



On Cellularity, Closure, Sobriety and Separation Axioms in the Lower Vietoris Topology

by

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I, Shimon Shay Corcos, declare that

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Abstract

This thesis introduces four new classes of hyperspaces with the Lower Vietoris Topology, allowing for a complete study of various properties in these hyperspaces. One consists of those satisfying the *Sub-base Condition*, such as the hyperspace 2^X of non-empty closed subsets of a topological space X . Another consists of those which are *almost finitely natural*, such as the hyperspace $\mathcal{F}_n(X)$ of n -element subsets of X , generalising the notion of *natural families* by Ivanova-Dimova.

Interestingly, these hyperspaces are used to provide new characterisations of the sober T_1 property in X as well as sobriety in the T_0 -identification of X . Sobriety and irreducibility are characterised in these hyperspaces in terms of X . It is shown that any pair of almost finitely natural hyperspaces have the same closure. A surprising relationship exists between irreducibility and closure for arbitrary hyperspaces.

When X is Hausdorff, Fedorchuk showed that $\mathcal{F}_n(X)$ with the Vietoris Topology has the same cellularity as X^n and a subspace $X^{[n]}$; Costantini et al. showed that the projection $\hat{j}_n : X^{[n]} \rightarrow \mathcal{F}_n(X)$ is a local homeomorphism. This thesis shows that $X^{[n]}$ always has cellularity at most that of X^n and that X being Hausdorff is (almost always) equivalent to not only the quotient but also the perfect covering property in \hat{j}_n . The quasi-open property of \hat{j}_n is characterised in X . In this case, if X has infinite cellularity, then the cellularity of every almost finitely natural hyperspace is the same as X^n and $X^{[n]}$. A non-Hausdorff space Y is constructed where \hat{j}_n is quasi-open and the Vietoris Topology is not the same as the Lower Vietoris Topology on $\mathcal{F}_n(Y)$.

Known results on the T_0 and T_1 properties and the T_0 -identification in the Lower Vietoris Topology are extended to arbitrary hyperspaces. When X is T_0 , the T_D property in 2^X is characterised in terms of X as a corollary of a more general new result. The preregular, T_D and T_0 properties in $\mathcal{F}_n(X)$ are characterised in X ; the latter two are shown to be equivalent in $\mathcal{F}_n(X)$. It is shown that $\mathcal{F}_n(X)$ is Hausdorff if and only if X is Hausdorff.

Table of Notations and Symbols

Notation or Symbol	Meaning
$:=$ or $==$:	Denotes "equals by definition".
iff	Abbreviates "if and only if".
■	Denotes the end of a proof.
$A \setminus B$	The complement of B relative to A , the set $\{x \in A : x \notin B\}$.
$\mathfrak{G} Y$	The set $\{S \cap Y : Y \in \mathfrak{G}\}$.
$f A$	The restriction of the function $f : X \rightarrow Y$ to $A \subseteq X$.
$ U $	The cardinality of the set U .
\mathbb{N}	The set of natural numbers.
\mathbb{R}	The set of real numbers.
τ	The topology on a set X .
$\text{Int}_\tau(U)$	The interior of a subset U of a topological space (X, τ) .
\bar{A} or $\text{Cl}_\tau(A)$	The closure of A in the topological space (X, τ) .
τA	The subspace topology $\{V \cap A : V \in \tau\}$ induced on A from (X, τ) .
$\mathcal{F}_\mathcal{G}$	If \mathcal{G} is a sub-base for the closed sets of X , $\mathcal{F}_\mathcal{G}$ denotes the set $\{\emptyset\} \cup \{\bigcup_{i=1}^m G_i : G_1, \dots, G_m \in \mathcal{G}, m \in \mathbb{N}\}$.
$\prod_{i \in I} X_i$	The Cartesian product of the family $\{X_i : i \in I\}$, i.e. the set $\{f : I \rightarrow \bigcup_{i \in I} X_i\}$.
π_j	Denotes the map $\pi_j : \prod_{i \in I} X_i \rightarrow X_j$ defined by $\pi_j(f) = f(j)$ for all $f \in \prod_{i \in I} X_i$.
X^I	The set $\prod_{i \in I} X = \{f : I \rightarrow X : f \text{ is a function}\}$.
$X^{[I]}$	The set $\{f \in X^I : i \neq j \implies f(i) \neq f(j)\}$.
$X_1 \times \dots \times X_n$	The set $\prod_{1 \leq i \leq n} X_i = \{(x_1, \dots, x_n) : x_1 \in X_1, \dots, x_n \in X_n\}$.
X^n	The set $\{(x_1, \dots, x_n) : x_1, \dots, x_n \in X\}$.
$X^{[n]}$	The set $\{(x_1, \dots, x_n) \in X^n : i \neq j \implies x_i \neq x_j\}$.

Notation or Symbol	Meaning
τ_g	For a surjective function $g : X \rightarrow Y$ between a topological space X and a set Y , $\tau_g = \{V \subseteq Y : g^{-1}(V) \in \tau\}$.
$\downarrow F$	For a subset F of a poset (P, \leq) , this is the set $\{y \in P : \text{there is } x \in F, y \leq x\}$.
\leq_τ	The specialisation pre-order on a topological space X , defined by $x \leq_\tau y$ iff $x \in \overline{\{y\}}$.
ZFC	Abbreviation for the theory of sets arising from the Zermelo-Fraenkel axioms together with the Axiom of Choice.
ZF	Abbreviation for the theory of sets arising from the Zermelo-Fraenkel axioms.
α, β, γ	Placeholders for arbitrary ordinals.
$\alpha < \beta$	For ordinals α, β , this is the well-order given by $\alpha \in \beta$.
ω	The first infinite ordinal.
$\lambda, \kappa, \mu, \rho$	Placeholders for arbitrary cardinals.
\aleph_0	The first infinite cardinal.
$\prod_{i \in I} \lambda_i$	The cardinal $ \prod_{i \in I} \lambda_i $ (if λ_i are cardinals).
λ^κ	The cardinal $ \lambda^\kappa $ (if λ, κ are cardinals).
$\lambda \cdot \kappa$	The cardinal $ \lambda \times \kappa $ (if λ, κ are cardinals).
$\lambda + \kappa$	The cardinal $ \lambda \times \{0\} \cup \kappa \times \{1\} $ (if λ, κ are cardinals).
$\sup \mathcal{A}$	The supremum of a family of cardinals \mathcal{A} .
$\neg \varphi$	The negation of a sentence φ .
$S + T$	The union of the sets of sentences S and T .
GCH	Abbreviation for the Generalised Continuum Hypothesis.
MA	Abbreviation for Martin's Axiom.

Notation or Symbol	Meaning
$\text{cell}(X)$	The cellularity or Souslin number of X , i.e. the smallest cardinal number for which $ \mathcal{U} \leq \text{cell}(X)$ whenever \mathcal{U} is a pairwise disjoint family of open subsets of X .
$\mathfrak{S}, \mathfrak{T}$	Sets of non-empty subsets of X .
2^X	The set of all non-empty closed subsets of X .
$\mathcal{P}_0(X)$	The set of all non-empty subsets of X .
$\mathcal{C}(X)$	The set of all non-empty compact subsets of X .
$\mathcal{F}(X)$	The set of all non-empty finite subsets of X .
$\mathcal{F}_n(X)$	The set of all non-empty finite subsets of X with exactly n elements for $n \in \mathbb{N}$.
$\mathcal{F}_{\leq n}(X)$	The set of all non-empty finite subsets of X with at most n elements for $n \in \mathbb{N}$.
$\mathcal{S}_n(X)$	The set of all non-empty subspaces of X with cellularity n for $n \in \mathbb{N}$.
$\mathcal{S}_{\leq n}(X)$	The set of all non-empty subspaces of X with cellularity at most n for $n \in \mathbb{N}$.
$\mathcal{D}_n(X)$	The set of all non-empty discrete subspaces of X with exactly n elements.
$\mathcal{D}_f(X)$	The set of all non-empty finite discrete subspaces of X .
$\mathbf{L}(\tau)$	The lower Vietoris topology on $\mathcal{P}_0(X)$.
$\langle U_1, U_2, \dots, U_n \rangle$	The set $\{W \in \mathcal{P}_0(X) : W \subseteq \bigcup_{i=1}^n U_i \text{ and for each } i \in \{1, 2, \dots, n\}, W \cap U_i \neq \emptyset\}$ for given $n \in \mathbb{N}$ and subsets U_1, \dots, U_n of X . In particular, for $U \subseteq X$, $\langle U \rangle = \{V \in \mathcal{P}_0(X) : V \subseteq U\}$.
c	The map $c : \mathcal{P}_0(X) \rightarrow \mathcal{P}_0(X)$ defined by $c(A) = \overline{A}$ for each $A \in \mathcal{P}_0(X)$.

Notation or Symbol	Meaning
\hat{i}	The map $\hat{i} : X^n \rightarrow \mathcal{F}_n(X)$ defined by $\hat{i}(x_1, \dots, x_n) = \{x_1, \dots, x_n\}$.
\hat{j}	The restriction map $\hat{i} X^{[n]}$.
Δ	The set of permutations on $\{1, \dots, n\}$.
δ	An element of Δ , i.e. a permutation on $\{1, \dots, n\}$.

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Chapter 1

Introduction

Various structures on sets of subsets of a topological space have been studied in the literature for more than a century. This began with the works of Pompeiu [48] in 1905 and Hausdorff [26] in 1914, who investigated metrics on the set of non-empty, closed and bounded subsets of a metric space. One of the most prevalent such structures in the literature is the *Vietoris Topology*. This topology was initially studied by Vietoris in 1922 [56, p. 259, \mathfrak{R} -Umgebungen] and was considered on the set 2^X of all non-empty closed subsets of a T_3 -space X (cf. [57, Definition 14.1] and [56, p. 258]).

Suppose X is a topological space and there is a topology on some set \mathfrak{S} of non-empty subsets of X . In that case, we refer to the topological space consisting of \mathfrak{S} with this topology as a *hyperspace* of X . In this case, we refer to X as the *base space*.

In 1951, Michael [44] carried out an extensive investigation of the Vietoris Topology on 2^X for an arbitrary topological space X (which he referred to as the *finite topology* on 2^X) and extended its definition to describe the Vietoris Topology on the set $\mathcal{P}_0(X)$ of all non-empty subsets of X (cf. [44, Definition 5.1]). Michael also stated that the Vietoris Topology on $\mathcal{P}_0(X)$ may be studied in terms of the *upper semi-finite topology* on $\mathcal{P}_0(X)$ and *lower semi-finite topology* on $\mathcal{P}_0(X)$ (cf. [44, Definition 9.1]), stating that their *join in the lattice of topologies* (cf. [57, 3G]) is, in fact, the Vietoris Topology on $\mathcal{P}_0(X)$.

The term “Lower Vietoris Topology” is the most prevalent term in the literature to refer to the aforementioned lower semi-finite topology; it is thus also the one we adopt in this thesis. It appears to have originated in 1985 (cf. [19] and [42]). Alternative terminologies for the Lower Vietoris Topology include the *lambda topology* (also referred to as λ topology) (cf. [17], [18], [39, Remark, p. 175]) and the *local topology* (cf. [12, p. 931]). Another term which is used in the literature on order theory is the *Hoare power space* when referring to 2^X - where X is an arbitrary topological space - with the Lower Vietoris Topology (cf. [22, Exercise 9.7.14]).

The first works in the literature dedicated to the study of the Lower Vietoris Topology on arbitrary hyperspaces are, to my knowledge, Feichtinger’s 1969 PhD thesis [17] and his 1970 conference paper [18]. These works focused on hyperspaces which contain all singleton subsets of the base space as elements; the properties of connectedness, separability, first countability and aspects of compactness were studied. For other notable studies of the Lower Vietoris Topology, see any one of [4, 5, 8, 49]. The Lower Vietoris Topology has found applications in order theory (cf. [51, Definition, p. 81], [51, Chapter 6]) and category theory (cf. [31] and [30]). One of its first applications was in the study of *lower semi-continuity* of multi-valued functions, especially by Kuratowski [39] and Michael [44].

The notion of an *irreducible* space (cf. [53]) is well-known in the literature; alternative names include “hyperconnected space” (cf. [52, p. 29]) or “D-space” (cf. [43]).

The origin of sobriety can be traced back to Grothendieck and Dieudonné [24, 2.1.1, page 49], who defined a *sobre* topological space as one in which every non-empty irreducible closed subspace is the closure of a unique singleton of the space. The term *sobre* is French, and so we rather adopt the English translation, i.e. the term *sober*. Sober spaces are also known as “pc-spaces” or “spectral spaces” (see [32, Definition 2.1]).

Sober spaces are always T_0 (cf. Proposition 3.4.3). The origin of the non- T_0 version of sobriety can be traced back to Hong [32, Definition 2.1] who defined the notion of a *quasi-sobre* topological space. This is equivalent to the notion of a *quasi-sober space*

which originated from [29, p. 154]. We will adopt the term *quasi-sober space*.

The notion of the *cellularity* of a topological space was introduced in the doctoral thesis of D. Kurepa [40] (see also [6, p. 41]). It is related to irreducibility because every irreducible space has cellularity 1, moreover I will show that finite cellularity has a relationship with irreducible components (cf. Propositions 3.3.6, 3.3.7, 3.3.8 and 3.3.11). It is also known as the *Souslin number*. A space having countable cellularity is said to have the *Souslin property* [13, p. 59]. The name “Souslin number” is due to its relevance to the *Souslin problem*, i.e. the question of whether or not there exists a *Souslin space*, i.e. a linearly ordered, non-separable space with countable cellularity. It is known that the Souslin problem and the statement “a finite product must have countable cellularity if each of its factors have countable cellularity” are both independent of ZFC [13, See the remarks in pp. 115-116]. The notion of cellularity is also related to the *κ -chain condition* (or the *κ -c.c.*) of a topological space (cf. Remark 3.2.2.1). Cellularity in the Vietoris Topology has been investigated by Fedorchuk [15] and jointly by Fedorchuk and Todorčević [16]. Other cardinal functions such as hereditary density, character and tightness have been investigated in the Lower Vietoris Topology by Costantini et al. [7].

To my knowledge, the notions of irreducibility and sobriety have not been considered for hyperspaces with the Lower Vietoris Topology other than 2^X and $\mathcal{P}_0(X)$. Moreover, the notion of cellularity has not, to my knowledge, been investigated in any hyperspace with the Lower Vietoris Topology.

In this thesis, I study cellularity, irreducibility and sobriety in hyperspaces with the Lower Vietoris Topology and obtain new results, some of which generalise known results in the literature. This is explained in more detail below. All results are new unless specified otherwise.

The main results concern three classes of hyperspaces which have not been previously investigated in the literature: those which are *almost natural*, *almost finitely natural* and *almost d -finitely natural*; listed in order from smallest to largest and include the well-

known hyperspaces $\mathcal{F}_n(X)$ of n -element subsets of X and $\mathcal{F}_{\leq n}(X)$ of subsets of X with at most n elements (cf. Definition 5.0.0.1).

Chapter 2 describes the theory of topological spaces and of orders necessary for this thesis.

Chapter 3 defines and studies the notions of cellularity, irreducible spaces, quasi-sober spaces and sober spaces.

- Section 3.3 provides new results on the relationship between irreducible components and finite cellularity: Proposition 3.3.6, Proposition 3.3.7 and Proposition 3.3.11.
- Section 3.5.2 provides new results on determining the irreducible components of product spaces and when they have finite cellularity. I correct the proof of a result from the literature (cf. Lemma 3.5.10) and I prove a more general result which is new and expands on known results from the literature for products with infinite cellularity (cf. Theorem 3.5.11). Specifically, it is shown that the cellularity of a product is finite iff there are a finite number of non-irreducible factors with finite cellularity, in which case the cellularity of the product, is the product of the cellularity of each of the factors.
- Section 3.6 provides new results on the cellularity of the augmented n -th power of a topological space. The main result (cf. Theorem 3.6.7) is that in general, $\text{cell}(X^n) \leq \text{cell}(X^{[n]})$. This expands upon a known result by Fedorchuk (cf. Proposition 3.6.1), stating that $\text{cell}(X^n) = \text{cell}(X^{[n]})$ when X is Hausdorff.

In Chapter 4, the Lower Vietoris Topology is defined and well-known properties are studied. The new results include a description of the T_0 -identification of \mathfrak{S} with the Lower Vietoris Topology (cf. Theorem 4.3.3), a characterisation of the T_0 , symmetry and T_1 properties are characterised in the Lower Vietoris Topology (cf. Theorem 4.4.1), and useful corollaries 4.4.2 and 4.4.3 which describe how our results generalise well-known results in the literature, namely: 2^X with the Lower Vietoris Topology is always a T_0 -

space, $\mathcal{P}_0(X)$ with the Lower Vietoris Topology is a T_0 -space iff X has the discrete topology and 2^X with the Lower Vietoris Topology is the T_0 -identification of $\mathcal{P}_0(X)$ with the Lower Vietoris Topology.

In Chapter 5, closure, irreducibility and quasi-sobriety is studied in the Lower Vietoris Topology.

- In Section 5.1, closure and irreducibility and their relationship is investigated in hyperspaces.
 - Section 5.1.1 is devoted to arbitrary hyperspaces. In general, closure in hyperspaces is determined by Proposition 5.1.1, its Corollary 5.1.2 and Proposition 5.1.3. Proposition 5.1.4 describes precisely when hyperspaces are irreducible and illustrates the relationship between closure and irreducibility in hyperspaces. Theorem 5.1.6 is more general than the previous result and is more useful for determining when hyperspaces are quasi-sober.
 - Corollaries 5.1.5, 5.1.7, 5.1.8, 5.1.9 and 5.1.10 are surprising results which describe how the property of being irreducible is somewhat upwards closed in hyperspaces.
 - Finally Theorem 5.1.14 demonstrates that those closed hyperspaces having cellularity n are those having exactly n maximal elements.
 - Section 5.2.1 is devoted to almost d-finitely natural hyperspaces. The main results on irreducibility are Theorems 5.1.16 and 5.1.17 and Corollaries 5.1.18, 5.1.19 and 5.1.20. With regards to closure, the main results are that $\mathcal{F}_n(U)$ and $\mathcal{S}_{\leq n}(U)$ have the same closure in $\mathbf{L}(\tau)$ (cf. Theorem 5.1.21) and so do $\mathcal{D}_n(U)$ and $\mathcal{S}_n(U)$ have the same closure in $\mathbf{L}(\tau)$ (cf. Theorem 5.1.22). Under a certain condition $\mathcal{D}_n(U)$ and $\mathcal{F}_n(U)$ also have the same closure in $\mathbf{L}(\tau)$ (cf. Theorem 5.1.23).
- In Section 5.2, quasi-sobriety is investigated in hyperspaces. Theorem 5.2.1 characterises in general when a hyperspace is quasi-sober.

- In Section 5.2.2, a necessary condition for an almost natural hyperspace to be quasi-sober is obtained (cf. Theorem 5.2.2) and quasi-sobriety is characterised in all almost finitely natural hyperspaces (cf. Theorem 5.2.3). Corollary 5.2.4 verifies that $\mathcal{S}_{\leq n}(X)$ is always quasi-sober for any $n \in \mathbb{N}$, generalising the well-known notion of the sobrification of a space (cf. Corollary 5.2.5, Remark 5.2.5.1 and Definition 5.2.5.1). In Theorem 5.2.6 the properties of $\mathcal{S}_{\leq n}(X)$ are explored further when $n > 1$ and it is shown that it is T_0 iff X is sober and T_1 . The main result is that if $c(\mathcal{F}_n(X)) \subseteq c(\mathfrak{G}) \subseteq c(\mathcal{F}_n(X))$, then \mathfrak{G} is quasi-sober iff X is quasi-sober (cf. Theorem 5.2.7 and Corollary 5.2.8).
- In Section 5.2.3, the Sub-base Condition is defined as a generalisation of the fact that $\langle A \rangle$ is irreducible for every closed set A in X - cf. [23, Proposition 3.2] and [32, pp. 313-314] (where the latter only states the forward direction). A general characterisation is obtained for when a hyperspace is quasi-sober and satisfies the Sub-base Condition (cf. Theorem 5.2.11). It is also seen that the Sub-base Condition is somewhat upwards-closed in hyperspaces (cf. Proposition 5.2.10). In specific cases, better characterisations are obtained (cf. Corollary 5.2.12, Theorem 5.2.15, Corollary 5.2.16) and it is determined that $\mathcal{P}_0(X)$, 2^X , $\mathcal{C}(X)$, $\mathcal{F}(X)$ and $\bigcup_{i=n}^{\infty} \mathcal{F}_i(X)$ satisfy the Sub-base Condition for every n (cf. Corollary 5.2.14). It is determined when $\mathcal{C}(X)$ and $\mathcal{F}(X)$ are quasi-sober (cf. Corollary 5.2.17).

In Chapter 6, the maps $\hat{j}_n : X^{[n]} \rightarrow \mathcal{F}_n(X)$ and $\hat{i}_n : X^n \rightarrow \mathcal{F}_{\leq n}(X)$ are studied (cf. Definition 6.0.0.1) to provide insights into the cellularity of all almost d-finitely natural hyperspaces, as well as to determine when the Lower Vietoris Topology on $\mathcal{F}_n(X)$, $\mathcal{F}_{\leq n}(X)$ coincides with the Vietoris Topology.

- A complete characterisation is given for \hat{j}_n to be a quasi-open map in terms of the properties of X (cf. Theorem 6.2.5). For almost all spaces, there are two equivalent characterisations of when \hat{j}_n is a quasi-open map; one is “every non-discrete open

subspace of X has cellularity at least n ”, while the other provides useful insights.

- If $\text{cell}(X)$ is infinite and every non-discrete open subspace of X has cellularity at least n , then any almost d -finitely natural hyperspace in X has the same cellularity as $\text{cell}(X^{[n]}) = \text{cell}(X^n)$ (cf. Theorem 6.3.1); this is a generalised version of an analogous result of Fedorchuk for the hyperspaces $\mathcal{F}_n(X)$, $\mathcal{F}_{\leq n}(X)$ with the Vietoris Topology (cf. Proposition 6.1.1).
- The implications of Theorem 6.3.1 are investigated under different set theoretical assumptions (cf. Corollaries 6.3.2, 6.3.3 and 6.3.4).
- A complete characterisation of when \hat{i}_n is a quotient map is given, which is surprisingly equivalent to the Lower Vietoris Topology on $\mathcal{F}_{\leq n}(X)$ being equal to the Vietoris Topology (cf. Theorem 6.4.1).
- Necessary conditions are provided for the Lower Vietoris Topology to be equal to the Vietoris Topology on $\mathcal{F}_n(X)$ (cf. Theorem 6.4.2).
- A complete characterisation of when \hat{j}_n is a quotient map (under certain conditions on X) which is surprisingly equivalent to the Lower Vietoris Topology on $\mathcal{F}_n(X)$ being equal to the Vietoris Topology, equivalent to X being Hausdorff and equivalent to \hat{j}_n being a perfect covering map (cf. Theorem 6.4.3). This generalises a result by Costantini et al. (cf. Proposition 6.1.2).
- An example is provided to comment on the results in this section (cf. Example 6.5.0.1).

In Chapter 7, sobriety and several separation axioms are studied in the hyperspaces $\mathcal{F}_n(X)$.

- In Section 7.1, the study of the T_0 separation axiom and the sober property in the Lower Vietoris Topology that was started in Chapter 4 and Chapter 5, Section 5.2 respectively is continued. The T_D separation axiom is also investigated in the Lower Vietoris Topology, due to its relationship with sobriety (cf. Proposition 7.1.1).

It is determined when $c(\mathfrak{S})$ is T_D in the case that X is T_0 and $c(\mathcal{D}_f(X)) \subseteq c(\mathfrak{S})$ (cf. Theorem 7.1.2). Assuming that X is T_0 , it is proven that X is T_D iff $c(\mathcal{F}(X))$ is T_D (cf. Corollary 7.1.3); moreover the T_D axiom in 2^X is characterised (cf. Theorem 7.1.4).

For each $n > 1$, it is shown that the T_0 and T_D axioms (and the “sober” and “sober and T_D ” properties) are equivalent in the hyperspace $\mathcal{F}_n(X)$ and the T_0 axiom in this hyperspace (resp. the “sober” property in this hyperspace) is characterised in terms of the properties of X (cf. Theorem 7.1.6, Corollary 7.1.7 (1) and (2) and Lemma 7.1.5). An interesting observation is noted in Corollary 7.1.7 (3) that if $n > 2$ and $\mathcal{F}_n(X)$ is T_0 (resp. sober) then $\mathcal{F}_m(X)$ is T_D (resp. sober and T_D) for all $m \in \mathbb{N}$.

Some examples are given at the end of this section (cf. Examples 7.1.7.1 - 7.1.7.5) to illustrate that these results cannot be improved.

- In Section 7.2.1, a necessary condition is found for $\mathcal{F}_n(X)$ to be a non-trivial symmetric space (cf. Theorem 7.2.4) and it is shown that $\mathcal{F}_n(X)$ is T_1 iff X is T_1 (cf. Theorem 7.2.5).
- In Section 7.2.2, necessary and sufficient conditions for $\mathcal{F}_n(X)$ to be a non-trivial preregular space are determined in terms of the properties of X (cf. Theorem 7.3.5, Corollary 7.3.6). Examples 7.3.6.1, 7.3.6.2 are provided to illustrate that the results cannot be improved. The section concludes with a verification that for each $n \in \mathbb{N}$, $\mathcal{F}_n(X)$ is Hausdorff iff X is Hausdorff (cf. Theorem 7.3.7). Every preregular space (resp. Hausdorff space) is quasi-sober (resp. sober) (cf. Proposition 7.3.1), explaining the interest in studying preregularity in hyperspaces.

Chapter 2

Preliminaries

In this chapter, standard definitions and results are provided from the theory of topological spaces and order theory that are used in this thesis. One may use any one of [9, 13, 21, 39, 45, 57] as a reference for any terms or symbols that appear in this thesis yet are undefined, or for more details on the results which are provided here.

Some topological and set-theoretical results that are used later in this thesis are proved in this chapter.

There are concepts which appear in either [57] or [45] for which an alternative notation is used:

1. \subseteq is used rather than \subset to denote the subset relation on a set X .
2. Given sets A, B , $A \setminus B$ is used to denote the set $\{x \in A : x \notin B\}$ rather than $A - B$ (i.e. the complement of B relative to A).
3. Whenever \mathcal{A} is a collection of sets, $\bigcup \mathcal{A}$ is used to denote the set $\bigcup_{A \in \mathcal{A}} A = \bigcup \{A : A \in \mathcal{A}\}$ and, if $\mathcal{A} \neq \emptyset$, $\bigcap \mathcal{A}$ is used to denote the set $\bigcap_{A \in \mathcal{A}} A = \bigcap \{A : A \in \mathcal{A}\}$.
4. If \mathfrak{S} is a collection of subsets of X and $Y \subseteq X$:

$$\mathfrak{S}|Y := \{S \cap Y : S \in \mathfrak{S}\}.$$

5. If $f : X \rightarrow Y$ is a function between sets X, Y , and A is a subset of X , $f|_A$ denotes the restriction of f to A .

Throughout this thesis, n, m, k denote natural numbers unless stated otherwise, i.e. $n, m, k \in \mathbb{N} = \{1, 2, 3, \dots\}$.

Definition 2.0.0.1. Elements x_1, \dots, x_n of a set X are *pairwise distinct* if for each $i, j \in \{1, \dots, n\}$ such that $i \neq j$, $x_i \neq x_j$.

Definition 2.0.0.2. A family \mathcal{A} of sets is said to be *pairwise disjoint* if every element of \mathcal{A} is non-empty and whenever V, W are distinct elements of \mathcal{A} then $V \cap W = \emptyset$.

Note that the empty family of sets is, by definition, a pairwise disjoint family.

Lemma 2.0.1. Let X be a set, A be a subset of X and let $f : X \rightarrow X$ be a function. If $f^{-1}(A) = A$, then for every subset B of X :

$$A \cap f(B) = f(A \cap B).$$

Proof. Suppose that $A = f^{-1}(A)$. Then $f(A) = f(f^{-1}(A)) \subseteq A$. Thus:

$$f(A \cap B) \subseteq f(A) \cap f(B) \subseteq A \cap f(B).$$

Conversely, let $x \in A \cap f(B)$. There is $y \in B$ such that $x = f(y)$. Moreover, $f(y) = x \in A$, hence $y \in f^{-1}(A) = A$. Thus $y \in A \cap B$, hence $x = f(y) \in f(A \cap B)$. Thus $A \cap f(B) \subseteq f(A \cap B)$. It follows that $A \cap f(B) = f(A \cap B)$. ■

Definition 2.0.1.1. If $f : X \rightarrow Y$ is a function then the sets $f^{-1}(\{y\})$ where $y \in Y$ are called *fibres* of f .

2.1 Introduction to Topological Spaces

Definition 2.1.0.1. [57, Definition 3.1]

A *topological space* is a pair (X, τ) where X is a set - referred to as the *underlying set* -

and τ is a collection of subsets of X - called the *topology* on X - such that the following conditions are satisfied:

- $X, \emptyset \in \tau$.
- If $\{U_i : i \in I\}$ is a sub-collection of τ , then $\bigcup_{i \in I} U_i \in \tau$.
- If $n \in \mathbb{N}$ and $U_1, \dots, U_n \in \tau$ then $\bigcap_{i=1}^n U_i \in \tau$.

The elements of τ are said to be *open* in (X, τ) (or open in X if no confusion arises) or *open subsets* of (X, τ) (or open subsets of X if no confusion arises).

If no confusion arises, it may be declared that X is a topological space without denoting its topology.

For the rest of this thesis, X, Y, Z denote topological spaces unless stated otherwise.

For this section, A and B denote subsets of X unless stated otherwise.

The sets X, \emptyset are called the *trivial opens* of X , all other members of τ are said to be *non-trivial opens*.

A *neighbourhood* of a point $x \in X$ is a subset U of X for which there is an open V in X where $x \in V \subseteq U$ [57, 4.1 Definition].

The *interior* of A in X is [57, Definition 3.9]:

$$\text{Int}_\tau(A) := \bigcup \{F : F \text{ is open in } X, F \subseteq A\}.$$

The subscript is dropped if no confusion arises.

Definition 2.1.0.2. [57, Definition 6.1]

$\tau|A = \{U \cap A : U \in \tau\}$ is the *subspace topology* induced on A from τ . It is said that A is a *subspace* of X to mean that A is considered as a topological space with the subspace topology $\tau|A$ induced on A from τ . In this case if $D \subseteq A$:

- D is open in A iff there is V open in X where $V \cap A = D$ [57, Theorem 6.3(a)].

- D is a subspace of X iff D is a subspace of A [57, Theorem 6.2(d)].
- If D is open in A and A is open in X then D is open in X [45, Part I, Chapter 2, §16, Lemma 16.2].

Hereafter it is assumed that any subset of a topological space carries the subspace topology unless otherwise stated.

Definition 2.1.0.3. The set of real numbers \mathbb{R} has the *usual topology* if it has the topology where U is open in \mathbb{R} iff $U = \bigcup \mathcal{B}$ for some collection \mathcal{B} of open intervals in \mathbb{R} (cf. [57, 3.2(b)]).

Definition 2.1.0.4. [57, Definition 3.3]

A is said to be *closed* in X (or is said to be a closed subset of X) if $A = X \setminus U$ for some $U \in \tau$.

X, \emptyset are said to be the *trivial closed sets* of X ; all other closed sets in X are said to be *non-trivial closed sets*.

The intersection of a non-empty family of closed sets is closed and a finite union of closed sets is closed [57, Theorem 3.4].

The *closure* of A in X is the set [57, Definition 3.5]:

$$\text{Cl}_\tau(A) := \bigcap \{F : F \text{ is closed in } X, A \subseteq F\}.$$

If no confusion arises, the alternative notation $\bar{A} := \text{Cl}_\tau(A)$ is used.

A is said to be *dense* in X if $\bar{A} = X$, equivalently, for every non-empty open subset U of X , $A \cap U \neq \emptyset$.

If $\{A_i : i \in F\}$ is a finite collection of subsets of X then [57, Theorem 3.7]:

$$\overline{\bigcup_{i \in F} A_i} = \bigcup_{i \in F} \bar{A}_i.$$

For $x \in X$, $x \in \bar{A}$ iff for every open subset V of X such that $x \in V$, $V \cap A \neq \emptyset$ [45, Part I, Chapter 2, §17, Theorem 17.5(a)].

If $D \subseteq A$, then:

- $\text{Cl}_{\tau|A}(D) = \overline{D} \cap A$ [57, Theorem 6.3(c)].
- D is closed in A iff there is a closed subset V of X such that $V \cap A = D$ [57, Theorem 6.3(b)].
- If D is closed in A and A is closed in X , then D is closed in X [45, Part I, Chapter 2, §17, Theorem 17.3].

Definition 2.1.0.5. X is said to be a:

- T_0 -space if whenever $x, y \in X$ and $\overline{\{x\}} = \overline{\{y\}}$ then $x = y$ [46, Chapter I, 1.1].
Every subspace of a T_0 -space is a T_0 -space [57, Problems 13B.1].
- *symmetric space* if whenever $x, y \in X$ and $x \in \overline{\{y\}}$ then $\overline{\{x\}} = \overline{\{y\}}$ [46, Chapter I, 2.3].

These are also called R_0 -spaces and were introduced as such by A.S. Davis in [10, Theorem 2]. The symmetric axiom is referred to in the same reference as a *regularity axiom*.

- T_1 -space if $\{x\} = \overline{\{x\}}$ for all $x \in X$ [46, Chapter I, 2.1].
Every subspace of a T_1 -space is a T_1 -space [57, Problems 13B.1].
 X is a T_1 -space iff X is both a T_0 -space and a symmetric space [46, Chapter I, Proposition 2.3.1].
- T_2 -space or a *Hausdorff space* if for any two distinct elements x, y of X there are disjoint open subsets U, V of X for which $x \in U$ and $y \in V$ [46, Chapter I, 3.1].
Every Hausdorff space is a T_1 -space [57, p. 86].
Every subspace of a Hausdorff space is a Hausdorff space [57, Theorem 13.8(a)].
- a *Tychonoff space* if X is a T_1 -space and whenever A is closed in X and $x \in X \setminus A$ there is a continuous function $f : X \rightarrow [0, 1]$ which assigns x to 0 and A to 1 [57, 14.8 Definition].

X may be said to be T_0 , T_1 , symmetric, T_2 , Hausdorff or Tychonoff in place of the terms T_0 -space, T_1 -space, symmetric space, T_2 -space, Hausdorff space or Tychonoff space respectively.

Definition 2.1.0.6. [57, Definition 17.1]

X is said to be *compact* if whenever there is a collection $\{U_i : i \in I\}$ of open subsets of X such that $\bigcup_{i \in I} U_i = X$, then there is a finite subset F of I such that $\bigcup_{i \in F} U_i = X$.

Note that:

- Every finite subspace of X is compact [45, Part I, Chapter 3, §26, Example 3].
- If X is compact then every closed subspace of X is compact [57, Theorem 17.5(a)].
- A finite union of compact subspaces of X is compact as a subspace of X [57, Problems 17B.2].
- A compact subspace of a Hausdorff space X is closed in X [57, Theorem 17.5(b)].

Definition 2.1.0.7. [57, 15G]

X is *extremally disconnected* if the closure of any open set in X is open.

Definition 2.1.0.8. The following are well-known topologies:

- The *discrete topology* on X is the topology on X consisting of every subset of X [57, Examples 3.2(c)]. X is said to be a *discrete space* or simply X is said to be *discrete* if X has the discrete topology.
- The *trivial topology* on X is the topology on X consisting of only X and \emptyset [57, Examples 3.2(d)]. X is said to be a *trivial space* or simply that X is said to be *trivial* to mean that X has the trivial topology. Note X is trivial iff $\overline{\{x\}} = X$ for all $x \in X$ [57, Examples 3.8 f)].
- The *co-finite topology* on an infinite set X is the one in which the finite subsets of X are the proper closed subsets of X [57, Examples 3.8(a)].
 X is T_1 but not Hausdorff [57, Examples 13.6(a)].

- If X is uncountable, then the *countable complement topology* on X is the one in which the countable subsets of X are the proper closed subsets of X [52, Part II, Section 20].

If $A \subseteq X$, then A is compact iff A is finite [52, Part II, Section 20, Point 2].

X is T_1 but not Hausdorff. [52, Part II, Section 20, Point 1]

Definition 2.1.0.9. [57, Definition 5.1]

A *base* for X (or a *base for the open sets* of X) is a family \mathcal{B} of opens in X such that every open U in X is a union of some subfamily of \mathcal{B} , equivalently, whenever U is open in X and $x \in U$ then there is $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

Proposition 2.1.1. [57, Theorem 5.3]

A family \mathcal{B} of subsets of X is a base for some topology on X iff the following two conditions hold:

1. $X = \bigcup \mathcal{B}$, and
2. whenever $B_1, B_2 \in \mathcal{B}$ and there is $x \in B_1 \cap B_2$, there is $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Definition 2.1.1.1. [57, Definition 5.5] A *sub-base* for X (or a *sub-base for the open subsets* of X) is a family \mathcal{S} of open sets in X such that the following family is a base for X , called the *base for X generated by \mathcal{S}* :

$$\mathcal{B}_{\mathcal{S}} := \left\{ \bigcap_{i=1}^n S_i : n \in \mathbb{N}, S_1, \dots, S_n \in \mathcal{S} \right\} \cup \{X\}.$$

It is also said that \mathcal{S} *generates* the topology on X .

Since $\mathcal{B}_{\mathcal{S}}|_A = \mathcal{B}_{\mathcal{S}|_A}$ is a base for A (cf. [57, Theorem 6.3(f)]), it follows that $\mathcal{S}|_A$ is a sub-base for A .

Proposition 2.1.2. [57, Theorem 5.6]

Any family of subsets of X is a sub-base for some topology on X .

Definition 2.1.2.1. [57, Problems 5E]; [39, Ch. 1, §5, XI, Remark 4]

A family \mathcal{F} of closed sets of X is called a *base for the closed sets* of X if for every proper closed set U of X there is a non-empty subfamily \mathcal{H} of \mathcal{F} such that $U = \bigcap \mathcal{H}$.

A family \mathcal{G} of closed sets of X is said to be a *sub-base for the closed sets* of X if the following family of subsets of X is a base for the closed sets of X , called the *base for the closed sets of X generated by \mathcal{G}* :

$$\mathcal{F}_{\mathcal{G}} := \left\{ \bigcup_{i=1}^n G_i : n \in \mathbb{N}, G_1, \dots, G_n \in \mathcal{G} \right\} \cup \{\emptyset\}$$

It is also said that \mathcal{G} *generates* the closed sets of X .

Proposition 2.1.3. [39, Ch. 1, §5, XI, Remark 4]

A family \mathcal{G} of subsets of X is a sub-base for the closed sets of X iff $\{X \setminus G : G \in \mathcal{G}\}$ is a sub-base for X .

Any family of subsets of X is a sub-base for the closed sets of some topology on X .

Definition 2.1.3.1. Let $f : X \rightarrow Y$ be a function. Then f is said to be:

- *continuous* if $f^{-1}(V)$ is open in X whenever V is open in Y [57, Definition 7.1, Theorem 7.2], equivalently:
 - whenever U is a closed subset of Y then $f^{-1}(U)$ is a closed subset of X [57, Theorem 7.2(c)].
 - whenever E is a subset of X then $f(\overline{E}) \subseteq \overline{f(E)}$ [57, Theorem 7.2(d)].
- *open* if whenever V is an open subset of X then $f(V)$ is an open subset of Y [57, Definition 8.5].
 - If f is continuous then f is open iff there is a base \mathcal{B} for X such that $f(U)$ is open in Y for every $U \in \mathcal{B}$ [13, Theorem 1.4.14].
- *closed* if whenever U is a closed subset of X then $f(U)$ is a closed subset of Y [57, Definition 8.5].

- If f is continuous then $f(\overline{A}) = \overline{f(A)}$; moreover
 f is closed iff for every $y \in Y$ and every open neighbourhood U of $f^{-1}(\{y\})$,
there is a neighbourhood V of y such that $f^{-1}(V) \subseteq U$ [13, Theorem 1.4.13].
- a *homeomorphism* if there is a continuous function $g : Y \rightarrow X$ such that $g \circ f = \text{id}_X$
is the identity function on X and $f \circ g = \text{id}_Y$ is the identity function on Y ; in this
case it is said that X and Y are homeomorphic [57, Definition 7.8].

If f is continuous then $f|_A : A \rightarrow f(A)$ is continuous [57, Theorem 7.5].

If X is homeomorphic to Y and Y is homeomorphic to Z , then X is homeomorphic to Z [57, p. 46].

If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are continuous, then the composite function $g \circ f : X \rightarrow Z$ is continuous [57, Theorem 7.3].

Lemma 2.1.4. *Let \mathcal{G} be a sub-base for the closed subsets of Y and let $f : X \rightarrow Y$ be a function. Then f is continuous iff $f^{-1}(A)$ is closed in X for all $A \in \mathcal{G}$.*

Proof. Only the backwards direction needs to be verified. For $G \in \mathcal{G}$, $X \setminus f^{-1}(G) = f^{-1}(Y \setminus G)$ is open in X . Since $\mathcal{S} := \{Y \setminus G : G \in \mathcal{G}\}$ is a sub-base for Y , the result follows from [45, p. 103]. ■

Lemma 2.1.5. *Let \mathcal{G} be a sub-base for the closed subsets of X . Let $f : X \rightarrow X$ be a function such that $f^{-1}(U) = U$ for all $U \in \mathcal{G}$. Then $f|_B : B \rightarrow f(B)$ is a continuous, closed, open and surjective map such that whenever A is open or closed in X :*

$$f(A \cap B) = A \cap f(B) \quad \text{and} \quad (f|_B)^{-1}(f(A \cap B)) = A \cap B.$$

Proof. f is continuous by the previous Lemma 2.1.4 hence $f|_B$ is continuous.

Let $U \in \mathcal{F}_{\mathcal{G}}$. Suppose that $U \neq \emptyset$. Then there are $V_1, \dots, V_n \in \mathcal{G}$ such that $U = \bigcup_{i=1}^n V_i$ and hence $f^{-1}(U) = f^{-1}(\bigcup_{i=1}^n V_i) = \bigcup_{i=1}^n f^{-1}(V_i) = \bigcup_{i=1}^n V_i = U$. This equality also holds if $U = \emptyset$.

Let A be a proper closed subset of X . There is a non-empty subfamily $\{V_i : i \in I\}$ of \mathcal{F}_{cg} such that $A = \bigcap_{i \in I} V_i$ and hence $f^{-1}(A) = f^{-1}(\bigcap_{i \in I} V_i) = \bigcap_{i \in I} f^{-1}(V_i) = \bigcap_{i \in I} V_i = A$. This equality also holds if $A = X$.

Hence $f^{-1}(A) = A$ for every closed subset A of X . Hence for every open subset U of X , $f^{-1}(X \setminus U) = X \setminus U$ and hence $f^{-1}(U) = f^{-1}(X \setminus (X \setminus U)) = X \setminus f^{-1}(X \setminus U) = X \setminus (X \setminus U) = U$. Thus by Lemma 2.0.1, it follows that for any subset A of X which is either closed or open in X , $f(A \cap B) = A \cap f(B)$ and thus:

$$\begin{aligned} (f|_B)^{-1}(f(A \cap B)) &= (f|_B)^{-1}(A \cap f(B)) = f^{-1}(A \cap f(B)) \cap B \\ &= f^{-1}(A) \cap f^{-1}(f(B)) \cap B \\ &= f^{-1}(A) \cap B \\ &= A \cap B. \end{aligned}$$

If C is closed in B , there is a closed subset A of X such that $C = A \cap B$ and so $f(C) = f(A \cap B) = A \cap f(B)$ is closed in $f(B)$. Thus $f|_B$ is a closed map. Similarly, it can be shown that this map is open. ■

2.1.1 Products, Quotient Spaces and the T_0 -identification

The Definition 2.1.5.1 below uses the notation $|I|$ for the cardinality of I ; this notion is explained in the next Chapter 3.

Definition 2.1.5.1. [57, Definition 8.1, Definition 8.3]

For this definition, X is considered a set.

If $\{X_i : i \in I\}$ is a family of sets then its *product* is the set:

$$\prod_{i \in I} X_i := \{f : I \rightarrow \bigcup_{i \in I} X_i : \text{for each } i, f(i) \in X_i\}.$$

When I is finite where $|I| = n \in \mathbb{N}$ then $I = \{i_1, \dots, i_n\}$ and:

$$X_{i_1} \times \dots \times X_{i_n} := \prod_{i \in I} X_i = \{(x_1, \dots, x_n) : \text{for all } i \in I, x_i \in X_i\}.$$

If $X_i = X$ for all $i \in I$ then the product $X^I := \prod_{i \in I} X$ is the set of all functions $f : I \rightarrow X$. The subset of X^I consisting of injective functions is what is denoted by $X^{[I]}$. This notation appears for the case where I is finite in [7, p. 246, 1. Preliminaries].

Moreover:

$$X^n := \{(x_1, \dots, x_n) : x_1, \dots, x_n \in X\} \quad \text{and}$$

$$X^{[n]} := \{(x_1, \dots, x_n) : x_1, \dots, x_n \in X \text{ and } x_i \neq x_j \text{ whenever } i \neq j\}.$$

X^n is called the n -th power of X and $X^{[n]}$ the *augmented n -th power* of X .

If for each $i \in I$, X_i is a topological space then the *product topology* or *Tychonoff topology* on the product $\prod_{i \in I} X_i$ is the one whose sub-basic opens have the following form: by specifying that for each $i \in I$ and each open $U \in X_i$, the set $\pi_i^{-1}(U)$ is open, where $\pi_i : \prod_{i \in I} X_i \rightarrow X_i$ is the i -th *projection map* defined by $\pi_i(f) = f(i)$ for each $f \in \prod_{i \in I} X_i$ (cf. Proposition 2.1.2). This map is both continuous and open [57, 8.6 Theorem].

Then the following family forms a base for $\prod_{i \in I} X_i$:

$$\left\{ \bigcap_{j=1}^n \pi_{i_j}^{-1}(U_{i_j}) : n \in \mathbb{N} \text{ and } i_1, \dots, i_n \in I \right\}.$$

Hereafter, $\prod_{i \in I} X_i$ will always be assumed to have the product topology if X_i is a topological space for every $i \in I$.

Definition 2.1.5.2. [57, Definition 9.1]

If Y is a set, and $g : X \rightarrow Y$ is a surjective map, then $\tau_g := \{V \subseteq Y : g^{-1}(V) \in \tau\}$ is the *quotient topology on Y induced by g* ; in this case, g is called a *quotient map*.

