is precisely the torsion subgroup of M. Thus M is singular if and only if M is a torsion abelian group, and M is nonsingular if and only if M is a torsion-free abelian group. Furthermore, if M is an abelian group and $t(M)$ denotes its torsion subgroup then $t(M/t(M)) = 0$, i.e., $M/t(M)$ is torsion-free. In general, the analogue of this property does not hold for the singular submodule. In other words, it is not always true that $Z(M/Z(M)) = 0$. Nevertheless, this very desirable property does hold whenever M is a left module over a left nonsingular ring R. For this reason we shall initially at least restrict ourselves to the case where R is left nonsingular. This is Goodearl's approach in [24].

Let R be a left nonsingular ring and $E(R)$ the injective hull of ${}_RR$. We now define a ring structure on $E(R)$. The reader should compare the details which follow, with those in the introductory section of this chapter. For $x \in E(R)$, define an R-homomorphism $\varphi_x : {}_RR \longrightarrow E(R)$ by $(r)\varphi_x = rx$. Then by injectivity of $E(R)$ there exists an R-homomorphism $\hat{\varphi}_x : E(R) \longrightarrow E(R)$ that extends φ_x. Since ${}_RR$ is nonsingular and $E(R)/{}_RR$ singular, $\hat{\varphi}_x$ is unique. Consider now the "addition" operation defined on $E(R)$ and define a "multiplication" operation $\star$ in $E(R)$ as follows: $x \star y = (1)(\hat{\varphi}_x \circ \hat{\varphi}_y)$ for all $x, y \in E(R)$.

Proposition 2.2.3 *Let R be a left nonsingular ring. Then E(R) is a ring and the* *ring structure agrees with the left R-module structure.*

Proof: It is easily checked that the properties of a ring are satisfied. We shall only establish associativity. To this end note that $\hat{\phi}_{(x\star y)} = \hat{\phi}_x \circ \hat{\phi}_y$. For let $r \in R$. Then $(r)(\hat{\phi}_x \circ \hat{\phi}_y) = (r)(\hat{\phi}_x)\hat{\phi}_y = ((r)\phi_x)\hat{\phi}_y = (rx)\hat{\phi}_y = r((x)\hat{\phi}_y) = r[(1x)\hat{\phi}_y] = r[((1)\phi_x)\hat{\phi}_y] = r[(1)(\hat{\phi}_x \circ \hat{\phi}_y)] = r(x \star y) = (r)\phi_{x\star y} = (r)\hat{\phi}_{x\star y}$. Thus $\hat{\phi}_x \circ \hat{\phi}_y$ and $\hat{\phi}_{x\star y}$ both extend $\phi_{x\star y}$ and so they must be equal. Let $t, p, q \in E(R)$. Then $(p \star q) \star t = (1)[\hat{\phi}_{p\star q} \circ \hat{\phi}_t] = (1)[(\hat{\phi}_p \circ \hat{\phi}_q) \circ \hat{\phi}_t] = (1)[\hat{\phi}_p \circ (\hat{\phi}_q \circ \hat{\phi}_t)] = p \star (q \star t)$.
Now show that this structure agrees with the R-module structure. The ring structure agrees with the left R-module structure because, by Proposition 1.2, we have that $(r) \wr \star y = (1)[\hat{\phi}_{(r)\wr} \circ \hat{\phi}_y] = [(1)\phi_{r\wr}]\hat{\phi}_y = [(1)(r\wr)]\hat{\phi}_y = (r)\hat{\phi}_y = (r)\phi_y = ry$. $\square$

Below we survey a construction of the maximal ring of quotients of an arbitrary (not necessary nonsingular) ring R. To this end and because of the need to have a "torsion-free" ring, we need to modify our notion of "torsion" by introducing the notion of "denseness". There is another change which needs also to be made, the multiplication operation will not be defined on the whole of $E(R)$ but on a "smaller hull" within $E(R)$.

A left ideal I of R is said to be *dense* in R if $Hom_R(R/I, E(R)) = 0$.

Lemma 2.2.4 [43] *Every dense left ideal of R is essential in R.*

Proof: Let $I \leq {}_RR$ be dense in R and consider $0 \neq K \leq {}_RR$ such that $I \cap K = 0$. Then $K \cong [I + K]/I \leq R/I$. By the injectivity of $E(R)$ we have the commutative diagram

$$\begin{tikzcd}
K \arrow[r] \arrow[d, "i"'] & R/I \arrow[dl, "f"] \\
E(R) &
\end{tikzcd}$$

with f extending i. This contradicts the denseness of I unless $K = 0$. $\square$

Note that if R is a left nonsingular ring the converse of Lemma 4 holds, i.e.:

Proposition 2.2.5 [43] *If R is left nonsingular then a left ideal I of R is dense in R if and only if I is essential in R.*

Proof: Let I be an essential left ideal of a left nonsingular ring R. Then by Proposition 2, R/I is singular. Since $E(R)$ is nonsingular we must have $Hom_R(R/I, E(R)) = 0$ meaning I is dense in R. $\square$

Proposition 2.2.6 [43] *Let I and J be left dense ideals of R. Then:*

(i) $I \cap J$ is dense in R;

(ii) if $I \subseteq K \leq {}_RR$, then K is also dense in R;

(iii) if $(K : b)$ is dense in R for all $b \in I$, then K is dense in R;

(iv) $(I : r)$ is dense in R for every $r \in R$.

Proof: (i) By the injectivity of $E(R)$, every R-homomorphism $\alpha : R/(I \cap K) \to E(R)$ can be extended to an R-homomorphism $\hat{\alpha} : R/I \to E(R)$. Therefore if $\hat{\alpha} = 0$ then $\alpha = 0$.

(ii) Let $\pi : R/I \longrightarrow R/K$ be the canonical epimorphism and $\varphi \in Hom_R(R/K, E(R))$. Then the composition $\pi\varphi \in Hom_R(R/I, E(R)) = 0$ since I is dense in R. Therefore $\varphi = 0$.

(iii) For all $b \in J$, we have $R/(I : b) \cong R/(0 : b + I) \cong R(b + I) \leq (I + J)/I$. If

$\varphi \in Hom_R(R/I, E(R))$ then $Ker\varphi \supseteq R(b+I)$ for all $b \in J$ because $Hom_R(R/(I:b), E(R)) = 0$, so $Ker\varphi \supseteq (I+J)/I$. Hence φ factors through $(R/I)/((I+J)/I) \cong R/(I+J)$. Since J is dense in R, $I+J$ is also dense in R by (i). It follows that $\varphi = 0$.

(iv) Let $r \in R$. We have that $R/(I:r) \cong R/(0:r+I) \cong R(r+I) \leq R/I$; so the R-homomorphism $\alpha : R/(I:r) \longrightarrow E(R)$ can be extended to an R-homomorphism $\hat{\alpha} : R/I \longrightarrow E(R)$. Therefore $\alpha = 0$ inasmuch as $\hat{\alpha} = 0$. $\square$

For each $M \in R-Mod$ we define $\delta(M)$ as follows:

$$\delta(M) = \{x \in M \mid (0:x) \text{ is a dense left ideal in } R\}.$$

Note that $\delta(M) \subseteq Z(M)$ by Lemma 4 and equality holds if R is left nonsingular by Proposition 5.

Proposition 2.2.7 *For an arbitrary ring R and left R-module M the following assertions are true:*

(i) $\delta(M)$ is a submodule of M;

(ii) $\delta(M/\delta(M)) = 0$;

(iii) $\delta({}_RR) = 0$.

Proof: (i) Obviously $\delta(M) \neq \emptyset$. Let $x, y \in \delta(M)$. By Proposition 6, we have that $(0:x+y)$ is dense in R since $(0:x)$ and $(0:y)$ are dense in R and $(0:x) \cap (0:y) \subseteq (0:x+y)$. Therefore $x+y \in \delta(M)$. It remains to show that $rx \in \delta(M)$ for every $r \in R$. Since $((0:x):r) = (0:rx)$ and $((0:x):r)$ is dense in R we have that $(0:rx)$ is dense in R by Proposition 6. Therefore $rx \in \delta(M)$.

(ii) Let $\bar{x} = x + \delta(M) \in M/\delta(M)$. Then $\bar{x} \in \delta(M/\delta(M))$ if $(0:\bar{x})$ is dense in R. But $(0:\bar{x}) = \{r \in R \mid rx \in \delta(M)\}$ and $(0:rx) = ((0:x):r)$. It follows from

Proposition 6 (iii) that if $((0:x):r)$ is dense in R then $(0:x)$ is also dense in R. Therefore $\bar{x} \in \delta(M/\delta(M))$ implies that $x \in \delta(M)$, i.e., $\bar{x}=0$.

(iii) $\delta({}_RR) = \{x \in R \mid Ix = 0 \text{ with } I \leq {}_RR \text{ such that } Hom_R(R/I, E(R)) = 0\}$. Assume that $\delta({}_RR) \neq 0$ and take $0 \neq t \in \delta({}_RR)$. Then $Hom_R(R/(0:t), E(R)) = 0$. Since $Rt \cong R/(0:t)$ we have that $Hom_R(Rt, E(R)) = 0$. On the other hand $Rt \hookrightarrow {}_RR \hookrightarrow E(R)$. Thus the equality $Hom_R(Rt, E(R)) = 0$ implies $Rt = 0$. Hence $\delta({}_RR) = 0$. ☐

In the light of Proposition 7 we see that $\delta(M)$ more closely resembles the "classical" torsion submodule $t(M)$ than the singular submodule $Z(M)$. To construct the maximal ring of quotients in the general case we need to define a "smaller hull" for ${}_RR$ within $E(R)$. We define a submodule $E_\delta(R)$ of $E(R)$ as follows: $E_\delta(R)/R = \delta(E(R)/R)$. Thus ${}_RR \subseteq E_\delta(R) \subseteq E(R)$. Let B be a left R-module and $A \leq B$. We say that A is a *dense* submodule of B if $\delta(B/A) = B/A$ (this is one of several equivalent formulations of denseness). The module $E_\delta(R)$ satisfies the following important relative injectivity condition:

Proposition 2.2.8 *Given a diagram*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[dr] & B \arrow[d, dotted, "g"] \\
 & & E_\delta(R)
\end{tikzcd}$$

with A a dense submodule of B, there exists a unique R-homomorphism $g : B \longrightarrow E_\delta(R)$ which makes the diagram commute.

Proof: Consider the following diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] & B \arrow[dl, "g"] \arrow[d, dotted, "h"] \\
E_\delta(R) \arrow[r, "i"'] & E(R)
\end{tikzcd}$$

Since $E(R)$ is injective there exists an R-homomorphism $h : B \longrightarrow E(R)$ which makes the diagram commute. We need to show that Imh is contained in $E_\delta(R)$. To this end consider $[B]h \cap E_\delta(R)$. Note that $[B]h/([B]h \cap E_\delta(R))$ is isomorphic to $([B]h + E_\delta(R))/E_\delta(R)$ which is contained in $E(R)/E_\delta(R)$. But $\delta(E(R)/E_\delta(R)) = 0$ since $E(R)/E_\delta(R) \cong (E(R)/R)/\delta(E(R)/R)$ and $\delta((E(R)/R)/\delta(E(R)/R)) = 0$. Since $[B]h/([B]h \cap E_\delta({}_R R))$ is isomorphic to a submodule of $E(R)/E_\delta(R)$, $\delta([B]h/([B]h \cap E_\delta(R))) = 0$. However, there is a canonical epimorphism $B/A \longrightarrow [B]h/([B]h \cap E_\delta(R))$ defined by $x + A \longrightarrow (x)h + ([B]h \cap E_\delta(R))$. This map is well-defined because $x \in A$ implies $(x)h = (x)f \in E_\delta(R)$, but $(x)h \in [B]h$, so $(x)h \in [B]h \cap E_\delta(R)$. Since A is dense in B and $[B]h/([B]h \cap E_\delta(R))$ is an epimorphic image of B/A we must have $[B]h \cap E_\delta(R)$ dense in $[B]h$. But earlier we showed that $\delta([B]h/([B]h \cap E_\delta(R))) = 0$. Whence $[B]h/([B]h \cap E_\delta(R)) = 0$, i.e., $[B]h \subseteq E_\delta(R)$. This means that there exists $g : B \longrightarrow E_\delta(R)$ such that $gi = h$. The map g makes the above diagram commute. Using the fact that B/A is singular and $E_\delta(R)$ is nonsingular it is easily verified that g is unique. ☐

With this result in hand the left maximal ring of quotients of R can be defined as in the case of a left nonsingular ring. For $x \in E_\delta(R)$ define $\varphi_x : R \longrightarrow E_\delta(R)$ by r $\longmapsto rx$ and take $\hat{\varphi}_x$ the unique R-endomorphism on $E_\delta(R)$ which extends φ_x. Now if $x, y \in E_\delta(R)$ put $x \star y = (1)(\hat{\varphi}_x \circ \hat{\varphi}_y)$. The verification that $E_\delta(R)$ is a ring under $+$ and $\star$, and the proof of the next proposition are left to the reader. We call the ring $E_\delta(R)$ the *left maximal ring of quotients of* R and denote it by $Q_{max}({}_RR)$.

Proposition 2.2.9 *Let R be an arbitrary ring. Then $E_\delta(R)$ is a ring and the ring structure agrees with the left R-module structure.* ☐

2.3 The maximal ring of quotients as an endomorphism ring

In [18], Golan adopts a superficially different approach and defines the left maximal ring of quotients of R to be $End_R E_\delta(R)$. We need to show that the two approaches yield the same result. For $r \in R$ define an R-homomorphism $\phi_r : {}_RR \longrightarrow {}_RR$ by $t \longmapsto (t)\phi_r = tr$. Since ${}_RR$ is dense in $E_\delta(R)$, it follows from Proposition 2.8 that there exists a unique $\hat{\phi}_r$ which makes the diagram

$$\begin{tikzcd}
{}_RR \arrow[r, "\phi_r"] \arrow[d] & {}_RR \arrow[d] \\
E_\delta(R) \arrow[r, "\hat{\phi}_r"] & E_\delta(R)
\end{tikzcd}$$

commute.

Lemma 2.3.1 *Let $r \in R$. The map $\phi : R \longrightarrow End_R E_\delta(R)$ defined by $r \longmapsto \hat{\phi}_r$ is a ring monomorphism.*

Proof: Let $r, r' \in R$. The endomorphisms $\hat{\phi}_{rr'}$ and $\hat{\phi}_r \circ \hat{\phi}_{r'}$ both extend $\phi_{rr'} : {}_RR \longrightarrow {}_RR$, so ϕ is multiplicative. A similar argument shows that ϕ is additive. Thus by uniqueness we have $\hat{\phi}_{rr'} = \hat{\phi}_r \circ \hat{\phi}_{r'}$. Obviously ϕ is a monomorphism. ☐

Since the ring R can be viewed as a subring of $End_R E_\delta(R)$ via the ring monomorphism ϕ, it is possible to define a left R-module structure on $End_R E_\delta(R)$ as follows: $r.\alpha = \hat{\phi}_r \circ \alpha$ for $r \in R$ and $\alpha \in End_R E_\delta(R)$.

Proposition 2.3.2 *$E_\delta(R)$ and $End_R E_\delta(R)$ are isomorphic as left R-modules.*

Proof: Consider the map $\beta : E_\delta(R) \longrightarrow End_R E_\delta(R)$ defined by $(x)\beta = \hat{\phi}_x$. Observe that β is an R-homomorphism since for $x, y \in E_\delta(R)$ the endomorphisms $\hat{\phi}_{x+y}$ and $\hat{\phi}_x + \hat{\phi}_y$ both extend $\phi_x + \phi_y : {}_RR \longrightarrow E_\delta(R)$. Thus they must be equal. Similarly, if $r \in R$ we must have $(rx)\beta = \hat{\phi}_{rx} = \hat{\phi}_r \circ \hat{\phi}_x = \hat{\phi}_r \circ (x)\beta = r[(x)\beta]$. If $\alpha \in End_R(E_\delta(R))$ then $\hat{\phi}_{(1)\alpha} = \alpha$, so β is an epimorphism. Finally, β is a monomorphism since if $0 \neq x \in E_\delta(R)$ then $(1)(x)\beta = (1)\hat{\phi}_x = (1)\phi_x = x \neq 0$. $\square$

If $E_\delta(R)$ is viewed as a ring with multiplication defined as in the previous section then the map β becomes a ring isomorphism.

Chapter 3

Torsion Theories

Torsion theories on $R - Mod$ can be described either by torsion radicals (kernel functors) on $R - Mod$ or by Gabriel topologies (idempotent topologizing filters) on R or by hereditary torsion classes of R-modules. Most of results of this chapter come from [18], [19], [43] and [45].

3.1 Preradicals

A functor $\tau : R - Mod \longrightarrow R - Mod$ is said to be a *preradical* if the following two conditions are satisfied:

P1 For all $M \in R - Mod$, $\tau(M) \leq M$.

P2 For every R-homomorphism $f : M \longrightarrow N$, $\quad [\tau(M)]f \subseteq \tau(N)$.

For example, the functors ξ and χ defined by $\xi(M) = 0$ and $\chi(M) = M$ for all $M \in R - Mod$ are preradicals referred to as the "trivial" preradicals. The class of all preradicals on $R - Mod$ is, in general, not a set and so cannot be a lattice in the strict sense of the word. It does, however, enjoy the properties of a complete lattice with partial order defined as follows: $\tau_1 \leq \tau_2$ if and only if $\tau_1(M) \leq \tau_2(M)$ for all $M \in R-Mod$. Moreover, for any class T of preradicals on $R-Mod$ the preradicals $\bigwedge T$ and $\bigvee T$ given by $(\bigwedge T)(M) = \bigcap_{\tau \in T} \tau(M)$ and $(\bigvee T)(M) = \sum_{\tau \in T} \tau(M)$ are respectively the greatest lower bound element and the least upper bound element of T. With every preradical τ on $R-Mod$ we associate two classes of left R-modules:

$\mathcal{T}_\tau = \{M \in R-Mod \mid \tau(M) = M\}$ and $\mathcal{F}_\tau = \{M \in R-Mod \mid \tau(M) = 0\}$ which satisfy the following properties:

Proposition 3.1.1 [45] *Let τ be a preradical on $R-Mod$. Then:*

(i) $\mathcal{T}_\tau$ is closed under taking quotients and direct sums;

(ii) $\mathcal{F}_\tau$ is closed under taking submodules and direct products;

(iii) $Hom_R(M,N) = 0$ for all $M \in \mathcal{T}_\tau$ and $N \in \mathcal{F}_\tau$.

Proof: (*i*) Let $M \in \mathcal{T}_\tau$ and $N \leq M$. Since there is a canonical epimorphism $\alpha : M \longrightarrow M/N$ it follows that $\tau(M/N) \geq [\tau(M)]\alpha = [M]\alpha = M/N$. Thus $M/N \in \mathcal{T}_\tau$.

Let now $\{M_i \mid i \in I\}$ be a family of elements of $\mathcal{T}_\tau$. Since each M_i embeds in $\bigoplus M_i$, $M_i = \tau(M_i) \leq \tau(\bigoplus M_i)$ for each $i \in I$, so $\bigoplus M_i = \tau(\bigoplus M_i)$.

(*ii*) Let $M \in \mathcal{F}_\tau$ and $N \leq M$ and $\wr : N \hookrightarrow M$ be the inclusion mapping. Then $[\tau(N)]\wr = \tau(N) \leq \tau(M) = 0$. Thus $N \in \mathcal{F}_\tau$.

Suppose $\{M_i \mid i \in I\} \subseteq \mathcal{F}_\tau$. For each $j \in I$, let $\pi_j : \prod M_i \longrightarrow M_j$ be the canonical projection. Then $[\tau(\prod M_i)]\pi_j \leq \tau(M_j) = 0$ for all $j \in I$. Therefore $\tau(\prod M_i) = 0$.

(*iii*) Let $M \in \mathcal{T}_\tau$, $N \in \mathcal{F}_\tau$ and $\alpha \in Hom_R(M,N)$. By (*i*), $M/Ker\alpha \in \mathcal{T}_\tau$. But $M/Ker\alpha \cong Im\alpha \leq N$ and since $\mathcal{F}_\tau$ is closed under submodules, $Im\alpha \in \mathcal{F}_\tau$. It follows that $Im\alpha = 0$ and so $\alpha = 0$. $\square$

If $\mathcal{C}$ is a class of left R-modules closed under taking quotients and direct sums then it is possible to associate with $\mathcal{C}$ a preradical σ on $R - Mod$ in the following way: for $M \in R - Mod$ denote by $\sigma(M)$ the sum of all submodules of M contained in $\mathcal{C}$, i.e., $\sigma(M) = \sum\{N \leq M \mid N \in \mathcal{C}\}$.

Since $\mathcal{C}$ is closed under taking quotients and direct sums, the canonical epimorphism $\bigoplus\{N \leq M \mid N \in \mathcal{C}\} \longrightarrow \sum\{N \leq M \mid N \in \mathcal{C}\} = \sigma(M)$ shows that $\sigma(M) \in \mathcal{C}$. To prove that σ is a preradical let $\alpha : M \longrightarrow M'$ be an R-homomorphism and take $N \leq M$ with $N \in \mathcal{C}$. $[N]\alpha \in \mathcal{C}$ because $\mathcal{C}$ is closed under quotients. Clearly this implies $[\sum\{N \leq M \mid N \in \mathcal{C}\}]\alpha \leq \sum\{N' \leq M' \mid N' \in \mathcal{C}\}$, i.e., $[\sigma(M)]\alpha \leq \sigma(M')$. Thus σ is a preradical. The association of σ with $\mathcal{C}$ is natural in the sense that $\mathcal{T}_\sigma = \mathcal{C}$. To see this note first that $\mathcal{C} \subseteq \mathcal{T}_\sigma$. It was noted above that $\sigma(M) \in \mathcal{C}$ for every $M \in R - Mod$. Thus $\mathcal{T}_\sigma \subseteq \mathcal{C}$ and so $\mathcal{T}_\sigma = \mathcal{C}$.

In general a preradical on $R - Mod$ need not be an exact functor. But in some cases a partial exactness can happen.

Proposition 3.1.2 [45] *The following conditions are equivalent for a preradical τ on $R - Mod$:*

(i) if $M \in R - Mod$ and $N \leq M$ then $\tau(N) = N \cap \tau(M)$;

(ii) τ is a left exact functor.

Proof: $(i) \Longrightarrow (ii)$ The short exact sequence $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ of left R-modules induces a sequence $0 \longrightarrow \tau(A) \xrightarrow{\tau(\alpha)} \tau(B) \xrightarrow{\tau(\beta)} \tau(C)$ which is exact at $\tau(A)$. We have exactness at $\tau(B)$ because $Ker(\tau(\beta)) = Ker\beta \cap \tau(B) = Im\alpha \cap \tau(B) = \tau(Im\alpha) = Im(\tau(\alpha))$.

$(ii) \Longrightarrow (i)$ Let $A \leq B$ be left R-modules. By (ii), the exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow B/A \longrightarrow 0$ induces the exact sequence $0 \longrightarrow \tau(A) \longrightarrow \tau(B) \longrightarrow \tau(B/A)$. The exactness at $\tau(B)$ implies $\tau(A) = A \cap \tau(B)$. □

A preradical τ satisfying the equivalent conditions of the above proposition is called a *torsion preradical*. In [22], Goldman calls such a preradical a *kernel functor*.

Note that if $N \leq M$ with $N \in \mathcal{T}_\tau$ then $N \subseteq \tau(M)$; but $\tau(M)$ need not be contained in $\mathcal{T}_\tau$ for a preradical τ. If τ is a torsion preradical then $\tau(M) \in \mathcal{T}_\tau$ because by Proposition 2, $\tau(\tau(M)) = \tau(M) \cap \tau(M) = \tau(M)$. For example, the functors $Soc(-)$ and $Z(-)$ which associate with every module its socle and its singular submodule respectively, can be regarded as torsion preradicals while the functor $Rad(-)$ which associates with every module its Jacobson radical can be viewed as a preradical on $R - Mod$.

Proposition 3.1.3 [45] *Let τ be a torsion preradical on $R - Mod$. Then:*

(i) $\mathcal{T}_\tau$ is closed under taking quotients, direct sums and submodules;

(ii) $\mathcal{F}_\tau$ is closed under taking submodules, direct products and essential extensions;

(iii) $Hom_R(M, E(N)) = 0$ for all $M \in \mathcal{T}_\tau$ and $N \in \mathcal{F}_\tau$.

Proof: For (i) and (ii) it suffices to prove that $\mathcal{T}_\tau$ and $\mathcal{F}_\tau$ are closed under taking submodules and essential extensions respectively. Let $N \leq M \in \mathcal{T}_\tau$. By Proposition 2, $\tau(N) = N \cap \tau(M) = N \cap M = N$. Thus $N \in \mathcal{T}_\tau$.

Let Q be an essential extension of $P \in \mathcal{F}_\tau$. Then $0 = \tau(P) = P \cap \tau(Q)$. Since P is essential in Q, $\tau(Q) = 0$.

We now prove (iii). By (ii), for any $N \in \mathcal{F}_\tau$, $E(N) \in \mathcal{F}_\tau$. By Proposition 1 (iii), we have that $Hom_R(M, E(N)) = 0$ for all $M \in \mathcal{T}_\tau$ and $N \in \mathcal{F}_\tau$. □

3.2 Torsion radicals

Let τ_1 and τ_2 be preradicals on $R-Mod$. We define $\tau_1 : \tau_2$ as follows: for any left R-module M put $(\tau_1 : \tau_2)(M)/\tau_1(M) = \tau_2(M/\tau_1(M))$.

We claim that $\tau_1 : \tau_2$ is a preradical. To see this take $M, N \in R-Mod$ and $\alpha \in Hom_R(M,N)$. Since $[\tau_1(M)]\alpha \leq \tau_1(N)$, there exists a well-defined R-homomorphism $\bar{\alpha} : M/\tau_1(M) \longrightarrow N/\tau_1(N)$ given by $(x + \tau_1(M))\bar{\alpha} = (x)\alpha + \tau_1(N)$ for $x \in M$. Since τ_2 is a preradical $[\tau_2(M/\tau_1(M))]\bar{\alpha} \leq \tau_2(N/\tau_1(N))$. Then $[(\tau_1 : \tau_2)(M)]\alpha \leq (\tau_1 : \tau_2)(N)$.

Obviously $(\tau_1 : \tau_2)(M) \leq M$. Thus the claim holds.

A preradical τ is said to be a *radical* if $\tau : \tau = \tau$; in other words if $\tau(M/\tau(M)) = 0$ for all $M \in R-Mod$. For example, $Rad(-)$ is a radical. But in general $Z(-)$ and $Soc(-)$ are not radicals.

From any preradical τ on $R-Mod$ the radical $\bar{\tau}$ is constructed as follows. We define a sequence of preradicals using the definition of $\tau : \tau$ and transfinite induction. Put $\tau_1 = \tau$. If i is not a limit ordinal put $\tau_i = \tau_{i-1} : \tau$, i.e., $\tau_i(M)/\tau_{i-1}(M) = \tau(M/\tau_{i-1}(M))$ for $M \in R-Mod$. If i is a limit ordinal put $\tau_i = \bigvee_{j<i} \tau_j$, i.e., $\tau_i(M) = \sum_{j<i} \tau_j(M)$ for $M \in R-Mod$. This gives rise to an ascending chain of preradicals $\{\tau_i\}$. For each left R-module M let $k(M)$ be the smallest ordinal for which $\tau_{k(M)} = \tau_{k(M)+1}$. Define $\bar{\tau}$ by $\bar{\tau}(M) = \tau_{k(M)}(M)$.

Proposition 3.2.1 [43] *$\bar{\tau}$ is the smallest radical on $R-Mod$ larger than τ.*

Proof: Let $M \in R-Mod$ and for simplicity put $k = k(M)$ with $k(M)$ defined as above. Put $L = M/\tau_k(M)$. We now prove using transfinite induction that $\tau_i(L) = 0$ for all ordinals i. This would then imply $\tau_{k(L)}(M/\tau_k(M)) = 0$, i.e., $\bar{\tau}$ is a radical.

