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AGE STRUCTURED MODELS OF MATHEMATICAL EPIDEMIOLOGY

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As the candidate's supervisor, I have approved this dissertation for submission.

Signed: .... Name: .... Date: ....

Preface

The work described in this thesis was carried out in the School of Mathematics, Statistics and Computer Science; University of KwaZulu-Natal, Durban, under the supervision of Professor Jacek Banasiak.

These studies represent original work by the author and have not otherwise been submitted in any form for any degree or diploma to any tertiary institution. Where use has been made of the work of others it is duly acknowledged in the text.

Abstract

We consider a mathematical model which describes the dynamics for the spread of a directly transmitted disease in an isolated population with age structure, in an invariant habitat, where all individuals have a finite life-span, that is, the maximum age is finite, hence the mortality is unbounded. We assume that infected individuals do not recover permanently, meaning that these diseases do not convey immunity (these could be: common cold, influenza, gonorrhoea) and the infection can be transmitted horizontally as well as vertically from adult individuals to their newborns. The model consists of a nonlinear and nonlocal system of equations of hyperbolic type. Note that the above-mentioned model has been already analysed by many authors who assumed a constant total population. With this assumption they considered the ratios of the density and the stable age profile of the population, see [16, 31]. In this way they were able to eliminate the unbounded death rate from the model, making it easier to analyse by means of the semigroup techniques. In this work we do not make such an assumption except for the error estimates in the asymptotic analysis of a singularly perturbed problem where we assume that the net reproduction rate $R \leq 1$.

For certain particular age-dependent constitutive forms of the force of infection term, solvability of the above-mentioned age-structured epidemic model is proven. In the intercohort case, we use the semigroup theory to prove that the problem is well-posed in a suitable population state space of Lebesgue integrable vector valued functions and has a unique classical solution which is positive, global in time and has the property of continuous dependence on the initial data. Further, we prove, under additional regularity conditions (composed of specific assumptions and compatibility conditions at the origin), that the solution is smooth. In the intracohort case, we have to consider a suitable population state space of bounded vector valued functions on which the (unbounded) population operator cannot generate a strongly continuous semigroup which, therefore, is not suitable for semigroup techniques–any strongly continuous semigroup on the space of bounded vector valued functions is uniformly continuous, see [6, Theorem 3.6]. Since, for a finite life-span of the population, the space of bounded vector valued functions is a subspace densely and continuously embedded in the state space of Lebesgue integrable vector valued functions, thus we can restrict the analysis of the intercohort case to the above-mentioned space of bounded vector valued functions. We prove that this state space is invariant under the action of the strongly continuous semigroup generated by the (unbounded) population operator on the state space of Lebesgue integrable vector valued functions. Further, we prove the existence and uniqueness of a mild solution to the problem.

In general, different time scales can be identified in age-structured epidemiological models. In fact, if the disease is not terminal, the process of getting sick and recovering is much faster than

a typical demographical process. In this work, we consider the case where recovering is much faster than getting sick, giving birth and death. We consider a convenient approach that carries out a preliminary theoretical analysis of the model and, in particular, identifies time scales of it. Typically this allows separation of scales and aggregation of variables through asymptotic analysis based on the Chapman-Enskog procedure, to arrive at reduced models which preserve essential features of the original dynamics being at the same time easier to analyse.

Declaration 1 - Plagiarism

I, Rodrigue Yves M'pika Massoukou, declare that

1. The research reported in this thesis, except where otherwise indicated, is my original research.

2. This thesis has not been submitted for any degree or examination at any other university.

3. This thesis does not contain other persons' data, pictures, graphs or other information, unless specifically acknowledged as being sourced from other persons.

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Declaration 2-Publications

Paper submitted

1. J. Banasiak, R. Y. M. Massoukou, A Singularly Perturbed Age Structured SISR Model with Fast Recovery.

Paper in preparation

1. J. Banasiak, R. Y. M. Massoukou, Solvability of Age Structured SIS Model: Semigroup Approach.

Signed: ....

I dedicate this dissertation to the Lord

God Almighty,

to my precious mother

Denise Massoukou

and to the memory of my father

Paulin Massoukou.

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Contents

1. Introduction

The idea that transmission and spread of infectious diseases follow laws that can be formulated in a mathematical framework arose in the eighteenth century. One of the great achievements was the work published in 1766 by Daniel Bernoulli, which described the effects of smallpox inoculation on life expectancy using mathematical life table analysis. In the course of time, many authors invested their efforts to understand the dynamics of infectious disease transmission and improve the field of mathematical modelling. In the beginning of the twentieth century, the reason for which an epidemic ended before all susceptibles were infected was understood by William Hamer in 1906, who become one of the first to recognize that it was the diminishing density of susceptibles alone that could bring the epidemic to a halt. Further in this century, Kermack and McKendrick, in 1927, described the dynamics of infectious disease transmission in terms of a system of differential equations. Only towards the end of the twentieth century did mathematical modelling come into more widespread use for public health policy making.

In this thesis, we are interested in a population that, in absence of epidemics, can be modelled by the linear McKendrick model describing the evolution in time of the density of the population with respect to age, determined by the vital rates.

In the sequel, we introduce models which describe: the dynamics of an infectious disease regardless of the vital rates, the vital dynamics of the population and the dynamics resulting from the interactions between the two previous dynamics. In addition to these, we present an asymptotic analysis of a singularly perturbed problem of the latest model and we give the outline of the thesis.

1.1 An Overview of Epidemic Models

We briefly introduce the mathematical modelling for communicable diseases such as influenza, measles, rubella (German measles), chicken pox and hepatitis, transmitted by viral agents; tuberculosis and meningitis, transmitted by bacteria.

It shall be noted that mathematical modelling of communicable diseases has provided many considerable insights concerning the epidemiology of infectious diseases. The most notable of these involve threshold conditions (that is the so-called basic reproductive number) that describe whether invasion and persistence of a disease is possible. Indeed, an infectious disease can spread in a susceptible population only if the basic reproduction number is above a threshold value.

The development of much of this theory has fluctuated around the use of extremely simple models, such as deterministic compartmental models, with the population of interest being subdivided into a small number of compartments based on the infection status (e.g. susceptible, infectious, recovered or exposed). The dynamics in each compartment (or class) is described by an ordinary differential equation (ODE) resulting from the rate of change of the corresponding individuals; hence the model will consist of a system of ODEs. The derivation of these equations typically involves a number of simplifying assumptions about the nature and rates of transfer from one compartment to another, an important example of this is the "mass-action" description of transmission, which will be discussed later in detail.

Since we are concerned with models for epidemics acting on a sufficiently rapid time scale, the demographic effects such as births, natural deaths, immigration and emigration of a population are not included. To model an epidemic of this type, the population of interest is split into three classes labelled S, I and R. We denote by $S(t)$ the number of individuals who are susceptible to the disease, i.e., individuals who can contract the disease under appropriate conditions at time t, but are not (yet) infected; by $I(t)$ the number of infective individuals who can spread the disease by contact with susceptible individuals; by $R(t)$ the number of individuals who had been infected and then 'removed' from the class of infective individuals without any chance to be reinfected. Removal is carried out through isolation from the rest of the population, through immunization against infection, through recovery from the disease with full immunity against reinfection, or through death caused by the disease. These characterizations of 'removed' members are different from an epidemiological perspective but are often equivalent from the modelling point of view that takes into account only the state of an individual with respect to the disease. We will use an appropriate acronym that provides information about the model structure. The resulting models are classified by a string of letters (I, R, S) that provides information about the model structure. We shall use SIR to describe the dynamics of an infectious disease that confers immunity against the reinfection, to indicate that the individuals move from the susceptible class S to the infective class I, then to the removed class R. Further, we use SIS to describe the dynamics of infectious disease which does not confer immunity against the reinfection, that is, to indicate that the individuals move from the susceptible class S to the infective class I, then back to the susceptible class S. In addition to this basic distinction between infectious diseases for which the recovery confers immunity against the reinfection and infectious diseases for which the recovered members are again susceptible to the reinfection, we have infectious diseases for which some of the recovered members move to the removed class R, while others become susceptible to the reinfection. Such a dynamics is described by $SIRS$. If we consider the class, E, of exposed (or latent) individuals (explicitly containing those infected not yet infectious) in each of the three previous model structure, we obtain the models classified by $SEIR$, $SEIS$ and $SEIRS$, respectively.

We are assuming that the epidemic process is deterministic, i.e., the behaviour of a population is determined completely by its history and by the rules that describe the model.

As mentioned above, one of the earliest achievements of the mathematical modelling was the formulation of the first epidemic model, namely the SIR model, by Kermack and McKendrick in 1927. Although simple, its predictions were very similar to the behaviour observed in many epidemics. This model was based on the following assumptions:

• cN is the number of contacts per unit time an infective makes, where N denotes the total population size,

• S/N is the probability that a random contact by an infective is with a susceptible individual,

• $(cN)(S/N)$ is the number of contacts with susceptibles in unit time per infective,

• pcS is the number of susceptibles who become infected in unit time per infective, where p denote the probability that a contact with a susceptible results in transmission,

• $\beta SI = pcSI$ is the rate of new infections (incidence), where $\beta = pc$,

• $\Lambda(t)S$ is the number of individuals who become infected per unit time, where $\Lambda(t) = \beta I(t)$ is known as the infection rate or the force of infection and β is the constant of proportionality called the transmission rate.

The number of new cases with a specific illness occurring during a given time period is called incidence. This depends on the assumption made on the form of the force of infection. In this case, it is called mass action and this type of incidence yields the following differential equation for the susceptible individuals:

$$\frac{dS}{dt}(t) = -\Lambda(t)S(t).$$

The susceptible individuals who contract the infection move to the class I while individuals who recover or die leave the class I at a constant per capita probability per unit time δ called the recovery rate. That is, infective individuals leave the class I at rate δI per unit time. Thus,

$$\frac{dI}{dt}(t) = \Lambda(t)S(t) - \delta I(t)\,, \quad \frac{dR}{dt}(t) = \delta I(t).$$

Hence, the model proposed by Kermack and McKendrick in 1927 is

$$\frac{dS}{dt}(t) = -\Lambda(t)S(t), \tag{1.1a}$$

$$\frac{dI}{dt}(t) = \Lambda(t)S(t) - \delta I(t), \tag{1.1b}$$

$$\frac{dR}{dt}(t) = \delta I(t). \tag{1.1c}$$

For this model the total population size N is given by $N(t) = S(t) + I(t) + R(t)$. Adding up Equations (1.1a)-(1.1c) leads to $N^{'}(t) = 0$ and so the total population size $N(t)$ is constant.

In the case of epidemics which do not convey immunity, the model is reduced to the following

$$\frac{dS}{dt}(t) = -\Lambda(t)S(t) + \delta I(t),$$
$$\frac{dI}{dt}(t) = \Lambda(t)S(t) - \delta I(t).$$

1.2 An Overview of the Linear McKendrick Model

Let us denote by $\beta(a)$ the age specific fertility, i.e., the average number of newborns, in one time unit, coming from an individual of age $(a, a + da)$ and by $\mu(a)$ the age specific mortality or the per capita mortality rate for individuals of age $(a, a + da)$. From these basic vital rates, we derive other biologically significant quantities. Namely

$$\Pi_\mu(a) = e^{-\int_0^a \mu(\sigma)\, d\sigma}, \tag{1.2}$$

known as the survival probability, see [8, 27], i.e. the probability for an individual to survive to age a;

$$R = \int_0^\omega \beta(a)\Pi_\mu(a)\, da \tag{1.3}$$

known as the net reproduction rate, i.e., the expected number of newborns to be produced by an individual in his reproductive life. ω is the maximum attainable age. Since, in a biological sense, no individual can live beyond ω, $\Pi_\mu(\omega) = 0$ which requires

$$\int_0^\omega \mu(\sigma)\, d\sigma = +\infty. \tag{1.4}$$

For convenience, in some papers, including [50], the maximum age is assumed infinite, $\omega = +\infty$; so that, from (1.4), μ could be either bounded or unbounded on $\mathbb{R}_+$. But, realistically speaking, this assumption does not have any biological meaning and therefore, in this thesis we make a biologically realistic assumption that $\omega < +\infty$; hence, from (1.4), μ cannot be bounded as $a \to \omega^-$. The latest assumption introduces another unbounded operator, multiplication by μ, in the system of differential equations that will be introduced later in this chapter.

The demographic process of the population is described by its age density function at time t:

$$p(a, t), \quad 0 \le a \le \omega,\ 0 \le t.$$

In the sequel, we assume that p is differentiable. Let

$$p(a, t)da \equiv \text{number of individuals of age } (a, a + da) \text{ at time } t.$$

If we consider the rate of change, $p(a,t)\Delta a$ of the number of individuals in a given age interval $(a, a+\Delta a)$, we may write

$$\frac{\partial}{\partial t}[p(a,t)\Delta a] = \begin{bmatrix} +\text{ rate of entry at } a \\ -\text{ rate of departure at } (a+\Delta a) \\ -\text{ rate of deaths} \end{bmatrix}$$

or

$$\frac{\partial p}{\partial t}(a,t)\Delta a = J(a,t) - J(a+\Delta a,t) - \mu(a)p(a,t)\Delta a,$$

where $J(a,t)$ is the 'flux' of individuals of age a at time t. Dividing by Δa yields

$$\frac{\partial p}{\partial t}(a,t) = -\left[\frac{J(a+\Delta a,t) - J(a,t)}{\Delta a}\right] - \mu(a)p(a,t),$$

which leads to

$$\frac{\partial p}{\partial t}(a,t) = -\frac{\partial J}{\partial a}(a,t) - \mu(a)p(a,t) \tag{1.5}$$

as $\Delta a \to 0$ and provided J is differentiable. This is know as a 'conservation law' for the density of individuals.

The 'flux', $J(a,t)$, of individuals is understood as the movement of individuals in age. Since all individuals age, we should expect this flux to be proportional to the density, $p(a,t)$, of individuals, with some characteristic velocity $v(a,t)$ of ageing:

$$J(a,t) \equiv p(a,t)v(a,t).$$

For most cases, ageing is just the passage of time:

$$v = \frac{da}{dt} = 1.$$

Hence, Equation (1.5) becomes

$$\frac{\partial p}{\partial t} + \frac{\partial p}{\partial a} = -\mu(a)p. \tag{1.6}$$

In the McKendrick-von Foerster model, the total birth rate, $B(t)$, is modelled as a boundary condition at age zero:

$$B(t) := p(0,t) = \int_0^{\omega} \beta(a)p(a,t)\,da. \tag{1.7}$$

To complete the model, we must also specify an initial age distribution,

$$p(a,0) = p_0(a). \tag{1.8}$$

The system of equations (1.6), (1.7), (1.8) is the basic model which describes the evolution in time of the density population $p(a,t)$ with respect to age $a \in [0,\omega]$, $\omega < +\infty$.

It is shown, in [27, p. 11-12], that

$$p(a,t) = \begin{cases} p_0(a-t)\frac{\Pi_\mu(a)}{\Pi_\mu(a-t)}, & a > t, \\ B(t-a)\Pi_\mu(a), & a < t. \end{cases} \tag{1.9}$$

Substituting (1.9) into (1.7) yields the following Volterra integral equation

$$B(t) = F(t) + \int_0^t K_\mu(t-s)B(s)\,ds, \tag{1.10}$$

with

$$F(t) = \int_t^\omega \beta(a)\frac{\Pi_\mu(a)}{\Pi_\mu(a-t)}p_0(a-t)\,da, \quad K_\mu(t) = \beta(t)\Pi_\mu(t), \tag{1.11}$$

where the functions $\beta(a)$, $\frac{\Pi_\mu(a)}{\Pi_\mu(a-t)}$, $p_0(a)$ are extended by zero for $a > \omega$.

Next, we present a model resulting from the interactions of the two models introduced above.

1.3 An Overview of the Age-Structured Epidemic Model

A long time ago, it was recognized that the age structure of a population affects the dynamics of the disease transmission. In fact, some basic age-structured models of epidemics were pioneered by Sharpe and Lotka [48] and by McKendrick [39]. Recently, great efforts were invested to determine the threshold conditions for the disease to become endemic, the stability of the steady state solutions, and the global behaviour of age-structured epidemic models; we are indebted to Cooke and Busenberg [18]; Busenberg, Iannelli and Thieme [16]; Anderson and May [3],[4]; Dietz and Schenzle [19]; Gripenberg [25]; Busenberg, Cooke and Iannelli [15]; Martcheva and Crispino-O'Connell [37]; Greenhalgh [23, 24]; Inaba [30, 31]; Li, Gupur and Zhu [36]; Feng, Huang and Castillo-Chavez [21]; Iannelli [27]; Webb [51]; Inaba [31]; Iannelli, Milner and Pugliese [28]; etc. It should be noted that in most of the papers cited above, it is assumed that the net reproduction rate, R, is equal to one and the total population has reached its steady-state distribution. This assumption enabled the unbounded function μ in the full system to be removed and yielded a normalized system that was easier to analyse. As we shall see, these assumptions are not always satisfied and we will have to treat the model (1.6) with some care.

We consider the general SIRS age-structured model introduced in [27], where the basic variables $s(a,t)$, $i(a,t)$ and $r(a,t)$ are the age-specific density of susceptible, infective and 'removed' individuals, respectively, of age a at time t, the basic parameters $\gamma(a)$ and $\delta(a)$ are age-specific removal and recovery rates, respectively (here the removal refers to the recovery with permanent immunity), and q, $w \in [0,1]$ are the coefficients of vertical transmission of infectiveness and immunity, respectively, namely the fraction of newborns who are born in the corresponding class of their parents. That is,

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)s(a,t)=-\mu(a)s(a,t)-\Lambda(a,i(\cdot,t))s(a,t)+\delta(a)i(a,t), \tag{1.12}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)i(a,t)=-\mu(a)i(a,t)+\Lambda(a,i(\cdot,t))s(a,t)-\big(\gamma(a)+\delta(a)\big)i(a,t), \tag{1.13}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)r(a,t)=-\mu(a)r(a,t)+\gamma(a)i(a,t), \tag{1.14}$$

$$s(0,t)=\int_0^\omega \beta(a)\big(s(a,t)+(1-q)i(a,t)+(1-w)r(a,t)\big)\,da, \tag{1.15}$$

$$i(0,t)=q\int_0^\omega \beta(a)i(a,t)\,da, \tag{1.16}$$

$$r(0,t)=w\int_0^\omega \beta(a)r(a,t)\,da, \tag{1.17}$$

$$s(a,0)=s_0(a)=\phi^s(a), \tag{1.18}$$

$$i(a,0)=i_0(a)=\phi^i(a), \tag{1.19}$$

$$r(a,0)=r_0(a)=\phi^r(a), \tag{1.20}$$

for $\omega<+\infty$. The function Λ denotes the infection rate (or force of infection), namely the per capita rate of acquisition of infection; which assumes one of the following particular age-dependent forms:

(i) $\Lambda(a,i(\cdot,t))=\int_0^\omega K(a,a^{'})i(a^{'},t)\,da^{'}$,

(ii) $\Lambda(a,i(\cdot,t))=K_0(a)i(a,t)$.

The form (ii) represents the situation in which individuals can be infected only by infectives of the same age, and the form (i) represents the situation where the age of infectives does not affect their contact rates with other individuals.

Let $u(a,t)$ be the age-density at time t of the host population. Thus, $u(a,t)$ is given by

$$u(a,t)=s(a,t)+i(a,t)+r(a,t).$$

Adding up, respectively, the Equations (1.12)-(1.14), (1.15)-(1.17) and (1.18)-(1.20) leads to the following linear McKendrick model:

$$\frac{\partial u}{\partial t}(a,t)+\frac{\partial u}{\partial a}(a,t)=-\mu\left(a\right)u(a,t), \tag{1.21a}$$

$$u\left(0,t\right)=\int_0^\omega \beta(a)u(a,t)\,da, \tag{1.21b}$$

$$u\left(a,0\right)=u_0(a)=s_0(a)+i_0(a)+r_0(a), \tag{1.21c}$$

for $\omega < +\infty$.

Note that in building (1.12)-(1.20), care should be taken to consider the discrepancy of time scales between the processes. Indeed, if we deal with a human population, then the vital rates are measured in units 1/year. On the other hand, if we model a disease such as flu or measles, then the removal and recovery rates are measured in units 1/day (average duration of the disease, which is the inverse of the recovery rate, is 2-7 days). Thus, if we are to use 1 year as the unit of (1.12)-(1.20), then the numerical values of δ and γ, used in the literature, should be multiplied by 365. For convenience, we instead multiply by $1/\epsilon$, where $\epsilon = 1/365$. Certainly, for other diseases, such as HIV / AIDS, the duration of the disease is at the same time scale as the vital dynamics and the above argument is not satisfied.

The scaling of the infection rate Λ depends on the constitutive law which governs Λ. In the case of a simple compartmental SIR model, regardless of the vital rates, the simplest constitutive law for Λ is given by $\Lambda = \beta I$, where β is a constant defined by $\beta = \gamma R_0/P$, where R_0 is the so-called basic reproduction rate and P is the size of the population. The constant R_0 for, say, measles, is 18 [13, p. 9], so the relative magnitude of β depends on the size of the population: if the population is large, then this term can be included in 'small' terms, while if it is small, it is a 'large' term. In this thesis we shall be concerned with the former case and we shall work with the singularly perturbed system of the form

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) s_\epsilon(a,t) = -\mu(a)s_\epsilon(a,t) - \Lambda(a, i_\epsilon(\cdot,t))s_\epsilon(a,t) + \frac{1}{\epsilon}\delta(a)i_\epsilon(a,t), \tag{1.22}$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) i_\epsilon(a,t) = -\mu(a)i_\epsilon(a,t) + \Lambda(a, i_\epsilon(\cdot,t))s_\epsilon(a,t) - \frac{1}{\epsilon}\big(\gamma(a) + \delta(a)\big)i_\epsilon(a,t), \tag{1.23}$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) r_\epsilon(a,t) = -\mu(a)r_\epsilon(a,t) + \frac{1}{\epsilon}\gamma(a)i_\epsilon(a,t), \tag{1.24}$$

$$s_\epsilon(0,t) = \int_0^\omega \beta(a)\big(s_\epsilon(a,t) + (1-q)i_\epsilon(a,t) + (1-w)r_\epsilon(a,t)\big)\,da, \tag{1.25}$$

$$i_\epsilon(0,t) = q\int_0^\omega \beta(a)i_\epsilon(a,t)\,da, \tag{1.26}$$

$$r_\epsilon(0,t) = w\int_0^\omega \beta(a)r_\epsilon(a,t)\,da, \tag{1.27}$$

$$s_\epsilon(a,0) = s_0(a) = \phi^s(a), \tag{1.28}$$

$$i_\epsilon(a,0) = i_0(a) = \phi^i(a), \tag{1.29}$$

$$r_\epsilon(a,0) = r_0(a) = \phi^r(a), \tag{1.30}$$

as $\epsilon \to 0$.

1.4 An Overview of the Asymptotic Analysis of Singularly Perturbed Problems

We re-write the singularly perturbed problem (1.22)-(1.30) in the form

$$\partial_t \mathbf{f}_\epsilon = \mathbf{S}\mathbf{f}_\epsilon + \mathbf{M}_\mu \mathbf{f}_\epsilon + \mathfrak{F}(\mathbf{f}_\epsilon) + \frac{1}{\epsilon}\mathbf{M}_\delta \mathbf{f}_\epsilon, \tag{1.31a}$$

$$\mathbf{f}_\epsilon(0,t) = \int_0^\omega \mathbf{B}(a)\mathbf{f}_\epsilon(a,t)\,da, \tag{1.31b}$$

$$\mathbf{f}_\epsilon(a,0) = \mathbf{f}^0(a), \tag{1.31c}$$

where $\epsilon \to 0$, $\mathbf{f}_\epsilon = (s_\epsilon, i_\epsilon, r_\epsilon)^T$, $\mathbf{S} := \operatorname{diag}\{-\partial_a, -\partial_a, -\partial_a\}$, $\mathbf{M}_\mu := \operatorname{diag}\{-\mu, -\mu, -\mu\}$

$$\mathfrak{F}(\mathbf{f}_\epsilon) := \begin{bmatrix} -\Lambda(\cdot, i_\epsilon)s_\epsilon \\ \Lambda(\cdot, i_\epsilon)s_\epsilon \end{bmatrix}, \ \mathbf{M}_\delta := \begin{bmatrix} 0 & -\delta & 0 \\ 0 & -(\delta+\gamma) & 0 \\ 0 & -\delta & 0 \end{bmatrix}, \text{ and } \mathbf{B} := \begin{bmatrix} \beta & (1-q)\beta & (1-w)\beta \\ 0 & q\beta & 0 \\ 0 & 0 & w\beta \end{bmatrix}.$$

In this thesis, we are only interested in analysing a simple SIS system. We shall prove that, for sufficiently small ϵ, the solution of the above-mentioned SIS system can be approximated by means of the solution of a single kinetic equation and an initial layer term. For the analysis of the full system we refer the reader to [11].

The above-mentioned kinetic equation to be used for the approximation is derived in a systematic way using a proper mathematical asymptotic analysis, see [1], consisting of:
(i) constructing a solution to the system (1.31a)-(1.31c) in the form of a truncated power series

$$\mathbf{f}_\epsilon^{(n)}(t) = \mathbf{f}_0(t) + \epsilon\mathbf{f}_1(t) + \epsilon^2\mathbf{f}_2(t) + \cdots + \epsilon^n\mathbf{f}_n(t),$$

and deriving an algorithm which provides a systematic way for approximating the family, $\mathbf{f}_0$, $\mathbf{f}_1$, $\mathbf{f}_2$, ..., $\mathbf{f}_n$. Then we say $\mathbf{f}_\epsilon^{(n)}$ is an approximation of order n to the solution $\mathbf{f}_\epsilon$ of the equation (1.31a) if we have

$$\left\|\mathbf{f}_\epsilon - \mathbf{f}_\epsilon^{(n)}\right\| = \mathbf{o}\left(\epsilon^n\right) \tag{1.32}$$

uniformly for $0 \leq t \leq T$.
(ii) proving the convergence of the solution $\mathbf{f}_\epsilon$ in the sense of (1.32).

To implement (i), we find the hydrodynamic space of $\mathbf{M}_\delta$ (that is, its null-space) and then, using the spectral projections onto this space and the projection onto the complementary space, we decompose the original problem into two problems, for the kinetic and hydrodynamic parts of the solution, respectively. These problems are still coupled but can be converted into a hierarchy of decoupled equations by using the asymptotic expansion of the solution. Here we focus on a modification of the classical Chapman-Enskog asymptotic procedure (see [10],[22],[41]), i.e., we expand the kinetic (or non-hydrodynamic) part in a power series of ϵ and the hydrodynamic part remains unexpanded. The advantage of this method is that the whole information conveyed by the hydrodynamic part is kept together and allows for a more accurate closure of the asymptotic hierarchy. Note that in most cases the Chapman-Enskog expansion fails to satisfy all boundary and initial conditions, (1.31b) and (1.31c), respectively. If the initial condition (1.31c) does not hold for the Chapman-Enskog approximation, then it is necessary to introduce the initial layer correction by using the Hilbert procedure with rescaled fast time $\tau = t/\epsilon$ to improve the convergence for small t and with the hydrodynamic part expanded in ϵ. Similarly, the approximation could fail close to $a = 0$ or close to $a = t = 0$. To improve accuracy in such cases, the so-called boundary correction and the corner correction are introduced, respectively. These are obtained by, respectively, rescaling age as $\alpha = a/\epsilon$ and simultaneously rescaling time and age as $\tau = t/\epsilon$ and $\alpha = a/\epsilon$, and repeating the Hilbert type expansion procedure.

1.5 Outline of the Thesis

The thesis is organized as follows:

In Chapter 2, we introduce some basic tools required for the analysis of our model. First, we give some definitions related to normed vector spaces and Banach spaces. Next, we provide basic facts on calculus in Banach spaces, namely on continuity, differentiability and integrability of Banach space valued functions. Further, we introduce some suitable results related to the generation of strongly continuous semigroups of (unbounded) operators. We also consider the Hille-Yosida theorem which is used for proving that the (unbounded) population operator, resulting from the linear part of our model, generates a strongly continuous semigroup. We finally recall some existing results from the theory of semilinear evolution equations, including results on the existence of mild and strong solutions of the abstract Cauchy problem of the original system.

In Chapter 3, we are concerned with the solvability of a general age-structured model for epidemics which do not confer immunity, namely the SIS epidemics, allowing for vertical and horizontal transmissions. First, we define an appropriate setting in which we shall analyse our model. Some important notation and assumptions are introduced. Next, we establish the well-posedness analysis of our model. The linear part of our model involves three main operators: the differentiation operator, $\mathbf{S}$, the unbounded death operator, $\mathbf{M}_\mu$, and the bounded 'recovery' operator, $\mathbf{M}_\delta$. Thanks to the diagonal structure of the operators $\mathbf{S}$ and $\mathbf{M}_\mu$, we see that the operator $\mathbf{A} := \mathbf{S} + \mathbf{M}_\mu$ is also diagonal and therefore can be analysed as in the scalar case. We use the Hille-Yosida theorem to show that the operator $\mathbf{A}$, on an appropriate domain $D(\mathbf{A})$, generates a strongly continuous semigroup and then we use the bounded perturbation theorem to show that the operator $\mathbf{A} + \mathbf{M}_\delta$ is also the generator of a strongly continuous semigroup.

Further, thanks to the structure of the operator $\mathbf{M}_\delta$, we are able to write explicitly the semigroup generated by $\mathbf{M}_\delta$ and see that the semigroup is positive. Hence, we show that the Trotter product formula is applicable to $\mathbf{A}+\mathbf{M}_\delta$ and then we show that the resulting semigroup is positive and quasicontractive. Further, we show that the semigroup generated by $\mathbf{A}+\mathbf{M}_\delta$ can be estimated by the solution of the scalar McKendrick problem. Next, we analyse the nonlinear problem. For the intercohort case, we consider the state space $\mathrm{L}^1 \times \mathrm{L}^1$ and we prove that the quadratic nonlinear term is locally Lipschitz continuous and continuously Fréchet differentiable. The local existence and the uniqueness of the classical solution follow from the standard results on semilinear evolution equations. Thanks to the positivity of the semigroup generated by the $\mathbf{A}+\mathbf{M}_\delta$, we prove that the local solution is nonnegative. In addition to this property, we prove that the classical solution is global in time, as long as the initial condition belongs to the positive cone of the domain of the unbounded operator. Under some additional assumptions on the demographic parameters, recovery rate and some relevant compatibility conditions (to be derived), we prove a higher regularity of the solution constructed above. For the intracohort case, we consider the state space $\mathrm{L}^\infty \times \mathrm{L}^\infty$, which suits the form of the nonlinear quadratic term resulting from the particular force of infection considered. Note that the unbounded operator $\mathbf{A}+\mathbf{M}_\delta$ does not generate a strongly continuous semigroup on this space. We consider a finite life-span ω, so $\mathrm{L}^\infty([0,\omega]) \subset \mathrm{L}^1([0,\omega])$ hold. Then we show that the semigroup generated by $\mathbf{A}+\mathbf{M}_\delta$ in $\mathrm{L}^1 \times \mathrm{L}^1$ leaves $\mathrm{L}^\infty \times \mathrm{L}^\infty$ invariant. Further, we prove, as in the previous case, that the quadratic nonlinear term is locally Lipschitz continuous and continuously Fréchet differentiable, and that it leaves $\mathrm{L}^\infty \times \mathrm{L}^\infty$ invariant. Hence, we are able to prove the local existence and the uniqueness of a mild solution to the problem in an appropriately chosen closed ball in the space of continuous functions with values in $\mathrm{L}^\infty \times \mathrm{L}^\infty$, for sufficiently small time interval.

In Chapter 4, we consider an age-structured model of SIS epidemics with two time scales: the recovery process is much faster (in 1/day units) than the infection process (in 1/year units) and demographic process (in 1/year units). As mentioned earlier, if we consider 1 year as the unit of time in this model, then the numerical value of δ, used in the literature, should be multiplied by 365, which corresponds to $1/\epsilon$ where $\epsilon = 1/365$. This leads to the singularly perturbed system. Following the description of the asymptotic analysis given earlier, we approximate the solution of the above-mentioned singularly perturbed system by means of the solution of a system of a suitable hydrodynamic equation and a kinetic equation, where the state variables represent, respectively, the global density of the population and the density of the infective individuals, and the initial and a boundary conditions are given. Due to the failure of the approximation for the times close to 0 and for the ages close to 0, we have the presence of the initial layer phenomenon and the boundary and/or the corner layer phenomenon, respectively. Hence, we supplement the approximation with the initial layer corrector and the corner layer corrector, but, as we shall see, we do not need the boundary layer correction. The reasons for this will be given later. Further, we show that the error of the approximation is of order ϵ, for ϵ sufficiently small.

Finally, in Chapter 5, we summarize the outcomes from Chapters 3 and 4.

2. Mathematical Tools

2.1 Normed Vector Spaces and Banach Spaces

The following definitions are based on [34].

Definition 2.1.1. A vector space (or linear space) over a field $\mathbb{K} = \mathbb{R}\,\text{or}\,\mathbb{C}$ is a nonempty set $\mathbf{X}$ of elements $\mathbf{x}, \mathbf{y}, \ldots$ (called vectors) together with two algebraic operations. These operations are called vector addition and multiplication of vectors by scalars, that is, by elements of $\mathbb{K}$.

Definition 2.1.2. A subset of a vector space $\mathbf{X}$ is a nonempty set $\mathbf{Y}$ of $\mathbf{X}$ such that for all $\mathbf{y}_1,\, \mathbf{y}_2 \in \mathbf{Y}$ and all scalars $\alpha,\, \beta$ we have $\alpha\mathbf{y}_1 + \beta\mathbf{y}_2 \in \mathbf{Y}$. Hence $\mathbf{Y}$ is itself a vector space, the two algebraic operators are those induced by $\mathbf{X}$.

Definition 2.1.3. A linear combination of vectors $\mathbf{x}_1, \ldots, \mathbf{x}_m$ of a vector space $\mathbf{X}$ is an expression of the form

$$\alpha_1\mathbf{x}_1 + \cdots + \alpha_m\mathbf{x}_m$$

where the coefficients $\alpha_1, \ldots, \alpha_m$ are any scalars.

Definition 2.1.4. For any nonempty subset $\mathbf{M} \subset \mathbf{X}$, the set of all linear combination of vectors of $\mathbf{M}$ is called the span of $\mathbf{M}$, written

$$\mathrm{span}\mathbf{M}.$$

Obviously, this is a subspace $\mathbf{Y}$ of $\mathbf{X}$, and we say that $\mathbf{Y}$ is spanned, or generated, by $\mathbf{M}$.

Definition 2.1.5. A vector space $\mathbf{X}$ is said to be finite dimensional if there is a positive integer n such that $\mathbf{X}$ contains a linearly independent set of n vectors whereas any set of $n+1$ vectors or more vectors of $\mathbf{X}$ is linearly dependent. The integer n is called the dimension of $\mathbf{X}$ and we denote this by $\dim \mathbf{X} = n$. By definition, $\mathbf{X} = \{\mathbf{0}\}$ is finite dimensional and $\dim \mathbf{X} = 0$. If $\dim \mathbf{X}$ is not finite dimensional, it is said to be infinite dimensional.

Note that if $\dim \mathbf{X} = n$, a linearly independent collection of n vectors of $\mathbf{X}$ is called a basis for $\mathbf{X}$ (or basis in $\mathbf{X}$). If $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a basis for $\mathbf{X}$, every $\mathbf{x} \in \mathbf{X}$ has a unique representation as a linear combination of the basis vectors:

$$\mathbf{x} = \sum_{i=1}^{n} \alpha_i \mathbf{e}_i \quad (\alpha_i \in \mathbb{K}).$$

Definition 2.1.6. A normed space $\mathbf{X}$ is a vector space with a norm defined on it. A Banach space is a complete normed space (complete in the metric defined by the norm). Here a norm on a (real or complex) vector space $\mathbf{X}$ is a real-valued function on $\mathbf{X}$ whose value at an $\mathbf{x} \in \mathbf{X}$ is denoted by $\|\mathbf{x}\|$ (read "norm of $\mathbf{x}$") and which has the properties

(N1) $\|\mathbf{x}\| \geq 0$,

(N2) $\|\mathbf{x}\| = 0 \Leftrightarrow \mathbf{x} = 0$,

(N3) $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$,

(N4) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ (Triangle inequality);

here $\mathbf{x}$ and $\mathbf{y}$ are arbitrary vectors in $\mathbf{X}$ and α is any scalar.

A norm on $\mathbf{X}$ defines a metric d on $\mathbf{X}$, which is given by

$$d(x, y) = \|\mathbf{x} - \mathbf{y}\|$$

and is called the metric induced by the norm.

The normed space $\mathbf{X}$ with norm $\|\cdot\|$ is denoted by $(\mathbf{X}, \|\cdot\|)$. However, if there is no misunderstanding, we simply write $\mathbf{X}$.

Remark 2.1.7. A normed space is said to be complete if every Cauchy sequence converges, and a Banach space is a normed linear space that is complete.

2.2 Subsets of a Normed Vector Space

In this section, the definitions are based on [12].

Definition 2.2.1. Let $\mathbf{X}$ be a normed vector space and $\mathbf{F}$ be a subset of $\mathbf{X}$.

(i) An element $\mathbf{f} \in \mathbf{X}$ is called a limit point of $\mathbf{F}$ if there exists a sequence $\{\mathbf{f}_n\} \subset \mathbf{F}$ such that $\mathbf{f}_n \to \mathbf{f}$ as $n \to \infty$.

(ii) The set of all limit points of $\mathbf{F}$ is called the closure of $\mathbf{F}$ and is denoted by $\overline{\mathbf{F}}$.

Definition 2.2.2. Let $\mathbf{X}$ be a normed vector space. The following assertions are equivalent

(i) A subset $\mathbf{F}$ of $\mathbf{X}$ is dense in $\mathbf{X}$ if $\overline{\mathbf{F}} = \mathbf{X}$.

(ii) A subset $\mathbf{F}$ of $\mathbf{X}$ is dense in $\mathbf{X}$ if every element $\mathbf{f}$ in $\mathbf{X}$ is the limit of a sequence of elements in $\mathbf{F}$.

Definition 2.2.3. Let $\mathbf{X}$ be a real vector space ($\mathbb{K} = \mathbb{R}$) of real-valued functions on a fixed domain $\Omega \subseteq \mathbb{R}$. The set

$$\mathbf{X}_+ = \{\mathbf{f} : \mathbf{f} \in \mathbf{X},\, \mathbf{f}(x) \geq 0 \text{ for all } x \in \Omega\}$$

is called the (closed) positive cone of $\mathbf{X}$.

Remark 2.2.4. We note from Definition 2.2.3:
R1: $\alpha\mathbf{f} + \beta\mathbf{g}$ for all $\mathbf{f}, \mathbf{g} \in \mathbf{X}_+$ and $\alpha \geq 0$, $\beta \geq 0$,
R2: if $\mathbf{f}$ is not the zero element of $\mathbf{X}$ and $\mathbf{f} \in \mathbf{X}_+$, then $-\mathbf{f} \notin \mathbf{X}_+$.

2.3 Calculus of Banach Space Valued Functions

In this section, definitions and theorems are based on [12]; if not, a reference is given.

2.3.1 Banach Space Valued Continuous Functions

Definition 2.3.1. Let $I \subset \mathbb{R}$, let $\mathbf{X}$ be a (real) normed vector space and let $\mathbf{u} : I \to \mathbf{X}$.

(i) $\mathbf{u}$ is strongly continuous at $a \in I$ if, for each $\epsilon > 0$, a $\delta = \delta(a, \epsilon) > 0$ can be found such that

$$\|\mathbf{u}(t) - \mathbf{u}(a)\| < \epsilon \quad \text{whenever} \quad t \in I \quad \text{and} \quad |t - a| < \delta. \tag{2.1}$$

(ii) $\mathbf{u}$ is strongly continuous on I if it is strongly continuous at a for every $a \in I$.

(iii) The set of all functions $\mathbf{u} : I \to \mathbf{X}$ which are strongly continuous on I will be denoted by $\mathcal{C}(I, \mathbf{X})$.

Remark 2.3.2. Throughout this thesis continuity will refer to the strong continuity.

Definition 2.3.3. Let $I \subset \mathbb{R}$ and let $\mathbf{X}$ be a (real) normed vector space. A vector-valued function $\mathbf{u} : I \to \mathbf{X}$ is (strongly) uniformly continuous on I if, for each $\epsilon > 0$, a $\delta = \delta(\epsilon) > 0$ can be found such that

$$\|\mathbf{u}(t) - \mathbf{u}(\tilde{t})\| < \epsilon \quad \text{whenever} \quad t, \tilde{t} \in I \quad \text{and} \quad |t - \tilde{t}| < \delta. \tag{2.2}$$

In contrast to (2.1), (2.2) provides a δ which does not involve the position of $t, \tilde{t} \in I$ as long as they fulfill $|t - \tilde{t}| < \delta$. That is, δ depends only on ϵ. Uniform continuity implies pointwise continuity as given by Definition 2.3.1 (ii), but the converse does not work in general.

Theorem 2.3.4. *Let $[a, b] \subset \mathbb{R}$ and let $\mathbf{X}$ be a (real) normed vector space. Any vector-valued function $\mathbf{u} \in \mathcal{C}([a, b], \mathbf{X})$ is uniformly continuous on $[a, b]$.*

Theorem 2.3.5. *Let $\mathbf{X}$ be a Banach space. Then $\mathcal{C}([a, b], \mathbf{X})$ is also a Banach space with respect to the norm*

$$\|\mathbf{u}\|_\infty = \sup_{t \in [a,b]} \|\mathbf{u}(t)\|.$$

Definition 2.3.6 (Lipschitz Continuity)**.** Let $\mathbf{X}$ be a Banach space and let $\mathfrak{F} : \mathcal{D}_0 \to \mathbf{X}$ with $\mathcal{D}_0 \subseteq \mathbf{X}$. Assume that $\mathcal{D}_0$ is an open subset.