$g : X \rightarrow (Y, \tau_g)$ is continuous and A is open (resp. closed) in Y iff $g^{-1}(A)$ is open (resp. closed) in X .

Proposition 2.1.6. [57, Theorem 9.2]

If $f : X \rightarrow Y$ is a continuous, surjective map which is either open or closed, then the given topology on Y is the quotient topology on Y induced by f .

Proposition 2.1.7. [57, Theorem 9.6]; [57, Theorem 9.7]

Let Y be a set and $f : X \rightarrow Y$ be a surjective function. Let $\mathcal{D} := \{f^{-1}(\{f(x)\}) : x \in X\}$ and let $g : X \rightarrow \mathcal{D}$ be the map defined by the formula $g(x) = f^{-1}(\{f(x)\})$ for each $x \in X$. Then (Y, τ_f) is homeomorphic to (\mathcal{D}, τ_g) .

Definition 2.1.7.1. [57, Examples 13.2(c) and Problems 13C]

The T_0 -identification of X is the topological space with underlying set $\{\overline{\{x\}} : x \in X\}$ and whose topology is the quotient topology induced by the map assigning each $x \in X$ to $\overline{\{x\}}$.

The T_0 -identification of X is a T_0 -space [57, Problems 13C.2]; [47, Theorem 3.19].

X is T_0 iff X is homeomorphic to its T_0 -identification [47, Corollary 3.20].

X is symmetric iff the T_0 -identification of X is T_1 [47, Section 4, Table 1].

Proposition 2.1.8. *If X and Y are homeomorphic then their T_0 -identifications are homeomorphic.*

Proof. Let $f : X \rightarrow Y$ be a homeomorphism. Let \hat{X}, \hat{Y} be the T_0 -identifications of X and Y respectively. Let $p_X : X \rightarrow \hat{X}, p_Y : Y \rightarrow \hat{Y}$ be the maps specified by Definition 2.1.7.1. By [47, Theorem 3.22], there are continuous maps $\hat{f} : \hat{X} \rightarrow \hat{Y}$ and $\hat{g} : \hat{Y} \rightarrow \hat{X}$ such that $\hat{f} \circ p_X = p_Y \circ f$ and $\hat{g} \circ p_Y = p_X \circ f^{-1}$.

Now, $(\hat{g} \circ \hat{f})(p_X(x)) = \hat{g}(p_Y(f(x))) = p_X(f^{-1}(f(x))) = p_X(x)$ for all $x \in X$; thus $\hat{g} \circ \hat{f}$ is the identity function on \hat{X} . Similarly, $\hat{f} \circ \hat{g}$ is the identity function on \hat{Y} . Thus \hat{f} is a homeomorphism. ■

2.2 Introductory Order Theory

Definition 2.2.0.1. [21, Definition O-1.1; Definition O-1.6]

A relation \leq on a set P is said to be a:

- *pre-order* if it satisfies the following conditions:

- Reflexivity: $x \leq x$ for all $x \in P$
- Transitivity: $x \leq y$ and $y \leq z$ implies $x \leq z$ for all $x, y, z \in P$
- *partial order* if it is a pre-order and satisfies the following condition:
 - Anti-symmetry: $x \leq y$ and $y \leq x$ implies $y = x$ for all $x, y \in P$.

Given a relation \leq on a set P , (P, \leq) is said to be a *pre-ordered set* (resp. *partially ordered set* - often referred to as *poset* for short) if \leq is a pre-order (resp. a partial order) on P . If there is no confusion, P is referred to as a pre-ordered/partially ordered set.

For $x, y \in P$, the statement $x < y$ denotes $x \leq y$ and $x \neq y$.

Definition 2.2.0.2. Suppose that P is a pre-ordered set.

- $A \subseteq P$ is *directed* with respect to \leq if for each $x, y \in A$ there is $z \in A$ such that $x \leq z$ and $y \leq z$ [21, Definition O-1.1].
- Given $A \subseteq P$:

$$\uparrow A := \{x \in P : \text{there is } a \in A \text{ such that } a \leq x\} \text{ and}$$

$$\downarrow A := \{x \in P : \text{there is } a \in A \text{ such that } x \leq a\}.$$

A is called an *upper set* (resp. *lower set*) of (P, \leq) if $A = \uparrow A$ (resp. $A = \downarrow A$) [21, Definition O-1.3].

In a particular context, the phrase “with respect to \leq ” may be removed if there is no confusion as to the fact that \leq is the considered pre-order on P in that context.

Definition 2.2.0.3. Suppose that (P, \leq) is a poset.

- [9, 1.23 Maximal and minimal elements] An element x of P is said to be:
 - *maximal* with respect to \leq , if for every $y \in P$ such that $x \leq y$ it must be the case that $x = y$.
 - *minimal* with respect to \leq , if for every $y \in P$ such that $y \leq x$ it must be the case that $x = y$.

- the *minimum* element of (P, \leq) if for every $y \in P$, $x \leq y$.
- the *maximum* element of (P, \leq) if for every $y \in P$, $y \leq x$.
- [21, Definition O-1.6] P is said to be:
 - an *antichain* if P is non-empty and for every $x, y \in P$, $x \leq y$ implies $x = y$.
 - *well-ordered* (equivalently \leq is a *well-order*) if every subset of P has a minimum element with respect to \leq .
- Let $Y \subseteq P$.
 - [35, p. 201] Y is *dense in* P if for every $x \in P$ there is $y \in Y$ such that $y \leq x$.
 - [35, p. 202] Y is a *filter* on P if:
 - * Y is a non-empty upper set in P and
 - * If $p, q \in Y$ then there is $r \in Y$ where $r \leq p$ and $r \leq q$.
- [9, 1.34 Definitions] A function $f : P \rightarrow Q$ where Q is a poset is called an *order-isomorphism* if $x \leq y$ iff $f(x) \leq f(y)$. If such a function exists then P and Q are said to be *order-isomorphic*.

Note that order-isomorphisms are necessarily bijections.

Proposition 2.2.1. [9, p. 16] *Let (P, \leq) be a finite poset. For each $x \in P$, there is a maximal element y of P such that $x \leq y$.*

Definition 2.2.1.1. [21, Definition O-5.4]

Let (P, \leq) be a poset. The *upper interval topology* on P is the one which has, as a subbase for its closed sets, the collection $\mathcal{G}_{UI} := \{\downarrow \{a\} : a \in P\} \cup \{P\}$ (cf. Proposition 2.1.3).

P with the upper interval topology is a T_0 -space [36, Proposition 1.8].

Moreover if $x \in P$ then $\overline{\{x\}} = \downarrow \{x\}$ [36, In the proof of Proposition 1.8], thus for any finite subset F of P , $\overline{F} = \downarrow F$.

Remark 2.2.1.1. Gierz *et al.* [21, Definition O-5.4] uses the term “upper topology” instead of “upper interval topology”. The latter term is chosen since this topology is associated with the well-known *interval topology* on a poset (cf. [21, Definition O-5.4]).

Definition 2.2.1.2. [27, Section 2.1, pp. 217-218] The *specialisation pre-order* on X is the pre-order \leq_τ on X defined by $x \leq_\tau y$ iff $x \in \overline{\{y\}}$ for each $x, y \in X$.

Proposition 2.2.2. 1. \leq_τ is a partial order iff X is a T_0 -space.

2. (X, \leq_τ) is an antichain iff X is a T_1 -space.

Proof. In [22, Definition 4.2.1], the specialisation pre-order is defined differently compared to Definition 2.2.1.2 (and is referred to as the “specialization quasi-order”). However, these definitions are equivalent as demonstrated in [22, Lemma 4.2.7]. Finally the result in this Proposition 2.2.2 follows from [22, Proposition 4.2.3]. ■

Hereafter, X, Y, Z will always denote infinite topological spaces unless otherwise specified.

Chapter 3

Cellularity and Sobriety

In this chapter, I define and study the notions of cellularity, irreducible spaces, quasi-sober spaces and sober spaces. The results presented in this chapter will be used to determine which hyperspaces with the Lower Vietoris Topology are irreducible or sober (cf. Chapter 5) as well as to study the cellularity of hyperspaces with the Lower Vietoris Topology (cf. Chapter 6).

Section 3.1 provides an overview of the notions of set-theory and cardinality which are essential for defining and discussing the cellularity of a topological space.

Section 3.2 provides an overview of the cellularity of a topological space. Unlike in the literature, the cellularity is allowed to be finite to accommodate for the fact that many hyperspaces with the Lower Vietoris Topology have finite cellularity. Results from the literature on cellularity are provided which will be used in the following sections and chapters.

Section 3.3 provides an overview of irreducible spaces, irreducible components of topological spaces and how the notion of irreducible components is related to the notion of finite cellularity. I believe Proposition 3.3.6, Proposition 3.3.7 Proposition 3.3.8 and Proposition 3.3.11 are new.

Section 3.4 provides an overview of sober spaces and quasi-sober spaces.

Section 3.5 provides an overview of the results in the literature on cellularity of products of topological spaces; these results are relevant to Chapter 6. The results from Section 3.5.1 will be utilised to expand on the new results achieved in Chapter 6 (cf. Theorem 6.3.1 and Corollaries 6.3.4 and 6.3.2). In 3.5.2, it is determined when product spaces have finite cellularity by utilising the relationship between cellularity and irreducibility established in Section 3.3.2. I correct the proof of a result from the literature (cf. Lemma 3.5.10) and I prove a more general result which is new and expands on known results from the literature for products with infinite cellularity (cf. Theorem 3.5.11). Specifically, the irreducible components of a general product are determined and it is shown that the cellularity of a product is finite iff there are a finite number of non-irreducible factors with finite cellularity, in which case the cellularity of the product, is the product of the cellularity of each of the factors.

Section 3.6 presents new results on the cellularity of the augmented n -th power of a topological space. The main result (cf. Theorem 3.6.7) states that in general, $\text{cell}(X^n) \leq \text{cell}(X^{[n]})$. This extends a known result by Fedorchuk (cf. Proposition 3.6.1), stating that $\text{cell}(X^n) = \text{cell}(X^{[n]})$ when X is Hausdorff. This result will be improved on in Chapter 6, where it will be shown that for a specific class of non-Hausdorff spaces, this equality still holds (cf. Theorem 6.3.1).

Throughout this chapter, unless otherwise specified, f, g, h represent functions.

3.1 Introduction to ZFC and Cardinals

This section provides a basic overview of set theory, cardinals and ordinals limited to definitions, notations and results used throughout this thesis. Some notions, such as that of an ordinal, are not defined here since a discussion would be out of the scope of this thesis. References are provided for further reading.

[35, p. 3, Axioms of Zermelo-Fraenkel] The theory of sets used in this thesis is known as ZFC; its axioms are known as the ZFC *axioms* and consists of:

- The *Zermelo-Fraenkel axioms* (or *ZF axioms*); the theory resulting from these axioms is called ZF.
- The *Axiom of Choice*.

The Axiom of Choice. [35, p. 47]

If \mathcal{A} is a family of non-empty sets then there is a function $f : \mathcal{A} \rightarrow \bigcup \mathcal{A}$ where $f(A) \in A$ for every $A \in \mathcal{A}$.

This statement is of course equivalent to the following: “If $\{X_i : i \in I\}$ is a family of non-empty sets, then $\prod_{i \in I} X_i$ is non-empty”.

In the ZF set theory the concept of an *ordinal number* (or simply an *ordinal*) can be defined as a special type of well-ordered set where the class of all ordinal numbers is well-ordered by the relation $\alpha < \beta \iff \alpha \in \beta$ (which is equivalent to $\alpha \subset \beta$). Thus, every family of ordinals has a least element. Moreover, every well-ordered set is order-isomorphic to a unique ordinal number. The Greek letters α, β, γ denote ordinal numbers. A *successor ordinal* is one that is equal to $\alpha \cup \{\alpha\}$ for some α ; any other non-zero ordinal is called a *limit ordinal*. The finite ordinals are the natural numbers and ω denotes the first infinite ordinal (which is also the first limit ordinal) and is order-isomorphic to the set of natural numbers \mathbb{N} [11, p. 66].

The statement “For any set, there is a well-order on that set” is equivalent to the Axiom of Choice in ZF (cf. [11, p. 58; p. 59, Theorem 2.7.3]). Thus in ZFC, whenever A is a set, there is a bijection $f : \alpha \rightarrow A$ where α is some ordinal (as every order-isomorphism is also a bijection). The *cardinal number* or *cardinal* of A is $|A| := \min\{\alpha : \text{there is a bijection } f : \alpha \rightarrow A\}$. The Greek letters $\lambda, \kappa, \mu, \rho$ denote cardinal numbers. Every finite ordinal is a cardinal and the finite cardinals are the finite ordinals, i.e. the natural numbers. A set is finite iff its cardinal is finite. The first infinite cardinal is

$\aleph_0 := |\omega| = \omega$ and every infinite cardinal is a limit ordinal [11, p. 76]. λ is said to be *countable* if $\lambda \leq \aleph_0$; otherwise λ is *uncountable* [11, p. 81].

The class of cardinals is well-ordered with respect to the inherited ordering from the class of ordinals. Thus any family of cardinals has a least element. Note that if \mathcal{A} is a family of cardinals, then its supremum in the class of all cardinals is equal to its supremum in the class of all ordinals and is equal to $\sup \mathcal{A} := \bigcup \mathcal{A}$ [35, p. 20, point (2.4); p. 29, Lemma 3.4(ii)].

Note that $\kappa \leq \lambda$ iff whenever $\kappa = |A|$ and $\lambda = |B|$ there is an injective function $f : A \rightarrow B$ [11, Lemma 3.6.2]. Moreover the following statement is true in ZF:

The Schröder-Bernstein Theorem. [11, p. 77]

If A and B are sets and there are injections $f : A \rightarrow B$ and $g : B \rightarrow A$, there is a bijection $h : A \rightarrow B$.

Thus $\kappa = \lambda$ iff whenever $\kappa = |A|$ and $\lambda = |B|$ there is a bijection $f : A \rightarrow B$. In this case A and B are said to have the same cardinality.

If no confusion arises, $\prod_{\alpha < \beta} \lambda_\alpha$ denotes the cardinal $|\prod_{\alpha < \beta} \lambda_\alpha|$ and $\sum_{\alpha < \beta} \lambda_\alpha$ denotes the cardinal $|\bigcup_{\alpha < \beta} \lambda_\alpha \times \{\alpha\}|$; κ^λ denotes the cardinal $\prod_{\alpha < \lambda} \kappa = |\kappa^\lambda|$. The notation $\kappa + \lambda$ is used for $|(\kappa \times \{0\}) \cup (\lambda \times \{1\})|$ and $\kappa \cdot \lambda$ is used for $|\kappa \times \lambda|$. This notation is defined in [11, Section 3.7].

For any cardinal λ , it is true that $\lambda < 2^\lambda = |\mathcal{P}(\lambda)|$ [11, Lemma 3.6.6, Lemma 3.9.1]. Thus the *successor* of λ may be defined as $\lambda^+ := \min\{\kappa : \lambda < \kappa\}$ [11, p. 80]. λ is a *successor cardinal* if $\lambda = \kappa^+$ for some cardinal κ ; otherwise λ is a *limit cardinal* [11, p. 88].

Let S be a set of sentences. S is said to be *consistent* iff there is no sentence φ where φ and its negation, i.e. $\neg\varphi$, are both provable from S . If S is consistent then all of its non-empty subsets are also consistent. A sentence φ is not provable from S iff $S + \neg\varphi := S \cup \{\neg\varphi\}$ is consistent [38, p. 6]. φ is said to be *independent* of S if both $S + \neg\varphi$ and $S + \varphi$ are

consistent.

The following statement is independent of ZFC (assuming ZFC is consistent, then ZFC+GCH is consistent [38, Ch VI, §4, Corollary 4.9, p. 175], while ZFC+¬ GCH is also consistent [38, Ch. VII, §5, Corollary 5.15; Ch. VII, §6, Theorem 6.18]):

The Generalised Continuum Hypothesis (GCH). [11, p. 97]

If λ is infinite then $2^\lambda = \lambda^+$.

Proposition 3.1.1. [11, Corollary 3.7.10]

Let λ be an infinite cardinal. The union of at most λ sets of cardinality at most λ has cardinality at most λ .

The *cofinality* of a limit ordinal γ , denoted by $\text{cf}(\gamma)$ or $\text{cf } \gamma$, is the smallest limit ordinal α such that there is an increasing sequence of ordinals $\{\gamma_\beta : \beta < \alpha\}$ where $\gamma = \sup_{\beta < \alpha} \gamma_\beta$. An infinite cardinal λ is said to be a *regular cardinal* if $\text{cf}(\lambda) = \lambda$, otherwise λ is a *singular cardinal* [11, p. 89].

For any limit ordinal λ , $\text{cf } \lambda$ is a regular cardinal and $\aleph_0 \leq \text{cf } \lambda \leq \lambda$ [11, Lemma 3.8.4].

An infinite cardinal λ is said to be a *weakly inaccessible cardinal* if λ is a regular, uncountable limit cardinal [11, p. 96]. The existence of such a cardinal is not provable in ZFC+GCH [38, Ch. VI, §5, Corollary 4.13, p. 177].

Proposition 3.1.2. [11, Theorem 3.8.6] *Every infinite successor cardinal is regular.*

Proposition 3.1.3. [35, p. 29, (3.7) and (3.8); Theorem 5.16]; [11, Theorem 3.9.3]

Let λ be infinite. Then:

1. $\lambda < \text{cf}(2^\lambda) \leq 2^\lambda = \kappa^\lambda$ whenever $2 \leq \kappa \leq \lambda$.

2. $\kappa \cdot \lambda = \max\{\kappa, \lambda\} = \kappa + \lambda$. [11, Corollary 3.7.8]

3. $(\kappa^\lambda)^\mu = \kappa^{\lambda \cdot \mu}$. [35, p. 29, (3.7)]

4. If $\kappa \leq \lambda$ then $\kappa^\mu \leq \lambda^\mu$. [35, p. 29, (3.8)]

Let κ be infinite. Then:

5. Suppose $\kappa > \lambda$ and $\mu^\lambda < \kappa$ whenever $\mu < \kappa$. Then if $\text{cf } \kappa > \lambda$ then $\kappa^\lambda = \kappa$ [35, Theorem 5.20 (iii)(a)].

Let $\{\lambda_i : i \in I\}$ be a family of cardinals (not necessarily infinite).

6. If $I = \bigcup_{j \in J} A_j$ is a disjoint union then $\prod_{i \in I} \lambda_i = \prod_{j \in J} \prod_{i \in A_j} \lambda_i$ [35, p. 53, 5.16].

Definition 3.1.3.1. [38, Ch II, §2, Definition 2.2]

A poset (P, \leq) satisfies the *countable chain condition* (abbreviated “ P is *ccc*”) if every antichain in P is countable.

Martin’s Axiom (MA). [35, Ch II, §2, Definition 2.5]

If (P, \leq) is a *ccc* poset and if \mathcal{A} is a family of dense subsets of P where $|\mathcal{A}| < 2^{\aleph_0}$ then there is a filter on P which meets every element of \mathcal{A} .

3.2 Cellularity

Definition 3.2.0.1. [15, p. 56],[6, p. 29, Definition]

A *cellular family* in X is a pairwise disjoint family of opens in X .

The following definition for the cellularity of a topological space is similar to Engelking’s definition [13, 1.7.12], but differs in an important aspect (cf. Remark 3.2.0.1).

Definition 3.2.0.2. The *cellularity* or the *Souslin number* (also *Suslin number*) of X is:

$$\text{cell}(X) := \sup\{|\mathcal{U}| : \mathcal{U} \text{ cellular in } X\}.$$

Note that $X = \emptyset$ iff $\text{cell}(X) = 0$ since $\emptyset = \{\}$ is a cellular family, moreover $\{X\}$ is always cellular so the converse holds.

Remark 3.2.0.1. Definition 3.2.0.2) differs from Engelking’s definition [13, 1.7.12] only in that Engelking assumes that $|\mathbb{N}| \leq \text{cell}(X)$. This assumption is also made in [14, p.

56]. This assumption is reasonable for when X is a Hausdorff space; in this case every infinite subspace has infinite cellularity.

Suppose that every cellular family in X has at most n elements. Then $\text{cell}(X) \leq n \leq |\mathbb{N}|$. Under the assumption that $|\mathbb{N}| \leq \text{cell}(X)$, it follows from the well-ordering of the cardinals that $\text{cell}(X) = |\mathbb{N}|$. Thus, under this assumption, one would not be able to distinguish between topological spaces in which every cellular family has at most a finite number of elements. This needs to be done in Section 3.6 of this chapter, Chapter 5, Chapter 6 and Chapter 7.

Since this distinction needs to be made, it is not assumed that $|\mathbb{N}| \leq \text{cell}(X)$. There are infinite topological spaces for which every subspace has finite cellularity (cf. Proposition 3.3.12 and Example 3.3.12.1) and there are infinite T_1 -spaces with finite cellularity (cf. Example 3.3.0.1).

Remark 3.2.0.2. X has a cellular family of cardinality $\text{cell}(X)$ if:

- $\text{cell}(X)$ is a successor cardinal [37, p. 11, Definition 1.22],
- $\text{cell}(X)$ is a singular cardinal [37, p. 84, Point 4.1] or
- if $\text{cell}(X) = \omega$ [6, p. 31, Lemma 2.10]

The remaining case is when $\text{cell}(X)$ is a regular uncountable limit cardinal, i.e. a weakly inaccessible cardinal; this is addressed in the next Remark 3.2.0.3.

Remark 3.2.0.3. [37, p. 147, point 7.6]

If λ is a weakly inaccessible cardinal and $D(\kappa)$ is the discrete space with underlying set κ (for $\kappa < \lambda$), then $\prod_{\kappa < \lambda} D(\kappa)$ has cellularity λ but no cellular family with cardinality λ .

Definition 3.2.0.3. [13, p. 12, p. 25]

The *weight* of X is $w(X) := \min\{|\mathcal{B}| : \mathcal{B} \text{ is a base for } X\}$.

X is *second-countable* if $w(X) \leq \aleph_0$.

The *density* of X is $d(X) := \min\{|A| : A \subseteq X \text{ and } \overline{A} = X\}$.

X is *separable* if $d(X) \leq \aleph_0$.

Proposition 3.2.1. *The following statements are true:*

1. [13, 2.7.9(d)] *If A is dense in X then $\text{cell}(A) = \text{cell}(X)$.*
2. [13, 1.7.12(b)] *If $f : X \rightarrow Y$ is continuous and onto then $\text{cell}(Y) \leq \text{cell}(X)$.*
3. [15, Proposition 1.2] *If $X = Y \cup Z$ then $\text{cell}(X) \leq \text{cell}(Y) + \text{cell}(Z)$.*
4. $\text{cell}(X) \leq d(X) \leq \min\{w(X), |X|\}$.
5. *If X is discrete then $\text{cell}(X) = |X|$.*

Proof. 4. The statement $\text{cell}(X) \leq d(X) \leq w(X)$ follows from [13, 1.7.12(a)]; moreover since X is dense in itself, $d(X) \leq |X|$.

5. $\{\{x\} : x \in X\}$ is cellular in X and $|X| = |\{\{x\} : x \in X\}| \leq \text{cell}(X)$. By the previous point and the well-ordering of the cardinals, the result follows. ■

Remark 3.2.1.1. It may be the case that $\text{cell}(X) < d(X)$ (i.e. the inequality in point 4 of the previous Proposition 3.2.1 may be strict); see Example 3.5.7.1.

Definition 3.2.1.1. • $f : X \rightarrow Y$ is said to be a *quasi-open map* if $\text{Int}(f(U)) \neq \emptyset$ whenever U is non-empty and open in X .

- [57, 20G] $f : X \rightarrow Y$ is a *perfect map* if f is continuous, closed and all the fibres of f are compact.

Example 3.2.1.1. [14, p. 52, Пример 1]

If λ is uncountable, there are Tychonoff spaces X and Y and an open perfect function $f : X \rightarrow Y$ with finite fibres such that $\text{cell}(Y) = \aleph_0 < \lambda = \text{cell}(X)$.

Proposition 3.2.2. [14, p. 51, Предложение 2]

Let $f : X \rightarrow Y$ be a quasi-open, continuous, onto function where there is $n \in \mathbb{N}$ such that $|f^{-1}(\{y\})| \leq n$ for all $y \in Y$. If $\text{cell}(X)$ is infinite then $\text{cell}(X) = \text{cell}(Y)$.

The following notion is often used in place of cellularity in the literature:

Definition 3.2.2.1. X has the κ -chain condition (or is κ -c.c.) if every cellular family in X has cardinality less than κ [54, p. 295].

X has the *countable chain condition* or is *ccc* if X is $(\aleph_0)^+$ -cc [38, Ch. II, §2, Definition 1.7, p. 50].

Remark 3.2.2.1. Note that $\text{cell}(X) \leq \kappa$ iff X is κ^+ -c.c.

The forward direction follows from the definitions, while if X is κ^+ -c.c. then by Remark 3.2.0.2, $\text{cell}(X) < \kappa^+$, hence $\text{cell}(X) \leq \kappa$.

3.3 Irreducible Spaces and Finite Cellularity

In this section, irreducible spaces are discussed and the relationship between irreducible components and finite cellularity is explored. I believe Proposition 3.3.6, Proposition 3.3.7, Proposition 3.3.8 and Proposition 3.3.11 are new.

3.3.1 Irreducible Spaces

The information contained here on irreducible spaces, irreducible components and Noetherian spaces can be found in [2, Ch. II, §4.1, §4.2, p. 94-98], unless otherwise stated.

Definition 3.3.0.1. X is said to be *irreducible* if $\text{cell}(X) \leq 1$, i.e. whenever V, W are non-empty open subsets of X , then $V \cap W \neq \emptyset$, i.e. every non-empty open subset of X is dense in X .

A is an *irreducible subset* of X if it is irreducible as a subspace of X , which is equivalent to the following statements:

1. whenever V, W are non-empty opens in X such that $V \cap A, W \cap A \neq \emptyset$ then $V \cap W \cap A \neq \emptyset$.

2. given opens V_1, \dots, V_n in X such that $V_i \cap A \neq \emptyset$ for all i , then $\bigcap_{i=1}^n V_i \cap A \neq \emptyset$.
3. \overline{A} is irreducible.
4. If V_1, \dots, V_n are closed in X and $A \subseteq \bigcup_{i=1}^n V_i$ then $A \subseteq V_i$ for some $i \in \{1, \dots, n\}$.
5. If V, W are closed in X and $A \subseteq V \cup W$ then $A \subseteq V$ or $A \subseteq W$.

Note that $\overline{\{x\}}$ is irreducible for any $x \in X$. If F is a non-empty finite subspace of X then F is irreducible iff $\overline{F} = \overline{\{x\}}$ for some $x \in F$.

The continuous image of any irreducible space is irreducible [2, Ch. II, §4.1, Proposition 4].

Example 3.3.0.1 (Irreducible subspaces of an infinite topological space with the co-finite topology).

Suppose X has the co-finite topology. If $X = F \cup G$ where F, G are closed in X , then one of F, G is infinite and so one of them is X by definition of the closed sets in this topology. Thus X is irreducible. Moreover, each infinite subset of X is dense in X and is thus irreducible. Thus X is a T_1 -space, each of whose infinite subspaces have cellularity 1.

Proposition 3.3.1. [27, Fact 2.6(i)] *If A is a subspace of X which is directed with respect to the specialisation pre-order on X then A is irreducible.*

Proposition 3.3.2. *Let U be a subspace of X .*

1. *Let \mathcal{B} be a base for the open subsets of X . Then U is irreducible iff whenever there are $V, W \in \mathcal{B}$ such that $U \cap V \neq \emptyset$ and $U \cap W \neq \emptyset$ then $U \cap V \cap W \neq \emptyset$.*
2. *Let \mathcal{S} be a sub-base for the open subsets of X . Then U is irreducible iff whenever there are $V_1, \dots, V_n \in \mathcal{S}$ such that $U \cap V_i \neq \emptyset$ for each $i \in \{1, \dots, n\}$, then $U \cap \bigcap_{i=1}^n V_i \neq \emptyset$.*

Proof. 1. \Leftarrow : Let V, W be open in X such that $U \cap V \neq \emptyset$ and $U \cap W \neq \emptyset$.

Then there are non-empty subfamilies $\{B_i : i \in I\}$ and $\{C_j : j \in J\}$ of \mathcal{B} such

that $V = \bigcup_{i \in I} B_i$ and $W = \bigcup_{j \in J} C_j$. Thus there is $(i, j) \in I \times J$ such that $B_i \cap U, C_j \cap U \neq \emptyset$. By the hypothesis $B_i \cap C_j \cap U \neq \emptyset$ and thus $V \cap W \cap U \neq \emptyset$. Thus U is irreducible.

2. \Leftarrow : Let $V, W \in \mathcal{B}_{\mathcal{S}}$ (i.e. the base for X generated by \mathcal{S}) such that $U \cap V \neq \emptyset$ and $U \cap W \neq \emptyset$. If $V = X$ or $W = X$, then $U \cap V \cap W \neq \emptyset$. Suppose that $V, W \neq X$. Then there are $S_1, \dots, S_n, T_1, \dots, T_m \in \mathcal{S}$ such that $V = \bigcap_{i=1}^n S_i$, $W = \bigcap_{i=1}^m T_i$. By the hypothesis, $U \cap V \cap W = U \cap \bigcap_{i=1}^n S_i \cap \bigcap_{i=1}^m T_i \neq \emptyset$. Thus in all possible cases, $U \cap V \cap W \neq \emptyset$.

Thus by the previous point of this Proposition 3.3.2, U is irreducible. \blacksquare

The following is easy to verify from the previous Proposition 3.3.2:

Corollary 3.3.3. *Let \mathcal{G} be a sub-base for the closed subsets of X . Then a subspace U of X is irreducible iff whenever there are $G_1, \dots, G_n \in \mathcal{G}$ such that $U \subseteq \bigcup_{i=1}^n G_i$ then $U \subseteq G_i$ for some $i \in \{1, \dots, n\}$.*

The result in the following Proposition 3.3.4, is stated in passing and without proof in [32, p. 313-314] (see the end of page 313 and the beginning of page 314) in the particular case when X is a hyperspace of some other topological space with the Lower Vietoris Topology (cf. Definition 4.2.0.1).

Proposition 3.3.4. *Let \mathcal{G} be a sub-base for the closed subsets of X such that $X \in \mathcal{G}$. Let A be a non-empty irreducible subspace of X . Then $\{G \in \mathcal{G} : A \subseteq G\} \neq \emptyset$ and $\bar{A} = \bigcap \{G \in \mathcal{G} : A \subseteq G\}$.*

Proof. Of course $\{G \in \mathcal{G} : A \subseteq G\} \neq \emptyset$ since $X \in \mathcal{G}$, moreover $\bar{A} \subseteq \bigcap \{G \in \mathcal{G} : A \subseteq G\}$. Equality occurs if A is dense in X ; suppose that $\bar{A} \neq X$. Then there is a non-empty subfamily $\{F_j : j \in J\}$ of $\mathcal{F}_{\mathcal{G}}$ - the base for the closed sets of X generated by \mathcal{G} - such that $\bar{A} = \bigcap_{j \in J} F_j$. Note that $F_j \neq \emptyset$ for all $j \in J$ as $A \neq \emptyset$ by the hypothesis.

Let $j \in J$. Then there is $n_j \in \mathbb{N}$ and $G_1^j, \dots, G_{n_j}^j \in \mathcal{G}$ such that $F_j = \bigcup_{i=1}^{n_j} G_i^j$. Now, $A \subseteq F_j = \bigcup_{i=1}^{n_j} G_i^j$. Since A is irreducible, it follows that there is $i \in \{1, \dots, n_j\}$ such that $A \subseteq G_i^j =: G_j$ and hence $\bar{A} \subseteq G_j$. Hence:

$$\bar{A} \subseteq \bigcap_{j \in J} G_j \subseteq \bigcap_{j \in J} F_j = \bar{A} \implies \bar{A} = \bigcap \{G \in \mathcal{G} : A \subseteq G\}.$$

■

3.3.2 Finite cellularity and Irreducible components

Definition 3.3.4.1. [2, Ch II, §4.1, Definition 2, p. 95]

An *irreducible component* of X is a maximal irreducible subspace of X .

Proposition 3.3.5. [2, Ch II, §4.1, Proposition 5, p.95]

1. *Every irreducible subspace of X is a subset of some irreducible component of X .*
2. *X is the union of its irreducible components.*
3. *All of the irreducible components of X are closed.*

Definition 3.3.5.1. [57, 20.2 Definition] A family of sets \mathcal{A} is a *locally finite family* in X if every every $x \in X$ has an open neighbourhood meeting only a finite number of members of \mathcal{A} .

The following two Propositions 3.3.6 and 3.3.7 explore the relationship between cellularity and irreducibility; I believe they are new. These results generalise [2, p. 96, Ch. II, §4.2, Proposition 6].

Proposition 3.3.6. *Suppose $X = \bigcup \mathcal{U}$ where \mathcal{U} is a family of closed irreducible subsets of X . Then $\text{cell}(X) \leq |\mathcal{U}|$ and if \mathcal{U} is finite then every irreducible component of X is a member of \mathcal{U} .*

Proof. Let \mathcal{V} be a cellular family in X . Let $U \in \mathcal{U}$, $V, V' \in \mathcal{V}$, such that $V \cap U \neq \emptyset$ and $V' \cap U \neq \emptyset$. Since U is irreducible, $V \cap V' \cap U \neq \emptyset$. Since \mathcal{V} is a cellular family it

follows that $V = V'$. Thus there is an injective function which assigns to each member V of \mathcal{V} a single member U of \mathcal{U} where $V \cap U \neq \emptyset$ (which exists since $X = \bigcup \mathcal{U}$). Thus $|\mathcal{V}| \leq |\mathcal{U}|$. Hence $\text{cell}(X) \leq |\mathcal{U}|$.

If \mathcal{U} is finite, then whenever V is an irreducible component of $X = \bigcup \mathcal{U}$ then $V \subseteq U$ for some $U \in \mathcal{U}$ by irreducibility of V and thus $V = U \in \mathcal{U}$ by maximality of V . ■

Proposition 3.3.7. *Suppose $X = \bigcup \mathcal{U}$ where \mathcal{U} is a family of closed irreducible subsets of X such that $U \not\subseteq V$ for distinct $U, V \in \mathcal{U}$. Then:*

1. *For any finite subfamily \mathcal{F} of \mathcal{U} , $\text{Int}(\bigcup \mathcal{F}) \subseteq X \setminus \bigcup \mathcal{U} \setminus \mathcal{F}$.*
2. *If \mathcal{U} is locally finite then $\{\text{Int}(U) : U \in \mathcal{U}\}$ is cellular in X , moreover $\text{Int}(\bigcup \mathcal{F}) = X \setminus \bigcup \mathcal{U} \setminus \mathcal{F}$ and $\text{cell}(X) = |\mathcal{U}|$.*
3. *If \mathcal{U} is finite, then it is the family of irreducible components of X .*

Proof. 1. Let V be a subset of $\bigcup \mathcal{F}$ which is open in X . Suppose to the contrary there is $U \in \mathcal{U} \setminus \mathcal{F}$ such that $V \cap U \neq \emptyset$. Since U is irreducible and $V \cap U$ is open in U , $U = \overline{V \cap U} \subseteq \overline{V} \subseteq \overline{\bigcup \mathcal{F}} = \bigcup \mathcal{F}$. Since U is irreducible, there is some $U' \in \mathcal{F}$ such that $U \subseteq U'$; by the hypothesis $U = U' \in \mathcal{F}$, a contradiction to the assumption that $U \in \mathcal{U} \setminus \mathcal{F}$. Thus, $V \subseteq X \setminus U$ for each $U \in \mathcal{U} \setminus \mathcal{F}$. Hence, $\text{Int}(\bigcup \mathcal{F}) \subseteq \bigcap \{X \setminus U : U \in \mathcal{U} \setminus \mathcal{F}\} = X \setminus \bigcup \mathcal{U} \setminus \mathcal{F}$.

2. Given $x \in X \setminus \bigcup \mathcal{U} \setminus \mathcal{F}$ there is $U \in \mathcal{U}$ where $x \in U$; thus $U \in \mathcal{F}$. Thus $X \setminus \bigcup \mathcal{U} \setminus \mathcal{F} \subseteq \bigcup \mathcal{F}$. Since $\mathcal{U} \setminus \mathcal{F}$ is a locally finite family of closed sets, its union is closed in X (cf. [57, Lemma 20.5]) and so $X \setminus \bigcup \mathcal{U} \setminus \mathcal{F}$ is open in X . Hence $X \setminus \bigcup \mathcal{U} \setminus \mathcal{F} \subseteq \text{Int}(\bigcup \mathcal{F})$. By the previous point, $X \setminus \bigcup \mathcal{U} \setminus \mathcal{F} = \text{Int}(\bigcup \mathcal{F})$.

If $U, V \in \mathcal{U}$, then $\bigcup \mathcal{U} \setminus \{U\} \cup \bigcup \mathcal{U} \setminus \{V\} = \bigcup \mathcal{U} = X$ and thus

$$\text{Int}(U) \cap \text{Int}(V) = X \setminus (\bigcup \mathcal{U} \setminus \{U\}) \cap X \setminus (\bigcup \mathcal{U} \setminus \{V\}) = \emptyset.$$

Thus, $\{\text{Int}(U) : U \in \mathcal{U}\}$ is cellular in X and has cardinality $|\mathcal{U}|$. Thus $|\mathcal{U}| \leq \text{cell}(X)$; from the previous Proposition 3.3.6 it follows that $|\mathcal{U}| = \text{cell}(X)$.

3. Let $U \in \mathcal{U}$ and let A be an irreducible subset of X such that $U \subseteq A$. Now, $A \subseteq X = \bigcup \mathcal{U}$, implying that $A \subseteq V$ for some $V \in \mathcal{U}$. Hence $U \subseteq V$, which implies that $U = V$ by the hypothesis. Thus $A = U$. Thus, U is an irreducible component of X . By the previous Proposition 3.3.6, it follows that \mathcal{U} is the family of irreducible components of X . ■

Example 3.3.7.1. Consider \mathbb{R} with the usual topology. Then $\text{cell}(\mathbb{R}) = \aleph_0 < |\mathbb{R}|$ as \mathbb{Q} is dense in \mathbb{R} (cf. Proposition 3.2.1 (4)). Moreover the irreducible components are the singletons (which is true for any Hausdorff space).

Proposition 3.3.8. *If $\text{cell}(X) = n$ then given any cellular family V_1, \dots, V_n in X , $\overline{V_1}, \dots, \overline{V_n}$ are the irreducible components of X .*

Proof. Let V_1, \dots, V_n be cellular in X . If U, W are open subsets of V_i for some i then $\{U, W\} \cup \{V_1, \dots, V_n\} \setminus \{V_i\}$ is not cellular so $U \cap W \neq \emptyset$, implying that V_i is irreducible. Thus $\overline{V_1}, \dots, \overline{V_n}$ are irreducible.

If U is open in X , then $\{U, V_1, \dots, V_n\}$ is not cellular in X so $U \cap (V_1 \cup \dots \cup V_n) \neq \emptyset$. Thus, $\overline{V_1} \cup \dots \cup \overline{V_n} = X$. For each $i \neq j$, $\overline{V_i} \not\subseteq \overline{V_j}$ as $V_i \cap V_j = \emptyset$. By Proposition 3.3.7, $\{\overline{V_1}, \dots, \overline{V_n}\}$ is the family of irreducible components of X . ■

The purpose of the following Lemma 3.3.9 is to encapsulate the information in the previous Propositions 3.3.6, 3.3.7 and 3.3.8 as well as for ease of reference later.

Lemma 3.3.9. *The following are equivalent for $U \subseteq X$:*

1. $\text{cell}(U) = n$.
2. $\overline{U} = U_1 \cup \dots \cup U_n$ where U_1, \dots, U_n are closed irreducible subspaces of X such that $U_i \not\subseteq U_j$ for $i \neq j$.
3. There is a cellular family $\{V_1, \dots, V_n\}$ in \overline{U} where $\overline{V_1}, \dots, \overline{V_n}$ are the irreducible components of \overline{U} and for each i , $\text{Int}_{\overline{U}}(\overline{V_i}) = V_i$.

4. \bar{U} has n irreducible components.

Proof. 1 \implies 2: By Proposition 3.2.1 (1), $\text{cell}(\bar{U}) = n$, so by Proposition 3.3.8, \bar{U} has n irreducible components so the result follows from Proposition 3.3.5.

2 \implies 3: By Proposition 3.3.7 (2) and (3), U_1, \dots, U_n are the irreducible components of \bar{U} and $\{\text{Int}_{\tau|\bar{U}}(U_i) : i = 1, \dots, n\}$ is a cellular family in \bar{U} ; for each i , $\overline{\text{Int}_{\tau|\bar{U}}(U_i)} = U_i$ as U_i is irreducible.

3 \implies 4: Obvious.

4 \implies 1: By Proposition 3.3.7 (2), $\text{cell}(\bar{U}) = n$ so it follows by Proposition 3.2.1 (1) that $\text{cell}(U) = n$. ■

Remark 3.3.9.1. Let F be a finite subspace of X . It is shown below that $\text{cell}(F) = |F|$ iff for all distinct $x, y \in F$, $x \notin \overline{\{y\}}$ iff F is discrete.

Suppose for each $x, y \in F$ where $x \neq y$, $x \notin \overline{\{y\}}$. Then F is discrete as for each $x \in F$, $\{x\} = F \setminus \overline{F \setminus \{x\}}$, so by Proposition 3.2.1 (5), $\text{cell}(F) = |F|$.

Conversely suppose $\text{cell}(F) = |F|$. Then \bar{F} has $|F|$ irreducible components (cf. Lemma 3.3.9). Let $x, y \in F$ with $x \neq y$. Then $\overline{F \setminus \{x\}}$ has $\text{cell}(F \setminus \{x\})$ irreducible components (cf. Lemma 3.3.9) which is at most $|F \setminus \{x\}| = |F| - 1$ (cf. Proposition 3.2.1 (4)). Thus $\bar{F} \neq \overline{F \setminus \{x\}}$. Thus $x \notin \overline{\{y\}}$. By the previous paragraph, F is discrete.

Thus the statement at the beginning of the remark has been verified.

Let F' be a minimal subset of F for which $F \subseteq \bar{F}'$. Then for any $x, y \in F'$, $x \notin \overline{\{y\}}$; thus F' is discrete. Since $\bar{F} = \bar{F}'$ it follows that $\text{cell}(F) = \text{cell}(F') = |F'|$ (cf. Proposition 3.2.1 (1)).

The following Lemma 3.3.10 follows from the previous Remark 3.3.9.1.

Lemma 3.3.10. *The following are equivalent:*

1. X is a T_1 -space.

2. Every finite subspace of X is discrete.

3. If $F \subseteq X$ is finite then $\text{cell}(F) = |F|$.

Remark 3.3.10.1. Let H, G be discrete subspaces of X for which $\overline{H} \cap G = \emptyset = H \cap \overline{G}$.

It is shown below that $H \cup G$ is a discrete subspace of X .

Let $x \in H \cup G$. Suppose without loss of generality that $x \notin \overline{G}$; there is an open set V for which $x \in V$ and $V \cap G = \emptyset$. Since H is discrete there is an open W in X for which $\{x\} = W \cap H$. Thus $\{x\} = (W \cap V) \cap (H \cup G)$ is open in $H \cup G$. Thus $H \cup G$ is a discrete subspace of X .

Proposition 3.3.11. *If $F \subseteq \overline{G}$ where $|F| \leq \text{cell}(G) < \aleph_0$ then there is a discrete subspace H of \overline{G} such that $|H| = \text{cell}(G)$ and $F \subseteq \overline{H}$.*

Proof. By Remark 3.3.9.1, there is a discrete subspace F' of F such that $F \subseteq \overline{F'}$. If there is a discrete subspace H of \overline{G} such that $|H| = \text{cell}(G)$ and $F' \subseteq \overline{H}$, then $F \subseteq \overline{F'} \subseteq \overline{H}$, hence proving the statement in question. Thus for the sake of convenience, it is assumed that F is discrete. If $|F| = \text{cell}(G)$ then one may let $H = F$; suppose that $|F| < \text{cell}(G)$ and let $k_1 := \text{cell}(G) - |F|$.

Let $n := \text{cell}(G)$ and let G_1, \dots, G_n be a cellular family in \overline{G} such that $\overline{G_1}, \dots, \overline{G_n}$ are the irreducible components of \overline{G} and $G_i = \text{Int}_{\tau|\overline{G}}(\overline{G_i})$ for each i (cf. Lemma 3.3.9).

F cannot intersect more than $|F|$ members of $\{G_1, \dots, G_n\}$.

Suppose without loss of generality that $F \cap \bigcup_{i=1}^{k_1} G_i = \emptyset$ (Note $|\{G_{k_1+1}, \dots, G_n\}| = n - k_1 = |F|$).

Hence $\overline{F} \cap \bigcup_{i=1}^{k_1} \overline{G_i} = \emptyset$ (since $\text{Cl}_{\overline{G}}(F) = \overline{F}$).

Let C_1 consist of a single element of each of G_1, \dots, G_{k_1} . Then C_1 is discrete, $|C_1| = k_1$ and $\overline{F} \cap C_1 = \emptyset$. Since F is discrete it follows by Remark 3.3.10.1 that $H_1 := (F \setminus \overline{C_1}) \cup C_1$ is discrete. Note that $F \subseteq \overline{H_1}$, $H_1 \subseteq \overline{G}$ and $|H_1| = |F \setminus \overline{C_1}| + k_1$.

If $F \cap \overline{C_1} = \emptyset$, then $\text{cell}(G) = n = |F| + k_1 = |H_1|$, so one may let $H = H_1$.

Suppose that $F \cap \overline{C_1} \neq \emptyset$. Now $F \setminus \overline{C_1}$ cannot intersect more than $|F \setminus \overline{C_1}|$ members of $\{G_{k_1+1}, \dots, G_n\}$ (which has $|F|$ elements; note now that $|F \setminus \overline{C_1}| < |F|$). Suppose without loss of generality that $(F \setminus \overline{C_1}) \cap \bigcup_{i=k_1+1}^{k_2} G_i = \emptyset$ where $k_2 := k_1 + |F| - |F \setminus \overline{C_1}|$. Then $\overline{F \setminus \overline{C_1}} \cap \bigcup_{i=k_1+1}^{k_2} G_i = \emptyset$, as the union is open in \overline{G} and $\overline{F \setminus \overline{C_1}} = \text{Cl}_{\overline{G}}(F \setminus \overline{C_1})$.

