EXPLORING GRADE 11 MATHEMATICS LEARNERS LEARNING OF TRIGONOMETRIC IDENTITIES

A CASE STUDY OF ONE SCHOOL IN KWAZULU-NATAL

By

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In fulfillment of the requirement for the

DEGREE OF MASTER OF EDUCATION

In the School of Education

(Mathematics Education)

College of Humanities

University of KwaZulu- Natal

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December 2019

DEDICATION

This work is dedicated to my late grandmother (Jabisile Hendrietta Mbuyisa) and my late aunt (Sphiwe Tinny Mbuyisa).

Acknowledgements

I wish to express my gratitude to the following people:

- Prof J. Naidoo, for her mentoring and guidance throughout the study.
- Sabelo Dlamini, for his endless support and motivation.
- My father, for raising me with the accurate values; from him I learnt that education is indeed a powerful weapon.
- My mother, for always being there for me and for her prayers.
- My daughter, for being understanding and bearing with my busy schedule during the course of the study.
- The participants of this study, for the contribution they made.

DECLARATION

I, Nontobeko Khuzwayo declare that:

1. The research report of this dissertation, except where otherwise stated, is my original work.
2. This dissertation has not been submitted for any examination or degree at any other university.
3. All sources used in this dissertation have been acknowledged.

Signed: …………………………. (Student)

As the candidate's Supervisor I agree to the submission of this dissertation.

Signed: ………………………….. (Supervisor)

ABSTRACT

The purpose of this study was to explore the learning of trigonometric identities in grade 11 mathematics. The study sought to discover and understand the learning of trigonometric identities by grade 11 mathematics learners. This study was guided by qualitative methods, framed within the interpretive paradigm and employed case study methods. The study was underpinned by the principles of Kilpatrick's strands of mathematical proficiency.

In the study, ten grade 11 mathematics learners were purposively selected at a high school in the province of KwaZulu-Natal, South Africa. The study generated primary data through semi-structured interviews and an activity worksheet. In interpreting data that was collected, the researcher applied both inductive and deductive approaches to data analysis. Thematic data analyses methods were employed for analysing data.

The findings in this case study indicated that learners had an understanding of the fundamental trigonometric identities, and were able to prove these identities making use of the unit circle. In addition, the results showed that learners experience challenges when they have to apply algebraic manipulations in the learning of trigonometric identities.

The findings of this study were similar to other research studies regarding the learning of trigonometry and how learners experience difficulty in this topic. Hence, making use of the unit circle when trigonometric identities are introduced was seen as a tool to develop conceptual and procedural understanding of mathematical concepts. Also, the use of knowledge from other topics when teaching trigonometric identities was seen as crucial.

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Chapter One

Overview and rationale of the study

1.1 Introduction

This study sought to explore the learning of trigonometric identities in grade 11 mathematics. The participants of the study were grade 11 mathematics learners. This chapter provides an outline of the study. In this chapter, the background of the study; purpose of the study and motivation of conducting the study are discussed. The research questions are introduced and the research methods and analysis of data are discussed. Ethical considerations and limitations of the study are also briefly discussed. Lastly, an overview of each chapter is briefly outlined.

1.2 Background of the study

In the school curriculum, different subjects play different academic roles in preparing learners for the future. However, learners view mathematics as an important yet difficult subject. Bhagat, Chang and Chang (2016) confirm this in stating that in the school curriculum, mathematics is among one the important subjects. In stating that mathematics is included in the curriculum of many countries because it forms the basis of other scientific disciplines; it is regarded as an effective tool for communicating and thinking; and an essential tool in many professions, Hernandez-Martinez and Vos (2018) confirm that mathematics is a gateway subject to numerous fields.

Even though mathematics is a gateway subject for numerous fields, many learners do still struggle in learning this subject. Brijlall and Niranjan (2015) posit that in South Africa, under-achievement in mathematics is a great concern and hence, the education system of the country requires urgent improvement. Yang and Sianturi (2017) are of the view that in doing mathematics, learners make use of complex, non- algorithmic thinking to solve problems, which could be a contributing factor to learners' under-achievement in mathematics.

This therefore suggests that the manner in which mathematics is taught in South Africa needs to be changed and improved, so that it will ultimately improve learners' level of competency such that learners will be able to compete with the rest of the world (Brijlall & Niranjan, 2015). In the mathematics curriculum, trigonometry is among the topics that learners do not like because they feel that it is too challenging. According to May and Courtney (2016), trigonometric concepts and ideas remain a crucial component of the high school mathematics curriculum, however, research shows that trigonometry is still a difficult topic for both teachers and learners. DeReu (2019) further suggests that even though teachers and learners do not like trigonometry, it is a crucial component to success in higher mathematics and other fields.

Niranjan (2013) is of the view that trigonometry is a mathematics topic that relies heavily on a learner's knowledge of geometry, which may be lacking and this could possibly be the reason why trigonometry challenges learners. When solving mathematical problems, learners may employ a number of problem- solving techniques. However, Mutodi (2015) suggests that in mathematics, the major approaches that are commonly used when solving problems are algebraic and graphical approaches, but algebraic problem solving is complex, hence, it is crucial for learners to understand a variety of representations in order to effectively solve problems.

Trigonometry has its roots embedded in other mathematics topics, hence it is crucial that learners master this topic in order to improve their achievement in other topics, and ultimately in mathematics. DeReu (2019) states that trigonometry is not learnt in isolation and its knowledge is required in pre-calculus and calculus; however, learners often experience challenges in recalling trigonometric skills. Exploring the learning of trigonometry, more specifically trigonometric identities was seen as crucial since trigonometry has a major impact in the achievement of good results in mathematics. Niranjan (2013) further states that in the high school curriculum, trigonometry may be regarded as an inseparable component of mathematics, much of which is a product of trigonometric relationships and algebraic techniques.

1.3 Statement of the problem

Kamber and Takaci (2018) stated that trigonometry is a crucial yet difficult component of the mathematics curriculum, which connects algebraic and geometric ways of thinking. However, the results of a study conducted by Rohimah and Prabawanto (2019) showed that learners experienced difficulty in solving trigonometric equation-problems such as having a challenge in factorising trigonometric equations in quadratic form. Thus, they suggested that how the learning of trigonometry is done in the classroom needs to be reviewed.

Among the factors that hinder learners from learning trigonometry effectively is that of memorising information without grounding concepts (Mutodi, 2015). Chigonga (2016) is of the view that learners experience difficulty in solving trigonometry problems because they only have procedural knowledge and lack the necessary conceptual knowledge. Rohimah and Prabawanto (2019) further state that in reviewing the learning of trigonometry, possible errors and misunderstandings need to be identified before learners are taught so as to overcome learners' difficulties in solving trigonometry problems.

DeReu (2019) is of the view that the way in which trigonometry is presented by teachers in mathematics classrooms could possibly be amongst the causes of the difficulty of learning trigonometry. Kamber and Takaci (2018) mentioned that trigonometry connects algebraic and geometric ways of thinking, such that if learners do not understand algebra they are likely to perform poorly in trigonometry. Sarría, Gonzάlez, Magreñάn, Narvάez and Orcos (2017) suggest that algebraic language promotes thinking based on pattern recognition and analysis; and problem solving and reasoning skills; however, some learners experience difficulty due to the high level of abstraction in algebra.

In stating that "it's no secret that right triangle trigonometry is a notoriously difficult subject that both teachers and learners struggle in", DeReu (2019, p. 2) further confirms that trigonometry is still a challenge for teachers and learners. Hence, the conduction of this study was seen as important. Mutodi (2015) further states that learners that have a well-developed conceptual understanding; procedural understanding and representation knowledge also know more than one problem-solving technique which helps them be successful in mathematics.

1.4 Purpose of the study

The purpose of this study was to explore the learning of trigonometric identities by grade 11 mathematics learners. The study intended to gain insight on learners' understanding of trigonometric identities. This includes finding out what learners know in trigonometric identities such as learners' strengths in solving problems related to trigonometric identities. The study also explored learners' encounters when learning trigonometric identities, these encounters included possible challenges; misconceptions and weaknesses in solving problems related to trigonometric identities.

Rohimah and Prabawanto (2019) stated that trigonometry is among the topics that are studied in high school mathematics, in which learners often encounter difficulties. According to Retnowati and Maulidya (2018), problem- solving in trigonometry is complex because it does not only involve trigonometric ratios, but involves other knowledge bases such as the Pythagoras theorem, which could be another factor contributing to learners' difficulties.

In a study conducted by Sholehwati and Wahyudin (2019) it became evident that learners lacked good critical thinking skills, but had good visual thinking skills in trigonometry; hence they suggested that teachers should find and employ learning methods that can best assist learners to understand the concepts of trigonometry well. This will in turn develop learners' enthusiasm for solving trigonometric problems and ultimately improve learners' performance in trigonometry.

Thus it was crucial that a study was conducted that aimed to explore the learning of trigonometric identities by grade 11 mathematics learners. The findings and recommendations on how learners learn trigonometric identities might be of good use to mathematics teachers in finding and employing teaching methods that will make the learning of trigonometric identities easier and a better experience for learners.

1.5 Location of the study

The study is conducted in one secondary school which is situated in the rural areas of Port Shepstone, in the Ugu district in KwaZulu-Natal. The school's learner enrollment is around eight hundred with staff members around thirty. The school is a no-fee school, and learners that attend the school come from different socio-economic backgrounds. The study was conducted over a period of one year.

1.6 Motivation of the study

This study was motivated by:

- The researchers' professional experiences
- The researchers' personal experiences
- Learners' difficulty in understanding trigonometric identities

1.6.1 Professional experiences

In the researcher's experience as a mathematics teacher and through discussions with other mathematics teachers, it was observed that most learners experience challenges in trigonometry, especially manipulations that require algebraic knowledge. Thus the researcher was interested in exploring the learning of trigonometry (trigonometric identities) by learners', so as to gain insight on their experiences and encounters. It was also observed that trigonometric identities are a branch of trigonometry that are not only learnt in isolation, but they have their roots in trigonometry in 2D and 3D and are also applied in double angle identities.

Trigonometric identities are also applied when working with the general solution, which suggests that they form an integral part of trigonometry. For examination purposes in mathematics, learners write two papers. Trigonometry is examined in paper two and constitutes about 40% of the paper. In the FET phase, learners are required to obtain a minimum of 30% to pass mathematics; this suggests that if learners master this topic, they are likely to pass paper two. It is therefore important that learners do well in trigonometry.

1.6.2 Personal experiences

As a high school learner, trigonometry became the researcher's favorite topic in mathematics when it was introduced in grade 10. Unfortunately this was not the case with the researcher's classmates whom the researcher would often assist because some of these classmates did not like mathematics. The researcher was among the few learners that liked mathematics and felt that it was an interesting subject. Similarly, DeReu (2019) confirms that most learners do not like trigonometry and feel that it is not an easy topic, which is what the researcher has also recognised based on past experiences.

This experience interested the researcher to find out what really makes learners dislike trigonometry and how learners learn trigonometry and what they encounter when learning trigonometry (trigonometric identities).

1.6.3 Learners' difficulty in understanding trigonometric identities

As a mathematics teacher, the researcher has observed that learners display a lack of knowledge when working with trigonometric identities. The learners' lack of understanding of trigonometric identities are rooted in a number of mathematics topics, this includes their lack of knowledge in algebra such as factorising; fractions; working with like and unlike terms; etc., and trigonometric ratios. In trigonometric ratios, learners are unable to recognise the hypotenuse; adjacent side, and opposite side in relation to a given angle.

As a result, learners end up memorising procedures. In addition, some learners do not understand that trigonometric ratios can only be applied in right-angled triangles; and instead they erroneously apply them in any triangle. In proving trigonometric identities, learners are required to make use of trigonometric ratios. This suggests that it is crucial that learners understand trigonometric ratios before they are introduced to trigonometric identities. This made exploring the learning of trigonometric identities a topic worth researching.

1.7 Objectives of the study

This study addresses the following key objectives:

1. To discover the learning of trigonometric identities by grade 11 mathematics learners.
2. To gain insight on the encounters of grade 11 mathematics learners when learning trigonometric identities.

The outlined key objectives were the focus of the study in order to explore and understand grade 11 mathematics learners learning of trigonometric identities.

1.8 Research questions

This study explored the learning of trigonometric identities in grade 11 mathematics, specifically focusing on the use of the two fundamental identities $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and $sin^2\theta +$

$cos^2\theta = 1$, that is deriving and using these identities in solving and proving mathematical problems, in order to answer the following key research questions:

1. How do grade 11 mathematics learners learn trigonometric identities?
2. What are the encounters of grade 11 mathematics learners when learning trigonometric identities?

1.9 Research methods

This study intended to explore the learning of trigonometric identities by grade 11 mathematics learners. In researching this study, qualitative research methods were employed. Niranjan (2013, p. 42) suggests that qualitative research methodology allows the use of various research strategies in collecting data, and allows the voice of the participants to be heard. Teherani, Martimianakis, Stenfors-Hayes, Wadhwa and Varpio (2015) also suggest that in qualitative studies researchers enquire the experiences of individuals on life aspects, in the case of this study, learners' learning of trigonometric identities was explored. Aspects relating to research methodology are discussed in chapter 4. For brevity, in this study, two data collection tools were used:

1.9.1 Worksheet

Each participant was asked to complete an activity worksheet on their own (see Appendix attached). The activity worksheet involved trigonometry problems with specific reference to trigonometric identities.

1.9.2 Semi-structured interviews

Each participant was interviewed individually (see Appendix attached). A few interview questions were based on participants' encounters with mathematics, and the rest of the questions were based on participants' encounters with trigonometric identities.

1.10 Sampling

Aspects relating to sampling are discussed in chapter 4, for brevity, ten learners were purposively selected to participate in the study. To increase the validity of the study, the worksheet and semi-structured interview questions were piloted with six grade 11 mathematics learners.

1.11 Analysis of data

Thematic data analysis methods were employed to analyse data. A brief discussion on the analysis of data from the activity worksheet and semi-structured interviews is outlined.

1.11.1 Worksheet

Themes were developed according to question numbers and participant's errors. Patterns were further identified among the categories of learners' responses.

1.11.2 Semi-structured interview

Semi-structured interviews were individually conducted with six grade 11 mathematics learners. The semi-structured interviews generated data on:

- Learners' understanding of concepts and calculations related to the learning of trigonometric identities in grade 11 mathematics
- Learners' encounters when learning trigonometric identities in grade 11 mathematics
- Learners' misconceptions when learning trigonometric identities in grade 11 mathematics

Audio-recorded responses of semi-structured interviews were transcribed, word for word, by the researcher. All responses were grouped to a particular question, thereafter, thematic analysis methods were undertaken.

1.12 Ethical considerations

Ethical considerations are discussed in chapter 4, for brevity the following key points are outlined:

- Gatekeepers' permission was sought.
- All participants were issued a letter of contest.
- Parents/ guardians were issued a letter of contest.
- The researcher was transparent with regards to participant's freedom of withdrawal.

- Participants' confidentiality was ensured through the use of pseudonyms.
- Full details of participants and school understudy were not provided.
- Participants were issued copies of the findings of the study.
- Worksheet and interview questions were piloted.
- Participants were constantly reminded of their role and rights.

1.13 Limitations of the study

Limitations of the study are discussed in chapter 6, for brevity the following key points are outlined:

- The study had a small sample size, and therefore generalisations could not be made.
- Only one school was studied.
- The study focused on one aspect of trigonometry, namely, trigonometric identities.
- Language barriers were amongst the limitations to this study.
- Withdrawal from participating in the study by one of the participants.
- Some of the participants were not willing to be audio-recorded during interview sessions.

1.14 Definition of trigonometric ratios

In relation to a right-angled triangle, there are six trigonometric ratios. These trigonometric ratios are ratios of two sides of a triangle (right –angled) and a related angle (Niranjan, 2013). In calculating the value of an unknown angle or the magnitude of an unknown angle, trigonometric ratios are applied (Simons, 2016). The six functions are related and are defined in terms of each other. Definition of trigonometric ratios:

- Sine: $\sin \Theta = \frac{opposite}{hypotenuse} = \frac{1}{cosec\,\Theta}$
- Cosine: $\cos \Theta = \frac{adjacent}{hypotenuse} = \frac{1}{\sec \Theta}$
- Tangent: $\tan \Theta = \frac{opposite}{adjacent} = \frac{1}{\cot \Theta}$
- Cosecant: $cosec\ \Theta = \frac{hypotenuse}{opposite} = \frac{1}{\sin \Theta}$
- Secant: $\sec \Theta = \frac{hypotenuse}{adjacent} = \frac{1}{\cos \Theta}$
- Cotangent: $\cot \Theta = \frac{adjacent}{opposite} = \frac{1}{\tan \Theta}$

The acronym SOHCAHTOA is used to recall the ratio of the three functions sine; cosine and tangent (Niranjan, 2013). The ratio of the functions sine; cosine, and tangent are used to prove the fundamental trigonometric identities.

1.15 Overview of the study

In this chapter, the key reasons for conducting this study have been outlined. In the subsequent chapters, further reasons are advanced and clarified in detail. This study is organised into six chapters with the following content:

Chapter1

In this chapter, the background; location, and purpose of the study are outlined. A discussion of the problem statement is made. The motivation of the study is also discussed. An outline of the objectives of the study; research questions, research methods, and sampling are made. This chapter also provides a brief discussion of data analysis, and an outline of the ethical considerations and limitations of the study.

Chapter 2

This chapter focuses on literature relevant to the learning of trigonometric identities in grade 11 mathematics. The chapter presents a literature review which supports this study. This chapter provides a discussion of the South African education; mathematics and trigonometric identities. Challenges and misconceptions are also outlined in this chapter.

Chapter 3

This chapter presents the conceptual framework for this study, which forms the basis for analysis and arguments brought forward. In this chapter, a brief description of mathematical proficiency; proficiency in mathematical activity and proficiency in the mathematical work of teaching is given. The chapter further provides a discussion of the five strands of mathematical proficiency as outlined by Kilpatrick and how each of the strands may be cultivated in the learning of trigonometric identities in grade 11 mathematics.

Chapter 4

This chapter begins with the reintroduction of the critical research questions, and then offers an overview of the methodology. In this chapter, the researcher describes the methodological framework applied to answer the critical questions, and the paradigm within which this study is located is discussed. The chapter also provides an explanation of the approach used to generate and analyse data, and the research style used. This chapter also discusses the sampling method and the ethical issues considered in the conduction of the study.

Chapter 5

This chapter offers a discussion of data that is analysed from the learners' written responses and their responses to semi-structured interviews.

Chapter 6

This chapter presents the general findings of the study, and also responds to the key questions of the study by providing a discussion of the findings. The study also outlines learners' suggestions on how the learning of trigonometric identities may be improved, provides the recommendations, suggestions for further studies, limitations of the study, and themes for further studies.

In the appendices section of this study, the activity worksheet, a semi-structured interview schedule, consent forms signed by parents and learners, gatekeeper's letter, research office ethical clearance certificate, editorial certificate, and a summary of the turn-it-in report, are included.

1.16 Synopsis

This chapter offered a brief discussion of the background and location and purpose of the study. A discussion of the rationale that stimulated the researcher's interest in conducting the study was also made. The chapter also outlined the objectives of the study, research questions, research methods, analysis of data and overview of the study. In the next chapter, a discussion of literature relevant to the learning of trigonometric identities is reviewed.

CHAPTER TWO
LITERATURE REVIEW

2.1 Introduction

This chapter focuses on literature relevant to the learning of trigonometric identities and which supports this study. Trigonometric identities are a branch of trigonometry which was, and still is a substantial component of the grade 12 mathematics examination (Niranjan, 2013). The education system of South Africa experienced change when policy-makers revised curricula in accordance with the Curriculum and Assessment Policy Statement (DoBE, 2012).

According to Van Laren (2012) the revision had a sole intention, which was that of improving teaching and learning of all subjects. Since there is a low pass percentage of grade 12 results in subjects such as mathematics (Ally, Brijlall & Maharaj, 2016; Butgereit, 2007), it becomes important to determine what some of the problems are in mathematics learning and hence this study aims to explore the learning of trigonometric identities in grade 11 mathematics.

In presenting the literature, an overview on the South African education system and an overview of the learning and the importance and challenges of mathematics and trigonometry are made. Finally, the learning of trigonometric identities in relation to the unit circle is outlined, and challenges related to the learning of trigonometric identities are also outlined.

2.2 South African education

Irrespective of the fact that more than twenty years have elapsed since the establishment of constitutional democracy, South Africa still faces an ongoing challenge of ensuring that the right to basic education in the whole country is realised (Lathlean, McConnachie & Sephton, 2014). A study conducted by Modisaotsile (2012) on the failing standard of basic education in South Africa revealed that there are many signs that there is a crisis in education, hence, the enrolment rate is increasing every year yet the grade 12 output, which is the performance of grade 12 learners on national examinations, is declining. However, the relevance of the findings of this study is a concern, hence individual NSC diagnostic reports have revealed a slight improvement in respective subjects over the past few years.

A study on comparative analysis of mathematics syllabi for secondary school and teacher training programs also discovered that there was an ongoing challenge in South Africa of delivering experienced and qualified teachers to teach mathematics and science (Biyela, Sibaya & Sibaya, 2016). This articulates that more concentration needs to be focused on the quality of education. Several factors or problems prevent learners from receiving a good standard education, but it remains the duty of the government to ensure that teachers are trained accordingly, and that schools have adequate basic resources for effective teaching and learning to take place (Modisaotsile, 2012).

2.2.1 Learning in South Africa

According to Christie (1998); Mason and Todd (2005); Msila (2007), and Ramnarian (2016), the legacy of apartheid education, which is very problematic, requires transformation is the lack of functioning of a large number of previously black schools. Ramnarian (2016, p. 598) further states that, "one of the key imperatives in the transformation of education in South Africa is the need to provide quality education for all". Education is regarded as a weapon of transformation and policies have been put in place so as to ensure the occurrence of this transformation (Msila, 2007).

South African education policies stipulate that all learners should be provided an opportunity to participate in all classroom activities, however, the lack of resources and the attitudes and actions of teachers in classrooms still hampers the implementation of inclusive education (Engelbrecht, Nel, Nel, & Tlale, 2015). In defining inclusive education, Engelbrecht, Nel, Nel, and Tlale (2015, p.1) state that "inclusive education is regarded as the right of every learner to be part of mainstream classrooms, which intends to educate learners irrespective of their gender; culture; ability; language; ethnicity and class".

According to Probyn (2015), in South Africa, English is the medium in which a majority of learners learn, irrespective of the fact that it is a home language to only a minority of learners and this lack of English proficiency often limits the access of learners to the curriculum. Similarly to a study conducted by Van der Berg (2008) on the effectiveness of poor schools which revealed that several international tests have confirmed that the South African educational quality lags far behind economically poorer countries; a study on the challenges of social justice also revealed that in South African schools the quality of education is substandard; this is evident by the dismal results as indicated by the second Annual National Assessment (ANA) results of learners in mathematics and literacy skills (Badat & Sayed, 2014).

Bush, Joubert, Kiggundu and van Rooyen (2009) state that for the highest possible standards of learner achievement to be promoted, principals need to ensure that leadership and management is provided in all areas of the school. They state that this is necessary to enable the creation of conditions under which high quality teaching and learning occurs. However, a study on enacting understanding of inclusion in complex contexts revealed that education changes often place new demands on the teacher, hence, classrooms contain a heterogeneous mix of learners from different backgrounds making effective teaching difficult (Engelbrecht, Nel, Nel, & Tlale, 2015). Bush, Joubert, Kiggundu and van Rooyen (2009) further state that the three fundamental requirements for developing effective teaching and learning are, sufficient and suitable learning material; sound classroom practises, and proactive management of learning.

2.2.2 Learning mathematics in South Africa

"In South Africa, mathematics is regarded as a crucial subject because competencies in this subject are a key requirement for economic development" (Butgereit, 2007, p.1). However, a study investigating the impact of e-learning support material found that learners are not performing well in this subject. The low pass percentage by learners who exit school at Grade 12 is a serious concern in South Africa (Ally, Brijlall & Maharaj, 2016; Butgereit, 2007). In contrast, the 2018 mathematics analysis document reveals that there has been a steady improvement in performance over the last few years, suggesting that there is now some degree of stability in the subject. According to research conducted by the Human Sciences Research Council (HSRS), South African grade 8 learners occupied the last position among 38 countries in the Third International Mathematics and Science Study (Brijlall & Niranjan, 2015).

This indicates that obtaining high pass rates in grade 12 mathematics might still be a challenge even in 2019, when these learners reach grade 12. In a study on difficult topics in junior secondary school mathematics, Yushau (2013) discovered that learners' low performance in mathematics could be attributed by the following factors: lack of mathematics foundation, lack of motivation from teachers and parents, and learners' beliefs about mathematics. The following table outlines learners' performance in the National Seniors Certificate (NSC) mathematics 2017 examination, in the nine provinces of South Africa.

Table 2.1: A summary of candidates' provincial performance in the 2017 mathematics NSC examination. Adapted from the Ugu district Further Education and Training, 2018 analysis document (p. 2).

Province	**Pass percentage in 2017**
Eastern Cape	42.3%
Free State	70.6%
Gauteng	67.7%
KwaZulu-Natal	41.6%
Limpopo	50.1%
Mpumalanga	47.8%
North West	61.2%
Northern Cape	57.4%
Western Cape	73.9%
National pass percentage: 51.9%	

From Table 2.1, it is evident that KwaZulu-Natal obtained the lowest pass percentage of all the provinces, which is below the national pass percentage. This therefore suggests that learners in KZN experienced challenges in answering the 2017 NSC mathematics examination questions. According to the 2018 moderators' report, learners' performance may be enhanced if teachers and learners pay attention to areas such as, strengthening the content knowledge in trigonometry and learners' exposure to complex and problem-solving type of questions, which should start in earlier grades.

During the teaching and learning process of mathematics, learners face several obstacles because problem solving skills are very complicated in this subject (Mensah, 2017). The challenges that learners experience at elementary school level, which further prevents their success in problem solving, as cited by Chapman (2015) are as follows:

- Lack of understanding of vocabulary used in problem solving
- Failure to understand a problem in whole or in part, due to lack of practice in solving problems
- Lack of methods of attacking problems
- Confusion of processes, leading to the random trialing of any processes

- Insufficiency of ability in fundamentals
- Lack of knowledge of important facts, rules, and formulas
- Ignoring principles, rules, or processes underlying the correct solution of problems
- Insufficient mastery of computational skills
- Inability to perform the computations involved, either through forgetting of the procedure or failure to learn it

In the 2018 moderators report, in the overview of learners' performance in paper 2, it became evident that candidates struggled with trigonometry and Euclidean geometry, and the number of candidates who did not attempt questions based on these topics is a cause for concern.

A deeper understanding of mathematics means that learners can represent mathematical ideas in many ways (numerically, graphically, symbolically, etc.) can make meaningful connections within mathematics, can understand the underpinnings of mathematical concepts, and can use appropriate mathematical vocabulary (Charles, 2015). Trigonometry is among the topics that seem to be challenging learners in mathematics.

Challenger (2009) and Aslan-Tutak and Sarac (2017) assert that learners hate trigonometry and feel that it is a complicated section of mathematics. An interlinked three-way relationship exists between a teacher, mathematics and learners, and any change in one of these interlinks influences the other two links.