(i) $\mathfrak{F}(\mathbf{f})$ fulfills a Lipschitz condition on $\mathcal{D}_0$ if there exists a positive constant L such that

$$\|\mathfrak{F}(\mathbf{f}) - \mathfrak{F}(\mathbf{g})\| \leq L\|\mathbf{f} - \mathbf{g}\|, \quad \mathbf{f}, \mathbf{g} \in \mathcal{D}_0.$$

(ii) $\mathfrak{F}(\mathbf{f})$ satisfies a local Lipschitz condition if, for any $\mathbf{u}_0 \in \mathcal{D}_0$, a closed ball $\overline{\mathbf{B}}(\mathbf{u}_0, \varepsilon) = \{\mathbf{f} \in \mathbf{X} : \|\mathbf{f} - \mathbf{u}_0\| \leq \varepsilon\}$ and a positive constant $l(\mathbf{u}_0, \varepsilon)$ exist such that

$$\|\mathfrak{F}(\mathbf{f}) - \mathfrak{F}(\mathbf{g})\| \leq l(\mathbf{u}_0, \varepsilon)\|\mathbf{f} - \mathbf{g}\| \quad \mathbf{f}, \mathbf{g} \in \overline{\mathbf{B}}(\mathbf{u}_0, \varepsilon). \tag{2.3}$$

2.3.2 Banach Space Valued Differentiable Functions

Since the derivative of a real-valued function f at a is defined as the limit of $\frac{1}{h}\{f(a+h) - f(a)\}$ as $h \to 0$, a similar expression can be investigated in the case of a vector-valued function $\mathbf{u} : I \to \mathbf{X}$, where $I \subset \mathbb{R}$. Since the quantity $\frac{1}{h}\{\mathbf{u}(a+h) - \mathbf{u}(a)\}$ belongs to $\mathbf{X}$, we may attempt to obtain the limit as $h \to 0$ via the norm on $\mathbf{X}$.

Definition 2.3.7. Let $I \subset \mathbb{R}$, let $\mathbf{X}$ be a (real) normed vector space and let $\mathbf{u} : I \to \mathbf{X}$.

(i) $\mathbf{u}$ is (strongly) differentiable at $a \in I$ if there exists an element $\mathbf{v} \in \mathbf{X}$ such that

$$\frac{1}{h}\{\mathbf{u}(a+h) - \mathbf{u}(a)\} \to \mathbf{v} \quad \text{as} \quad h \to 0,$$

that is, given $\epsilon > 0$, a positive $\delta = \delta(a, \epsilon)$ can be found such that

$$\left\|\frac{1}{h}\{\mathbf{u}(a+h) - \mathbf{u}(a)\} - \mathbf{v}\right\| < \epsilon \quad \text{whenever} \quad a + h \in I \quad \text{and} \quad 0 < |h| < \delta.$$

(ii) $\mathbf{u}$ is (strongly) differentiable on I if $\mathbf{u}$ is (strongly) differentiable at a for all $a \in I$.

(iii) The set of all functions $\mathbf{u} : I \to \mathbf{X}$ which are (strongly) differentiable on I will be denoted by $\mathcal{C}^1(I, \mathbf{X})$. Hence each $\mathbf{u} \in \mathcal{C}^1(I, \mathbf{X})$ has a strong derivative $\mathbf{u}'$, which is strongly continuous on I.

Remark 2.3.8. [12, p. 18]
R1: Throughout this thesis, differentiability will refer to the strong differentiability.
R2: $\mathcal{C}^1(I, \mathbf{X}) \subset \mathcal{C}(I, \mathbf{X})$, as in the case of $\mathbf{X} = \mathbb{R}$, and results such as

$$(\mathbf{u} + \mathbf{v})'(t) = \mathbf{u}'(t) + \mathbf{v}'(t), \qquad (\alpha\mathbf{u})'(t) = \alpha\mathbf{u}'(t) \quad (t \in I)$$

are valid for $\mathbf{u}, \mathbf{v} \in \mathcal{C}^1(I, \mathbf{X})$, $\alpha \in \mathbb{R}$. Hence $\mathcal{C}^1(I, \mathbf{X})$ is a vector subspace of $\mathcal{C}(I, \mathbf{X})$.
R3: In contrast to the above rules, the product rule for differentiation is meaningless except if a third algebraic operation, "multiplication" (this stands for multiplication of two vectors), is available in $\mathbf{X}$. This third operation is not available in general (e.g. when $\mathbf{X} = \mathrm{L}^1(\Omega)$), but is available in certain very important special cases (e.g. when $\mathbf{X} = \mathbb{R}$ and $\mathbf{X} = \mathcal{C}([a, b])$).

Definition 2.3.9 (Fréchet Differentiability)**.** Let $\mathbf{X}$ be a Banach space and let $\mathfrak{F} : \boldsymbol{\mathcal{D}}_0 \to \mathbf{X}$ with $\boldsymbol{\mathcal{D}}_0 \subseteq \mathbf{X}$. Assume that $\boldsymbol{\mathcal{D}}_0$ is an open subset and for any $\mathbf{f} \in \boldsymbol{\mathcal{D}}_0$ consider an open ball $\mathbf{B}(\mathbf{0}, \rho)$ such that $\mathbf{f} + \mathbf{B}(\mathbf{0}, \rho) \subset \boldsymbol{\mathcal{D}}_0$. If a bounded linear operator $\mathfrak{F}_{\mathbf{f}}$ exists on $\mathbf{X}$ such that

$$\mathfrak{F}(\mathbf{f} + \boldsymbol{\delta}) = \mathfrak{F}(\mathbf{f}) + \mathfrak{F}_{\mathbf{f}}\boldsymbol{\delta} + \mathbf{G}(\mathbf{f}, \boldsymbol{\delta}), \tag{2.4}$$

for $\boldsymbol{\delta} \in \mathbf{B}(\mathbf{0}, \rho)$, where the remainder $\mathbf{G}$ satisfies

$$\lim_{\|\boldsymbol{\delta}\| \to 0} \frac{\left\|\mathbf{G}(\mathbf{f}, \boldsymbol{\delta})\right\|}{\|\boldsymbol{\delta}\|} = 0,$$

then we say that $\mathfrak{F}$ is Fréchet differentiable at $\mathbf{f} \in \boldsymbol{\mathcal{D}}_0$ and $\mathfrak{F}_{\mathbf{f}}$ is the Fréchet derivative of $\mathfrak{F}$ at $\mathbf{f}$.

The operator $\mathfrak{F}$ is said to be Fréchet differentiable on $\boldsymbol{\mathcal{D}}_0$ if it is Fréchet differentiable at any $\mathbf{f} \in \boldsymbol{\mathcal{D}}_0$.

2.3.3 Banach Space Valued Integrable Functions

With regard to the integration of vector-valued functions, we consider $\mathbf{u} : [\alpha, \beta] \to \mathbf{X}$, where $\mathbf{X}$ is a normed space, and define the partition P_n,

$$P_n : \alpha = t_0 < t_1 < \cdots < t_{n-1} < t_n = \beta$$

of $[\alpha, \beta]$ into n sub-intervals $[t_{k-1}, t_k]$ $(i = 1, 2, \ldots, n)$. Then define

$$\mathcal{S}(\mathbf{u}, P_n) = \sum_{k=1}^{n} (t_k - t_{k-1})\mathbf{u}(\tilde{t}_k) \in \mathbf{X}, \tag{2.5}$$

called the Riemann sum corresponding to P_n. In (2.5), $\tilde{t}_k$ may be arbitrarily chosen in the subinterval $[t_{k-1}, t_k]$. Let

$$\|P_n\| = \max_{1 \le k \le n} (t_k - t_{k-1}).$$

We define

$$\int_{\alpha}^{\beta} \mathbf{u}(t)\, dt = \lim_{\|P_n\| \to 0} \mathcal{S}(\mathbf{u}, P_n) \text{ in } \mathbf{X} \tag{2.6}$$

whenever the limit exists (and is independent of the sequence $\{P_n\}$ of partitions and the choice of the intermediate $\tilde{t}_k$).

Theorem 2.3.10. *Let* $\mathbf{X}$ *be a* (*real*) *Banach space and let* $\mathbf{u} : [\alpha, \beta] \to \mathbf{X}$ *be strongly continuous on* $[\alpha, \beta]$. *Then the* (*strong*) *Riemann integral* $\int_\alpha^\beta \mathbf{u}(t)\, dt$ *exists as an element of* $\mathbf{X}$ *and is obtained via* (2.5) *and* (2.6).

Theorem 2.3.11. *Let* $\mathbf{X}$ *be a* (*real*) *Banach space and let* $\mathbf{u} : [\alpha, \beta] \to \mathbf{X}$ *be continuous. Then*

$$\left\| \int_\alpha^\beta \mathbf{u}(t)\, dt \right\| \leq \int_\alpha^\beta \|\mathbf{u}(t)\|\, dt$$

where on the right-hand side we have the usual Riemann integral of the continuous function $\|\mathbf{u}\| : [\alpha, \beta] \to [0, \infty)$.

Theorem 2.3.12. *Let* $\mathbf{X}$ *be a Banach space and let* $\mathbf{u} : [\alpha, \infty) \to \mathbf{X}$ *be continuous. If* $\int_a^\infty \|\mathbf{u}(t)\|\, dt$ *exists* (*as a real number*) *then* $\int_\alpha^\infty \mathbf{u}(t)\, dt$ *exists as an element of* $\mathbf{X}$ *and*

$$\left\| \int_\alpha^\infty \mathbf{u}(t)\, dt \right\| \leq \int_\alpha^\infty \|\mathbf{u}(t)\|\, dt.$$

Theorem 2.3.13. *Let* $\mathbf{X}$ *be a Banach space and let* $\mathbf{u} : [\alpha, \beta] \to \mathbf{X}$ *be strongly continuous. Then for each* $t \in [\alpha, \beta]$, *the strong Riemann integral* $\int_\alpha^t \mathbf{u}(s)\, ds$ *exists in* $\mathbf{X}$ *and*

$$\frac{d}{dt}\left[\int_\alpha^t \mathbf{u}(s)\, ds \right] = \mathbf{u}(t).$$

2.4 Linear Operators

Definition 2.4.1. Let $(\mathbf{X}, \|\cdot\|_\mathbf{X})$ and $(\mathbf{Y}, \|\cdot\|_\mathbf{Y})$ be normed spaces and $\mathfrak{T} : \mathbf{X} \to \mathbf{Y}$ be a linear operator. The operator $\mathfrak{T}$ is said to be bounded if there exists a constant M such that for all $\mathbf{u} \in \mathbf{X}$,

$$\|\mathfrak{T}(\mathbf{u})\|_\mathbf{Y} \leq M \|\mathbf{u}\|_\mathbf{X}. \tag{2.7}$$

Definition 2.4.2. Let $(\mathbf{X}, \|\cdot\|_\mathbf{X})$ and $(\mathbf{Y}, \|\cdot\|_\mathbf{Y})$ be normed spaces.

(i) $\mathbb{B}(\mathbf{X}, \mathbf{Y})$ is the set of bounded linear operators $\mathfrak{T} : \mathbf{X} \to \mathbf{Y}$.

(ii) When $\mathbf{X} = \mathbf{Y}$, the space $\mathbb{B}(\mathbf{X}, \mathbf{Y})$ is denoted by $\mathbb{B}(\mathbf{X})$.

Definition 2.4.3. The norm of an operator $\mathfrak{T} : \mathbf{X} \to \mathbf{Y}$ is given by

$$\|\mathfrak{T}\|_{\mathbb{B}(\mathbf{X},\mathbf{Y})} = \sup_{\mathbf{u} \neq 0} \frac{\|\mathfrak{T}\mathbf{u}\|_\mathbf{Y}}{\|\mathbf{u}\|_\mathbf{X}} = \sup_{\|\mathbf{u}\|_\mathbf{X} = 1} \|\mathfrak{T}\mathbf{u}\|_\mathbf{Y}. \tag{2.8}$$

Proposition 2.4.4. *Let* $(\mathbf{X}, \|\cdot\|_\mathbf{X})$ *be a normed space and* $(\mathbf{Y}, \|\cdot\|_\mathbf{Y})$ *be a Banach space. Then* $\mathbb{B}(\mathbf{X}, \mathbf{Y})$ *is a Banach space.*

Theorem 2.4.5. *Let* $(\mathbf{X}, \|\cdot\|_{\mathbf{X}})$ *and* $(\mathbf{Y}, \|\cdot\|_{\mathbf{Y}})$ *be normed spaces and* $\mathfrak{T} : \mathbf{X} \to \mathbf{Y}$ *be a linear operator. Then:*

(i) $\mathfrak{T}$ *is continuous if and only if* $\mathfrak{T}$ *is bounded.*

(ii) *If* $\mathfrak{T}$ *is continuous at a single point, it is continuous.*

Definition 2.4.6. A linear operator is called unbounded if it is not bounded.

Remark 2.4.7. Since differential operators in L^p spaces are unbounded, thus the larger part of the linear operators that arise in the study of PDEs are unbounded operators.

Notation 2.4.8. *Though it is artificial to reduce the domain of a bounded operator to some subspace of* $\mathbf{X}$*, this is most certainly not the case for unbounded operators, for which the domain is an integral part of the definition. Strictly speaking, an unbounded operator is not specified until we have given its domain. Thus an unbounded operator* $\mathbf{A}$ *is really a pair* $(\mathbf{A}, D(\mathbf{A}))$*, where* $D(\mathbf{A})$ *is the domain of definition for* $\mathbf{A}$*. However, for convenience, we simply use the notation* $\mathbf{A}$*.*

Note that the presentation above is based on [45, p. 79]. Moreover, following [38], we suppose that $\mathbf{X}$ is a normed space and we denote by $\mathcal{O}(\mathbf{X})$ the class of operators $(\mathbf{A}, D(\mathbf{A}))$. A member $\mathbf{A}$ of $\mathcal{O}(\mathbf{X})$ is said to be closed if the graph $\{(\mathbf{x}, \mathbf{A}\mathbf{x}) : \mathbf{x} \in D(\mathbf{A})\}$ is closed in the space $\mathbf{X} \times \mathbf{X}$. Clearly $\mathbf{A} \in \mathcal{O}(\mathbf{X})$ is closed only if for each sequence $\{\mathbf{x}_n\}_{n=1}^{\infty}$ in $D(\mathbf{A})$ such that $\lim_{n\to\infty} \mathbf{x}_n = \mathbf{x}$ and $\lim_{n\to\infty} \mathbf{A}\mathbf{x}_n = \mathbf{y}$, it follows that $\mathbf{x} \in D(\mathbf{A})$ and $\mathbf{A}\mathbf{x} = \mathbf{y}$. Some simple results concerning closed operators are given by the following lemma

Lemma 2.4.9. *Suppose that* $\mathbf{A} \in \mathcal{O}(\mathbf{X})$*.*

(i) *If* $\mathbf{A}$ *is continuous and* $D(\mathbf{A})$ *is closed, then* $\mathbf{A}$ *is closed.*

(ii) *If* $\mathbf{A}$ *is invertible in* $\mathcal{O}(\mathbf{X})$ *and* $\mathbf{A}^{-1}$ *is closed, then* $\mathbf{A}$ *is closed.*

(iii) *If* $\mathbf{A}$ *is closed and* $\mathbf{B} \in \mathbb{B}(\mathbf{X})$*, then* $\mathbf{A} + \mathbf{B}$ *is closed.*

2.5 Banach Lattices and Positive Operators

The following presentation is based on [9]

Definition 2.5.1. Let $\mathbf{X}$ be an arbitrary set. A partial order (or simply, an order) on $\mathbf{X}$ is a binary relation, denoted here by '$\geq$', which is reflexive, transitive, and antisymetric, that is,

(1) $\mathbf{x} \geq \mathbf{x}$ for each $\mathbf{x} \in \mathbf{X}$;

(2) $\mathbf{x} \geq \mathbf{y}$ and $\mathbf{y} \geq \mathbf{x}$ imply $\mathbf{x} = \mathbf{y}$ for any $\mathbf{x}, \mathbf{y} \in \mathbf{X}$;

(3) $\mathbf{x} \geq \mathbf{y}$ and $\mathbf{y} \geq \mathbf{z}$ imply $\mathbf{x} \geq \mathbf{z}$ for any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{X}$.

Definition 2.5.2. An ordered vector space is a vector space $\mathbf{X}$ equipped with a partial order which is compatible with its vector structure in the sense that

(4) $\mathbf{x} \geq \mathbf{y}$ implies $\mathbf{x} + \mathbf{z} \geq \mathbf{y} + \mathbf{z}$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{X}$;

(5) $\mathbf{x} \geq \mathbf{y}$ implies $\alpha\mathbf{x} \geq \alpha\mathbf{y}$ for any $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and $\alpha \geq 0$.

Definition 2.5.3. A vector space $\mathbf{X}$ is a lattice if every pair of elements (and so every finite collection of them) has both supremum and infimum.

If the ordered vector space $\mathbf{X}$ is also a lattice, then it is called a vector lattice or a Riesz space.

Example 2.5.4. Typical examples of Riesz spaces are provided by function spaces. If $\mathbf{X}$ is a vector space of real-valued functions on a set Ω, then we can introduce a pointwise order in $\mathbf{X}$ by saying that $\mathbf{f} \leq \mathbf{g}$ in $\mathbf{X}$ if $\mathbf{f}(\mathbf{x}) \leq \mathbf{g}(\mathbf{x})$ for any $\mathbf{x} \in \mathbf{S} \subset \Omega$. Equipped with such an order, $\mathbf{X}$ becomes an ordered vector space. Let us define on $\mathbf{X} \times \mathbf{X}$ the operations $\mathbf{f} \vee \mathbf{g}$ and $\mathbf{f} \wedge \mathbf{g}$ by taking pointwise maxima and minima; that is, for any $\mathbf{f}, \mathbf{g} \in \mathbf{X}$,

$$\big(\mathbf{f} \vee \mathbf{g}\big)(\mathbf{x}) := \max\{\mathbf{f}(\mathbf{x}), \mathbf{g}(\mathbf{x})\},$$
$$\big(\mathbf{f} \wedge \mathbf{g}\big)(\mathbf{x}) := \min\{\mathbf{f}(\mathbf{x}), \mathbf{g}(\mathbf{x})\}.$$

We say that $\mathbf{X}$ is a function space if $\mathbf{f} \vee \mathbf{g}, \mathbf{f} \wedge \mathbf{g} \in \mathbf{X}$, whenever $\mathbf{f}, \mathbf{g} \in \mathbf{X}$. A function space with pointwise ordering is a Riesz space. Examples of function spaces are offered by the spaces of all real functions $\mathbb{R}^{\Omega}$ or all real bounded functions $M(\Omega)$ on a set Ω, and by spaces $\mathcal{C}(\Omega)$, $\mathcal{C}(\overline{\Omega})$, or l^p , $1 \leq p \leq \infty$.

If Ω is a measure space, then all the above considerations are valid when the pointwise order is replaced by $\mathbf{f} \leq \mathbf{g}$ if $\mathbf{f}(\mathbf{x}) \leq \mathbf{g}(\mathbf{x})$ almost everywhere. With this understanding, $\mathrm{L}^0(\Omega)$ (defined as the space of equivalence classes of all measurable real functions on Ω, see [9, p. 11]) and $\mathrm{L}^p(\Omega)$ spaces with $1 \leq p \leq \infty$ become function spaces and are thus Riesz spaces.

For an element $\mathbf{x}$ in a Riesz space $\mathbf{X}$ we can define its positive and negative part, and its absolute value, respectively, by

$$\mathbf{x}_+ = \sup\{\mathbf{x}, \mathbf{0}\}, \quad \mathbf{x}_- = \sup\{-\mathbf{x}, \mathbf{0}\}, \quad |\mathbf{x}| = \sup\{\mathbf{x}, -\mathbf{x}\}.$$

The functions $(\mathbf{x}, \mathbf{y}) \to \sup\{\mathbf{x}, \mathbf{y}\}$, $(\mathbf{x}, \mathbf{y}) \to \inf\{\mathbf{x}, \mathbf{y}\}$, $\mathbf{x} \to \mathbf{x}_\pm$ and $\mathbf{x} \to |\mathbf{x}|$ are collectively referred to as the lattice operations of a Riesz space. The relation between them is given in the next proposition.

Proposition 2.5.5. *If* $\mathbf{x}$ *is an element of a Riesz space, then*

$$\mathbf{x} = \mathbf{x}_+ - \mathbf{x}_-, \quad |\mathbf{x}| = \mathbf{x}_+ + \mathbf{x}_-.$$

Thus, in particular, the positive cone in a Riesz space is generating.

Now, we investigate the relation between the lattice structure and the norm, when $\mathbf{X}$ is both a normed and an ordered vector space.

Definition 2.5.6. A norm on a vector lattice $\mathbf{X}$ is called a lattice norm if

$$|\mathbf{x}| \leq |\mathbf{y}| \text{ implies } \|\mathbf{x}\| \leq \|\mathbf{y}\|. \tag{2.9}$$

A Riesz space $\mathbf{X}$ complete under the lattice norm is called a Banach lattice.

Property (2.9) gives the important identity:

$$\|\mathbf{x}\| = \||\mathbf{x}|\|, \quad \mathbf{x} \in \mathbf{X}.$$

Proposition 2.5.7. *If* $\mathbf{X}$ *is a normed lattice, then all lattice operations are uniformly continuous in the norm of* $\mathbf{X}$ *with respect to all variables involved.*

In the next step, we introduce positive operators.

Definition 2.5.8. A linear operator $\mathbb{A}$ from a Banach lattice $\mathbf{X}$ into a Banach lattice $\mathbf{Y}$ is called positive, denoted by $\mathbf{A} \geq 0$, if $\mathbf{A}\mathbf{x} \geq 0$ for any $\mathbf{x} \geq 0$.

Positive operators are fully determined by their behaviour on the positive cone. Precisely speaking, we have the following theorem.

Theorem 2.5.9. *If* $\mathbf{A} : \mathbf{X}_+ \to \mathbf{Y}_+$ *is additive, then* $\mathbf{A}$ *extends uniquely to a positive linear operator from* $\mathbf{X}$ *to* $\mathbf{Y}$ *. Keeping the notation* $\mathbf{A}$ *for the extension, we have, for each* $\mathbf{x} \in \mathbf{X}$*,*

$$\mathbf{A}\mathbf{x} = \mathbf{A}\mathbf{x}_+ - \mathbf{A}\mathbf{x}_-.$$

Theorem 2.5.10. *If* $\mathbf{A}$ *is an everywhere defined positive operator from a Banach lattice to a normed Riesz space, then* $\mathbf{A}$ *is bounded.*

Theorem 2.5.11. *If* $\mathbf{A}$ *is positive, then*

$$\|\mathbf{A}\| = \sup_{\mathbf{x} \geq 0, \|\mathbf{x}\| \leq 1} \|\mathbf{A}\mathbf{x}\|. \tag{2.10}$$

2.6 Elementary Spectral Theory

The following presentation is based on [12].

Let $\mathbf{X}$ be a Banach space and let $\mathbf{A} : \mathbf{X} \supseteq D(\mathbf{A}) \to R(\mathbf{A}) \subseteq \mathbf{X}$ be a linear operator. A crucial role in spectral theory is played by the resolvent operator

$$\mathcal{R}(\lambda, \mathbf{A}) \equiv \left(\lambda\mathbf{I} - \mathbf{A}\right)^{-1}, \tag{2.11}$$

whenever it exists.

In finite dimensions, where $\mathbf{A}$ is a matrix, $\mathcal{R}(\lambda, \mathbf{A})$ exists if and only if the complex number λ is not an eigenvalue of $\mathbf{A}$ (i.e. if and only if $\det\left(\lambda\mathbf{I} - \mathbf{A}\right)^{-1} \neq 0$). Also, the properties of "one-to-one" and "onto" go together.

In infinite dimensions, life gets much more complicated. The operator $\lambda\mathbf{I} - \mathbf{A}$ may be one-to-one on its domain $D(\lambda\mathbf{I} - \mathbf{A}) = D(\lambda\mathbf{I}) \cap D(\mathbf{A}) = D(\mathbf{A})$ but its range may well not be all of $\mathbf{X}$. Even when the range equals $\mathbf{X}$, the inverse operator may be "badly behaved".

It is traditional to split the complex plane into two disjoint subsets.

Definition 2.6.1. Let $\mathbf{X}$ be a Banach space and let $\mathbf{A} : \mathbf{X} \supseteq D(\mathbf{A}) \to R(\mathbf{A}) \subseteq \mathbf{X}$ be a linear operator.

(i) The resolvent set, $\rho(\mathbf{A})$, of $\mathbf{A}$ is given by

$$\rho(\mathbf{A}) := \left\{\lambda \in \mathbb{C} : \left(\lambda\mathbf{I} - \mathbf{A}\right)^{-1} \in \mathbb{B}(\mathbf{X})\right\}. \tag{2.12}$$

(ii) The spectrum, $\sigma(\mathbf{A})$, of $\mathbf{A}$ is the complement of $\rho(\mathbf{A})$, i.e.

$$\sigma(\mathbf{A}) := \left\{\lambda \in \mathbb{C} : \lambda \notin \rho(\mathbf{A})\right\}.$$

Note that, given $\lambda \in \rho(\mathbf{A})$ and $\mathbf{g} \in \mathbf{X}$, the equation

$$(\lambda\mathbf{I} - \mathbf{A})\mathbf{f} = \mathbf{g} \quad (\mathbf{f} \in D(\mathbf{A})) \tag{2.13}$$

has a unique solution $\mathbf{f} = \left(\lambda\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{g}$. Further, if $\mathbf{g}_1$ is close to $\mathbf{g}$ in the sense that $\|\mathbf{g} - \mathbf{g}_1\|_{\mathbf{X}}$ is small, then the solution $\mathbf{f}_1 \in D(\mathbf{A})$ of $(\lambda\mathbf{I} - \mathbf{A})\mathbf{f}_1 = \mathbf{g}_1$ will be "close" to $\mathbf{f}$ since $\left(\lambda\mathbf{I} - \mathbf{A}\right)^{-1}$ is bounded. Thus, when $\lambda \in \rho(\mathbf{A})$, equation (2.13) is as well behaved as possible since we get existence, uniqueness and stability of our solution. The problem can be resolved. Hence, the operator (2.11) is called the resolvent operator (with parameter λ) and the set (2.12) is called the resolvent set.

2.7 Laplace Transform and Convolution

Let $\mathbf{X}$ be a Banach space and $\mathbf{f} \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}_+, \mathbf{X})$. For $\lambda \in \mathbb{C}$ we consider the Laplace integral

$$\widehat{\mathbf{f}}(\lambda) = \lim_{a\to\infty} \int_0^a e^{-\lambda t}\mathbf{f}(t)\,dt.$$

We define the abscissa of convergence of $\mathbf{f}$ by

$$\sigma = \inf\left\{\Re\lambda \,|\, \widehat{\mathbf{f}}(\lambda) \text{ exists}\right\}.$$

It follows, [5, Proposition 1.4.1], that the Laplace integral $\widehat{\mathbf{f}}(\lambda)$ converges if $\lambda > \sigma$ and diverges if $\lambda < \sigma$. We say that $\sigma = -\infty$ if the Laplace integral exists for any $\lambda \in \mathbb{C}$, and $\sigma = +\infty$ if the domain of convergence is empty.

Let $\sigma < +\infty$. The function

$$\lambda \to \widehat{\mathbf{f}}(\lambda), \quad \Re\lambda > \sigma$$

is called the Laplace transform of $\mathbf{f}$. It is an analytic function of λ, [5, Theorem 1.5.1].

We define the exponential growth bound of $\mathbf{f}$ by

$$\xi(\mathbf{f}) = \inf\left\{\xi \in \mathbb{R} \,|\, \sup_{t\geq 0}\left\|e^{-\xi t}\mathbf{f}(t)\right\| < \infty\right\}.$$

For the inversion of the Laplace transform $\widehat{\mathbf{f}}$, we shall use the following result which is a simplified version of [5, Theorem 1.5.1].

Theorem 2.7.1. *If $\mathbf{f} \in \mathrm{L}^\infty(\mathbb{R}_+)$ and $\mathbf{F}(t) = \int_0^t \mathbf{f}(s)\,ds$, then*

$$\mathbf{F}(t) = \lim_{k\to\infty} \frac{1}{2\pi i}\int_{c-ik}^{c+ik} e^{\lambda t}\widehat{\mathbf{f}}(\lambda)\frac{d\lambda}{\lambda}, \tag{2.14}$$

where the limit is uniform for $t \in [0, a]$ for any $a > 0$ and $c > 0$ is arbitrary.

We note that if $\widehat{\mathbf{f}}$ is absolutely integrable along an imaginary line $\xi \pm i\infty$ with $\xi > \xi(\mathbf{f})$, then the derivative of the integrand with respect to $t > 0$ is Bochner integrable. Thus, by differentiating (2.14) with respect to t and shifting the result by ξ, we obtain

$$\mathbf{f}(t) = \lim_{k\to\infty} \frac{1}{2\pi i}\int_{\xi-ik}^{\xi+ik} e^{\lambda t}\widehat{\mathbf{f}}(\lambda)d\lambda.$$

The notion of convolution and its Laplace transform are highly important in applications. We have the following result,

Proposition 2.7.2. *Let* $\mathbf{f} \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}_+, \mathbf{X})$ *and let* $\mathbf{F} : \mathbb{R}_+ \to \mathbb{B}(\mathbf{X}, \mathbf{Y})$ *be a strongly continuous function. Then the convolution*

$$\big(\mathbf{F} * \mathbf{f}\big)(t) := \int_0^t \mathbf{F}(t-s)\mathbf{f}(s)\,ds \tag{2.15}$$

exists and is a continuous function from $\mathbb{R}_+$ *to* $\mathbf{Y}$*. Moreover, if* $1 \leq p, q, r \leq \infty$ *satisfy* $1/p + 1/q = 1 + 1/r$, $\int_0^\infty \|\mathbf{F}(t)\|^p\,dt < \infty$, *and* $\mathbf{f} \in \mathrm{L}^q(\mathbb{R}_+, \mathbf{X})$*, then* $\mathbf{F} * \mathbf{f} \in \mathrm{L}^r(\mathbb{R}_+, \mathbf{Y})$ *and the Young inequality is valid:*

$$\|\mathbf{F} * \mathbf{f}\|_r \leq \|\mathbf{f}\|_q \|\mathbf{F}\|_p$$

Remark 2.7.3. The convolution defined by (2.15) is a particular case, for functions with support in $\mathbb{R}_+$, of the convolution over the whole real line. The Young inequality also holds in the latter, more general, case.

We state the following result on the Laplace transform of the convolution.

Proposition 2.7.4. *Assume that* $\mathbf{F} : \mathbb{R}_+ \to \mathbb{B}(\mathbf{X})$ *is a strongly continuous function with the exponential bound* $\xi(\mathbf{F}) < \infty$ *and* $\mathbf{f}$ *satisfies* $e^{-\xi t}\mathbf{f}(t) \in \mathrm{L}^1(\mathbb{R}_+, \mathbf{X})$ *for any* $\xi > \xi(\mathbf{F})$*. Then* $\xi(\mathbf{F} * \mathbf{f}) \leq \xi(\mathbf{F})$ *and*

$$\widehat{(\mathbf{F} * \mathbf{f})}(\lambda) = \widehat{\mathbf{F}}(\lambda)\widehat{\mathbf{f}}(\lambda), \quad \lambda > \xi(\mathbf{F}).$$

Note that the propositions presented in this section are based on [9].

2.8 Projections

Let $\mathbf{X}$ be a normed space. A linear mapping $\boldsymbol{\mathcal{P}} : \mathbf{X} \to \mathbf{X}$ is called a projection in $\mathbf{X}$ if

$$\boldsymbol{\mathcal{P}}^2 = \boldsymbol{\mathcal{P}},$$

i.e., $\boldsymbol{\mathcal{P}}\big(\boldsymbol{\mathcal{P}}\mathbf{x}\big) = \boldsymbol{\mathcal{P}}\mathbf{x}$ for every $\mathbf{x} \in \mathbf{X}$.

Suppose that $\boldsymbol{\mathcal{P}}$ is a projection in $\mathbf{X}$, with null space $\mathscr{N}\big(\boldsymbol{\mathcal{P}}\big)$ and range $\mathscr{R}\big(\boldsymbol{\mathcal{P}}\big)$. The following facts are obvious:

(a) $\mathscr{R}\big(\boldsymbol{\mathcal{P}}\big) = \mathscr{N}\big(\boldsymbol{\mathcal{I}} - \boldsymbol{\mathcal{P}}\big) = \{\mathbf{x} \in \mathbf{X} : \boldsymbol{\mathcal{P}}\mathbf{x} = \mathbf{x}\}$;

(b) $\mathscr{N}\big(\boldsymbol{\mathcal{P}}\big) = \mathscr{R}\big(\boldsymbol{\mathcal{I}} - \boldsymbol{\mathcal{P}}\big)$;

(c) $\mathscr{R}\big(\boldsymbol{\mathcal{P}}\big) \cap \mathscr{N}\big(\boldsymbol{\mathcal{P}}\big) = \{\mathbf{0}\}$ and $\mathbf{X} = \mathscr{R}\big(\boldsymbol{\mathcal{P}}\big) \oplus \mathscr{N}\big(\boldsymbol{\mathcal{P}}\big)$;

(d) If $\mathbf{M}$ and $\mathbf{N}$ are subspaces of $\mathbf{X}$ such that $\mathbf{M} \cap \mathbf{N} = \{\mathbf{0}\}$ and $\mathbf{X} = \mathbf{M} \oplus \mathbf{N}$, then there is a unique projection $\boldsymbol{\mathcal{P}}$ in $\mathbf{X}$ with $\mathbf{M} = \mathscr{R}\big(\boldsymbol{\mathcal{P}}\big)$ and $\mathbf{N} = \mathscr{N}\big(\boldsymbol{\mathcal{P}}\big)$.

The proofs of these assertions are based on [47, p. 133].

The following result holds.

Theorem 2.8.1. [22, p. 10] *Let* $\mathbf{X}$ *be a normed space,* $\mathbf{v} \in \mathbf{X}$ *and* $\mathbf{f} \in \mathbf{X}^*$*, (where* $\mathbf{X}^*$ *denotes the dual space of* $\mathbf{X}$ *), be given. Then the operator* $\boldsymbol{\mathcal{P}}$ *defined by* $\boldsymbol{\mathcal{P}}\mathbf{u} := (\mathbf{u}, \mathbf{f})\mathbf{v}$ *for all* $\mathbf{u} \in \mathbf{X}$ *is a projection if and only if* $(\mathbf{v}, \mathbf{f}) = 1$*. In this case* $\boldsymbol{\mathcal{P}}\mathbf{X}$ *is the one-dimensional subspace* $[\mathbf{v}]$ *spanned by* $\mathbf{v}$*, and* $(\boldsymbol{\mathcal{I}} - \boldsymbol{\mathcal{P}})$ *is the linear subspace of* $\mathbf{X}$ *consisting of all* $\mathbf{u} \in \mathbf{X}$ *with* $(\mathbf{u}, \mathbf{f}) = 0$ *(where* $(\mathbf{u}, \mathbf{f})$ *denotes the image of* $\mathbf{u}$ *under* $\mathbf{f}$ *).*

2.9 Linear Semigroups

Definition 2.9.1. [42] Let $\mathbf{X}$ be a Banach space. A one parameter family $(\boldsymbol{\mathcal{T}}(t))_{t\geq 0}$ of bounded linear operators from $\mathbf{X}$ into $\mathbf{X}$ is called a semigroup of bounded linear operators on $\mathbf{X}$ if the conditions:

(i) $\boldsymbol{\mathcal{T}}(0) = \mathbf{I}$ ($\mathbf{I}$ is the identity operator on $\mathbf{X}$),

(ii) $\boldsymbol{\mathcal{T}}(t+s) = \boldsymbol{\mathcal{T}}(t)\boldsymbol{\mathcal{T}}(s)$, for all $t, s \geq 0$ (the semigroup property),

are satisfied. Moreover, a semigroup $(\boldsymbol{\mathcal{T}}(t))_{t\geq 0}$ of bounded linear operators on $\mathbf{X}$ is called a strongly continuous semigroup of bounded linear operators if

$$\lim_{t\to 0} \boldsymbol{\mathcal{T}}(t)\mathbf{x} = \mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbf{X}$$

and the linear operator $\mathbf{A}$ defined on the domain

$$D(\mathbf{A}) = \left\{ \mathbf{x} \in \mathbf{X} : \lim_{t\to 0} \frac{\boldsymbol{\mathcal{T}}(t)\mathbf{x} - \mathbf{x}}{t} \text{ exists} \right\}$$

by

$$\mathbf{A}\mathbf{x} = \lim_{t\to 0} \frac{\boldsymbol{\mathcal{T}}(t)\mathbf{x} - \mathbf{x}}{t}$$

is called the generator of the semigroup $\boldsymbol{\mathcal{T}}(t)$.

Proposition 2.9.2. [42, p. 5] *If* $\mathbf{A}$ *is the generator of a strongly continuous semigroup* $(\boldsymbol{\mathcal{T}}(t))_{t\geq 0}$*, then* $D(\mathbf{A})$ *is dense in* $\mathbf{X}$ *and* $\mathbf{A}$ *is a closed linear operator.*

The rest of the presentation is based on [12].

Theorem 2.9.3. *Let* $(\boldsymbol{\mathcal{T}}(t))_{t\geq 0}$ *be a strongly continuous semigroup on the Banach space* $\mathbf{X}$*. Then there exist constants* $M > 0$ *and* $\omega \geq 0$ *such that*

$$\|\boldsymbol{\mathcal{T}}(t)\| \leq M e^{\omega t} \quad \text{for all } t \geq 0. \tag{2.16}$$

Definition 2.9.4. For real numbers $M > 0$ and $\omega \geq 0$, let $\mathcal{G}(M, \omega; \mathbf{X})$ denote the set of generators of strongly continuous semigroups $\{\boldsymbol{\mathcal{T}}(t)\}_{t\geq 0}$ on a Banach space $\mathbf{X}$ which satisfy (2.16).

Theorem 2.9.5 (The Hille − Yosida Theorem)**.** $\mathbf{A} \in \mathcal{G}(M, \omega, \mathbf{X})$ *if and only if*

(i) $\mathbf{A}$ *is a closed linear operator whose domain* $D(\mathbf{A})$ *is dense in* $\mathbf{X}$*, and*

(ii) *for all complex numbers* λ *with* $\Re\lambda > \omega$, $\lambda \in \rho(\mathbf{A})$*, and*

$$\left\| \left[\mathcal{R}(\lambda, \mathbf{A})\right]^n \right\| \leq \frac{M}{(\Re\lambda - \omega)^n}$$

for $n = 1, 2, \ldots$.

2.10 A Classical Perturbation Result

Let $\mathbf{A}$ be the generator of a strongly continuous semigroup on a Banach space $\mathbf{X}$ and $\mathbf{B}$ be another operator in $\mathbf{X}$. The aim of the perturbation theory is to come up with conditions that guarantee that there is an extension $\mathbf{G}$ of $\mathbf{A}+\mathbf{B}$ that generates a strongly continuous semigroup on $\mathbf{X}$ and characterize this extension. For the linear part of the problem we are working with, we are only interested in the simplest result on perturbation by a bounded linear operator, namely the bounded perturbation theorem:

Theorem 2.10.1. *Let* $\mathbf{A} \in \mathcal{G}(M, \eta; \mathbf{X})$ *and* $\big(\boldsymbol{\mathcal{S}}_A(t)\big)_{t\geq 0}$ *be the semigroup generated by* $\mathbf{A}$*. If* $\mathbf{B} : \mathbf{X} \to \mathbf{X}$ *is linear and bounded, then* $\big(\mathbf{K}, D(\mathbf{K})\big) = \big(\mathbf{A}+\mathbf{B}, D(\mathbf{A})\big) \in \mathcal{G}\big(M, \eta + M\|\mathbf{B}\|_{\mathbb{B}(\mathbf{X})}; \mathbf{X}\big)$.

Proof. Cf. [9, Theorem 4.9]. □

In many cases the Bounded Perturbation Theorem does not provide enough information. So, we can combine this with the Trotter product formula, [20, 42]. We make assumption that $\mathbf{K}_0$ is of type $(1, \eta_0)$, and $\mathbf{K}_1$ is of type $(1, \eta_1)$, $\eta_0, \eta_1 \in \mathbb{R}$. Further, we assume that $(\mathbf{K}_1 + \mathbf{K}_2)$ is closed and the range, $R(\lambda\mathbf{I} - \mathbf{K}_1 - \mathbf{K}_2)$, of $\lambda\mathbf{I} - \mathbf{K}_1 - \mathbf{K}_2$ satisfies $R(\lambda\mathbf{I} - \mathbf{K}_1 - \mathbf{K}_2) = \mathbf{X}$. If we know that $\big(\mathbf{K}, D(\mathbf{K}_0) \cap D(\mathbf{K}_1)\big) = \big(\mathbf{K}_0 + \mathbf{K}_1, D(\mathbf{K}_0) \cap D(\mathbf{K}_1)\big)$ is the generator of a semigroup, then

$$\mathbf{G}_{\mathbf{K}}(t)\mathbf{x} = \lim_{n\to\infty} \big(\mathbf{G}_{\mathbf{K}_0}(t/n)\mathbf{G}_{\mathbf{K}_1}(t/n)\big)^n \mathbf{x}, \quad \mathbf{x} \in \mathbf{X}, \tag{2.17}$$

uniformly in t on compact intervals and $\mathbf{K}$ is of type $(1, \eta)$ with $\eta = \eta_0 + \eta_1$. Moreover, if both semigroups $\big(\mathbf{G}_{\mathbf{K}_0}(t)\big)_{t\geq 0}$ and $\big(\mathbf{G}_{\mathbf{K}_1}(t)\big)_{t\geq 0}$ are positive, then $\big(\mathbf{G}_{\mathbf{K}}(t)\big)_{t\geq 0}$ is positive.

2.11 Semilinear Evolution Equations

In this section, the definitions and theorems are based on [42]. $\mathfrak{F}(\mathbf{u}(t))$ and $\mathcal{Q}$ correspond, respectively, to $f(t, u(t))$ and A in [42]. In addition to this, we take $t_0 = 0$.

The aim is to study the following semilinear initial value problem

$$\begin{cases} \dfrac{d\mathbf{u}}{dt}(t) &= \mathcal{Q}\mathbf{u}(t) + \mathfrak{F}(\mathbf{u}(t)), \quad (t > 0), \\ \mathbf{u}(0) &= \mathbf{u}_0, \end{cases} \tag{2.18}$$

where $\mathcal{Q}$ is the generator of a strongly continuous semigroup $(\mathcal{T}(t))_{t\geq 0}$ on a Banach space $\mathbf{X}$ and $\mathfrak{F} : \mathbf{X} \to \mathbf{X}$ satisfies a Lipschitz condition in $\mathbf{u}$.

We begin by precisely defining the notions of mild and classical solution of (2.18).

Definition 2.11.1 (Classical Solution)**.** A function $\mathbf{u} : [0, \infty) \to \mathbf{X}$ is a classical solution of (2.18) on $[0, \infty)$ if $\mathbf{u}$ is continuous on $[0, \infty)$, continuously differentiable on $(0, \infty)$, $\mathbf{u}(t) \in D(\mathcal{Q})$ for $t > 0$ and (2.18) is satisfied on $[0, \infty)$.