Let C_2 consist of a single element of each of $G_{k_1+1}, \dots, G_{k_2}$. Then C_2 is discrete. Since $\bigcup_{i=1}^{k_1} G_i, \bigcup_{j=k_1+1}^{k_2} G_j$ are disjoint opens in \overline{G} , it is true that $C_1 \cap \overline{C_2} = \emptyset = \overline{C_1} \cap C_2$. Thus $C_1 \cup C_2$ is discrete by Remark 3.3.10.1 and $|C_1 \cup C_2| = k_2$. By the last sentence in the previous paragraph, $\overline{F \setminus \overline{C_1}} \cap C_2 = \emptyset$. It was also shown earlier that $\overline{F} \cap C_1 = \emptyset$. Thus $\overline{F \setminus \overline{C_1}} \cap (C_1 \cup C_2) = \emptyset$. Since F is discrete it follows by Remark 3.3.10.1 that $H_2 := (F \setminus \overline{C_1 \cup C_2}) \cup C_1 \cup C_2$ is discrete.

Note that $F \subseteq \overline{H_2}, H_2 \subseteq \overline{G}$ and $|H_2| = |F \setminus \overline{C_1 \cup C_2}| + k_2$.

If $F \cap \overline{C_1 \cup C_2} = \emptyset$, then $\text{cell}(G) = n = |H_2|$ and so one may let $H = H_2$.

Suppose this is not the case, i.e. $F \cap \overline{C_1 \cup C_2} \neq \emptyset$. Then this process is repeated until subsets C_1, \dots, C_s of \overline{G} are defined where $s \leq n$ such that $F \cap \overline{C_1 \cup \dots \cup C_s} = \emptyset$, in which case one may let $H = (F \setminus \overline{C_1 \cup \dots \cup C_s}) \cup C_1 \cup \dots \cup C_s$. ■

3.3.3 Noetherian Spaces

Definition 3.3.11.1. [2, Ch. II, §4.2, Definition 3, p. 97] X is a *Noetherian space* (or simply X is *Noetherian*) if whenever $\{A_n : n \in \mathbb{N}\}$ is a family of closed sets where $A_n \supseteq A_{n+1}$ for each $n \in \mathbb{N}$, then there is $N \in \mathbb{N}$ where $A_n = A_N$ for all $n \geq N$.

Proposition 3.3.12. *Every subspace of a Noetherian space has finite cellularity.*

Proof. From [2, Ch. II, §4.2, Proposition 10, p.98], it follows that every subspace of a Noetherian space has a finite number of irreducible components; it follows from Lemma 3.3.9 that every subspace of a Noetherian space has finite cellularity. ■

Example 3.3.12.1. X is Noetherian iff $\mathcal{P}_0(X)$ with the Lower Vietoris Topology is Noetherian iff 2^X with the Lower Vietoris Topology is Noetherian (cf. [23, Proposition 3.2] and [22, Exercise 9.7.14]); the latter spaces are also irreducible (cf. Definition 4.2.0.1 and Corollary 5.1.13).

3.4 Quasi-Sober Spaces and Sober Spaces

The origin of sobriety can be traced back to Grothendieck and Dieudonné [24, 2.1.1, page 49], whose definition of a *sobre* topological space is equivalent to Definition 3.4.0.1 (2). *Sobre* is the French translation of the term *sober*.

Sober spaces are always T_0 (cf. Proposition 3.4.3 (2)). The origin of the non- T_0 version of sobriety can be traced back to Hong [32, Definition 2.1] who defined the notion of a *quasi-sobre* topological space. This notion is equivalent to the notion of a *quasi-sober space* as defined in Definition 3.4.0.1 (1). The term *quasi-sober space* originated from [29, p. 154].

Definition 3.4.0.1. [29, p. 145; p. 154]

1. X is said to be a *quasi-sober space* (or simply “ X is *quasi-sober*”) if for every non-empty, irreducible, closed subspace A of X there is $x \in X$ for which $A = \overline{\{x\}}$.
2. X is said to be a *sober space* (or simply “ X is *sober*”) if for every non-empty, irreducible, closed subspace A of X there is a unique $x \in X$ for which $A = \overline{\{x\}}$.

Proposition 3.4.1. *If X is quasi-sober then for each subset U of X , $\text{cell}(U) = n$ only if $\overline{U} = \overline{F}$ for some $F \subseteq U$ where $|F| = n = \text{cell}(F)$.*

Proof. By Lemma 3.3.9, there are closed irreducible subsets U_1, \dots, U_n of X such that $\overline{U} = U_1 \cup \dots \cup U_n$ and for $i \neq j$, $U_i \not\subseteq U_j$. Since X is quasi-sober, there are elements $x_1 \in U_1, \dots, x_n \in U_n$ such that $U_1 = \overline{\{x_1\}}, \dots, U_n = \overline{\{x_n\}}$. Thus $\overline{U} = \overline{F}$ where $F := \{x_1, \dots, x_n\}$. By Proposition 3.2.1 (1), $\text{cell}(F) = \text{cell}(U) = n = |F|$. ■

Proposition 3.4.2. [21, Exercise O-5.16]

Let Y be a closed subspace of X . If X is quasi-sober then Y is quasi-sober.

Quasi-sober spaces and sober spaces are related in the following way; in fact, Hong [32, Definition 2.1] defines a sober space as a quasi-sober T_0 -space.

Proposition 3.4.3. 1. X is quasi-sober iff its T_0 -identification is sober.

2. X is sober iff X is T_0 and quasi-sober.

Proof. 1. Proved in [32, Proposition 2.2].

2. Follows from the definitions. ■

Example 3.4.3.1 (Trivial spaces are irreducible and quasi-sober but not T_0).

If X is trivial, then X is the only non-empty closed subset of itself and for each $x \in X$, $X = \overline{\{x\}}$. Thus X is irreducible and quasi-sober and not T_0 .

Example 3.4.3.2 (A sober space which is not T_1). [28, Remark 2.6]; [57, Examples 3.2(e)]

Consider the *Sierpiński space* $X := \{a, b\}$ (where $a \neq b$) whose only non-trivial open is $\{a\}$. Then X is a sober T_D space which is not T_1 (cf. Definition 7.1.0.1).

Example 3.4.3.3 (A T_1 -space which is not a sober space).

Suppose X has the co-finite topology. Then X is T_1 and irreducible (cf. Example 3.3.0.1), while for each $x \in X$, $\overline{\{x\}} = \{x\} \neq X$. Thus X is neither quasi-sober nor sober.

Example 3.4.3.4 (A sober T_1 -space which is not Hausdorff). [28, Theorem 2.4, Theorem 2.10]

Let $X := [0, 1]$ be the space where:

U is closed in X iff

- $U \in \{\emptyset, X\}$, or

- $0 \in U$ and U is finite, or
- $0 \notin U$ and $U = [0, 1] \cap A$ where A is closed in \mathbb{R} with the usual topology.

Then X is a sober T_1 -space which is not Hausdorff.

3.5 Cellularity of Products

In this section, the cellularity of product spaces is discussed.

Firstly in 3.5.1, an overview is provided of results from the literature on cellularity of products of spaces (each with infinite cellularity); this study depends on the axioms of set theory that one employs. The results from 3.5.1 will be utilised to expand on the new results achieved in Chapter 6 (cf. Theorem 6.3.1 and Corollaries 6.3.4 and 6.3.2).

In 3.5.2, it is determined when product spaces have finite cellularity by utilising the relationship between cellularity and irreducibility established in Section 3.3.2. I correct the proof of a result from the literature (cf. Lemma 3.5.10) and I prove a more general result which is new and expands on known results from the literature for products with infinite cellularity (cf. Theorem 3.5.11). Specifically, the irreducible components of a general product are determined and it is shown that the cellularity of a product is finite iff there are a finite number of non-irreducible factors with finite cellularity, in which case the cellularity of the product, is the product of the cellularity of each of the factors.

3.5.1 Infinite Cellularity

In this sub-section 3.5.1 it is assumed that all spaces have infinite cellularity.

The study of cellularity of product spaces depends on the axioms of set theory one employs. The following result appears to be the most general result on cellularity of products in the literature:

Proposition 3.5.1. [37, p. 106, point 5.6; p. 107, Lemma; p. 115, Corollary]

Let $\{X_i : i \in I\}$ be a family of topological spaces. Let $\text{csup} := \sup_{i \in I} \text{cell}(X_i)$. Then:

$$\begin{aligned} \text{cell}\left(\prod_{i \in I} X_i\right) &= \sup\{\text{cell}\left(\prod_{i \in F} X_i\right) : F \subseteq I \text{ is finite}\} \\ &\leq \min\{2^{\text{csup}}, \sup_{i \in I} d(X_i)\}. \end{aligned}$$

Corollary 3.5.2. For any infinite ordinal α :

$$\sup_{n < \omega} \text{cell}(X^n) = \text{cell}(X^\omega) = \text{cell}(X^\alpha) \leq \min\{2^{\text{cell}(X)}, d(X)\}.$$

Corollary 3.5.3. Whenever X_1, \dots, X_n are topological spaces with $n > 1$ then

$$\text{cell}(X_1 \times \dots \times X_{n-1}) \leq \text{cell}(X_1 \times \dots \times X_n) \leq \max_{i=1, \dots, n} d(X_i).$$

Remark 3.5.3.1. If one of the X_i are second-countable, say X_n , then

$$\text{cell}(X_1 \times \dots \times X_{n-1}) = \text{cell}(X_1 \times \dots \times X_n) \text{ (cf. [16, Corollary 2.9])}.$$

Corollary 3.5.4. For any $n < \omega$, $\text{cell}(X^n) \leq \text{cell}(X^{n+1})$.

Determining whether the inequalities in the previous results are strict was first studied by Kurepa [41, §5. Cartesian multiplication of topological spaces, pp. 132-136]. Note that:

- The statement “if $\{X_i : i \in I\}$ is a finite family of topological spaces with countable cellularity then $\prod_{i \in I} X_i$ has countable cellularity” is independent of ZFC [13, Remarks on p. 115-116]. Assuming it holds, then “finite” may be replaced with arbitrary (cf. Proposition 3.5.7).
- If one seeks to show the inequality in Corollary 3.5.3 is strict, one needs to look at extremally disconnected compact Hausdorff spaces (cf. Proposition 3.5.5).
- In ZFC there is a cardinal and a compact Hausdorff space where the inequality in Corollary 3.5.4 is strict (cf. Remark 3.5.6.1 and Proposition 3.5.6).

- Assuming MA to be true but the GCH to be false, it is true that if all the factors of an arbitrary product have countable cellularity then the product has countable cellularity (cf. Proposition 3.5.8), so the inequality in Corollary 3.5.3 becomes an equality.
- Assuming GCH, one may take any infinite cardinal λ and find an extremally disconnected Hausdorff space X where $\text{cell}(X^n) = \lambda < \text{cell}(X^{n+1})$ (cf. Proposition 3.5.9), in which case the inequality in Corollary 3.5.4 is strict.

Proposition 3.5.5. [20, p. 36, Theorem 2.8] *For any infinite cardinal λ , the following are equivalent:*

1. *There are spaces X, Y where $\text{cell}(X) = \text{cell}(Y) = \lambda < \text{cell}(X \times Y)$.*
2. *There are extremally disconnected compact Hausdorff spaces X, Y where $\text{cell}(X) = \text{cell}(Y) = \lambda < \text{cell}(X \times Y)$.*

Proposition 3.5.6. [55, p. 361, Corollary 1] *Let $\kappa_0 := (2^{\aleph_0})^+$ and for each $0 < n < \omega$ let $\kappa_n := (\kappa_{n-1})^+$. Let $\lambda := \sup_{n < \omega} \kappa_n$. For each $1 \leq n < \omega$, there is a compact Hausdorff space X for which $\text{cell}(X^n) = \lambda < \text{cell}(X^{n+1})$.*

The previous Proposition 3.5.6 is stated differently in [55, p. 361, Corollary 1] as Todorčević states that for each $1 \leq n < \omega$, there is a compact Hausdorff space X for which $\text{cell}(X^n) = (2^{\aleph_0})^{+\omega}$ but $\text{cell}(X^{n+1}) > (2^{\aleph_0})^{+\omega}$. This statement is made with the underlying assumption that this cardinal $(2^{\aleph_0})^{+\omega}$ satisfies the conditions 1 and 2 stated below in Remark 3.5.6.1. I have not seen a definition of the cardinal $(2^{\aleph_0})^{+\omega}$ in the literature, but I believe it is the cardinal described in the previous Proposition 3.5.6. In Remark 3.5.6.1 below, I show that the cardinal λ in Proposition 3.5.6 satisfies these conditions stated by Todorčević.

Remark 3.5.6.1. Todorčević [55, p. 361] gave an example of a cardinal denoted by $\kappa := (2^{\aleph_0})^{+\omega}$ with cofinality μ (Todorčević uses λ for this) that is supposed to satisfy the following two conditions:

1. [55, p. 359] There is a sequence $\{\kappa_\alpha : \alpha < \mu\}$ of infinite cardinals such that

$$\sup_{\alpha < \mu} \kappa_\alpha = \kappa \text{ and}$$

$$\left| \prod_{\alpha < \beta} \kappa_\alpha \right| < \kappa_\beta = \text{cf } \kappa_\beta \quad \text{for all } \beta < \mu.$$

2. [55, p. 360] $\rho^\mu < \kappa$ for all $\rho < \kappa$.

Let $\kappa_0 := (2^{\aleph_0})^+$ and for each $0 < n < \omega$ let $\kappa_n := (\kappa_{n-1})^+$.

Let $\lambda := \sup_{n < \omega} \kappa_n$.

By Proposition 3.1.2, $\kappa_n = \text{cf } \kappa_n$ for all $n < \omega$. Moreover if $0 < n < \omega$ then by Proposition 3.1.3 (2):

$$\left| \prod_{i < n} \kappa_i \right| = \sup_{i < n} \kappa_i = \kappa_{n-1} < \kappa_n = \text{cf } \kappa_n.$$

If $n = 0$ then $|\prod_{i < n} \kappa_i| = 0 < \kappa_0$.

Since $\text{cf } \lambda = \omega$, it follows that λ satisfies Condition 1.

In verifying that λ satisfies Condition 2, it will be verified by induction that $(\kappa_n)^\omega = \kappa_n$ for all $n < \omega$.

By Proposition 3.1.3 (2) and (3), $(2^{\aleph_0})^\omega = 2^{\aleph_0}$.

If $\mu < \kappa_0$ then $\mu \leq 2^{\aleph_0}$ implying that $\mu^\omega \leq (2^{\aleph_0})^\omega = 2^{\aleph_0} < \kappa_0$ (cf. Proposition 3.1.3 (4)).

Moreover, $\omega < 2^{\aleph_0} < \kappa_0 = \text{cf } \kappa_0$ (cf. Proposition 3.1.3 (1)). Thus by Proposition 3.1.3 (5), $(\kappa_0)^\omega = \kappa_0$.

Suppose that $(\kappa_n)^\omega = \kappa_n$ for some $n < \omega$. Now if $\mu < \kappa_{n+1}$ then $\mu \leq \kappa_n$ hence $\mu^\omega \leq (\kappa_n)^\omega = \kappa_n < \kappa_{n+1}$. Moreover, $\omega < 2^{\aleph_0} < \kappa_{n+1} = \text{cf } \kappa_{n+1}$. Thus $(\kappa_{n+1})^\omega = \kappa_{n+1}$.

Thus it has been shown by induction that $(\kappa_n)^\omega = \kappa_n$ for all $n < \omega$.

For each $\mu < \lambda$, there is $n < \omega$ such that $\mu < \kappa_n$ and hence $\mu^{\text{cf } \lambda} = \mu^\omega \leq (\kappa_n)^\omega = \kappa_n < \lambda$.

Thus λ satisfies Conditions 1 and 2.

Remark 3.5.6.2. According to [16, p. 136, Example 3.1], Proposition 3.5.6 should

read “For each $1 \leq n < \omega$, there is a compact Hausdorff space X for which $\text{cell}(X) = \text{cell}(X^n) = \lambda < \text{cell}(X^{n+1})$ ” and the authors claim this is implicit in the construction from [55, p. 360, Theorem 1; p. 361, Corollary 1]. I have not seen a verification of this remark.

Proposition 3.5.7. [38, Ch II, §2, Theorem 1.9, p. 51]

If $\{X_i : i \in I\}$ is a family of topological spaces such that $\text{cell}(\prod_{i \in F} X_i) = \aleph_0$ whenever F is a finite subset of I then $\text{cell}(\prod_{i \in I} X_i) = \aleph_0$.

Example 3.5.7.1. [38, Ch II, §2, p. 51] For $\kappa > 2^\omega$, the space $\{0, 1\}^\kappa$ (where $\{0, 1\}$ is discrete) satisfies the antecedent of the previous Proposition 3.5.7 and is not separable.

The consistency of ZFC implies the consistency of ZFC + MA + $\aleph_0^+ < 2^{\aleph_0}$ [6, 2nd Theorem on p. 201].

Proposition 3.5.8. [6, Point (c) of the last Theorem on p. 201]

(MA + $\aleph_0^+ < 2^{\aleph_0}$) If $\{X_i : i \in I\}$ is a family of topological spaces such that $\text{cell}(X_i) = \aleph_0$ for each $i \in I$ then $\text{cell}(\prod_{i \in I} X_i) = \aleph_0$.

Proposition 3.5.9. [20, p. 49, Corollary 4.9]

Let λ be an infinite cardinal such that $\lambda^+ = 2^\lambda$. Then for each $n < \omega$, there is an extremally disconnected compact Hausdorff space X such that $\text{cell}(X^n) = \lambda < \text{cell}(X^{n+1})$.

Remark 3.5.9.1. In the references to Proposition 3.5.5, Proposition 3.5.6, Proposition 3.5.7, Proposition 3.5.8 and Proposition 3.5.9, the authors phrase their results in terms of chain conditions instead of cellularity. In this case one must refer to Remark 3.2.2.1 to see the relationship between the two notions.

3.5.2 Finite cellularity

The result that I prove in the following Lemma 3.5.10 was proved incorrectly in [53, Theorem 11]. The author argued that, given a family of irreducible spaces, their product

must be irreducible because any two sub-basic open sets of the product have non-empty intersection. However, there exist non-irreducible spaces where the latter holds true, as evidenced by the following Example 3.5.9.1:

Example 3.5.9.1. Consider \mathbb{R} with the topology generated by the opens $\{(0, 1), (-1, 1/2), (1/2, 2) \cup (-1, -1/2)\}$. Then every two sub-basic opens must meet but \mathbb{R} is not irreducible as the intersection of all 3 opens is empty.

Lemma 3.5.10. *Let $\{X_\alpha : \alpha \in A\}$ be a family of topological spaces. X_α is irreducible for every $\alpha \in A$ iff $\prod_{\alpha \in A} X_\alpha$ is irreducible.*

Proof. \implies : Let $\mathcal{V}_1, \dots, \mathcal{V}_n$ be non-empty sub-basic opens in $\prod_{\alpha \in A} X_\alpha$ and let $\mathcal{V} := \bigcap_{i=1}^n \mathcal{V}_i$. Then there are $\alpha_1, \dots, \alpha_n \in A$ (not necessarily pairwise-distinct) where for each $i \in \{1, \dots, n\}$, $\mathcal{V}_i := \pi_{\alpha_i}^{-1}(V_i)$ where V_i is an open subset of X_{α_i} . Let $\{\beta_1, \dots, \beta_m\}$ be the set of pairwise-distinct elements of the collection $\alpha_1, \dots, \alpha_n$. For $i = 1, \dots, m$, $\pi_{\beta_i}(\mathcal{V}) = \bigcap \{V_j : 1 \leq j \leq n \text{ and } \alpha_j = \beta_i\}$ is non-empty because X_{β_i} is an irreducible space by the hypothesis. Moreover, $\pi_\beta(\mathcal{V}) = X$ for $\beta \notin \{\beta_1, \dots, \beta_m\}$. By the Axiom of Choice, \mathcal{V} is non-empty. By Proposition 3.3.2 (2), $\prod_{\alpha \in A} X_\alpha$ is irreducible.

\impliedby : For each α , X_α is irreducible as the continuous image (under π_α) of the irreducible space $\prod_{\alpha \in A} X_\alpha$. ■

As far as I am aware, the following Theorem 3.5.11 is new and generalises the previous Lemma 3.5.10. It also shows that the general result of cellularity in products (cf. Proposition 3.5.1) does not apply to the finite case. It also expands on the work of Kurepa [41] and Todorčević [55] on cellularity in products.

Theorem 3.5.11. *Let $\{X_\alpha : \alpha < \beta\}$ be a family of topological spaces and \mathcal{U}_α be the family of irreducible components of X_α for each $\alpha < \beta$. Let $X := \prod_{\alpha < \beta} X_\alpha$. Let β_0 be a non-zero ordinal such that X_α is irreducible iff $\alpha \geq \beta_0$. Then:*

1. $\prod_{\alpha < \beta} \text{cell}(X_\alpha) = \prod_{\alpha < \beta_0} \text{cell}(X_\alpha)$.

2. The family of irreducible components of X is

$$\{\bigcap_{\alpha < \beta_0} \pi_\alpha^{-1}(U_\alpha) : \forall \alpha < \beta_0, U_\alpha \in \mathcal{U}_\alpha\}, \text{ which has cardinality } |\prod_{\alpha < \beta_0} \mathcal{U}_\alpha|.$$

3. $\text{cell}(X)$ is finite iff β_0 is finite and $\text{cell}(X_\alpha)$ is finite for each $\alpha < \beta_0$.

$$\text{In this case, } \text{cell}(X) = \prod_{\alpha < \beta_0} \text{cell}(X_\alpha).$$

Proof. 1. The equality is obvious if $\beta_0 = \beta$. Suppose $\beta_0 < \beta$. Then

$$\begin{aligned} \prod_{\alpha < \beta} \text{cell}(X_\alpha) &= \prod_{\alpha < \beta_0} \text{cell}(X_\alpha) \cdot \prod_{\beta_0 \leq \alpha < \beta} \text{cell}(X_\alpha) \quad (\text{cf. Proposition 3.1.3 (6)}) \\ &= \sup\left\{ \prod_{\alpha < \beta_0} \text{cell}(X_\alpha), \prod_{\beta_0 \leq \alpha < \beta} \text{cell}(X_\alpha) \right\} \quad (\text{cf. Proposition 3.1.3 (2)}) \\ &= \sup\left\{ \prod_{\alpha < \beta_0} \text{cell}(X_\alpha), \prod_{\beta_0 \leq \alpha < \beta} 1 \right\} \\ &= \prod_{\alpha < \beta_0} \text{cell}(X_\alpha). \end{aligned}$$

2. Let $U_\alpha \in \mathcal{U}_\alpha$ for each $\alpha < \beta_0$. Then $\bigcap_{\alpha < \beta_0} \pi_\alpha^{-1}(U_\alpha)$ is irreducible as a subspace of X by Lemma 3.5.10.

Suppose \mathcal{V} is an irreducible subspace of X where $\bigcap_{\alpha < \beta_0} \pi_\alpha^{-1}(U_\alpha) \subseteq \mathcal{V}$.

Let $\alpha < \beta_0$. Then $\pi_\alpha(\mathcal{V}) \supseteq U_\alpha$ and $\pi_\alpha(\mathcal{V})$ is irreducible as the continuous image of the irreducible space \mathcal{V} . Thus $U_\alpha = \pi_\alpha(\mathcal{V})$ as U_α is an irreducible component of X_α by the hypothesis. So $\mathcal{V} \subseteq \pi_\alpha^{-1}(U_\alpha)$.

Thus $\mathcal{V} \subseteq \bigcap_{\alpha < \beta_0} \pi_\alpha^{-1}(U_\alpha)$. Since it was assumed that $\bigcap_{\alpha < \beta_0} \pi_\alpha^{-1}(U_\alpha) \subseteq \mathcal{V}$, it must be the case that $\bigcap_{\alpha < \beta_0} \pi_\alpha^{-1}(U_\alpha) = \mathcal{V}$. Hence, $\bigcap_{\alpha < \beta_0} \pi_\alpha^{-1}(U_\alpha)$ is an irreducible component of X .

Suppose \mathcal{V} is an irreducible subspace of X . For each $\alpha < \beta_0$, $\pi_\alpha(\mathcal{V})$ is an irreducible subspace of X_α as a continuous image of an irreducible space; so $\pi_\alpha(\mathcal{V}) \subseteq U_\alpha$ for some $U_\alpha \in \mathcal{U}_\alpha$. Thus $\mathcal{V} \subseteq \bigcap_{\alpha < \beta_0} \pi_\alpha^{-1}(U_\alpha)$.

Hence, $\{\bigcap_{\alpha < \beta_0} \pi_\alpha^{-1}(U_\alpha) : \forall \alpha < \beta_0, U_\alpha \in \mathcal{U}_\alpha\}$ is the family of irreducible components of X and has cardinality $|\prod_{\alpha < \beta_0} \mathcal{U}_\alpha|$.

3. \implies : Suppose $\text{cell}(X)$ is finite. Then the family of irreducible components of X is finite and has cardinality $\text{cell}(X)$ (cf. Proposition 3.3.8). By the second point of this Proposition, for every $\alpha < \beta_0$, \mathcal{U}_α is finite; by Proposition 3.3.7, $|\mathcal{U}_\alpha| = \text{cell}(X_\alpha)$. By the first and second points of this Proposition, $\text{cell}(X) = |\prod_{\alpha < \beta_0} \mathcal{U}_\alpha| = \prod_{\alpha < \beta_0} \text{cell}(X_\alpha)$ is finite.

Since $\text{cell}(X) = \prod_{\alpha < \beta_0} \text{cell}(X_\alpha) \geq 2^{|\beta_0|} > |\beta_0|$, it follows that β_0 is finite.

\impliedby : By the hypothesis, for every $\alpha < \beta_0$, $\text{cell}(X_\alpha)$ is finite and so the family \mathcal{U}_α of irreducible components of X_α has cardinality $\text{cell}(X_\alpha) = |\mathcal{U}_\alpha|$ (cf. Proposition 3.3.8). By the previous point of this Proposition, $\text{cell}(X) = \prod_{\alpha < \beta_0} \text{cell}(X_\alpha)$; this must be a finite cardinal as β_0 is finite by the hypothesis. ■

Corollary 3.5.12. *For any ordinal α , $\text{cell}(X^\alpha)$ is finite iff either X is irreducible (in which case X^α is irreducible) or both α and $\text{cell}(X)$ are finite (in which case $\text{cell}(X^\alpha) = \text{cell}(X)^\alpha$).*

3.6 Cellularity of the Augmented n -th power of X

The results in this section are new unless stated otherwise; the main result is that when $\text{cell}(X)$ is infinite then $\text{cell}(X^n) \leq \text{cell}(X^{[n]})$ (cf. Theorem 3.6.7); expanding on the result below by Fedorchuk (cf. Proposition 3.6.1) that $\text{cell}(X^n) = \text{cell}(X^{[n]})$ whenever X is Hausdorff. This result by Fedorchuk will be improved on in Chapter 6, where it will be shown that $\text{cell}(X^n) = \text{cell}(X^{[n]})$ is true for a certain class of non-Hausdorff spaces.

Proposition 3.6.1. [15]

If X is Hausdorff then:

1. $X^{[n]}$ is open in X^n .
2. $\text{cell}(X^{[n]}) \leq \text{cell}(X^{n+1})$.

3. $\text{cell}(X^{[n]}) = \text{cell}(X^n)$.

Proof. The first point is stated in the proof of [15, Proposition 1.8]. The second point is proved in [15, Proposition 1.8] and the third point is proved in [15, Proposition 1.9]. ■

Point 1 of the previous Proposition 3.6.1 cannot be extended beyond Hausdorff spaces, as when X is not a Hausdorff space then $X^{[n]}$ is no longer open in X^n :

Proposition 3.6.2. *For $n \geq 2$, $X^{[n]}$ is open in X^n iff X is Hausdorff.*

Proof. \implies : Let $x_1, \dots, x_n \in X$ be pairwise distinct. Then $(x_1, \dots, x_n) \in X^{[n]}$ so there are open sets V_1, \dots, V_n in X where $(x_1, \dots, x_n) \in V_1 \times \dots \times V_n \subseteq X^{[n]}$. Suppose to the contrary there is $i \neq j$ such that there is $x \in V_i \cap V_j$. Let $\underline{y} \in X^n$ such that $y_i = y_j = x$ and $y_k \in V_k$ for $k \neq i, j$. Then $\underline{y} \in V_1 \times \dots \times V_n$ but $\underline{y} \notin X^{[n]}$, a contradiction. Thus V_1, \dots, V_n is cellular in X and $x_1 \in V_1, \dots, x_n \in V_n$. Thus X is Hausdorff.

\impliedby : Let $(x_1, \dots, x_n) \in X^{[n]}$. Since X is Hausdorff, there is a cellular family V_1, \dots, V_n in X where $x_1 \in V_1, \dots, x_n \in V_n$. Hence $(x_1, \dots, x_n) \in V_1 \times \dots \times V_n \subseteq X^{[n]}$. Thus $X^{[n]}$ is open in X^n . ■

Lemma 3.6.3. *Let $n \geq 2$ and $V_1, \dots, V_n \subseteq X$ such that whenever $|F| < n$ and $\emptyset \neq F \subseteq \{1, \dots, n\}$, $\prod_{i \in F} V_i \cap X^{[F]} \neq \emptyset$.*

If $(V_1 \times \dots \times V_n) \cap X^{[n]} = \emptyset$ then $|\bigcup_{i=1}^n V_i| = n - 1$.

Proof. Let $(x_1, \dots, x_{n-1}) \in V_1 \times \dots \times V_{n-1} \cap X^{[n-1]}$.

By the hypothesis, $(V_1 \times \dots \times V_n) \cap X^{[n]} = \emptyset$ and thus $V_n \subseteq \{x_1, \dots, x_{n-1}\}$.

Suppose $x_{i_1} \in V_{i_2}, \dots, x_{i_m} \in V_{i_{m+1}}, x_{i_{m+1}} \in V_n$ where $1 \leq i_1 < \dots < i_{m+1} < n$. If there is $y \in V_{i_1} \setminus \{x_1, \dots, x_{n-1}\}$ then letting $z_{i_1} := y \in V_{i_1}, z_{i_2} := x_{i_1} \in V_{i_2}, \dots, z_{i_{m+1}} := x_{i_m} \in V_{i_{m+1}}, z_n := x_{i_{m+1}} \in V_n$ and $z_j := x_j \in V_j$ for $j \in \{1, \dots, n\} \setminus \{i_1, \dots, i_{m+1}, n\}$ then it must be the case that $(z_1, \dots, z_n) \in (V_1 \times \dots \times V_n) \cap X^{[n]}$, a contradiction to the hypothesis. Thus $V_{i_1} \subseteq \{x_1, \dots, x_{n-1}\}$.

Suppose without loss of generality there is a least index i_1 for which if $j \geq i_1$ then there is a sequence $i_1 \leq j < k_1 < \dots < k_m < n$ where $x_j \in V_{k_1}, x_{k_1} \in V_{k_2}, \dots, x_{k_{m-1}} \in V_{k_m}, x_{k_m} \in V_n$.

For $k_1 \geq i_1$ it follows by the previous two paragraphs that $V_{k_1} \subseteq \{x_1, \dots, x_{n-1}\}$. Now if $x_j \in V_{k_1}$ where $j < i_1$ then since $k_1 \geq i_1$ there is a sequence $j < k_1 < \dots < k_m < n$ where $x_j \in V_{k_1}, x_{k_1} \in V_{k_2}, \dots, x_{k_{m-1}} \in V_{k_m}, x_{k_m} \in V_n$, a contradiction to the minimality of i_1 . Thus $\bigcup_{j \geq i_1} V_j = \{x_{i_1}, \dots, x_{n-1}\}$, hence $|\bigcup_{j \geq i_1} V_j| = n - i_1$. If $i_1 > 1$, then by the hypothesis, $\prod_{j \geq i_1} V_j \cap X^{[n-i_1]} \neq \emptyset$ and so $|\bigcup_{j \geq i_1} V_j| \geq n - i_1 + 1$, a contradiction. Thus $i_1 = 1$. Hence $|\bigcup_{i=1}^n V_i| = n - 1$. ■

Corollary 3.6.4. *Given $V_1, \dots, V_n \subseteq X$ where $n \geq 2$, then:*

1. $(V_1 \times \dots \times V_n) \cap X^{[n]} \neq \emptyset$ iff whenever $\emptyset \neq F \subseteq \{1, \dots, n\}$, $|\bigcup_{i \in F} V_i| \geq |F|$.
2. If $(V_1 \times \dots \times V_n) \cap X^{[n]} \neq \emptyset$ then $\prod_{i \in F} V_i \cap X^{[F]} \neq \emptyset$ whenever $\emptyset \neq F \subseteq \{1, \dots, n\}$.
3. If $(V_1 \times \dots \times V_n) \cap X^{[n]} \neq \emptyset$ and there is $V \subseteq X$ such that $|V| \geq m$ and $V \supseteq \bigcup_{i=1}^n V_i$ then $(V_1 \times \dots \times V_n \times V \times \dots \times V) \cap X^{[m]} \neq \emptyset$.
4. If $(V_1 \times \dots \times V_n) \cap X^{[n]} \neq \emptyset$ and there is $V \subseteq X$ such that $|V \setminus \bigcup_{i=1}^n V_i| \geq n - m$ then $(V_1 \times \dots \times V_n \times V \times \dots \times V) \cap X^{[m]} \neq \emptyset$.

Proof. 1. \implies : By the hypothesis there is $(x_1, \dots, x_n) \in (V_1 \times \dots \times V_n) \cap X^{[n]}$ and hence x_1, \dots, x_n are pairwise distinct. Thus whenever $\emptyset \neq F \subseteq \{1, \dots, n\}$ it follows $\{x_i : i \in F\} \subseteq \bigcup_{i \in F} V_i$ and so $|\bigcup_{i \in F} V_i| \geq |F|$.

\impliedby : Follows from the previous Lemma 3.6.3.

2. Follows from the previous point.

3. Follows from the first point.

4. Follows from the first point. ■

The following Proposition 3.6.5 shows that one cannot use Proposition 3.2.1 (1) to show that $\text{cell}(X^{[n]}) = \text{cell}(X^n)$.

Proposition 3.6.5. *For $n \geq 2$,*

$$\begin{aligned} \overline{X^{[n]}} &= \{(x_1, \dots, x_n) \in X^n : m \leq n \text{ and } |\{x_{i_1}, \dots, x_{i_m}\}| < m \\ &\implies \{x_{i_1}, \dots, x_{i_m}\} \text{ is not open}\}. \end{aligned}$$

Proof. Let $(x_1, \dots, x_n) \in X^n$ and suppose there is $m \leq n$ and a sequence $1 \leq i_1 < \dots < i_m \leq n$ where $F := \{x_{i_1}, \dots, x_{i_m}\}$ is open in X and $|F| < m$. Now let \underline{F} be the subset of X^n where $\pi_i(\underline{F}) = F$ when $x_i \in F$ and $\pi_i(\underline{F}) = X$ otherwise. Then \underline{F} is open in X^n , $(x_1, \dots, x_n) \in \underline{F}$ but $|\{i : x_i \in F\}| = m$ while

$$\left| \bigcup \{\pi_i(\underline{F}) : x_i \in F\} \right| = |F| = |\{x_{i_1}, \dots, x_{i_m}\}| < m,$$

so by Corollary 3.6.4 (1), $\underline{F} \cap X^{[n]} = \emptyset$. Hence $(x_1, \dots, x_n) \notin \overline{X^{[n]}}$.

Let $(x_1, \dots, x_n) \notin \overline{X^{[n]}}$. There are opens V_1, \dots, V_n in X where $(x_1, \dots, x_n) \in V_1 \times \dots \times V_n \subseteq X^n \setminus \overline{X^{[n]}}$. Then $(V_1 \times \dots \times V_n) \cap X^{[n]} = \emptyset$ so it follows by Corollary 3.6.4 (1) that there are $1 \leq i_1, \dots, i_m \leq n$ such that $|\bigcup_{i=i_1}^{i_m} V_i| < m$. Hence $\bigcup_{i=i_1}^{i_m} V_i = \{x_{i_1}, \dots, x_{i_m}\}$ is open while $|\{x_{i_1}, \dots, x_{i_m}\}| < m$. ■

The following Lemma 3.6.6 shows that point 2 of Proposition 3.6.1 remains true for any topological space:

Lemma 3.6.6. *For $n \geq 2$, $\text{cell}(X^{[n]}) \leq \text{cell}(X^{[n+1]})$; moreover $\text{cell}(X) \leq \text{cell}(X^{[2]})$.*

Proof. Let $\{\mathcal{V}_\alpha : \alpha \in A\}$ be a cellular family in $X^{[n]}$; for each $\alpha \in A$ there are opens $V_1^\alpha, \dots, V_n^\alpha$ in X where $\emptyset \neq (V_1^\alpha \times \dots \times V_n^\alpha) \cap X^{[n]} \subseteq \mathcal{V}_\alpha$.

Let $\alpha \neq \beta$. Then:

$$\begin{aligned} (V_1^\alpha \cap V_1^\beta) \times \dots \times (V_n^\alpha \cap V_n^\beta) \cap X^{[n]} &= (V_1^\alpha \times \dots \times V_n^\alpha) \cap X^{[n]} \cap (V_1^\beta \times \dots \times V_n^\beta) \\ &= \emptyset. \end{aligned}$$

By Corollary 3.6.4 (2):

$$\begin{aligned}\emptyset &= (V_1^\alpha \cap V_1^\beta) \times \cdots \times (V_n^\alpha \cap V_n^\beta) \times X \cap X^{[n+1]} \\ &= (V_1^\alpha \times \cdots \times V_n^\alpha \times X) \cap X^{[n+1]} \cap (V_1^\beta \times \cdots \times V_n^\beta \times X).\end{aligned}$$

Since $(V_1^\alpha \times \cdots \times V_n^\alpha) \cap X^{[n]} \neq \emptyset$ it follows by Corollary 3.6.4 (3) that

$$(V_1^\alpha \times \cdots \times V_n^\alpha \times X) \cap X^{[n+1]} \neq \emptyset.$$

Thus $\{(V_1^\alpha \times \cdots \times V_n^\alpha \times X) \cap X^{[n+1]} : \alpha \in A\}$ is cellular in $X^{[n+1]}$ and has the same cardinality as \mathcal{V}_α . Thus $|\mathcal{V}_\alpha| \leq \text{cell}(X^{[n+1]})$.

Hence $\text{cell}(X^{[n]}) \leq \text{cell}(X^{[n+1]})$.

If $\{V_\alpha : \alpha \in A\}$ is cellular in X then $\{(V_\alpha \times X) \cap X^{[2]} : \alpha \in A\}$ is obviously cellular in $X^{[2]}$. Thus $\text{cell}(X) \leq \text{cell}(X^{[2]})$. ■

The following Theorem 3.6.7 is the main result of this section. The proof is inspired by an unverified remark in the proof of [15, Proposition 1.9] that $X^n \setminus X^{[n]}$ is a finite union of subspaces homeomorphic to X^{n-1} (where X is assumed to be Hausdorff).

Theorem 3.6.7. *If $\text{cell}(X)$ is infinite then $\text{cell}(X^n) \leq \text{cell}(X^{[n]})$; otherwise $\text{cell}(X)^n \leq \binom{n(n-1)}{2} + 1 \cdot \text{cell}(X^{[n]})$.*

Proof. By Proposition 3.2.1 (3),

$$\text{cell}(X^n) \leq \text{cell}(\overline{X^{[n]}}) + \text{cell}(X^n \setminus \overline{X^{[n]}}) = \text{cell}(X^{[n]}) + \text{cell}(X^n \setminus \overline{X^{[n]}}).$$

It will be verified that $\text{cell}(X^{[n]}) \geq \text{cell}(X^n \setminus \overline{X^{[n]}})$ by induction on $n \geq 2$.

Let $D := X^2 \setminus X^{[2]} = \{(x, x) : x \in X\}$. The map $f : X \rightarrow D$ defined by $f(x) = (x, x)$ is continuous as $f^{-1}((U \times V) \cap D) = U \cap V$ is open whenever U, V are open in X . Thus $\text{cell}(D) \leq \text{cell}(X) \leq \text{cell}(X^{[2]})$ by Proposition 3.2.1 (2) and the previous Lemma 3.6.6. Since $X^2 \setminus \overline{X^{[2]}}$ is an open subspace of D , $\text{cell}(X^2 \setminus \overline{X^{[2]}}) \leq \text{cell}(D) \leq \text{cell}(X^{[2]})$.

Suppose $n > 2$ and suppose that whenever $2 \leq m < n$, $\text{cell}(X^m) \leq \text{cell}(X^{[m]})$.

For $1 \leq i < j \leq n$, define $f_{i,j} : X^{n-1} \rightarrow X^n \setminus X^{[n]}$ by $f_{i,j}(x_1, \dots, x_{n-1}) = (y_1, \dots, y_n)$ where:

$$\text{for } k < j, y_k = x_k; \quad y_j = x_i; \quad \text{for } k > j, y_k = x_{k-1}.$$

Let $D_{i,j} := \{(y_1, \dots, y_n) \in X^n : y_i = y_j\}$ denote the image of $f_{i,j}$. Clearly $f_{i,j}$ is a bijection. Now $f_{i,j}$ is continuous as whenever U_1, \dots, U_n are opens in X then $f_{i,j}^{-1}((U_1 \times \dots \times U_n) \cap D_{i,j}) = V_1 \times \dots \times V_{n-1}$ is open in X^{n-1} where:

$$\text{For } k < j, \quad V_k = U_k, \quad \text{and} \quad \text{for } k \geq j, \quad V_k = U_{k+1}.$$

Moreover, whenever V_1, \dots, V_{n-1} are opens in X , $f_{i,j}(V_1 \times \dots \times V_{n-1}) = U_1 \times \dots \times U_n \cap D_{i,j}$ is open in $D_{i,j}$ where:

$$\text{For } k < j, \quad U_k = V_k; \quad U_j = V_i; \quad \text{and} \quad \text{for } k \geq j, \quad U_{k+1} = V_k.$$

Thus $f_{i,j}$ is a homeomorphism, so $\text{cell}(X^{n-1}) = \text{cell}(D_{i,j})$. Now $X^n \setminus X^{[n]} = \bigcup_{1 \leq i < j \leq n} D_{i,j}$ and so:

$$\begin{aligned} \text{cell}(X^n \setminus \overline{X^{[n]}}) &\leq \text{cell}(X^n \setminus X^{[n]}) \quad (\text{As } X^n \setminus \overline{X^{[n]}} \text{ is open in } X^n \setminus X^{[n]}) \\ &\leq \sum_{1 \leq i < j \leq n} \text{cell}(D_{i,j}) \quad (\text{By Proposition 3.2.1(3)}) \\ &= \sum_{1 \leq i < j \leq n} \text{cell}(X^{n-1}) \\ &= \frac{n(n-1)}{2} \cdot \text{cell}(X^{n-1}) \quad (\text{cf. Remark 3.6.7.1}) \\ &\leq \frac{n(n-1)}{2} \cdot \text{cell}(X^{[n-1]}) \quad (\text{By the induction hypothesis}) \\ &\leq \frac{n(n-1)}{2} \cdot \text{cell}(X^{[n]}) \quad (\text{By the previous Lemma 3.6.6}). \end{aligned}$$

Thus, for all $n > 1$:

$$\text{cell}(X^n) \leq \text{cell}(X^{[n]}) + \text{cell}(X^n \setminus \overline{X^{[n]}}) \leq \left(\frac{n(n-1)}{2} + 1 \right) \cdot \text{cell}(X^{[n]}).$$

If $\text{cell}(X)$ is infinite, then by Corollary 3.5.12, $\text{cell}(X^n)$ is infinite and so by Proposition 3.1.3 (2), $(\frac{n(n-1)}{2} + 1) \cdot \text{cell}(X^{[n]}) = \text{cell}(X^{[n]})$ and so $\text{cell}(X^n) \leq \text{cell}(X^{[n]})$.

If $\text{cell}(X)$ is finite, then by Corollary 3.5.12, $\text{cell}(X^n) = \text{cell}(X)^n$ and so $\text{cell}(X)^n \leq (\frac{n(n-1)}{2} + 1) \cdot \text{cell}(X^{[n]})$. ■

Remark 3.6.7.1. Note that for $n \geq 2$, the sum $\sum\{1 : 1 \leq i < j \leq n\}$ has the same value as:

$$\begin{aligned} & |\{(i, j) : 1 \leq i < j \leq n\}| \\ &= |\{(1, j) : 2 \leq j \leq n\}| + |\{(2, j) : 3 \leq j \leq n\}| + \cdots + |\{(n-1, j) : j = n\}| \\ &= (n-1) + (n-2) + \cdots + 1 \\ &= \frac{n(n-1)}{2} \quad (\text{Easily verified by induction}). \end{aligned}$$

Chapter 4

Introduction to the Lower Vietoris

Topology

For the rest of this thesis, \mathfrak{S} and \mathfrak{T} both denote sets of non-empty subsets of X .

In this chapter the Lower Vietoris Topology on \mathfrak{S} is defined (cf. Definition 4.2.0.1). It is assumed thereafter that any set of non-empty subsets of X is a topological space equipped with the Lower Vietoris Topology.

In Section 4.1 relevant notation is defined and basic results are proved. I believe that the hyperspaces $\mathcal{D}_n(X)$, $\mathcal{S}_n(X)$, $\mathcal{S}_{\leq n}(X)$, $\mathcal{D}_f(X)$ have not been studied previously in the literature.

In Section 4.2 some well-known properties of the Lower Vietoris Topology are proved.

In Section 4.3 the set of closures in X of elements of \mathfrak{S} is shown to be the T_0 -identification of \mathfrak{S} (cf. Theorem 4.3.3). I believe this result is new and generalises a remark by Hong (see the paragraph preceding Theorem 4.3.3).

In Section 4.4 a new result is proved: necessary and sufficient conditions are determined for \mathfrak{S} to be a T_0 -space or a symmetric space (cf. Theorem 4.4.1). Useful corollaries are also given (cf. Corollary 4.4.2 and Corollary 4.4.3). Some of these generalise known

results.