2.3 Mathematics

Wilson (2015), posits that there are three main ways to explain the origin of mathematics, namely, foundational philosophies; humanistic philosophies of mathematics and mathematical platonism. The foundational philosophies' point of view is that mathematics develops from axioms and is defined using logic; the humanistic philosophies perspective is that humans are the source of math and are the ones who invent it; and the mathematical platonism' view is that "mathematics exists 'out there' to be discovered (Wilson, 2015, p.2). Similarly, to Wilson (2015), Baki and Gürsoy (2018, p.78) state that "mathematics is always considered as something waiting to be discovered".

According to Hernandez-Martinez and Vos (2018) the primary reason most countries include mathematics in their curriculum is that it is useful in societies, hence it serves as an effective tool for thinking and communicating, and that it is essential in many professions. Bhagat, Chang and Chang (2016) also suggest that in the school curriculum, mathematics is one of the important subjects, and in fact mathematical knowledge is needed by people to make informed decisions both as citizens and as workers. Hernandez-Martinez and Vos (2018) further state that one of the reasons mathematics is taught is that it teaches learners to think logically.

However, a study on candidates' views on using the history of mathematics revealed that learners' are of the view that mathematics is disconnected from their daily lives and other disciplines, which causes a gap between them and understanding it (Baki and Gürsoy, 2018). Rohimah and Prabawanto (2019) are of the view that learners have low abilities in other mathematical fields because they experience difficulties in learning mathematics. The abstract nature of mathematics is one of the reasons learners develop fear of it, thus, in addition to teaching mathematics, teachers are confronted with the challenge of eliminating negative thoughts that learners have about mathematics (Baki and Gürsoy, 2018).

Table 2.2: An outline of content progression in trigonometry, Grade 10 to 12. Adapted from the Mathematics CAPS document, FET phase (p.23; p.33 & p.42).

TRIGONOMETRY		
GRADE 10	**GRADE 11**	**GRADE 12**
i) Defining the trigonometric ratios sin ϴ, cos ϴ and tan ϴ, using right-angled triangles. ii) Extension of the definitions of sin ϴ, cos ϴ and tan ϴ, in the interval: $0° \leq ϴ \leq 360°$ iii) Defining the reciprocals of the trigonometric ratios cosec ϴ, sec ϴ and cot ϴ, using right-angled triangles (the reciprocals cosec ϴ, sec ϴ and cot ϴ are only examinable in grade 10). iv) Deriving the values of the trigonometric ratios for the special cases (without the use of a calculator), ϴ ϵ {0°, 30°, 45°, 60°, 90°}. v) Solving two- dimensional problems in right-angled triangles. vi) Solving simple trigonometric equations for angles in the interval 0° to 90°. vii) Using diagrams to determine the numerical value of the ratios for angles between 0° and 360°.	i) Deriving and using the identities: $\tan \Theta = \frac{\sin \Theta}{\cos \Theta}$ and $sin^2\Theta + cos^2\Theta = 1$ ii) Deriving and using reduction formulae to simplify expressions. iii) Determining which values of a variable an identity holds. iv) Determining the general solution of trigonometric equations (also, solutions in specific time intervals).	i) Compound angle identities. ii) Double angle identities.
viii) Solving problems in two dimensions.	v) Proving and applying the sine, cosine and area rule, and solving problems in two dimensions using these rules.	iii) Solving problems in two and three dimensions.

Table 2.2 shows the progression of content as outlined in the CAPS document. The CAPS document does not only provide an outline of content progression, but also provides clarity on the content to be taught. Shield and Dole (2013) posits that in the mathematics curriculum documents that the importance of equipping learners with in-depth knowledge of the curriculum is emphasised as opposed to a superficial coverage of the curriculum.

2.3.1 Trigonometry in Mathematics

Trigonometry is a branch of mathematics that deals with the relationship between sides of a triangle and angles formed at the vertices of those triangles. Charles (2015); Gür (2009), and Mensah (2017) stated that the word trigonometry originates from the Greek words trigonon which refers to triangles, and metron, which refers to the science of measuring. Trigonometry is usually used when new ideas and concepts are being defined and used to form in mathematical explanations, for example, trigonometric ratios are used when describing the relationship between angles and the sides of a right-angled triangle (Simons, 2016).

In the South African mathematics curriculum, trigonometry is introduced in grade10. Learners progress with this topic to grades 11 and 12. In the Curriculum Assessment Policy Statement (CAPS) document, a specification of content progression for trigonometry from grade 10 to grade 12 is clearly outlined (Brijlall & Maharaj 2014; Naidoo & Naidoo, 2016; Simons, 2016).

2.3.2 The importance of trigonometry in Mathematics

In mathematics education, trigonometry is an important component. Sections that are particularly important in trigonometry include transformations and trigonometric identities, but learners and even teachers have trouble in articulating and justifying trigonometric concepts (Bornstein, 2017). Trigonometry is not only common in pure mathematical studies, but is also widespread in physical applications (Bornstein, 2017; Simons, 2016). In trigonometry, many sophisticated mathematical concepts are required. Real world problems involving trigonometry are common in fields such as engineering, construction, design, and physics (Simons, 2016).

It is therefore crucial that learners develop basic trigonometric knowledge in high school to be able to pursue their studies in fields such as the ones mentioned above. Furthermore, a good understanding of trigonometry is crucial in the solving of advanced mathematics tasks, since it equips learners with comprehensive knowledge of the necessary mathematical concepts (Koyunkaya, 2016).

2.3.3 Challenges in the learning of trigonometry in Mathematics

Regardless of being challenging, trigonometry remains an important topic in the curriculum in secondary schools because it assists learners in developing intellectual tactics (Aslan-Tutak & Sarac, 2017). The following sub-headings are discussed: the inadequate training of mathematics teachers; exploring the language barriers when teaching trigonometry in mathematics; exploring the use of fractions in trigonometry; the lack of resource when teaching trigonometry in mathematics and overcrowding in mathematics classrooms in the learning of trigonometry.

2.3.3.1 The inadequate training of Mathematics teachers

O'Meara, Fitzmaurice and Johnson (2017) are of the view that barriers encountered by learners in mathematics are as a result of teaching knowledge and methodologies of their teachers or built in the topic itself. It has been discovered that both learners and teachers do not enjoy certain topics in mathematics because they believe that they are difficult to learn or to teach (Yushau, 2013). According to Brijlall and Niranjan (2015), mathematics teachers are frequently reminded of the low performance of learners in mathematics.

Challenges that lead to high failure rates in mathematics include the fact that there are not enough adequately trained teachers to teach the subject (Biyela, Sibaya & Sibaya, 2016; Butgereit, 2007; Chikiwa & Schäfer, 2017). Thus, the South African Minister of Education is of the view that inadequacies in teacher knowledge are a major cause of serious problems in the South African education system (Modisaotsile, 2012). However, currently, the issue of inadequate training of mathematics teachers may necessarily not account for failure rates in mathematics, hence studies such as trends in qualification of South African mathematics teachers have revealed that indeed many schools are well supplied with qualified mathematics teachers (Long & Wendt, 2019).

Yushau (2013) claims that continuous low levels of performance in mathematics are attributed to deficient instructional techniques and inadequate teachers' content knowledge of the subject matter. Another factor that might contribute to inadequate teachers' content knowledge is that the university content provided to student-teachers is not completely in line with the syllabi that are offered in schools. The syllabi in public schools are designed by the National Department of Basic Education (DBE) and the South African education faculties design syllabi which are then offered to pre-service teachers. Biyela, Sibaya and Sibaya (2016) confirm that there is no consensus with regards to the basic knowledge that student teachers receive during their training, which is a problem for learners and teachers. However, it cannot be overlooked that most universities work with schools at district level through the conduction of workshops for teachers, this means that there is a link between universities and school based curriculum.

A recent study conducted in Turkey investigated the relationship between teacher efficacy to student trigonometry self-efficacy and student trigonometry. The results showed how important it is for teachers to show enthusiasm when teaching trigonometry for student trigonometry self-efficacy (Aslan-Tutak & Sarac, 2017). Phorabatho and Mafora (2015) also mention that teachers need to be updated timeously with developments through continuing professional development since they play a crucial role in the effective implementation of the curriculum.

2.3.3.2 Exploring the language barriers when teaching trigonometry in Mathematics

In the 1990s, language barriers were one of the factors hindering the effective learning of mathematical concepts. Scholars such as Capps and Cox (1991) assert that irrespective of learners' abilities, mathematics teachers' must recognise that content cannot be taught without language. In fact, language issues remain a barrier in the effective learning of content. This is made evident by Chikiwa (2015) who stated that trigonometry consists of technical mathematical terms that are not usually used in everyday life of the learners and teachers. Capps and Cox (1991) mention that it is crucial that teachers succeed in making learners understand the symbols and the language of the specific content area they are teaching for every lesson taught.

A study conducted by Chikiwa (2015) on teaching trigonometry in a grade 11 multilingual mathematics class further supports this claim by revealing that in trigonometry a lot of symbols and specific language are used. Furthermore, it very important for teachers to unpack concepts in their content area. In the high school curriculum, trigonometry is a mathematical rich sub-register that connects concepts in geometry and other mathematics sub-registers such as proofs and arithmetic. Trigonometry has numerous situations whereby concepts and processes are embedded in one object; this makes it very important that teachers be careful with their language use when explaining processes and concepts.

With trigonometric concepts, a majority of learners in township and rural schools are taught in a language that is not commonly spoken in the communities they live in and it is therefore not their first language (Biyela, Sibaya & Sibaya, 2016; Butgereit, 2007; Chikiwa & Schäfer, 2017; Kersaint, Thompson, & Petkova, 2013). It is crucial to note that even though English is not a home language for a majority of learners in townships and rural schools, these learners' encounter with the language is now not only at school level, hence they watch television shows that are broadcasted in English, thus suggesting that they are gradually getting used to the language. Adler (2001) suggests that teacher's language practices must scaffold learners' entry into mathematical discourse. Code switching, which refers to the use of two or more languages when communicating, is used by teachers to incorporate learners' first language during the teaching and learning process (Chikiwa, 2015).

"Code switching enables learners to carry out intellectually challenging tasks or to convey the meaning of concepts and ideas more precisely" (Naidoo, 2016, p. 375). Chikiwa and Schäfer (2017) further propose that another factor that contributes to challenges associated with language is the fact that most mathematics classes in South Africa are multilingual, which presents teachers with numerous tasks, such as that of translating some concepts to the learners' first language to ensure that learners understand concepts.

2.3.3.3 Exploring the use of fractions in trigonometry

Torbeyns, Schneider, Xin and Siegler (2015) state that fractions are very important in numerical development however, according to the integrated theory of numerical development fractions are complex because they are a ratio of two whole numbers. In stating that fractions are one of the fundamental topics in mathematics that have been reported as difficult for learners to learn and for teachers to teach, Victoria, Fauzi and Ananda (2017) agrees that fractions are challenging to learners.

In studying the centrality of understanding fractions to mathematics achievement in students from three different continents, Torbeyns, Schneider, Xin and Siegler (2015) discovered that it was easier to acquire arithmetic skills and numerical understanding for whole numbers, as compared to fractions. Irrespective of the presentation of various strategies for fractions to be taught, the learners ability to work with fractions has not improved (Victoria, Fauzi & Ananda; 2017). However, mathematical achievement and the magnitude to which fractions are understood are correlated, this therefore supports the essential role of fractions in the achievement of mathematics understanding (Torbeyns, Schneider, Xin & Siegler, 2015).

A study on students' full understanding of trigonometric ratios revealed that for learners, ratios are extremely difficult to understand, which could possibly be one of the reasons why learners find it difficult to understand trigonometry (Jung-A, Jae-Geun & Kyeong Hwa, 2013). A right-angled triangle is a three-sided figure that is made up of three angles, two of which are acute and one of which has a magnitude of ninety degrees (Yushau, 2013). In a right-angled triangle, the longest side that is opposite the right angle is called the hypotenuse. From the acute angle Θ, the other two sides are the adjacent and hypotenuse sides, to their position to Θ on the triangle (Pienaar, 2014).

The trigonometric functions are tangent; cosine; sine, and their inverses are cotangent; secant and cosecant, respectively. In the South African curriculum, cotangent, secant, and cosecant are only learnt in grade 10 (CAPS document). Sine and cosine are used to express all other trigonometric functions. SOH CAH TOA is a mnemonic that is used as a memory device to remember the definitions for trigonometric ratios (Jung-A, Jae-Geun & Kyeong Hwa, 2013).

2.3.3.4 Exploring the lack of resources when teaching trigonometry in Mathematics

Kagenyi (2016, p.40) states that "the importance of the use of teaching aids cannot be ignored in the teaching and learning of trigonometry in secondary schools", hence it plays a crucial role in the facilitation of teaching and learning concepts and it increases effectiveness on information processing thus giving meaning to words. In addition, it helps in attracting learners' attention therefore increasing their level of focus and assisting the teacher in relating abstractness to concreteness.

In concurrence with the above statement, Khumalo and Mji (2014) found that lack of infrastructure and facilities has a negative effect on learning in South African schools, in a study on teachers' perceptions on the impact of poor infrastructure on teaching and learning in rural South African schools. Since a majority of learners' regard trigonometry as a difficult topic in mathematics, the use of appropriate teaching aids is very important (Kagenyi, 2016). In consensus with Kegenyi (2016), Visser, Juan and Feza (2015) state that the creation of an environment conducive for learning is very important for learners' academic achievement and that most South African studies have revealed that availability or scarcity of school resources has a great impact on educational outcomes.

In fact, studies have found that a high availability of resources is linked to better educational outcomes. Furthermore, Kagenyi (2016) suggests that trigonometry can be made interesting if teaching aids were to include activities that enhance learner's participation in the teaching and learning process, since this would enrich teacher-learner participation in trigonometry. However, it is crucial to note that as teachers are becoming technologically advanced, the use of teaching aids has also improved thus enhancing the teaching and learning process.

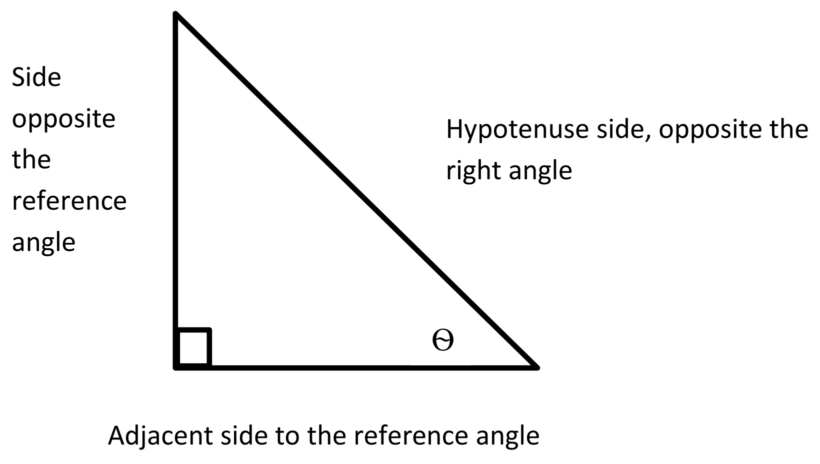
2.3.3.5 Overcrowding in Mathematics classrooms in the learning of trigonometry

In some South African schools, overcrowded classrooms are a major problem with broad impacts for both teachers and learners (Marais, 2016). Bansilal and Rosenberg (2016) suggest that irrespective of the dawn of democracy that South Africa experienced in 1994, producing improved learning outcomes has remained a struggle in the education system and in most cases, teachers teach under difficult conditions, partly because of overcrowding.

Marais (2016) further states that overcrowded classrooms prevent proactive teaching and create difficult learning environments. In fact, effective teaching and assessment are not practiced because teachers are not able to use a variety of methods such as active learning approaches and higher order questioning due to the large number of learners. George and Adu (2018) suggest that the environment in which learners are taught mathematics is very important because learners' performance depends on it and both teachers and learners are motivated by a good learning environment.

Learners perform better in classes that do not have a large number of learners because in such classes they receive more individualised instruction and support from the teachers whereas larger classes are noisier and, as a result, teachers spend most of their time disciplining and trying to control learners, thus losing valuable teaching time (Marais, 2016). George and Adu (2018) shares the same sentiment as Marais (2016) in stating that research studies indicate that a few manageable learners in a classroom do perform better than a large number of learners who are overcrowded in a classroom.

In a study conducted by Bansilal and Rosenberg (2016) to identify problems of practice reported by teachers that teach under disadvantaged conditions in South Africa found that most teachers cited the issue of large classes as a problem, with some having as many as 78 learners in their classrooms. The teachers stated that large classes made the assessment of classwork and homework a very stressful task. Marais (2016) suggests that training institutions for student teachers need to ensure that they offer programs that will enable student teachers to cope with the teaching profession demands such as teaching in overcrowded classrooms.

Ratio	**Right- angled triangle**
	
tan Θ	$\frac{o}{a} = \frac{opposite}{adjacent} = \frac{y}{x}$
cos Θ	$\frac{a}{h} = \frac{adjacent}{hypotenuse} = \frac{x}{r}$
sin Θ	$\frac{o}{h} = \frac{opposite}{hypotenuse} = \frac{y}{r}$

cot Θ	$\frac{a}{o} = \frac{adjacent}{opposite} = \frac{x}{y}$
sec Θ	$\frac{h}{a} = \frac{hypotenuse}{adjacent} = \frac{r}{x}$
cosec Θ	$\frac{h}{o} = \frac{hypotenuse}{opposite} = \frac{r}{y}$

Figure 2.1: Trigonometric ratios. Adapted from Pienaar (2014, p.44).

For many high school learners, learning trigonometry is a challenge and prevents their access to careers in mathematics, technology, science and engineering (Bornstein, 2017; Simons, 2016). Koyunkaya (2016) makes it evident that research on learners' understanding of trigonometry is very limited, even though many studies have presented useful findings related to teachers' and learners' difficulties with the concept. The research suggests that it is not only the learners' who have trouble when dealing with trigonometry, but also the teachers who often have a limited and shallow knowledge of trig. Hence this study will study in depth the grade 11 learners' understanding of trigonometric identities.

In a study exploring the use of manipulatives to support an embodied approach to learning trigonometry it became evident that manipulatives were important mediating tools in the development of conceptual and procedural understanding of mathematical concepts (Brijlall & Niranjan, 2015). Yushau (2013) cites the following as problems that also contribute to the ineffective learning of trigonometry (sine, cosine and tangent of a triangle):

- Teachers' inability to define the concepts sine, cosine and tangent appropriately
- Large class syndrome
- Inadequate teaching and learning resources
- Lack of a variety of techniques that can be used by teachers to teach learners how to calculate sine, cosine and tangent of an acute angle using a right-angled triangle
- Unsatisfactory presentation of instructions

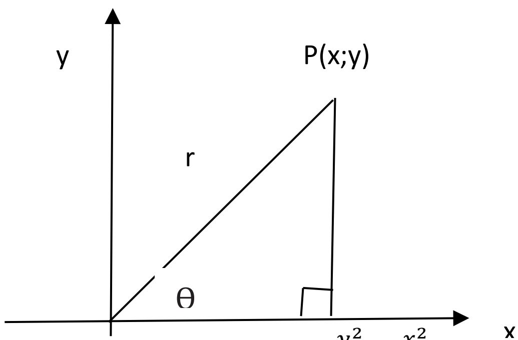
2.4 Trigonometric identities

Learners are at the centre on the process of mathematics learning. In the development of mathematical thinking in classrooms, the assumption is that learners are active cognitive subjects (Valero, 2005). As outlined in the CAPS document, learners are introduced to trigonometric identities in grade 11.

Aird and van Duyn (2012) define a trigonometric identity as a mathematical statement that is true for all values of the angle except those for which the statement is not defined. The two standard trigonometric identities that are learnt in grade 11 are $\tan\theta = \frac{\sin\theta}{cos\theta}$ and $cos^2\theta + sin^2\theta = 1$ (Bradley, Campbell and McPetrie, 2012). An identity consists of two sides, namely, the Left-Hand Side (LHS) and the Right-Hand Side (RHS), and to prove an identity you must show that the RHS=LHS (Aird & van Duyn, 2012).

Proof of the two standard trigonometric equations:

Proof of $\tan\theta = \frac{\sin\theta}{cos\theta}$



LHS: $\tan\theta = \frac{y}{x}$ by definition

RHS: $\frac{sin\theta}{cos\theta} = \frac{\frac{y}{r}}{\frac{x}{r}}$ by definition

$= \frac{y}{r} \times \frac{r}{x}$

$= \frac{y}{x}$

Proof of $cos^2\theta + sin^2\theta = 1$

$LHS = \frac{y^2}{r^2} + \frac{x^2}{r^2}$ by definition

$= \frac{y^2+x^2}{r^2}$ common denominator

but $y^2 + x^2 = r^2$ by Pythagoras theorem

$= \frac{r^2}{r^2} = 1$

Figure 2.2: Proof of the fundamental identities. Adapted from Aird & van Duyn (2012, p.226) and Bradley et al (2012, p.142).

According to Bradley et al (2012, p.142), the following steps may be followed when proving trigonometric identities:

- Make $\tan\theta = \frac{\sin\theta}{cos\theta}$ or $\frac{1}{tan\theta} = \frac{\cos\theta}{sin\ \theta}$ so that all the ratios are in terms of sin and cos
- Consider the LHS or RHS as an algebraic expression and use algebraic manipulations to simplify further
- If there are fractions, find the Lowest Common Denominator (LCD) and add
- Factorise where possible
- Use the square identity where possible in any of the forms: $cos^2\theta + sin^2\theta = 1$ or $cos^2\theta = 1 - sin^2\theta$ or $sin^2\theta = 1 - cos^2\theta$
- Simplify both sides of the identity as far as possible

In the curriculum, trigonometric identities link algebraic and geometric reasoning (Koyunkaya, 2016). Hence, it is crucial that learners have a good understanding of algebra and geometry so that they will not experience challenges when solving mathematical problems related to trigonometric identities that require both geometric and algebraic reasoning.

It is also crucial to note that the 2018 moderators report revealed that mathematics cannot be studied in compartments and that learners' are expected to be able to apply knowledge from one section to another section of work. Jung-A, Jae-Geun and Kyeong-Hwa (2013, p.45) support this in stating that "growth of understanding is not linear, higher level of understanding needs the lower level of understanding".

2.4.1 Exploring the conceptual change approach

Teachers are confronted with the problem of ascertaining what learners understand in mathematics and how this understanding grows and changes over time; the kind of learning that is required when learners' prior knowledge comes into conflict with new information to be learnt in the classroom is called conceptual change (Price and van Jaarsveld, 2017). Multiple choice questions and short response items are typical ways of assessing conceptual understanding.

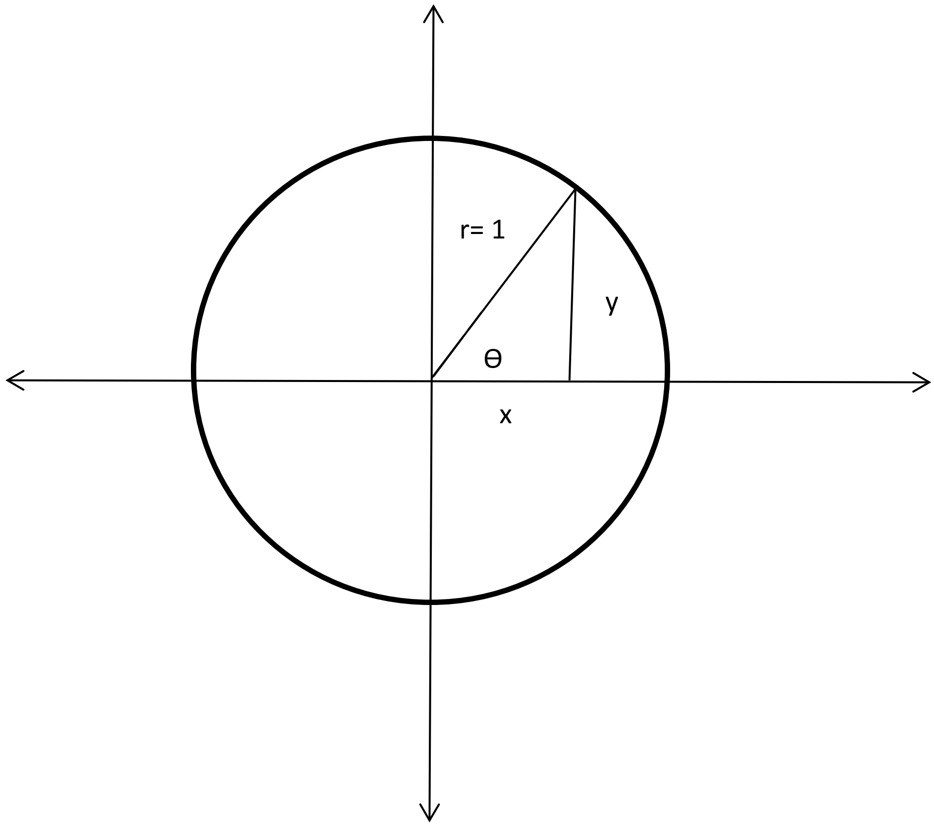
It is in situations of such conflict of knowledge that major reorganisation of prior knowledge is required; even though conceptual change is a form of learning, it is crucial to note that it is different from other forms of learning in the sense that it utilises different instructional interventions (O'Meara, Fitzmaurice & Johnson, 2017). In a study conducted by Price and van Jaarsveld (2017) on the use of open- response tasks to reveal learners' conceptual understanding, it became evident that whilst most learning is based on the enrichment of existing knowledge, conceptual change cannot be accomplished through additive mechanisms, and in fact one of the major causes of misconceptions is the use of additive mechanisms in situations that require conceptual change.

The 2018 moderator's report also revealed that learners' struggled with concepts in the curriculum that required deeper conceptual understanding, and that many of the errors made by learners had their origins in a poor understanding of the basics and foundational competencies taught in earlier grades.

2.4.2 Exploring the unit-circle in the learning of trigonometric identities

A study conducted by Koyunkaya (2016) on mathematics education graduate students' understanding of trigonometric ratios discovered that the unit circle may be utilised as a concept to cultivate conceptual change when trigonometric identities are introduced. Similarly to Koyunkaya's (2016) findings, Mickey and McClelland (2017) also found that the unit circle acts as an integrated conceptual structure that supports solving problems encountered during learning, in a study on making use of the unit circle as a grounded conceptual structure in Pre-Calculus trigonometry. Mutodi (2015) is of the view that the unit circle is an important presentation to solving trigonometry-related problems.

Figure 2.3: Proof of the square identity using the unit circle (moving from the known to the new concept):



Polar coordinates:

$x = r\cos\Theta$ $\quad$ $y = r\sin\Theta$

$= (1)\cos\Theta$ $\quad$ $= (1)\sin\Theta$

$= \cos\Theta$ $\quad$ $= \sin\Theta$

From the theorem of Pythagoras

$r^2 = x^2 + y^2$

$(1)^2 = (\cos\Theta)^2 + (\sin\Theta)^2$

$\therefore 1 = cos^2\Theta + sin^2\Theta$

According to Johnson (2016), a circle that is centered at the origin and has a radius of 1 unit is called the unit circle. By applying the formulae for the distance between two points, the origin $(0; 0)$, and $(x; y)$ the endpoint of the radial line, and by considering that by the basic definition of the unit circle $r = 1$, leads to the Pythagoras theorem $r^2 = x^2 + y^2$ which learners' are introduced to in grade 8 (Mickey & McClelland, 2017).