Definition 2.11.2 (Mild Solution)**.** Let $\mathcal{Q}$ be the generator of a strongly continuous semigroup $(\mathcal{T}(t))_{t\geq 0}$. Let $\mathbf{u}_0 \in \mathbf{X}$ and $\mathfrak{F} \in \mathrm{L}^1(\mathbf{X})$. The function $\mathbf{u} \in \mathcal{C}([0, \infty), \mathbf{X})$, satisfying

$$\mathbf{u}(t) = \mathcal{T}(t)\mathbf{u}_0 + \int_0^t \mathcal{T}(t - s)\mathfrak{F}(\mathbf{u}(s))\, ds, \quad t \geq 0, \tag{2.19}$$

is called the mild solution of the initial value problem (2.18) on $[0, \infty)$.

Next, we present the existence and uniqueness results of solution to (2.18) and hence also to (2.19). Note that the properties of the nonlinear function $\mathfrak{F}$ are essential for solvability of semilinear evolution equations. If we assume that $\mathbf{u}_0 \in \mathbf{X}$ and $\mathfrak{F}$ is globally Lipschitz in $\mathbf{u}$, a proof of existence and uniqueness of the global mild solution of (2.18) can be done via the contraction argument (Contraction Mapping Theorem). Namely, from (2.19), we define the map $\mathcal{J} : \mathcal{C}\big([0, \infty); \mathbf{X}\big) \to \mathcal{C}\big([0, \infty); \mathbf{X}\big)$ by

$$\left(\mathcal{J}\mathbf{u}\right)(t) = \mathcal{T}(t)\mathbf{u}_0 + \int_0^t \mathcal{T}(t - s)\mathfrak{F}(\mathbf{u}(s))\, ds \tag{2.20}$$

and we prove that it has a unique fixed point $\mathbf{u} \in \mathcal{C}\big([0, \infty); \mathbf{X}\big)$. Note that a treatment of a semilinear initial value problem, with an autonomous perturbation function $f(t, u(t))$ defined on $[0, T] \times \mathbf{X}$, was developed in [42]. Due to the non-autonomous perturbation function $\mathfrak{F}(\mathbf{u}(t))$, defined on $\mathbf{X}$, in the semilinear initial value problem (2.18), here we can consider the time interval $[0, \infty)$. Hence, we state the following result:

Theorem 2.11.3 (Existence and Uniqueness of Mild Solution)**.** *Let $\mathfrak{F}: \mathbf{X} \to \mathbf{X}$ be uniformly Lipschitz continuous (with constant L) on $\mathbf{X}$. If $\mathcal{Q}$ is the generator of a strongly continuous semigroup $(\mathcal{T}(t))_{t\geq 0}$ on $\mathbf{X}$, then for every $\mathbf{u}_0 \in \mathbf{X}$ the semilinear ACP (2.18) has a unique mild solution $\mathbf{u} \in \mathcal{C}\big([0,\infty),\mathbf{X}\big)$. Moreover, the mapping $\mathbf{u}_0 \to \mathbf{u}$ is Lipschitz continuous from $\mathbf{X}$ into $\mathcal{C}\big([0,\infty),\mathbf{X}\big)$.*

We state the following result:

Theorem 2.11.4 (Existence and Uniqueness of Classical Solution)**.** *If the assumptions of Theorem 2.11.3 are satisfied and, in addition, $\mathfrak{F}: \mathbf{X} \to \mathbf{X}$ is Fréchet continuously differentiable, then the mild solution of (2.18) with $\mathbf{u}_0 \in D(\mathcal{Q})$ is a classical solution of the initial value problem (2.18).*

If we assume that $\mathbf{u}_0 \in \mathbf{X}$ and $\mathfrak{F}$ is only locally Lipschitz continuous in $\mathbf{u}$, the previous approach only yields the existence and the uniqueness of the local mild solution of (2.18). Hence we shall consider the space $\mathbf{Y} = \mathcal{C}\big([0,\tau]; \overline{\mathbf{B}}(\mathbf{u}_0,\rho)\big)$ endowed with the usual sup norm $\|\mathbf{u}\|_{\mathbf{Y}} = \sup\limits_{t\in[0,\tau]} \|\mathbf{u}(t)\|_{\mathbf{X}}$. Next, from the local Lipschitz property of $\mathfrak{F}$, we show that $\mathcal{J}\big(\mathbf{Y}\big) \subset \mathbf{Y}$. If we use the Contraction Mapping Theorem, the strategy is to choose $\tau > 0$ small enough such that

$$\sup_{t\in[0,\tau]} \big\|\big(\mathcal{J}\mathbf{u}\big)(t) - \mathbf{u}_0\big\|_{\mathbf{X}} \leq \rho$$

(so that $\mathcal{J}\big(\mathbf{Y}\big) \subset \mathbf{Y}$), and then further restrict τ to ensure that there exists a constant $0 < \zeta(\tau) < 1$ such that

$$\big\|\big(\mathcal{J}\mathbf{u}\big) - \big(\mathcal{J}\mathbf{v}\big)\big\|_{\mathbf{Y}} \leq \zeta(\tau)\|\mathbf{u}-\mathbf{v}\|_{\mathbf{Y}}, \qquad \mathbf{u},\, \mathbf{v} \in \mathbf{Y}.$$

Next, we introduce one more basic result on local mild solutions.

Theorem 2.11.5. *Let $\mathbf{X}$ be a Banach space and let $\mathfrak{F}: \mathbf{X} \to \mathbf{X}$ be locally Lipschitz continuous in $\mathbf{u}$. If $\mathcal{Q}$ is the generator of a strongly continuous semigroup $\big(\mathcal{T}(t)\big)_{t\geq 0}$ on $\mathbf{X}$, then for every $\mathbf{u}_0 \in \mathbf{X}$, there is a $t_{\max} \leq \infty$ such that the initial value problem (2.18) has a unique mild solution $\mathbf{u}$ on $[0, t_{\max})$. Moreover, if $t_{\max} < +\infty$ then*

$$\lim_{t\to t_{\max}^-} \big\|\mathbf{u}(t)\big\|_{\mathbf{X}} = +\infty.$$

If, in addition, we assume that $\mathfrak{F}$ is continuously Fréchet differentiable on $\mathbf{X}$, then the result of Theorem 2.11.4 holds on $[0, t_{\max})$.

3. Solvability of Age Structured SIS Epidemiological Models

To simplify further considerations, we reduce the SIRS model (1.12)-(1.20), analysed in ([11]), to an SIS model by assuming $\gamma = 0$ and $r_0(a) = 0$ in (1.12)-(1.20); hence we obtain

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) s(a,t) = -\mu(a)s(a,t) - \Lambda(a, i(\cdot,t))s(a,t) + \delta(a)i(a,t) \tag{3.1a}$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) i(a,t) = -\mu(a)i(a,t) + \Lambda(a, i(\cdot,t))s(a,t) - \delta(a)i(a,t) \tag{3.1b}$$

$$s(0,t) = \int_0^\omega \beta(a)\left\{s(a,t) + (1-q)i(a,t)\right\}\, da \tag{3.1c}$$

$$i(0,t) = q\int_0^\omega \beta(a)i(a,t)\, da, \tag{3.1d}$$

$$s(a,0) = s_0(a) = \phi^s(a) \tag{3.1e}$$

$$i(a,0) = i_0(a) = \phi^i(a), \tag{3.1f}$$

for $0 \le t \le T < +\infty$, $0 \le a \le \omega < +\infty$, which is a typical model that describes the evolution of epidemics which do not convey any immunity. In the sequel, we consider the SIS model (3.1).

The constitutive form of the force of infection $\Lambda(a, i(\cdot,t))$ depends on the mechanisms of transmission (intercohort case or intracohort case); the concrete assumptions will be introduced when needed. In both cases we deal with a semilinear problem; that is, with a nonlinear (algebraic) perturbation of a linear problem. The decisive role is played by the semigroup generated by the linear part of the problem.

Problems like (3.1a)-(3.1f) have been relatively well-researched, including the cases where μ and β are nonlinear functions depending on the total population, see [16, 50] and references therein. Since the results are scattered and refer to two distinct cases (with the maximum age $\omega < +\infty$ and $\omega = +\infty$), we summarize basic facts in the form suitable to the problem at hand. Though the model most resembles that discussed in [50], the main difference is that in *op. cit.* the maximum age ω is infinite, μ is bounded. However, as we mentioned earlier, a biologically realistic assumption is that $\omega < +\infty$. This is in contrast with the case $\omega = +\infty$, and introduces another unbounded operator μ in (3.1a)-(3.1b). In the work by Inaba [29], this was circumvented by assuming that there is a maximum reproductive age $a_r < \omega$, so that the fertility β satisfies $\beta(a) = 0$ for $a > a_r$ and hence, ignoring the post-reproductive population, doing the analysis for $a \in [0, a_r]$. The analysis of the model without any simplification in the scalar and linear case was done in [27] by reducing it to an integral equation by integrating along characteristics. It can be proved that the solution of such a problem is given by a strongly continuous semigroup. Here we shall prove this directly by refining the argument of [29].

Notation and Assumptions

We denote $\mathrm{I} = (0, \omega)$ and $\bar{\mathrm{I}} = [0, \omega]$ and we consider the state space $\mathbf{X} = \left(\mathrm{L}^1(\mathrm{I})\right)^2$ with the norm $\|(p_1, p_2)\|_{\mathbf{X}} = \|p_1\|_1 + \|p_2\|_1$, where the norm $\|\cdot\|_1$ refers to the norm in $\mathrm{L}^1(\mathrm{I})$; the norm in $\mathbb{R}^2$ will be denoted by $\|\cdot\|$ such that $\|(x, y)\| = |x| + |y|$ for all $x, y \in \mathbb{R}$. For any measurable function α on $[0, \omega]$ we introduce

$$\overline{\alpha} = \operatorname*{esssup}_{a \in [0,\omega]} \alpha(a), \quad \underline{\alpha} = \operatorname*{essinf}_{a \in [0,\omega]} \alpha(a)$$

Then we make the following assumptions (cf. [27]) on the coefficients of (3.1).

(H1) $\mu \in \mathrm{L}^\infty_{\mathrm{loc}}([0, \omega))$, satisfying $\int_0^\omega \mu(r)\, dr = +\infty$, with $0 < \underline{\mu} \leq \mu(a)$ (since the death rate of population of age a cannot be zero);

(H2) $0 \leq \beta \in \mathrm{L}^\infty(\bar{\mathrm{I}})$ with $\beta(a) \leq \overline{\beta}$;

(H3) $0 \leq \delta \in \mathrm{W}^{1,\infty}(\bar{\mathrm{I}})$ with $\underline{\delta} \leq \delta(a) \leq \overline{\delta}$.

Let $\left(\mathrm{W}^{1,1}(\bar{\mathrm{I}})\right)^2$ be the Sobolev space of vector valued functions with integrable first derivatives. Further, we define
$\mathbf{S} = \mathrm{diag}\left\{-\partial_a, -\partial_a\right\}$ on $D(\mathbf{S}) = \left(\mathrm{W}^{1,1}\left(\bar{\mathrm{I}}\right)\right)^2$, $\mathbf{M}_\mu = \mathrm{diag}\left\{-\mu, -\mu\right\}$ on $D(\mathbf{M}_\mu) = \{\boldsymbol{\varphi} \in \mathbf{X} :\ \mu\boldsymbol{\varphi} \in \mathbf{X}\}$,

$$\mathbf{M}_\delta = \begin{bmatrix} 0 & \delta \\ 0 & -\delta \end{bmatrix} \tag{3.2}$$

on $\mathbb{B}(\mathbf{X})$. Note that this dual use of notation, $\mathbf{M}_\mu$ and $\mathbf{M}_\delta$, shall be understood as two 2×2 matrices where μ and δ are respectively the only non-zero entries. Further

$$\mathbf{B} = \begin{bmatrix} \beta & (1-q)\beta \\ 0 & q\beta \end{bmatrix} \tag{3.3}$$

with

$$\boldsymbol{\mathcal{B}}\boldsymbol{\varphi} = \int_0^\omega \mathbf{B}(a)\boldsymbol{\varphi}(a)\, da;$$

the operator $\boldsymbol{\mathcal{B}}$ belongs to $\mathbb{B}(\mathbf{X}, \mathbb{R}^2)$ and satisfies $\|\boldsymbol{\mathcal{B}}\|_{\mathbb{B}(\mathbf{X},\mathbb{R}^2)} \leq \overline{\beta}$. Moreover, we introduce the linear operator $\mathbf{A}$ defined on the domain

$$D\left(\mathbf{A}\right) = \{\boldsymbol{\varphi} \in D(\mathbf{S}) \cap D(\mathbf{M}_\mu);\ \boldsymbol{\varphi}(0) = \boldsymbol{\mathcal{B}}\boldsymbol{\varphi}\} \tag{3.4}$$

by

$$\mathbf{A} = \mathbf{S} + \mathbf{M}_\mu. \tag{3.5}$$

Let $\boldsymbol{\mathcal{Q}}$ be the linear operator defined on the domain $D(\boldsymbol{\mathcal{Q}}) = D(\mathbf{A})$ by $\boldsymbol{\mathcal{Q}} = \mathbf{A} + \mathbf{M}_\delta$ with $\mathbf{M}_\delta \in \mathbb{B}(\mathbf{X})$.

Using this notation, we re-write (3.1a)-(3.1f) in the following compact form

$$\partial_t \mathbf{u} = \boldsymbol{\mathcal{Q}}\mathbf{u} + \mathfrak{F}(\mathbf{u}), \tag{3.6a}$$

$$\mathbf{u}(0,t) = \int_0^\omega \mathbf{B}(a)\mathbf{u}(a,t)\,da, \tag{3.6b}$$

$$\mathbf{u}(a,0) = \mathbf{u}_0(a) = \boldsymbol{\varphi}(a), \tag{3.6c}$$

where $\mathbf{u} = (s,i)^T$ and $\mathfrak{F}$ is a nonlinear function defined by

$$\mathfrak{F}(\mathbf{u}) = \begin{bmatrix} -\Lambda(\cdot,i)s \\ \Lambda(\cdot,i)s \end{bmatrix}. \tag{3.7}$$

3.1 The Linear Part

To prove that (3.1a)-(3.1f) is well-posed in $\mathbf{X}$, first we show that the linear operator $\boldsymbol{\mathcal{Q}}$ on $D(\boldsymbol{\mathcal{Q}}) = D(\mathbf{A})$ generates a strongly continuous semigroup in $\mathbf{X}$.

Theorem 3.1.1. *The linear operator $\boldsymbol{\mathcal{Q}}$ generates a strongly continuous positive semigroup $(\boldsymbol{\mathcal{T}}(t))_{t\geq 0}$ in $\mathbf{X}$.*

A version of the proof of this theorem based on the Hille-Yosida theorem is given in [29]. The author considers in that paper the realistic assumption $\omega < +\infty$. However, to avoid dealing with unbounded μ, he introduces the maximum reproductive age $a_r < \omega$ and, by restricting the analysis to $[0, a_r]$, leaves out the singularity of μ at ω. Hence, in his case the operator $\boldsymbol{\mathcal{Q}}$ consists of one unbounded (differentiation) operator and two bounded operators. By contrast, in our work we allow $\omega = a_r$ and thus we will have to handle the sum of two unbounded operators which requires more care. In our version of the proof of Theorem 3.1.1, it is sufficient to prove the generation result for the unbounded operator $\mathbf{A}$ and use Theorem 2.10.1 (the bounded perturbation theorem) to prove the generation for $\boldsymbol{\mathcal{Q}}$; then we use formula (2.17) to show that the semigroup generated by $\boldsymbol{\mathcal{Q}}$ is positive. Using the argument developed in [27], we shall be able to obtain a better estimate of the semigroup generated by $\boldsymbol{\mathcal{Q}}$.

Lemma 3.1.2. *The linear operator $\mathbf{A}$ generates a strongly continuous positive semigroup $\left(e^{t\mathbf{A}}\right)_{t\geq 0}$ in $\mathbf{X}$ such that*

$$\|e^{t\mathbf{A}}\|_{\mathbb{B}(\mathbf{X})} \leq e^{(\overline{\beta}-\underline{\mu})t}. \tag{3.8}$$

To prove the lemma we construct and estimate the resolvent of $\mathbf{A}$. First we introduce the survival rate matrix $\mathbf{L}(a)$, which corresponds to the survival probability $\Pi_\mu(a)$ in a single population. $\mathbf{L}(a)$ is a solution of the matrix differential equation:

$$\frac{d\mathbf{L}}{da}(a) = \mathbf{M}_\mu(a)\mathbf{L}(a), \quad \mathbf{L}(0) = \mathbf{I}, \tag{3.9}$$

where $\mathbf{I}$ denotes the 2×2 identity matrix. The solution of (3.9) is a diagonal matrix given by

$$\mathbf{L}(a) = \begin{bmatrix} e^{-\int_0^a \mu(r)\,dr} & 0 \\ 0 & e^{-\int_0^a \mu(r)\,dr} \end{bmatrix} = e^{-\int_0^a \mu(r)\,dr} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = e^{-\int_0^a \mu(r)\,dr}\mathbf{I}. \tag{3.10}$$

We see that

$$\mathbf{L}^{-1}(a) = e^{\int_0^a \mu(r)\,dr}\mathbf{I} \tag{3.11}$$

and

$$\frac{d\mathbf{L}^{-1}}{da}(a) = -\mathbf{M}_\mu(a)\mathbf{L}^{-1}(a), \quad \mathbf{L}^{-1}(0) = \mathbf{I}. \tag{3.12}$$

Hence, we can define the fundamental matrix $\mathbf{L}(a, b)$ as

$$\mathbf{L}(a, b) = \mathbf{L}(a)\mathbf{L}^{-1}(b).$$

Lemma 3.1.3. *Let $\mathbf{L}(a, b)$, $a > b$, be the transition matrix. Then the following are satisfied:*

1. *$\mathbf{L}(a, b)$ is nonnegative,*

2. *$\big\|\mathbf{L}(a, b)\big\|_{\mathbb{B}(\mathbb{R}^2)} \leq e^{-\underline{\mu}(a-b)}$.*

Proof. We have $\mathbf{L}(a, b) = \mathbf{L}(a)\mathbf{L}^{-1}(b) = e^{-\int_b^a \mu(r)\,dr}\mathbf{I}$. Thus, $\mathbf{L}(a, b)$ is nonnegative. Moreover, taking the norm in $\mathbb{B}(\mathbb{R}^2)$, we obtain

$$\|\mathbf{L}(a, b)\|_{\mathbb{B}(\mathbb{R}^2)} = e^{-\int_b^a \mu(r)\,dr} \leq e^{-\underline{\mu}(a-b)},$$

by (H1). This completes the proof. □

In the next result, we prove the existence of the resolvent operator and provide its explicit formula. This is an extension of the result by Inaba [29] but with the setting, where the maximum life span, ω, of the population is finite, the mortality $\mu(a)$ is unbounded, the fertility $\beta(a)$ is bounded, on the whole age interval is $[0, \omega]$. The proof of this result needs more care because of the unbounded coefficient $\mu(a)$. But thanks to the diagonal structure of the survival rate function $\mathbf{L}(a)$, we are able to handle the terms which are multiplied by $\mu(a)$.

Lemma 3.1.4. *If $\lambda > \overline{\beta} - \underline{\mu}$, then $(\lambda \mathbf{I} - \mathbf{A})^{-1}$ is given by*

$$\boldsymbol{\varphi} = (\lambda \mathbf{I} - \mathbf{A})^{-1} \boldsymbol{\psi} = e^{-\lambda a}\mathbf{L}(a) \left(\mathbf{I} - \int_0^\omega e^{-\lambda\sigma} \mathbf{B}(\sigma)\mathbf{L}(\sigma)\, d\sigma \right)^{-1} \int_0^\omega e^{-\lambda a}\mathbf{B}(a)\mathbf{L}(a) \times \int_0^a e^{\lambda\sigma}\mathbf{L}^{-1}(\sigma)\boldsymbol{\psi}(\sigma)\, d\sigma\, da + e^{-\lambda a}\mathbf{L}(a) \int_0^a e^{\lambda\sigma}\mathbf{L}^{-1}(\sigma)\boldsymbol{\psi}(\sigma)\, d\sigma, \tag{3.13}$$

for some $\boldsymbol{\psi} \in \mathbf{X}$.

Proof. Let $\lambda > \overline{\beta} - \underline{\mu}$. A function $\boldsymbol{\varphi} \in D(\mathbf{A})$ if and only if $\boldsymbol{\varphi} \in \left(\mathrm{W}^{1,1}(\bar{I})\right)^2$,

$$\lambda\boldsymbol{\varphi}(a) + \frac{d}{da}\boldsymbol{\varphi}(a) - \mathbf{M}_\mu(a)\boldsymbol{\varphi}(a) = \boldsymbol{\psi}(a), \tag{3.14a}$$

$$\boldsymbol{\varphi}(0) = \int_0^\omega \mathbf{B}(a)\boldsymbol{\varphi}(a)\, da, \tag{3.14b}$$

$$\mu\boldsymbol{\varphi} \in \mathbf{X}, \tag{3.14c}$$

for some $\boldsymbol{\psi} \in \mathbf{X}$. By Duhamel's formula, (3.14a) leads to

$$\begin{aligned}\boldsymbol{\varphi}(a) &= e^{-\lambda a}\mathbf{L}(a,0)\boldsymbol{\varphi}(0) + \int_0^a e^{-\lambda(a-s)}\mathbf{L}(a,s)\boldsymbol{\psi}(s)\, ds \\ &= e^{-\lambda a - \int_0^a \mu(r)\, dr}\boldsymbol{\varphi}(0) + \int_0^a e^{-\lambda(a-s) - \int_s^a \mu(r)\, dr}\boldsymbol{\psi}(s)\, ds \end{aligned}\tag{3.15}$$

for some unspecified as yet initial condition $\boldsymbol{\varphi}(0)$. Since $\boldsymbol{\psi} \in \mathbf{X}$ and $\mathbf{L}(a,s) = e^{-\int_s^a \mu(r)\, dr}$ is differentiable almost everywhere on $\bar{\mathrm{I}}$, hence $\boldsymbol{\varphi}(a)$ is differentiable almost everywhere on $\bar{\mathrm{I}}$.

For a fixed $\boldsymbol{\varphi}(0)$, we denote $\boldsymbol{\varphi} := \boldsymbol{\mathcal{R}}_{\boldsymbol{\varphi}(0)}(\lambda)\boldsymbol{\psi}$; we see that

$$\left(\lambda\mathbf{I} - \mathbf{S} - \mathbf{M}_\mu\right)\boldsymbol{\mathcal{R}}_{\boldsymbol{\varphi}(0)}(\lambda)\boldsymbol{\psi} = \boldsymbol{\psi}, \tag{3.16}$$

for a.a. $a \in [0, \omega)$. The unknown $\boldsymbol{\varphi}(0)$ can be determined from (3.14b) by substituting (3.15). First, we get

$$\boldsymbol{\varphi}(0) = \int_0^\omega e^{-\lambda a - \int_0^a \mu(r)\, dr}\mathbf{B}(a)\boldsymbol{\varphi}(0)\, da + \int_0^\omega e^{-\lambda a - \int_0^a \mu(r)\, dr}\mathbf{B}(a)\left(\int_0^a e^{\lambda s + \int_0^s \mu(r)\, dr}\boldsymbol{\psi}(s)\, ds \right) da. \tag{3.17}$$

Now, we see

$$\left\| \int_0^\omega e^{-\lambda a - \int_0^a \mu(r)\, dr}\mathbf{B}(a)\, da \right\|_{\mathbb{B}(\mathbb{R}^2)} \leq \overline{\beta} \int_0^\omega e^{-(\lambda + \underline{\mu})a}\, da = \frac{\overline{\beta}}{\lambda + \underline{\mu}}\left(1 - e^{-(\lambda+\underline{\mu})\omega}\right) \leq \frac{\overline{\beta}}{\lambda + \underline{\mu}} < 1 \tag{3.18}$$

for $\lambda > \overline{\beta} - \underline{\mu}$. Thus,

$$\left(\mathbf{I} - \int_0^\omega e^{-\lambda a - \int_0^a \mu(r)\, dr}\mathbf{B}(a)\, da \right)^{-1}$$

exists by the Neumann series, see [12, p. 40]; hence (3.17) is solvable,

$$\boldsymbol{\varphi}(0) = \left(\mathbf{I} - \int_0^\omega e^{-\lambda a - \int_0^a \mu(r)\,dr}\mathbf{B}(a)\,da\right)^{-1} \int_0^\omega e^{-\lambda a - \int_0^a \mu(r)\,dr}\mathbf{B}(a)\left(\int_0^a e^{\lambda s + \int_0^s \mu(r)\,dr}\boldsymbol{\psi}(s)\,ds\right)da \tag{3.19}$$

and we can substitute (3.19) in (3.15) to define

$$\begin{aligned}\boldsymbol{\mathcal{R}}(\lambda)\boldsymbol{\psi}(a) := e^{-\lambda a - \int_0^a \mu(r)\,dr}\left(\mathbf{I} - \int_0^\omega e^{-\lambda s - \int_0^s \mu(r)\,dr}\mathbf{B}(s)\,ds\right)^{-1}\int_0^\omega e^{-\lambda a - \int_0^a \mu(r)\,dr}\mathbf{B}(a) \\ \times \int_0^a e^{\lambda s + \int_0^s \mu(r)\,dr}\boldsymbol{\psi}(s)\,ds\,da + e^{-\lambda a - \int_0^a \mu(r)\,dr}\int_0^a e^{\lambda s + \int_0^s \mu(r)\,dr}\boldsymbol{\psi}(s)\,ds.\end{aligned} \tag{3.20}$$

To prove the one-to-one property of $\lambda\mathbf{I} - \mathbf{A}$, let $\boldsymbol{\psi} = \mathbf{0}$ in (3.14a). Then

$$\boldsymbol{\varphi}(a) = e^{-\lambda a - \int_0^a \mu(r)\,dr}\boldsymbol{\varphi}(0).$$

Substituting this in (3.14b) yields

$$\left(\mathbf{I} - \int_0^\omega e^{-\lambda s - \int_0^s \mu(r)\,dr}\mathbf{B}(s)\,ds\right)\boldsymbol{\varphi}(0) = 0$$

for $\lambda > \overline{\beta} - \underline{\mu}$, this leads to $\boldsymbol{\varphi}(0) = \mathbf{0}$, hence $\boldsymbol{\varphi}(a) = \mathbf{0}$ for all $a \in [0, \omega]$ and thus the operator $\lambda\mathbf{I} - \mathbf{A}$ is one-to-one, for $\lambda > \overline{\beta} - \underline{\mu}$.

In the next step, we aim to prove that the operator $\boldsymbol{\mathcal{R}}(\lambda)$, defined in (3.20), is indeed the resolvent operator. To do this, we shall check whether $\boldsymbol{\mathcal{R}}(\lambda)$ satisfies the properties of the resolvent operator. We have

$$\begin{aligned}\left\|\boldsymbol{\mathcal{R}}(\lambda)\boldsymbol{\psi}\right\|_{\mathbf{X}} &\le \int_0^\omega e^{-\lambda a - \int_0^a \mu(r)\,dr}\left\|\left(\mathbf{I} - \int_0^\omega e^{-\lambda s - \int_0^s \mu(r)\,dr}\mathbf{B}(s)\,ds\right)^{-1}\right\|_{\mathbb{B}(\mathbb{R}^2)} \\ &\quad \times \int_0^\omega e^{-\lambda v - \int_0^v \mu(r)\,dr}\left\|\mathbf{B}(v)\right\|_{\mathbb{B}(\mathbb{R}^2)}\int_0^v e^{\lambda s + \int_0^s \mu(r)\,dr}\left\|\boldsymbol{\psi}(s)\right\|\,ds\,dv\,da \\ &\qquad + \int_0^\omega e^{-\lambda a - \int_0^a \mu(r)\,dr}\int_0^a e^{\lambda s + \int_0^s \mu(r)\,dr}\left\|\boldsymbol{\psi}(s)\right\|\,ds\,da, \\ &\le \overline{\beta}\cdot\frac{\lambda + \underline{\mu}}{\lambda - (\overline{\beta} - \underline{\mu})}\int_0^\omega e^{-\lambda a - \int_0^a \mu(r)\,dr} \\ &\quad \times \int_0^\omega\int_0^v e^{-\lambda(v-s) - \int_s^v \mu(r)\,dr}\left\|\boldsymbol{\psi}(s)\right\|\,ds\,dv\,da \\ &\qquad + \int_0^\omega\int_0^a e^{-\lambda(a-s) - \int_s^a \mu(r)\,dr}\left\|\boldsymbol{\psi}(s)\right\|\,ds\,da,\end{aligned}$$

$$\begin{aligned}
&\leq \overline{\beta} \cdot \frac{\lambda + \underline{\mu}}{\lambda - (\overline{\beta} - \underline{\mu})} \int_0^\omega e^{-(\lambda+\underline{\mu})a} \int_0^\omega e^{(\lambda+\underline{\mu})s} \|\boldsymbol{\psi}(s)\| \\
&\quad \times \int_s^\omega e^{-(\lambda+\underline{\mu})v} \, dv \, ds \, da \\
&\qquad\qquad + \int_0^\omega e^{(\lambda+\underline{\mu})s} \|\boldsymbol{\psi}(s)\| \int_s^\omega e^{-(\lambda+\underline{\mu})a} \, da \, ds, \\
&\leq \frac{\overline{\beta}}{\lambda - (\overline{\beta} - \underline{\mu})} \int_0^\omega e^{-(\lambda+\underline{\mu})a} \|\boldsymbol{\psi}\|_{\mathbf{X}} \, da + \frac{1}{\lambda + \underline{\mu}} \|\boldsymbol{\psi}\|_{\mathbf{X}}.
\end{aligned}$$

Hence,

$$\|\boldsymbol{\mathcal{R}}(\lambda)\boldsymbol{\psi}\|_{\mathbf{X}} \leq \frac{1}{\lambda - (\overline{\beta} - \underline{\mu})} \|\boldsymbol{\psi}\|_{\mathbf{X}}, \tag{3.21}$$

that is, $\boldsymbol{\mathcal{R}}(\lambda) : \mathbf{X} \to \mathbf{X}$.

Furthermore, we have

$$\begin{aligned}
\int_0^\omega \|\mu(a)\boldsymbol{\mathcal{R}}(\lambda)\boldsymbol{\psi}(a)\| \, da &\leq \int_0^\omega \mu(a) e^{-\lambda a - \int_0^a \mu(r)\,dr} \left\| \left(\mathbf{I} - \int_0^\omega e^{-\lambda s - \int_0^s \mu(r)\,dr} \mathbf{B}(s) \, ds \right)^{-1} \right\|_{\mathbb{B}(\mathbb{R}^2)} \\
&\quad \times \int_0^\omega e^{-\lambda v - \int_0^v \mu(r)\,dr} \|\mathbf{B}(v)\|_{\mathbb{B}(\mathbb{R}^2)} \int_0^v e^{\lambda s + \int_0^s \mu(r)\,dr} \|\boldsymbol{\psi}(s)\| \, ds \, dv \, da \\
&\qquad + \int_0^\omega \mu(a) e^{-\lambda a - \int_0^a \mu(r)\,dr} \int_0^a e^{\lambda s + \int_0^s \mu(r)\,dr} \|\boldsymbol{\psi}(s)\| \, ds \, da.
\end{aligned}$$

First, we have

$$\begin{aligned}
&\int_0^\omega \mu(a) e^{-\lambda a - \int_0^a \mu(r)\,dr} \left\| \left(\mathbf{I} - \int_0^\omega e^{-\lambda s - \int_0^s \mu(r)\,dr} \mathbf{B}(s) \, ds \right)^{-1} \right\|_{\mathbb{B}(\mathbb{R}^2)} \\
&\quad \times \int_0^\omega e^{-\lambda v - \int_0^v \mu(r)\,dr} \|\mathbf{B}(v)\|_{\mathbb{B}(\mathbb{R}^2)} \int_0^v e^{\lambda s + \int_0^s \mu(r)\,dr} \|\boldsymbol{\psi}(s)\| \, ds \, dv \, da \\
&\qquad \leq \overline{\beta} \cdot \frac{\lambda + \underline{\mu}}{\lambda - (\overline{\beta} - \underline{\mu})} \left(\int_0^\omega \frac{d}{da} \left(-e^{-\int_0^a \mu(r)\,dr} \right) e^{-\lambda a} \, da \right) \\
&\qquad\quad \times \int_0^\omega \int_0^v e^{-\lambda(v-s) - \int_s^v \mu(r)\,dr} \|\boldsymbol{\psi}(s)\| \, ds \, dv \\
&\qquad \leq \overline{\beta} \frac{\lambda + \underline{\mu}}{\lambda - (\overline{\beta} - \underline{\mu})} \left(\int_0^\omega \frac{d}{da} \left(-e^{-\int_0^a \mu(r)\,dr} \right) e^{-\lambda a} \, da \right) \\
&\qquad\quad \times \int_0^\omega e^{(\lambda+\underline{\mu})s} \|\boldsymbol{\psi}(s)\| \int_s^\omega e^{-(\lambda+\underline{\mu})v} \, dv \, ds \\
&\qquad \leq \overline{\beta} \frac{\lambda + \underline{\mu}}{\lambda - (\overline{\beta} - \underline{\mu})} \left(1 - e^{-\int_0^\omega \mu(r)\,dr} e^{-\lambda\omega} - \lambda \int_0^\omega e^{-\int_0^a \mu(r)\,dr} e^{-\lambda a} \, da \right) \\
&\qquad\quad \times \int_0^\omega e^{(\lambda+\underline{\mu})s} \|\boldsymbol{\psi}(s)\| \int_s^\omega e^{-(\lambda+\underline{\mu})v} \, dv \, ds
\end{aligned}$$

$$\leq \frac{\overline{\beta}}{\lambda - (\overline{\beta} - \underline{\mu})}\left(1 - \lambda\int_0^\omega e^{-\lambda a - \int_0^a \mu(r)\,dr}\,da\right)\|\boldsymbol{\psi}\|_{\mathbf{X}}$$
$$\leq \frac{\overline{\beta}}{\lambda - (\overline{\beta} - \underline{\mu})}\left(1 + \lambda\int_0^\omega e^{-\lambda a - \int_0^a \mu(r)\,dr}\,da\right)\|\boldsymbol{\psi}\|_{\mathbf{X}}.$$

Hence,

$$\int_0^\omega \mu(a)e^{-\lambda a - \int_0^a \mu(r)\,dr}\left\|\left(\mathbf{I} - \int_0^\omega e^{-\lambda s - \int_0^s \mu(r)\,dr}\mathbf{B}(s)\,ds\right)^{-1}\right\|_{\mathbb{B}(\mathbb{R}^2)}$$
$$\times \int_0^\omega e^{-\lambda v - \int_0^v \mu(r)\,dr}\big\|\mathbf{B}(v)\big\|_{\mathbb{B}(\mathbb{R}^2)}\int_0^v e^{\lambda s + \int_0^s \mu(r)\,dr}\big\|\boldsymbol{\psi}(s)\big\|\,ds\,dv\,da$$
$$\leq \frac{2\overline{\beta}}{\lambda - (\overline{\beta} - \underline{\mu})}\|\boldsymbol{\psi}\|_{\mathbf{X}}$$

as

$$\lambda\int_0^\omega e^{-\lambda a - \int_0^a \mu(r)\,dr}\,da \leq \frac{\lambda}{\lambda + \underline{\mu}} < 1.$$

Next,

$$\int_0^\omega \mu(a)e^{-\lambda a - \int_0^a \mu(r)\,dr}\int_0^a e^{\lambda s + \int_0^s \mu(r)\,dr}\big\|\boldsymbol{\psi}(s)\big\|\,ds\,da$$
$$= \int_0^\omega \left(\frac{d}{da}\left(-e^{-\int_0^a \mu(r)\,dr}\right)e^{-\lambda a}\int_0^a e^{\lambda s + \int_0^s \mu(r)\,dr}\big\|\boldsymbol{\psi}(s)\big\|\,ds\right)da$$
$$= -\lim_{\overline{\omega}\to\omega^-}\left(e^{-\lambda\overline{\omega}}e^{-\int_0^{\overline{\omega}} \mu(r)\,dr}\int_0^{\overline{\omega}} e^{\lambda s}e^{\int_0^s \mu(r)\,dr}\big\|\boldsymbol{\psi}(s)\big\|\,ds\right)$$
$$- \lambda\int_0^\omega e^{-\lambda a}e^{-\int_0^a \mu(r)\,dr}\int_0^a e^{\lambda s}e^{\int_0^s \mu(r)\,dr}\big\|\boldsymbol{\psi}(s)\big\|\,ds\,da + \int_0^\omega \big\|\boldsymbol{\psi}(a)\big\|\,da$$
$$\leq \big\|\boldsymbol{\psi}\big\|_{\mathbf{X}} + \lim_{\overline{\omega}\to\omega^-}\left(-e^{-\lambda\overline{\omega}}e^{-\int_0^{\overline{\omega}} \mu(r)\,dr}\int_0^{\overline{\omega}} e^{\lambda s}e^{\int_0^s \mu(r)\,dr}\big\|\boldsymbol{\psi}(s)\big\|\,ds\right)$$
$$+ \lambda\int_0^\omega e^{\lambda s}e^{\int_0^s \mu(r)\,dr}\big\|\boldsymbol{\psi}(s)\big\|\int_s^\omega e^{-\lambda a}e^{-\int_0^a \mu(r)\,dr}\,da\,ds$$
$$\leq 3\big\|\boldsymbol{\psi}\big\|_{\mathbf{X}}.$$

Indeed, from the calculations of (3.21),

$$\lambda\int_0^\omega e^{\lambda s + \int_0^s \mu(r)\,dr}\big\|\boldsymbol{\psi}(s)\big\|\int_s^\omega e^{-\lambda a}e^{-\int_0^a \mu(r)\,dr}\,da\,ds \leq \frac{\lambda}{\lambda + \underline{\mu}}\big\|\boldsymbol{\psi}\big\|_{\mathbf{X}} < \big\|\boldsymbol{\psi}\big\|_{\mathbf{X}}$$

and

$$\lim_{\overline{\omega}\to\omega^-} e^{-\lambda\overline{\omega}}e^{-\int_0^{\overline{\omega}} \mu(r)\,dr}\int_0^{\overline{\omega}} e^{\lambda s}e^{\int_0^s \mu(r)\,dr}\big\|\boldsymbol{\psi}(s)\big\|\,ds \leq \lim_{\overline{\omega}\to\omega^-} e^{-\lambda\overline{\omega}}e^{-\int_0^{\overline{\omega}} \mu(r)\,dr}e^{\lambda\overline{\omega}}e^{\int_0^{\overline{\omega}} \mu(r)\,dr}\int_0^{\overline{\omega}}\big\|\boldsymbol{\psi}(s)\big\|\,ds$$
$$= \lim_{\overline{\omega}\to\omega^-}\int_0^{\overline{\omega}}\big\|\boldsymbol{\psi}(s)\big\|\,ds = \int_0^\omega\big\|\boldsymbol{\psi}(s)\big\|\,ds = \big\|\boldsymbol{\psi}\big\|_{\mathbf{X}}.$$

Hence, we obtain

$$\|\mu\boldsymbol{\mathcal{R}}(\lambda)\boldsymbol{\psi}\|_{\mathbf{X}} \leq \left(3 + \frac{2\overline{\beta}}{\lambda - (\overline{\beta} - \underline{\mu})}\right)\|\boldsymbol{\psi}\|_{\mathbf{X}},$$

that is, $\boldsymbol{\mathcal{R}}(\lambda) : \mathbf{X} \to D(\mathbf{M}_\mu)$.

Since for any $\boldsymbol{\psi} \in \mathbf{X}$, $\boldsymbol{\varphi} = \boldsymbol{\mathcal{R}}(\lambda)\boldsymbol{\psi}$ satisfies

$$\lambda\boldsymbol{\mathcal{R}}(\lambda)\boldsymbol{\psi} + \frac{d}{da}\boldsymbol{\mathcal{R}}(\lambda)\boldsymbol{\psi} - \mathbf{M}_\mu\boldsymbol{\mathcal{R}}(\lambda)\boldsymbol{\psi} = \boldsymbol{\psi}$$

almost everywhere on $\bar{\mathrm{I}}$, we have

$$\frac{d}{da}\boldsymbol{\mathcal{R}}(\lambda)\boldsymbol{\psi} = \boldsymbol{\psi} - \lambda\boldsymbol{\mathcal{R}}(\lambda)\boldsymbol{\psi} + \mathbf{M}_\mu\boldsymbol{\mathcal{R}}(\lambda)\boldsymbol{\psi}, \tag{3.22}$$

where, by the above estimates, all terms on the right hand side are in $\mathbf{X}$. Hence $\boldsymbol{\mathcal{R}}(\lambda)\boldsymbol{\psi} \in \left(\mathrm{W}^{1,1}(\bar{\mathrm{I}})\right)^2$.

Since $\boldsymbol{\mathcal{R}}(\lambda)\boldsymbol{\psi}$ satisfies the boundary condition (3.14b), using the above estimates, we see that $\boldsymbol{\mathcal{R}}(\lambda) : \mathbf{X} \to D(\mathbf{A})$. Hence $\boldsymbol{\mathcal{R}}(\lambda)$ is the right inverse of $(\lambda\mathbf{I} - \mathbf{A}, D(\mathbf{A}))$. To prove that it is also the left inverse, we repeat the standard argument. Assume that for some $\boldsymbol{\varphi} \in D(\mathbf{A})$ we can find $\widetilde{\boldsymbol{\varphi}} \in D(\mathbf{A})$, $\widetilde{\boldsymbol{\varphi}} \neq \boldsymbol{\varphi}$, that satisfies $\boldsymbol{\mathcal{R}}(\lambda)(\lambda\mathbf{I} - \mathbf{A})\boldsymbol{\varphi} = \widetilde{\boldsymbol{\varphi}}$. Since $\boldsymbol{\mathcal{R}}(\lambda) : \mathbf{X} \to D(\mathbf{A})$ is the right inverse of $\lambda\mathbf{I} - \mathbf{A}$, we can write

$$(\lambda\mathbf{I} - \mathbf{A})\boldsymbol{\varphi} = (\lambda\mathbf{I} - \mathbf{A})\boldsymbol{\mathcal{R}}(\lambda)(\lambda\mathbf{I} - \mathbf{A})\boldsymbol{\varphi} = (\lambda\mathbf{I} - \mathbf{A})\widetilde{\boldsymbol{\varphi}}.$$

Further, since the linear operator $\lambda\mathbf{I} - \mathbf{A}$ is one-to-one for $\lambda > \overline{\beta} - \underline{\mu}$, it follows that $\boldsymbol{\varphi} = \widetilde{\boldsymbol{\varphi}}$; hence $\boldsymbol{\mathcal{R}}(\lambda) = \left(\lambda\mathbf{I} - \mathbf{A}\right)^{-1}$ for $\lambda > \overline{\beta} - \underline{\mu}$. □

We also state the following:

Lemma 3.1.5. $\mathbf{A}$ *is a closed linear operator in* $\mathbf{X}$*.*

Proof. Since we have $\mathbf{A} = -(\lambda\mathbf{I} - \mathbf{A}) + \lambda\mathbf{I}$ and we already showed that the resolvent operator $(\lambda\mathbf{I} - \mathbf{A})^{-1} : \mathbf{X} \to D(\mathbf{A}) \subset \mathbf{X}$ is bounded, the resolvent operator $(\lambda\mathbf{I} - \mathbf{A})^{-1}$ is continuous, by Theorem 2.4.5 (i). Hence, from Lemma 2.4.9 (i), it follows that the resolvent operator $(\lambda\mathbf{I} - \mathbf{A})^{-1}$ is closed. Then, the closedness of the operator $\lambda\mathbf{I} - \mathbf{A}$ follows from Lemma 2.4.9 (ii). Thanks to the boundedness of the operator $\lambda\mathbf{I}$, from Lemma 2.4.9 (iii), we are able to show that the operator $\mathbf{A} = -(\lambda\mathbf{I} - \mathbf{A}) + \lambda\mathbf{I}$ is closed. □

The following lemma shows that the operator $\mathbf{A}$ is densely defined on $\mathbf{X}$. A proof of this result (with gaps) is provided in [29, p. 60]. A more comprehensive proof can be found in [50]. We present a much simpler proof which, moreover, allows for an approximation of $\mathbf{f} \in \mathbf{X}_+$ by elements of $D(\mathbf{A})_+$.