For $\tau_i = \tau$ the result is immediate, for $\tau(M/\tau_k(M)) = \tau_{k+1}(M)/\tau_k(M) = 0$. Let k' be an ordinal and assume the result holds for all $i < k'$. If k' is not a limit ordinal then $\tau_{k'}(L)/\tau_{k'-1}(L) = \tau(L/\tau_{k'-1}(L))$. By the induction hypothesis, $\tau_{k'-1}(L) = 0$ and so $\tau_{k'}(L) = \tau(L) = 0$. If k' is a limit ordinal then $\tau_{k'}(L) = \sum_{i<k'} \tau_i(L)$. But since each $\tau_i(L) = 0$ we have $\tau_{k'}(L) = 0$. Thus the result holds for k'. By transfinite induction $\tau_i(L) = 0$ for all ordinals i.

Now let σ be a radical on $R - Mod$ such that $\sigma \geq \tau$. Then $\sigma_i = \sigma$ for all ordinals i. It follows $\sigma = \sigma_i \geq \tau_i$ for all ordinals i; in particular $\sigma \geq \bar{\tau}$. □

Remark 3.2.2 If τ is a left exact preradical and M a left R-module, then $\tau(M)$ is an essential submodule of $\bar{\tau}(M)$. For suppose $L \subseteq \bar{\tau}(M)$ and $L \cap \tau(M) = 0$. Then $\tau(L) = L \cap \tau(M) = 0$ from which we infer $\bar{\tau}(L) = 0$. But $\bar{\tau}(L) = L \cap \bar{\tau}(M) = L$, so $L = 0$.

Lemma 3.2.3 [45] *Let τ be a radical on $R - Mod$ and M a left R-module.*
If $N \subseteq \tau(M)$ then $\tau(M/N) = \tau(M)/N$.

Proof: If $N \subseteq \tau(M)$ the canonical epimorphism $\alpha : M \longrightarrow M/N$ induces an R-homomorphism $\bar{\alpha} : \tau(M) \longrightarrow \tau(M/N)$ with $Ker\bar{\alpha} = N$; so $\tau(M)/N \subseteq \tau(M/N)$. The canonical R-homomorphism $\beta : M/N \longrightarrow M/\tau(M)$ also induces a homomorphism $\bar{\beta} : \tau(M/N) \longrightarrow \tau(M/\tau(M))$. Since $\tau(M/\tau(M)) = 0$, $Ker\beta = \tau(M/N)$. Then $\tau(M/N) \subseteq Ker\beta = \tau(M)/N$. Thus $\tau(M/N) = \tau(M)/N$. □

Proposition 3.2.4 [45] *If τ is a radical on $R - Mod$, then $\mathcal{F}_\tau$ is closed under taking submodules and direct products. Conversely, if $\mathcal{D}$ is a class of left R-modules closed under taking submodules and direct products there exists a unique radical λ with $\mathcal{F}_\lambda = \mathcal{D}$.*

Proof: By Proposition 1.1, $\mathcal{F}_\tau$ is closed under taking submodules and direct products. If $\mathcal{D}$ is a class of left R-modules closed under taking submodules and direct products, we define λ as follows: for $M \in R-Mod$ put:

$$\lambda(M) = \bigcap\{N \leq M \mid M/N \in \mathcal{D}\}.$$

We need to prove that λ is a radical. Let $M, M' \in R-Mod$ and $\alpha \in Hom_R(M, M')$. Take $N' \in \{N' \leq M' \mid M'/N' \in \mathcal{D}\}$. There is a canonical monomorphism $M/[N']\alpha^{-1} \hookrightarrow M'/N'$ defined by $x + [N']\alpha^{-1} \longmapsto (x)\alpha + N'$ with $x \in M$. Since $\mathcal{D}$ is closed under submodules it follows that $M/[N']\alpha^{-1} \in \mathcal{D}$ and so $[N']\alpha^{-1} \in \{N \in M \mid M/N \in \mathcal{D}\}$. This implies that $\bigcap\{N' \leq M' \mid M'/N' \in \mathcal{D}\} \supseteq \bigcap\{[N]\alpha \leq M' \mid M/N \in \mathcal{D}\} = [\bigcap\{N \leq M \mid M/N \in \mathcal{D}\}]\alpha$. Thus λ is a preradical.

Clearly $\mathcal{D} \subseteq \mathcal{F}_\lambda$. But the canonical monomorphism $M/\lambda(M) = M/\bigcap\{N \leq M \mid M/N \in \mathcal{D}\} \hookrightarrow \prod\{M/N \mid M/N \in \mathcal{D}\}$ for $M \in R-Mod$ shows that $M/\lambda(M) \in \mathcal{D}$ from which we infer $\mathcal{D} \supseteq \mathcal{F}_\lambda$. Hence $\mathcal{D} = \mathcal{F}_\lambda$. Since $M/\lambda(M) \in \mathcal{F}_\lambda$, λ is a radical. It remains to prove the uniqueness. Suppose σ is a radical and let λ be the radical associated with $\mathcal{F}_\sigma$. Let $M \in R - Mod$. Since σ is a radical, $M/\sigma(M) \in \mathcal{F}_\sigma$ and so $\sigma(M) \in \{N \leq M \mid M/N \in \mathcal{F}_\sigma\}$. Therefore $\lambda(M) \subseteq \sigma(M)$. Hence by Lemma 3, $\sigma(M/\lambda(M)) = \sigma(M)/\lambda(M)$. But $M/\lambda(M) \in \mathcal{F}_\sigma$. Thus $\sigma(M)/\lambda(M) = 0$ and so $\sigma = \lambda$. □

The above proposition informs us that if σ is a radical on $R - Mod$ we can recover it from $\mathcal{F}_\sigma$. The next proposition follows.

Proposition 3.2.5 [43] *There is a bijective correspondence between radicals on $R - Mod$ and classes of left R-modules closed under taking submodules and direct products.* □

A torsion preradical that is a radical is called a *torsion radical*. Goldman [22] calls such a preradical an *idempotent kernel functor*. If τ is a torsion radical, a left R-module M is called *τ-torsion* if $M \in \mathcal{T}_\tau$, i.e., $\tau(M) = M$ and *τ-torsion-free* if $M \in \mathcal{F}_\tau$, i.e., $\tau(M) = 0$.

Proposition 3.2.6 [45] *Let τ be a torsion radical on $R - Mod$. Then:*

(i) $\mathcal{T}_\tau = \{M \in R - Mod \mid Hom_R(M, E(N)) = 0$ for all $N \in \mathcal{F}_\tau\}$;

(ii) $\mathcal{F}_\tau = \{N \in R - Mod \mid Hom_R(M, E(N)) = 0$ for all $M \in \mathcal{T}_\tau\}$.

Proof: (i) Let $M \in \mathcal{T}_\tau$. Then $Hom_R(M, E(N)) = 0$ for all $N \in \mathcal{F}_\tau$ by Proposition 1.3. Thus $\mathcal{T}_\tau \subseteq \{M \in R - Mod \mid Hom_R(M, E(N)) = 0$ for all $N \in \mathcal{F}_\tau\}$.
Conversely, suppose $M \in R - Mod$ and $Hom_R(M, E(N)) = 0$ for all $N \in \mathcal{F}_\tau$. Since τ is a radical $M/\tau(M) \in \mathcal{F}_\tau$ and hence $Hom_R(M, E(M/\tau(M))) = 0$. This means that $M/\tau(M) = 0$, i.e., $M = \tau(M)$. Thus $M \in \mathcal{T}_\tau$.
(ii) Clearly $\mathcal{F}_\tau \subseteq \{N \in R - Mod \mid Hom_R(M, E(N)) = 0$ for all $M \in \mathcal{T}_\tau\}$. Assume now that $N \in R - Mod$ and $Hom_R(M, E(N))$ for all $M \in \mathcal{T}_\tau$. Since τ is a torsion radical $\tau(N) \in \mathcal{T}_\tau$. Hence $Hom_R(\tau(N), E(N)) = 0$. This holds if and only if $\tau(N) = 0$. Thus $N \in \mathcal{F}_\tau$. □

A class $\mathcal{C}$ of left R-modules is said to be *closed under taking module extensions* if given any exact sequence $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ with $L, N \in \mathcal{C}$, then $M \in \mathcal{C}$.

Proposition 3.2.7 [45] *Let τ be a torsion radical on $R - Mod$. Then:*

(i) $\mathcal{T}_\tau$ is closed under taking quotients, direct sums, submodules and module extensions.

(ii) $\mathcal{F}_\tau$ *is closed under taking submodules, direct products, essential extensions and module extensions.*

Proof: By Proposition 1.3, it remains to prove that $\mathcal{T}_\tau$ and $\mathcal{F}_\tau$ are closed under taking module extensions. Let $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ be an exact sequence. By the left exactness of τ the sequence induces the exact sequence $0 \longrightarrow \tau(L) \longrightarrow \tau(M) \longrightarrow \tau(N)$.

If $L \in \mathcal{T}_\tau$ then $L \leq \tau(M)$. Thus $\tau(M/L) = \tau(M)/L$. On the other hand $M/L \cong N \in \mathcal{T}_\tau$. It follows that $\tau(M)/L = M/L$ and so $\tau(M) = M$. Therefore (i) holds.

If $\tau(L) = 0$ and $\tau(N) = 0$ then $\tau(M) = 0$. Thus $M \in \mathcal{F}_\tau$. This completes the proof. □

In the paragraph following Proposition 1.1, we showed that if $\mathcal{C}$ is a class of left R-modules closed under taking quotients and direct sums then there exists a preradical σ such that $\mathcal{T}_\sigma = \mathcal{C}$. Suppose that $\mathcal{C}$ is, in addition, closed under taking submodules and module extensions. Let $M \in R-Mod$ and choose $L \leq M$. Since $\sigma(M) \in \mathcal{C}$ and $\mathcal{C}$ is closed under taking submodules, $\sigma(M) \cap L \in \mathcal{T}_\sigma$ and hence $\sigma(L) = L \cap \sigma(M)$, i.e., σ is a torsion preradical. We show that σ is a radical, i.e., $\sigma(M/\sigma(M)) = 0$. To this end put $\sigma(M/\sigma(M)) = N/\sigma(M)$. Consider the exact sequence $0 \longrightarrow \sigma(M) \longrightarrow N \longrightarrow N/\sigma(M) \longrightarrow 0$. Since $\sigma(M), N/\sigma(M) \in \mathcal{T}_\sigma = \mathcal{C}$ which is closed under taking module extensions, it follows that $N \in \mathcal{T}_\sigma$. Thus $N \leq \sigma(M)$. Therefore $\sigma(M/\sigma(M)) = 0$. Hence σ is a torsion radical with $\mathcal{T}_\sigma = \mathcal{C}$. Furthermore, suppose τ is an arbitrary torsion radical on $R-Mod$ and let σ be the torsion radical associated with $\mathcal{T}_\tau$. Choose $M \in R - Mod$. Since $\tau(M)$ contains every submodule of M contained in $\mathcal{T}_\tau$, we have that $\sigma(M) \leq \tau(M)$. However, since τ is a torsion radical $\tau(M) \in \mathcal{T}_\tau$ and so $\tau(M) \leq \sigma(M)$. Thus $\sigma = \tau$. This means that if τ is a torsion radical it is possible to recover τ from $\mathcal{T}_\tau$.

In Proposition 4 we proved that if $\mathcal{D}$ is a class of left R-modules closed under taking submodules and direct products there exists a unique radical λ with $\mathcal{F}_\lambda = \mathcal{D}$. Let us prove now that if $\mathcal{D}$ is, in addition, closed under taking essential extensions and module extensions, λ is a torsion radical. Let $M \in R-Mod$ and $L \leq M$. Put $L' = L \cap \lambda(M)$ and consider the diagram

$$\begin{tikzcd}
L' \arrow[r] \arrow[d, "\beta"'] & \lambda(M) \arrow[dl, "\alpha"] \\
E(L'/\lambda(L')) &
\end{tikzcd}$$

where β is the composition of the canonical epimorphism $L' \to L'/\lambda(L')$ and the canonical monomorphism $L'/\lambda(L') \hookrightarrow E(L'/\lambda(L'))$. Since $E(L'/\lambda(L'))$ is injective there exists $\alpha : \lambda(M) \longrightarrow E(L'/\lambda(L'))$ that makes the diagram commute. On the other hand $E(L'/\lambda(L')) \in \mathcal{F}_\lambda$ because λ is a radical and $\mathcal{D} = \mathcal{F}_\lambda$ is closed under essential extensions. Now let $\lambda(\lambda(M)) = K$. Consider the exact sequence $0 \longrightarrow \lambda(M)/K \longrightarrow M/K \longrightarrow M/\lambda(M) \longrightarrow 0$. Since $\lambda(M)/K, M/\lambda(M) \in \mathcal{F}_\lambda$ we must have $M/K \in \mathcal{F}_\lambda = \mathcal{D}$. Therefore $K = \lambda(\lambda(M)) \in \{N \leq M \mid M/N \in \mathcal{D}\}$. Clearly this implies that $\lambda(M) = \lambda(\lambda(M))$ and hence $\lambda(M) \in \mathcal{T}_\lambda$. But $E(L'/\lambda(L')) \in \mathcal{F}_\lambda$; so by Proposition 1.3 (*iii*), $Hom_R(\lambda(M), E(L'/\lambda(L'))) = 0$ implying $\alpha = 0$ and so $L'/\lambda(L') = 0$. Thus $L \cap \lambda(M) \in \mathcal{T}_\lambda$. It follows that $\lambda(L) = L \cap \lambda(M)$, i.e., λ is a torsion radical. Thus if τ is a torsion radical it can be recovered from $\mathcal{T}_\tau$ or $\mathcal{F}_\tau$. The next proposition follows.

Proposition 3.2.8 [43] *There are bijective correspondences between:*

(i) torsion radicals on $R - Mod$;

(ii) classes $\mathcal{C}$ of left R-modules closed under taking quotients, direct sums, submodules and modules extensions;

(iii) classes $\mathcal{D}$ of left R-modules closed under taking submodules, direct products, essential extensions and module extensions. □

3.3 Gabriel topologies

In Chapter 2 we introduced the notion of relative denseness. There is a similar notion of denseness associated with every torsion preradical. More precisely, let σ be a torsion preradical on $R - Mod$ and M be a left R-module. A submodule N of M is said to be *σ-dense* in M if $M/N \in \mathcal{T}_\sigma$. A left ideal I of R is called σ-dense if it is σ-dense as a submodule of ${}_RR$.

Proposition 3.3.1 [19] *Let τ be a torsion preradical on $R - Mod$ and M a left R-module. Then:*

(i) if N is a τ-dense submodule of M and $N \subseteq N' \leq M$ then N' is τ-dense in M;

(ii) if N and N' are τ-dense in M then $N \cap N'$ is τ-dense in M;

(iii) if N is τ-dense in M and $m \in M$ then $(N : m)$ is a τ-dense left ideal of R.

Proof: (i) Let N be a τ-dense submodule of M. Since $M/N' \cong (M/N)/(N'/N)$ and $\mathcal{T}_\tau$ is closed under quotients, $M/N' \in \mathcal{T}_\tau$, so N' is τ-dense in M.

(ii) Consider the canonical R-monomorphism $\alpha : M/(N \cap N') \longrightarrow M/N \oplus M/N'$ defined by $[m + N \cap N']\alpha = (m + N, m + N')$ where $m \in M$ and N, N' are τ-dense submodules of M. Since $\mathcal{T}_\tau$ is closed under quotients and direct sums, $M/N \oplus M/N' \in \mathcal{T}_\tau$. Then $M/(N \cap N') \in \mathcal{T}_\tau$ because it is isomorphic to a submodule of $M/N \oplus M/N'$.

(iii) Consider now the map $\beta : R/(N : m) \longrightarrow M/N$ defined by $[r + (N : m)]\beta = rm + N$. It is easy to check that β is a well-defined R-homomorphism. Then $R/(N : m)$ is isomorphic to a submodule of $M/N \in \mathcal{T}_\tau$ and thus $R/(N : m) \in \mathcal{T}_\tau$. ☐

We shall now focus on the set of all τ-dense left ideals of R which will be denoted by Φ_τ, i.e., $\Phi_\tau = \{I \leq {}_RR \mid \tau(R/I) = R/I\}$. Observe that $R \in \Phi_\tau$.

Proposition 3.3.2 [45] *Let τ be a torsion preradical on $R-Mod$ and M be a left R-module. Then $\tau(M) = \{m \in M \mid Am = 0$ for some $A \in \Phi_\tau\} = \{m \in M \mid (0 : m) \in \Phi_\tau\}$.*

Proof: Clearly $\{m \in M \mid Am = 0$ for some $A \in \Phi_\tau\} = \{m \in M \mid (0 : m) \in \Phi_\tau\}$. If $m \in \tau(M)$, then $Rm \subseteq \tau(M)$ and so Rm is τ-torsion. Thus $R/(0 : m) \cong Rm$ is τ-torsion from which we infer $(0 : m)$ is τ-dense in ${}_RR$. Conversely, if $(0 : m) \in \Phi_\tau$ for $m \in M$, then $R/(0 : m)$ is τ-torsion and hence so is Rm. Therefore $m \in Rm \subseteq \tau(M)$ since τ is a torsion preradical. ☐

This proposition informs us that Φ_τ completely determines τ. The following gives a necessary and sufficient condition for a set Φ of left ideals of R to be of the form Φ_τ for some torsion preradical τ on $R - Mod$.

A nonempty set Φ of left ideals of R is called a *left topologizing filter* if it satisfies the following conditions:

T_1 If $A \in \Phi$ and $A \subseteq B \leq {}_RR$ then $B \in \Phi$.

T_2 If $A, B \in \Phi$ then $A \cap B \in \Phi$.

T_3 If $A \in \Phi$ and $r \in R$ then $(A : r) \in \Phi$.

Note that T_3 implies that Φ contains R. If Φ satisfies, in addition,

T_4 If $(A : b) \in \Phi$ for all $b \in J \in \Phi$ then $A \in \Phi$,

then Φ is called a *left Gabriel topology*. For example, the family of dense left ideals of R is a Gabriel topology by Proposition 2.2.6. The next proposition asserts that if Φ satisfies T_3 and T_4 then it also satisfies T_1 and T_2.

Proposition 3.3.3 [43] *A family of left ideals of a ring R is a left Gabriel topology if and only if it satisfies T_3 and T_4.*

Proof: Let Φ be a family of left ideals of R satisfying T_3 and T_4. Let $I \in \Phi$ and suppose $I \subseteq J$ in R. If $a \in I$ then $(J : a) = R \in \Phi$. By T_4, $J \in \Phi$. Thus T_1 holds. Now take $I, J \in \Phi$ and let $a \in I$. We have that $(I \cap J : a) = (I : a) \cap (J : a) = (J : a)$ because $(I : a) = R$. By T_3, $(J : a) \in \Phi$ and finally by T_4, $I \cap J \in \Phi$. $\square$

Note that for a left Gabriel topology Φ, if $I, J \in \Phi$ then $IJ \in \Phi$. For if $b \in J$ then $(IJ : b) \supseteq I \in \Phi$. Thus by $T_1, (IJ : b) \in \Phi$ and so $IJ \in \Phi$ by T_4.

Example 3.3.4 By Proposition 1.4.2, the family $\mathcal{E}$ of essential left ideals of R is a left topologizing filter. In general, $\mathcal{E}$ is not a Gabriel topology. For example, the unique proper nonzero ideal of the ring $\mathbb{Z}_4$ of integers modulo 4, is essential and nilpotent. Since a Gabriel topology is closed under products (as noted above) and the zero ideal is not essential, it follows that the topologizing filter of all essential ideals of $\mathbb{Z}_4$ is not a Gabriel topology.

It is possible to associate with $\mathcal{E}$ a Gabriel topology $\mathcal{J}(\mathcal{E}) = \{I \leq {}_R R | \text{there exists} J \in \mathcal{E}$ such that $I \subseteq J$ and $(I : r) \in \mathcal{E}$ for all $r \in J\}$ [43, Proposition VI.6.3]. One can show that $\mathcal{J}(\mathcal{E})$ is the smallest left Gabriel topology on R containing $\mathcal{E}$ [43, Proposition VI.5.4]. $\mathcal{J}(\mathcal{E})$ is known as the *Goldie topology*.

Proposition 3.3.5 [45] *Let Φ be a nonempty set of left ideals of R. Then the following conditions are equivalent:*

(i) Φ is a left Gabriel topology;

(ii) there is a torsion radical τ on $R - Mod$ such that $\Phi_\tau = \Phi$.

Proof: $(i) \Longrightarrow (ii)$ For a left R-module M, put $\tau(M) = \{m \in M \mid Am = 0 \text{ for some } A \in \Phi\}$. Since $(0 : m + m') \supseteq (0 : m) \cap (0 : m')$ and $(0 : rm) \supseteq (0 : m)$ for all $m, m' \in M$ and $r \in R$, we have that $m + m' \in \tau(M)$ and $rm \in \tau(M)$ by T_1 whenever $m, m' \in \tau(M)$. Thus $\tau(M)$ is a submodule of M. If $f : M \longrightarrow M'$ is an R-homomorphism then $Am = 0$ implies $A((m)f) = [Am]f = 0$ for $m \in M$ and so $[\tau(M)]f \subseteq \tau(M')$. This shows that τ is a preradical on $R - Mod$.

Take a submodule N of M. If $n \in \tau(N)$ then there exists $A \in \Phi$ such that $An = 0$. Thus $n \in \tau(M)$. It follows that $\tau(N) \subseteq N \cap \tau(M)$. Obviously $N \cap \tau(M) \subseteq \tau(N)$. Thus τ is a torsion preradical. It remains to show that τ is a radical, i.e., $\tau(M/\tau(M)) = 0$ for all $M \in R - Mod$. Let $m + \tau(M) \in \tau(M/\tau(M))$ and put $A = (0 : m + \tau(M)) \in \Phi$. Then $A = (\tau(M) : m) = \{r \in R \mid rm \in \tau(M)\} = \{r \in R \mid (0 : rm) \in \Phi\} = \{r \in R \mid ((0 : m) : r) \in \Phi\}$. Thus $A \in \Phi$ and for all $a \in A, ((0 : m) : a) \in \Phi$, so $(0 : m) \in \Phi$. But then $m \in \tau(M)$. Hence $\tau(M/\tau(M)) = 0$.

$(ii) \Longrightarrow (i)$ The fact that Φ_τ is a left topologizing filter has been shown in Proposition 1. We need to show that Φ_τ satisfies T_4. Let B be a left ideal of R and $A \in \Phi_\tau$ such that $(B : a) \in \Phi_\tau$ for all $a \in A$. Consider the exact sequence $0 \longrightarrow A/(A \cap B) \longrightarrow R/B \longrightarrow R/(A + B) \longrightarrow 0$. Now A being τ-dense implies $A + B \supseteq A$ is τ-dense, so $R/(A + B) \in T_\tau$.

For $a \in A, (0 : a + (A \cap B)) = ((A \cap B) : a) = (B : a) \in T_\tau$ and so $a + (A \cap B) \in$

$\tau(A/(A \cap B))$. Thus $A/(A \cap B) \in T_\tau$. Since T_τ is closed under module extensions $R/B \in T_\tau$. Therefore $B \in \Phi_\tau$. ☐

Remark 3.3.6 The correspondence $\tau \longmapsto \Phi_\tau$ is one-to-one for it is clear that $\Phi_\tau = \Phi_{\tau'}$ if and only if $\tau = \tau'$.

Proposition 3.3.7 [43] *There are bijective correspondences between:*

(i) torsion radicals on $R - Mod$;

(ii) left Gabriel topologies on R;

(iii) classes $\mathcal{C}$ of left R-modules closed under taking quotients, direct sums, submodules and modules extensions;

(iv) classes $\mathcal{D}$ of left R-modules closed under taking submodules, direct products, essential extensions and module extensions.

Proof: follows from Propositions 5 and 2.8. ☐

Since every left Gabriel topology on R is a *set* of left ideals, the collection of torsion radicals on $R - Mod$ is also a set.

3.4 Hereditary pretorsion classes

The following definition due to Dickson [12] is given by Stenström [43, page 139]. A *torsion theory* on $R - Mod$ is a pair $(\mathcal{T}, \mathcal{F})$ of classes of left R-modules satisfying the following three conditions:

(i) $Hom_R(M, N) = 0$ for all $M \in \mathcal{T}$ and $N \in \mathcal{F}$;

(ii) if $Hom_R(M, N) = 0$ for all $M \in \mathcal{T}$ then $N \in \mathcal{F}$;

(iii) if $Hom_R(M, N) = 0$ for all $N \in \mathcal{F}$ then $M \in \mathcal{T}$.

$\mathcal{T}$ is called the *class of torsion modules* and $\mathcal{F}$ the *class of torsion-free modules* of the torsion theory $(\mathcal{T}, \mathcal{F})$.

Proposition 3.4.1 [43] *Let $(\mathcal{T}, \mathcal{F})$ be a torsion theory on $R - Mod$. Then:*

(i) $\mathcal{T}$ is closed under taking quotients, direct sums and module extensions;

(ii) $\mathcal{F}$ is closed under taking submodules, direct products and module extensions.

Proof: (i) Let $M' \leq M \in \mathcal{T}$. Suppose $Hom_R(M, N) = 0$ for any $N \in \mathcal{F}$. Since $Hom_R(M/M', N) \hookrightarrow Hom_R(M, N)$ it follows that $Hom_R(M/M', N) = 0$ for $N \in \mathcal{F}$. Thus $M/M' \in \mathcal{T}$. Since $Hom_R(\bigoplus M_i, N) \cong \prod Hom_R(M_i, N)$ for every family of left R-modules $\{M_i \mid i \in I\}$, $\mathcal{T}$ is closed under taking direct sums. Finally, let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be an exact sequence of left R-modules with $A, C \in \mathcal{T}$. Let $N \in \mathcal{F}$ and $f : B \longrightarrow N$ an R-homomorphism. Then f is zero on A and so f factors over C. But $Hom_R(C, N) = 0$, so $f = 0$. Thus $B \in \mathcal{T}$.

(ii) Obviously $\mathcal{F}$ is closed under taking submodules. Since $Hom_R(M, \prod N_i) \cong \prod Hom_R(M, N_i)$ for every family of left R-modules $\{M_i \mid i \in I\}$, $\mathcal{F}$ is closed under taking direct products.

Consider the exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ with A, $C \in \mathcal{F}$. We have that $0 \longrightarrow Hom_R(M, A) \longrightarrow Hom_R(M, B) \longrightarrow Hom_R(M, C)$ is exact. Thus if $Hom_R(M, A) = Hom_R(M, C) = 0$, then $Hom_R(M, B) = 0$. □

In general, $\mathcal{T}$ need not be closed under submodules. A torsion theory $(\mathcal{T}, \mathcal{F})$ is said to be *hereditary* if the class $\mathcal{T}$ of torsion modules is closed under submodules. For instance the pair $(\mathcal{T}_\tau, \mathcal{F}_\tau)$ determined by a torsion radical τ is a hereditary torsion theory on $R - Mod$.

Proposition 3.4.2 [43] *A torsion theory $(\mathcal{T}, \mathcal{F})$ is hereditary if and only if $\mathcal{F}$ is* *closed under taking essential extensions.*

Proof: Assume that $\mathcal{T}$ is closed under taking submodules and let N' be an essential extension of a torsion-free module N. Let $\alpha \in Hom_R(M, N')$ with $M \in \mathcal{T}$. Since $\mathcal{T}$ is closed under taking quotients and submodules $[M]\alpha, ([M]\alpha \cap N) \in \mathcal{T}$. Therefore $[M]\alpha = 0$ because $\mathcal{F}$ is closed under taking submodules and N is essential in N', so $\alpha = 0$. Thus $N' \in \mathcal{F}$. Conversely if $N \in \mathcal{F}$ then $E(N) \in \mathcal{F}$ since $\mathcal{F}$ is closed under taking essential extensions. Let M' be a submodule of $M \in \mathcal{T}$. By the injectivity of $E(N)$ the R-homomorphism $\alpha : M' \longrightarrow E(N)$ can be extended to an R-homomorphism $M \longrightarrow E(N)$. But $Hom_R(M, E(N)) = 0$, so $\alpha = 0$. Therefore $M' \in \mathcal{T}$. $\square$

Proposition 3.4.3 [45] *The following conditions are equivalent for any class $\mathcal{C}$ of left R-modules:*

(i) $\mathcal{C}$ is closed under taking quotients, direct sums, submodules and module extensions;

(ii) $\mathcal{C}$ is a torsion class for some hereditary torsion theory.