Rohmah and Ekawati (2018) describe a proof as information that shows that a mathematical statement is true. In cultivating conceptual change, in the sense of moving from what learners know to introducing a new concept, Mickey and McClelland (2017) suggest that in grade 11 when learners are introduced to the fundamental trigonometric identities, it's a matter of replacing x with $cos\ \Theta$ and y with $sin\ \Theta$ to obtain the square identity $cos^2\Theta + sin^2\Theta = 1$ using the idea of the square identity, $r^2 = x^2 + y^2$, of the Pythagoras theorem.

In grade 11, when learners are working with problems that involve trigonometric identities they are usually required to make use of algebraic skills to simplify or to prove that the Left-Hand Side is equal to the Right-Hand Side. Mickey and McClelland (2017) are in agreement with this idea and state that algebraic manipulations may be used to derive additional key trigonometric identities. This idea of applying skills of one topic to another topic usually becomes a challenge for most learners. This is also made evident by Burhanzade and Aygör (2015) who found in a study they did that for the questions that combined two topics, learners had difficulty in solving problems; these authors also cite misconceptions; process errors, and lack of information as problems learners face in solving problems.

Bradley et al (2012) mentioned that the square identity comes in three forms:
$cos^2\Theta + sin^2\Theta = 1$ which is the basic identity, or $cos^2\Theta = 1 - sin^2\Theta$ which is obtained by transposing $sin^2\Theta$ or $sin^2\Theta = 1 - cos^2\Theta$ which is obtained by transposing $cos^2\Theta$. It usually becomes a challenge for learners to apply the difference of two squares in trigonometry, for example, being able to note that $(1 - \cos\Theta)(1 + \cos\Theta)$ is the expanded form of $1 - cos^2\Theta$ and $(1 - \sin\Theta)(1 + \sin\Theta)$ is the expanded form of $1 - sin^2\Theta$.

In Burhanzade and Aygör's (2015) study, it became evident that learners have major inadequacies in mathematical fundamentals, factorisation, and are therefore not able to reach conclusions when solving problems even if they know the solutions, they do not have the required understanding. Learners also experience challenges in choosing the correct identities when simplifying. Other misconceptions include thinking that $sin\ \Theta + cos\ \Theta = 1$, $cos\ \Theta = 1 - sin\ \Theta$ and $sin\ \Theta = 1 - cos\ \Theta$. Rohmah and Ekawati (2018) state that in trigonometry that learners usually experience challenges in manipulating formulae.

Burhanzade and Aygör (2015) mentioned that some of the learners that participated in their study were not able to realise when they had to factorise by using identities. As a result, they came to the conclusion that learners lacked major knowledge in the fundamentals of mathematics. In preliminary studies that were reviewed by Mickey and McClelland (2017), it became evident that learners' who made use of the unit circle to visualise trigonometric quantities performed better than learners' who didn't make use of the model.

2.4.3 Misconceptions

Misinterpretations and confusions that arise from incorrect meanings are called misconceptions (Ojose, 2015). In mathematics, problem solving is a skill that is very complicated and as a result, learners' face numerous obstacles during the teaching and learning process (Mensah, 2017). Ojose (2015) further states that in mathematics, learners have misconceptions regarding various concepts in all grades. According to Mensah (2017), studies that have been conducted have shown that learners lack the development of concepts in trigonometry.

Durkin and Rittle-Johnson (2014) posit that assessing learners is required during the conceptual change process, in diagnosing misconceptions and how knowledge changes. The researcher has observed that little is being said about the challenges experienced by learners when learning trigonometry. Koyunkaya (2016) states that irrespective of the importance of trigonometry in advanced mathematical thinking, interest in the study of learners' trigonometric concepts has been minimal; one of the reasons being that trigonometry forms a small part in the mathematics curriculum. Hence, this study explored challenges experienced by grade 11 mathematics learners when learning trigonometric identities.

The NSC mathematics 2018 moderator's report revealed that learners' performance in mathematics may be enhanced if content knowledge is strengthened in trigonometry and learners' exposure to complex and problem solving type questions, thus suggesting these areas are of a great concern. Exposing learners' to complex questions and problem solving across all topics in the curriculum should start in earlier grades.

According to the NSC mathematics 2019 moderator's report, the questions that tested learners' trigonometric skills were poorly answered. Learners' failed to work with fractions in trigonometry, with others not attempting the questions at all. Simplification of trigonometric ratios posed problems to learners, this therefore means that there is a continuous need for revision of concepts such as: trigonometric ratios, co-ratios and subtraction/ addition of trigonometry fractions.

At the end of the study, recommendations will be made on how educators can address challenges experienced by learners when learning trigonometry. The findings and recommendations might assist mathematics educators to teach in a manner that will minimise challenges experienced by learners' when learning trigonometric identities. According to Brijlall and Niranjan (2015), the way in which teachers teach mathematics has to change so as to improve learners' performance such that they are able to keep up with the rest of the world, and so that they will all have access to learn mathematics in higher grades. For learners to have equal access to learn mathematics, teachers and school managers must work together more effectively (Aluka and Shonubi, 2014).

2.5 Conclusion

Chapter two offered an overview of literature relevant to the learning of trigonometric identities in grade 11 mathematics. In this chapter, a discussion of the South African education; mathematics, which includes trigonometry and its importance; and the learning of trigonometric identities has been provided. Chapter three focuses on the conceptual framework of this study.

CHAPTER THREE
CONCEPTUAL FRAMEWORK

3.1 Introduction

Chapter two offered an overview of literature relevant to the learning of trigonometric identities in grade 11 mathematics. This chapter begins with an outline of what a conceptual framework is; the framework that underlies this study and its significance to the study. The researcher then provides a detailed discussion of the framework that has been applied in this study. This study has been framed by the principles of Kilpatrick's strands of mathematical proficiency.

The researcher also provides a discussion on how each strand is relevant to the learning of trigonometric identities in grade 11 mathematics. As Simons (2016) suggests, trigonometry requires a number of sophisticated mathematical concepts and is regularly used in mathematical explanations and definitions of new ideas and concepts, it therefore crucial that educators equip learners with the necessary mathematical skills during the learning of mathematics.

In this study the strands of mathematical proficiency are used to explore whether learners possess the necessary trigonometric proficiencies. Kilpatrick, Swafford and Findell (2001) proposed five strands of mathematical proficiency, which are, conceptual understanding, procedural fluency, adaptive reasoning, strategic competency, and productive disposition. In this chapter, the five strands of mathematical proficiency and how they may be enhanced during the teaching and learning process is discussed in detail.

3.2 Conceptual framework

Bertram and Christiansen (2014, p. 117) describe a theory as "a well-developed, coherent explanation for an event that provides a possible explanation for why things happen". Similarly, Rule and John (2011) as cited in Mthembu (2015, p. 33) suggest that a theory may be defined as a set of ideas that intends to provide an exposition about a particular phenomenon. In defining a conceptual framework, Bell (2005) as cited in Niranjan (2013, p. 32) stated that a conceptual framework is a structure that provides the necessary grounds on which a study may be constructed.

Mthembu (2015) also suggest that a good framework is simple, coherent and falsifiable, and gives an account of the phenomenon under study. Niranjan (2013, p. 32) further suggests that "a conceptual framework should make sense and facilitate contextual understanding of the findings of a research study for other researchers". Bertram and Christiansen (2014) further suggest that some researchers conduct their study with the aim of proving whether a theory is true or false, whilst others use a theory to frame their study. In this study, a conceptual framework is used to frame the study and to assess grade 11 learners' mathematical proficiency

3.3 Defining mathematical proficiency

Wilson and Heid (2011) define mathematical proficiency as the dynamic use of the mathematical knowledge a person possesses. Alex and Mammem (2018, p. 46) state that "learning with understanding has increasingly received attention from teachers, and has progressively been elevated to one of the most important goals for all learners". Mathematical proficiency comprises aspects of mathematical knowledge and ability such as conceptual understanding and procedural fluency needed by learners in mathematics (Wilson & Heid, 2011).

Mabotja, Chuene, Maoto and Kibirige (2018), suggest that it is important that learners' do not only develop a good conceptual understanding of geometric properties, but also be able to apply such properties in solving geometric problems. Wilson and Heid (2011) further mention that mathematical proficiency for secondary schools is different from the mathematical proficiency needed in primary schools because topics studied at a secondary level have a wider content range; there is a greater emphasis on mathematical proofs; and greater attention is paid to mathematical structure and abstraction, such as identities; domain; inverses; etc. Figure 3 shows the three components of proficiency.

Mathematical Proficiency for Teaching (MPT) is said to have three overlapping components, namely, Mathematical Activity (MA); Mathematical Proficiency (MP) and Mathematical Work for Teaching (MWT).

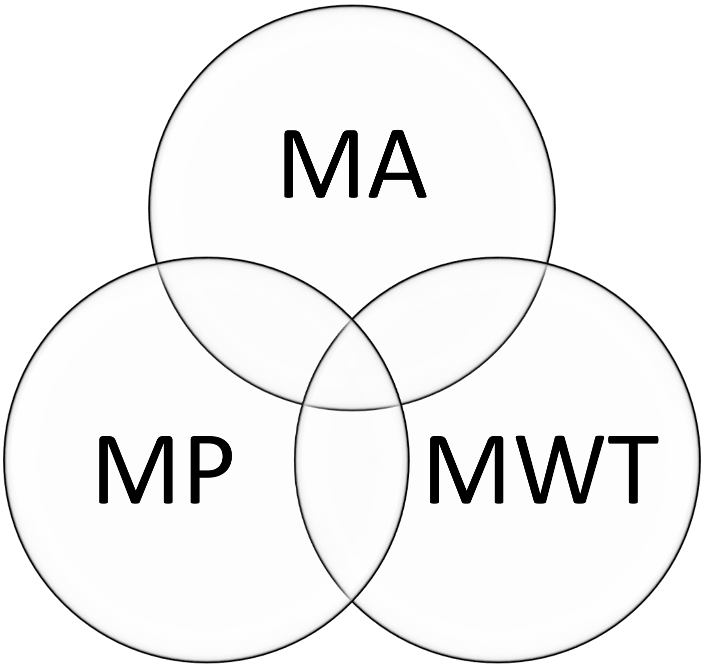


Figure 3.1: The three components of Mathematical proficiency for teaching. Adapted from Wilson and Heid (2011, p. 3)

3.3.1 Mathematical Proficiency

Learners' development of mathematical proficiency is usually highly dependent on how well developed their teachers' proficiency is (Wilson & Heid, 2011). Groves (2012) suggests that mathematical proficiency comprises more than just skill or understanding, hence it is important for learners to develop all five proficiency components simultaneously. To be able to foster mathematical proficiency in their learners', mathematics secondary school teachers need to have a well-developed mathematical proficiency and their learners should have learned enough in primary school to have a good mathematical proficiency for when they take mathematics or any subject related to it, at higher grade levels and at university (Wilson & Heid, 2011).

3.3.2 Proficiency in Mathematical Activity

Proficiency in mathematical activity may be defined as doing mathematics; this proficiency is encountered on a daily basis as teachers and learners engage in the teaching and learning process of mathematics (Wilson & Heid, 2011). In the learning of mathematics, mathematical problem solving is central (Groves, 2012). Proficiency in mathematical activity overlaps the mathematical proficiency component, but greatly emphasises activities that teachers want their learners to learn, examples include representing objects and operations; connecting concepts; and justifying mathematical arguments in solving and proving trigonometric identities (Wilson & Heid, 2011).

3.3.3 Proficiency in the Mathematical Work of Teaching

Wilson and Heid (2011) suggest that one of the aspects of proficiency in the mathematical work of teaching is understanding learners' mathematical thinking such as recognising the mathematical nature of their errors and misconceptions. Proficiency in the mathematical work of teaching is seen as applicable in this study, hence, one of the objectives of this study is to understand the learning of trigonometric identities by grade 11 mathematics learners, which includes recognising misconceptions they may have.

Groves (2012) is of the view that teachers need to design mathematical problems that support learners' mathematical proficiency. Wilson and Heid (2011) further suggest that teachers also need to be able to decide whether a mathematical proof is wrong or incomplete, how a well suggested solution satisfies the conditions of a problem, and whether an alternative solution is equivalent to any proposed solution.

3.4 The principles of Kilpatricks's strands of mathematical proficiency

Wilson, Heid, Zbiek, Wilson, O'Kelley, McClintock and Gleason (2010) suggest that mathematics learnt in secondary schools goes far beyond the learning of facts, routines and strategies; but includes to a great extent interrelated mathematical concepts, ways to represent and communicate those concepts, and tools for solving all kinds of mathematical problems. As described by Kilpatrick, Swafford, and Findell (2001), mathematical proficiency can be used as a tool to capture what is necessary for anyone to learn mathematics successfully.
Hence in this study, mathematical proficiency is used as a framework tool in the learning of trigonometric identities in grade 11 mathematics. Furthermore, mathematics learnt in secondary schools requires reasoning and creativity, and provides learners with mathematical knowledge while also laying a foundation for further studies in mathematics and other disciplines (Wilson et al, 2010).

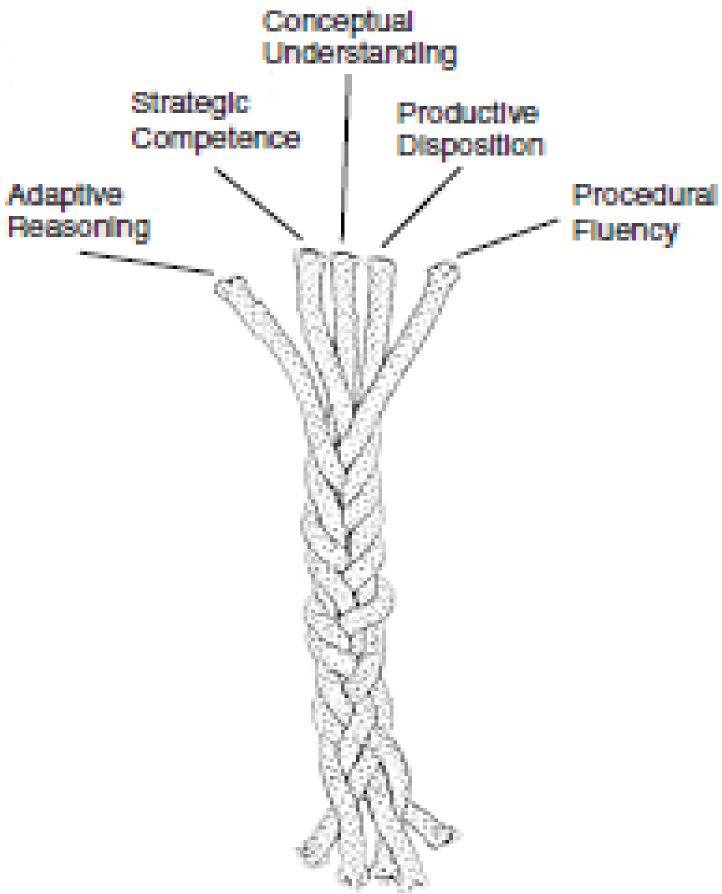


Figure 3.2: The intertwined strands of Mathematical proficiency. Adapted from Groves (2012, p. 121)

As illustrated in figure 3.2, the five strands of mathematical proficiency are conceptual understanding, procedural fluency, strategic competency, adaptive reasoning, and productive disposition (Kilpatrick, Swafford & Findell, 2001; Kilpatrick, 2009). Chapman (2015) mentions that the five strands of mathematical proficiency are not independent, but interdependent in the development of proficiency, hence they represent aspects of a complex whole. Chapman (2015, p. 21) also states that "Mathematical proficiency is not a one dimensional trait, and cannot be achieved by focusing on just one or two of the five strands of Mathematical proficiency".

3.4.1 Conceptual understanding

Conceptual understanding refers to comprehension of mathematical concepts, operations, and relations (Kilpatrick, Swafford & Findell, 2001; Kilpatrick, 2009). For instance, in proving trigonometric identities, learners are expected to be able to comprehend the necessary mathematical concepts, such as the Pythagoras theorem; adjacent; hypotenuse and opposite, with reference to sides of a triangle. Simons (2016) asserts that trigonometry is a branch of mathematics that deals with relationships of sides and angles in triangles. Kilpatrick et al. (2001) is of the view that conceptual understanding refers to an integrated and functional grasp of mathematical ideas, hence learners with conceptual understanding know more than isolated facts and methods.

Wilson and Heid (2011) suggest that conceptual understanding may be described as the "knowing why" of mathematical proficiency, which is demonstrated if a person is able to derive formulae without memorising them; evaluate the correctness of an answer and understand mathematical connections. In this study, learners may demonstrate conceptual understanding if they are able to prove the fundamental identities, $\tan\Theta = \frac{\sin\Theta}{\cos\Theta}$ and $sin^2\Theta + cos^2\Theta = 1$, making use of the relevant diagram.

Kilpatrick et al. (2001) further mention that learners that have a conceptual understanding understand why a mathematical idea is important and the kinds of contexts in which it is useful; their knowledge is also organised into a coherent whole, which enables them to learn new ideas by connecting those ideas to what they already know.

In this study, conceptual understanding is seen is applicable and in the review of literature it became evident that trigonometry consists of many sophisticated mathematical concepts which in addition to a good understanding of trigonometry is crucial in the solving of advanced mathematics tasks (Koyunkaya, 2016). It is therefore crucial that learners develop conceptual understanding in the learning of trigonometric identities in grade 11 mathematics.

3.4.2 Procedural fluency

Procedural fluency refers to skill in carrying out procedures flexibly, accurately, efficiently, and appropriately (Kilpatrick, Swafford & Findell, 2001; Kilpatrick, 2009). According to Wilson and Heid (2011) if a learner is able to identify the conditions for when and how a procedure may be applied and can apply it competently, that learner is said to be procedurally fluent, however it doesn't allow deriving new uses for a previously learned procedure.

Sometimes learners are required to perform routine procedure when solving problems in mathematics, however misconceptions may prevail. For example, in trigonometric identities it would be incorrect to assume that $sin\Theta + \cos\Theta = 1$ because $sin^2\Theta + cos^2\Theta = 1$. Wilson and Heid (2011) further suggest that procedural fluency is important because being able to recall and accurately carry out procedures assists in finding solutions to mathematical problems.

When learners are examined in mathematics, the level of questioning is set in accordance with the four cognitive levels based on Bloom's taxonomy. One of the cognitive levels is knowledge, which is defined as testing situations that emphasise recalling. This cognitive level is closely related to procedural fluency, which refers to carrying out procedures accurately (Kilpatrick et al., 2001).

Chapman (2015) cites the inability to perform computations involved, either through forgetting of the procedure or failure to learn it, as one of the factors that prevent learners' from successfully solving mathematical problems. Furthermore, in the 2018 moderator's report, it was evident that learners' had a poor understanding of the basics and foundational competencies in l topics such as trigonometry. It is therefore crucial to test whether learners' possess procedural knowledge and fluency in the learning of trigonometric identities in grade 11 mathematics.

3.4.3 Adaptive reasoning

Adaptive reasoning refers to the capacity for logical thought, reflection, explanation, and justification (Kilpatrick, Swafford & Findell, 2001; Kilpatrick, 2009). In solving trigonometric identity problems, sometimes learners are required to logically use aspects from other mathematics topics. According to Groves (2012), successful mathematical learning comprises much more than just knowledge of skills and procedures. Adapting reasoning may also be referred to as the ability to recognise existing assumptions, and adjust to changes in assumptions and conventions; this involves the comparison of assumptions and working in different, mathematical contexts (Wilson & Heid, 2011). For example being able to recognise that algebraic manipulations applied in factorising:

- $x^2 + x = x(x + 1)$
- $x^2 - 1 = (x + 1)(x - 1)$

- $x^2 + 2xy + y^2 = (x + y)(x + y)$

are the same manipulations applied when factorising in a different mathematical context, namely, trigonometry:

- $cos^2\theta + \cos\theta = \cos\theta\ (\cos\theta + 1)$
- $1 - sin^2\theta = (1 - \sin\theta)(1 + \sin\theta)$
- $1 + 2\sin\theta\cos\theta = sin^2\theta + 2\sin\theta\cos\theta + cos^2\theta = (\sin\theta + \cos\theta)\ (\sin\theta + \cos\theta)$

The 2018 mathematics analysis document is highlighted that learners' algebraic skills are poor, and that they lack fundamental and basic mathematical competencies which they are supposed to have acquired in the lower grades (that is grades lower than grade 12). The 2018 mathematics analysis document also mentioned that the integration of mathematics topics is a challenge for many learners, and it is expected that learners be able to apply knowledge from one section to another section since mathematics topics should not be learnt in isolation.

Maoto, Masha and Mokwana (2018) state that proof and reasoning should be an ongoing part of classroom discussions, irrespective of the mathematics topic being learnt. This means that learners should be encouraged to engage in proofs since providing relevant reasons in all the topics could possibly enhance their mathematical skills. In the review of literature, it became evident that amongst the factors that prevent learners' success in problem solving is that they lack knowledge of important facts, rules, and formulas (Chapman, 2015).

As stated by Kilpatrick et al (2001), adaptive reasoning is the ability to think logically about the connections among situations and concepts. However, Chapman (2015) mentions that learners ignore principles, rules or processes underlying the solutions to problems. Furthermore, Kilpatrick et al (2001) suggests that adaptive reasoning is referred to as the "glue" that holds everything together in the learning of mathematics. It is therefore important to test whether grade 11 learners possess adaptive reasoning in the learning of trigonometric identities.

3.4.4 Strategic competency

According to Kilpatrick et al (2001), strategic competence may be defined as the ability to formulate, represent and solve mathematical problems; this strand is said to be similar to problem solving and problem formulation, hence the characteristics of learners' that are good problem solvers are also discussed. Additionally, learners must be able to show strategic competencies when solving problems, such as, for example, being able to see that $(1 + \sin x)(1 - \sin x)$ is actually an expansion (factorised form) of $1 - sin^2 x$.

Kilpatrick et al. (2001) further suggest that in order for successful problem solving to take place, teachers need to create a classroom climate that supports problem solving. In other words problems given to learners need to be sufficiently challenging to interest them, but not so difficult as to frustrate them. Wilson and Heid (2011) suggest that strategic competency requires both conceptual understanding and procedural fluency since this gives learners the ability to generate, evaluate and implement problem-solving strategies.

According to Chapman (2015) it is crucial that teachers possess knowledge of mathematical problem solving, not only because they are problem solvers but to assist learners to acquire relevant problem solving skills. Wilson and Heid (2011, p. 8) further suggest that strategic competency may also be described as “knowing how” but differs from procedural fluency in that it requires added creativity and flexibility.

3.4.4.1 Characteristics of good problem-solvers

According to Chapman (2015, p. 28) the characteristics of good problem solvers are the following:

- The ability to understand how the problem is structured.
- The ability to visualise and interpret facts and relationships.
- The ability to apprehend the structural features of the problem.
- Having a generalised memory of mathematical relations and proofs.
- The ability to understand mathematical terms, to note differences and likenesses, and to select correct procedures.
- The ability to evaluate, select, and make use of alternative solutions.
- The ability to analyse a problem before doing any calculations.
- The ability to demonstrate great flexibility in conducting mathematical mental processes.
- The tendency to strive for clarity, simplicity, and rationality in solutions.

- The ability to analyse information in a problem quickly, accurately, and with greater confidence than unsuccessful problem solvers.

In the review of literature, Hernandez-Martinez and Vos (2018) made it evident that one of the reasons that mathematics is taught is that it teaches learners to think logically. Groves (2012), states that problem solving is central to the learning of mathematics. However, in the 2018 Analysis document, it is suggested that learners' have to be exposed to problem solving across all topics in the mathematics curriculum in order for performance to be enhanced. Problem solving and strategic competency are similar, since they both help learners to be able to formulate, problems, represent, and solve mathematical problems (Kilpatrick et al., 2001), and therefore strategic competency is applicable in this study.

Groves (2012) further suggests that if learners accept and embrace the challenge of tackling unfamiliar problems for which they do not know the solutions, that this can be viewed as genuine problem solving. Rohmah and Ekawati (2018), state that trigonometry learners usually experience challenges in manipulating formula. It is therefore crucial to test whether learners possess strategic competency in the learning of trigonometric identities in grade 11 mathematics.

3.4.5 Productive disposition

Productive disposition refers to the habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy (Kilpatrick, Swafford & Findell, 2001; Kilpatrick, 2009). Productive disposition goes beyond performing procedures or coming up with strategies when solving problems. Wilson and Heid (2011, p. 8) state that learners' that possess productive disposition "are curious and enthusiastic about mathematics and are therefore motivated to see a problem through to its conclusion, even if that involves thinking about the problem for an extended time so as to make progress".

Kilpatrick et al. (2001, p. 131) state that "if students are to develop the five strands of Mathematical proficiency, they must believe that mathematics is understandable and that with diligent effort, it can be learned and used". In the review of literature, it is highlighted that trigonometry is not only common in pure mathematical studies, but is also widespread in physical applications (Bornstein, 2017; Simons, 2016). It is also evident that real world problems involving trigonometry are common in fields such as engineering, construction, design, and physics (Simons, 2016).

It is therefore important that during the teaching and learning process, learners become equipped with mathematical skills that will assist them beyond the school level. This therefore suggests that learners must be exposed to challenging problems that improve their mathematical ability rather than only being exposed to problems that require routine procedures. Productive disposition is seen as applicable in this study, hence, it is a strand that if a learner possesses, he or she is able to notice mathematics in the world around him or her, and is able to apply mathematical principles to - situations outside of the classroom (Wilson & Heid, 2011).

3.5 Creating a learning environment that cultivates mathematical proficiency

Schoenfeld and Kilpatrick (2008) suggest that a proficient teachers' knowledge should be both broad and deep, in the sense that the teacher should be able to conceptualise grade-level content knowledge in a number of ways, represent knowledge in different ways, understand the key aspect of different topics, and see connections with other topics. If teachers are proficient with the mathematical content taught in schools, they can equip their learners with good skills, thus cultivating different strands of mathematical proficiency.

In each of the following sub-headings, the researcher provides a discussion of how conceptual understanding, procedural fluency, adaptive reasoning, strategic competency, and productive disposition may be cultivated in the learning of trigonometric identities in grade 11 mathematics.

3.5.1 Cultivating conceptual understanding

Alex and Mammem (2018) suggest that in order for mathematics, in general, and geometry in particular, to be taught meaningfully, the development of learners' conceptual understanding is crucial. Rittle-Johnson and Schneider (2015) suggest that mathematical competency relies on the development of conceptual knowledge, which typically involves proving mathematical definitions and explanations. In the learning of trigonometric identities in grade 11 mathematics, conceptual knowledge may be cultivated when trigonometric identities are introduced. Examples of how this can be done are given in the discussion below.

After explaining and making use of the unit circle to introduce and prove the fundamental identities $\tan\Theta = \frac{\sin\Theta}{\cos\Theta}$ and $sin^2\Theta + cos^2\Theta = 1$, in the square identity, instead of merely making $sin^2\Theta$ or $cos^2\Theta$ the subject of the formula, learners may be asked to make use of the unit circle to prove that $cos^2\Theta = 1 - sin^2\Theta$ or to prove that $sin^2\Theta = 1 - cos^2\Theta$. Another method that is regarded as a good measure to enhance learning that may be used to cultivate conceptual understanding in the learning of trigonometric identities, as suggested by Alex and Mammem (2018) is to identify learners' prior knowledge before teaching commences, for example revisiting trigonometric ratios learnt in grade 10, namely, the concepts of sine; cosine, and tangent.