Lemma 3.1.6. $\overline{D(\mathbf{A})_+} = \mathbf{X}_+$.

Proof. Fix $\mathbf{f} \in \mathbf{X}_+$. For any given $\epsilon > 0$ there exists a so-called mollifier function $\boldsymbol{\varphi} \in \left(\mathcal{C}_0^\infty(\mathrm{I})\right)^2$, which is positive, such that $\|\mathbf{f} - \boldsymbol{\varphi}\|_{\mathbf{X}} \le \epsilon$, see [14, p. 90]. We see that $\boldsymbol{\varphi} \in D(\mathbf{M}_\mu)$, but

$$0 = \boldsymbol{\varphi}(0) \neq \int_0^\omega \mathbf{B}(a)\boldsymbol{\varphi}(a)\, da > 0$$

unless supports of $\mathbf{B}$ and $\boldsymbol{\varphi}$ are disjoint. Take a nonnegative function $\eta \in \mathcal{C}_0^\infty\big([0,\omega)\big)$ with $\eta(0) = 1$ and let $\eta_\epsilon(a) = \eta(a/\epsilon)$, then $\eta(a/\epsilon) = 0$ for $a > \epsilon\omega$ and hence $\operatorname{supp}\eta_\epsilon \subseteq [0, \epsilon\omega]$. Further, let $\boldsymbol{\alpha}$ be a vector and consider

$$\boldsymbol{\psi} = \boldsymbol{\varphi} + \eta_\epsilon \boldsymbol{\alpha}. \tag{3.23}$$

We see that $\boldsymbol{\psi} \in \left(\mathrm{W}^{1,1}(\bar{\mathrm{I}})\right)^2 \cap D(\mathbf{M}_\mu)$. Now, we need to find $\boldsymbol{\alpha}$ such that $\boldsymbol{\psi}$ satisfies the compatibility condition

$$\boldsymbol{\psi}(0) = \int_0^\omega \mathbf{B}(a)\boldsymbol{\psi}(a)\, da. \tag{3.24}$$

Since $\operatorname{supp}\eta_\epsilon \subseteq [0, \epsilon\omega]$, from (3.24) and (3.23), we get

$$\boldsymbol{\alpha} = \int_0^\omega \mathbf{B}(a)\boldsymbol{\varphi}(a)\, da + \left(\int_0^{\epsilon\omega} \eta_\epsilon(a)\mathbf{B}(a)\, da\right)\boldsymbol{\alpha}, \tag{3.25}$$

where

$$\eta_\epsilon(a)\mathbf{B}(a) = \begin{bmatrix} \beta(a)\eta_\epsilon(a) & (1-q)\beta(a)\eta_\epsilon(a) \\ 0 & q\beta(a)\eta_\epsilon(a) \end{bmatrix}. \tag{3.26}$$

Now, since

$$0 \le \int_0^{\epsilon\omega} \beta(a)\eta_\epsilon(a)\, da = \epsilon\int_0^\omega \beta(\epsilon s)\eta(s)\, ds \le \epsilon\overline{\beta}\int_0^\omega \eta(s)\, ds = \epsilon\overline{\beta}\|\eta\|_1,$$

the matrix l_1-norm of $\int_0^{\epsilon\omega} \eta_\epsilon(a)\mathbf{B}(a)\, da$ satisfies

$$\left\| \int_0^\omega \begin{bmatrix} \epsilon\beta(\epsilon s)\eta(s) & \epsilon(1-q)\beta(\epsilon s)\eta(s) \\ 0 & \epsilon q\beta(\epsilon s)\eta(s) \end{bmatrix} ds \right\|_{\mathbb{B}(\mathbb{R}^2)} \le \epsilon\overline{\beta}\|\eta\|_1.$$

With the same argument as used earlier for (3.17), we can see that (3.25) is solvable for sufficiently small ϵ and $\boldsymbol{\alpha}$ is nonnegative if $\boldsymbol{\varphi} > 0$ with

$$\|\boldsymbol{\alpha}\| \le \left\|\int_0^\omega \mathbf{B}(a)\boldsymbol{\varphi}(a)\, da\right\| (1 - \epsilon\overline{\beta}\|\eta\|_1)^{-1} \le C$$

for some constant C, which is independent of ϵ for sufficiently small ϵ.

It follows that

$$\|\mathbf{f} - \boldsymbol{\psi}\|_{\mathbf{X}} = \|(\mathbf{f} - \boldsymbol{\varphi}) + (\boldsymbol{\varphi} - \boldsymbol{\psi})\|_{\mathbf{X}} \le \|\mathbf{f} - \boldsymbol{\varphi}\|_{\mathbf{X}} + \epsilon\|\boldsymbol{\alpha}\|\|\eta\|_1 \le (1 + C\|\eta\|_1)\epsilon.$$

□

Proof of Lemma 3.1.2. Using the above lemmas with the estimate (3.21), we can see that $\mathbf{A}$ satisfies the assumptions of the Hille-Yosida theorem. Hence, it generates a strongly continuous semigroup $\left(e^{t\mathbf{A}}\right)_{t\geq 0}$ satisfying (3.8). Since the resolvent is positive, the semigroup is positive as well. □

Proof of Theorem 3.1.1. Since $\mathbf{M}_\delta(a) \in \mathbb{B}(\mathbf{X})$, with $\left\|\mathbf{M}_\delta(a)\right\|_{\mathbb{B}(\mathbf{X})} = 2\delta(a) \leq 2\overline{\delta}$, thus Theorem 2.10.1 (Bounded Perturbation Theorem) is applicable and states that the linear operator $(\boldsymbol{\mathcal{Q}}, D(\mathbf{A}))$ generates a strongly continuous semigroup $\left(\boldsymbol{\mathcal{T}}(t)\right)_{t\geq 0}$. Using the estimate (3.8), we have:

$$\|\boldsymbol{\mathcal{T}}(t)\|_{\mathbb{B}(\mathbf{X})} \leq e^{t(\overline{\beta}-\underline{\mu}+2\overline{\delta})}.$$

Thanks to the structure of $\mathbf{M}_\delta$, we can improve the above estimate and also show that the semigroup $\left(\boldsymbol{\mathcal{T}}(t)\right)_{t\geq 0}$ generated by $\boldsymbol{\mathcal{Q}}$ is positive. Since the variable a plays in $\mathbf{M}_\delta$ the role of a parameter, we can consider an abstract Cauchy problem of the form

$$\mathbf{u}^{'}(t) = \mathbf{M}_\delta \mathbf{u}(t), \quad \mathbf{u}(0) = \mathbf{u}_0,$$

where $\mathbf{u} = (s, i)^T$, which corresponds to the following system of scalar equations

$$\begin{cases} s^{'}(t) &= \delta(a)i(t), \qquad s(0) &= s_0, \\ i^{'}(t) &= -\delta(a)i(t), \qquad i(0) &= i_0. \end{cases} \tag{3.27}$$

Hence, we obtain

$$\begin{bmatrix} s(t) \\ i(t) \end{bmatrix} = \begin{bmatrix} 1 & 1-e^{-t\delta(a)} \\ 0 & e^{-t\delta(a)} \end{bmatrix} \begin{bmatrix} s_0 \\ i_0 \end{bmatrix}.$$

This shows that the semigroup generated by the operator $\mathbf{M}_\delta$ is defined by

$$e^{t\mathbf{M}_\delta} = \begin{bmatrix} 1 & 1-e^{-t\delta(a)} \\ 0 & e^{-t\delta(a)} \end{bmatrix},$$

and so we have

$$\left\|e^{t\mathbf{M}_\delta}\right\|_{\mathbb{B}(\mathbf{X})} = 1.$$

We also see that $\left(e^{t\mathbf{M}_\delta}\right)_{t\geq 0}$ is positive. Hence, by (2.17), we obtain

$$\|\boldsymbol{\mathcal{T}}(t)\|_{\mathbb{B}(\mathbf{X})} \leq e^{t(\overline{\beta}-\underline{\mu})} \tag{3.28}$$

and $\left(\boldsymbol{\mathcal{T}}(t)\right)_{t\geq 0}$ is positive. □

Remark 3.1.7. The estimates (3.8) and (3.28) are not optimal. In fact, for the scalar linear McKendrick problem

$$\frac{\partial u}{\partial t}(a,t) = -\frac{\partial u}{\partial a}(a,t) - \mu(a)u(a,t), \quad t > 0,\, a \in (0,\omega), \tag{3.29a}$$

$$u(0,t) = \int_0^\omega \beta(a)u(a,t)\,da, \tag{3.29b}$$

$$u(a,0) = u_0(a), \tag{3.29c}$$

it is proved, see [27], that there exists a unique dominant eigenvalue λ^* of (3.29) satisfying

$$1 = \int_0^\omega \beta(a) e^{-\lambda^* a - \int_0^a \mu(s)\,ds}\, da$$

such that the solution, u, to (3.29) satisfies the estimate

$$\|u(t)\|_1 \leq N e^{t\lambda^*} \|u_0\|_1,$$

for some constant N, with $(A, D(A))$ being the generator of the semigroup $\left(e^{tA}\right)_{t\geq 0}$ for the scalar McKendrick problem (3.29), defined analogously to (3.4). Hence, we arrive at

$$\|e^{tA}\|_{\mathbb{B}(\mathrm{L}^1)} \leq N e^{t\lambda^*}. \tag{3.30}$$

This eigenvalue is, respectively, positive, zero or negative if and only if the net reproduction rate

$$R = \int_0^\omega \beta(a) e^{-\int_0^a \mu(s)\,ds}\, da$$

is bigger, equal or smaller, than 1.

Consider now an initial condition $(s_0, i_0) \in D(\mathbf{A})_+$. Since the semigroup $\left(\boldsymbol{\mathcal{T}}(t)\right)_{t\geq 0}$ is positive, the strict solution (s, i) of the linear part of (3.1) is nonnegative and the total population $0 \leq s(a,t) + i(a,t) = u(a,t)$ satisfies (3.29). Using nonnegativity, we find $s(a,t) \leq u(a,t)$ and $i(a,t) \leq u(a,t)$ and consequently

$$\|\boldsymbol{\mathcal{T}}(t)(s_0, i_0)\|_{\mathbf{X}} \leq N e^{t\lambda^*} \|(s_0, i_0)\|_{\mathbf{X}}$$

for $(s_0, i_0) \in D(\mathbf{A})_+$. However, by Lemma 3.1.6, the above estimate can be extended to $\mathbf{X}_+$ and, by Theorem 2.5.11, to

$$\|\boldsymbol{\mathcal{T}}(t)\|_{\mathbb{B}(\mathbf{X})} \leq N e^{t\lambda^*}. \tag{3.31}$$

Note that the crucial role in the above argument is played by the fact that (s, i) satisfies the differential equation (3.1)–if it was only a mild solution, it would be difficult to directly prove that the sum $s + i$ is the mild solution to (3.29).

3.2 The Nonlinear Problem

For the semilinear initial value problem we are working with, the nonlinear term depends on the form of the infection rate $\Lambda(a, i(\cdot, t))$ which corresponds to the mechanism of transmission of the disease in the population, namely intracohort transmission and intracohort transmission, as mentioned earlier. These are possibly the most common in infectious disease transmission. To make our analysis more general, we shall consider both cases.

Intercohort Transmission

In the case of intercohort transmission, individuals of any age can infect individuals of any age, though with possibly different intensity. Then

$$\Lambda(a, i(\cdot,t)) = \int_0^{\omega} K(a,a^{'})i(a^{'},t)\, da^{'}, \tag{3.32}$$

where $K(a,a^{'})$ is a nonnegative bounded function on $[0,\omega]\times[0,\omega]$ which accounts for the probability of an individual of age a becoming infected through contact with an infected individual of age $a^{'}$.

Consider the assumption

(H4) $0 \le K \in \mathrm{L}^{\infty}(\bar{\mathrm{I}}^2)$

and the following notation

$$\begin{aligned}\overline{\mathbf{B}}(\mathbf{u}_0,\varrho) &= \{\mathbf{r}\in\mathbf{X} : \|\mathbf{r}-\mathbf{u}_0\|_{\mathbf{X}} \le \varrho\}, \text{ for some constant } \varrho,\\ \|\cdot\|_{\infty} &= \|\cdot\|_{\mathrm{L}^{\infty}([0,\omega]^2)},\\ \chi_{\varrho,\mathbf{u}_0} &= 2\|K\|_{\infty}(\varrho + \|\mathbf{u}_0\|_{\mathbf{X}}).\end{aligned}$$

We show that the nonlinear quadratic term $\mathfrak{F}$, defined in (3.7), has the following properties:

Proposition 3.2.1. *$\mathfrak{F}$ is locally Lipschitz continuous on $\mathbf{X}$.*

Proof. Let $\mathbf{u}_0 = (s_0, i_0)^T \in \mathbf{X}$, and $\mathbf{u}_1 = (s_1,i_1)^T$, $\mathbf{u}_2 = (s_2,i_2)^T \in \overline{\mathbf{B}}(\mathbf{u}_0,\varrho)$, then

$$\begin{aligned}\Big\|\mathfrak{F}(\mathbf{u}_1)-\mathfrak{F}(\mathbf{u}_2)\Big\|_{\mathbf{X}} &\le 2\|K\|_{\infty}\Big[\big\|i_1\big\|_{\mathrm{L}^1}\big\|s_1-s_2\big\|_{\mathrm{L}^1} + \big\|s_2\big\|_{\mathrm{L}^1}\big\|i_1-i_2\big\|_{\mathrm{L}^1}\Big]\\ &\le 2\|K\|_{\infty}\Big[\big\|\mathbf{u}_1\big\|_{\mathbf{X}}\big\|s_1-s_2\big\|_{\mathrm{L}^1} + \big\|\mathbf{u}_2\big\|_{\mathbf{X}}\big\|i_1-i_2\big\|_{\mathrm{L}^1}\Big]\\ &\le 2\|K\|_{\infty}\big(\varrho+\big\|\mathbf{u}_0\big\|_{\mathbf{X}}\big)\big\|\mathbf{u}_1-\mathbf{u}_2\big\|_{\mathbf{X}},\end{aligned}$$

as $\big\|\mathbf{u}_i\big\|_{\mathbf{X}} = \big\|\mathbf{u}_i-\mathbf{u}_0+\mathbf{u}_0\big\|_{\mathbf{X}} \le \big\|\mathbf{u}_i-\mathbf{u}_0\big\|_{\mathbf{X}} + \big\|\mathbf{u}_0\big\|_{\mathbf{X}} \le \varrho + \big\|\mathbf{u}_0\big\|_{\mathbf{X}}$.

We write

$$\Big\|\mathfrak{F}(\mathbf{u}_1)-\mathfrak{F}(\mathbf{u}_2)\Big\|_{\mathbf{X}} \le \chi_{\varrho,\mathbf{u}_0}\big\|\mathbf{u}_1-\mathbf{u}_2\big\|_{\mathbf{X}},$$

hence $\mathfrak{F}$ is locally Lipschitz continuous on $\mathbf{X}$. □

Proposition 3.2.2. *$\mathfrak{F}$ is continuously Fréchet differentiable with respect to $\boldsymbol{\phi}\in\mathbf{X}$ and for any $\boldsymbol{\phi} = (\phi^s,\phi^i)^T$, $\boldsymbol{\psi} = (\psi^s,\psi^i)^T \in \mathbf{X}$ the Fréchet derivative at $\boldsymbol{\phi}$, $\mathfrak{F}_{\boldsymbol{\phi}}$, is given by*

$$\Big(\mathfrak{F}_{\boldsymbol{\phi}}\boldsymbol{\psi}\Big)(a) := \begin{bmatrix} -\psi^s(a)\int_0^{\omega}K(a,a^{'})\phi^i(a^{'})\,da^{'} - \phi^s(a)\int_0^{\omega}K(a,a^{'})\psi^i(a^{'})\,da^{'} \\ \\ \psi^s(a)\int_0^{\omega}K(a,a^{'})\phi^i(a^{'})\,da^{'} + \phi^s(a)\int_0^{\omega}K(a,a^{'})\psi^i(a^{'})\,da^{'}\end{bmatrix}. \tag{3.33}$$

Proof. Let $\boldsymbol{\phi}$, $\boldsymbol{\psi} \in \mathbf{X}$, then

$$\mathfrak{F}\left(\boldsymbol{\phi}+\boldsymbol{\psi}\right)(a) = \mathfrak{F}\left(\boldsymbol{\phi}\right)(a) + \mathfrak{F}_{\boldsymbol{\phi}}\left(\boldsymbol{\psi}\right)(a) + \mathbf{G}\left(\boldsymbol{\psi},\boldsymbol{\psi}\right)(a),$$

where

$$\mathbf{G}\left(\boldsymbol{\psi},\boldsymbol{\psi}\right)(a) = \begin{bmatrix} -\psi^s(a)\int_0^\omega K(a,a^{'})\psi^i(a^{'})\,da^{'} \\ \\ \psi^s(a)\int_0^\omega K(a,a^{'})\psi^i(a^{'})\,da^{'} \end{bmatrix},$$

and so

$$\begin{aligned} \Big\|\mathbf{G}\left(\boldsymbol{\psi},\boldsymbol{\psi}\right)\Big\|_{\mathbf{X}} &= \int_0^\omega \Big| -\psi^s(a)\int_0^\omega K(a,a^{'})\psi^i(a^{'})\,da^{'}\Big|\,da, \\ &\quad + \int_0^\omega \Big|\psi^s(a)\int_0^\omega K(a,a^{'})\psi^i(a^{'})\,da^{'}\Big|\,da, \\ &\le 2\|K\|_\infty\|\psi^i\|_1\|\psi^s\|_1. \end{aligned}$$

Thus we obtain

$$\|\mathbf{G}\left(\boldsymbol{\psi},\boldsymbol{\psi}\right)\|_{\mathbf{X}} \le \|K\|_\infty\|\boldsymbol{\psi}\|^2_{\mathbf{X}}.$$

It follows that

$$\frac{\|\mathbf{G}\left(\boldsymbol{\psi},\boldsymbol{\psi}\right)\|_{\mathbf{X}}}{\|\boldsymbol{\psi}\|_{\mathbf{X}}} \le \|K\|_\infty\|\boldsymbol{\psi}\|_{\mathbf{X}} \to 0 \text{ as } \|\boldsymbol{\psi}\|_{\mathbf{X}} \to 0.$$

This shows that $\mathfrak{F}$ is Fréchet differentiable at each $\boldsymbol{\phi} = (\phi^s,\phi^i)^T \in \mathbf{X}$ and its Fréchet derivative $\mathfrak{F}_{\boldsymbol{\phi}}$ at $\boldsymbol{\phi}$ is given by (3.33) and satisfies

$$\|\mathfrak{F}_{\boldsymbol{\phi}}\boldsymbol{\psi}\|_{\mathbf{X}} \le \chi_{\varrho,\mathbf{u}_0}\|\boldsymbol{\psi}\|_{\mathbf{X}}\,, \quad \forall\boldsymbol{\psi}\in\mathbf{X},\ \boldsymbol{\phi}\in\overline{\mathbf{B}}(\mathbf{u}_0,\varrho).$$

Hence,

$$\mathfrak{F}_{\boldsymbol{\phi}} \in \mathbb{B}(\mathbf{X}) \text{ and } \|\mathfrak{F}_{\boldsymbol{\phi}}\|_{\mathbb{B}(\mathbf{X})} \le \chi_{\varrho,\mathbf{u}_0}.$$

Moreover, for $\boldsymbol{\phi}_1$, $\boldsymbol{\phi}_2$, $\boldsymbol{\psi} \in \mathbf{X}$, we have

$$\begin{aligned} \Big\|\mathfrak{F}_{\boldsymbol{\phi}_1}\boldsymbol{\psi} - \mathfrak{F}_{\boldsymbol{\phi}_2}\boldsymbol{\psi}\Big\|_{\mathbf{X}} &= \int_0^\omega \Big| -\boldsymbol{\psi}^s(a)\int_0^\omega K(a,a^{'})\left[\phi_1^i(a^{'})-\phi_2^i(a^{'})\right]\,da^{'} \\ &\qquad - \left[\phi_1^s(a)-\phi_2^s(a)\right]\int_0^\omega K(a,a^{'})\psi^i(a^{'})\,da^{'}\Big|\,da \\ &\qquad + \int_0^\omega \Big|\psi^s(a)\int_0^\omega K(a,a^{'})\left[\phi_1^i(a^{'})-\phi_2^i(a^{'})\right]\,da^{'} \\ &\qquad + \left[\phi_1^s(a)-\phi_2^s(a)\right]\int_0^\omega K(a,a^{'})\psi^i(a^{'})\,da^{'}\Big|\,da \\ &\le 2\|K\|_\infty\Big(\|\phi_1^i-\phi_2^i\|_1\|\psi^s\|_1 + \|\phi_1^s-\phi_2^s\|_1\|\psi^i\|_1\Big) \\ &\le 2\|K\|_\infty\|\boldsymbol{\psi}\|_{\mathbf{X}}\|\boldsymbol{\phi}_1-\boldsymbol{\phi}_2\|_{\mathbf{X}}. \end{aligned}$$

Taking supremum over $\{\boldsymbol{\psi} \in \mathbf{X} : \|\boldsymbol{\psi}\|_{\mathbf{X}} = 1\}$, we get

$$\left\|\mathfrak{F}_{\boldsymbol{\phi}_1} - \mathfrak{F}_{\boldsymbol{\phi}_2}\right\|_{\mathbb{B}(\mathbf{X})} \leq 2\|K\|_\infty \|\boldsymbol{\phi}_1 - \boldsymbol{\phi}_2\|_{\mathbf{X}}$$

and we see that

$$\left\|\mathfrak{F}_{\boldsymbol{\phi}_1} - \mathfrak{F}_{\boldsymbol{\phi}_2}\right\|_{\mathbb{B}(\mathbf{X})} \to 0 \text{ as } \|\boldsymbol{\phi}_1 - \boldsymbol{\phi}_2\|_{\mathbf{X}} \to 0.$$

Hence, the Fréchet derivative $\mathfrak{F}_{\boldsymbol{\phi}}$ is uniformly continuous with respect to $\boldsymbol{\phi}$. □

Remark 3.2.3. From (3.33), we have

$$\Big(\big[\mathfrak{F}_{\boldsymbol{\phi}\boldsymbol{\phi}}\boldsymbol{\psi}\big]\boldsymbol{\delta}\Big)(a) := \begin{bmatrix} -\psi^s(a)\int_0^\omega K(a,a')\delta^i(a')\,da' - \delta^s(a)\int_0^\omega K(a,a')\psi^i(a')\,da' \\ \\ \psi^s(a)\int_0^\omega K(a,a')\delta^i(a')\,da' + \delta^s(a)\int_0^\omega K(a,a')\psi^i(a')\,da' \end{bmatrix}$$

for any $\boldsymbol{\delta} = (\delta^s, \delta^i)^T \in \mathbf{X}$. This shows that $\big[\mathfrak{F}_{\boldsymbol{\phi}\boldsymbol{\phi}}\boldsymbol{\psi}\big]\boldsymbol{\delta}$ is constant, since it does not depend on $\boldsymbol{\phi}$; hence, the higher order Fréchet derivatives are zero.

3.2.1 The Local Existence, Uniqueness, and Positivity of Solution

The above results, together with Theorem 2.11.5, enable us to state that for each $\mathbf{u}_0 = (s_0, i_0)^T \in \mathbf{X}$, there is a $t(\mathbf{u}_0)$ such that the problem (3.1) has a unique mild solution on $[0, t(\mathbf{u}_0)) \ni t \to \mathbf{u}(t)$, and Theorem 2.11.4 to ensure that this solution is a classical solution if $\mathbf{u}_0 \in D(\mathbf{A})$.

We recall that the proof consists in showing that the Picard iterates

$$\begin{aligned} \mathbf{u}^0 &= \mathbf{u}_0, \\ \mathbf{u}^n(t) &= \boldsymbol{\mathcal{T}}(t)\mathbf{u}_0 + \int_0^t \boldsymbol{\mathcal{T}}(t-s)\mathfrak{F}(\mathbf{u}^{n-1}(s))\,ds \end{aligned} \tag{3.34}$$

converge in $\mathcal{C}\big([0, t(\mathbf{u}_0)), \overline{\mathbf{B}}(\mathbf{u}_0, \rho)\big)$.

Since we proved that the nonlinear term $\mathfrak{F}$ is only locally Lipschitz continuous, the question whether the solution can be extended to $[0, \infty)$ requires employing positivity techniques.

Since $\mathfrak{F}$ is not positive on $\mathbf{X}_+$, we cannot claim that the constructed local solution is non-negative, as the iterates defined by (3.34) need not to be positive, even if we start with $\mathbf{u}_0 \geq 0$. Hence, we re-write (3.1) in the following equivalent form

$$\left\{ \begin{array}{rcl} \dfrac{d\mathbf{u}}{dt} &=& (\boldsymbol{\mathcal{Q}} - \kappa\mathbf{I})\,\mathbf{u} + (\kappa\mathbf{I} + \mathfrak{F})\,(\mathbf{u}), \quad t > 0, \\ \mathbf{u}(0) &=& \mathbf{u}_0, \end{array} \right. \tag{3.35}$$

for some $\kappa \in \mathbb{R}_+$ to be determined later. Denote $\boldsymbol{\mathcal{Q}}_\kappa = \boldsymbol{\mathcal{Q}} - \kappa\mathbf{I}$; then $\big(\boldsymbol{\mathcal{T}}_\kappa(t)\big)_{t\geq 0} = \big(e^{-\kappa t}\boldsymbol{\mathcal{T}}(t)\big)_{t\geq 0}$ is the semigroup generated by $\boldsymbol{\mathcal{Q}}_\kappa$ and hence $\big(\boldsymbol{\mathcal{T}}_\kappa(t)\big)_{t\geq 0}$ is positive.

Thus the mild solution satisfies

$$\mathbf{u}(t) = e^{-\kappa t}\boldsymbol{\mathcal{T}}(t)\mathbf{u}_0 + \int_0^t e^{-\kappa(t-s)}\boldsymbol{\mathcal{T}}(t-s)\big(\kappa\mathbf{I} + \mathfrak{F}\big)\mathbf{u}(s)\,ds, \quad 0 \le t < t(\mathbf{u}_0). \tag{3.36}$$

The following result holds.

Lemma 3.2.4. *For any ϱ there exists κ such that $\big(\kappa\mathbf{I} + \mathfrak{F}\big)(\mathbf{X}_+ \cap \overline{\mathbf{B}}(\mathbf{u}_0, \varrho)) \subset \mathbf{X}_+$.*

Proof. Taking $\mathbf{u} = (s,i)^T \in \mathbf{X}_+ \cap \overline{\mathbf{B}}(\mathbf{u}_0, \varrho)$, we have

$$\left(\kappa\begin{bmatrix} s \\ i \end{bmatrix} + \mathfrak{F}\begin{bmatrix} s \\ i \end{bmatrix}\right)(a,t) = \begin{bmatrix} \Big(\kappa - \int_0^\omega K(a,a^{'})i(a^{'},t)da^{'}\Big)s(a,t) \\ \\ \kappa i(a,t) + s(a,t)\int_0^\omega K(a,a^{'})i(a^{'},t)da^{'} \end{bmatrix}$$

and so it is easy to see that

$$\int_0^\omega K(a,a^{'})i(a^{'},t)da^{'} \le \big\|K\big\|_\infty \big\|i\,(\cdot,t)\,\big\|_1 \le \frac{1}{2}\chi_{\varrho,\mathbf{u}_0}.$$

Therefore,

$$\kappa - \int_0^\omega K(a,a^{'})i(a^{'},t)da^{'} \ge \kappa - \frac{1}{2}\chi_{\varrho,\mathbf{u}_0} \ge 0$$

provided $\kappa \ge \chi_{\varrho,\mathbf{u}_0}$. Hence $\big(\kappa\mathbf{I} + \mathfrak{F}\big)\mathbf{u} \ge 0$ if $\mathbf{u} \ge 0$. □

The following result holds.

Corollary 3.2.5. *Assume that $\mathbf{u}_0 \in \mathbf{X}_+$ and let $\mathbf{u} : [0, t_{\max}) \to \mathbf{X}$ be the unique mild solution of (3.6). Then this solution is nonnegative on the maximal interval of its existence.*

Proof. From the positivity of the semigroup $\big(\boldsymbol{\mathcal{T}}(t)\big)_{t\ge 0}$ and Lemma 3.2.4, the Picard iterates for (3.36) are nonnegative, provided $\mathbf{u}_0 \in \mathbf{X}_+$. Thus the mild solution is nonnegative whenever it exists. □

3.2.2 Global Existence

We need to show that the solutions to (3.6) are global in time. Note that, even for ordinary differential equations, the solution with quadratic nonlinearity can blow up in finite time. Here we use positivity to show that positive solutions exist globally in time. For this, we have to show that $t \to \|\mathbf{u}(t)\|_{\mathbf{X}}$ does not blow up in finite time. We state the following result:

Theorem 3.2.6. *For any $\mathbf{u}_0 \in D(\mathbf{A}) \cap \mathbf{X}_+$, the problem (3.6) has a unique classical positive solution $\mathbf{u}(t)$ defined on the whole time interval $[0,\infty)$.*

Proof. The proof follows the same idea as in Remark 3.1.7. Under the assumptions, we have a locally defined positive strict solution $\mathbf{u}(t) = \big(s(t), i(t)\big)^T$ to (3.6), and hence of (3.1), in $\mathbf{X}$. Thus, we obtain

$$\begin{aligned}\big\|\mathbf{u}(t)\big\|_{\mathbf{X}} &= \|s(t)\|_1 + \|i(t)\|_1 \\ &= \int_0^\omega |s(a,t)|\,da + \int_0^\omega |i(a,t)|\,da \\ &= \int_0^\omega \left(s(a,t) + i(a,t)\right)\,da = \int_0^\omega u(a,t)\,da,\end{aligned}$$

where $u(t)$ is the solution to the McKendrick equation (3.29). Using the definition of the L^1-norm and the positivity of the solution $u(t)$, we have

$$\int_0^\omega u(a,t)\,da \leq Ne^{\lambda^* t}\int_0^\omega u_0(a)\,da$$

and hence

$$\|\mathbf{u}(t)\|_{\mathbf{X}} \leq Ne^{\lambda^* t}\|\mathbf{u}_0\|_{\mathbf{X}} \tag{3.37}$$

as long as $t \in [0,\infty)$. Accordingly, $\|\mathbf{u}(t)\|_{\mathbf{X}}$ does not blow up in finite time and hence the solution is global. □

Corollary 3.2.7. *For any $\mathbf{u}_0 \in \mathbf{X}_+$, the problem (3.6) has a unique mild positive solution $\mathbf{u}(t)$ defined on the whole time interval $[0,\infty)$.*

Proof. If we take $\mathbf{u}_0 \in \mathbf{X}_+$, then we have a mild solution $\mathbf{u}(t,\mathbf{u}_0)$. We suppose that $\mathbf{u}(t,\mathbf{u}_0)$ is only defined on $[0,t_{\max})$ with $t_{\max} < \infty$, that is, $\lim\limits_{t\to t_{\max}^-} \|\mathbf{u}(t,\mathbf{u}_0)\|_{\mathbf{X}} \to \infty$. Further, from Lemma 3.1.6, there is a sequence $(\mathbf{v}_n) \in D(\mathbf{A})_+$ such that $\lim\limits_{n\to\infty} \mathbf{v}_n = \mathbf{u}_0$. Since $\mathbf{v}_n \in D(\mathbf{A})_+$, it follows that $\mathbf{u}(t,\mathbf{v}_n)$ are classical solutions to (3.6), defined on $[0,\infty)$. Hence, there is M_3 such that $\|\mathbf{u}(t,\mathbf{v}_n)\|_{\mathbf{X}} \leq N\max\left\{1, e^{\lambda^* t_{\max}}\right\}\|\mathbf{v}_n\|_{\mathbf{X}} \leq M_3$ for $t \in [0,t_{\max}]$, see proof of Theorem 3.2.6. We can find $T < t_{\max}$ such that $\|\mathbf{u}(T,\mathbf{u}_0)\|_{\mathbf{X}} \geq 2M_3$ and in particular, $\|\mathbf{u}(T,\mathbf{v}_n)\|_{\mathbf{X}} \leq M_3$ holds.

Hence

$$\begin{aligned}\|\mathbf{u}(T,\mathbf{u}_0) - \mathbf{u}(T,\mathbf{v}_n)\|_{\mathbf{X}} &\geq \big|\|\mathbf{u}(T,\mathbf{u}_0)\|_{\mathbf{X}} - \|\mathbf{u}(T,\mathbf{v}_n)\|_{\mathbf{X}}\big| \\ &= \|\mathbf{u}(T,\mathbf{u}_0)\|_{\mathbf{X}} - \|\mathbf{u}(T,\mathbf{v}_n)\|_{\mathbf{X}} \geq M_3.\end{aligned} \tag{3.38}$$

Let $M_1 := \sup\limits_{[0,T]}\|\mathbf{u}(t,\mathbf{u}_0)\|_{\mathbf{X}} \geq 2M_3$. The next part follows from the continuous dependence of solutions on initial conditions. We shall recall the proof for the case at hand, as it is simpler than in the general case.

Denote

$$\mathbf{w}_n(t) := \mathbf{u}(t, \mathbf{v}_n) - \mathbf{u}(t, \mathbf{u}_0), \quad \overset{o}{\mathbf{w}}_n := \mathbf{v}_n - \mathbf{u}_0. \tag{3.39}$$

Then the equation

$$\mathbf{w}_n(t) = \boldsymbol{\mathcal{T}}(t)\overset{o}{\mathbf{w}}_n + \int_0^t \boldsymbol{\mathcal{T}}(t-s)\Big(\mathfrak{F}\big(\mathbf{u}(s,\mathbf{v}_n)\big) - \mathfrak{F}\big(\mathbf{u}(s,\mathbf{u}_0)\big)\Big)\, ds$$

is satisfied. Now, we consider the ball $\mathbf{B}(0, M_1)$ and the Lipschitz constant $l(M_1) := l(0, M_1)$, see (2.3). We note that $l(M_1)$ is independent of $\mathbf{v}_n$ and $\mathbf{u}(t, \mathbf{u}_0)$, $\mathbf{u}(t, \mathbf{v}_n) \in \mathbf{B}(0, M_1)$ for $t \in [0, T]$. Hence, after taking the norm in $\mathbf{X}$ and using the estimate (3.31), we get

$$\begin{aligned}\|\mathbf{w}_n(t)\|_{\mathbf{X}} &\leq Ne^{\lambda^* t}\|\overset{o}{\mathbf{w}}_n\|_{\mathbf{X}} + N\int_0^t e^{\lambda^*(t-s)}\left\|\mathfrak{F}\big(\mathbf{u}(s,\mathbf{v}_n)\big) - \mathfrak{F}\big(\mathbf{u}(s,\mathbf{u}_0)\big)\right\|_{\mathbf{X}} ds \\ &\leq Ne^{\lambda^* t}\|\overset{o}{\mathbf{w}}_n\|_{\mathbf{X}} + l(M_1)Ne^{\lambda^* t}\int_0^t e^{-\lambda^* s}\|\mathbf{w}_n(s)\|_{\mathbf{X}}\, ds.\end{aligned}$$

This implies

$$e^{-\lambda^* t}\|\mathbf{w}_n(t)\|_{\mathbf{X}} \leq N\|\overset{o}{\mathbf{w}}_n\|_{\mathbf{X}} + l(M_1)N\int_0^t e^{-\lambda^* s}\|\mathbf{w}_n(s)\|_{\mathbf{X}}\, ds$$

and, by Gronwall's lemma, we obtain

$$\|\mathbf{w}_n(t)\|_{\mathbf{X}} \leq N\max\left\{1, e^{(\lambda^*+l(M_1)N)T}\right\}\|\overset{o}{\mathbf{w}}_n\|_{\mathbf{X}} = M_J\|\overset{o}{\mathbf{w}}_n\|_{\mathbf{X}}, \tag{3.40}$$

where $M_J = Ne^{(\lambda^*+l(M_1)N)T}$ is independent of $\mathbf{v}_n$. Thus, by (3.39),

$$\lim_{n\to\infty}\|\mathbf{u}(t,\mathbf{v}_n) - \mathbf{u}(t,\mathbf{u}_0)\|_{\mathbf{X}} \to 0, \quad \text{as } \lim_{n\to\infty}\|\mathbf{v}_n - \mathbf{u}_0\|_{\mathbf{X}} \to 0$$

for $t \in [0, T]$. This shows that the local solutions of (3.6) depend continuously on the initial conditions. Hence, we can take n such that

$$\|\mathbf{u}(t,\mathbf{v}_n) - \mathbf{u}(t,\mathbf{u}_0)\|_{\mathbf{X}} \leq \frac{M_3}{2}$$

for $t \in [0, T]$. This contradicts (3.38). □

3.2.3 Further Regularity of the Solution

In this subsection $\mathcal{C}(\bar{\mathrm{I}})$ represents the space of continuous vector-valued functions on $\bar{\mathrm{I}}$. First, we consider the following result:

Lemma 3.2.8. *Let $\mathbf{u} \in \mathrm{W}^{1,p}(\mathrm{I})$ with $1 \leq p \leq \infty$, and I bounded or unbounded; then there exists a function $\tilde{\mathbf{u}} \in \mathcal{C}(\bar{\mathrm{I}})$ such that*

$$\mathbf{u} = \tilde{\mathbf{u}} \ \ a.e. \text{ on } \bar{\mathrm{I}}, \quad \text{and} \quad \tilde{\mathbf{u}}(b) - \tilde{\mathbf{u}}(a) = \int_a^b \mathbf{u}'(s)\, ds \quad \forall\, a, b \in \bar{\mathrm{I}}$$

where $\mathbf{u}'$ is the generalized derivative of $\mathbf{u}$.

Proof. Cf. proof of Theorem 8.2 in [14, p. 204]. □

Since $D(\mathbf{A}) \subset \big(\mathrm{W}^{1,1}(\bar{\mathrm{I}})\big)^2$, from Lemma 3.2.8 we have $D(\mathbf{A}) \subset \big(\mathrm{W}^{1,1}(\bar{\mathrm{I}})\big)^2(\bar{\mathrm{I}}) \subset \mathcal{C}(\bar{\mathrm{I}})$; hence

$$\mathcal{C}\big([0,T]; D(\mathbf{A})\big) \subset \mathcal{C}\big([0,T]; \mathcal{C}(\bar{\mathrm{I}})\big).$$

Let us consider the following abstract form of (3.6):

$$\begin{cases} \dfrac{d\mathbf{u}}{dt} &= \mathfrak{Q}\mathbf{u} + \mathfrak{F}(\mathbf{u}), \quad t > 0, \\ \mathbf{u}(0) &= \mathbf{u}_0. \end{cases} \tag{3.41}$$

The mild solution of (3.41) satisfies the integral equation

$$\mathbf{u}(t) = \boldsymbol{\mathcal{T}}(t)\mathbf{u}_0 + \int_0^t \boldsymbol{\mathcal{T}}(t-s)\mathfrak{F}(\mathbf{u}(s))\, ds \quad (0 \le t < t(\mathbf{u}_0)). \tag{3.42}$$

During the last two decades, several numerical methods have been proposed for the approximation of solutions of the age-structured demographic models, for instance the finite difference method and the finite element method. In particular, see [32], the finite difference approach requires the solution to be at least three times differentiable. Here we shall show the semigroup type conditions which ensure such a regularity of the solution to (3.1). In the sequel we assume that the following holds:

(H5) $0 \le \mathbf{u}_0 \in \mathcal{C}^3\left([0,\omega]\right)$, β, $\delta \in \mathcal{C}^2([0,\omega])$ and $\mu \in \mathcal{C}^2([0,\omega))$.

Denote $\mathbf{M} = \mathbf{M}_\mu + \mathbf{M}_\delta$. Similarly to the result in [32, Theorem 2.2, p. 164], we state the result below.