Proof: $(ii) \Longrightarrow (i)$ follows from Proposition 1.

$(i) \Longrightarrow (ii)$ Put $\mathcal{F} = \{N \in R-Mod \mid Hom_R(M,N) = 0 \text{ for all } M \in \mathcal{C}\}$ and $\mathcal{T} = \{M \in R-Mod \mid Hom_R(M,N) = 0 \text{ for all } N \in \mathcal{F}\}$. We need to show that $\mathcal{C} = \mathcal{T}$. Suppose $Hom_R(M,N) = 0$ for all $N \in \mathcal{F}$. Since $\mathcal{C}$ is closed under taking quotients and direct sums there exists a largest submodule V of M belonging to $\mathcal{C}$. The proof will be done if $V = M$, i.e., if $M/V \in \mathcal{F}$. Now let $\alpha \in Hom_R(V', M/V)$ with $V' \in \mathcal{C}$. Then $Im\alpha \in \mathcal{C}$. Put $Im\alpha = M'/V$ with $V \subseteq M' \leq M$. Since $M'/V, V \in \mathcal{C}$ and $\mathcal{C}$ is closed under taking module extensions we must have $M' \in \mathcal{C}$.

Then $M' \subseteq V$, so $Im\alpha = M'/V = 0$, i.e., $\alpha = 0$. Therefore $M/V \in \mathcal{F}$ and hence $V = M$. ☐

We have the following dual result. We omit the proof which makes use of arguments which are dual to those used in the previous proposition.

Proposition 3.4.4 [43] *The following conditions are equivalent for a class $\mathcal{D}$ of left R-modules:*

(i) $\mathcal{D}$ is closed under taking submodules, direct products, essential extensions and module extensions;

(ii) $\mathcal{D}$ is a torsion-free class for some hereditary torsion theory. ☐

Remark 3.4.5 If $(\mathcal{T}, \mathcal{F})$ is a hereditary torsion theory, then

$\mathcal{F} = \{N \in R - Mod \mid Hom_R(M, E(N)) = 0$ for all $M \in \mathcal{T}\}$ and

$\mathcal{T} = \{M \in R - Mod \mid Hom_R(M, E(N)) = 0$ for all $N \in \mathcal{F}\}$.

We point out that Golan [19] adopts a different approach to that of Dickson and defines (hereditary) torsion theories via equivalence classes of injective modules. Our next objective is to briefly survey Golan's approach and to then show its equivalence with Dickson's. To this end some results are needed.

Let $\mathcal{C}$ be a class of left R-modules. An element $A \in \mathcal{C}$ is said to be a *generator* of $\mathcal{C}$ if every module in $\mathcal{C}$ is the homomorphic image of a direct sum of copies of A. Dually an element A of $\mathcal{C}$ is said to be a *cogenerator* of $\mathcal{C}$ if every element of $\mathcal{C}$ is isomorphic to a submodule of a direct product of copies of A. For example, ${}_RR$ is a (projective) generator for $\mathcal{C} = R - Mod$ while $E(\bigoplus S_i)$ with $\{S_i \mid i \in I\}$ a representative set of simple left R-modules is an (injective) cogenerator for $\mathcal{C} = R - Mod$. (The latter result is a consequence of the fact that every module is a subdirect product of subdirectly irreducible modules - see [3, page 95].)

Proposition 3.4.6 [43] *Let $(\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory on $R - Mod$. Then:*

(i) $\mathcal{T}$ is generated by the direct sum of a representative set of cyclic left R-modules in $\mathcal{T}$.

(ii) $\mathcal{F}$ is cogenerated by an injective left R-module.

Proof: (i) Let C be the direct sum of a representative set of cyclic left R-modules in $\mathcal{T}$. Take $M \in \mathcal{T}$. Since $\mathcal{T}$ is closed under taking submodules, Rx is a cyclic left R-module in $\mathcal{T}$ for every $x \in M$. We thus obtain a pair of R-epimorphisms $\bigoplus_{x \in M} C \longrightarrow \bigoplus_{x \in M} Rx \longrightarrow \sum\{Rx \mid x \in M\} = M$. We conclude that C is a generator for $\mathcal{T}$.

(ii) Put $E = \prod E(R/I)$ where the product is taken over all left ideals I of R such that $R/I \in \mathcal{F}$. Let $0 \neq N \in \mathcal{F}$. We shall demonstrate that N is embeddable in a direct product of copies of E. Take $0 \neq x \in N$. Since $\mathcal{F}$ is closed under taking submodules $Rx \in \mathcal{F}$. Therefore there exists a monomorphism from Rx to E which extends by injectivity to an R-homomorphism $\alpha : N \longrightarrow E$. This shows that for every nonzero $x \in N$ there exists $\alpha \in Hom_R(N, E)$ such that $x \notin Ker\alpha$. We conclude that $K = \bigcap\{Ker\alpha \mid \alpha \in Hom_R(N, E)\} = 0$. The canonical monomorphism $N = N/K \hookrightarrow \prod E$ yields the required embedding. $\square$

Golan defines two injective left $R-$modules E and E' to be *equivalent* if each of them can be embedded in (and hence is isomorphic to a direct summand of) a direct product of copies of the other. This is equivalent to saying that E and E'

cogenerate each other. The above gives rise to an equivalence relation on the class of all injective left R-modules. An equivalence class is called a *torsion theory* (in the sense of Golan) and shall be denoted by a lower case Greek letter, for example τ. The next two propositions show that Dickson's notion and Golan's notion of a torsion theory, coincide in essence.

Proposition 3.4.7 [19] *For injective left R-modules E and E' the following conditions are equivalent:*
(i) E and E' are equivalent;
(ii) E and E' cogenerate the same class $\mathcal{F}$ of left R-modules;
(iii) $\{M \in R-Mod \mid Hom_R(M,E) = 0\} = \{M \in R-Mod \mid Hom_R(M,E') = 0\}$.

Proof: Obviously $(i) \Longrightarrow (ii) \Longrightarrow (iii)$.
$(iii) \Longrightarrow (i)$ Let I be the set $Hom_R(E,E')$ and consider $\psi : E \longrightarrow (E')^I$ defined by $x \longmapsto \{(x)\alpha \mid \alpha \in I\}$. If $K = Ker\psi$ then $Hom_R(K,E') = 0$ because by the injectivity of E' any R-homomorphism from $K \longrightarrow E'$ can be extended to an R-homomorphism from $E \longrightarrow E'$. Then by (iii) $Hom_R(K,E) = 0$. Since K is a submodule of E this implies $K = 0$. Thus E can be embedded in a direct product of copies of E'. Similarly E' can be also embedded in a direct product of copies of E. □

Note that the class cogenerated by E or E' satisfies the conditions of Proposition 4. Therefore we have:

Proposition 3.4.8 [43] *There is a one-to-one correspondence between classes of left R-modules cogenerated by an injective left R-module and hereditary torsion theories on $R - Mod$.* □

We summarize the different results obtained in this chapter in the following.

Proposition 3.4.9 *There are bijective correspondences between:*

(i) hereditary torsion theories on $R - Mod$;

(ii) torsion radicals on $R - Mod$;

(iii) left Gabriel topologies on R;

(iv) equivalence classes of injective left R-modules;

(v) classes of left R-modules closed under taking quotients, direct sums, submodules and module extensions;

(vi) classes of left R-modules closed under taking submodules, direct products, essential extensions and module extensions. ☐

Since the collection of torsion radicals on $R - Mod$ is a set, the collection of hereditary torsion theories on $R - Mod$ is also a set. We shall denote by $R - tors$ the set of all hereditary torsion theories on $R - Mod$. The adjective "hereditary" will henceforth be omitted. For any $\tau \in R - tors$, we denote by $\mathcal{T}_\tau, \mathcal{F}_\tau, t_\tau(-)$ the torsion class, the torsion-free class and the torsion radical respectively associated with τ, i.e.,

$\mathcal{T}_\tau = \{M \in R - Mod \mid t_\tau(M) = M\}$ and $\mathcal{F}_\tau = \{M \in R - Mod \mid t_\tau(M) = 0\}$.

Remark 3.4.10 The torsion radical $t_\tau(-)$ is left exact but is, in general, not exact. By [18, Proposition 5.5], $t_\tau(-)$ is exact if and only if $\mathcal{F}_\tau$ is closed under taking quotients .

$R - tors$ is a lattice with the partial order defined by $\tau_1 \leq \tau_2$ if $\mathcal{T}_{\tau_1} \subseteq \mathcal{T}_{\tau_2}$ (equivalently, $\mathcal{F}_{\tau_1} \supseteq \mathcal{F}_{\tau_2}$). We define $gen(\tau)(R) = \{\sigma \in R - tors \mid \sigma \geq \tau\}$ and simply write $gen(\tau)$ if there is no ambiguity about rings. An element of $gen(\tau)$ is called a *generalization* of τ. We shall describe $\wedge$ and $\vee$ on $R-tors$ in detail in Chapter 5 but we note here that $R-tors$ is a complete lattice (see [19] for further details). We now examine how a class of left R-modules "generates" or "cogenerates" a torsion theory on $R-Mod$. Let $\mathcal{C}$ be a nonempty class of left R-modules. Define $\mathcal{F} = \{N \in R-Mod \,|\, Hom_R(M, E(N)) = 0 \text{ for all } M \in \mathcal{C}\}$.

It is easily checked that $\mathcal{F}$ is closed under taking submodules. $\mathcal{F}$ is closed under taking essential extensions for if N is an essential extension of N' then $E(N) \cong E(N')$. Hence $Hom_R(M, E(N')) = 0$ implies $Hom_R(M, E(N)) = 0$. Assume that $\{N_i \mid i \in I\}$ is a family of left R-modules contained in $\mathcal{F}$. If $M \in \mathcal{C}$ then $Hom_R(M, E(\prod N_i)) \hookrightarrow Hom_R(M, \prod E(N_i)) \cong \prod Hom(M, N_i) = 0$. Therefore $\mathcal{F}$ is closed under taking direct products. Let $0 \longrightarrow L \longrightarrow P \longrightarrow Q \longrightarrow 0$ be a short exact sequence with $L, Q \in \mathcal{F}$. Choose $P' \in \mathcal{C}$. Let L' be an orthogonal complement of L in P. Then $L \oplus L'$ is essential in P and so $E(P) = E(L) \oplus E(L')$. Thus $Hom_R(P', E(P)) = Hom_R(P', E(L)) \oplus Hom_R(P', E(L')) = Hom_R(P', E(L'))$ since $Hom_R(P', E(L)) = 0$. Moreover, since $L' \cong (L \oplus L')/L \leq P/L \cong Q \in \mathcal{F}$, $Hom_R(P', E(L')) = 0 = Hom_R(P', E(P))$, so $P \in \mathcal{F}$, i.e., $\mathcal{F}$ is closed under taking module extensions. By Proposition 4, $\mathcal{F}$ is a torsion-free class of some hereditary torsion theory τ on $R-Mod$ with $\mathcal{T}_\tau = \{M \in R-Mod \,|\, Hom_R(M, E(N)) = 0 \text{ for all } N \in \mathcal{F}\}$ and $\mathcal{F}_\tau = \mathcal{F}$. The torsion theory τ is said to be *generated* by the class $\mathcal{C}$ and we denote it by $\xi(\mathcal{C})$.

Dually, define $\mathcal{T} = \{M \in R-Mod \,|\, Hom_R(M, E(N)) = 0 \text{ for all } N \in \mathcal{C}\}$.

If $0 \longrightarrow L \longrightarrow P \longrightarrow Q \longrightarrow 0$ is an exact sequence then by the injectivity of $E(N)$ it induces the exact sequence $0 \longrightarrow Hom_R(Q, E(N)) \longrightarrow Hom_R(P, E(N)) \longrightarrow Hom_R(L, E(N)) \longrightarrow 0$ for any $N \in \mathcal{C}$. It follows that if $P \in \mathcal{T}$ then $L, Q \in \mathcal{T}$

and conversely, if $L, Q \in \mathcal{T}$ then $P \in \mathcal{T}$. Therefore $\mathcal{T}$ is closed under taking submodules, quotients and module extensions. Suppose $\{M_i \mid i \in I\}$ is a family of left R-modules contained in $\mathcal{T}$. Choose $N \in \mathcal{C}$. Then $Hom_R(\bigoplus M_i, E(N)) \cong \prod Hom_R(M_i, E(N)) = 0$. Thus $\mathcal{T}$ is closed under taking direct sums. Also by Proposition 3, $\mathcal{T}$ is the torsion class of some hereditary torsion theory τ, i.e., $\mathcal{T}_\tau = \mathcal{T}$ and $\mathcal{F}_\tau = \{N \in R - Mod \mid Hom_R(M, E(N)) = 0$ for all $M \in \mathcal{T}\}$. The torsion theory τ is said to be *cogenerated* by $\mathcal{C}$ and we denote it by $\chi(\mathcal{C})$. If $\mathcal{C}$ is a singleton, say $\{M\}$, then we shall write $\xi(M)$ (*resp.*, $\chi(M)$) instead of $\xi(\mathcal{C})$ (*resp.*, $\chi(\mathcal{C})$). The torsion theory cogenerated by the class $\mathcal{C} = \{0\}$ is called the *improper torsion theory* and is denoted by χ and the torsion theory cogenerated by all injective left R-modules is the *trivial torsion theory* denoted by ξ. Note that χ and ξ are the maximal element and the minimal element of the lattice $R - tors$ respectively.

Proposition 3.4.11 [45] *Let $\mathcal{C}$ be a class of left R-modules. Then:*

(i) $\xi(\mathcal{C})$ is the smallest torsion theory on $R - Mod$ whose torsion class contains $\mathcal{C}$;

(ii) $\chi(\mathcal{C})$ is the largest torsion theory on $R - Mod$ whose torsion-free class contains $\mathcal{C}$.

Proof : (*i*) Let $\tau \in R - tors$ and suppose $\mathcal{T}_\tau \supseteq \mathcal{C}$. Then by Remark 5, $\mathcal{F}_\tau = \{N \in R - Mod \mid Hom_R(M, E(N)) = 0$ for all $M \in \mathcal{T}_\tau\} \subseteq \{N \in R - Mod \mid Hom_R(M, E(N)) = 0$ for all $M \in \mathcal{C}\} = \mathcal{F}_{\xi(\mathcal{C})}$. Thus $\mathcal{F}_\tau \subseteq \mathcal{F}_{\xi(\mathcal{C})}$ and so $\tau \geq \xi(\mathcal{C})$.

(*ii*) Let $\tau \in R - tors$ and suppose $\mathcal{F}_\tau \supseteq \mathcal{C}$. Referring once more to Remark 5 we have that $\mathcal{T}_\tau = \{M \in R - Mod \mid Hom_R(M, E(N)) = 0$ for all $N \in \mathcal{F}_\tau\} \subseteq \{M \in R - Mod \mid Hom_R(M, E(N)) = 0$ for all $N \in \mathcal{C}\} = \mathcal{T}_{\chi(\mathcal{C})}$ and so $\tau \leq \chi(\mathcal{C})$. □

We close this section with a study of a particular torsion theory, namely, in general, the *Goldie torsion theory*. Consider the family $\mathcal{E}$ of Example 3.4. Since $\mathcal{E}$ is not a Gabriel topology, the torsion preradical Z associated with $\mathcal{E}$ need not be a torsion radical. But by Proposition 2.1, there exists a smallest torsion radical $\bar{Z}$ larger than Z. We now show that $\bar{Z} = Z : Z$. By Remark 2.2, $Z(M)$ is essential in $\bar{Z}(M)$ for every left R-module M. It follows then that $\bar{Z}(M)/Z(M)$ is singular. Therefore $\bar{Z}(M)/Z(M) \subseteq Z(M/Z(M)) = (Z : Z)(M)/Z(M)$. Thus $\bar{Z}(M) = (Z : Z)(M)$ for all $M \in R - Mod$, whence $\bar{Z} = Z : Z$.

The torsion theory corresponding with the torsion radical $\bar{Z}$ is called the *Goldie torsion theory* and is denoted by $\tau_g(R)$ or simply τ_g if there is only one ring involved. Note that a left R-module is τ_g-torsion-free if and only if it is nonsingular. Observe also that since $\bar{Z}$ is the smallest torsion radical larger than Z and every singular left R-module is isomorphic to M/N where $N \leq_e M \in R - Mod$, we must have $\tau_g = \xi(\{M/N \mid M \in R - Mod \text{ and } N \leq_e M\})$.

For a commutative integral domain R, a left R-module M is τ_g-torsion-free if and only if $rm \neq 0$ for every $0 \neq m \in M$ and $0 \neq r \in R$. This is the classic notion of "torsion-free" as it originated in the theory of abelian groups.

By Proposition 2.2.6, the family $\mathcal{D} = \{I \leq {}_RR \mid Hom_R(R/I, E(R)) = 0\}$ of dense left ideals of R is a left Gabriel topology on R. The torsion theory corresponding with $\mathcal{D}$ is precisely the one cogenerated by $E(R)$ and is called the *Lambek torsion theory*. It is denoted by $\chi({}_RR)$. If ${}_RR$ is nonsingular, the two torsion theories coincide by Proposition 2.2.5.

3.5 Pure submodules and τ-injective modules

Let $\tau \in R - tors$. A submodule N of a left R-module M is said to be *τ-pure* if M/N is τ-torsion-free, i.e., $t_\tau(M/N) = 0$. Stenström [43, page 207] uses the term *τ-saturated* instead of τ-pure. For instance, $t_\tau(M)$ is a τ-pure submodule of M. Note that if N is a τ-pure submodule of M then N contains $t_\tau(M)$ for otherwise $[t_\tau(M) + N]/N$ would be a nonzero τ-torsion submodule of M/N.

Lemma 3.5.1 [19] *Let M be a left R-module. The intersection of all τ-pure submodules of M is τ-pure.*

Proof: Let $\{A_i \mid i \in I\}$ be a family of τ-pure submodules of M. Then there exists a canonical monomorphism $\alpha : M/\bigcap A_i \longrightarrow \prod M/A_i$. Since $\mathcal{F}_\tau$ is closed under taking direct products and submodules, the image of $M/\bigcap A_i$ under α is τ-torsion-free. Therefore $\bigcap A_i$ is τ-pure in M. □

Observe that any submodule N of a left R-module M is contained in at least one τ-pure submodule, namely M itself and so using Lemma 1, one can show that there exists a unique minimal element of family of all τ-pure submodules of M containing N. This element is called the *τ-purification* of N in M. Other authors such as Zelmanowitz [50] speak of the *τ-closure* instead of the τ-purification. The τ-purification of N will be denoted by N^c and one can show that $N^c/N = t_\tau(M/N)$. To see this, let $t_\tau(M/N) = K/N$. Since τ is a radical, $(M/N)/(K/N) \cong M/K$ is τ-torsion-free. Therefore K is τ-pure in M, so $N^c \subseteq K$. This implies that $N^c/N \subseteq t_\tau(M/N)$. Conversely, K/N^c is τ-torsion-free because it is a submodule of M/N^c. Furthermore, it is τ-torsion because it is a homomorphic image of K/N which is τ-torsion by definition. Thus $K/N^c = 0$, i.e., $K = N^c$. This establishes equality.

By the previous lemma the family of all τ-pure submodules of M, denoted by $\mathcal{P}_\tau(M)$ is closed under taking intersections. A complete lattice structure can be defined on $\mathcal{P}_\tau(M)$ by setting $\bigwedge \mathcal{A} = \bigcap \mathcal{A}$ and $\bigvee \mathcal{A} = (\sum \mathcal{A})^c$ for every family $\mathcal{A}$ of elements of $\mathcal{P}_\tau(M)$ (for more details on properties of the closure operator c and the lattice $\mathcal{P}_\tau(M)$, see [19] and [38]).

Let $\tau \in R-tors$. A left R-module M is said to be *τ-injective* if given any diagram

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r] \arrow[dr, "f"'] & N \arrow[d, dotted, "g"] \\
 & & M
\end{tikzcd}$$

with L τ-dense in N there exists an R-homomorphism $g : N \longrightarrow M$ which makes the diagram commute. Obviously any injective left R-module is τ-injective. Since any essential left ideal of R is τ_g-dense in R, a left R-module is τ_g-injective if and only if it is injective. To see this, let E be a τ_g-injective left R-module. Consider an R-homomorphism $\alpha : I \longrightarrow E$ with I a left ideal of R. Choose an orthogonal complement J of I in R. Then $\alpha' : I \oplus J \longrightarrow E$ defined by $[(i, j)]\alpha' = (i)\alpha$ extends α. But since $I \oplus J$ is essential in ${}_RR$, α' can be extended to an R-homomorphism from R into E. Therefore E is injective by Baer's Criterion.

Note that the class of τ-injective left R-modules is closed under taking direct products, direct summands, extensions, τ-pure submodules and finite sums [19, Proposition 8.4].

Proposition 3.5.2 [19] *Let τ be a torsion theory on $R-Mod$. For a left R-module M the following conditions are equivalent:*

(i) M is τ-injective;

(ii) M is τ-pure in its injective hull $E(M)$;

(iii) for any diagram

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[dr, "\alpha"'] & {}_RR \arrow[d, dotted, "\beta"] \\ & & M \end{tikzcd}$$

with I a τ-dense left ideal in R, there exists an R-homomorphism $\beta : {}_RR \longrightarrow M$ which makes the diagram commute.

(iv) M is a direct summand of a τ-pure submodule of any left R-module containing it.

Proof: $(i) \Longrightarrow (ii)$ Let M^c be the τ-purification of M in $E(M)$. We need to show that $M^c = M$. By (i) the identity map $M \longrightarrow M$ can be extended to an R-homomorphism $\beta : M^c \longrightarrow M$. Since M is essential in $E(M)$ it is essential in M^c and so β is a monomorphism. Since the restriction of β to M is the identity map it follows that the inclusion $M \hookrightarrow M^c$ must be an epimorphism. Hence $M^c = M$. Then $t_\tau(E(M)/M) = 0$.

$(ii) \Longrightarrow (iii)$ Let I be a τ-dense left ideal of R and let $\alpha : I \longrightarrow M$ be an R-homomorphism. By the injectivity of $E(M)$ there exists an R-homomorphism φ making the diagram

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d, "\alpha"'] & R \arrow[d, "\varphi"] \\ & M \arrow[r] & E(M) \end{tikzcd}$$

commute. Set $x = (1)\varphi$. Then $[Rx + M]/M \cong R/(M : x)$ and $(M : x)$ is τ-dense since it contains I. Therefore $[Rx + M]/M$ is τ-torsion. By (*ii*) this implies that $x \in M$ and so $[R]\varphi \subseteq M$.

(*iii*) $\Longrightarrow$ (*i*) Let L be a τ-dense submodule of N and $\alpha : L \longrightarrow M$ an R-homomorphism. Consider the set of all pairs (W, β) where W is a submodule of N containing L and $\beta : W \longrightarrow M$ an R-homomorphism extending α to W. Partially order this set by putting $(W, \beta) \leq (W', \beta')$ if and only if $W \leq W'$ and β is the restriction of β' to W. If $\{(W_i, \beta_i) \mid i \in I\}$ is a chain in this partially ordered set, say P, then $(\bigcup_i W_i, \bigcup_i \beta_i) \in P$ is an upper bound for this chain and by Zorn's Lemma it has a maximal element, say (W_o, β_o). The proof will be done if we show that $W_o = N$. Assume the contrary and let $x \in N \setminus W_o$. Set $I = (W_o : x)$. Then $(L : x) \subseteq I$ and so I is a τ-dense left ideal of R. Consider the R-homomorphism $\varphi : I \longrightarrow M$ defined by $(a)\varphi = (ax)\beta_o$. By (*iii*), φ can be extended to an R-homomorphism $\hat{\varphi} : {}_RR \longrightarrow M$. Now define $\beta_1 : W_o + Rx \longrightarrow M$ by $(w_o + rx)\beta_1 = (w_o)\beta_o + (r)\hat{\varphi}$. The map β_1 is well-defined for if $w_o + rx = 0$ then $rx = -w_o \in W_o$, so $r \in (W_o : x)$. Therefore $(r)\hat{\varphi} = (r)\varphi = (rx)\beta_o = -(w_o)\beta_o$. It follows that $(w_o)\beta_o + (r)\hat{\varphi} = 0$. Thus the β_1-value of $w_o + rx$ doesn't depend on the choice of w_o in W_o and r in R. Clearly, β_1 is an R-homomorphism which properly extends β_o. But this contradicts the maximality of (W_o, β_o). Thus $W_o = N$.

(*ii*) $\Longrightarrow$ (*iv*) Let P be a left R-module such that $M \leq P$. Then $E(P) = E(M) \oplus N$ for some submodule N of $E(P)$. If $Q = P \cap N$ then $P/(M \oplus Q)$ is isomorphic to a submodule of $E(P)/(M \oplus N)$ and furthermore $E(P)/(M \oplus N) \cong$ $(E(M) \oplus N)/(M \oplus N) \cong E(M)/M$ which is τ-torsion-free by (*ii*). Therefore $M \oplus Q$ is τ-pure in P.

(iv) $\Longrightarrow$ *(ii)* By assumption there exists a submodule N of $E(M)$ such that $M \oplus N$ is τ-pure in $E(M)$. But M is essential in $E(M)$ and so we must have $N = 0$. $\square$

Proposition 3.5.3 [19] *Let M be a left R-module.*

(i) If $\tau \leq \sigma$ are torsion theories on $R - Mod$ and if M is σ-injective then it is τ-injective.

(ii) If U is a nonempty set of torsion theories on $R - Mod$ then M is $(\bigvee U)$-injective if and only if it is τ-injective for every $\tau \in U$.

Proof: *(i)* Let M be a σ-injective left R-module and I be a σ-dense left ideal of R. For any R-homomorphism $\alpha : I \longrightarrow M$ there exists an R-homomorphism $\hat{\alpha} : {}_RR \longrightarrow M$ which extends α. Since $\tau \leq \sigma$ any τ-dense left ideal of R is σ-dense. It follows that M is τ-injective.

(ii) Let $\emptyset \neq U \subseteq R - tors$. For any $\tau \in U$, $\tau \leq \bigvee U$ and so by *(i)* every $(\bigvee U)$-injective left R-module is τ-injective. Conversely assume that M is τ-injective for every $\tau \in U$. Let I be a $(\bigvee U)$-dense left ideal of R and let $\alpha : I \longrightarrow M$ be an R-homomorphism. Consider the set of all pairs (J, β), where J is a left ideal containing I (and hence $(\bigvee U)$-dense in R) and $\beta : J \longrightarrow M$ an R-homomorphism which extends α. Partially order this set by setting $(J, \beta) \leq (J', \beta')$ if and only if $J \leq J'$ and β is the restriction of β' to J. This set is inductive and so, by Zorn's Lemma, it has a maximal element, say (J_o, β_o). We claim that $J_o = R$. To this end assume that $J_o \neq R$. Then J_o is a proper $(\bigvee U)$-dense left ideal of R and so there exists an element τ of U such that J_o is τ-dense and therefore not τ-pure in R. Let J_o^c be the τ-purification of J_o in R. Then since M is τ-injective, β_o can be extended to an R-homomorphism from J_o^c to M. This contradicts the maximality

of J_o. Therefore M is $(\bigvee U)$-injective by Proposition 2. $\square$

A torsion theory τ on $R - Mod$ is said to be *stable* if the class of torsion left R-modules is closed under taking injective hulls. For example, the Goldie torsion theory τ_g is stable. For let M' be an essential extension of $M \in \mathcal{T}_{\tau_g}$. We have that M'/M is singular. Since $\mathcal{T}_{\tau_g}$ is closed under taking module extensions, it follows from the short exact sequence $0 \longrightarrow M \longrightarrow M' \longrightarrow M'/M \longrightarrow 0$ that $M' \in \mathcal{T}_{\tau_g}$.