Figure 2.1 outlined the definition of trigonometric ratios and figure 2.2 outlined how trigonometric identities are proved using trig identities. For example, a learner that understands the concept of trig ratios, the Pythagoras theorem and the square identity, when required to prove that $sin^2\Theta = 1 - cos^2\Theta$ is expected to come up with the following workings:

LHS: $sin^2\Theta$	RHS: $1 - cos^2\Theta$
$= (\frac{y}{r})^2$	$= 1 - (\frac{x}{r})^2$
$= \frac{y^2}{r^2}$	$= 1 - \frac{x^2}{r^2}$
	$= \frac{r^2 - x^2}{r^2}$
	$= \frac{y^2}{r^2}$ (from the Pythagoras theorem: $r^2 - x^2 = y^2$)

Alex and Mammem (2018, p. 46), further mention that "mathematics education should include appropriate emphasis on the teaching of conceptual understanding of mathematics". It is therefore crucial that learners understand the necessary concepts when trigonometric identities are introduced in grade 11.

In this study, when trigonometric identities were introduced, learners' were asked to identify the adjacent, opposite, and hypotenuse side in relation to theta, in a right angled triangle. Learners were then asked to define the trigonometric functions sine; cosine and tangent. In building up from learners' prior knowledge, the quotient identity $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and the square identity $sin^2\theta + cos^2\theta = 1$ were then introduced and proven. Also, the first two questions of the worksheet used to collect data were directed at determining whether learners' possess conceptual understanding in the learning of trigonometric identities in grade 11 mathematics.

Niranjan (2013) is of the view that learners' often rely on pre-existing knowledge when they are developing new knowledge; however, as previously stated in the review of literature, studies that have been conducted in the past revealed that learners' lack the development of concepts in trigonometry (Mensah, 2017). Furthermore, Chapman (2015) cited the lack of understanding of vocabulary used in problem solving as one of the factors that hinder learners' success in problem solving.

It therefore becomes crucial to test whether learners do understand trigonometric concepts, and if learners do not understand concepts, to make recommendations for how teachers may cultivate conceptual understanding in the learning of trigonometric identities. Hence, the cultivation of conceptual understanding is seen as important in this study.

3.5.2 Cultivating procedural fluency

According to Rittle-Johnson and Schneider (2015), procedural knowledge includes to a great extent the solving of problems in which learners need to get accurate answers or follow the correct procedure. Maoto, Masha and Mokwana (2018) suggest that there are many processes to be experienced and developed in the learning of mathematics. Rittle-Johnson and Schneider (2015) further mention that procedural tasks are the type of tasks learners have solved before and should therefore already know procedures for solving such problems.

In the learning of trigonometric identities in grade 11 mathematics, problems that require basic procedural knowledge are, for example, problems that require direct substitution of $\tan\Theta$ when given $\frac{\sin\Theta}{\cos\Theta}$; substituting 1 when given $sin^2\Theta + cos^2\Theta$ or substituting $sin^2\Theta$ when given $1 - cos^2\Theta$. Procedural fluency is useful in the learning of mathematics because if learners are able to quickly recall and accurately perform procedures this significantly assists them in finding solutions to problems (Wilson & Heid, 2011). Sometimes procedural tasks may include problems that learners have not solved before but require recognition of a well-known procedure (Rittle-Johnson & Schneider, 2015).

In this study, in the learning of trigonometric identities, learners were exposed to mathematical problems that required them to carry out procedure accurately and flexibly. Also, on the activity worksheet completed by learners, questions 2.1 and 2.2 were designed to determine whether learners have the necessary skills to carrying out the appropriate procedures. In the review of literature, it became evident that learners fail to understand mathematical problems due to a lack of practise in problem solving; and confusion regarding mathematical processes, and a lack fundamental knowledge (Chapman, 2015).

It is therefore crucial to check whether learners possess the necessary skills to carry out procedures accurately and flexibly when solving problems in the learning of trigonometric identities. Hence, the cultivation of procedural fluency is seen as important in this study.

3.5.3 Cultivating adaptive reasoning

Maoto et al (2018, p. 61) states that, "the development of attributes for proof and reasoning require a culture of multiplicity in the resolution of problems". Wilson and Heid (2011) are of the view that a learner with adaptive reasoning is able to compare assumptions and work in a variety of mathematical systems. Maoto et al (2018) further suggest that in solving mathematical problems, it is not the final answer that is essential but the variety of ways that may be used to get to the solution, and it is thus important that both teachers and learners appreciate this. In the event of cultivating adaptive reasoning in the learning of trigonometric identities, it is crucial that teachers encourage learners to use a variety of problem solving methods.

In the review of literature it became evident that in the mathematics curriculum trigonometric identities link algebraic and geometric reasoning (Koyunkaya, 2016). However, in the 2018 analysis document it became evident that in answering trigonometric questions, learners' demonstrated poor algebraic manipulation skills when squaring fractions and removing common fractions. It is crucial to note that Torbeyns et al (2015) stated that fractions are complex because they are a ratio of two whole numbers and that ratios are extremely difficult to understand, which could possibly be one of the reasons why learners find it difficult to understand trigonometry (Jung-A et al, 2013).

In the 2018 analysis document, it was suggested that learners' need to exercise caution with algebraic manipulation skills since overlooking certain basic principles made them o lose marks unnecessarily. Since adaptive reasoning refers to the capacity for logical thought, reflection, explanation, and justification (Kilpatrick et al, 2001 & Kilpatrick, 2009), cultivation of adaptive reasoning is therefore seen as important in this study.

In the activity worksheet that was completed by learners, some of the questions required learners to apply algebraic skills and other questions required them to have skills in working with fractions. These questions were designed to explore whether learners' possess the necessary skills that require their reasoning to be adaptive when solving trigonometric problems.

3.5.4 Cultivating strategic competency

Chapman (2015) is of the view that a positive classroom climate that supports problem solving must be created by teachers in order for successful problem solving to take place. Jones (2002) as cited in Alex and Mammem (2018, p. 46) suggests that geometry helps learners to develop visualisation skills, critical thinking, problem solving, and deductive reasoning. Hence, it is crucial that strategic competency is enhanced in the learning of trigonometric identities in grade 11. Chapman (2015) further suggests that in cultivating strategic competency, teachers may give learners challenging problems, challenging in the sense that the problems arouse interest and not frustration.

Coute and Vale (2014) suggest that the development of geometrical thought is important for solving problems in learners' daily lives. When giving learners challenging problems to solve, teachers should assist learners when they get stuck, however they must ensure that learners retain ownership of those problems, and avoid providing them with solutions (Chapman, 2015). Alex and Mammem (2018, p. 46) state that "correct terminology of concepts is necessary to avoid misconceptions and confusion", hence, in assisting learners, teachers should always attempt to build on what learners have already discovered, providing corrections only where necessary.

In this study, when trigonometric identities were introduced, the unit circle was used to build up on learners' prior knowledge. As stated in the literature review, the unit circle may be utilised as a concept to cultivate conceptual change when trigonometric identities are introduced (Koyunkaya, 2016). Also, learners were exposed to different levels of questioning, that is, they were given problems to work that ranged from easy to very difficult. The reason they were exposed to very difficult problems was to test how strategic they can be in terms of their reasoning when trying to solve the problems.

Wilson and Heid (2011) suggest that a learner that demonstrates strategic competency is able to generate problem-solving strategies; this includes, making use of a known formula or deriving a new formula; then evaluating the effectiveness of those strategies and accurately implementing the best strategy. Strategic competency is therefore seen as an important part of this study.

3.5.5 Cultivating productive disposition

Once learners have developed conceptual understanding, procedural fluency, strategic competency, and adaptive reasoning abilities, they should realise that mathematics is understandable, and that with effort it can be learnt and applied to real life situations (Kilpatrick et al, 2001 & Kilpatrick, 2009). According to Wilson and Heid (2011), learners with a productive disposition are curious and enthusiastic about mathematics, and are therefore most likely to explore different solutions to problems.

Kilpatrick et al. (2001) and Kilpatrick (2009) also suggest that to develop a productive disposition, opportunities to make sense of mathematics are frequently required; this allows learners to recognise the benefits of perseverance and to experience sense-making in mathematics.

Wilson and Heid (2011) further suggest that learners with a productive disposition are motivated to see mathematical problems through to their conclusion, even if it means they have to think about a problem for an extended time in order to make progress. In the review of literature, it is evident that mathematics performance in KwaZulu-Natal is below national level (Mathematics analysis document, 2018), the researcher is therefore of the view that in order for performance to be improved, learners need to be motivated in all possible ways, hence the cultivation of productive disposition is important in this study.

Kilpatrick et al (2001) and Kilpatrick (2009) further suggest that many learners see mathematics as being totally divorced from their real life experiences; as a result they fail to see that mathematics is a powerful tool in everyday life. It is therefore important that teachers teach in a manner that cultivates productive disposition and that they make connections between the content and its use in the real world.

3.6 Conclusion

This chapter offered an overview of the conceptual framework employed in this study, the principles of Kilpatrick's strands of mathematical proficiency. The chapter began by providing a brief discussion of what a conceptual framework is and its significance in the study. mathematical proficiency has also been defined, providing a discussion of the three components of mathematical proficiency for teaching.

The five strands of mathematical proficiency were then discussed, providing a brief discussion on how each strand may be cultivated in the learning of trigonometric identities in grade 11 mathematics. Chapter four offers an overview of the research methodology employed in this study.

CHAPTER FOUR
RESEARCH METHODOLOGY

4.1 Introduction

Chapter three offered an overview of the conceptual framework applied in this study and a discussion of how the five strands may be enhanced in the learning of trigonometric identities and how these strands are relevant in this study was made. This chapter begins with the reintroduction of the critical research questions, and then offers an overview of the methodology. According to Chilisa and Kawulich (2012), the methodology provides a summary of the research process, which refers to how the research proceeds.

This chapter includes a discussion of the methodological framework that was applied to answer the critical questions, and the paradigm within which this study was located. Furthermore, the researcher explains how the paradigm that was used fits the study. The chapter also provides a detailed explanation on the approach that was used to generate and analyse data, and the research style that was used. This chapter also discusses the sampling method and why the participants were chosen to participate in the study. Lastly, the chapter includes a discussion of the ethical issues that were considered in the conduction of the study.

4.2 Critical research questions

This study sought to explore the learning of trigonometric identities in grade 11 mathematics. The study addressed the following key questions:

1. How do grade 11 mathematics learners learn trigonometric identities?
2. What are the encounters of grade 11 mathematics learners when learning trigonometric identities?

4.3 Geographic location

This study was conducted at a school situated in the rural area of Port Shepstone, in the Ugu district, KwaZulu-Natal, South Africa, since the school fit the purpose of this study. The school in which the study was conducted is a no-fee school, and learners that attend the school come from different backgrounds. For the purpose of the interview sessions, the researcher and the participants met six times, and for the purpose of the completion of the worksheet, the researcher and the participants met two times. The contact times between the researcher and the participants did not affect the teaching and learning process and during these times, participants were also provided with refreshments.

4.4 Qualitative research methodology

Dhlamini (2012) states that a research method refers to a strategy followed in studying a phenomenon in order to obtain information. According to Niranjan (2013, p. 42) qualitative research methodology allows the use of various research strategies for collecting data, and allows the voice of the participants to be heard. Hancock and Algozzine (2006) suggest that quantitative research usually involves the use of instruments such as tests and surveys with the aim of measuring defined variables from a large group of people, whereas qualitative research makes use of observations, focus groups, interviews, and document reviews, in order to obtain detailed information that is used to understand a phenomenon.

In researching a particular study, a researcher may employ quantitative; qualitative, or both methods depending on the aim and fitness of the study (Dhlamini, 2012). Kawulich (2012) posits that qualitative methods are utilised by the interpretive paradigm; however, there are instances where a researcher may use quantitative methods to pursue an interpretative study.

One of the key objectives of this study was to understand the learning of trigonometric identities by grade 11 mathematics learners. Since the objective of the study was to understand a phenomenon, the interpretive paradigm was employed; hence qualitative research methods were also employed. Teherani, Martimianakis, Stenfors-Hayes, Wadhwa and Varpio (2015) confirm that in studies that employ qualitative methods, researchers enquire about the experiences of individuals on life aspects, how people or groups of people behave, the functioning of organisations, and how relationships are shaped by various interactions.

4.4.1 Research design

A research design may be defined as an overview and guide as to how the researcher will systematically collect and analyse data that is needed in order to answer the research question or questions being asked (Bertram & Christiansen, 2014; Lewis, 2015). The following table outlines the steps followed by the researcher in answering the research question in this study.

Table 4.1: An outline of the research design. Adapted from Dhlamini (2012, p.45)

Outline the aims and objectives of the study
↓
Design the methodology of the study
↓
Develop research instruments
↓
Select actual participants and participants for the pilot study
↓
Conduct pilot study
↓
Make alterations to research instruments (if necessary)
↓
Conduct research
↓
Analyse the data
↓
Write a report

In researching this study, the researcher first outlined the aims and objectives of the study and then developed the methodology of the study. The researcher then developed the research instruments, which were used to collect data that were used to answer the critical research questions. The researcher next selected participants for both the pilot and actual study, and thereafter conducted the pilot study so as to ensure that the research instrument did not have errors and was not ambiguous.

The research instruments did not have errors, and thus did not need to be altered. The researcher conducted the study; analysed the data and thereafter wrote a report. The research methodology is discussed in detail.

4.4.2 Research paradigm

A paradigm may be defined as a view of what constitutes knowledge and truth, which guides thoughts, beliefs and assumptions that researchers have about themselves and societies (Babbie, 2007; Chilisa & Kawulich, 2012). The classification of paradigms differs from scholar to scholar, hence in the 1990's, scholars such as Lincoln and Guba (1994) classified paradigms into four main categories, namely, critical theory, constructivism, positivism, and post positivism. As the years progressed, scholars such as Creswell (2007) became of the view that the four main paradigms were pragmatism, social constructivism, post positivism, and advocacy/ participatory. Scotland (2012) was later of the view that the three major educational research paradigms are the interpretive, critical, and scientific paradigm.

This study was researched through the lens of the interpretive paradigm. Bertram and Christiansen (2014) state that within the interpretive paradigm, the purpose of conducting research is to develop a greater understanding of how people make sense of the contexts they live and work in. This study aimed to explore grade 11 mathematics learners' learning of trigonometric identities, and the interpretive paradigm was regarded as suitable for this study since in the process of exploring a particular phenomenon, insight is gained.

The interpretive paradigm also assumes that there are many truths, and as a result, reality is socially constructed (Bertram & Christiansen, 2014). Hence, in interacting with the participants, their responses were not the same since they conceptualised the learning of trigonometric identities in different ways. Furthermore, the interpretive paradigm was a good fit for this study because interviews and a written task were employed as data collection methods, and since Cohen (2011) is of the view that the interpretive approach relies much on observations, analysing written texts, and interviews.

Kawulich (2012) proposes that within the interpretive paradigm, the purpose of conducting research is to understand and describe human nature. Similarly, Scotland (2012) stated that a crucial characteristic of the interpretive paradigm is directed at understanding of a phenomenon from an individual's perspective. This study had the intention of understanding how grade 11 mathematics learners learn trigonometric identities, to explore what challenges they encounter and also to gain more knowledge on their strengths in so far as problem solving is concerned.

Kawulich and Chilisa (2012) posit that researchers' that employ the interpretive paradigm conduct research in order to gain knowledge on peoples' encounters. Epistemology, ontology, methodology, and methods are the four components a paradigm consists of (Scotland, 2012). Methods are the means that are used to collect data and constitute a crucial part of the methodology (Kawulich & Chilisa, 2012). Paradigms have different ontological and epistemological views and therefore have different assumptions of knowledge and truth, which further underpin their approach to research (Scotland, 2012). According to Scotland (2012), every paradigm is based on its ontological and epistemological assumptions; ontology may be defined as a researcher's view on how things work and their standpoint on what constitutes reality whereas, epistemology may be defined as a researcher's view on how people create, acquire, and communicate knowledge.

Interpretivists assume that an individual's perception of truth lies within that individual's experience, hence knowledge is socially constructed and mind independent and therefore subjective (Kawulich & Chilisa, 2012). Scotland (2012) states that the ontological position of the interpretive paradigm is relativism, which refers to the subjectivity of reality, thus suggesting that individuals have their own different realities. In this study, the researcher's ontological position was that in the learning of trigonometric identities, learners' have different conceptual understanding and thus use different problem solving techniques because the way they think is different and subject to their previous encounters.

4.4.3 Research style

In researching this topic, a case study method was employed. Bertram and Christiansen (2014, pg. 33) state that "a case study is an in-depth study of one particular case, where the case may be a person, a group of people, a community, or an organization". This was a study of a group of grade 11 learners, which aimed to explore how they learn trigonometric identities, hence the use of a case study method. The following table shows a comparison of the use of case study methods in different paradigms. In the table, the researcher defines a case study in the context of the paradigm it is used within, and the data collection method that is used thereafter.

Table 4.2: A comparison of the use of case study methods in different paradigms. Adapted from Yazan (2015, p.148 to p.150).

Paradigm	Positivism	Constructivism and existentialism	Constructivism

Defining a case study	A case study is an empirical inquiry conducted to investigate a phenomenon.	Qualitative case study is a study conducted with the aim of understanding the particularity and complexity of a single case.	Qualitative case is an intensive description and analysis of a bounded phenomenon such as a person, institution, program, etc.
Data collection	Researchers' make use of six data collection methods: physical artefacts, participant observation, direct observation, documentation, archival records and interviews	Researchers' utilise observation, interview and document review as data collection tools.	Researchers' make use of three data collection techniques: analysing documents, observations, and conducting interviews.

It is crucial to note that Kawulich (2012, pg.9) states that "Constructivism and interpretivism are related concepts that address understanding the world as others experience it". From the table 2, the researcher observed that case study approaches are employed in qualitative studies. Yazan (2015) is also of the view that in qualitative research methodologies, a case study is commonly used in educational research.

Also, researchers' that employ case study methods use three data collection techniques, namely, analyzing documents, making observations, and conducting interviews. In this study two of these data collection techniques were utilised. A case study method was seen as suitable because the study was based on learners' experiences with learning trigonometric identities and learners' responses were framed by their experiences which differed from learner to learner. According to Bertram and Christiansen (2014) case studies aim to describe what it is like to be in any particular situation, and participants live the experiences and thoughts about a particular deed.

4.5 Research sampling

A population refers to the total number of a defined class of people, events or objects. In researching any population, researchers aim to gather information from every element within a population (O'leary, 2004). The process of selecting representatives of a whole population is called sampling; larger-scale studies employ larger samples of a population, whereas in-depth studies use small, defined and accessible samples of a population (Scotland, 2012).

Bertram and Christiansen (2014) are of the view that sampling refers to making a decision regarding the selection of the participants that are going to participate in the study, and that there are two different types of sampling methods, namely random (probability) sampling and purposive (non-probability) sampling. Random sampling refers to the process by which every element in a population has an equal chance of being selected to be part of a sample, this process is also said to eliminate researcher bias (O'leary, 2004).

Purposive sampling refers to the process by which participants are chosen based on their relevance to the study; the researcher makes specific choices about which people to include in the sample (Babbie, 2007; Bertram & Christiansen, 2014; Scotland, 2012). In this study purposive sampling was employed, hence participants were chosen on the basis that they were grade 11 mathematics learners as the study aimed to explore the learning of trigonometric identities in grade 11 mathematics. The researcher's positionality in the study is that of being a teacher at the school, teaching mathematics in both grade 11 and 12, the sampling was therefore also convenient in that the participants were in the same environment as the researcher. According to Strydom and Delport (2005) qualitative studies usually employ sampling methods that are non-probability, which is what this study does in using purposive sampling.

Scotland (2012) states that it is unlikely for researchers to access every element of a population, hence the need to gather information from a sample and apply the findings to a broader population. According to O'leary (2004) sample size may be defined as the number of people that are needed in a study and it is informed by the nature of the research that is conducted; this includes the generalizability and transferability of the study and the type of data a researcher aims to collect. Since this study was in-depth, ten learners were purposively selected to provide detailed data. This was believed to be suitable since having many grade 11 mathematics learners as participants was thought to be irrelevant because no generalizations were going to be made based on the study. Scotland (2012) stated that researchers can collect qualitative data from an in-depth study with the aim of understanding rather than looking for representatives.

4.6 Data collection methods

According to Bertram and Christiansen (2014), data is the information or evidence that researchers' collect with the aim of answering particular questions and data takes many forms. Furthermore how data is collected is guided by the research question, research paradigm, and research design. Niranjan (2013) mentions that the nature of qualitative research methodology allows the use of different data collection methods; it also allows the voice of participants to be heard. Exploring the learning of trigonometric identities was a qualitative study, researched through the lens of the interpretive paradigm, and hence qualitative methods of data collection were employed.

4.6.1 Data collection instruments

In terms of data generation, this study generated primary data through semi-structured interviews and worksheets. The researcher used these data generation methods because this study was in-depth in nature, as its main objective was to explore the learning of trigonometric identities by grade 11 mathematics learners. Using semi-structured interviews as a data collection instrument became useful because the study aimed to collect descriptive and detailed data. During the interview sessions the researcher conducted the interviews, this enabled the researcher to ask further question(s) on response(s) that needed to be clarified or probed.

Bertram and Christiansen (2014) mention that some of the advantages of conducting interviews is that interviews allow researchers to seeking clarity by asking questions to get detailed information on responses that are not clear. Giving learners worksheets to complete enabled the researcher to identify questions that learners answered incorrectly and was therefore able to provide insight into what the learners found challenging. The worksheet also enabled the researcher to identify questions that learners answered correctly and was therefore able to show what learners' knew about trigonometric identities.

4.6.2 Data collection procedures

The learners were taught trigonometric identities in four 60-minute lessons, thereafter the data collection process began. When trigonometric identities were introduced, geometry learnt in grades 8 to 10 was reviewed, and the unit circle was utilised to integrate learners' prior knowledge with the new concept that was being introduced, namely, the two fundamental identities $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and $sin^2\theta + cos^2\theta = 1$. The lesson was not a normal chalk and chalkboard lesson and instead charts were used as teaching aids.

Semi-structured interviews and learners' written responses to an activity worksheet were the data collection procedures used in this study. Each data collection method is discussed in the following sub-sections.

4.6.2.1 Trigonometry activity worksheet

The learners' were given a trigonometry activity worksheet to complete after four 60-minute lessons. The questions on the worksheet were designed to explore learners' understanding on proving trigonometric identities and solving problems. The questions on the worksheet were structured such that they moved from lower order to higher order thinking, and each question required the comprehension of mathematical concepts; additionally, learners were also expected carry out routine and non-routine procedures when completing the activities and solving the problems on the worksheet.

In a study conducted by Siagan, Saragih and Sinaga (2019) on the development of learning materials oriented on problem- based learning to improve students' mathematical problem solving ability and metacognition ability, it is highlighted that problem solving has not only been the main thing in learning mathematics, but has long been seen as an important aspect of learning mathematics. Hence, in the study the questions on the activity worksheet were not only structured to examine learners' problem solving skills, but from their solutions misconceptions might have emancipated and ultimately reflect learners' encounters when learning trigonometric identities.

The first question required participants to prove the fundamental identities, in the second question participants were required to simplify as far as possible, making use of the fundamental identities and their knowledge of basic algebra. In the last question, participants were required to prove identities; questions from question three required learners to have thorough understanding of algebraic manipulations. A total of eight learners completed the trigonometry activity worksheet.

4.6.2.2 Semi-structured interviews

After the participants completed the trigonometry activity worksheet, phase two of the data collection process began, which was the commencement of the semi-structured interviews. The aim of the interviews was to understand and interpret the participants' responses; and most of the questions were designed to probe and clarify responses.

4.6.3 Data analysis methods

Data analysis refers to the process of making meaning out of raw data. Bertram and Christiansen (2014) cite two main approaches to qualitative data analysis, namely the inductive and deductive approach. According to Arma, Assarroudi, Rad, Sharifi and Heydari (2018) both approaches are used simultaneously in the process of analysing qualitative data, with the inductive approach employed when previous research theories or findings are limited, and the deductive approach employed when previous research theories or findings on the phenomenon under study exist.

In interpreting data that was collected, the researcher applied both inductive and deductive approaches to data analysis. From the information that was collected during the interviews, the researcher looked for patterns in the participants responses so as to generate conclusions. Bertram and Christiansen (2014) also state that deductive reasoning works from the more general to the more specific, based on theory, data is classified in order to look for patterns and connections. In comparison, Arma et al (2018) states that inductive reasoning works from specific observations to broader generalisations in the sense that raw data is collected, patterns and regularities are detected and in the end a general conclusion is reached.

4.6.3.1 Thematic data analysis method

The thematic data analysis method is a technique that is employed in analysing textual data (Braun & Clarke, 2006; Vaismoradi, Jones, Turunen & Sherril, 2016).In employing this method, the following steps were followed:

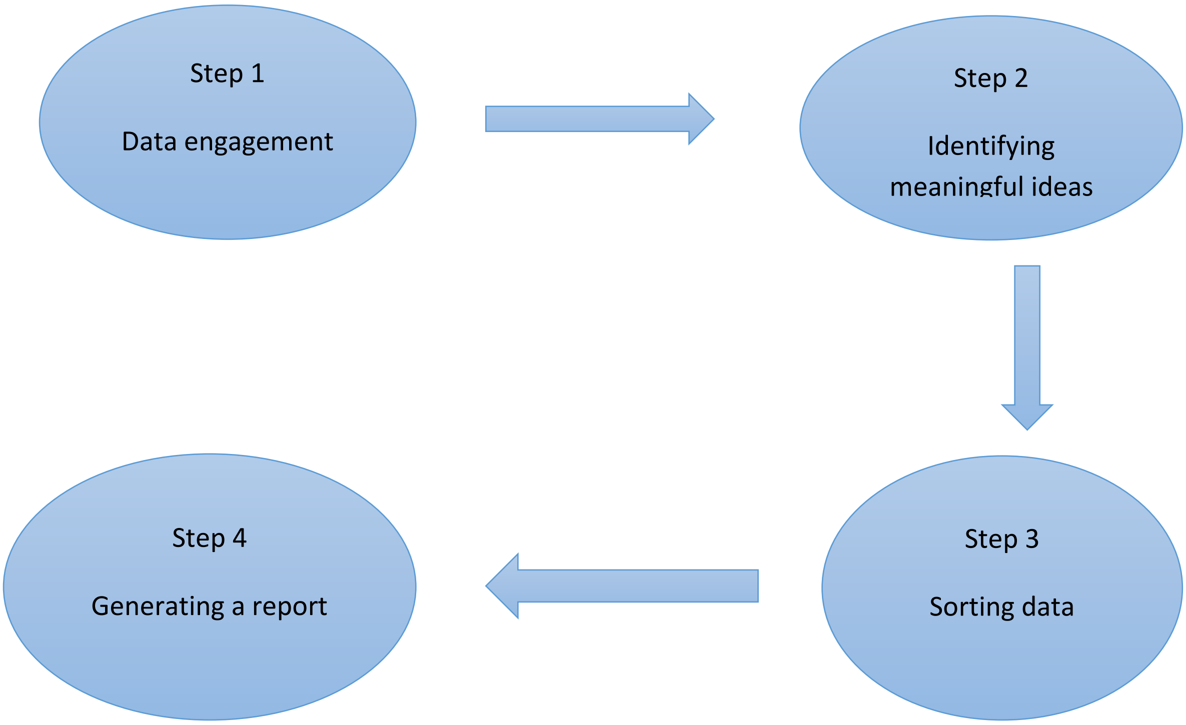


Figure 4.1: The four data analysis steps as constructed by the researcher

- In the first step, the researcher listened to audio recorded interviews and transcribed the responses of the participants to each question. After transcribing, participants were given transcripts to read and check that what was transcribed was really what they had said in answering the interview questions.
- In the second step, the researcher read the transcripts and highlighted data (using different coloured highlighters) that presented meaningful ideas.
- In the third step, the researcher sorted data into different categories. Data that was highlighted with the same colour formed a category, until all the colours used were grouped together.
- In the fourth step, the researcher generated themes using the different coloured categories. From each theme, the researcher generated a report, quoting participants' responses and linking these with the relevant literature.