Theorem 3.2.9. (Regularity) *Let* $\mathbf{u}_0 \in D(\mathbf{A}) \cap \mathbf{X}^+$ *and the assumptions* (H1)-(H5) *hold. If the compatibility conditions*

$$\begin{aligned} \mathbf{u}_0'(0) &= \Big(\mathbf{M}(0) - \mathbf{B}(0)\Big)\mathbf{u}_0(0) + \mathfrak{F}\left(\mathbf{u}_0(0)\right) - \int_0^\omega \Big(\mathbf{B}'(a) + \mathbf{B}(a)\mathbf{M}(a)\Big)\mathbf{u}_0(a)\, da \\ &\quad - \int_0^\omega \mathbf{B}(a)\mathfrak{F}\left(\mathbf{u}_0(a)\right)\, da, \end{aligned} \tag{3.43}$$

$$\begin{aligned} \mathbf{u}_0''(0) &= \Big(2\mathbf{M}(0) - \mathbf{B}(0)\Big)\mathbf{u}_0'(0) + \Big(\mathbf{M}'(0) - \mathbf{M}^2(0) + \mathbf{B}(0)\mathbf{M}(0)\Big)\mathbf{u}_0(0) - \mathbf{M}(0)\mathfrak{F}\left(\mathbf{u}_0(0)\right) \\ &\quad + \int_0^\omega \Big(\mathbf{B}''(a) + 2\mathbf{B}'(a)\mathbf{M}(a) + \mathbf{B}(a)\mathbf{M}'(a) + \mathbf{B}(a)\mathbf{M}^2(a)\Big)\mathbf{u}_0(a)\, da \\ &\qquad + \int_0^\omega \mathbf{B}(a)\mathbf{M}(a)\mathfrak{F}\left(\mathbf{u}_0(a)\right)\, da, \end{aligned} \tag{3.44}$$

$$
\begin{aligned}
\mathbf{u}_0'''(0) = &\Big(3\mathbf{M}(0) - 3\mathbf{B}'(0)\mathbf{M}(0) - \mathbf{B}(0)\Big)\mathbf{u}_0''(0) + \Big(\mathbf{B}'(0) - 3\mathbf{M}^2(0) + 3\mathbf{M}'(0) \\
&+ 3\mathbf{B}(0)\mathbf{M}(0)\Big)\mathbf{u}_0'(0) - \Big(\mathbf{B}''(0) + 3\mathbf{B}(0)\mathbf{M}^2(0) - \mathbf{M}^3(0) + \mathbf{M}'(0)\mathbf{M}(0) \\
&+ 3\mathbf{M}(0)\mathbf{M}'(0)\Big)\mathbf{u}_0(0) + \Big(\mathbf{B}'(0) - \mathbf{B}(0)\mathbf{M}(0) + \mathbf{M}^2(0)\Big)\mathfrak{F}\left(\mathbf{u}_0(0)\right) \\
&- \mathbf{B}(0)\frac{d\mathfrak{F}}{da}\left(\mathbf{u}_0(a)\right)\Big|_{a=0} - \frac{d}{da}\big(\mathbf{M}(a)\mathfrak{F}\left(\mathbf{u}_0(a)\right)\big)\Big|_{a=0} + \frac{d^2\mathfrak{F}}{da^2}\left(\mathbf{u}_0(a)\right)\Big|_{a=0} \\
&- \int_0^\omega \Big(3\frac{d}{da}\big(\mathbf{B}(a)\mathbf{M}(a)\big) + 3\frac{d}{da}\big(\mathbf{B}(a)\mathbf{M}^2(a)\big) + \mathbf{B}(a)\mathbf{M}''(a) + \mathbf{B}(a)\mathbf{M}^3(a) \\
&-\mathbf{B}(a)\mathbf{M}'(a)\mathbf{M}(a) - 3\mathbf{B}(a)\mathbf{M}(a)\mathbf{M}'(a)\Big)\,\mathbf{u}_0(a)\,da \\
&+ \int_0^\omega \Big(\mathbf{B}''(a) - \mathbf{B}(a)\mathbf{M}^2(a) - \mathbf{B}'(a)\mathbf{M}(a)\Big)\,\mathfrak{F}\left(\mathbf{u}_0(a)\right)\,da
\end{aligned} \tag{3.45}
$$

are satisfied, then the solution $\mathbf{u} = (s, i)^T$ *of* (3.6a)-(3.6c) *belongs to* $\mathcal{C}^3\left([0,\omega]\times[0,T]\right)$ *for some constant* $0 < T < \infty$.

Proof. First, we want to prove that if we take $\mathbf{u}_0 \in D(\mathbf{A})$ then the solution $\mathbf{u}$ to (3.6) is continuous on $[0,\omega]\times[0,T]$.

We claimed earlier that if we take $\mathbf{u}_0 \in D(\mathbf{A})$, the problem (3.6), and thus ACP (3.41), has a unique classical solution $\mathbf{u}$ given by (3.42), see Theorem 2.11.4. Clearly we have $\mathbf{u} \in \mathcal{C}\big([0,T]; D(\mathbf{A})\big)$, see Definition 2.11.1. Therefore, from Lemma 3.2.8, we get $\mathbf{u} \in \mathcal{C}\big([0,T]; \mathcal{C}\big([0,\omega]\big)\big)$; that is, $\forall\, t > 0\ \forall\, \epsilon > 0\ \exists\, \eta_1 = \eta_1(t,\epsilon) > 0$ such that if $|h_1| < \eta_1$ then

$$\sup_{0\le a\le\omega} \|\mathbf{u}(a, t+h_2) - \mathbf{u}(a,t)\| < \frac{\epsilon}{2} \tag{3.46}$$

Moreover, for each t, $\mathbf{u}$ is continuous in a; hence there exists, $\eta_2 = \eta_2(a,t,\epsilon) > 0$, for given t and a, such that if $|h_2| < \eta_2$, then

$$\|\mathbf{u}(a+h_1, t) - \mathbf{u}(a,t)\| < \frac{\epsilon}{2}. \tag{3.47}$$

Now, we are ready to prove that $\mathbf{u} \in \mathcal{C}\big([0,T]\times[0,\omega]\big)$; it is sufficient to show that for arbitrary a, t and $\epsilon > 0$, we can find $\eta = \eta(a,t,\epsilon) > 0$ such that if $|h_1|$, $|h_2| < \eta$, then

$$\|\mathbf{u}(a+h_1, t+h_2) - \mathbf{u}(a,t)\| < \epsilon.$$

We have

$$
\begin{aligned}
\|\mathbf{u}(a+h_1, t+h_2) - \mathbf{u}(a,t)\| &\le \|\mathbf{u}(a+h_1, t+h_2) - \mathbf{u}(a+h_1,t)\| + \|\mathbf{u}(a+h_1, t) - \mathbf{u}(a,t)\| \\
&\le \sup_{0\le a\le\omega} \|\mathbf{u}(a, t+h_2) - \mathbf{u}(a,t)\| + \|\mathbf{u}(a+h_1, t) - \mathbf{u}(a,t)\|.
\end{aligned}
$$

Taking $\eta = \min\{\eta_1, \eta_2\}$, from (3.46) and (3.47) we get

$$\|\mathbf{u}(a+h_1, t+h_2) - \mathbf{u}(a,t)\| < \epsilon$$

for $|h_1|$, $|h_2| < \eta$. Hence

$$\mathbf{u} \in \mathcal{C}\big([0,\omega]\times[0,T]\big).$$

Next, since $\mathbf{u}$ is a classical solution, $\mathbf{u}^{'}$ exists and

$$\frac{d\mathfrak{F}}{dt}(\mathbf{u}) = \mathfrak{F}_{\mathbf{u}}(\mathbf{u})\mathbf{u}^{'}.$$

Hence, all the terms on the right hand side in (3.42) are differentiable with respect to t. Thus $\mathbf{u}^{'}(t)$ satisfies the equation, see [42],

$$\mathbf{u}^{'}(t) = \boldsymbol{\mathcal{T}}(t)\big(\boldsymbol{\mathcal{Q}}\mathbf{u}_0 + \mathfrak{F}(\mathbf{u}_0)\big) + \int_0^t \boldsymbol{\mathcal{T}}(t-s)\mathfrak{F}_{\mathbf{u}}\big(\mathbf{u}(s)\big)\mathbf{u}^{'}(s)\,ds.$$

Denote $\mathbf{v} = \mathbf{u}^{'}$; then write

$$\mathbf{v}(t) = \boldsymbol{\mathcal{T}}(t)\mathbf{v}_0 + \int_0^t \boldsymbol{\mathcal{T}}(t-s)\mathfrak{F}_{\mathbf{u}}\big(\mathbf{u}(s)\big)\mathbf{v}(s)\,ds, \tag{3.48}$$

where $\mathbf{v}_0 = \boldsymbol{\mathcal{Q}}\mathbf{u}_0 + \mathfrak{F}(\mathbf{u}_0)$ satisfies $\mathbf{v}_0 \in D(\mathbf{A})$ by (3.43). Hence Remark 3.2.3 allows for applying Theorem 2.11.4 to claim that the abstract initial value problem

$$\begin{cases} \mathbf{v}_t &= \boldsymbol{\mathcal{Q}}\mathbf{v} + \mathfrak{F}_{\mathbf{u}}(\mathbf{u})\mathbf{v}, \qquad t>0,\\ \mathbf{v}(0) &= \mathbf{v}_0,\\ \mathbf{v} &\in D(\mathbf{A}) \end{cases} \tag{3.49}$$

has a unique classical solution given by (3.48). Thus $\mathbf{v} \in \mathcal{C}\big([0,T]; D(\mathbf{A})\big)$, i.e., $\mathbf{u}^{'} \in \mathcal{C}\big([0,T]; D(\mathbf{A})\big) \subset \mathcal{C}\big([0,T]; \mathcal{C}([0,\omega])\big)$. So, as in the first part, $\dfrac{\partial \mathbf{u}}{\partial t}$ exists and is continuous. Then, the continuity of $\dfrac{\partial \mathbf{u}}{\partial a} = -\dfrac{\partial \mathbf{u}}{\partial t} - \mathbf{M}\mathbf{u} + \mathfrak{F}\big(\mathbf{u}\big)$ is guaranteed by the fact that all the terms on the right hand side are continuous, where $\mathbf{M}$ is continuous by assumption (H5). This shows that

$$\mathbf{u} \in \mathcal{C}^1\big([0,\omega]\times[0,T]\big).$$

Applying the same argument as above to $\mathbf{v}$ shows that $\mathbf{v}^{'}$ exists and satisfies

$$\mathbf{v}^{'}(t) = \boldsymbol{\mathcal{T}}(t)\Big(\boldsymbol{\mathcal{Q}}^2\boldsymbol{\varphi} + \boldsymbol{\mathcal{Q}}\mathfrak{F}\left(\boldsymbol{\varphi}\right) + \mathfrak{F}_{\mathbf{u}}(\mathbf{u}_0)\mathbf{u}_0^{'}\Big) + \int_0^t \boldsymbol{\mathcal{T}}(t-s)\mathfrak{F}_{\mathbf{u}\mathbf{u}}\big(\mathbf{u}(s)\big)\mathbf{v}^2(s)\,ds$$
$$+ \int_0^t \boldsymbol{\mathcal{T}}(t-s)\mathfrak{F}_{\mathbf{u}}\big(\mathbf{u}(s)\big)\mathbf{v}^{'}(s)\,ds.$$

Denote $\mathbf{w} = \mathbf{v}'$; then write

$$\mathbf{w}(t) = \boldsymbol{\mathcal{T}}(t)\mathbf{w}_0 + \int_0^t \boldsymbol{\mathcal{T}}(t-s)\mathfrak{F}_{\mathbf{uu}}\big(\mathbf{u}(s)\big)\mathbf{v}^2(s)\,ds + \int_0^t \boldsymbol{\mathcal{T}}(t-s)\mathfrak{F}_{\mathbf{u}}\big(\mathbf{u}(s)\big)\mathbf{w}(s)\,ds, \quad (3.50)$$

where $\mathbf{w}_0 = \boldsymbol{\mathcal{Q}}^2\mathbf{u}_0 + \boldsymbol{\mathcal{Q}}\mathfrak{F}(\mathbf{u}_0) + \mathfrak{F}_{\mathbf{u}}(\mathbf{u}_0)\mathbf{u}_0'$ satisfies $\mathbf{w}_0 \in D(\mathbf{A})$ by (3.44). Since $\mathfrak{F}_{\mathbf{uu}}(\mathbf{u})$ is continuously Fréchet differentiable by Remark 3.2.3, by Theorem 2.11.4, the abstract initial value problem

$$\begin{cases} \mathbf{w}_t &= \boldsymbol{\mathcal{Q}}\mathbf{w} + \mathfrak{F}_{\mathbf{u}}(\mathbf{u})\mathbf{w} + \mathfrak{F}_{\mathbf{uu}}(\mathbf{u})\mathbf{v}^2, \quad t > 0, \\ \mathbf{w}(0) &= \mathbf{w}_0, \\ \mathbf{w} &\in D(\mathbf{A}) \end{cases}$$

has a unique classical solution given by (3.50). Thus $\mathbf{w} \in \mathcal{C}\big([0,T]; D(\mathbf{A})\big)$, i.e., $\mathbf{u}'' \in \mathcal{C}\big([0,T]; D(\mathbf{A})\big) \subset \mathcal{C}\big([0,T], \mathcal{C}([0,\omega])\big)$. So, as in the first and second parts, $\dfrac{\partial^2\mathbf{u}}{\partial t^2}$ exists and is continuous. The continuity of $\dfrac{\partial \mathbf{u}^2}{\partial t\partial a}$, $\dfrac{\partial \mathbf{u}^2}{\partial a\partial t}$ and $\dfrac{\partial \mathbf{u}^2}{\partial a^2}$ results from the fact that:

(i) as the classical solution $\mathbf{u}$ is given by the ACP (3.49), we write

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathbf{u}}{\partial t}\right) = -\frac{\partial}{\partial a}\left(\frac{\partial \mathbf{u}}{\partial t}\right) + \mathbf{M}\frac{\partial \mathbf{u}}{\partial t} + \mathfrak{F}_{\mathbf{u}}(\mathbf{u})\frac{\partial \mathbf{u}}{\partial t}.$$

This implies that

$$\frac{\partial^2\mathbf{u}}{\partial a\partial t} = -\frac{\partial^2\mathbf{u}}{\partial t^2} + \mathbf{M}\frac{\partial \mathbf{u}}{\partial t} + \mathfrak{F}_{\mathbf{u}}(\mathbf{u})\frac{\partial \mathbf{u}}{\partial t},$$

which concludes the continuity of $\dfrac{\partial^2\mathbf{u}}{\partial a\partial t}$.

(ii) as the classical solution $\mathbf{u}$ satisfies the ACP (3.41), we write

$$\frac{\partial \mathbf{u}}{\partial t} = -\frac{\partial \mathbf{u}}{\partial a} + \mathbf{M}\mathbf{u} + \mathfrak{F}(\mathbf{u}), \quad \text{or} \quad \frac{\partial \mathbf{u}}{\partial a} = -\frac{\partial \mathbf{u}}{\partial t} + \mathbf{M}\mathbf{u} + \mathfrak{F}(\mathbf{u}),$$

where the right hand side is continuously differentiable with respect to t and a. It follows that

$$\frac{\partial \mathbf{u}^2}{\partial t^2} = -\frac{\partial \mathbf{u}^2}{\partial t\partial a} + \mathbf{M}\frac{\partial \mathbf{u}}{\partial t} + \mathfrak{F}_{\mathbf{u}}(\mathbf{u})\frac{\partial \mathbf{u}}{\partial t}$$

and

$$\frac{\partial \mathbf{u}^2}{\partial a\partial t} = -\frac{\partial \mathbf{u}^2}{\partial a^2} + \frac{\partial \mathbf{M}}{\partial a}\mathbf{u} + \mathbf{M}\frac{\partial \mathbf{u}}{\partial a} + \mathfrak{F}_{\mathbf{u}}(\mathbf{u})\frac{\partial \mathbf{u}}{\partial a}.$$

From the first part, we obtain

$$\frac{\partial \mathbf{u}^2}{\partial t\partial a} = -\frac{\partial \mathbf{u}^2}{\partial t^2} + \mathbf{M}\frac{\partial \mathbf{u}}{\partial t} + \mathfrak{F}_{\mathbf{u}}(\mathbf{u})\frac{\partial \mathbf{u}}{\partial t}$$

and

$$\frac{\partial \mathbf{u}^2}{\partial a^2} = -\frac{\partial \mathbf{u}^2}{\partial a\partial t} + \frac{\partial \mathbf{M}}{\partial a}\mathbf{u} + \mathbf{M}\frac{\partial \mathbf{u}}{\partial a} + \mathfrak{F}_{\mathbf{u}}(\mathbf{u})\frac{\partial \mathbf{u}}{\partial a},$$

respectively. These show that $\dfrac{\partial \mathbf{u}^2}{\partial t\partial a}$ and $\dfrac{\partial \mathbf{u}^2}{\partial a^2}$ are continuous.

Hence

$$\mathbf{u} \in \mathcal{C}^2\big([0,\omega] \times [0,T]\big).$$

Repeating the same argument as above and setting $\mathbf{z} = \mathbf{w}'$, we have

$$\mathbf{z}(t) = \boldsymbol{\mathcal{T}}(t)\mathbf{z}_0 + \int_0^t \boldsymbol{\mathcal{T}}(t-s)\Big\{\mathfrak{F}_{\mathbf{uu}}\big(\mathbf{u}(s)\big)\Big[\mathbf{w}(s)\mathbf{v}(s) + \mathbf{v}(s)\mathbf{w}(s) + \mathbf{w}(s)\Big] + \mathfrak{F}_{\mathbf{u}}\big(\mathbf{u}(s)\big)\mathbf{z}(s)\Big\}\, ds. \tag{3.51}$$

where $\mathbf{z}_0 = \boldsymbol{\mathcal{Q}}^3\mathbf{u}_0 + \boldsymbol{\mathcal{Q}}^2\mathfrak{F}(\mathbf{u}_0) + \boldsymbol{\mathcal{Q}}\mathfrak{F}_{\mathbf{u}}(\mathbf{u}_0)\mathbf{u}_0' + \mathfrak{F}_{\mathbf{uu}}(\mathbf{u}_0)\big[\mathbf{u}_0'\big]^2 + \mathfrak{F}_{\mathbf{u}}(\mathbf{u}_0)\mathbf{u}_0''$ satisfies $\mathbf{z}_0 \in D(\mathbf{A})$ by (3.45). In a similar way as above, we claim that the abstract initial value problem

$$\begin{cases} \mathbf{z}_t &= \boldsymbol{\mathcal{Q}}\mathbf{z} + \mathfrak{F}_{\mathbf{uu}}(\mathbf{u})\big[\mathbf{wv} + \mathbf{vw} + \mathbf{w}\big] + \mathfrak{F}_{\mathbf{u}}(\mathbf{u})\mathbf{z}\,, \quad t > 0, \\ \mathbf{z}(0) &= \mathbf{z}_0, \\ \mathbf{z} &\in D(\mathbf{A}), \end{cases}$$

has a unique classical solution given by (3.51). Thus, $\mathbf{z} \in \mathcal{C}\big([0,T]; D(\mathbf{A})\big)$, i.e., $\mathbf{u}''' \in \mathcal{C}\big([0,T]; D(\mathbf{A})\big)$ and $\mathbf{u} \in \mathcal{C}^3\big([0,T]; D(\mathbf{A})\big)$. It follows that $\dfrac{d^3\mathbf{u}}{dt^3}$ exists and is continuous. Likewise as above, it is easy to show that the continuity of the partial derivatives

$$\frac{\partial \mathbf{u}^3}{\partial a^3}, \frac{\partial \mathbf{u}^3}{\partial t \partial a^2}, \frac{\partial \mathbf{u}^3}{\partial a \partial t^2} \frac{\partial \mathbf{u}^3}{\partial t \partial a \partial t}, \frac{\partial \mathbf{u}^3}{\partial a \partial t \partial a}, \frac{\partial \mathbf{u}^3}{\partial a^2 \partial t} \text{ and } \frac{\partial \mathbf{u}^3}{\partial t^2 \partial a}$$

holds. Hence

$$\mathbf{u} \in \mathcal{C}^3\big([0,\omega] \times [0,T]\big).$$

□

Intracohort Transmission

In this section, we are concerned with the situation where the disease transmission interactions are restricted to individuals of the same age. A constitutive form of the infection rate relative to this mechanism of transmission is provided in [15, p. 1381]:

$$\Lambda(a, i(\cdot, t)) = K_0(a) i(a,t). \tag{3.52}$$

Therefore, the nonlinear term $\mathfrak{F}$ in (3.6) is defined by

$$\mathfrak{F}\big(\mathbf{u}(a,t)\big) = \begin{bmatrix} -K_0(a)s(a,t)i(a,t) \\ K_0(a)s(a,t)i(a,t) \end{bmatrix}, \quad \mathbf{u} = (s,i)^T.$$

In the sequel, we consider the assumption

(H6) $0 \le K_0 \in \mathrm{L}^\infty(\bar{\mathrm{I}})$

and the notation

$$\begin{aligned} \mathbf{X}_1 &= \left(\mathrm{L}^1(\bar{\mathrm{I}})\right)^2, \\ \mathbf{X}_\infty &= \left(\mathrm{L}^\infty(\bar{\mathrm{I}})\right)^2, \\ \mathbf{Y}_\infty &= \mathcal{C}\big([0,T], \mathbf{X}_\infty\big), \text{ for given } 0 < T < \infty. \end{aligned}$$

The main problem with the intracohort transmission is that, in general, $\mathfrak{F}(\mathbf{u}) \notin \mathbf{X}_1$ for $\mathbf{u} \in \mathbf{X}_1$. Multiplication is well defined in $\mathrm{L}^\infty(\bar{\mathrm{I}})$ but then the latter space is not suitable for the semigroup techniques – any strongly continuous semigroup on $\mathrm{L}^\infty(\bar{\mathrm{I}})$ is uniformly continuous, [6, Theorem 3.6]. To handle this nonlinearity, we use the fact that for $\omega < \infty$, $\mathbf{X}_\infty$ is a subspace densely and continuously embedded in $\mathbf{X}_1$ and show that we can restrict the analysis performed in the previous section to $\mathbf{X}_\infty$.

Let $\mathbf{M}_\delta(a)$, $\mathbf{B}(a)$, $\mathbf{L}(a)$, $\mathbf{M}_\mu(a)$, and $\mathbf{L}(a,b)$, denote the recovery matrix, the fertility matrix, the survival rate matrix, the mortality matrix, and the transition rate matrix, respectively, as defined in the previous section, respectively, in (3.2), in (3.3), in (3.10), as $\operatorname{diag}\{-\mu(a), -\mu(a)\}$, and as $\mathbf{L}(a)\mathbf{L}^{-1}(b)$ where $\mathbf{L}^{-1}(b)$ is defined in (3.11).

First we note the following result.

Proposition 3.2.10. *For any* $t \ge 0$, $\boldsymbol{\mathcal{T}}(t)\left(\mathbf{X}_\infty\right) \subset \mathbf{X}_\infty$ *with* $\|\boldsymbol{\mathcal{T}}(t)\mathbf{u}_0\|_{\mathbf{X}_\infty} \le \overline{\beta}\omega e^{(\overline{\beta}-\underline{\mu})t}\|\mathbf{u}_0\|_{\mathbf{X}_\infty}$ *for all* $\mathbf{u}_0 \in \mathbf{X}_\infty$.

Proof. We consider the explicit representation of the semigroup $(\boldsymbol{\mathcal{T}}(t))_{t\ge 0}$ below (cf.[43, p. 69],[29, p. 62]):

$$\boldsymbol{\mathcal{T}}(t)\mathbf{u}_0(a) = \begin{cases} \mathbf{L}(a, a-t)\mathbf{u}_0(a-t), & a \in (t, \omega), \\ \mathbf{L}(a, 0)\mathbf{b}(t-a; \mathbf{u}_0), & a \in (0, t), \end{cases} \tag{3.53}$$

where $\mathbf{b}(t;\mathbf{u}_0)$ is the solution of the integral equation

$$\mathbf{b}(t;\mathbf{u}_0) = \mathbf{J}(t) + \int_0^t \mathbf{B}(s)\mathbf{L}(s)\mathbf{b}(t-s;\mathbf{u}_0)\,ds, \tag{3.54}$$

with

$$\mathbf{J}(t) = \int_t^\omega \mathbf{B}(s)\mathbf{L}(s, s-t)\mathbf{u}_0(s-t)\,ds = \int_t^\omega \beta(s)\frac{\Pi_\mu(s)}{\Pi_\mu(s-t)}\begin{bmatrix} 1 & 1-q \\ 0 & q \end{bmatrix}\mathbf{u}_0(s-t)\,ds, \tag{3.55}$$

for $t \ge 0$, where $\Pi_\mu(a)$ is the probability for an individual to survive to age a.

Using the statement (2) of Lemma 3.1.3, equation (3.55) yields

$$\|\mathbf{J}(t)\| \le \overline{\beta}\omega e^{-\underline{\mu}t}\|\mathbf{u}_0\|_{\mathbf{X}_\infty}.$$

Substituting the latest estimate into (3.54) results in

$$\|\mathbf{b}(t;\mathbf{u}_0)\| \leq \overline{\beta}\omega e^{-\underline{\mu}t}\|\mathbf{u}_0\|_{\mathbf{X}_\infty} + \overline{\beta}e^{-\underline{\mu}t}\int_0^t e^{\underline{\mu}s}\|\mathbf{b}(s;\mathbf{u}_0)\|\,ds,$$

which leads to

$$e^{\underline{\mu}t}\|\mathbf{b}(t;\mathbf{u}_0)\| \leq \overline{\beta}\omega\|\mathbf{u}_0\|_{\mathbf{X}_\infty} + \overline{\beta}\int_0^t e^{\underline{\mu}s}\|\mathbf{b}(s;\mathbf{u}_0)\|\,ds.$$

By Gronwall's lemma, we get

$$\|\mathbf{b}(t;\mathbf{u}_0)\| \leq \overline{\beta}\omega e^{(\overline{\beta}-\underline{\mu})t}\|\mathbf{u}_0\|_{\mathbf{X}_\infty}$$

and, using (3.53), we obtain

$$\begin{aligned}\|\mathbf{L}(a,0)\mathbf{b}(t-a;\mathbf{u}_0)\| &\leq \overline{\beta}\omega e^{\overline{\beta}(t-a)}e^{-\underline{\mu}t}\|\mathbf{u}_0\|_{\mathbf{X}_\infty}\\ &= \overline{\beta}\omega e^{(\overline{\beta}-\underline{\mu})t}e^{-\overline{\beta}a}\|\mathbf{u}_0\|_{\mathbf{X}_\infty} \leq \overline{\beta}\omega e^{(\overline{\beta}-\underline{\mu})t}\|\mathbf{u}_0\|_{\mathbf{X}_\infty}, \quad \text{for } a > t\end{aligned}$$

and

$$\|\mathbf{L}(a,a-t)\mathbf{u}_0(a-t)\| \leq e^{-\underline{\mu}t}\sup_{s\in[0,a]}\|\mathbf{u}_0(s)\| \leq e^{-\underline{\mu}t}\|\mathbf{u}_0\|_{\mathbf{X}_\infty}, \quad \text{for } a < t.$$

Hence,

$$\|\boldsymbol{\mathcal{T}}(t)\mathbf{u}_0\|_{\mathbf{X}_\infty} \leq \sup_{a\in[0,\omega]}\|\mathbf{L}(a,0)\mathbf{b}(t-a;\mathbf{u}_0)\| \leq \overline{\beta}\omega e^{(\overline{\beta}-\underline{\mu})t}\|\mathbf{u}_0\|_{\mathbf{X}_\infty}.$$

□

In the next step, take $\mathbf{u} = (s,i)^T \in \mathbf{X}_\infty$ and re-write (3.1), with (3.52), in the following abstract form

$$\begin{cases}\dfrac{d\mathbf{u}}{dt} = \mathfrak{Q}\mathbf{u} + \mathfrak{F}(\mathbf{u}), & t > 0,\\ \mathbf{u}(0) = \mathbf{u}_0.\end{cases} \tag{3.56}$$

3.2.4 Some Properties of $\mathfrak{F}$

In this section, we aim to prove that the nonlinear term $\mathfrak{F}$ has the following properties:

Proposition 3.2.11. $\mathfrak{F}(\mathbf{X}_\infty) \subset \mathbf{X}_\infty$ *with* $\|\mathfrak{F}(\mathbf{u})\|_{\mathbf{X}_\infty} \leq \|K_0\|_\infty\|\mathbf{u}\|^2_{\mathbf{X}_\infty}$.

Proof. For $\mathbf{u} = \big(s,i\big)^T \in \mathbf{X}_\infty$, we have

$$\begin{aligned}\big\|\mathfrak{F}(\mathbf{u}(a))\big\| &= 2|K_0(a)s(a)i(a)|\\ &\leq 2\|K_0\|_\infty|s(a)||i(a)|\\ &\leq \|K_0\|_\infty\|\mathbf{u}(a)\|^2\\ &\leq \|K_0\|_\infty\|\mathbf{u}\|^2_{\mathbf{X}_\infty},\end{aligned}$$

and hence

$$\left\|\mathfrak{F}(\mathbf{u})\right\|_{\mathbf{X}_\infty} \leq \|K_0\|_\infty \|\mathbf{u}\|^2_{\mathbf{X}_\infty}. \tag{3.57}$$

□

Proposition 3.2.12. *$\mathfrak{F}$ is locally Lipschitz on $\mathbf{X}_\infty$.*

Proof. For $\mathbf{u}_1 = (s_1, i_1)^T$, $\mathbf{u}_2 = (s_2, i_2)^T \in \overline{\mathbf{B}}(\mathbf{u}_0, \rho)$, we have

$$\begin{aligned}
\left\|\mathfrak{F}(\mathbf{u}_1(a)) - \mathfrak{F}(\mathbf{u}_2(a))\right\| &= 2\left|K_0(a)s_1(a)i_1(a) - K_0(a)s_2(a)i_2(a)\right| \\
&\leq 2\|K_0\|_\infty \{|s_1(a)||i_1(a) - i_2(a)| \\
&\qquad + |i_2(a)||s_1(a) - s_2(a)|\} \\
&\leq 2\|K_0\|_\infty \{\|s_1\|_\infty |i_1(a) - i_2(a)| \\
&\qquad + \|i_2\|_\infty |s_1(a) - s_2(a)|\} \\
&\leq 2\|K_0\|_\infty \{\|\mathbf{u}_1\|_{\mathbf{X}_\infty} |i_1(a) - i_2(a)| \\
&\qquad + \|\mathbf{u}_2\|_{\mathbf{X}_\infty} |s_1(a) - s_2(a)|\} \\
&\leq 2\left(\rho + \|\mathbf{u}_0\|_{\mathbf{X}_\infty}\right)\|K_0\|_\infty \|\mathbf{u}_1(a) - \mathbf{u}_2(a)\|.
\end{aligned}$$

Hence, we obtain

$$\left\|\mathfrak{F}(\mathbf{u}_1(a)) - \mathfrak{F}(\mathbf{u}_2(a))\right\| \leq 2\left(\rho + \|\mathbf{u}_0\|_{\mathbf{X}_\infty}\right)\|K_0\|_\infty \|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathbf{X}_\infty},$$

which yields

$$\left\|\mathfrak{F}(\mathbf{u}_1) - \mathfrak{F}(\mathbf{u}_2)\right\|_{\mathbf{X}_\infty} \leq \chi_{\rho,\mathbf{u}_0} \|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathbf{X}_\infty}. \tag{3.58}$$

□

Proposition 3.2.13. *$\mathfrak{F}$ is continuously Fréchet differentiable with respect to $\boldsymbol{\varphi} \in \mathbf{X}_\infty$ and for any $\boldsymbol{\varphi} = (\varphi^s, \varphi^i)^T$, $\boldsymbol{\psi} = (\psi^s, \psi^i)^T \in \mathbf{X}_\infty$ the Fréchet derivative at $\boldsymbol{\varphi}$, $\mathfrak{F}_{\boldsymbol{\varphi}}$, is given by*

$$\left(\mathfrak{F}_{\boldsymbol{\varphi}}\boldsymbol{\psi}\right)(a) := \begin{bmatrix} -K_0(a)\psi^s(a)\varphi^i(a) - K_0(a)\varphi^s(a)\psi^i(a) \\ K_0(a)\psi^s(a)\varphi^i(a) + K_0(a)\varphi^s(a)\psi^i(a) \end{bmatrix}.$$

Proof. Let $\boldsymbol{\varphi}$, $\boldsymbol{\psi} \in \mathbf{X}_\infty$, then

$$\mathfrak{F}\left(\boldsymbol{\varphi} + \boldsymbol{\psi}\right)(a) = \mathfrak{F}\left(\boldsymbol{\varphi}\right)(a) + \mathfrak{F}_{\boldsymbol{\varphi}}\left(\boldsymbol{\psi}\right)(a) + \mathbf{G}\left(\boldsymbol{\psi}, \boldsymbol{\psi}\right)(a),$$

where

$$\mathbf{G}\left(\boldsymbol{\psi}, \boldsymbol{\psi}\right)(a) = \begin{bmatrix} -K_0(a)\psi^s(a)\psi^i(a) \\ K_0(a)\psi^s(a)\psi^i(a) \end{bmatrix},$$

and so follows

$$\left\|\mathbf{G}\left(\boldsymbol{\psi},\boldsymbol{\psi}\right)(a)\right\| \leq 2\|K_0\|_\infty|\psi^i(a)||\psi^s(a)| \leq \|K_0\|_\infty\|\boldsymbol{\psi}(a)\|^2 \leq \|K_0\|_\infty\|\boldsymbol{\psi}\|^2_{\mathbf{X}_\infty}.$$

Taking essential supremum over $[0,\omega]$, we obtain

$$\|\mathbf{G}\left(\boldsymbol{\psi},\boldsymbol{\psi}\right)\|_{\mathbf{X}_\infty} \leq \|K_0\|_\infty\|\boldsymbol{\psi}\|^2_{\mathbf{X}_\infty}$$

and so

$$\frac{\|\mathbf{G}\left(\boldsymbol{\psi},\boldsymbol{\psi}\right)\|_{\mathbf{X}_\infty}}{\|\boldsymbol{\psi}\|_{\mathbf{X}_\infty}} \leq \|K_0\|_\infty\|\boldsymbol{\psi}\|_{\mathbf{X}_\infty} \to 0 \text{ as } \|\boldsymbol{\psi}\|_{\mathbf{X}_\infty} \to 0.$$

This shows that $\mathfrak{F}$ is Fréchet differentiable at each $\boldsymbol{\varphi} = (\varphi^s,\varphi^i)^T \in \mathbf{X}_\infty$ and its Fréchet derivative $\mathfrak{F}_{\boldsymbol{\varphi}}$ at $\boldsymbol{\varphi}$ satisfies

$$\|\mathfrak{F}_{\boldsymbol{\varphi}}\boldsymbol{\psi}\|_{\mathbf{X}_\infty} \leq \chi_{\rho,\mathbf{u}_0}\|\boldsymbol{\psi}\|_{\mathbf{X}_\infty}, \quad \forall\boldsymbol{\psi} \in \mathbf{X}_\infty,\ \boldsymbol{\varphi} \in \overline{\mathbf{B}}(\mathbf{u}_0,\rho).$$

Moreover, for $\boldsymbol{\varphi}_1$, $\boldsymbol{\varphi}_2$, $\boldsymbol{\psi} \in \mathbf{X}_\infty$, we have

$$\begin{aligned}\left\|\left(\mathfrak{F}_{\boldsymbol{\varphi}_1}\boldsymbol{\psi}\right)(a) - \left(\mathfrak{F}_{\boldsymbol{\varphi}_2}\boldsymbol{\psi}\right)(a)\right\| &\leq 2\left\|K_0\right\|_\infty\left\|\boldsymbol{\psi}(a)\right\|\left\|\boldsymbol{\varphi}_1(a)-\boldsymbol{\varphi}_2(a)\right\| \\ &\leq 2\left\|K_0\right\|_\infty\left\|\boldsymbol{\psi}\right\|_{\mathbf{X}_\infty}\left\|\boldsymbol{\varphi}_1-\boldsymbol{\varphi}_2\right\|_{\mathbf{X}_\infty}.\end{aligned}$$

Taking supremum over $\{\boldsymbol{\psi} \in \mathbf{X}_\infty :\ \|\boldsymbol{\psi}\|_{\mathbf{X}_\infty} = 1\}$, we get

$$\left\|\mathfrak{F}_{\boldsymbol{\varphi}_1} - \mathfrak{F}_{\boldsymbol{\varphi}_2}\right\|_{\mathbb{B}(\mathbf{X}_\infty)} \leq 2\left\|K_0\right\|_\infty\left\|\boldsymbol{\varphi}_1-\boldsymbol{\varphi}_2\right\|_{\mathbf{X}_\infty}$$

and thus

$$\left\|\mathfrak{F}_{\boldsymbol{\varphi}_1} - \mathfrak{F}_{\boldsymbol{\varphi}_2}\right\|_{\mathbb{B}(\mathbf{X}_\infty)} \leq 2\left\|K_0\right\|_\infty\left\|\boldsymbol{\varphi}_1-\boldsymbol{\varphi}_2\right\|_{\mathbf{X}_\infty} \to 0 \text{ as } \|\boldsymbol{\varphi}_1-\boldsymbol{\varphi}_2\|_{\mathbf{X}_\infty} \to 0.$$

Hence, the Fréchet derivative $\mathfrak{F}_{\boldsymbol{\varphi}}$ is uniformly continuous with respect to $\boldsymbol{\varphi}$. □

3.2.5 The Existence and Uniqueness of Solution

Using the above properties of $\mathfrak{F}$ and Proposition 3.2.10, we claim that the Picard iterates of the integral formulation of (3.56):

$$\mathbf{u}(t) = \boldsymbol{\mathcal{T}}(t)\mathbf{u}_0 + \int_0^t \boldsymbol{\mathcal{T}}(t-s)\mathfrak{F}\big(\mathbf{u}(s)\big)\,ds, \quad 0 < t < T, \tag{3.59}$$

stay in an appropriately chosen closed ball in $\mathbf{Y}_\infty$ and converge to a unique fixed point which is thus a mild solution of (3.56). To prove this statement, we denote by $\overline{\mathbf{B}}(0,R)$ a closed ball in $\mathbf{Y}_\infty$ defined by

$$\overline{\mathbf{B}}(0,R) = \{\mathbf{v} \in \mathbf{Y}_\infty : \|\mathbf{v}\|_{\mathbf{Y}_\infty} \leq R\} \tag{3.60}$$

and we introduce the integral operator $\mathcal{J}$ defined on $\overline{\mathbf{B}}(0,R)$ by

$$(\boldsymbol{\mathcal{J}}\mathbf{u})\,(t) = \boldsymbol{\mathcal{T}}(t)\mathbf{u}_0 + \int_0^t \boldsymbol{\mathcal{T}}(t-s)\mathfrak{F}\big(\mathbf{u}(s)\big)\,ds, \quad 0<t<T, \tag{3.61}$$

for sufficiently small T. We aim to show that $\boldsymbol{\mathcal{J}}$ has a fixed point. To this aim, for a fixed initial datum $\mathbf{u}_0 \in \mathbf{X}_\infty$ we take R such that

$$\frac{R}{2L} > e^{(\overline{\beta}-\underline{\mu})T}\|\mathbf{u}_0\|_{\mathbf{X}_\infty}, \tag{3.62}$$

where $L = \overline{\beta}\omega$.

For the following result we assume $\mathbf{u}_0 \neq \mathbf{0}$. The case $\mathbf{u}_0 = \mathbf{0}$ can be done in a similar way.

Lemma 3.2.14. *Let $\overline{\mathbf{B}}(0,R)$ be defined as in (3.60) with R satisfying (3.62). Then the integral operator $\boldsymbol{\mathcal{J}}$, defined in (3.61), maps $\overline{\mathbf{B}}(0,R)$ into itself. Moreover, for $\mathbf{u}$, $\mathbf{v} \in \overline{\mathbf{B}}(0,R)$, $t\in[0,T]$, we have*

$$\|\boldsymbol{\mathcal{J}}\mathbf{u} - \boldsymbol{\mathcal{J}}\mathbf{v}\|_{\mathbf{Y}_\infty} \le C_R T \|\mathbf{u}-\mathbf{v}\|_{\mathbf{Y}_\infty}, \tag{3.63}$$

where $C_R = \frac{\|K_0\|_\infty}{\|\mathbf{u}_0\|_{\mathbf{X}_\infty}}R^2$.

Proof. Let $\mathbf{u}$, $\mathbf{v} \in \overline{\mathbf{B}}(0,R)$, we have

$$\begin{aligned}\Big\|\big(\boldsymbol{\mathcal{J}}\mathbf{u}\big)(a,t) - \big(\boldsymbol{\mathcal{J}}\mathbf{v}\big)(a,t)\Big\| &= \Big\|\int_0^t \boldsymbol{\mathcal{T}}(t-s)\Big(\mathfrak{F}\big(\mathbf{u}(a,s)\big) - \mathfrak{F}\big(\mathbf{v}(a,s)\big)\Big)\,ds\Big\| \\ &\le L e^{(\overline{\beta}-\underline{\mu})T}\int_0^t \Big\|\mathfrak{F}\big(\mathbf{u}(s)\big) - \mathfrak{F}\big(\mathbf{v}(s)\big)\Big\|_{\mathbf{X}_\infty} ds \\ &\le \frac{R}{2\|\mathbf{u}_0\|_{\mathbf{X}_\infty}}\int_0^t \Big\|\mathfrak{F}\big(\mathbf{u}(s)\big) - \mathfrak{F}\big(\mathbf{v}(s)\big)\Big\|_{\mathbf{X}_\infty} ds.\end{aligned}$$

Taking essential supremum over $\overline{\mathrm{I}}$, we get

$$\Big\|\big(\boldsymbol{\mathcal{J}}\mathbf{u}\big)(t) - \big(\boldsymbol{\mathcal{J}}\mathbf{v}\big)(t)\Big\|_{\mathbf{X}_\infty} \le \frac{R}{2\|\mathbf{u}_0\|_{\mathbf{X}_\infty}}\int_0^t \Big\|\mathfrak{F}\big(\mathbf{u}(s)\big) - \mathfrak{F}\big(\mathbf{v}(s)\big)\Big\|_{\mathbf{X}_\infty} ds$$

and, from (3.63),follows

$$\begin{aligned}\Big\|\big(\boldsymbol{\mathcal{J}}\mathbf{u}\big)(t) - \big(\boldsymbol{\mathcal{J}}\mathbf{v}\big)(t)\Big\|_{\mathbf{X}_\infty} &\le \frac{\chi_{R,0}R}{2\|\mathbf{u}_0\|_{\mathbf{X}_\infty}}\int_0^t \|\mathbf{u}(s)-\mathbf{v}(s)\|_{\mathbf{X}_\infty}\,ds \\ &= \frac{\|K_0\|_\infty}{\|\mathbf{u}_0\|_{\mathbf{X}_\infty}}R^2\int_0^t \|\mathbf{u}(s)-\mathbf{v}(s)\|_{\mathbf{X}_\infty}\,ds \\ &= C_R\int_0^t \|\mathbf{u}(s)-\mathbf{v}(s)\|_{\mathbf{X}_\infty}\,ds.\end{aligned}$$

Using (3.61), we show that

$$\Big\|\big(\boldsymbol{\mathcal{J}}\mathbf{u}\big)(a,t)\Big\| \le L e^{(\overline{\beta}-\underline{\mu})T}\|\mathbf{u}_0\|_{\mathbf{X}_\infty} + L e^{(\overline{\beta}-\underline{\mu})T}\int_0^t \big\|\mathfrak{F}\big(\mathbf{u}(s)\big)\big\|_{\mathbf{X}_\infty} ds.$$

Taking essential supremum over $\bar{\mathrm{I}}$, we have

$$\left\|(\mathcal{J}\mathbf{u})(t)\right\|_{\mathbf{X}_\infty} \leq Le^{(\overline{\beta}-\underline{\mu})T}\|\mathbf{u}_0\|_{\mathbf{X}_\infty} + Le^{(\overline{\beta}-\underline{\mu})T}\int_0^t \left\|\mathfrak{F}(\mathbf{u}(s))\right\|_{\mathbf{X}_\infty} ds,$$

hence, from (3.57) and (3.62), we obtain

$$\begin{aligned}\left\|(\mathcal{J}\mathbf{u})(t)\right\|_{\mathbf{X}_\infty} &\leq \frac{R}{2}\left(1 + \frac{\|K_0\|_\infty R}{\|\mathbf{u}_0\|_{\mathbf{X}_\infty}}\int_0^t ds\right)\\ &\leq \frac{R}{2}\left(1 + \frac{\|K_0\|_\infty R}{\|\mathbf{u}_0\|_{\mathbf{X}_\infty}}t\right)\\ &\leq \frac{R}{2}\left(1 + \frac{\|K_0\|_\infty R}{\|\mathbf{u}_0\|_{\mathbf{X}_\infty}}T\right)\\ &\leq \frac{R}{2}(1+1) = R,\end{aligned}$$

provided

$$T \leq \frac{1}{R}\cdot\frac{\|\mathbf{u}_0\|_{\mathbf{X}_\infty}}{\|K_0\|_\infty}. \tag{3.64}$$

Therefore, T and R have to satisfy the constraints (3.62) and (3.64). It follows that T satisfies $LTe^{(\overline{\beta}-\underline{\mu})T}\|K_0\|_\infty < \frac{1}{2}$ and R satisfies $2Le^{(\overline{\beta}-\underline{\mu})T}\|\mathbf{u}_0\|_{\mathbf{X}_\infty} < R \leq \frac{1}{T}\cdot\frac{\|\mathbf{u}_0\|_{\mathbf{X}_\infty}}{\|K_0\|_\infty}$.