4.1 Notation and Set-Theoretic Results

Definition 4.1.0.1. The following are sets of non-empty subsets of X :

- $\mathcal{P}_0(X)$ will denote the set of all non-empty subsets of X [5, Section 1].
- 2^X will denote the set of all non-empty closed subsets of X [44, Notation 1.4].
- $\mathcal{C}(X)$ will denote the set of all non-empty compact subspaces of X .
- $\mathcal{F}(X)$ will denote the set of all non-empty finite subsets of X [5, Section 1].
- $\mathcal{F}_n(X)$ (resp. $\mathcal{F}_{\leq n}(X)$) will denote the set of all non-empty finite subsets of X with exactly (resp. at most) n elements.
- $\mathcal{S}_n(X)$ (resp. $\mathcal{S}_{\leq n}(X)$) will denote the set of all non-empty subsets of X with cellularity exactly (resp. at most) n .

The reason behind this notation is that the cellularity of X is also known as the ‘‘Souslin number’’ of X (see Definition 3.2.0.2). This notation also avoids confusion with the set $\mathcal{C}(X)$ defined earlier.

- $\mathcal{D}_n(X)$ will denote the set of all non-empty discrete subspaces of X with exactly n elements.
- $\mathcal{D}_f(X) := \bigcup_{n \in \mathbb{N}} \mathcal{D}_n(X)$.

Remark 4.1.0.1. I believe that the hyperspaces $\mathcal{D}_n(X)$, $\mathcal{S}_n(X)$, $\mathcal{S}_{\leq n}(X)$, $\mathcal{D}_f(X)$ have not been studied previously in the literature.

Remark 4.1.0.2. It seems there is no standard notation for these sets. Some authors use $\mathcal{F}_n(X)$ [44, Notation 1.4], [5, Section 1] to denote my use of $\mathcal{F}_{\leq n}(X)$. Costantini et al. use $\mathcal{E}_n(X)$, $\mathcal{H}(X)$ and $\text{CL}_\emptyset(X)$ to denote my use of $\mathcal{F}_n(X)$, $\mathcal{C}(X)$ and 2^X respectively

[7, 1. Preliminaries, p. 246]. Michael uses $\mathcal{A}(X)$ to denote my use of $\mathcal{P}_0(X)$ [44, p. 152]. Bourbaki uses $\mathfrak{P}_0(X)$, $\mathfrak{F}(X)$ and $\mathfrak{C}(X)$ to denote my use of $\mathcal{P}_0(X)$, 2^X and $\mathcal{F}(X)$ respectively [3, Ch. I, Exercises for §2, Exercise 5 and Exercises for §8, Exercise 12].

The following notation is due to [44, Definition 5.1.1] and is also used in [4, Definition 1]:

Definition 4.1.0.2. [44, Definition 5.1.1] Given $n \in \mathbb{N}$ and subsets U_1, \dots, U_n of X :

$$\langle U_1, \dots, U_n \rangle := \{W \in \mathcal{P}_0(X) : W \subseteq \bigcup_{i=1}^n U_i \text{ and for } 1 \leq i \leq n, W \cap U_i \neq \emptyset\}.$$

The notation $\langle \mathcal{F} \rangle$ may be used in place of $\langle U_1, \dots, U_n \rangle$ when $\mathcal{F} = \{U_1, \dots, U_n\}$.

Remark 4.1.0.3. For subsets U, U_1, \dots, U_n of X :

$$\langle U \rangle = \{V \in \mathcal{P}_0(X) : V \subseteq U\} \quad \text{and}$$

$$\langle X, U_1, \dots, U_n \rangle = \{V \in \mathcal{P}_0(X) : \text{for each } i \in \{1, \dots, n\}, V \cap U_i \neq \emptyset\} = \bigcap_{i=1}^n \langle X, U_i \rangle.$$

Thus:

$$\langle U_1, \dots, U_n \rangle = \left\langle \bigcup_{i=1}^n U_i \right\rangle \cap \langle X, U_1, \dots, U_n \rangle = \left\langle \bigcup_{i=1}^n U_i \right\rangle \cap \bigcap_{i=1}^n \langle X, U_i \rangle.$$

Common alternative notation in the literature is given by $\square U := \langle U \rangle$ and $\diamond U := \langle X, U \rangle$, see [22, Definition 8.2.17], [22, Exercise 9.7.14], [27, Section 2.3] and [58, p. 4]. Although this notation is more recent, the notation in Definition 4.1.0.2 appears to be more convenient to work with and easier to read.

Remark 4.1.0.4. Let $\{V_i : i \in I\}$ be a non-empty family of subsets of X . For any $A \in \mathcal{P}_0(X)$, $A \in \bigcap_{i \in I} \langle V_i \rangle$ iff for each $i \in I$, $A \subseteq V_i$ iff $A \subseteq \bigcap_{i \in I} V_i$ iff $A \in \langle \bigcap_{i \in I} V_i \rangle$. Thus $\bigcap_{i \in I} \langle V_i \rangle = \langle \bigcap_{i \in I} V_i \rangle$.

The result of the following Proposition 4.1.1 is stated (without proof) in the proof of [44, Proposition 2.1]:

Proposition 4.1.1. *Given subsets U_1, \dots, U_n and V_1, \dots, V_m of X , if $U := \bigcup_{i=1}^n U_i$ and $V := \bigcup_{j=1}^m V_j$ then:*

$$\langle U_1, \dots, U_n \rangle \cap \langle V_1, \dots, V_m \rangle = \langle U_1 \cap V, \dots, U_n \cap V, V_1 \cap U, \dots, V_m \cap U \rangle.$$

Proof. Let A be an element of the left-hand side. Then $A \subseteq U$ and $A \subseteq V$ so $A \subseteq U \cap V$. Now $\bigcup_{i=1}^n (U_i \cap V) = \bigcup_{j=1}^m (V_j \cap U) = U \cap V$. Thus $A \subseteq \bigcup_{i=1}^n U_i \cap \bigcup_{j=1}^m V_j = \bigcup_{i=1}^n (U_i \cap V) \cap \bigcup_{j=1}^m (V_j \cap U)$. Because A is an element of the left-hand side, $A \cap V_j \neq \emptyset$ for each $1 \leq j \leq m$ and $A \cap U_i \neq \emptyset$ for each $1 \leq i \leq n$. Since $A \subseteq U$, $A \cap U \cap V_j = A \cap V_j \neq \emptyset$ for all $1 \leq j \leq m$. Since $A \subseteq V$, $A \cap V \cap U_i = A \cap U_i \neq \emptyset$ for all $1 \leq i \leq n$. Hence A is an element of the right-hand side.

If A is an element of the right-hand side, then $A \cap U_i \neq \emptyset$ for every $1 \leq i \leq n$, $A \cap V_j \neq \emptyset$ for every $1 \leq j \leq m$ and $A \subseteq \bigcup_{i=1}^n (U_i \cap V) \cap \bigcup_{j=1}^m (V_j \cap U) = \bigcup_{i=1}^n U_i \cap \bigcup_{j=1}^m V_j$ and thus A is an element of the left-hand side. \blacksquare

Corollary 4.1.2. *Given subsets U, U_1, \dots, U_n and V_1, \dots, V_m of X :*

$$\langle X, U_1, \dots, U_n \rangle \cap \langle X, V_1, \dots, V_m \rangle = \langle X, U_1, \dots, U_n, V_1, \dots, V_m \rangle, \text{ and}$$

$$\langle U \rangle \cap \langle X, V_1, \dots, V_m \rangle = \langle U, U \cap V_1, \dots, U \cap V_m \rangle.$$

The following Proposition 4.1.3 is stated without proof in [44, Converse of Lemma 2.3.1].

Proposition 4.1.3. [44, Converse of Lemma 2.3.1] *Let U_1, \dots, U_n and V_1, \dots, V_m be subsets of X . Suppose that $\bigcup_{i=1}^n U_i \subseteq \bigcup_{j=1}^m V_j$ and for all $1 \leq j \leq m$ there is $1 \leq i \leq n$ such that $U_i \subseteq V_j$. Then*

$$\langle U_1, \dots, U_n \rangle \subseteq \langle V_1, \dots, V_m \rangle.$$

Proof. Let $A \in \langle U_1, \dots, U_n \rangle$. Then $A \subseteq \bigcup_{i=1}^n U_i \subseteq \bigcup_{j=1}^m V_j$ (by the hypothesis). Let $j \in \{1, \dots, m\}$. By the hypothesis, there is $1 \leq i \leq n$ such that $U_i \subseteq V_j$. Moreover, since $A \in \langle U_1, \dots, U_n \rangle$, $A \cap U_i \neq \emptyset$ and thus $A \cap V_j \neq \emptyset$. Hence, $A \in \langle V_1, \dots, V_m \rangle$. \blacksquare

Remark 4.1.3.1. For any subset U of X , if $W := \bigcup(\langle U \rangle \cap \mathfrak{S})$, then:

$$\langle U \rangle \cap \mathfrak{S} = \langle W \rangle \cap \mathfrak{S} \text{ and } \bigcup(\langle W \rangle \cap \mathfrak{S}) = W.$$

Lemma 4.1.4. *Given non-empty subsets V_1, \dots, V_m of X , there is $1 \leq s \leq m$ and a pairwise disjoint family U_1, \dots, U_s of non-empty subsets of X where:*

- for each $1 \leq i \leq s$, $U_i = \bigcap_{j \in J} V_j$ for some $\emptyset \neq J \subseteq \{1, \dots, m\}$, and
- for each $1 \leq i \leq m$, there is $1 \leq j \leq s$ such that $U_j \subseteq V_i$.

Hence:

$$\langle X, U_1, \dots, U_s \rangle \subseteq \langle X, V_1, \dots, V_m \rangle$$

Proof. Consider $\mathscr{W} := \{\mathscr{A} \subseteq \{V_1, \dots, V_m\} : \mathscr{A} \neq \emptyset \text{ and } \bigcap \mathscr{A} \neq \emptyset\}$, which is non-empty as $\{V_i\} \in \mathscr{W}$ for each $1 \leq i \leq m$.

Claim: there is $1 \leq s \leq m$ and elements $\mathscr{V}_1, \dots, \mathscr{V}_s$ of \mathscr{W} such that $\{\bigcap \mathscr{V}_1, \dots, \bigcap \mathscr{V}_s\}$ is a pairwise disjoint family and $\bigcup_{i=1}^s \mathscr{V}_i = \{V_1, \dots, V_m\}$.

If $m = 1$, this is trivial. Suppose that $m > 1$.

Suppose to the contrary that for each $1 \leq s \leq m$ and for any elements $\mathscr{V}_1, \dots, \mathscr{V}_s$ of \mathscr{W} , it is either the case that $\bigcup_{i=1}^s \mathscr{V}_i \neq \{V_1, \dots, V_m\}$ or $\{\bigcap \mathscr{V}_1, \dots, \bigcap \mathscr{V}_s\}$ is not a pairwise disjoint family. (*)

Since (\mathscr{W}, \subseteq) is a non-empty finite poset, there is a maximal element \mathscr{V}_1 of (\mathscr{W}, \subseteq) such that $\{V_1\} \subseteq \mathscr{V}_1$, i.e. $V_1 \in \mathscr{V}_1$.

By the assumption (*), $\mathscr{V}_1 \neq \{V_1, \dots, V_m\}$ (any singleton is a pairwise disjoint family).

Let $i_2 := \min\{j : V_j \notin \mathscr{V}_1\}$. Let \mathscr{V}_2 be a maximal element of (\mathscr{W}, \subseteq) such that $V_{i_2} \in \mathscr{V}_2$.

Now $\mathscr{V}_1 \subset \mathscr{V}_1 \cup \mathscr{V}_2$ so by maximality of \mathscr{V}_1 , $\bigcap \mathscr{V}_1 \cap \bigcap \mathscr{V}_2 = \bigcap(\mathscr{V}_1 \cup \mathscr{V}_2) = \emptyset$. Thus $\{\bigcap \mathscr{V}_1, \bigcap \mathscr{V}_2\}$ is a pairwise disjoint family.

Let $1 < k \leq m$ and suppose that this process is continued to define k maximal elements $\mathscr{V}_1, \dots, \mathscr{V}_k$ of (\mathscr{W}, \subseteq) such that $\{\bigcap \mathscr{V}_1, \dots, \bigcap \mathscr{V}_k\}$ is a pairwise disjoint family.

By the assumption (*), $\bigcup_{l=1}^k \mathscr{V}_l \neq \{V_1, \dots, V_m\}$.

Let $i_{k+1} := \min\{j : V_j \notin \bigcup_{l=1}^k \mathscr{V}_l\}$. Let \mathscr{V}_{k+1} be a maximal element of (\mathscr{W}, \subseteq) such that $V_{i_{k+1}} \in \mathscr{V}_{k+1}$. For $1 \leq l \leq k$, it follows by maximality of \mathscr{V}_l that $\bigcap \mathscr{V}_{k+1} \cap \bigcap \mathscr{V}_l = \bigcap(\mathscr{V}_l \cup \mathscr{V}_{k+1}) = \emptyset$. Thus $\{\bigcap \mathscr{V}_1, \dots, \bigcap \mathscr{V}_{k+1}\}$ is a pairwise disjoint family.

This process may be continued so that there are $m + 1$ maximal elements $\mathcal{V}_1, \dots, \mathcal{V}_{m+1}$ of (\mathcal{W}, \subseteq) such that $\{\bigcap \mathcal{V}_1, \dots, \bigcap \mathcal{V}_{m+1}\}$ is a pairwise disjoint family. By the construction, for every $1 \leq i \leq m + 1$ there is $1 \leq j \leq m$ such that $V_j \in \mathcal{V}_i$. By the Pigeonhole Principle, there are distinct $i, i' \in \{1, \dots, m + 1\}$ such that there is $1 \leq j \leq m$ such that $V_j \in \mathcal{V}_i \cap \mathcal{V}_{i'}$; this would imply that $\bigcap \mathcal{V}_i \cap \bigcap \mathcal{V}_{i'} \supseteq V_j \neq \emptyset$.

Thus, the assumption (*) leads to a contradiction.

Thus, there must be $1 \leq s \leq m$ and elements $\mathcal{V}_1, \dots, \mathcal{V}_s$ of \mathcal{W} such that $\bigcup_{i=1}^s \mathcal{V}_i = \{V_1, \dots, V_m\}$ and that $\{\bigcap \mathcal{V}_1, \dots, \bigcap \mathcal{V}_m\}$ is a pairwise disjoint family. ■

Lemma 4.1.5. *Suppose the family of irreducible components of X is $\{U_1, \dots, U_n\}$. Let $\mathcal{V} = \{V_1, \dots, V_m\}$ be a family of opens in X .*

There is a cellular family $\{W_1, \dots, W_n\}$ in X such that for all i , $W_i = \text{Int}(U_i) \cap \bigcap \mathcal{F}_i$ for some finite subset \mathcal{F}_i of \mathcal{V} . Moreover, $\{\overline{W}_1, \dots, \overline{W}_n\}$ is the family of irreducible components of X and:

$$\langle X, W_1, \dots, W_n \rangle \subseteq \langle X, V_1, \dots, V_m \rangle.$$

Proof. By Proposition 3.3.7, $\{\text{Int}(U_1), \dots, \text{Int}(U_n)\}$ is cellular in X . Since $X = U_1 \cup \dots \cup U_n$ it follows there is i such that $\bigcup \mathcal{V} \cap U_i \neq \emptyset$ and hence $\bigcup \mathcal{V} \cap \text{Int}(U_i) \neq \emptyset$ (as U_i is irreducible and $\text{Int}(U_i)$ is non-empty).

Let $1 \leq j \leq n$. If $\text{Int}(U_j) \cap \bigcup \mathcal{V} = \emptyset$, let $W_j := \text{Int}(U_j)$. Otherwise, consider the subfamily \mathcal{V}_j of \mathcal{V} such that $V \cap \text{Int}(U_j) \neq \emptyset$ for all $V \in \mathcal{V}_j$. Considering that U_j is irreducible it follows that $\text{Int}(U_j) \cap \bigcap \mathcal{V}_j \neq \emptyset$ and so let $W_j := \text{Int}(U_j) \cap \bigcap \mathcal{V}_j$.

Now $\{W_1, \dots, W_n\}$ is cellular in X . Moreover, for all $1 \leq j \leq n$, $\overline{W}_j = U_j$ as U_j is irreducible. Finally,

$$\langle X, W_1, \dots, W_n \rangle \subseteq \langle X, V_1, \dots, V_m \rangle.$$

■

4.2 Definition and Properties

Definition 4.2.0.1. The *Lower Vietoris Topology* on $\mathcal{P}_0(X)$ is generated by $\{\langle X, V \rangle : V \in \tau\}$ and is denoted by $\mathbf{L}(\tau)$.

$\{\langle X, V \rangle \cap \mathfrak{S} : V \in \tau\}$ is a sub-base for the subspace topology $\mathbf{L}(\tau)|\mathfrak{S}$ induced on \mathfrak{S} from $\mathbf{L}(\tau)$. The *Lower Vietoris Topology on \mathfrak{S}* is $\mathbf{L}(\tau)|\mathfrak{S}$.

It is assumed hereafter that any set of non-empty subsets of X is a topological space equipped with the Lower Vietoris Topology, unless otherwise stated.

Definition 4.2.0.2. [34, Definition 2.1] The *Vietoris Topology* on $\mathcal{P}_0(X)$ is generated by $\{\langle X, V \rangle : V \in \tau\} \cup \{\langle V \rangle : V \in \tau\}$. The Vietoris Topology will be discussed in Chapter 5.

Remark 4.2.0.1. In some works such as [17], [58, p. 4] and [33, Definition 2.4], the Lower Vietoris Topology on \mathfrak{S} is first defined in terms of its sub-base as given in Definition 4.2.0.1.

Were it not for the results in the following Proposition 4.2.1 and Corollary 4.2.2, the above Definition 4.2.0.1 would cause confusion: for instance if Y is a subspace of X , then the Lower Vietoris Topology on $\mathcal{P}_0(Y)$ is either $\mathbf{L}(\tau|Y)$ (from the first paragraph of Definition 4.2.0.1) or $\mathbf{L}(\tau)|\mathcal{P}_0(Y)$ (from the third paragraph of Definition 4.2.0.1); however these are equal:

Proposition 4.2.1. [3, Chapter I, Exercises for §3, Exercise 10]

For every subset Y of X , $\mathbf{L}(\tau)|\mathcal{P}_0(Y) = \mathbf{L}(\tau|Y)$.

Proof. $\{\langle X, V \rangle \cap \mathcal{P}_0(Y) : V \in \tau\}$ is a sub-base for $\mathbf{L}(\tau)|\mathcal{P}_0(Y)$ and by Proposition 4.1.1 this is equal to $\{\langle Y, V \cap Y \rangle : V \in \tau\} = \{\langle Y, U \rangle : U \in \tau|Y\}$, which is a sub-base for $\mathbf{L}(\tau|Y)$. Thus the result follows. ■

Corollary 4.2.2. *If $Y \subseteq X$ and $\mathfrak{S} \subseteq \mathcal{P}_0(Y)$ then $\mathbf{L}(\tau|Y)|\mathfrak{S} = \mathbf{L}(\tau)|\mathfrak{S}$.*

Proof. By Proposition 4.2.1, $\mathbf{L}(\tau|Y)|\mathfrak{S} = (\mathbf{L}(\tau)|\mathcal{P}_0(Y))|\mathfrak{S} = \mathbf{L}(\tau)|\mathfrak{S}$. ■

Proposition 4.2.3. *The following collection is a base for the open subsets of \mathfrak{S} :*

$$\{\langle X, V_1, \dots, V_n \rangle \cap \mathfrak{S} : n \in \mathbb{N}, V_1, \dots, V_n \text{ are open subsets of } X\}.$$

Proof. The base for \mathfrak{S} generated by its sub-base $\{\langle X, V \rangle \cap \mathfrak{S} : V \text{ is an open subset of } X\}$ is given by:

$$\begin{aligned} & \left\{ \bigcap_{i=1}^n \langle X, V_i \rangle \cap \mathfrak{S} : n \in \mathbb{N}, V_1, \dots, V_n \in \tau \right\} \\ & = \{ \langle X, V_1, \dots, V_n \rangle \cap \mathfrak{S} : n \in \mathbb{N}, V_1, \dots, V_n \in \tau \} \text{ (By Corollary 4.1.2)} \end{aligned}$$

■

The following Proposition 4.2.4 describes a well-known sub-base for the closed subsets of \mathfrak{S} ; in some works such as [3, Chapter I, Exercises for §2, Exercise 7(a)] and [39, Remark, p.175], the Lower Vietoris Topology on \mathfrak{S} is defined in terms of this sub-base when $\mathfrak{S} = 2^X$ or $\mathfrak{S} = \mathcal{P}_0(X)$.

Proposition 4.2.4.

The collection $\{\langle U \rangle \cap \mathfrak{S} : U \text{ is a closed subset of } X\}$ is a sub-base for the closed sets of the Lower Vietoris Topology on \mathfrak{S} ; moreover, this collection has \emptyset and \mathfrak{S} as elements.

Proof.

$$\begin{aligned} \text{If } U \subseteq X, \langle U \rangle &= \{A \in \mathcal{P}_0(X) : A \subseteq U\} = \{A \in \mathcal{P}_0(X) : A \cap X \setminus U = \emptyset\} \\ &= \{A \in \mathcal{P}_0(X) : A \notin \langle X, X \setminus U \rangle\} \\ &= \mathcal{P}_0(X) \setminus \langle X, X \setminus U \rangle. \end{aligned}$$

Thus the following family is a sub-base for the closed sets of \mathfrak{S} and has $\emptyset = \langle \emptyset \rangle \cap \mathfrak{S}$ and $\mathfrak{S} = \langle X \rangle \cap \mathfrak{S}$ as elements:

$$\{\mathfrak{S} \setminus \langle X, V \rangle : V \text{ open in } X\} = \{\mathfrak{S} \setminus \langle X, X \setminus U \rangle : U \text{ closed in } X\} = \{\langle U \rangle \cap \mathfrak{S} : U \text{ closed in } X\}.$$

■

The following definition is adapted from [44, p. 153]:

Definition 4.2.4.1. A given topology $\exp \tau$ on $\mathcal{P}_0(X)$ is said to be *admissible topology* with respect to τ if the map $i : X \rightarrow (\mathcal{F}_1(X), \exp \tau|_{\mathcal{F}_1(X)})$ defined by $i(x) = \{x\}$ for all $x \in X$, is a homeomorphism. The terminology of [8, Paragraph after Corollary 1.3] is adopted; i is referred to as the *canonical map* from X to $\mathcal{P}_0(X)$.

The result in the following Proposition 4.2.5 is well-known. This result is stated without proof in [44, Definition 9.1] and [3, Chapter I, §2, Exercise 7(a)]. The proof for this result is hinted in [17, Proposition III.1], and it is proved for the case when X is a Tychonoff space in [8, Proposition 1.4] (however, their given proof is different from the one presented below).

Proposition 4.2.5. $L(\tau)$ is admissible with respect to τ .

Proof. For any closed subset U of X ,

$i^{-1}(\langle U \rangle \cap \mathcal{F}_1(X)) = \{x \in X : \{x\} \in \langle U \rangle \cap \mathcal{F}_1(X)\} = U$ is closed in X . By Lemma 2.1.4 and Proposition 4.2.4, $i : X \rightarrow \mathcal{F}_1(X)$ is continuous. For any closed subset U of X , $i(U) = \{\{x\} : x \in U\} = \langle U \rangle \cap \mathcal{F}_1(X)$ is closed in $\mathcal{F}_1(X)$ (cf. Proposition 4.2.4). Thus $i : X \rightarrow \mathcal{F}_1(X)$ is a closed map. Thus $i : X \rightarrow \mathcal{F}_1(X)$ is a homeomorphism. ■

Proposition 4.2.6. For any $A \in \mathfrak{S}$, $\text{Cl}_{L(\tau)|\mathfrak{S}}(\{A\}) = \langle \bar{A} \rangle \cap \mathfrak{S}$.

Proof. Let $A \in \mathcal{P}_0(X)$. Then $A \in \langle \bar{A} \rangle$ and if there is a closed subset B of X for which $A \in \langle B \rangle$, then $A \subseteq B$, so $\bar{A} \subseteq B$ and hence $\langle \bar{A} \rangle \subseteq \langle B \rangle$.

Thus $\langle \bar{A} \rangle = \bigcap \{\langle B \rangle : B \text{ closed in } X, A \in \langle B \rangle\}$.

By Proposition 4.2.4, $\{\langle B \rangle : B \text{ closed in } X\}$ is a sub-base for the closed sets of $\mathcal{P}_0(X)$, which has $\mathcal{P}_0(X)$ as an element. It follows from Proposition 3.3.4 that $\langle \bar{A} \rangle = \text{Cl}_{L(\tau)}(\{A\})$.

Thus for any $A \in \mathfrak{S}$, $\text{Cl}_{L(\tau)|\mathfrak{S}}(\{A\}) = \text{Cl}_{L(\tau)}(\{A\}) \cap \mathfrak{S} = \langle \bar{A} \rangle \cap \mathfrak{S}$. ■

Recall from Definition 2.2.1.2 that $\leq_{L|\mathfrak{S}}$ is the specialisation pre-order on \mathfrak{S} . Corollary

4.2.7 (1) is well-known for the case when $\mathfrak{S} = \mathcal{P}_0(X)$ (cf. [50, Proposition 6.4.1(i)] and [23, First sentence of Section 3.2]).

Corollary 4.2.7. *Let $A, B \in \mathfrak{S}$:*

1. $A \leq_{\mathbf{L}(\tau)|\mathfrak{S}} B$ iff $A \in \langle \bar{B} \rangle \cap \mathfrak{S}$ iff $A \subseteq \bar{B}$.
2. $\text{Cl}_{\mathbf{L}(\tau)|\mathfrak{S}}(\{A\}) = \text{Cl}_{\mathbf{L}(\tau)|\mathfrak{S}}(\{B\})$ iff $\bar{A} = \bar{B}$.

Proof. 1. By Proposition 4.2.6, $\langle \bar{B} \rangle \cap \mathfrak{S} = \text{Cl}_{\mathbf{L}(\tau)|\mathfrak{S}}(\{B\})$. Hence:

$$A \leq_{\mathbf{L}(\tau)|\mathfrak{S}} B \iff A \in \text{Cl}_{\mathbf{L}(\tau)|\mathfrak{S}}(\{B\}) \iff A \in \langle \bar{B} \rangle \cap \mathfrak{S} \iff A \subseteq \bar{B}.$$

$$2. \text{Cl}_{\mathbf{L}(\tau)|\mathfrak{S}}(\{A\}) = \text{Cl}_{\mathbf{L}(\tau)|\mathfrak{S}}(\{B\})$$

$$\iff A \leq_{\mathbf{L}(\tau)|\mathfrak{S}} B \text{ and } B \leq_{\mathbf{L}(\tau)|\mathfrak{S}} A$$

$$\iff A \subseteq \bar{B} \text{ and } B \subseteq \bar{A} \text{ (By the previous point of this Corollary 4.2.7)}$$

$$\iff \bar{A} = \bar{B}. \quad \blacksquare$$

4.3 The T_0 -identification

Definition 4.3.0.1. The map $c : \mathcal{P}_0(X) \rightarrow \mathcal{P}_0(X)$ is defined by the formula $c(A) = \bar{A}$ for all $A \in \mathcal{P}_0(X)$.

Feichtinger [17, Proposition III.19] showed that $c : \mathcal{P}_0(X) \rightarrow 2^X$ is a *retraction* (cf. [57, Problems 7J]); the continuity of this map is also a well-known result (cf. [3, Chapter I, Exercises for §2, Exercise 7(e)]). The following Lemma 4.3.1 generalises these results:

Lemma 4.3.1. *The map $c|\mathfrak{S} : \mathfrak{S} \rightarrow c(\mathfrak{S})$ is a closed, continuous, open and surjective map such that for every subset \mathcal{A} of $\mathcal{P}_0(X)$ which is closed or open in $\mathcal{P}_0(X)$, the following two equations are true:*

$$c(\mathcal{A} \cap \mathfrak{S}) = \mathcal{A} \cap c(\mathfrak{S}) \text{ and } (c|\mathfrak{S})^{-1}(c(\mathcal{A} \cap \mathfrak{S})) = \mathcal{A} \cap \mathfrak{S}.$$

Proof. If U is closed in X , then:

$$c^{-1}(\langle U \rangle) = \{A \in \mathcal{P}_0(X) : \bar{A} \in \langle U \rangle\} = \{A \in \mathcal{P}_0(X) : \bar{A} \subseteq U\} = \langle U \rangle$$

Thus the result follows from Proposition 4.2.4 and Lemma 2.1.5. ■

Corollary 4.3.2. *For every closed subset U of X :*

$$c(\langle U \rangle \cap \mathfrak{S}) = \langle U \rangle \cap c(\mathfrak{S}) \quad \text{and} \quad (c|\mathfrak{S})^{-1}(c(\langle U \rangle \cap \mathfrak{S})) = \langle U \rangle \cap \mathfrak{S}.$$

Proof. Follows from Proposition 4.2.4 and the previous Lemma 4.3.1. ■

It is remarked in [32, Remark 2.4(1)] (without proof) that the T_0 -identification of $\mathcal{P}_0(X)$ is essentially 2^X . The result in the following Theorem 4.3.3 generalises this statement; I believe this result is new.

Theorem 4.3.3. *$c(\mathfrak{S})$ is homeomorphic to the T_0 -identification of \mathfrak{S} .*

Proof. For each $A \in \mathfrak{S}$:

$$\begin{aligned} (c|\mathfrak{S})^{-1}(\{(c|\mathfrak{S})(A)\}) &= \{B \in \mathfrak{S} : (c|\mathfrak{S})(B) = (c|\mathfrak{S})(A)\} \\ &= \{B \in \mathfrak{S} : \bar{B} = \bar{A}\} \\ &= \{B \in \mathfrak{S} : \text{Cl}_{\mathbf{L}(\tau)|\mathfrak{S}}(\{B\}) = \text{Cl}_{\mathbf{L}(\tau)|\mathfrak{S}}(\{A\})\} \quad (\text{By Corollary 4.2.7 (2)}). \end{aligned}$$

Now, the T_0 -identification of \mathfrak{S} is the topological space whose underlying set is given by:

$$\begin{aligned} \hat{\mathfrak{S}} &:= \{\{B \in \mathfrak{S} : \text{Cl}_{\mathbf{L}(\tau)|\mathfrak{S}}(\{B\}) = \text{Cl}_{\mathbf{L}(\tau)|\mathfrak{S}}(\{A\})\} : A \in \mathfrak{S}\} \\ &= \{(c|\mathfrak{S})^{-1}(\{(c|\mathfrak{S})(A)\}) : A \in \mathfrak{S}\} \end{aligned}$$

and whose topology \mathbf{T} is the quotient topology on $\hat{\mathfrak{S}}$ induced by the map $p : \mathfrak{S} \rightarrow \hat{\mathfrak{S}}$ which is defined by the following formula for each $A \in \mathfrak{S}$:

$$p(A) = \{B \in \mathfrak{S} : \text{Cl}_{\mathbf{L}(\tau)|\mathfrak{S}}(\{A\}) = \text{Cl}_{\mathbf{L}(\tau)|\mathfrak{S}}(\{B\})\} = (c|\mathfrak{S})^{-1}(\{(c|\mathfrak{S})(A)\}).$$

By Lemma 4.3.1, $c|\mathfrak{S} : \mathfrak{S} \rightarrow c(\mathfrak{S})$ is a continuous, closed, surjective map. Hence the Lower Vietoris Topology on $c(\mathfrak{S})$ is the quotient topology on $c(\mathfrak{S})$ induced by the map $c|\mathfrak{S}$. Thus by Proposition 2.1.7, $c(\mathfrak{S})$ is homeomorphic to $(\hat{\mathfrak{S}}, \mathbf{T})$. ■

The following Corollary 4.3.4 is interesting as it allows one to define the T_0 -identification of any topological space in terms of the Lower Vietoris Topology without relying on the theory of quotient spaces (compare to Definition 2.1.7.1):

Corollary 4.3.4. $c(\mathcal{F}_1(X))$ is homeomorphic to the T_0 -identification of X .

Proof. By Proposition 4.2.5, X is homeomorphic to $\mathcal{F}_1(X)$. Hence the T_0 -identification of X is homeomorphic to the T_0 -identification of $\mathcal{F}_1(X)$ (by Proposition 2.1.8), and the latter is homeomorphic to $c(\mathcal{F}_1(X))$ (by the previous Theorem 4.3.3). ■

The fact that 2^X is a T_0 -space (cf. Corollary 4.3.5 (1)) is a well-known result (cf. [4, Proposition 2], [8, Corollary 1.3(a)] and [17, Proposition III.2]).

Corollary 4.3.5 (3) is also well-known for the case when $c(\mathfrak{S}) = 2^X$ (cf. [4, Proposition 1]).

Corollary 4.3.5. *The following are true:*

1. $c(\mathfrak{S})$ is a T_0 -space; in particular $2^X = c(\mathcal{P}_0(X))$ is a T_0 -space.
2. $\leq_{\mathbf{L}(\tau)|c(\mathfrak{S})}$ is the subset relation on $c(\mathfrak{S})$.

Proof. 1. Follows from the previous Theorem 4.3.3.

2. For $A, B \in c(\mathfrak{S})$, $A \leq_{\mathbf{L}(\tau)|c(\mathfrak{S})} B$ iff $A \subseteq \overline{B}$ (by Corollary 4.2.7 (1)) iff $A \subseteq B$ (as $B = \overline{B}$). ■

4.4 The T_0 , Symmetry and T_1 Separation Axioms

I believe that the following result is new:

Theorem 4.4.1. \mathfrak{S} is a:

1. T_0 -space iff distinct elements of \mathfrak{S} have distinct closures in X .

2. symmetric space iff $(c(\mathfrak{S}), \subseteq)$ is an antichain iff $c(\mathfrak{S})$ is a T_1 -space.

Proof. 1. \implies : If $A, B \in \mathfrak{S}$ where $\overline{A} = \overline{B}$, then by Corollary 4.2.7 (2), $\text{Cl}_{\mathbf{L}(\tau)|\mathfrak{S}}(\{A\}) = \text{Cl}_{\mathbf{L}(\tau)|\mathfrak{S}}(\{B\})$ and so $A = B$.

\impliedby : $c|\mathfrak{S} : \mathfrak{S} \rightarrow c(\mathfrak{S})$ is an injective map and thus by Lemma 4.3.1 is a homeomorphism. Since $c(\mathfrak{S})$ is T_0 by Corollary 4.3.5 (1), it follows that \mathfrak{S} is T_0 .

2. \mathfrak{S} is symmetric iff its T_0 -identification is T_1 iff $c(\mathfrak{S})$ is T_1 (cf. Theorem 4.3.3) iff $(c(\mathfrak{S}), \leq_{\mathbf{L}(\tau)|c(\mathfrak{S})})$ is an antichain (By Proposition 2.2.2 (2)) iff $(c(\mathfrak{S}), \subseteq)$ is an antichain (By Corollary 4.3.5 (3)). ■

Corollary 4.4.2 (1) generalises a result in [32, p. 314].

Corollary 4.4.2. *The following statements are true:*

1. If $2^X \subset \mathfrak{S}$ then \mathfrak{S} is T_0 iff X is discrete;
in particular [32, p. 314] $\mathcal{P}_0(X)$ is T_0 iff X is discrete.
2. If X is a Hausdorff space then $\mathcal{C}(X)$ is a T_0 -space.
3. If $n > 1$ then $\mathcal{F}_{\leq n}(X)$ is T_0 iff X is T_1 .

Proof. 1. \implies : Given $\emptyset \neq A \subseteq X$, $\overline{A} = \overline{A} \in c(\mathcal{P}_0(X))$ so $\overline{A} = A$ by Theorem 4.4.1 (1). Thus X is discrete. The converse is obvious.

2. A compact subspace of a Hausdorff space is closed so the result follows from Theorem 4.4.1 (1).

3. Suppose to the contrary that $\mathcal{F}_{\leq n}(X)$ is T_0 while there is $y \in X$ where $\{y\}$ is not closed. Now $\mathcal{F}_1(X)$ is T_0 so by Proposition 4.2.5, X is T_0 . Thus there is $z \in \overline{\{y\}}$ where $y \notin \overline{\{z\}}$. Now $\overline{\{y, z\}} = \overline{\{y\}}$ while $\{y, z\} \neq \{y\}$ and $\{y, z\}, \{y\} \in \mathcal{F}_{\leq n}(X)$, a contradiction to Theorem 4.4.1. Thus if $\mathcal{F}_{\leq n}(X)$ is T_0 then X must be T_1 . The converse follows from Theorem 4.4.1. ■

The following example shows the converse of Corollary 4.4.2 (2) is false in general: an example of a non-Hausdorff topological space X for which $\mathcal{C}(X)$ is a T_0 -space is presented.

Example 4.4.2.1.

Suppose X is uncountable and has the countable complement topology. Then X is T_1 but not Hausdorff, moreover $\mathcal{C}(X) = \mathcal{F}(X) = c(\mathcal{F}(X))$ is T_0 by Corollary 4.3.5 (1).

Corollary 4.4.3 (2), (3) and (4) generalise the well-known fact that neither 2^X nor $\mathcal{P}_0(X)$ are T_1 in general (cf. [8, Corollary 1.3(a)], [17, Proposition III.2], [39, Remark, p. 175], [44, Definition 9.1] and [3, Chapter I, Exercises for §8, Exercise 12(a)]). I believe that Corollary 4.4.3 is a new result.

Corollary 4.4.3. *The following statements are true:*

1. \mathfrak{S} is T_1 only if $(\mathfrak{S}, \subseteq)$ is an antichain.
2. If $X \in c(\mathfrak{S})$, then \mathfrak{S} is symmetric iff every element of \mathfrak{S} is dense in X .
3. $\mathcal{P}_0(X)$ is symmetric iff 2^X is T_1 iff X is trivial.
4. Suppose that $\mathfrak{S} \in \{\mathcal{P}_0(X), \mathcal{C}(X), \mathcal{F}(X)\} \cup \{\mathcal{F}_{\leq n}(X) : n \in \mathbb{N}, n > 1\}$. Then \mathfrak{S} is T_1 iff X is a singleton set.

Proof. 1. Let $A, B \in \mathfrak{S}$ such that $A \subseteq B$. Then $\overline{A} \subseteq \overline{B}$ and so $\overline{A} = \overline{B}$ (cf. Theorem 4.4.1 (2)). By Theorem 4.4.1 (1), $A = B$.

2. Easily seen from Theorem 4.4.1 (2).

3. Easily follows from previous point.

4. Follows from the first point. ■

Chapter 5

Closure, Irreducibility and Quasi-Sobriety in the Lower Vietoris Topology

This chapter deals with closure, irreducibility (covered in 5.1) and quasi-sobriety (covered in 5.2) in hyperspaces with the Lower Vietoris Topology. Theorem 5.1.23 in this chapter will be used to prove Theorem 6.3.1 in Chapter 6; moreover Theorem 5.2.7 in this chapter will be used to study preregularity and sobriety in $\mathcal{F}_n(X)$ in Chapter 7.

It is helpful to keep in mind that $c(\mathcal{S}_n(X)) \subseteq \mathcal{S}_n(X)$ for any $n \in \mathbb{N}$.

The following definition is made to allow the results that appear in this chapter and the following chapters to be effectively paraphrased.

Definition 5.0.0.1. I say that \mathfrak{S} is:

- *almost natural in X of degree n* if $c(\mathcal{F}_n(X)) \subseteq c(\mathfrak{S})$.
- *almost finitely natural in X of degree n* if $c(\mathcal{F}_n(X)) \subseteq c(\mathfrak{S}) \subseteq \mathcal{S}_{\leq n}(X)$.
- *almost d -finitely natural in X of degree n* if $c(\mathcal{D}_n(X)) \subseteq c(\mathfrak{S}) \subseteq \mathcal{S}_{\leq n}(X)$.

- *almost natural in X* (resp. *almost finitely natural in X*) (resp. *almost d -finitely natural in X*) if \mathfrak{S} is almost natural in X (resp. almost finitely natural in X) (resp. almost d -finitely natural in X) of some degree.

This terminology extends the term *natural family in X* coined by Ivanova-Dimova [33, Definition 2.14], to refer to the case when $\mathcal{F}_1(X) \subseteq \mathfrak{S}$.

5.1 Irreducibility and Closure

5.1.1 General Hyperspaces

In this section, closure and irreducibility and their relationship is investigated in general hyperspaces.

In general, closure in hyperspaces is determined by Proposition 5.1.1, its Corollary 5.1.2 and Proposition 5.1.3.

Proposition 5.1.4 describes precisely when hyperspaces are irreducible and illustrates the relationship between closure and irreducibility in hyperspaces.

Theorem 5.1.6 is more general than the previous result and is more useful for determining when hyperspaces are quasi-sober.

Corollaries 5.1.5, 5.1.7, 5.1.8, 5.1.9 and 5.1.10 are surprising results which describe how, under different conditions, the property of being irreducible is somewhat upwards closed in hyperspaces.

Finally Theorem 5.1.14 demonstrates that those closed hyperspaces having cellularity n are those having exactly n maximal elements.

Remark 5.1.0.1. If U is closed in X then by Corollary 4.3.2:

$$\overline{\bigcup \langle U \rangle \cap \mathfrak{S}} = \overline{\bigcup c(\langle U \rangle \cap \mathfrak{S})} = \overline{\bigcup \langle U \rangle \cap c(\mathfrak{S})}.$$

Proposition 5.1.1. *If $\mathfrak{S} \subseteq \downarrow c(\mathfrak{T})$ then $\mathfrak{S} \subseteq \text{Cl}_{\mathbf{L}(\tau)}(\mathfrak{T})$*

Proof. Let $A \in \mathfrak{S}$ and V_1, \dots, V_m be open in X such that $A \in \langle X, V_1, \dots, V_m \rangle$. Then there is $B \in \mathfrak{T}$ such that $A \subseteq \overline{B}$ and hence $B \in \langle X, V_1, \dots, V_m \rangle$. Thus $A \in \text{Cl}_{\mathbf{L}(\tau)}(\mathfrak{T})$. ■

Corollary 5.1.2. *If $\downarrow c(\mathfrak{S}) = \downarrow c(\mathfrak{T})$ then $\text{Cl}_{\mathbf{L}(\tau)}(\mathfrak{S}) = \text{Cl}_{\mathbf{L}(\tau)}(\mathfrak{T})$.*

Proof. Follows from the previous Proposition 5.1.1 and the fact that

$$\mathfrak{S} \subseteq \downarrow c(\mathfrak{T}) \text{ iff } c(\mathfrak{S}) \subseteq \downarrow c(\mathfrak{T}). \quad \blacksquare$$

Example 5.1.2.1. From Theorem 5.1.22, $\text{Cl}_{\mathbf{L}(\tau)}(\mathcal{F}_n(X)) = \text{Cl}_{\mathbf{L}(\tau)}(\mathcal{S}_{\leq n}(X))$. However in general, $\mathcal{S}_n(X) \not\subseteq \downarrow c(\mathcal{F}_n(X))$. For instance if X has the co-finite topology then $\mathcal{S}_1(X) = \{X\} \cup \mathcal{F}_1(X) \neq \mathcal{F}_1(X) = \downarrow c(\mathcal{F}_1(X))$. Thus the converse of Corollary 5.1.2 is not true in general.

If X is quasi-sober, then by Proposition 3.4.1, $c(\mathcal{S}_n(X)) = c(\mathcal{F}_n(X))$ for every n .

Proposition 5.1.3.

$$\mathcal{F}_1(\overline{\bigcup \mathfrak{S}}) \cup \bigcup_{A \in \mathfrak{S}} \langle \overline{A} \rangle \subseteq \text{Cl}_{\mathbf{L}(\tau)}(\mathfrak{S}) = \text{Cl}_{\mathbf{L}(\tau)}(c(\mathfrak{S})) \subseteq \langle \overline{\bigcup \mathfrak{S}} \rangle.$$

Proof. Since $c(\mathfrak{S}) = c(c(\mathfrak{S}))$, the equality $\text{Cl}_{\mathbf{L}(\tau)}(\mathfrak{S}) = \text{Cl}_{\mathbf{L}(\tau)}(c(\mathfrak{S}))$ follows from Corollary 5.1.2. By Proposition 4.2.6, for all $A \in \mathfrak{S}$, $\langle \overline{A} \rangle = \text{Cl}_{\mathbf{L}(\tau)}(\{A\}) \subseteq \text{Cl}_{\mathbf{L}(\tau)}(\mathfrak{S})$. Since $\mathcal{F}_1(\overline{\bigcup \mathfrak{S}}) \subseteq \downarrow c(\mathfrak{S})$, it must be the case that $\mathcal{F}_1(\overline{\bigcup \mathfrak{S}}) \subseteq \text{Cl}_{\mathbf{L}(\tau)}(\mathfrak{S})$ by Proposition 5.1.1. Since $\mathfrak{S} \subseteq \langle \overline{\bigcup \mathfrak{S}} \rangle$ and the latter is closed by Proposition 4.2.4, it follows that $\text{Cl}_{\mathbf{L}(\tau)}(c(\mathfrak{S})) = \text{Cl}_{\mathbf{L}(\tau)}(\mathfrak{S}) = \subseteq \langle \overline{\bigcup \mathfrak{S}} \rangle$. ■

Remark 5.1.3.1. If U is closed then by the previous Proposition 5.1.3 and Corollary 4.3.2:

$$\text{Cl}_{\mathbf{L}(\tau)}(\langle U \rangle \cap \mathfrak{S}) = \text{Cl}_{\mathbf{L}(\tau)}(c(\langle U \rangle \cap \mathfrak{S})) = \text{Cl}_{\mathbf{L}(\tau)}(\langle U \rangle \cap c(\mathfrak{S})).$$

In particular if $c(\mathfrak{T}) \subseteq c(\mathfrak{S})$ then $c(\mathfrak{S} \cup \mathfrak{T}) = c(\mathfrak{S})$ and thus:

$$\text{Cl}_{\mathbf{L}(\tau)}(\langle U \rangle \cap \mathfrak{S}) = \text{Cl}_{\mathbf{L}(\tau)}(\langle U \rangle \cap (\mathfrak{S} \cup \mathfrak{T})).$$

Example 5.1.3.1. If $\bigcup \mathfrak{S} = \overline{\bigcup \mathfrak{S}}$ then $\mathcal{F}_1(\overline{\bigcup \mathfrak{S}}) \subseteq \bigcup_{A \in \mathfrak{S}} \langle \overline{A} \rangle$.