Proposition 3.5.4 [19] *For a torsion theory τ on $R - Mod$, the following conditions are equivalent:*

(i) τ is stable;

(ii) $t_\tau(M)$ is a direct summand of every injective left R-module M;

(iii) $t_\tau(M)$ is a direct summand of every τ-injective left R-module M.

Proof: $(i) \Longrightarrow (ii)$ If M is an injective left R-module then $E(t_\tau(M)) \subseteq M$. By (i), $E(t_\tau(M))$ is τ-torsion and so $E(t_\tau(M)) \subseteq t_\tau(E(M)) = t_\tau(M)$ because M is injective, whence $E(t_\tau(M)) = t_\tau(M)$. Since $t_\tau(M)$ is injective, $t_\tau(M)$ is a direct summand of M.

$(ii) \Longrightarrow (iii)$ Let M be a τ-injective left R-module. Then it is τ-pure in $E(M)$ and so $t_\tau(E(M))$ is contained in M. Thus $t_\tau(E(M)) = t_\tau(M)$. By assumption $t_\tau(E(M))$ is a direct summand of $E(M)$ and so is injective. Therefore $t_\tau(E(M)) = t_\tau(M)$ is a direct summand of M.

$(iii) \Longrightarrow (i)$ If M is a τ-torsion left R-module then by (iii), $t_\tau(E(M))$ is a direct summand of $E(M)$ which contains M. But M being essential in $E(M)$ implies that $t_\tau(M) = E(M)$. Thus $E(M) = t_\tau(E(M))$ is τ-torsion. $\square$

Corollary 3.5.5 [19] *Let τ be a stable torsion theory on $R - Mod$. Then any*

τ-torsion τ-injective left R-module is injective.

Proof: If M is a τ-torsion τ-injective left R-module then by the previous proposition, $M = t_\tau(E(M)) = E(M)$. ☐

By Proposition 2, for any left R-module M, the τ-purification of M in $E(M)$ is a τ-injective submodule of $E(M)$ containing M as essential submodule and indeed it is the minimal such submodule of $E(M)$. We call it the *τ-injective hull* of M and denote it by $E_\tau(M)$. Note that $E_\tau(M)/M = t_\tau(E(M)/M)$. Thus M is τ-pure in $E(M)$ if and only if $M = E_\tau(M)$.

Proposition 3.5.6 [19] *If M is a τ-torsion left R-module, then $E_\tau(M)$ is also τ-torsion.*

Proof: Consider the short exact sequence $0 \to M \to E_\tau(M) \to E_\tau(M)/M \to 0$ in which M is τ-torsion by assumption. Since $N^c/N = t_\tau(M/N)$ for any $M \in R-Mod$ and $N \leq M$, $E_\tau(M)/M$ is τ-torsion, so $E_\tau(M)$ is also τ-torsion since the class of τ-torsion modules is closed under taking module extensions. ☐

As a consequence we have:

Corollary 3.5.7 [19] *For any torsion theory τ on $R-Mod$, the following conditions are equivalent:*

(i) every τ-torsion left R-module is semisimple;

(ii) every τ-torsion left R-module is τ-injective.

Proof: $(i) \Longrightarrow (ii)$ Let M be a τ-torsion left R-module. By Proposition 6, $E_\tau(M)$ is also τ-torsion and so is semisimple by (i). Thus M is a direct summand of $E_\tau(M)$. On the other hand M is essential in $E_\tau(M)$. Hence $E_\tau(M) = M$, i.e., M

is τ-injective.

$(ii) \Longrightarrow (i)$ Let N be a submodule of a τ-torsion left R-module M. By assumption N is τ-injective and so the exact sequence $0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$ splits. Thus N is a direct summand of M. Therefore M is semisimple. □

Let τ be a torsion theory on $R-Mod$. A left R-module M is said to be *absolutely τ-pure* if M is τ-torsion-free and τ-injective. Stenström [43, page 198] uses the term *τ-closed* instead of absolutely τ-pure.

Proposition 3.5.8 [19] *The following conditions for a torsion theory τ on $R-Mod$ are equivalent:*

(i) every absolutely τ-pure left R-module is injective;

(ii) every τ-torsion-free left R-module is τ-dense in its injective hull.

Proof: $(i) \Longrightarrow (ii)$ If M is a τ-torsion-free left R-module then $E_\tau(M) \leq E(M)$ and so by assumption $E_\tau(M) = E(M)$ since $E_\tau(M)$ is injective. Because M is τ-dense in $E_\tau(M)$, (ii) holds.

$(ii) \Longrightarrow (i)$ Let M be an absolutely τ-pure left R-module. Then $E(M)/M$ is τ-torsion by (ii). On the other hand, M being τ-injective implies that $E_\tau(M) = M$ from which we infer $E(M)/M$ is τ-torsion-free. Therefore $E(M) = M$. □

It has been noted that if N is a submodule of a nonsingular left R-module M such that M/N is singular, then N is essential in M (see Proposition 2.2.5). Similarly it is easily seen that every τ-dense submodule of a τ-torsion-free left R-module M is essential in M. But the converse does not, in general, hold. When this occurs, the left R-module M is said to be *τ-full*. To be more precise we have the following definition. Let $\tau \in R-tors$. A left R-module M is said to be *τ-full* if it is τ-

torsion-free and if N is a submodule of M, N is τ-dense if and only if it is essential in M.

Proposition 3.5.9 [19] *Let τ be a torsion theory on $R-Mod$. A τ-torsion-free left R-module M is τ-full if and only if the lattice $\mathcal{P}_\tau(M)$ is complemented.*

Proof: Assume that M is τ-full and let N be a τ-pure submodule of M. Let K be an orthogonal complement of N in M. Then $K \oplus N$ is essential in M and also τ-dense since M is τ-full. Furthermore $N \cong [N \oplus K]/K \leq_e M/K$. Since N is τ-torsion-free and the class of all τ-torsion-free left R-modules is closed under taking essential extensions, M/K must be τ-torsion-free. Hence K is τ-pure in M from which we infer $\mathcal{P}_\tau(M)$ is complemented.
Conversely, let N be an essential submodule of M. Since $\mathcal{P}_\tau(M)$ is complemented there exists, by Zorn's Lemma, a τ-pure submodule K of M, maximal such that $K \cap N^c = 0$ and $K \oplus N^c$ τ-dense in M. But since N is essential in M and $N \subseteq N^c$, N^c is essential in M. Thus $K = 0$ and so N^c is τ-dense in M, i.e., $t_\tau(M/N^c) = M/N^c$. This occurs only if $N^c = M$ because N^c is τ-pure in M. Hence $t_\tau(M/N) = N^c/N = M/N$. □

Theorem 3.5.10 [19] *Let τ be a torsion theory on $R-Mod$ and M be a τ-full τ-injective left R-module. Then every τ-pure submodule of M is a direct summand of M.*

Proof: Let W be a τ-pure submodule of M. By Proposition 9, $\mathcal{P}_\tau(M)$ is complemented and so there exists a submodule K of M maximal such that $W \cap K = 0$ and the τ-purification of $K \oplus W$ is M. Since M is τ-injective there exists an endomorphism α of M which extends the canonical projection $\nu : K \oplus W \longrightarrow W$.

Moreover, α induces an R-homomorphism $\alpha' : M/[K \oplus W] \longrightarrow M/W$. Since M/W is τ-torsion-free it follows that $[M]\alpha \subseteq W$. Thus α is an R-epimorphism from M onto W. If $m \in M$ then $(m\alpha)\alpha = (m\alpha)\nu = m\alpha$ and so $\alpha\alpha = \alpha$. Therefore $M = Ker\alpha \oplus Im\alpha = Ker\alpha \oplus W$. □

3.6 Rings and modules of quotients

The construction of a localization functor on $R - Mod$ described below is due to Golan [18] and it leans on the notion of "torsion-freeness" and "injectivity" relative to a torsion theory on $R - Mod$.

Proposition 3.6.1 [18] *Let τ be a torsion theory on $R-Mod$. For a left R-module M the following conditions are equivalent:*

(i) M is absolutely τ-pure;

(ii) given any diagram of the form

$$\begin{tikzcd}
0 \arrow[r] & N' \arrow[r] \arrow[dr, "f"'] & N \arrow[d, dotted, "g"] \\
 & & M
\end{tikzcd}$$

with N' τ-dense in N there exists a unique R-homomorphism $g : N \longrightarrow M$ which makes the diagram commute.

Proof: $(i) \Longrightarrow (ii)$ Let M be τ-torsion-free and τ-injective. Since M is τ-injective g exists. Assume that there exist β and $\hat{\beta}$ which make the diagram commute. Then $Ker(\beta - \hat{\beta}) \supseteq N'$ and therefore $\beta - \hat{\beta}$ factors through $N/N' \longrightarrow M$. But N/N' is τ-torsion, so $Im(\beta - \hat{\beta}) \cong N/Ker(\beta - \hat{\beta})$ is τ-torsion. On the other hand

$Im(\beta - \hat{\beta})$ being contained in M is τ-torsion-free. Thus $Im(\beta - \hat{\beta}) = 0$ from which we infer $\beta = \hat{\beta}$.

$(ii) \Longrightarrow (i)$ By (ii), M is τ-injective. Again by (ii) the zero map $0 \longrightarrow M$ has a unique extension $t_\tau(M) \longrightarrow M$, which must consequently also be the zero map. Thus $t_\tau(M) = 0$, i.e., M is τ-torsion-free. ☐

Define a functor $Q_\tau(-) : R - Mod \longrightarrow R - Mod$ as follows:

(i) For any $M \in R - Mod$ put $Q_\tau(M) = E_\tau(M/t_\tau(M))$.

(ii) If $\alpha \in Hom_R(M, N)$ then since $[t_\tau(M)]\alpha \subseteq t_\tau(N)$, α induces an R-homomorphism $\bar{\alpha} : M/t_\tau(M) \longrightarrow N/t_\tau(N)$. Since $M/t_\tau(M)$ is τ-dense in $E_\tau(M/t_\tau(M))$ and $E_\tau(N/t_\tau(N))$ is absolutely τ-pure, it follows from Proposition 1 that there exists a unique β making the diagram

$$\begin{tikzcd}
M/t_\tau(M) \arrow[r, "\bar{\alpha}"] \arrow[d, "\wr"'] & N/t_\tau(N) \arrow[d, "\wr"] \\
E_\tau(M/t_\tau(M)) \arrow[r, "\beta"'] & E_\tau(N/t_\tau(N))
\end{tikzcd}$$

commute. Set $Q_\tau(\alpha) = \beta$.

$Q_\tau(-)$ is called the $\tau -$ *localization* functor on $R - Mod$ and $Q_\tau(M)$ the *module of quotients of M*. We shall denote $Q_\tau(M)$ by M_τ. We define an R-homomorphism $\hat{\tau}_M : M \longrightarrow Q_\tau(M)$ to be the composition of the canonical projection $M \longrightarrow M/t_\tau(M)$ and the canonical embedding $M/t_\tau(M) \hookrightarrow E_\tau(M/t_\tau(M))$. Observe that the maps $\hat{\tau}_M$ yield a natural transformation: $1_{R-Mod} \longrightarrow Q_\tau(-)$. It is easily seen that $Ker(\hat{\tau}_M) = t_\tau(M)$ [43, Lemma IX.1.2].

Now take $M = {}_RR$. A ring structure can be defined on $E_\tau(R/t_\tau(R))$.

Proposition 3.6.2 [18] *Let τ be a torsion theory on $R - Mod$. Then the endomorphism ring $End_R E_\tau(R/t_\tau(R))$ is canonically a left R-module which is isomorphic* *to* $E_\tau(R/t_\tau(R))$.

Proof: Similar to Proposition 2.3.2. □

Inasmuch as $End_R Q_\tau({}_RR)$ is a ring, the existence of a left R-module isomorphism $\rho : Q_\tau({}_RR) \longrightarrow End_R Q_\tau({}_RR)$ allows us to view $Q_\tau({}_RR)$ as a ring with multiplication given by $xy = ((x)\rho(y)\rho)\rho^{-1}$ for $x, y \in Q_\tau({}_RR)$. If $Q_\tau({}_RR)$ is viewed as a ring in this way, ρ becomes a ring isomorphism and $\hat{\tau}_R : R \longrightarrow Q_\tau({}_RR)$ a ring homomorphism which we shall henceforth denote by $\hat{\tau}$. The *ring* $Q_\tau({}_RR)$ is called the *localization of* R *with respect to the torsion theory* τ or the *localization of* R *at* τ, and is denoted by R_τ.

Take $\tau = \chi({}_RR)$ the Lambek torsion theory. Since $t_\tau({}_RR) = 0$, the ring homomorphism $R \longrightarrow R_\tau$ becomes a ring monomorphism and R_τ coincides with $Q_{max}({}_RR)$, the left maximal ring of quotients of R (see Section 2.2).

Proposition 3.6.3 [45] *Let* τ *be a torsion theory on* $R - Mod$. *Then:*

(i) Every absolutely τ*-pure left* R*-module has a unique left* R_τ*-module structure which extends its* R*-module structure.*

(ii) If $\alpha : M \longrightarrow N$ *is an* R*-homomorphism between left* R_τ*-modules such that* $R_\tau([M]\alpha)$ *is* τ*-torsion-free as a left* R*-module then* α *is naturally an* R_τ*-homomorphism.*

(iii) Any R*- homomorphism between absolutely* τ*-pure left* R*-modules is an* R_τ*-homomorphism.*

Proof: (i) Let M be an absolutely τ-pure left R-module. For each $m \in M$ let $\delta_m : {}_RR \longrightarrow M$ be the R-homomorphism defined by $(r)\delta_m = rm$ for $r \in R$. Since M is τ-torsion-free, $t_\tau(R) \subseteq Ker\delta_m$ and so δ_m induces an R-homomorphism $\bar{\delta}_m : R/t_\tau(R) \longrightarrow M$ with $(r + t_\tau(R))\bar{\delta}_m = rm$. By Proposition 1, $\bar{\delta}_m$ can be uniquely extended to an R-homomorphism $\hat{\delta}_m : R_\tau \longrightarrow M$. For $\alpha \in R_\tau$ and $m \in M$, define $\alpha.m = \alpha\hat{\delta}_m$. It is easily verified that $1.m = m$, $(\alpha + \beta).m = \alpha.m + \beta.m$ and $\alpha.(m + n) = \alpha.m + \alpha.n$ for $\alpha, \beta \in R_\tau$ and $m, n \in M$. We show that $(\alpha\beta).m = \alpha.(\beta.m)$. Put $\beta\hat{\delta}_m = n \in M$. Then $\alpha.(\beta.m) = \alpha.n = \alpha\hat{\delta}_n$. Observe that $(r + t_\tau(R))\hat{\delta}_n = (r + t_\tau(R))\bar{\delta}_n = rn = r\beta\hat{\delta}_m = (r + t_\tau(R))\beta\hat{\delta}_m$ for all $r \in R$, in other words, $\hat{\delta}_n$ and $\beta\hat{\delta}_m$ agree on $R/t_\tau(R)$ and so $\alpha.(\beta.m) = \alpha\beta\hat{\delta}_m = (\alpha\beta).m$. Now let $r \in R$ and put $\alpha = \hat{\tau}(r)$. Observe that $\alpha(r' + t_\tau(R)) = r'(r + t_\tau(R))$ and so $\alpha.m = \alpha\hat{\delta}_m = (r + t_\tau(R))\hat{\delta}_m = (r + t_\tau(R))\bar{\delta}_m = rm$. Thus the R_τ-module structure naturally extends the R-module structure. It remains to show the uniqueness. Suppose there exists another multiplication $\star : R_\tau \times M \longrightarrow M$ which extends the R-module structure of M. Let $m \in M$ and define a function $\psi : R_\tau \longrightarrow M$ by $(\alpha)\psi = \alpha.m - \alpha\star m$ for all $\alpha \in R_\tau$. Since both multiplications extend the R-module structure of M, $r[(\alpha)\psi] = (\hat{\tau}(r)\alpha).m - (\hat{\tau}(r).\alpha)\star m = (r\alpha).m - (r\alpha)\star m = (r\alpha)\psi$. This shows that ψ is an R-homomorphism. Since $\hat{\tau}(r).m = \hat{\tau}(r)\star m$ for all $r \in R$, it follows that $Im\hat{\tau} \subseteq Ker\psi$. We obtain then a canonical epimorphism $R_\tau/Im\hat{\tau} \longrightarrow M/Ker\psi \cong Im\psi \in \mathcal{F}_\tau$. On the other hand the R-isomorphism $R_\tau \longrightarrow End_R R_\tau$ induces an isomorphism from $End_R R_\tau/Im\hat{\tau} \cong R_\tau/Im\hat{\tau}$ to $R_\tau/(R/t_\tau(R)) \in \mathcal{T}_\tau$ and so $Im\psi = 0$. Thus $\alpha.m = \alpha\star m$ for all $\alpha \in R_\tau$. This completes the proof of (i).

(ii) Let $\alpha \in Hom_R(M, N)$. For each $m \in M$ define $\alpha_m : R_\tau/Im\hat{\tau} \longrightarrow R_\tau([M]\alpha)$ by $(r + Im\hat{\tau})\alpha_m = (rm)\alpha - r(m\alpha)$ for all $r \in R_\tau$. It is easily verified that α_m is an R-homomorphism. But $R_\tau/Im\hat{\tau}$ is τ-torsion and $R_\tau([m]\alpha)$ is τ-torsion-free by hypothesis, so α_m must be the zero map. It follows that α is an R_τ-homomorphism.

(iii) This is a particular case of (ii). □

It follows from the above results that if $M, N \in R - Mod$ and $\alpha \in Hom_R(M, N)$, then $Q_\tau(M)$ and $Q_\tau(N)$ can be regarded as objects in $R_\tau - Mod$ and $Q_\tau(\alpha)$ as an R_τ-homomorphism. The τ-localization functor $Q_\tau(-)$ may thus be regarded as a functor from $R - Mod$ to $R_\tau - Mod$. The full subcategory of $R - Mod$ whose objects are the τ-torsion-free τ-injective left R-modules is called the *category of modules of quotients* or *the quotient category* of $R - Mod$ with respect to τ, and is denoted by $(R, \tau) - Mod$. Observe that $(R, \tau) - Mod \subseteq R_\tau - Mod$. The category $(R, \tau) - Mod$ is an example of a *Grothendieck* category. We refer the interested reader to Stenström [43] for information on Grothendieck categories.

Now let us give some important homological results which concern the passage of the injectivity condition between $R - Mod$ and $R_\tau - Mod$.

Proposition 3.6.4 [18] *Let τ be a torsion theory on $R - Mod$ and $M \in R_\tau - Mod$ that is τ-torsion-free as a left R-module. The following conditions are equivalent:*

(i) M is injective as a left R-module;

(ii) M is injective as a left R_τ-module.

Proof: $(i) \Longrightarrow (ii)$ Let $E(M)$ be the injective hull of M in $R_\tau - Mod$. Since ${}_RM$ is injective there exists an R-homomorphism $\alpha : E(M) \longrightarrow M$ which restricted to M is the identity map. By Proposition 3 (ii), α is an R_τ-homomorphism and so M is a direct summand of $E(M)$ in $R_\tau - Mod$. Thus $E(M) = M$.

$(ii) \Longrightarrow (i)$ Let $E(M)$ be the injective hull of M in $R - Mod$. Then M is a τ-torsion-free left R-module and hence so is $E(M)$. Therefore $E(M)$ is absolutely τ-pure. This implies that $E(M) = Q_\tau(E(M))$. Again by Proposition 3 (ii), the inclusion map $M \hookrightarrow E(M)$ is an R-homomorphism. It follows by (ii) that there exists an R_τ-homomorphism $\alpha : E(M) \longrightarrow M$ having the identity map as restriction to M.

Thus M is a direct summand of $E(M)$ in $R-Mod$ and so $M = E(M)$. $\square$

Corollary 3.6.5 [18] *Let τ be a torsion theory on $R-Mod$ and $M \in R_\tau - Mod$. If M is τ-torsion-free as a left R-module then $E(_RM) = E(_{R_\tau}M)$.*

Proof: Since $t_\tau(_RM) = 0$, $E(_RM)$ is absolutely τ-pure and is injective as a left R_τ-module by Proposition 4. But M being essential in $E(_RM)$ implies that $E(_RM)$ is the injective hull of M in $R_\tau - Mod$. Thus $E(_RM) = E(_{R_\tau}M)$. $\square$

Next we examine how a torsion theory on $R-Mod$ induces a torsion theory on $R_\tau - Mod$. Let M be a left R-module and L an R-submodule of M_τ. We have the following commutative diagram with exact rows

$$\begin{tikzcd}
 & M \arrow[r, "\hat{\tau}_M"] \arrow[d] & M_\tau \arrow[r] \arrow[d] & M_\tau/Im\hat{\tau}_M \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & M/[L]\hat{\tau}_M^{-1} \arrow[r] & M_\tau/L \arrow[r] & M_\tau/Im\hat{\tau}_M \arrow[r] & 0
\end{tikzcd}$$

It follows that L is τ-dense in M_τ if and only if $[L]\hat{\tau}_M^{-1}$ is τ-dense in M. If $\mathcal{F}$ is the left Gabriel topology corresponding with τ on $R-Mod$, then $\mathcal{F}^\star = \{U \leq {}_{R_\tau}R_\tau \mid [U]\hat{\tau}^{-1} \in \mathcal{F}\}$ is a Gabriel topology on $R_\tau - Mod$. There then exists a torsion theory, $\tau^\star$ say, corresponding with $\mathcal{F}^\star$. Observe that $_{R_\tau}M \in \mathcal{T}_{\tau^\star}$ if and only if $[M]\hat{\tau}_M^{-1} \in \mathcal{T}_\tau$. It is easily checked that if E is an injective R-module such that $\tau = \chi(_RE)$ then $\tau^\star = \chi(_{R_\tau}E)$.

We now focus on the lattices $\mathcal{P}_\tau(M)$ and $\mathcal{P}_\tau(M_\tau)$.

Proposition 3.6.6 [43] *Let τ be a torsion theory on $R-Mod$ and M an absolutely τ-pure left R -module. Then the following conditions are equivalent for a submodule L of M:*

(i) L is τ-injective;

(ii) L is τ-pure in M.

Proof: For every τ-dense left ideal I of R and submodule L of M we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & Hom_R(R,L) \arrow[r] \arrow[d, "\lambda"] & Hom_R(R,M) \arrow[r] \arrow[d, "\mu"] & Hom_R(R,M/L) \arrow[r] \arrow[d, "\eta"] & 0 \\
0 \arrow[r] & Hom_R(I,L) \arrow[r] & Hom_R(I,M) \arrow[r] & Hom_R(I,M/L) &
\end{tikzcd}$$

with exact rows and canonical homomorphisms. Using the fact that M is τ-torsion-free and τ-injective, it is easily shown that μ is an isomorphism. It is also easy to verify that λ is an epimorphism if and only if η is a monomorphism, i.e., L is τ-injective if and only if M/L is τ-torsion-free. $\square$

Proposition 3.6.7 [43] *Let τ be a torsion theory on $R-Mod$. For every left R-module M there is a lattice isomorphism $\mathcal{P}_\tau(M) \longrightarrow \mathcal{P}_\tau(M_\tau)$ given by $L \longmapsto L_\tau$.*

Proof: Let $L \leq M$. If $\imath : L \hookrightarrow M$ denotes the inclusion map, it is easily verified that $Q_\tau(\imath) : L_\tau \longrightarrow M_\tau$ is an R-monomorphism. Since M_τ is absolutely τ-pure and L_τ is τ-injective, it follows from the previous proposition that $L_\tau \in \mathcal{P}_\tau(M_\tau)$. Consider the canonical R-homomorphism $\hat{\tau}_M : M \longrightarrow M_\tau$. If $N \in \mathcal{P}_\tau(M_\tau)$, then $[N]\hat{\tau}_M^{-1}$ is τ-pure in M because $M/[N]\hat{\tau}_M^{-1}$ may be considered as a submodule of M_τ/N, and is therefore τ-torsion-free. We claim that $N \longmapsto [N]\hat{\tau}_M^{-1}$ is the inverse of the map $L \longmapsto L_\tau$. Since for each $L \leq M$, $L_\tau \in \mathcal{P}_\tau(M_\tau)$, we have that $[L_\tau]\hat{\tau}_M^{-1}$ is τ-pure in M and, indeed, it is the smallest τ-pure submodule of M containing L, so $[L_\tau]\hat{\tau}_M^{-1} = L^c$. It follows that $[L_\tau]\hat{\tau}_M^{-1} = L$ when L is τ-pure in M. On the other hand, for every absolutely τ-pure submodule N of M_τ we have $([N]\hat{\tau}_M^{-1})_\tau = N$. The map $L \longmapsto L_\tau$ thus defines an isomorphism of complete lattices. $\square$

We close the section with an example of localization.

Example 3.6.8 Let P be a prime ideal of a commutative ring R. It is easily verified that the family $\mathcal{F}_P = \{I \trianglelefteq R | I \nsubseteq P\}$ is a Gabriel topology on R. Denote by R_P the localization of R at P and $S = R \backslash P$ (see page 14). Let τ be the torsion theory on $R - Mod$ associated with $\mathcal{F}_P$. We need to show that the functors $S^{-1}(-)$ and $Q_\tau(-)$ are naturally equivalent. It shall follow that $R_\tau \cong R_P$. If $M \in R - Mod$, then $t_\tau(M) = \{m \in M \mid (0 : m) \in \mathcal{F}_P\}$ or, equivalently, $t_\tau(M) = \{m \in M \mid (0 : m) \subseteq S\} = \{m \in M \mid sm = 0 \text{ for some } s \in S\}$. One can show that the mapping $m + t_\tau(M) \longrightarrow \frac{m}{1}$ is an R-isomorphism from $M/t_\tau(M)$ to a submodule $\{\frac{m}{1} | m \in M\}$ of $S^{-1}M$. Moreover, the canonical R-homomorphism $\hat{S}_M : M \longrightarrow S^{-1}M$ is the composition of the canonical epimorphism $M \longrightarrow M/t_\tau(M)$ and the embedding $M/t_\tau(M) \hookrightarrow S^{-1}M$. We claim that $S^{-1}M$ is a τ-injective hull for $M/t_\tau(M)$. Since $s(\frac{m}{s}) = \frac{m}{1} \in M/t_\tau(M)$ for $s \in S$, it follows that $M/t_\tau(M)$ is essential in $S^{-1}M$. Therefore $M/t_\tau(M) \leq S^{-1}M \leq E(M/t_\tau(M))$. Now consider $S^{-1}M/[M/t_\tau(M)] \leq E(M/t_\tau(M))/[M/t_\tau(M)]$. If $\frac{m}{s} + M/t_\tau(M) \in S^{-1}M/[M/t_\tau(M)]$, then $s(\frac{m}{s} + M/t_\tau(M)) = 0$. Whence $\frac{m}{s} + M/t_\tau(M) \in t_\tau(E(M/t_\tau(M))/[M/t_\tau(M)])$. Conversely, if $m + M/t_\tau(M) \in t_\tau(E(M/t_\tau(M))/[M/t_\tau(M)])$, then $s(m + M/t_\tau(M)) = 0$ for some $s \in S$ and so $sm \in M/t_\tau(M)$, i.e., $sm = \frac{n}{1}$ for some $n \in M$. We have thus $s(m - \frac{n}{s}) = 0$, which implies that $m - \frac{n}{s} \in t_\tau(E(M/t_\tau(M)))$. But $E(M/t_\tau(M))$ is τ-torsion-free, so we must have $m - \frac{n}{s} = 0$ and $m + M/t_\tau(M) \in S^{-1}M/[M/t_\tau(M)]$. Thus $t_\tau(E(M/t_\tau(M))/[M/t_\tau(M)]) = S^{-1}M/[M/t_\tau(M)]$ and so the claim holds. Since the τ-injective hull is unique, there exists an R-isomorphism

$\eta(M) : S^{-1}M \longrightarrow Q_{\tau}(M)$ which makes the diagram

$$\begin{tikzcd}
M \arrow[r, "\hat{S}_M"] \arrow[d, "\hat{\tau}_M"'] & S^{-1}M \arrow[dl, "\eta(M)"] \\
Q_{\tau}(M) &
\end{tikzcd}$$

commute. The functors $Q_{\tau}(-)$ and $S^{-1}(-)$ are thus naturally equivalent. Taking $M = {}_RR$, the R-module isomorphism $\eta(M) : S^{-1}M \longrightarrow Q_{\tau}(M)$ actually induces a ring isomorphism from R_P to R_{τ}. To see this note that in the diagram below, $\hat{S}$ and $\hat{\tau}$ are ring homomorphisms, so $\eta(R)$ is a ring isomorphism.

$$\begin{tikzcd}
R \arrow[r, "\hat{S}"] \arrow[d, "\hat{\tau}"'] & R_P \arrow[dl, "\eta(R)"] \\
R_{\tau} &
\end{tikzcd}$$

Chapter 4

The Spectral Category $(R, \tau) - Mod$

This chapter is devoted to the study of spectral torsion theories. One of our tasks shall be to show how properties of the ring of quotients of R can be characterized in terms of spectral torsion theories on $R - Mod$.

