

**AN APOS ANALYSIS OF THE TEACHING AND LEARNING
OF FACTORISATION OF QUADRATIC EXPRESSIONS IN
GRADE 10 MATHEMATICS CLASSROOMS**

By

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ABSTRACT

The South African Curriculum and Assessment Policy Statements (CAPS) document, for the Further Education and Training Phase (FET) Mathematics Grades 10-12 (2011) shows that the factorisation of algebraic quadratic expressions or equations pervades the mathematics of the secondary school. As a result, for learners to be successful at mathematics in Grade 12, they need to know a great deal of algebra, particularly the factorisation of quadratics. It is therefore important for us as mathematics educators to identify areas in the factorization of quadratics that teachers and learners are struggling to learn and apply. With this in mind, the study sets to embark on an APOS analysis of the teaching and learning of factorisation of quadratic expressions in Grade 10 mathematics classrooms.

Following on from the research questions, this study is located within the principles of the mixed methods case study approach. The combination of methodologies has allowed me to identify broad trends across the groups of learners and those of educators as a whole as well as differences within the participants of the groups themselves. The participants of the study were the groups of Grade 10 learners from the two participating schools, as well as the Grade 10 mathematics teachers from the two circuits of Ilembe District. Five sources of data were used. Firstly, data were generated from 25 teachers from the two circuits who participated in the teachers' questionnaires. A second data collection instrument was the classroom lessons' observations of the six teachers. A third data source was the learner group activity and learners' interviews administered to 12 learners. A fourth data source was the unstructured interviews with six teachers. The final instrument was the analysis of the 205 Grade 10 mathematics 2019 March common paper learners' scripts.

This study was guided by the theory of constructivism and more specifically Action, Process, Object, Schema (APOS) theory which views learning as changes in conception. As an individual engages with a concept, the conception changes from an initial external view towards seeing the concept as a totality upon which other Actions and Processes can act.

This study has found that, firstly, teachers and learners tend to rely too much on the use of rules in factoring certain quadratics. In so doing, a prototype of the quadratic expression concept is perceived which consists of isolated and disconnected concepts. As a result, most learners were not able to factor the trinomial quadratic of $a \neq 1$, since they perceived the factoring of $ax^2 + bx + c$ with $a = 1$ and that of $a \neq 1$ as two different procedures. Secondly, there are

also students whose mental constructions (conception) are limited to Action levels in terms of APOS theory.

The findings of the study suggest that teachers and learners should be able to consider quadratic expressions as one big idea and follow the fundamental considerations when factoring the quadratic expressions. Furthermore the use of multi-methods in factoring quadratics is encouraged and needed for students to better understand the connections between different methodologies for conceptual development.

DECLARATION

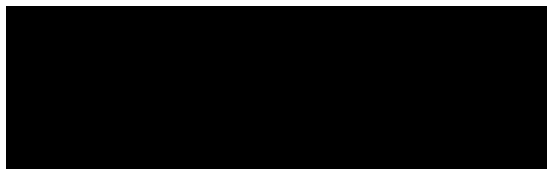
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Signed: 

Prof. S. Bansilal

Supervisor

DEDICATION

This work is dedicated to my late wife Zanele Ntombenhle Vilakazi (MaGumede) and may her soul rest in eternal peace. While she was suffering from cancer, she was always there for me, and bearing with my busy schedule during the course of this study. I thank her for affording me her time in ensuring that I succeed in completing this assignment, and also for encouraging me to finalize this thesis project for her benefit, and the benefit for the whole family. Furthermore, this work is dedicated to the generations of Nozipho, Amanda, and as well as to my little two sons, Melokuhle and Sanele, for they are still undergoing the primary school education, to consider education as the key towards their success. Finally, I dedicate this work to my entire extended family members.

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ABBREVIATIONS

ACE Teaching Cycle	: Activities; Classroom Discussions; Exercises
ANA	: Annual National Assessment
APOS	: Action Process Object Schema
CAPS	: Curriculum and Assessment Policy Statement
DH	: Departmental Head
FET	: Further Education and Training
GD	: Genetic Decomposition
GET	: General Education and Training
KZN	: KwaZulu-Natal Province of the Republic of South Africa
MCK	: Mathematics Content Knowledge
MPCK	: Mathematical Pedagogical Content Knowledge
NSC	: National Senior Certificate
OECD/PISA	: Organisation for Economic Co-operation and Development/ Programme for International Student Assessment
PCK	: Pedagogical Content Knowledge
SA	: Republic of South Africa
SACMEQ	: The Southern and Eastern Africa Consortium for Monitoring Educational Quality
SCK	: Subject Content Knowledge
SP	: Senior Phase
TIMSS	: Trends in International Mathematics and Science Study
UKZN	: University of KwaZulu-Natal

TABLE OF CONTENTS

TITLE	i
ABSTRACT	ii
DECLARATION	iii
DEDICATION	iv
ACKNOWLEDGEMENTS	v
ABBREVIATIONS	vii
CONTENTS	viii
LIST OF TABLES USED IN THIS STUDY	ix
LIST OF FIGURES USED IN THIS STUDY	xii
LIST OF APPENDICES USED IN THIS STUDY	xiii
CHAPTERS	xiv
REFERENCES	312
APPENDICES	328

TABLES USED IN THIS THESIS

Table 1.1	Mathematics content clarification on Grade 9 factorisation of quadratics	3
Table 1.2	Mathematics content clarification on Grade 10 factorisation of quadratics	4
Table 2.1	Methods of factoring quadratic expressions as adapted in Mart 2019	36
Table 3.1	Reflective Abstraction Indicators	65
Table 3.2	Characteristic Differences of APOS Theory and Traditional Approach	66
Table 4.1	APOS classification of the tasks in the teachers' and learners' activity on factoring quadratics	81
Table 5.1	Summary of practices on classroom lessons observations	140
Table 6.1	Summary on how learners defined the concept "factorisation	145
Table 7.1	Consolidated summary of interview responses on Task 1 of learners' activity on factorization of quadratics in terms of correct/ incorrect; valid/ invalid reasons	162
Table 7.2	Consolidated summary of interview responses on Task 2 of learners' activity on factorization of quadratics in terms of correct/ incorrect; valid/ invalid reasons	163
Table 7.3	Consolidated summary of interview responses on Task 3 of learners' activity on factorization of quadratics in terms of correct/ incorrect; valid/ invalid reasons	168
Table 7.4	Consolidated summary of interview responses on Task 4 of learners' activity on factorization of quadratics in terms of correct/ incorrect; valid/ invalid reasons	171
Table 7.5	Consolidated summary of interview responses on Task 5 of learners' activity on factorization of quadratics in terms of correct/ incorrect; valid/ invalid reasons	175
Table 7.6	Consolidated summary of interview responses on Task 6 of learners' activity on factorization of quadratics in terms of correct/ incorrect; valid/ invalid reasons	177

Table 7.7	Consolidated summary of interview responses on Task 7 of learners' activity on factorization of quadratics in terms of correct/ incorrect; valid/ invalid reasons	181
Table 7.8	Learners' methods of factoring quadratic expressions (interviews)	182
Table 7.9	Learners' placement along APOS indicators(learners from the T-School)	183
Table 7.10	Learners' placement along APOS indicators (learners from the R-School)	184
Table 7.11	Description of questions in the grade 10 2019 March common test	189
Table 7.12	Marks per sub question in a grade 10 2019 March common test	190
Table 7.13	distribution of marks for Question 1.1 1 of the grade 10 2019 March common test	193
Table 7.14	Distribution of marks for Question 2.3 of the grade 10 2019 March common test	194
Table 7.15	Performance of learners from the T-School in Question 1.1.1 and Question 2. 3 and their location to APOS levels	199
Table 7.16	Performance of learners from the R-School in Question 1.1.1 and Question 2. 3 and their location to APOS levels	199
Table 8.1	Teachers' teaching qualifications	201
Table 8.2	Number of years teaching mathematics at grade 10	201
Table 8.3	The use of mathematics textbooks in schools.	202
Table 8.4	Teachers' Continuous Professional Development (CPD) activities	204
Table 8.5	Tracking the Mathematics Curriculum coverage	205
Table 8.6	Mathematics teachers' School Environment	207
Table 8.7	Grade 10 Mathematics Curriculum	208
Table 8.8	Consolidated summary of questionnaire responses on Task 1 of teachers' activity on factorization of quadratics in terms of correct/ incorrect; valid/ invalid reasons	221
Table 8.9	Consolidated summary of questionnaire responses on Task 2 of teachers' activity on factorization of quadratics in terms of correct/ incorrect; valid/ invalid reasons	222

Table 8.10	Consolidated summary of questionnaire responses on Task 3 of teachers' activity on factorization of quadratics in terms of correct/incorrect; valid/ invalid reasons	223
Table 8.11	Consolidated summary of questionnaire responses on Task 4 of teachers' activity on factorization of quadratics in terms of correct/incorrect; valid/ invalid reasons	224
Table 8.12	Consolidated summary of questionnaire responses on Task 5 of teachers' activity on factorization of quadratics in terms of correct/incorrect; valid/ invalid reasons	225
Table 8.13	Consolidated summary of questionnaire responses on Task 6 of teachers' activity on factorization of quadratics in terms of correct/incorrect; valid/ invalid reasons	226
Table 8.14	Consolidated summary of questionnaire responses on Task 7 of teachers' activity on factorization of quadratics in terms of correct/incorrect; valid/ invalid reasons	226
Table 8.15	Teachers' methods of factoring quadratics (questionnaires)	227
Table 9.1	Teachers' methods of factoring quadratics (interviews)	275

LIST OF FIGURES USED IN THIS THESIS

Figure 1.1	How the quadratic expression is formulated	9
Figure 1.2	How the monomial quadratic expression is formulated	10
Figure 1.3	How the binomial quadratic expression of $c = 0$ is formulated	10
Figure 1.4	How the binomial quadratic expression when $b = 0$ is formulated	11
Figure 1.5	How the trinomial quadratic expression is formulated	11
Figure 2.1	Guess and Check method of factoring quadratic expressions	37
Figure 2.2	Cross method of factoring quadratic expressions	38
Figure 2.3	Grouping method of factoring quadratic expressions	39
Figure 2.4	Algebra tiles method of factoring quadratic expressions	40
Figure 2.5	Algebra tiles used in factoring $2x^2 + 7x + 3$	41
Figure 2.6	Algebra tiles used in factoring $4x^2 + 12x + 9$	41
Figure 2.7	The Autrey's and Austin Method used in factoring $4x^2 - 16x - 9$	44
Figure 2.8	Slip-Slide method of factoring quadratic expressions	44
Figure 3.1	The Research Framework as adapted in Asiala et al. (1997)	58
Figure 3.2	Comparison of flexible algebraic action and algebraic action adapted in Block, J. (2016)	68
Figure 7.1	Mean percent per sub question per school	190
Figure 7.2	Graph of Zeros for the T-School	191
Figure 7.3	Graph of Zeros for the R-School	192
Figure 7.4	Question1.1.1 of the grade 10 2019 March common test	192
Figure 7.5	Question 2.3 of the grade 10 2019 March common test	193

LIST OF APPENDICES USED IN THIS THESIS

Appendix A	Ethical Clearance Letter from the UKZN	328
Appendix B	Permission Letter from HOD KZN Department of Education	329
Appendix C	Informed Consent Teachers	330
Appendix D	Informed Consent Parents	331
Appendix E	Informed Consent Learners	332
Appendix F	Classroom Lesson Observation Schedule	333
Appendix G	2019 Grade 10 Mathematics Common Test	336
Appendix H	Learners' Focused-Activity	338
Appendix I	Learners' Interview Schedule	339
Appendix J	Teachers' Questionnaire	340
Appendix K	Teachers' Interview Schedule	346
Appendix L	Teachers' Responses from the Questionnaire The activity	348
Appendix M	Editor's Certificate	353
Appendix N	Turnitin Report	354

CHAPTER 1: INTRODUCTION

1.1 Overview.	1
1.2 Background.	1
1.3 Purpose of the study.	2
1.4 Research Focus.	5
1.4.1 Research Questions.	5
1.4.2 Research Objectives.	6
1.5 Motivation of the study.	6
1.6 Algebra.	7
1.7 Quadratics.	8
1.7.1 The quadratic expression – the definition.	8
1.7.2 The birth of the quadratic expression.	8
1.7.3 Types of quadratics.	9
1.7.4 Factorisation of quadratic expression.	12
1.7.4.1 Fundamental considerations when factoring quadratic expressions.	12
1.8 Action-Process-Object-Schema (A-P-O-S).	13
1.9 Benefits of the study.	13
1.10 Overview of this study.	14

CHAPTER 2: LITERATURE REVIEW

2.1 Introduction.	17
2.2 Research about the learning of algebra.	17
2.2.1 Research about teaching and learning of algebra – globally.	18
2.2.2 Research about teaching and learning of algebra in South Africa.	25
2.3 Studies about learning of factorisation and quadratic functions.	31
2.4 Approaches to the teaching of factorisation and quadratic expressions.	35
2.4.1 Factoring quadratic expressions by “Guess and Check” Method or Trial and Error Method.	36
2.4.2 Factoring quadratic expressions by Cross Method.	37
2.4.3 Factoring quadratic expressions by Grouping Method.	39
2.4.4 Factoring quadratic expressions by Tool-based Geometric rectangular algebra tiles.	39
2.4.5 Factoring quadratic expressions by Autrey’s and Austin’s Method (A-A Method).	43
2.4.6 Factoring quadratic expressions by Slip-Slide Method.	44
2.5 Research about mathematics teachers’ knowledge and practice.	44
2.5.1 Mathematics Practices.	45
2.5.2 Instructional practices.	46
2.5.2.1 Traditional teaching approaches to teaching algebra.	46
2.5.2.2 Reform teaching approaches to teaching algebra.	48
2.5.2.3 Shifting from traditional teaching approaches to reform teaching approaches.	48
2.5.3 Teaching practices.	49
2.6 Conclusion.	53

CHAPTER 3: THEORETICAL FRAMEWORK

3.1 Introduction.	54
3.2 The constructivist paradigm.	54
3.3 Algebraic reasoning.	55
3.4 The importance of knowledge of factoring quadratic expressions in the mathematics curriculum	57
3.5 The Research Framework.	57
3.5.1 Theoretical Analysis.	58
3.5.2. Why use APOS Theory.	59
3.5.3 Descriptions of the mental constructions in APOS theory.	60
3.6 Description of the mental mechanisms.	62
3.7 The Initial Genetic Decomposition for the construction of the concept of factoring quadratics.	63
3.8 APOS – directed instructional approaches.	65
3.9 The Triad mental mechanisms.	69
3.10 Conclusion.	69

CHAPTER 4: METHODOLOGY

4.1 Introduction.	70
4.2 Context of the Study.	70
4.3 The Critical Research Questions.	70
4.3.1 Research Questions.	71
4.4 Research Design.	71
4.4.1 Mixed Methods research.	72
4.4.2 The Interpretive paradigm.	73
4.5 The Participants and the Sampling.	74
4.6 Phases of Data Collection.	77
4.6.1 Phase 1 (2018).	77
4.6.2 Phase 2 (2019).	77
4.7 Data Collection Instruments.	78
4.7.1 Classroom Lessons Observations.	78
4.7.2 Learners' Scripts Analysis.	80
4.7.3 Learner activity on factoring quadratics.	80
4.7.4 Learner Semi-Structured Interviews.	82
4.7.5 Teacher Open-Form Questionnaire.	83
4.7.6 Teacher Semi-Structured Interviews.	84
4.8 Data Analysis.	85
4.8.1 Qualitative Data Analysis.	85
4.8.2 Quantitative Data Analysis.	85
4.8.3 Triangulation.	86
4.9 Validity.	86

4.10 Ethical considerations.	88
4.11 Limitations of the study.	89
4.12 Conclusion.	89

CHAPTER 5: CLASSROOM LESSON OBSERVATIONS

5.1 Introduction.	90
5.2 Case study one: Teacher Rakhulu (Teacher RT).	90
5.2.1 The Biography of Teacher Rakhulu.	90
5.2.2 Description of Teacher Rakhulu's Lessons.	91
5.2.2.1 Lesson one of Teacher Rakhulu: Factorisation of quadratic expressions.	91
5.2.2.2 Lesson two with Teacher Rakhulu: Factorisation of quadratic expressions –the grouping method.	96
5.3 Case study two: Teacher Ndulini (Teacher ND).	99
5.3.1 The Biography of Teacher Ndulini.	99
5.3.2 Description of Teacher Ndulini's Lessons.	100
5.3.2.1 Lesson one of Teacher Ndulini: Factorising using common factor.	100
5.3.2.2 Lesson two of Teacher Ndulini: Factorising Quadratic Expression.	102
5.4 Case study three: Teacher Hlophe (Teacher HP).	107
5.4.1 The Biography of Teacher Hlophe.	107
5.4.2 Description of Teacher Hlophe's Lessons.	108
5.4.2.1 Lesson one of Teacher Hlophe: Product of binomial and binomials.	108
5.4.2.2 Lesson two of Teacher Hlophe: Products of binomials and trinomials.	113
5.5 Case study four: Teacher Mzimela (Teacher MZ).	117
5.5.1 The Biography of Teacher Mzimela.	117
5.5.2 Description of Teacher Mzimela's Lessons.	118
5.5.2.1 Lesson one of Teacher Mzimela: Factorisation of quadratics of the difference of two squares.	118
5.5.2.2 Lesson two of Teacher Mzimela: Factorization of quadratic trinomials.	122

5.6 Case study five: Teacher Kweyama (Teacher KW).	126
5.6.1 The Biography of Teacher Kweyama.	126
5.6.2 Description of Teacher Kweyama's Lessons.	127
5.6.2.1 Lesson one of Teacher Kweyama: Factorisation of quadratics of the difference of two squares.	127
5.6.2.2 Lesson two of Teacher Kweyama: Factorisation of quadratic trinomial expression.	130
5.7 Case study six: Teacher Gwamanda (Teacher GS).	132
5.7.1 The Biography of Teacher Gwamanda.	132
5.7.2 The description of Teacher Gwamanda's lessons.	133
5.7.2.1 Lesson one of Teacher Gwamanda: addition and subtraction of algebraic fractions.	133
5.7.2.2 Lesson two of Teacher Gwamanda: Simplification of algebraic expression.	135
5.8 Summary on lesson observations.	140
5.9 Conclusion.	141

CHAPTER 6: LEARNERS' NARRATIVES ABOUT GENERAL PRACTICES IN THE GRADE 10 MATHEMATICS CLASSROOM - RESPONSES TO THE INTERVIEWS

6.1 Introduction.	142
6.2 On the understanding of the concept “algebra”.	142
6.3 On the definition of the concept “factorisation.	143
6.4 On the relation between the amounts of work required for the factorisation of quadratics in Grade 10 mathematics, and the amount of time for teaching.	145
6.5 On whether the teachers of mathematics in their previous grades prepared them for Grade 10 mathematics.	146
6.6 On whether learners solicit support from sources other than their teachers.	148
6.7 On whether the content on the factorisation of quadratics is appropriate for Grade 10 learners.	150
6.8 On whether every learner can understand mathematics.	151
6.9 On whether their teachers give them enough opportunities to explore Mathematics.	153
6.10 On the role they expect from their mathematics teachers in order for them to learn mathematics better.	155
6.11 On motivating a learner who is experiencing some challenges regarding the concept of factorization of quadratics.	157
6.12 Conclusion.	159

CHAPTER 7: LEARNERS' INTERVIEW RESPONSES TO THE ACTIVITY OF FACTORING QUADRATICS, AND THE ANALYSIS OF THEIR PERFORMANCE IN THE MARCH 2019 GRADE 10 COMMON TEST.

7.1 Introduction.	160
7.2 Learners responding to items on factoring quadratics.	160
7.2.1 Task 1: Factorisation of x^2 .	160
7.2.2 Task 2: Factorisation of $x^2 - 4$.	162
7.2.3 Task 3: Factorisation of $x^2 + 4$.	164
7.2.4 Task 4: Factorisation of $x^2 - 4x$.	168
7.2.5 Task 5: Factorisation of $x^2 - 4y$.	172
7.2.6 Task 6: Factorisation of $x^2 - x - 6$.	175
7.2.7 Task 7: Factorisation of $6x^2 + 7x - 20$.	177
7.3 Summary of responses across the items.	181
7.3.1 Summary of methods used as learners engaged in the activity of factoring quadratics.	181
7.3.2 Summary according to APOS genetic decomposition.	182
7.4 On items that were challenging in the activity of factoring quadratics.	185
7.5 What learners found easy to grasp in the factorisation of quadratics.	187
7.6 Performance of learners in Grade 10 common test (algebra).	189
7.6.1 Overall summary of the test.	189
7.6.2 A closer look at the items based on quadratic factorization.	192
7.7 Detailed qualitative analysis of the responses of the 10 interviewed participants to Question 1.1.1 and Question 2. 3.	194
7.7.1 How the interviewed learners from the T-School performed in Question 1.1.1 and Question 2. 3.	195

7.7.2 How the interviewed learners from the R-School performed in Question 1.1.1 and Question 2. 3.	197
7.7.3 Summary on how the ten interviewed learners performed in Question 1.1.1 and Question 2. 3; and their location according to APOS levels.	198
7.8 Conclusion.	200

CHAPTER 8: TEACHERS' VIEWS ON GENERAL PRACTICES IN THE GRADE 10 MATHEMATICS CLASSROOM - RESPONSES TO THE QUESTIONNAIRE

8.1 Introduction.	201
8.2 Biographical information of teachers.	201
8.3 The use of mathematics textbooks in schools.	202
8.4 Teachers' Continuous Professional Development (CPD) activities.	203
8.5 Tracking the Mathematics Curriculum coverage.	205
8.6 Mathematics teachers' School Environment.	206
8.7 Grade 10 Mathematics Curriculum.	208
8.8 Teachers' responses to the activity on factoring quadratics that appeared in the questionnaire.	210
8.8.1 Task 1: Factorisation of x^2 .	210
8.8.2 Task2: Factorisation of $x^2 - 4$.	211
8.8.3 Task 3: Factorisation of $x^2 + 4$.	213
8.8.4 Task 4: I Factorisation of $x^2 - 4x$.	215
8.8.5 Task 5: Factorisation of $x^2 - 4y$.	216
8.8.6 Task 6: Factorisation of $x^2 - x - 6$.	217
8.8.7 Task 7: Factorisation of $6x^2 + 7x - 20$.	219
8.9 Summary of teachers' responses to items on factoring quadratics that appeared in the teachers' questionnaire.	220
8.9.1 Summary according to correct/incorrect and valid/invalid reasons when teachers responded to items of the activity.	220
8.9.2 Summary according to methods followed by teachers in factoring quadratics	227
8.9.3 Summary of how teachers' responses to activity of factoring quadratics are	

located according to APOS.	227
8.10 Conclusion.	228

CHAPTER 9: TEACHERS' NARRATIVES ON GENERAL PRACTICES IN THE GRADE 10 MATHEMATICS CLASSROOM - RESPONSES TO THE ACTIVITY ON FACTORING QUADRATICS - TEACHERS' INTERVIEWS - THE RESULTS

9.1 Introduction.	229
9.2 Teacher narratives on the general practices of Grade 10 mathematics teachers.	229
9.2.1 On the understanding of the concept “algebra”.	229
9.2.2 On the definition of the concept “factorisation.	230
9.2.3 On the role of the teaching experience in teaching of factorization of algebraic quadratic expressions.	232
9.2.4 On the area in the factorisation of quadratic expressions that is challenging to teach at a Grade 10 level.	234
9.2.5 On the area that is easy to teach in the factorisation of quadratic expressions at Grade 10 level.	236
9.2.6 On how they prepare their lessons on the factorisation of quadratic expressions for their diversely-talented Grade 10 learners.	237
9.2.7 On how they manage their Grade 10 work while also teaching higher grades.	238
9.2.8 On the extent at which they incorporate the context-based activities in their teaching of factorisation of quadratics in Grade 10.	240
9.2.9 On the extent at which technological advances can assist in enabling Grade 10 learners perform successfully when dealing with the factorisation of quadratics.	241
9.2.10 On the extent to which English, as the language of instruction, can impact on the success of Grade 10 learners when they deal with the factorisation of quadratics.	242
9.2.11 Dialogue on matters within the factorisation curriculum - the Grade 10 content factorisation of quadratics.	243
9.2.11.1 Question 1: Do you think the number of concepts on factorisation of quadratics	

in Grade 10 is adequate for Grade 10 learners to progress the FET band?	243
9.2.11.2 Question 2: Do you see the Grade 10 content of factorisation of quadratics being appropriate for Grade 10 learners?	244
9.2.11.3 Question 3: What is your view regarding the amount of content required for Grade 10 factorisation of quadratics in relation to the expected pace of instruction?	245
9.2.12 On their views on the use of short-cuts by teachers in solving problems in mathematics.	246
9.2.13 On their views on whether teacher knowledge impacts on learner performances.	250
9.2.14 On whether their teaching practices offer learners opportunities to explore mathematics.	251
9.3 Teachers' responses to the activity on the factorisation of quadratics.	254
9.3.1 Task 1: Factorisation of x^2 .	254
9.3.2 Task 2: Factorisation of $x^2 - 4$.	256
9.3.3 Task 3: Factorisation of $x^2 + 4$.	261
9.3.4 Task 4: Factorisation of $x^2 - 4x$.	264
9.3.5 Task 5: Factorisation of $x^2 - 4y$.	266
9.3.6 Task 6: Factorisation of $x^2 - x - 6$.	270
9.3.7 Task 7: Factorisation of $6x^2 + 7x - 20$.	273
9.4 Summary of various strategies used by teachers in factoring quadratics.	274
9.5 Conclusion.	276

CHAPTER 10: FINDINGS AND DISCUSSIONS

10.1 Introduction.	277
10.2 Answers to RQ 1: What does the analysis of a common test reveal about the Grade 10 learners' understanding of factorisation of quadratic expressions?	277
10.3 Answers to RQ2: What levels of APOS thinking do teachers engage in when dealing with items based on the concept of factorization of quadratic expressions.	278
10.3.1 Summary of teachers' written responses per item.	279
10.3.1.1 Task 1: Teachers factoring the monomial quadratic expression of x^2 .	279
10.3.1.2 Task 2: Teachers factorisation of a binomial quadratic expression with $b = 0$: $x^2 - 4$.	279
10.3.1.3 Task 3: Teachers' understanding of the non-factorisation of a binomial quadratic expression where $b = 0$: $x^2 + 4$.	280
10.3.1.4 Task 4: Teachers understanding in factoring a binomial quadratic expression where $c = 0$: $x^2 - 4x$.	280
10.3.1.5 Task 5: Teachers' understanding of the non-factorisation of a binomial quadratic expression where $c = 0$: $x^2 - 4y$.	280
10.3.1.6 Task 6: Teachers factoring a trinomial quadratic expression where $a = 1$: $x^2 - x - 6$.	281
10.3.1.7 Task 7: Teachers factorising a trinomial quadratic expression where $a \neq 1$: $6x^2 + 7x - 20$.	281
10.3.2 Summary of teachers' APOS mental constructions as revealed by the written responses for all items in the activity sheet.	281
10.4 Answers to RQ 3: What levels of APOS thinking do the learners engage in when dealing with items based on the concept of factorisation of quadratic expressions.	282

10.4.1 Summary of learners' performance per item.	282
10.4.1.1 Task 1: Learners' factorisation of a monomial quadratic expression of x^2 .	282
10.4.1.2 Task 2: Learners' understanding of factorisation of a binomial quadratic expression where $b = 0$: $x^2 - 4$.	283
10.4.1.3 Task 3: Learners' understanding of the non-factorisation of a binomial quadratic expression where $b = 0$: $x^2 + 4$.	283
10.4.1.4 Task 4: Learners' understanding in factoring a binomial quadratic expression where $c = 0$: $x^2 - 4x$.	283
10.4.1.5 Task 5: Learners' understanding of the non-factorisation of a binomial quadratic expression where $c = 0$: $x^2 - 4y$.	284
10.4.1.6 Task 6: Learners' understanding of the factorisation a trinomial quadratic expression where $a = 1$: $x^2 - x - 6$.	284
10.4.1.7 Task 7: Learners' understanding in factoring a trinomial quadratic expression where $a \neq 1$: $6x^2 + 7x - 20$.	284
10.4.2. Summary of learners' APOS mental constructions across the items.	284
10.5 Further issues arising from the items analysed for RQ 2 and RQ 3.	285
10.5.1 Most teachers found it challenging to explain how the trinomial quadratic expression of $a \neq 1$, could be factorised.	285
10.5.2 Teachers found it challenging to provide sound algebraic arguments why certain quadratics could not be factorised.	285
10.5.3 Most learners found it challenging to factorise the trinomial quadratic expression of $a \neq 1$.	286
10.5.4 Most learners found it challenging to provide sound algebraic arguments why certain quadratics could not be factorised.	286

10.5.5 A research perspective on RQ 2 and RQ 3.	286
10.6 Answers to RQ 4(a): How do mathematics teachers approach the teaching of factorisation of quadratic expressions with their Grade 10 learners?	289
10.6.1 Teachers follow the traditional approach to teaching factorisation of quadratic expressions – where they use examples as a means of teaching the concept.	289
10.6.2 Teachers opted to use short cuts when factoring the quadratic expressions.	291
10.6.3 Teachers believe that different prototypes of quadratic expressions can be factored by different approaches.	293
10.6.4 Teachers taught the factoring of quadratic expressions by building on factorisation as the reverse process of the expansion of products.	293
10.6.5 Teachers displayed knowledge of multiple approaches to teaching the factoring of quadratic expressions.	294
10.6.5.1 Teachers factoring of quadratic expressions by the Trial and Error approach and the Cross Method.	295
10.6.5.2 Teachers factoring quadratic expressions by the grouping method.	296
10.7 Answers to RQ 4(b) about why teachers use the approaches that they do in RQ 4(a).	297
10.7.1 Teachers found the short-cuts easy to teach when factoring quadratic expressions.	297
10.7.2 In CAPS, the focus when factoring quadratics is on the answer not on the pathway.	298
10.7.3 The use of short–cuts is encouraged in the ATP.	299
10.7.4 The use of short-cuts is encouraged in the prescribed mathematics textbooks.	300
10.8 Further issues arising from the study.	301
10.8.1 Learners use the same strategies as their teachers when factorising quadratics.	301
10.8.1.1 Learners found it easy to use the short-cuts in factoring quadratics of the difference of two squares type.	301

10.8.1.2 Learners factorising quadratic expressions by trial and error method.	301
10.8.2 The fifth operation “cancel” has emerged.	302
10.8.3 Teachers do not focus on the work in Grade 10 as much as they do in Grade 12.	303
10.8.4 Teachers offer few opportunities for learners to explore mathematics.	304
10.8.5 Teachers and learners are driven by the “push to solve” goal.	305
10.8.6 There is limited attention given to checking whether the answers are correct.	305
10.8.7 Learners factoring of quadratic expressions following the Autrey’s and Austin’s method (A-A method).	307
10.8.8 Appropriate content in an inappropriate grade.	308
10.8.8.1 Teachers spoke about the graphical solution in factoring quadratics at Grade 10 level.	308
10.8.8.2 Teachers spoke about the use of the quadratic formula in factoring quadratics at Grade 10 level.	308
10.9 The implications and the recommendations of the study.	309
10.10 Themes for further research.	310
10.11 Conclusion.	310

CHAPTER 1

INTRODUCTION

1.1 Overview.

In this chapter, I set out to present the milestones on how the research process unfolded. The motivation for doing this research, as well as the background and purpose for embarking on this study, are presented in this chapter. The research questions and key terms in the study are introduced, and this is followed by the significance of the results of this study. The anticipated benefits in the South African mathematical landscape are also indicated. Later in this chapter, summaries of successive chapters are presented.