However suppose that X is T_1 and A is a proper dense subspace of X . Then:

$$\mathcal{F}_1(\overline{\bigcup \mathcal{F}(A)}) = \mathcal{F}_1(\overline{A}) = \mathcal{F}_1(X) \not\subseteq \mathcal{F}(A) = \bigcup_{F \in \mathcal{F}(A)} \langle \overline{F} \rangle.$$

Note in this case that $\text{Cl}_{\mathbf{L}(\tau)}(\mathcal{F}(A)) = \langle \overline{A} \rangle = \mathcal{P}_0(X)$ by Proposition 5.1.11, which is not the same as the union $\mathcal{F}(A) \cup \mathcal{F}_1(X) = \bigcup_{F \in \mathcal{F}(A)} \langle \overline{F} \rangle \cup \mathcal{F}_1(\overline{\bigcup \mathcal{F}(A)})$. Thus the lower bound in Proposition 5.1.3 is not always equal to $\text{Cl}_{\mathbf{L}(\tau)}(\mathfrak{S})$.

The following Proposition 5.1.4 not only characterises irreducibility for hyperspaces but points 1 and 3 respectively inform that:

- \mathfrak{S} is irreducible iff its closure is a member of the sub-base for the closed sets of $\mathcal{P}_0(X)$ as described in Proposition 4.2.4.
- Every closed irreducible subspace of \mathfrak{S} is a member of the sub-base for the closed sets of \mathfrak{S} as described in Proposition 4.2.4. This result is stated in passing in [32, p. 313-314] for the case $\mathfrak{S} = \mathcal{P}_0(X)$ without proof - see the end of page 313 and the beginning of page 314 in that reference.

Proposition 5.1.4. 1. *The following are equivalent:*

- (a) \mathfrak{S} is irreducible.
- (b) $\text{Cl}_{\mathbf{L}(\tau)}(\mathfrak{S}) = \langle \overline{\bigcup \mathfrak{S}} \rangle$.
- (c) $\text{Cl}_{\mathbf{L}(\tau)}(\mathfrak{S}) = \text{Cl}_{\mathbf{L}(\tau)}(\{\bigcup \mathfrak{S}\})$.
- (d) $\bigcup \mathfrak{S} \in \text{Cl}_{\mathbf{L}(\tau)}(\mathfrak{S})$.
- (e) $\overline{\bigcup \mathfrak{S}} \in \text{Cl}_{\mathbf{L}(\tau)}(\mathfrak{S})$.

2. *If $\bigcup \mathfrak{S}$ is irreducible then so is \mathfrak{S} .*

3. *A closed subspace \mathfrak{T} of \mathfrak{S} is irreducible only if $\mathfrak{T} = \langle \overline{\bigcup \mathfrak{T}} \rangle \cap \mathfrak{S}$.*

Proof. 1. $a \implies b \implies c$: Suppose that \mathfrak{S} is irreducible. Then by Proposition 4.2.4 and Proposition 3.3.4:

$$\text{Cl}_{\mathbf{L}(\tau)}(\mathfrak{S}) = \bigcap \{ \langle U \rangle : U \text{ closed and } \mathfrak{S} \subseteq \langle U \rangle \}$$

$$\begin{aligned}
&= \bigcap \{ \langle U \rangle : U \text{ closed and } \bigcup \mathfrak{S} \subseteq U \} \\
&= \langle \bigcap \{ U : U \text{ closed and } \bigcup \mathfrak{S} \subseteq U \} \rangle \\
&= \overline{\langle \bigcup \mathfrak{S} \rangle} \\
&= \text{Cl}_{\mathbf{L}(\tau)}(\langle \bigcup \mathfrak{S} \rangle) \quad (\text{By Proposition 4.2.6}).
\end{aligned}$$

$c \implies d$: Obvious.

$d \implies e$: By Proposition 4.2.6, $\overline{\bigcup \mathfrak{S}} \in \langle \overline{\bigcup \mathfrak{S}} \rangle = \text{Cl}_{\mathbf{L}(\tau)}(\langle \bigcup \mathfrak{S} \rangle) \subseteq \text{Cl}_{\mathbf{L}(\tau)}(\mathfrak{S})$.

$e \implies a$: By Proposition 4.2.6, $\langle \overline{\bigcup \mathfrak{S}} \rangle = \text{Cl}_{\mathbf{L}(\tau)}(\langle \overline{\bigcup \mathfrak{S}} \rangle) \subseteq \text{Cl}_{\mathbf{L}(\tau)}(\mathfrak{S})$; thus by Proposition 5.1.3, $\text{Cl}_{\mathbf{L}(\tau)}(\mathfrak{S}) = \langle \overline{\bigcup \mathfrak{S}} \rangle = \text{Cl}_{\mathbf{L}(\tau)}(\langle \overline{\bigcup \mathfrak{S}} \rangle)$, hence \mathfrak{S} is irreducible.

2. Let V_1, \dots, V_m be non-empty and open in X such that $\bigcup \mathfrak{S} \in \langle X, V_1, \dots, V_m \rangle$.

Then for all i , $\bigcup \mathfrak{S} \cap V_i \neq \emptyset$. By the hypothesis $\bigcup \mathfrak{S} \cap \bigcap_{i=1}^m V_i \neq \emptyset$. Thus, $\langle X, \bigcap_{i=1}^m V_i \rangle \cap \mathfrak{S} \neq \emptyset$. Hence, $\bigcup \mathfrak{S} \in \text{Cl}_{\mathbf{L}(\tau)}(\mathfrak{S})$. By the first point, \mathfrak{S} is irreducible.

3. Let \mathfrak{T} be a closed subspace of \mathfrak{S} which is irreducible. Then $\text{Cl}_{\mathbf{L}(\tau)}(\mathfrak{T}) = \langle \overline{\bigcup \mathfrak{T}} \rangle$ by the first point, hence $\mathfrak{T} = \text{Cl}_{\mathbf{L}(\tau)|_{\mathfrak{S}}}(\mathfrak{T}) = \langle \overline{\bigcup \mathfrak{T}} \rangle \cap \mathfrak{S}$. ■

Remark 5.1.4.1. The converse of point 2 of the previous Proposition 5.1.4 is false in general; whenever $\mathcal{F}(X) \subseteq \text{Cl}_{\mathbf{L}(\tau)}(\mathfrak{S})$ it is true that \mathfrak{S} is irreducible (cf. Proposition 5.1.11) regardless of the properties of $\overline{\bigcup \mathfrak{S}} = X$.

Corollary 5.1.5. *If $\mathfrak{S} \subseteq \text{Cl}_{\mathbf{L}(\tau)}(\mathfrak{T})$, $\overline{\bigcup \mathfrak{S}} = \overline{\bigcup \mathfrak{T}}$ and \mathfrak{S} is irreducible then $\text{Cl}_{\mathbf{L}(\tau)}(\mathfrak{T}) = \langle \overline{\bigcup \mathfrak{S}} \rangle$ and \mathfrak{T} is irreducible.*

Proof. By Proposition 5.1.4, $\overline{\bigcup \mathfrak{T}} = \overline{\bigcup \mathfrak{S}} \in \text{Cl}_{\mathbf{L}(\tau)}(\mathfrak{S}) \subseteq \text{Cl}_{\mathbf{L}(\tau)}(\mathfrak{T})$, so the result follows by the same Proposition. ■

Theorem 5.1.6. *The following are equivalent:*

1. $\text{Cl}_{\mathbf{L}(\tau)}(\langle U \rangle \cap \mathfrak{S}) = \langle \overline{U} \rangle$.
2. $U \in \text{Cl}_{\mathbf{L}(\tau)}(\langle U \rangle \cap \mathfrak{S})$.
3. $\langle U \rangle \cap \mathfrak{S}$ is irreducible and $\overline{U} = \overline{\langle U \rangle \cap \mathfrak{S}}$.

Proof. 1 \implies 2: Obvious. ■

2 \implies 3: Then $\bigcup(\langle U \rangle \cap \mathfrak{S}) \in \langle \bar{U} \rangle = \text{Cl}_{\mathbf{L}(\tau)}(\{U\}) \subseteq \text{Cl}_{\mathbf{L}(\tau)}(\langle U \rangle \cap \mathfrak{S})$. By Proposition 5.1.4 (1), $\langle U \rangle \cap \mathfrak{S}$ is irreducible and $\text{Cl}_{\mathbf{L}(\tau)}(\langle U \rangle \cap \mathfrak{S}) = \overline{\bigcup(\langle U \rangle \cap \mathfrak{S})}$. The latter has U as an element by the hypothesis, thus $\overline{\bigcup(\langle U \rangle \cap \mathfrak{S})} = \bar{U}$.

3 \implies 1: Follows from Proposition 5.1.4 (1). ■

Corollary 5.1.7. *Suppose $\bar{U} = \overline{\bigcup(\langle U \rangle \cap \mathfrak{S})}$, $\langle U \rangle \cap \mathfrak{S}$ is irreducible and $\mathfrak{S} \subseteq \mathfrak{T}$. Then $\text{Cl}_{\mathbf{L}(\tau)}(\langle U \rangle \cap \mathfrak{T}) = \langle \bar{U} \rangle$, $\bar{U} = \overline{\bigcup(\langle U \rangle \cap \mathfrak{T})}$ and $\langle U \rangle \cap \mathfrak{T}$ is irreducible.*

Proof. By Theorem 5.1.6 and Proposition 5.1.3, $\langle \bar{U} \rangle = \text{Cl}_{\mathbf{L}(\tau)}(\langle U \rangle \cap \mathfrak{S}) \subseteq \text{Cl}_{\mathbf{L}(\tau)}(\langle U \rangle \cap \mathfrak{T}) \subseteq \langle \bar{U} \rangle$. Thus by Theorem 5.1.6, $\bar{U} = \overline{\bigcup(\langle U \rangle \cap \mathfrak{T})}$ and $\langle U \rangle \cap \mathfrak{T}$ is irreducible. ■

Corollary 5.1.8. *Suppose $\bar{U} = \overline{\bigcup(\langle U \rangle \cap \mathfrak{S})}$, $\langle U \rangle \cap \mathfrak{S}$ is irreducible, $\mathfrak{S} \subseteq \downarrow \mathfrak{T}$ and $\downarrow(\langle U \rangle \cap \mathfrak{T}) = \langle U \rangle \cap \downarrow \mathfrak{T}$. Then $\text{Cl}_{\mathbf{L}(\tau)}(\langle U \rangle \cap \mathfrak{T}) = \langle \bar{U} \rangle$, $\bar{U} = \overline{\bigcup(\langle U \rangle \cap \mathfrak{T})}$ and $\langle U \rangle \cap \mathfrak{T}$ is irreducible.*

Proof. By Corollary 5.1.7, $\langle U \rangle \cap \downarrow \mathfrak{T}$ is irreducible, $\text{Cl}_{\mathbf{L}(\tau)}(\langle U \rangle \cap \downarrow \mathfrak{T}) = \langle \bar{U} \rangle$ and $\bar{U} = \overline{\bigcup(\langle U \rangle \cap \downarrow \mathfrak{T})}$. By the hypothesis and Corollary 5.1.2, $\text{Cl}_{\mathbf{L}(\tau)}(\langle U \rangle \cap \mathfrak{T}) = \text{Cl}_{\mathbf{L}(\tau)}(\downarrow(\langle U \rangle \cap \mathfrak{T})) = \text{Cl}_{\mathbf{L}(\tau)}(\langle U \rangle \cap \downarrow \mathfrak{T}) = \langle \bar{U} \rangle$.

The rest follows from Theorem 5.1.6. ■

Corollary 5.1.9. *Suppose $U = \overline{\bigcup(\langle U \rangle \cap \mathfrak{S})}$, $\langle U \rangle \cap \mathfrak{S}$ is irreducible and $c(\mathfrak{S}) \subseteq c(\mathfrak{T})$. Then $\text{Cl}_{\mathbf{L}(\tau)}(\langle U \rangle \cap \mathfrak{T}) = \langle U \rangle$, $U = \overline{\bigcup(\langle U \rangle \cap \mathfrak{T})}$ and $\langle U \rangle \cap \mathfrak{T}$ is irreducible.*

Proof. By Remark 5.1.3.1, the closure of $\langle U \rangle \cap \mathfrak{T}$ is the same as the closure of $\langle U \rangle \cap (\mathfrak{S} \cup \mathfrak{T})$, which is equal to $\langle U \rangle$ by Corollary 5.1.7 (Considering that $\mathfrak{S} \subseteq \mathfrak{S} \cup \mathfrak{T}$). The rest follows from Theorem 5.1.6. ■

Corollary 5.1.10. *Suppose $U = \overline{\bigcup(\langle U \rangle \cap \mathfrak{S})}$, $\langle U \rangle \cap \mathfrak{S}$ is irreducible, $\mathfrak{S} \subseteq \downarrow c(\mathfrak{T})$ and $\downarrow(\langle U \rangle \cap c(\mathfrak{T})) = \langle U \rangle \cap \downarrow c(\mathfrak{T})$.*

Then $\text{Cl}_{\mathbf{L}(\tau)}(\langle U \rangle \cap \mathfrak{T}) = \langle \bar{U} \rangle$, $\bar{U} = \overline{\bigcup(\langle U \rangle \cap \mathfrak{T})}$ and $\langle U \rangle \cap \mathfrak{T}$ is irreducible.

Proof. Since U is closed it follows by Corollary 4.3.2 that

$\langle U \rangle \cap \downarrow c(\mathfrak{T}) = \downarrow (\langle U \rangle \cap c(\mathfrak{T})) = \downarrow c(\langle U \rangle \cap \mathfrak{T})$ which has the same closure as $\langle U \rangle \cap \mathfrak{T}$ by Corollary 5.1.2. By Corollary 5.1.7, $\text{Cl}_{\mathbf{L}(\tau)}(\langle U \rangle \cap \mathfrak{T}) = \text{Cl}_{\mathbf{L}(\tau)}(\langle U \rangle \cap \downarrow c(\mathfrak{T})) = \langle \overline{U} \rangle$. The rest of the result follows from Theorem 5.1.6. ■

In [3, Ch. I, Exercises for §2, 7)b)] it is stated that if A is dense in X then $\text{Cl}_{\mathbf{L}(\tau)}(\mathcal{F}(A)) = \mathcal{P}_0(X)$. The second point in the following Proposition 5.1.11 restates this in a different way so that one may apply the result to the study of quasi-sobriety.

Proposition 5.1.11. 1. For each $A \in \mathfrak{S}$, $\langle A \rangle \cap \mathfrak{S}$ is irreducible.

2. For each $A \subseteq X$, $\langle A \rangle$ is irreducible and $\text{Cl}_{\mathbf{L}(\tau)}(\mathcal{F}(A)) = \langle \overline{A} \rangle$.

Proof. 1. Since $A \in \langle A \rangle \cap \mathfrak{S}$, from Theorem 5.1.6 it must be the case that $\langle A \rangle \cap \mathfrak{S}$ is irreducible.

2. Note that if V_1, \dots, V_m are opens in X such that $A \in \langle X, V_1, \dots, V_m \rangle$, then there is a finite subset F of A consisting of a single element of each of V_1, \dots, V_m , hence $F \in \langle X, V_1, \dots, V_m \rangle \cap \mathcal{F}(A)$. Hence $A \in \text{Cl}_{\mathbf{L}(\tau)}(\mathcal{F}(A))$. By Theorem 5.1.6, $\text{Cl}_{\mathbf{L}(\tau)}(\mathcal{F}(A)) = \langle \overline{A} \rangle$. ■

Proposition 5.1.12. \mathfrak{S} is quasi-sober iff $c(\mathfrak{S})$ is sober.

Proof. Follows from Proposition 3.4.3 (1) and Theorem 4.3.3. ■

It is known already that $\mathcal{P}_0(X)$ is quasi-sober [32, p. 313] and that 2^X is sober [31, Proposition 2.9(1)], [23, Lemma 3.1].

Corollary 5.1.13. $\text{Cl}_{\mathbf{L}(\tau)}(\mathfrak{S})$ is always quasi-sober;

in particular $\mathcal{P}_0(X)$ is an irreducible quasi-sober space and 2^X is an irreducible sober space.

Proof. If \mathcal{A} is a non-empty, closed and irreducible subspace of $\mathcal{P}_0(X)$, then $\mathcal{A} = \text{Cl}_{\mathbf{L}(\tau)}(\{\bigcup \mathcal{A}\})$ by Proposition 5.1.4 (3). Hence, $\mathcal{P}_0(X)$ is quasi-sober; it is irreducible as $\mathcal{P}_0(X) = \text{Cl}_{\mathbf{L}(\tau)}(\{X\})$. By Proposition 3.4.2, $\text{Cl}_{\mathbf{L}(\tau)}(\mathfrak{S})$ is quasi-sober for every $\mathfrak{S} \subseteq \mathcal{P}_0(X)$. Since $2^X = c(\mathcal{P}_0(X))$ it follows by Proposition 5.1.3 that $\text{Cl}_{\mathbf{L}(\tau)}(2^X) = \mathcal{P}_0(X)$ and so 2^X is irreducible and sober by Proposition 5.1.12. \blacksquare

Theorem 5.1.14. *The following are equivalent:*

1. $\text{cell}_{\mathbf{L}(\tau)}(\mathfrak{S}) = n$.
2. *there are closed subsets U_1, \dots, U_n of X such that $U_i \not\subseteq U_j$ whenever $i \neq j$ and*

$$\text{Cl}_{\mathbf{L}(\tau)}(\mathfrak{S}) = \langle U_1 \rangle \cup \dots \cup \langle U_n \rangle.$$
3. $(\text{Cl}_{\mathbf{L}(\tau)}(\mathfrak{S}), \subseteq)$ *has exactly n maximal elements.*

Proof. 1 \implies 2: Since $\mathcal{P}_0(X)$ is a quasi-sober space it follows by Proposition 3.4.1 that there are $U_1, \dots, U_n \in \mathcal{P}_0(X)$ such that $\text{cell}_{\mathbf{L}(\tau)}(\{U_1, \dots, U_n\}) = n$ and $\text{Cl}_{\mathbf{L}(\tau)}(\mathfrak{S}) = \text{Cl}_{\mathbf{L}(\tau)}(\{U_1, \dots, U_n\}) = \bigcup_{i=1}^n \langle \overline{U_i} \rangle$. By Remark 3.3.9.1, for $i \neq j$ it must be the case that $U_i \notin \text{Cl}_{\mathbf{L}(\tau)}(\{U_j\}) = \langle \overline{U_j} \rangle$, i.e. $U_i \not\subseteq \overline{U_j}$ and hence $\overline{U_i} \not\subseteq \overline{U_j}$.

2 \implies 3: U_1, \dots, U_n are the maximal elements sought.

3 \implies 1: Let U_1, \dots, U_n be the maximal elements of $(\text{Cl}_{\mathbf{L}(\tau)}(\mathfrak{S}), \subseteq)$. For each i , $\langle \overline{U_i} \rangle = \text{Cl}_{\mathbf{L}(\tau)}(\{U_i\}) \subseteq \text{Cl}_{\mathbf{L}(\tau)}(\mathfrak{S})$, so $U_i = \overline{U_i}$ by maximality of U_i . Then $\text{Cl}_{\mathbf{L}(\tau)}(\mathfrak{S}) = \bigcup_{i=1}^n \langle U_i \rangle = \text{Cl}_{\mathbf{L}(\tau)}(\{U_1, \dots, U_n\})$ (cf. Proposition 4.2.6). For $i \neq j$, $U_i \not\subseteq U_j$ hence $U_i \notin \langle U_j \rangle = \text{Cl}_{\mathbf{L}(\tau)}(\{U_j\})$. Thus $\text{cell}(\mathfrak{S}) = \text{cell}(\{U_1, \dots, U_n\}) = n$ by Remark 3.3.9.1. \blacksquare

5.1.2 Almost d-finitely natural hyperspaces

In this section, almost d-finitely natural hyperspaces are considered. Let $U \subseteq X$.

Theorem 5.1.16 determines that $\mathcal{D}_n(U)$ is irreducible under a certain condition and verifies

its closure in $\mathbf{L}(\tau)$; this condition is shown to be essential by Example 5.1.23.1 in the next section. This example also verifies that the antecedents of Corollaries 5.1.7 and 5.1.10 respectively cannot be removed.

The previous result is used to study irreducibility and closure in subspaces of almost d -finitely natural hyperspaces in X - see Theorem 5.1.17 and Corollaries 5.1.18, 5.1.19 and 5.1.20.

It is determined that $\mathcal{F}_n(U)$ and $\mathcal{S}_{\leq n}(U)$ have the same closure in $\mathbf{L}(\tau)$ (cf. Theorem 5.1.21) and so do $\mathcal{D}_n(U)$ and $\mathcal{S}_n(U)$ have the same closure in $\mathbf{L}(\tau)$ (cf. Theorem 5.1.22). Under a certain condition $\mathcal{D}_n(U)$ and $\mathcal{F}_n(U)$ also have the same closure in $\mathbf{L}(\tau)$ (cf. Theorem 5.1.23).

Lemma 5.1.15. *If $\text{cell}(U) > n$ then $\mathcal{S}_{\leq n}(U)$ is not irreducible.*

Proof. By the hypothesis there is a family V_1, \dots, V_{n+1} of $n + 1$ opens in X such that $\{U \cap V_1, \dots, U \cap V_{n+1}\}$ is cellular in U , and hence for each i , $\langle X, V_i \rangle \cap \mathcal{S}_{\leq n}(U) \neq \emptyset$ while (cf. Corollary 4.1.2):

$$\begin{aligned} \langle X, V_1, \dots, V_{n+1} \rangle \cap \mathcal{S}_{\leq n}(U) &= \langle X, V_1, \dots, V_{n+1} \rangle \cap (\langle U \rangle \cap \mathcal{S}_{\leq n}(X)) \\ &= \langle U, U \cap V_1, \dots, U \cap V_{n+1} \rangle \cap \mathcal{S}_{\leq n}(X) \\ &= \emptyset. \end{aligned}$$

■

Example 5.1.15.1. If X has the co-finite topology then X is irreducible and every finite subset of X is discrete, so for every $n > 1$, $\bigcup \mathcal{S}_n(X) = X$ and so $\mathcal{S}_n(X)$ is irreducible (cf. Theorem 5.1.17). Thus $\mathcal{S}_n(X)$ being irreducible does not imply that $\text{cell}(X) = n$.

Theorem 5.1.16. *If every discrete subset of U with at most n elements is contained in a subset of U with cellularity n , then $U \subseteq \bigcup c(\mathcal{D}_n(U))$ and the following are equivalent:*

1. $\text{cell}(U) \leq n$.

2. $U \in \text{Cl}_{\mathbf{L}(\tau)}(\mathcal{D}_n(U))$.
3. $\text{Cl}_{\mathbf{L}(\tau)}(\mathcal{D}_n(U)) = \overline{\langle U \rangle}$.
4. $\mathcal{D}_n(U)$ is irreducible.

Proof. Let $x \in U$. By the hypothesis, $x \in G$ for some subset G of U with cellularity n . By Proposition 3.3.11, there is a discrete subset H of $\text{Cl}_U(G)$ such that $|H| = n$ and $x \in \overline{H}$. Thus

$$U \subseteq \bigcup \{ \overline{H} : H \in \mathcal{D}_n(U) \} = \bigcup c(\mathcal{D}_n(U)) \subseteq \overline{\bigcup \mathcal{D}_n(U)}.$$

Thus $\overline{U} = \overline{\bigcup \mathcal{D}_n(U)}$. Hence by Theorem 5.1.6: (2), (3) and (4) are equivalent.

1 \implies 2: Let $k := \text{cell}(U)$. Let V_1, \dots, V_m be non-empty opens in X such that $U \in \langle X, V_1, \dots, V_m \rangle$. By Lemma 4.1.5, there is a cellular family $\{W_1, \dots, W_k\}$ of opens in U such that $\langle U, W_1, \dots, W_k \rangle \subseteq \langle U, U \cap V_1, \dots, U \cap V_m \rangle = \langle U \rangle \cap \langle X, V_1, \dots, V_m \rangle$ (cf. Corollary 4.1.2 and Proposition 4.1.3).

Let F be a subset of U consisting of exactly one element from each of W_1, \dots, W_k . Then F is discrete. By the hypothesis, F is a subset of some $G \subseteq U$ with cellularity n . By Proposition 3.3.11, there is a discrete subset H of $\text{Cl}_U(G)$ such that $|H| = n$ and $F \subseteq \text{Cl}_U(H)$. Thus $H \in \langle U, W_1, \dots, W_k \rangle \cap \mathcal{D}_n(X)$, which implies that $\langle X, V_1, \dots, V_m \rangle \cap \mathcal{D}_n(U) \neq \emptyset$. Hence, $U \in \text{Cl}_{\mathbf{L}(\tau)}(\mathcal{D}_n(U))$.

2 \implies 4: Already stated above.

4 \implies 1: Since $\overline{\bigcup \mathcal{S}_{\leq n}(U)} = \overline{U} = \overline{\bigcup \mathcal{D}_n(U)}$ it follows by Corollary 5.1.7 that $\mathcal{S}_{\leq n}(U)$ is irreducible so the result follows from Lemma 5.1.15. ■

Theorem 5.1.17. *For each $k \in \mathbb{N}$ let $\mathcal{D}_k(X) \subseteq \mathfrak{S}_k \subseteq \mathcal{S}_{\leq k}(X)$.*

The following are equivalent:

1. $\text{cell}(U) \leq n$.
2. There is $k \leq n$ for which $U \in \text{Cl}_{\mathbf{L}(\tau)}(\langle U \rangle \cap \mathfrak{S}_k)$.

3. There is $k \leq n$ for which $\text{Cl}_{\mathbf{L}(\tau)}(\langle U \rangle \cap \mathfrak{S}_k) = \langle \bar{U} \rangle$.

In addition, if $\overline{\bigcup(\langle U \rangle \cap \mathfrak{S}_k)} = \bar{U}$, then these statements are equivalent to:

4. There is $k \leq n$ for which $\langle U \rangle \cap \mathfrak{S}_k$ is irreducible.

Proof. 1 \implies 2: If $k := \text{cell}(U)$ then by Theorem 5.1.16, $U \in \text{Cl}_{\mathbf{L}(\tau)}(\mathcal{D}_k(U)) \subseteq \text{Cl}_{\mathbf{L}(\tau)}(\langle U \rangle \cap \mathfrak{S}_k)$.

2 \implies 3 and 3 \implies 4 follow from Theorem 5.1.6.

3 \implies 1: Then $\text{Cl}_{\mathbf{L}(\tau)}(\mathcal{S}_{\leq k}(U)) = \langle \bar{U} \rangle$ (cf. Proposition 5.1.3), so by Theorem 5.1.6, $\mathcal{S}_{\leq k}(U)$ is irreducible and thus $\text{cell}(U) \leq k \leq n$ by Lemma 5.1.15.

Suppose now that $\overline{\bigcup(\langle U \rangle \cap \mathfrak{S}_k)} = \bar{U}$.

4 \implies 1: Then $\mathcal{S}_{\leq k}(U)$ is irreducible (cf. Corollary 5.1.7) hence by Lemma 5.1.15, $\text{cell}(U) \leq k \leq n$. ■

Corollary 5.1.18. *The following are equivalent when $|U| \geq n$ and*

$\mathcal{F}_n(X) \subseteq \mathfrak{S} \subseteq \mathcal{S}_{\leq n}(X)$:

1. $\text{cell}(U) \leq n$.

2. $U \in \text{Cl}_{\mathbf{L}(\tau)}(\langle U \rangle \cap \mathfrak{S})$.

3. $\text{Cl}_{\mathbf{L}(\tau)}(\langle U \rangle \cap \mathfrak{S}) = \langle \bar{U} \rangle$.

4. $\langle U \rangle \cap \mathfrak{S}$ is irreducible.

Proof. Follows from the previous Theorem 5.1.17 and the fact that $\bigcup(\langle U \rangle \cap \mathfrak{S}) = U$ as $\mathcal{F}_n(X) \subseteq \mathfrak{S}$ and $|U| \geq n$. ■

Corollary 5.1.19. *For each $k \in \mathbb{N}$ let $c(\mathcal{D}_k(X)) \subseteq c(\mathfrak{S}_k) \subseteq \mathcal{S}_{\leq k}(X)$.*

The following are equivalent if U is closed:

1. $\text{cell}(U) \leq n$.

2. There is $k \leq n$ for which $U \in \text{Cl}_{\mathbf{L}(\tau)}(\langle U \rangle \cap \mathfrak{S}_k)$.

3. There is $k \leq n$ for which $\text{Cl}_{\mathbf{L}(\tau)}(\langle U \rangle \cap \mathfrak{S}_k) = \langle \bar{U} \rangle$.

In addition, if $\overline{\bigcup(\langle U \rangle \cap c(\mathfrak{S}_k))} = \bar{U}$, then these statements are equivalent to:

4. There is $k \leq n$ for which $\langle U \rangle \cap \mathfrak{S}_k$ is irreducible.

Proof. For each $k \in \mathbb{N}$, it follows by Remark 5.1.3.1 that $\langle U \rangle \cap (\mathfrak{S}_k \cup \mathcal{D}_k(X))$ has the same closure as $\langle U \rangle \cap \mathfrak{S}_k$. Moreover, $\mathfrak{S}_k \subseteq \mathcal{S}_{\leq k}(X)$, thus $\mathfrak{S}_k \cup \mathcal{D}_k(X) \subseteq \mathcal{S}_{\leq k}(X)$.

1 \implies 2: By Theorem 5.1.17, there is $k \leq n$ for which $U \in \text{Cl}_{\mathbf{L}(\tau)}(\langle U \rangle \cap (\mathfrak{S}_k \cup \mathcal{D}_k(X))) = \text{Cl}_{\mathbf{L}(\tau)}(\langle U \rangle \cap \mathfrak{S}_k)$.

2 \implies 3 and 3 \implies 4 follow from Theorem 5.1.6.

3 \implies 1: Then $\text{Cl}_{\mathbf{L}(\tau)}(\langle U \rangle \cap (\mathfrak{S}_k \cup \mathcal{D}_k(X))) = \langle \bar{U} \rangle$ by the remark at the beginning of this proof, so by Theorem 5.1.17, $\text{cell}(U) \leq n$.

Suppose now that $\overline{\bigcup(\langle U \rangle \cap c(\mathfrak{S}_k))} = \bar{U}$. Then $\overline{\bigcup(\langle U \rangle \cap \mathfrak{S}_k)} = \bar{U}$ (cf. Remark 5.1.0.1).

4 \implies 1: Then $\langle U \rangle \cap (\mathfrak{S}_k \cup \mathcal{D}_k(X))$ is irreducible by the remark at the beginning of the proof, so by Theorem 5.1.17, $\text{cell}(U) \leq k \leq n$. ■

Corollary 5.1.20. *The following are equivalent when U is closed, $|U| \geq n$ and $c(\mathcal{F}_n(X)) \subseteq c(\mathfrak{S}) \subseteq \mathcal{S}_{\leq n}(X)$:*

1. $\text{cell}(U) \leq n$.
2. $U \in \text{Cl}_{\mathbf{L}(\tau)}(\langle U \rangle \cap \mathfrak{S})$.
3. $\text{Cl}_{\mathbf{L}(\tau)}(\langle U \rangle \cap \mathfrak{S}) = \langle \bar{U} \rangle$.
4. $\langle U \rangle \cap \mathfrak{S}$ is irreducible.

Proof. Follows from the previous Corollary 5.1.19 and the fact that $\bigcup(\langle U \rangle \cap c(\mathfrak{S})) = U$ as $c(\mathcal{F}_n(X)) \subseteq c(\mathfrak{S})$ and $|U| \geq n$. ■

Theorem 5.1.21. *If $|U| \geq n$ then $\text{Cl}_{\mathbf{L}(\tau)}(\mathcal{F}_n(U)) = \text{Cl}_{\mathbf{L}(\tau)}(\mathcal{S}_{\leq n}(U))$.*

Proof. For any $A \in \mathcal{S}_{\leq n}(U)$, it follows by Corollary 5.1.18 that $A \in \text{Cl}_{\mathbf{L}(\tau)}(\mathcal{F}_n(A)) \subseteq \text{Cl}_{\mathbf{L}(\tau)}(\mathcal{F}_n(U))$. ■

Theorem 5.1.22. $\text{Cl}_{\mathbf{L}(\tau)}(\mathcal{D}_n(U)) = \text{Cl}_{\mathbf{L}(\tau)}(\mathcal{S}_n(U))$.

Proof. For any $A \in \mathcal{S}_n(U)$ it follows by Theorem 5.1.16 that $A \in \text{Cl}_{\mathbf{L}(\tau)}(\mathcal{D}_n(A)) \subseteq \text{Cl}_{\mathbf{L}(\tau)}(\mathcal{D}_n(U))$. ■

Theorem 5.1.23. *If U is closed and every discrete subset of U with at most n elements is contained in a subset of U with cellularity n , then:*

$$\text{Cl}_{\mathbf{L}(\tau)}(\mathcal{D}_n(U)) = \text{Cl}_{\mathbf{L}(\tau)}(\mathcal{F}_n(U)).$$

Proof. Let $F \in \mathcal{F}_n(U)$. Then $\text{cell}(F) \leq n$ so by Theorem 5.1.16, $F \in \text{Cl}_{\mathbf{L}(\tau)}(\mathcal{D}_n(F)) \subseteq \text{Cl}_{\mathbf{L}(\tau)}(\mathcal{D}_n(U))$. Thus the result follows. ■

5.1.3 An Example

This example illustrates that there is an irreducible, T_D , non-sober space X which has a proper subset Y_2 where for $n > 2$:

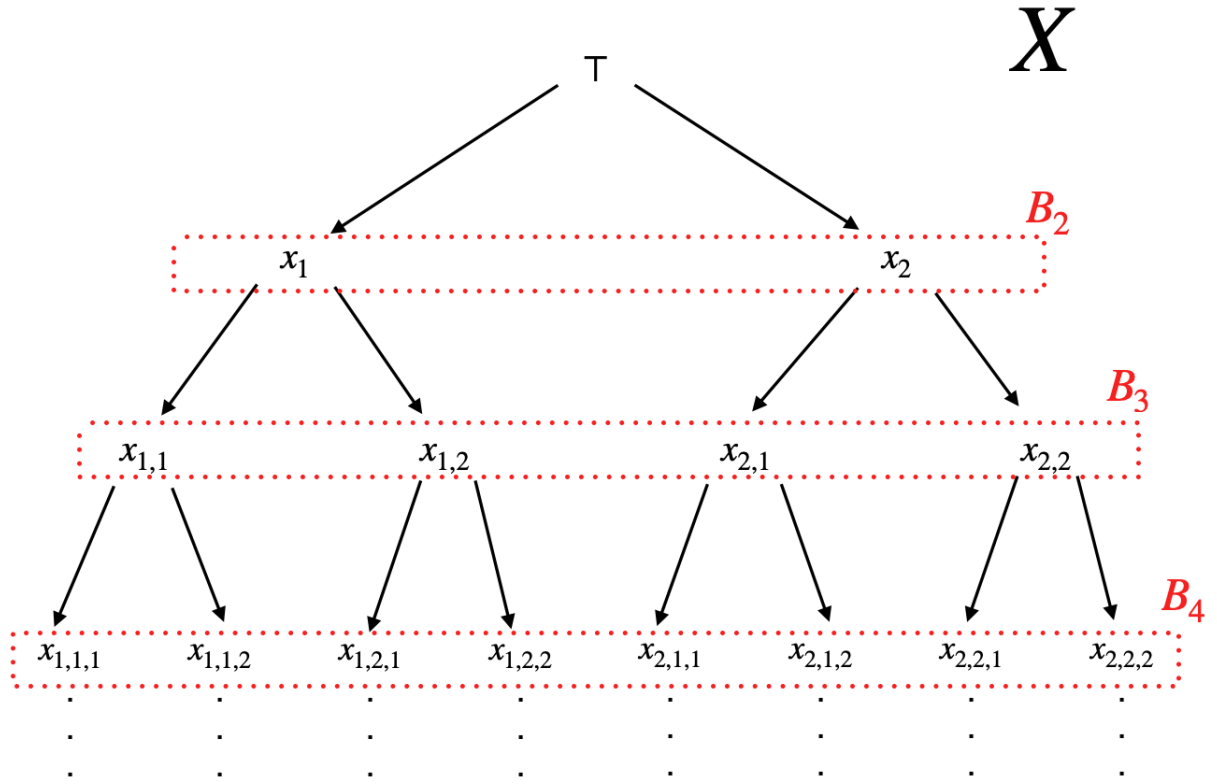
1. $\mathcal{S}_n(X) \subseteq \downarrow \mathcal{S}_2(X)$ but $\mathcal{D}_2(X) \not\subseteq \downarrow \mathcal{S}_n(X)$ as B_2 is a subset of Y_2 which is not contained in any subset of X with cellularity n (so the antecedent of Theorem 5.1.16 is not always satisfied).
2. $\mathcal{D}_n(X) = \mathcal{D}_n(Y_2)$ and $\mathcal{S}_n(X) = \mathcal{S}_n(Y_2)$.
3. If $c(\mathcal{D}_n(X)) \subseteq c(\mathfrak{S}) \subseteq \mathcal{S}_n(X)$ then \mathfrak{S} is not irreducible.
4. $\bigcup c(\mathcal{D}_n(Y_2)) = Y_2$.
5. $\text{cell}(X) = 1 < n$ and $\text{cell}(Y_2) = 2 < n$.
6. $\mathcal{S}_3(Y_3) \subset \downarrow \mathcal{S}_2(Y_3) = \downarrow c(\mathcal{S}_2(Y_3))$ and $\mathcal{S}_3(Y_3)$ is irreducible while $\mathcal{S}_2(Y_3)$ is not irreducible; moreover $\downarrow \mathcal{S}_2(Y_3) = \downarrow c(\mathcal{S}_2(Y_3)) \subset \langle Y_3 \rangle \downarrow \mathcal{S}_2(X) = \langle Y_3 \rangle \downarrow c(\mathcal{S}_2(X))$.

This shows that the conditions “ $\downarrow (\langle U \rangle \cap \mathfrak{T}) = \langle U \rangle \cap \downarrow \mathfrak{T}$ ” and “ $\downarrow (\langle U \rangle \cap c(\mathfrak{T})) = \langle U \rangle \cap \downarrow c(\mathfrak{T})$ ” in Corollaries 5.1.7 and 5.1.10 respectively cannot be removed.

Example 5.1.23.1. Let X be the poset defined as follows. Let \top be the top element of X and let $\downarrow \top \setminus \{\top\} = \downarrow \{x_1, x_2\}$. For $m \geq 1$ and $i_1, \dots, i_m \in \{1, 2\}$, let:

$$\downarrow \{x_{i_1, \dots, i_m}\} \setminus \{x_{i_1, \dots, i_m}\} = \downarrow \{x_{i_1, \dots, i_m, 1}, x_{i_1, \dots, i_m, 2}\}.$$

Then X is an infinite binary tree. Consider the upper interval topology on X . Consult the following picture for ease of visualisation:



For $2 \leq m < \omega$, let $B_m := \{x_{i_1, \dots, i_{m-1}} : i_1, \dots, i_{m-1} \in \{1, 2\}\}$, which is the m -th level of the tree. Thus $B_2 = \{x_1, x_2\}$, $B_3 = \{x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}\}$, etc. Thus $X \setminus \{\top\} = \bigcup_{2 \leq m < \omega} B_m$. Note that B_m is discrete in X and thus has cellularity m (cf. Remark 3.3.9.1). Let $Y_m := \downarrow B_m$, which also has cellularity m . Given $x \in B_m$, the following are closed in X :

$$\downarrow \{x\} \setminus \{x\} = Y_{m+1} \cap \downarrow \{x\} \quad \text{and} \quad Y_m \setminus \{x\} = \downarrow (B_m \setminus \{x\}) \cup (Y_{m+1} \cap \downarrow \{x\}).$$

Of course $\downarrow \{\top\} \setminus \{\top\} = Y_2$ is closed in X . Thus X is a T_D -space (cf. Definition 7.1.0.1).

Let $n > 2$. Let U be a subset of X with cellularity n . Then $\downarrow U \neq X$ and $\downarrow U \neq Y_2$. Thus $\top \notin U$ and either $x_1 \notin U$ or $x_2 \notin U$. Thus $U \subseteq Y_2 \setminus \{x_1\}$ or $U \subseteq Y_2 \setminus \{x_2\}$. Hence $\mathcal{S}_n(X) = \mathcal{S}_n(Y_2)$ and $\mathcal{D}_n(X) = \mathcal{D}_n(Y_2)$ (cf. Remark 3.3.9.1). Moreover B_2 is not contained in any subset of X or Y_2 with cellularity n (so the antecedent of Theorem 5.1.16 is not satisfied). Moreover:

$$\mathcal{S}_n(X) \subseteq \langle Y_2 \setminus \{x_1\} \rangle \cup \langle Y_2 \setminus \{x_2\} \rangle.$$

Now

$$\{x_1, x_{2,1}, x_{2,2,1}, \dots, x_{\substack{2, \dots, 2 \\ n-1 \text{ times}}, 1}\}$$

is a discrete subset of X with n elements and is not a subset of $Y_2 \setminus \{x_2\}$. Thus $\mathcal{D}_n(X) \not\subseteq \langle Y_2 \setminus \{x_2\} \rangle$ and $x_1 \in \bigcup \mathcal{D}_n(X)$. Similarly it is true that $\mathcal{D}_n(X) \not\subseteq \langle Y_2 \setminus \{x_1\} \rangle$ and $x_2 \in \bigcup \mathcal{D}_n(X)$. Since $Y_2 \setminus \{x_1\}, Y_2 \setminus \{x_2\}$ are closed in X , it follows that $\langle Y_2 \setminus \{x_1\} \rangle, \langle Y_2 \setminus \{x_2\} \rangle$ are closed in $\mathcal{P}_0(X)$ (cf. Proposition 4.2.4).

Thus $\mathcal{S}_n(X)$ is not irreducible. Note also that since $x_1, x_2 \in \bigcup \mathcal{D}_n(X)$ it must be the case that $Y_2 = \downarrow \{x_1, x_2\} \subseteq \bigcup c(\mathcal{D}_n(X)) \subseteq \bigcup c(\mathcal{S}_n(X))$. Since $\bigcup c(\mathcal{D}_n(X)) \subseteq \bigcup c(\mathcal{S}_n(X)) = \bigcup c(\mathcal{S}_n(Y_2)) \subseteq Y_2$ it follows that $Y_2 = \bigcup c(\mathcal{D}_n(X)) = \bigcup c(\mathcal{S}_n(X))$. Thus $\overline{Y_2} = \overline{\bigcup \mathcal{D}_n(X)} = \overline{\bigcup \mathcal{S}_n(X)}$. If $c(\mathcal{D}_n(X)) \subseteq c(\mathfrak{S}) \subseteq \mathcal{S}_n(X)$ then $\overline{\bigcup \mathfrak{S}} = \overline{\bigcup c(\mathfrak{S})} = \overline{\bigcup c(\mathcal{D}_n(X))} = \overline{\bigcup \mathcal{D}_n(X)} = \overline{\bigcup \mathcal{S}_n(X)}$ and thus by Corollary 5.1.9, \mathfrak{S} is not irreducible (this can also be verified directly in a way similar to my workings above). Note that $\text{cell}(X), \text{cell}(Y_2) < n$. I have already stated that B_2 is not contained in any subset of X with cellularity n .

Note that $\mathcal{S}_n(X) \subseteq \downarrow \mathcal{S}_2(X)$ (since I showed that any subset U of X with cellularity n is a subset of Y_2 which has cellularity 2) but $\mathcal{D}_2(X) \not\subseteq \downarrow \mathcal{S}_n(X)$ (as B_2 is not contained in any subset of X with cellularity n).

Note that $\mathcal{S}_3(Y_3) = \langle Y_3 \rangle \cap \mathcal{S}_3(Y_3) = \text{Cl}_{\mathbf{L}(\tau)|_{\mathcal{S}_3(Y_3)}}(\{Y_3\})$ is irreducible. However for any $y, z \in B_3$, $\mathcal{S}_2(Y_3) \not\subseteq \langle \downarrow \{y, z\} \rangle$ (as $B_3 \setminus \{y, z\}$ is an element of the former but not the

latter) while:

$$\mathcal{S}_2(Y_3) \subseteq \bigcup \{ \langle \downarrow \{y, z\} \rangle : y, z \in B_3 \}.$$

Thus $\mathcal{S}_2(Y_3)$ is not irreducible. Moreover $\mathcal{S}_3(Y_3) \subseteq \downarrow \mathcal{S}_2(Y_3) = \downarrow c(\mathcal{S}_2(Y_3))$ (the latter equation is due to the fact that $c(\mathcal{S}_2(Y_3)) \subseteq \mathcal{S}_2(Y_3)$). Note also that $\downarrow c(\mathcal{S}_2(Y_3)) = \downarrow \mathcal{S}_2(Y_3) \subset \langle Y_3 \rangle = \langle Y_3 \rangle \cap \downarrow \mathcal{S}_2(X) = \langle Y_3 \rangle \cap \downarrow c(\mathcal{S}_2(X))$.

Consider the following subspace of X :

$$K := \{x_1, x_{2,1}, x_{2,2,1}, x_{2,2,2,1}, \dots\}$$

For any finite subset F of X , $K \not\subseteq \downarrow F$ and hence $\overline{K} \not\subseteq \downarrow F$. Thus by Corollary 3.3.3, \overline{K} is an irreducible closed subspace of X which is not the closure of any point in X . Thus X is not sober.

5.2 Quasi-Sobriety

Theorem 5.2.1 characterises in general when a hyperspace is quasi-sober. In Section 5.2.2, a necessary condition for an almost natural hyperspace to be quasi-sober is obtained (cf. Theorem 5.2.2) and quasi-sobriety is characterised in all almost finitely natural hyperspaces (cf. Theorem 5.2.3). Corollary 5.2.4 verifies that $\mathcal{S}_{\leq n}(X)$ is always quasi-sober for any $n \in \mathbb{N}$, generalising the well-known notion of the sobrification of a space (cf. Corollary 5.2.5, Remark 5.2.5.1 and Definition 5.2.5.1). In Theorem 5.2.6 the properties of $\mathcal{S}_{\leq n}(X)$ are explored further when $n > 1$ and it is shown that it is T_0 iff X is sober and T_1 . The main result is that if $c(\mathcal{F}_n(X)) \subseteq c(\mathfrak{S}) \subseteq c(\mathcal{F}_n(X))$, then \mathfrak{S} is quasi-sober iff X is quasi-sober (cf. Theorem 5.2.7 and Corollary 5.2.8).

In section 5.2.3, the Sub-base Condition is defined as a generalisation of the fact that $\langle A \rangle$ is irreducible for every closed set A in X - cf. [23, Proposition 3.2] and [32, p. 313-314] (where the latter only states the forward direction). A general characterisation is obtained for when a hyperspace is quasi-sober and satisfies the Sub-base Condition (cf.