4.6.3.2 Analysing the worksheet

In analysing the worksheet, the researcher went through all the participants' written responses and identified all errors and noted which of these were common errors. The researcher also identified responses and answers that were correct. In generating a report, the researcher discussed how the participants' could have arrived at the wrong answer, drawing from the relevant literature and linking this to the conceptual framework. The researcher also generated a report on what participants understood on trigonometric identities, drawing from the questions answered correctly and relating this to relevant literature and the conceptual framework. Examples for both correct and incorrect responses were extracted.

4.6.4 Data evaluation

Table 4.3: An outline of data evaluation as constructed by the researcher

Research questions	Research instruments	Data analysis
1. How do Grade 11 Mathematics learners learn trigonometric identities?	1. Worksheet 2. Semi-structured interview schedule	In conjunction with the worksheets (incorrect substitution, simplification and proof of formulae); the researcher read the transcripts to discover how learners learn trigonometric identities, applying thematic data analysis method.
2. What are the encounters of Grade 11 Mathematics learners when learning trigonometric identities?	• Semi-structured interview schedule	The researcher read the transcripts and applied the thematic data analysis method, and looked for patterns from learners' responses on their encounters when learning trigonometric identities.
Audio recordings were transcribed by the researcher. Data that was gathered using the two research instruments (semi-structured interviews and worksheet) were used to draw conclusive findings on the learning of trigonometric identities by grade 11 mathematics learners.		

4.7 Ethical considerations

Ethical issues refer to behavior that is considered as right or wrong (Bertram and Christiansen, 2014). In addressing ethical issues, the study observed the following three basic principles.

4.7.1 Autonomy

In this study, the issue of autonomy was observed in that all participants were issued a letter of contest, which clearly stated the objectives of the study and what was expected from the participants. Since grade 11 learners are minors, a letter of consent which clearly stated the objectives of the study and how the learners' were going to contribute to the study was issued to parents/guardians. Parents/guardians were also at liberty to agree or disagree that their children be photographed; video recorded or voice recorded during the interview sessions.

Parents/guardians were also made aware that their children had the freedom to withdraw from the study at any stage. Since the worksheets were completed and interviews conducted on the premises of the school, gatekeepers' permission was sought Thus a letter requesting permission for the study to be conducted was issued to the school principal. In agreeing, the school principal signed and stamped a letter granting permission for the study to be conducted. Academic activities were not disrupted in any way while the study was being done.

As Bertram and Christiansen (2014) state that one of the key elements of autonomy is that participants must participate in the study voluntarily and must be granted freedom to withdraw at any stage; participants were therefore informed on their ability to stop participating in the study at any stage.

4.7.2 Non-maleficence

Non-maleficence refers to the assurance of participants' confidentiality and that participants must not be harmed in any manner (Bertram & Christiansen, 2014). In this study, non-maleficence was observed in that all participants' names were changed to pseudonyms. The activity worksheet that participants completed and the interview transcripts contained three letter names that were not the participants' actual names. Also, full details of the school in which participants were chosen from was not given. Confidentiality was observed in that interview transcripts and completed worksheets were used for the purpose of this study only.

All safety majors were observed when this study was conducted. Interviews were conducted during break times; the worksheet was also completed during break times, thus ensuring learners' safety. The study did not result in any form of abuse (physical; emotional or social harm).

4.7.3 Beneficence

Bertram and Christiansen (2014) state that beneficence is about the benefit of the study. In this study, participants were given copies of the findings of the study, which enabled them to gain better knowledge of the study. Completed worksheets were scanned and stored in a storage device, and hard copies were destroyed. Interview recordings and scanned worksheets were stored in the researcher's supervisor office where no one will have access to them, and these will be destroyed after 5 years.

4.7.4 Piloting instruments

Bertram and Christiansen (2014, p.7) "describe piloting as the trialing of questions with individuals who are similar to the actual respondents of a study", with the aim of finding problems they may encounter when answering those questions. This study was piloted with six grade 11 mathematics learners, in order to check that all the interview and worksheet questions were clear and not ambiguous. The learners which the study was piloted with were not part of the participants that later participated in the study.

4.7.5 Debriefing

The researcher made contact with the participants throughout the study, constantly reminding them of their roles and the rights they had for the whole duration of the study, thus ensuring they were well informed on the aims and objectives of the study and on what was expected of them.

4.7.6 Limitation of the researcher's role

In this study, the role of the researcher was limited in the sense that the researcher could not lead participants into answering questions in a particular manner. The researcher only probed and provided clarity during the conduction of interviews and only if participants were not answering in full (in the case of probing) or provided clarity if the participants did not understand a question.

4.8 Trustworthiness

In ensuring the trustworthiness of this study, three techniques are discussed, namely, credibility, triangulation, and reliability.

4.8.1 Credibility

When conducting research, researchers need to ensure that their results are credible. Credibility generally involves validity or authenticity, which refers to ensuring that research processes can be reproduced by others (Bertram & Christiansen, 2014). According to Scotland (2012), researchers also need to consider if and to what extent their findings may be applicable to studies outside their frame of reference.

In in-depth studies, this is called transferability and refers to what extent the findings of a study may be applicable to another study (O'leary, 2004). In larger-scale studies this is called generalizability, and refers to whether or not the findings of a study may be applicable to any other population (Bertram & Christiansen, 2014). In this study, authenticity was observed in that the researcher used detailed descriptive data, and during the data analysis phase the researcher ensured non-biasness by being objective.

4.8.2 Triangulation

According to Hussein (2009) triangulation refers to the use of more than one methodological approach; data source; analysis method or theoretical perspective in one study so as to increase the validity of the study. In increasing the validity of this study, data triangulation was enhanced through the use of two data collection methods, namely, semi-structured interviews and the worksheet. The data collection methods that were used were further analysed in different ways.

4.8.3 Reliability

It has been mentioned in Table 2 that reliability refers to the notion of obtaining the same results if a study is done by another researcher. In this study, reliability was ensured in that all the accepted research processes were followed in conducting the study. This included the study undergoing a proposal review by experts in the field and when the study was conducted the relevant literature was reviewed and the relevant participants were selected.

Furthermore, in ensuring the validity and reliability of the study, trustworthiness was adhered to in the following ways:

- Audio recording devices were used to capture responses of the participants during the interview sessions.
- Participants were provided with copies of transcripts to check that what was transcribed was really what they had said.
- During the data analysis phase, the researcher had a prolonged engagement with data, reading, and coding.

4.9 Limitations of the study

This study was done over a period of one year, so the first and foremost limitation of the study was the time frame. As a result, only one school in the Ugu district was studied. It would have been more appropriate to study a greater number of schools in the district so as to increase the validity of the results. Since one school was studied, the issue of transferability was also questionable because other schools might not necessarily have the same contextual factors as the one that was studied.

The study aimed to explore the learning of trigonometric identities by grade 11 mathematics learners. It is crucial to note that trigonometry is a very broad topic which covers aspects such as trigonometric ratios, trigonometric identities, trigonometry in two-dimensional and three-dimensional shapes, the general solution, reduction formulae trigonometric functions, etc. Due to time constraints, this study did not cover all these aspects but only focused on trigonometric identities.

In researching this study, the interpretive paradigm was employed, and hence qualitative research methods were also employed. The study therefore had a small sample size, the findings of the study were therefore not generalisable.

Another limitation to the study was the withdrawal of participation by one of the participants. Participants were made aware that their participation was voluntary and that they had the freedom to withdraw from the study at any stage. When the participants were requested to complete worksheets, one of them did not show up and the researcher ascertained that this learner was no longer willing to participate in the study. Some of the remaining participants were willing to be interviewed but did not want to be audio recorded, which was a challenge since interviewing and capturing their responses simultaneously would have resulted in more complete capturing of their responses.

In the school where the study was conducted, English is done as a first additional language but most learners are not fluent in the language and lack basic communication skills. The researcher minimised this barrier by using the language of communication that a participant would say he or she is more comfortable communicating in.

4.10 Conclusion

This chapter considered the methodological approaches that were employed in researching this study. The critical research questions were outlined, and geographic location of the study was discussed. The research design has been outlined and briefly discussed, followed by a detailed discussion of the research paradigm, research style, and theoretical perspective. The data collection and sampling methods that were employed were in line with qualitative research methods. Ethical considerations that were observed in researching this study and how trustworthiness was adhered to, have been discussed as well as the limitations of the study. The next chapter entails an analysis and discussion of data that was collected in conduction of this study.

CHAPTER FIVE

ANALYSIS OF WRITTEN RESPONSES AND INTERVIEWS

5.1 Introduction

Chapter four offered an overview of the methodological framework that was applied to answer the critical questions of this study and discussed the paradigm within which the study was located; the approach that was used to generate and analyse data; and the research style and sampling method that were employed in the study. This chapter provides an analysis of the written responses (worksheet) and interviews.

This study employed qualitative methods; in collecting data an activity worksheet was completed and semi-structured interviews were conducted. The questions on the activity worksheet were designed to gain insight into the encounters and knowledge of grade 11 mathematics learners when answering questions based on trigonometric identities. The semi-structured interview questions were designed to establish how these mathematics learners learn trigonometric identities. A selection of learners' responses to the activity worksheet and semi-structured interviews is analysed. The analysis sample is extracted in both the activity worksheet and interviews.

5.2 Analysis of the activity worksheet

Each question on the activity worksheet is referred to according to the number it appears as on the activity worksheet.

5.2.1 Exploring proofs of the fundamental trigonometric identities

3 out of 8 learners' made use of the Cartesian plane and were able to prove the quotient identity $\tan\theta = \frac{\sin\theta}{\cos\theta}$. Whereas 5 out of 8 learners did not make use of the Cartesian plane and were still able to prove the quotient identity by making use of trig definitions. Learners' responses are shown.

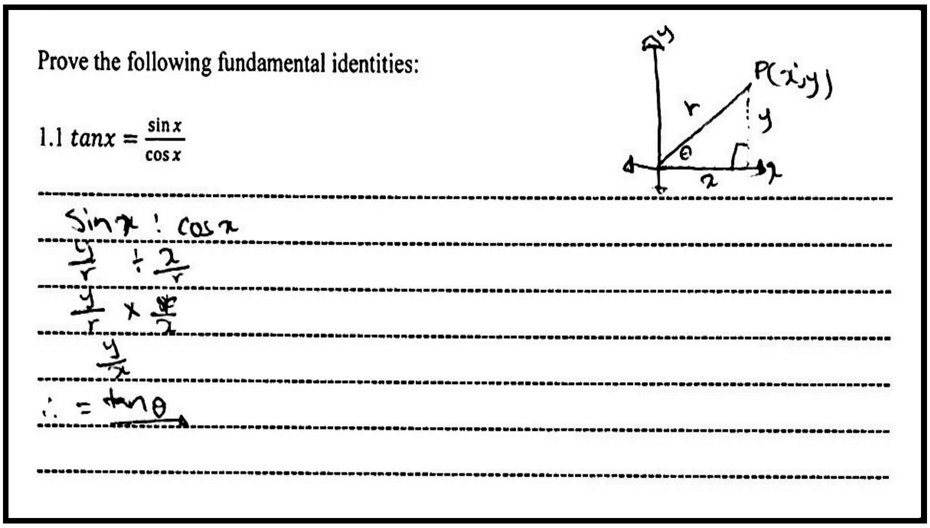


Figure 5.1: Asy’s response to Q 1.1

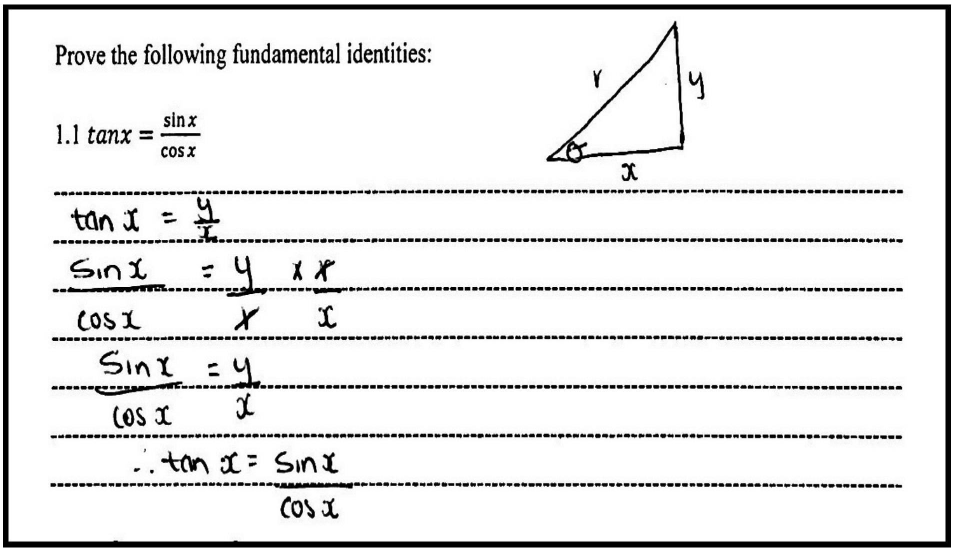


Figure 5.2: Any’s response to Q 1.1

From learners’ responses, it is evident that they were able to grasp trigonometric definitions, hence even without the aid of a diagram proof of the quotient identity didn’t pose a challenge. This therefore suggests that to a certain degree, learners were able to acquire conceptual understanding in the learning of trigonometry. In preliminary studies that were reviewed, it became evident that learners’ who made use of the unit circle to visualise trigonometric quantities performed better than learners’ who didn’t make use of the model, Mickey and McClelland (2017) confirmed that making use of a model, in this case making constructions of a triangle and labeling $x; y$ and r to prove trigonometric identities, confirmed that the use of models cultivates learners’ understanding of a certain concept.

2 out of 8 learners in this study were able to prove the identity $sin^2x = 1 - cos^2x$ by making use of trigonometry definitions.

Q 1.2 required learners' to prove that $sin^2x = 1 - cos^2x$. 5 out of 8 learners' did not prove what they were required to prove, and instead they proved the fundamental square identity $sin^2x + cos^2x = 1$. This demonstrated that learners' lack conceptual understanding, as a result they simply memorise proofs; this could be a possible reason why they did not prove what they were required to prove. Learners' responses are shown.

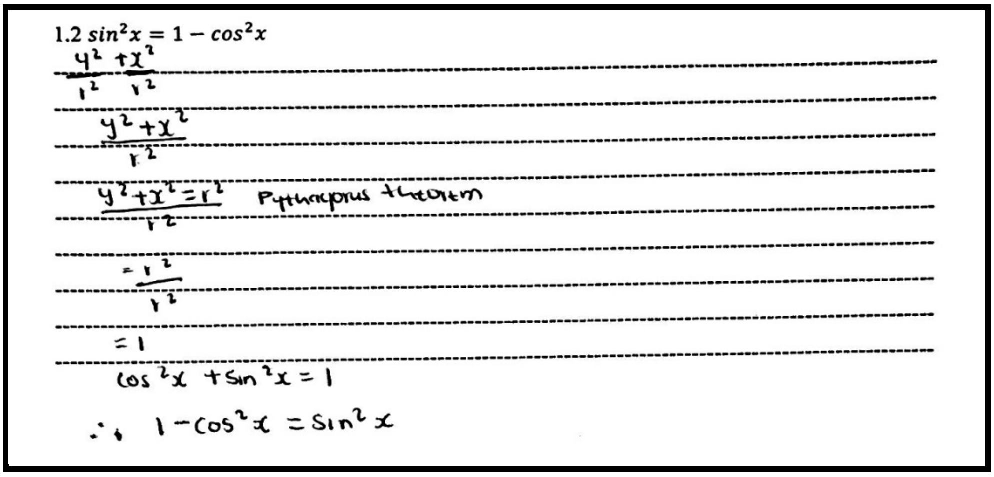


Figure 5.3: Any's response to Q 1.2

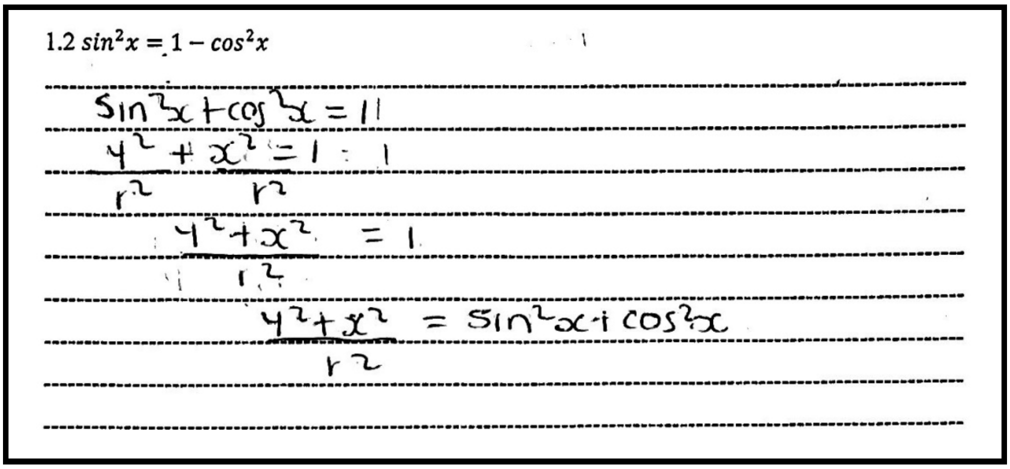


Figure 5.4: Apy's response to Q1.2

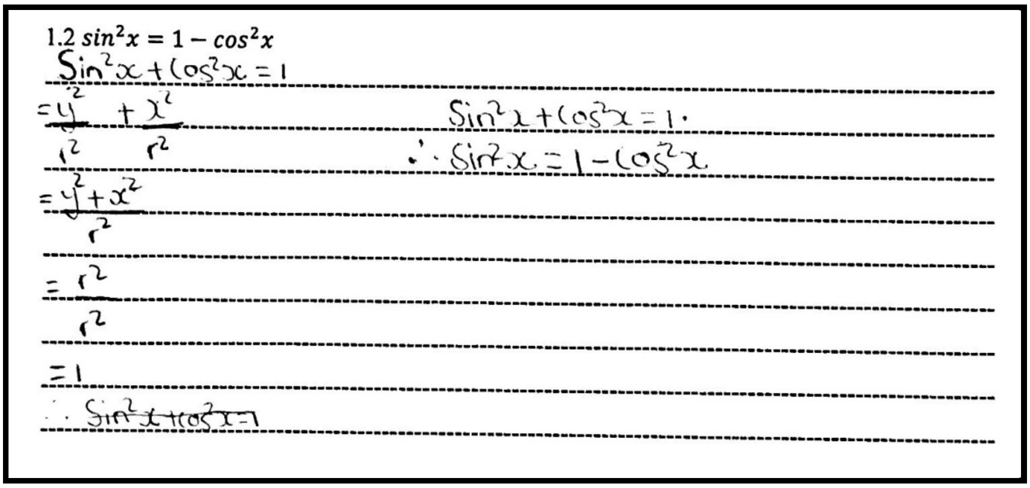
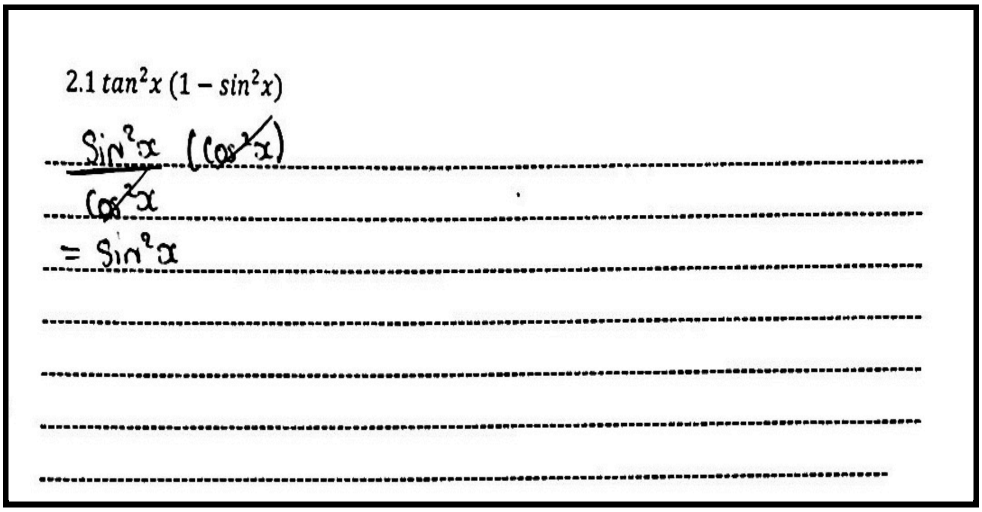


Figure 5.5: Ajy’s response to Q 1.2

Learners’ responses reveal that higher levels of conceptual understanding is still of a great concern, hence they were not able to do what was required of them and opted to bring what they had been taught as it was. This suggests that learners were not able to think out of the box and apply what they have been taught on something that had been slightly twisted. Rohimah and Prabawanto (2019) state that learners’ often encounter difficulties when learning trigonometry as a result of incomprehension of trigonometric concepts. May and Courtney (2016) are of the view that trigonometric concepts and ideas are an important component of the high school mathematics curriculum. Rohimah and Prabawanto (2019) further suggest that one of the sections that is considered difficult for learners in trigonometry is the similarity and proof of trigonometric identities, because their require understanding of concepts and the highest level of accuracy in their application.

5.2.2 Solving basic problems related to trigonometric identities

6 out of 8 learners were able to simplify basic trigonometric problems that required direct substitution of trigonometric identities. This demonstrated that learners have basic understanding of the application of trigonometric identities. Learners’ responses are shown.

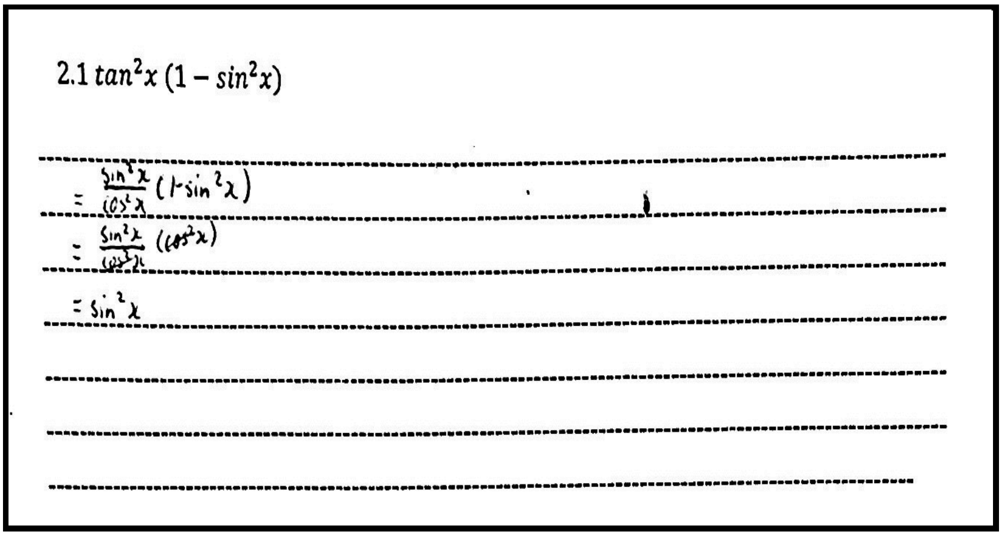


2.1 $tan^2x\,(1 - sin^2x)$

$\frac{\sin^2 x}{\cos^2 x}(\cos^2 x)$

$= \sin^2 x$

Figure 5.6: Avy's response to Q 2.1



2.1 $tan^2x\,(1 - sin^2x)$

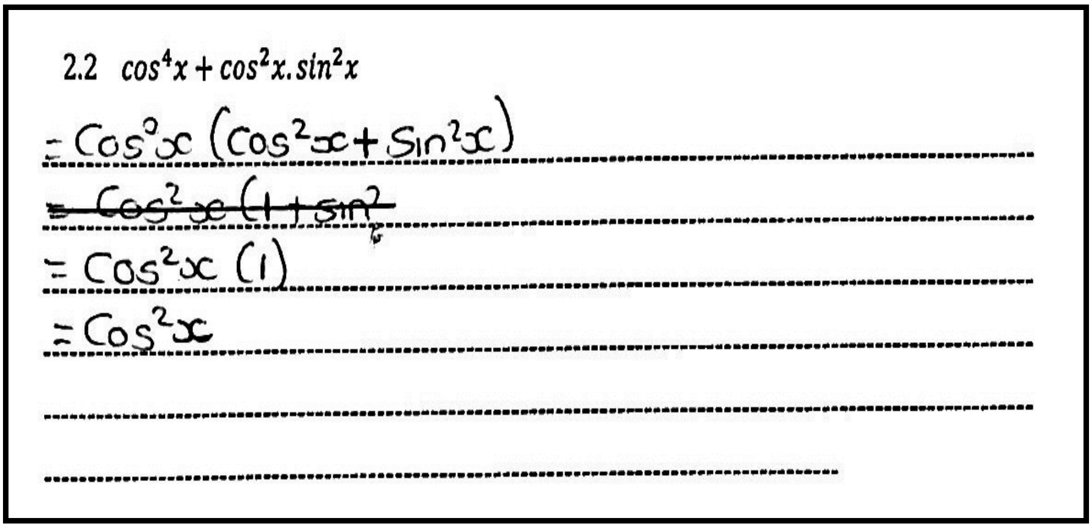
$= \frac{\sin^2 x}{\cos^2 x}(1 - \sin^2 x)$

$= \frac{\sin^2 x}{\cos^2 x}(\cos^2 x)$

$= \sin^2 x$

Figure 5.7: Aly's response to Q 2.1

5 out of 8 learners were also able to simplify basic trigonometric problems that required direct factorisation and direct substitution of trigonometric identities.