4.1 Definition of a spectral torsion theory

A Grothendieck category is said to be *spectral* if every short exact sequence in the category splits. One can show that $R - Mod$ is spectral if and only if R is a semisimple ring [43, page 130]. If M is an object of a spectral category, then $End_R M$ is a regular left self-injective ring [43, Proposition XII.1.2]. A torsion theory τ on $R - Mod$ is called *spectral* if the category $(R, \tau) - Mod$ is spectral.

Lemma 4.1.1 [4] *The following conditions are equivalent for a torsion theory τ on $R - Mod$:*

(i) τ is spectral;

(ii) every absolutely τ-pure left R-module is injective;

(iii) every τ-torsion-free left R-module is τ-dense in its injective hull.

Proof: $(i) \Longleftrightarrow (ii)$ By definition, τ is spectral if every short exact sequence in $(R,\tau) - Mod$ splits. This occurs if and only if every module of $(R,\tau) - Mod$ is injective by Proposition 1.5.1.

$(ii) \Longleftrightarrow (iii)$ Proposition 3.5.8. ☐

Note that if τ is spectral then M_τ is injective for every left R-module M.

Proposition 4.1.2 [23] *The Goldie torsion theory τ_g is spectral.*

Proof: Since every τ_g-injective left R-module is injective (see page 54) the result follows from Lemma 1 ($(i) \Longleftrightarrow (ii)$). ☐

Proposition 4.1.3 [4] *Let τ be a spectral torsion theory on $R - Mod$. Then every generalization of τ is spectral.*

Proof: If $\sigma \geq \tau$ then every σ-torsion-free left R-module is τ-torsion-free and every τ-dense submodule is also a σ-dense submodule. It follows from Lemma 1 ($(i) \Longleftrightarrow (ii)$) that σ is spectral. ☐

It follows from the two previous results that every generalization of the Goldie torsion theory is spectral. But later we shall see an example of a torsion theory τ which is spectral without being greater than τ_g. The functor $Q_\tau(-) : R - Mod \longrightarrow R - Mod$ defined by $M \longmapsto Q_\tau(M) = M_\tau$ is known to be left exact [22]. Moreover, when the torsion theory τ is spectral, the functor $Q_\tau(-)$ is also right exact and so exact. To see this let $f : M \longrightarrow N$ be an R-epimorphism. Consider the commutative diagram

$$\begin{tikzcd}
M \arrow[r] \arrow[d, "\hat{\tau}_M"'] & N \arrow[d, "\hat{\tau}_N"] \\
M_\tau \arrow[r] & N_\tau
\end{tikzcd}$$

where $f_\tau = Q_\tau(f)$. Since τ is spectral, $Kerf_\tau = (Kerf)_\tau$ is injective and so $Kerf_\tau$ is a direct summand of M_τ. Thus Imf_τ is also injective by Proposition 1.3.7. On the other hand $Im(\hat{\tau}_M f_\tau) = Im(f\hat{\tau}_N) = N/t_\tau(N)$ implies $Im(\hat{\tau}_M f_\tau)$ is essential in N_τ because $N/t_\tau(N)$ is τ-dense in N_τ. Therefore Imf_τ is essential in N_τ. Thus by Theorem 1.4.1, $Imf_\tau = N_\tau$.

Lemma 4.1.4 [23] *Let τ be a spectral torsion theory on $R-Mod$ and $f : M \longrightarrow N$ an R-epimorphism from a τ-torsion-free injective left R-module M onto a τ-torsion-free left R-module N. Then N is injective.*

Proof: With the previous diagram and notation, we have $f\hat{\tau}_N = \hat{\tau}_M f_\tau$ with $\hat{\tau}_M$ an isomorphism because M is absolutely τ-pure. Since τ is spectral, f_τ is an epimorphism and since N is τ-torsion-free, $\hat{\tau}_N$ is a monomorphism. It follows that $\hat{\tau}_N$ is also an epimorphism and hence $N \cong N_\tau$ is injective. □

Proposition 4.1.5 [4] *Let τ be a torsion theory on $R - Mod$. The following conditions are equivalent:*

(i) every absolutely τ-pure left R-module is injective;

(ii) every τ-torsion-free left R-module is τ-full;

(iii) if M is absolutely τ-pure and N a submodule of M, then N is τ-pure if and only if N is a direct summand of M;

(iv) $\mathcal{P}_\tau(M)$ is complemented for every τ-torsion-free left R-module M;

(v) τ is spectral.

Proof: $(i) \Longrightarrow (ii)$ Let M be a τ-torsion-free left R-module and N be an essential submodule of M. Since $E_\tau(M)$ is injective by (i), $E_\tau(M) = E(M)$ and so M is τ-dense in $E(M)$. Furthermore, $N \leq_e M$ implies $E(N) = E(M)$ by Proposition 1.4.4. Hence $E(M)/N \in T_\tau$. It follows that $M/N \leq E(M)/N$ is τ-torsion and so N is τ-dense. Thus M is τ-full.

$(ii) \Longrightarrow (iii)$ Let M be an absolutely τ-pure left R-module. By (ii), M is τ-full. Therefore every τ-pure submodule of M is a direct summand by Proposition 3.5.10. Conversely if $M = N \oplus K$, then $M/N \cong K$. Since M is τ-torsion-free and the class of τ-torsion-free left R-modules is closed under taking submodules, K is τ-torsion-free and so N is τ-pure.

$(iii) \Longrightarrow (iv)$ Let M be a τ-torsion-free left R-module. Since M_τ is absolutely τ-pure, by (iii), $\mathcal{P}_\tau(M_\tau)$ is complemented. By Proposition 3.6.7, $\mathcal{P}_\tau(M) \cong \mathcal{P}_\tau(M_\tau)$ as lattices. Hence $\mathcal{P}_\tau(M)$ is complemented.

$(iv) \Longrightarrow (v)$ Let M be an absolutely τ-pure left R-module. $E(M)$ is τ-torsion-free since the class of τ-torsion-free left R-modules is closed under taking essential extensions. Thus by (iv), M is complemented in $E(M)$ from which we infer $E(M) = M$, i.e., M is injective.

$(v) \Longrightarrow (i)$ is Lemma 1. □

4.2 Relative essential submodules

Let $\tau \in R-tors$ and $M \in R-Mod$. A submodule N of M is said to be *τ-essential* in M if for any submodule L of M, $L \cap N \in T_\tau$ implies $L \in T_\tau$. Note that if $\tau = \xi$, then the τ-essential submodules of M are just the essential submodules.

Proposition 4.2.1 [23] *Let τ be a torsion theory on $R-Mod$ and N a submodule of a left R-module M. The following conditions are equivalent:*

(i) N is τ-essential in M;

(ii) N^c is τ-essential in M;

(iii) N^c is an essential element of the lattice $\mathcal{P}_\tau(M)$;

(iv) N_τ is an essential submodule of M_τ in $(R,\tau)-Mod$;

(v) $(N+t_\tau(M))/t_\tau(M)$ is essential in $M/t_\tau(M)$.

Proof: $(i) \Longrightarrow (ii)$ Let $L \leq M$ such that $N^c \cap L \in T_\tau$. Then $N^c \cap L \subseteq t_\tau(M)$. Therefore $(N \cap L^c)^c = (N^c \cap L)^c = t_\tau(M)$ and so $N \cap L^c \subseteq t_\tau(M)$ from which we infer $L^c \subseteq t_\tau(M)$ since N is τ-essential. Thus $L \subseteq t_\tau(M)$.

$(ii) \Longrightarrow (iii)$ By definition, N^c is τ-pure in M, i.e., $N^c \in \mathcal{P}_\tau(M)$. Let $L \leq M$ such that $N^c \cap L = 0$. By (ii), $L \subseteq t_\tau(M) \subseteq N^c$, so $L = 0$. Thus (iii) holds.

$(iii) \Longrightarrow (i)$ If $L \leq M$ is such that $N \cap L \subseteq t_\tau(M)$, then $N^c \cap L^c = (N \cap L)^c = t_\tau(M)$ and thus $L^c = t_\tau(M)$ from which we infer $L \subseteq t_\tau(M)$.

$(iii) \Longleftrightarrow (iv)$ Since $(N^c)_\tau = N_\tau$ the result follows from Proposition 3.6.7.

$(iv) \Longleftrightarrow (v)$ follows from the embedding $M/t_\tau(M) \hookrightarrow M_\tau$. □

Note that if $N \leq M$ such that $t_\tau(M) \subseteq N$ then N is essential in M whenever N is τ-essential in M. But the converse is not true since, for example, $t_\tau(M)$ may be essential in M but is never τ-essential in M unless $M = t_\tau(M)$.

For any $M \in R - Mod$, put

$$\mathcal{E}_\tau(M) = \{N \leq M \mid N \text{ is } \tau\text{-essential in } M\}.$$

Denote by $\mathcal{E}_\tau$ the set of all τ-essential left ideals of R. Observe that $\mathcal{E}_\xi = \mathcal{E}$ (see Example 3.3.4). Note also that $\mathcal{E}_\tau(M)$ contains every τ-dense submodule of M.

Indeed, if $L \leq M$ is τ-dense, then $L^c = M$ which is trivially τ-essential in M, so $L \in \mathcal{E}_\tau(M)$ by Proposition 1. This implies that $t_\tau(M) \subseteq \mathcal{E}_\tau(M)$.

Proposition 4.2.2 [23, Proposition 2.4] *Let τ be a torsion theory on $R - Mod$. The family $\mathcal{E}_\tau$ of τ-essential left ideals of R is a left topologizing filter on R.* $\square$

In general, $\mathcal{E}_\tau$ is not a Gabriel topology. Let Z_τ be the torsion preradical on $R - Mod$ defined by $Z_\tau(M) = \{m \in M \mid (0 : m) \in \mathcal{E}_\tau\}$ for every $M \in R - Mod$ and $\bar{Z}_\tau$ the smallest torsion radical larger than Z_τ.

Lemma 4.2.3 [23] *Let N be a τ-essential submodule of a left R-module M. Then M/N is Z_τ-torsion.*

Proof: It suffices to prove that if $f \in Hom_R(A, B)$ then $[L]f^{-1}$ is τ-essential in A whenever L is a τ-essential submodule of B. The argument we use is similar to that used in "essential" case (see [24, Proposition 1.20]). Suppose L is τ-essential in B. By Proposition 1, L^c is an essential element of $\mathcal{P}_\tau(B)$. It suffices, in view of Proposition 1, to show that $[L^c]f^{-1} = ([L]f^{-1})^c$ is an essential element of $\mathcal{P}_\tau(A)$. Let $K \in \mathcal{P}_\tau(A)$ be such that $K \cap [L^c]f^{-1} = t_\tau(A)$. Thus $(K \cap [L^c]f^{-1})f \subseteq [K]f \cap L^c \subseteq [t_\tau(A)]f \subseteq t_\tau(B)$ and so $([K]f)^c \cap L^c = ([K]f \cap L)^c = t_\tau(B)$. Therefore since L^c is an essential element in $\mathcal{P}_\tau(B)$ we have that $([K]f)^c = t_\tau(B)$. Hence $K \subseteq [t_\tau(B)]f^{-1} \subseteq [L^c]f^{-1}$ and so $K = K \cap [L^c]f^{-1} = t_\tau(A)$. $\square$

Lemma 4.2.4 [23] *Let M be a left R-module. Then:*

(i) $Z_\tau(M/t_\tau(M)) = Z_\tau(M)/t_\tau(M)$;

(ii) $Z_\tau(M) \in \mathcal{E}_\tau(\bar{Z}_\tau(M))$;

(iii) $\bar{Z}_\tau = Z_\tau : Z_\tau$.

Proof: (i) Let $\bar{m} = m + t_\tau(M) \in M/t_\tau(M)$. Then $(0 : \bar{m}) = \{r \in R \mid rm \in t_\tau(M)\}$. Since $(0 : m)^c/(0 : m) = t_\tau(R/(0 : m)) = t_\tau(Rm) = Rm \cap t_\tau(M)$, we have that $(0 : \bar{m}) = (0 : m)^c$. It follows that $\bar{m} \in Z_\tau(M/t_\tau(M))$ if and only if $m \in Z_\tau(M)$. Then the result follows.

(ii) By Lemma 3.2.3, we have $\bar{Z}_\tau(M/t_\tau(M)) = \bar{Z}_\tau(M)/t_\tau(M)$. Since $Z_\tau(M/t_\tau(M))$ is essential in $\bar{Z}_\tau(M/t_\tau(M))$ by Remark 3.2.2, it follows from (i) that $Z_\tau(M)/t_\tau(M)$ is essential in $\bar{Z}_\tau(M)/t_\tau(M)$. By Proposition 1 ($(i) \Longleftrightarrow (v)$), $Z_\tau(M)$ is τ-essential in $\bar{Z}_\tau(M)$, i.e., $Z_\tau(M) \in \mathcal{E}_\tau(\bar{Z}_\tau(M))$.

(iii) Since $Z_\tau(M)$ is τ-essential in $\bar{Z}_\tau(M)$, $\bar{Z}_\tau(M)/Z_\tau(M)$ is Z_τ-torsion. Thus $\bar{Z}_\tau(M) = (Z_\tau : Z_\tau)(M)$. □

In the following, we study the relationship between the torsion theory τ and the torsion theory corresponding with the torsion radical $\bar{Z}_\tau$ which we shall henceforth denote by $\tilde{\tau}$. Let $M \in R-Mod$ and $L \leq M$. We call $K \leq M$ a *τ-complement* of L in M if $L \cap K$ is τ-torsion and K is maximal among the submodules of M with this property.

Lemma 4.2.5 [23] *Let L be a submodule of a left R-module M. The following conditions are equivalent for a submodule K of M:*

(i) K is a τ-complement of L in M;

(ii) K is the pseudocomplement of L^c in $\mathcal{P}_\tau(M)$.

Proof: $(i) \Longrightarrow (ii)$ Let $K \leq M$ be a τ-complement of L in M. Then $K \cap L \subseteq t_\tau(M)$; hence $K^c \cap L \subseteq K^c \cap L^c = (K \cap L)^c = t_\tau(M)$. Thus by the maximality of K, we have that $K = K^c$ and so K is τ-pure. It is easily shown that K is a pseudocomplement of L^c in $\mathcal{P}_\tau(M)$.

$(ii) \implies (i)$ Let $K \in \mathcal{P}_\tau(M)$ be such that $K \cap L^c = t_\tau(M)$ and K is maximal with this property. Then if $K \subseteq K_1$ and $K_1 \cap L \subseteq t_\tau(M)$, we have that $K_1^c \cap L^c = (K_1 \cap L)^c = t_\tau(M)$. It follows that $K = K_1^c$ and so $K = K_1$. □

This lemma implies, in particular, that every τ-complement K of L in M is τ-pure, whence $L^c \cap K = t_\tau(M)$.

Let K be a τ-complement of L in M. Then by the previous lemma K is a pseudo-complement of L^c in $\mathcal{P}_\tau(M)$. Inasmuch as $(K + L^c)^c$ corresponds with the join of K and L^c in $\mathcal{P}_\tau(M)$, it follows that $(K+L^c)^c$ is an essential element of $\mathcal{P}_\tau(M)$ (see [43, Proposition III.6.4]). Now we have that $(K+L^c)^c = (K+L)^c$. Since $(K+L)^c$ is essential in $\mathcal{P}_\tau(M)$, $K + L$ is τ-essential in M by Proposition 1 ($(i) \iff (iii)$). We have thus proved:

Lemma 4.2.6 [23] *Let K be a τ-complement of L in M. Then $K + L \in \mathcal{E}_\tau(M)$.*□

Proposition 4.2.7 [23] *Let τ be a torsion theory on $R - Mod$. Then the torsion theory $\tilde{\tau}$ is spectral.*

Proof: We have to show that every absolutely $\tilde{\tau}$-pure left R-module M is injective. Let I be a left ideal of R and $f \in Hom_R(I, M)$. Let $J \leq {}_RR$ be a τ-complement of I in R. Then $I \cap J \subseteq t_\tau(I)$ and $I + J \in \mathcal{E}_\tau$ by Lemma 6. Since M is $\tilde{\tau}$-torsion-free, it is also τ-torsion-free because $\tau \leq \tilde{\tau}$ and so we have that $[I \cap J]f \subseteq [t_\tau(I)]f \subseteq t_\tau(M) = 0$. Thus if $x \in I, y \in J$, we can define $h : I + J \longrightarrow M$ by $(x+y)h = (x)f$. Clearly h is an R-homomorphism extending f to $I + J$. On the other hand, since M is $\tilde{\tau}$-injective and $I + J \in \mathcal{E}_\tau$ implies $R/(I + J)$ is Z_τ-torsion by Lemma 3, h has can be extended to ${}_RR$. Therefore by Baer's Criterion, M is injective. □

Proposition 4.2.8 [23] *Let τ be a torsion theory on $R-Mod$. Then $t_\tau = Z_\tau$, i.e., $\tau = \tilde{\tau}$ if and only if τ is spectral.*

Proof: The implication in one direction is obvious in view of Proposition 7. Conversely, it follows from Proposition 1.5 that τ is spectral if and only if, for every τ-torsion-free $M \in R-Mod$, $\mathcal{P}_\tau(M)$ is complemented. Since $\mathcal{P}_\tau(M)$ is pseudocomplemented, this happens if and only if $\mathcal{P}_\tau(M)$ has no essential elements different from M, and this is equivalent to saying that $\mathcal{E}_\tau(M) = \{N \leq M | t_\tau(M/N) = M/N\}$ by Proposition 1. Taking $M = {}_RR$, τ being spectral implies $Z_\tau = t_\tau$. $\square$

Theorem 10 gives a characterization of a spectral torsion theory τ in terms of the localization ring R_τ. Before stating this theorem we shall discuss the connection between an arbitrary spectral torsion theory τ on $R-Mod$ and the Goldie torsion theory τ_g on $R/t_\tau(R)-Mod$. We show in the next proposition that an investigation on an arbitrary spectral torsion theory on $R-Mod$ reduces to the study of the Goldie torsion theory τ_g on the category of modules over the factor ring $R/t_\tau(R)$.

Proposition 4.2.9 *Let τ be a spectral torsion theory on $R-Mod$. Let $\mathcal{S}$ and $\mathcal{N}$ denote the classes of singular and nonsingular left $R/t_\tau(R)$-modules, respectively. Then:*

(i) $R/t_\tau(R)$ is a left nonsingular ring;

(ii) $\mathcal{T}_\tau \cap R/t_\tau(R) - Mod = \mathcal{S}$ and if σ is any torsion theory on $R-Mod$ for which $t_\sigma(R) = t_\tau(R)$ and $\mathcal{T}_\sigma \cap R/t_\sigma(R) - Mod = \mathcal{S} = \mathcal{T}_\tau \cap R/t_\tau(R) - Mod$, then $\sigma \leq \tau$;

(iii) $\mathcal{F}_\tau = \mathcal{N}$.

Proof: Let τ be an arbitrary torsion theory on $R-Mod$. We interpret $R/t_\tau(R) - Mod$ as a subcategory of $R-Mod$ by identifying $R/t_\tau(R) - Mod$ with

$\{M \in R - Mod \mid t_\tau(R)M = 0\}$. Note that $t_\tau(R)M \subseteq t_\tau(M)$, so that $\mathcal{F}_\tau \subseteq R/t_\tau(R) - Mod$. Furthermore, $R/t_\tau(R) - Mod \cap \mathcal{T}_\tau$ is a hereditary torsion class of $R/t_\tau(R) - Mod$, so there is a torsion theory τ' on $R/t_\tau(R) - Mod$ such that $\mathcal{T}_{\tau'} = R/t_\tau(R) - Mod \cap \mathcal{T}_\tau$.

Let $\mathcal{F}_{\tau'}$ denote the torsion-free class associated with τ'. We claim that $\mathcal{F}_{\tau'} = \mathcal{F}_\tau$. Obviously $\mathcal{F}_\tau \subseteq \mathcal{F}_{\tau'}$ since $\mathcal{T}_{\tau'} \subseteq \mathcal{T}_\tau$. Suppose the inclusion is strict and let $N \in \mathcal{F}_{\tau'} \setminus \mathcal{F}_\tau$. There then exist $M \in \mathcal{T}_\tau$ and a nonzero $R/t_\tau(R)$-homomorphism $\alpha : M \longrightarrow E(N)$. But $\mathcal{T}_\tau = \mathcal{T}_{\tau'}$ in $R/t_\tau(R) - Mod$. Thus α must be zero because $N \in \mathcal{F}_{\tau'}$, a contradiction. Our claim is thus established.

Now suppose τ is a spectral torsion theory on $R - Mod$.

(*i*) Since τ is spectral, $Z_\tau(-) = t_\tau(-)$ by Proposition 8. It follows then by the above claim that $R/t_\tau(R)$ is left nonsingular.

(*ii*) Let $\mathcal{F}$ be the left Gabriel topology associated with τ. It follows that $\mathcal{F} = \{K \leq {}_RR \mid K \text{is } \tau\text{-essential in } {}_RR\} = \{K \leq {}_RR \mid [K + t_\tau(R)]/t_\tau(R) \text{ is essential in } R/t_\tau(R)\}$. Let $\mathcal{L} = \{K \leq {}_RR \mid K \supseteq t_\tau(R)\}$ be the left topologizing filter which corresponds with $R/t_\tau(R) - Mod$. Observe that $\mathcal{F} \cap \mathcal{L} = \{K \leq {}_RR \mid K \supseteq t_\tau(R) \text{ and } K/t_\tau(R) \leq_e R/t_\tau(R)\}$. It follows then that $\mathcal{T}_\tau \cap R/t_\tau(R) - Mod = \mathcal{T}_{\tau'}$ is precisely the class of all singular left $R/t_\tau(R)$-modules. Furthermore, it can be checked that $\mathcal{F}$ is the unique largest left topologizing filter on R for which $\mathcal{F} \cap \mathcal{L} = \{K \leq {}_RR \mid K \supseteq t_\tau(R) \text{ and } K/t_\tau(R) \leq_e R/t_\tau(R)\}$. Thus $\mathcal{T}_\tau$ is the unique largest hereditary pretorsion class for which $\mathcal{T}_\tau \cap R/t_\tau(R) - Mod$ coincides with the class of all singular left $R/t_\tau(R)$-modules.

(*iii*) follows from the claim made in the paragraph preceding the proof of (*i*) and statement (*ii*). □

It follows from Proposition 2.2.5 that since $R/t_\tau(R)$ is left nonsingular, $\mathcal{S}$ is cogenerated by $E(R/t_\tau(R))$, or equivalently, $\mathcal{N}$ is the smallest torsion-free class of $R/t_\tau(R) - Mod$ containing $R/t_\tau(R)$. But since $\mathcal{F}_\tau = \mathcal{N}$, $\mathcal{N}$ is also the smallest torsion-free class of $R - Mod$ which contains $R/t_\tau(R)$. We conclude that $\tau = \chi(R/t_\tau(R))$.

A torsion theory τ on $R - Mod$ which is cogenerated by $E(R/t_\tau(R))$ is said to be *saturated.*

Theorem 4.2.10 [23] *Let τ be a torsion theory on $R - Mod$. The following conditions are equivalent:*

(i) R_τ is a regular left self-injective ring and τ is the largest torsion theory σ on $R - Mod$ such that $R_\sigma = R_\tau$;

(ii) R_τ is left nonsingular, ${}_RR_\tau$ is injective and τ is cogenerated by ${}_RR_\tau$;

(iii) R_τ is left nonsingular and τ is saturated;

(iv) $R/t_\tau(R)$ is left nonsingular and τ is saturated;

(v) τ is spectral.

Proof: $(i) \Longrightarrow (ii)$ It is known that every regular ring is nonsingular. Since R_τ is left self-injective, it is also injective as a left R-module by Proposition 3.6.4, so ${}_RR_\tau = E({}_RR_\tau)$. Let σ be the torsion theory on $R - Mod$ cogenerated by ${}_RR_\tau$. Then $R/t_\tau(R)$ is σ-torsion-free and $t_\sigma(R) \subseteq t_\tau(R)$. On the other hand since σ is the largest torsion theory relative to which ${}_RR_\tau$ is torsion-free, it follows that $\tau \leq \sigma$ and so $t_\sigma(R) = t_\tau(R)$, i.e., $R/t_\sigma(R) = R/t_\tau(R)$. Since $R/t_\tau(R)$ is τ-dense in $E(R/t_\tau(R))$ we must have that $R/t_\sigma(R)$ is σ-dense in $E(R/t_\tau(R))$, so $R_\sigma = E(R/t_\tau(R)) = R_\tau$. By (i), $\tau \geq \sigma$, whence $\tau = \sigma$.

$(ii) \Longrightarrow (iii)$ Since ${}_RR_\tau$ is injective, ${}_RR_\tau = E(R/t_\tau(R))$ and by (ii), τ is cogenerated by $E(R/\tau(R))$, i.e., τ is saturated.

$(iii) \Longrightarrow (iv)$ Since $R/t_\tau(R) \leq_e {}_RR_\tau$, the left singularity of R_τ implies that $R/t_\tau(R)$ is left nonsingular.

$(iv) \Longrightarrow (v)$ Put $\bar{R} = R/t_\tau(R)$ and let $Z_{\bar{R}}$ denote the singular torsion preradical on $\bar{R} - Mod$. Since $\bar{R}$ is left nonsingular it follows from Proposition 2.2.5 that $Z_{\bar{R}}$ is the torsion radical which corresponds with the Lambek torsion theory $\chi({}_{\bar{R}}\bar{R})$ on $\bar{R} - Mod$. If $Z_{\bar{R}}$ is interpreted as a preradical on $R - Mod$, then $Z_{\bar{R}}$ coincides with Z_τ (because, by Proposition 1, K is a Z_τ-dense left ideal of R if and only if $[K + t_\tau(R)]/t_\tau(R)$ is an essential left ideal of $\bar{R}$). If $\chi({}_{\bar{R}}\bar{R})$ is interpreted as a torsion theory on $R - Mod$ then it coincides with $\chi(R/t_\tau(R))$. But τ is saturated by (iv), so $\tau = \chi(R/t_\tau(R))$. We conclude that $\tau = Z_\tau$, so τ is spectral by Proposition 8.

$(v) \Longrightarrow (i)$ R_τ is regular and left self-injective since the endomorphism ring of every object in a spectral Grothendieck category is regular and self-injective. Since τ is spectral, $\tau = \tilde{\tau}$ and $R/t_\tau(R)$ is left nonsingular. Using the preceding notation it follows from Proposition 2.2.5 that $\tilde{\tau} = \chi({}_{\bar{R}}\bar{R})$, i.e., τ is saturated and so cogenerated by $E(R/t_\tau(R))$. If σ is a torsion theory such that $R_\sigma = R_\tau$, then $E(R/t_\tau(R))$ is σ-torsion-free and $\sigma \leq \tau$. □

The following example shows that if one of the conditions given in (i) of the preceding theorem fails, τ may not be spectral.