Theorem 5.2.11). It is also seen that the Sub-base Condition is somewhat upwards-closed in hyperspaces (cf. Proposition 5.2.10). In specific cases, better characterisations are obtained (cf. Corollary 5.2.12, Theorem 5.2.15, Corollary 5.2.16) and it is determined that $\mathcal{P}_0(X)$, 2^X , $\mathcal{C}(X)$, $\mathcal{F}(X)$ and $\bigcup_{i=n}^{\infty} \mathcal{F}_i(X)$ satisfy the Sub-base Condition for every n (cf. Corollary 5.2.14). It is determined when $\mathcal{C}(X)$ and $\mathcal{F}(X)$ are quasi-sober (cf. Corollary 5.2.17).

5.2.1 General Results

Recall Proposition 5.1.12 which states that \mathfrak{S} is quasi-sober iff $c(\mathfrak{S})$ is sober. The following characterises when \mathfrak{S} is quasi-sober.

Theorem 5.2.1. *The following are equivalent:*

1. \mathfrak{S} is quasi-sober.
2. for every $U \subseteq X$ where $\langle U \rangle \cap \mathfrak{S}$ is irreducible and non-empty, it must be the case that $\overline{\bigcup \langle U \rangle \cap \mathfrak{S}} \in c(\mathfrak{S})$.
3. for every closed U in X where $\langle U \rangle \cap c(\mathfrak{S})$ is irreducible and non-empty, it must be the case that $\overline{\bigcup \langle U \rangle \cap c(\mathfrak{S})} \in c(\mathfrak{S})$.

Proof. 1 \implies 2: Let $U \subseteq X$ where $\langle U \rangle \cap \mathfrak{S}$ is irreducible and non-empty. Then $\bigcup \langle U \rangle \cap \mathfrak{S} \subseteq \bigcup \overline{\langle U \rangle \cap \mathfrak{S}} \cap \mathfrak{S} \subseteq \overline{\bigcup \langle U \rangle \cap \mathfrak{S}}$. Thus $\overline{\bigcup \langle U \rangle \cap \mathfrak{S}} \cap \mathfrak{S} = \overline{\bigcup \langle U \rangle \cap \mathfrak{S}}$. By Corollary 5.1.5, $\overline{\langle U \rangle \cap \mathfrak{S}} \cap \mathfrak{S}$ is non-empty, irreducible and is closed in \mathfrak{S} (cf. Proposition 4.2.4). Since \mathfrak{S} is quasi-sober, $\overline{\langle U \rangle \cap \mathfrak{S}} \cap \mathfrak{S} = \text{Cl}_{\mathbf{L}(\tau)|\mathfrak{S}}(\{A\}) = \langle \bar{A} \rangle \cap \mathfrak{S}$ for some $A \in \mathfrak{S}$ (cf. Proposition 4.2.6). Hence $\overline{\bigcup \langle U \rangle \cap \mathfrak{S}} = \overline{\bigcup \langle \bar{A} \rangle \cap \mathfrak{S}} = \bar{A} \in c(\mathfrak{S})$.

2 \implies 3: Let U be closed in X such that $\langle U \rangle \cap c(\mathfrak{S})$ is irreducible and non-empty. By Remark 5.1.3.1, $\langle U \rangle \cap \mathfrak{S}$ is irreducible and non-empty. By Remark 5.1.0.1 and the hypothesis, $\overline{\bigcup \langle U \rangle \cap c(\mathfrak{S})} = \overline{\bigcup \langle U \rangle \cap \mathfrak{S}} \in c(\mathfrak{S})$.

3 \implies 1: Let \mathcal{A} be non-empty irreducible and closed in $c(\mathfrak{S})$. Then $\mathcal{A} = \langle \overline{\bigcup \mathcal{A}} \rangle \cap c(\mathfrak{S})$ (cf. Proposition 5.1.4). Thus $\bigcup \mathcal{A} = \bigcup \langle \overline{\bigcup \mathcal{A}} \rangle \cap c(\mathfrak{S}) \neq \emptyset$. By the hypothesis, $\overline{\bigcup \mathcal{A}} = \overline{\bigcup \langle \overline{\bigcup \mathcal{A}} \rangle \cap c(\mathfrak{S})} \in c(\mathfrak{S})$. Thus $\text{Cl}_{\mathbf{L}(\tau)|c(\mathfrak{S})}(\{\overline{\bigcup \mathcal{A}}\}) = \langle \overline{\bigcup \mathcal{A}} \rangle \cap c(\mathfrak{S}) = \mathcal{A}$. Thus $c(\mathfrak{S})$ is quasi-sober; since it is also T_0 (cf. Corollary 4.3.5), it follows that $c(\mathfrak{S})$ is sober. By Proposition 5.1.12, \mathfrak{S} is quasi-sober. \blacksquare

5.2.2 Almost Natural Hyperspaces

Theorem 5.2.2. *Suppose $c(\mathcal{F}_n(X)) \subseteq c(\mathfrak{S})$.*

If \mathfrak{S} is quasi-sober then whenever U is closed in X such that $|U| \geq n$ and $\text{cell}(U) \leq n$, it must be the case that $U \in c(\mathfrak{S})$.

Proof. Let U be closed in X such that $|U| \geq n$ and $\text{cell}(U) \leq n$. By Corollary 5.1.20, $U = \overline{\bigcup \langle U \rangle \cap \mathfrak{S}}$ and $\langle U \rangle \cap \mathfrak{S}$ is irreducible. By Theorem 5.2.1, $U \in c(\mathfrak{S})$. \blacksquare

Theorem 5.2.3. *Suppose $c(\mathcal{F}_n(X)) \subseteq c(\mathfrak{S}) \subseteq \mathcal{S}_{\leq n}(X)$.*

Then \mathfrak{S} is quasi-sober iff whenever U is closed in X such that $|U| \geq n$ and $\text{cell}(U) \leq n$, it must be the case that $U \in c(\mathfrak{S})$.

Proof. \implies : Follows from the previous Theorem 5.2.2.

\impliedby : Let U be closed in X such that $\langle U \rangle \cap c(\mathfrak{S})$ is non-empty and irreducible. Then $|U| \geq n$ and $\langle U \rangle \cap \mathfrak{S}$ is irreducible (cf. Remark 5.1.3.1). By Corollary 5.1.20, $\text{cell}(U) \leq n$ and so by the hypothesis, $\overline{\bigcup \langle U \rangle \cap c(\mathfrak{S})} = U \in c(\mathfrak{S})$. By Theorem 5.2.1, \mathfrak{S} is quasi-sober. \blacksquare

Corollary 5.2.4. *$\mathcal{S}_{\leq n}(X)$ is quasi-sober.*

Proof. Follows from the previous Theorem 5.2.3 and the fact that $c(\mathcal{S}_{\leq n}(X)) \subseteq \mathcal{S}_{\leq n}(X)$. \blacksquare

Corollary 5.2.5. 1. *$\mathcal{S}_1(X)$ is quasi-sober.*

2. X is quasi-sober iff X is homeomorphic to $\mathcal{S}_1(X)$.

3. [25, 4.2.1] X is sober iff X is homeomorphic to $c(\mathcal{S}_1(X))$.

Remark 5.2.5.1. Corollary 5.2.5 is a well-known result (see [25, 4.2.1], [21, Exercise V-4.9], [22, Corollary 8.2.23, Fact 8.2.24]).

The following Definition 5.2.5.1 is the definition of the sobrification of a space given by Grothendieck and Verdier [25, 4.2.1]. In their definition, the Lower Vietoris Topology on $c(\mathcal{S}_1(X))$ was used without being named. The same definition is given in [22, Definition 8.2.17], where it is acknowledged that the topology used was indeed the Lower Vietoris Topology. Other definitions for the sobrification exist; for instance, see [21, Remark for Proposition V-4.7 and Exercise V-4.9].

Definition 5.2.5.1. [25, 4.2.1] The *sobrification* of X is $c(\mathcal{S}_1(X))$.

Theorem 5.2.6. *The following are equivalent for $n > 1$:*

1. $\mathcal{S}_{\leq n}(X)$ is T_0 .
2. $\mathcal{S}_{\leq n}(X)$ is sober.
3. X is sober and T_1 .
4. For $m \leq n$, $\mathcal{F}_m(X) = \mathcal{S}_m(X)$.

Proof. 1 \implies 2: Follows from Corollary 5.2.4.

2 \implies 3: Since $\mathcal{F}_{\leq 2}(X) \subseteq \mathcal{S}_{\leq n}(X)$ it follows that $\mathcal{F}_{\leq 2}(X)$ is T_0 and hence by Corollary 4.4.2 (3), X is T_1 . Let U be a non-empty irreducible closed subset of X . Suppose to the contrary that $U \neq \{x\}$ for any $x \in U$. For any $x \in U$, $U \setminus \{x\}$ is open in U and so $\overline{U \setminus \{x\}} = U$ (as U is irreducible). Thus $\overline{U \setminus \{x\}}$ is irreducible so $U \setminus \{x\} \in \mathcal{S}_{\leq n}(X)$. Since $\mathcal{S}_{\leq n}(X)$ is T_0 by the hypothesis, there is a contradiction to Theorem 4.4.1 as $U, U \setminus \{x\}$ are distinct members of $\mathcal{S}_{\leq n}(X)$ with the same closure in X . Thus $U = \{x\}$ for some $x \in U$. Hence X is a sober space.

3 \implies 4: In this case if $U \subseteq X$ such that $\text{cell}(U) = n$, then $\overline{U} = \overline{F} = F$ for some finite subset F of U where $|F| = n$ (cf. Proposition 3.4.1). Thus, $U = F$. Hence for $m \leq n$, $\mathcal{S}_m(X) = \mathcal{F}_m(X)$.

4 \implies 1: Let $A \in \mathcal{S}_{\leq n}(X)$. Then there is $m \leq n$ where $A \in \mathcal{S}_m(X) = \mathcal{F}_m(X)$ so $|A| = m$. Then $\overline{A} \in \mathcal{S}_m(X) = \mathcal{F}_m(X)$ so $|\overline{A}| = m = |A|$. Thus $\overline{A} = A$ since they are both finite. Thus $\mathcal{S}_{\leq n}(X) = c(\mathcal{S}_{\leq n}(X))$ is T_0 by Corollary 4.3.5 (1). \blacksquare

Theorem 5.2.7. *Suppose $c(\mathcal{F}_n(X)) \subseteq c(\mathfrak{S}) \subseteq c(\mathcal{F}_{\leq n}(X))$.*

Then \mathfrak{S} is quasi-sober iff X is quasi-sober.

Proof. \implies : Let A be an irreducible closed non-empty subset of X . If A is finite then $A = \overline{\{x\}}$ for some $x \in A$. Suppose that A is infinite. By Theorem 5.2.3 and the hypothesis, $A \in c(\mathfrak{S})$. By the hypothesis there is $B \in \mathcal{F}_{\leq n}(X)$ where $A = \overline{B}$. Then B is irreducible so $A = \overline{B} = \overline{\{x\}}$ for some $x \in B$.

Thus X is quasi-sober.

\impliedby : Let U be closed in X such that $|U| \geq n$ and $\text{cell}(U) \leq n$. By Proposition 3.4.1, $U = \overline{F}$ for some subset F of U with $|F| \leq n$. Thus $U = \overline{G}$ for some n -element subset G of U ; thus $U \in c(\mathcal{F}_n(X)) \subseteq c(\mathfrak{S})$.

By Theorem 5.2.3, \mathfrak{S} is quasi-sober. \blacksquare

Corollary 5.2.8. *If there is n such that one of $\mathcal{F}_n(X)$, $\mathcal{F}_{\leq n}(X)$ or X is quasi-sober, then they are all quasi-sober.*

5.2.3 The Sub-base Condition

It is already known that a closed subspace \mathcal{A} of $\mathcal{P}_0(X)$ is irreducible iff \mathcal{A} is an element of the sub-base for the closed subsets of $\mathcal{P}_0(X)$ as described in Proposition 4.2.4 - cf. [23, Proposition 3.2] and [32, p. 313-314] (where the latter only states the forward direction).

Thus the following definition is made to study such hyperspaces.

The Sub-base Condition. *For every closed subset V of X , $\langle V \rangle \cap \mathfrak{S}$ is an irreducible subspace of \mathfrak{S} .*

Remark 5.2.8.1. If \mathfrak{S} satisfies the Sub-base Condition, then by Proposition 5.1.4, a closed subspace \mathcal{A} of \mathfrak{S} is irreducible iff there is a closed subset V of X such that $\mathcal{A} = \langle V \rangle \cap \mathfrak{S}$. This is true iff \mathcal{A} is an element of the sub-base for the closed subsets of \mathfrak{S} as described in Proposition 4.2.4. In other words, every element of the aforementioned sub-base for the closed subsets of \mathfrak{S} is irreducible.

Proposition 5.2.9. *\mathfrak{S} satisfies the Sub-base Condition iff $c(\mathfrak{S})$ satisfies the Sub-base Condition.*

Proof. Given a closed subset V of X , it follows by Proposition 5.1.3 that $\langle V \rangle \cap \mathfrak{S}$ is irreducible iff $c(\langle V \rangle \cap \mathfrak{S}) = \langle V \rangle \cap c(\mathfrak{S})$ is irreducible (cf. Corollary 4.3.2). Thus the result follows. ■

Corollary 5.2.10. *If \mathfrak{S} satisfies the Sub-base Condition, $U = \overline{\langle U \rangle \cap \mathfrak{S}}$ for any closed set U in X , and $c(\mathfrak{S}) \subseteq c(\mathfrak{T})$, then \mathfrak{T} satisfies the Sub-base Condition.*

Proof. Let U be closed in X . By the hypothesis, $\langle U \rangle \cap \mathfrak{S}$ is irreducible and is also closed in \mathfrak{S} (cf. Proposition 4.2.4). By Corollary 5.1.9, $\langle U \rangle \cap \mathfrak{T}$ is irreducible. The result follows. ■

Remark 5.2.10.1. If $U \subseteq X$ then:

$$\langle U \rangle \cap \mathfrak{S} = \langle \bigcup \langle U \rangle \cap \mathfrak{S} \rangle \cap \mathfrak{S}.$$

Now if U is closed in X :

$$\langle U \rangle \cap \mathfrak{S} = \overline{\langle \bigcup \langle U \rangle \cap \mathfrak{S} \rangle} \cap \mathfrak{S}.$$

Theorem 5.2.11. *The following are equivalent:*

1. \mathfrak{S} is quasi-sober and satisfies the Sub-base Condition.

2. for every $U \subseteq X$ where $\bigcup \langle U \rangle \cap \mathfrak{S} \neq \emptyset$, it must be the case that $\overline{\bigcup \langle U \rangle \cap \mathfrak{S}} \in c(\mathfrak{S})$.

3. for every closed U in X where $\bigcup \langle U \rangle \cap c(\mathfrak{S}) \neq \emptyset$, it must be the case that $\overline{\bigcup \langle U \rangle \cap c(\mathfrak{S})} \in c(\mathfrak{S})$.

Proof. 1 \implies 2: Let $U \subseteq X$ where $\bigcup \langle U \rangle \cap \mathfrak{S} \neq \emptyset$. Then $\overline{\langle \bigcup \langle U \rangle \cap \mathfrak{S} \rangle} \cap \mathfrak{S}$ is non-empty and irreducible (by the hypothesis). By Theorem 5.2.1, $\overline{\langle \bigcup \langle U \rangle \cap \mathfrak{S} \rangle} \in c(\mathfrak{S})$.

2 \implies 3: Let U be closed in X such that $\bigcup \langle U \rangle \cap c(\mathfrak{S}) \neq \emptyset$. Then $\bigcup \langle U \rangle \cap \mathfrak{S} \neq \emptyset$ and so by Remark 5.1.0.1 and the hypothesis, $\overline{\bigcup \langle U \rangle \cap c(\mathfrak{S})} = \overline{\bigcup \langle U \rangle \cap \mathfrak{S}} \in c(\mathfrak{S})$.

3 \implies 1: Let U be closed in X . If $\langle U \rangle \cap c(\mathfrak{S}) = \emptyset$ then it is trivially irreducible. Suppose $\langle U \rangle \cap c(\mathfrak{S}) \neq \emptyset$. Then $\bigcup \langle U \rangle \cap c(\mathfrak{S}) \neq \emptyset$. By the hypothesis $\overline{\bigcup \langle U \rangle \cap c(\mathfrak{S})} \in c(\mathfrak{S})$ and thus $\langle U \rangle \cap c(\mathfrak{S}) = \overline{\langle \bigcup \langle U \rangle \cap c(\mathfrak{S}) \rangle} \cap c(\mathfrak{S}) = \text{Cl}_{\mathbf{L}(\tau)|c(\mathfrak{S})}(\{\overline{\bigcup \langle U \rangle \cap c(\mathfrak{S})}\})$ (cf. Proposition 4.2.6 and Remark 5.2.10.1). Hence $\langle U \rangle \cap c(\mathfrak{S})$ is irreducible in all cases and thus $c(\mathfrak{S})$ satisfies the Sub-base Condition. By Proposition 5.2.9, \mathfrak{S} satisfies the Sub-base Condition.

By Theorem 5.2.1, \mathfrak{S} is quasi-sober. ■

Corollary 5.2.12. *Suppose $c(\mathcal{F}_n(X)) \subseteq c(\mathfrak{S})$. Then the following are equivalent:*

1. \mathfrak{S} is quasi-sober and \mathfrak{S} satisfies the Sub-base Condition.

2. Whenever U is closed in X and $|U| \geq n$, $U \in c(\mathfrak{S})$.

Proof. Whenever U is closed in X and $|U| \geq n$, $\overline{\bigcup \langle U \rangle \cap c(\mathfrak{S})} = U \neq \emptyset$ as $c(\mathcal{F}_n(X)) \subseteq c(\mathfrak{S})$ by the hypothesis. Thus the result follows from the equivalence of 1 and 3 in Theorem 5.2.11. ■

Proposition 5.2.13. *If $c(\mathfrak{S})$ is closed under non-empty finite unions, then \mathfrak{S} satisfies the Sub-base Condition.*

Proof. Let V be closed in X . For any $A, B \in \langle V \rangle \cap c(\mathfrak{S})$, $A \cup B \in \langle V \rangle \cap c(\mathfrak{S})$. Thus $\langle V \rangle \cap c(\mathfrak{S})$ is directed with respect to $\subseteq = \leq_{\mathbf{L}(\tau)|c(\mathfrak{S})}$ (recall Definition 2.2.1.2 and Corollary

4.3.5). By Proposition 3.3.1, $\langle V \rangle \cap c(\mathfrak{S})$ is irreducible. Thus $c(\mathfrak{S})$ satisfies the Sub-base Condition. By Proposition 5.2.9, the result follows. \blacksquare

As explained in the beginning of this section, the following Corollary is well-known for $\mathcal{P}_0(X)$:

Corollary 5.2.14. $\mathcal{C}(X), \mathcal{F}(X), \mathcal{P}_0(X), 2^X$ all satisfy the Sub-base Condition. For every $n \in \mathbb{N}$, $\bigcup_{i=n}^{\infty} \mathcal{F}_i(X)$ satisfies the Sub-base Condition.

Example 5.2.14.1. It may be the case that $c(\mathfrak{S})$ is closed under non-empty finite unions, while \mathfrak{S} is not closed under non-empty finite unions (The converse is trivially true).

Consider \mathbb{R} with the usual topology. Then $2^{\mathbb{R}} \cup \{(0, 1)\}$ is not closed under finite unions as $(0, 1), \{0\} \in 2^{\mathbb{R}} \cup \{(0, 1)\}$ but $(0, 1) \cup \{0\} = [0, 1] \notin 2^{\mathbb{R}} \cup \{(0, 1)\}$. However $c(2^{\mathbb{R}} \cup \{(0, 1)\}) = 2^{\mathbb{R}}$ is clearly closed under non-empty finite unions.

In general the smallest hyperspace which is closed under non-empty finite unions and contains $c(\mathcal{F}_n(X))$ is $c(\bigcup_{i=n}^{\infty} \mathcal{F}_i(X))$.

Theorem 5.2.15. Suppose $c(\bigcup_{i=n}^{\infty} \mathcal{F}_i(X)) \subseteq c(\mathfrak{S})$. Then \mathfrak{S} satisfies the Sub-base Condition. Moreover, \mathfrak{S} is quasi-sober iff every closed subset U of X for which $|U| \geq n$ satisfies $U \in c(\mathfrak{S})$.

Proof. Since $\bigcup_{i=n}^{\infty} \mathcal{F}_i(X)$ satisfies the Sub-base Condition (cf. Corollary 5.2.14) it follows by Corollary 5.2.10 that \mathfrak{S} also satisfies the Sub-base Condition. Thus the result follows from Corollary 5.2.12. \blacksquare

Corollary 5.2.16. Suppose $c(\mathcal{F}(X)) \subseteq c(\mathfrak{S})$. Then \mathfrak{S} satisfies the Sub-base Condition. Moreover, \mathfrak{S} is quasi-sober iff $c(\mathfrak{S}) = 2^X$.

Corollary 5.2.17. The following are true:

1. $\mathcal{C}(X)$ is quasi-sober iff every closed subset of X is the closure of some compact subspace of X ; this is true if X is compact.

2. $\mathcal{F}(X)$ is quasi-sober iff X is quasi-sober and every subset of X has finite cellularity.

Proof. The first statement follows directly from the previous Corollary. Only the second statement will be proved.

\implies : Let $U \subseteq X$. By Corollary 5.2.16 there is a finite subset F of X where $\overline{U} = \overline{F}$, so $\text{cell}(U) = \text{cell}(F) \leq |F|$. Moreover every non-empty irreducible closed set A is the closure of some finite subspace of X so $A = \overline{\{x\}}$ for some $x \in A$. Thus X is quasi-sober.

\impliedby : Let U be a closed subset of X . Then U has finite cellularity and X is quasi-sober by the hypothesis, so by Proposition 3.4.1, $U = \overline{F}$ for some finite subset F of U . Thus $\mathcal{F}(X)$ is quasi-sober by Corollary 5.2.16. ■

Chapter 6

Infinite Cellularity in the Lower Vietoris Topology

For this Chapter, $n > 1$ unless specified otherwise.

In this Chapter, new results are proved which generalise or extend some results from the literature (cf. Section 6.1) - all results are new unless otherwise stated. In particular, the maps $\hat{j}_n : X^{[n]} \rightarrow \mathcal{F}_n(X)$ and $\hat{i}_n : X^n \rightarrow \mathcal{F}_{\leq n}(X)$ are studied (cf. Definition 6.0.0.1) to provide insights into the cellularity of all almost d-finitely natural hyperspaces, as well as to determine when the Lower Vietoris Topology on $\mathcal{F}_n(X), \mathcal{F}_{\leq n}(X)$ coincides with the Vietoris Topology.

- A complete characterisation is given for \hat{j}_n to be a quasi-open map in terms of the properties of X (cf. Theorem 6.2.5). For almost all spaces, there are two equivalent characterisations of when \hat{j}_n is a quasi-open map; one is “every non-discrete open subspace of X has cellularity at least n ”, while the other provides useful insights.
- If $\text{cell}(X)$ is infinite and every non-discrete open subspace of X has cellularity at least n , then any almost d-finitely natural hyperspace in X has the same cellularity as $\text{cell}(X^{[n]}) = \text{cell}(X^n)$ (cf. Theorem 6.3.1); this is a generalised version of an

analogous result of Fedorchuk for the hyperspaces $\mathcal{F}_n(X)$, $\mathcal{F}_{\leq n}(X)$ with the Vietoris topology (cf. Proposition 6.1.1).

- The implications of Theorem 6.3.1 are investigated under different set theoretical assumptions (cf. Corollaries 6.3.2, 6.3.3 and 6.3.4).
- A complete characterisation of when \hat{i}_n is a quotient map is given, which is surprisingly equivalent to the Lower Vietoris Topology on $\mathcal{F}_{\leq n}(X)$ being equal to the Vietoris Topology (cf. Theorem 6.4.1).
- Necessary conditions are provided for the Lower Vietoris Topology to be equal to the Vietoris Topology on $\mathcal{F}_n(X)$ (cf. Theorem 6.4.2).
- A complete characterisation of when \hat{j}_n is a quotient map (under certain conditions on X) which is surprisingly equivalent to the Lower Vietoris Topology on $\mathcal{F}_n(X)$ being equal to the Vietoris Topology, equivalent to X being Hausdorff and equivalent to \hat{j}_n being a perfect covering map (cf. Theorem 6.4.3). This generalises a result by Costantini et al. (cf. Proposition 6.1.2).
- An example is provided to comment on the results in this section (cf. Example 6.5.0.1).

Definition 6.0.0.1. Let $n > 1$ and $\hat{i}_n : X^n \rightarrow \mathcal{F}_{\leq n}(X)$ be defined by

$$\hat{i}_n(x_1, \dots, x_n) = \{x_1, \dots, x_n\} \text{ for any } x_1, \dots, x_n \in X.$$

Note that $\hat{i}_n^{-1}(\mathcal{F}_n(X)) = X^{[n]}$. Let $\hat{j}_n := \hat{i}_n|_{X^{[n]}}$; again the subscript n is dropped unless more clarification is needed.

Let Δ be the set of permutations on $\{1, \dots, n\}$.

Given $x_1, \dots, x_n \in X$, $\hat{i}_n^{-1}(\{x_1, \dots, x_n\})$ has at most n^n elements.

If x_1, \dots, x_n are pairwise distinct then $\hat{j}_n^{-1}(\{x_1, \dots, x_n\}) = \{(x_{\delta(1)}, \dots, x_{\delta(n)}) : \delta \in \Delta\}$ has $n!$ elements.

Remark 6.0.0.1. It is easy to verify the following:

1. $\hat{i}_n^{-1}(\langle V \rangle) = V^n$.
2. $\hat{i}_n^{-1}(\langle X, V \rangle) = \bigcup_{j=1}^n \pi_j^{-1}(V)$ and for any $1 \leq j \leq n$,
 $\hat{i}_n(\pi_j^{-1}(V)) = \langle X, V \rangle \cap \mathcal{F}_{\leq n}(X)$.
- 3.

$$\begin{aligned} \hat{i}_n(V_1 \times \cdots \times V_n) &= \{\{x_1, \dots, x_n\} : x_1 \in V_1, \dots, x_n \in V_n\} \\ &\subseteq \langle V_1, \dots, V_n \rangle \cap \mathcal{F}_{\leq n}(X), \text{ and} \end{aligned}$$

$$\begin{aligned} \hat{j}_n(V_1 \times \cdots \times V_n \cap X^{[n]}) &= \{\{x_1, \dots, x_n\} \in \mathcal{F}_n(X) : x_1 \in V_1, \dots, x_n \in V_n\} \\ &\subseteq \langle V_1, \dots, V_n \rangle \cap \mathcal{F}_n(X). \end{aligned}$$

Hence:

$$\begin{aligned} \hat{i}_n^{-1}(\hat{i}_n(V_1 \times \cdots \times V_n)) &= \bigcup_{\delta \in \Delta} V_{\delta(1)} \times \cdots \times V_{\delta(n)}, \text{ and} \\ \hat{j}_n^{-1}(\hat{j}_n(V_1 \times \cdots \times V_n \cap X^{[n]})) &= \bigcup_{\delta \in \Delta} V_{\delta(1)} \times \cdots \times V_{\delta(n)} \cap X^{[n]}. \end{aligned}$$

Note that if $x_1 \in \bigcap_{i=1}^n V_i$ but $x_2, \dots, x_n \in V_1 \setminus \bigcup_{i=2}^n V_i$, where x_1, \dots, x_n are pairwise distinct then $\{x_1, \dots, x_n\} \in \langle V_1, \dots, V_n \rangle \cap \mathcal{F}_n(X)$ but $\{x_1, \dots, x_n\} \notin \hat{i}_n(\bigcup_{\delta \in \Delta} V_{\delta(1)} \times \cdots \times V_{\delta(n)})$. This means that both inequalities in this point may be strict.

4. If V_1, \dots, V_n are pairwise disjoint, then:

$$\langle V_1, \dots, V_n \rangle \cap \mathcal{F}_n(X) = \{\{x_1, \dots, x_n\} \in \mathcal{F}_n(X) : x_1 \in V_1, \dots, x_n \in V_n\}$$

and so by the previous point:

$$\hat{j}_n^{-1}(\langle V_1, \dots, V_n \rangle \cap \mathcal{F}_n(X)) = V_1 \times \cdots \times V_n \cap X^{[n]}.$$

6.1 Results from the Literature

A generalised version of an analogue of the following Proposition 6.1.1 (a result for the Vietoris Topology) by Fedorchuk will be proved for the Lower Vietoris Topology in The-

orem 6.2.5 and Theorem 6.3.1. A result by Costantini (cf. Proposition 6.1.2) will be generalised in Theorem 6.4.3.

Proposition 6.1.1. [15, Proposition 1.11] *Suppose $\mathcal{F}_n(X)$ and $\mathcal{F}_{\leq n}(X)$ have the Vietoris Topology and X is Hausdorff. Then for all $n > 1$:*

1. \hat{j}_n is an open map and
2. $\text{cell}(\mathcal{F}_{\leq n}(X)) = \text{cell}(X^n) = \text{cell}(X^{[n]}) = \text{cell}(\mathcal{F}_n(X))$.

Definition 6.1.1.1. [57, 9K] A continuous map $f : X \rightarrow Y$ is said to be a *local homeomorphism* if whenever $x \in X$ there is an open neighbourhood U of x such that $f(U)$ is open and $f|U : U \rightarrow f(U)$ is a homeomorphism.

In [7, p. 252, Proposition 1.7], it is proved that if X is Hausdorff then \hat{j}_n is a local homeomorphism for all $n \in \mathbb{N}$. However, from their proof it is clear that these maps actually satisfy the stronger property of being a covering map:

Definition 6.1.1.2. [57, 9K] A continuous map $f : X \rightarrow Y$ is said to be a (*k-fold covering map*) if for all $y \in Y$ there is an open neighbourhood U of y such that there is a cellular family \mathcal{V} in X so that $f^{-1}(U) = \bigcup \mathcal{V}$ and for all $V \in \mathcal{V}$, the restriction $f|V : V \rightarrow U$ is a homeomorphism. If, in addition, there is $k \in \mathbb{N}$ such that the fibers of f each have exactly k elements, then f is said to be a *k-fold covering map* [45, p. 341].

Proposition 6.1.2. [7, p. 252, Proposition 1.7]

If X is Hausdorff then:

1. \hat{j}_n is a $n!$ -fold covering map for all $n > 1$.
2. $\{\langle V_1, \dots, V_n \rangle \cap \mathcal{F}_n(X) : V_1, \dots, V_n \text{ cellular in } X\}$ is a base for $\mathcal{F}_n(X)$.
3. $\{V_1 \times \dots \times V_n : V_1, \dots, V_n \text{ cellular in } X\}$ is a base for $X^{[n]}$.

The previous Proposition 6.1.2 will be generalised in Theorem 6.4.3.

6.2 When \hat{j}_n is quasi-open

This section develops the proof of Theorem 6.2.5, characterising when \hat{j}_n is a quasi-open map, which is a new result and verifies a generalised analogue of a result for the Vietoris Topology by Fedorchuk (cf. Proposition 6.1.1).

Lemma 6.2.1. 1. *If X is trivial then \hat{i}_n, \hat{j}_n are open.*

2. *If X has a largest non-trivial open subspace V_U which is discrete and $|V_U| < n$ then \hat{j}_n is quasi-open but \hat{i}_n is not quasi-open.*
3. *If \hat{j}_n is quasi-open and every non-trivial open subspace of X has less than n elements, then X has a largest non-trivial open subspace V_U which is discrete and $|V_U| < n$.*

Proof. 1. If X has the trivial topology, the map \hat{j}_n is open as $X^{[n]}$ is the only open subset of itself and $\hat{j}_n(X^{[n]}) = \mathcal{F}_n(X)$ is open and non-empty. Similarly for \hat{i}_n .

2. Suppose X has a largest non-trivial open subspace V_U which is discrete and $|V_U| < n$.

Let $V_U = \{x_1, \dots, x_m\}$.

Now $\hat{i}_n(\{x_1\} \times \dots \times \{x_1\}) = \{\{x_1\}\}$ has empty interior, otherwise there are opens W_1, \dots, W_l in X where $\langle X, W_1, \dots, W_l \rangle \cap \mathcal{F}_{\leq n}(X) = \{\{x_1\}\}$ (cf. Proposition 4.2.3), but then if $y \in X \setminus \{x_1\}$ then $\{y, x_1\} \in \langle X, W_1, \dots, W_l \rangle \cap \mathcal{F}_{\leq n}(X)$, a contradiction. Thus \hat{i}_n is not quasi-open.

Let $V_1 \times \dots \times V_n \cap X^{[n]} \neq \emptyset$ where V_1, \dots, V_n are open in X . By Corollary 3.6.4 (1), $|\bigcup_{i=1}^n V_i| \geq n > m = |V_U|$, so there is $k < n$ such that $V_{k+1}, \dots, V_n = X$ and $V_1, \dots, V_k \subseteq V_U$. By Corollary 3.6.4 (2), $V_1 \times \dots \times V_k \cap X^{[k]} \neq \emptyset$; suppose without loss of generality that it has (x_1, \dots, x_k) as a member. Then $\langle X, \{x_1\}, \dots, \{x_k\} \rangle \cap \mathcal{F}_n(X)$ is a non-empty open subset of $\hat{j}_n(V_1 \times \dots \times V_n \cap X^{[n]})$ (cf. Proposition 4.2.3). Thus \hat{j}_n is quasi-open.

3. Let V_U be the union of the non-trivial opens in X . If $|V_U| \geq n$ then $V_U = X$; moreover there is $x_1, \dots, x_n \in V_U$ and U_1, \dots, U_n open in X where $|U_1|, \dots, |U_n| < n$

and $x_1 \in U_1, \dots, x_n \in U_n$. Thus $\bigcup_{i=1}^n U_i = V_U = X$ but $|X| \leq \sum_{i=1}^n |U_i| < n^n$, contradicting our assumption that X is infinite. Thus $|V_U| < n$.

Let $V_U := \{x_1, \dots, x_{|V_U|}\}$ and let V_1, \dots, V_k be the minimal open subsets of V_U . Suppose to the contrary that V_U is not discrete. Consider the following non-empty open subset of $X^{[n]}$: $V_U \times \dots \times V_U \times X \times \dots \times X \cap X^{[n]}$. Consider its image under \hat{j}_n , i.e. $\mathcal{V} := \{F \in \mathcal{F}_n(X) : V_U \subseteq F\}$.

This set has non-empty interior as \hat{j}_n is quasi-open, moreover $\mathcal{V} \not\subseteq \mathcal{F}_n(X)$, so there are non-trivial opens W_1, \dots, W_m in X for which $\mathcal{V} \supseteq \langle X, W_1, \dots, W_m \rangle \cap \mathcal{F}_n(X) \neq \emptyset$ (cf. Proposition 4.2.3). Hence $W_1, \dots, W_m \subseteq V_U$ as the latter is the largest proper open subset of X . For each $i \in \{1, \dots, m\}$, there is $j \in \{1, \dots, k\}$ such that $W_i \supseteq V_j$. By Proposition 4.1.3, it follows that $\mathcal{V} \supseteq \langle X, W_1, \dots, W_m \rangle \cap \mathcal{F}_n(X) \supseteq \langle X, V_1, \dots, V_k \rangle \cap \mathcal{F}_n(X)$.

If $k < |V_U|$, then $\langle X, V_1, \dots, V_k \rangle \cap \mathcal{F}_n(X)$ has an element F where $V_U \not\subseteq F$, contradicting the definition of \mathcal{V} .

Thus $k = |V_U|$, implying that V_U is discrete. ■

Lemma 6.2.2. *If $\emptyset \neq \bigcup(\langle X, W_1, \dots, W_m \rangle \cap \mathcal{F}_n(X)) \neq X$,*

then $m = n$, W_1, \dots, W_m is a cellular family in X and:

$$\langle W_1, \dots, W_m \rangle \cap \mathcal{F}_n(X) = \langle X, W_1, \dots, W_m \rangle \cap \mathcal{F}_n(X).$$

Moreover if $F \in \langle X, W_1, \dots, W_m \rangle \cap \mathcal{F}_n(X)$, then $F = \bigcup_{i=1}^m (F \cap W_i)$ and $|F \cap W_i| = 1$ for all i .

Proof. Since $\bigcup(\langle X, W_1, \dots, W_m \rangle \cap \mathcal{F}_n(X)) \neq X$ it follows that $\langle X, W_1, \dots, W_m \rangle \cap \mathcal{F}_{n-1}(X) =$

\emptyset . Let $F \in \langle X, W_1, \dots, W_m \rangle \cap \mathcal{F}_n(X)$. For any $x \in F$,

$F \setminus \{x\} \notin \langle X, W_1, \dots, W_m \rangle \cap \mathcal{F}_{n-1}(X)$. Thus both of the following statements are true:

1. $F \subseteq \bigcup_{j=1}^m W_j$.

2. For all $x, y \in F$, there is j such that $x \in W_j$ and $y \notin W_j$.

By the first point, $\langle X, W_1, \dots, W_m \rangle \cap \mathcal{F}_n(X) = \langle W_1, \dots, W_m \rangle \cap \mathcal{F}_n(X)$. Suppose to the contrary there is $x \in W_i \cap W_j$ for some $i \neq j$. Consider $G := \{x\} \cup F \setminus (W_i \cup W_j)$. Now $G \in \langle X, W_1, \dots, W_m \rangle \cap \mathcal{F}_{\leq n-1}(X)$, which contradicts our assumption that $\langle X, W_1, \dots, W_m \rangle \cap \mathcal{F}_{n-1}(X) = \emptyset$. Hence $\{W_1, \dots, W_m\}$ is cellular in X . Thus $m \leq n$, otherwise $\langle X, W_1, \dots, W_m \rangle \cap \mathcal{F}_n(X) = \emptyset$. For each $x \in F$, it is true by the second point above that $\{x\} = \bigcap \{W_j : x \in W_j\} \cap F = W_i \cap F$ for some i . Thus $m = n$. \blacksquare

Lemma 6.2.3. *If \hat{j}_n is quasi-open and X has a proper open subset with at least n elements, then given open subsets V_1, \dots, V_p of X where $p \leq n$, $\bigcup_{i=1}^p V_i$ is a proper subset of X and $V_1 \times \dots \times V_p \cap X^{[p]} \neq \emptyset$, there is a cellular family U_1, \dots, U_p in X where $U_i \subseteq V_i$ for all i .*

Proof. Let V_1, \dots, V_n be open subsets of X where $\bigcup_{i=1}^n V_i$ is a proper subset of X and $V_1 \times \dots \times V_n \cap X^{[n]} \neq \emptyset$. Then the following has non-empty interior:

$$\mathcal{V} := \hat{j}_n(V_1 \times \dots \times V_n \cap X^{[n]}) = \{\{x_1, \dots, x_n\} \in \mathcal{F}_n(X) : x_1 \in V_1, \dots, x_n \in V_n\}.$$

Hence there are opens W_1, \dots, W_m in X where

$\emptyset \neq \langle X, W_1, \dots, W_m \rangle \cap \mathcal{F}_n(X) \subseteq \mathcal{V}$ (cf. Proposition 4.2.3). Now $\bigcup \mathcal{V} \subseteq \bigcup_{i=1}^n V_i \neq X$ so $\bigcup (\langle X, W_1, \dots, W_m \rangle \cap \mathcal{F}_n(X)) \neq X$. Let $F := \{x_1, \dots, x_n\} \in \langle X, W_1, \dots, W_m \rangle \cap \mathcal{F}_n(X)$.

By Lemma 6.2.2, $m = n$ and W_1, \dots, W_m is a cellular family in X where (without loss of generality) $\{x_i\} = F \cap W_i$ for all i and $F = \bigcup_{i=1}^m F \cap W_i$. Since $F \in \mathcal{V}$, there is $\delta \in \Delta$ such that $(x_{\delta(1)}, \dots, x_{\delta(n)}) \in V_1 \times \dots \times V_n$. For each i , let $U_i := W_{\delta(i)} \cap V_i$. Then the family U_1, \dots, U_n is cellular in X and $U_i \subseteq V_i$ for all i .

Given a proper open subspace V of X where $|V| \geq n$ then $V \times \dots \times V \cap X^{[n]} \neq \emptyset$ by Corollary 3.6.4 (1) and so by the workings thus far, $\text{cell}(V) \geq n$. Hence, $\text{cell}(X) \geq n$. Thus every open subset of X with at least n elements has cellularity at least n .

Let V_1, \dots, V_p be open subsets of X where $p < n$, $V_1 \times \dots \times V_p \cap X^{[p]} \neq \emptyset$ and $V := \bigcup_{i=1}^p V_i$ is a proper subset of X .

If $|V| \geq n$ then $V_1 \times \cdots \times V_p \times V \times \cdots \times V \cap X^{[n]} \neq \emptyset$ by Corollary 3.6.4 (3).

Suppose that $m := |V| < n$. If $|X \setminus \bar{V}| < n - m$ then X has cellularity at most $\text{cell}(\bar{V}) + \text{cell}(X \setminus \bar{V}) = \text{cell}(V) + \text{cell}(X \setminus \bar{V}) \leq |V| + |X \setminus \bar{V}| < n$ (cf. Proposition 3.2.1), a contradiction to the fact that $\text{cell}(X) \geq n$ (which was proved earlier). Thus $|X \setminus \bar{V}| \geq n - m$. By Corollary 3.6.4 (4), $V_1 \times \cdots \times V_p \times X \setminus \bar{V} \times \cdots \times X \setminus \bar{V} \cap X^{[n]} \neq \emptyset$.

In either case, it follows by our argument above that there is a cellular family U_1, \dots, U_p in X where $U_1 \subseteq V_1, \dots, U_p \subseteq V_p$. ■

Lemma 6.2.4. *Suppose that whenever V is an open, non-discrete subspace of X then $\text{cell}(V) \geq n$. Then \hat{j}_n is quasi-open.*

Proof. Let \mathcal{V} be non-empty and open in $X^{[n]}$ and let V_1, \dots, V_n be open in X such that $\emptyset \neq V_1 \times \cdots \times V_n \cap X^{[n]} \subseteq \mathcal{V}$. Suppose without loss of generality that there is $0 \leq q \leq n$ where $\text{cell}(V_i) < n$ iff $i \leq q$. By the hypothesis, V_1, \dots, V_q are discrete and so $|V_i| = \text{cell}(V_i) < n$ for $1 \leq i \leq q$. Moreover each of V_{q+1}, \dots, V_n have cellularity $\geq n$ so for each $i \in \{q+1, \dots, n\}$, there is a cellular family W_1^i, \dots, W_n^i in V_i .

Since $V_1 \times \cdots \times V_q \cap X^{[q]} \neq \emptyset$, there is $(x_1, \dots, x_q) \in V_1 \times \cdots \times V_q \cap X^{[q]}$. By the hypothesis, V_1, \dots, V_q are all open and discrete subspaces of X , thus $\{\{x_1\}, \dots, \{x_q\}\}$ is cellular in X and $\{x_i\} \subseteq V_i$ for $i \in \{1, \dots, q\}$.

If $q = n$ then $\langle X, \{x_1\}, \dots, \{x_n\} \rangle \cap \mathcal{F}_n(X) = \{\{x_1, \dots, x_n\}\}$ is an open subset of $\{\{y_1, \dots, y_n\} : y_1 \in V_1, \dots, y_n \in V_n\} = \hat{j}_n(V_1 \times \cdots \times V_n \cap X^{[n]}) \subseteq \hat{j}_n(\mathcal{V})$ (cf. Remark 6.0.0.1 and Proposition 4.2.3).

Suppose that $q < n$. Then $\{x_1, \dots, x_q\}$ intersects at most q members of $\{W_j^i : 1 \leq j \leq n\}$ (as the latter family is cellular in V_i) for each $i \in \{q+1, \dots, n\}$. Suppose without loss of generality that $\{x_1, \dots, x_q\} \cap \bigcup_{q+1 \leq i, j \leq n} W_j^i = \emptyset$.

Let us consider the following non-empty poset (as it has $\{W_{q+1}^{q+1}\}$ as an element) with the

order being \subseteq :

$$\mathcal{W} := \{\mathcal{A} \subseteq \{W_j^i : q+1 \leq i, j \leq n\} : \emptyset \neq \mathcal{A} \text{ and } \bigcap \mathcal{A} \neq \emptyset\}.$$

Let \mathcal{W}_1 be a maximal element of \mathcal{W} for which $\{W_{q+1}^{q+1}\} \subseteq \mathcal{W}_1$, i.e. $W_{q+1}^{q+1} \in \mathcal{W}_1$.

If there is $j \in \{q+1, \dots, n\}$ for which $W_j^{q+2} \in \mathcal{W}_1$ then there is only one such j as $\{W_t^{q+2} : 1 \leq t \leq n\}$ is cellular in V_{q+2} and $\bigcap \mathcal{W}_1 \neq \emptyset$. Suppose without loss of generality that $W_{q+2}^{q+2} \notin \mathcal{W}_1$. Let \mathcal{W}_2 be a maximal element of \mathcal{W} for which $W_{q+2}^{q+2} \in \mathcal{W}_2$. Then $\mathcal{W}_2 \cup \mathcal{W}_1 \notin \mathcal{W}$ by maximality of \mathcal{W}_1 hence $\bigcap \mathcal{W}_2 \cap \bigcap \mathcal{W}_1 = \bigcap (\mathcal{W}_2 \cup \mathcal{W}_1) = \emptyset$.

If there is $j \in \{q+1, \dots, n\}$ for which $W_j^{q+3} \in \mathcal{W}_1 \cup \mathcal{W}_2$ then there are at most 2 such j as $\{W_t^{q+3} : 1 \leq t \leq n\}$ is cellular in V_{q+3} and $\bigcap \mathcal{W}_1, \bigcap \mathcal{W}_2 \neq \emptyset$. Suppose without loss of generality that $W_{q+3}^{q+3} \notin \mathcal{W}_1 \cup \mathcal{W}_2$. Let \mathcal{W}_3 be a maximal element of \mathcal{W} for which $W_{q+3}^{q+3} \in \mathcal{W}_3$. Then $\mathcal{W}_3 \cup \mathcal{W}_1, \mathcal{W}_3 \cup \mathcal{W}_2 \notin \mathcal{W}$ by maximality of \mathcal{W}_1 and \mathcal{W}_2 , hence $\bigcap \mathcal{W}_3 \cap (\bigcap \mathcal{W}_1 \cup \bigcap \mathcal{W}_2) = \bigcap (\mathcal{W}_3 \cup \mathcal{W}_1) \cup \bigcap (\mathcal{W}_3 \cup \mathcal{W}_2) = \emptyset$.