2.2 $cos^4x + cos^2x.sin^2x$

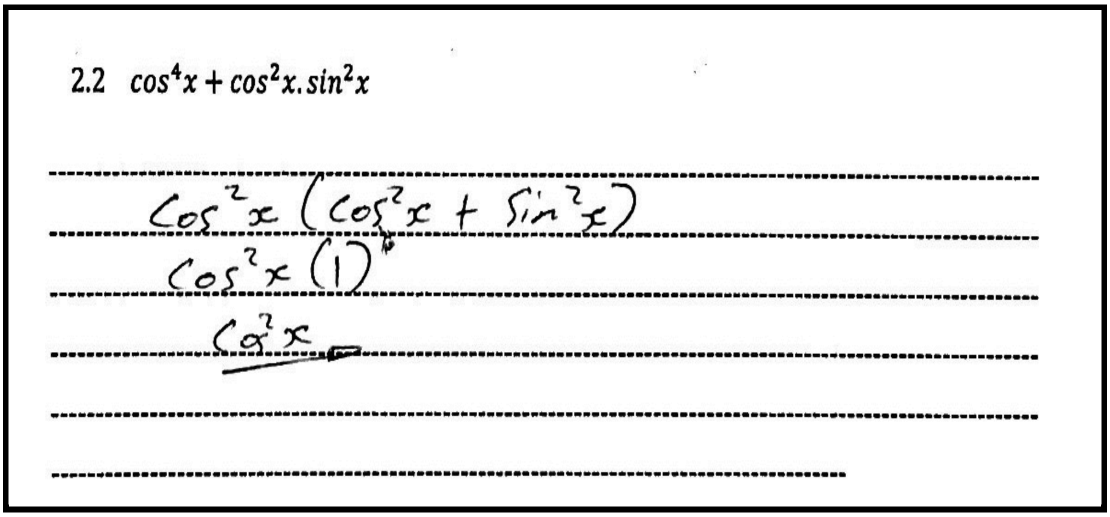
$= Cos^2x\,(cos^2x + sin^2x)$

~~$= Cos^2x\,(1 + sin^2$~~

$= Cos^2x\,(1)$

$= Cos^2x$

Figure 5.8: Apy’s response to Q 2.2



2.2 $cos^4x + cos^2x.sin^2x$

$Cos^2x\,(cos^2x + sin^2x)$

$Cos^2x\,(1)$

Cos^2x

Figure 5.9: Amy’s response to Q 2.2

Learners’ responses in both questions 2.1 and 2.2 demonstrate that they are able to apply basic mathematical computations in problems that routine procedures. This suggests that learners’ were able to grasp lower and medium levels of procedural fluency in the learning of trigonometric identities.

5.2.3 Exploring the use of fractions in the learning of trigonometric identities in Grade 11 Mathematics

6 out of 8 learners were not able to apply the relevant algebraic manipulations when working with trigonometric identity problems that involved fractions. Learners’ did not work out the fractions by writing them under one denominator but instead that divided the co-functions with

one another (cancelled), which demonstrated a lack of understanding or inability of working with fractions. Aby; Amy and Agy's responses are shown.

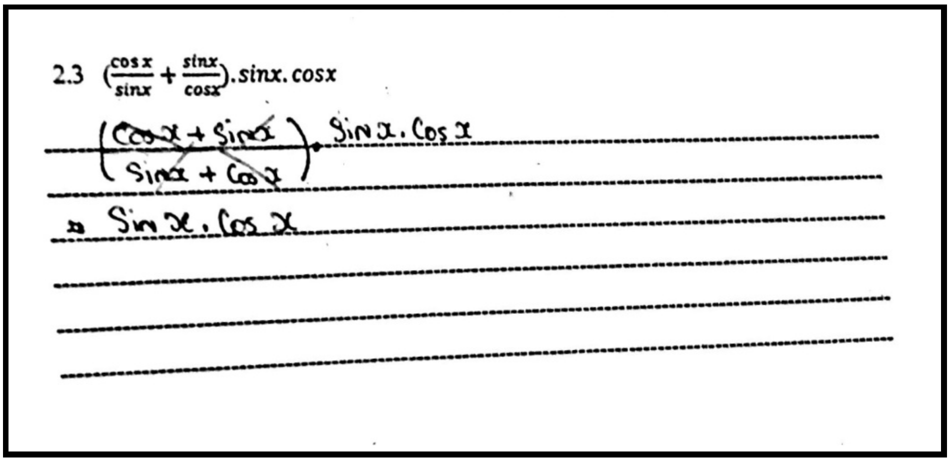


Figure 5.10: Aby's response to Q 2.3

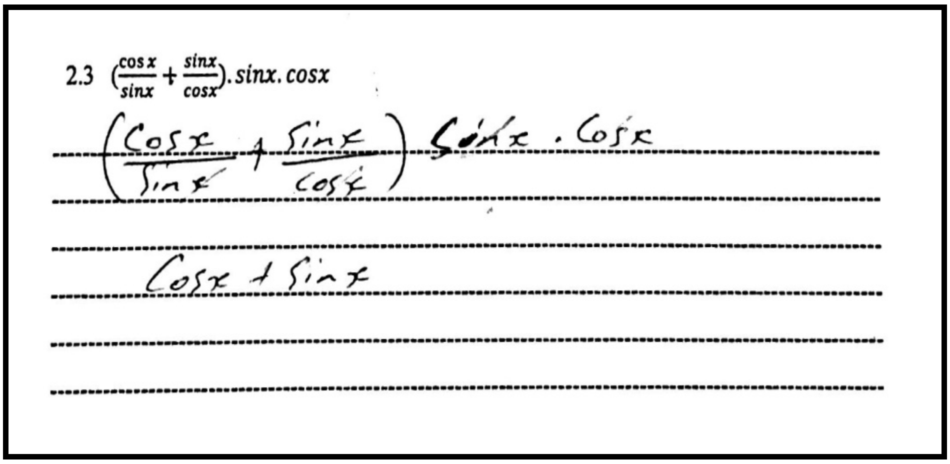


Figure 5.11: Amy's response to Q 2.3

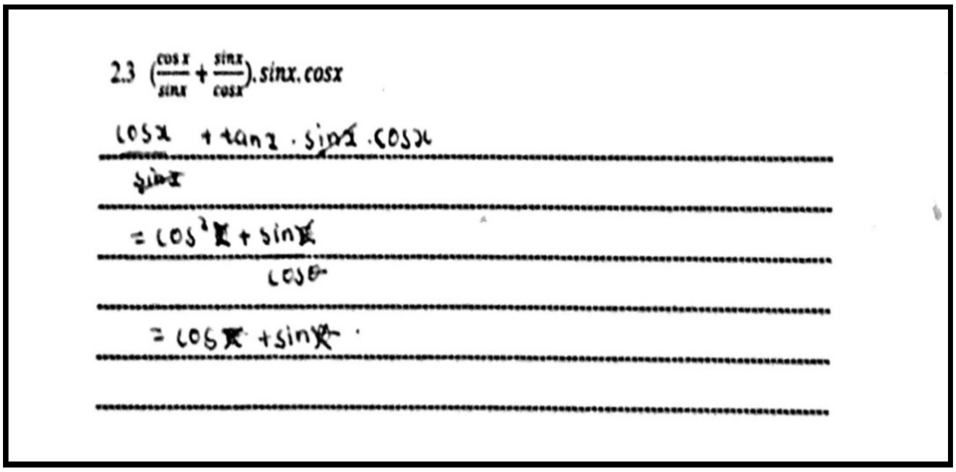


Figure 5.12: Agy's response to Q 2.3

Learners' responses revealed that fractions in trigonometry pose a great challenge to them, this suggests that they may have not grasped them in earlier grades, hence working with fractions in trigonometry seems difficult. This suggests that learners' adaptive reasoning is of a great concern. Torbeyns, Schneider, Xin and Siegler (2015) stated that fractions are very important in numerical development and fractions are complex because they are a ratio of two whole numbers. Victoria, Fauzi and Ananda (2017) agrees that fractions are challenging for learners to learn and teachers to teach, yet fractions are one of the fundamental topics in mathematics For learners, ratios are extremely difficult to understand, which could possibly be one of the reasons why learners find it difficult to understand trigonometry (Jung-A, Jae-Geun & Kyeong Hwa, 2013).

Learners ability to work with fractions has not improved irrespective of the use of various strategies for teaching fractions (Victoria, Fauzi & Ananda; 2017). However, mathematical achievement and the magnitude to which fractions are understood are correlated; this therefore supports the belief that fractions play an essential role in the understanding of mathematics (Torbeyns, Schneider, Xin & Siegler, 2015).

5.2.4 Exploring the difference of two squares in the learning of trigonometric identities in Grade 11 Mathematics

In problems that required learners to either multiply by inspection or to be able to recognise an expanded form of the difference of two squares, learners' opted to multiply out and were able to do it correctly (a few learners had incorrect solutions of the first step). However, 6 out of 8 learners were not able to simplify further, and they either didn't continue with the solution or subtracted the coefficient of $sin^2\Theta$ from the constant. This could possibly be caused by learners not knowing when they should or should not subtract or add any two given terms, which could be a result of them not understanding the basics of mathematics, such as working with like and unlike terms. Aly; Amy and Asy's responses are shown.

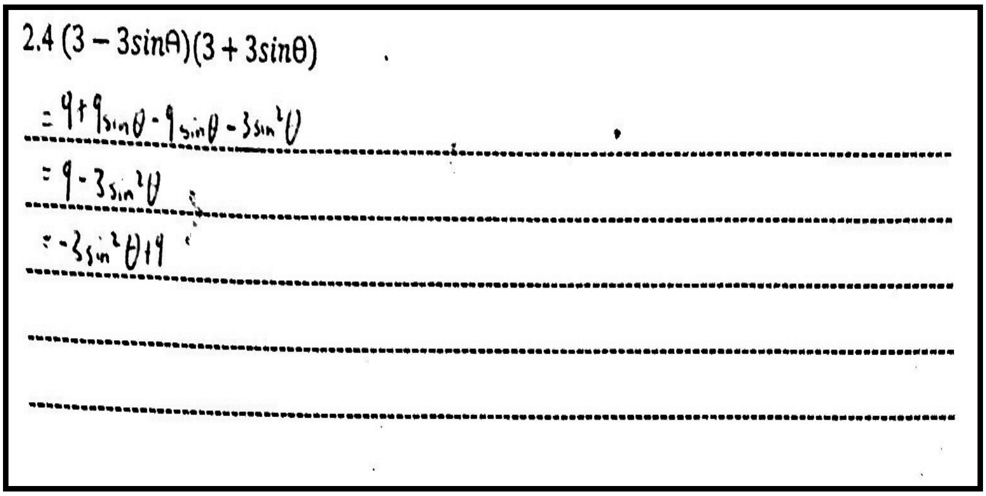


Figure 5.13: Aly's response to Q 2.4

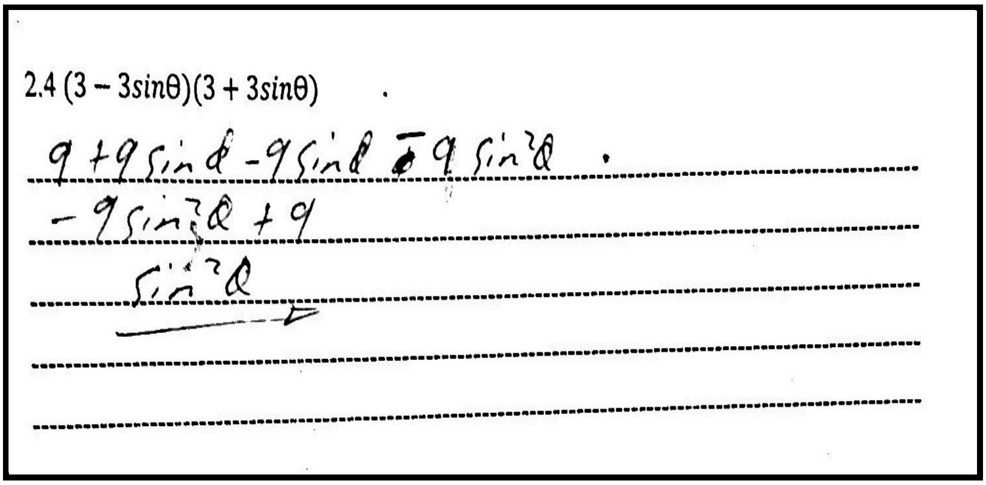


Figure 5.14: Amy's response to Q 2.4

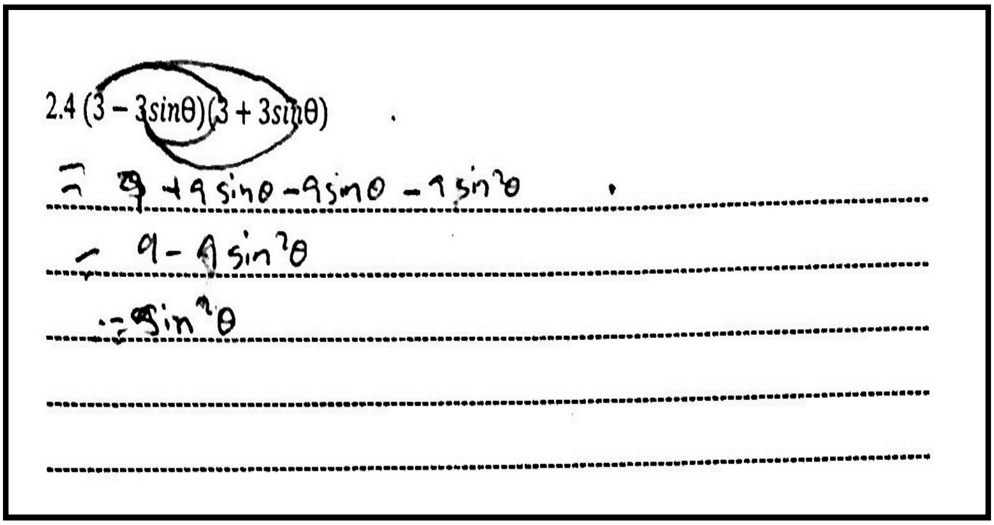


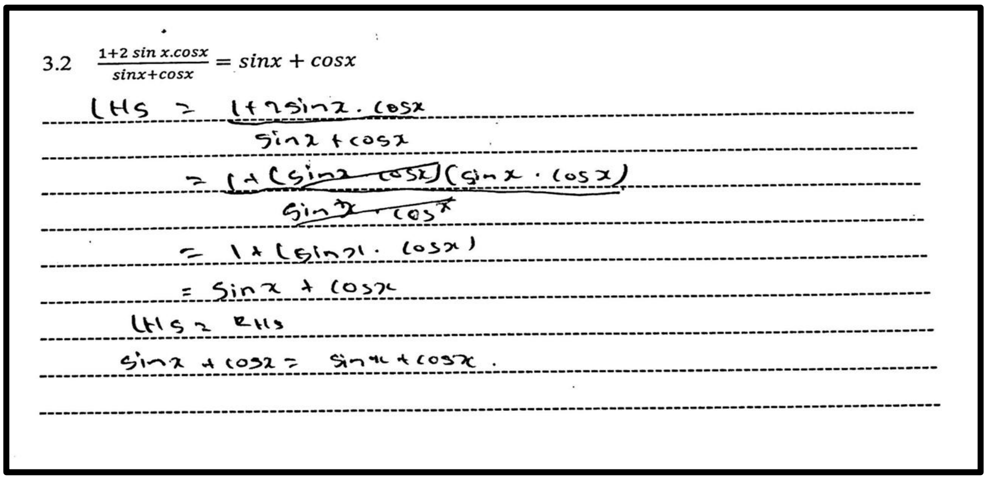
Figure 5.15: Asy's response to Q 2.4

Siew; Geofrey and Lee (2016) suggest that algebraic language promotes reasoning about pattern recognition, problem-solving and thinking skills, however, this may be difficult for some learners because algebra has high levels of abstraction. Jupri and Drijvers (2016) posit that the application of arithmetic operations, understanding the notation of variables, and algebraic expressions are amongst the difficulties experienced by learners' in algebra.

Hence, learners were not able to simplify trigonometric expressions that required basic knowledge of algebraic computations. It has also been found in the Trend in International Mathematics and Science Study (TIMSS) that learners' experience difficulty in providing solutions to questions that require the application and understanding of algebraic expressions to perform complex procedures (Siew et al, 2016).

5.2.5 Exploring factorising a trinomial in the learning of trigonometric identities in Grade 11 Mathematics

5 out of 8 learners were not able to apply the relevant algebraic manipulations when working with trigonometric identity problems that involved factorisation. Learners' did not apply the square identity: $1 = sin^2\theta + cos^2\theta$, so as to obtain a trinomial and thereafter apply factorisation. Learners simply divided terms of the numerator by terms of the denominator. This could possibly be caused by learners' inability to apply knowledge of one mathematical topic to another topic, in this case, learners apply factorisation when solving for an unknown (usually x) in algebra but experience difficulty in factorising a trigonometric expression. Asy, Amy and Agy's response are shown:


3.2 $\frac{1+2\,sin\,x.cosx}{sinx+cosx} = sinx + cosx$

$LHS = \frac{1+2sinx.cosx}{sinx+cosx}$

$= \frac{1+(sinx\ cosx)(sinx \cdot cosx)}{sinx+cosx}$

$= 1+(sinx \cdot cosx)$

$= sinx + cosx$

LHS = RHS

$sinx + cosx = sinx + cosx$.

Figure 5.16: Asy's response to Q 3.2

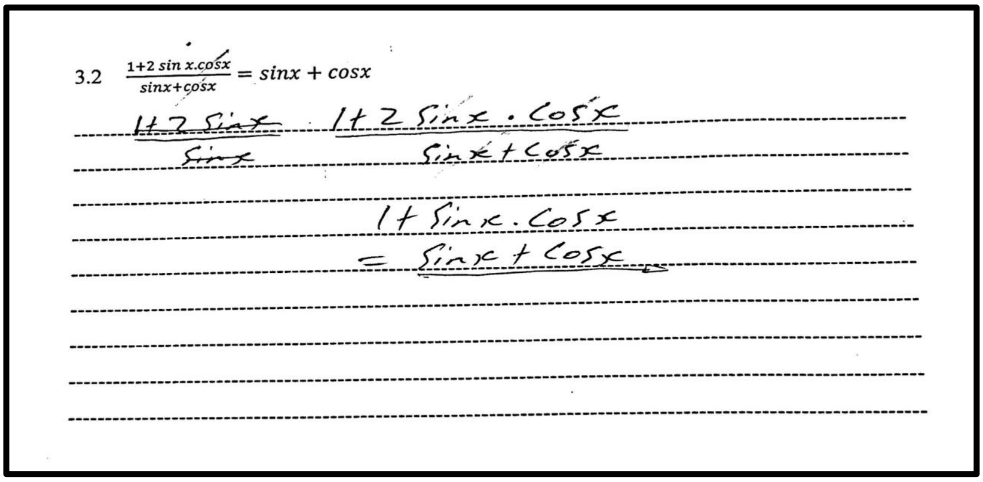


Figure 5.17: Amy’s response to Q 3.2

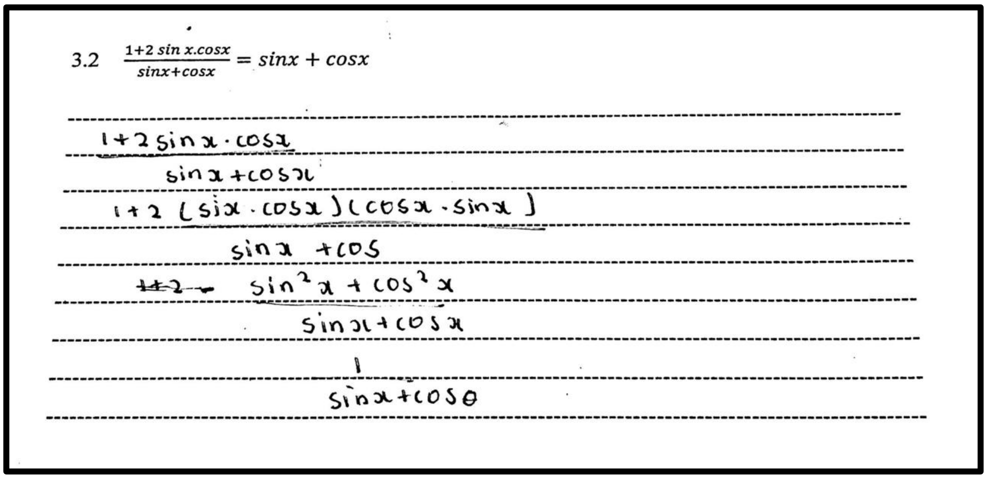


Figure 5.18: Agy’s response to Q 3.2

Learners’ responses revealed that they experience great difficulty when mathematical topics are integrated. This suggests that learners are only comfortable with working on routine procedures problems, and therefore aren’t able to think strategically. This suggests that in the learning of trigonometric identities, strategic competency is of a great concern. According to Koyunkaya (2016), in the curriculum, trigonometric identities link algebraic and geometric reasoning. Hence, it is crucial that learners have a good understanding of algebra and geometry so that they will not experience challenges when solving mathematical problems related to trigonometric identities that require geometric and algebraic reasoning. It is also crucial to note that the 2018 moderators report revealed that mathematics cannot be studied in compartments and that learners’ are expected to be able to apply knowledge from one section to another section of work.

Jung-A, Jae-Geun and Kyeong-Hwa (2013) support this idea in stating that “growth of understanding is not linear, higher level of understanding needs the lower level of understanding”. Learners need to grow from a lower to a higher level of understanding in mathematics. And learners need to be exposed to problems related to the integration of different mathematics topic during the teaching and learning process.

5.2.6 Exploring working with squared brackets in the learning of trigonometric identities in Grade 11 Mathematics

In problems that required learners to either expand and multiply (by inspection) squared brackets or work (simplify) within the brackets and then square the numerator and denominator, 5 out of 8 learners were able to multiply out, but experienced challenges in simplifying further. Learners either left the problem incomplete or wrote squared solutions that had no logical explanation on how they were obtained, and were mathematically incorrect.

This could possibly be caused by learners’ inability to apply knowledge of one mathematical topic to another topic; In this case, learners’ usually were able to expand in algebra but experienced difficulty in expanding (working with square) trigonometric expressions. Learners’ solutions also demonstrated that they lack basic mathematical computations, and hence some of them had solutions that had no logical explanations. Learners’ responses are shown.

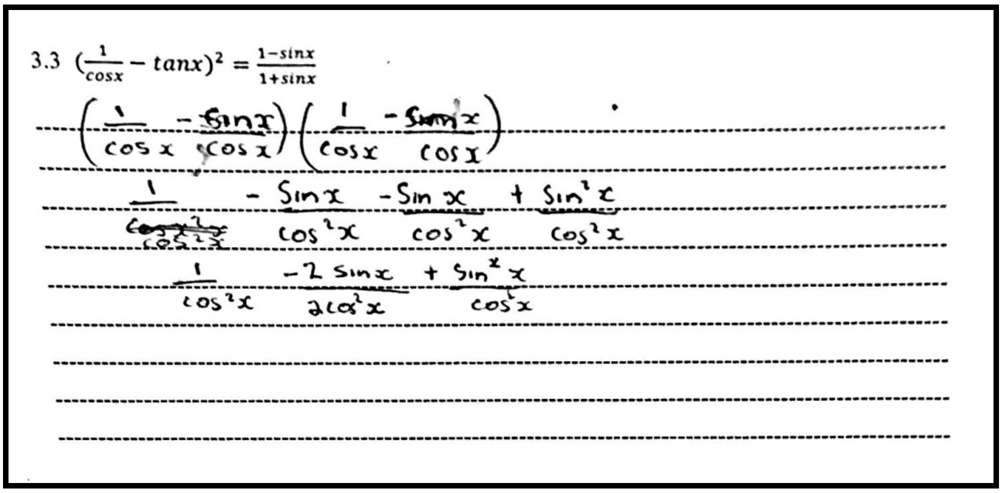


Figure 5.19: Any’s respose to Q 3.3

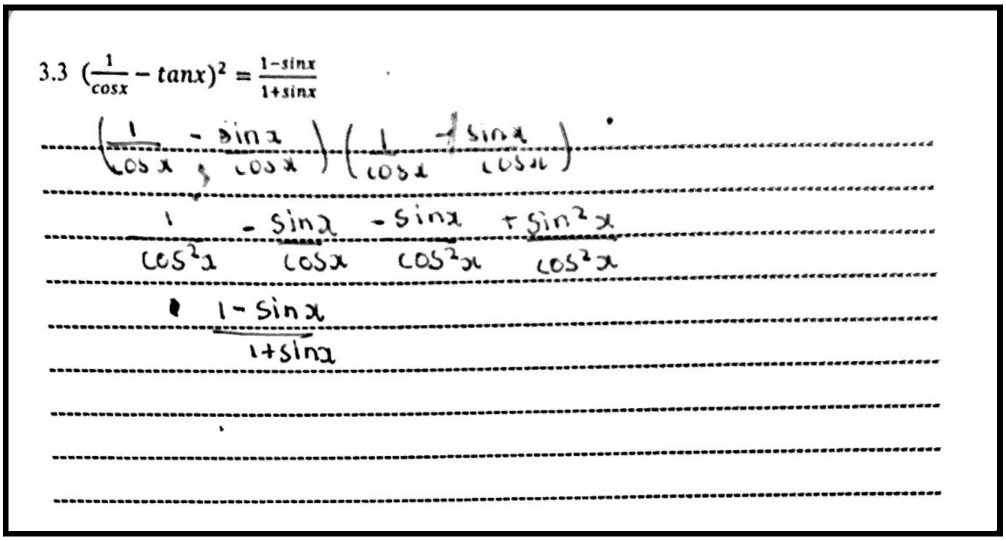


Figure 5.20: Agy's response to Q 3.3

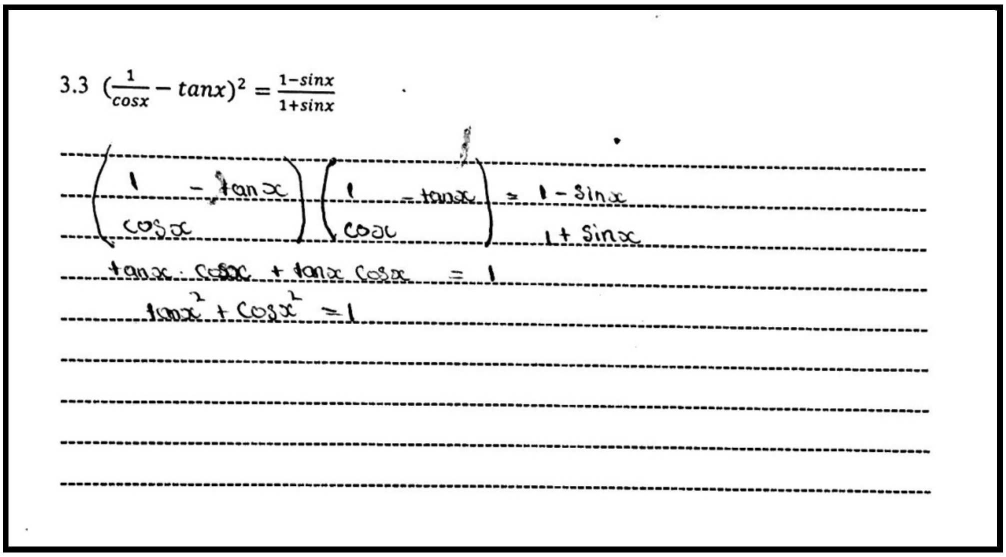


Figure 5.21: Ady's response to Q 3.3

Yang and Sianturi (2017) state that trigonometry is a branch of mathematics that takes aspects of arithmetic and geometry as sources, and is a result of algebraic techniques, geometrical realities, and trigonometric relationships. Generally, if learners are not able to connect algebraic reasoning and geometry in learning trigonometry, they will experience difficulty in solving trigonometric problems (Rohimah & Prabawanto, 2019).

Thus, learners need to be given an opportunity to build knowledge with geometric proofs, to learn through exploration, to make interconnections among different mathematics topics, and to gain better comprehension of definitions (Jung, 2015).