1.2 Background.

My involvement in secondary school mathematics extended over the course of two years during which I practiced as a secondary school mathematics teacher; the Departmental Head (DH) for Mathematics and Sciences for the General Education and Training (GET), and the Further Education and Training (FET) phases; Subject Advisor for the FET Mathematics; as well as Tutor for FET Mathematics at the University of KwaZulu-Natal (UKZN).

As a practicing teacher of mathematics, and based on my analysis of the mathematics curriculum, I am aware that algebra pervades the curriculum as prescribed in the two South African Curriculum and Assessment Policy Statements (CAPS) documents, the Senior Phase (SP) Mathematics Grades 7-9 (2011b) document, and the Further Education and Training Phase (FET) Mathematics Grades 10-12 (2011a) document. Again, I have noted the factorisation of algebraic quadratic expressions or equations pervading the mathematics of the South African secondary school. As a result, for learners to be successful at mathematics in Grade 12, they fundamentally need to know a great deal of algebra, particularly the factorisation of quadratics. The South African National Senior Certificate (NSC) mathematics paper one consists mainly of algebra, with topics, namely, Algebra & Equations; Functions & Graphs; and Differential Calculus. These topics in nature are categorised as being fully-fledged algebraic topics and contribute about 32% of the whole Grade 12 mathematics paper. In the Grade 12 mathematics paper two, learners are also required to have sound background knowledge of algebra since most topics require applications of algebra. For example, in trigonometry, areas such as proving

the identities and finding the general solutions of trigonometric equations require some levels of competence in the concept of factorisation of quadratics.

This dominance of factorisation of quadratics in the NSC mathematics paper was noted in the examination written in 2014 and was presented by Govender (2015). In his paper on the assessment of quadratic equations and functions in the Grade 12 mathematics papers, Govender (2015, p.175) noted that factorisation of quadratics dominated the NSC mathematics paper. It appeared from his presentation that paper 1 of 2014 had sums on quadratics amounting to 43 marks out of 150, which equates to 14% of the entire NSC paper. Govender (2015, p.179) further presented that paper 2 of 2014 had tasks of a quadratic nature that amounted to 24 marks out of 150, which equates to 8% of the entire NSC paper. Summarily, the 2014 NSC mathematics paper comprised of questions involving quadratics which contributed almost 22% of the paper as a whole, and this trend has been holding even during the year 2020. However, despite the portion covered by quadratics in matric, numerous studies bear evidence that most learners lose marks on topics that are centred within the factorisation of quadratics, such as quadratic inequalities in the Grade 12 NSC examinations (Godden, Mbekwa and Julie, 2013; Makonye, 2015).

1.3 Purpose of the study.

As teachers of mathematics, we expect learners to perform better in the concept of factorisation of quadratics. As pointed out earlier, the preparation of learners begins in Grade 9. The DBE (2011b, pp. 142-143) outlines and clarifies factorisation in the Grade 9 content as presented in Table 1.1.

Table 1.1: Mathematics content clarification of Grade 9 factorisation of quadratic expressions.

Content Area	Patterns, functions, and algebra
Topic	2.3 Algebraic expressions
Concepts And Skills	Factorise algebraic expressions - Factorise algebraic expressions that involve: -- common factors -- difference of two squares -- trinomials of the form: ♦♦ $x^2 + bx + c$ ♦♦ $ax^2 + bx + c$, where a is a common factor - Simplify algebraic expressions that involve the above factorisation processes - Simplify algebraic fractions using factorisation
Some Clarification Notes Or Teaching Guidelines	Factorising expressions - Make sure learners understand that factorising is the reverse of expanding an expression through multiplication e.g. Expand: $2x(x + 3) = 2x^2 + 6$ Factorise: $2x^2 + 6 = 2x(x + 3)$ - Note that 1 and -1 are common factors of every expression e.g. Expand: $a - 4b = 1(a - 4b)$ Factorise: $4b - a = -1(a - 4b)$ Examples of expressions with common factors that can be factorised a) $6a^4 - 4a^2$; b) $ax - bx + 2a - 2b$; c) $2x(a - b) - 3(a - b)$ d) $2x(a - b) - 3(b - a)$; e) $(a + b)^2 - 5(a + b)$ Examples of expressions with a difference of two squares that can be factorised: a) $25a^2 - 1$; b) $a^4 - b^4$; c) $9(a + b)^2 - 1$; d) $3x^3 - 27$ Examples of algebraic fractions that can be factorised: a) $\frac{2x+6y}{x+3y}$; b) $\frac{3x-3y}{6x-6y}$; c) $\frac{9a^2-1}{3a+1}$ Examples of trinomials that can be factorised: (a) $x^2 + 5x + 6$; b) $x^2 - 5x + 6$; c) $x^2 - x - 6$; d) $x^2 - 6x + 9$; e) $2x^2 + 10x + 12$

At Grade 9 level, the term quadratics is not used, however learners are taught the technique to factorise particular forms or types of quadratics. The term “quadratics” is introduced for the first time in Grade 10. The DBE (2011a, pp.13 & 21) provides an overview of the topics for Grade 10 mathematics, in particular clarification on the factorisation of quadratics as presented in Table 1.2 that follows.

Table 1.2: Mathematics content clarification on Grade 10 factorisation of quadratics expressions.

Topic	Algebraic Expressions
Curriculum statement	5. Factorisation to include types taught in Grade 9 and: • Trinomials • grouping in pairs • sum and difference of two cubes 6. Simplification of algebraic fractions using factorisation with denominators of cubes (limited to sum and difference of cubes).
Clarification	Where an example is given, the cognitive demand is suggested: knowledge (K), routine procedure (R), complex procedure (C) or problem-solving (P) Examples to illustrate the different cognitive levels involved in factorisation: 1. Factorise fully: 1.1. $m^2 - 2m + 1$ (revision) Learners must be able to recognise the simplest perfect squares. (R) 1.2. $2x^2 - x - 3$ This type is routine and appears in all texts. (R) 1.3. $\frac{y^2}{2} - \frac{13y}{2} + 18$ Learners are required to work with fractions and identify when an expression has been “fully factorised”. (R) 2. Simplify $\frac{1-2x}{4x^2-1} - \frac{x+4}{2x^2-3x+1} + \frac{1}{1-x}$ (C)

The outlines presented in Table 1.1 and Table 1.2 indicate the point from which subject teachers of mathematics may accompany learners on their journey to a better understanding of the factorisation of quadratics. The main purpose in this study is to conduct a fine-grained analysis of the learning and teaching of factorisation of quadratic expressions by considering the perspectives, understanding, and the teaching approaches of teachers toward the concept, as well as the perspective of learners and their understanding of the concept of factorisation of quadratic expressions.

As contemplated by Makonye and Mashaka (2016, p. 289), research that aims to assist learners to understand algebra has the potential to contribute to improving achievement, not only in mathematics, but in many other subjects in which mathematics is used as a tool to understand and solve problems such as science, economics and information and communication technologies. Hence, this study seeks, with its focus on quadratic expressions, to benefit the mathematics community overall.

1.4 Research Focus.

Even though the concept of factorisation of quadratics is introduced as early as Grade 9, it is Grade 10 that marks the entry to comprehensive and holistic factorisation, and where learners are first introduced to quadratic expressions of $a \neq 1$. However, many learners perform poorly as they exit the matric level where the very same Grade 10 concept is examined. This study focuses on the journey to understanding factorisation of quadratics embarked on by both teachers and learners as a milestone towards their grasping of how to solve quadratic equations. The study was based on the APOS (Action-Process-Objects-Schema) approach in exploring the conceptual understanding regarding the factorisation of quadratics displayed in Grade 10 classrooms of both teachers and learners from two Secondary Schools in the Ilembe Education District of the Province of KwaZulu-Natal (KZN), in the Republic of South Africa (SA).

Hence, this study is titled:

An APOS analysis of the teaching and learning of factorisation of quadratic expressions in Grade 10 mathematics classrooms.

The overall aim of the study is to explore how APOS theory can be used to analyse the teaching and learning of the factorisation of quadratic expression in Grade 10.

1.4.1 Research Questions.

The study has the following questions:

Research Question One (RQ1): How do Grade 10 learners perform in algebra in the KZN provincial common test?

Research Question Two (RQ2): What levels of APOS thinking do teachers engage in when dealing with items based on the concept of factorisation of quadratic expressions?

Research Question Three (RQ3): What levels of APOS thinking do learners engage in when dealing with items based on the concept of factorisation of quadratic expressions?

Research Question Four (RQ4): A) How do mathematics teachers approach the teaching of factorisation of quadratic expressions with their Grade 10 learners?

Research Question Four (RQ4): B) Why do they use the approaches they do?

1.4.2 Research Objectives.

The objectives of this current study were:

- to determine how Grade 10 learners perform in algebra in KZN provincial common tests.
- to explore the levels of APOS thinking that teachers engage in when dealing with items based on the concept of factorisation of quadratic expressions.
- to explore the levels of APOS thinking that learners engage in when dealing with items based on the concept of factorisation of quadratic expressions.
- to explore mathematics teacher's approaches to the teaching of factorisation of quadratic expressions with their Grade 10 learners.

These objectives were accomplished by gaining a better understanding on how both teachers and learners conceptualise the process of factorisation of quadratics. Again, I wanted to establish whether both teachers and learners were able to draw a bigger picture, or whether they were treating factorisation of quadratics as an isolated entity within the wider area of mathematics.

1.5 Motivation of the study.

This study was motivated by various factors. Firstly, the ranking of South Africa in the Trends in International Mathematics and Science Study (TIMSS) 2015 mathematics (Reddy et al., 2016). Comparing South Africa with other countries, I present the TIMSS 2015 mathematics. The top five ranked countries were from East Asia – Singapore (621), the Republic of Korea (606), Chinese Taipei (599), Hong Kong SAR (594), and Japan (587). The five lowest performing countries were Botswana (391), Jordan (386), Morocco (384), South Africa (372), and Saudi Arabia (368). Of these five countries, the only country with a statistically significantly different score to that of South Africa was Botswana. Secondly, my personal experience and exposure to the world of secondary school mathematics. The third motivating

factor was teachers' difficulties in leading learners to a better understanding of the factorisation of quadratics. Lastly, difficulties experienced by learners in conceptualising the factorisation of quadratics.

1.6 Algebra.

Any discussion of mathematical skills and knowledge necessary for success in mathematics tasks is not complete without a discussion of the role played by algebra in the solution of such tasks. Vygotsky (as cited in Schmittau, 2005, p. 16) shares similar sentiments as he asserts that “the student who had mastered algebra had attained “a new higher plane of thought”, a level of abstraction and generalisation which transformed the meaning of the lower (arithmetic) level”. Hence this study is aimed at exploring the teaching and learning of the factorisation of quadratic algebraic expressions in the Grade 10 mathematics curriculum.

Sfard (1995, p. 18) defines algebra as a “science of generalised computations”. Sfard (1995) uses the term algebra with respect to any kind of mathematical endeavour concerned with generalised computational processes, whatever the tools used to convey this generality. A similar perspective is embraced in the pronouncement made by RAND Mathematics Study Panel (2001) who said algebra provided the tools (language and structure) for analysing and representing quantitative relationships, for modelling situations, for solving problems, and for stating and proving generalisations. The position taken by Sfard (1995), together with her proponents, is that of the mathematical problem-solver seeking the generalistic approach to problem-solving for transference of employment techniques from one situation to another.

Star et al., (2015) further alluded that “understanding algebra is a key for success in future mathematics courses, including geometry and calculus. Many mathematics experts also consider algebra knowledge and skills important for post-secondary success as well as for producing a skilled workforce for scientific and technical careers”.

This understanding about algebra is central to advocating mathematics in South Africa, as the DBE (2011a, p. 10) prescribes “In the FET Phase, learners should be exposed to mathematical experiences that give them many opportunities to develop their mathematical reasoning and creative skills in preparation for more abstract mathematics in Higher/Tertiary Education

institutions”. The founding reason to that perspective is that algebra requires proficiency with multiple representations, including symbols, equations, and graphs, as well as the ability to reason logically, both of which play crucial roles in advanced mathematics courses. RAND (2001) went further in unearthing the apex role played by algebra as they argued, “algebra, and more generally, the broad mathematical skills that algebra encompasses, are critical both to mathematical proficiency and to equity in the achievement of proficiency” (RAND Mathematics Study Panel 2001, p.78).

Furthermore, according to Star et al., (2015) algebra moves students beyond an emphasis on arithmetic operations to focus on the use of symbols to represent numbers and express mathematical relationships. Again, algebra is often the first mathematics subject that requires extensive abstract thinking, a challenging new skill for many students. This description of algebra being the vehicle that moves students from arithmetic to use of symbols, fits well within the context of Grade 10 mathematics content in South Africa. Makonye and Mashaka (2016, p. 289) shared a similar view as they were quoted as saying “algebra forms the basis of high school mathematics and various topics and concepts are dependent on it. Learners who do not have algebraic competency are precluded from passing mathematics”.

1.7 Quadratics.

1.7.1 The quadratic expression – the definition.

The quadratic algebraic expression is identified by the presence of the variable of the second power as the highest power in the expression. The internationally-recognised general formula of the quadratic expression is $ax^2 + bx + c$, where a, b and c are called the coefficients. Factorisation, according to Kotsopoulos (2007), refers to the decomposition of an object into a product of other objects, or factors, which yield the original when multiplied together. Therefore, factoring of quadratic expressions is the writing of polynomial as a product of other polynomials. It is also a process that advocates the finding of the products within the multiplication table.

1.7.2 How the quadratic expression is formulated.

The product of any two linear expressions results in a quadratic expression. The linear expression by definition is the algebraic expression with the variable of the power one as the

highest power in the expression. In trying to understand the conceptualisation of the concept of factorisation of a quadratic expression, let us look at how the quadratic expressions can be conceived.

Consider the two linear algebraic expressions:

$kx + t$ and $mx + n$, where k, t, m and n are real numbers and $k \neq 0$ and $m \neq 0$

Their product will be:

$$\begin{aligned} & (kx + t)(mx + n) \\ & kmx^2 + knx + tmx + tn \\ & kmx^2 + (kn + tm)x + tn \end{aligned}$$

Figure 1.1 How the quadratic expression is formulated.

The final expression in figure 1.1 is commonly presented as $ax^2 + bx + c$ which is referred to as the general formula for the algebraic quadratic expression. This means that for any quadratic expression, $a = km$; $b = kn + tm$ and $c = tn$ where

$$ax^2 + bx + c = (kx + t)(mx + n)$$

1.7.3 Types of quadratics.

We consider seven special cases of quadratic expressions $(ax^2 + bx + c)$ based on whether $b = 0$, $c = 0$, and $a = 1$ or not $a \neq 1$. The numerical values of k, t, m and n respectively, dictate and decide upon the type of quadratics to be produced. Basically, at the Grade 10 level, there are three types of algebraic quadratics, namely, the monomial quadratic; the binomial quadratic (where $b = 0$); the binomial quadratic (where $c = 0$); and the trinomial quadratic. It is, however, of paramount importance to always make mention of those quadratics with $a = 1$ across the full spectrum of different types of quadratics, since they portray the special behaviours as well as the special characteristics. Such special behaviours, especially that of a trinomial quadratics, tend to shift the trinomial quadratic $a = 1$ from the type in which it belongs, and render and confine it to quadratic of another type, as if it does not belong to a same type as that of $a \neq 1$. Across all the quadratics it is critical to look at the one where $a = 1$ as it has the special behaviour. In figure 1.2, I present two cases of the monomial quadratic

expression characterised by having $b = 0$, and $c = 0$, where case 1 is of $a = 1$, and case 2 is of $a \neq 1$.

Case1: Monomial quadratic 1

$k \text{ and } m$	$t \text{ and } n$	General form	a, b, c	Example
$k = 1; m = 1$	$t = 0, n = 0$	ax^2	$a = 1, b = 0, c = 0$	x^2

Case 2: Monomial quadratic 2

$k \text{ and } m$	$t \text{ and } n$	General form	a, b, c	Examples
$k \neq 1 \text{ or } m \neq 1$	$t = 0 \text{ and } n = 0$	ax^2	$a \neq 1, b = 0, c = 0$	$6x^2; -5x^2; -x^2$

Figure 1.2 How the monomial quadratic expression is formulated.

The next two cases are that of a binomial quadratic expression characterised by having $c = 0$ and is presented in figure 1.3.

Case3: Binomial quadratic 1

$k \text{ and } m$	$t \text{ and } n$	General form	a, b, c	Examples:
$k = 1; m = 1$	$t = 0 \text{ or } n = 0$	$ax^2 + bx$	$a = 1, b \in R, c = 0$	$x^2 + 3x; x^2 - 4x$

Case 4: Binomial quadratic 2

$k \text{ and } m$	$t \text{ and } n$	General form	a, b, c	Examples:
$k \neq 1 \text{ or } m \neq 1$	$t = 0 \text{ or } n = 0$	$ax^2 + bx$	$a \neq 1, b \in R, c = 0$	$5x^2 + 3x;$ $-9x^2 + 4x;$ $-x^2 - x$

Figure 1.3 How the binomial quadratic expression of $c = 0$ is formulated.

There is a second binomial quadratic expression characterised by having $b = 0$ and is presented in figure 1.4.

Case 5: Binomial quadratic 3

k and m	t and n	General form	a, b, c	Examples:
$k = 1; m = 1$	$t = -n$	$ax^2 + c$	$a = 1; b = 0; c \in R$	$x^2 - 4; x^2 - 25;$ $x^2 + 4; x^2 - 5$

Case 6: Binomial quadratic 4

k and m	t and n	General form	a, b, c	Examples:
$k \neq 1; m \neq 1$	$t < 0$ and $n > 0$ or $t > 0$ and $n < 0$	$ax^2 + c$	$a \neq 1; b = 0; c \in R$	$4x^2 - 25;$ $-x^2 - 16$

Figure 1.4 How the binomial quadratic expression when $b = 0$ is formulated.

There is a trinomial quadratic expression characterised by having $b \neq 0$, and $c \neq 0$ as presented in figure 1.5.

Case 7: Trinomial quadratic 1

k and m	t and n	General form	a, b, c	Examples:
$k = 1; m = 1$	$t \neq 1; n \neq 1$	$ax^2 + bx + c$	$a = 1; b \in R; c \in R$	$x^2 + 3x - 6; x^2 + 4x + 7$

Case 8: Trinomial quadratic 2

k and m	t and n	General form	a, b, c	Examples:
$k \neq 1$ or $m \neq 1$	$t \neq 1; n \neq 1$	$ax^2 + bx + c$	$a \neq 1; b \in R; c \in R$	$5x^2 + 3x - 6;$ $-x^2 + 4x + 7$

Figure 1.5 How the trinomial quadratic expression is formulated.

1.7.4 Factorisation of a quadratic expression.

The process of factorisation of a quadratic expression is a journey by the problem-solver, of rewriting a general form $ax^2 + bx + c$ of the quadratic expression, as the format of $kmx^2 + (kn + tm)x + tn$. Basically, in this view, factorisation is the breaking down of the quadratic expression $ax^2 + bx + c$ by means of searching for the numerical values of k, t, m and n , as they are represented in $kmx^2 + (kn + tm)x + tn$, such that in return, they are able to satisfy the expression $ax^2 + bx + c$. This view is also shared by Kotsopoulos (2007) as she describes the factoring of quadratics as the rewriting of the polynomials as a product of their constituent polynomials, which are linear polynomials. Understanding the factorisation of quadratic expressions as a concept is crucial to learners in becoming successful in mathematics in the South African dispensation and elsewhere. Factorisation is also considered the cornerstone to understanding advanced mathematics in countries such as Indonesia, as Rahmawati, Budayasa and Ekawati (2018, p. 205) contends “factorisation is one of the materials that is included in algebra knowledge which is important to be mastered by students in order to understand advanced mathematics”.

1.7.4.1 Fundamental considerations when factoring quadratic expressions.

In my experiences as a mathematics teacher, I observed that every quadratic expression can be factored by using any single approach, once having been considered in its general form $ax^2 + bx + c$. For the purposes of this thesis, I termed this way of looking at quadratics the “fundamental considerations when factoring quadratic expressions”. It is fundamental in a sense that it can be applied to work in factoring all types of quadratic expressions. How fundamental considerations are followed in factoring quadratic expressions, in this case, where the *grouping* method, as discussed in section 2.4.3 of this thesis, is used. In factoring all the quadratic expressions we consider them as represented in $ax^2 + bx + c$

(i) Factoring a trinomial quadratic expression of $a \neq 1$

Example 1: Factorise $4x^2 - 20x + 21$.

We follow the grouping method to get $(2x - 3)(2x - 7)$

(ii) Example 2: Factoring a trinomial quadratic expression of $a = 1$

Example 1: Factorise $x^2 - 2x - 15$,

We follow the grouping method to get $(x - 5)(x + 3)$

(iii) Factoring a binomial quadratic expression of $c = 0$

Example 3: Factorise $3x^2 - 5x$,

We consider $3x^2 - 5x$ as $3x^2 - 5x + 0$ and follow the grouping method to get $x(3x - 5)$

(iii) Factoring a binomial quadratic expression of $b = 0$

Example 4: Factorise $x^2 - 9$.

We consider $x^2 - 9$ as $x^2 + 0x - 9$ and follow the grouping method to get $(x - 3)(x + 3)$.

1.8 Action-Process-Object-Schema (A-P-O-S).

APOS is an acronym for Action, Process, Object, Schema, the mental constructions that students are likely to make in formulating their understanding of mathematical concepts (Weller et al., 2000). APOS Theory is a constructivist theory of how learning a mathematical concept might take place. According to Dubinsky (2000, p. 2) APOS Theory is based on the following hypothesis about the nature of mathematical knowledge and how it is developed: “An individual’s mathematical knowledge is her or his tendency to respond to perceived mathematical situations and their solutions by reflecting on them in a social context and constructing or reconstructing mathematical actions, processes, and objects and organising these in schemas to use in dealing with the situations”. More details about APOS theory are provided in Chapter 3.

1.9 Benefits of the study.

This study takes the posture of embarking on a journey of the professional development of practising mathematics teachers, at its core, the methodologies on the factorisation of quadratic expressions. Even and Ball (2009, p. 2) outline three reasons why practising teachers need a developed mathematics. Firstly, mathematics students’ learning compels attention to teachers, and to what the work of teaching demands. Secondly, no effort to improve students’ opportunities to learn mathematics can succeed without parallel attention to their teachers’ opportunities for learning. The third reason is that the notion of teacher education is rapidly expanding. In teaching reforms in the United States of America (U.S.A), the RAND (2001,

p.78) furthers this notion of development by suggesting the following focus areas, if teachers are to be developed for mathematical proficiency: Firstly, to develop teachers' mathematical knowledge in ways that are directly useful for teaching. Secondly, the development of teaching and learning skills for mathematical thinking and problem-solving, in teachers. Lastly, to develop teachers in teaching and learning of algebra. I find the abovementioned reasons relevant to this study as it aims to contribute to addressing the methodologies regarding the factorisation of quadratic expressions, for the benefit of the overall mathematics community.

As previously stated, algebra has for years created considerable difficulties for students and teachers alike (Booth, 1990). It is clear that if similar tensions are left unattended, learners will experience difficulty as they make their way in the FET phase. This study therefore seeks reasons why South African learners are performing poorly in mathematics, with particular reference to the factorisation of quadratic expressions. It is hoped that researching learners' perceptions on learning the manipulation of algebraic expressions, particularly factorisation, will assist teachers to understand the problems faced by learners in encroaching the landscape of the FET mathematics algebraic field, and motivate them to become aware of their own mathematical thinking styles. It may help curriculum planners and teachers design the relevant Annual Teaching Plans, coupled with suggested activities useful to learners.