Taking supremum over $[0,T]$ we arrive at

$$\|\mathcal{J}\mathbf{u}\|_{\mathbf{Y}_\infty} \leq R, \tag{3.65}$$

that is, $\mathcal{J}\mathbf{u} \in \overline{\mathbf{B}}(0,R)$, $\forall\,\mathbf{u} \in \overline{\mathbf{B}}(0,R)$; and so $\mathcal{J}$ maps $\overline{\mathbf{B}}(0,R)$ into itself. □

Consequently, we have the following result:

Theorem 3.2.15. *For R and T satisfying* (3.62)*, the integral operator $\mathcal{J} : \overline{\mathbf{B}}(0,R) \to \overline{\mathbf{B}}(0,R)$ given by* (3.61) *has a unique fixed point.*

Proof. Let us denote by $\mathcal{J}^N$ N compositions of $\mathcal{J}$, with N a positive integer. For $\mathbf{u}, \mathbf{v} \in \overline{\mathbf{B}}(0,R)$, we have

$$\begin{aligned}\left\|(\mathcal{J}^N\mathbf{v})(t) - (\mathcal{J}^N\mathbf{u})(t)\right\|_{\mathbf{X}_\infty} &\leq C_R\int_0^t \left\|(\mathcal{J}^{N-1}\mathbf{v})(\sigma) - (\mathcal{J}^{N-1}\mathbf{u})(\sigma)\right\|_{\mathbf{X}_\infty} d\sigma\\ &\leq C_R^2\int_0^t\int_0^\sigma \left\|(\mathcal{J}^{N-2}\mathbf{v})(\alpha) - (\mathcal{J}^{N-2}\mathbf{u})(\alpha)\right\|_{\mathbf{X}_\infty} d\alpha\, d\sigma\\ &\vdots\\ &\leq C_R^N \underbrace{\int_0^t\int_0^\sigma \cdots \int_0^\gamma}_{N \text{ times}} \|\mathbf{v}(\gamma) - \mathbf{u}(\gamma)\|_{\mathbf{X}_\infty} \underbrace{d\gamma \ldots d\alpha\, d\sigma}_{N \text{ times}}.\end{aligned}$$

Taking supremum over $[0,T]$, we obtain

$$\left\|\boldsymbol{\mathcal{J}}^N\mathbf{v}-\boldsymbol{\mathcal{J}}^N\mathbf{u}\right\|_{\mathbf{Y}_\infty}\leq\frac{T^N}{N!}C_R^N\left\|\mathbf{v}-\mathbf{u}\right\|_{\mathbf{Y}_\infty}, \tag{3.66}$$

where $\frac{T^N}{N!}C_R^N<1$, provided N is sufficiently large. Hence, $\boldsymbol{\mathcal{J}}^N$ is a contraction on $\overline{\mathbf{B}}(0,R)$, and so it possesses a fixed point $\mathbf{u}\in\overline{\mathbf{B}}(0,R)$, that is, $\mathbf{u}=\boldsymbol{\mathcal{J}}^N\mathbf{u}$, see [45, p. 44]. Denote $\widetilde{\boldsymbol{\mathcal{J}}}:=\boldsymbol{\mathcal{J}}^N$, it follows that $\widetilde{\boldsymbol{\mathcal{J}}}\mathbf{u}=\mathbf{u}$. Hence $\widetilde{\boldsymbol{\mathcal{J}}}^n\mathbf{u}=\mathbf{u}$. For every $\hat{\mathbf{u}}\in\overline{\mathbf{B}}(0,R)$,

$$\widetilde{\boldsymbol{\mathcal{J}}}^n\hat{\mathbf{u}}\to\mathbf{u}\qquad\text{as } n\to 0.$$

For $\hat{\mathbf{u}}=\boldsymbol{\mathcal{J}}\mathbf{u}$, we have

$$\mathbf{u}=\lim_{n\to\infty}\widetilde{\boldsymbol{\mathcal{J}}}^n\left(\boldsymbol{\mathcal{J}}\mathbf{u}\right)=\lim_{n\to\infty}\boldsymbol{\mathcal{J}}\left(\widetilde{\boldsymbol{\mathcal{J}}}^n\mathbf{u}\right)=\boldsymbol{\mathcal{J}}\mathbf{u}.$$

In addition, if $\mathbf{u},\mathbf{w}\in\overline{\mathbf{B}}(0,R)$, $\mathbf{u}\neq\mathbf{w}$, are both fixed points of $\boldsymbol{\mathcal{J}}^N$, we show that

$$\|\mathbf{u}-\mathbf{w}\|_{\mathbf{Y}_\infty}=\|\boldsymbol{\mathcal{J}}^N\mathbf{u}-\boldsymbol{\mathcal{J}}^N\mathbf{w}\|_{\mathbf{Y}_\infty}\leq\frac{T^N}{N!}C_R^N\|\mathbf{u}-\mathbf{w}\|_{\mathbf{Y}_\infty},$$

which is a contradiction since we chose T so that $\frac{T^N}{N!}C_R^N<1$. Thus, we proved that $\mathbf{u}=\mathbf{w}$, and this guarantees the uniqueness of the fixed point of $\boldsymbol{\mathcal{J}}^N$. Since every fixed point of $\boldsymbol{\mathcal{J}}$ is also a fixed point of $\boldsymbol{\mathcal{J}}^N$, we see that $\boldsymbol{\mathcal{J}}$ cannot have more than one fixed point. □

4. Asymptotic Analysis of Singularly Perturbed SIS Epidemiological Models

4.1 Introduction

In this chapter we consider the asymptotic behaviour of solutions to

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)s(a,t)=-\mu(a)s(a,t)-\Lambda(a,i(\cdot,t))s(a,t)+\frac{1}{\epsilon}\delta(a)i(a,t), \tag{4.1a}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)i(a,t)=-\mu(a)i(a,t)+\Lambda(a,i(\cdot,t))s(a,t)-\frac{1}{\epsilon}\delta(a)i(a,t), \tag{4.1b}$$

$$s(0,t)=\int_0^\omega \beta(a)\left\{s(a,t)+(1-q)i(a,t)\right\}da, \tag{4.1c}$$

$$i(0,t)=q\int_0^\omega \beta(a)i(a,t)\,da, \tag{4.1d}$$

$$s(a,0)=s^0(a)=\varphi^s(a), \tag{4.1e}$$

$$i(a,0)=i^0(a)=\varphi^i(a), \tag{4.1f}$$

where

$$\Lambda(a,i(\cdot,t))=\int_0^\omega K(a,a^{'})i(a^{'},t)\,da^{'}, \tag{4.2}$$

as $\epsilon\to 0$.

An important consequence of the structure of problem (4.1) and the classical solvability for $\mathbf{f}^0=(s^0,i^0)^T\in D(\mathbf{A})$, (as given in (3.4), in the previous chapter), is that, by adding the equations as well as the boundary and the initial condition in (4.1) and denoting $\varrho(a,t)=s(a,t)+i(a,t)$, we obtain

$$\frac{\partial\varrho}{\partial t}(a,t)+\frac{\partial\varrho}{\partial a}(a,t)=-\mu(a)\varrho(a,t), \tag{4.3a}$$

$$\varrho(0,t)=\int_0^\omega \beta(a)\varrho(a,t)\,da, \tag{4.3b}$$

$$\varrho(a,0)=s^0(a)+i^0(a)=\varrho^0(a), \tag{4.3c}$$

where $\varrho^0\in D(A)$ with $\big(A,D(A)\big)$ being the generator of the strongly continuous semigroup $\left(e^{tA}\right)_{t\geq 0}$ for the scalar McKendrick problem (4.3), defined analogously to (3.4).

Since for $\mathbf{f}^0\geq 0$ we have $\mathbf{f}=(s,i)^T\geq 0$, we see that each term is controlled for all $t\geq 0$ and any $\epsilon>0$ by an ϵ-independent solution ϱ of the scalar McKendrick problem (4.3):

$$0\leq s(a,t)\leq\varrho(a,t)\quad\text{and}\quad 0\leq i(a,t)\leq\varrho(a,t) \tag{4.4}$$

for each $t \geq 0$ and almost every $a \in [0, \omega]$. Note that these inequalities can be extended to mild solutions by continuous dependence of the solutions on the initial data for both linear and nonlinear problems. Inequalities (4.4) allow for much better control of the solutions to (4.1) due to $\left(e^{tA}\right)_{t\geq 0}$ having the property of asynchronous exponential growth on $\mathrm{L}^1\big([0,\omega]\big)$ as time approaches infinity, [7, 27]. In particular, the population $t \to \varrho(\cdot, t)$ exponentially (in $\mathrm{L}^1\big([0,\omega]\big)$) tends to zero, tends to the stable distribution or grows with an exponential bound, if and only if the net reproduction rate R, see (1.3), is, respectively, smaller, equal or greater than 1.

4.2 The Chapman-Enskog Procedure

Using notation introduced in the previous chapter, we write (4.1) as

$$\partial_t \mathbf{f} = \mathbf{S}\mathbf{f} + \mathbf{M}_\mu \mathbf{f} + \mathfrak{F}(\mathbf{f}) + \frac{1}{\epsilon}\mathbf{M}_\delta \mathbf{f}, \tag{4.5a}$$

$$\mathbf{f}(0,t) = \int_0^\omega \mathbf{B}(a)\mathbf{f}(a,t)\,da, \tag{4.5b}$$

$$\mathbf{f}(a,0) = \mathbf{f}^0(a), \tag{4.5c}$$

where $\mathbf{f} = (s,i)^T$, $\mathbf{f}^0 \in D(\mathbf{A})$, as given in (3.4). Following the general approach of asymptotic analysis, we are looking for the so-called hydrodynamic space V of the singularly perturbed equation (4.5) which, in this case, is given by the null-space of $\mathbf{M}_\delta$. Since the variable a plays in the following decomposition the role of a parameter, the null-space of $\mathbf{M}_\delta$ is determined by

$$\mathbf{M}_\delta \mathbf{u} = \mathbf{0}. \tag{4.6}$$

We then obtain $i = 0$, and so $\mathbf{u} = u_1\boldsymbol{\psi}_1$, where u_1 is arbitrary and $\boldsymbol{\psi}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. So, for a fixed $a \in \mathbb{R}_+$, V is a one dimensional subspace defined by

$$V = \left\{\mathbf{u} \in \mathbb{R}^2;\, \mathbf{u} = u_1(a)(1,0),\, a \in \mathbb{R}^+\right\} = \operatorname{span}\{\boldsymbol{\psi}_1\}.$$

The complementary spectral subspace W, called the kinetic space, corresponding to the eigenvalue $\lambda(a) = -\delta(a)$, is spanned by

$$\boldsymbol{\psi}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Moreover, we can see that

$$\mathbf{e} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

is the left eigenvector corresponding to the dominant eigenvalue 0 (while $\boldsymbol{\psi}_1$, defined earlier, is its corresponding right eigenvector). In addition, $\mathbf{e}$ satisfies

$$\mathbf{e}\cdot\boldsymbol{\psi}_1 = 1 \quad \text{and} \quad \mathbf{e}\cdot\boldsymbol{\psi}_2 = 0.$$

Using Theorem 2.8.1, we are able to find the decomposition

$$\mathbf{f} = \mathbf{Pf} + \mathbf{Qf}$$

into the hydrodynamic and kinetic space, where

$$\mathbf{Pf} = (\mathbf{f}\cdot\mathbf{e})\,\boldsymbol{\psi}_1 = (s+i)\,\boldsymbol{\psi}_1, \tag{4.7}$$

and

$$\mathbf{Qf} = \mathbf{f} - (\mathbf{f}\cdot\mathbf{e})\,\boldsymbol{\psi}_1 = i\boldsymbol{\psi}_2. \tag{4.8}$$

Next, we use this decomposition to change variables in (4.1). Since $\varrho = s + i$, see (4.3), we define $m = i$. This yields

$$\frac{\partial\varrho}{\partial t} + \frac{\partial\varrho}{\partial a} = -\mu\varrho, \tag{4.9a}$$

$$\varrho(0,t) = \int_0^\omega \beta(a)\varrho(a,t)\,da, \tag{4.9b}$$

$$\varrho(a,0) = s^0(a) + i^0(a) = \varrho^0(a) \tag{4.9c}$$

and

$$\frac{\partial m}{\partial t} + \frac{\partial m}{\partial a} = -\mu m + (\varrho - m)\int_0^\omega Km\,da' - \frac{1}{\epsilon}\delta m, \tag{4.10a}$$

$$m(0,t) = q\int_0^\omega \beta(a)m(a,t)\,da, \tag{4.10b}$$

$$m(a,0) = i^0(a) = m^0, \tag{4.10c}$$

where $m^0 \in D(A)$. Thus, $\varrho\boldsymbol{\psi}_1$ belongs to the hydrodynamic space and $m\boldsymbol{\psi}_2$ to the kinetic space. Since the total population ϱ decouples from the system, there is no need to approximate it and in (4.10a) it can be treated as a known function. Thus, we consider the bulk part approximation $\bar{m}$ of m and we write

$$m \approx \bar{m},$$

where the approximation equality symbol $\approx$ accounts for the fact that we only consider the first terms of the asymptotic expansion. Following the Chapman-Enskog procedure, we consider the expansion

$$\bar{m} = \bar{m}_0 + \epsilon\bar{m}_1 + \cdots\,. \tag{4.11}$$

Substituting (4.11) into (4.10a), we get:

$$\begin{aligned}\epsilon\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)(\bar{m}_0+\epsilon\bar{m}_1+\cdots) &= -\epsilon\mu(\bar{m}_0+\epsilon\bar{m}_1+\cdots)\\ &+\epsilon(\varrho-\bar{m}_0-\epsilon\bar{m}_1-\cdots)\int_0^\omega K(\bar{m}_0+\epsilon\bar{m}_1+\cdots)\,da'\\ &-\delta(\bar{m}_0+\epsilon\bar{m}_1-\cdots).\end{aligned}\tag{4.12a}$$

Comparing the coefficients at like powers of ϵ, we obtain:

$$\bar{m}_0 = 0, \tag{4.13a}$$

$$\bar{m}_1 = 0, \tag{4.13b}$$

and, by induction, $\bar{m}_n = 0$ for $n = 2, 3, \ldots$.

Hence, denoting by $\bar{m}$ the bulk approximation of m, we arrive at the (formal) bulk approximation equations

$$\frac{\partial\varrho}{\partial t}+\frac{\partial\varrho}{\partial a} = -\mu\varrho, \tag{4.14a}$$

$$\bar{m} = 0, \tag{4.14b}$$

with boundary and initial conditions:

$$\varrho(0,t) = \int_0^\omega \beta(a)\varrho(a,t)\,da, \quad \bar{m}(0,t) = 0, \tag{4.15}$$

$$\varrho(a,0) = \varrho^0(a), \quad \bar{m}(a,0) = m^0(a). \tag{4.16}$$

Thus the bulk approximation $\bar{m} = 0$ is at least one reason why the error cannot be of order ϵ – the initial condition m^0 is of order 1.

To cater for this, we have to introduce the initial layer correction.

4.3 The Initial Layer Correction

Let ϱ^* and m^* be the actual solutions of (4.9a) and (4.10a), respectively. We approximate ϱ^*, m^* as the sums of the bulk solutions ϱ, $\bar{m}$ and the initial layer solutions $\tilde{\varrho}$, $\tilde{m}$:

$$\varrho^* = \varrho + \tilde{\varrho}, \quad m^* \approx \bar{m} + \tilde{m} = \tilde{m}, \tag{4.17}$$

by solving, respectively, Equations (4.9a) and (4.10a) with the time rescaled accordingly to $\tau = t/\epsilon$.

By rescaling the time in (4.9a) and (4.10a), we have

$$\frac{\partial \varrho}{\partial \tau} + \epsilon \frac{\partial \varrho}{\partial a} = -\epsilon \mu \varrho \tag{4.18}$$

and

$$\frac{\partial m}{\partial \tau} + \epsilon \frac{\partial m}{\partial a} = -\epsilon \mu m + \epsilon(\varrho - m) \int_0^\omega K m \, da' - \delta m, \tag{4.19}$$

respectively.

Equation (4.18) for ϱ is linear, so we can look for $\tilde{\varrho}$ separately. Further, Equation (4.19) for m does not depend on the bulk part approximation. Hence, we now expand the initial layer solutions in the power series of ϵ:

$$\tilde{\varrho} = \tilde{\varrho}_0 + \epsilon \tilde{\varrho}_1 + \cdots , \quad \tilde{m} = \tilde{m}_0 + \epsilon \tilde{m}_1 + \cdots . \tag{4.20}$$

Inserting (4.20) into (4.18)-(4.19) and comparing coefficients at like powers of ϵ, we obtain

$$\frac{\partial \tilde{\varrho}_0}{\partial \tau} = 0, \tag{4.21a}$$

$$\frac{\partial \tilde{m}_0}{\partial \tau} = -\delta \tilde{m}_0. \tag{4.21b}$$

We assume that the initial layer solutions vanish at $\tau \to \infty$:

$$\lim_{\tau \to \infty} \tilde{\varrho}_0(\tau) = 0.$$

Since $\tilde{m}$ vanishes at ∞, the choice of its initial condition is arbitrary. We assume the initial condition:

$$\tilde{m}(a, 0) = m^0(a).$$

Thus, we obtain

$$\tilde{\varrho}_0(\tau) = 0, \qquad \tilde{m}_0(\tau) = m^0 e^{-\delta \tau}. \tag{4.22}$$

The approximation with the initial layer correction is given by

$$\varrho^* = \varrho, \quad m^* \approx \tilde{m}_0$$

with

$$\varrho^*(0) = \varrho^0, \quad m^*(0) = \tilde{m}_0(0) = m^0(0),$$

The error, v, of the approximation of the solution m^* is given by

$$m^* = \tilde{m}_0 + \epsilon v \tag{4.23}$$

with

$$v(0) = 0.$$

Since m^* is a classical solution to (4.10a), we insert (4.23) into it and we get

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right)(\tilde{m}_0 + \epsilon v) = -\mu\,(\tilde{m}_0 + \epsilon v) - \frac{\delta}{\epsilon}\,(\tilde{m}_0 + \epsilon v)$$
$$+ (\bar{\varrho} - \tilde{m}_0 - \epsilon v)\int_0^\omega K\,(\tilde{m}_0 + \epsilon v)\;da'. \qquad (4.24)$$

Since $m^0 \in D(A)$, as mentioned earlier, we also have $\tilde{m}_0 \in D(A)$, see (4.22). Further, we have $v \in D(A)$, see (4.23). This allows to split the terms $\partial_t(\tilde{m}_0 + \epsilon v)$, $\partial_a(\tilde{m}_0 + \epsilon v)$ and $\mu\,(\tilde{m}_0 + \epsilon v)$ in (4.24) and conclude that v is a classical solution to

$$\frac{\partial v}{\partial t} + \frac{\partial v}{\partial a} = -\epsilon^{-1}\delta v - \left(\mu + \int_0^\omega K\tilde{m}_0\,da'\right)v + (\varrho - \tilde{m}_0)\int_0^\omega Kv\,da'$$
$$- \epsilon v\int_0^\omega Kv\,da' - \epsilon^{-1}\Delta, \qquad (4.25)$$

with the boundary condition

$$v(0,t) = q\int_0^\omega \beta(a)v(a,t)\,da + \frac{1}{\epsilon}\left(-\tilde{m}_0(0,t/\epsilon) + q\int_0^\omega \beta(a)\tilde{m}_0(a,t/\epsilon)\,da\right) \qquad (4.26)$$

and the initial condition

$$v(a,0) = v^0(a) = 0, \qquad (4.27)$$

where

$$\Delta = \frac{\partial\tilde{m}_0}{\partial a} + \mu\tilde{m}_0 - (\varrho - \tilde{m}_0)\int_0^\omega K\tilde{m}_0\,da'.$$

We see that we still have terms which are potentially of lower order on the right hand side in (4.26). Fortunately, the inhomogeneities on the boundary both have exponential decay in $t \to \infty$ and $\epsilon \to 0$. Thus, we should not need the boundary layer correction but only the corner layer correction, obtained by simultaneous rescaling of time and age according to $\tau = t/\epsilon$, $\alpha = a/\epsilon$.

4.4 The Corner Layer Correction

Similarly to the initial layer correction, to get rid of the initial layer contribution in (4.26), we introduce the corner layer correction; that is, we look for the approximation

$$\varrho^*(a,t) \approx \varrho(a,t) + \breve{\varrho}(\alpha,\tau), \quad m^*(a,t) \approx \tilde{m}_0(a,\tau) + \breve{m}(\alpha,\tau), \qquad (4.28)$$

where $\breve{\varrho}$, $\breve{m}_0$ are determined by solving the Equations (4.9a)-(4.10a) with time and age rescaled according to $\tau = t/\epsilon$, $\alpha = a/\epsilon$, while $\tilde{m}_0$ is given by (4.22). Thus

$$\frac{\partial \breve{\varrho}}{\partial \tau} + \frac{\partial \breve{\varrho}}{\partial \alpha} = -\epsilon\mu\breve{\varrho}, \tag{4.29a}$$

$$\frac{\partial \breve{m}}{\partial \tau} + \frac{\partial \breve{m}}{\partial \alpha} = -\epsilon\mu\breve{m} + \epsilon(\breve{\varrho} - \breve{m})\int_0^{\omega} K\breve{m}\, da^{'} - \delta\breve{m}. \tag{4.29b}$$

After defining the following expansions

$$\breve{\varrho} = \breve{\varrho}_0 + \epsilon\breve{\varrho}_1 + \cdots, \quad \breve{m} = \breve{m}_0 + \epsilon\breve{m}_1 + \cdots,$$
$$\delta(\epsilon\alpha) = \delta(0) + \epsilon\alpha\frac{d\delta}{da}(0) + \cdots, \quad \mu(\epsilon\alpha) = \mu(0) + \epsilon\alpha\frac{d\mu}{da}(0) + \cdots,$$

and inserting them into (4.29a)-(4.29b), we compare coefficients at like powers of ϵ and we obtain the corner layer equations

$$\frac{\partial \breve{\varrho}_0}{\partial \tau} + \frac{\partial \breve{\varrho}_0}{\partial \alpha} = 0, \tag{4.30a}$$

$$\frac{\partial \breve{m}_0}{\partial \tau} + \frac{\partial \breve{m}_0}{\partial \alpha} = -\delta(0)\breve{m}_0. \tag{4.30b}$$

We consider the boundary and the initial conditions for $\breve{\varrho}_0$ which satisfy

$$\breve{\varrho}_0(0,\tau) = \breve{\varrho}_0(\alpha, 0) = 0. \tag{4.31}$$

Thus, from (4.30a), we get

$$\breve{\varrho}_0(\alpha,\tau) = \begin{cases} \breve{\varrho}_0(\alpha-\tau, 0), & \alpha > \tau, \\ B_{\breve{\varrho}_0}(\tau - \alpha), & \alpha < \tau, \end{cases}$$

where $B_{\breve{\varrho}_0}$, defined by $B_{\breve{\varrho}_0}(t) := \breve{\varrho}_0(0, t/\epsilon)$, is the total birth rate corresponding to the age-density population $\breve{\varrho}_0$. From, (4.31), we then obtain

$$\breve{\varrho}_0 = 0.$$

Unfortunately, a standard approach to the corner layer is not sufficient here as the corner layer equation (4.30b) does not incorporate the unbounded operator $\mathcal{M}_0 = \mu$. In general, the corner layer $\breve{m}_0$ does not belong to $D(\mathcal{M}_0)$. Thus, instead of the above standard corner layer corrector, we define our corner corrector to be the solution to the full linear part of (4.29). Therefore, the corner layer corrector $\breve{m}_0$ is given by

$$\frac{\partial \breve{m}_0}{\partial t} + \frac{\partial \breve{m}_0}{\partial a} = -\mu(a)\breve{m}_0 - \frac{1}{\epsilon}\delta(a)\breve{m}_0. \tag{4.32}$$

Let u be the error of the approximation (4.28). We have

$$m^*(a,t) = \tilde{m}_0(a, t/\epsilon) + \breve{m}_0(a/\epsilon, t/\epsilon) + \epsilon u(a,t). \tag{4.33}$$

In the next step, from (4.33), we derive the boundary condition for the error function u. We have

$$\begin{aligned}
u(0,t) &= \frac{1}{\epsilon}\left(m^*(0,t) - \tilde{m}_0(0,t/\epsilon) - \breve{m}_0(0,t/\epsilon)\right)\\
&= \frac{q}{\epsilon}\int_0^\omega \beta(a) m^*(a,t)\,da - \frac{1}{\epsilon}\tilde{m}_0(0,t/\epsilon) - \frac{1}{\epsilon}\breve{m}_0(0,t/\epsilon)\\
&= \frac{q}{\epsilon}\int_0^\omega \beta(a)\big(\tilde{m}_0(a,t/\epsilon) + \breve{m}_0(a/\epsilon,t/\epsilon) + \epsilon u(a,t)\big)\,da\\
&\quad - \frac{1}{\epsilon}\tilde{m}_0(0,t/\epsilon) - \frac{1}{\epsilon}\breve{m}_0(0,t/\epsilon)\\
&= q\int_0^\omega \beta(a) u(a,t)\,da - \frac{1}{\epsilon}\tilde{m}_0(0,t/\epsilon) - \frac{1}{\epsilon}\breve{m}_0(0,t/\epsilon)\\
&\quad + \frac{q}{\epsilon}\left(\int_0^\omega \beta(a)\tilde{m}_0(a,t/\epsilon)\,da + \int_0^\omega \beta(a)\breve{m}_0(a/\epsilon,t/\epsilon)\,da\right).
\end{aligned}$$

To get rid of the troublesome terms containing $\tilde{m}_0$ introduced by the initial layer, we impose the following boundary condition for $\breve{m}_0$:

$$\breve{m}_0(0,t/\epsilon) = q\int_0^\omega \beta(a)\breve{m}_0(a/\epsilon,t/\epsilon)\,da + q\int_0^\omega \beta(a)\tilde{m}_0(a,t/\epsilon)\,da - \tilde{m}_0(0,t/\epsilon). \tag{4.34}$$

We supplement Equations (4.32) and (4.34) by the initial condition

$$\breve{m}_0(a/\epsilon, 0) = 0.$$

Thus, from (4.33), we get

$$u(a,0) = 0.$$

We see that, our corner layer corrector is the solution of the following problem

$$\frac{\partial \breve{m}_0}{\partial t} + \frac{\partial \breve{m}_0}{\partial a} = -\mu(a)\breve{m}_0 - \frac{1}{\epsilon}\delta(a)\breve{m}_0, \tag{4.35a}$$

$$\breve{m}_0(0,t/\epsilon) = q\int_0^\omega \beta(a)\breve{m}_0(a/\epsilon,t/\epsilon)\,da + q\int_0^\omega \beta(a)\tilde{m}_0(a,t/\epsilon)\,da - \tilde{m}_0(0,t/\epsilon), \tag{4.35b}$$

$$\breve{m}_0(a/\epsilon,0) = 0. \tag{4.35c}$$

Next, we shall investigate the solvability of (4.35) as in [27] where (4.35) corresponds to a scalar McKendrick problem of the form

$$\frac{\partial n}{\partial t} + \frac{\partial n}{\partial a} = -\mu(a) n, \tag{4.36a}$$

$$n(0,t) = q\int_0^\omega \beta(a) n(a,t)\,da, \tag{4.36b}$$

$$n(a,0) = n_0(a), \tag{4.36c}$$

which has a classical solution (as proved by Iannelli [27]). We shall adapt some results by Iannelli [27], namely Theorem 4.1 and Theorem 4.2, which pertain only to the homogeneous case (4.36). We present a more detailed proof for each one. In the sequel, ψ will denote the inhomogeneous term in (4.35b).

Lemma 4.4.1. *Let* (H2) *and* (H3) *be satisfied. Then*

$$\psi \in W^{1,\infty}(\mathbb{R}_+) \cap \mathcal{C}^1(\mathbb{R}_+). \tag{4.37}$$

Proof. From (4.35b), we set

$$\psi(t) = -m^0(0)e^{-\frac{t}{\epsilon}\delta(0)} + q\int_0^\omega \beta(a)m^0(a)e^{-\frac{t}{\epsilon}\delta(a)}\,da \tag{4.38}$$

and we see that the left hand side is differentiable on $\mathbb{R}_+$ since the right hand side is differentiable on $\mathbb{R}_+$. Hence, $\psi'(t)$ exists and is given by

$$\psi'(t) = \frac{1}{\epsilon}m^0(0)\delta(0)e^{-\frac{t}{\epsilon}\delta(0)} - \frac{q}{\epsilon}\int_0^\omega \beta(a)m^0(a)\delta(a)e^{-\frac{t}{\epsilon}\delta(a)}\,da. \tag{4.39}$$

From (4.38), we get

$$\begin{aligned}
|\psi(t)| &= \left|-m^0(0)e^{-\frac{t}{\epsilon}\delta(0)} + q\int_0^\omega \beta(a)m^0(a)e^{-\frac{t}{\epsilon}\delta(a)}\,da\right| \\
&\le e^{-\frac{\delta}{\epsilon}t}\left|m^0(0)\right| + q\overline{\beta}e^{-\frac{\delta}{\epsilon}t}\left\|m^0\right\|_1 \\
&\le e^{-\frac{\delta}{\epsilon}t}\left(\left|m^0(0)\right| + q\overline{\beta}\left\|m^0\right\|_{\mathrm{W}^{1,1}([0,\omega])}\right).
\end{aligned} \tag{4.40}$$

Since $m^0 \in D(A)$, we have $m^0 \in \mathrm{W}^{1,1}([0,\omega])$. From Lemma 3.2.8, we have

$$m^0(a) - m^0(0) = \int_0^a \frac{dm^0}{d\sigma}(\sigma)\,d\sigma \quad \forall\, a \in [0,\omega].$$

This implies

$$m^0(0) = m^0(a) - \int_0^a \frac{dm^0}{d\sigma}(\sigma)\,d\sigma.$$

We have

$$\int_0^\omega |m^0(0)|\,da \le \int_0^\omega |m^0(a)|\,da + \int_0^\omega \left(\int_0^a \left|\frac{dm^0}{d\sigma}(\sigma)\right|\,d\sigma\right)\,da,$$

and thus

$$\omega|m^0(0)| \le \|m^0\|_1 + \omega\left\|\frac{dm^0}{da}\right\|_1.$$

We obtain

$$|m^0(0)| \leq \omega^{-1}\|m^0\|_1 + \left\|\frac{dm^0}{da}\right\|_1 \leq \max\{1,\omega^{-1}\}\left(\|m^0\|_1 + \left\|\frac{dm^0}{da}\right\|_1\right) = k\|m^0\|_{\mathrm{W}^{1,1}([0,\omega])},$$

for $\omega \neq 0$, where $k = \max\{1,\omega^{-1}\}$; and hence, from (4.40),

$$|\psi(t)| \leq (k + q\overline{\beta})e^{-\frac{\delta}{\epsilon}t}\|m^0\|_{\mathrm{W}^{1,1}([0,\omega])} = C_i e^{-\frac{\delta}{\epsilon}t}$$

for $t \in [0,\infty)$, $\epsilon > 0$ and some constant C_i depending on $\mathrm{W}^{1,1}([0,\omega])$-norm of m^0 (as well as on the coefficients of the equations). In a similar way, from (4.39), we have

$$\begin{aligned}\left|\psi^{'}(t)\right| &= \left|\frac{1}{\epsilon}m^0(0)\delta(0)e^{-\frac{t}{\epsilon}\delta(0)} - \frac{q}{\epsilon}\int_0^\omega \beta(a)m^0(a)\delta(a)e^{-\frac{t}{\epsilon}\delta(a)}\,da\right| \\ &\leq \epsilon^{-1}(k+q\overline{\beta})\overline{\delta}e^{-\frac{\delta}{\epsilon}t}\|m^0\|_{\mathrm{W}^{1,1}([0,\omega])} = \epsilon^{-1}C_i\overline{\delta}e^{-\frac{\delta}{\epsilon}t}\end{aligned}$$

for $t \in [0,\infty)$, $\epsilon > 0$ and some constant C_i. Taking the supremum over $\mathbb{R}_+$, we get

$$\sup_{t\in\mathbb{R}_+}\left|\psi(t)\right| \leq C_i < \infty, \quad \sup_{t\in\mathbb{R}_+}\left|\psi^{'}(t)\right| \leq \epsilon^{-1}C_i\overline{\delta} < \infty,$$

for fixed $\epsilon > 0$. Hence, $\psi,\ \psi^{'} \in \mathrm{L}^1(\mathbb{R}_+)$. Further, from (4.38), we have

$$\begin{aligned}|\psi(t)-\psi(t_0)| &\leq k\left|e^{-\frac{t}{\epsilon}\delta(0)} - e^{-\frac{t_0}{\epsilon}\delta(0)}\right|\left\|m^0\right\|_{\mathrm{W}^{1,1}([0,\omega])} + q\overline{\beta}\int_0^\omega \left|m^0(a)\right|\left|e^{-\frac{t}{\epsilon}\delta(a)} - e^{-\frac{t_0}{\epsilon}\delta(a)}\right|\,da \\ &\leq \overline{\delta}\left(k+q\overline{\beta}\right)\frac{e^{-\frac{\delta}{\epsilon}\alpha}}{\epsilon}\,|t-t_0|\left\|m^0\right\|_{\mathrm{W}^{1,1}([0,\omega])}\end{aligned}$$

for $t, t_0 \in \mathbb{R}_+$, $\alpha \in (t_0,t)$ and fixed $\epsilon > 0$. It follows that

$$|\psi(t)-\psi(t_0)| \to 0 \quad \text{as} \;\; t \to t_0.$$

This shows that ψ is continuous on $\mathbb{R}_+$. Moreover, from (4.39), we have

$$\begin{aligned}|\psi^{'}(t)-\psi^{'}(t_0)| &\leq \frac{\overline{\delta}}{\epsilon}\left(k\left|e^{-\frac{t}{\epsilon}\delta(0)} - e^{-\frac{t_0}{\epsilon}\delta(0)}\right|\left\|m^0\right\|_{\mathrm{W}^{1,1}([0,\omega])} + q\overline{\beta}\int_0^\omega \left|m^0(a)\right|\left|e^{-\frac{t}{\epsilon}\delta(a)} - e^{-\frac{t_0}{\epsilon}\delta(a)}\right|\,da\right) \\ &\leq \overline{\delta}^2\left(k+q\overline{\beta}\right)\frac{e^{-\frac{\delta}{\epsilon}\alpha}}{\epsilon^2}\,|t-t_0|\left\|m^0\right\|_{\mathrm{W}^{1,1}([0,\omega])}\end{aligned}$$

for $t, t_0 \in \mathbb{R}_+$, $\alpha \in (t_0,t)$ and fixed $\epsilon > 0$. It follows that

$$|\psi^{'}(t)-\psi^{'}(t_0)| \to 0 \quad \text{as} \;\; t \to t_0.$$

This shows that $\psi^{'}$ is continuous on $\mathbb{R}_+$. Hence $\psi \in \mathcal{C}^1(\mathbb{R}_+)$.

Since $\psi \in \mathcal{C}^1(\mathbb{R}_+) \cap \mathrm{L}^\infty(\mathbb{R}_+)$ and $\psi^{'} \in \mathrm{L}^\infty(\mathbb{R}_+)$, from Remark 2 [14, p. 202], we have $\psi \in \mathrm{W}^{1,\infty}(\mathbb{R}_+) \cap \mathcal{C}^1(\mathbb{R}_+)$. Moreover, the usual derivative of ψ coincides with its derivative in the $\mathrm{W}^{1,\infty}$ sense. □

Let B_ψ be the total birth rate of the age-density function $\breve{m}_0(a/\epsilon, t/\epsilon)$, which is given by $\breve{m}_0(0, t/\epsilon) = B_\psi(t)$. Thus, by integration along the characteristics lines $a - t =$ constant, the solution to (4.35) is given by

$$\breve{m}_{0,\epsilon}(a,t) = \begin{cases} 0, & a > t \\ B_\psi\,(t-a)\,\Pi_\mu(a) e^{-\frac{1}{\epsilon}\int_0^a \delta(r)\,dr}, & a < t, \end{cases} \tag{4.41}$$

where

$$B_\psi(t) = \psi(t) + q\int_0^{\min\{t,\omega\}} e^{-\frac{1}{\epsilon}\int_0^{t-s}\delta(r)\,dr} K_\mu(t-s) B_\psi(s)\,ds \tag{4.42}$$

with $\Pi_\mu(a)$ given in (1.2) and $K_\mu(a) = \beta(a)\Pi_\mu(a)$ for $a \in [0,\omega]$. This is the basic tool to investigate (4.35), the connection being provided by the formula (4.41). In the sequel, most of the results will follow from Iannelli [27], though the function $\psi(t)$ in (4.42) does not have the same definition as $F(t)$ in (1.10) and therefore their properties differ. But what matters is that we were able to construct a Volterra integral equation; therefore it is possible to refer to [27]. The following result shows the existence and uniqueness of solution to (4.42) and states some of its properties.

Theorem 4.4.1. *Let* (H1)-(H3) *hold and let* $\psi(t) \geq 0$. *Then equation* (4.42) *has a unique solution* $B_\psi \in \mathcal{C}(\mathbb{R}_+)$ *such that* $B_\psi(t) \geq 0$ *for all* $t > 0$. *Moreover,* $B_\psi \in \mathcal{C}^1([0,T])$ *for any* $0 < T < \infty$, *provided* (4.37) *holds.*

Proof. Let us define

$$\Gamma_\epsilon(t) := q e^{-\frac{1}{\epsilon}\int_0^t \delta(r)\,dr} K_\mu(t)$$

extended by zero for $t > \omega$. It follows that

$$\|\Gamma_\epsilon\|_{L^1(\mathbb{R}_+)} \leq \frac{\epsilon q\overline{\beta}}{\epsilon\underline{\mu} + \underline{\delta}} \leq \epsilon\frac{q\overline{\beta}}{\underline{\delta}} < 1 \tag{4.43}$$

for sufficiently small ϵ. Thus the solution of (4.42) results from the Picard iterations procedure

$$\begin{cases} B_\psi^0(t) = \psi(t) \\ B_\psi^{k+1}(t) = \psi(t) + \int_0^t \Gamma_\epsilon(t-s) B_\psi^k(s)\,ds. \end{cases} \tag{4.44}$$

Since $\psi \in \mathcal{C}(\mathbb{R}_+)$, it follows that $B_\psi^k \in \mathcal{C}([0,T])$, for any $T > 0$. If $\psi(t) \geq 0$ then $B_\psi^k(t) \geq 0$. Moreover, we write B_ψ^k as

$$\begin{aligned} B_\psi^k &= B_\psi^k - B_\psi^{k-1} + B_\psi^{k-1} - B_\psi^{k-2} + \cdots - B_\psi^1 + B_\psi^1 \\ &= B_\psi^1 + \sum_{j=1}^{k-1}\left(B_\psi^{j+1} - B_\psi^j\right). \end{aligned}$$

For any positive integer l such that $k > l$, we write

$$B_\psi^k = B_\psi^l + \sum_{j=l}^{k-1} \left(B_\psi^{j+1} - B_\psi^j \right).$$

Since $B_\psi^k \in \mathcal{C}([0,T])$, by using the sup norm, we have

$$\left\| B_\psi^k - B_\psi^l \right\|_{\mathcal{C}([0,T])} \leq \sum_{j=l}^{k-1} \left\| B_\psi^{j+1} - B_\psi^j \right\|_{\mathcal{C}([0,T])}.$$

Since

$$\left| B_\psi^{j+1}(t) - B_\psi^j(t) \right| \leq \int_0^t \left| \Gamma_\epsilon(t-s) \right| \left| B_\psi^j(s) - B_\psi^{j-1}(s) \right| ds,$$

taking supremum over $[0,T]$, for any $T > 0$, leads to

$$\left\| B_\psi^{j+1} - B_\psi^j \right\|_{\mathcal{C}([0,T])} \leq \left\| \Gamma_\epsilon \right\|_{\mathrm{L}^1(\mathbb{R}_+)} \left\| B_\psi^j - B_\psi^{j-1} \right\|_{\mathcal{C}([0,T])} \tag{4.45}$$

and thus

$$\left\| B_\psi^{j+1} - B_\psi^j \right\|_{\mathcal{C}([0,T])} \leq \left\| \Gamma_\epsilon \right\|_{\mathrm{L}^1(\mathbb{R}_+)}^j \left\| B_\psi^1 - B_\psi^0 \right\|_{\mathcal{C}([0,T])} \leq \left\| \Gamma_\epsilon \right\|_{\mathrm{L}^1(\mathbb{R}_+)}^{j+1} \left\| \psi \right\|_{\mathcal{C}([0,T])}$$

since

$$\left\| B_\psi^1 - B_\psi^0 \right\|_{\mathcal{C}([0,T])} \leq \left\| \Gamma_\epsilon \right\|_{\mathrm{L}^1(\mathbb{R}_+)} \left\| \psi \right\|_{\mathcal{C}([0,T])}.$$

Hence,

$$\begin{aligned} \left\| B_\psi^k - B_\psi^l \right\|_{\mathcal{C}([0,T])} &\leq \|\psi\|_{\mathcal{C}([0,T])} \sum_{j=l}^{k-1} \left\| \Gamma_\epsilon \right\|_{\mathrm{L}^1(\mathbb{R}_+)}^{j+1} \\ &\leq \|\psi\|_{\mathcal{C}([0,T])} \left(\frac{\left\| \Gamma_\epsilon \right\|_{\mathrm{L}^1(\mathbb{R}_+)}^l - \left\| \Gamma_\epsilon \right\|_{\mathrm{L}^1(\mathbb{R}_+)}^k}{1 - \left\| \Gamma_\epsilon \right\|_{\mathrm{L}^1(\mathbb{R}_+)}} \right) \to 0 \quad \text{as } k, l \to \infty, \end{aligned}$$

provided estimate (4.43) holds; hence B_ψ^k is a Cauchy sequence in $\mathcal{C}([0,T])$ and thus it converges uniformly in $\mathcal{C}([0,T])$ to a solution B_ψ given by

$$B_\psi(t) = \psi(t) + \int_0^t \Gamma_\epsilon(t-s) B_\psi(s)\, ds. \tag{4.46}$$

We have $B_\psi \in \mathcal{C}([0,T])$. Moreover, if $\psi(t) \geq 0$ then $B_\psi(t) \geq 0$.