5.2.7 Exploring misconceptions in the learning of trigonometric identities in Grade 11 Mathematics

6 out of 8 learners had a conception that $sin^2\theta + cos^2\theta = 1$ is the same as $\sin\theta + \cos\theta = 1$ and that $1 + \cos\theta = \sin\theta$. This could possibly be caused by learners' lack of conceptual understanding in mathematical topics. This could also be caused by learners' tendency to not follow the correct steps when solving problems, but rather to create their own ways of arriving at answers, most of the time the steps they follow are incorrect. This is evident in Aly and Apy's response to Q 3.4, and in Ady's response to Q 3.2

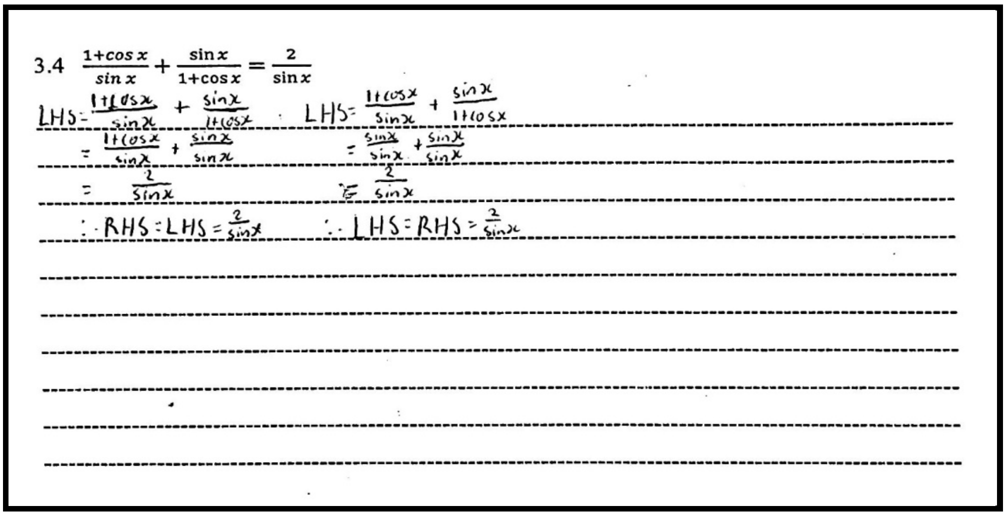


Figure 5.22: Aly's response to Q 3.4

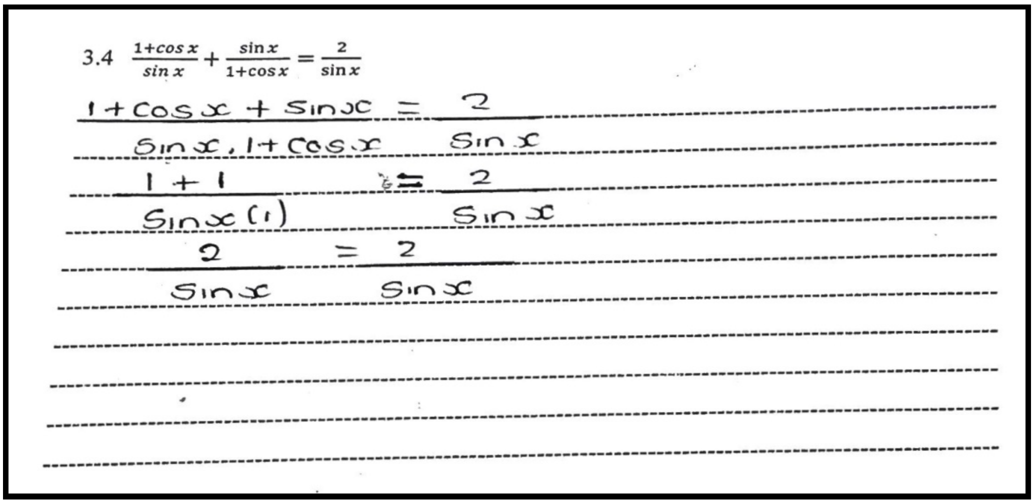


Figure 5.23: Apy's response to Q 3.4

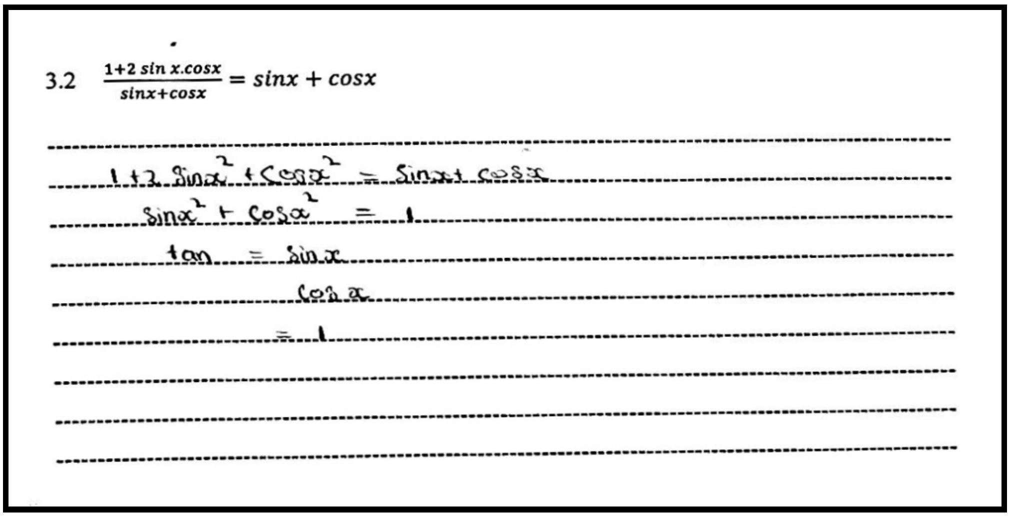


Figure 5.24: Ady’s response to Q 3.2

Misinterpretations and confusions that arise from incorrect meanings are called misconceptions (Ojose, 2015). In mathematics, problem solving is a skill that is very complicated; s a result, learners’ faced numerous obstacles during the teaching and learning process (Mensah, 2017). Ojose (2015) further states that in mathematics, learners have misconceptions regarding various concepts in all grades.

According to Mensah (2017), studies have shown that learners lack the development of concepts in trigonometry. Durkin and Rittle-Johnson (2014) posit that assessing learners is required during the conceptual change process in order to diagnose misconceptions and assess how knowledge changes. Chapman (2015) is of the view that some of the elements that prevent learners’ from solving mathematical problems successfully include:

- Lack of understanding of vocabulary used in problem solving
- Lack of methods of attacking problems
- Confusion of processes, leading to the random trialing of any processes
- Insufficiency of ability in fundamentals
- Lack of knowledge of important facts, rules, and formulas
- Ignoring principles, rules, or processes underlying the correct solution of problems

5.3 Analysis of semi-structured interviews

Interview responses are classified according to themes.

5.3.1 Exploring challenging topics in Grade 11 Mathematics

Topics that gave learners a challenge in learning mathematics in grade 11 include Euclidean geometry and trigonometry. 2 out of 8 learners' mentioned functions and probability amongst the topics they found to be challenging. Any and Apy's response are shown.

> Any: *"The two topics that give me much of a challenge are Euclidean geometry and Trigonometry, because these two topics require much proof…"*

Apy's response is captured in the following quotation:

> Apy: *"The topics that give me a challenge are Trigonometry and Euclidean geometry…"*

The findings of the 2018 moderators report confirmed that Euclidean geometry and trigonometry were amongst the most poorly answered questions confirming these findings, thus suggesting that these topics are challenging to learners'.

5.3.2 Exploring real life applications of trigonometry

6 out of 8 learners were not able to relate trigonometry to their daily encounters. This could be caused by not relating the subject matter to learners' daily encounters when learners are taught. Learners' responses are shown.

> *Ajy: "Eeeyyy I don't know."*

Any's response is captured in the following quotation:

> *Any: "Trigonometry, eishhh I really don't know where trigonometry may be applied in real life situations, but I do know that Euclidean geometry deals with the measurement of things, for example when houses are being built, one part must be equal to another part, angles all that, but with trigonometry, I don't know."*

Ady's response is captured in the following quotation:

> *Ady: "I don't know where trigonometry may be applied, I may perhaps think of other things that may be applied, for example in Euclidean geometry may be applied in building, like a roof, then the person may calculate angles so as to be able to know how*

planks will be joined, so I think Euclidean geometry may be applied in architecture, but with trigonometry I honestly don't know."

Hernandez-Martinez and Vos (2018) assert that many learners do not experience Mathematics as relevant. Hence, they view it as a subject that is disconnected from their daily encounters.

However, it cannot be overlooked that 2 out of 8 learners were able to make relevant examples on application of trigonometry outside the classroom. Avy and Apy's response are shown.

Avy: "I usually hear doctors talking about degrees when perhaps referring to how broke say for example a patients arm is, and also in construction when working with bridges."

Apy's response is captured in the following quotation:

Apy: "I see the application of trigonometry in roofs of buildings and in bridges."

5.3.3 Exploring the introducing of trigonometric identities in Grade 11 Mathematics

When trigonometric identities were introduced, the fundamental identities were proven. Learners were thereafter taught how to make sin or cos the subject of the formula in the square identity. Learners' responses are shown.

Apy explains how trigonometric identities were introduced.

Apy: *"The first trigonometric identity is that of* $\tan$ *which is sine over cosine, and then we also have the identity that is equal to one, which is* $sin^2\Theta + cos^2\Theta = 1$. *These identities were explained and proven, it was proven how we arrive at these identities, we were also shown that the identity that is equal to one can also be written as* $1 - sin^2\Theta$ *and* $1 - cos^2\Theta$*."*

Any's response is captured in the following quotation:

Any: *"What I remember, firstly is that we spoke of* $\tan\Theta$ *being equal to sine…, it was said they were rules, I can't remember what these rules are called, but it was said that* $sin\,\Theta$ *is equal to* cos, $tan\Theta = \frac{sin\,\Theta}{cos\,\Theta}$, *it was also said that* $sin^2\Theta + cos^2\Theta = 1$ *there was also another rule that states that* $1 - cos^2\Theta = sin^2\Theta$, *jahhhh."*

5.3.4 Exploring learners understanding of trigonometric ratios in Grade 11 Mathematics

Learners' demonstrated a lack of understanding of trigonometric ratios (defining and applying them). In defining trigonometric ratios in terms of x; y and r, 5 out of 8 learners' confused trigonometric definitions, for example, they defined $sine$ as x over r and $cosine$ as y over r. Another misconception that learners had was that trigonometric ratios may be used to calculate the area. This could possibly be caused by learners' misunderstanding of the Cartesian plane in relation to the opposite, adjacent, and hypotenuse sides and not knowing when they may apply each trigonometric ratio. The following quotation reveals Avy's view:

> Avy: *"When given a shape in the Cartesian plane, with sine, when looking at x values and r, the hypotenuse and then I calculate what I need to calculate, I use cosine when I have the value of y and the hypotenuse…"*

Ady's view is captured in the following quotation:

> Ady: *"Sine is used if we want to calculate a side, and then cosine is used if we want to calculate an angle, hhhmmm tan is used if we want to calculate the area, I don't know, I think we use tan if we're calculating the area."*

Any's view is captured in the following quotation:

> Any: *"When referring to sine, opposite over adjac over hypotenuse, eeehyyy, I don't know"*

In a study by Jung (2015) on learners work involving geometric concepts, it became evident that learners lack conceptual understanding of trigonometric definitions, hence some learners' struggled to use trigonometric definitions appropriately. Retnowati and Maulidya (2018) suggest that trigonometry problem solving is seen as complex since it not only involves trigonometric ratios but other knowledge bases as well, such as the Pythagoras theorem. This therefore suggests that when dealing with right-angled triangles, it is crucial that learners connect prior knowledge to new concepts, that they understand definitions; and can generalise mathematical ideas (Jung, 2015).

5.3.5 Exploring the experiences of Grade 11 Mathematics learners when learning trigonometric identities

In learning trigonometric identities, learners described their experience as difficult. This could possibly be caused by learners' fear of trigonometry and not being exposed to different levels of questioning. The following quotation reveals Any's view:

Any: *"Eeeyyy it is difficult, because most of the time I do not understand, if you do not understand something then doing it doesn't become good, you do not enjoy it, but it is not that difficult because I try to work hard, that's what I can say, but doing something you do not understand is not nice. So my experience is not too good."*

Asy's view is captured in the following quotation:

Asy: *"My experience is that in the beginning I found trigonometric identities difficult, as time went by they became better, but the level of how questions were asked was upped, they became more and more difficult."*

Ady's view is captured in the following quotation:

Ady: *"I may say I experienced challenges in learning trigonometric identities, if for example when writing, especially the identity of* $\tan$*, I have a problem with applying this identity…"*

Challenger (2009) and Aslan-Tutak and Sarac (2017) assert that learners hate trigonometry and feel that it is a complicated section of mathematics. Rohimah and Prabawanto (2019) state that some learners often encounter difficulties when learning trigonometry, and that the difficulties are a result of learners' incomprehension of trigonometric concepts. The results of a study conducted by Sholehawati and Wahyudin (2019) on critical mathematical thinking, visual thinking, and self-efficacy in trigonometry showed that learners lack critical thinking skills and lack confidence in their mathematical skills.

Rohimah and Prabawanto (2019) further suggest that learner' experience challenges in solving trigonometric equations problems because they fail to describe the nature of a problem, have difficulty in factorising a quadratic equation in trigonometric form, and have difficulty in using basic trigonometric equation solving.

5.3.6 Exploring challenges encountered by Grade 11 Mathematics learners when learning trigonometric identities

Challenges encountered by grade 11 mathematics learners when solving problems related to trigonometric identities include proving that the Left Hand Side is equal to the Right Hand Side and simplifying the expressions.

5.3.6.1 Exploring proving that the Left Hand Side is equal to the Right Hand Side

Learners experience difficulty when proving that the LHS is equal to the RHS. This could be due to learners' confusion on whether they should simplify one side or both sides. In some instances, learners are able to identify a side that requires simplification but fail to choose and substitute the relevant trigonometric identity. The following quotation reveals Ajy's view:

> Ajy: *"Eeehhh one of the challenges I came across, is to, as I've said before, is to prove that the left hand side is equal to the right hand side…"*

Any's view is captured in the following quotation:

> Any: *"… For example, if you have to prove that the one side is equal to the other side, in trigonometry, that becomes difficult for me…"*

Ady's view is captured in the following quotation:

> Ady: *"In trigonometry, I have a challenge with proving identities, when proving I sometimes know what to do only in the first and second step, and then I get stuck."*

According to Siew; Geofrey and Lee (2016), the main factor for learners that experience difficulties in applying equality in solving problems is the lack of proficiency in mastering the properties of equality. Jupri and Drijvers (2016) are of the view that mastering algebraic expressions requires an understanding of the order in which algebraic expressions should be processed, the expectation to get a numeric outcome rather than an expression, and the discomfort in dealing with expressions that cannot be simplified further.

Siew et al (2016) further state that skills attained in algebra are based on the fact that equations can be manipulated by execution of the same operations on both sides to form another equation written in a different manner but with the same value.

5.3.6.2 Exploring simplification of expressions

Learners' experience difficulty when simplifying trigonometric expressions. This could possibly be caused by learners' lack of basic understanding of trigonometric identities, as a result they become confused when choosing an identity to substitute in an expression.

The following quotation reveals Avy's view:

Avy: *"the challenges I experience is that I sometimes don't know how to continue solving a problem, say for example I have to take out the common factor, say maybe I'm given* cos^2*, like in the problem on the worksheet, after taking out the common factor I get stuck, not know how to solve the problem further… I basically have a challenge with solving problems until the last step…"*

Ady's view is captured in the following quotation:

Ady: *"Not knowing how to prove the identity or how to apply it… When it comes to simplifying, I experience great difficulty…With simplifying, I get confused as to which identity to substitute, say for example you choose the correct identity, if you somehow end up having a fraction, I have a problem with fractions, so once I see a fraction I simply don't know what to do"*

In a study conducted by Jung (2015) on learners work involving geometric concepts, it became evident that learners lack prior knowledge and connection, and thus learners' were unsure about the relationships between the length of sides and their corresponding angles. Rohimah and Prabawanto (2019) state that learners experience challenges in solving trigonometric equations because they have difficulty in applying general trigonometric formulas and experience difficulty in doing algebraic computations in trigonometry. Jung (2015) also suggests that it is important that algebraic contexts are included when learners learn trigonometry.

5.3.7 Exploring learners' views on how the learning of trigonometric identities may be improved

Learners had different views on how the learning of trigonometric identities may be improved. 6 out of 8 learners were of the view that if they received thorough practice of different problems (different levels of questioning) that their understanding of trigonometric identities might improve. 2 out of 8 learners were of the view that the use of mnemonics, relating trigonometric identities to learners' daily encounters, and getting a clear understanding of algebra, may also improve their knowledge of trigonometric identities.

Asy's response is captured in the following quotation:

Asy: *"I think in order for the learning of trigonometric identities to be improved, is that if a teacher may teach basics until learners understand, and to also help them practice frequently before learners write an exam, the teachers shouldn't give learners the same* *type of questions but give them different questions, so that they will be prepared for the exam because in the exam the questions are asked in a different manner, which is a bit different from what was learnt in class."*

Apy's response is captured in the following quotation:

> Apy: *"Maybe it may be improved if when trigonometry is learnt, real life examples may be made. Also, if a teacher explains, he or she must try to explain in all possible ways so as to ensure that we understand. The teacher should not move to the next topic if we do not all understand, we should learn until we fully understand, we can also be given different ways on how the topic is going to be assessed; we should also practice with different question papers."*

2 out of 8 learners suggested that the use of mnemonics in the learning of trigonometric identities could also assist them. Ady and Any's responses are captured.

> Ady: *"... I think the reason I why I encounter challenges when I have to simplify is because I have a problem with algebra, so if maybe I may improve on algebra, like identifying the mistakes I make when factorising and expanding, then when I need to apply this knowledge on trigonometry it will be much easier, then I might be able to improve."*

Any's response is captured in the following quotation:

> Any: *"Maybe if there could be an easier way that could be used for us to understand, an easier way that could be used to explain so that I can understand how to work with these rules without memorising them, for example the mnemonic All Sciences Teachers Cheat (CAST) used to remember positive trig ratios in each quadrant, something similar to help me remember, because if you memorise something, sometimes you end up forgetting."*

5.4 Conclusion

Chapter five provided an analysis of the written responses (worksheet) and interviews. The activity worksheet aimed to discover the encounters of grade 11 mathematics learners when learning trigonometric identities. The interviews that were conducted aimed to understand how these learners learn trigonometric identities.

The learners' responses discussed in this chapter indicated that some of the learners have an understanding of trigonometric ratios and the fundamental trigonometric ratios, however, some of the learners' demonstrated confusion in trigonometric ratios. The discussion also indicated that learners experience challenges when applying algebraic manipulations in trigonometric expressions, therefore, most learners were unable to correctly factorise trigonometric equations and were not able to simplify trigonometric expressions. The discussion also indicated that learners have misconceptions in trigonometric identities.

The next chapter concludes the study by responding to the critical research questions by providing a discussion of the findings, recommendations, researcher's thoughts, and the limitations of the study.

CHAPTER SIX

FINDINGS, RECOMMENDATIONS AND LIMITATIONS

6.1 Introduction

This study explored grade 11 mathematics learners' learning of trigonometric identities. Qualitative methods were employed and the study was researched through the lens of the interpretive paradigm and using the case study method.

The study was underpinned by the principles of Kilpatrick's (2001) strands of mathematical proficiency. In this study, ten learners were purposively selected to participate; interviews were conducted and a worksheet was completed to collect data. Thematic data analysis methods were used to analyse data.

The study addressed the following key questions:

1. What are the encounters of grade 11 mathematics learners when learning trigonometric identities?
2. How do grade 11 mathematics learners learn trigonometric identities?

Chapter five provided an analysis of the written responses (worksheet) and interviews. This chapter provides general findings of the study and responds to the key research questions by presenting the findings. This chapter also outlines learners' suggestions on how the learning of trigonometric identities may be improved, provides the conclusions of the study, recommendations, suggestions for further studies, limitations of the study, and themes for further studies.

6.2 Findings of the study

The interview questions were designed to explore general findings of the study, such as topics that give learners a challenge in mathematics; and learners' understanding of $sine$, $cosine$ and $tangent$. One of the interview questions was also designed to answer the key question, how do grade 11 mathematics learners learn trigonometric identities? The questions on the activity worksheet were designed to explore learners' understanding of trigonometric identities and learners' encounters when learning trigonometric identities, so as to answer the question, what are the encounters of grade 11 mathematics learners when learning trigonometric identities? A further discussion of the findings is presented.

6.2.1 General findings of the interview

The general findings of the interview are presented.

6.2.1.1 Challenging topics in Grade 11 Mathematics and real life applications of trigonometry

The most common topics that appeared as challenging to learners' when learning mathematics in grade 11 were Euclidean geometry and trigonometry. A few learners also mentioned probability and functions as challenging topics.

Most learners did not know the real life applications of trigonometry. This suggests that when trigonometric identities were presented to learners, that connections with learners' daily encounters were not made.

It can therefore be concluded that, Euclidean geometry and trigonometry gave learners a challenge in grade 11 mathematics and it can also be concluded that learners cannot relate trigonometric identities to things around them in everyday life.

6.2.1.2 Learners' understanding of trigonometric identities

Learners demonstrated a clear understanding of trigonometric identities, hence five of them were able to prove the fundamental identities even without the aid of a diagram. This may therefore suggest that in learning trigonometric identities, learners were able to grasp the strand conceptual understanding which refers to comprehension of mathematical concepts, in this regard, understanding trigonometric ratios and their definitions. Learners' were also able to apply trigonometric identities to basic trigonometric expressions which demonstrated an understanding of the identities. It can therefore be concluded that learners were able to grasp the formulae of basic trigonometric identities. It is crucial to note that a few learners demonstrated that they do memorise proofs by proving what was not required in the activity worksheet. This therefore suggests that procedural fluency which refers to carrying out procedures still poses a challenge for some learners.

6.2.2 How do Grade 11 Mathematics learners learn trigonometric identities?

In response to the key question, how do grade 11 mathematics learners learn trigonometric identities, learners were introduced to trigonometric identities and the two fundamental identities $sin^2\theta + cos^2\theta = 1$ and $\tan\theta = \frac{\sin\theta}{\cos\theta}$ were explained and proven. In proving the fundamental identities, definitions of trigonometric ratios were used. Learners were also taught that the identity $sin^2\theta + cos^2\theta = 1$ can also be written as $sin^2\theta = 1 - cos^2\theta$ or $cos^2\theta = 1 - sin^2\theta$.

Learners were thereafter given class work and home work on trigonometric identities.

6.2.3 What are the encounters of Grade 11 Mathematics learners when learning trigonometric identities?

In response to the key question, what are the encounters of grade 11 mathematics learners when learning trigonometric identities, a number of factors contributed to making trigonometric identities difficult for learners. These factors include working with trigonometric identity problems that involved fractions, difference of two squares, factorising trinomials, working with like and unlike terms, and expanding expressions by multiplying out. This demonstrates a deficiency in learners' adaptive reasoning, which refers to application of mathematical knowledge of a topic in another topic.

It can therefore be concluded in this study that learners encountered challenges when learning trigonometric identities. These challenges were a result of their inability to perform algebraic manipulations in trigonometry. Price and van Jaarsveld (2018) confirmed these findings by stating that trigonometry is a topic that is frequently misunderstood because it has rich connections between other mathematical aspects such as algebra; geometry and graphical ways of thinking.

In stating that one of the aspects in trigonometry that learners find difficult is similarity and proof of trigonometric identity because it requires an understanding of the correct concepts and high accuracy in their application; Rohimah and Prabawanto (2019) confirmed that proofs are not easy for learners, hence learners' end up memorising them rather than understanding them.

Learners' strategic competency which refers to applying various methods in solving mathematical problems which may not be necessarily straightforward, is also of a great concern, hence, learners struggled to work on questions such as 3.2 and 3.4 on the activity worksheet.

The difficulties that learners experienced when learning trigonometric identities made their encounter with trigonometry to be negative. In fact, most learners felt that learning something that they did not understand was not a pleasant experience although some learners did feel that trigonometry was a nice topic but they still maintained that it was challenging for them.

6.3 Learners' suggestions on how the learning of trigonometric identities may be improved

Learners had different views on how the learning of trigonometric identities may be improved. However, the most common view was that of being taught the basics until all learners have a clear understanding of the concept being introduced. Learners further suggested that once they understand basics (formulation and application) that this is when basic calculations may be introduced and thereafter upper levels of questioning may be introduced.

Learners also suggested that being exposed to how the topic will be assessed could also assist them. This demonstrated that when learners were also not exposed to different ways of questioning when they were taught trigonometric identities in class.

The fact that learners state that they require an understanding of basics suggests that their understanding of trigonometric concepts is of a great concern, and Prabawanto (2019) confirm that understanding and mastering trigonometric concepts is crucial in the learning of trigonometry else learners struggle with the topic.

Some learners were of the view that their poor performance in trigonometric identities is a result of their poor knowledge in algebra. This suggested that in order for the learning of trigonometric identities to be improved, the learners need to understand algebra quite well.

6.4 Conclusions of the study

In this study, it can be concluded that learners were able to prove the fundamental identities, which suggests that learners' conceptual knowledge in the learning of trigonometric identities is adequately developed. Learners were also able to apply basic substitution of trigonometric identities and basic algebraic manipulations in trigonometric expressions, such as taking out the common factor. As mentioned in the framework of the study the procedural knowledge includes to a great extent solving problems, resulting in learners getting accurate answers or following the correct procedure (Rittle-Johnson & Schneider, 2015), this suggests that learners' procedural knowledge in the learning of trigonometric identities is well developed.

Learners experienced difficulty in proving that the LHS is equal to the RHS, which resulted from them not being able to simplify trigonometric expressions by applying algebraic manipulations, which includes factorising a trinomial, working with like and unlike terms, etc. This suggests that learners experience difficulty in applying knowledge from one topic to another topic, thus implying that learners have poor adaptive reasoning. Learners also experienced difficulty in simplifying algebraic expressions, which resulted from not being able to choose the relevant identity to substitute nor correctly applying algebraic manipulations such as factorization. These findings imply that learners' have poor adaptive reasoning.

Another mathematical aspect that gave learners a challenge is that of working with trigonometric expressions in the form of fractions, which resulted from their knowledge gap in working with fractions. Learners also experienced difficulty in working with squared trigonometric expressions written in the form of fractions which resulted from not being able to multiply expressions by inspection. It is mentioned in the framework of the study that in cultivating strategic competency, teachers may give learners challenging problems; challenging in the sense that the problems arouse interest and not frustration (Chapman, 2015). Learners' inability to work with such problems demonstrated poor strategic competency.

Some of the learners did not demonstrate enthusiasm in exploring different solutions to problems; learners' were also not able to make connections between their daily encounters and trigonometric content learnt in the classroom, which suggests that learners' productive disposition is limited.

6.5 Recommendations

Based on the findings of this study, it is recommended that learners require exposure to different levels of questioning. This may begin in the classroom, where learners may be given questions of different levels as an activity. Mathematics should not be taught in compartments and, in fact, when learners are being taught a particular topic, an integration of other topics should also be made.

When learners are introduced to a particular topic, connections between the content of the topic and the world around them should be made. This will ensure that learners do not view mathematics as a subject that is disconnected from their encounters outside the classroom. Workshops also need to be conducted to introduce teachers to innovative methods of teaching trigonometric identities.

Mathematics teachers as key elements in quality assurance of mathematics education, should not only possess the scope of the curriculum, but should also have adequate knowledge beyond the scope.