1.10 Overview of this study.

I established and adopted the structure I deemed suitable for this thesis. This thesis is comprised of ten chapters, the bibliography and the appendices. The chapters are as follows:

Chapter One introduces the background and purpose of this study. Furthermore, the motivation for doing this research is outlined. The research questions and key terms in the study are introduced. This chapter concludes by giving the anticipated benefits of this study, drawing particular reference to the South African mathematical landscape.

Chapter Two presents an overview of studies by different researchers that are pertinent to this study which focuses on the teaching and learning of factorisation, with particular attention to

the factorisation of quadratics expressions. In this study, I look at the experiences showcased by learners and teachers, both in South Africa as well as elsewhere. Studies based on APOS in the South African context are summarised.

Chapter Three presents the theoretical framework for this research study. The theory that impacts on this study is discussed. More specifically, the APOS theory, as an extension to reflective abstractions, forms the framework for this research study. The triad mechanism that is used to explain students' understanding is presented. The relevance of APOS to the study is clearly indicated.

Chapter Four focuses on the research design, the research methodology, and the procedures undertaken to conduct this study. The outline, the summary of the research design, and the research instruments used are presented. The preliminary process which involved the pilot study, research paradigm and how it fits with the study, data sources, data collection, and analysis methods together with the study limitations, are also discussed.

Chapter Five focuses on classroom lesson observations. The focus is on how teachers embed their lessons and how learners respond to teachers' instructions.

Chapter Six focuses on learner interviews and particularly on general practices in the mathematics classroom.

Chapter Seven focuses on learners' interviews, with emphasis on the discussion on the video recordings of learner interviews on the mathematical activity. There is also a discussion on the statistical analysis done on learners' work – the March 2019 Grade 10 common test, the findings of which are presented in Chapter 10.

Chapter Eight discusses the teachers' written questionnaires on the mathematical activity, as well as on general practices in the mathematics classroom.

Chapter Nine discusses the video recordings of the teachers' interviews on the mathematical activity, as well as on general practices in the mathematics classroom.

Chapter Ten presents the findings and the conclusions that were drawn based on the overall study. The recommendations of the study are also presented. In the following chapter, the literature reviewed is discussed while the theory that informs this study is reviewed.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction.

In this chapter, I present an overview of studies by different researchers that are pertinent to this study which focuses on the teaching and learning of factorisation, with particular attention to the factorisation of quadratics expressions. In this study, I look at the experiences showcased by learners and teachers, both in South Africa as well as elsewhere. As previously mentioned in chapter 1 Table 1.1 and Table 1.2, the factorisation of quadratics as a practice is rarely examined as a stand-alone item in the South African schooling system, instead, it is always applied in various areas of mathematics. These areas include, for example, the simplification of the algebraic fractions; the solution of quadratic equations; etc., hence it can be concluded that quadratic expressions and equations are applied to a much wider field in mathematics. In this literature review, I first look at research based on the learning of algebra generally, before focusing on research issues regarding factorisation and quadratic functions. Thereafter, I present some pertinent research about approaches to the teaching of algebra. The final section is focused on general teaching of mathematics and presents research about mathematics teachers' knowledge and practice.

2.2 Research about the learning of algebra.

Any discussion of mathematical skills and knowledge necessary for success in mathematics tasks is not complete without a discussion of the role played by algebra in the solution of such tasks. In order to attain mathematical proficiency, the problem-solver has to be fluent in various algebraic procedures because algebra is the most commonly used tool employed to solve mathematical problems. According to Vermeulen (2007), algebra is described as mathematical language that enables us to express generalisations, to investigate and describe patterns, relationships and procedures by appropriate manipulation. "Algebra is also viewed as the language for investigating and communicating most of mathematics. Algebra can be seen as generalised arithmetic, and can be extended to the study of functions and other relationships between variables" (Vermeulen, 2007, p. 15). Barnard (2002, p. 15) shares similar sentiments where he proposes that "basically, algebra illuminates the workings of mathematics and provides a means of understanding and explaining mathematical phenomena". According to RAND Mathematics Study Panel (2001, p. xx) "algebra is regarded as foundational in all areas

of mathematics because it provides the tools (language and structure), for analysing and representing quantitative relationships, for modelling situations, for solving problems, and for stating and proving generalisations”. Again, Barnard 2002, proceed to describe algebra as a language, and so, learning rules like $a^5/a^3=a^2$ is part of learning that language, and is as crucial to mathematics as correct spelling and punctuation is to English.

In the South African mathematics classroom, mathematics teaching is rolled out by means of the DBE (2011a). According to DBE (2011a) mathematics is defined as:

A language that makes use of symbols and notations for describing numerical, geometric, and graphical relationships. It is a human activity that involves observing, representing and investigating patterns and qualitative relationships in physical and social phenomena and between mathematical objects themselves. It helps to develop mental processes that enhance logical and critical thinking, accuracy and problem-solving that will contribute in decision-making. Mathematical problem-solving enables us to understand the world (physical, social and economic) around us, and, most of all, to teach us to think creatively (DBE, 2011a, p. 8).

There is a strong relationship between the definition of mathematics offered in CAPS and the description of algebra as offered by scholars. The implications of applying this definition of mathematics to the classroom is that teachers of mathematics, in executing the pedagogical contract, need to help learners develop a better understanding of algebra, which will lead to a better understanding of mathematics. However, for years, algebra has created considerable difficulties for students and teachers alike (Booth, 1990). It is therefore worth reporting on research covering issues experienced by learners in the learning of algebra locally and globally. The detailed discussion is presented in the oncoming sections.

2.2.1 Research about teaching and learning of algebra – globally.

There is considerable literature depicting the global and local scholarly work conducted in secondary school algebra, focusing on all grades. Various researchers have reported their findings, which notably indicate that students experience similar challenges across all grades. Teaching and learning algebra has long been seen as a source of difficulty; and that has been of serious concern to all stakeholders in the mathematics community (de Lima & Tall, 2006).

In this section, I present the research findings on the teaching of algebra, delving into the successes and challenges, as noted and observed by mathematics practitioners, with focus on secondary school mathematics within the global community.

Several studies conducted indicate that the challenges associated with teaching and learning algebra begin at earlier grades. Matzin and Shahrill (2015) conducted a study with 78 Year 7 students in Darussalam which aimed at identifying areas in solving algebra problems the students found most difficult or challenging. Two important findings were derived from the study. Firstly, many of the Year 7 mathematics students demonstrated a poor grasp of basic algebraic knowledge and skills. Secondly, the students still experienced difficulties when solving problems that involved manipulation of algebraic expressions and equations. These findings corroborate the findings that were noted in the study conducted by Jupri, Drijvers and van den Heuvel-Panhuizen (2014), in Indonesia with Grade 7 students which found that students had difficulties in initial algebra and in solving equations in one variable.

Studies conducted abroad also reveal that Grade 10 learners are challenged particularly in algebra. The study conducted by Norton and Irvin (2007) on how to solve equations with variables on both sides in Year 9 classes reveals that much of the students' struggle was associated with a lack of understanding of arithmetical concepts, including those associated with equivalence, operations with negative integers, and the distributive law and fraction concepts. Citing similar challenges, the study conducted by Filloy, Rojano and Solares (2004, p. 397) suggested that "dealing with problems and equations involving two unknowns quantities imply the necessity to re-elaborate: (a) the notion of *mathematical unknown*, (b) the notion of *algebraic equality*, and (c) the notion of *algebraic representation of the unknown*." For Filloy et al., (2004), traditional teaching models introduce the problems with several unknown quantities, assuming that, once the first elements of the algebraic language are acquired, the further development of algebra will be carried out through a simple extension of what was previously learned. However, the study reveals that the students in fact find algebra challenging when dealing with equality between "unknown things".

Further studies reveal the students' difficulties in dealing with the factorisation of quadratic expressions. Yahya and Shahrill (2015) conducted the study for the Year 11 mathematics students in Brunei, with the aim of gaining a better understanding on why, and what, makes students choose a certain method or strategy in solving problems in algebra. The study identified four categories of learners' barriers to learning factorisation of quadratic expressions. These barriers are: errors in understanding factorising quadratic expression; errors in multiplication of factors; errors in addition of integers; and errors in the formation of solutions for factorisation. A closer look at the errors identified, shows there is standing evidence that more groundwork has to be done for learners at lower grades.

I argue that, if the students' inabilities to deal with algebra at early grades are left unattended, similar tensions will therefore be evident in the upper grades. In supporting this argument, I followed the study conducted by Güner (2017) on investigating the achievements of 10th Grade students in Istanbul, in solving quadratic equations. The purpose of the study was to examine the students' tendency to use different ways to solve problems such as completing square, factorisation and quadratic formula, and determine their errors. The findings of the study revealed that most students had difficulty in solving quadratic equations and made various errors. According to Güner (2017), the students' errors in solving quadratic equations were due to a weakness in mastering topics such as algebra, fractions, integers, the rules of quadratic equations' solution methods, calculation and algebraic simplification.

Similar challenges associated with quadratic equations were also observed in a study conducted by de Lima and Tall (2010), whose focus was on the performance of students on quadratic equations after they had developed procedural methods of solving linear equations. The results showed that, in the case of linear equations, the participants did not build on embodied models such as a balance; they did not encapsulate processes as mental objects, nor did they use formal principles such as 'do the same thing to both sides'. Instead of a conceptual embodiment underlying the symbolism of solving equations, such as a balance, they developed a procedural embodiment of symbol shifting in the mind and on paper to 'move towards' achieving an answer. Furthermore, the study by de Lima and Tall (2010, p. 17) found that "the procedural embodiments as anticipated in linear equations are now for quadratic equations extended with a new procedural embodiment to 'move the power over the other side and change it to a root', which then resulted in participants finding only the positive root."

Challenges associated with algebra are also noted when students enter higher institutions of learning. In Maldives, Naseer (2015) conducted a study with the main purpose of identifying the common errors and misconceptions of students enrolled in pre-University (Foundation Level) mathematics courses through analysis of their examination papers. The findings of the study reported that students have misconceptions, firstly with number sense; secondly with order of operations; thirdly with fractions; fourthly with indices; and fifthly with errors due to a lack of technical vocabulary. For students to be successful in algebra at higher institutions of learning, “Primary and lower secondary school mathematics teachers are responsible for laying a solid foundation to ensure students’ later success in mathematics. Identification of common errors and misconceptions, how they are caused, and how they can be prevented and remedied need to be included in the initial teacher training curriculum to ensure teachers are aware of these” (Naseer, 2015, p. 38). A similar call has been made by Snipes and Finkelstein (2015, p. 1), when they said, “algebra matters. Students who struggle with math in middle school and high school have a lower chance of meeting eligibility requirements in California’s public universities; four years of math in high school, with a strong foundation in algebra that builds from middle school, is key to higher education access”.

Grade 8 is an important year for mathematics for students since this is the year where students transition from arithmetic to algebra. “Algebra can be particularly challenging not only because it introduces more abstract representations and more complex relationships between quantities, but also because it can magnify the misconceptions that have their roots in earlier instruction” (Booth et al., 2014, p.1). According to Booth et al., (2014), students hold many misconceptions as they transition from arithmetic to algebraic thinking, which can hinder their performance and learning in the subject. The line of distinction between the origins of misconceptions and mistakes is articulated by Holmes et al., (2013, p.30) as they said “misconceptions are derived from problems due to conceptual misunderstandings. Mistakes are derived from computational or minor mishaps”.

Challenges of transition from arithmetic to algebra were also noted in the study conducted in Norway for 11th graders by Breiteig and Grevholm (2006). During the study students were given the following task:

“Eva is thinking of two numbers. The sum of them is 19. The difference between them is 5.

a Find the numbers.

b How can you find the numbers?

c Why is it always possible to find the two numbers when we know their sum and difference?”

The analysis of the study revealed that students resisted the shift from arithmetic to algebra. Even though they were able to arrive at the correct solutions, they had challenges in providing the reasons, explanations, arguments, generalisations, and justifications on how they arrived at the solutions.

Algebra is a language as asserted in Vermeulen (2007). In pursuit of further identifying the underlying causes and reasons of the difficulties depicted, I followed the study conducted by MacGregor and Stacey (2007) on the students’ understanding of algebraic notation. The aim of the study was on investigating the cognitive and linguistic demands of learning algebra in secondary school. The study revealed that students of earlier grades, 13 to 15-year-olds had difficulties in learning to use algebraic notations. According to MacGregor and Stacey (2007), the difficulties in learning to use algebraic notation have several origins, and noted as follows: Firstly, intuitive assumptions and sensible, pragmatic reasoning about an unfamiliar notation system; secondly, analogies with symbol systems used in everyday life, in other parts of mathematics or in other school subjects; thirdly, the interference in new learning in mathematics; and lastly, poorly-designed and misleading teaching materials

Solving problems in mathematical situations where algebra is embedded in context requires students to be able to mathematise the context. A study aimed at understanding the problems students experienced related to the algebraic “mathematisation” process, meaning, was conducted in Indonesia (Muchoko, Jupri & Prabawanto, 2019). Mathematisation as a process is where students need to convert real life story problems into mathematical tasks. The study adopted the use of “visual” representations such as diagrams and schemes to facilitate understanding (of fractions) in the factorisation or simplification of algebraic terms. The results showed that Grade 8 students had difficulty in visualising algebraic forms and applying the associatively and distributive properties of algebraic expressions when they had to factorise or simplify algebraic terms.

As highlighted in preceding paragraphs, issues experienced by students arising out of their inability to transition from arithmetic to algebra; their understanding of algebraic notation; their

inability to mathematise, all converge, resulting in their failure to address other concepts. A study on the performance and difficulties of students in formulating and solving quadratic equations with one unknown was conducted in the Middle East (Didis and Erbas, 2015). The aim of the study was to investigate the performance of 10th Grade students in solving quadratic equations with one unknown, using symbolic equation and word-problem representations. The participants of the study took part in the administration of the open-ended test, open-ended questionnaire, and eventually on the semi-structured interviews. The findings of the study revealed that although students have difficulties in solving both symbolic quadratic equations and quadratic word problems, they performed better in the context of symbolic equations compared with word problems. According to Didis and Erbas (2015) student difficulties in solving symbolic problems were mainly associated with arithmetic and algebraic manipulation errors. In the word problems, however, students had difficulties comprehending the context and were therefore unable to formulate the equation to be solved. Thus, it is concluded that the differences in the structural properties of the symbolic equations and word problem representations, affected student performance in formulating and solving quadratic equations with one unknown.

Generalising is part of the tactics in solving problems in mathematics. One of the challenges experienced by students is related to the generalisation that one is required to make using algebra. A study conducted with 9th Graders in the US, focused on the generalisation strategies of beginning high school algebra (Becker & Rivera, 2005). This study identified three types of generalisation, which were, firstly, there are students who use numerical generalisation and they employ trial and error as a similarity strategy with no sense of what the coefficients in the linear pattern represent. The variables are used merely as placeholders with no meaning except as a generator for linear sequences of numbers, with lack of representational fluency. Secondly, there are students who use figural generalisation and employ perceptual similarity strategies in which the focus is on relationships among numbers in the linear sequence. Variables are seen as not only placeholders, but within the context of a functional relationship. Thirdly, there are students who use pragmatic generalisation and employ both numerical and figural strategies and are representationally fluent; that is, they see sequences of numbers as consisting of both properties and relationships (Becker & Rivera, 2005, p. 128).

Mathematical competencies are the mathematical processes that students apply as they attempt to solve problems. For students to be able to compete in a changing economy, they need to be mathematical competent and able to adapt the knowledge they are acquiring - they need to be mathematically proficient (Kilpatrick, 2001). Mathematical proficiency is defined in terms of five separate, but interwoven strands which are, firstly, *conceptual understanding*, which refers to the student's comprehension of mathematical concepts, operations, and relations. Secondly, there is *procedural fluency*, or the student's skill in carrying out mathematical procedures flexibly, accurately, efficiently, and appropriately. Thirdly, *strategic competence*, which is the student's ability to formulate, represent, and solve mathematical problems. Fourthly, *adaptive reasoning*, is the capacity for logical thought and for reflection on, explanation of, and justification of mathematical arguments; and, lastly, a *productive disposition*, which includes the student's habitual inclination to see mathematics as a sensible, useful, and worthwhile subject to be learned, coupled with a belief in the value of diligent work and in one's own efficacy as a doer of mathematics (Kilpatrick, 2001). The study on the students' difficulties in applying conceptual knowledge, procedural skills, strategic competence, and algebraic thinking in solving routine (instructional) algebraic problems, was conducted in Toronto (Egodawatte & Stoilescu, 2015). The purpose of the study was to investigate Grade 11 university/college stream mathematics, and was advocated by students writing standardised tests, followed up by interviews. The study found: Firstly, students lack *application* skills such as strategic skills, procedural skills, and metacognitive skills. Secondly, they lack proper *relational* skills such as translation of one form to another, usage of the equal sign etc. Lastly, they did not show a proper understanding of the *structural* features of algebra such as properties, axioms, and other required conceptual features. It is important to note that these three areas are not mutually exclusive.

Factorisation of quadratic expressions plays a significant role and contributes to the better understanding of the solution of quadratic equations. A qualitative study conducted for 11th Grade high school students in Turkey aimed at exploring the conceptions high school students have concerning quadratic equations with one unknown (Kabar, 2018). The study sought to address the following questions: Firstly, how do high school students define the concept of the quadratic equation with one unknown? Secondly, what are high school students' concept images of the quadratic equation with one unknown? Thirdly, which conceptual problems do high school students have while working with quadratic equations? The analysis of the data

revealed that none of the students could provide an exact and correct definition of a quadratic equation and their attempted definitions were not consistent with the formal (standard) definition of quadratic equation. Factorisation of quadratic expressions, again, plays a significant role in students' mastering of the concept of the solution of the limits of algebraic function. Nurhayati and Retnowati (2019) from Indonesia, conducted a study on an analysis of errors in solving limits of algebraic function. The study reported the students' errors as belonging to errors of principle and procedure. However, the deep analysis revealed that central to the errors, learners were found to be weak in addressing factorisation, particularly that of dealing with quadratics

Challenges associated with the learning of algebra seem to be incremental in that they are found in almost all grades, as students migrate from grade to grade. The work of Daud and Ayub (2019) in corroborating that of Yahya and Shahrill (2015) also shows that students reach year 11 of their studies with challenges associated with the factorisation of algebraic quadratics. In qualifying this claim, a comparative quantitative study was conducted for 465 students in Year 9 classes in Thailand, Year 10 students in Brunei Darussalam, and second-year university students in the United States, who attempted to solve the same quadratic equations (Vaiyavutjamai, Ellerton & Clements, 2005). The analyses of responses to two apparently straightforward quadratic equations suggested that in the Thai (Grade 9) and Brunei (Year 10) subsamples, more than 50 percent of the students were confused with respect to the concept of a variable as it manifested itself in the quadratic equations. Again, the analysis of data suggests that a sizeable proportion of the US (second-year) subsample was also confused on the same matter.

2.2.2 Research about teaching and learning of algebra in South Africa.

A number of studies in algebra at secondary school mathematics level have been conducted in South Africa. In this section I present, for the purpose of this study, the research findings with a focus on secondary school mathematics. According to DBE (2011a, p.13), one of the key objectives of learning algebra in Grade 10 is to manipulate algebraic expressions by: "simplifying, adding and subtracting algebraic fractions with denominators of cubes (limited to sum and difference of cubes)". Also, Table 1.2 shows that the factorisation of quadratic expressions embedded in fractions belongs to the complex procedure level, which is the object

level in terms of APOS. Hence, the research on algebraic fractions at Grade 10 mathematics is fundamental, and is a huge contribution to the FET mathematics body of knowledge.

Learning from research reports, learners enter the FET phase already having some challenges on matters pertaining to the manipulation of algebraic variables and expressions. Several studies conducted in the GET phase mathematics presented evidence that learners of Grade 9 held misconceptions on certain concepts. One study was conducted by Dhlamini and Kibirige (2014) on errors and misconceptions that Grade 9 learners exhibited when adding simple fractions, and fractions of algebraic numerators and denominators, respectively. The findings revealed that when adding algebraic fractions, learners produce errors and reveal misconceptions as a result of short cuts they perceive as easier methods of adding fractions, instead of following the apparently long and cumbersome yet algorithmically correct method of adding fractions. Similar findings were also observed on the study by Mashazi (2014), whose focus explored the thinking underlying Grade 9 learner errors in introductory algebra. The results of that study indicated that various aspects contributed to learners' errors including task instructions, new knowledge, ignoring of the letter, and replacing letters with numeric values.

A study was conducted on a content analysis of the three Grade 9 Mathematics Annual National Assessment (ANA) papers, paying particular attention to cognitive demand and levels of difficulty (Pournara, Mpofu & Sanders, 2015). This study was driven by the following reasons: Firstly, the poor performance in higher grades (9-12) was linked to poor performance in algebra. Secondly, the Annual National Assessments (ANA) introduced to address South Africa's poor performance in international mathematics assessments, such as TIMSS and SACMEQ. The content analysis involved the ANA papers written in the years 2012, 2013 and 2014. It appeared that the 2013 paper was easier than the other two years. This is evident in the national ANA reports, where, in 2012, the national average was 12.7 percent. In 2013, the average increased to 13.9%, but then dropped to 10.8% in 2014. The study found that the overall picture is one of very poor performance (Pournara, et al., 2015),

Algebra is language inclined, as was evident in the study conducted by Adu and Olaoye (2015a), whose focus was to investigate language proficiency and methods of teaching as determinant of Grade 9 students' academic performance in algebra. Such language paramouncy

in the study of mathematics is a contributory factor in learner errors as language plays a pivotal role in task instructions.

A study, focused on Grade 10 learners to elicit learner errors and misconceptions in simplifying rational algebraic expressions to improve teaching and learning was conducted with two main objectives. Firstly, to identify the errors and misconceptions that Grade 10 learners make when simplifying rational algebraic expressions. Secondly, to discover what extent teaching, focused on the identified errors and misconceptions, helps reduce the identified errors and misconceptions (Makonye &Stepwell 2016). The results from the study showed that teaching directed at identified errors and misconceptions helped on average to reduce the errors learners make. However, there were some errors that remained difficult to remediate as they kept appearing, even among strong mathematics learners. Common errors observed were: *Conjoining Errors; Incorrect Cancelling; Incomplete Cancelling; De-fractionalisation; Like Term Error; Addition or Subtraction of the Numerators*. These findings are noted to contribute to the work initially conducted by Makonye and Khanyile (2015) where they uncovered the serious mathematical errors encountered as Grade 10 learners simplify algebraic fractions.

A study focussed on dealing with grade 10 learners' misconceptions and errors when simplifying algebraic fractions, was conducted in the Eastern Cape Province in South Africa (Baidoo, 2019). The results from the study showed that learners, "conceptual understanding of simplifying algebraic fractions was severely flawed. Furthermore, "the results of the study showed that conceptual, mathematical language, procedural and application errors hinder learners' appropriate understanding as well as the application of algebraic fractions. Notable among the four errors were conceptual and procedural errors that seemed to permeate attempts that learners made to simplify algebraic fractions" (Baidoo, 2019, p. 54).

Another study on understanding Grade 10 learner errors and misconceptions in elementary algebra was conducted by Makonye and Mashaka (2016). The study was undertaken in three steps: Firstly, learners were given a pre-test so that co-researchers could locate their errors in the simplification of algebraic fractions. The second stage was a pedagogical intervention that interrogated identified learner errors in the pre-test through classroom discussions and dialogue, with the focus of the discussions being the identified errors. At the third stage, a post-test was administered to measure the extent to which the intervention helped these learners

address their errors. Again, the common error that resurfaced was that of ‘senseless’ cancelling. According to Makonye and Mashaka (2016), students just cancelled at will, whenever they saw any similar terms in the numerator and the denominator. As a result, students were unable to factorise the algebraic fractions.

Another study aimed at measuring the proficiency Grade 10 mathematics learners have in the simplification of rational algebraic expressions, was conducted by Mhakure, Jacobs and Julie (2014). This study noted the existence of misconceptions related to rational algebraic fractions. The nature of the students’ difficulties with respect to rational algebraic fractions were noted to be conceptual, in that students operated on rational algebraic fractions without necessarily understanding or justifying what they were doing.

Algebra is one of the mathematics topics that many learners find challenging (Mhakure et al., 2014). I strongly agree with this sentiment as I have observed the common factor in errors identified throughout the discussion points to the students’ inability to manipulate basic algebra. As teachers of secondary school mathematics, we have a pedagogical contract to prepare and improve learners’ proficiency when it comes to algebra. We do this, bearing in mind that as students move to higher learning institutions in South Africa, their acceptance into particular study areas requires mathematics, which has many algebraic aspects.

Solving equations in mathematics is one area that advocates the use of algebra. A study on investigating the errors that a class of Grade 11 learners revealed when they solved quadratic equations tasks through factorisation was conducted in the Gauteng Province by Makonye and Nhlanhla (2014). Participants of the study were presented with quadratic equations tasks which they were asked to solve. Learners’ scripts were scrutinised for errors. Selected learners were then probed and pressed about their errors so as to manifest their thinking thereof. The study found the errors were mainly systematic, conceptual, and procedural in nature. Types of errors in solving quadratic equation were categorised as follows: Firstly, there are *applicability errors*. Makonye and Nhlanhla (2014) citing Clark (1973) referred to these as misuse of the rules of algebra. For example, the incorrect use of the distributive rule in a problem that does not even require the use of the rule. Secondly, there are *procedural errors*: An example of these is a *transpose error* where learners fail to transpose correctly by changing the sign of the term when it is transposed from one side to the other. Thirdly, there are *conceptual errors*: These are errors

where learners do not comprehend the ideas subsumed by the task, such as handling a quadratic equation as though it were a simple linear equation.

More work needs to be done at Grade 10 level because learners migrate with misconceptions and errors from Grade 10 to Grade 11 Makonye and Nhlanhla (2014, p. 634) were quoted as saying: "...notice some patterns in the mathematical errors and conceptions that learners have, particularly in factorising algebraic expressions and equations. The errors Grade 11 learners make are similar to the errors that Grade 10 learners make. We have realised that even though learners are taught to factorise in Grade 9 and Grade 10, they still struggle to factorise in Grade 11. We have wondered: Why are they still not mastering the notion of factorising? We have realised that learners have problematic schemas on these tasks. We argue that until we know why learners hold these errors, we will not be able to help them much".

This grade to grade migration of challenges facing learners was also noted in the study aimed at tracking a cohort from Grade 9 to Grade 11 on a diagnostic algebra test, conducted by Pournara, et al., (2016). The study drew on quantitative and qualitative data to illuminate the kinds of errors that learners make in relation to basic algebra, as they progress from grade to grade. Pournara, et al., (2016, p. 9) presents the findings of the study as follows: Firstly, the existence of conjoining errors becomes less frequent by Grade 11, but they are still persistent. Secondly, the inappropriate application of the distributive law increased in Grade 10 but had dropped off by Grade 11. Thirdly, learners' performance on items involving subtraction and negatives improved substantially from Grade 10 to Grade 11. Fourthly, the interviews show evidence of learners talking of subtraction as 'take away', even when dealing with algebraic terms. Lastly, there is widespread evidence in the tests and interviews of learners evaluating letters rather than accepting expressions such as $8 + g$ as a final answer. The most common evaluation strategy was equal splits and it was persistent up to Grade 11. This suggests that learners assume the letters must have values which may be a curriculum effect of the emphasis on equation solving.