Example 4.2.11 Let p be a prime integer and $\mathbb{Z}_{(p)} = \{\frac{m}{n} \in \mathbb{Q} \mid (n,p) = 1\}$ be the localization of $\mathbb{Z}$ at the prime ideal $p\mathbb{Z}$. Consider $\mathbb{Z}_{p^\infty} = \mathbb{Q}_p/\mathbb{Z}$ where $\mathbb{Q}_p$ is the subgroup of $\mathbb{Q}$ generated by $\{\frac{1}{p^n} \mid n \in \mathbb{Z}\}$. It is known that $\mathbb{Z}_{p^\infty}$ is the injective hull of $\mathbb{Z}_{(p)}$ as $\mathbb{Z}$-modules. Put $R = \mathbb{Z}_{(p)} \oplus \mathbb{Z}_{p^\infty}$ and define a multiplication operation on

R by $(\lambda, x).(\mu, y) = (\lambda\mu, \lambda y + \mu x)$. Then R is an injective cogenerator in $R - Mod$ [13, page 214]. Hence if $\tau = \xi$ then $R \cong R_\tau$ is regular and left self-injective. Nevertheless τ is not spectral since ξ is spectral if and only if R is semisimple as we shall see in Chapter 6 (Proposition 6.1.3).

On the other hand, if R is a regular non left self-injective ring and τ denotes the torsion theory on $R - Mod$ which is cogenerated by $E(R)$ and $E(E(R)/R)$ (this is the so-called "canonical" torsion theory [43, page 205]), then ${}_RR$ is absolutely τ-pure but not injective. Therefore τ cannot be spectral by Proposition 1.5. This shows that it is not possible to do without the self-injectivity of R in the previous theorem.

Corollary 4.2.12 [23] *There is a bijective correspondence between spectral torsion theories on $R - Mod$ and left rings of quotients of R that are regular and left self-injective.* ☐

As noted in the sequel to Proposition 1.3, every $\tau \in gen(\tau_g)$ is spectral. But, as the following examples show, a torsion theory τ can be spectral without being larger than τ_g.

Example 4.2.13 Let R be a commutative ring with an essential prime ideal P such that R_P, the localization of R at P, is a field. For instance, let F be a field and $F_n = F$ for $n = 1, 2, 3, \ldots$. Consider $P = \bigoplus F_n$. Denote by R the F-subalgebra of $\prod F_n$ generated by 1 and P. It is easily checked that R is a commutative regular ring [25, pages 4 & 5] and P is an essential prime ideal of R. The family $\mathcal{F}_P = \{I \trianglelefteq R \mid I \nsubseteq P\}$ is a Gabriel topology on R (see Example 3.6.8). Since R_P is a field, it is semisimple and self-injective and so the torsion theory, say τ, determined by $\mathcal{F}_P$ is spectral by Proposition 6.1.3. Clearly P belongs to the Goldie topology on R (Example 3.3.4) but not to $\mathcal{F}_P$. Thus τ is spectral without being greater than τ_g.

Example 4.2.14 A second illustration is given by taking

$$R = \begin{pmatrix} K & K \\ 0 & K \end{pmatrix} \text{ where } K \text{ is a field.}$$

Let

$$I = \begin{pmatrix} 0 & K \\ 0 & K \end{pmatrix} \text{ and } \mathcal{F} = \{I, R\}.$$

If τ denotes the torsion theory corresponding with $\mathcal{F}$, it is easily checked that

$$t_\tau(R) = \begin{pmatrix} K & K \\ 0 & 0 \end{pmatrix}.$$

Inasmuch as

$${}_RR = \begin{pmatrix} K & K \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 0 & K \end{pmatrix}$$

we have that

$$E_\tau(R/t_\tau(R)) = \begin{pmatrix} 0 & 0 \\ 0 & K \end{pmatrix}.$$

We thus have $R_\tau \cong K$ and, again by Proposition 6.1.3, τ is spectral.

Chapter 5

Simple Modules and Spectral Torsion Theories

In this chapter, we show how the notion of a semiartinian ring may be characterized by means of spectral torsion theories. We start by defining the "pseudocomplement" of a torsion theory and constructing a special torsion radical, namely $Soc_C(-)$.

5.1 The lattice $R - tors$

Let $R - tors$ denote the set of all hereditary torsion theories on $R - Mod$. One can show that $R - tors$ is a subset of the "big" lattice of preradicals on $R - Mod$, which is closed under meets ($\wedge$) but not generally closed under joins ($\vee$); so $R - tors$ is not a sublattice. Nevertheless a lattice structure on $R - tors$ can be defined by introducing a new join (least upper bound). For a nonempty subset T of $R - tors$ define $\bigvee T = \bigwedge \{\sigma \mid \sigma \geq \tau \text{ for all } \tau \in T\}$. By convention put $\bigwedge \emptyset = \chi$ and $\bigvee \emptyset = \xi$.

In this way $R - tors$ is a complete lattice. Moreover, the meets distribute over infinite joins, i.e., $\tau \wedge (\bigvee T) = \bigvee \{\tau \wedge \sigma \mid \sigma \in T\}$ for every element τ of $R - tors$ and subset T of $R - tors$ [19, Proposition 29.1]. A lattice with this property is called a *frame* or a *Brouwerian* lattice. The following Lemma provides us with some useful results.

Lemma 5.1.1 [45] *Let T be a subset of $R - tors$. Then:*
(i) $\bigvee T = \xi(\bigcup_{\tau \in T} \mathcal{T}_\tau)$;
(ii) $\bigwedge T = \chi(\bigcup_{\tau \in T} \mathcal{F}_\tau)$;
(iii) $\mathcal{T}_{\bigwedge T} = \bigcap_{\tau \in T} \mathcal{T}_\tau$;
(iv) $\mathcal{F}_{\bigvee T} = \bigcap_{\tau \in T} \mathcal{F}_\tau$.

Proof: (i) Observe that if $\sigma \in R - tors$, then $\sigma \geq \tau$ for all $\tau \in T$ if and only if $\mathcal{T}_\tau \supseteq \bigcup_{\tau \in T} \mathcal{T}_\tau$. By Proposition 3.4.11 (i), $\sigma = \xi(\bigcup_{\tau \in T} \mathcal{T}_\tau)$ is the smallest torsion theory for which $\mathcal{T}_\sigma \supseteq \bigcup_{\tau \in T} \mathcal{T}_\tau$ or, equivalently, for which $\sigma \leq \tau$ for all $\tau \in T$. But by definition, this torsion theory is $\bigvee T$, and so $\xi(\bigcup_{\tau \in T} \mathcal{T}_\tau) = \bigvee T$.

(ii) If $\sigma, \tau \in R - tors$, then $\sigma \leq \tau$ if and only if $\mathcal{F}_\sigma \supseteq \mathcal{F}_\tau$. It follows then that $\sigma \leq \tau$ for all $\tau \in T$ if and only if $\mathcal{F}_\sigma \supseteq \bigcup_{\tau \in T} \mathcal{F}_\tau$. Referring on Proposition 3.4.11 (ii), $\sigma = \chi(\bigcup_{\tau \in T} \mathcal{F}_\tau)$ is the largest torsion theory for which $\mathcal{F}_\sigma \supseteq \bigcup_{\tau \in T} \mathcal{F}_\tau$ or, equivalently, for which $\sigma \leq \tau$ for all $\tau \in T$. Hence $\chi(\bigcup_{\tau \in T} \mathcal{F}_\tau) = \bigwedge T$.

(iii) Since $\bigwedge T \leq \tau$ for all $\tau \in T$, we have that $\mathcal{T}_{\bigwedge T} \subseteq \bigcap_{\tau \in T} \mathcal{T}_\tau$. Conversely, suppose that $M \in \bigcap_{\tau \in T} \mathcal{T}_\tau$. Then $Hom_R(M, E(N)) = 0$ for all $N \in \bigcup_{\tau \in T} \mathcal{F}_\tau$, and so M is $\chi(\bigcup_{\tau \in T} \mathcal{F}_\tau)$-torsion. By (ii), $M \in \mathcal{T}_{\bigwedge T}$. Thus $\mathcal{T}_{\bigwedge T} = \bigcap_{\tau \in T} \mathcal{T}_\tau$.

(iv) Since $\bigvee T \leq \tau$ for all $\tau \in T$, it follows that $\mathcal{F}_{\bigvee T} \subseteq \bigcap_{\tau \in T} \mathcal{F}_\tau$. Conversely, if $N \in \bigcap_{\tau \in T} \mathcal{F}_\tau$, then $Hom_R(M, E(N)) = 0$ for all $M \in \bigcup_{\tau \in T} \mathcal{T}_\tau$ and so N is

$\xi(\bigcup_{\tau\in T}\mathcal{T}_\tau)$-torsion-free. By (i), $N \in \mathcal{F}_{\bigvee T}$. Thus $\mathcal{F}_{\bigvee T} = \bigcap_{\tau\in T}\mathcal{F}_\tau$. ☐

Let τ be an element of $R - tors$. Then the set $T = \{\sigma \in R - tors \mid \tau\wedge\sigma = \xi\}$ is nonempty since $\xi \in T$. Since $R - tors$ is Brouwerian, we have that $\tau \wedge (\bigvee T) = \bigvee\{\tau\wedge\sigma \mid \sigma \in T\} = \xi$ and so T has a unique maximal element, namely $\bigvee T$. This element, which is a *unique* pseudocomplement of τ, is denoted by $\tau^\perp$. Note that a left R-module M is $\tau^\perp$-torsion if and only if its every homomorphic image is τ-torsion-free. In particular this implies that every $\tau^\perp$-torsion left R-module has no proper τ-dense submodule [19, page 280]. It is not, in general, true that $\tau^{\perp\perp} = \tau$ but the map $\tau \longmapsto \tau^{\perp\perp}$ is a closure operator on $R - tors$ which preserves finite meets, i.e., $(\sigma\wedge\tau)^{\perp\perp} = \sigma^{\perp\perp}\wedge\tau^{\perp\perp}$ for all $\sigma, \tau \in R - tors$.

Lemma 5.1.2 [19] *Let τ be a stable torsion theory on $R-Mod$. Then the following statements are equivalent for a left R-module M:*

(i) M is $\tau^\perp$-torsion-free;

(ii) M can be embedded in a direct product of τ-torsion left R-modules.

Proof: $(i) \Longrightarrow (ii)$ It is clear from the definition that $\tau^\perp$ is the largest torsion theory for which every member of $\mathcal{T}_\tau$ is $\tau^\perp$-torsion-free. Hence $\tau^\perp = \chi(\mathcal{T}_\tau)$. By (i), $0 = \tau^\perp(M) = \bigcap\{Ker\,\alpha \mid \alpha \in Hom_R(M, E(N))$ and $N \in \mathcal{T}_\tau\}$. Since τ is stable, $E(N)$ is τ-torsion whenever $N \in \mathcal{T}_\tau$, so $Ker\,\alpha$ is τ-dense in M for $\alpha \in Hom_R(M, E(N))$. It follows that the intersection of all τ-dense submodules of M is zero, whence (ii). $(ii) \Longrightarrow (i)$ Let M be such that $M \hookrightarrow \prod T_i$ where $T_i \in \mathcal{T}_\tau$. Then for a nonzero submodule N of M, M/N is τ-torsion. Whence M is $\tau^\perp$-torsion-free. ☐

Denote by $R - Simp$ a complete set of representatives of the isomorphism classes of simple left R-modules.

Proposition 5.1.3 [19] *Let τ be a torsion theory on $R-Mod$. Then* $\tau^{\perp} = \chi(\{S \mid S \in R-Simp \cap \mathcal{T}_{\tau}\})$.

Proof: Put $\mathcal{T} = R-Simp \cap \mathcal{T}_{\tau}$. Since $\xi(\mathcal{T}) \leq \tau$, $\xi(\mathcal{T}) \wedge \tau^{\perp} \leq \tau \wedge \tau^{\perp} = \xi$. Therefore $\xi(\mathcal{T}) \wedge \tau^{\perp} = \xi$ and so $\xi(\mathcal{T}) \nleq \tau^{\perp}$. This implies that every $S \in \mathcal{T}$ is not $\tau^{\perp}$-torsion, so every $S \in \mathcal{T}$ is $\tau^{\perp}$-torsion-free. It follows that $\chi(\mathcal{T}) \geq \tau^{\perp}$. Now let $\tau' = \chi(R-Simp \cap F_{\tau})$. Clearly $\tau \leq \tau'$. Moreover, $\tau \wedge \chi(\mathcal{T}) \leq \tau' \wedge \chi(\mathcal{T}) = \chi(R-Simp) = \xi$. Whence $\chi(\mathcal{T}) \leq \tau^{\perp}$. Thus $\tau^{\perp} = \chi(\{S \mid S \in R-Simp \cap \mathcal{T}_{\tau}\})$.□

5.2 The torsion radical $Soc_{\mathcal{C}}(-)$

In the sequel $\tau^{\perp}$ denotes the pseudocomplement of a torsion theory τ in the frame $R-tors$.

Proposition 5.2.1 [4] *Let τ be a spectral torsion theory on $R-Mod$ and $M \in R-Mod$. If $M \in \mathcal{T}_{\tau^{\perp}}$, then M is a semisimple module.*

Proof: Let $N \leq_e M \in \mathcal{T}_{\tau^{\perp}}$. Since τ is spectral, it follows from Proposition 4.1.5 that M is τ-full, hence $M/N \in \mathcal{T}_{\tau}$. But $M/N \in \mathcal{F}_{\tau}$ because every homomorphic image of M is τ-torsion-free. Thus $M/N = 0$ and so $M = N$. Therefore M has no proper essential submodule, i.e., M is semisimple. □

Let $\mathcal{C} \subseteq R-Simp$ and $M \in R-Mod$. We set

$$Soc_{\mathcal{C}}(M) = \sum\{S \leq M \mid S \text{ is isomorphic to some element of } \mathcal{C}\}.$$

As usual, $Soc(M)$ denotes the socle of M.

Proposition 5.2.2 [4] *Let τ be a spectral torsion theory on $R-Mod$ and $\mathcal{C} = R-Simp \cap \mathcal{F}_{\tau}$. Then $\tau^{\perp} = \xi(\mathcal{C})$ and $t_{\tau^{\perp}}(-) = Soc_{\mathcal{C}}(-)$.*

Proof: Observe that if $S \in R-Simp$, then $S \in \mathcal{T}_{\tau^\perp}$ if and only if $S \in \mathcal{C}$ from which we infer $\xi(\mathcal{C}) \leq \tau^\perp$ by Proposition 3.4.11 (i). On the other hand by Proposition 1 and since τ is spectral, $\tau^\perp \leq \xi(\mathcal{C})$. Thus $\tau^\perp = \xi(\mathcal{C})$. Finally, since the $\tau^\perp$-torsion modules are semisimple, we have that $t_{\tau^\perp}(M) = Soc_{\mathcal{C}}(M)$ for every left R-module M. □

Lemma 5.2.3 [24] *The following statements are equivalent for a simple left R-module S:*

(i) S is nonsingular;

(ii) S is projective.

Proof: Note first that $I \leq_e {}_RR$ if and only if R/I is singular.

$(i) \implies (ii)$ Since S is simple $S \cong R/M$ for some maximal left ideal M of R. Furthermore, S simple nonsingular implies that S is not singular. It follows in view of the above observation that M is not essential in R and therefore there exists a left ideal K of R such that ${}_RR = K \oplus M$. By Proposition 1.3.1, S is projective.

$(ii) \implies (i)$ Suppose S is projective. Again by Proposition 1.3.1 we have ${}_RR = K \oplus M$ for some left ideal K of R. Whence M is not essential in ${}_RR$. It follows that S is not singular, so S is nonsingular since it is simple. □

Put

$$\mathcal{P} = \{S \in R - Simp \mid S \text{ is projective}\}.$$

Corollary 5.2.4 [4] *$\tau_g^\perp = \xi(\mathcal{P})$ and $t_{\tau_g^\perp}(-) = Soc_{\mathcal{P}}(-)$.*

Proof: The result holds by Proposition 2 since τ_g is spectral and by the previous lemma the simple projective left R-modules are precisely the simple nonsingular left R-modules. $\square$

Remark 5.2.5 If τ is spectral then $\mathcal{F}_\tau$ coincides with the class of all nonsingular left $R/t_\tau(R)$-modules. It follows from Lemma 3 that $\mathcal{C} = R - Simp \cap \mathcal{F}_\tau$ is precisely the class of all simple projective left $R/t_\tau(R)$-modules. We warn the reader that the members of $\mathcal{C}$ need not be projective in $R - Mod$.

In general, for an arbitrary ring R and $\mathcal{C} \subseteq R - Simp$, $Soc_{\mathcal{C}}(-)$ is a torsion preradical but not necessarily a radical. Proposition 2 shows that if τ is spectral and $\mathcal{C} = R - Simp \cap \mathcal{F}_\tau$, then $Soc_{\mathcal{C}}(-)$ is a torsion radical. The notion of a spectral torsion theory can be used to describe those subsets $\mathcal{C}$ of $R - Simp$ for which $Soc_{\mathcal{C}}(-)$ is a radical.

Proposition 5.2.6 [4] *Let τ be a spectral torsion theory on $R - Mod$ and $\mathcal{C} = R - Simp \cap \mathcal{F}_\tau$. If $0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$ is an exact sequence of τ-torsion-free left R-modules, then the sequence $0 \longrightarrow Soc_{\mathcal{C}}(N') \longrightarrow Soc_{\mathcal{C}}(N) \longrightarrow Soc_{\mathcal{C}}(N'') \longrightarrow 0$ is exact.*

Proof: As noted in Remark 5, $\mathcal{C}$ consists of simple projective left $R/t_\tau(R)$-modules, i.e., nonsingular left $R/t_\tau(R)$-modules and so $\mathcal{F}_{\tau_g^\perp}$ is closed under taking quotients. Since $Soc_{\mathcal{C}}(-) = t_{\tau_g^\perp}(-)$ on $R/t_\tau(R) - Mod$, it follows by Remark 3.4.10 that $Soc_{\mathcal{C}}(-)$ is an exact torsion radical. $\square$

Proposition 5.2.7 [4] *Let $\emptyset \neq \mathcal{C} \subseteq R - Simp$. The following conditions are equivalent:*

(i) $\chi(\mathcal{C})$ is spectral;

(ii) 1. $Soc_{\mathcal{C}}(-)$ is a torsion radical on $R - Mod$ and,

2. if $0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$ *is an exact sequence of* $\chi(\mathcal{C})$*-torsion-free modules, then the sequence* $0 \longrightarrow Soc_{\mathcal{C}}(N') \longrightarrow Soc_{\mathcal{C}}(N) \longrightarrow Soc_{\mathcal{C}}(N'') \longrightarrow 0$ *is exact.*

Proof: *(i)* $\Longrightarrow$ *(ii)* Observe that $\mathcal{C} = R - Simp \cap \mathcal{F}_{\chi(\mathcal{C})}$. Then (1) follows from Proposition 2 and (2) follows from Proposition 6.

(ii) $\Longrightarrow$ *(i)* By Proposition 4.1.5, it suffices to prove that every $\chi(\mathcal{C})$-torsion-free module is $\chi(\mathcal{C})$-full. We claim that if $N \leq_e M \in \mathcal{F}_{\chi(\mathcal{C})}$, then $Soc_{\mathcal{C}}(M/N) = 0$. By (2), $Soc_{\mathcal{C}}(M/N) \cong Soc_{\mathcal{C}}(M)/Soc_{\mathcal{C}}(N)$. But since $N \leq_e M$, $N \supseteq Soc(M) \supseteq Soc_{\mathcal{C}}(M)$ by Proposition 1.5.2 *(iv)*, so $Soc_{\mathcal{C}}(M) = Soc_{\mathcal{C}}(N)$, whence $Soc_{\mathcal{C}}(M/N) = 0$ as claimed. We now show that if $Soc_{\mathcal{C}}(L) = 0$ then $L \in \mathcal{T}_{\chi(\mathcal{C})}$. Suppose, on the contrary, that $L \notin \mathcal{T}_{\chi(\mathcal{C})}$. Then there exists $S \in \mathcal{C}$ and a nonzero R-homomorphism $f : L \longrightarrow E(S)$. If $Kerf \cap L' = 0$ with $L' \leq L$ then $L' \leq L/Kerf \cong Imf \subseteq E(S)$. Since $Soc(L) = 0$ we must have $L' = 0$. Hence $Kerf \leq_e L$. By the above argument, $Soc_{\mathcal{C}}(L/Kerf) = Soc_{\mathcal{C}}(Imf) = 0$, a contradiction. We thus have that if $N \leq_e M \in \mathcal{F}_{\chi(\mathcal{C})}$, then $M/N \in \mathcal{T}_{\chi(\mathcal{C})}$, i.e., M is $\chi(\mathcal{C})$-full, as required. □

The following result gives necessary and sufficient conditions on τ and $\mathcal{C}$ in order that $Soc_{\mathcal{C}}(-)$ be an exact torsion radical.

Proposition 5.2.8 [4] *Let* τ *be a spectral torsion theory on* $R - Mod$ *and* $\mathcal{C} = R - Simp \cap \mathcal{F}_\tau$*. Then the following conditions are equivalent:*

(i) $Soc_{\mathcal{C}}(-)$ *is an exact torsion radical;*

(ii) $\mathcal{C}$ *consists of projective modules;*

(iii) $\tau^{\perp\perp} \geqslant \tau_g$.

Proof: $(i) \Longrightarrow (ii)$ Let $S \in \mathcal{C}$ and $0 \longrightarrow J \longrightarrow R \xrightarrow{\pi} S \longrightarrow 0$ be a presentation for S. By (i), the sequence $0 \longrightarrow Soc_{\mathcal{C}}(J) \longrightarrow Soc_{\mathcal{C}}(R) \xrightarrow{\bar{\pi}} Soc_{\mathcal{C}}(S) \rightarrow 0$ is exact where $\bar{\pi}$ is a restriction map of π to $Soc_{\mathcal{C}}(R)$. Since $Soc_{\mathcal{C}}(R)$ is semisimple the sequence splits. Therefore there exists a nonzero R-homomorphism $i : S \longrightarrow Soc_{\mathcal{C}}(R)$ such that $i \circ \bar{\pi} = 1_S$. Since $\bar{\pi}$ is the restriction of π to $Soc_{\mathcal{C}}(R)$, we have that the original sequence splits and S is projective.

$(ii) \Longrightarrow (i)$ Proposition 2.2 implies that $t_{\tau^{\perp}}(-) = Soc_{\mathcal{C}}(-)$. Condition (ii) implies that $\mathcal{F}_{\tau^{\perp}}$ is closed under taking quotients. Therefore $Soc_{\mathcal{C}}(-)$ is an exact torsion radical by Remark 3.4.10.

$(ii) \Longleftrightarrow (iii)$ Note that $\tau^{\perp\perp} = \chi(\mathcal{C})$ since $\tau^{\perp} = \chi(\{S \mid S \in R - Simp \cap \mathcal{T}_{\tau}\})$ implies $\tau^{\perp\perp} = \chi(\{S \mid S \in R - Simp \cap \mathcal{F}_{\tau}\})$. By Lemma 2.3, every member of $\mathcal{C}$ is projective if and only if every member of $\mathcal{C}$ is nonsingular. The latter condition is equivalent to $\chi(\mathcal{C}) \geq \tau_g$, i.e., $\tau^{\perp\perp} \geqslant \tau_g$. $\square$

5.3 Semiartinian rings

A left R-module is said to be *semiartinian* if its every nonzero homomorphic image has a nonzero socle. For example, a $\mathbb{Z}$-module M is semiartinian if and only if M is a torsion abelian group [19, page 338]. Obviously every artinian module is semiartinian. A ring R is *left semiartinian* if R is semiartinian as a left module over itself. It can be shown that if R is left semiartinian then every left R-module is semiartinian. Moreover, R is left semiartinian if and only if every torsion theory on $R - Mod$ is generated by a class of simple modules [42, Proposition 3.12]. This result is expressed more precisely in the following two propositions.

Proposition 5.3.1 [4] *Let R be a left semiartinian ring. Then* $\xi(R-Simp\cap\mathcal{T}_\tau)=\tau=\chi(R-Simp\cap\mathcal{F}_\tau)$ *for every torsion theory τ on $R-Mod$.*

Proof: Let $\tau\in R-tors$ and $\mathcal{C}=R-Simp\cap\mathcal{F}_\tau$. Clearly $\tau\leqslant\chi(\mathcal{C})$. Assume that the inequality is strict. There then exists a nonzero left R-module N which is $\chi(\mathcal{C})$-torsion and τ-torsion-free. Since R is semiartinian, N has a simple submodule N' which is also τ-torsion-free and so is isomorphic to an element of $\mathcal{C}$. But in that case N' is not $\chi(\mathcal{C})$-torsion which is a contradiction. Therefore $\tau=\chi(\mathcal{C})$. □

Proposition 5.3.2 [4] *If* $\xi(R-Simp\cap\mathcal{T}_\tau)=\tau=\chi(R-Simp\cap\mathcal{F}_\tau)$ *for some spectral torsion theory τ on $R-Mod$, then R is left semiartinian.*

Proof: Put $\mathcal{C}=\xi(R-Simp\cap\mathcal{F}_\tau)$ and let $0\neq M\in R-Mod$. We shall prove that $Soc(M)\neq 0$. If $Soc(M)=0$, then, as shown in the proof of Proposition 2.7, we must have $M\in\mathcal{T}_{\chi(\mathcal{C})}=\mathcal{T}_\tau$. Since $\tau=\xi(R-Simp\cap\mathcal{T}_\tau)$, necessarily $Soc(M)\neq 0$. We conclude that R is left semiartinian. □

Corollary 5.3.3 [4] *Let $\mathcal{P}$ denote the class of all simple projective left R-modules. Then the following conditions are equivalent:*

(i) R is left semiartinian;

(ii) $\tau_g=\xi(\{\textit{singular simple left R-modules}\})=\chi(\mathcal{P})$.

Proof: The result is a consequence of the preceding proposition since τ_g is spectral and any singular simple module is τ_g-torsion and any projective simple module is τ_g-torsion-free by Lemma 2.3. □

Semisimple rings may be characterized in terms of spectral torsion theories as the following result shows.

Proposition 5.3.4 [4] *The following conditions are equivalent for a ring R:*

(i) every torsion theory τ on $R - Mod$ is spectral;

(ii) ξ is spectral;

(iii) R is a semisimple ring.

Proof: $(i) \iff (ii)$ Obvious by Proposition 4.1.3.

$(ii) \Longrightarrow (iii)$ Since ξ is spectral, every left R_ξ-module M is injective. But $R_\xi = R$, so every left R-module is injective. Thus R is semisimple by Proposition 1.5.1.

$(iii) \Longrightarrow (ii)$ If R is semisimple, then every short exact sequence splits in $R-Mod$, i.e., every left R-module is injective. But $R - Mod$ coincides with the class of all absolutely ξ-pure left R-modules. Thus ξ is spectral. □

Proposition 5.3.5 [4] *Let τ be a torsion theory on $R - Mod$. If τ and $\tau^\perp$ are spectral, then R is a left semiartinian ring.*

Proof: We first prove that $\tau^\perp \vee \tau^{\perp\perp} = \chi$. Obviously $\tau^\perp \vee \tau^{\perp\perp} \leq \chi$. Assume that the inequality is strict and let $M \in R - Mod$ be such that $0 \neq M \in \mathcal{F}_{\tau^\perp \vee \tau^{\perp\perp}}$. It follows that $M \in \mathcal{F}_{\tau^\perp}$ and $M \in \mathcal{F}_{\tau^{\perp\perp}}$ by Lemma 1.1 (iv). Take $\mathcal{C} = R - Simp \cap \mathcal{F}_\tau$. Since τ is spectral, $Soc_{\mathcal{C}}(M) = 0$ by Proposition 2.2. This implies, as shown in the proof of Proposition 2.7, that $M \in \mathcal{T}_{\chi(\mathcal{C})} = \mathcal{T}_{\tau^{\perp\perp}}$. But this contradicts the fact that $M \in \mathcal{F}_{\tau^{\perp\perp}}$. Thus $\tau^\perp \vee \tau^{\perp\perp} = \chi$.