If B_ψ and B_ψ^* are solutions of (4.46), by similar calculations to the estimate (4.45), we have

$$\left\| B_\psi - B_\psi^* \right\|_{\mathcal{C}([0,T])} \leq \left\| \Gamma_\epsilon \right\|_{\mathrm{L}^1(\mathbb{R}_+)} \left\| B_\psi - B_\psi^* \right\|_{\mathcal{C}([0,T])}.$$

Thus, we obtain

$$\left(1 - \left\|\Gamma_\epsilon\right\|_{\mathrm{L}^1(\mathbb{R}_+)}\right) \left\|B_\psi - B_\psi^*\right\|_{\mathcal{C}([0,T])} \leq 0,$$

where $\left(1 - \left\|\Gamma_\epsilon\right\|_{\mathrm{L}^1(\mathbb{R}_+)}\right) > 0$ by (4.43), for sufficiently small ϵ. Hence,

$$\left\|B_\psi - B_\psi^*\right\|_{\mathcal{C}([0,T])} = 0,$$

and so we obtain

$$B_\psi = B_\psi^* \ \text{ in } \mathcal{C}\left([0,T]\right)$$

for any $T > 0$. This concludes the uniqueness of the solution of (4.42).

Since $\psi \in \mathrm{W}^{1,\infty}(\mathbb{R}_+) \cap \mathcal{C}^1(\mathbb{R}_+)$, by (4.43), the iterates B_ψ^k given by (4.44) satisfy

$$B_\psi^k \in \mathcal{C}^1([0,T]) \tag{4.47}$$

for any $0 < T < \infty$.

In the next step, we define a function

$$V_\psi^k(t) := \frac{d}{dt} B_\psi^k(t)$$

for $t \in [0,T]$, $T > 0$, which is given by the iterates

$$\begin{cases} V_\psi^0(t) & = \psi'(t) \\ V_\psi^{k+1}(t) & = \psi'(t) + \Gamma_\epsilon(t) B_\psi^k(0) + \int_0^t \Gamma_\epsilon(t-s) V_\psi^k(s)\, ds \end{cases}$$

with $B_\psi^k(0) = \psi(0)$, where $\psi(0) = -m^0(0) + q\int_0^\omega \beta(a) m^0(a)\, ds$. Since $m^0 \in D(A)$, thus $\psi(0) = 0$; hence

$$\begin{cases} V_\psi^0(t) & = \psi'(t) \\ V_\psi^{k+1}(t) & = \psi'(t) + \int_0^t \Gamma_\epsilon(t-s) V_\psi^k(s)\, ds. \end{cases} \tag{4.48}$$

We recall that B_ψ^k are differentiable, see (4.47), and converge uniformly on $[0,T]$; hence $B_\psi^k(t)$ converge pointwise for any $t \in [0,T]$. From (4.48), we can also prove, similarly as in the case of B_ψ^k, that V_ψ^k is a Cauchy sequence in $\mathcal{C}([0,T])$ and thus it converges uniformly in $\mathcal{C}([0,T])$. Hence, from Theorem 7.17 in [46], we have

$$\frac{d}{dt} B_\psi(t) = \lim_{n\to\infty} V_\psi^k(t) = \lim_{n\to\infty} \frac{d}{dt} B_\psi^k(t), \qquad (0 \leq t \leq T).$$

It follows that

$$B_\psi' \in \mathcal{C}([0,T]),$$

and hence we arrive at

$$B_\psi \in \mathcal{C}^1([0,T]) \tag{4.49}$$

for any $0 < T < \infty$. Moreover, from (4.48), we obtain

$$B_{\psi}^{'}(t) = \psi^{'}(t) + \int_0^t \Gamma_\epsilon(t-s)B_{\psi}^{'}(s)\,ds. \tag{4.50}$$

□

In the following result, we show that $\breve{m}_0$ is a classical solution to (4.35). For convenience, we denote $\breve{m}_{0,\epsilon}(a,t) := \breve{m}_0(a/\epsilon, t/\epsilon)$.

Theorem 4.4.2. *Let* (H1)-(H3), (4.37) *hold and let* $\breve{m}_{0,\epsilon}(a,t)$ *be defined by* (4.41)*, where* $B_\psi(t)$ *is the solution of* (4.42)*. If* $\psi \geq 0$ *and* $\psi(0) = 0$ *hold, then:*

$$\breve{m}_{0,\epsilon} \in \mathcal{C}([0,\omega]\times\mathbb{R}_+),\ \breve{m}_{0,\epsilon}(a,t) \geq 0,\ \mu(\cdot)\breve{m}_{0,\epsilon}(\cdot,t) \in \mathrm{L}^1([0,\omega]),\ \forall\, t > 0; \tag{4.51}$$

$\breve{m}_{0,\epsilon}$ *is a locally Lipschitz continuous function on* $[0,\omega]\times\mathbb{R}_+$; $\dfrac{\partial}{\partial t}\breve{m}_{0,\epsilon}(a,t)$, $\dfrac{\partial}{\partial a}\breve{m}_{0,\epsilon}(a,t)$ *exist a.e. on* $[0,\omega]\times\mathbb{R}_+$, *belong to* $\mathrm{L}^1([0,\omega])$ *and coincide with the distributional derivatives.* (4.52)

Proof. Using formula (4.41) and the properties of B_ψ stated in Theorem 4.4.1, for $a < t$, we have

$$\begin{aligned}|\breve{m}_{0,\epsilon}(a+h_1,t+h_2) - \breve{m}_{0,\epsilon}(a,t)| \leq{}& |\breve{m}_{0,\epsilon}(a+h_1,t+h_2) - \breve{m}_{0,\epsilon}(a,t+h_2)| \\ &+ |\breve{m}_{0,\epsilon}(a,t+h_2) - \breve{m}_{0,\epsilon}(a,t)|,\end{aligned}$$

where $a + h_1 < t + h_2$. We show that

$$\begin{aligned}&\left|\breve{m}_{0,\epsilon}(a+h_1,t+h_2) - \breve{m}_{0,\epsilon}(a,t+h_2)\right| \\ &\quad= |B_\psi(t+h_2-a-h_1)\Pi_{\mathcal{M}_\delta}(a+h_1) - B_\psi(t+h_2-a)\Pi_{\mathcal{M}_\delta}(a)|, \\ &\quad= |\Pi_{\mathcal{M}_\delta}(a)|\left|B_\psi(t+h_2-a-h_1)e^{-\int_a^{a+h_1}(\mu(s)+\frac{1}{\epsilon}\delta(s))\,ds} - B_\psi(t+h_2-a)\right|, \\ &\quad\leq e^{-(\underline{\mu}+\frac{1}{\epsilon}\underline{\delta})a}\left|\left(e^{-\int_a^{a+h_1}(\mu(r)+\frac{1}{\epsilon}\delta(r))\,dr} - 1\right)B_\psi(t-a+h_2-h_1)\right| \\ &\qquad+ e^{-(\underline{\mu}+\frac{1}{\epsilon}\underline{\delta})a}|B_\psi(t-a+h_2-h_1) - B_\psi(t-a+h_2)|\end{aligned}$$

and

$$\begin{aligned}&\left|\breve{m}_{0,\epsilon}(a,t+h_2) - \breve{m}_{0,\epsilon}(a,t)\right| \\ &\quad= |B_\psi(t+h_2-a)\Pi_{\mathcal{M}_\delta}(a) - B_\psi(t-a)\Pi_{\mathcal{M}_\delta}(a)|, \\ &\quad= |\Pi_{\mathcal{M}_\delta}(a)||B_\psi(t+h_2-a) - B_\psi(t-a)|, \\ &\quad\leq e^{-(\underline{\mu}+\frac{1}{\epsilon}\underline{\delta})a}|B_\psi(t+h_2-a) - B_\psi(t-a)|,\end{aligned}$$

where $\mathcal{M}_\delta := \mu + \frac{1}{\epsilon}\delta$ and $\Pi_{\mathcal{M}_\delta}(a) := e^{-\int_0^a \mathcal{M}_\delta(s)\,ds}$.
Using the continuity of B_ψ with respect to t, we clearly see that

$$|\breve{m}_{0,\epsilon}(a+h_1, t+h_2) - \breve{m}_{0,\epsilon}(a, t+h_2)|\,,\ \left|\breve{m}_{0,\epsilon}(a, t+h_2) - \breve{m}_{0,\epsilon}(a,t)\right| \to 0 \text{ as } h_1, h_2 \to 0$$

and we arrive at

$$\lim_{\substack{h_1\to 0\\ h_2\to 0}} \left|\breve{m}_{0,\epsilon}(a+h_1, t+h_2) - \breve{m}_{0,\epsilon}(a,t)\right| = 0.$$

This shows that $\breve{m}_{0,\epsilon}$ is continuous for $a < t$. Moreover, at the origin $(0,0)$, the continuity is ensured by $\psi(0) = 0$ which we obtained before. In fact, $\psi(0) = 0$ implies $B_\psi(0) = 0$ and hence $\breve{m}_{0,\epsilon}(0,0) = 0$. Therefore the continuity of $\breve{m}_{0,\epsilon}$ at the origin $(0,0)$ holds. Now, we shall prove that the continuity of $\breve{m}_{0,\epsilon}$ holds across the diagonal $a = t$. Since the maps $t \mapsto a - t$ and $t \mapsto B_\psi(t)$ are both continuous on $[0,T]$, it follows that

$$\begin{aligned}
\lim_{t\to a^-} \breve{m}_{0,\epsilon}(a,t) &= \lim_{t\to a^-} B_\psi(t-a)\Pi_\mu(a)e^{-\frac{1}{\epsilon}\int_0^a \delta(r)\,dr} \\
&= B_\psi\big(\lim_{t\to a^-}(t-a)\big)\Pi_\mu(a)e^{-\frac{1}{\epsilon}\int_0^a \delta(r)\,dr} \\
&= B_\psi(0)\Pi_\mu(a)e^{-\frac{1}{\epsilon}\int_0^a \delta(r)\,dr} = 0.
\end{aligned}$$

Hence, $\breve{m}_{0,\epsilon}(a,t)$ is a continuous function on $a \le t$.
Since $\breve{m}_{0,\epsilon} = 0$ for $a > t$, this is sufficient to state the continuity of $\breve{m}_{0,\epsilon}$ across the diagonal $a = t$. Hence, we have

$$\breve{m}_{0,\epsilon} \in \mathcal{C}([0,\omega]\times[0,T]), \qquad \text{for any } T > 0. \tag{4.53}$$

Further, since we proved earlier that $B_\psi \ge 0$ if $\psi \ge 0$, see Theorem 4.4.1, then from (4.41) we see that $\breve{m}_{0,\epsilon}(a,t) \ge 0$.

For arbitrary $T > 0$ and $t \in [0,T]$, we have:

$$\begin{aligned}
\int_0^\omega \mu(a)\breve{m}_{0,\epsilon}(a,t)\,da &= \int_0^{\min\{t,\omega\}} \mu(a)e^{-\int_0^a\left(\mu(r)+\frac{1}{\epsilon}\delta(r)\right)dr}B_\psi(t-a)\,da, \\
&\le \left(\max_{s\in[0,t]} B_\psi(s)\right)\int_0^{\min\{t,\omega\}} \frac{d}{da}\left(-e^{-\int_0^a \mu(r)\,dr}\right)e^{-\frac{a}{\epsilon}\underline{\delta}}\,da, \\
&\le \left(\max_{s\in[0,T]} B_\psi(s)\right)\left(1 - \underline{\delta}\int_0^{\min\left\{\frac{t}{\epsilon},\frac{\omega}{\epsilon}\right\}} e^{-\int_0^{\epsilon\sigma}\mu(r)\,dr}e^{-\sigma\underline{\delta}}\,d\sigma\right), \\
&\le \max_{s\in[0,T]} B_\psi(s) < \infty,
\end{aligned}$$

hence $\mu(\cdot)\breve{m}_{0,\epsilon}(\cdot,t) \in \mathrm{L}([0,\omega])$ for all $t > 0$.

In the rest of this, we prove the existence of first order partial derivatives of $\breve{m}_{0,\epsilon}$. Since the survival probability function, $\Pi_\mu(a)$, is a.e. differentiable with bounded derivatives on $[0,\tilde{\omega}]$ for any $\tilde{\omega} < \omega$, thus

$$\Pi_\mu(a) e^{-\frac{1}{\epsilon}\int_0^a \delta(r)\,dr} \in \mathcal{C}([0,\omega]) \cap \mathcal{C}^1([0,\omega)) \cap \mathrm{W}^{1,1}([0,\omega]) \cap \mathrm{W}^{1,\infty}_{\mathrm{loc}}([0,\omega)). \tag{4.54}$$

We see that $\breve{m}_{0,\epsilon}$ is a.e. differentiable for $t > a$ and $t < a$. Next, we show that $\breve{m}_{0,\epsilon}$ is locally Lipschitz continuous across the diagonal. Consider arbitrary $R = \{(a,t) : \underline{a} \le a \le \overline{a},\, \underline{t} \le t \le \overline{t}\}$ with $\overline{a} < \omega$. Let $p_0 = (a_0, t_0)$, $p_1 = (a_1, t_1)$ and $p_2 = (a_2, t_2)$ be three co-linear points in R so that $a_0 \le \max\{a_1, a_2\} \le \overline{a} < \omega$ and p_0 on the diagonal. We show that

$$|\breve{m}_{0,\epsilon}(p_1) - \breve{m}_{0,\epsilon}(p_2)| \le |\breve{m}^u_{0,\epsilon}(p_1) - \breve{m}^u_{0,\epsilon}(p_0)| + |\breve{m}^d_{0,\epsilon}(p_0) - \breve{m}^d_{0,\epsilon}(p_2)|, \tag{4.55}$$

where $\breve{m}^u_{0,\epsilon}(a,t) = B_\psi(t-a)\Pi_\mu(a)e^{-\frac{1}{\epsilon}\int_0^a \mu(r)\,dr}$ and $\breve{m}^d_{0,\epsilon}(a,t) = 0$, see (4.41). Moreover, $\breve{m}^u_{0,\epsilon}(p_0) = \breve{m}^d_{0,\epsilon}(p_0) = 0$ by continuity, see (4.53). Hence, from (4.55), we have

$$|\breve{m}_{0,\epsilon}(p_1) - \breve{m}_{0,\epsilon}(p_2)| \le |\breve{m}^u_{0,\epsilon}(p_1)|.$$

For convenience, we write

$$|\breve{m}_{0,\epsilon}(p_1) - \breve{m}_{0,\epsilon}(p_2)| \le |\breve{m}^u_{0,\epsilon}(p_1) - \breve{m}^u_{0,\epsilon}(p_0)|.$$

Since $\breve{m}_{0,\epsilon}$ is a.e. differentiable for $(a,t) \in R$ and the derivatives are bounded for $a \le \overline{a} < \omega$, then $\breve{m}^u_{0,\epsilon}$ is Lipschitz continuous in R. Hence

$$|\breve{m}_{0,\epsilon}(p_1) - \breve{m}_{0,\epsilon}(p_2)| \le M_1 \|p_1 - p_0\|,$$

where $0 < M_1 < \infty$, see (4.49) and (4.54). We have

$$|\breve{m}_{0,\epsilon}(p_1) - \breve{m}_{0,\epsilon}(p_2)| \le M_1 \|p_1 - p_0\| + M_2 \|p_0 - p_2\|$$

for any $0 < M_2 < \infty$. From the co-linearity of p_0, p_1 and p_2, it follows that

$$|\breve{m}_{0,\epsilon}(p_1) - \breve{m}_{0,\epsilon}(p_2)| \le \max\{M_1, M_2\}\, \|p_1 - p_2\|.$$

This shows that $\breve{m}_{0,\epsilon}$ is Lipschitz on $(0,\omega) \times \mathbb{R}_+$. Then to prove that the partial derivatives coincide with distributional derivatives on $(0,\omega)\times\mathbb{R}_+$, let $\varphi \in \mathcal{C}^\infty_0\big((0,\omega)\times\mathbb{R}_+\big)$. We assume that $\mathrm{supp}\varphi$ intersects the diagonal (otherwise the statement is obvious). We provide calculations for the derivative with respect to t. Assume $\mathrm{supp}\varphi \subset [\underline{a}, \overline{a}] \times [\underline{t}, \overline{t}] \subset (0,\omega)\times\mathbb{R}_+$, we have

$$\int_{(0,\omega)\times\mathbb{R}_+} \breve{m}_{0,\epsilon}(a,t) \frac{\partial \varphi}{\partial t}(a,t)\, da\, dt = \int_{\underline{a}}^{\overline{a}} \left(\int_a^{\overline{t}} \breve{m}^u_{0,\epsilon}(a,t) \frac{\partial \varphi}{\partial t}(a,t)\, dt \right) da.$$

Since $\breve{m}_{0,\epsilon}$ is differentiable with respect to t for $t \geq a$, it follows that

$$\int_{(0,\omega)\times\mathbb{R}_+} \breve{m}_{0,\epsilon}(a,t)\frac{\partial\varphi}{\partial t}(a,t)\,da\,dt = \int_{\underline{a}}^{\overline{a}} \bigg(\breve{m}^u_{0,\epsilon}(a,\bar{t})\varphi(a,\bar{t}) - \breve{m}^u_{0,\epsilon}(a,a)\varphi(a,a) - \int_a^{\bar{t}} \frac{\partial \breve{m}^u_{0,\epsilon}}{\partial t}(a,t)\varphi(a,t)\,dt\bigg)\,da.$$

Since $\varphi(a,\bar{t}) = 0$, as $\mathrm{supp}\varphi \subset R$, and $\breve{m}^u_{0,\epsilon}(a,a) = 0$ by continuity, see (4.53), we obtain

$$\begin{aligned}\int_{(0,\omega)\times\mathbb{R}_+} \breve{m}_{0,\epsilon}(a,t)\frac{\partial\varphi}{\partial t}(a,t)\,da\,dt &= -\int_{\underline{a}}^{\overline{a}}\int_a^{\bar{t}} \frac{\partial \breve{m}^u_{0,\epsilon}}{\partial t}(a,t)\varphi(a,t)\,dt\,da \\ &= \int_{\underline{a}}^{\overline{a}}\int_a^{\bar{t}} \psi(a,t)\varphi(a,t)\,dt\,da\end{aligned}$$

where

$$\psi(a,t) := \begin{cases} \frac{\partial \breve{m}^u_{0,\epsilon}}{\partial t}(a,t), & a < t \\ 0, & a > t. \end{cases}$$

Hence ψ is the distributional derivative of

$$\breve{m}_{0,\epsilon}(a,t) = \begin{cases} \breve{m}^u_{0,\epsilon}(a,t), & a < t \\ 0, & a > t \end{cases}$$

and coincides with pointwise partial derivative of $\breve{m}_{0,\epsilon}$ with respect to t. □

Next we find the estimate of the corner corrector $\breve{m}_{0,\epsilon}$, which is essential for the rest of our analysis. We start by deriving estimates in the general case for the solution to a scalar McKendrick problem of the form

$$\frac{\partial\phi}{\partial t} = \frac{\partial\phi}{\partial a} - \mathcal{M}(a)\phi, \tag{4.56a}$$

$$\phi(0,t) = \int_0^\omega \beta(a)\phi(a,t)\,da + \Phi(t), \tag{4.56b}$$

$$\phi(a,0) = 0, \tag{4.56c}$$

where Φ is the inhomogeneous term. Also, $\mathcal{M} \geq 0$ is a function bounded on each interval $[0,a]$ with $a < \omega$ and such that

$$\Pi_{\mathcal{M}}(a) := e^{-\int_0^a \mathcal{M}(r)\,dr}$$

satisfies $\Pi_{\mathcal{M}}(\omega) = 0$. Further, $\Phi(t) = O\left(e^{-ct}\right)$ for some constant c. Then the solution to (4.56) is given by

$$\breve{m}_{0,\epsilon}(a,t) = \begin{cases} 0, & a > t, \\ B_\Phi\left(t-a\right)\Pi_{\mathcal{M}}(a), & a < t, \end{cases} \tag{4.57}$$

with

$$B_\Phi(t) = \Phi(t) + \int_0^{\min\{t,\omega\}} K_{\mathcal{M}}(t-a)B_\Phi(a)\,da, \tag{4.58}$$

where $K_{\mathcal{M}}(a) = \beta(a)\Pi_{\mathcal{M}}(a)$ and $R_{\mathcal{M}} = \int_0^\omega K_{\mathcal{M}}(a)\,da$. We have $K_{\mathcal{M}}(a) \leq \overline{\beta}e^{-\underline{\mathcal{M}}a}$ where $\underline{\mathcal{M}} = \inf\limits_{a\in[0,\omega]} \mathcal{M}(a)$.

Since $\Phi(t) = O\left(e^{-ct}\right)$, the function $\Phi(t)$ is absolutely Laplace transformable. Further, since $|K_{\mathcal{M}}(t)| < \overline{\beta}e^{-\underline{\mathcal{M}}t}$, thus $K_{\mathcal{M}}(t)$, extended by zero for $t > \omega$, is absolutely Laplace transformable. Using, respectively, the notation $\widehat{\Phi}(\lambda)$ and $\widehat{K}_{\mathcal{M}}(\lambda)$ for the Laplace transforms of $\Phi(t)$ and $K_{\mathcal{M}}(t)$, we have

$$\widehat{\Phi}(\lambda) = \int_0^\infty e^{-\lambda a}\Phi(a)\,da, \quad \widehat{K}_{\mathcal{M}}(\lambda) = \int_0^\infty e^{-\lambda a}K_{\mathcal{M}}(a)\,da. \tag{4.59}$$

Taking the Laplace transform of (4.58) we obtain

$$\widehat{B}_\Phi(\lambda) = \widehat{\Phi}(\lambda) + \widehat{B}_\Phi(\lambda)\widehat{K}_{\mathcal{M}}(\lambda).$$

Hence,

$$\widehat{B}_\Phi(\lambda) = \frac{\widehat{\Phi}(\lambda)}{1-\widehat{K}_{\mathcal{M}}(\lambda)} = \widehat{\Phi}(\lambda) + \frac{\widehat{\Phi}(\lambda)\widehat{K}_{\mathcal{M}}(\lambda)}{1-\widehat{K}_{\mathcal{M}}(\lambda)}. \tag{4.60}$$

Note that contrary to the case considered by Iannelli [27], $\Phi(t)$ does not vanish for $t > \omega$. Thus, $\widehat{\Phi}(\lambda)$ is not an entire function but it is analytic in $\Re\lambda > -c$, so the only singularities of $\widehat{B}_\Phi(\lambda)$ are either the roots of the equation $\widehat{K}_{\mathcal{M}}(\lambda) = 1$ or are located in the half-plane $\{\Re\lambda \leq -c\}$.
The functions $\widehat{\Phi}(\lambda)$ and $\widehat{K}_{\mathcal{M}}(\lambda)$ tend to zero as $|\lambda| \to \infty$ in any half plane $\Re\lambda > \nu$, $\nu > -c$. A proof of this result, for L^1-functions, is given in [26, p. 66-67]. However, if Φ and $K_{\mathcal{M}}$ are differentiable a.e. with bounded derivative on $[0,\infty)$ and $[0,\omega)$, respectively, then the proof of this fact is elementary and we give it below. Since $\Phi(t) = O\left(e^{-ct}\right)$, we have $|\Phi(t)| = O\left(e^{-ct}\right)$. Further, we assume that $\left|\Phi'(t)\right| = O\left(e^{-ct}\right)$. For $\lambda = x + iR$, we have

$$\begin{aligned}\widehat{\Phi}(x+iR) &= \int_0^\infty e^{-(x+iR)s}\Phi(s)\,ds \\ &= \frac{\Phi(0)}{x+iR} + \frac{1}{x+iR}\int_0^\infty e^{-(x+iR)s}\Phi'(s)\,ds.\end{aligned}$$

Taking the absolute value on the left and right hand sides, we get

$$\begin{aligned}\left|\widehat{\Phi}(x+iR)\right| &\leq \left|\frac{\Phi(0)}{x+iR}\right| + \left|\frac{1}{x+iR}\int_0^\infty e^{-(x+iR)s}\Phi'(s)\,ds\right| \\ &\leq \frac{1}{\sqrt{x^2+R^2}}\left(|\Phi(0)| + \int_0^\infty e^{-xs}\left|\Phi'(s)\right|\,ds\right) \\ &\leq \frac{1}{|\lambda|}\left(|\Phi(0)| + \int_0^\infty e^{-xs}\left|\Phi'(s)\right|\,ds\right) \\ &\leq \frac{1}{|\lambda|}\left(L_1 + L_2\int_0^\infty e^{-(x+c)s}\,ds\right) \leq \frac{1}{|\lambda|}\cdot\left(L_1 + \frac{L_2}{\sigma_1+c}\right),\end{aligned} \tag{4.61}$$

for some constants L_1, L_2, where $\sigma_1 < \Re\lambda < \sigma_2$, as given in (4.64). Thus $\widehat{\Phi}(\lambda) \to 0$ as $|\lambda| \to \infty$, uniformly in x. Since $\mathrm{supp} K_{\mathcal{M}} = [0,\omega]$ and $K_{\mathcal{M}}(s)$ is differentiable on $[0,\omega)$ with bounded derivative, we have

$$\begin{aligned}\widehat{K}_{\mathcal{M}}(x+iR) &= \int_0^\omega e^{-(x+iR)s}K_{\mathcal{M}}(s)\,ds \\ &= \frac{\beta(0)}{x+iR} + \frac{1}{x+iR}\int_0^\omega e^{-(x+iR)s}K'_{\mathcal{M}}(s)\,ds \\ &= \frac{1}{x+iR}\left[\beta(0) + \int_0^\omega e^{-(x+iR)s}e^{-\int_0^s \mathcal{M}(r)\,dr}\Big(\beta'(s) - \beta(s)\mathcal{M}(s)\Big)\,ds\right].\end{aligned}$$

Denote $\tilde{\beta} = \operatorname*{esssup}_{a\in[0,\omega]} \beta'(a)$. Taking the absolute value on the left and right hand sides, we get

$$\begin{aligned}\left|\widehat{K}_{\mathcal{M}}(x+iR)\right| &\leq \frac{1}{\sqrt{x^2+R^2}}\left[\overline{\beta} + \tilde{\beta}\int_0^\omega e^{-xs}e^{-\underline{\mathcal{M}}s}\,ds + \overline{\beta}\int_0^\omega e^{-xs}\frac{d}{ds}\left(-e^{-\int_0^s \mathcal{M}(r)\,dr}\right)\,ds\right] \\ &\leq \frac{1}{|\lambda|}\left[2\overline{\beta} + \left(\tilde{\beta} + x\overline{\beta}\right)\int_0^\omega e^{-(x+\underline{\mathcal{M}})s}\,ds\right] \\ &\leq \frac{1}{|\lambda|}\Big[2\overline{\beta} + \frac{\tilde{\beta}+\sigma\overline{\beta}}{\underline{\mathcal{M}}+\sigma_1}\Big],\end{aligned} \tag{4.62}$$

which shows that $\widehat{K}_{\mathcal{M}}(\lambda) \to 0$ as $|\lambda| \to \infty$, uniformly in x.

On the other hand, as $K_{\mathcal{M}}(t)$ vanishes for $t > \omega$, $\widehat{K}_{\mathcal{M}}$ is an entire function and thus the roots of the following equation

$$\widehat{K}_{\mathcal{M}}(\lambda) = 1 \tag{4.63}$$

are isolated of finite order and give rise to the poles of $\widehat{B}_\Phi(\lambda)$. As in the treatment of problem (4.56) for $\Phi = 0$, [27], the characteristic equation (4.63) has only one real root, $\lambda = \lambda_{\mathcal{M}}$, of algebraic multiplicity 1 and all other roots λ_j of (4.63) arise as complex conjugates and fulfil $\Re\lambda_j < \lambda_{\mathcal{M}}$. Moreover, since $\widehat{K}_{\mathcal{M}}(\lambda) \to 0$ as $|\lambda| \to \infty$, within any strip

$$-c < \sigma_1 < \Re\lambda < \sigma_2 \tag{4.64}$$

there is at most a finite number of the roots because otherwise $\widehat{K}_{\mathcal{M}}(\lambda)$ would vanish identically. Finally, $\lambda_{\mathcal{M}} > 0$, $\lambda_{\mathcal{M}} = 0$ and $\lambda_{\mathcal{M}} < 0$ if and only if, respectively, $R_{\mathcal{M}} > 1$, $R_{\mathcal{M}} = 1$ and $R_{\mathcal{M}} < 1$. Now, we make the assumption

$$-c < \lambda_{\mathcal{M}} \tag{4.65}$$

so that $\lambda_{\mathcal{M}}$ is in the domain of analyticity of $\widehat{\Phi}(\lambda)$.

We take into account the last term in (4.60),

$$\widehat{H}(\lambda) = \frac{\widehat{\Phi}(\lambda)\widehat{K}_{\mathcal{M}}(\lambda)}{1 - \widehat{K}_{\mathcal{M}}(\lambda)}.$$

Hence, on each line $\{\sigma + iy;\, y \in \mathbb{R}\}$ which does not meet any root of (4.63), we have

$$\inf_{y\in\mathbb{R}} \left|1 - \widehat{K}_{\mathcal{M}}(\sigma + iy)\right| = m_\sigma > 0$$

and

$$\int_{-\infty}^{+\infty} \left| \frac{\widehat{\Phi}(\sigma + iy)\widehat{K}_{\mathcal{M}}(\sigma + iy)}{1 - \widehat{K}_{\mathcal{M}}(\sigma + iy)} \right| dy < \infty.$$

The last inequality follows from the fact that we have:

$$\int_{-\infty}^{+\infty} \left| \frac{\widehat{\Phi}(\sigma + iy)\widehat{K}_{\mathcal{M}}(\sigma + iy)}{1 - \widehat{K}_{\mathcal{M}}(\sigma + iy)} \right| dy \leq \frac{1}{m_\sigma} \int_{-\infty}^{+\infty} \left|\widehat{\Phi}(\sigma + iy)\right| \left|\widehat{K}_{\mathcal{M}}(\sigma + iy)\right| dy. \tag{4.66}$$

Since

$$\begin{aligned}\widehat{K}_{\mathcal{M}}(\sigma + iy) &= \int_0^\infty e^{-(\sigma+iy)t} K_{\mathcal{M}}(t)\, dt \\ &= \int_0^\infty e^{-iyt} e^{-\sigma t} K_{\mathcal{M}}(t)\, dt \\ &= \int_{-\infty}^\infty e^{-iyt} k_\sigma(t)\, dt,\end{aligned}$$

where

$$k_\sigma(t) := \begin{cases} e^{-\sigma t} K_{\mathcal{M}}(t), & t \geq 0 \\ 0, & t < 0, \end{cases}$$

we hence write

$$\widehat{K}_{\mathcal{M}}(\sigma + iy) = \check{k}_\sigma(y)$$

where $\check{k}_\sigma(y)$ is the Fourier transform of $k_\sigma(t)$.

Likewise, we show that

$$\widehat{\Phi}(\sigma+iy)=\check{\phi}_\sigma(y)$$

where $\check{\phi}_\sigma(y)$ is the Fourier transform of $\phi_\sigma(t)$ defined by

$$\phi_\sigma(t):=\begin{cases} e^{-\sigma t}\Phi(t), & t\geq 0\\ 0, & t<0,\end{cases}$$

Moreover, since

$$\left(\int_{-\infty}^{\infty}|k_\sigma(t)|^2\,dt\right)^{\frac{1}{2}}<\frac{\overline{\beta}}{\sqrt{2(\sigma+\underline{\mathcal{M}})}},$$

it follows that $k_\sigma\in \mathrm{L}^2(\mathbb{R})$; hence, from the Plancherel theorem, we have $\check{k}_\sigma\in \mathrm{L}^2(\mathbb{R})$. Likewise, we also show that $\check{\phi}_\sigma\in \mathrm{L}^2(\mathbb{R})$. Hence (4.66) becomes

$$\int_{-\infty}^{+\infty}\left|\frac{\widehat{\Phi}(\sigma+iy)\widehat{K}_{\mathcal{M}}(\sigma+iy)}{1-\widehat{K}_{\mathcal{M}}(\sigma+iy)}\right|dy\leq\frac{1}{m_\sigma}\int_{-\infty}^{+\infty}|\check{\phi}_\sigma(y)||\check{k}_\sigma(y)|\,dy$$

and it follows, from Cauchy-Schwarz inequality, that

$$\begin{aligned}\int_{-\infty}^{+\infty}\left|\frac{\widehat{\Phi}(\sigma+iy)\widehat{K}_{\mathcal{M}}(\sigma+iy)}{1-\widehat{K}_{\mathcal{M}}(\sigma+iy)}\right|dy&\leq\frac{1}{m_\sigma}\int_{-\infty}^{+\infty}|\check{\phi}_\sigma(y)|\check{k}_\sigma(y)|\,dy\\&\leq\frac{1}{m_\sigma}\|\check{k}_\sigma\|_{\mathrm{L}^2(\mathbb{R})}\|\check{\phi}_\sigma\|_{\mathrm{L}^2(\mathbb{R})}<\infty.\end{aligned}$$

Taking the inverse Laplace transform of $\widehat{H}(\lambda)$, we obtain

$$H(t)=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\widehat{\Phi}(\sigma+iy)\widehat{K}_{\mathcal{M}}(\sigma+iy)}{1-\widehat{K}_{\mathcal{M}}(\sigma+iy)}e^{(\sigma+iy)t}\,dy \tag{4.67}$$

for any $\sigma>\lambda_{\mathcal{M}}$. Hence

$$B_\Phi(t)=\Phi(t)+H(t).$$

To estimate $H(t)$ we note that, by properties of $\widehat{H}$, we can shift the line of integration to $\{\sigma_1+iy;\,y\in\mathbb{R}\}$ where $\zeta<\sigma_1<\lambda_{\mathcal{M}}$, $\zeta=\max\{\Re\lambda_1,-c\}$ and λ_1 is the first eigenvalue such that $\Re\lambda_1<\lambda_{\mathcal{M}}$. By application of the Cauchy theorem over the closed region
$\{\sigma_1\leq\Re z\leq\sigma,\,-R\leq\Im z\leq R;\}$, we have

$$\begin{aligned}&\int_{-R}^{R}\frac{\widehat{\Phi}(\sigma+iy)\widehat{K}_{\mathcal{M}_\delta}(\sigma+iy)}{1-\widehat{K}_{\mathcal{M}}(\sigma+iy)}e^{(\sigma+iy)t}\,dy+\int_{\sigma}^{\sigma_1}\frac{\widehat{\Phi}(x+iR)\widehat{K}_{\mathcal{M}}(x+iR)}{1-\widehat{K}_{\mathcal{M}}(x+iR)}e^{(x+iR)t}\,dx\\&+\int_{+R}^{-R}\frac{\widehat{\Phi}(\sigma_1+iy)\widehat{K}_{\mathcal{M}}(\sigma_1+iy)}{1-\widehat{K}_{\mathcal{M}}(\sigma_1+iy)}e^{(\sigma_1+iy)t}\,dy+\int_{\sigma_1}^{\sigma}\frac{\widehat{\Phi}(x-iR)\widehat{K}_{\mathcal{M}}(x-iR)}{1-\widehat{K}_{\mathcal{M}}(x-iR)}e^{(x-iR)t}\,dx\\&=i2\pi\underset{\lambda=\lambda_{\mathcal{M}}}{res}\frac{e^{\lambda t}\widehat{\Phi}(\lambda)\widehat{K}_{\mathcal{M}}(\lambda)}{1-\widehat{K}_{\mathcal{M}}(\lambda)}.\end{aligned} \tag{4.68}$$

From the second term on the left hand side in (4.68), we obtain

$$\left|\int_{\sigma}^{\sigma_1} \frac{\widehat{\Phi}(x+iR)\widehat{K}_{\mathcal{M}}(x+iR)}{1-\widehat{K}_{\mathcal{M}}(x+iR)} e^{(x+iR)t}\,dx\right| \le \int_{\sigma}^{\sigma_1} \frac{|\widehat{\Phi}(x+iR)\widehat{K}_{\mathcal{M}}(x+iR)|}{|1-\widehat{K}_{\mathcal{M}}(x+iR)|} e^{xt}\,dx$$
$$\le e^{\sigma_1 t}\int_{\sigma}^{\sigma_1} \frac{|\widehat{\Phi}(x+iR)\widehat{K}_{\mathcal{M}}(x+iR)|}{|1-\widehat{K}_{\mathcal{M}}(x+iR)|}\,dx$$
$$\le \frac{e^{\sigma_1 t}}{m_{\sigma_1}}\int_{\sigma}^{\sigma_1} |\widehat{\Phi}(x+iR)||\widehat{K}_{\mathcal{M}}(x+iR)|\,dx.$$

Since, $\widehat{\Phi}(x+iR), \widehat{K}(x+iR) \to 0$ as $R\to\infty$ uniformly for x in any bounded interval, and, from estimates (4.61) and (4.62),

$$|\widehat{\Phi}(x+iR)||\widehat{K}_{\mathcal{M}}(x+iR)| \le \frac{1}{x^2+R^2}\left(L_1+\frac{L_2}{c+\sigma_1}\right)\left(2\overline{\beta}+\frac{\tilde{\beta}+\sigma\overline{\beta}}{\underline{\mathcal{M}}+\sigma_1}\right)$$

for x in any bounded interval, we get, by the dominated convergence theorem,

$$\int_{\sigma}^{\sigma_1} |\widehat{\Phi}(x+iR)||\widehat{K}_{\mathcal{M}}(x+iR)|\,dx \to 0 \quad \text{as } R\to\infty,$$

hence,

$$\left|\int_{\sigma}^{\sigma_1} \frac{\widehat{\Phi}(x+iR)\widehat{K}_{\mathcal{M}}(x+iR)}{1-\widehat{K}_{\mathcal{M}}(x+iR)} e^{(x+iR)t}\,dx\right| \le \frac{e^{\sigma_1 t}}{m_{\sigma_1}}\int_{\sigma}^{\sigma_1} |\widehat{\Phi}(x+iR)||\widehat{K}_{\mathcal{M}}(x+iR)|\,dx \to 0 \quad \text{as } R\to\infty.$$

Likewise, we show that

$$\left|\int_{\sigma_1}^{\sigma} \frac{\widehat{\Phi}(x-iR)\widehat{K}_{\mathcal{M}}(x-iR)}{1-\widehat{K}_{\mathcal{M}}(x-iR)} e^{(x-iR)t}\,dx\right| \to 0 \qquad \text{as } R\to\infty.$$

Taking the limit as $R\to\infty$ in (4.68), we have

$$H(t) = H_1(t) + H_2(t),$$

where

$$H_1(t) = \underset{\lambda=\lambda_{\mathcal{M}}}{res}\frac{e^{\lambda t}\widehat{\Phi}(\lambda)\widehat{K}_{\mathcal{M}}(\lambda)}{1-\widehat{K}_{\mathcal{M}}(\lambda)} = B_0 e^{\lambda_{\mathcal{M}} t} \tag{4.69}$$

with

$$B_0 = \frac{\int_0^\infty e^{-\lambda_{\mathcal{M}} a}\Phi(a)\,da}{\int_0^\omega a e^{-\lambda_{\mathcal{M}} a} K_{\mathcal{M}}(a)\,da} \tag{4.70}$$

and H_2 satisfies the estimate

$$\big|H_2(t)\big| \le \frac{e^{\sigma_1 t}}{2\pi}\int_{-\infty}^{+\infty}\left|\frac{\widehat{\Phi}(\sigma_1+iy)\widehat{K}_{\mathcal{M}}(\sigma_1+iy)}{1-\widehat{K}_{\mathcal{M}}(\sigma_1+iy)}\right| dy,$$
$$\le Ce^{\sigma_1 t}\int_{-\infty}^{+\infty}\big|\check{\phi}_{\sigma_1}(y)\check{k}_{\sigma_1}(y)\big|\,dy \le B_1 e^{\sigma_1 t},$$

where B_1 satisfies $B_1 \leq C\big\|\check{\phi}_{\sigma_1}\big\|_{\mathrm{L}^2(\mathbb{R})}\big\|\check{k}_{\sigma_1}\big\|_{\mathrm{L}^2(\mathbb{R})}$ for some constant C. Thus, we arrive at the representation

$$B_\Phi(t) = e^{\lambda_{\mathcal{M}} t} B_0 + \Phi(t) + e^{\sigma_1 t} B_1,$$

or, taking into account that $\sigma_1 < \lambda_{\mathcal{M}}$, we obtain

$$|B_\Phi(t)| \leq e^{\lambda_{\mathcal{M}} t} B_2 + |\Phi(t)|, \tag{4.71}$$

with $B_2 = |B_0| + |B_1|$. From (4.57), we have

$$\begin{aligned}
\big\|\phi(t)\big\|_1 &= \int_0^{\min\{t,\omega\}} |B_\Phi(t-s)|\Pi_{\mathcal{M}}(s)\,ds \\
&\leq B_2 \int_0^{\min\{t,\omega\}} e^{\lambda_{\mathcal{M}}(t-s)}\Pi_{\mathcal{M}}(s)\,ds + \int_0^{\min\{t,\omega\}} |\Phi(t-s)|\Pi_{\mathcal{M}}(s)\,ds \\
&\leq B_2 e^{\lambda_{\mathcal{M}} t} \int_0^{\min\{t,\omega\}} e^{-(\lambda_{\mathcal{M}}+\underline{\mathcal{M}})s}\,ds + \int_0^{\min\{t,\omega\}} |\Phi(t-s)|\Pi_{\mathcal{M}}\,ds \\
&\leq \frac{B_2 e^{\lambda_{\mathcal{M}} t}}{\lambda_{\mathcal{M}} + \underline{\mathcal{M}}}\left(1 - e^{-(\lambda_{\mathcal{M}}+\underline{\mathcal{M}})\min\{t,\omega\}}\right) + \int_0^{\min\{t,\omega\}} |\Phi(t-s)|\Pi_{\mathcal{M}}(s)\,ds \\
&\leq \frac{B_2 e^{\lambda_{\mathcal{M}} t}}{\lambda_{\mathcal{M}} + \underline{\mathcal{M}}} + \int_0^{\min\{t,\omega\}} |\Phi(t-s)|\Pi_{\mathcal{M}}(s)\,ds.
\end{aligned} \tag{4.72}$$

Now, we need to specify the estimate (4.72) for (4.35). To do this, first let us consider $\bar{\mathcal{M}}$ such that $\mathcal{M} \leq \bar{\mathcal{M}}$. Then, since $\lambda_{\bar{\mathcal{M}}}$ and $\lambda_{\mathcal{M}}$ are the unique real solutions to (4.63), from

$$e^{-\int_0^a \bar{\mathcal{M}}(s)\,ds - \lambda a} \leq e^{-\int_0^a \mathcal{M}(s)\,ds - \lambda a},$$

it follows

$$\lambda_{\bar{\mathcal{M}}} \leq \lambda_{\mathcal{M}}.$$

Let $\lambda_{\mathcal{M}_\theta}$ be the (dominant) eigenvalue corresponding to $\mathcal{M} = \mu + \theta/\epsilon$ for any age-dependent positive function or positive constant θ. Let also $\hat{\delta} < \underline{\delta}$. Then, since

$$e^{-\int_0^a \mu(s)\,ds - \frac{1}{\epsilon}\int_0^a \delta(s)\,ds - \lambda a} \leq e^{-\int_0^a \mu(s)\,ds - \frac{\hat{\delta}}{\epsilon}a - \lambda a},$$

we get

$$\lambda_{\mathcal{M}_\delta} < \lambda_{\mathcal{M}_{\hat{\delta}}} = \lambda_{\mathcal{M}_0} - \frac{\hat{\delta}}{\epsilon}, \tag{4.73}$$

where $\lambda_{\mathcal{M}_0}$ is the eigenvalue corresponding to $\mathcal{M}_0 = \mu$; that is, to the problem (4.3).