The implication from these findings is the need to work on areas of algebra as a way to prepare learners for Grade 12. In the quest towards gaining a better understanding of how the competency in algebra may impact on learners' future endeavours, we navigated the study conducted by Makgakga and Makwakwa (2016) in Limpopo Province, South Africa. The purpose of this study is to report on **Grade 12** learners' academic achievements and difficulties

when finding the derivatives of functions using first principles and rules of differentiation. A diagnostic test was administered to the participants, with the aim of measuring the level of achievement between the learners finding the derivatives of functions using first principles, and those using rules of differentiation. The semi-structured interviews were constructed following the learners' responses in the diagnostic test. The findings revealed that learners performed more poorly in finding the derivatives using first principles than using the rules of differentiation. Central to the poor performance was: Firstly, learners experienced difficulties in dealing with differentiation when given functions involving fractions. Secondly, learners performed badly due to the teachers' teaching approaches and learners' lack of procedural and conceptual knowledge. Thirdly, learners made incorrect choices of the first principles formula, with incorrect substitution, and incorrect computations. These findings partially corroborate the findings of the study conducted by Luneta and Makonye (2010) on Grade 12 learner errors and misconceptions in elementary analysis.

The Grade 12 students' inability to apply basic algebra was also noted in the study conducted by Chigonga (2016) in the Limpopo Province of South Africa. The focus of the study was on learners' errors when solving trigonometric equations and suggested interventions from Grade 12 mathematics teachers. Teachers were interviewed, and part of the findings of the study revealed that: Firstly, students struggled to solve the trigonometric equations. Secondly, students failed to distinguish whether the equation was linear or quadratic in form, in order to determine the solution method e.g., if only one trigonometric function is present, use the linear method; and if the equation is in quadratic form, factorise or use the quadratic formula. Thirdly, students are encouraged to try using identities to change the form of the equation in terms of a single trigonometric function. It may be helpful to square both sides of the equation first. If this is done, check for extraneous solutions. The findings of this study are crucial since the trigonometry content area has a weighting of 40% in the Grade 12 Mathematics Paper 2. As noted in Sasman (2011), failure to understand trigonometry concepts on the part of the learners has a ripple effect on the success rate of Grade 12 mathematics learners.

Challenges associated with basic algebra manifest themselves even at the higher institutions of learning. In 2011, the UKZN School of Mathematics, Sciences and Technology Education remarked on the poor performance of students in their first year mathematics module. From the analysis it was affirmed that students lack basic skills and knowledge, referring to, the

prerequisites for the study of first year university mathematics modules. These basic skills and knowledge were detected to belong to the following sections: (1) algebra, (2) functions, and (3) reasoning using symbols and connectives (Maharaj, Brijlall & Narain, 2015). The implication is that students who have not studied algebra, or who have a weak proficiency in algebra find themselves excluded from completing particular courses. For future studies and careers, a sound understanding of algebra is essential for employment purposes. Employers prefer to employ creative people who are able to solve problems (Makonye and Stepwell, 2016).

2.3 Studies about learning of factorisation and quadratic functions.

The poor performance of pupils in mathematics and the factorisation of quadratics in particular are of great concern to mathematics educators, and all interested parties, globally and locally. From the literature, there is evidence that there are very few studies conducted with a focus on the factorisation of quadratics as a stand-alone item. However, most studies focus attention on the application of the factorisation of quadratics, particularly, in quadratic functions. In this section, I navigate and report on how the learning of this special portion of algebra has progressed, both internationally and locally.

Several studies have shown that learners find it challenging to work with the factorisation of quadratic expressions (Naseer, 2015; Kotsopoulos, 2007). Internationally, there is evidence that most countries follow the use of traditional methods in the factorisation of quadratic expressions. In Indonesia, Saputro et al., (2018b) on the 8th Grade students' learning obstacles in factoring a quadratic form. The 8th Grade students were taught using the cross method of factoring quadratics, which was then followed by learners writing a test. Each student was then interviewed to determine the learning obstacle category. After conducting interviews with students who made mistakes, the study obtained two types of learning obstacles, that is didactical obstacle and epistemological obstacle. According to Saputro et al., (2018b, pp. 902-903), "Didactical obstacle occurs due to two factors. The first factor is the design of the cross method which gives less affirmation of pairs of factors obtained to be arranged into the multiplication of two linear factors. The second factor is the teacher presents the cross-method design by not involving the variable x while running the procedure. Epistemological obstacle

occurs because students are not able to use existing procedures on the type of problem that he has never met”.

Similar findings were corroborated in the findings from the study on the use of the cross method conducted in Singapore by Leong, Yap et al., (2010). In Bukit View Secondary, the quadratic factorisation was usually taught to students through the familiar “cross-method”. The team found that a majority of students was unable to factorise quadratic expressions correctly. According to Leong, Yap et al., (2010, p. 20), the problems that students experienced regarding quadratic factorisation were: Firstly, the students’ common conception of algebra as being “abstract” and thus beyond them. Secondly, the students’ lack of pre-requisite algebraic skills (such as simplification of like terms). Thirdly, students’ perception of the “cross method” as arbitrary, and the subsequent failure to make sense; and lastly, factorisation is seen as an isolated skill to be learned without a broader context.

Knowledge and skills on the factorisation of quadratics are needed and demanded in the understanding of other sections of algebra, through the application of the factorisation of quadratics. The implication is that there are other sections of algebra that will be difficult to master without the knowledge of the factorisation of quadratics. For example, the use of knowledge on the factorisation of quadratics is crucial in the solution of quadratic equations. Recent studies show that learners find it challenging to work with quadratic equations (Makonye & Matuku, 2016). A study on the effects of classroom instruction on students’ understanding of quadratic equations was conducted in Thailand (Vaiyavutjamai & Clements, 2006). The aim of the study was to investigate how traditional lessons on quadratic equations, in which the emphasis was on teaching Grade 9 students to solve quadratic equations by factorisation (and application of the null factor law), by “completing the square” and by the quadratic formula, influenced student development of knowledge, skills, concepts, and understanding with respect to quadratic equations. According to Vaiyavutjamai and Clements (2006, p. 47), “in traditional lessons the teacher talks more than the students and classroom discourse patterns feature many sequences of chorused answers, with teachers channelling students’ thinking towards answers that they want the students to give”. The analysis of interview data revealed that many interviewees who obtained correct solutions actually had serious misconceptions about what quadratic equations actually are. Their answers were correct but, from a mathematical point of view, they did not know what they were talking about.

Similar findings were also noted in another study conducted to determine students' errors in learning quadratic equations for the Form Three students from a secondary school in Indonesia (Zakaria, 2010). The purpose of the study was: firstly, to determine the type of students' errors in using factorisation, and secondly, to determine the type of students' errors in using the completing the square method. It found that students made errors at the process skill level, especially in the expansion of quadratics expression. Again, in computation, students made mistakes in positive and negative signs when developing algebraic expressions. Errors also occurred when replacing values that had a negative sign. The similar findings as presented by Zakaria (2010) were also noted in another study aimed at investigating students' mathematical difficulties with quadratic equations by O'Connor and Norton (2016). In their report, it was found that students attempted to rearrange quadratic equations as if they were linear equations. These attempts suggested that students had misconceptions regarding the nature of quadratic equations and, in their attempts to rearrange the expressions, students also demonstrated various misconceptions regarding algebraic conventions.

Challenges associated with students' inability to work with factorisation of quadratics have been noted to prevail in another dimension. A study aimed at investigating the difficulties of students in integrating procedural and conceptual knowledge on quadratic equations to solve realistic problems was conducted in Vietnam by Phuong (2019). A questionnaire consisting of six questions affiliated to Physics, Technology, Engineering and Finance, was administered to 10th graders who were taught quadratic equations. The findings show that students made their difficulties in "mathematising" "real situations" to build a correct "mathematical model", and in "interpreting" "mathematical solutions" to "realistic solutions" for realistic problems. The core reason is the integrating of conceptual and procedural knowledge on a specific mathematical knowledge (e.g. on quadratic equations) to solve realistic problems.

According to DBE (2011a), as presented in Table 1.2, in Grade 10, the quadratic expression $ax^2 + bx + c$ is treated with an inclusion of examples of $a \neq 1$, and the application of the factorisation techniques is also introduced to the simplification of algebraic fractions. Several studies conducted in South Africa revealed the challenges experienced by Grade 10 students as they work with the simplification of algebraic fractions (Mhakure et al., 2014). To remedy the situation, Makonye and Khanyile (2015) conducted a study with the purpose of exploring

to what extent probing learners' errors on simplifying algebraic fractions would help them reduce those errors. The findings reported that students were able to overcome their errors in that they were able to factorise a trinomial. Secondly, they reduced careless mistakes. Thirdly, they avoided obtaining answers using incorrect mathematical rules. Lastly, they could work out algebraic tasks relationally.

Factorisation of quadratic expression has been noted to play a crucial role in the solution of quadratic inequalities. A study on the investigation of the obstacles faced by Grade 11 learners in the learning of quadratic inequalities was conducted in the Gauteng Province in South Africa by Makonye and Shingirayi (2014). Learners responded to a written task consisting of quadratic inequalities tasks in order to establish the kinds of errors they held and the misconceptions thereof. The written tasks, together with the interview responses were analysed to determine the types of errors as well as the possible reasons behind those errors. The results of the study noted that *procedural and conceptual errors linked to algebraic processes, as well as meaning systems confusing equations and inequalities were prevalent*. Solving quadratic inequalities involves many steps, which includes writing the expression in standard form, factorising and or removing the brackets, finding the critical values, reading and interpreting the solution. The quadratic factorisation procedure is one that needs to be done with some understanding. According to Makonye and Shingirayi (2014, p. 723), "It was clear that many learners had difficulties with the first step concerning factorisation of the given quadratic inequalities so as to establish the critical values. Therefore, far from exploring errors in solving quadratic inequalities per se, we found ourselves lodged in exploring errors in the "pathway processes" to solving inequalities such as factorisation and dealing with solutions to quadratic equations. Learners showed clearly that their understanding of factorisation was in the main incomplete".

The application of the knowledge of factorisation of quadratics is extensively used in areas of algebra in Grade 12. In the study conducted by Chigonga (2016) in Limpopo Province on exploring Grade 12 learners' errors when solving trigonometric equations, the crucial role played by the factorisation of quadratics was made explicit. The participants in the study were Grade 12 mathematics teachers. The study followed an explanatory sequential mixed-methods design where the quantitative data was collected first by means of a diagnostic test, followed by qualitative data by means of the semi-structured interviews. Results of the study reflected

the inability of students to deal with the factorisation of quadratics, as is central in addressing issues around the trigonometric equations.

2.4 Approaches to the teaching of factorisation and quadratic expressions.

In this section, I present the research reports that have been gathered, globally and locally, on the teaching of factorisation and that of quadratic functions. Throughout this discussion, I will consider the quadratic expression in its general form $ax^2 + bx + c$.

In Chapter 1, I presented four types of quadratic expressions as prevalent in the secondary school mathematics. Several studies conducted point to the factorisation of quadratic expressions as the most conceptually challenging aspect to teaching and learning algebra (Chua, 2017; Kotsopoulos, 2007; Leong Yap et al., 2010). Patrick (2012, p. 47) agrees and is specifically quoted as saying “factoring quadratic trinomial, which requires to rewrite $ax^2 + bx + c$ into the form $(mx + n)(px + q)$, is one of the most difficult algebraic challenges for students of the high school mathematics curriculum”. To be precise, the most challenging part is to work with quadratic trinomials of $a \neq 1$. A term that is constructed by means of combining two symbols is called “juxtaposition”. According to Patrick (2012), it is very difficult for teachers to imagine why some students cannot handle the term, for an example, $6x^2$ since it can be given a variety of different interpretations in different contexts. For an example, $6x^2$ can be viewed as $6 \times x^2$ or $3 \times 2 \times x^2$ or $6 \times x \times x$ or $3 \times 2 \times x \times x$. This poses a challenge as French (2002, p. 11) laments “...how do we interpret the two symbols 2 and x when they stand next to each other as in $2x$?”

Having observed the challenge of working with quadratic expression with $a \neq 1$, mathematics practitioners have developed a variety of techniques and strategies in order to address these challenges, aimed at facilitating the meaningful teaching and learning of factorisation of quadratic expression. The use of various approaches in teaching mathematics is advocated in Ma (1999), as they contend that teachers’ mathematics content knowledge has to be broad, deep, and thorough; that is, teachers should have profound understanding of fundamental mathematics which is attuned to, and usable in teaching. This understanding includes connectedness, multiple perspectives, fundamental ideas, and longitudinal coherence. Kilpatrick, Swafford and Findell (2001, p. 428) share sentiments similar to that of Ma (1999)

as they suggest that mathematics teachers need specialised knowledge that, “includes an integrated knowledge of mathematics”. Table 2.1 adapted in Mart (2019) provides the various methods employed in factoring the quadratic expressions. For the purpose of this study we will discuss a few of the methods as they fit this study viz. Factoring by “Guess and Check” Method; Factoring by Cross Method; Factoring by “Grouping” or “Long” Method; Factoring by Algebra Tiles; Factoring by Autrey’s and Austin’s Method; and Factoring by Slip-Slide Method. Mart’s currently used methods of solving quadratic equations are presented in Table 2.1 that follows.

Table 2.1 Methods of factoring quadratic expressions as adapted in Mart 2019.

1. The Quadratic Formula
2. Completing the Square
3. Factoring Special Cases
3.1 The Square of a Binomial
3.2 The Difference of Two Squares
3.3 Greatest Common Factor
4. Factoring by “Guess and Check” Method
5. Factoring by Cross Method
6. Factoring by Vedic Method
7. Factoring by “Grouping” or “Long” Method
8. Factoring by Baker’s Method
9. Factoring by: Moskol’s Method
10. Factoring by Savage’s Method
11. Factoring by Algebra Tiles
12. Factoring by X-Method or Diamond Method
13. Factoring by Box Method
14. Factoring by Autrey’s and Austin’s Method
15. Factoring by Slip-Slide Method

2.4.1 Factoring quadratic expressions by “Guess and Check” Method or Trial Error Method.

How “Guess and Check” is followed in factoring $6x^2 - 19x + 10$.

1. Since a leading coefficient is 6, therefore, there are two possibilities:

$$(1x - p)(6x - q) \text{ or } (2x - p)(3x - q)$$

2. The constant is 10, so the product of p and q is 10, which gives us two possibilities: or 1×10 or 2×5

The list of all possibilities and outcomes

$$(1x - 1)(6x - 10) = 6x^2 - 16x + 10 \quad (2x - 1)(3x - 10) = 6x^2 - 23x + 10$$

$$(1x - 10)(6x - 1) = 6x^2 - 61x + 10 \quad (2x - 10)(3x - 1) = 6x^2 - 32x + 10$$

$$(1x - 2)(6x - 5) = 6x^2 - 17x + 10 \quad (2x - 2)(3x - 5) = 6x^2 - 16x + 10$$

$$(1x - 5)(6x - 2) = 6x^2 - 32x + 10 \quad (2x - 5)(3x - 2) = 6x^2 - 19x + 10$$

Only the last combination produced the giving trinomial.

$$\text{Therefore, } 6x^2 - 19x + 10 = (2x - 5)(3x - 2)$$

Figure 2.1 Factoring quadratic expressions by “Guess and Check” Method as adapted in Mart 2019.

2.4.2 Factoring quadratic expressions by Cross Method.

One of the methods used in quadratic factoring is the Cross Method (Patrick, 2012). Other sectors of the land follow the Vedic Method, the method that employs the similar technique to that of Cross Method as outlined by Nataraj and Thomas (2006). In getting to better understand the Cross Method, Saputro et al., (2018b, p. 3) describe the flow of events as one approach of using the cross method in factorising the quadratic expression, using $2x^2 + 7x + 3$ as an example.

Step 1: Determine the possible factors of the terms in x^2 and also those of the constant term.

$$\begin{array}{ccc} 2x^2 & + & 7x & + & 3 \\ \uparrow & & \uparrow & & \\ x \times 2x & & (+1) \times (+3) & & \end{array}$$

Step 2: Write the factors vertically as shown.

$$\begin{array}{cc|c} x & +1 & \\ 2x & +3 & \\ \hline \end{array}$$

Step 3: Cross-multiply the factors and write the product in the last column.

$$\begin{array}{r|l}
 \begin{array}{r} x \\ 2x \end{array} & \begin{array}{r} +1 \\ +3 \end{array} \\
 \hline
 & \begin{array}{r} 2x \\ 3x \end{array}
 \end{array}$$

Step 4: Add the last column. Reject the work if the sum is not equal to the term x in the given expression.

$$\begin{array}{r|l}
 \begin{array}{r} x \\ 2x \end{array} & \begin{array}{r} +1 \\ +3 \end{array} \\
 \hline
 & \begin{array}{r} 2x \\ 3x \end{array}
 \end{array}
 \left. \vphantom{\begin{array}{r|l} \begin{array}{r} x \\ 2x \end{array} \\ \begin{array}{r} +1 \\ +3 \end{array} \end{array}} \right\} \text{Add}$$

$$\begin{array}{r|l}
 & \begin{array}{r} 5x \neq 7x \\ \therefore \text{rejected} \end{array}
 \end{array}$$

Step 5: Exchange place for 1 and 3. Cross-multiply and add the last column again.

$$\begin{array}{r|l}
 \begin{array}{r} x \\ 2x \end{array} & \begin{array}{r} +3 \\ +1 \end{array} \\
 \hline
 & \begin{array}{r} 6x \\ x \end{array}
 \end{array}$$

$$\begin{array}{r|l}
 & 7x \text{ Accepted}
 \end{array}$$

Step 6: As step 5 is accepted, the factors of the quadratic expression are those “circled”

$$\begin{array}{r|l}
 \begin{array}{r} (x) \\ (2x) \end{array} & \begin{array}{r} (+3) \\ (+1) \end{array} \\
 \hline
 & \begin{array}{r} 6x \\ x \end{array}
 \end{array}$$

$$\begin{array}{r|l}
 & 7x
 \end{array}$$

Figure 2.2 Factoring quadratic expressions by Cross Method :

$$2x^2 + 7x + 3 = (2x + 3)(x + 1).$$

Quadratic factoring ties the arithmetical processes (involve factoring of coefficients) and the formal algebraic manipulations (involve understanding of $ax^2 + bx + c$ together. Based on this description, Patrick (2012) is adamant that the cross method is destined to fail the system and has tabled the reasons why the cross method is unable to address the challenges of factoring the quadratic trinomial. Furthermore, Patrick (2012) is quoted as saying:

Cross method mainly focus on seeking the correct combination of the factors of the x^2 term and constant term, it means that cross method involves operational thinking intensively. But cross method itself is not effective to facilitate a transition from an operational to a structural mode of students' thinking, which limits the reification of

quadratic factoring by students. Fail in reification made students fail to develop conceptual understanding of quadratic factoring” (Patrick, 2012, p. 51).

2.4.3 Factoring quadratic expressions by Grouping Method.

According to Patrick (2012), the grouping method is also known as the “A-C Method”, “X-Box Method” even “Diamond Method” in the teaching materials of the United States. According to Bruns (2009), this grouping has the additional benefit of promoting a better understanding of the process of factorisation in general, even the monic case. Furthermore, this algorithm works for polynomials over any unique factorisation domain. The work by Bernard, Ramirez and Villalobos (2017) on Using Technology to Determine Factorability or Non-factorability of Quadratic Algebraic Trinomials details how technology follows the a-c method in producing the factors of the quadratic expression, which is an indication that the grouping method is precise. Figure 2.3 as adapted from Patrick (2012, p. 52) shows how the grouping method can be used step-by-step to factorise, for an example, the quadratic $2x^2 + 5x + 3$

Step	Procedure	Example
1	Find the product ac	$(2)(3) = 6$
2	Express ac as multiple of two factors	$6 = 1 \times 6;$ $6 = 2 \times 3$
3	Sum each pair of two factors	$1 + 6 = 7;$ $2 + 3 = 5$
4	Which pair of factors can give a sum equal to b ? Rewrite b as the sum of these factors	$2x^2 + 5x + 3$ $= 2x^2 + 2x + 3x + 3$
5	Group the expression into two parts and find the common factor of each group if any	$2x^2 + 2x + 3x + 3$ $2x(x + 1) + 3(x + 1)$
6	Take out the common factor of two parts again and write the answers	$(2x)(x + 1) + (3)(x + 1)$ $= (2x + 3)(x + 1)$

Figure 2.3: Steps in factoring quadratic expression $2x^2 + 5x + 3$ by Grouping Method.

2.4.4 Factoring quadratic expressions by Tool-based Geometric rectangular algebra tiles.

Several researchers, (Irianti and Qohar, 2019; Leong, Yap et al., 2010; Malim, 2018; Miranda and Adler, 2010) have adopted the use of algebra tiles to address the conceptual challenges associated with the factorisation of quadratic expressions. Leitz and Kitt (2000) describe the

algebra tiles, which students help make, as consisting of small squares, large squares, and rectangles. The number 1, the unit tile, is represented by the small square; the large square represents x^2 ; and the rectangle represents x . The sizing of these homemade tiles is similar to that of commercially made tiles. The side of the x^2 tile is equal to the length of the x tile. The length of the x tile is not an integral multiple of the side length of the unit tile. The width of the x tile is equal to the side length of the unit tile. Commercially made plastic algebra tiles are constructed with one side of one colour and the other side of another colour.

Martinez et al., (2017) describe the algebra tile as a geometric tool based on the sum of areas for operations of multiplication, division, and some polynomials factorisation. Furthermore, for some clarity, Martinez et al., (2017, p. 7) provides a five-step procedure for developing the generic rectangle, and is presented in figure 2.4 that follows.

- Step 1: draw a rectangle and divide it into four parts.
- Step 2: Place the quadratic term ax^2 in the lower left corner and the independent term c in the upper right corner.
- Step 3: Find two numbers m and n such that $mn = ac$ and $m \pm n = b$.
- Step 4: mx and nx terms are placed in the remaining corners, so that the algebraic sum of them match the linear term.
- Step 5: Find the common factor for both columns and rows. Thus, the sides of the rectangle are obtained (the values of the constants correspond to those of m and n) and thus the result of the factorisation is the area of the rectangle.

Figure 2.4. Steps in factoring quadratic expressions by Tool-based Geometric rectangular algebra tiles as adapted in Avalos et al. (2017).

Patrick (2012, p. 55) supports the five-step procedure provided by Martinez et al., (2017) by providing the illustrations through the use of the diagrams as they are shown in figure 2.5 and figure 2.6.

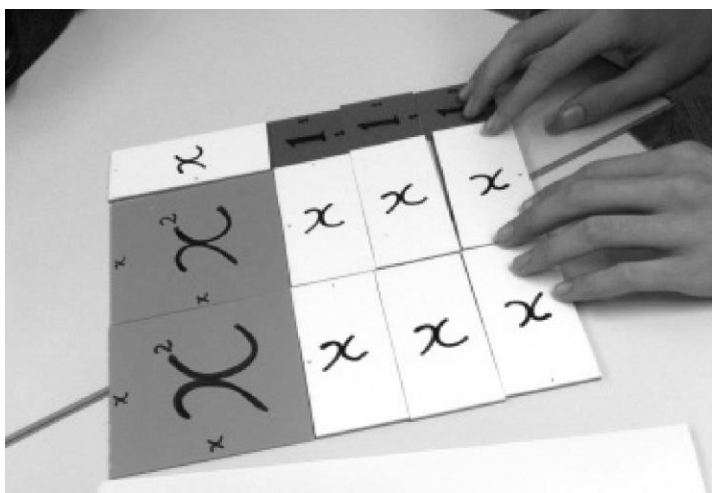


Figure 2.5: Factoring $2x^2 + 7x + 3$ using algebra tiles.

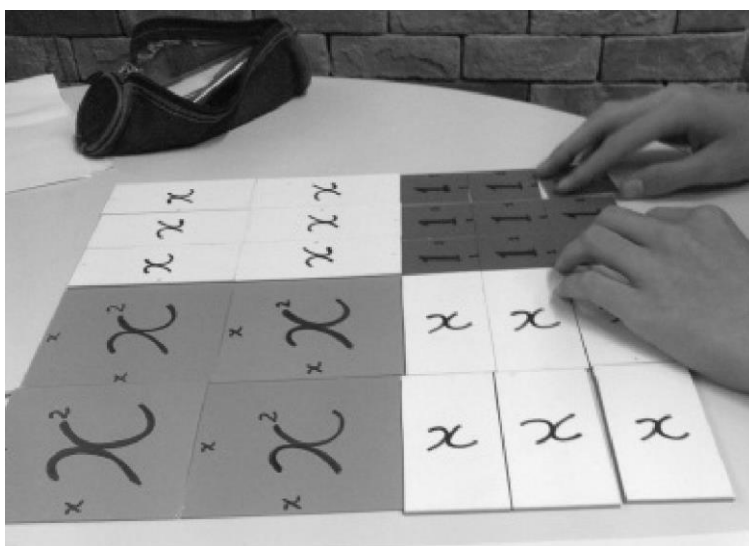


Figure 2.6: Factoring $4x^2 + 12x + 9$ using algebra tiles.

There are a number of studies (Leong, Yap et al., 2010; Martinez et al., 2017; Strickland, 2014) that sought benefits in the use of the Generic Rectangle tiles in the factoring of quadratic expressions. Chua (2017) contends that, for example, the expansion of two linear binomials can be introduced using concrete algebra tiles. The use of algebra tiles helps students to appreciate the geometrical meaning of the algebraic expressions involved. A study conducted on the cognitive process of understanding quadratic equations for 16 students in Bengkulu City, Indonesia, shows that the cognitive process of students who learn to use learning media based on square and rectangular, enables them to reach a high level, the trans-level (Herawaty et al., 2021).

According to Latiff et al., (2017) learners have to be exposed to a variety of methods of factorisation of quadratics for them to improve their understanding. Latiff et al., (2017) conducted the study, in Brunei, where they used the concrete model to enhance conceptual knowledge of low ability Year 10 students in factorising quadratic expressions. The results of the study revealed that: Firstly, learning factorisation with the aid of the concrete model does have a positive effect on students' performances. After the interventions, most of them could factorise the quadratic expressions correctly using methods they were comfortable with. Secondly, they found that students learning mathematics with the aid of the concrete model found it more enjoyable and it made mathematics concepts easier to grasp. Students also gained more confidence in factorising quadratic expressions.

Much work has been achieved in the factorisation of quadratic expressions where the factoring methods are deduced through the concrete material (Dorado & Torres, 2019). Norton and Irvin (2007) conducted a study on the use of the concrete approach to teaching algebra in a Year 9 class. The study was advocated by embarking on a case study that used design based research, in so much as cycles of design, enactment, analysis and re-enactment, analysis, and further design taking place. The study results indicated that much of the struggle experienced by students was associated with a lack of understanding of arithmetic concepts, including those associated with equivalence, operations with negative integers, and the distributive law, and fraction concepts. These barriers were later addressed by means of supporting students through the introduction and the use of appropriate language. Norton and Irvin (2007, p. 558) are quoted, "The results support the notion that the essence of learning algebra, like that of arithmetic, is to make connections between materials, patterns and symbolic meaning through the medium of language". In contrast, the action research study conducted by Latiff and colleagues in (Latiff et al., 2017) for Year 10 low ability classes in Brunei, revealed that the use of concrete models in the teaching and learning of factorisation enhanced conceptual knowledge of low ability students in factorising quadratic expressions, in particular (Latiff et al., 2017). The study reported two important findings. Firstly, based on the analysis of students' achievement tests, learning factorisation with the aid of the concrete model has a positive effect on students' performances in the post-test where 81.8% showed improvements from their pre-test. After the interventions, most of them could factorise the quadratic expressions correctly using methods they were comfortable with. Secondly, it was found that learning mathematics with the aid of the concrete model was more enjoyable and made mathematics concepts easier to grasp. They also gained more confidence in factorising quadratic expressions. However,

there were a few concerns regarding the use of algebra tiles. When higher coefficients are involved, it may cause difficulty and confusion to rearrange/draw the algebra tiles into a rectangular shape because there are so many tiles involved. The students were afraid this manipulation of algebra tiles would not be acceptable during exams.