Let $k < n$ and suppose maximal elements $\mathcal{W}_1, \dots, \mathcal{W}_k$ of \mathcal{W} have been defined for which $\bigcap \mathcal{W}_1, \dots, \bigcap \mathcal{W}_k$ is a pairwise disjoint family of open subsets of V_{q+1}, \dots, V_{q+k} , respectively. If there is $j \in \{q+1, \dots, n\}$ for which $W_j^{q+k+1} \in \bigcup_{i=1}^k \mathcal{W}_i$ then there are at most k such j as $\{W_t^{q+k+1} : 1 \leq t \leq n\}$ is cellular in V_{q+k+1} and $\bigcap \mathcal{W}_1, \dots, \bigcap \mathcal{W}_k \neq \emptyset$. Suppose without loss of generality that $W_{q+k+1}^{q+k+1} \notin \bigcup_{i=1}^k \mathcal{W}_i$. Let \mathcal{W}_{k+1} be a maximal element of \mathcal{W} for which $W_{q+k+1}^{q+k+1} \in \mathcal{W}_{k+1}$. Then for each $1 \leq i \leq k$, $\mathcal{W}_{k+1} \cup \mathcal{W}_i \notin \mathcal{W}$ by maximality of \mathcal{W}_i and so $\bigcap \mathcal{W}_{k+1} \cap \bigcup_{i=1}^k \bigcap \mathcal{W}_i = \bigcup_{i=1}^k \bigcap (\mathcal{W}_{k+1} \cup \mathcal{W}_i) = \emptyset$. Thus $\{\bigcap \mathcal{W}_1, \dots, \bigcap \mathcal{W}_{k+1}\}$ is a pairwise disjoint family of open subsets of $V_{q+1}, \dots, V_{q+k}, V_{q+k+1}$, respectively.

One may continue this process until maximal elements $\mathcal{W}_1, \dots, \mathcal{W}_{n-q}$ of \mathcal{W} are found such that $U_{q+1} := \bigcap \mathcal{W}_1, \dots, U_n := \bigcap \mathcal{W}_{n-q}$ is a pairwise disjoint family of open subsets of V_{q+1}, \dots, V_n respectively. Thus

$$\langle X, \{x_1\}, \dots, \{x_q\}, U_{q+1}, \dots, U_n \rangle \cap \mathcal{F}_n(X) = \langle \{x_1\}, \dots, \{x_q\}, U_{q+1}, \dots, U_n \rangle \cap \mathcal{F}_n(X)$$

is an open subset of $\{\{y_1, \dots, y_n\} : y_1 \in V_1, \dots, y_n \in V_n\} = \hat{j}_n(V_1 \times \dots \times V_n \cap X^{[n]}) \subseteq \hat{j}_n(\mathcal{V})$ (cf. Proposition 4.2.3).

Hence \hat{j}_n is a quasi-open map. ■

Theorem 6.2.5. *If X is trivial then \hat{i}_n, \hat{j}_n are open.*

If X has a largest non-trivial open subspace V_U which is discrete and $|V_U| < n$ then \hat{j}_n is quasi-open but \hat{i}_n is not quasi-open.

If X has neither of these properties, the following are equivalent (in which case X has a proper open subset with at least n elements):

1. \hat{j}_n is quasi-open.
2. If V is an open, non-discrete subspace of X then $\text{cell}(V) \geq n$.
3. If V is an open, non-discrete subspace of X with finite cellularity, then there is a discrete (and finite) open subspace Y of V such that $|Y| \geq n$ and $\overline{V} = \overline{Y}$. Moreover, if $F \subseteq Y$ then:

$$V \setminus \overline{Y \setminus F} = \text{Int}(\overline{V} \setminus (Y \setminus F)) = F \quad \text{if } |F| < n \text{ and}$$

$$\text{Int}(\overline{V} \setminus (Y \setminus F)) \subseteq F \cup \bigcap_{y_1, \dots, y_k \in F} \overline{\{y_1, \dots, y_k\}} \quad \text{if } |F| = (n-1) + k, \quad k \geq 1.$$

In particular, if $k = |Y| - n + 1$:

$$\text{Int}(\overline{V}) \subseteq Y \cup \bigcap_{y_1, \dots, y_k \in Y} \overline{\{y_1, \dots, y_k\}}.$$

Proof. The first two statements in this Theorem follow from Lemma 6.2.1 (1) and (2). Suppose that X does not have the trivial topology nor does it have a largest proper open subspace V_U which is discrete and $|V_U| < n$.

1 \implies 2: Suppose \hat{j}_n is quasi-open. Since X does not have a largest non-trivial open subspace V_U which is discrete and $|V_U| < n$, it follows by Lemma 6.2.1 (3) that X has a non-trivial open subspace with at least n elements. Given a proper open subspace V of X where $|V| \geq n$ then $V \times \cdots \times V \cap X^{[n]} \neq \emptyset$ by Corollary 3.6.4 (1) and so by Lemma 6.2.3, $\text{cell}(V) \geq n$. Hence, $\text{cell}(X) \geq n$. Thus every open subset of X with at least n elements has cellularity at least n .

Consider a finite open subset F of X with $|F| \leq n$. Then $F \times \cdots \times F \cap X^{[F]} \neq \emptyset$ so by Lemma 6.2.3, there is a cellular family $U_1, \dots, U_{|F|}$ in F , in which case $U_1, \dots, U_{|F|}$ are the singleton subsets of F . Thus F is discrete.

2 \implies 3: Let V be a non-discrete open subspace of X which has cellularity $m < \aleph_0$. By the hypothesis, $|V| > n$ and $m \geq n$. Now $\bar{V} = \bigcup_{i=1}^m \bar{U}_i$ where U_1, \dots, U_m is a cellular family of irreducible subspaces of \bar{V} and $\text{Int}_{\bar{V}}(\bar{U}_i) = U_i$ for each i (cf. Lemma 3.3.9). For each i , $U_i \cap V$ is an open subspace of \bar{U}_i and \bar{U}_i is irreducible, so $\overline{U_i \cap V} = \bar{U}_i$. Hence $\bar{V} = \bigcup_{i=1}^m \overline{U_i \cap V}$, moreover for each i , $U_i \cap V$ is an irreducible open subspace of X . Now $\text{cell}(\bigcup_{i=1}^{n-1} U_i \cap V) = n - 1 < n$, so by the hypothesis, $\bigcup_{i=1}^{n-1} U_i \cap V$ is discrete and hence $|\bigcup_{i=1}^{n-1} U_i \cap V| = n - 1$. Hence $|U_1 \cap V|, \dots, |U_{n-1} \cap V| = 1$. Applying this process repeatedly (on each repetition increasing the initial and final index of the union by 1) it follows that for all $i \in \{1, \dots, m\}$, $U_i \cap V = \{y_i\}$ for some y_i . Thus V has a discrete open subspace $Y := \{y_1, \dots, y_m\}$ where $\overline{\{y_i\}} = \bar{U}_i$ for all i . Note that $\bar{Y} = \bar{V}$.

Let $F \subseteq Y$ and $G_F := \text{Int}(\bar{V} \setminus (Y \setminus F))$.

Then:

1. F is open in X and is a subset of $\bar{V} \setminus (Y \setminus F)$, thus $F \subseteq G_F$.
2. If H is a non-empty open subset of G_F , then $H \cap \bar{Y} = H \cap \bar{V} \neq \emptyset$, so $H \cap Y \neq \emptyset$, however $H \cap Y \setminus F = \emptyset$, thus $H \cap F \neq \emptyset$.
3. Note that $V \setminus \overline{Y \setminus F}$ is an open subset of G_F for which $F \subseteq V \setminus \overline{Y \setminus F}$.

Suppose $|F| < n$. Suppose to the contrary that $G_F \setminus F \neq \emptyset$. Then by Corollary 3.6.4:

$$G_F \times F \times \cdots \times F \cap X^{[|F|+1]} \neq \emptyset$$

By Lemma 6.2.4, \hat{j}_n is quasi-open. Moreover $G_F \cup F \neq X$ as $(G_F \cup F) \cap Y \setminus F = \emptyset$ and $Y \setminus F \neq \emptyset$. Thus by Lemma 6.2.3 there is a non-empty open subset H of G_F for which $H \cap F = \emptyset$, a contradiction to the 2nd point above. Thus, $F = G_F$ and so $F = V \setminus \overline{Y \setminus F}$.

Let $|F| = (n - 1) + k$ where $1 \leq k$ (noting $k \leq m - (n - 1)$ as $|F| \leq m$).

Suppose to the contrary that $G_F \setminus F \not\subseteq \bigcap_{y_1, \dots, y_k \in F} \overline{\{y_1, \dots, y_k\}}$.

Then $G_F \setminus (F \cup \overline{\{y_1, \dots, y_k\}}) \neq \emptyset$ for some $y_1, \dots, y_k \in F$. Now $|F \setminus \{y_1, \dots, y_k\}| \geq n - 1$ as $|F| = (n - 1) + k$. By Corollary 3.6.4:

$$G_F \setminus \overline{\{y_1, \dots, y_k\}} \times F \setminus \{y_1, \dots, y_k\} \times \cdots \times F \setminus \{y_1, \dots, y_k\} \cap X^{[n]} \neq \emptyset.$$

By Lemma 6.2.4 and the hypothesis, \hat{j}_n is quasi-open. Moreover, $G_F \setminus \overline{\{y_1, \dots, y_k\}} \cup F \setminus \{y_1, \dots, y_k\} \neq X$ as the former does not contain $\{y_1, \dots, y_k\}$. Since F is discrete as a subspace of X , it follows by Remark 3.3.9.1 that $x \notin \overline{\{y_1, \dots, y_k\}}$ whenever $x \in F \setminus \{y_1, \dots, y_k\}$ and thus $F \setminus \{y_1, \dots, y_k\} = F \setminus \overline{\{y_1, \dots, y_k\}}$ is open in X . Thus by Lemma 6.2.3 there is a non-empty open subset H of $G_F \setminus \overline{\{y_1, \dots, y_k\}}$ for which $H \cap F \setminus \{y_1, \dots, y_k\} = \emptyset$. It follows that $H \cap F = \emptyset$, a contradiction.

Thus $G_F \subseteq F \cup \bigcap_{y_1, \dots, y_k \in F} \overline{\{y_1, \dots, y_k\}}$.

In particular, $|Y| = m = (n - 1) + (m - n + 1)$; thus by our workings:

$$\text{Int}(\overline{V}) \subseteq Y \cup \bigcap_{y_1, \dots, y_{m-(n-1)} \in Y} \overline{\{y_1, \dots, y_{m-(n-1)}\}}.$$

3 \implies 1: It is clear that 3 \implies 2 is true, and so the result follows from Lemma 6.2.4. \blacksquare

6.3 Infinite Cellularity in Hyperspaces

Theorem 6.3.1. *Suppose that whenever V is a non-discrete open subspace of X then $\text{cell}(V) \geq n$. Suppose that $\text{cell}(X)$ is infinite. If either of the following are true:*

1. $c(\mathcal{F}_n(X)) \subseteq c(\mathfrak{G}) \subseteq \mathcal{S}_{\leq n}(X)$, or
2. $c(\mathcal{D}_n(X)) \subseteq c(\mathfrak{G}) \subseteq \mathcal{S}_{\leq n}(X)$ and every discrete subset of X with at most n elements is contained in a subset of X with cellularity n ,

Then:

$$\text{cell}(\mathfrak{G}) = \text{cell}(X^{[n]}) = \text{cell}(X^n) \geq \text{cell}(X).$$

Proof. By Theorem 6.2.5, \hat{j}_n is a quasi-open map. By Corollary 3.5.4 and Theorem 3.6.7, $\text{cell}(X) \leq \text{cell}(X^n) \leq \text{cell}(X^{[n]})$. By the hypothesis, $\text{cell}(X^{[n]})$ is infinite. From Proposition 3.2.2, $\text{cell}(\mathcal{F}_n(X)) = \text{cell}(X^{[n]})$. From Theorem 5.1.21, $\text{Cl}_{\mathbf{L}(\tau)}(\mathcal{F}_n(X)) = \text{Cl}_{\mathbf{L}(\tau)}(\mathcal{F}_{\leq n}(X))$. Thus $\text{cell}(\mathfrak{S}) = \text{cell}(\mathcal{F}_{\leq n}(X)) = \text{cell}(\mathcal{F}_n(X)) = \text{cell}(X^{[n]})$. Since \hat{i}_n is continuous it follows by Proposition 3.2.1 (2) that $\text{cell}(\mathcal{F}_{\leq n}(X)) \leq \text{cell}(X^n)$. Thus $\text{cell}(\mathcal{F}_n(X)) = \text{cell}(X^n) = \text{cell}(X^{[n]})$. Now:

1. For the first point, by Theorem 5.1.21, $\text{Cl}_{\mathbf{L}(\tau)}(\mathfrak{S}) = \text{Cl}_{\mathbf{L}(\tau)}(\mathcal{F}_n(X))$ and thus $\text{cell}(\mathfrak{S}) = \text{cell}(X^n) = \text{cell}(X^{[n]})$.
2. For the second point, by Theorem 5.1.23, $\text{Cl}_{\mathbf{L}(\tau)}(\mathfrak{S}) = \text{Cl}_{\mathbf{L}(\tau)}(\mathcal{F}_n(X))$ and thus $\text{cell}(\mathfrak{S}) = \text{cell}(X^n) = \text{cell}(X^{[n]})$.

■

Corollary 6.3.2.

Suppose that whenever V is an open subspace of X which is not discrete then $\text{cell}(V) \geq n$. Suppose $\text{cell}(X)$ is infinite and for $1 \leq n < \omega$ let $c(\mathcal{F}_n(X)) \subseteq c(\mathfrak{S}_n) \subseteq \mathcal{S}_{\leq n}(X)$. Then:

1. For $1 \leq n < \omega$, $\text{cell}(\mathfrak{S}_n) \leq \text{cell}(\mathfrak{S}_{n+1})$.
2. If X is second-countable, then $\text{cell}(\mathfrak{S}_n) = \aleph_0$ for all $1 \leq n < \omega$.
3. (MA + $\aleph_0^+ < 2^{\aleph_0}$) If $\text{cell}(X) = \aleph_0$ then $\text{cell}(\mathfrak{S}_n) = \aleph_0$ for all $1 \leq n < \omega$.

Proof. 1. Follows from Theorem 6.3.1 and Corollary 3.5.4.

2. Follows from Theorem 6.3.1 and Remark 3.5.3.1.

3. Follows from Theorem 6.3.1 and Proposition 3.5.8.

■

Corollary 6.3.3.

Suppose that whenever V is an open subspace of X which is not discrete then $\text{cell}(V) \geq n$. Suppose every discrete subset of X with at most n elements is contained in a subset of X

with cellularity n . Suppose $\text{cell}(X)$ is infinite and for $1 \leq n < \omega$ let $c(\mathcal{D}_n(X)) \subseteq c(\mathfrak{S}_n) \subseteq \mathcal{S}_{\leq n}(X)$. Then:

1. For $1 \leq n < \omega$, $\text{cell}(\mathfrak{S}_n) \leq \text{cell}(\mathfrak{S}_{n+1})$.
2. If X is second-countable, then $\text{cell}(\mathfrak{S}_n) = \aleph_0$ for all $1 \leq n < \omega$.
3. (MA + $\aleph_0^+ < 2^{\aleph_0}$) If $\text{cell}(X) = \aleph_0$ then $\text{cell}(\mathfrak{S}_n) = \aleph_0$ for all $1 \leq n < \omega$.

Proof. Same proof as the previous Corollary. ■

Corollary 6.3.4.

1. For each $n < \omega$, there is an infinite, singular cardinal κ and a compact Hausdorff space X such that whenever $c(\mathcal{D}_n(X)) \subseteq c(\mathfrak{S}_n) \subseteq \mathcal{S}_{\leq n}(X)$ and $c(\mathcal{D}_{n+1}(X)) \subseteq c(\mathfrak{S}_{n+1}) \subseteq \mathcal{S}_{\leq n+1}(X)$ it is true that $\text{cell}(\mathfrak{S}_n) = \lambda < \text{cell}(\mathfrak{S}_{n+1})$.
2. (GCH) For each $n < \omega$ and for every infinite cardinal λ , there is an extremally disconnected compact Hausdorff space X such that whenever $c(\mathcal{D}_n(X)) \subseteq c(\mathfrak{S}_n) \subseteq \mathcal{S}_{\leq n}(X)$ and $c(\mathcal{D}_{n+1}(X)) \subseteq c(\mathfrak{S}_{n+1}) \subseteq \mathcal{S}_{\leq n+1}(X)$ it is true that $\text{cell}(\mathfrak{S}_n) = \lambda < \text{cell}(\mathfrak{S}_{n+1})$.

Proof. 1. Follows from Theorem 6.3.1 and Proposition 3.5.6.

2. Follows from Theorem 6.3.1 and Proposition 3.5.9. ■

6.4 When \hat{j}_n, \hat{i}_n are quotient maps

As is evident from the new result below, the property of \hat{i}_n being open or even a quotient map is almost never satisfied.

Theorem 6.4.1. *The following are equivalent for $n > 1$:*

1. \hat{i}_n is a quotient map.
2. The Lower Vietoris Topology on $\mathcal{F}_{\leq n}(X)$ is equal to the Vietoris Topology.
3. X is trivial.

Proof. 1 \implies 2: Let V_1, \dots, V_n be open in X . Then

$\hat{i}_n^{-1}(\hat{i}_n(V_1 \times \dots \times V_n)) = \bigcup_{\delta \in \Delta} V_{\delta(1)} \times \dots \times V_{\delta(n)}$ is open in X^n (cf. Remark 6.0.0.1). Since \hat{i}_n is a quotient map by the hypothesis, $\hat{i}_n(V_1 \times \dots \times V_n)$ is open. Since $\{W_1 \times \dots \times W_n : W_1, \dots, W_n \text{ open in } X\}$ is a base for X^n , \hat{i}_n is an open map.

Thus for any open subset U of X , $\langle U \rangle \cap \mathcal{F}_{\leq n}(X) = \hat{i}_n(U^n)$ is open because U^n is open in X^n (cf. Remark 6.0.0.1). Hence the result follows.

2 \implies 3: Suppose to the contrary there is $x \in X$ such that there is $y \in X \setminus \overline{\{x\}}$. By the hypothesis, $\langle X \setminus \overline{\{x\}} \rangle \cap \mathcal{F}_{\leq n}(X)$ is open, moreover it has $\{y\}$ as an element. Since $y \in \overline{\{x, y\}}$ and $X \setminus \overline{\{x\}}$ is open it follows that $\overline{\{x, y\}} \in \langle X \setminus \overline{\{x\}} \rangle \cap \mathcal{F}_{\leq n}(X)$, a contradiction. Hence, for every $x \in X$, $\overline{\{x\}} = X$. Thus, X has the trivial topology.

3 \implies 1: Then \hat{i}_n is both continuous and open and so is a quotient map. ■

The following new result, Theorem 6.4.2, is needed to prove Theorem 6.4.3:

Theorem 6.4.2. *If the Lower Vietoris Topology on $\mathcal{F}_n(X)$ is equal to the Vietoris Topology then:*

1. Every non-trivial open subspace of X with at least n elements is Hausdorff.
2. If F is a finite subset of X where $|\overline{F}| \geq n$ then X is the only open neighbourhood of the set of minimal elements of \overline{F} with respect to \leq_τ .
3. If $z_1 >_\tau \dots >_\tau z_n$ is a strictly decreasing sequence in X then X is the only open neighbourhood of z_n .

Proof. 1. Let $x, y \in W$ where W is a non-trivial open subspace of X . Let F be an n -element subset of W where $x, y \in F$. Since $\langle W \rangle \cap \mathcal{F}_n(X)$ is open in $\mathcal{F}_n(X)$, there is W_1, \dots, W_m open in X such that $F \in \langle X, W_1, \dots, W_m \rangle \cap \mathcal{F}_n(X) \subseteq \langle W \rangle$.

Since $W \neq X$, it is true that $\bigcup \langle X, W_1, \dots, W_m \rangle \cap \mathcal{F}_n(X) \neq X$.

By Lemma 6.2.2, W_1, \dots, W_m is cellular in X , $m = n$ and $F = \bigcup_{i=1}^m F \cap W_i$ where $|F \cap W_i| = 1$ for all i . Let W_x, W_y be the members of $\{W_1, \dots, W_m\}$ where $\{x\} = W_x \cap F$, $\{y\} = W_y \cap F$. Then $W_x \cap W_y = \emptyset$.

Thus W is Hausdorff.

2. Let M be the set of minimal elements of \overline{F} with respect to \leq_τ . If V is a non-trivial open in X where $M \subseteq V$ then $\overline{F} \subseteq V$ and so by the previous point V is Hausdorff, a contradiction. Thus the result follows.

3. Follows from the previous point. ■

The following new result, Theorem 6.4.3, generalises Proposition 6.1.2 by Costantini et al.

Theorem 6.4.3. *Suppose that whenever $x, y \in X$ are distinct, there is a non-trivial open subset V of X with at least n elements such that $x, y \in V$. Then the following are equivalent:*

1. \hat{j}_n is a quotient map.
2. The Lower Vietoris Topology on $\mathcal{F}_n(X)$ is equal to the Vietoris Topology.
3. X is Hausdorff.
4. $\{\langle V_1, \dots, V_n \rangle \cap \mathcal{F}_n(X) : V_1, \dots, V_n \text{ cellular in } X\}$ is a base for $\mathcal{F}_n(X)$ and $\{V_1 \times \dots \times V_n : V_1, \dots, V_n \text{ cellular in } X\}$ is a base for $X^{[n]}$.
5. \hat{j}_n is a perfect $(n!)$ -fold covering map.

Proof. 1 \implies 2: Similar to the proof of 1 \implies 2 in Theorem 6.4.1.

2 \implies 3: Follows from the hypothesis and Theorem 6.4.2 (1).

3 \implies 4: See Proposition 6.1.2.

4 \implies 5: By Proposition 6.1.2 \hat{j}_n is a $(n!)$ -fold covering map. Since \hat{j}_n has compact fibers, the result will follow if \hat{j}_n is a closed map, which is shown below.

Let $\{x_1, \dots, x_n\} \in \mathcal{F}_n(X)$ and let \mathcal{U} be open in $X^{[n]}$ such that $\hat{j}_n^{-1}(\{x_1, \dots, x_n\}) \subseteq \mathcal{U}$.

Let $\delta \in \Delta$. There is a cellular family $V_1^\delta, \dots, V_n^\delta$ in X such that $(x_{\delta(1)}, \dots, x_{\delta(n)}) \in V_1^\delta \times \dots \times V_n^\delta \subseteq \mathcal{U}$ (By the hypothesis). For each $i \in \{1, \dots, n\}$, $x_i = x_{\delta^{-1}(i)} \in V_{\delta^{-1}(i)}^\delta$.

Let $V_i := \bigcap_{\delta \in \Delta} V_{\delta^{-1}(i)}^\delta$. Then V_i is open (as Δ is finite), moreover $x_i \in V_i$.

Hence for each $\delta \in \Delta$, $(x_{\delta(1)}, \dots, x_{\delta(n)}) \in V_{\delta(1)} \times \dots \times V_{\delta(n)}$.

Thus, $\hat{j}_n^{-1}(\langle V_1, \dots, V_n \rangle \cap \mathcal{F}_n(X)) = \bigcup_{\delta \in \Delta} V_{\delta(1)} \times \dots \times V_{\delta(n)} \subseteq \mathcal{U}$ (cf. Remark 6.0.0.1) and $\{x_1, \dots, x_n\} \in \langle V_1, \dots, V_n \rangle \cap \mathcal{F}_n(X)$.

Thus \hat{j}_n is closed. ■

6.5 An Example

The following Example 6.5.0.1 shows that in Theorem 6.3.1, one cannot drop the assumption that X has infinite cellularity. In particular X may be T_0 , non-sober, non- T_1 and have finite cellularity while the following are true:

- \hat{j}_n is quasi-open.
- $\text{cell}(X^n) > \text{cell}(X) > \text{cell}(\mathcal{F}_n(X)) = 1$.
- X satisfies the 3rd point of Theorem 6.2.5 in a non-trivial fashion.
- For any $m > 1$, \hat{j}_m is not a quotient map and the Lower Vietoris Topology on $\mathcal{F}_m(X)$ is not equal to the Vietoris Topology.

Example 6.5.0.1. Let $X := Y \cup Z$ where $Y := \{y_1, \dots, y_n\}$ and Z is infinite. Suppose X has the partial order defined by: $z \leq y_1, \dots, y_n$ for all $z \in Z$. Consider the upper interval topology on X . Given $1 \leq i \leq n$, $\{y_i\} = X \downarrow \{y_j : j \neq i\}$ is open and so Y is a discrete open subspace of X where $\overline{Y} = \downarrow Y = X$. Since $\text{cell}(Y) = n$ (cf. Remark 3.3.9.1), it follows that $\text{cell}(X) = n$.

Let W be a discrete, open subspace of X . Then $|W| = \text{cell}(W) \leq \text{cell}(X) = n$. If $W \cap Z \neq \emptyset$ then $W \cap Z \not\subseteq \overline{W \cap Y}$ (as W is discrete, see Remark 3.3.9.1) and so $W \cap Y = \emptyset$; this would mean that $X \setminus W = Y \cup (Z \setminus W)$ is closed so $Y \cup (Z \setminus W) = \downarrow (Y \cup (Z \setminus W)) = X$, a contradiction. Thus $W \subseteq Y$.

Let V be a non-discrete open subspace of X . Then $V \not\subseteq Y$ so $V \cap Z \neq \emptyset$. Since $Z = \bigcap_{i=1}^n \downarrow \{y_i\} = \bigcap_{i=1}^n \overline{\{y_i\}}$, it follows that $Y \subseteq V$. Thus $\overline{V} = X = \overline{Y}$ and $\text{cell}(V) = n$.

Thus every open set T in X must satisfy $Y \subseteq T$ or $T \subseteq Y$. Hence:

1. $V \setminus \overline{Y \setminus \emptyset} = \emptyset = \text{Int}(\overline{V} \setminus (Y \setminus \emptyset))$.
2. If $1 \leq |F| < n$ then $V \setminus \overline{Y \setminus F} = V \setminus (Z \cup (Y \setminus F)) = F = \text{Int}(\overline{V} \setminus (Y \setminus F))$: let T be an open subset of X that is a subset of $\overline{V} \setminus (Y \setminus F)$, then $T \subseteq Y$ (from the workings), so $T \subseteq F$.
3. $\text{Int}(\overline{V}) = X = Y \cup Z = Y \cup \bigcap_{x \in Y} \overline{\{x\}}$.

Thus by Theorem 6.2.5, \hat{j}_n is a quasi-open map.

Note that $Z = X \setminus Y$ is closed. Given $x_1, \dots, x_m \in X$, either:

1. $x_1, \dots, x_m \in Z$ in which case $\downarrow \{x_1, \dots, x_m\} = \{x_1, \dots, x_m\} \not\subseteq Z$ or
2. $x_i \in Y$ for some i , in which case, $\downarrow \{x_1, \dots, x_m\} \supseteq Z \cup \{x_i\} \supset Z$.

Thus Z is irreducible (cf. Corollary 3.3.3). However Z is not the closure of any point in X ; thus X is not a sober space and is also not a T_1 -space as $\downarrow \{y_i\} \neq \{y_i\}$ for any i .

Now $\text{cell}(X^n) = \text{cell}(X)^n = n^n > 1 = \text{cell}(\mathcal{F}_n(X))$ (cf. Corollary 3.5.12 and Corollary

5.1.18). Thus the result in Theorem 6.3.1 is not always true when $\text{cell}(X)$ is finite, even if the condition “whenever V is a non-discrete open subspace of X then $\text{cell}(V) \geq n$ ” is satisfied.

Let $x, y \in X$ be distinct. For any $z \in Z$ where $x, y \neq z$ then $x, y \in X \setminus \{z\} = X \setminus \overline{\{z\}}$ and $X \setminus \{z\}$ is infinite. Thus for any $m > 1$, it follows by Theorem 6.4.3 that \hat{j}_m is not a quotient map and the Lower Vietoris Topology on $\mathcal{F}_m(X)$ is not equal to the Vietoris Topology.

Chapter 7

Sobriety and Separation Axioms in the Lower Vietoris Topology

7.1 T_0 , Sobriety and T_D

In this section, the study of the T_0 separation axiom and the sober property in the Lower Vietoris Topology that was started in Chapter 4 and Chapter 5, Section 5.2 respectively is continued. The T_D separation axiom is also investigated in the Lower Vietoris Topology, due to its relationship with sobriety (cf. Proposition 7.1.1). The new results are summarised below.

It is determined when $c(\mathfrak{S})$ is T_D in the case that X is T_0 and $c(\mathcal{D}_f(X)) \subseteq c(\mathfrak{S})$ (cf. Theorem 7.1.2). Assuming that X is T_0 , it is proven that X is T_D iff $c(\mathcal{F}(X))$ is T_D (cf. Corollary 7.1.3); moreover the T_D axiom in 2^X is characterised (cf. Theorem 7.1.4).

For each $n > 1$, it is shown that the T_0 and T_D axioms (and the “sober” and “sober and T_D ” properties) are equivalent in the hyperspace $\mathcal{F}_n(X)$ and the T_0 axiom in this hyperspace (resp. the “sober” property in this hyperspace) is characterised in terms of the properties of X (cf. Theorem 7.1.6, Corollary 7.1.7 (1) and (2) and Lemma 7.1.5).

An interesting observation is noted in Corollary 7.1.7 (3) that if $n > 2$ and $\mathcal{F}_n(X)$ is T_0 (resp. sober) then $\mathcal{F}_m(X)$ is T_D (resp. sober and T_D) for all $m \in \mathbb{N}$.

Some examples are given at the end of this section (cf. Examples 7.1.7.1 - 7.1.7.5) to illustrate that these results cannot be improved.

Definition 7.1.0.1. [1, Definition 3.1] X is a T_D -space (or simply X is T_D) if for each $x \in X$, $\overline{\{x\}} \setminus \{x\}$ is closed in X , equivalently, $\{x\}$ is open in $\overline{\{x\}}$.

Every T_D -space is a T_0 -space and every T_1 -space is a T_0 -space ([1, p. 34]).

See Definition 7.1.4.1 for an example of a T_D -space which is not a T_1 -space and an example of a T_0 -space which is not a T_D -space.

Examples 7.1.7.1 and 7.1.7.2 describe T_D -spaces which are not sober. See [46, Chapter I, 3.1, 3.2, 3.3] for an example of a sober space which is not T_D .

Remark 7.1.0.1. If X is T_D then every subspace is also T_D :

If A is a subspace of X , then for each $x \in A$, $\text{Cl}_A(\{x\}) \setminus \{x\} = (\overline{\{x\}} \setminus \{x\}) \cap A$ is closed in A because $\overline{\{x\}} \setminus \{x\}$ is closed in X .

Proposition 7.1.1. [28, Theorem 2.2 (b),(d)] *The following are equivalent:*

1. *Every subspace of X is sober.*
2. *X is a sober T_D -space.*

Theorem 7.1.2. *Suppose that $c(\mathcal{D}_f(X)) \subseteq c(\mathfrak{S})$ and X is T_0 .*

The following are equivalent:

1. *$c(\mathfrak{S})$ is T_D .*
2. *For each $A \in c(\mathfrak{S})$, there is a discrete, open finite subset F of A such that $A = \overline{F}$.*

Proof. 1 \implies 2: Let $A \in c(\mathfrak{S})$. By the hypothesis there are V_1, \dots, V_m open in X such that:

$$\{A\} = \langle X, V_1, \dots, V_m \rangle \cap \text{Cl}_{\mathbf{L}(\tau)|c(\mathfrak{S})}(\{A\})$$

$$\begin{aligned}
&= \langle X, V_1, \dots, V_m \rangle \cap \langle A \rangle \cap c(\mathfrak{S}) \quad (\text{cf. Proposition 4.2.6}) \\
&= \langle A, A \cap V_1, \dots, A \cap V_m \rangle \cap c(\mathfrak{S}). \quad (\text{cf. Corollary 4.1.2})
\end{aligned}$$

By Lemma 4.1.4, there is a cellular family $\{U_1, \dots, U_k\}$ in A such that:

$$\emptyset \neq \langle A, U_1, \dots, U_k \rangle \cap c(\mathfrak{S}) \subseteq \langle A, A \cap V_1, \dots, A \cap V_m \rangle \cap c(\mathfrak{S}).$$

Thus $\langle A, U_1, \dots, U_k \rangle \cap c(\mathfrak{S}) = \{A\}$. Let $x, y \in U_i$ for some i . Then $x, y \notin \overline{\bigcup_{j \neq i} U_j}$ as $U_i \cap \bigcup_{j \neq i} U_j = \emptyset$ and U_i is open. If $\overline{\{x\}} \neq \overline{\{y\}}$ then without loss of generality, $y \notin \overline{\{x\}}$ so $y \notin \overline{\{x\}} \cup \overline{\bigcup_{j \neq i} U_j}$ hence $\overline{\{x\}} \cup \overline{\bigcup_{j \neq i} U_j} \neq A$. However, $U_i \cap \overline{\{x\}} \cup \overline{\bigcup_{j \neq i} U_j} \neq \emptyset$ for all t , a contradiction. Thus $\overline{\{x\}} = \overline{\{y\}}$ whenever $x, y \in U_i$ for some i . Since X is T_0 by the hypothesis it follows that for each i , $U_i = \{x_i\}$ for some $x_i \in U_i$.

Let $F = \bigcup_{i=1}^k U_i$. Then F is finite and discrete as a subspace of X so $\overline{F} \in c(\mathcal{D}_f(X)) \subseteq c(\mathfrak{S})$. Hence:

$$\overline{F} \in \langle A, U_1, \dots, U_k \rangle \cap c(\mathfrak{S}) = \{A\}.$$

Thus $A = \overline{F}$. Moreover F is open in A as a union of open sets in A .

2 \implies 1: Let $A \in c(\mathfrak{S})$. By the hypothesis, there is a discrete open finite subset F of A such that $\overline{F} = A$. Let $\{x_1, \dots, x_k\}$ be the elements of F ; since F is discrete, $\{x_i\}$ is open in F and hence in A for each i (as F is open in A). For each i , let V_i be an open subset of X for which $V_i \cap A = \{x_i\}$. If there is $G \in c(\mathfrak{S})$ for which $G \subseteq A$ and $x_1, \dots, x_k \in G$, then $F \subseteq G$ and hence $G = \overline{G} = A$. Thus:

$$\begin{aligned}
\langle X, V_1, \dots, V_m \rangle \cap \text{Cl}_{\mathbf{L}(\tau)|c(\mathfrak{S})}(\{A\}) &= \langle A, \{x_1\}, \dots, \{x_k\} \rangle \cap c(\mathfrak{S}) \quad (\text{See working above}) \\
&= \{A\}.
\end{aligned}$$

Thus $c(\mathfrak{S})$ is T_D . ■

Remark 7.1.2.1. The T_0 assumption of the previous Theorem is essential to verify that the F constructed in the forward direction of the previous proof is actually finite.

Corollary 7.1.3. *If X is T_0 then X is T_D iff $c(\mathcal{F}(X))$ is T_D .*

Proof. \implies : Suppose X is T_D . Let F be a finite subset of X . By Remark 3.3.9.1 there is a discrete subset G of F where $\overline{G} = \overline{F}$. Since X is T_D , $\overline{\{x\}} \setminus \{x\}$ is closed in X for each $x \in G$. Since G is discrete, it follows by Remark 3.3.9.1 that $G \cap \bigcup_{x \in G} \overline{\{x\}} \setminus \{x\} = \emptyset$, i.e. $\overline{G} \setminus G \supseteq \bigcup_{x \in G} \overline{\{x\}} \setminus \{x\}$. This is clearly an equality, i.e. $\overline{G} \setminus G = \bigcup_{x \in G} \overline{\{x\}} \setminus \{x\}$. Thus G is open in \overline{G} .

By the previous Theorem 7.1.2, $c(\mathcal{F}(X))$ is T_D .

\impliedby : Suppose $c(\mathcal{F}(X))$ is T_D . Then $c(\mathcal{F}_1(X))$ is T_D as a subspace of a T_D -space (cf. Remark 7.1.0.1). By Corollary 4.3.4 (2) and the hypothesis, X is homeomorphic to $c(\mathcal{F}_1(X))$. Thus X is T_D . ■

Corollary 7.1.4. *If X is T_0 then:*

2^X is T_D

iff 2^X is sober and T_D

iff $c(\mathcal{F}(X))$ is sober and T_D .

iff X is T_D and every closed set is the closure of some finite set in X .

Proof. Suppose 2^X is T_D . It is also sober by Corollary 5.1.13. Hence every subspace of 2^X is sober (cf. Proposition 7.1.1) and T_D (cf. Remark 7.1.0.1), including $c(\mathcal{F}(X))$ and $c(\mathcal{F}_n(X))$ for every $n \in \mathbb{N}$. Since X is T_0 by the hypothesis, the previous Corollary 7.1.3 implies that X is T_D . By Corollary 5.2.17, X is sober and every subset of X has finite cellularity. Thus every closed set in X is the closure of some finite set in X (cf. Proposition 3.4.1).

Conversely, if X is T_D and every closed set is the closure of some finite set in X then by Corollary 7.1.3, $2^X = c(\mathcal{F}(X))$ is T_D . ■

Definition 7.1.4.1. [1, Definition 3.2, Theorem 3.2] X is a T_F -space (or simply X is T_F) if $y \in \overline{\{x\}} \setminus \{x\}$ implies $\{y\}$ is closed for all $x \in X$.

Every T_1 -space is T_F and every T_F -space is T_0 ([1, p. 34]).

The T_D and T_F axioms are independent of each other as illustrated by [1, Example 4.1]

(which describes a T_F -space that is not a T_D -space) and [1, Example 4.4] (which describes a T_D -space that is not a T_F -space).

Remark 7.1.4.1. If X is a T_0 -space where all point-closures are finite, then X is also a T_D -space: given $x \in X$, $\overline{\{x\}} \setminus \{x\} = \bigcup_{y \in \overline{\{x\}} \setminus \{x\}} \{y\} = \bigcup_{y \in \overline{\{x\}} \setminus \{x\}} \overline{\{y\}}$ is closed in X .

Lemma 7.1.5. *If $\mathcal{F}_n(X)$ is T_0 for some $n > 1$ then X is T_D and T_F and $|\overline{\{x\}}| \leq 2$ for every $x \in X$.*

Proof. Let $y \in X$ such that $\{y\}$ is not closed.

Suppose to the contrary that $\overline{\{y\}} \setminus \{y\}$ has at least two distinct elements z_1, z_2 . Let $x_1, \dots, x_{n-2} \in X \setminus \{y, z_1, z_2\}$ be pairwise distinct. Then $\{y, z_1\} \cup \{x_1, \dots, x_{n-2}\}$ and $\{y, z_2\} \cup \{x_1, \dots, x_{n-2}\}$ are distinct members of $\mathcal{F}_n(X)$ with the same closure in X , which contradicts Theorem 4.4.1.

Thus $\overline{\{y\}} \setminus \{y\}$ consists of a single element, z .

Suppose to the contrary that $\{z\}$ is not closed. Then $\overline{\{z\}} = \{y, z\} = \overline{\{y\}}$. Let $x_1, \dots, x_{n-1} \in X \setminus \{y, z\}$ be pairwise distinct. Then $\{y\} \cup \{x_1, \dots, x_{n-1}\}$, $\{z\} \cup \{x_1, \dots, x_{n-1}\}$ are distinct members of $\mathcal{F}_n(X)$ with the same closure in X , which contradicts Theorem 4.4.1.

Thus $\{z\}$ is closed.

Hence X is a T_F -space and $|\overline{\{x\}}| \leq 2$ for all $x \in X$. By Remark 7.1.4.1, X is a T_D -space. ■

Theorem 7.1.6. *1. The following are equivalent:*

- (a) $\mathcal{F}_2(X)$ is T_0 .
- (b) X is T_0 and for every $x \in X$, $|\overline{\{x\}}| \leq 2$.
- (c) $\mathcal{F}_2(X)$ is T_D .

2. If $n > 2$, then the following are equivalent:

(a) $\mathcal{F}_n(X)$ is T_0 .

(b) X is T_1 or there is $c \in X$ such that $\{c\} = \overline{\{y\}} \setminus \{y\}$ for all $y \in X$ such that $|\overline{\{y\}}| > 1$.

(c) $\mathcal{F}_n(X)$ is T_D .

Proof. 1. $a \implies b$: Follows from the previous Lemma 7.1.5.

$b \implies c$: Suppose that $\{x, y\} \neq \{z, w\}$, where $x \neq y$ and $z \neq w$. Suppose without loss of generality that $z \notin \{x, y\}$.

If $|\overline{\{x, y\}}| = 2$, then $\overline{\{x, y\}} = \{x, y\} \neq \overline{\{z, w\}}$ (since $z \notin \{x, y\}$).

Suppose that $|\overline{\{x, y\}}| > 2$.

If $\overline{\{z\}} \cap \{x, y\} \neq \emptyset$, then since X is a T_0 -space by the hypothesis, either $z \notin \overline{\{x\}}$ or $z \notin \overline{\{y\}}$ and thus $z \in \overline{\{z, w\}} \setminus \overline{\{x, y\}}$, thus $\overline{\{z, w\}} \neq \overline{\{x, y\}}$.

Suppose that $x, y \notin \overline{\{z\}}$. Since $|\overline{\{w\}}| \leq 2$ by the hypothesis and $|\overline{\{x, y\}}| > 2$, it must be the case that $\overline{\{x, y\}} \not\subseteq \overline{\{w\}}$ and hence either $x \notin \overline{\{w\}}$ or $y \notin \overline{\{w\}}$, respectively implying that $x \in \overline{\{x, y\}} \setminus \overline{\{z, w\}}$ or $y \in \overline{\{x, y\}} \setminus \overline{\{z, w\}}$. In either case, $\overline{\{x, y\}} \neq \overline{\{z, w\}}$.

Thus in all cases, $\overline{\{x, y\}} \neq \overline{\{z, w\}}$.

By Theorem 4.4.1, $\mathcal{F}_2(X)$ is T_0 . Hence $\mathcal{F}_2(X)$ is homeomorphic to $c(\mathcal{F}_2(X))$ by Theorem 4.3.3.

By Lemma 7.1.5, X is a T_D -space. By Corollary 7.1.3, $c(\mathcal{F}(X))$ is a T_D -space; by Remark 7.1.0.1, $c(\mathcal{F}_2(X))$ is a T_D -space. Thus $\mathcal{F}_2(X)$ is T_D (by the previous paragraph).

$c \implies a$: Every T_D -space is T_0 (cf. Definition 7.1.0.1).

2. $a \implies b$: By the previous Lemma 7.1.5, X is a T_F -space and $|\overline{\{x\}}| \leq 2$ for every $x \in X$.

Suppose that X is not a T_1 -space. If there is only one element y of X where $\{y\}$ is not closed then $\overline{\{y\}} \setminus \{y\} = \{c\}$ for some $c \in X$ and so the conclusion is satisfied.

Let $y, z \in X$ be distinct such that $|\overline{\{y\}}|, |\overline{\{z\}}| > 1$. Since $|\overline{\{y\}}|, |\overline{\{z\}}| \leq 2$ and X is T_F , there are $k, l \in X$ such that $\{k\} = \overline{\{y\}} \setminus \{y\}$ and $\{l\} = \overline{\{z\}} \setminus \{z\}$ are closed in X . Thus $l \neq y$ and $k \neq z$. Suppose to the contrary that $k \neq l$. Let $x_1, \dots, x_{n-3} \in X \setminus \{y, z, k, l\}$ be pairwise distinct. Then the sets $\{y, z, k\} \cup \{x_1, \dots, x_{n-3}\}$ and $\{y, z, l\} \cup \{x_1, \dots, x_{n-3}\}$ are distinct members of $\mathcal{F}_n(X)$ with the same closures in X , which contradicts Theorem 4.4.1. Thus, $k = l$ and so $\overline{\{y\}} \setminus \{y\} = \overline{\{z\}} \setminus \{z\}$; moreover these sets consist of a single element.

$b \implies c$: Suppose that $F \neq G$ where $|F| = n = |G|$.

If X is a T_1 -space then $\overline{F} = F \neq G = \overline{G}$.

Suppose X is not a T_1 -space. By the hypothesis there is $c \in X$ such that $\{c\} = \overline{\{y\}} \setminus \{y\}$ whenever $y \in X$ and $|\overline{\{y\}}| > 1$.

Since G, F are distinct finite sets with the same cardinality it follows that $G \setminus \{c\} \neq F \setminus \{c\}$.

Let $x \in \overline{G} \setminus \{c\}$. There is $z \in G$ such that $x \in \overline{\{z\}}$. If $z \neq x$ then $|\overline{\{z\}}| > 1$ so $\overline{\{z\}} = \{z, c\}$, hence $x = c$, a contradiction to the assumption that $x \neq c$. Thus $x = z \in G \setminus \{c\}$. Thus $\overline{G} \setminus \{c\} = G \setminus \{c\}$. Similarly it is true that $\overline{F} \setminus \{c\} = F \setminus \{c\}$.

Thus $\overline{G} \setminus \{c\} \neq \overline{F} \setminus \{c\}$ and hence $\overline{G} \neq \overline{F}$.

Thus $\mathcal{F}_n(X)$ is a T_0 -space by Theorem 4.4.1. Hence $\mathcal{F}_n(X)$ is homeomorphic to $c(\mathcal{F}_n(X))$ by Theorem 4.3.3.

By Lemma 7.1.5, X is a T_D -space. By Corollary 7.1.3, $c(\mathcal{F}(X))$ is a T_D -space; by Remark 7.1.0.1, $c(\mathcal{F}_n(X))$ is a T_D -space. Thus $\mathcal{F}_n(X)$ is T_D (by the previous paragraph).

$c \implies a$: Every T_D -space is T_0 (cf. Definition 7.1.0.1). ■

Corollary 7.1.7. *Let $n > 1$.*

1. $\mathcal{F}_n(X)$ is T_0 iff $\mathcal{F}_n(X)$ is T_D .
2. $\mathcal{F}_n(X)$ is sober iff $\mathcal{F}_n(X)$ is sober and T_D .
3. If $n > 2$ and $\mathcal{F}_n(X)$ is T_0 (resp. sober) then $\mathcal{F}_m(X)$ is T_D (resp. sober and T_D) for all $m \in \mathbb{N}$.
4. If $\mathcal{F}_n(X)$ is sober then X is sober, T_D , T_F and $|\overline{\{x\}}| \leq 2$ for all $x \in X$.
5. The following are equivalent:
 - (a) $\mathcal{F}_2(X)$ is sober.
 - (b) X is sober and for every $x \in X$, $|\overline{\{x\}}| \leq 2$.
6. If $n > 2$, then the following are equivalent:
 - (a) $\mathcal{F}_n(X)$ is sober.
 - (b) X is sober and either X is T_1 or there is $c \in X$ such that $\{c\} = \overline{\{y\}} \setminus \{y\}$ for all $y \in X$ such that $|\overline{\{y\}}| > 1$.