Teachers should also register and become active members of AMESA or any other mathematics association, where they can be able to share ideas with other colleagues on pedagogical matters.

It is recommended that mathematics teachers enroll for qualifications such as ACE in order to enhance their own professional development through academic study. It is also recommended that these teachers embrace difficult topics and learn how to teach them concretely and explicitly.

6.6 Suggestions for further research

Generalisations cannot be made since the sample consisted of only ten participants. Future research may be conducted with a larger sample of participants which would increase the validity of the findings and results. In addition, trigonometry is a very broad topic and this study only focused on trigonometric identities. It would be useful if future studies are conducted which focus on other sections of trigonometry.

In terms of data collection instruments, more non-routine problems may be used in future research studies. This study made use of two data collection instruments, namely, interviews and an activity worksheet. Future studies may include observations; this will be of good use if added to interviews and worksheets since it can be used for a study's triangulation. Furthermore, this study explored the learners' learning of trigonometric identities and, in fact, it may be useful then for future studies to also explore teachers' understanding of trigonometric identities. Future studies may also explore the subject advisors role in the learning of challenging topics such as trigonometry and Euclidean geometry.

6.7 Limitations of the study

The study followed an interpretive paradigm and therefore had a small sample size, which further implies that generalisations of the findings could not be made. Only one school in the Ugu district was studied. It would have been more appropriate to study a greater number of schools in the district so as to increase the validity of the results. The researcher's positionality in the study also posed a limitation, in that learners may have been intimidated by being interviewed by their teacher. Learners were assured that participation in the study was solely for the purpose of the research.

Trigonometry is a very broad topic which covers a number of aspects, such as trigonometry in two-dimensional and three-dimensional shapes, the general solution; reduction formulae, trigonometric functions, etc. This study did not cover all those aspects but only focused on trigonometric identities. Language barriers were also amongst the limitations to this study. In the school that the study was conducted, English is done as a first additional language.

Another limitation to the study was withdrawal of participation by one of the participants. Some of the learners were willing to complete the activity worksheet and to be interviewed but were not willing to be recorded during the interview sessions. As a result, not all ten participants were interviewed because capturing all the information they were going to provide was going to be a challenge.

6.8 Themes for further studies

The following themes are suggested for other research studies:

- A similar study on the application of trigonometric studies in other fields, such as engineering, physics, etc. may be conducted.
- A similar study on the use of visuals in the learning of trigonometric identities may be conducted.
- A similar study on the use of blended teaching and learning in trigonometric identities.
- A similar study on other identities in trigonometry, such as compound angle identities, may be conducted.

6.9 Synopsis

This study intended to make a contribution to the body of knowledge related to the learning of trigonometric identities in grade 11 mathematics. During the interview sessions, some of the learners showed frustration and were demotivated towards trigonometric identities. They felt incompetent and found it difficult to work with trigonometric identity problems, especially problems that required algebraic manipulations.

Novice mathematics teachers should receive support from their departmental heads and from the district from their subject advisors within their work place since this will greatly assist them in dealing with content challenges they might encounter.

Teachers should meet often in their respective clusters to assist one another, to share ideas and different teaching strategies. This will better equip them with different teaching methods which may be of good use in the classroom.

Teachers should also receive proper professional development. The researcher is of the view that workshops and trainings should be constantly offered to teachers to improve their content knowledge and teaching methods.

Even though mathematics results continue to be poor in South Africa, mathematics teachers must not lose hope but must instead continue to explore innovative techniques to improve learners’ understanding since this will eventually result in obtaining better mathematics results.

References

1. Aird, J. & van Duyn, J. (2012). *Clever: keeping maths simple*. South Africa: Macmillan.
2. Alder, J. (2001). A language of teaching dilemmas: unlocking the complex multilingual secondary mathematics classroom. *For the learning of mathematics*, 18(1), 24-33.
3. Alex, J. K., & Mammem, K. J. (2018). Students' understanding of geometry terminology through the lens of Van Hiele theory. *Pythagoras,* 39(1), a376. (page 45-52)
4. Ally,N., Brijall, D., & Maharaj, A. (2016). Investigating the impact of e-learning support material on first year engineering students' mathematics Learning. *Journal of Communication*, 7(2), 296-303.
5. Armat, M., Assarroudi, A., Rad, M., Sharifi, H., & Heydari, A. (2018). Inductive and deductive: Ambiguous labels in qualitative content analysis. *The Qualitative Report*, 23(1), 219-221.
6. Aslan- Tutak, F., & Sarac, A. (2017). The relationship between teacher efficacy, and students' trigonometry self-efficacy and achievement. *International Journal for Mathematics Teaching and Learning*, 18(1), 66- 83.
7. Babbie, E. (2007). *The practice of social research*. Thomson Wadsworth: Belmont, CA
8. Badat, S., & Sayed, Y. (2014). Post-1994 South African education: The challenge of social justice. *The Annals of the American Academy*, 127- 148.
9. Bansilal, S., & Rosenberg, T. (2016). Reflections of South African teachers teaching under adversarial conditions. *Problems of education in the 21st century*, 70, (34).
10. Bertram, C., & Christiansen, I. (2014). Understanding research. (1st ed.). Pretoria: Van Schaik Publishers.
11. Bhagat, K. K., Chang, C. N., & Chang, C. Y. (2016). The impact of the flipped classroom on mathematics concept learning in high school. *Journal of Educational Technology & Society*, 19 (3), 134- 142.
12. Biyela, K., Sibaya, D., & Sibaya, P. (2016). Comparative analysis of mathematics syllabi for secondary school and teacher training programs. *International Journal of Educational Sciences*, 13(3), 325-334.
13. Bornstein, N. M. (2017). Music of the triangles: How students come to understand trigonometric identities and transformations.

14. Braun, V. & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology*, 3(2), 77- 101.
15. Brijlall, D., & Maharaj, D. (2014). Exploring support strategies for high school mathematics teachers from underachieving schools. *International Journal of Educational Science*, 7(1), 99-107.
16. Brijlall, D., & Niranjan, C. (2015). Using manipulatives to support an embodied approach to learning trigonometry in a South African school: a case study, *Africa Education Review*, 12(3), 361-380.
17. Burhanzadea, H., & Aygöra, N. (2015). Difficulties that students face during factorisation questions. *Procedia - Social and Behavioral Sciences*, 191, 859 – 864.
18. Bush, T., Joubert, R., Kiggundu, E., & van Rooyen, J. (2009). Managing teaching and learning in South African schools, *International Journal of Educational Development*, 1- 7.
19. Butgereit, L. (2007). Math on MXit: Using MXit as a medium for mathematics education. In *Meraka INNOVATE Conference for Educators*, 18- 20.
20. Capps, L. R., & Cox, L. S. (1991). Improving the learning of mathematics in our schools. Focus on exceptional children, 23(9), 1- 8.
21. Carraher, D. W., Schliemann, A. D., & Schwartz, J. L. (2017). Early algebra is not the same as algebra early. In *Algebra in the early grades* (pp. 235- 272). Routledge.
22. Challenger, M. (2009). From triangles to a concept: a phenomenographic study of A-level students' development of the concept of trigonometry. Submitted for the degree of PhD.
23. Chapman, O. (2015). Mathematics teachers' knowledge for teaching problem solving. *LUMAT*, 3(1), 19- 36.
24. Charles, C. (2015). Teaching the trigonometric ratios through embodiment, symbolism and formalism. *Journal of the Mathematics Council of the Alberta Teachers' Association*, 52(2), 31- 36.
25. Charmaine Price & Pieter van Jaarsveld (2017). Using open-response tasks to reveal the conceptual understanding of learners- learners teaching the teacher what they know about trigonometry. *African Journal of Research in Mathematics, Science and Technology Education*, 21(2), 159-175.

26. Chigonga, B. (2016). Learners errors when solving trigonometric equations and suggested interventions from grade 12 mathematics teachers.
27. Chikiwa, C. (2015). Teaching trigonometry in a grade 11 multilingual mathematics class: A focus teacher code switching practices
28. Chikiwa, C., & Schäfer, M. (2017). Promoting critical thinking in multilingual mathematics classes through questioning: Are teachers truly cultivating it?
29. Creswell, J.W. (2007). *Qualitative inquiry and research design. Choosing among five approaches*. (2nd ed.). Thousand Oaks, London and New Delhi: SAGE Publications.
30. Department of Basic Education. (2016). *National Senior Certificate: Diagnostic Report.* Republic of South Africa.
31. DeReu, E. (2019). Student's self-discovery of right triangle trigonometry.
32. Dhlamini, S. S. (2012). An investigation into grade 12 teachers' understanding of Euclidean geometry.
33. Durkin, K., & Rittle- Johnson, B. (2015). Diagnosing misconceptions: Revealing changing decimal fraction knowledge. *Learning and Instruction*, 37, 21-29.
34. Engelbrecht, P., Nel, M., Nel, N., & Tlale, D. (2015). Enacting understanding of inclusion in complex contexts: classroom practices of South African teachers. *South African Journal of Education*, 35(3), 1- 10.
35. Fuchs, L. S., Malone, A. S., Schumacher, R. F., Namkung, J., & Wang, A. (2017). Fraction intervention for students with mathematics difficulties: Lessons learned from five randomised controlled trials. *Journal of learning disabilities*, 50(6), 631-639.
36. George, A., & Adu, E. O. (2018). Motivation and attitude of grade nine learners towards mathematics in King Williams town education district, South Africa. *Ghana Journal of Development Studies*, 15 (1), 135- 149.
37. Groves, S. (2012). Developing mathematical proficiency. *Journal of science and mathematics education in Southeast Asia*, 35(2), 119-145.
38. Gür, H. (2009). Trigonometry learning. New horizons in education, 57(1), 67- 80.
39. Hernandez-Martinez, P., & Vos, P.(2018). "Why do I have to learn this?" A case study on students' experiences of the relevance of mathematical modeling activities. *ZDM*, 50, 245- 257.

40. Hussein, A. (2009). The use of triangulation in social sciences research: Can qualitative and quantitative methods be combined? *Journal of Comparative Social Work*, 1, 1- 12.

41. Johnson, L. (2016). Trigonometry: An overview of important topics. *All Capstone Projects*, Paper 173, 1- 62.
42. Jones, K. (2002). Issues in the teaching and learning of geometry. *Aspects of teaching secondary mathematics: Perspectives on practice*, 12- 139.
43. Jung, H. (2015). Analysing student work involving geometric concepts. *Mathematics Teaching.*
44. Jung-A, Y., Jae-Geun, Y., & Kyeong Hwa, L. (2013). Toward students' full understanding of trigonometric ratios. *Journal of the Korean Society of Mathematical Education Series D*, 17(1), 63-78.
45. Jupri, A., & Drijvers, P. H. M. (2016). Student difficulties in mathematising word problems in algebra. *Eurasia Journal of Mathematics, Science and Technology Education*, 12(9), 2481- 2502.
46. Kagenyi, D. G. (2016). Pedagogical factors affecting the learning of trigonometry in secondary schools in Gatundu North Sub-County, Kiambu County, Kenya.
47. Kamber, D., & Takaci, D. (2018). On problematic aspects in learning trigonometry, *International Journal of Mathematical Education in Science and Technology*, 49(2), 161-175.
48. Kawulich, B. (2012). Selecting a research approach: paradigm, methodology and methods. *Field Methods Journal,* 23(2), 1-22.
49. Kersaint, G., Thompson, D., & Petkova, M. (2013). Teaching mathematics to english language learners. New York: Routledge.
50. Khumalo, B., & Mji, A. (2014). Exploring educators' perceptions of the impact of poor infrastructure on learning and teaching in rural South African schools. *Mediterranean Journal of Social Sciences*, 5(20), 1521- 1532.

51. Kilpatrick, J. (2009). The mathematics teacher and curriculum change. *PNA*, 3(3), 107-121.
52. Kilpatrick, J., Swafford, J. & Findell, B. (2001). *Adding It Up: helping children learn mathematics*. Washington, DC: National Academies Press.
53. Koyunkaya, M. Y. (2016). Mathematics education graduate students' understanding of trigonometric ratios. *International Journal of Mathematical Education in Science and Technology*, 47(7), 1028-1047.

54. Lincon, Y.S. & Guba, E.G. (1994). Competing paradigms in qualitative research. *Handbook of qualitative research*, 2, 163-194.

55. Maoto, S., Masha, K., & Mokwana, L. (2018). Teachers' learning and assessing of mathematical processes with emphasis on representations, reasoning and proof. *Pythagoras,* 39(1), 53-62.

56. Marias, P. (2016). "We can't believe what we see": Overcrowded classrooms through the eyes of student teachers. *South African Journal of Education*, 36(2), 1- 10.

57. Mason, M. B., & Todd, A. (2005). Enhancing learning in South African schools: strategies beyond outcomes-based education. *International Journal of Educational Development*, 25(3), 221-235.

58. May, V., & Courtney, S. (2016). Developing meaning in trigonometry. *Illinois Mathematics Teacher*, 63 (1), 25- 33.

59. Mensah, F. S. (2017). Ghanaian senior high school students' error in learning of trigonometry. *International Journal of Environmental & Science Education*, 12(8), 1709-1717.

60. Mickey, K. & McClelland, J. L. (2017). The unit circle as a grounded conceptual structure in Pre-Calculus Trigonometry. *Acquisition of Complex Arithmetic Skills and Higher-Order Mathematics Concepts*. Elsevier/Academic Press

61. Msila, V. (2007). From apartheid education to the revised National Curriculum Statement: Pedagogy for identity formation and nation building in South Africa. *Nordic Journal of African Studies*, 16(2), 146–160.

62. Mthembu, S. G. (2015). Subject advisors as instructional leaders: a case study of six subject advisors in ilembe district.

63. Mutodi, P. (2015). Exploring grade 12 learners' approaches for solving trigonometric equations.

64. Naidoo, J. (2016). Language matters! Exploring promise's use of pedagogic strategies in her mathematics classroom. *Language Matters*, 47(3), 372- 392.

65. Naidoo, K., & Naidoo, R. (2016). From 2D to MD: Implications for teaching secondary school undergraduate students in a pre-service mathematics course at a university of technology. *International Journal of Engineering Technology Science and Research*, 3(9), 46- 50.

66. Niranjan, C. (2013). Using artefacts to support an embodied approach to learning trigonometry: A case study of Grade 10 learners

67. O'leary, Z. (2004). The essential guide to doing research. SAGE Publications, London.

68. O'Meara, N., Fitzmaurice, O., & Johnson, P. (2017). Old habits die hard: An uphill struggle against rules without reason in mathematics teacher education. *European Journal of Science and Mathematics Education*, 5(1), 91-109.

69. Ojose, B. (2015). Students' misconceptions in mathematics: Analysis of remedies and what research says. *Ohio Journal of School Mathematics*, 72, 30- 34.

70. Phorabatho, T. & Mafora, P.(2015). The role of curriculum coordinators in managing teachers' continuing professional development. *International Journal of Educational Sciences*, 8(1), 93- 101.

71. Probyn, M. (2015). Pedagogical translanguaging: bridging discourses in South African science classrooms, *Language and Education*, 29(3), 218-234.

72. Ramnarain, U. (2016). Understanding the influence of intrinsic and extrinsic factors on inquiry-Based science education at township schools in South Africa. *Journal of Research in Science Teaching*, 53 (4), 598– 619.

73. Retnowati, E., & Maulidya, S. R. (2018). Designs of goal- free problems for trigonometry learning. In *Journal of Physics: Conference Series*, 983 (1). IOP Publishing.

74. Rittle- Johnson, B., & Schneider, M. (2015). Developing conceptual and procedural knowledge of mathematics. *Oxford handbook of numerical cognition*, 1118- 1134.

75. Rohimah, S. M., & Prabawanto, S. (2019). Students' difficulty identification in completing the problem of equation and trigonometry identities. *International Journal of Trends in Mathematics Education Research,* 2 (1), 34-36.

76. Rohmah, N., & Ekawati, R. (2018). Proof profile of students with various adversity quotients in trigonometry. *Jurnal Ilmiah Pendidikan Matematika*, 1(7), 121- 127.

77. Sarría, Í., Gonzàlez, R., Magreñàn, Á. A., Narvàez, S. P., & Orcos, L. (2017). Games math. Adaptive video game to evaluate basic mathematic concepts. In *International Workshop on Learning Technology for Education in Cloud* (pp. 27- 35). Springer, Cham.

78. Saxena, R. C., & Singh, B. (2018). Unit-1 Nature, need and place of mathematics in school curriculum. IGNOU.

79. Schoenfeld, A. H. & Kilpatrick, J. (2008). Toward a theory of proficiency in teaching mathematics. *International Handbook of Mathematics Teacher Education,* 2, 1- 35.

80. Scotland, J. (2012). Exploring the philosophical underpinnings of research: Relating ontology and epistemology to the methodology and methods of the scientific, interpretive, and critical research paradigms. *English Language Teaching*, 5(9), 9- 16.

81. Shield, M., & Dole, S. (2013). Assessing the potential of mathematics textbooks to promote deep learning. *Educational Study Mathematics*, 82, 183- 199.

82. Sholehawati, R., & Wahyudin, W. (2019). Investigation of critical mathematical thinking ability, visual thinking and self-efficacy students' in trigonometry. In *Journal of Physics: Conference Series,* 1157 (3). IOP Publishing.

83. Siagan, M., Saragih, S., & Sinaga, B. (2019). Development of learning materials oriented on problem- based learning model to improve students' mathematical problem solving ability and metacognition ability. *International Electronic Journal of Mathematics Education*, 14(2), 331- 340.

84. Simons, M. D. (2016). An Ethnomethodological analysis of learners' ways of working in a high-stakes grade 12 mathematics National Senior Certificate (NSC) examination: The case of trigonometry.

85. Siew, N. M., Geofrey, J., & Lee, B. N. (2016). Students' algebraic thinking and attitudes towards algebra: the effects of game- based learning using Dragonbox 12+ App. *The Research Journal of Mathematics and Technology*, 5(1), 66- 79.

86. Teherani, A., Martimianakis, T., Stenfors-Hayes, T., Wadhwa, A. & Varpio, L. (2015). Choosing a qualitative research approach. *Journal of Graduate Medical Education*, 669- 670.

87. Torbeyns, J., Schneider, M., Xin, Z., & Siegler, R. S. (2015). Bridging the gap: Fraction understanding is central to mathematics achievement in students from three different continents. *Learning and Instruction*, 37, 5- 13.

88. Usiskin, Z. (2015). What does it mean to understand some Mathematics? *In Selected regular lectures from the 12th international congress on mathematical education*, 821- 841. Springer, Cham.

89. Vaismoradi, M., Jones, J., Turunen, H., & Sherrill, S. (2016). Theme development in qualitative content analysis and thematic analysis. *Journal of Nursing Education and Practice*, 6(5), 100- 110.
90. Valero, P. (2005). The myth of the active learner: From cognitive to socio-political interpretations of students in mathematics classrooms. *Proceedings of the Third International Mathematics Education and Society Conference.* Danmarks Pædagogiske Universitet, 489- 500.
91. van der Berg, S.(2008). How effective are poor schools? Poverty and educational outcomes in South Africa, *Center for European Governance and Economic Development Research*, 69, 1- 36, Göttingen.
92. Vince, J. (2018). Trigonometric identities using complex numbers. In *Imaginary Mathematics for Computer Science*, 189- 193. Springer, Cham.
93. Visser, M., Juan, A., & Feza, N. (2015). Home and school resources as predictors of mathematics performance in South Africa. *South African Journal of Education*, 35 (1), 1- 10.
94. Vitoria, L., Fauzu, F., & Ananda, N. (2017). Students' performance in solving problems involving fractions. *Proceedings of AICS- Social Sciences*, 7, 707- 714.
95. Wilson, P. S. & Heid, M. K. (2011). Framework for mathematical proficiency for teaching.
96. Yang, D. C., & Sianturi, I. A. (2017). An analysis of Singaporean versus Indonesian textbooks based on trigonometry content. *EURASIA Journal of Mathematics Science and Technology Education,* 13(7), 3829- 3848.
97. Yazan, B. (2015). Three approaches to case study methods in education: Yin, Merriam, and Stake. *The Qualitative Report*, 20(2), 134- 152.
98. Yi, J. A., Yoo, J. G., & Lee, K. H. (2013). Toward students full understanding of trigonometric ratios. *Research in Mathematical Education*, 17(1), 63- 78.
99. Yusha'u, M. A. (2013). Difficult topics in junior secondary school mathematics: practical aspect of teaching and learning trigonometry. *Scientific Journal of Pure and Applied Sciences*, 2(4), 161-174.

Appendices

Appendix 1

School of Education,
College of Humanities,
University of KwaZulu-Natal,
Edgewood Campus,

Dear Participant

INFORMED CONSENT LETTER

My name is Nontobeko Khuzwayo. I am a Master's degree (Mathematics Education) candidate studying at the University of KwaZulu-Natal, Edgewood campus, South Africa. I am interested in exploring the learning of trigonometric identities in grade 11 mathematics learners. To gather the information, I am interested in asking you some questions.
Please note that:

- Your confidentiality is guaranteed as your inputs will not be attributed to you in person, but reported only as a population member opinion.
- The interview may last for about 45 minutes to 1 hour.
- Any information given by you cannot be used against you, and the collected data will be used for purposes of this research only.
- Data will be stored in secure storage and destroyed after 5 years.
- You have a choice to participate, not participate or stop participating in the research. You will not be penalized for taking such an action.
- Your involvement is purely for academic purposes only, and there are no financial benefits involved.
- If you are willing to be interviewed, please indicate (by ticking as applicable) whether or not you are willing to allow the interview to be recorded by the following equipment:

Equipment	**Willing**	**Not willing**
Audio equipment		
Photographic equipment		
Video equipment		

I can be contacted at:
Email: khuzwayontobe@gmail.com
Cell: 078 694 5971

My supervisor is Dr. Jayaluxmi Naidoo who is located at the School of Education, Edgewood campus of the University of KwaZulu-Natal.
Contact details: email: naidooj2@ukzn.ac.za Phone number: +27312601127.

You may also contact the Research Office through:
Ms P Ximba (HSSREC Research Office)
Tel: 031 260 3587
Email: ximbap@ukzn.ac.za)
Thank you for your contribution to this research.

DECLARATION

I………………………………………………………………………… (full names of participant) hereby confirm that I understand the contents of this document and the nature of the research project, and I consent to participating in the research project.

I understand that I am at liberty to withdraw from the project at any time, should I so desire.

SIGNATURE OF PARTICIPANT DATE

……………………………………… …………………

SIGNATURE OF PARENT (If participant is a minor) DATE

………………………………………… …………………..

Appendix 2

School of Education,
College of Humanities,
University of KwaZulu-Natal,
Edgewood Campus,

Dear Parent(s) / Guardian(S)

INFORMED CONSENT LETTER

My name is Nontobeko Khuzwayo. I am a Master's degree (Mathematics Education) candidate studying at the University of KwaZulu-Natal, Edgewood campus, South Africa. I am interested in exploring the learning of trigonometric identities in grade 11 mathematics learners. To gather the information, I am interested in asking your child some questions. Please note that:

- Your child's confidentiality is guaranteed as his/her inputs will not be attributed to him/her in person, but reported only as a population member opinion.
- The interview may last for about 45 minutes to 1 hour.
- Any information given by your child cannot be used against him/her, and the collected data will be used for purposes of this research only.
- Data will be stored in secure storage and destroyed after 5 years.
- Your child has a choice to participate, not participate or stop participating in the research. He/ she will not be penalized for taking such an action.
- Your child's involvement is purely for academic purposes only, and there are no financial benefits involved.
- If you are allowing your child to be interviewed, please indicate (by ticking as applicable) whether or not you are willing to allow the interview to be recorded by the following equipment:

Equipment	**Willing**	**Not willing**
Audio equipment		
Photographic equipment		
Video equipment		

I can be contacted at:
Email: khuzwayontobe@gmail.com
Cell: 078 694 5971

My supervisor is Dr. Jayaluxmi Naidoo who is located at the School of Education, Edgewood campus of the University of KwaZulu-Natal.
Contact details: email: naidooj2@ukzn.ac.za   Phone number: +27312601127.

You may also contact the Research Office through:
Ms P Ximba (HSSREC Research Office)
Tel: 031 260 3587
Email: ximbap@ukzn.ac.za)
Thank you for your contribution to this research.

DECLARATION

I……………………………………………………………………………… (full names of parent/guardian) hereby confirm that I understand the contents of this document and the nature of the research project, and I consent to my child's participation in the research project.

I understand that my child is at liberty to withdraw from the project at any time, should he/she so desire.

SIGNATURE OF PARENT / GUARDIAN DATE

………………………………………… …………………..

Appendix 3

Grade 11 activity worksheet: Trigonometric identities

Instructions and information:

- **The following questions are design to explore your understanding on solving problems using trigonometric identities.**
- **Do not write your name on any of these pages.**
- **Please answer all the questions.**
- **When answering questions, please clearly show how you arrived at your answer.**

Question 1

Prove the following fundamental identities:

1.1 $tanx = \frac{\sin x}{\cos x}$

1.2 $sin^2x = 1 - cos^2x$

Question 2

Simplify the following as far as possible

2.1 $tan^2x\ (1 - sin^2x)$

2.2 $cos^4x + cos^2x.sin^2x$

2.3 $(\frac{\cos x}{sinx} + \frac{sinx}{cosx}).sinx.cosx$

2.4 $(3 - 3sin\Theta)(3 + 3sin\Theta)$

Question 3

Prove the following identities:

3.1 $\frac{sin^2x+sinx.cosx}{cosx.sinx+cos^2x} = tanx$

3.2 $\frac{1+2\ sin\ x.cosx}{sinx+cosx} = sinx + cosx$

3.3 $(\frac{1}{cosx} - tanx)^2 = \frac{1-sinx}{1+sinx}$

3.4 $\frac{1+cos\,x}{sin\,x} + \frac{\sin x}{1+\cos x} = \frac{2}{\sin x}$

Appendix 4

Interview questions

1. Which topics do you find challenging when learning Mathematics in Grade 11? Why?
2. Please site any real life situations in which trigonometry may be applied in.
3. In your understanding, what is the meaning of sine; cosine and tangent?
4. Kindly explain how trigonometric identities were introduced to you
5. Please describe your encounters/ experiences when learning trigonometric identities in Grade 11 Mathematics? Please elaborate.
6. What challenges do you encounter when answering trigonometric identities related problems in Grade 11 Mathematics? Please explain.
7. In your opinion, how can the learning of trigonometric identities in Grade 11 Mathematics be improved?
8. Is there any information you would like to share with me? Please explain.

Thank you for your time

Appendix 5

CERTIFICATE OF ENGLISH EDITING

This certificate confirms that the thesis entitled below has been edited by an English expert with a Ph.D. who has professional review and editing experience. The following errors were checked for and corrected: grammar, punctuation, spelling, word choice, sentence structure, and clarity.

Thesis title: EXPLORING GRADE 11 MATHEMATICS LEARNERS LEARNING OF TRIGONOMETRIC IDENTITIES

A CASE STUDY OF ONE SCHOOL IN KWAZULU-NATAL

Author: Nontobeko Khuzwayo

Date issued: 27 November, 2019

Editor: Dr. Rae Osborn

(A.A.S., A.A.S., B.Sc., B.Sc. (Hons), M.Sc., Ph.D.)

Appendix 6

Turnitin Certificate

Exploring grade 11 Mathematics learners learning of trigonometric identities: A case study of one school in KZN

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