Martinez et al., (2017, p. 96) speaking from the tertiary level, agreed and is quoted as saying “...the results point out that if we want to have students with high academic performances we have to focus on encouraging them to unleash their potential and let them try/play with new methodologies. So, it can be said that the generic rectangle promotes meaningful learning in factoring quadratics and has a strong positive attitude toward math”. However, there were a few concerns regarding the use of algebra tiles. For example, Leong, Yap et al., (2010) is more concerned about the time it takes to complete the task, and is quoted, “we were glad that even though the lesson study process was indeed time-consuming, there was observable positive effect on some students, which made the effort worthwhile” (Leong, Yap et al., 2010, p. 24). Again, as has been highlighted by Latiff et al., (2017), it creates difficulty and confusion to rearrange/draw the algebra tiles into a rectangular shape, if higher coefficients are involved in the factoring of quadratic expressions.

2.4.5 Factoring quadratic expressions by Autrey’s and Austin’s Method (A-A Method).

This method is developed on the knowledge that the transformation factorisation of a quadratic expression will result in the product of two linear expressions. According to Saputro et al., (2018a) the strategy embraced in the A-A Method of factoring of quadratics is that the problem-solver has to find the factors m and n of ac such that $mn=ac$ and $m+n=b$. The requirement is to simplify the expression $\frac{(ax+m)(ax+n)}{a}$ by factoring the common factors of a and m and of a and n .

Step 1	Multiply a and c (4 x -9) = -36 and find the two numbers whose product is -36 and whose sum is -16.	<table><tr><td colspan="3">Product</td><td colspan="3">Sum</td></tr><tr><td>-18</td><td>+2</td><td>-36</td><td>-18</td><td>+2</td><td>-16</td></tr></table>	Product			Sum			-18	+2	-36	-18	+2	-16
Product			Sum											
-18	+2	-36	-18	+2	-16									
Step 2	open the brackets and write in <i>ax</i> and divide by <i>a</i>	$\rightarrow \frac{(4x \quad)(4x \quad)}{4}$												
Step 3	put in the factors -18 and +2 without any order	$\rightarrow \frac{(4x - 18)(4x + 2)}{4}$												
Step 4	work out to eliminate the divisor by taking out the common factor above and then divide	$\rightarrow \frac{2(2x - 9)2(2x + 1)}{4}$ $\rightarrow \frac{4(2x - 9)(2x + 1)}{4}$ $\rightarrow (2x - 9)(2x + 1)$												

Figure 2.7: Steps in factoring quadratic expression $4x^2 - 16x - 9$ using the A-A Method.

2.4.6 Factoring quadratic expressions by Slip-Slide Method.

In factoring $6x^2 - 19x + 10$	
Step 1: Let's multiply the leading coefficient by the constant: $x^2 - 19x + 60$	
Step 2: We are looking for two numbers with a sum of -19 and a product of 60.	
Step 3: Both numbers must be negative: -15 and -4. So, $(x - 15)(x - 4)$.	
Step 4: "Slide" back the leading coefficient: $(x - \frac{15}{6})(x - \frac{4}{6}) = (x - \frac{5}{2})(x - \frac{2}{3})$	
Step 5: "Slide" back the denominators in front of each x: $(2x - 5)(3x - 2)$	
Therefore, $6x^2 - 19x + 10 = (2x - 5)(3x - 2)$	

Figure 2.8 Steps in factoring quadratic expression $6x^2 - 19x + 10$ by Slip-Slide Method as adapted in Mart 2019.

2.5 Research about mathematics teachers' knowledge and practice.

Curriculum reforms in South Africa through the introduction of CAPS, has necessitated a variety of changes in relation to teaching and learning mathematics. The DBE (2011a, p. 8), prescribes eight specific aims under which the teaching and learning of mathematics should be advocated. Three of which are:

- To provide the opportunity to develop in learners the ability to be methodical, to generalise, make conjectures and try to justify or prove them.

- To develop problem-solving and cognitive skills. Teaching should not be limited to “how” but should rather feature the “when” and “why” of problem types. Learning procedures and proofs without a good understanding of why they are important will leave learners ill-equipped to use their knowledge in later life.
- To prepare the learners for further education and training as well as the world of work.

In order to achieve these aims of mathematics, a great deal of change in teachers’ practices is inevitable. These changes include how teachers’ practices influence learners’ contributions and interactions in mathematics classrooms (Molefe et al., 2010). For the purpose of this study, I will take the position as taken by Molefe et al., (2010) in that there are two kinds of practices that are critical to the mathematics classroom: mathematics practices, and teaching practices. Mathematical practices are the “basic contexts in which individuals gain their experience and in which mathematical objects emerge. Consequently, the object here acquires a status derived from the practice that precedes it” (Font, Godino & Gallardo, 2013, p. 104). According to Font et al., (2013), “one way of conceptualising mathematical practices is to regard them as the combination of an operative practice through which mathematical texts are read and produced, and a discursive practice, which enables reflection upon the operative practice. Both these practices would be able to be recognised as mathematical by an expert observer. This way of understanding mathematical practice requires consideration of the personal and institutional facets, among which complex dialectical relationships are established and whose study is essential for mathematics education”. On the other hand, teaching practices are termed by Molefe et al., (2010) as involving particular approaches or methods that teachers employ in their classrooms in order to teach mathematics or develop mathematical practices. Basically, in mathematics classrooms, the two sets of practices are related.

2.5.1 Mathematics Practices.

From the South African perspective and elsewhere, most studies demonstrated that pre-service teachers possess inadequate conceptual understanding of the mathematics content they are expected to teach upon qualification (Ndlovu, Amin & Samuel, 2017). The findings of the study by Okoye-Ogbalu (2019) on exploring pre-service mathematics teachers’ knowledge of logarithms in one of the universities in KwaZulu-Natal, revealed that most pre-service mathematics teachers do not have a good knowledge of logarithm. The findings also reveal that

pre-service mathematics teachers do not have good knowledge of logarithm due to how the concept of logarithm was introduced to them in secondary school. Similar findings were also noted in a recent study by Ubah and Bansilal (2018) on pre-service mathematics teachers' knowledge of mathematics for teaching quadratic functions. According to the study, some students identified different forms that the equation of a quadratic function could be expressed as but were unable to apply this to derive the equation. These results indicate that these students are not yet ready to teach these school level concepts even though they have studied advanced mathematics topics as part of their pre-service training. Similar results seem to be true for practising teachers. The study by Bansilal, Mkhwanazi and Brijlall (2014) on the exploration of the common content knowledge of high school math teachers revealed that there are instances where teachers don't know well enough, the mathematics their learners need to learn. Similar findings were also noted in the study in Malaysia by Leong, Meng and Rahim (2015) on understanding Malaysian pre-service teachers' mathematical content knowledge, and pedagogical content knowledge. The main objective of this study was to investigate the level of teaching knowledge attained by the Malaysian pre-service primary and secondary teachers. Results of analysis showed that Malaysian pre-service teachers at the primary and secondary performed below the international average for both mathematics content knowledge (MCK) and mathematics pedagogical content knowledge (MPCK).

2.5.2 Instructional practices.

Classroom practices reflect either student-centered or teacher-centered pedagogies, and teachers are always identified using those instructional practices. The teacher-centered pedagogy is always referred to as the traditional teaching approach, whereas the student-centered pedagogy is the "reform approach" (Polly et al., 2013).

2.5.2.1 Traditional teaching approaches to teaching algebra.

According to Vaiyavutjamai (2004) in traditional lessons, the teacher talks more than the students and classroom discourse patterns feature many sequences of chorused answers, with teachers channelling students' thinking towards answers that they want the students to give. This approach of teaching tends to place emphasis on teachers imparting knowledge to students rather than on the students exploring and acquiring knowledge for themselves. While such a teacher-centred approach is useful to convey knowledge, it does not, however, seem to promote much critical thinking in students (Chua, 2017).

Vaiyavutjamai and Clements (2006) conducted a study in Thailand for Grade 9 students on the effects of classroom instruction on students' understanding of quadratic equations. Students were engaged on the pre-test, teaching, post-test, and then interviews. The analysis of interview data revealed that many interviewees who obtained correct solutions actually had serious misconceptions about what quadratic equations actually are. Their answers were correct but, from a mathematical point of view, they did not know what they were talking about.

A study conducted by Ilyas et al., (2013, p. 152) reveals that, "it is not only the developing countries lack in human and material resources that are required for effective teaching of algebra, but multiple studies reveal that students of developed countries also fall behind proper understanding of algebraic concepts because the teachers mostly deal with algebraic variables mechanically, without explaining real meaning in social context" To that extent, the Government of Pakistan had recognised that the quality of its teachers is abysmally low. They argue that there is little difference between professionally-trained and untrained teachers, citing that the teaching in the country revolves around rote based knowledge and learning. Furthermore, "their existing teaching allows little room for social constructivism, as public school teachers often use scolding and corporal punishment within classroom teaching" (Ilyas et al., 2013, p. 152).

Similar tensions are also felt at the higher levels of education as Dubinsky (1997) alerted. Speaking of his experiences about algebra at college level, Dubinsky (1997, pp. 10-11) pointed out three sources of students' difficulties in learning algebra which I perceive as relevant to the context of teaching algebra at secondary school. Firstly, the overall pedagogical approach in linear algebra is that of telling students about mathematics and showing how it works. The weakness in this approach is that the algorithms and procedures work even if their meanings are not understood. Secondly, students lack the understanding of the background concepts that are not part of linear algebra, but it is important to learn it. Thirdly, there is a lack of pedagogical strategies that give students a chance to construct their own ideas about concepts in linear algebra.

2.5.2.2 Reform teaching approaches to teaching algebra.

A call to shift from the traditional teaching approach to the reform teaching approach has been made, both locally and globally. The sense of urgency for reform has been heightened by competition from alternative routes to teaching that jump straight to practice, often criticised for foregoing essential knowledge and theory (Waddell & Vartuli, 2015). According to Chua (2017), a teacher-centred approach has to change if teaching is to move towards empowering students to be active learners engaged in discovering and generating new knowledge and in taking charge of their own learning.

Part of the reform in teaching advocates the engagement of teachers to reflections. Chapman (2015) conducted a study on reflective awareness in mathematics teachers' learning and teaching. The findings indicated that engaging in self-based and meaning-based questioning and creating pedagogical models were central to teachers' learning and use of reflective awareness. Their knowledge of reflective awareness as a way of knowing was important to their development of an inquiry stance and knowledge of mathematics for teaching and mathematics pedagogy.

A study was conducted on the association between teachers' beliefs, enacted practices, and student learning in mathematics by Polly et al., (2013). Data was generated from 35 teachers and 494 elementary school students. Findings of the study reported that significant associations were found between teachers' dispositions to mathematics teaching and learning, and their enacted instructional practices. Furthermore, "students whose teachers had reported teacher-centered beliefs and teacher-centered practices, had significantly lower gain scores on curriculum-based assessments" (Polly et al., 2013, p. 23).

2.5.2.3 Shifting from traditional teaching approaches to reform teaching approaches.

Balancing theory and practice seems to be challenging when teachers embed mathematical practices. According to Schoenfeld (2020, p. 1163), "challenges the field faces in conceptualising mathematics instruction include: What can be warranted as legitimate mathematical practices? Which aspects of mathematical practices are relevant and appropriate for K-16 instruction? What kinds of support are necessary? What is viable at scale?" The similar findings were also noted in the study on investigating teachers' knowledge for teaching mathematics by Chapman (2013). On the extent at which teachers are shifting from traditional

knowledge-based tendency to thinking of teachers' knowledge as knowledge generated by others, teachers' traditional-oriented instructional way of thinking and teaching was still more dominant. For example, although they used a problem-solving heuristic model that could allow students to think for themselves, they engaged students in a traditional manner by helping them to arrive at answers in a specified way that limited their understanding of the answers and the process used to get the answers. According to Chapman (2013, p. 242), "the most profound challenges that emerged for teachers were the following: to make personal sense of constructivism; to reorientate the culture of the classroom to accommodate constructivist philosophy; and to deal with conservatism that works against teaching for understanding".

There are however researchers who opt to take centre stage. Recent calls to find a balance between allegedly opposite instructional approaches—commonly dichotomised as "traditional" and "reform" approaches—have at times posited these extreme approaches as complementary, suggesting that the two may productively interact (Metz et al., 2016). It is argued that "it is not enough to combine instructional approaches; rather, it is necessary to transcend the borders of restrictive dichotomies" ((Metz et al., 2016, p. 1257). This notion is also supported by Kotsopoulos (2007) as she argues the need to make intentional and informed decisions about the choices teachers make in teaching. This may imply that in some instances procedural knowledge may be emphasised more heavily (i.e., basic fact retrieval, multiplication, etc.) than conceptual knowledge, but not to the exclusion of conceptual knowledge. For example, "in the case of multiplication, it is useful for students to have an understanding of repeated addition as the conceptual underpinnings of basic multiplication — particularly for when long-term semantic memory fails. However, as teachers, we understand how timely this can be when students resort to repeated addition in order to multiply. Therefore, in the case of number facts, procedural knowledge is important" (Kotsopoulos, 2007, p. 21).

2.5.3 Teaching practices.

Studies show that teaching practices are a determining factor in addressing classroom expectations. The study conducted by Molefe et al., (2010) on the examination of teaching practices of two Grade 10 mathematics teachers as they began implementing the new FET curriculum, reported that even though teachers of the study were more "reform-oriented", some of the practices they used needed development. Benjamin (2020) conducted the study on the exploration of pre-service teachers' mathematics knowledge for teaching in trigonometry at a

higher education institution. The study was composed of fifteen mathematics final-year pre-service teachers who were registered for a Bachelor of Education (B Ed) degree programme at a rural-based institution of higher learning in South Africa. The study found the following: Firstly, teachers' mastery of pedagogical content knowledge in trigonometry was limited. Moreover, the extent of pre-service teachers' development of mathematical knowledge for teaching based on results from classroom practices was sub-standard. Secondly, the traditional teaching methods and learner-misconceptions never left pre-service teachers all through the four years of teacher education. Basically, there is some work needed to ensure the total shift in teachers of mathematics from being "traditional oriented" to being "reform-oriented"

Studies also show that the teaching practices are largely informed by teachers' beliefs. The study conducted by Siswono, Kohar and Hartono (2017) pointed out that teachers' beliefs have a strong relationship with teachers' knowledge about problem-solving. The similar findings were also corroborated in the study by Polly et al., (2013) on the association between teachers' beliefs, enacted practices, and student learning in mathematics. According to Polly et al., (2013), there are significant associations between teachers' dispositions to mathematics teaching and learning and their enacted instructional practices. The study by Mapolelo and Akinsola (2015) on the preparation of mathematics teachers, lessons from review of literature on teachers' knowledge, beliefs, and teacher education, also alluded to this view. According to Mapolelo and Akinsola (2015), a teacher's memories from their school years are a central influencing factor that affects their mathematics related beliefs, hence there is a need to enhance pre-service teachers' positive attitude towards mathematics during training.

One of the specific aims of mathematics as elaborated upon earlier in this section stresses the need for teachers to develop in learners problem-solving and cognitive skills. This call suggests that for teachers to be able to embed the mandate of teaching successfully, they have to ensure a balance between the classroom practices and their content knowledge base. For example, the study by Lee, Capraro and Capraro (2018) on mathematics teachers' subject matter knowledge and pedagogical content knowledge in problem posing, showed that there is a disconnect between teachers' teacher knowledge and their teaching practices. Shulman (1986) provides a clear distinction as he provides a description between Subject Matter Knowledge (SMK) and the Pedagogical Content Knowledge (PCK). According to Shulman (1986, p. 9), "subject matter knowledge (SMK) is knowledge of the content of the discipline per se, consisting both

of substantive knowledge (the key facts, concepts, principles and explanatory frameworks in a discipline) and syntactic knowledge (the nature of enquiry in the field, and how new knowledge is introduced and accepted in that community). Pedagogical content knowledge (PCK) is particularly difficult to define and characterise, conceptualising both the link and the distinction between knowing something for oneself and being able to enable others to know it". These descriptions of SMK & PCK are bringing forth a clear indication that the aspects of mathematical knowledge and that of pedagogical knowledge are intertwined.

Furthering the argument, the study by Pournara (2014) on the insights from the case of annuities, identified the "contextual knowledge" as another important aspect, in the mathematics of finance in this case, for the teachers to consider in their practices. The South African dispensation, the DBE (2011a, pp. 11 & 16) provides the following curricular requirements: teaching guidelines; sequencing of topics per term; the pacing of topics over the year; and the weighting of content areas. These curricular knowledge requirements serve as a vehicle towards teachers' success in addressing issues of content as well as those of pedagogy.

The seminal work of (Rowland, Huckstep & Thwaites, 2003; Rowland, Huckstep & Thwaites 2005; Rowland & Turner, 2007; Rowland, Jared & Thwaites, 2011) identified the "Knowledge Quartets", with four broad dimensions, or 'units', as describing ways in which the teacher's content knowledge for teaching is enacted in the classroom. The knowledge quartet is comprehensive as a tool for thinking about the ways that subject knowledge comes into play in the classroom. These four units are named as follows: foundation; transformation; connection; and contingency. The implication is that, central to the lesson planning, teachers contemplate and conceptualise how the teaching of the factorisation of quadratic expression concepts in their Grade 10 classroom will be advocated. However, despite such rigorous planning, Rowland notes the existence of the knowledge Quartet on our teachers as posing a detrimental impact in unlocking the potential in our learners.

A study focusing on mathematics lessons prepared and conducted by pre-service elementary teachers towards the end of their initial training at one university was conducted by Rowland, et al., (2003). The aim of the study was to locate ways in which they drew on their knowledge of mathematics and mathematics pedagogy in their teaching. The focus was on ways that

elementary trainees' mathematics content knowledge, both SMK and PCK, could be seen to contribute to their teaching during the 'practical' element of their training – the school-based placements. One objective was to develop an empirically based conceptual framework for productive discussion of mathematics content knowledge between teacher educators, trainees and teacher-mentors, in the context of school-based placements. The findings of the study revealed that teachers' practices were deeply immersed and defined in "Knowledge Quartets". According to Rowland et al., (2003), a knowledge quartet is defined as the comprehensive tool for thinking about the ways that subject knowledge comes into play in the classroom.

A number of studies, (Rowland et al., 2007; Rowland, 2005) found the "knowledge quartet" as being defined in four broad, superordinate categories, or 'units', which they named: Firstly, the foundation: adheres to textbook; awareness of purpose; concentration on procedures; identifying errors; overt subject knowledge; theoretical underpinning; use of terminology. Secondly, the transformation: choice of examples; choice of representation; demonstration. Thirdly, the connection: anticipation of complexity; decisions about sequencing; making connections; recognition of conceptual appropriateness. Lastly, the contingency: deviation from agenda; responding to children's ideas; use of opportunities. According to Rowland, naturally, as teachers of mathematics, we are immersed in the process from which the aforementioned units emerged. For example, Davis and Johnson (2008) noted the pedagogic practices across the five schools they were studying, being dominated by a proceduralisation of school mathematics that uses the iconic features of mathematical quadratic expressions. In other words, teachers are training students to observe the special features of the expression, which they called "character distribution matrix", with such features foretelling the outcomes of factorisation.

A character distribution matrix makes the notational aspects of the mathematical expressions (trinomial aspect) as well as symbol distribution rules primary while the mathematical ideas and principle-based reasoning (with respect to the quadratic aspect) remain secondary or implicit. In other words, the character distribution matrix as resource is a mechanism that enables the teacher to direct the student to perform procedures that are, essentially, rehearsals of basic arithmetic: finding factors and sums of whole numbers (addition and multiplication of whole numbers) (Johnson and Davis, 2010, p. 141)

The study conducted by Johnson and Davis (2010) in the Eastern Cape Province in South Africa, for Grade 10 on the notion of a character distribution matrix as a pedagogic resource, particularly in the factorisation of quadratics revealed the following: Firstly, the mathematical content, techniques and procedures become context dependent and so relies on standard question and solution “types”. Secondly, mathematical propositional and deductive reasoning never emerges as the primary resource for regulating mathematical activity. According to Johnson and Davis (2010), this kind of approach to factorisation of quadratic expressions is mathematically incorrect, since it embraces the non-mathematical journey to reach the desired destination.

2.6 Conclusion.

This chapter has been able to present the research findings of the studies conducted on algebra. The theoretical framework adopted in this thesis is presented in Chapter 3 of this thesis.

CHAPTER 3

THEORETICAL FRAMEWORK

3.1 Introduction.

This study explores learners' mental constructions in the learning of factorisation of quadratic expressions in the Grade 10 mathematics classroom. This chapter begins with a discussion of the concept of constructivism and its components. Further discussions will be made in an attempt to understand the ideas of Piaget concerning reflective abstraction. A description of the heart of the theoretical framework used in this study, the APOS theory, will be presented. A theoretical framework within which this study is situated is discussed. I then present a genetic decomposition of the concept of factorisation of quadratic expressions, followed by a discussion of the implications of a teaching approach guided by APOS theory.

3.2 The constructivist paradigm.

Due to the nature of the study and the research questions, this study was guided by the theory of constructivism. Constructivism has been described by two hypotheses: Firstly, knowledge is actively constructed by the cognising subject, not passively received from the environment ((Caffarella & Merriam, 1999). Secondly, coming to know is an adaptive process that organises one's experiential world; it does not discover an independent, pre-existing world outside the mind of the knower (Panasuk & Lewis, 2012). The implication derived from this description is the recognition that the role of the teacher is not to "pour" knowledge in the minds of learners, but for learners to actively create and own their learning. Bodner (1986, p. 4) agreed to this notion and remarked that the "constructivist model can be summarised in the statement: Knowledge is constructed in the mind of the learner". He went on to say: "In the constructivist model, knowledge is assumed to fit reality the way a key fits a lock".

The constructivist theory is rooted in the work of Jean Piaget cited in Bodner (1986) who believed that knowledge is acquired as the result of a life-long constructive process in which we try to organise, structure, and restructure our experiences in light of existing schemes of thought, and thereby gradually modify and expand these schemes. Piaget further unpacked the theory on constructivism in his theory of accommodation and assimilation when he proposed that for intellectual development, knowledge is constructed, as the learner strives to organise his or her experiences in terms of pre-existing mental structures or schemes. This argument is furthered by Barman and Bhattacharyya (2015, p. 69) who are quoted as saying, "constructivist

teaching method is such a method which draws on students' existing knowledge, beliefs, and skills. With a constructivist approach, students synthesise new understanding from prior learning and new information." I strongly agree with this view as it recognises the prior knowledge as the building block towards understanding the new knowledge.

There are benefits associated with the use of the theory of constructivism in the mathematics classroom. Pugalee (2001) commented that constructivist teaching requires the use of non-traditional approaches to mathematics as it advocates the effective instructional practices, which promote a student's active construction of knowledge. As also perceived by Cobb, Wood and Yackel (1991), the constructivist view places the teacher in a position of implementing instructional methodologies that enable students to interact in the learning environment, and to construct meanings about mathematics. In such learning situations, individuals construct mathematical knowledge as they interact with their peers and teachers. Again, in the constructivist teaching process, the teacher assumes the role of facilitator by being responsible for the creation of climate in which exploration, problematising, and discourse guide the students in actively refining and constructing knowledge of mathematics (Pugalee, 2001). In addition, it is argued that the "teacher's initiation and guidance" offers students the opportunities to "mathematise their experiences" as they contribute to their learning (Cobb, Yackel & Wood, 1992).

3.3 Algebraic reasoning.

Factorisation of quadratics as a phenomenon progresses and develops in a hierarchical manner. Saminanto (2014) contends that hierarchy material characteristics are appropriate to use in a constructivist learning model; therefore, learners need to pay attention in the learning process by always linking and using what they already have, to solve the problems encountered. Makonye and Khanyile (2015) share similar sentiments and are quoted as saying, "understanding a mathematical concept refers to the ability to integrate the concept and connect it to other mathematics concepts like it, and also to differentiate and separate it from mathematical concepts which are unlike it". Mathematical concepts, structures, and ideas have been invented as tools to organise the phenomena of the natural, social, and mental world. However, in context-based situations, problems do not arise in ways that allow the logical application of content strands as they are found in the mathematics curriculum (OECD/PISA, 2003, p. 34). The implication is, for the problem-solver in mathematics to be successful, there

is a need for knowledge acquisition in terms of mathematical content within all walks of life; hence the understanding of a mathematical concept is critical.

Speaking on the Korean mathematics curriculum, Kieran (2004) defines algebra “as a subject dealing with expressions with symbols and the extended numbers beyond the whole numbers in order to solve equations, to analyse functional relations, and to determine the structure of the representational system, which consists of expressions and relations” (Kieran, 2004, p. 146)”. Part of this definition of algebra describes the activities that unfold as the student embarks on the journey of factoring the quadratic expression. In order for a student to engage with algebra meaningfully, a great deal of algebraic thinking is advocated. According to Kieran (2004, p. 149), “algebraic thinking in the early grades involves the development of ways of thinking within activities for which letter-symbolic algebra can be used as a tool, but which are not exclusive to algebra and which could be engaged in without using any letter-symbolic algebra at all, such as, analysing relationships between quantities, noticing structure, studying change, generalising, problem-solving, modelling, justifying, proving, and predicting”.

Windsor (2010, p. 671) contends that one of the distinctive characteristics of the teaching approach advocating the development of algebraic thinking, is that it requires teachers to adapt and change the problems, yet maintain mathematical generalisations present within the problems. The definition of algebra offered by Caspi and Sfard (2012) puts the emphasis of drawing on generalisations when dealing with algebra. According to Caspi and Sfard (2012), algebra is defined as a discourse, that is, a form of communication, and the algebraic thinking begins when one starts scrutinising numerical relations and processes in the search for generalisation or in an attempt to solve equations. Sfard and Linchevski (1994, p. 191) share similar sentiments, and go on to say, “the nature and growth of algebraic thinking is first analysed from an epistemological perspective supported by historical observations, and eventually, its development is presented as sequence of ever more advanced transitions from operational to structural outlook.” According to Schmittau (2005), these three essential characteristics of the Vygotskian approach to the development of algebraic thinking: initial development from the general structural foundation, ascent from the abstract to the concrete, and appropriation of psychological tools, are all interrelated and function synergistically.

In working with algebraic concepts, Kaput (1993) identified a critical dimension, the “algebraic reasoning”, which he viewed as a process of construction and representation of patterns and regularities, deliberate generalisation, and active exploration and conjecture. According to Kaput (1993), algebraic reasoning is embedded in the five identified major strands of algebraic thinking which alternately focus on mathematical process (generalisation, formalisation, manipulation), content (structures, functions), and language. Smith and Thompson (2007, p. 10) share a similar view and are quoted as saying, “algebraic reasoning is characterised by its generality and by the role that symbolic expressions play in stating general relationships, comparing and manipulating them, and facilitating many numerical evaluations”.

3.4 The importance of knowledge of factoring quadratic expressions in the mathematics curriculum.

As discussed in Chapter 2 of this thesis, factoring of quadratic expressions is important in building an understanding of the variety of concepts in the mathematics curriculum. A number of researchers have shown that almost every algebraic concept requires knowledge of the factoring of quadratics which is needed when working with quadratic equations. For example, a study in Antalya, Turkey examining 10th Grade students’ procedures for solving quadratic equations with one unknown, revealed that the students’ understanding in solving quadratic equations was instrumental or procedural, rather than relational or conceptual (Didiş, Baş & Erbaş, 2011). The implications of these findings indicate that learners need to master the factorisation of quadratic expressions to be able to master the solution of quadratic equations. Factoring of quadratics is also needed when working with quadratic inequalities (Makonye & Shingirayi, 2014). Again, the knowledge of factoring quadratics is a requirement when working with functions (algebraic and/or trigonometric). The work by Govender (2015) on analysing the Grade 12 mathematics papers (one and two) has provided consolidated input on the extent to which knowledge of factoring quadratics is required when working with a number of mathematical concepts.