Let λ^* be the solution to (4.63) with the largest real part $\Re\lambda^* = \sigma^*$ that is smaller than $\lambda_{\mathcal{M}_0}$. Then the eigenvalue $\lambda^* - \hat{\delta}/\epsilon$ satisfies

$$\widehat{K}_{\mathcal{M}_{\hat{\delta}}}(\lambda) = 1 \tag{4.74}$$

in such a way that we cannot find other solutions in the strip

$$\left\{\lambda;\, \sigma^* - \frac{\hat{\delta}}{\epsilon} < \Re\lambda < \lambda_{\mathcal{M}_{\hat{\delta}}} = \lambda_{\mathcal{M}_0} - \frac{\hat{\delta}}{\epsilon}\right\}.$$

Next, we let ϕ_θ be the solution to (4.56) with $\mathcal{M}_\theta$. It follows that

$$\phi_{\theta_1}(a,t) \leq \phi_{\theta_2}(a,t), \qquad \text{for a.a. } (a,t) \in [0,\omega] \times \mathbb{R}_+ \tag{4.75}$$

whenever $\theta_2 \leq \theta_1$.

In the next step, we apply the general results obtained from (4.56) to the case at hand, namely, to the problem (4.35). We see that, from the proof of the Lemma 4.4.1, the inhomogeneity in (4.35), denoted by ψ, satisfies

$$|\psi(t)| \leq C_i e^{-\frac{\underline{\delta}}{\epsilon}t}. \tag{4.76}$$

Hence $\widehat{\psi}$ is analytic for $\Re\lambda > -\frac{\underline{\delta}}{\epsilon}$ and, by (4.73), in general $\lambda_{\mathcal{M}_\delta}$ will not belong to the domain of analyticity of $\widehat{\psi}$. Thus we have to settle for a slightly weaker estimate. Using (4.75), we have

$$\phi_\delta \leq \phi_{\hat{\delta}} \tag{4.77}$$

and we carry out the estimate for $\phi_{\hat{\delta}}$. Then we have

$$\begin{aligned}\int_0^\infty e^{-\lambda_{\mathcal{M}_{\hat{\delta}}}a}|\psi(a)|\,da &\leq C_i \int_0^\infty e^{-\left(\lambda_{\mathcal{M}_0} - \frac{\hat{\delta}}{\epsilon}\right)a} e^{-\frac{\underline{\delta}}{\epsilon}a}\,da = C_i \int_0^\infty e^{-\lambda_{\mathcal{M}_0}a} e^{-\frac{(\underline{\delta}-\hat{\delta})}{\epsilon}a}\,da \\ &= \frac{\epsilon C_i}{\epsilon\lambda_{\mathcal{M}_0} + (\underline{\delta} - \hat{\delta})},\end{aligned}$$

provided $\epsilon < -(\underline{\delta} - \hat{\delta})/\lambda_{\mathcal{M}_0}$ (this condition is relevant if $\lambda_{\mathcal{M}_0} < 0$). Likewise,

$$\begin{aligned}\int_0^\omega a e^{-\lambda_{\mathcal{M}_{\hat{\delta}}}a} K_{\mathcal{M}_{\hat{\delta}}}\,da &= \int_0^\omega a e^{-\left(\lambda_{\mathcal{M}_0} - \frac{\hat{\delta}}{\epsilon}\right)a} \beta(a) e^{-\int_0^a \mu(s)\,ds - \frac{\hat{\delta}}{\epsilon}a}\,da \\ &= \int_0^\omega a e^{-\lambda_{\mathcal{M}_0}a} e^{-\int_0^a \mu(s)\,ds}\,da.\end{aligned}$$

Hence, from (4.70) we obtain

$$|B_0| \leq \epsilon C \tag{4.78}$$

for some constant C. Next, we shall estimate B_1. To make calculations easier for $\sigma^* < 0$, we make assumption that

$$\epsilon < -(\underline{\delta} - \hat{\delta})/\sigma^*; \tag{4.79}$$

otherwise ϵ can be arbitrary. Then both $\lambda_{\mathcal{M}_{\hat{\delta}}}$ and $\lambda^* - \hat{\delta}/\epsilon$ belong to the domain of analyticity of $\widehat{\psi}$.

We point out that $\inf_{y\in\mathbb{R}}|1-\widehat{K}_{\mathcal{M}_{\hat\delta}}(\sigma_\epsilon+iy)|\geq m_\sigma>0$ independently of ϵ, where σ_ϵ is between $\lambda_{\mathcal{M}_{\hat\delta}}$ and the real part of the next solution to (4.4). In fact, by the considerations above, we can take

$$\sigma_\epsilon=\tilde\sigma-\frac{\hat\delta}{\epsilon}$$

where $\sigma^*<\tilde\sigma<\lambda_{\mathcal{M}_0}$ is independent of ϵ. Then we complete the estimate of B_1 by the following calculations

$$\begin{aligned}\int_0^\omega\left|e^{-\frac{\delta}{\epsilon}a}e^{-\sigma_\epsilon a}K_{\mathcal{M}_{\hat\delta}}(a)e^{-\sigma_\epsilon a}\right|da &= \int_0^\omega\left|\beta(a)e^{-\int_0^a\mu(s)\,ds-\frac{\hat\delta}{\epsilon}a}e^{\left(-2\tilde\sigma+2\frac{\hat\delta}{\epsilon}\right)a}e^{-\frac{\delta}{\epsilon}a}\right|da\\ &= \int_0^\omega\left|\beta(a)e^{-\int_0^a\mu(s)\,ds-\frac{\underline{\delta}-\hat\delta}{\epsilon}a-2\tilde\sigma a}\right|da\\ &\leq C_2\int_0^\omega e^{-\frac{\underline{\delta}-\hat\delta}{\epsilon}a}\,da\leq\epsilon C_3,\end{aligned}\tag{4.80}$$

where C_3 is bounded independently of ϵ.

We see that for our particular problem, namely (4.35), we shall write (4.72) as an estimate of $\breve m_{0,\epsilon}$. Thus, the estimate related to the last term in (4.72) is obtained as follows

$$\int_0^{\min\{t,\omega\}}|\psi(t-s)|\Pi_{\mathcal{M}_{\hat\delta}}(s)\,ds=C_ie^{-\frac{\delta}{\epsilon}t}\int_0^{\min\{t,\omega\}}e^{\frac{\underline{\delta}-\hat\delta}{\epsilon}s}\Pi_{\mathcal{M}_0}(s)\,ds,\tag{4.81}$$

$$\begin{aligned}&\leq\frac{\epsilon}{\underline{\delta}-\hat\delta}C_ie^{-\frac{\delta}{\epsilon}t}\begin{cases}e^{\frac{\underline{\delta}-\hat\delta}{\epsilon}t}-1, & \text{for } t<\omega,\\ e^{\frac{\underline{\delta}-\hat\delta}{\epsilon}\omega}-1, & \text{for } t\geq\omega\end{cases}\\ &\leq\frac{\epsilon}{\underline{\delta}-\hat\delta}C_ie^{-\frac{\hat\delta}{\epsilon}t}\begin{cases}1-e^{-\frac{\underline{\delta}-\hat\delta}{\epsilon}t}, & \text{for } t<\omega,\\ e^{\frac{\underline{\delta}-\hat\delta}{\epsilon}(\omega-t)}-e^{-\frac{\hat\delta}{\epsilon}t}, & \text{for } t\geq\omega\end{cases}\\ &\leq\epsilon C_5e^{-\frac{\hat\delta}{\epsilon}t}.\end{aligned}\tag{4.82}$$

Hence, taking into account that $\breve m_{0,\epsilon}(0)=0$, (4.78) and (4.80), the estimate (4.72) leads to

$$\|\breve m_{0,\epsilon}(t)\|_1\leq\epsilon C_4e^{(\lambda_{\mathcal{M}_0}-\frac{\hat\delta}{\epsilon})t}+\epsilon C_5e^{-\frac{\hat\delta}{\epsilon}t}\leq\epsilon C_ie^{-\frac{\hat\delta}{\epsilon}t}\max\left\{e^{\lambda_{\mathcal{M}_0}t},1\right\}.\tag{4.83}$$

Since, by (4.51) and (4.52), $\breve m_{0,\epsilon}$ is a classical solution to (4.35), we can insert (4.33) into (4.9c) and we get

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)(\tilde m_0+\breve m_0+\epsilon u)=-\mu\left(\tilde m_0+\breve m_0+\epsilon u\right)-\frac{\delta}{\epsilon}\left(\tilde m_0+\breve m_0+\epsilon u\right)\tag{4.84}$$

$$+\left(\varrho-\tilde m_0-\breve m_0-\epsilon u\right)\int_0^\omega K(a,a')\left(\tilde m_0+\breve m_0+\epsilon u\right)\,da'.\tag{4.85}$$

Since $\tilde{m}_0 \in D(A)$, Equation (4.85) yields

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} = -\left(\mu + \frac{1}{\epsilon}\delta\right) u - \mathcal{F}(u) - \psi, \tag{4.86}$$

with the boundary condition

$$u(0,t) = q\int_0^\omega \beta(a)u(a,t)\,da \tag{4.87}$$

and the initial condition

$$u(a,0) = 0, \tag{4.88}$$

where

$$\mathcal{F}(u) := -u\int_0^\omega K(a,a^{'})(\tilde{m}_0 + \breve{m}_0)\,da^{'} + (\varrho - \tilde{m}_0 - \breve{m}_0)\int_0^\omega K(a,a^{'})u\,da^{'} - \epsilon u\int_0^\omega K(a,a^{'})u\,da^{'}, \tag{4.89}$$

$$\begin{aligned}\psi =& \frac{1}{\epsilon}\left(\frac{\partial \tilde{m}_0}{\partial a} + \mu\tilde{m}_0 + \tilde{m}_0\int_0^\omega K(a,a^{'})\tilde{m}_0\,da^{'}\right) + \frac{1}{\epsilon}\left(\breve{m}_0\int_0^\omega K(a,a^{'})\breve{m}_0\,da^{'}\right)\\ &+ \frac{1}{\epsilon}\left(-\varrho\int_0^\omega K(a,a^{'})(\tilde{m}_0+\breve{m}_0)\,da^{'} + \breve{m}_0\int_0^\omega K(a,a^{'})\tilde{m}_0\,da^{'} + \tilde{m}_0\int_0^\omega K(a,a^{'})\breve{m}_0\,da^{'}\right),\\ =:& \psi_{1,\epsilon} + \psi_{2,\epsilon} + \psi_{3,\epsilon}.\end{aligned} \tag{4.90}$$

4.5 The Error Estimates

In this section we deal with the error equations (4.86)-(4.90). We aim to prove that the error of the approximation of the solution to our problem, which we constructed in the previous section, is of order $O(\epsilon)$ with exponential decay in time t. Denote $A = -d/da - \mu$ and $Q_\epsilon = A + M_{\delta,\epsilon}$ with $M_{\delta,\epsilon} = -\delta/\epsilon$. Note that A and Q_ϵ generate, respectively, strongly continuous semigroups $\left(e^{tA}\right)_{t\geq 0}$ and $\left(e^{tQ_\epsilon}\right)_{t\geq 0}$, which satisfies the estimates (3.30) and (3.31), see Remark 3.1.7. That is,

$$\left\|e^{tA}\right\|_{\mathbb{B}(\mathrm{L}^1)} \leq Ne^{\lambda^* t} \tag{4.91}$$

and

$$\left\|e^{tQ_\epsilon}\right\|_{\mathbb{B}(\mathrm{L}^1)} \leq Ne^{\lambda^* t}. \tag{4.92}$$

If the net reproduction rate R satisfies

$$\int_0^\omega \beta(a)e^{-\int_0^a \mu(s)\,ds}\,da \leq 1,$$

we have $\lambda^* \leq 0$; hence (4.91) and (4.92) become

$$\|e^{tA}\|_{\mathbb{B}(\mathrm{L}^1)} \leq N \tag{4.93}$$

and

$$\|e^{tQ_\epsilon}\|_{\mathbb{B}(\mathrm{L}^1)} \leq N. \tag{4.94}$$

In the sequel, we will work with the assumption $R \leq 1$.
Now, we make the following claim:

Lemma 4.5.1. *Let* ϱ^* *and* m^* *be the solutions to* (4.9) *and* (4.10)*, respectively. If* u *is defined in* (4.33)*, then there exists a positive constant* M *such that for any given* $t \in [0, \infty]$ *and* $\epsilon > 0$*:*

$$\sup_{t\in\overline{\mathbb{R}}_+} \|\epsilon u(t)\|_1 \leq M. \tag{4.95}$$

Proof. It follows, from $\tilde{m}_0\left(a, \frac{t}{\epsilon}\right) = m^0(a)e^{-\frac{t}{\epsilon}\delta(a)}$, that

$$\|\tilde{m}_0\left(t/\epsilon\right)\|_1 \leq \|m^0\|_1 e^{-\frac{\delta}{\epsilon}t}. \tag{4.96}$$

On the other hand ϱ^*, as a solution to (4.3), satisfies (4.93). Then we can write

$$\|\varrho^*(t)\|_1 \leq N\|\varrho^0\|_1. \tag{4.97}$$

For $\varrho^* = s + i$ and $m^* = i$, $\|i(t)\|_1 \leq \|s(t) + i(t)\|_1$ holds, since $\big(s, i\big)^T \geq 0$. Thus,

$$\|m^*(t)\|_1 \leq \|\varrho^*(t)\|_1. \tag{4.98}$$

From (4.97) and (4.98), we obtain

$$\|m^*(t)\|_1 \leq N\|\varrho^0\|_1, \tag{4.99}$$

for any $t \in [0, \infty]$.

Using equation (4.33), we have the following

$$\|\epsilon u(t)\|_1 \leq \|m^*(t)\|_1 + \|\tilde{m}_0(t/\epsilon)\|_1 + \|\breve{m}_{0,\epsilon}(t)\|_1,$$

and from (4.83), (4.96) and (4.99) it follows that

$$\begin{aligned}\|\epsilon u(t)\|_1 &\leq \|m^0\|_1 e^{-\frac{\delta}{\epsilon}t} + N\|\varrho^0\|_1 + \epsilon C_i e^{-\frac{\hat{\delta}}{\epsilon}t} \\ &\leq \max\{1, N\}\left(\left\|m^0\right\|_1 + \|\varrho^0\|_1\right) + \epsilon C_i e^{-\frac{\hat{\delta}}{\epsilon}t} \\ &= C_0\|\mathbf{g}^0\|_{\mathbf{X}} + \epsilon C_i e^{-\frac{\hat{\delta}}{\epsilon}t},\end{aligned}$$

where $C_0 = \max\{1, N\}$. Taking supremum over $\overline{\mathbb{R}}_+$, we obtain

$$\sup_{t\in\overline{\mathbb{R}}_+} \|\epsilon u(t)\|_1 \leq C_0\|\mathbf{g}^0\|_{\mathbf{X}} + \frac{C_i}{\sigma^*}(\hat{\delta} - \underline{\delta}) = M > 0,$$

where $\mathbf{g}^0 = (\varrho^0, m^0)^T$, provided (4.79) holds. $\square$

Lemma 4.5.2. *Let* (H4) *be satisfied and let* $\tilde{m}_0$ *be defined by* (4.22). *Then*

$$\left\|\tilde{m}_{0,a}(s/\epsilon)\right\|_1 \leq \left\|(m^0)'\right\|_1 e^{-\frac{\delta}{\epsilon}s} + \left\|\delta'\right\|_\infty \left\|m^0\right\|_1 \frac{s}{\epsilon} e^{-\frac{\delta}{\epsilon}s}. \tag{4.100}$$

Proof. Given $\tilde{m}_0\left(a, \frac{s}{\epsilon}\right) = m^0(a)e^{-\frac{s}{\epsilon}\delta(a)}$, we have

$$\frac{\partial \tilde{m}_0}{\partial a}\left(a, \frac{s}{\epsilon}\right) = e^{-\frac{s}{\epsilon}\delta(a)}\frac{dm^0}{da}(a) - e^{-\frac{s}{\epsilon}\delta(a)}\frac{s}{\epsilon}m^0(a)\frac{d\delta}{da}(a).$$

It follows that

$$\begin{aligned}
\int_0^\omega \left|\frac{\partial \tilde{m}_0}{\partial a}\left(a, \frac{s}{\epsilon}\right)\right| da &\leq e^{-\frac{s}{\epsilon}\delta}\int_0^\omega \left|\frac{dm^0}{da}(a)\right| da + \int_0^\omega \left|\frac{s}{\epsilon}m^0(a)e^{-\frac{s}{\epsilon}\delta(a)}\frac{d\delta}{da}(a)\right| da \\
&\leq e^{-\frac{\delta}{\epsilon}} \left\|\frac{dm^0}{da}\right\|_1 + \frac{s}{\epsilon}e^{-\frac{\delta}{\epsilon}s}\left\|\frac{d\delta}{da}\right\|_\infty \int_0^\omega |m^0(a)|\, da \\
&\leq e^{-\frac{\delta}{\epsilon}s}\left\|\frac{dm^0}{da}\right\|_1 + \frac{s}{\epsilon}e^{-\frac{\delta}{\epsilon}s}\left\|\frac{d\delta}{da}\right\|_\infty \left\|m^0\right\|_1.
\end{aligned}$$

This completes the proof of the lemma. □

Similarly as in the proof of Lemma 3.1.2, we see that A generates a strongly continuous positive semigroup of contractions $\left(e^{tA}\right)_{t\geq 0}$.

Since the variable a plays in $M_{\delta,\epsilon}$ the role of a parameter, we get

$$\left\|e^{tM_{\delta,\epsilon}}\right\|_{\mathbb{B}(\mathrm{L}^1)} \leq e^{-\frac{\hat{\delta}}{\epsilon}t}.$$

and we can see that the semigroup $\left(e^{tM_{\delta,\epsilon}}\right)_{t\geq 0}$ is positive; hence, from (2.17), we arrive at

$$\left\|e^{tQ_\epsilon}\right\|_{\mathbb{B}(\mathrm{L}^1)} \leq e^{-\frac{\hat{\delta}}{\epsilon}t}$$

with $\left(e^{tQ_\epsilon}\right)_{t\geq 0}$ positive.

Consider the strongly continuous positive semigroup $\left(e^{tQ_\epsilon}\right)_{t\geq 0}$. By Duhamel's formula, the solution to (4.86)-(4.90) is given by

$$\begin{aligned}
u(t) &= -\int_0^t e^{(t-s)Q_\epsilon}\Big(\mathcal{F}(u(s)) + \psi(s)\Big)\, ds \\
&= -\int_0^t e^{(t-s)Q_\epsilon}\Big(\mathcal{F}(u(s)) + \psi_{1,\epsilon}(s) + \psi_{2,\epsilon}(s) + \psi_{3,\epsilon}(s)\Big)\, ds, \quad t \geq 0.
\end{aligned} \tag{4.101}$$

Now, we state the following result:

Theorem 4.5.1. *Let* $\mathrm{g}_\epsilon(a,t) = \varrho_\epsilon \boldsymbol{\psi}_1 + m_\epsilon \boldsymbol{\psi}_2$ *be the solution to* (4.5). *Then there exists a constant* $C(\mu, \delta, K, m^0)$ *such that for any* $m^0 \in D(A)$, *we have*

$$\left\|m_\epsilon(\cdot, t) - e^{-\frac{t}{\epsilon}\delta(\cdot)}m^0(\cdot)\right\|_1 \leq \epsilon C(\mu, \delta, K, m^0)e^{-\frac{\hat{\delta}}{2\epsilon}t} \tag{4.102}$$

for any $t \in [0, \infty)$.

Proof. From (4.101), we have

$$\begin{aligned}
\|u(t)\|_1 &= \left\|- \int_0^t e^{(t-s)Q_\epsilon}\Big(\mathcal{F}(u(s)) + \psi(s)\Big)\, ds\right\|_1 \\
&\le \bigg(\int_0^t e^{-\frac{\hat{\delta}}{\epsilon}(t-s)} \left\|\mathcal{F}(u(s))\right\|_1 \, ds + \int_0^t e^{-\frac{\hat{\delta}}{\epsilon}(t-s)} \left\|\psi_{1,\epsilon}(s)\right\|_1 \, ds \\
&\qquad + \int_0^t e^{-\frac{\hat{\delta}}{\epsilon}(t-s)} \left\|\psi_{2,\epsilon}(s)\right\|_1 \, ds + \int_0^t e^{-\frac{\delta}{\epsilon}(t-s)} \left\|\psi_{3,\epsilon}(s)\right\|_1 \, ds\bigg) \\
&\le e^{-\frac{\hat{\delta}}{\epsilon}t} \bigg(\int_0^t e^{\frac{\hat{\delta}}{\epsilon}s} \left\|\mathcal{F}(u(s))\right\|_1 \, ds + \int_0^t e^{\frac{\hat{\delta}}{\epsilon}s} \left\|\psi_{1,\epsilon}(s)\right\|_1 \, ds \\
&\qquad + \int_0^t e^{\frac{\hat{\delta}}{\epsilon}s} \left\|\psi_{2,\epsilon}(s)\right\|_1 \, ds + \int_0^t e^{\frac{\hat{\delta}}{\epsilon}s} \left\|\psi_{3,\epsilon}(s)\right\|_1 \, ds\bigg).
\end{aligned}$$

It follows that

$$\begin{aligned}
e^{\frac{\hat{\delta}}{\epsilon}t}\|u(t)\|_1 \le \int_0^t e^{\frac{\hat{\delta}}{\epsilon}s} \left\|\mathcal{F}(s)\right\|_1 \, ds + \int_0^t e^{\frac{\hat{\delta}}{\epsilon}s} \left\|\psi_{1,\epsilon}(s)\right\|_1 \, ds + \int_0^t e^{\frac{\hat{\delta}}{\epsilon}s} \left\|\psi_{2,\epsilon}(s)\right\|_1 \, ds \\
+ \int_0^t e^{\frac{\hat{\delta}}{\epsilon}s} \left\|\psi_{3,\epsilon}(s)\right\|_1 \, ds, \qquad (4.103)
\end{aligned}$$

for any $t \in [0, \infty)$.

The estimates for the integrals, in (4.103), involving $\mathcal{F}$, $\psi_{1,\epsilon}$, $\psi_{2,\epsilon}$ and $\psi_{3,\epsilon}$, respectively, are obtained below.

Firstly,

$$\begin{aligned}
\int_0^t e^{\frac{\hat{\delta}}{\epsilon}s} \left\|\mathcal{F}(s)\right\|_1 \, ds &= \int_0^t e^{\frac{\hat{\delta}}{\epsilon}s} \bigg(\int_0^\omega \bigg| -u(a,s) \int_0^\omega K(a,a^{'})\Big(\tilde{m}_0(a^{'}, s/\epsilon) + \breve{m}_0(a^{'}/\epsilon, s/\epsilon)\Big)\, da^{'} \\
&\quad + \Big(\varrho(a,s) - \tilde{m}_0(a, s/\epsilon) - \breve{m}_{0,\epsilon}(a,s)\Big) \int_0^\omega K(a,a^{'})u(a^{'},s)\, da^{'} \\
&\quad - \epsilon u(a,s) \int_0^\omega K(a,a^{'})u(a^{'},s)\, da^{'} \bigg|\, da\bigg)\, ds, \\
&\le \|K\|_\infty \int_0^t e^{\frac{\hat{\delta}}{\epsilon}s} \big(\left\|\tilde{m}_0(s/\epsilon)\right\|_1 + \left\|\breve{m}_{0,\epsilon}(s)\right\|_1 + N\left\|\varrho(s)\right\|_1 \\
&\quad + \left\|\tilde{m}_0(s/\epsilon)\right\|_1 + \left\|\breve{m}_{0,\epsilon}(s)\right\|_1 + M\big) \left\|u(s)\right\|_1 \, ds, \\
&\le \|K\|_\infty \int_0^t e^{\frac{\hat{\delta}}{\epsilon}s} \big(2\left\|\tilde{m}_0(s/\epsilon)\right\|_1 + N\left\|\varrho^0\right\|_1 \\
&\quad + 2\left\|\breve{m}_{0,\epsilon}(a,s)\right\|_1 + M\big) \left\|u(s)\right\|_1 \, ds, \\
&\le 3M\|K\|_\infty \int_0^t e^{\frac{\hat{\delta}}{\epsilon}s} \left\|u(s)\right\|_1 \, ds = \widetilde{C} \int_0^t e^{\frac{\hat{\delta}}{\epsilon}s} \left\|u(s)\right\|_1 \, ds. \qquad (4.104)
\end{aligned}$$

Secondly,

$$\begin{aligned}\int_0^t e^{\frac{\hat{\delta}}{\epsilon}s}\left\|\psi_{1,\epsilon}(s)\right\|_1\,ds &\le \frac{1}{\epsilon}\int_0^t e^{\frac{\hat{\delta}}{\epsilon}s}\left(\left\|\frac{\partial\tilde{m}_0}{\partial a}(s/\epsilon)\right\|_1 + \int_0^\omega e^{-\frac{s}{\epsilon}\delta(a)}\Big|\mu(a)m^0(a)\right.\\
&\qquad\left. + m^0(a)\int_0^\omega K(a,a^{'})m^0(a^{'})e^{-\frac{s}{\epsilon}\delta(a^{'})}\,da^{'}\Big|\,da\right)\,ds,\\
&\le \frac{1}{\epsilon}\int_0^t e^{\frac{\hat{\delta}}{\epsilon}s}\left(\left\|\frac{\partial\tilde{m}_0}{\partial a}(s/\epsilon)\right\|_1 + e^{-\frac{s}{\epsilon}\underline{\delta}}\int_0^\omega\left|\mu(a)m^0(a)\right|da\right.\\
&\qquad\left. + e^{-\frac{s}{\epsilon}\underline{\delta}}\left\|m^0\right\|_1\int_0^\omega\left|K(a,a^{'})m^0(a^{'})\right|da^{'}\right)\,ds,\\
&\le \frac{1}{\epsilon}\int_0^t\left(\left\|\frac{dm^0}{da}\right\|_1 + \frac{s}{\epsilon}\left\|\frac{d\delta}{da}\right\|_\infty\left\|m^0\right\|_1\right.\\
&\qquad\left. + \int_0^\omega\left|\mu(a)m^0(a)\right|da + \|K\|_\infty\left\|m^0\right\|_1^2\right)e^{-\frac{s}{\epsilon}(\underline{\delta}-\hat{\delta})s}\,ds\\
&\le \frac{1}{(\underline{\delta}-\hat{\delta})^2}\max\left\{(\underline{\delta}-\hat{\delta}),\left\|\frac{d\delta}{da}\right\|_\infty\right\}\left\|m^0\right\|_{\mathrm{W}^{1,1}([0,\omega])}\\
&\qquad + \frac{1}{\underline{\delta}}\left(\gamma + \|K\|_\infty\left\|m^0\right\|_1\right)\left\|m^0\right\|_{\mathrm{W}^{1,1}([0,\omega])} = C_i^{'},\end{aligned} \tag{4.105}$$

where

$$\gamma = \frac{1}{\left\|m^0\right\|_1}\int_0^\omega\left|\mu(a)m^0(a)\right|da = \frac{\left\|\mu m^0\right\|_1}{\left\|m^0\right\|_1} < \infty, \quad (\text{where } m^0 \neq 0),$$

since $m^0 \in D(A)$.

Thirdly,

$$\begin{aligned}\int_0^t e^{\frac{\hat{\delta}}{\epsilon}s}\left\|\psi_{2,\epsilon}(s)\right\|_1\,ds &\le \frac{1}{\epsilon}\int_0^t e^{\frac{\hat{\delta}}{\epsilon}s}\left(\int_0^\omega\left|\breve{m}_{0,\epsilon}(a,s)\int_0^\omega K(a,a^{'})\breve{m}_{0,\epsilon}(a^{'},s)\,da^{'}\right|da\right)\,ds\\
&\le \frac{1}{\epsilon}\|K\|_\infty\int_0^t e^{\frac{\hat{\delta}}{\epsilon}s}\left\|\breve{m}_{0,\epsilon}(s)\right\|_1^2\,ds\\
&\le \epsilon C_i^2\big\|K\big\|_\infty\int_0^t e^{\frac{\hat{\delta}}{\epsilon}s}e^{-\frac{2\hat{\delta}}{\epsilon}s}\,ds = \epsilon C_i^2\big\|K\big\|_\infty\int_0^t e^{-\frac{\hat{\delta}}{\epsilon}s}\,ds\\
&\le \frac{\hat{\delta}}{2}C_i^2\big\|K\big\|_\infty\int_0^t e^{-2s}\,ds\\
&\le \frac{\hat{\delta}}{4}C_i^2\big\|K\big\|_\infty = C_i^{''},\end{aligned} \tag{4.106}$$

for $\epsilon < \frac{\hat{\delta}}{2}$.

Finally,

$$\begin{aligned}
\int_0^t e^{\frac{\hat{\delta}}{\epsilon}s} \left\|\psi_{3,\epsilon}(s)\right\|_1 \, ds &\leq \frac{1}{\epsilon}\int_0^t e^{\frac{\hat{\delta}}{\epsilon}s}\Bigg(\int_0^\omega \Bigg| -\tilde{m}_0(a,s/\epsilon)\int_0^\omega K(a,a^{'})\breve{m}_{0,\epsilon}(a^{'},s)\,da^{'} \\
&\qquad - \varrho(a,s)\int_0^\omega K(a,a^{'})\big(\tilde{m}_0(a^{'},s/\epsilon)+\breve{m}_{0,\epsilon}(a^{'},s)\big)\,da^{'} \\
&\qquad\quad + \breve{m}_{0,\epsilon}(a,s)\int_0^\omega K(a,a^{'})\tilde{m}_0(a^{'},s/\epsilon)\,da^{'}\Bigg|\,da\Bigg)\,ds, \\
&\leq \frac{1}{\epsilon}\left\|K\right\|_\infty \int_0^t e^{\frac{\hat{\delta}}{\epsilon}s}\Big[2\left\|\tilde{m}_0(s/\epsilon)\right\|_1\left\|\breve{m}_{0,\epsilon}(s)\right\|_1 \\
&\qquad + N\left\|\varrho(s)\right\|_1\Big(\left\|\tilde{m}_0(s/\epsilon)\right\|_1+\left\|\breve{m}_0(s)\right\|_1\Big)\Big]\,ds \\
&\leq \frac{1}{\epsilon}\int_0^t e^{\frac{\hat{\delta}}{\epsilon}s}\Big(2\epsilon C_i e^{-\frac{\hat{\delta}}{\epsilon}s}\max\{1,N\}\left(\left\|m^0\right\|_1+\left\|\varrho^0\right\|_1\right) \\
&\qquad + Ne^{-\frac{\underline{\delta}}{\epsilon}s}\left\|\varrho^0\right\|_1\left\|m^0\right\|_{\mathrm{W}^{1,1}([0,\omega])}\Big)\left\|K\right\|_\infty\,ds \\
&\leq 2C_i\max\{1,N\}\left(\left\|m^0\right\|_1+\left\|\varrho^0\right\|_1\right)\left\|K\right\|_\infty\int_0^t ds \\
&\qquad + \frac{N}{\epsilon}\left\|K\right\|_\infty\left\|\varrho^0\right\|_1\left\|m^0\right\|_{\mathrm{W}^{1,1}([0,\omega])}\int_0^t e^{-\frac{s}{\epsilon}(\underline{\delta}-\hat{\delta})}\,ds \\
&\leq 2C_i\max\{1,N\}\left(\left\|m^0\right\|_1+\left\|\varrho^0\right\|_1\right)\left\|K\right\|_\infty t \\
&\qquad + \frac{N}{\underline{\delta}-\hat{\delta}}\left\|K\right\|_\infty\left\|\varrho^0\right\|_1\left\|m^0\right\|_{\mathrm{W}^{1,1}([0,\omega])}\left(1-e^{-\frac{t}{\epsilon}(\underline{\delta}-\hat{\delta})}\right) \\
&\leq 2C_i\max\{1,N\}\left(\left\|m^0\right\|_1+\left\|\varrho^0\right\|_1\right)\left\|K\right\|_\infty t \\
&\qquad + \frac{N}{\underline{\delta}-\hat{\delta}}\left\|K\right\|_\infty\left\|\varrho^0\right\|_1\left\|m^0\right\|_{\mathrm{W}^{1,1}([0,\omega])} \\
&= C_i^{'''}t + C_i^{''''}. \qquad (4.107)
\end{aligned}$$

Note that the constants $C_i^{'}$, $C_i^{''}$, $C_i^{'''}$ and $C_i^{''''}$ depend on the $\mathrm{W}^{1,1}([0,\omega])$ norm of the initial condition m^0 (as well as on other parameters).

Substituting (4.104)-(4.107) into (4.103), we get

$$e^{\frac{\hat{\delta}}{\epsilon}t}\left\|u(t)\right\|_1 \leq \left(L_i + tC_i^{'''}\right) + \widetilde{C}\int_0^t e^{\frac{\hat{\delta}}{\epsilon}s}\left\|u(s)\right\|_1\,ds,$$

where $L_i = C_i^{'} + C_i^{''} + C_i^{''''}$, and it follows, by Gronwall's Lemma, that

$$e^{\frac{\hat{\delta}}{\epsilon}t}\left\|u(t)\right\|_1 \leq \left(L_i + tC_i^{'''}\right) + e^{\widetilde{C}t}\int_0^t e^{-\widetilde{C}s}\left(L_i + sC_i^{'''}\right)ds,$$

which yields

$$\left\|u(t)\right\|_1 \leq e^{-\frac{\hat{\delta}}{\epsilon}t}\left\{\left(L_i + tC_i^{'''}\right) + e^{\widetilde{C}t}\int_0^t e^{-\widetilde{C}s}\left(L_i + sC_i^{'''}\right)ds.\right\},$$

After integrating by parts, we obtain

$$\|u(t)\|_1 \leq e^{-\frac{\hat{\delta}}{\epsilon}t}\left\{\big(L_i + tC_i'''\big) + \frac{1}{\widetilde{C}}\big(L_i\big)\big(e^{\widetilde{C}t}-1\big) + \frac{C_i'''}{\widetilde{C}^2}\big(e^{\widetilde{C}t} - t\widetilde{C} - 1\big)\right\},$$

that is,

$$\begin{aligned}\|u(t)\|_1 &\leq \frac{1}{\widetilde{C}^2}\left\{\widetilde{C}^2\big(L_i + tC_i'''\big) + \widetilde{C}L_i e^{\widetilde{C}t} + C_i''' e^{\widetilde{C}t}\right\} e^{-\frac{\hat{\delta}}{\epsilon}t},\\ &\leq \frac{1}{\widetilde{C}^2}\left\{\widetilde{C}^2\big(L_i + tC_i'''\big)e^{-\widetilde{C}t} + \widetilde{C}L_i + C_i'''\right\} e^{-\frac{t}{\epsilon}(\hat{\delta}-\epsilon\widetilde{C})},\\ &\leq \frac{1}{\widetilde{C}^2}\left\{\widetilde{C}^2 L_i e^{-\widetilde{C}t} + \widetilde{C}\big(L_i + C_i'''\widetilde{C}te^{-\widetilde{C}t}\big) + C_i'''\right\} e^{-\frac{t}{\epsilon}(\hat{\delta}-\epsilon\widetilde{C})},\\ &\leq \frac{1}{\widetilde{C}^2}\left\{\widetilde{C}^2 L_i + \widetilde{C}\big(L_i + \overline{C}C_i'''\big) + C_i'''\right\} e^{-\frac{\hat{\delta}}{2\epsilon}t}, \end{aligned} \tag{4.108}$$

for $t \in [0,\infty)$ and $\epsilon < \frac{\hat{\delta}}{2\widetilde{C}}$, where $\widetilde{C}te^{-\widetilde{C}t} \leq \overline{C}$. This shows that u is bounded and its bound has exponential decay in time t.

Since

$$\begin{aligned}\big|m_\epsilon(a,t) - \tilde{m}_0(a,t/\epsilon)\big| &= \big|\big(m_\epsilon(a,t) - \tilde{m}_0(a,t/\epsilon) - \breve{m}_{0,\epsilon}(a,t)\big) + \breve{m}_{0,\epsilon}(a,t)\big|\\ &\leq \big|m_\epsilon(a,t) - \tilde{m}_0(a,t/\epsilon) - \breve{m}_{0,\epsilon}(a,t)\big| + \big|\breve{m}_{0,\epsilon}(a,t)\big|\\ &\leq \epsilon\big|u(a,t)\big| + \big|\breve{m}_{0,\epsilon}(a,t)\big|,\end{aligned}$$

from (4.108) it follows that

$$\begin{aligned}\big\|m_\epsilon(\cdot,t) - \tilde{m}_0(\cdot,t/\epsilon)\big\|_1 &\leq \epsilon\big\|u(t)\big\|_1 + \big\|\breve{m}_{0,\epsilon}(t)\big\|_1\\ &\leq \epsilon\,\|u(t)\|_1 + \big\|\breve{m}_{0,\epsilon}(t)\big\|_1\\ &\leq \epsilon\,\|u(t)\|_1 + \epsilon C_i e^{-\frac{\hat{\delta}}{\epsilon}t}\\ &\leq \frac{\epsilon}{\widetilde{C}^2}\left\{\widetilde{C}^2 L_i + C_i''' + \widetilde{C}\big(L_i + \overline{C}C_i'''\big)\right\} e^{-\frac{\hat{\delta}}{2\epsilon}t}.\end{aligned}$$

Hence we can write

$$\big\|m_\epsilon(\cdot,t) - \tilde{m}_0(\cdot,t/\epsilon)\big\|_1 \leq \epsilon C(\mu,\delta,K,m^0)e^{-\frac{\hat{\delta}}{2\epsilon}t}, \tag{4.109}$$

for $t \in [0,\infty)$, where

$$C(\mu,\delta,K,m^0) = \frac{1}{\widetilde{C}^2}\left\{\widetilde{C}^2\big(C_i + L_i\big) + C_i''' + \widetilde{C}\big(L_i + \overline{C}C_i'''\big)\right\}.$$

□

5. Conclusion

In this work, an asymptotic analysis of a singularly perturbed age structured SIS model with fast recovery is given. In contrast with the linear age-structured multiscale population model analysed in [22], here we dealt with a nonlinear age-structured multiscale population model containing two unbounded operators. Precisely, we considered the general model for the dynamics of epidemics, introduced in [15],[27],[28],[31], which do not convey immunity, with a population dynamics described by the linear Lotka-McKendrick equation, determined by the vital rates $\beta(a)$ and $\mu(a)$. Note that the processes in our model have two different time scales. The slow dynamics in our model is described by mortality, birth and the infection processes, while in the linear population model analysed in [22] the slow dynamics was restricted to the demographic process. On the other hand, the fast dynamics in our model is represented by the recovery process while in the former model it was represented by the migration processes.

The solvability of the above-mentioned class of age-structured epidemics was presented in Chapter 3, where we extended some former results on well-posedness analysis of the problem. Further, we proved the nonnegativeness and the global existence of solution of the problem. Next, we investigated further regularity of the solution using a result similar to Theorem 2.2 in [32], where we made additional assumption on the vital rates and on the recovery rate, and we derived the compatibility conditions up to the third order in order that the solution $\mathbf{u}$ of the problem belongs to $\mathcal{C}^3\big([0,\omega]\times[0,T]\big)$. A comprehensive proof of this result is also provided.

In Chapter 4, our main tool was the Chapman-Enskog procedure for asymptotic analysis, developed in [1],[10],[22] for age-structured population model, and in [41] for the Carleman model of the Boltzmann equation. It enabled us to systematically aggregate the variables and yielded a good approximated formula with layer corrections, namely with the initial and corner correctors. We note that we needed to develop a special corner layer equation instead of the standard one. Since the corner layer term decays exponentially fast in both a/ϵ and t/ϵ, while the initial layer only does so in t/ϵ, using the $\big(\mathrm{L}^1([0,\omega])\big)^2$–norm, we can neglect the corner layer term and therefore obtain the approximation in terms of the initial layer term. Furthermore, the error estimate of our approximation was achieved.

Although our classical technique of asymptotic analysis follows the framework developed earlier for Boltzmann and linear age-structured population equations, however, its application in the context of nonlinear age-structured epidemiological models is new.

As pointed out previously, this work is close in the spirit to [10],[22], although the model considered in these references is linear and has different features, as mentioned earlier. In both cases, the existence of classical solutions of the perturbed system and aggregated system requires that the nonlocal compatibility conditions be satisfied, that is to say, the initial data have to satisfy nonlocal compatibility conditions.

Various interesting open problems can be considered in the future. It could be interesting to consider a similar problem with age-dependent demographic parameters specific to each epidemiological class of the population or to consider a similar problem where the slow dynamics represents the recovery and the demographic processes while the infection represents the fast dynamics.

One of the advantage of this asymptotic approach is that it can handle both linear and nonlinear age-structured multiscale population models.

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