Since τ and $\tau^\perp$ are spectral, $\mathcal{T}_{\tau^\perp}$ and $\mathcal{T}_{\tau^{\perp\perp}}$ consist of semisimple modules by Proposition 2.1. Thus for any nonzero $M \in R - Mod$, we have that $t_{\tau^\perp}(M) \neq 0$ or $t_{\tau^{\perp\perp}}(M) \neq 0$ because $\tau^\perp \vee \tau^{\perp\perp} = \chi$. Therefore $Soc(M) \neq 0$ and so R is a left semiartinian ring. □

In general, the torsion class $\mathcal{T}_\tau$ of a torsion theory τ on $R-Mod$ need not be closed under direct products. If this holds, τ is said to be *jansian* or *torsion-torsion-free (TTF)*. If Φ_τ is the Gabriel topology corresponding with a torsion theory τ on $R-Mod$, then it can be shown that τ is jansian if and only if Φ_τ is closed under arbitrary intersections or, equivalently, if Φ_τ contains a unique smallest member. For example, the torsion theory $\chi(\mathcal{P})$ where $\mathcal{P}$ denotes the class of all projective simple left R-modules, is jansian. For let $M \in R-Mod$. Certainly if $M \in \mathcal{T}_{\chi(\mathcal{P})}$ then $Soc_{\mathcal{C}}(M) = 0$ while the converse holds as shown in the proof of Proposition 2.7. It follows that $M \in \mathcal{T}_{\chi(\mathcal{P})}$ if and only if $Soc_{\mathcal{P}}(M) = 0$. Therefore if $\{M_i \mid i \in I\}$ is a family of $\chi(\mathcal{P})$-torsion left R-modules, then $[Soc_{\mathcal{P}}(\prod M_i)]\pi_i \subseteq Soc_{\mathcal{P}}(M_i)$ where π_i is the canonical projection $\prod M_i \to M_i$. But for each $i \in I$, $Soc_{\mathcal{P}}(M_i) = 0$, so $Soc_{\mathcal{P}}(\prod M_i) = 0$, i.e., $\prod M_i \in \mathcal{T}_{\chi(\mathcal{P})}$. Thus $\mathcal{T}_{\chi(\mathcal{P})}$ is closed under direct products. Denote by $\Phi_{\chi(\mathcal{P})}$ the Gabriel topology associated with $\chi(\mathcal{P})$. Then $\Phi_{\chi(\mathcal{P})} = \{I \leq {}_RR \mid t_{\chi(\mathcal{P})}(R/I) = R/I\} = \{I \leq {}_RR \mid Soc_{\mathcal{P}}(R/I) = 0\}$. This implies that the unique smallest member of $\Phi_{\chi(\mathcal{P})}$ is $\bigcap\{I \leq {}_RR \mid Soc_{\mathcal{P}}(R/I) = 0\} = Soc_{\mathcal{P}}({}_RR)$.

Proposition 5.3.6 [4] *Let τ_{sp} be the torsion theory on $R-Mod$ whose torsion class consists of the projective semisimple left R-modules. The following conditions are equivalent:*

(i) τ_{sp} is spectral;

(ii) R is left semiartinian and $R/Soc_{\mathcal{P}}(R)$ is a semisimple ring;

(iii) the torsion theory τ_g is jansian and $\mathcal{T}_{\tau_g}$ consists of semisimple modules;

(iv) the torsion theory τ_g is jansian and $\mathcal{T}_{\tau_g}$ consists of injective modules.

Proof: *(i)* $\Longrightarrow$ *(ii)* By Corollary 2.4, ${\tau_g}^{\perp} = \tau_{sp}$. Since τ_g and ${\tau_g}^{\perp}$ are spectral, R is left semiartinian by Proposition 5. It follows from Proposition 1 that $\tau_g =$

$\chi(R - Simp \cap \mathcal{F}_{\tau_g}) = \chi(\mathcal{P})$. Let Φ_{τ_g} denote the Gabriel topology associated with τ_g. Since $\tau_g = \chi(\mathcal{P})$, as noted above, Φ_{τ_g} has a minimal member $Soc_{\mathcal{P}}(R)$. Then $R/Soc_{\mathcal{P}}(R)$ is τ_g-torsion and thus it is $\tau_{sp}^{\perp}$-torsion because $\tau_{sp}^{\perp} = \tau_g^{\perp\perp} \geq \tau_g$. Therefore by Proposition 2.1, $R/Soc_{\mathcal{P}}(R)$ is a semisimple ring.

(ii) $\Longrightarrow$ *(iii)* Since R is left semiartinian, the torsion theory τ_g is jansian because $\tau_g = \chi(\mathcal{P})$. Since $Soc_{\mathcal{P}}(R)M \subsetneq Soc_{\mathcal{P}}(M)$, it follows that if $0 \neq M \in \mathcal{T}_{\tau_g}$, then $Soc_{\mathcal{P}}(R)M = 0$ and therefore M is semisimple as a left $R/Soc_{\mathcal{P}}(R)$-module. Hence M is semisimple as a left R-module.

(iii) $\Longrightarrow$ *(iv)* Let $M \in \mathcal{T}_{\tau_g}$. Since τ_g is a stable torsion theory, $E(M) \in \mathcal{T}_{\tau_g}$ and by *(iii)*, $E(M)$ is semisimple. Thus $M = E(M)$ because M is essential in $E(M)$, so M is injective.

(iv) $\Longrightarrow$ *(i)* It suffices to prove that each τ_{sp}-torsion-free module is injective. To this end let $M \in \mathcal{F}_{\tau_{sp}}$. Then since τ_g is stable $M \in \mathcal{F}_{\tau_g^{\perp}} = \{N \in R-Mod \mid N \hookrightarrow \prod T_\alpha;\ T_\alpha \in \mathcal{T}_{\tau_g}\}$ by Lemma 1.2. By *(iv)*, $M \in \mathcal{T}_{\tau_g}$ and hence M is injective . $\square$

Corollary 5.3.7 [4] *If the torsion theory τ_{sp} is spectral, then for every essential left ideal I of R, the module R/I is semisimple and injective.* $\square$

Chapter 6

Rings of Quotients and Spectral Torsion Theories

This chapter is concerned with the question of determining when the ring of quotients R_τ is respectively semisimple, isomorphic to a direct product of left full linear rings and flat as a right R-module.

6.1 Semisimple rings of quotients

A set S of left ideals of R is said to be *cofinally finite* (resp., *cofinally principal*) if given any $I \in S$ there exists a finitely generated (resp., principal) left ideal $J \subseteq I$ with $J \in S$.

Proposition 6.1.1 [43] *Let τ be a torsion theory on $R - Mod$. The following statements are equivalent for a left R-module M:*

(i) the lattice $\mathcal{P}_\tau(M)$ of τ-pure submodules of M is compact;

(ii) M_τ is a finitely generated object in the category $(R, \tau) - Mod$, i.e., the lattice

of subobjects of M_τ in $(R,\tau)-Mod$ is compact;

(iii) for every τ-dense $N \leq M$ there exists a finitely generated τ-dense $L \leq M$ such that $L \subseteq N$.

Proof: $(i) \iff (ii)$ Denote by $L(M_\tau)$ the lattice of subobjects of M_τ in $(R,\tau)-Mod$. Since M_τ is a τ-torsion-free τ-injective left R-module, a submodule of M_τ is τ-pure if and only if it is τ-injective by Proposition 3.6.6. There then exists a lattice isomorphism between $L(M_\tau)$ and $\mathcal{P}_\tau(M_\tau)$. It follows that $L(M_\tau)$ is isomorphic to $\mathcal{P}_\tau(M)$. Thus the equivalence holds.

$(i) \implies (iii)$ Let $N \leq M$ be such that $t_\tau(M/N) = M/N$. Then $N^c = M$. Since $\mathcal{P}_\tau(M)$ is compact by (i), given a family $\{L'_i \mid i \in I\}$ of τ-pure submodules of M, if $M = \bigvee L'_i = (\sum L'_i)^c$ then $M = (\sum_{i\in J} L'_i)^c = (L')^c$ for some finite subset J of I. This implies that L' is finitely generated and τ-dense in M. Since the class of τ-dense submodules is closed under intersections we have that $L = L' \cap N$ is τ-dense in M, so L satisfies the required conditions.

$(iii) \implies (i)$ Let $\{M_i \mid i \in I\}$ be a family of τ-pure submodules of M such that $M = \bigvee M_i = (\sum M_i)^c$. Then $\sum M_i$ is τ-dense; so by (iii) there exists a finitely generated τ-dense $L \leq M$ such that $L \subseteq \sum M_i$. It follows that $L \subseteq M_i$ for some i and then $M = L^c \subseteq {M_i}^c = M_i$ from which we infer $M = M_i$. We conclude that $\mathcal{P}_\tau(M)$ is compact. □

Remark 6.1.2 Put $\Phi_\tau = \{I \leq {}_RR \mid t_\tau(R/I) = R/I\}$, the Gabriel topology associated with the torsion theory τ. If $M = {}_RR$ in the above proposition, statement (iii) is equivalent to Φ_τ being cofinally finite.

The following result is due to Zelmanowitz [50].

Proposition 6.1.3 [50] *Let τ be a torsion theory on $R-Mod$. The following conditions are equivalent:*

(i) R_τ is a left semisimple ring;

(ii) R satisfies the ascending chain condition (a.c.c) on τ-pure left ideals and τ is spectral;

(iii) Φ_τ is cofinally finite and τ is spectral.

Proof: *(i)* $\Longrightarrow$ *(ii)* Since R_τ is left artinian, it is left noetherian [34, page 59]. Then every finitely generated left R_τ-module satisfies the a.c.c on submodules. In particular, R_τ satisfies the a.c.c as a left R_τ-module. Since every τ-pure submodule of R_τ is an R_τ-submodule of R_τ, it follows that the lattice $\mathcal{P}_\tau(R_\tau)$ satisfies the a.c.c. Therefore $\mathcal{P}_\tau(R)$ satisfies the a.c.c by Proposition 3.6.7. Moreover, R_τ being semisimple implies that every left R_τ-module is injective by Proposition 1.5.1. Then by Proposition 3.6.4, every left R_τ-module which is τ-torsion-free as a left R-module is injective. Thus τ is spectral.

(ii) $\Longrightarrow$ *(iii)* A consequence of the previous proposition and the fact that every complete lattice which satisfies the a.c.c is compact.

(iii) $\Longrightarrow$ *(i)* Since τ is spectral, $\mathcal{P}_\tau(R_\tau)$ is complemented by Proposition 4.1.5. By Proposition 1, $\mathcal{P}_\tau(R_\tau)$ is also compact. It follows that $End_R R_\tau$ is a semisimple ring. But R_τ and $End_R R_\tau$ are isomorphic as rings, so R_τ is semisimple. □

Corollary 6.1.4 *Let τ be a spectral torsion theory on $R-Mod$. Then the following conditions are equivalent:*

(i) R_τ is a semisimple ring;

(ii) $\mathcal{P}_\tau(R)$ satisfies the ascending chain condition;

(iii) Φ_τ *is cofinally finite.* □

A left R-module M is said to be *finite dimensional* if M contains no infinite direct sum of nonzero submodules. A ring R is *left finite dimensional* if R is finite dimensional as a left module over itself.

Proposition 6.1.5 [50] *Let* R *be a left finite dimensional ring and* τ *be a spectral torsion theory on* $R-Mod$. *Then* R_τ *is a semisimple ring and* $Q_{max}(R/t_\tau(R)) = R_\tau$.

Proof: First observe that if ${}_RR$ is finite dimensional, then R satisfies the a.c.c on τ-pure left ideals. Let $I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots$ be an ascending chain of τ-pure left ideals of R. Choose L_i a pseudocomplement of I_i in I_{i+1}. Note that $L_i = L_{i+1}$ if and only if $I_i = I_{i+1}$. By definition $L_i \cap I_i = 0$ and $L_i \subseteq I_k$ for $k > i$. Thus for $n \in \mathbb{N}$ we have that $L_{n+1} \cap (L_1 + L_2 + \cdots + L_n) \subseteq L_{n+1} \cap I_{n+1} = 0$. Therefore the sum $\sum_{i\in\mathbb{N}} L_i$ is a direct sum (see page 3). Since ${}_RR$ is finite dimensional there exists $k \in \mathbb{N}$ such that $L_k = L_{k+1} = \cdots = 0$. Whence $I_k = I_{k+1} = \dots$, i.e., the ascending chain of τ-pure left ideals of R terminates. It follows from Corollary 4 that R_τ is semisimple. Since τ is spectral, τ corresponds with the Goldie torsion theory τ_g on the left nonsingular ring $R/t_\tau(R)$ and the ring of quotients with respect to τ_g for a left nonsingular ring is just the maximal ring of quotients, i.e., $R_\tau = Q_{\max}(R/t_\tau(R))$. □

The next result provides conditions on a torsion theory τ in order that R_τ corresponds with the classical ring of quotients of R with respect to some left denominator subset of R. Let S be a left denominator subset of a ring R (see Section 1.7). It is easily shown as in the commutative case (Example 3.6.8) that

$\mathcal{F} = \{I \leq {}_RR \mid I \cap S \neq \emptyset\}$ is a left Gabriel topology on R and $S^{-1}R$ coincides with the localization of R at the torsion theory corresponding with $\mathcal{F}$ [43, Proposition XI.6.4]. Note that $\mathcal{F}$ is cofinally principal since for $I \in \mathcal{F}$, we have a τ-dense principal left ideal Rs contained in I where $s \in I \cap S$ and τ is the torsion theory corresponding with $\mathcal{F}$.

Proposition 6.1.6 [50] *Let τ be a spectral torsion theory on $R-Mod$ and suppose that Φ_τ, the Gabriel topology associated with τ, is cofinally principal. Then R_τ is a semisimple ring and $R_\tau = S^{-1}R$ where $S = \{s \in R \mid Rs \in \Phi_\tau\}$.*

Proof: By Corollary 4, R_τ is semisimple. We now prove that S is a left denominator subset of R. Let $s, t \in S$ and $a \in R$. We have $(Rt : a) \subseteq (Rts : as)$. Since $(Rt : a) \in \Phi_\tau$, $(Rts : as) \in \Phi_\tau$. It follows from Property T_4 of a Gabriel topology that $Rts \in \Phi_\tau$ because $as \in R \in \Phi_\tau$ and so $ts \in S$. Thus S is multiplicatively closed. Next let $s \in S$ and $a \in R$. Then $Rs \in \Phi_\tau$ and $(Rs : a) \in \Phi_\tau$ by Property T_3 of a Gabriel topology. But, since Φ_τ is cofinally principal, we must have $Rt \subseteq (Rs : a)$ for some $t \in S$. Thus there exists $b \in R$ such that $bs = at$. Now suppose that $as = 0$ holds for some $a \in R$ and $s \in S$. Put $\bar{R} = R/t_\tau(R)$. We claim that $\bar{S} = S + t_\tau(R)$ has a trivial right annihilator in $\bar{R}$. For suppose $\bar{b} = b + t_\tau(R) \in \bar{R}$ is such that $sb \in t_\tau(R)$. Consider the short exact sequence $0 \longrightarrow Rsb \longrightarrow Rb \longrightarrow Rb/Rsb \longrightarrow 0$. Since $sb \in t_\tau(R)$, Rsb is τ-torsion. Observe that there is a canonical epimorphism from R/Rs to Rb/Rsb defined by $r + Rs \longmapsto rb + Rsb$. Since $Rs \in \Phi_\tau$ it follows that Rb/Rsb is τ-torsion. Since the class of all τ-torsion modules is closed under taking module extensions, we must have Rb is τ-torsion, i.e., $b \in t_\tau(R)$, whence $\bar{b} = 0$. This establishes our claim. Since $\bar{R}$

is a subring of the semisimple ring R_τ, $\bar{S}$ must also have a trivial left annihilator in $\bar{R}$. Inasmuch as $\bar{a}\bar{s} = 0$, we conclude that $\bar{a} = 0$, i.e., $a \in t_\tau(R)$. Since Φ_τ is cofinally principal, we have that $ta = 0$ for some $t \in S$. This shows that S is a left denominator subset of R.

It remains to show that Φ_τ is the Gabriel topology corresponding with the left denominator set S, i.e., $\Phi_\tau = \{I \leq {}_R R \mid I \cap S \neq \emptyset\}$. Let I be a left ideal of R such that $I \cap S \neq \emptyset$. There then exists $s \in I \cap S$ such that $Rs \subseteq I$ and $Rs \in \Phi_\tau$ by definition of S. It follows from Property T_1 of a Gabriel topology that $I \in \Phi_\tau$. Thus $\{I \leq {}_R R \mid I \cap S \neq \emptyset\} \subseteq \Phi_\tau$. The reverse inclusion is easily checked. □

6.2 Regular self-injective rings of quotients

Recall that if $\tau \in R - tors$, then $gen(\tau)(R)$ denotes the set of $\sigma \in R - tors$ such that $\sigma \geq \tau$, i.e., the interval $[\tau, \chi]$ in $R - tors$. Put $\bar{R} = R/t_\tau(R)$. For each $\sigma \in gen(\tau)(R)$ let σ' denote the torsion theory on $\bar{R} - Mod$ induced by σ. If $\mathcal{T}_{\sigma'}$ is the class of all σ'-torsion left $\bar{R}$-modules and $\mathcal{T}_\sigma$ the class of all σ-left R-modules, then $\mathcal{T}_{\sigma'} = \bar{R} - Mod \cap \mathcal{T}_\sigma$ (see Section 4.2).

Proposition 6.2.1 [4] *Let τ be a torsion theory on $R - Mod$. Put $\bar{R} = R/t_\tau(R)$. If $\sigma \in gen(\tau)(R)$, then:*

(i) the class of all σ-torsion-free left R-modules coincides with the class of all σ'-torsion-free left $\bar{R}$-modules;

(ii) if ${}_R E$ is an injective cogenerator for σ then ${}_{\bar{R}} E$ is an injective cogenerator for σ' and the correspondence $\chi({}_R E) \longmapsto \chi({}_{\bar{R}} E)$ defines a lattice isomorphism from $gen(\tau)(R)$ to $gen(\tau')(\bar{R})$.

Proof: (i) follows from Proposition 4.2.9.

(ii) Let $\sigma \in gen(\tau)(R)$ and ${}_RE$ an injective cogenerator for σ. Since $\mathcal{F}_\sigma = \mathcal{F}_{\sigma'}$, ${}_{\bar{R}}E$ is σ'-torsion-free, so $\sigma' \leq \chi({}_{\bar{R}}E)$. Now let $N \in \mathcal{F}_{\sigma'}$. Then $Hom_R(M, E(N)) = 0$ for all $\chi({}_RE)$-torsion left $\bar{R}$-modules M because $\mathcal{T}_{\sigma'} = \mathcal{T}_\sigma \cap \bar{R} - Mod$. It follows that $Hom_R(M, E(N)) = 0$ for $M \in \mathcal{T}_{\chi({}_{\bar{R}}E)}$ and so $\mathcal{F}_{\sigma'} \subseteq \mathcal{F}_{\chi({}_{\bar{R}}E)}$, i.e., $\sigma' \geq \chi({}_{\bar{R}}E)$. Hence $\sigma' = \chi({}_{\bar{R}}E)$. This implies that $\sigma' \in gen(\tau')(\bar{R})$ whenever $\sigma \in gen(\tau)(R)$.

If $\chi({}_{\bar{R}}E_1) = \chi({}_{\bar{R}}E_2)$, then E_1 is cogenerated by E_2 and vice-versa as left $\bar{R}$-modules. Therefore as left R-modules each one is cogenerated by the other one; then $\chi({}_RE_1) = \chi({}_RE_2)$.

If $\sigma' \in gen(\tau')(\bar{R})$ and $\sigma' = \chi({}_{\bar{R}}E)$ with ${}_{\bar{R}}E$ τ-torsion-free injective $\bar{R}$-module, then $\chi({}_RE) \in gen(\tau)(R)$. It remains to show that the correspondence preserves order. Let $\sigma_1 = \chi({}_RE_1) \leq \sigma_2 = \chi({}_RE_2)$ in $gen(\tau)(R)$. We have that E_2 is cogenerated by E_1 as left R-modules and so as left $\bar{R}$-modules. Therefore ${\sigma'}_1 \leq {\sigma'}_2$. □

Proposition 6.2.2 [4] *Let* τ *be a spectral torsion theory on* $R - Mod$*. Then the correspondence* $gen(\tau)(R) \longrightarrow gen(\tau_g)(R_\tau)$ *defined by* $\chi({}_RE) \longmapsto \chi({}_{R_\tau}E)$ *is a lattice isomorphism.*

Proof: Observe that if E is an injective τ-torsion-free left R-module, then ${}_{R_\tau}E$ is injective by Proposition 3.6.4 and τ-torsion-free. By observations on page 67, the correspondence $\sigma = \chi({}_RE) \longmapsto \sigma^\star = \chi({}_{R_\tau}E)$ defines a lattice isomorphism between $gen(\tau)(R)$ and $gen(\tau^\star)(R_\tau)$. Since τ is spectral, the Gabriel topology corresponding with τ is $\Phi_\tau = \{I \leq {}_RR \mid I \text{ is } \tau\text{-essential left ideal}\}$. By Proposition 4.2.1, $\Phi_\tau = \{I \leq {}_RR \mid {}_{R_\tau}I_\tau \leq_e {}_{R_\tau}R_\tau\}$. It follows that the class of $\tau^\star$-torsion-free left R-modules coincides with the class of nonsingular left R_τ-modules. Therefore

${}_{R_\tau}E \in \mathcal{F}_{\tau_g(R_\tau)}$, i.e., $\chi({}_{R_\tau}E) \in gen(\tau_g)(R_\tau)$. Conversely, if ${}_{R_\tau}E$ is injective and nonsingular, we have that ${}_RE$ is injective and τ-torsion-free. The result now follows by Proposition 1. $\square$

Lemma 6.2.3 [36] *Let R be a regular left self-injective ring and $B(R)$ the complete boolean lattice of central idempotent elements of R. Then the correspondence $\varphi : B(R) \longrightarrow gen(\tau_g)(R)$ defined by $\varphi(e) = \chi(R(1-e))$ is a complete lattice isomorphism.*

Proof: It is easily verified that φ is a morphism of complete lattices. Since R is regular it is nonsingular and so $\tau_g(R) = \chi(R) \leq \chi(R(1-e))$. Thus $\chi(R(1-e)) \in gen(\tau_g)(R)$. On the other hand, $t_{\chi(R(1-e))}(R) = \{r \in R \mid Ir = 0$ for some $I \leq {}_RR$ such that $Hom_R(R/I, R(1-e)) = 0\} = (0 : R(1-e)) = Re$, so for $e, f \in B(R)$; $e \neq f$, we have $\varphi(e) \neq \varphi(f)$, i.e., φ is a monomorphism. It remains to show that φ is surjective. Consider $\sigma \in gen(\tau_g)(R)$ and an injective left R-module E such that $\sigma = \chi(E)$. Let $J = \sum\{Imf \mid f \in Hom_R(E,R)\}$. It is easily shown that J is a two-sided ideal of R. By Proposition 1.6.1 there exists a unique $e \in B(R)$ such that J is an essential ideal of Re. E is cogenerated by R since it is nonsingular; hence $\chi(E) \geq \chi({}_RR)$ from which we infer $\chi(Re) \leq \chi(E)$. Both E and R_τ are nonsingular injective modules which implies that for any $f \in Hom_R(E,R)$, the induced map $\alpha : E \longrightarrow Imf$ splits. Since R is nonsingular, Imf is nonsingular and hence $Ker\alpha$ has no proper essential extension in E. But since E is injective, we have that $Ker\alpha$ is a direct summand of E; so α splits. It follows that Imf is an injective R-module and is $\chi(E)$-torsion-free. Therefore for each family $\{f_i \mid 1 \leq i \leq n\} \subseteq Hom_R(E,R)$ we have that $Imf_1, Imf_2, \ldots, Imf_n$ are direct summands of E. In view of the observations on page 19, since E is nonsingular and injective $Imf_1 + Imf_2 + \cdots + Imf_n$ is a direct summand of E. Moreover, since $\bigoplus_{i=1}^{n} Imf_i$ is nonsingular injective and $\sum_{i=1}^{n} Imf_i$ nonsingular the R-epimorphism $\bigoplus_{i=1}^{n} Imf_i \longrightarrow \sum_{i=1}^{n} Imf_i$ splits. Therefore $\sum_{i=1}^{n} Imf_i$ is $\chi(E)$-torsion-free and injective. Hence J is $\chi(E)$-torsion-free because every finitely generated submodule of J is contained in $\sum_{i=1}^{n} Imf_i$ for suitable f_i and so Re is $\chi(E)$-torsion-free. Therefore $\chi(E) \leq \chi(Re)$. Thus $\chi(E) = \chi(Re)$. □

Proposition 6.2.4 [5] *Let τ be a spectral torsion theory on $R-Mod$. The following statements are equivalent:*

(i) $gen(\tau)(R)$ is an atomic lattice;

(ii) R_τ is isomorphic to a direct product of prime left self-injective rings.

Proof: Note first that since τ is spectral, R_τ is regular and left self-injective by Theorem 4.2.10.

$(i) \Longrightarrow (ii)$ It follows from Proposition 2 and Lemma 3 that $B(R_\tau)$ is atomic. By Proposition 1.6.3, statement (ii) holds.

$(ii) \Longrightarrow (i)$ By Proposition 1.6.3, $B(R_\tau)$ is atomic. Then (i) follows from Proposition 2 and Lemma 3. □

The following result can be proved similarly using Proposition 2, Lemma 3 and Proposition 1.6.2.

Proposition 6.2.5 [4] *Let τ be a spectral torsion theory on $R-Mod$. The following conditions are equivalent:*

(i) R_τ is indecomposable;

(ii) R_τ is a prime ring;

(iii) τ *is a coatom of* $R-tors$. ☐

The *jansian-hull* of a torsion theory τ on $R-Mod$ is the meet of all jansian torsion theories greater than or equal to τ. Since the class of jansian torsion theories on $R-Mod$ is closed under taking meets, the jansian-hull of τ is the smallest jansian generalization of τ. For example, the jansian-hull of τ_g is $\chi(\mathcal{P}) = \tau_g^{\perp\perp}$ where $\mathcal{P}$ is the class of simple projective left R-modules. To see this note first that a simple left R-module S is projective if and only if it is nonsingular. Hence $\mathcal{P}$ is contained in the class of all nonsingular left R-modules. As $\chi(\mathcal{P})$ is the unique largest torsion theory for which every member of $\mathcal{P}$ is torsion-free, we have that $\chi(\mathcal{P}) \geq \tau_g$. Since the Gabriel topology corresponding with a jansian torsion theory is closed under arbitrary intersections and $Soc(M) = \bigcap\{N \leq_e M\}$, we have that if σ is a jansian torsion theory such that $\sigma \geq \tau_g$, then $M/Soc(M)$ is σ-torsion for all $M \in R-Mod$. It follows, by considering the short exact sequence $0 \to Soc(M) \to M \to M/Soc(M) \to 0$, that if σ is a jansian generalization of τ_g and $Soc_{\mathcal{P}}(M) = 0$ (thus $Soc(M)$ singular), then M is σ-torsion. But as shown in Proposition 5.2.7 $Soc_{\mathcal{P}}(M) = 0$ if M is $\chi(\mathcal{P})$-torsion. Hence $\sigma \geq \chi(\mathcal{P})$. Therefore the jansian-hull of τ_g is $\chi(\mathcal{P})$.

Let $\tau \in R-tors$. A left R-module M is said to be τ-*cocritical* if M is τ-torsion-free and every nonzero submodule of M is τ-dense, i.e., $M \in \mathcal{F}_\tau$ and $M/N \in \mathcal{T}_\tau$ whenever $0 \neq N \leq M$. Observe that M is τ-cocritical if and only if M is τ-torsion-free and M has no nonzero τ-pure submodules. In [22], Goldman calls a τ-cocritical module a *supporting* module for a kernel functor τ and notes that a kernel functor need not have a supporting module. A torsion theory τ is said to be *prime* if there exists a τ-cocritical module M such that $\tau = \chi(M)$. If $M \in R-Mod$ is simple and

$\tau = \chi(M)$, then M is τ-cocritical and therefore τ is prime. Thus every maximal left ideal I of R gives rise to a prime torsion theory $\tau = \chi(R/I)$.