3.5 The Research Framework.

This study is founded on the APOS theory which according to Dubinsky (1997), has its specific framework for research and curriculum development in mathematics education. This framework consists of three components: theoretical analysis; the instructional treatment; and

the observations and assessment of students. The triangular relationship amongst these three components is presented in figure 3.1 as postulated in Asiala et al., (1997).

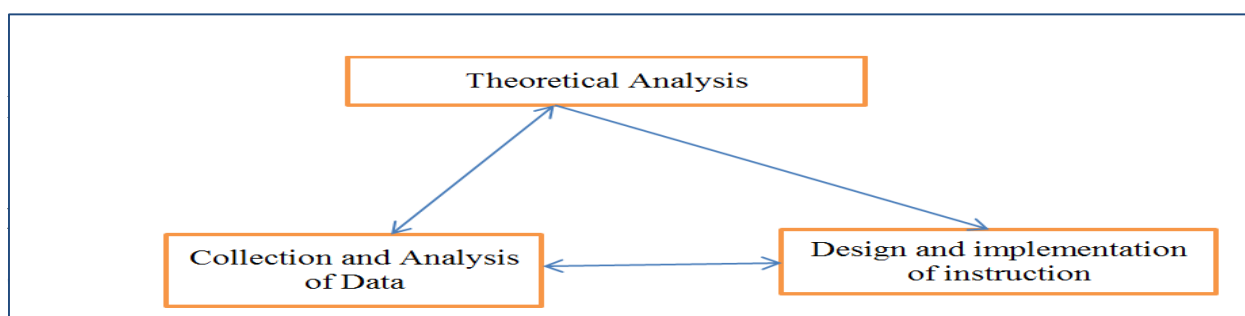


Figure 3.1: The Research Framework as adapted in Asiala et al., (1997).

According to Dubinsky (1997), the theoretical analysis produces assertions about mental constructions that can be made in order to learn a particular mathematical topic; the instruction tries to create situations which can foster making these constructions; and the evaluation tries to determine if the constructions appear to have been made. The evaluation also tries to determine the extent to which the students actually learned and is used formatively to reconsider both the theory and the instruction. These three components are cyclic in their step-wise operation and influence one another in that they are expected to be repeated until satisfactory results regarding students' learning and understanding of the content are obtained. The research process is cyclical: analysis of the data leads to the collection of more data or to a redesign of the instructional methods. In this study I focus on the two aspects of the ACE cycle – that of the theoretical analysis and the data collection, and analysis of the data. I hope the details of these processes will feed into the refinement of instructional treatments of the factorisation of quadratic expressions by mathematics teachers.

3.5.1 Theoretical Analysis.

As contemplated in Asiala et al., (1997), APOS adopts the use of the theoretical framework that is presented in Figure 3.1. The discussion on APOS as a theory is presented in details in section 3.5.2 of this study. Furthermore, Dubinsky (2000, p. 12) comments that, “the purpose of the theoretical analysis is to propose the specific mental constructions, the initial genetic decomposition, through which a student might learn the concept under construction”. This revelation by Dubinsky (2000) clarifies the reasons why the theoretical analysis presented in

figure 3.1 was used in this study. The proposed initial genetic decomposition for this study is presented in Section 3.7 of this thesis.

3.5.2. Why use APOS Theory?

The educational reforms in South Africa gave birth to (CAPS) Grade 7-9 Mathematics, of which one among many of its principles is to encourage an “active and critical learning and critical approach to learning, rather than rote and uncritical learning of giving truths” (DBE, 2011b, p. 4). Furthermore, the NCS aims to, “produce learners that are able to identify and solve problems and make decisions using critical and creative thinking” (DBE, 2011b, p. 5). Again, as presented in Table 1.2, the CAPS curriculum suggests that Grade 10 learners are expected to operate at the object level when faced with the concept of factoring quadratic expressions after instruction.

The material on the factorisation of quadratic expressions/equations is one of the mathematical concepts that must be studied by all students in secondary education. According to Cahyani and Rahaju (2019), studying mathematical material on the manipulation of quadratic expressions/equations, requires the deep understanding of concepts associated with quadratics. In Chapter 2, I presented difficulties that students encounter in factoring quadratic expressions/equations, namely errors of transformation and process skills. This phenomenon bears the evidence that students are inexperienced in combining and connecting the various concepts when working with quadratic expressions.

To make the necessary connections, learners need an understanding of the mathematical concepts underpinning the factorisation of quadratics. According to Cahyani and Rahaju (2019, pp. 2-3), “concepts are abstract ideas that can be used to classify the similarities of ideas, events, or objects that are related or share similarities. A concept is formed in the process of abstraction and classification, where abstraction is a conscious activity of thinking the similarity of ideas of students in classifying an object based on the similarity of the nature of new experiences formed, and classification is gathering experiences that have similarities from the results of abstraction”. In order to achieve what is advocated in the CAPS, the practice in the South African mathematics classroom should consider the Action-Process-Object-Schema (APOS) approach, rather than the traditional approach. According to Dubinsky and McDonald (2001),

“APOS theory proposes that an individual has to have appropriate mental structures to make sense of a given mathematical concept. The research based on this theory requires that for a given concept, the likely mental structures need to be detected, and then suitable learning activities should be designed to support the construction of these mental structures.” APOS Theory is a constructivist theory rooted in the work of Jean Piaget on the concept of reflective abstraction (Clark et al., 2000). Piaget (1985) found that reflective abstraction involves two steps. The first step includes reflecting on existing knowledge, while in the second step; existing knowledge is projected into a higher plane of thought. Cahyani and Rahaju (2019) commented that the reflective abstraction is students' understanding of mathematics through problems that reflect and reconstruct mental structures, and then the scheme is used to solve problems. Dubinsky's belief that reflective abstraction could serve as a powerful tool to describe the mental development of more advanced mathematical concepts, led to the development of the APOS theory. It is my belief that the philosophy behind the APOS theory is well-suited to this study which focuses on the learning and teaching of quadratic expressions.

3.5.3 Descriptions of the mental constructions in APOS theory.

The APOS framework describes the development in understanding of mathematics concepts through the hierarchical development of mental constructions, namely action, process, objects and schema. The theory is used to propose models of what might be taking place in the mind of a Grade 10 learner as he or she engages with mathematical concepts related to factorisation of quadratic expressions. It involves general descriptions of the mental structures and mental mechanisms (DeVries & Arnon, 2004). Dubinsky (1991) identified five types of mental mechanisms, namely interiorisation, coordination, reversal, encapsulation and thematisation, which enable the construction of mental structures of actions, processes, objects and schemas. The mental structures necessary for the construction of mathematics concepts are described below.

Action: An **action** is any transformation of objects according to an explicit algorithm in order to obtain other objects, and is somewhat externally driven. It requires specific instructions or rules as well as the ability to perform each step of the transformation explicitly. Thus a learner can think about the problem in a step-by-step manner and look at one step at a time. Skemp (1976) argued that there are two types of mathematics understanding and classified them as instrumental understanding, and relational understanding. According to Skemp (1976), a

student with instrumental understanding can only use rules without reasons. This description fits the student located at the action level of the concept.

Process: As an action is repeated and the individual reflects upon it, it may be interiorised into a mental **process**. An important characteristic of a process is that the individual is able to describe, or reflect upon the steps of the transformation in her/his mind without actually performing those steps. Additionally, once a mental process exists, it is possible for an individual to think of it in reverse and possibly construct a new process (a reversal of the original process). With the process conception of a factoring quadratic expression, an individual can link two or more processes that lead to a solution, or reverse the process by undergoing a journey on how the particular quadratic expression was obtained.

Object: When an individual becomes aware of the process as a totality and is able to transform it by some action, we say that the process has been **encapsulated** as an **object**. When necessary, an individual may **de-encapsulate** an object back to its underlying process. In other situations, the individual may think of the transformation in terms of actions (Font et al., 2011). Here, the student can reflect on operations applied to a particular process, realise those transformations and is able to construct such transformations.

The term that Sfard (1991) uses for this cognitive phenomenon is reification, which Font et al., (2011) argued is similar to Dubinsky's construct of encapsulation. According to Sfard (1991), reification is a term used in philosophy which means, etymologically, — to treat something like a thing. In the processes of mathematical reification, abstract notions are conceived like objects. To reify (or encapsulate) is to regard or treat an abstraction as if it had a concrete or material existence. "Reification therefore, is defined as an ontological shift - a sudden ability to see something familiar in a totally new light" (Sfard, 1991, p. 19). Hence, the cognitive process referred to as encapsulation, is no different from the process of reification.

Schema: Asiala et al., (2004) asserts that a **schema** for a certain mathematical concept is an individual's collection of actions, processes, objects and other schemas linked consciously or unconsciously in a coherent framework in the individual's mind. For example an individual may have a series of schema of working with quadratic expressions, in the following range: that of $a = 1$; and that of $a \neq 1$; trinomials; binomials; and monomials. Students operating at the schema level are described in OECD/PISA (2003), as students of the *highest proficiency*

level. At this level, students take a more creative and active role in their approach to mathematical problems. They typically interpret more complex information and negotiate a number of processing steps. They produce a formulation of a problem and often develop a suitable model that facilitates its solution. Students at this level typically identify and apply relevant tools and knowledge in an unfamiliar problem context. Likewise, they demonstrate insight in identifying a suitable solution strategy, and display other higher-order cognitive processes such as generalisation, reasoning, and argumentation to explain or communicate results. A student operating at the schema level is described in Skemp (1976) as having a relational understanding of a concept in that he knows both what to do and why.

3.6 Description of the mental mechanisms.

Dubinsky (1991) outlines five kinds of construction in reflective abstraction:

1. Interiorisation: Here, actions and objects are interiorised into a system of operations. During this phase, a learner becomes familiar with a process and can carry it out through mental representations.
2. Co-ordination: Here, two or more processes are co-ordinated in order to form a new process. For example, when simplifying an algebraic expression, a learner would coordinate knowledge of factorising, with that of manipulating the fractions.
3. Encapsulation: Here, the construction of mathematical understanding extends from one level to the other, where new forms of the process are built, drawing from the previous ones to form an object. Accordingly, I draw on the work of Sfard, to help me better understand students' difficulties in algebra. Sfard and Linchevski (1994, p. 194) describe reification "as the ability of minds' eyes to envision the results of processes as permanent entities in their own right". Under this metaphor, "the permanence of *having* gives way to the constant flux of *doing*" (Matos, Powell & Sztajn, 2009, p.169). According to Bowie (2000), reification involves a qualitative shift in understanding, which occurs when the student is able to detach the notion from the processes that produced it and see it as an object.
4. Generalisation: An existing schema is applied to a wide range of contexts. In this case, the learner may notice, for example, that within an algebraic fraction expression, both the numerator and the denominator may have common expressions or factor.

5. Reversal: This is an interiorised process that can be thought of in reverse. In this case, in order to work with the factorisation of any expression, a learner has to see the expression as being formed by the product of certain terms.

3.7 The Initial Genetic Decomposition (G.D.) for the construction of the concept of factoring quadratics.

According to Maharaj (2010, p. 41) a “genetic decomposition of a concept is a structured set of mental constructs which might describe how the concept can develop in the mind of an individual”. Furthermore, a genetic decomposition postulates the particular actions, processes, and objects that play a role in the construction of a mental schema for dealing with a given mathematical situation. In this study, I postulated the construction that a Grade 10 learner should make in order to understand the factorisation of quadratic expressions $ax^2 + bx + c$. Since the curriculum progression involves starting with quadratic expressions where $a = 1$, our genetic decomposition is based on this assumption. The schema for the factorisation of quadratic expressions has been considered as follows:

Action: At the action level, a student is able to carry out rote procedures focused on factorising a quadratic expression of the form $x^2 + bx + c$. This is done by identifying the factors of c which, when added together, give the sum b . For example given the quadratic expression, $x^2 - x - 6$, as a first step the learner will identify the factors of -6, which could be { 6, -1 } or { -1, 6 } or { 3, -2 } or { -3, 2 }. The second step would involve considering each factor pair and working out the sum of the factor pair until one obtains the sum -1, which is true for { -3, 2 }. The next step is to write down the factorised expression as $(x - 3)(x + 2)$. The learner then perceives the rules in manipulating these expressions as not connected.

Process. The action is interiorised into a process when the individual is able to work out the product mentally or make predictions about what the product would look like, without having to go through each step, e.g. being able to run through all possible factor pairs for c in one's mind, applicable for the expression $x^2 + bx + c$ and select the pair whose sum is b , without having to physically check the sum of each factor pair. At this stage, the learner would be able to recognise special examples such as $(x - 3)(x + 3) = x^2 - 9$ or $(x + 2)(x + 2) = x^2 + x + 4$ without having to perform each step explicitly. Here, an individual can see that the

reversal of the product results in the factorisation. For example, recognising that the expression $x^2 - 4$ is made up from the product of $(x - 2)$ and $(x + 2)$.

Object. An object is “constructed from a process when the learner becomes aware of the process as a totality and realises that transformations can act on it, and can actually compose such transformations; then we say the individual has condensed the process into a cognitive object” (Maharaj, 2010, p. 43). At this stage, the learner is able to perform other actions such as factoring for example quadratic expressions where $a \neq 1$, by extending the steps to consider the factors of a as well as the factors of c in $ax^2 + bx + c$ for example, $6x^2 - x - 2$. At this stage, the process of finding the product of algebraic expressions is encapsulated into an object upon which other transformations or actions can be effected, for example, to simplify the rational expression such as $\frac{x^2-x-6}{2x-6}$, by first decomposing the numerator and denominator into factors. Learners at this stage can distinguish between objects arising from similar processes and can argue why, for example $x^2 - 4$ is factorable, when $x^2 + 4$ is not.

Schema: At schema level, learners develop more flexible methods of handling the factorisation of quadratic expressions by following the fundamental approach to factorisation of quadratics. Subsequently, learners are able recognise that all quadratic expressions can be factorised by following a single universal approach.

The summary of the Genetic Decomposition is presented in Table 3.1 of the Reflective Abstraction Indicators, as presented in Rahmawati et al., (2018). For the purpose of this study, this table of Reflective Abstraction Indicators, has been used, and I effected a few modifications on the Reflective Abstraction Indicators to suit this study.

Table 3.1: Reflective Abstraction Indicators.

Reflective Abstraction	
Stages	Indicators
Action	• Connecting the mathematics knowledge that belongs to the factorisation content to find the answers. • Using some mathematics knowledge about factorisation to find the answers.
Process	• Making some connections within knowledge related to the factorisation content to find the answers. • Explain how the answers were obtained.
Object	• Connecting the mathematics knowledge with the factorisation knowledge which was wholly owned. • Being aware of process as a whole by realising the transformation done and being able to explain links between the rules used.
Schema	• Connecting actions, processes, objects, and leads to formation of a factorisation scheme. • Using schemes that have been formed through actions, processes, objects in mind to form new objects that can be used in accomplishing more complex tasks.

3.8 APOS – directed instructional approaches.

In improving learner ability in algebra, the instruction that is based on APOS theory has some advantages when compared to the traditional approach. Arnawa et al., (2007) outline the characteristic differences between the two approaches, as shown in the Table 3.1.

Table 3.2. Characteristic Differences of APOS Theory and Traditional Approach (Arnawa et al., 2007, p. 145).

No.	APOS Theory Approach	Traditional Approach
1	Topics are designed regarding the mental construction steps: actions, processes, objects, and schemas. Students are involved actively in learning. Mathematics ideas (definitions, lemmas, and theorems) are discovered by students through fact findings in laboratory activities.	Topics are not designed specifically, generally refers to text-books or lectures' class notes. Students receive information passively. Mathematics ideas are given in a ready-made fashion.
2	The role of lecturer/instructor as a facilitator, i.e. supports guidance to students, group or entire class through scaffolding, e.g. questioning, giving hints.	The role of lecturer/instructor as a knowledge transformer, i.e. the lecturer/instructor directly explains mathematics ideas.
3	There is multi-direction interaction among students, as well as students and lecturer. Students learn from peers through work group, discussion.	One or two-way interaction involving the teacher.

The relevance of the study on APOS within the South African perspective can be linked to the shortage of such studies that have been conducted. Most studies on APOS that have been conducted in the South African perspective have concerned Mathematics in Grade 12 and at tertiary level (Jojo, 2011; Jojo, Maharaj and Brijlall, 2012; Maharaj et al., 2015). Those studies conducted at secondary school level focused on other areas and other grades other than Grade 10. For example, there are few studies on APOS conducted at Grade 12 calculus (Brijlall & Ndlovu, 2013; Dlamini, Brijlall & Jojo, 2017). Another contribution at Grade 11 level was the study conducted on the analysis of Grade 11 learners' levels of understanding of functions in terms of APOS theory (Chimhande, Naidoo & Stols, 2017). Another study on APOS was conducted for Grade 8 by Mphuthi and Machaba (2018) who followed the APOS exploration of the conceptual understanding of algebraic expressions.

In the light of this, it would be compelling to explore APOS practice at the secondary level of schooling in South Africa, where the shortage of studies that explore the APOS theory in Grade 10 mathematics classrooms necessitated the present study. This study intends to find solutions, among many, to the poor attainment in national and international mathematics assessments, as

pointed out in the literature review. The use of APOS theory in the study provides an opportunity to examine whether the teaching of the subject is designed so that learners' understanding of the concepts is progressively strengthened. An APOS- based instructional strategy will focus on helping learners to develop action conceptions, which must then be transformed to process conceptions, and then to object-level conceptions. The action, processes, and objects are linked into a schema, which the learner uses to make sense of the content.

According to Dubinsky (2000, p. 12), “the role of the instructional treatment is to get students to make the proposed mental constructions and use them to construct an understanding of the concept as well as apply it in both mathematical and non-mathematical situations”. APOS adopts the use of the ACE Teaching Cycle as the instructional approach that attempts to foster the development of certain mental constructions by de-emphasising the lecture method in favour of problem-solving, cooperative learning, and the use of mathematical programming (Clark et al., 2000). The ACE stands for the Activities to be done on the computer; Classroom discussions; Exercises to be done in class. According to Dubinsky (2000, p. 13), the ACE Teaching Cycle fosters the mental constructions called for by the theoretical analysis. Furthermore, through cooperative learning, the social context is provided in which students can engage in the reflections. Again, students engage in active learning to figure things out on their own rather than relying on the instructor for explanations and information. This is different from an approach that merely seeks to teach concepts without making connections between them.

Liljedahl (2008) contends that most teaching practices are a reflection of the teachers' beliefs. In his view, a teacher with a view of mathematics as a *toolbox* will teach mathematics as such. This will mean an emphasis on rules, formula, and procedures with an abundance of practice to enforce memorisation and mastery. A teacher with a *systems* view will make extensive use of definitions and proofs both as a pedagogical strategy and as content to be acquired. Finally, a teacher with a *process* view will incorporate progressive constructivist teaching methodologies into their teaching in order for their students to experience the 'doing' of mathematics. The implication of this argument is that teachers use different approaches to transfer knowledge during classroom teaching. The researcher strongly believes that any interventions designed to improve the performance of learners must be informed by research

focused on the actual mathematical practices that are prevalent in classrooms. It is hoped that this study can provide significant information that can lead to better targeted interventions.

Block (2016) contends that teachers need to expose students to a variety of approaches, for learners to be flexible when embarking on problem-solving. Block (2016, p. 392) defines “flexible algebraic action” as the ability to choose an adequate processing method depending on the specific features of the task, and the abilities of the individual. This definition refers to the concept of flexibility in mental calculation”. Figure 3.2 shows the Comparison of flexible algebraic action and algebraic action as adapted in Block (2016, p. 391)

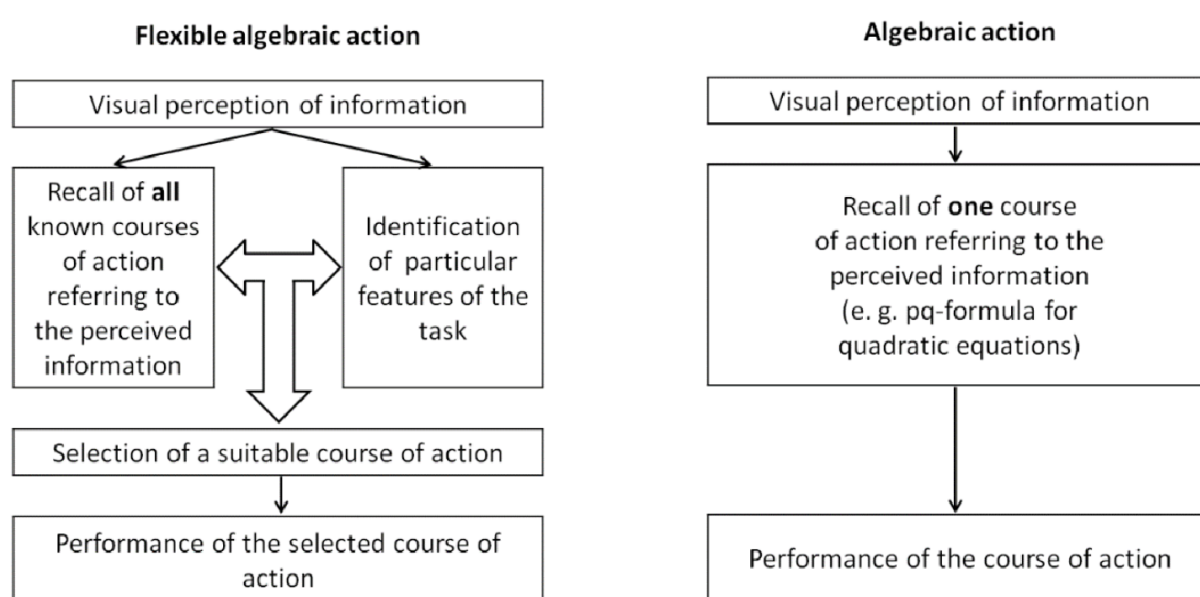


Figure 3.2: Comparison of flexible algebraic action and algebraic action adapted in Block, J. (2016).

According to Block (2016), when professional mathematicians solve quadratic equations like (i) $x^2 + x - 6 = 0$, (ii) $x^2 + 2x = 0$, (iii) $(x - 3)(x + 5)(x - 3)$ or (iv) $(x - 3)(x + 5) = 7$, they use different methods to find the solutions in effective and less error-prone ways. They do so, because they recognise different features of the equations and they are able to draw appropriate conclusions for solving the equations.

3.9 The Triad mental mechanisms.

On refining the APOS theory, Dubinsky and McDonald (2001) elaborated on the triad mechanism as consisting of three stages, referred to as *Intra*, *Inter*, and *Trans*. Even though this refinement was aimed at gaining a better understanding of other mathematical concepts, other than the factorisation of quadratic expression, I found it fitting for this study.

Firstly, the *Intra* stage of schema development is characterised by a focus on individual actions, processes, and objects in isolation from other cognitive items of similar nature. For example, in the factorisation of quadratic expression concept, a student would tend to focus on a single type of expression and the various activities that he or she could perform with it. Secondly, the *Inter* stage is characterised by the construction of relationships and transformations among these cognitive entities. At this stage, an individual may begin to group items together and even call them by the same name. A student may begin to understand that all types of quadratic expressions can be factorised using the single approach, yielding the intended outcomes. Finally, at the *Trans* stage, the individual constructs an implicit or explicit underlying structure through which the relationships developed in the *Inter* stage are understood and which gives the schema a coherence by which the individual can decide what is in the scope of the schema and what is not. For example, an individual at the *Trans* stage for the factorisation of quadratic expression concept could construct various systems of transformations of expressions such as those of $a \neq 1$. He may even attempt to argue if the quadratic expression is factorisable or not.

3.10 Conclusion.

This chapter has been able to delve on the theoretical framework adopted in this study. Chapter 4 presents the methodology adopted in this study.

CHAPTER 4

METHODOLOGY

4.1 Introduction.

This chapter will introduce detailed discussions of the research methodology adopted. These discussions include the context in which the study was focused; the design within which the research was located; the data collection instruments; data analysis; trustworthiness criteria; ethical considerations, as well as the limitations of the study.

4.2 Context of the Study.

The study was conducted at two secondary schools in one of the Education Circuits around the Ilembe Education District in the province of KwaZulu-Natal (KZN). The focus of the study was two types of schools: (i) one school from the rural setting (R-School); (ii) one school from the township (T-School). The performance in mathematics at matric level of the two schools identified is presented. For the academic years 2019 and 2020, the T-School has produced 60.7% and 72.3% consecutively. For the academic years 2019 and 2020, the R-School has produced 24.47% and 42.9% consecutively. These results indicate that both schools show improvement in matric mathematics, with the T-School performing better than the R-School. The sampling and selection of the participants for this study is outlined in Section 4.4 of this thesis.

4.3 The Critical Research Questions.

The study proposed the APOS approach to exploring the conceptual understanding displayed by Grade 10 mathematics learners from the two secondary schools selected, as they engaged with factorisation of quadratic expressions. The main issue in this study is to identify how students and teachers conceptualise quadratic expressions, with specific reference to factorisation. Furthermore, the study focuses on how both teachers and learners embark on the journey of factorisation of quadratic expressions, as a milestone towards their understanding of factorisation in general and how it is applied in the manipulation of algebraic expressions in many other sections of mathematics.

This study is titled:

An APOS analysis of the teaching and learning of factorisation of quadratics in Grade 10 mathematics classrooms.

4.3.1 Research Questions.

The study asks the following questions:

Research Question One (RQ1): How do Grade 10 learners perform in algebra in the KZN provincial common test?

Research Question Two (RQ2): What levels of APOS thinking do Grade 10 teachers engage in when dealing with items based on the concept of factorisation of quadratic expressions?

Research Question Three (RQ3): What levels of APOS thinking do Grade 10 learners engage in when dealing with items based on the concept of factorisation of quadratic expressions?

Research Question Four (RQ4): A) How do mathematics teachers approach the teaching of factorisation of quadratic expressions with their Grade 10 learners?

Research Question Four (RQ4): B) Why do they use the approaches that they do?

Further questions were also identified as a way of directly and indirectly addressing the main research question. These questions appeared on both the questionnaire, as well as in the interviews of both teachers and learners which are dealt with in later chapters of this thesis.

4.4 Research Design.

Different researchers have a variety of research approaches as is attributed to what Henning (2004, p. 1), describes as the foundations of research traditions. Research traditions are founded on the researchers' beliefs and intra-questions, which include the questions of ontology; epistemology; as well as those of methodology. Engelhard (1991) shares similar sentiments as she suggests that:

Different research traditions imply different assumptions and different ways of viewing measurement and social science research. The problems selected for study, the statistical models used to analyse the data, the results of the inquiry and the policy implications drawn from the research depend on the measurement models used (Engelhard, 1991, p. 1).

4.4.1 Mixed Methods Research.

Following on from the research questions, this study is located within the principles of the mixed methods case study approach. Creswell (2008, p. 552), defines mixed methods research as a procedure for collecting, analysing and “mixing” both quantitative and qualitative data at some stage of the research process within a single study to understand a research problem completely. According to Tashakkori and Teddlie (cited in Ivankova et al., 2007) the mixed methods researcher combines quantitative and qualitative strategies within one study, collects both numeric data (numbers e.g. scores on the survey instrument or ratings) and text data (words e.g. open-ended interviews or observations) concurrently or in sequence, and chooses variables and units of analysis which are most appropriate for addressing the study’s purpose and finding answers to the research questions. Libarkin and Kurdziel (2002, p. 80) agree in that the mixed methodology research is a research plan that can incorporate both qualitative and quantitative data.