Proof. Points 1, 2 and 3 follow from Theorem 7.1.6. Point 4 follows from Lemma 7.1.5 and Theorem 5.2.7. Points 5 and 6 follow from Theorem 5.2.7 and Theorem 7.1.6. ■

Example 7.1.7.1 (X is not sober but is T_D and T_F while for every $n > 1$, $\mathcal{F}_n(X)$ is T_D but is neither sober nor T_1).

Suppose that $X = Y \cup \{c\}$ where Y is infinite and consider the partial order \leq on X where $c \leq y$ for all $y \in Y$. Consider the upper interval topology on X . Now X is an irreducible T_0 -space as for $x_1, \dots, x_n \in X$, $X \neq \bigcup_{i=1}^n \overline{\{x_i\}}$ (cf. Corollary 3.3.3). Thus by Theorem 7.1.6 $\mathcal{F}_n(X)$ is T_D for every $n > 1$. By Lemma 7.1.5, X is T_D and T_F . Since $X \neq \downarrow x = \overline{\{x\}}$ for any $x \in X$, it follows that X is not sober. By Corollary 7.1.7, $\mathcal{F}_n(X)$ is not sober.

Let $y_1, \dots, y_n \in Y$. Then $\{y_1, \dots, y_{n-1}, c\}$ and $\{y_1, \dots, y_n\}$ are distinct members of $\mathcal{F}_n(X)$ where the closure of the former in X is contained in the closure of the latter in X . Thus $(c(\mathcal{F}_n(X)), \subseteq)$ is not an antichain so by Corollary 4.4.3, $\mathcal{F}_n(X)$ is not T_1 .

Example 7.1.7.2 (X is not sober but is T_D and T_F , $\mathcal{F}_2(X)$ is T_D but neither sober nor T_1 and $\mathcal{F}_n(X)$ is not a T_0 -space for $n > 2$).

Let $X = \mathbb{N} \cup (\mathbb{N} \times \mathbb{N})$ and consider the partial order defined by $x \leq (x, i)$ whenever $i \leq x$ and $i, x \in \mathbb{N}$. Consider the upper interval topology on X . For each $x, i \in \mathbb{N}$, $\overline{\{x\}} = \{x\}$ and $\overline{\{(x, i)\}} = \{(x, i), x\}$. Since X is T_0 it follows by Theorem 7.1.6 that $\mathcal{F}_2(X)$ is a T_D -space. By Lemma 7.1.5, X is T_F and T_D . However for $n > 2$, if x, y are distinct natural numbers then $\overline{\{(x, i)\}} \setminus \{(x, i)\} = \{x\} \neq \{y\} = \overline{\{(y, i)\}} \setminus \{(y, i)\}$; hence $\mathcal{F}_n(X)$ is not a T_0 -space by Theorem 7.1.6. Using a similar argument as in Example 7.1.7.1, X is not a sober space and thus by Corollary 7.1.7, $\mathcal{F}_2(X)$ is not a sober space.

$\{(1, 1), 1\}$ and $\{(1, 1), (2, 1)\}$ are distinct members of $\mathcal{F}_2(X)$ where the closure of the former in X is contained in the closure of the latter in X . Thus $c(\mathcal{F}_2(X))$ is not an antichain so by Corollary 4.4.3, $\mathcal{F}_2(X)$ is not T_1 .

Example 7.1.7.3 (X is sober, T_D , T_F and not T_1 while $\mathcal{F}_n(X)$ is sober, T_D and not T_1 for every $n > 1$).

Consider $X := \mathbb{R} \cup \{\perp\}$ where the open sets are the same as in \mathbb{R} with the usual topology. Then $\overline{\{x\}} = \{x, \perp\}$ for each $x \neq \perp$ and $\overline{\{\perp\}} = \{\perp\}$. Thus X is not T_1 . Let A be a non-empty closed irreducible subset of X . Suppose to the contrary A contains real numbers $x < y$. Then $(y, \infty) \cap (-\infty, x) \cap A = \emptyset$, a contradiction to the irreducibility of A . Thus A must contain at most one real number. Hence $A = \overline{\{x\}}$ for some real number x or $A = \{\perp\} = \overline{\{\perp\}}$. Thus X is sober. By Corollary 7.1.7 (5) and (6), $\mathcal{F}_n(X)$ is sober and T_D for every $n > 1$. By Corollary 7.1.7, X is T_D and T_F .

Let $n > 1$. Then $\{1, \dots, n-1, \perp\}$ and $\{1, \dots, n\}$ are distinct members of $\mathcal{F}_n(X)$ where the closure of the former in X is contained in the closure of the latter in X . Thus $c(\mathcal{F}_n(X))$ is not an antichain so by Corollary 4.4.3, $\mathcal{F}_n(X)$ is not T_1 .

Example 7.1.7.4 (X is sober, T_D , T_F but $\mathcal{F}_n(X)$ is not T_0 for $n > 1$).

Adjust the previous Example 7.1.7.3 by adding the point \perp' to X . Then X is still sober, T_D , T_F and not T_1 while by Lemma 7.1.5, $\mathcal{F}_n(X)$ is not T_0 for any $n > 1$ (as if $x \in \mathbb{R}$

then $\overline{\{x\}} = \{x, \perp, \perp'\}$ has 3 elements).

Example 7.1.7.5 (X is sober, $T_D, T_F, \mathcal{F}_2(X)$ is sober T_D but not T_1 , $\mathcal{F}_n(X)$ is not T_0 for $n > 2$).

Consider X as in the previous Example 7.1.7.4. Consider the topology on X generated by:

$$\{U \cup \{\perp\} : U \text{ open in } \mathbb{R}, (0, \infty) \subseteq U\} \cup \{U \cup \{\perp'\} : U \text{ open in } \mathbb{R}, (-\infty, 0) \subseteq U\}.$$

If U is open in X and $U \ni \perp$ then there are sub-basic opens V_1, \dots, V_m where $U \supseteq \bigcap_{i=1}^m V_i \ni \perp$, hence $U \supseteq \bigcap_{i=1}^m V_i \supseteq (0, \infty)$.

Similarly any open in X that has \perp' as an element must contain $(-\infty, 0)$.

Hence for $x < 0$, $\overline{\{x\}} \setminus \{x\} = \{\perp'\}$ and for $x > 0$, $\overline{\{x\}} \setminus \{x\} = \{\perp\}$.

Moreover, $\{0\}, \{\perp\}, \{\perp'\}$ are closed.

By a similar argument as in Example 7.1.7.3, it can be verified that X is sober but not T_1 . By Corollary 7.1.7 (2) and (5), $\mathcal{F}_2(X)$ is sober and T_D but by Theorem 7.1.6 $\mathcal{F}_n(X)$ is not T_0 for any $n > 2$. By Lemma 7.1.5, X is T_D and T_F .

Using a similar argument as in Example 7.1.7.3, one can verify that $\mathcal{F}_2(X)$ is not T_1 .

7.2 Symmetry and T_1 separation axioms in $\mathcal{F}_n(X)$

In this section, a necessary condition is found for $\mathcal{F}_n(X)$ to be a non-trivial symmetric space (cf. Theorem 7.2.4) and it is shown that $\mathcal{F}_n(X)$ is T_1 iff X is T_1 (cf. Theorem 7.2.5); these are new results.

Lemma 7.2.1. *For $U \subseteq X$, $\mathcal{F}_n(U)$ is a trivial space iff $U \subseteq \overline{F}$ for every n -element subset F of U . Moreover if $V \subseteq U$ and $\mathcal{F}_n(U)$ is a trivial space then $\mathcal{F}_n(V)$ is a trivial space.*

Proof. Note that for an n -element subset F of X :

$$\begin{aligned}
\text{Cl}_{\mathbf{L}(\tau)|\mathcal{F}_n(X)}(\{F\}) = \mathcal{F}_n(X) &\iff \langle \overline{F} \rangle \cap \mathcal{F}_n(X) = \mathcal{F}_n(X) \quad (\text{By Proposition 4.2.6}) \\
&\iff \mathcal{F}_n(X) \subseteq \langle \overline{F} \rangle \\
&\iff X \subseteq \overline{F} \\
&\iff X = \overline{F}
\end{aligned}$$

Thus $\mathcal{F}_n(X)$ is a trivial space

iff for every n -element subset F of X , $\text{Cl}_{\mathbf{L}(\tau)|\mathcal{F}_n(X)}(\{F\}) = \mathcal{F}_n(X)$

iff for every n -element subset F of X , $X = \overline{F}$.

Thus, if $U \subseteq X$, then $\mathcal{F}_n(U)$ is a trivial space

iff for any n -element subset F of U , $\text{Cl}_{\tau|U}(F) = U$

iff for any n -element subset F of U , $U \subseteq \overline{F}$.

Now let $V \subseteq U$ and suppose $\mathcal{F}_n(U)$ is a trivial space. For any n -element subset G of V , G is an n -element subset of U so $V \subseteq U \subseteq \overline{G}$ by the workings so far. By the workings so far, $\mathcal{F}_n(V)$ is a trivial space. ■

Definition 7.2.1.1. For each $n \in \mathbb{N}$, let $H_n := \bigcap \{\overline{F} : F \subseteq X \text{ and } |F| = n - 1\}$.

Lemma 7.2.2. *Let $n \in \mathbb{N}$. Suppose $\mathcal{F}_n(X)$ is a symmetric space and that $H_n = \emptyset$. Then X is a symmetric space.*

Proof. Let $x, y \in X$ such that $x \in \overline{\{y\}}$.

Let F be an $n - 1$ -element subset of X such that $x \notin \overline{F}$ (which exists as $H_n = \emptyset$ by the hypothesis). Hence $y \notin \overline{F}$ as $x \in \overline{\{y\}}$. Thus $\{x\} \cup F, \{y\} \cup F \in \mathcal{F}_n(X)$.

Now, $\overline{\{x\} \cup F} \subseteq \overline{\{y\} \cup F}$ because $x \in \overline{\{y\}}$.

Since $\mathcal{F}_n(X)$ is a symmetric space (by the hypothesis), it follows by Theorem 4.4.1 (2) that $(c(\mathcal{F}_n(X)), \subseteq)$ is an antichain. Thus $\overline{\{x\} \cup F} = \overline{\{y\} \cup F}$ and hence $y \in \overline{\{y\} \cup F} = \overline{\{x\} \cup F} = \overline{\{x\}} \cup \overline{F}$. Since $y \notin \overline{F}$, it follows that $y \in \overline{\{x\}}$.

Thus, $\overline{\{y\}} = \overline{\{x\}}$.

Hence X is a symmetric space. ■

Remark 7.2.2.1. Let n be the smallest natural number where $\mathcal{F}_n(X)$ is a trivial space.

Note X is trivial iff $n = 1$ (cf. Lemma 7.2.1). Suppose $n > 1$.

Then $\mathcal{F}_{n-1}(X)$ is non-trivial so by Lemma 7.2.1, there is an $n - 1$ -element subset F of X where $\overline{F} \neq X$. Then F is closed (otherwise $\overline{F} = \overline{F \cup \{k\}} = X$ for any $k \in \overline{F} \setminus F$). Moreover, $F \cup \overline{\{x\}} = X$ for any $x \notin F$. Since X is infinite, $\overline{\{x\}}$ is infinite. Since $\mathcal{F}_n(X)$ is trivial it follows by Lemma 7.2.1 that $\overline{\{x\}} = X$.

Thus X is neither T_0 nor symmetric and $H_n = F$.

Lemma 7.2.3. *Suppose $\mathcal{F}_n(X)$ is a symmetric space. Then:*

1. $|H_n| \geq n$ iff $H_n = X$ iff $\mathcal{F}_{n-1}(X)$ is a trivial space.
2. $H_n \neq \emptyset$ iff $\mathcal{F}_n(X)$ is a trivial space.

Proof. Let F be an $n - 1$ -element subset of X for which $H_n \not\subseteq F$; let $x \in H_n \setminus F$. Then $x \in \overline{F}$.

Let $y \in X \setminus F$. Then $\overline{\{x\} \cup F} = \overline{F} \subseteq \overline{\{y\} \cup F}$. Now $\mathcal{F}_n(X)$ is a symmetric space by the hypothesis so by Theorem 4.4.1 (2), $c(\mathcal{F}_n(X))$ is an antichain. Hence $\overline{F} = \overline{\{x\} \cup F} = \overline{\{y\} \cup F}$. Thus $y \in \overline{F}$.

Thus $X = \overline{F}$ whenever F is an $n - 1$ -element subset of X such that $H_n \not\subseteq F$. This implies that:

$$\begin{aligned} H_n &= \bigcap \{ \overline{F} : H_n \subseteq F \subseteq X \text{ and } |F| = n - 1 \} \\ &= \bigcap \{ F : H_n \subseteq F \subseteq X \text{ and } |F| = n - 1 \}. \end{aligned}$$

1. \implies : If $|H_n| \geq n$, then $\{F : H_n \subseteq F \subseteq X \text{ and } |F| = n - 1\} = \emptyset$. Hence, $H_n = X$ and thus by Lemma 7.2.1, $\mathcal{F}_{n-1}(X)$ is a trivial space.

\Leftarrow : If $\mathcal{F}_{n-1}(X)$ is a trivial space, then by Lemma 7.2.1, $\overline{F} = X$ for every $n - 1$ element subset F of X ; thus $H_n = X$ so $|H_n| \geq n$.

2. \Rightarrow : Let F be an n -element subset of X . Since $H_n \neq \emptyset$ by the hypothesis, there is $x \in H_n$, so $H_n \not\subseteq F \setminus \{x\}$ and hence $\overline{F \setminus \{x\}} = X$ (since $|F \setminus \{x\}| = n - 1$ and by a previous argument in this proof). Thus, $\overline{F} = X$.

By Lemma 7.2.1, $\mathcal{F}_n(X)$ is a trivial space.

\Leftarrow : Follows from Remark 7.2.2.1. ■

Thus,

Theorem 7.2.4. *If $\mathcal{F}_n(X)$ is a non-trivial symmetric space, then $H_n = \emptyset$ and X is a symmetric space.*

Proof. Follows from the previous two Lemmas 7.2.2 and 7.2.3. ■

Theorem 7.2.5. *$\mathcal{F}_n(X)$ is a T_1 -space iff X is a T_1 -space.*

Proof. \Rightarrow : Since $\mathcal{F}_n(X)$ is a T_0 -space, X is also a T_0 -space by Theorem 7.1.6. Moreover, $\mathcal{F}_n(X)$ is a non-trivial symmetric space, so X is a symmetric space by the previous Theorem 7.2.4. Thus X is a T_1 -space.

\Leftarrow : Let X be a T_1 -space. Then $\mathcal{F}_n(X) = c(\mathcal{F}_n(X))$ so $(c(\mathcal{F}_n(X)), \subseteq) = (\mathcal{F}_n(X), \subseteq)$ is an antichain. Hence, by Corollary 4.4.3 (1), $\mathcal{F}_n(X) = c(\mathcal{F}_n(X))$ is a T_1 -space. ■

7.3 Preregular and Hausdorff properties in $\mathcal{F}_n(X)$

In this section, necessary and sufficient conditions for $\mathcal{F}_n(X)$ to be a non-trivial preregular space are determined in terms of the properties of X (cf. Theorem 7.3.5, Corollary 7.3.6). Examples 7.3.6.1, 7.3.6.2 are provided to illustrate that the results cannot be improved.

The section concludes with a verification that for each $n \in \mathbb{N}$, $\mathcal{F}_n(X)$ is Hausdorff iff X is Hausdorff (cf. Theorem 7.3.7).

The interest in studying preregular spaces is that every preregular (resp. Hausdorff) space is quasi-sober (resp. sober).

Definition 7.3.0.1. [47, p. 17] X is said to be a *preregular space* if whenever $x, y \in X$ such that $\overline{\{x\}} \neq \overline{\{y\}}$ then there are disjoint open subsets V, W of X such that $x \in V$ and $y \in W$.

A preregular space is also called an R_1 -space and was introduced as such by A.S. Davis in [10, p. 889], where the preregularity separation axiom is called a *regularity axiom*. See also [46, Chapter I, 4.5].

Remark 7.3.0.1. If X is trivial, then X is an irreducible preregular space from the definitions. Conversely if X is an irreducible, preregular space, then no two opens of X are disjoint, so X cannot have two elements with distinct closures and thus X is a trivial space.

Proposition 7.3.1. 1. *Every subspace of a preregular space is preregular.*

2. *Every preregular space is quasi-sober and symmetric.*

3. [36, 1.6 Lemma (ii), p. 43] *Every Hausdorff space is sober.*

4. [46, 4.5.2] *X is Hausdorff iff X is preregular and T_0 .*

Proof. 1. Let U be a subspace of a preregular space X . Let $x, y \in U$ such that $\text{Cl}_{\tau|U}(\{x\}) \neq \text{Cl}_{\tau|U}(\{y\})$. Then $\overline{\{x\}} \cap U \neq \overline{\{y\}} \cap U$ so $\overline{\{x\}} \neq \overline{\{y\}}$. Since X is preregular by the hypothesis there are disjoint open sets V, W such that $x \in V$, $y \in W$. Hence $V \cap U, W \cap U$ are disjoint opens in U where $x \in V \cap U$ and $y \in W \cap U$. Thus U is preregular.

2. Let A be a non-empty closed irreducible subspace of a preregular space X . Now A is preregular by the previous point, so A must be a trivial space by Remark 7.3.0.1.

Hence there is some $x \in A$ where $A = \text{Cl}_{\tau|_A}(\{x\}) = \overline{\{x\}} \cap A$, which is equal to $\overline{\{x\}}$ as A is closed. Thus X is quasi-sober. The fact that a preregular space is symmetric is given in [10, p. 889,890] and [46, p. 9]. ■

The result in the following Proposition 7.3.2 is a generalisation of the result in [3, Chapter I, §8.1, Proposition 3].

Proposition 7.3.2. *Suppose that X is a preregular space and that \mathcal{B} is a base for X . Given elements x_1, \dots, x_n of X where $n > 1$ and $\overline{\{x_i\}} \neq \overline{\{x_j\}}$ whenever $i \neq j$, there is a cellular subfamily $\{V_1, \dots, V_n\}$ of \mathcal{B} such that $x_i \in V_i$ for all $i \in \{1, \dots, n\}$.*

Proof. The result is proved by induction on $n \geq 2$.

For $n = 2$: Let $x_1, x_2 \in X$ such that $\overline{\{x_1\}} \neq \overline{\{x_2\}}$. Since X is a preregular space by the hypothesis, there are disjoint open subsets V, W of X such that $x_1 \in V$ and $x_2 \in W$. There are $B_1, B_2 \in \mathcal{B}$ such that $x_1 \in B_1 \subseteq V$ and $x_2 \in B_2 \subseteq W$. Hence, $B_1 \cap B_2 = \emptyset$ since $V \cap W = \emptyset$.

Suppose that $n > 2$. Suppose that for every $k \in \mathbb{N}$ such that $2 \leq k < n$, if x_1, \dots, x_k are elements of X such that $\overline{\{x_i\}} \neq \overline{\{x_j\}}$ whenever $i \neq j$, then there is a cellular subfamily $\{V_1, \dots, V_k\}$ of \mathcal{B} such that $x_i \in V_i$ for all $i \in \{1, \dots, k\}$.

Let x_1, \dots, x_n be elements of X such that $\overline{\{x_i\}} \neq \overline{\{x_j\}}$ whenever $i \neq j$. By the induction hypothesis, there is a cellular subfamily $\{V_1, \dots, V_{n-1}\}$ of \mathcal{B} such that $x_i \in V_i$ for all $i \in \{1, \dots, n-1\}$. For each $i \in \{1, \dots, n-1\}$, $\overline{\{x_n\}} \neq \overline{\{x_i\}}$, so by the induction hypothesis there is a cellular subfamily $\{U_i, W_i\}$ of \mathcal{B} such that $x_n \in U_i$ and $x_i \in W_i$. Hence $x_n \in \bigcap_{i=1}^n U_i$, so there is $U \in \mathcal{B}$ such that $x_n \in U \subseteq \bigcap_{i=1}^n U_i$. For each $i \in \{1, \dots, n-1\}$, since $x_i \in W_i \cap V_i$, there is $Y_i \in \mathcal{B}$ for which $x_i \in Y_i \subseteq W_i \cap V_i$. By construction, $\{Y_1, \dots, Y_{n-1}, U\}$ is a subfamily of \mathcal{B} ; it needs to be verified that this is a cellular family. For distinct $i, j \in \{1, \dots, n-1\}$, $V_i \cap V_j = \emptyset$ by assumption, thus:

$$Y_i \cap Y_j \subseteq (W_i \cap V_i) \cap (W_j \cap V_j) = (W_i \cap W_j) \cap (V_i \cap V_j) = (W_i \cap W_j) \cap \emptyset = \emptyset,$$

hence

$$Y_i \cap Y_j = \emptyset.$$

For $i \in \{1, \dots, n-1\}$, $W_i \cap U_i = \emptyset$ by assumption, so $W_i \cap U = \emptyset$ (since $U \subseteq U_i$). Thus:

$$Y_i \cap U \subseteq W_i \cap V_i \cap U = \emptyset.$$

Thus:

$$Y_i \cap U = \emptyset.$$

Thus $\{Y_1, \dots, Y_{n-1}, U\}$ is a cellular subfamily of \mathcal{B} such that for each $i \in \{1, \dots, n-1\}$, $x_i \in Y_i$ and $x_n \in U$.

Thus it has been proved by induction that for every $n \in \mathbb{N}$ such that $n > 1$, if x_1, \dots, x_n are elements of X such that $\overline{\{x_i\}} \neq \overline{\{x_j\}}$ whenever $i \neq j$, then there is a cellular subfamily $\{V_1, \dots, V_n\}$ of \mathcal{B} such that $x_i \in V_i$ for all $i \in \{1, \dots, n\}$. ■

Lemma 7.3.3. *Suppose $\mathcal{F}_n(X)$ is a preregular space. The following statements are equivalent for $U \subseteq X$:*

1. $\mathcal{F}_n(U)$ is a trivial space.
2. $\text{cell}(U) \leq n$.
3. $\overline{U} = \overline{F}$ for some $F \subseteq U$ such that $|F| \leq n$.
4. $U \subseteq \overline{F}$ for some $F \subseteq X$ such that $|F| = n$.

Proof. 1 \implies 2: By Lemma 7.2.1, for every n -element subset F of U , $U = \text{Cl}_{\tau|U}(F)$ and hence $\text{cell}(U) = \text{cell}(F) \leq |F| = n$.

2 \implies 3: By Proposition 7.3.1, $\mathcal{F}_n(X)$ is a quasi-sober space and hence by Corollary 5.2.8 X is a quasi-sober space. Hence this statement follows from Proposition 3.4.1.

3 \implies 4: Obvious.

4 \implies 1: Note that $\text{cell}(\overline{F}) \leq |F| = n$. By Corollary 5.1.18, $\mathcal{F}_n(\overline{F})$ is an irreducible space, moreover it is prerregular by Proposition 7.3.1 (1). Thus by Remark 7.3.0.1, $\mathcal{F}_n(\overline{F})$ is a trivial space. By Lemma 7.2.1, $\mathcal{F}_n(U)$ is a trivial space. \blacksquare

Recall that Remark 7.2.2.1 shows that X may not be symmetric while $\mathcal{F}_n(X)$ is trivial (and thus prerregular). Thus the requirement that $\mathcal{F}_n(X)$ be non-trivial in the following Lemma 7.3.4, Theorem 7.3.5 and Corollary 7.3.6 is essential.

Lemma 7.3.4. *If $\mathcal{F}_n(X)$ is a non-trivial prerregular space then X is a prerregular space.*

Proof. Suppose to the contrary that X is not a prerregular space while $\mathcal{F}_n(X)$ is a non-trivial prerregular space. Then there are $x, y \in X$ where $\overline{\{x\}} \neq \overline{\{y\}}$ and whenever V, W are open subsets of X such that $x \in V$ and $y \in W$ then $V \cap W \neq \emptyset$. (*)

By Proposition 7.3.1 (2), $\mathcal{F}_n(X)$ is a symmetric space. Since this space is non-trivial (by the hypothesis) it follows by Theorem 7.2.4 that X is a symmetric space.

Suppose to the contrary that $|X \setminus \overline{\{x\}}| < n$. Then $\{x\} \cup (X \setminus \overline{\{x\}})$ has at most n elements and is dense in X . But by Lemma 7.3.3, there is no subset F of X for which $|F| \leq n$ and $\overline{F} = X$, a contradiction. Thus, $|X \setminus \overline{\{x\}}| \geq n$.

Suppose without loss of generality that $x \notin \overline{\{y\}}$ and let F be an $n - 1$ -element subset of $X \setminus (\overline{\{x\}} \cup \{y\})$. Since X is symmetric, $\overline{F} \cap \overline{\{x\}} = \emptyset$. Thus, $F \cup \{x\}, F \cup \{y\} \in \mathcal{F}_n(X)$. Moreover, $\overline{F \cup \{x\}} \neq \overline{F \cup \{y\}}$ because the latter does not contain x . By Corollary 4.2.7, $\text{Cl}_{\mathbf{L}(\tau)|_{\mathcal{F}_n(X)}}(\{F \cup \{x\}\}) = \text{Cl}_{\mathbf{L}(\tau)|_{\mathcal{F}_n(X)}}(\{F \cup \{y\}\})$.

By Proposition 4.2.3, the following collection is a base for the Lower Vietoris Topology on $\mathcal{F}_n(X)$:

$$\{\langle X, V_1, \dots, V_s \rangle \cap \mathcal{F}_n(X) : s \in \mathbb{N}, V_1, \dots, V_s \text{ are open subsets of } X\}.$$

Since $\mathcal{F}_n(X)$ is prerregular by the hypothesis, it follows by Proposition 7.3.2 that there is $m, k \in \mathbb{N}$ such that $m < k$ and open subsets V_1, \dots, V_m and V_{m+1}, \dots, V_k of X such that:

$$F \cup \{x\} \in \langle X, V_1, \dots, V_m \rangle \cap \mathcal{F}_n(X) \quad \text{and} \quad (7.1)$$

$$F \cup \{y\} \in \langle X, V_{m+1}, \dots, V_k \rangle \cap \mathcal{F}_n(X) \quad \text{while} \quad (7.2)$$

$$\langle X, V_1, \dots, V_m \rangle \cap \mathcal{F}_n(X) \cap \langle X, V_{m+1}, \dots, V_k \rangle \cap \mathcal{F}_n(X) = \emptyset. \quad (7.3)$$

By (7.3) and (7.2), $F \cup \{y\} \notin \langle X, V_1, \dots, V_m \rangle \cap \mathcal{F}_n(X)$. Thus there is $i \in \{1, \dots, m\}$ such that $V_i \cap (F \cup \{y\}) = \emptyset$, in which case $V_i \cap (F \cup \{x\}) \neq \emptyset$ (By (7.1)) and thus $x \in V_i$.

Similarly, it can be shown there is $j \in \{m+1, \dots, k\}$ such that $y \in V_j$.

Thus the sets $I_x := \{i \in \{1, \dots, m\} : x \in V_i\}$ and $I_y := \{j \in \{m+1, \dots, m+k\} : y \in V_j\}$ are non-empty. Let $U := \bigcap_{i \in I_x} V_i$ and $W := \bigcap_{j \in I_y} V_j$. Then U, W are open subsets of X such that $x \in U$ and $y \in W$. By the assumption (*), there is $u \in U \cap W$.

Let $i \in \{1, \dots, k\}$.

1. Suppose that $i \in I_x \cup I_y$. Then $x \in V_i$ or $y \in V_i$, from which it follows that $U \subseteq V_i$ or $W \subseteq V_i$, respectively (by definition of U, W). Hence, $u \in U \cap W \subseteq V_i$. Thus, $(F \cup \{u\}) \cap V_i \neq \emptyset$.
2. Suppose that $i \notin I_x \cup I_y$.
 - (a) Suppose that $i \in \{1, \dots, m\}$. By (7.1), $(F \cup \{x\}) \cap V_i \neq \emptyset$. Since $i \notin I_x$, it is true that $x \notin V_i$ and hence $F \cap V_i \neq \emptyset$.
 - (b) Suppose that $i \in \{m+1, \dots, k\}$. By (7.2), $(F \cup \{y\}) \cap V_i \neq \emptyset$. Since $i \notin I_y$, it is true that $y \notin V_i$ and hence $F \cap V_i \neq \emptyset$.

Thus in all cases $F \cap V_i \neq \emptyset$. This implies that $(F \cup \{u\}) \cap V_i \neq \emptyset$.

Thus $(F \cup \{u\}) \cap V_i \neq \emptyset$ for all i .

Hence:

$$\begin{aligned} F \cup \{u\} &\in \langle X, V_1, \dots, V_m, V_{m+1}, \dots, V_k \rangle \cap \mathcal{F}_n(X) \\ &= \langle X, V_1, \dots, V_m \rangle \cap \langle X, V_{m+1}, \dots, V_k \rangle \cap \mathcal{F}_n(X) \quad (\text{By Corollary 4.1.2}) \\ &= (\langle X, V_1, \dots, V_m \rangle \cap \mathcal{F}_n(X)) \cap (\langle X, V_{m+1}, \dots, V_k \rangle \cap \mathcal{F}_n(X)) \\ &= \emptyset \quad (\text{By (7.3)}). \end{aligned}$$

There is a contradiction.

Thus X must be a preregular space. ■

Theorem 7.3.5. *Suppose $\mathcal{F}_n(X)$ is a non-trivial space. If X is a preregular space and for every $F, G \in \mathcal{F}_n(X)$, if $\overline{F} \neq \overline{G}$ then $\text{cell}(F \cup G) > n$, then it must be the case that $\mathcal{F}_n(X)$ is a preregular space.*

Proof. \implies : By Lemma 7.3.4, X is a preregular space. Let $F, G \in \mathcal{F}_n(X)$ such that $\overline{F} \neq \overline{G}$. Suppose without loss of generality that $\overline{G} \setminus \overline{F} \neq \emptyset$. Then $\overline{F} \neq \overline{F \cup G}$. By Lemma 7.2.1, $\mathcal{F}_n(F \cup G)$ is not a trivial space. Thus by Lemma 7.3.3, $\text{cell}(F \cup G) > n$.

\impliedby : Let $F, G \in \mathcal{F}_n(X)$ such that $\text{Cl}_{\mathbf{L}(\tau)|_{\mathcal{F}_n(X)}}(\{F\}) \neq \text{Cl}_{\mathbf{L}(\tau)|_{\mathcal{F}_n(X)}}(\{G\})$. By Corollary 4.2.7, $\overline{F} \neq \overline{G}$. By the hypothesis, $\text{cell}(F \cup G) > n$. By Remark 3.3.9.1, there are elements x_1, \dots, x_{n+1} of $F \cup G$ such that $\overline{\{x_i\}} \neq \overline{\{x_j\}}$ for distinct $i, j \in \{1, \dots, n+1\}$. Since X is a preregular space by the hypothesis, it follows by Proposition 7.3.2 there is a pairwise disjoint family V_1, \dots, V_{n+1} of open subsets of X such that $x_i \in V_i$ for each $i \in \{1, \dots, n+1\}$.

Since $|F| = |G| = n$, $\{x_1, \dots, x_{n+1}\} \cap F \neq \emptyset$ and $\{x_1, \dots, x_{n+1}\} \cap G \neq \emptyset$. Thus:

$$F \in \langle \{X\} \cup \{V_i : x_i \in F\} \rangle =: \mathcal{A}_F \quad \text{and} \quad G \in \langle \{X\} \cup \{V_i : x_i \in G\} \rangle =: \mathcal{A}_G \quad \text{while}$$

$$\begin{aligned} (\mathcal{A}_F \cap \mathcal{F}_n(X)) \cap (\mathcal{A}_G \cap \mathcal{F}_n(X)) &= (\mathcal{A}_F \cap \mathcal{A}_G) \cap \mathcal{F}_n(X) \\ &= \langle X, V_1, \dots, V_{n+1} \rangle \cap \mathcal{F}_n(X) \quad (\text{By Corollary 4.1.2}) \\ &= \emptyset \end{aligned}$$

By Proposition 4.2.3, $\mathcal{A}_F \cap \mathcal{F}_n(X)$, $\mathcal{A}_G \cap \mathcal{F}_n(X)$ are non-empty open subsets of $\mathcal{F}_n(X)$.

Thus $\mathcal{F}_n(X)$ is a preregular space. ■

Corollary 7.3.6. *The following statements are true:*

1. *Suppose X is a preregular space. Suppose there is $n \in \mathbb{N}$ such that for every $F, G \subseteq X$ where $|F| = |G| = n$, if $\overline{F} \neq \overline{G}$ then $\text{cell}(F \cup G) > n$. Then $\mathcal{F}_n(X)$ is a preregular space.*

2. If there is $n \in \mathbb{N}$ such that $\mathcal{F}_n(X)$ is a non-trivial preregular space, then X must be a preregular space and for every $F, G \subseteq X$ where $|F| = |G| = n$, if $\overline{F} \neq \overline{G}$ then $\text{cell}(F \cup G) > n$.

The conditions in Corollary 7.3.6 are independent of each other, as are shown here.

Example 7.3.6.1. The condition “for every $F, G \in \mathcal{F}_n(X)$, if $\overline{F} \neq \overline{G}$ then $\text{cell}(F \cup G) > n$ ” is obviously true for every $n \in \mathbb{N}$ when X is a T_1 -space, in particular when X is an infinite set with the co-finite topology. In this case, X is irreducible and non-trivial and thus X is not a preregular space. Moreover $\mathcal{F}_n(X)$ is a non-trivial space. Thus by Corollary 7.3.6 (2), $\mathcal{F}_n(X)$ is not a preregular space.

Example 7.3.6.2. Let $n \in \mathbb{N}$ such that $n > 1$. Consider X as the topological space with \mathbb{N} as an underlying set and whose closed sets are generated by $\{1, \dots, n\}$ and its complement in X ; these are then the only non-trivial closed sets in X as well as the only non-trivial open sets in X . If $p, q \in X$ and $\overline{\{p\}} \neq \overline{\{q\}}$, then without loss of generality, $\overline{\{p\}} = \{1, \dots, n\}$ and $\overline{\{q\}} = X \setminus \{1, \dots, n\}$; these are disjoint open subsets of X . Thus X is a preregular space with irreducible components $\{1, \dots, n\}$ and $X \setminus \{1, \dots, n\}$.

The non-trivial closed sets are irreducible while X has cellularity 2 (cf. Lemma 3.3.9). Hence, the union of the closures of any n -element subset cannot have cellularity greater than 2, and hence not greater than n . Thus the condition “for every $F, G \in \mathcal{F}_n(X)$, if $\overline{F} \neq \overline{G}$ then $\text{cell}(F \cup G) > n$ ” is not satisfied for $n > 1$.

Thus, X may be a non-trivial preregular space while the condition “for every $F, G \in \mathcal{F}_n(X)$, if $\overline{F} \neq \overline{G}$ then $\text{cell}(F \cup G) > n$ ” is not satisfied for any $n > 1$; in this case $\mathcal{F}_n(X)$ is a non-trivial space which is not a preregular space (cf. Corollary 7.3.6 (2)).

Theorem 7.3.7. *The following statements are true:*

1. If X is a Hausdorff space, then $\mathcal{F}_n(X)$ is a Hausdorff space for every $n \in \mathbb{N}$.
2. If $\mathcal{F}_n(X)$ is a Hausdorff space for some $n \in \mathbb{N}$, then X must be a Hausdorff space.

Proof. 1. Suppose X is a Hausdorff space. By Lemma 3.3.10, if $F, G \subseteq X$ and $|F| = |G| = n$ where $\overline{F} \neq \overline{G}$ then F, G are discrete subsets of X and so $\text{cell}(F \cup G) = |F \cup G| > n$. By Proposition 7.3.1 (4), X is a preregular T_0 space. By Corollary 7.3.6 (1), $\mathcal{F}_n(X)$ is a non-trivial preregular space. Moreover by Theorem 7.2.5, $\mathcal{F}_n(X)$ is a T_1 -space and thus a T_0 -space. Thus, $\mathcal{F}_n(X)$ is a Hausdorff space by Proposition 7.3.1 (4).

2. Suppose $\mathcal{F}_n(X)$ is a Hausdorff space for some $n \in \mathbb{N}$. Then $\mathcal{F}_n(X)$ is a T_1 -space, so by Theorem 7.2.5, X is a T_1 -space and hence a T_0 -space. Moreover, $\mathcal{F}_n(X)$ is a non-trivial preregular space by Proposition 7.3.1 (4). Thus X is a preregular space by Corollary 7.3.6 (1). By Proposition 7.3.1 (4), X is a Hausdorff space.

■

Corollary 7.3.8. X is T_2 iff $\mathcal{F}_n(X)$ is T_2 for all $n \in \mathbb{N}$.

Glossary

T_D -space A topological space X where $\overline{\{x\}} \setminus \{x\}$ is closed for each $x \in X$. 115

T_F -space A topological space X where $y \in \overline{\{x\}} \setminus \{x\}$ implies $\{y\}$ is closed. 117

κ -c.c. See κ -chain condition. 3, 32

κ -chain condition A topological space X has the κ -chain condition if $\text{cell}(X) < \kappa$, i.e. if every cellular family in X has less than κ elements. 3, 32

(k -fold) covering map A continuous function $f : X \rightarrow Y$ where for any $y \in Y$ there is an open neighbourhood U of y in Y such that there is a cellular family \mathcal{V} in X so that $f^{-1}(U) = \bigcup \mathcal{V}$ and for all $V \in \mathcal{V}$, the restriction $f|_V : V \rightarrow U$ is a homeomorphism. If, in addition, there is $k \in \mathbb{N}$ such that the fibers of f each have exactly k elements, then f is said to be a k -fold covering map. 98

admissible topology A topology, say $\exp \tau$, on $\mathcal{P}_0(X)$ (for X a topological space) for which the map $i : X \rightarrow \mathcal{F}_1(X)$, defined by $i(x) = \{x\}$ for all $x \in X$, is a homeomorphism. 65

almost d-finitely natural in X The property that is satisfied by a set of non-empty subsets \mathfrak{S} of a topological space X when \mathfrak{S} is almost d-finitely natural in X of some degree $n \in \mathbb{N}$. 72

almost d-finitely natural in X of degree n The property that is satisfied by a set of non-empty subsets \mathfrak{S} of a topological space X when \mathfrak{S} is almost natural in X

of degree n and every member of \mathfrak{S} has the same closure in X as some n -element subset of X , for n a natural number. 71

almost finitely natural in X The property that is satisfied by a set of non-empty subsets \mathfrak{S} of a topological space X when \mathfrak{S} is almost finitely natural in X of some degree $n \in \mathbb{N}$. 72

almost finitely natural in X of degree n The property that is satisfied by a set of non-empty subsets \mathfrak{S} of a topological space X when \mathfrak{S} is almost natural in X of degree n and every member of \mathfrak{S} has the same closure in X as some subspace of X with cellularity at most n . 71

almost natural in X The property that is satisfied by a set of non-empty subsets \mathfrak{S} of a topological space X when \mathfrak{S} is almost natural in X of some degree $n \in \mathbb{N}$. 72

almost natural in X of degree n The property that is satisfied by a set of non-empty subsets \mathfrak{S} of a topological space X when $|X| \geq n$ and every n -element subset of X has the same closure in X as some element of \mathfrak{S} , for n a natural number. 71

augmented n -th power For a set X , this is the subset $X^{[n]}$ of X^n where if $(x_1, \dots, x_n) \in X^{[n]}$ where $x_i \neq x_j$ for all $i \neq j$. 19

base space See hyperspace. 1

ccc See countable chain condition. 29, 32

cellular family A pairwise disjoint family of open subsets of a topological space. 29

cellularity For a given topological space X , it is denoted by $c(X)$ and is the smallest cardinal number for which $|\mathcal{U}| \leq c(X)$ whenever \mathcal{U} is a pairwise disjoint family of open subsets of X . 3, 29

co-finite topology The topology on an infinite set X given by $\{X \setminus F : F \subseteq X \text{ and } F \text{ finite}\}$.

cofinality For a limit ordinal γ , the cofinality of γ is $\text{cf } \gamma$, the smallest limit ordinal α such that there is an increasing sequence of ordinals $\{\gamma_\beta : \beta < \alpha\}$ where $\gamma = \sup_{\beta < \alpha} \gamma_\beta$.

28

consistent A set of sentences S is consistent if there is no sentence φ where both φ and its negation, $\neg\varphi$ is provable from S . 27

countable chain condition A poset (P, \leq) satisfies the countable chain condition if every antichain in P is countable. A topological space X has the countable chain condition if every cellular family in X is countable. 29, 32

countable complement topology The topology on an uncountable set X given by $\{X \setminus U : U \subseteq X \text{ and } U \text{ countable}\}$. 15

density For a topological space X , the density of X is $d(X) = \min\{|A| : \overline{A} = X\}$. 30

finite topology Alternative term for the Vietoris Topology used by Ernest Michael [44].

1

Hoare power space Alternative term for 2^X with the lower Vietoris Topology. 2

hyperspace For a topological space X , this is a topological space whose underlying set is a set of non-empty subsets of X . Then X is called the base space of the hyperspace.

1

independent A sentence φ is independent of a set of sentences S if both $S + \varphi = S \cup \{\varphi\}$ and $S + \neg\varphi$ are consistent. 27

irreducible A topological space in which every two non-empty open subsets meet. 32

irreducible component A maximal irreducible subset of a topological space. 35

lambda topology Alternative term for the lower Vietoris Topology used by Feichtinger [17]. 2

limit cardinal A cardinal that is not a successor cardinal. 27

limit ordinal An ordinal that is not a successor ordinal. 26

local homeomorphism A continuous function $f : X \rightarrow Y$ where for any $x \in X$ there is an open neighbourhood U of x such that the restriction $f|_U : U \rightarrow f(U)$ is a homeomorphism and $f(U)$ is open in Y . 98

local topology Alternative term for the lower Vietoris Topology used by Effros [12, p. 931]. 2

locally finite family A family of subsets \mathcal{A} of a topological space X where every $x \in X$ has an open neighbourhood meeting only a finite number of members of \mathcal{A} . 35

lower semi-finite topology Alternative term for the lower Vietoris Topology used by Ernest Michael [44, Definition 9.1]. 1

Lower Vietoris Topology For a set \mathfrak{S} of non-empty subsets of a topological space X , this is the topology with the sub-base $\{\langle X, V \rangle \cap \mathfrak{S} : V \in \tau\}$, equivalently it is the subspace topology $\mathbf{L}(\tau)|_{\mathfrak{S}}$ induced on \mathfrak{S} from the topology $\mathbf{L}(\tau)$ on $\mathcal{P}_0(X)$, the latter having the collection $\{\langle X, V \rangle : V \in \tau\}$ as a sub-base. 63

natural family in X The property that is satisfied by a set of non-empty subsets \mathfrak{S} of a topological space X when \mathfrak{S} is natural in X of degree 1, a term coined in [33, Definition 2.14]. 72

Noetherian space A topological space X where if $A_1 \supseteq A_2 \supseteq A_3 \supseteq \dots$ is a descending sequence of closed sets then there is $N \in \mathbb{N}$ such that $A_n = A_N$ for all $n \geq N$. 40

perfect map A continuous, closed function all of whose fibers are compact. 31

preregular space A topological space X in which $\overline{\{x\}} \neq \overline{\{y\}}$ implies there are disjoint opens V, W in X where $x \in V$ and $y \in W$. 127

quasi-open map A function $f : X \rightarrow Y$ where $\text{Int}_Y(f(U)) \neq \emptyset$ whenever U is non-empty and open in X . 31

quasi-sober space A topological space X where for every non-empty, irreducible, closed subspace A of X , there is $x \in X$ such that $A = \overline{\{x\}}$. 41

regular cardinal An infinite cardinal λ such that $\lambda = \text{cf } \lambda$. 28

singular cardinal A non-regular infinite cardinal. 28

sober space A topological space X where for every non-empty, irreducible, closed subspace A of X , there is a unique $x \in X$ such that $A = \overline{\{x\}}$. 41

Souslin number See cellularity. 3, 29

Souslin problem The statement that there is a Souslin space. 3

Souslin property A topological space has the Souslin property if it has countable cellularity. 3

Souslin space a non-separable linearly ordered space with countable cellularity. 3

specialisation pre-order If X is a topological space then this is the pre-order \leq_τ on X defined by $x \leq_\tau y$ iff $x \in \overline{\{y\}}$ for all $x, y \in X$. 23

successor cardinal A cardinal of the form λ^+ for some cardinal λ , i.e. the smallest cardinal greater than λ . 27

successor ordinal An ordinal of the form $\alpha \cup \{\alpha\}$ for some ordinal α , called the successor of α . 26

Suslin number see cellularity. 29

symmetric space A topological space X in which $x \in \overline{\{y\}}$ implies that $\overline{\{x\}} = \overline{\{y\}}$ for all $x, y \in X$. 13

The Sub-base Condition The property satisfied by a set \mathfrak{S} of non-empty subsets of a topological space X when $\langle V \rangle \cap \mathfrak{S}$ is an irreducible subspace of \mathfrak{S} with the lower Vietoris topology whenever V is a closed subset of X . xi, 90

upper interval topology The topology on a partially ordered set (P, \leq) which has, as a sub-base for its closed sets, the family $\{\downarrow \{a\} : a \in P\} \cup \{P\}$. 22

upper semi-finite topology Alternative term for the Upper Vietoris Topology used by Ernest Michael [44, Definition 9.1]. 1

Vietoris Topology For a set \mathfrak{S} of non-empty subsets of a topological space X , this is the topology on \mathfrak{S} which has, as a sub-base, the collection $\{\langle X, V \rangle \cap \mathfrak{S} : V \in \tau\} \cup \{\langle V \rangle \cap \mathfrak{S} : V \in \tau\}$. 63

weakly inaccessible cardinal An uncountable, regular, limit cardinal. 28

weight The weight of a topological space X is $w(X) = \min\{|\mathcal{B}| : \mathcal{B} \text{ is a base for } X\}$.

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