Lemma 6.2.6 [37] *Let R be a regular left self-injective ring. A left R-module M is τ_g-cocritical if and only if M is simple and projective.*

Proof: Obviously if M is a simple and projective left R-module, then M is nonsingular and τ_g-cocritical. Conversely, suppose M is τ_g-cocritical. Inasmuch as a simple module is projective if and only if it is nonsingular, it suffices to prove that M is simple. Since M is τ_g-cocritical, M is uniform, i.e., every nonzero submodule N of M is essential in M and so M/N is singular. Since R is regular and left self-injective all finitely generated nonsingular left R-modules are projective (see page 19) and therefore every nonsingular module is injective by Proposition 1.5.1. Then we have that $E(N) = E(M) = M$. It follows that $Hom_R(M/N, M) = 0$. This implies that $M = N$, whence M is simple. ☐

Remark 6.2.7 If S is simple and projective we know that M is $\chi(S)$-torsion if and only if $Soc_S(M) = 0$. It follows that M is $\chi(S)$-torsion-free if and only if for every nonzero $L \leq M$, $Soc_S(L) \neq 0$, i.e., $Soc_S(M)$ is essential in M.

Lemma 6.2.8 [37] *The following statements are equivalent for a regular left self-injective ring R:*

(i) τ_g is prime;

(ii) τ_g is jansian and a coatom of $R-tors$;

(iii) R is isomorphic to a left full linear ring.

Proof: *(i)* $\Longrightarrow$ *(iii)* Suppose $\tau_g = \chi(S)$ with S a nonsingular uniform left R-module. By Lemma 6, S is simple and projective. Since R is left nonsingular

$\tau_g = \chi({}_RR) = \chi(S)$. Hence ${}_RR$ is $\chi(S)$-torsion-free. It follows from the above remark that $Soc_S({}_RR)$ is essential as a left ideal in R. Observe that 0 and 1 are the only central idempotents of R, for if $e \in B(R)$ such that $e \neq 0, 1$ then $Soc_S({}_RR) \subseteq Re$ or $Soc_S({}_RR) \subseteq R(1-e)$ which contradicts the fact that $Soc_S({}_RR)$ is essential in ${}_RR$. By Proposition 1.6.2, R is a prime ring. Since $Soc_S({}_RR) \leq_e {}_RR$ it follows from Proposition 1.6.4 that R is a left full linear ring.

$(iii) \Longrightarrow (ii)$ By Proposition 1.6.5 $Soc_S({}_RR)$ is a left essential ideal of R. Since $Soc_S({}_RR)$ coincides with the intersection of all essential left ideals of R, we infer that R has a unique smallest essential left ideal. On the other hand since R is left nonsingular the set of all essential left ideals of R is precisely the Gabriel topology associated with τ_g. We conclude that τ_g is jansian. Since R is prime, $B(R) = \{0, 1\}$. It follows from Lemma 3 that τ_g is a coatom of $R - tors$.

$(ii) \Longrightarrow (i)$ Since the jansian-hull of τ_g is $\chi(\mathcal{P})$ where $\mathcal{P}$ is the class of all simple projective left R-modules, we must have $\tau_g = \chi(S)$ for some $S \in \mathcal{P}$. Clearly S is τ_g-cocritical and so τ_g is prime. □

Proposition 6.2.9 [5] *Let τ be a spectral torsion theory on $R - Mod$. Then the following conditions are equivalent:*

(i) R_τ is isomorphic to a left full linear ring;

(ii) τ is a coatom of $R - tors$ and τ is prime.

Proof: $(i) \Longrightarrow (ii)$ If R_τ is isomorphic to a left full linear ring then R_τ is a prime ring and therefore τ is a coatom of $R - tors$ by Proposition 5. It remains to prove that τ is prime. By Lemma 8, statement (i) implies that τ_g is prime on $R_\tau - Mod$. Thus there exists a τ_g-cocritical left R_τ-module S such that $\tau_g = \chi(S)$. Since S

is simple and projective it is a direct summand of ${}_{R_\tau}R_\tau$. Hence S is a τ-torsion-free and τ-injective left R-module. By Proposition 3.6.4, S is injective as a left R-module. Since S is a simple left R_τ-module, we have that $\mathcal{P}_\tau(S) = \{S, 0\}$. To see this observe that if $L \in \mathcal{P}_\tau(S)$ then L is τ-injective by Proposition 3.6.6. By Proposition 3.6.3 (iii), if $\imath : L \longrightarrow S$ denotes the inclusion map then $\imath$ is an R_τ-homomorphism, i.e., L is an R_τ-submodule of S. But ${}_{R_\tau}S$ is simple, so $L = 0$ or S. Since S is τ-torsion-free and $\mathcal{P}_\tau(S) = \{S, 0\}$, S is τ-cocritical and $\tau \leq \chi(S) < \chi$. But τ being a coatom implies that $\tau = \chi(S)$, i.e., τ is prime in $R - tors$.

$(ii) \Longrightarrow (i)$ Since τ is spectral and a coatom, R_τ is a prime regular left self-injective ring by Proposition 5. To establish the result, it suffices in view of Proposition 1.6.4 to prove that $Soc({}_{R_\tau}R_\tau) \neq 0$. Since τ is a coatom and prime, $\tau = \chi(C)$ for some τ-cocritical left R-module C. Note that $\tau = \chi(C) = \chi({}_RC_\tau)$. Since τ is spectral, by Proposition 2, $\chi({}_{R_\tau}C_\tau) = \tau_g$ on $R_\tau - Mod$. Thus C_τ is a nonsingular left R_τ-module. Hence ${}_{R_\tau}C_\tau$ is projective and so $Soc({}_{R_\tau}R_\tau) \neq 0$. □

A torsion theory τ on $R - Mod$ is said to be *semiprime* if it is a meet of prime torsion theories. Obviously any prime torsion theory is semiprime.

Lemma 6.2.10 [37] *The following statements are equivalent for a regular left self-injective ring R:*

(i) τ_g *is semiprime;*

(ii) R is isomorphic to a direct product of left full linear rings;

(iii) τ_g *is jansian.*

Proof: $(i) \Longrightarrow (iii)$ If τ_g is semiprime, then $\tau_g = \bigwedge_{\alpha\in\Gamma} \chi(C_\alpha)$ with each C_α cocritical. Since $\chi(C_\alpha) \geq \tau_g$, C_α is uniform and nonsingular and so τ_g-cocritical. It follows from Lemma 6 that C_α is simple and projective. Therefore $\chi(C_\alpha)$ is jansian (see page 95). Whence τ_g is jansian because the class of jansian torsion theories is closed under meets.

(iii) $\Longrightarrow$ *(i)* follows from the fact that the jansian hull of τ_g is $\chi(\mathcal{P})$ where $\mathcal{P}$ is the class of simple projective left R-modules.

(ii) $\Longrightarrow$ *(iii)* follows from Proposition 8.

(iii) $\Longrightarrow$ *(ii)* Since τ_g is jansian the Gabriel topology Φ_{τ_g} is closed under intersections. Moreover, Φ_{τ_g} consists of essential left ideals of R because R is nonsingular. It follows that $Soc({}_RR) \leq_e {}_RR$, so by Proposition 1.6.5, R is isomorphic to a direct product of left full linear rings. □

Proposition 6.2.11 [5] *The following statements are equivalent for a spectral torsion theory τ on $R - Mod$:*

(i) τ is semiprime;

(ii) R_τ is isomorphic to a direct product of left full linear rings.

Proof: By Lemma 10, R_τ is isomorphic to a direct product of left full linear rings if and only if $\tau_g(R_\tau)$ is semiprime. By Proposition 2, it remains to prove that $\tau_g(R_\tau)$ is semiprime if and only if τ is semiprime. Since a lattice isomorphism preserves meets it suffices to prove that if $\chi({}_{R_\tau}E)$ is prime then so is $\chi({}_RE)$. Suppose $\chi({}_{R_\tau}E)$ is prime. Then $\chi({}_{R_\tau}E) \geq \tau_g(R_\tau)$. It follows that E is an uniform nonsingular left R_τ-module, so E is a simple object in $(R, \tau) - Mod$. This implies that ${}_RE$ does not contain proper nontrivial τ-pure submodules. Whence ${}_RE$ is τ-cocritical. We conclude that $\chi({}_RE)$ is prime. □

6.3 Flat modules of quotients

A right R-module B is said to be *flat* if, for every monomorphism $\alpha : M \longrightarrow N$ of left R-modules the map $1_B \otimes \alpha : B \otimes_R M \longrightarrow B \otimes_R N$ is also a monomorphism. For example, any projective right R-module is flat [34, page 91]. It is known that a ring R is regular if and only if all right (left) R-modules are flat [25, Corollary 1.13].

Lemma 6.3.1 [3, 19.17 The Flat Test Lemma] *The following statements about a right R-module V are equivalent:*

(i) V is flat;

(ii) for every $I \leq {}_RR$ *the sequence* $0 \longrightarrow V \otimes_R I \xrightarrow{V \otimes_R \iota} V \otimes_R R$ *with* $\iota : I \longrightarrow {}_RR$ *the inclusion map, is exact;*

(iii) for each (finitely generated) left ideal I of R the $\mathbb{Z}$*-epimorphism* $\mu_I : V \otimes_R I \longrightarrow VI$ *with* $\mu_I(v \otimes a) = va$ *is monic.* ☐

Let $\tau \in R\!-\!tors$. A left R-module M is said to be *τ-finitely generated* if M contains a finitely generated τ-dense submodule. Obviously every finitely generated left R-module is τ-finitely generated.

Remark 6.3.2 Let $\tau \in R - tors$ and $N \leq M \in R - Mod$. If N and M/N are both τ-finitely generated, so is M. Moreover, if M is τ-finitely generated then so is M/N [19, Proposition 17.4].

A left R-module M is called *τ-finitely presented* if there exist a finitely generated free left R-module H and an epimorphism $f : H \longrightarrow M$ with $Ker f$ a τ-finitely generated submodule of H.

Lemma 6.3.3 [19] *Let τ be a torsion theory on $R-Mod$ and M a finitely generated left R-module. If M is τ-finitely presented and $0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$ is a short exact sequence in which F is τ-finitely generated then K is τ-finitely generated.*

Proof: Let $\beta : F \longrightarrow M$ be an R-epimorphism from a τ-finitely generated left R-module F onto M. Put $K = Ker\beta$. Since M is τ-finitely presented there exists a finitely generated free left R-module F' and an R-epimorphism $\beta' : F' \longrightarrow M$ with $Ker\beta' = K'$ τ-finitely generated. Set $W = \{(x, x') \in F \oplus F' \mid (x)\beta = (x')\beta'\}$. We have the following canonical short exact sequences:

$0 \longrightarrow K' \longrightarrow W \longrightarrow F \longrightarrow 0$ and $0 \longrightarrow K \longrightarrow W \longrightarrow F' \longrightarrow 0$.

By Remark 2, it follows from the former sequence that W is τ-finitely generated. Since the left R-module F' is free (thus projective) the latter short exact sequence splits, so K is a homomorphic image of W. Therefore by Remark 2, K is τ-finitely generated. ☐

Note that the converse of Lemma 3 is also true.

For the sequel we need to introduce the following notation and lemma. Let $\tau \in R-tors$. The functor $R_\tau \otimes_R (-) : R-Mod \longrightarrow R-Mod$ is called the *τ-prelocalization functor* on $R - Mod$. The localization functor $Q_\tau(-)$ actually factors through $R_\tau \otimes_R (-)$ in the sense that for every $M \in R - Mod$ we have the commutative diagram

$$\begin{tikzcd} M \arrow[rr, "\hat{\tau}_M"] \arrow[dr, "\zeta^\tau_M"'] & & Q_\tau(M) = M_\tau \\ & R_\tau \otimes_R M \arrow[ur, "\Theta^\tau_M"'] & \end{tikzcd}$$

in which the R-homomorphisms ζ^τ_M and Θ^τ_M are defined by $m \longmapsto 1 \otimes m$ and $\Sigma(q_i \otimes m_i) \longmapsto \Sigma q_i(m_i\hat{\tau}_M)$ respectively.

Lemma 6.3.4 [19] $Ker\Theta^{\tau}_{M} = t_{\tau}(R_{\tau} \otimes_R M)$.

Proof: Since $[t_{\tau}(R_{\tau} \otimes_R M)]\Theta^{\tau}_{M} \subseteq t_{\tau}(M_{\tau})$ and M_{τ} is τ-torsion-free we have that $t_{\tau}(R_{\tau} \otimes_R M) \subseteq Ker\Theta^{\tau}_{M}$. Moreover, $[t_{\tau}(M)]\zeta^{\tau}_{M} \subseteq t_{\tau}(R_{\tau} \otimes_R M)$. It follows that the inclusion map $\wr : M/t_{\tau}(M) \longrightarrow M_{\tau}$ factors through $R_{\tau} \otimes_R M/t_{\tau}(R_{\tau} \otimes_R M)$, i.e., the diagram

$$\begin{tikzcd}
M/t_{\tau}(M) \arrow[rr, "\wr"] \arrow[dr, "\alpha"'] & & M_{\tau} \\
& R_{\tau} \otimes_R M/t_{\tau}(R_{\tau} \otimes_R M) \arrow[ur, "\beta"'] &
\end{tikzcd}$$

in which α, β are induced by ζ^{τ}_{M} and Θ^{τ}_{M} respectively, commutes. We need to prove that β is a monomorphism. Put $\bar{R} = R/t_{\tau}(R)$. For $m \in M$, the R-homomorphism $\varphi_m : R_{\tau}/\bar{R} \longrightarrow R_{\tau} \otimes_R M/Im\zeta^{\tau}_{M}$ defined by $q + \bar{R} \longmapsto (q \otimes m) + Im\zeta^{\tau}_{M}$ is well-defined because for $r \in \bar{R}$ we have that $(r + \bar{R}) \otimes_R m = 1 \otimes_R (rm + Im\zeta^{\tau}_{M}) \in Im\zeta^{\tau}_{M}$. Therefore $\varphi = \bigoplus_{m\in M} \varphi_m : \bigoplus_{M}(R_{\tau}/\bar{R}) \longrightarrow R_{\tau} \otimes_R M/Im\zeta^{\tau}_{M}$ is an R-epimorphism. Since $\bar{R}$ is τ-dense in R_{τ} and the class of τ-torsion modules closed under taking direct sums and homomorphic images $R_{\tau} \otimes_R M/Im\zeta^{\tau}_{M}$ is τ-torsion. It follows from the canonical R-epimorphism from $R_{\tau} \otimes_R M/Im\zeta^{\tau}_{M}$ onto $[R_{\tau} \otimes_R M/t_{\tau}(R_{\tau} \otimes_R M)]/Im\alpha$ that $[R_{\tau} \otimes_R M/t_{\tau}(R_{\tau} \otimes_R M)]/Im\alpha$ is τ-torsion, so $Im\alpha$ is τ-dense in $R_{\tau} \otimes_R M/t_{\tau}(R_{\tau} \otimes_R M)$ which is τ-torsion-free. Thus $Im\alpha$ is essential in $R_{\tau} \otimes_R M/t_{\tau}(R_{\tau} \otimes_R M)$. Since $\alpha\beta$ is a monomorphism, $Im\alpha \cap Ker\beta = 0$. Thus $Ker\beta = 0$, so β is monomorphism. □

The next proposition gives a characterization of τ-finitely presented modules. The following remark states a result due to Cateforis [10, Proposition 1.5] which constitutes a particular case of Proposition 6 when the ring R is left nonsingular and

$\tau = \tau_g$.

Remark 6.3.5 If R is left nonsingular then a finitely generated nonsingular left R-module M is essentially finitely presented if and only if $Z(Q_{max}({}_RR)\otimes_R M) = 0$.

Proposition 6.3.6 [23] *Let τ be a spectral torsion theory on $R - Mod$ and M a finitely generated left R-module. Then M is τ-finitely presented if and only if $t_\tau(R_\tau \otimes_R M) = 0$.*

Proof: Consider a presentation of M of the form $0 \longrightarrow K \rightarrow R^n \rightarrow M \rightarrow 0$. We obtain the following commutative diagram:

$$\begin{tikzcd}
 & R_\tau \otimes_R K \arrow[r] \arrow[d, "\Theta_K^\tau"] & R_\tau^n \arrow[r] \arrow[d, "\wr"] & R_\tau \otimes_R M \arrow[r] \arrow[d, "\Theta_M^\tau"] & 0 \\
0 \arrow[r] & K_\tau \arrow[r] & R_\tau^n \arrow[r] & M_\tau \arrow[r] & 0
\end{tikzcd}$$

Since the functor $R_\tau \otimes_R (-)$ is right exact the top row is exact. The bottom row is also exact because the localization functor $Q_\tau(-)$ is exact (τ is spectral). Since $Ker\Theta_M^\tau = t_\tau(R_\tau \otimes_R M)$, we have that $t_\tau(R_\tau \otimes_R M) = 0$ if and only if Θ_K^τ is an epimorphism (see page 3).

Now assume that M is τ-finitely presented. Then K is τ-finitely generated by Lemma 3 since R^n is finitely generated. It follows that there exists an exact sequence $0 \rightarrow K' \rightarrow K \rightarrow K/K' \rightarrow 0$ with K' finitely generated and K/K' τ-torsion because K' is τ-dense in K. Therefore $K'_\tau \cong K_\tau$. Since K' is finitely generated there exists an epimorphism $R^m \rightarrow K'$. We thus obtain the commutative diagram:

$$\begin{tikzcd}
R_\tau^m \arrow[r] \arrow[d, "\wr"'] & R_\tau \otimes_R K' \arrow[r] \arrow[d, "\Theta^\tau_{K'}"] & R_\tau \otimes_R K \arrow[d, "\Theta^\tau_K"] \\
R_\tau^m \arrow[r, "f"] & K'_\tau \arrow[r, "\cong"] & K_\tau
\end{tikzcd}$$

where f is an epimorphism because τ is spectral. Thus $\Theta^\tau_{K'}$ is an epimorphism and hence Θ^τ_K is also an epimorphism.

Conversely, suppose that Θ^τ_K is an epimorphism. Consider the short exact sequence $0 \longrightarrow K_\tau \longrightarrow R_\tau^n \longrightarrow M_\tau \longrightarrow 0$. Since τ is spectral, K_τ is an injective R-module and therefore it is also an injective R_τ-module by Proposition 3.6.4. Hence K_τ is a finitely generated R_τ-module because it is a direct summand of R_τ^n. We claim that there exists a finitely generated R-submodule K' of K such that the homomorphism $R_\tau \otimes_R K' \longrightarrow R_\tau \otimes_R K \xrightarrow{\Theta^\tau_K} K_\tau$ is an epimorphism. To see this consider $H = [K_\tau]\hat{\tau}_K^{-1}$. If $\{k_1, k_2, ..., k_n\}$ is a generating set of K_τ then each $k_i = \sum_j r_j[(a_{ij})\hat{\tau}_K]$ for $r_j \in R_\tau$ and $a_{ij} \in H$. Therefore $K' = \Sigma_{i,j} Ra_{ij}$ is a finitely generated R-submodule of K. By the construction of K' and the fact that Θ^τ_K is an epimorphism by hypothesis, the claim holds. Since the diagram

$$\begin{tikzcd}
R_\tau \otimes_R K' \arrow[r] \arrow[d, "\Theta^\tau_{K'}"'] & R_\tau \otimes_R K \arrow[d, "\Theta^\tau_K"] \\
K'_\tau \arrow[r] & K_\tau
\end{tikzcd}$$

is commutative, we have that the canonical morphism $K'_\tau \to K_\tau$ is an epimorphism. Therefore $(K/K')_\tau \cong K_\tau/K'_\tau \cong 0$ by the exactness of the localization functor $Q_\tau(-)$, so K/K' is τ-torsion. It follows that K is τ-finitely generated and so M is τ-finitely presented. $\square$

A ring R is said to be left *τ-coherent* if every finitely generated left ideal is τ-finitely presented.

In [11] and [10], Cateforis gives characterizations of when $Q_{max}({}_RR)$ is flat as a right R-module and when R is τ_g-coherent. These results are stated in the following two remarks.

Remark 6.3.7 Let R be left nonsingular. Then the following statements are equivalent:

(i) $Q_{max}({}_RR)$ is flat as a right R-module;

(ii) every finitely generated left ideal I of R is essentially presented;

(iii) for any finitely generated left ideal I of R and any element a of R, $(I : a)$ is essentially finitely generated [11, Theorem 2.1].

Remark 6.3.8 Let R be left nonsingular. Then the following statements are equivalent:

(i) if M is a finitely generated nonsingular left R-module then $Z(Q_{max}({}_RR) \otimes_R M) = 0$;

(ii) $Q_{max}({}_RR)$ is flat as a right R-module and $Z(Q_{max}({}_RR) \otimes_R Q_{max}({}_RR)) = 0$;

(iii) $(R : q)$ is essentially finitely generated for every $q \in Q_{max}({}_RR)$ [10, Theorem 1.6].

Proposition 6.3.9 [23] *Let τ be a spectral torsion theory on $R - Mod$. The following conditions are equivalent:*

(i) R is left τ-coherent;

(ii) $(R_\tau)_R$ is flat;

(iii) every right R_τ-module is flat as a right R-module.

Proof: $(i) \Longrightarrow (ii)$ By the Flat Test Lemma (see Lemma 1) it suffices to show that if I is a finitely generated left ideal of R with $\wr : I \longrightarrow {}_R R$ the canonical inclusion map, then $R_\tau \otimes_R \wr : R_\tau \otimes_R I \to R_\tau \otimes_R R$ is monic. Consider the commutative diagram

$$\begin{tikzcd}
R_\tau \otimes_R I \arrow[r, "R_\tau \otimes_R \wr"] \arrow[d, "\Theta_I^\tau"'] & R_\tau \otimes_R R \arrow[d, "\Theta_R^\tau"] \\
I_\tau \arrow[r, "Q_\tau(\wr)"] & R_\tau
\end{tikzcd}$$

Note that Θ_R^τ is an isomorphism and $Q_\tau(\wr)$ a monomorphism. By hypothesis I is τ-finitely presented, so by Proposition 6, $t_\tau(R_\tau \otimes_R I) = Ker\Theta_I^\tau = 0$, i.e., Θ_I^τ is a monomorphism. It follows that $R_\tau \otimes_R \wr$ is monic, as required.

$(ii) \Longrightarrow (i)$ If $(R_\tau)_R$ is flat then $R_\tau \otimes_R \wr$ is monic. It follows from the above diagram that Θ_I^τ must also be monic, i.e., $t_\tau(R_\tau \otimes_R I) = 0$. By Proposition 6, I is τ-finitely presented. We conclude that R is left τ-coherent.

$(iii) \Longleftrightarrow (ii)$ Obviously if every right R_τ-module is flat as a right R-module, then $(R_\tau)_R$ is flat. Conversely, since τ is spectral, R_τ is a regular ring and hence every right R_τ-module M is flat. Since $(R_\tau)_R$ is flat, the functors $M_{R_\tau} \otimes_{R_\tau} ((R_\tau)_R \otimes_R -)$ and $M_R \otimes_R -$ are naturally equivalent. Thus M_R is flat. □

Let $\varphi : R \longrightarrow S$ be a ring homomorphism, φ is a *ring epimorphism* if for any ring T and homomorphisms $\alpha, \beta : S \longrightarrow T$, $\alpha\varphi = \beta\varphi$ implies $\alpha = \beta$. An epimorphism of rings $\varphi : R \longrightarrow S$ is said to be *left flat* if S is a flat right R-module. We state a part of [43, Proposition XI.1.2] because of its use in the sequel.

Lemma 6.3.10 [43, Proposition XI.1.2] *Let $\varphi : R \longrightarrow S$ be a ring homomorphism. Then φ is a ring epimorphism if and only if $\Theta_S : S \otimes_R S \longrightarrow S$ is an R-isomorphism.* □

Proposition 6.3.11 [23] *Let τ be a spectral torsion theory on $R - Mod$. The following conditions are equivalent:*

(i) $\hat{\tau} : R \longrightarrow R_\tau$ is a left flat epimorphism of rings;

(ii) if M is a left ideal of R or a left R-submodule of R_τ, then $t_\tau(R_\tau \otimes_R M) = 0$;

(iii) R is left τ-coherent and every finitely generated left R-submodule of R_τ is τ-finitely presented.

Proof: $(i) \Longrightarrow (ii)$ Let I be a left ideal of R with $\wr : I \longrightarrow {}_RR$ the inclusion map. Consider the commutative diagram

$$\begin{tikzcd}
R_\tau \otimes_R I \arrow[r, "R_\tau \otimes \wr"] \arrow[d, "\Theta_I^\tau"'] & R_\tau \otimes R \arrow[d, "\Theta_R^\tau"] \\
I_\tau \arrow[r, "Q_\tau(\wr)"] & R_\tau
\end{tikzcd}$$

As shown in the proof of the preceding Proposition 9 ($(ii) \Longrightarrow (i)$) the flatness of $(R_\tau)_R$ implies that Θ_I^τ is a monomorphism. It follows that $t_\tau(R_\tau \otimes_R I) = Ker\Theta_I^\tau = 0$. Let M be a left R-submodule of R_τ with $\wr : M \longrightarrow R_\tau$ the inclusion map. Since $(R_\tau)_R$ is flat, $R_\tau \otimes_R \wr : R_\tau \otimes M \longrightarrow R_\tau \otimes_R R_\tau$ is a monomorphism. Since $\hat{\tau} : R \longrightarrow R_\tau$ is an epimorphism of rings the canonical homomorphism $\mu : R_\tau \otimes R_\tau \longrightarrow R_\tau$ is an isomorphism by Lemma 10. Then $R_\tau \otimes_R M$ embeds in R_τ as a left R-module. Whence $t_\tau(R_\tau \otimes_R M) = 0$ because R_τ is τ-torsion-free.

$(ii) \Longrightarrow (iii)$ If $t_\tau(R_\tau \otimes_R M) = 0$ for any left R-submodule M of R_τ or left ideal of R, then the equality holds in particular for any finitely generated left R-submodule of R_τ or left ideal of R. Therefore by Proposition 6, R_τ is τ-finitely presented, R is left τ-coherent and every finitely generated left R-submodule of R_τ is τ- finitely presented.

$(iii) \Longrightarrow (i)$ By Proposition 9, we have that $(R_\tau)_R$ is flat. To prove that $\hat{\tau} : R \to R_\tau$ is an epimorphism of rings, it suffices in view of Lemma 10 to show that the canonical homomorphism $\mu : R_\tau \otimes_R R_\tau \to R_\tau$ is an isomorphism. Let $x \in R_\tau \otimes_R R_\tau$ be such that $(x)\mu = 0$. There exist a finitely generated left R-submodule M of R_τ and an element $y \in R_\tau \otimes_R M$ which is mapped to x by the canonical homomorphism $R_\tau \otimes_R \wr : R_\tau \otimes_R M \longrightarrow R_\tau \otimes_R R_\tau$ where $\wr : M \longrightarrow R_\tau$ denotes the inclusion map. To see this write $x = \Sigma a_i \otimes b_i$ with $a_i, b_i \in R_\tau$. Take M to be the left R-submodule of R_τ generated by the b_i and $y = \Sigma a_i \otimes b_i$ (in $R_\tau \otimes_R M$). By hypothesis, M is τ-finitely presented, so $t_\tau(R_\tau \otimes_R M) = 0$ by Proposition 6. Therefore $\Theta^\tau_M : R_\tau \otimes_R M \to M_\tau$ is a monomorphism. Consider the commutative diagram

$$\begin{tikzcd}
R_\tau \otimes_R M \arrow[r, "R_\tau \otimes_R \wr"] \arrow[d, "\Theta^\tau_M"'] & R_\tau \otimes_R R_\tau \arrow[d, "\Theta^\tau_{R_\tau}"] \\
M_\tau \arrow[r, "Q_\tau(\wr)"] & R_\tau
\end{tikzcd}$$

It is easily checked that $\Theta^\tau_{R_\tau} = \mu$. Since Θ^τ_M and $Q_\tau(\wr)$ are both monomorphisms, we conclude that μ is a monomorphism. Hence $x = 0$. Therefore μ is monic and thus an isomorphism. $\square$

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