I draw on two of the many main advantages of the multi-method approach in social research. Firstly, the single observation provides only a limited view of the complexity of human behaviour and of situations in which human beings interact. It has been observed that as research methods act as filters through which the environment is selectively experienced, they are never atheoretical or neutral in representing the world of experience. Exclusive reliance on one method, therefore, may bias or distort the researcher’s picture of the particular aspect of reality being investigated. Again, the more the methods contrast with each other, the greater the researcher’s confidence. If for example, the outcomes of a questionnaire survey correspond to those of an observational study of the same phenomena, the more the researcher will be confident about the findings (Cohen, Manion & Morrison, 2018; Libarkin & Kurdziel, 2002, p. 80).

Jones (1997, p. 1) further justifies the use of this approach by bringing forth that, “the crucial aspect in justifying a mixed methodology research design is that both single methodology approaches (qualitative only and quantitative only) have strengths and weaknesses. The combination of methodologies on the other hand, can focus on their relevant strengths”. The researcher should aim to achieve the situation where “blending qualitative and quantitative methods of research can produce a final product which can highlight the significant contributions of both, where “qualitative data can support and explicate the meaning of quantitative research” It is held therefore that a mixed method approach is relevant for this study as the subjects’ views will be captured from different perspectives, in order for the researcher to gain a better understanding about the learners’ and teachers’ perceptions about, and performance in, mathematical tasks based on algebraic concepts. In this study, the combination of methodologies has allowed me to identify broad trends across the group as a whole as well as differences within the participants of the group itself.

4.4.2 The Interpretive Paradigm.

This study consists of an interpretive type of research. Although some quantitative data was gathered in the form of questionnaires and script analysis, the data was used to complement and supplement the other qualitative data sources (the interviews). “Interpretive researchers start out with the assumption that access to reality (given or socially constructed) is only through social constructions such as language, consciousness and shared meanings). This kind of research does not predefine dependent and independent variables, but focuses on the full complexity of human sense-making as the situation emerges” (Maree & van der Westhuizen, 2007, p. 3). They further point out that an interpretive framework has potential as it requires going into participants’ natural setting and experiencing the environment in which these participants create their reality. I found the particular approach of engaging learners and teachers in reflections as being relevant in my study, where my intention was to find meaning within social interactions as the participants were interacting with the context through problem-solving.

Within the interpretative paradigm, I opted to utilise a case study approach. According to Nieuwenhuis (2007, p. 75) the term “case study” can be used to describe a unit of analysis (e.g. a case study of a particular organisation) or to describe a research method. Yin (2018) agrees as he describes a case study as an empirical enquiry that investigates a contemporary

phenomenon within its real-life context, especially when boundaries between phenomenon and context are not clearly evident. Guba and Lincoln (1988, p. 370) offers an extended definition as they defines the case study as, “an intensive or complete examination of a facet, an issue, or perhaps the events of a geographic setting over time”. The case in this study is the group of learners and teachers emanating from the Grade 10 mathematics classrooms of the two secondary schools in the Ilembe District. This study sought to explore learners’ and teachers’ perceptions on how the APOS analysis informs the teaching of factorisation of quadratics in Grade 10 mathematics. I therefore found the case study methodology relevant in this study as the theoretical framework stresses the significance of context in problem-solving.

4.5 The Participants and the Sampling.

The Participants for this research project are described in a specified order. Firstly, in 2018, two learners from the R-School, Sthembiso (learner SK), male and Nobuhle (learner SZ), female, participated in the Pilot test. Also, one teacher from the R-School, Rakhulu (teacher RT), female, participated in the piloting of the study. In the piloting session, the classroom lessons’ observations were conducted, for only teacher Rakhulu. In the year 2019, the teachers’ questionnaire was administered to 25 mathematics teachers of the circuit. Teachers’ responses from the questionnaire are presented in Appendix L, and the following are the teachers’ codes: GS; KW; MZ; RT;MZ;ND; WMAN1; WMAN2; WMAN3; WMAN4; WMAN5; WMAN6; WMAN7; WMAN8; WMAN9; WMAN10;UDU1; UDU2; UDU3; UDU4; UDU5; MPQ1; MPQ2; MPQ3; MPQ4. From the two participating schools, all Grade 10 mathematics learners for the academic year 2019 were part of the study in that all their March common test scripts were analysed. There were 205 learners who participated, with 124 from T-School and 81 from R-School.

Twelve learners from the two participating schools were invited to participate in the interviews: one was the T School (township school), and the other the R School (school from the rural setting). The reason for choosing these two types of schools was so that the research was not confined to one context. The teachers were asked to volunteer names of learners who were willing to take part in the research. Seven learners were from the R-School, while the other five were from the T-School. The following is the list of learners interviewed, on their pseudo names with their codes:

From the R-School

- Learners participated in the Pilot test
 - Sthembiso (learner SK), male
 - Nobuhle (learner SZ), female
- Learners who enrolled in the Grade10 class of 2019
 - Sandile (learner LH), male
 - Bongekile (learner BT), female
 - Skhumbuzo (learner XD), male
 - Bayanda (learner DB), female
 - Gugulethu (learner GD), female

From the T-School, learners who enrolled in the Grade10 class of 2019 were

- Ntombizodwa (learner MW), female
- Enhle (learner GE), female
- Bonke (learner SB), male
- Thembani (learner TM), female
- Nomfundo (learner NQ), female

A total of twenty-five teachers teaching Grade 10 mathematics for the academic year 2019, comprising teachers of the participating schools and teachers from neighbouring schools within the circuit to which the participating schools belong, agreed to participate by responding to the questionnaire. Lastly, six teachers in total who volunteered to participate, were interviewed, three from each school. The following is the list of six teachers interviewed, on their pseudo names with their codes:

From the R-School

- Teacher participated in interviews in the Pilot (2018)
 - Rakhulu (teacher RT), female
- Teachers participated in interviews in 2019
 - Ndulini (teacher ND), female
 - Hlophe (teacher HP), male

From the T-School,

All the three teachers participated in the interviews in 2019

- Mzimela (teacher MZ), male
- Kweyama (teacher KW), male
- Gwamanda (teacher GS), male

For the three data collection methods, the teachers' questionnaires, the teachers' classroom lessons' observations, and the learners' scripts analysis, I adopted the convenience sampling method. Maree and Pietersen (2007a, p. 177) point out that this sampling method is only used based on the fact that the elements of the study are easily and conveniently available. We therefore engaged the whole group of Grade 10 mathematics teachers in our proximity as participants of the study, because these were the participants available to us. Again, we managed to analyse the scripts of Grade 10 learners who were enrolled at the participating schools in the 2019 academic year.

For the processes of interviews for learners and teachers; and the learner group activity, I embarked on a purposive sampling. According to Maree and Pietersen (2007a, p.178) "purposive sampling is the method of sampling used in special situations where the sampling is done with a particular purpose in mind." Purposive sampling is critical in this case for two particular reasons. Firstly, from each of the participating schools, the interview focus was on the teachers teaching mathematics in Grade 10 in the current year. Furthermore, for the learners' group activity coupled with interviews I opted for the purposive sampling where I was assisted by teachers in selecting learners in terms of their performance. Hence, I was given ten students from the two (2) high performing secondary schools in Grade 10 mathematics, five learners were from a school situated in the rural setting (R-School), and five learners were from a school from the township (T-School). My aim was to gain a better understanding on what informs good performance standing from the different socio economic backgrounds.

4.6 Phases of Data Collection.

The study was composed of two phases. All the activities in Phase 2 took place in term 1 of 2019. The full details of the activities entailed in the data collection instruments outlined in these phases are presented in Section 4.7 of this thesis.

4.6.1 Phase 1 (2018).

Phase 1 was the pilot study which was conducted for the purposes of validating the data collection instruments, the teachers' questionnaire, and the learners' activity with learners' clinical interviews, for the purposes of the main study. The participants in this phase were two Grade 10 learners of mathematics and one Grade 10 mathematics teacher from the R-School of the class of 2018. The teacher questionnaire as it appears in Appendix J was administered to one teacher. In addition, the learner- activity as it appears in Appendix H was administered to two learners. The results derived from the pilot study constitute the results of the entire study and are discussed in full in Chapter 10.

4.6.2 Phase 2 (2019).

There were no revisions done on the questionnaire and the learner-activity used in the Pilot study. This phase began with classroom lesson observations. The observation schedule appears in Appendix F. I conducted a workshop based on the teaching of factorisation of quadratics to Grade 10 learners with teachers from the R-School and the T-School. Twenty-five Grade 10 mathematics teachers from the two circuits of the Stanger CMC where the participating schools emanate, volunteered to take part by completing the questionnaire. Mathematics teachers from the two participating schools (R-School and T-School) also form part of this cohort of twenty-five teachers who completed the questionnaire. Selected teachers from the participating schools were interviewed. Learners' scripts were analysed, paying special attention to certain items taken from the 2019 March Grade 10 mathematics common paper. Those items appear in Appendix G. Lerner-activity was administered to ten learners from the participating schools, and was followed by interviews of learners. The interview schedule appears in Appendix I.

4.7 Data Collection Instruments.

The use of multiple methods or a multi-method approach in the social sciences is an attempt to map out, or explain fully, the richness and complexity of human behaviour by studying it from

more than one standpoint (Cohen et al., 2018,). Hence, the study utilised six data collection instruments viz. classroom observations; learners' scripts analysis; learners' activity; learners' semi-structured interviews; teacher questionnaires; and teacher semi-structured interviews.

4.7.1 Classroom Lessons Observations.

In order to gather in-depth data, I embarked on classroom observation during term 1 when the teachers were teaching quadratics. Critical to this observation was learner participation, as well as the manner in which the lesson was driven. A total of six teachers were drawn from the participating schools to participate in the study. Three teachers were from School T, and the other three were from School R. Each teacher was observed on two occasions, resulting in a total of twelve classroom observations conducted.

This study is located within the constructivist paradigm. According to the constructivist approach, students should be given the opportunity to discover and construct their own mathematical meanings. It follows that teachers should play the role of facilitator, refraining from providing direct instruction or telling students anything they might be able to discover for themselves (Allen, 2011). This type of teaching, according to Allen (2011), is referred to as the "democratic mathematics education", in that it acknowledges that students do not learn the same way, maximise interaction and teamwork, and focus on the mathematical means or processes rather than the ends or answers.

Observing teachers embedding lessons is conducted for various reasons. Mainly, researchers contend that teachers are the source of learners' errors. According to Moru, Qhobela, Wetsi, and Nchejane (2014, p. 11), "learners' errors are complex because they are a function of many variables. They may originate from the learners themselves, the teachers, the curriculum, the environment, or the interaction amongst these variables." Teachers are encouraged to engage in error analysis continuously in order to enrich knowledge of teaching, familiarisation with student errors, and error analysis from the literature, as some errors in mathematics are shared across contexts, where there is a need to make an effort to gain an understanding about learning theories, in particular the constructivist view (Moru et al., 2014). Similar sentiments were also

shared by Shalem, Sapire and Sorto (2014, as they engaged in a study on teachers' explanations of learners' errors in standardised mathematics assessments.

Other researchers claim that one of the causes of poor attainment in mathematics is teachers' weak knowledge of mathematics, and propose that improving teachers' mathematical knowledge would improve learner attainment (Bansilal et al., 2014; Makgakga 2016; Makonye and Matuku, 2016; Mji and Makgato, 2006; Pournara, Adler, Pillay and Hodgen, 2015). A study was conducted to investigate the preparedness of teachers to teach Grade 12 mathematics in the Pinetown District, of KwaZulu-Natal, South Africa (Maharaj, 2018). The teachers were given tests in algebra, trigonometry, and calculus. Those tests were based on common errors, misconceptions, or difficulties relating to school mathematics made by university students and those reported in the examiners' reports for the national senior certificate examinations in mathematics. The written responses of those teachers were analysed. This study showed that for the sample, there was a relationship between mathematics teachers' content knowledge and learner errors, misconceptions or difficulties reported on in the mathematics examiners' reports.

Another reason for observing lessons was so that it would allow me to identify commonalities in the teaching approaches and the learner approaches to factorising quadratic expressions. A previous study, Adu and Olaoye (2015) using quasi-experimental factorial design, found the method of teaching affects student's performance in algebra. In this study, I use an interpretative approach to study the teachers' approaches to teaching the content, as well as the learners' approaches towards, and performance in solving problems based on factorisation of quadratic expressions. The observations allow me to gain a deeper perspective of the relationship between the teachers' approaches and understanding of the concept, and that of the learners' approaches. The report on the classroom observations is presented in Chapter 5 of this thesis.

4.7.2 Learners' Script Analysis.

The Grade 10 learners of mathematics in the Province of KwaZulu-Natal in South Africa, write the common paper at the end of each term, for the first consecutive three terms. The purpose of these papers is to expose learners to standardised testing, as well as to prepare them for the

final examination written in term 4. My aim was to analyse learners' performance on these tests, due to their quality in that they are moderated at provincial level, so it was feasible to conduct an informed learner baseline assessment. I conducted the scripts analysis in the year 2019, analysing the 2019 Grade 10 mathematics March common paper, as it appears in Appendix G. The choice of the March common test was guided by fact that the teaching of the factorisation of quadratic expression in Grade 10 is done in term 1. Using general statistical methods, I managed to analyse 205 scripts comprising of 81 learners' scripts from the R-School and 124 learners' scripts from T-School, and my focus was on items involving the manipulation of algebraic expressions. The full details of the statistical reports for the scripts analysis appear in Chapter 7 of this thesis.

4.7.3 Learners' activity on factoring quadratics.

Twelve learners, who participated in the interviews, were first required to complete the learner activity. This twelve included the two learners from the R-School who participated in the pilot; five learners from R-School; as well as five learners from T-School, the latter both participated in the main study. To set up for the interview session, all learners had to work on the activity that was based on the factorisation of quadratic expressions. The activity appears in Appendix H of this thesis. This activity was aimed at setting up the agenda for the issues to be addressed during the interviews. According to Vygotsky (1978), constructivism refers to how learners socially construct knowledge linguistically through interacting with signs and symbols. Vygotsky introduced two important concepts to constructivism: the first is the Zone of Proximal Development (ZPD). Vygotsky described the ZPD as the distance between the actual development level as determined by independent problem-solving and the level of potential development as determined through problem-solving under adult guidance. Potential development is the learners' ability to solve mathematical problems with the help of a more knowledgeable peer or adult, and actual development is the complete developmental cycle which is shown when learners solve mathematical problems independently (Vygotsky, 1978). The second concept Vygotsky raised was scaffolding, which refers to the help given to learners by a teacher or more knowledgeable peer at the beginning of a learning activity and is suddenly withdrawn to allow the learner to build mental structures in collaboration with signs and symbols (Vygotsky, 1978).

This task asked whether certain quadratic expressions could be factorised and comprised quadratic expressions that are factorable and some that are non-factorable. Different researchers use non-examples as means to clarify concepts in mathematics (Herawaty et al., 2021). Chua (2017) shares a similar view as she argues that the use of examples and non-examples brings more clarity in understanding the factorisation of quadratic expressions. Again, the use of examples and non-examples as a road to understanding the concept of function in the study conducted by Wulan and El Milla (2020) produced the critical findings. The study found that the learners' understanding was limited to procedural knowledge such as mental representations and incomplete concept images. It follows that learners failed to be comprehensively linked to the definition of the concept of function. For the purposes of data analysis, the use of non-examples probes problem-solvers into thinking deeply by arguing why the non-example misfits the examples. Jojo (2015, p. 280) argues that problem-solvers who demonstrate, argue and explain their solutions to others, and demonstrate ability to fuse, connect, and relate the concept learnt to other concepts, operate on the Object stage of APOS. Each task on the activity was allocated to an APOS theory conception level, and is presented in Table 4.1 of this thesis, as was contemplated in the genetic decomposition as discussed in section 3.7 of this thesis.

Table 4.1: Content analysis of the tasks in teachers' and learners' activity on factoring quadratics.

Quadratic Expression	APOS level/s	Factorability
x^2	Action	Factorable
$x^2 - 4$	Process	Factorable
$x^2 + 4$	Object	Non-factorable in terms of x
$x^2 - 4x$	Process	Factorable
$x^2 - 4y$	Object	Non-factorable in terms of x
$x^2 - x - 6$	Action	Factorable
$6x^2 + 7x - 20$	Process	Factorable

4.7.4 Learner Semi-Structured Interviews.

All the interviews were conducted within the schools where learners are studying, respectively, and each interview lasted for about 30 minutes. Once an activity based on the factorisation of algebraic quadratic expressions was administered to the group of learners, all learners were then invited to participate in the 30 minutes interviews, for each learner, on the same day as

the activity. This type of engagement between the researcher and the interviewee is termed by Chimhande et al., (2017) as the clinical interviews in that it is in-depth, open-ended, task-based, and qualitative. Cahyani and Rahaju (2019) share similar sentiments as they contend that the clinical interviews allow the researcher to dig deeper into the subject's thinking. These interviews were semi-structured to allow for open-ended responses. The schedule for the learner interviews appears in Appendix I. Such an interview is flexible, and allows the interviewer to go into more depth if desired, or to clear up misunderstandings. It also encourages cooperation and helps establish rapport (Cohen et al., 2018). The purpose of the learner interviews in this study was to seek greater clarity on learners' responses from the group activity. Learner interviews were guided and informed by the learners' responses from the learner group activity. During the interviews, learners were encouraged to discuss their solutions from the given task. Furthermore, learners were expected to explain and describe the procedures they followed in coming up with the solutions. On both interviews, for learners and for teachers, discussed in section 4.7.6, participants were expected to reflect on their journeys respectively as they engaged the activity on factoring quadratic expressions. The kind of reflection the twelve learners and six teachers were expected to provide was to be in the form of writing, or in the form of narrative. I then analysed their reflections. Hatton and Smith, (cited in Rosenberg, 2009, p. 13) describes reflection as a reflection-in-action which is our 'on the spot experimentation' where teachers and learners think and respond moment-to-moment as needed in the teaching and learning of a lesson. This is where they show an ability to think on their feet in response to the challenges of the unexpected that the dynamics of the classroom throw at them. Rosenberg (2009) identifies four levels of reflection described as follows: Firstly, the descriptive writing (reports on events with no reason given for events); secondly, the descriptive reflection (provides reasons based on personal judgement); thirdly, the dialogic reflection (a form of dialogue 'talking' with oneself and explaining your actions by referring to particular educational theories); And finally, the critical reflection (where we justify our actions on the basis of broader historical, social, political contexts). All interview sessions with learners were video-recorded as per a consent agreement. The full analysis of the learner responses from the interviews was analysed and are presented in full in Chapter 7 of this thesis.

4.7.5 Teacher Open-Form Questionnaire.

One set of the open-form questionnaire as it appears in Appendix J was administered to and reached 25 Grade 10 mathematics teachers from the participating schools, as well as from neighbouring schools. This questionnaire was administered in term 1 of 2019, where with the assistance of the subject advisor of mathematics of the circuit, I was able to reach the Grade 10 mathematics teachers after their routine workshop. The questionnaire was designed such that it engaged teachers on various avenues: their biographical information and experience in teaching Grade 10 mathematics; their methodological expertise in the area of the factorisation of quadratic expressions; their continuous Professional Development (CPD) in the field of algebra, and finally, on managerial matters particularly the mathematics Grade 10 curriculum coverage. The purpose of the questionnaire was to seek insight from the teachers of Grade 10 mathematics regarding their methodological practices as well as their pedagogical practices, in as far as dealing with the manipulation of algebra in their classroom is concerned.

Cohen et al., (2018) is of the view that rather than forcing respondents to choose between rigidly limited responses, the open-form questionnaire permits them to answer freely and fully in their own words and their own frame of reference. Furthermore, this method of collecting data gives the subjects an opportunity to reveal their motives or attitudes and specify the background or provisional conditions upon which their answers are based. Cohen et al., (2018) highlight some advantages of the questionnaire over the interview. They are of the idea that a questionnaire is more reliable; it also encourages greater honesty since it is anonymous; it is more economical in terms of time and money. Maree and Pietersen (2007b) further view the use of the open-form questionnaire as being vital in that thematic analysis of responses yields extremely interesting information, categories and subcategories since complex questions are adequately answered. The notions of validity and reliability are conserved as well (Cohen et al., 2018).

However, much research has shown that there are common errors in designing the questionnaire. For example, in the case of this study some teachers responded to the questionnaire as if ‘adequate’ and ‘necessary’ were synonymous. Furthermore, if subjects are not highly literate and willing to give considerable time and critical thought to questions, they

cannot provide useful data (Cohen et al., 2018; Maree & Pietersen, 2007a). The full analysis of the teacher responses to the questionnaire is presented in Chapter 8 of this thesis.

4.7.6 Teacher Semi-Structured Interviews.

Six teachers (three from the T-School and three from R-School) participated in the one hour semi-structured interviews, for each teacher. The interview schedule of questions appears in Appendix K. The purpose of the teacher interviews in this study was to seek clarity on the teachers' responses provided in the teacher questionnaire. The interviews were conducted on different days, and in different settings, as agreed by individual participants. All interviews conducted were video-recorded with all ethical issues taken into consideration during the course of the interview as per informed consent agreement. All the interviews were conducted within the schools where teachers are teaching, respectively.

The interviews were only semi-structured to allow for open-ended responses. Such an interview is flexible and allows the interviewer to go into more depth if she desired, or to clear up misunderstandings. It also encourages cooperation and helps establish rapport (Cohen et al., 2018). This kind of interview was used in order to probe the teachers' approach to solving problems in relation to their background algebraic knowledge. According to Libarkin and Kurdziel (2002, p. 198), interviews elicit qualitative data. This kind of interview allowed me to achieve the same level of knowledge and understanding as those of participants (Henning, 2004, p. 75). Again, according to Henning (2004, p. 6) interviews "elicit thick data, which is the kind of data that gives an account of the phenomenon that is coherent; giving more than facts and empirical content; as well as interpreting the information in the light of other empirical information in the same study as well as from the basis of a theoretical framework that locates the study". The full analysis of the teacher responses from the interviews was analysed and is presented in full in Chapter 9 of this thesis.

4.8 Data Analysis.

As contemplated in Asiala et al., (1996) and presented in figure 3.1, the study adopts the use of APOS as a framework of analysis. According to Dubinsky (2000), this method of analysing

data looks at the interview transcripts, and APOS offers explanations of student successes and failures. As this study adopted the APOS theory, APOS as a framework for data analysis has been used. Furthermore, I used the genetic decomposition, as presented in section 3.7 of this thesis, to analyse the responses. Lastly, my data analysis was further informed by the use of Reflective Abstraction Indicators, as presented in Table 3.1 of this thesis

4.8.1 Qualitative Data Analysis.

Data was captured by taking field notes, voice recordings, and reading responses from the questionnaires. The following techniques were used to analyse data: Conceptualisation, Coding & Categorising (CCC), and examining relationships. The data collected from interviews was transcribed, familiarised, introduced into themes, coded, elaborated, and interpreted. Once the data was categorised into conceptualised themes, codes were allocated and then categorised according to their similarities, thereby reducing the amount of data. Themes with no similarities were set aside as stand-alone. Central to the teachers' strategies envisaged, was to observe if teachers are making efforts in ascertaining that when they are advocating their teaching practices, they are guided by the contentions made by Ma (1999), as well as by Kilpatrick et al., (2001) discussed in section 2.4 of this thesis. My strategy was therefore informed by the common view held by the two aforementioned mathematics educationists that says, teachers of mathematics should have the specialised knowledge that includes connectedness, multiple perspectives, fundamental ideas, and longitudinal coherence.

4.8.2 Quantitative Data Analysis.

Statistical methods were used to analyse the learners' responses to the 2019 Grade 10 mathematics March common test. The marks obtained by each learner for each item were transcribed into Excel and then imported into SPSS. Counts of marks, averages and medians were then calculated for each of the items. These were used to generate graphs from Excel as well as SPSS to illustrate the broad trends in performance across both schools. A particular challenge was to find a way to represent the large rates of zeros and blanks in the items. Graphs were selected to depict this phenomenon.

4.8.3 Triangulation.

The data gathered from the teachers' interviews; learners' interviews; classroom lessons observations; learners' scripts analysis; and teachers' questionnaires; were analysed following two different approaches and further discussed to see if they correlated. These approaches were mainly, the qualitative approach; and the quantitative approach. The simultaneous use of the approaches to data analysis is called triangulation. There are three main advantages of using multi methods in research as outlined in Mundia (2017). Firstly, it incorporates the strengths of both qualitative and quantitative approaches. Cohen et al., (2018) are of similar idea as they suggest that, by using multiple methods one attempts to try to map out the richness and complexity human behaviour by studying it from different approaches. Mundia (2017) contends that triangulation provides a more comprehensive view of the problem or phenomena being studied. De Vos (2006) agrees and further advocate the use of multiple methods in that the phenomena which are investigated in the social sciences are so enmeshed, that a single approach can most certainly not succeed in encompassing human beings in their full complexity. Lastly, triangulation does not limit the data being collected. Again, Creswell (2008) brings forward that the concept of triangulation is based on the assumption that any bias inherent in a particular data source, investigator, and method would be neutralised when used in conjunction with other data sources, investigators, and methods

4.9 Validity.

To ensure the validity and credibility of the findings in this research, "the methods for collecting data will be triangulated in order to determine if there are any discrepancies in the findings" (Guba and Lincoln, 1988, p. 186; Maree, 2007, p. 297; Maree and van der Westhuizen, 2007, p. 33; Nieuwenhuis, 2007, p. 80). Cohen et al., (2018) views triangulation as, the use of two or more methods when collecting data collection for a study which deals with aspects of human behaviour. According to Creswell (2008), the concept of triangulation is based on the assumption that any bias inherent in a particular data source, investigator, and method would be neutralised when used in conjunction with other data sources, investigators, and methods. De Vos (2006) advocate the use of multiple methods in that the phenomenon investigated in the social sciences are so enmeshed that a single approach can most certainly not succeed in encompassing human beings in their full complexity.

The instruments that were used included classroom observations; learners' scripts analysis; learners' activity; learners semi-structured interviews; teacher questionnaires; and teacher semi-structured interviews. Cohen et al., (2018) are of the opinion that the use of triangulation should be used when we need a techniques are suitable when a complete, rounded view of educational outcomes, or when we study a complex phenomenon which needs to be unpacked. Furthermore, triangulation is useful when an established approach yields a limited and frequently distorted picture". Here, one approach was used to justify the other approach. Libarkin and Kurdziel (2002, p. 80) agree, in that the use of multiple data sets can inform the research, yielding insight, and methodological changes that improve the study and strengthen findings. Cohen et al., (2018) argue that the use of triangular techniques, can help reduce the limitation posed by using a single method and shows the value of combining qualitative and quantitative methods. It is particularly useful where a researcher is engaged in a case study (Cohen et al., 2018)

Furthermore, credibility was ensured through the prolonged engagement by myself at the site of inquiry. Guba and Lincoln (1989, p. 237) are of the idea that "substantial involvement at the site of the inquiry, in order to overcome the effects of misinformation, distortion...build the trust necessary to uncover constructions, and to facilitate immersing oneself in and understanding the context's culture". I satisfied this criterion of prolonged engagement through the use of multi- instruments at the research site.

During the research process, the study was continuously submitted to the study supervisor for cross-examination in order to confirm or clarify issues that arose. Confirmability of the findings were ensured by the use of verbatim accounts. Confirmability is to make it possible for the results to be confirmed or corroborated by other researchers (Lincoln & Guba, 1985). This means that data can be tracked to its source, and that the logic used to assemble the interpretations into structurally coherent and corroborating wholes is both explicit and implicit. Thus, both the "raw products" and the "processes used to compress them" are available to be inspected and confirmed by outside reviewers of the study (Guba & Lincoln, 1989, p. 243).

4.10 Ethical considerations.

This study was granted full approval by the Ethics Committee of the UKZN. The Ethical Clearance letter appears in Appendix A of this thesis. This study was granted permission to conduct research in the KwaZulu-Natal Department of Education Institutions, and the conditions of the approval appear in Appendix B of this thesis. Furthermore, letters requesting access to the research sites were sent to the following stakeholders: Ilembe District Director; CES Circuit Cluster Management of the participating schools; and the principals of the participating schools.

Separate letters of Informed Consent were sent to the following stakeholders: participating teachers (Appendix C); participating learners (Appendix D); as well as to parents of the participating learners (Appendix E), to observe ethical issues. Participants were informed that their participation was voluntary in that they could withdraw from participating in the study at any given point in time. Participants were also made aware that their confidentiality will be ensured by not making their participation will be a matter between myself and them, respectively. The use of informed consent in this research was aimed at sharing openly with participants, information about the research goals, the process and the outcomes. Furthermore, “informed consent ensures the full knowledge and cooperation of subjects, while also resolving, or at least relieving, any possible tension, aggression, resistance or insecurity of the subjects” (Strydom, 2006, pp. 57-60). In this regard, participants become part of the research, as they are to perceive its validity.

4.11 Limitations of the study.

My purpose was to conduct a study to explore the teaching and learning of factorisation of quadratic expressions in the Grade 10 mathematics classroom. I did this because I believe that such developments will dramatically impact on the mathematical landscape yet hold significant implications for the way in which algebra is utilised. The participants of this study are the teachers and learners of Grade 10 mathematics in one circuit in the Ilembe District.

There are however, various issues which mark the limitations of this research project. Firstly, the huge limitation is the use of assessment tasks for research purposes. Examination stress might have affected learners' responses which would not have been the case if they were responding to the research instrument. Even though the test items were designed for assessment purposes, I used them for research purposes as they provided a rich source of data. Secondly, I, as the researcher, have been a mathematics teacher in this circuit for the past seven years. The implications on the study of my double role (as teacher and researcher) could have unduly influenced participants of the study to say things they wanted me to hear, particularly in the interviews. Thirdly, part of the analysis in this study is quantitative; the study was focused only on the two schools (2%) out of 122 secondary schools that wrote the Grade 10 March 2019 mathematics common test, which makes it difficult to generalise the findings.

4.12 Conclusion.

The data gathered from the teachers' interviews; learners' interviews; classroom lessons observations; learners' scripts analysis; and teachers' questionnaires; with the aim of gaining a better understanding of the teaching and learning of the factoring of quadratic expression was consolidated. The findings from this exercise are dealt with, and discussed in detail in chapter 10 of this thesis.

CHAPTER 5

CLASSROOM LESSONS OBSERVATIONS

5.1 Introduction.

This chapter presents notes from classroom lesson observations. Six teachers from the two participating schools were observed in their classroom lesson presentations. Three teachers were from the R-School, while the other three were from the T-School. The lesson observation schedule appears in Appendix F of this thesis. All classroom lesson observations were conducted from the schools in which the participating teachers are teaching, and were video-recorded as per the signed teachers' consent agreement.

5.2 Case study one: Teacher Rakhulu (Teacher RT).

5.2.1 The Biography of Teacher Rakhulu.

Teacher Rakhulu is a qualified mathematics teacher with a three-year Teacher Diploma in Education, majoring in mathematics and life sciences. She furthered her studies to achieve an Honours degree in Education, majoring in mathematics for the FET phase. She has twenty-three years of experience in teaching mathematics at secondary school level, precisely at the R-School in this study, and at present teaches Grade 10 and Grade 12 mathematics. Currently, she teaches mathematics to the whole of Grade 10 which consists of two grade sections. When preparing her lessons, teacher Rakhulu sometimes writes her own notes for her learners when teaching Grade 10 algebraic quadratic expressions. Again, when she shares the section with other teachers, she emphasises the use of various textbooks as she feels that textbooks need to supplement each other. Since she is the Departmental Head (DH) at the school, the only opportunity she gets for her personal continuous professional development, is when she attends cluster meetings and workshops organised by the subject adviser. Teacher Rakhulu is able to track the mathematics curriculum coverage using various practices and documents. She uses the CAPS guidelines; the PILO Jika Imfundo trackers; and the KZN annual teaching plan (ATP), and her work is continuously quality-checked by the school's deputy principal. Considering the dynamics and the workload involved in teaching mathematics at Grade 10 level, she feels there is a high possibility of not completing all the topics in the mathematics curriculum. According to teacher Rakhulu, mathematics teachers at her school are supportive of one another in that all Grade 10 teachers do the same assessment for all Grade 10 mathematics classes. However, teamwork is compromised in that Grade 10 teachers do not

teach according to the same common teaching plan, hence she is solely responsible for what she is teaching in her class, and it is of note that she is displeased with her teaching pace of the Grade 10 mathematics curriculum. Speaking from her experience, she is of the opinion that algebraic expressions form the base for any mathematical problem in Grade 11 and 12, hence the sequencing of topics in Grade 10 mathematics makes sense, and is necessary.

She is the only teacher who was observed during the pilot stage in 2018. Even though she taught two sections of Grade 10 mathematics classes, we agreed to observe only one of her classes. Teacher Rakhulu agreed to be observed while delivering the lesson to her Grade 10 class. The date and the period was agreed to by both the participant and myself as being accessible and convenient. The observation of the classroom lesson lasted about an hour. This session was video-recorded as per agreement in the informed consent documentation.

5.2.2 Description of Teacher Rakhulu's Lessons.

5.2.2.1 Lesson one of Teacher Rakhulu: Factorisation of quadratic expressions.

I was introduced to teacher Rakhulu's class as a person interested in teaching. My aim was to observe the lesson on the factorisation of quadratic expressions; however, at the time of my visit, this section had already been taught and the lesson turned out to be more of a revision class. Teacher Rakhulu started her lesson by writing down the topic of the day, which is factorisation, and presented the example on the chalkboard $x^2 - y^2$ which learners were asked to factorise.

RT: What type of expression is $x^2 - y^2$?

Learner 1: Difference of two squares.

RT: How do we factorise the difference of two squares?

Learner 2: $(x + y)(x - y)$.

Together with the class, teacher Rakhulu led the chorus on how $(x + y)(x - y)$ is achieved, they said: *"We open the two sets of brackets; we look at the factors of the first term and the factors of the second term, then we have to use different signs which are negative and positive in separating the terms"*.

Another example, $2x^2 - 8y^2$, was given and she then explained that in factorising the given expression, the first step you had to do was look if there was anything common between the terms, then you could simply take out the highest common factor; then the factorisation.

RT: Is there any common factor in $2x^2 - 8y^2$?

Class in a chorus: We have 2 as the common factor.

RT: What do we need to do then?

Class in a chorus: We take out the common factor of 2 and get $2(x^2 - 4y^2)$.

RT: What do we do next?

Learner 3: We look for the factors of the difference of two squares in brackets and get

$2(x - 2y)(x + 2y)$.

RT: Now, I want us to look at the factorisation of the trinomial. What is the trinomial?

There was no response from the class.

Teacher Rakhulu then led the discussion by giving the definition of a trinomial. According to her, a trinomial is an expression with three terms where the highest exponent is two. “Now let us factorise a trinomial or a quadratic expression. We call it a trinomial because it has three terms and quadratic because it has the highest exponent of 2.

Let us look at an example of $x^2 + 7x + 12$. For us to factorise this expression, we have to open up two sets of brackets remember we did the factorisation on binomials? ... same as we did on binomials we look at the factors of the first term and factors of the last term ... so the key here is that you have to check the signs ... if the sign of the last term is positive, it means your signs of the brackets must be the same, either a positive sign or a negative sign. For this example, it means, we will have positive and positive because the middle term is positive. What are the factors of the first term?

Class in a chorus: x and x .

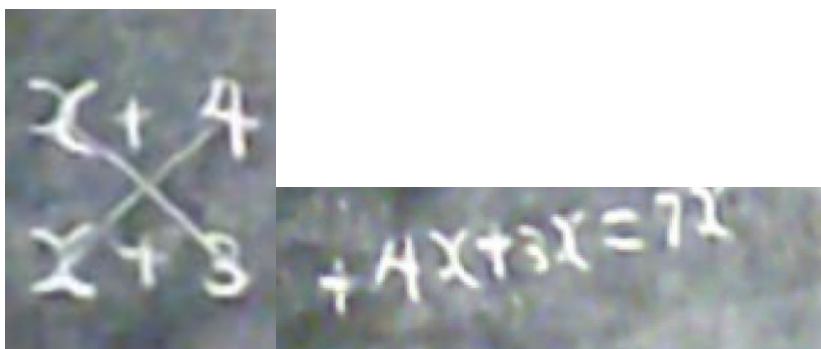
RT: What are the factors of the last term 12?

The class gave different responses simultaneously: some said 3 and 4; others said 6 and 2, and others said 12 and 1.

RT: Which set of factors can give us the middle term, when added?

Learner 1: 4 and 3.

RT: The reason why we take 4 *and* 3 is that they can give us 7 when added, and 7 is the coefficient of our middle term ... we said the sign has to be the same, and they will be positive and positive. We then cross multiply the factors of the first term with factors of the last term to test if their sum will give us $7x$, our middle term.



She wrote the answer as $(x + 4)(x + 3)$

Another example of $5x^2 + 35x + 50$ was given to the class.

RT: Do we have any common factor?

Class: Yes, 5.

RT: So, we need to take out 5 as the common factor. How can we do it?

She gave the class a few minutes to do the factorisation independently in their own workbooks before discussing it together with learners. Thereafter, together with the class, she went on to elaborate the procedure on what is expected to be done ...take out 5 as the common factor...open one set of brackets... $5x^2 \div 5 = x^2$; $35x \div 5 = 7x$; $50 \div 5 = 10$... hence we get $5(x^2 + 7x + 10)$...what do we do then?

Class: We then factorise $(x^2 + 7x + 10)$ and get $(x + 5)(x + 2)$ and finally the answer is $5(x + 5)(x + 2)$.

Another example of $4x^2 - 14x + 12$ was given to the class.

RT: Do we have the common factor here?

Class: 2.

RT: This means that we can rewrite it as $2(2x^2 - 7x + 6)$...then our signs will be both negative because we said the sign of the last term tells us the nature of the signs in the factors. Then we check the factors of the first term which happens to be $2x$ and x . What are factors of last term?

Class: 6 and 1, 2 and 3.

RT: Let us first take 6 and 1, and cross-multiply with factors of $2x^2$, and add their products, this will be $x \times 6 = 6x$ and $2x \times 1 = 2x$, then $6x + 2x = 8x$, do we get a middle term?

Class: No.

RT: This means that we have to change the factors ... how do we change the factors?

Class: 3 and 2.

RT: Let us cross-multiply our new factors: $2x \times -2 = -4x$ and $x \times -3 = -3x$, then we have to add them $-4x - 3x = -7x$ which is our middle term ... so what are our factors?

Class: $2(x - 2)(2x - 3)$.

Another example of $x^2 + x - 12$ was also given.

RT: Do we have a common factor here in $x^2 + x - 12$?

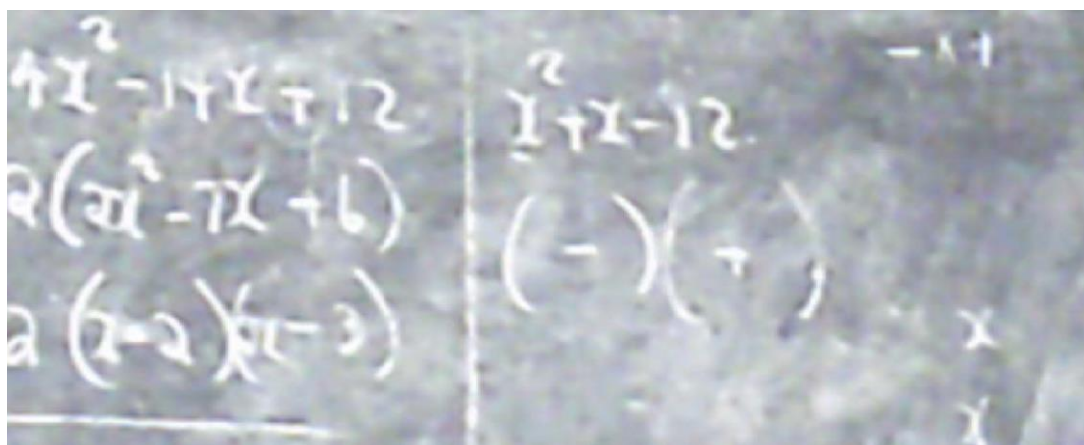
Class: No.

RT: For us to find the correct signs of our factors, we have to check the sign of the last term ... our last term is negative which means we have to have negative and positive as our signs. We only get a minus when we multiply a negative with a positive, and this means that for factors of this expression we must have a negative and a positive sign. The factors of x^2 is x and x then, what are the factors of 12?

Class: 6 and 1, 2 and 3.

She further clarified the solution to the problem by following the cross method which advocates the trial and error method and led the students to the solution $(x - 3)(x + 4)$.

Another example $4x^2 - 14x - 10$ was also given to the class. Again, she followed the use of trial and error though the cross method and reached the solution $2(x - 3)(2x - 3)$.



Classwork was then given out, and was titled “Fully factorise each of the following expressions.”

1. $4k^2 - 37k + 9$
2. $3x^2 - 3x - 18$
3. $4x^2 + 11x + 6$
4. $6m^2 - 23m - 4$
5. $4a^2 - 14ab + 12b^2$

Two learners were afforded the opportunity to present their solutions on the chalkboard. Both learners showed a full understanding of the use of the cross method. Learner 1 was able to factorise $4k^2 - 37k + 9$ to get $(4k - 1)(k - 9)$. In her explanation it appeared that she followed the trial and error approach in that she mentioned the sets of factors of $4k^2$ as $4k$ and k ; $2k$ and $2k$ as well as the sets of factors of 9 as 9 and 1; 3 and 3.

$$4k^2 - 37k + 9$$

$$(4k-1)(1k-9)$$

$$\begin{array}{r} 4k-1 \\ 1k-9 \\ \hline 4k^2 - 36k - 1k + 9 \\ 4k^2 - 37k + 9 \end{array}$$

Learner 2 was also able to factorise $4x^2 + 11x + 6$ and achieved $(4x + 3)(x + 2)$. Same as learner1, learner 2 also followed the trial and error approach as he mentioned in his explanation the sets of factors of factors of $4x^2$ as $4x$ and x ; $2x$ and $2x$ as well as the sets of factors of 6 as 6 and 1; 2 and 3.

$$3. 4x^2 + 11x + 6$$

$$(4x+3)(x+2)$$

$$\begin{array}{r} 4x+3 \\ x+2 \\ \hline 4x^2 + 8x + 3x + 6 \\ 4x^2 + 11x + 6 \end{array}$$

The lesson was brought to an end and the rest of the classwork exercises were converted to homework.

5.2.2.2 Lesson two with Teacher Rakhulu: Factorisation of quadratic expressions –the grouping method.

The second visit to teacher Rakhulu's class took place the following day. The teaching period was 30 minutes. Learners were afforded the opportunity to work on corrections of the work that had been given as homework. After ascertaining that all corrections were correctly done, learners were prepared for the new lesson of the day. On this day, teacher Rakhulu decided to demonstrate to her learners the method of grouping as the alternate method to follow when

factorising quadratics. She spotted some of the expressions and began to demonstrate to learners how the grouping method can be followed. She began with the example $x^2 + x - 12$.

Teacher Rakhulu elaborated on the following steps:

Step 1: You have to consider any quadratic expression as $ax^2 + bx + c$, which is the general form. Meaning, for our expression $x^2 + x - 12$, $a = 1$; $b = 1$; $c = 12$...

Step 2: Multiply a and c which is $1 \times 12 = 12$ and find the factors of 12 that will give coefficient of the middle term 1...

She explored all the sets of factors of 12 which are 12 and 1; 6 and 2; 4 and 3 and came to conclusion that the correct factors are +4 and - 3, hence the middle term can be rewritten as $+4x - 3x$.

Step 3: Rewrite the $x^2 + x - 12$ as a $x^2 + 4x - 3x - 12$.

Step 4: Begin grouping $x^2 + 4x - 3x - 12$ to be $(x^2 + 4x)(-3x - 12)$ and factorise each set to get $x(x + 4) - 3(x + 4)$.

Step 5: Take out the common factor and get $(x + 4)(x - 3)$ as the final answer.

RT: Can you compare the factors from what we achieved yesterday. Are they the same?

Class: Yes.

Teacher Rakhulu presented another example of $4k^2 - 37k + 9$ and led the class to its factorisation.

RT: Step 1. Remember that the general quadratic expression is $ax^2 + bx + c$, and in this case our $a = 4$; $b = -37$; $c = 9$...

Step 2: Multiply a and c which is $4 \times 9 = 36$ and find the factors of 36 that will give coefficient of the middle term -37...

She explored all the sets of factors of 36 which are 36 and 1; 18 and 2; 12 and 3; 9 and 4 and came to the conclusion that the correct factors are -36 and - 1, hence the middle term can be rewritten as $-36k - k$.

Step 3: Rewrite the $4k^2 - 37k + 9$ as a $4k^2 - 36k - 1k + 9$.

Step4: Begin grouping $4k^2 - 36k - 1k + 9$ to be $(4k^2 - 36k)(-1k + 9)$ and factorise each set to get $4k(k - 9) - 1(k - 9)$.

Step 5: Take out the common factor and get $(k - 9)(4k - 1)$ as the final answer.

RT: Is this answer the same as what we achieved yesterday?

Class: Yes.

RT: Let us do the last one. Here is the example $4x^2 + 11x + 6$.

RT: Step 1. The general quadratic expression is given as $ax^2 + bx + c$, and hence in this case our $a = 4$; $b = 11$; $c = 6$...

Step 2: Multiply a and c which is $4 \times 6 = 24$ and find the factors of 24 that will give coefficient of the middle term 11...

She explored all the sets of factors of 24 which are 24 and 1; 12 and 2; 8 and 3; 6 and 4 and came to conclusion that the correct factors are 8 and 3, hence the middle term can be rewritten as $8x + 3x$.

Step 3: Rewrite the expression $4x^2 + 11x + 6$ as a $4x^2 + 8x + 3x + 6$.

Step 4: Begin the grouping $4x^2 + 8x + 3x + 6$ to be $(4x^2 + 8x)(+3x + 6)$ and factorise each set to get $4x(x + 2) + 3(x + 2)$.

Step 5: Take out the common factor and get $(x + 2)(4x + 3)$ as the final answer.

RT: Is this answer the same as the one achieved yesterday?

Class: Yes.

Learners were then asked to try the method of factorisation by grouping at home for the following expressions:

1. $6m^2 - 23m - 4$

2. $4a^2 - 14ab + 12b^2$

The lesson was then brought to an end. It was good to note that teacher Rakhulu has exposed learners to multi-methods of factoring the quadratic expressions.

5.3 Case study two: Teacher Ndulini (Teacher ND).

5.3.1 The Biography of Teacher Ndulini.

Teacher Ndulini is a qualified mathematics teacher with a Bachelor of Education degree majoring in mathematics and physical sciences for the FET phase. She has three years of mathematics teaching experience at secondary school level, precisely the R-School in this study, and presently teaches mathematics at Grade 10 level as well as physical sciences at Grade 12 level. She currently shares the Grade 10 mathematics with one other teacher, with each teacher teaching a separate grade section. When she prepares her lessons, teacher Ndulini sometimes writes her own notes for her learners when teaching Grade 10 algebraic quadratic expressions. She recommends the use of different textbooks which, she claims, expose teachers and learners to different methods of solving problems. Teacher Ndulini has never been exposed to any form of continuous professional development particularly on the topic of algebraic quadratic expressions. She makes use of different practices and documents to track the mathematics curriculum coverage. She uses CAPS guidelines; the KZN annual teaching plan (ATP), and her work is checked by the departmental head (DH) of the school to maintain quality work. She sometimes uses the PILO Jika Imfundo trackers on her classwork. Considering the dynamics and the workload involved in teaching mathematics at Grade 10 level, she feels there is a small chance of completing all the topics in the Grade 10 mathematics curriculum. According to teacher Ndulini, mathematics teachers at her school work as a team supporting one another in that all Grade 10 teachers do the same assessment for all Grade 10 mathematics classes. It must be noted that teamwork is compromised in that Grade 10 teachers do not teach according to the same common teaching plan; hence she is solely responsible for what she teaches in her class. Teacher Ndulini is not happy with the pace of her teaching the Grade 10 mathematics curriculum. Teacher Ndulini believes all Grade 10 mathematics content on algebraic expressions needs to be covered in order to provide learners with the basic knowledge in preparation for Grade 11 and Grade 12 mathematics.

Teacher Ndulini agreed to be observed while delivering a lesson to her Grade 10 class, and the date and period was agreed to by both the participant and myself as being accessible and convenient. The classroom lesson observation lasted for about an hour. This session was video recorded as per agreement in the informed consent documentation.

5.3.2 Description of Teacher Ndulini's Lessons.

5.3.2.1 Lesson one of Teacher Ndulini: Factorising using common factor.

This is the first lesson observed for the main study in the year 2019. As a researcher, I was introduced to the class and the class was prepared for the lesson. Teacher Ndulini began the lesson by giving learners a chance to correct the homework given in their previous lesson. Together with the learners teacher Ndulini checked if the corrections were correct and learners were required to write the corrections in their exercise books. Teacher Ndulini then introduced the topic of factorisation of quadratic expression by removing the common factor. She presented the example $6x - 10y$ for the class work on.

ND: What is a common factor?

Learner: Common factor is the number that can be found as a multiple of a number or a term.

She then led the learners by saying that a common factor is a number that is found in both or all terms of expression.

ND: Is there any number that can be a multiple of both $6x$ and $-10y$?

Class: Yes ... 2.

She then emphasised that there might be many common factors but one must always take out the highest common factor, because if any common factor was used, one would discover another common factor in the solution, which would mean that the first common factor was not the highest, and as a result, the problem would remain unsimplified.

ND: What is the highest common factor in this case?

Class: 2.

ND: What is the value of $\frac{6x}{2}$?

Class: $3x$.

ND: What is the value of $\frac{10y}{2}$?

Class: $5y$.

She eventually arrived at the solution of $2(3x - 5y)$. Teacher Ndulini then emphasised to learners that the answer could be checked by removing the brackets.

Another example was given to the class $3a^2 - 9a + 12ab$.

ND: What is the highest common factor among the coefficients of the terms?

Class: 3.

ND: What is the highest common factor in variables?

Class: a .

Teacher Ndulini, together with learners, concluded that $3a$ was a common factor for this expression.

ND: Why don't we take b as our common factor?

Learner: Because it does not exist in other terms.

ND: Okay, let us come to our division part ... $\frac{3a^2}{3a}$ which will be equal to a , because in the numerator we have $a \times a$ and in the denominator we have one a , one a of the two a in the numerator will cancel the bottom a and we will achieve $\frac{3a^2}{3a} = a$. What is the value of $\frac{9a}{3a}$?

Class: 3.

ND: What is the value of $\frac{12ab}{3a}$?

Class: $4b$.

ND: Then the answer will be $3a(a - 3 + 4b)$.

Together with learners, they checked the solution using the product rule which proved them right.

Another example was given: $3a(p - 2q) - 2b(p - 2q)$.

ND: Do we have common factor here $3a(p - 2q) - 2b(p - 2q)$?

Class: Yes, $(p - 2q)$.

ND: Let us use the procedure we used in the previous example we say in $\frac{3a(p-2q)}{(p-2q)}$ the denominator $(p - 2q)$ will cancel the numerator $(p - 2q)$, and then we remain with $3a$... and also say in $\frac{-2b(p-2q)}{(p-2q)}$ we get $-2b$... finally we get the answer $(p - 2q)(3a - 2b)$.

Learners were then given homework.

1. $4a + 8b$
2. $p^2q^2 - pq^2$
3. $3x^2 - 6x^3 + 9x^4$
4. $3a(p - 3q) - 2b(p - 3q)$
5. $2p(a - b) - (a - b)$

5.3.2.2 Lesson two with Teacher Ndulini: Factorising Quadratic Expression.

Teacher Ndulini began the lesson by giving learners a chance to write corrections on the board. In factorising $4a + 8b$ one learner got $2(2a + 4b)$. Together with learners, teacher Ndulini noticed that the learner had given a partially correct answer and went to assist the learner. According to teacher Ndulini, in the brackets of the answer that was given by the learner, there was still another common factor which made the answer of the learner partially correct.

ND: The correct answer is 4 as it is the highest common factor. After removing four there are no possible factors. So $4a + 8b$ becomes $4(a + 2b)$.

Other examples were done correctly on the board, which showed that learners understood the concept that was taught on factorisation of algebraic expressions. She then wrote down the topic of the day.

ND: What is a trinomial?

Class: It is an expression with 3 terms.

ND: What is a quadratic?

Learner: It is the expression or equation with two possible answers.

She presented the example $x^2 + 5x + 6$ for the class to factorise.

ND: Okay, if we were to factorise this expression, we can factorise it in one of two ways.... please note the coefficient of x^2 what is the coefficient of x^2 in $x^2 + 5x + 6$?

Class: 1.

ND: We will do both ways, since we have coefficient of 1. What are the factors that give x^2 when multiplied?

Class: x and x .

She opened two sets of brackets without giving an explanation about the set of brackets.

ND: What two numbers can be multiplied to get 6 ... the last term?

Class: 2 and 3.

ND: Notice that we only talked about first and last term, we didn't talk about the middle term ... how are the signs separating terms in brackets going to be like?

Class: Positive and positive.

ND: Why both positive?

Learner: A positive 3 and positive 3 when added they give us positive 5 which is a middle term.

ND: Basically our factors are $(x + 3)(x + 2)$... so in order to find the middle term we looked at the sign of the last terms in the brackets.

On another approach to factorising $x^2 + 5x + 6$ we will open two sets of brackets... What is the coefficient of x^2 ?

Class: 1.

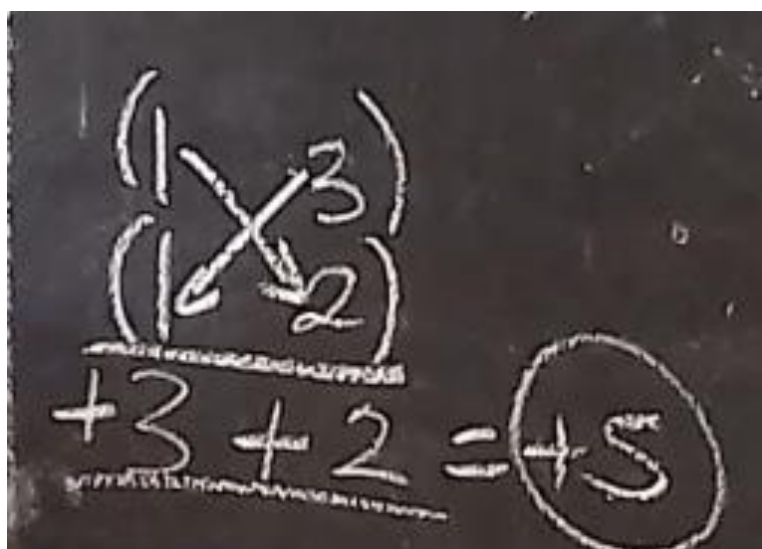
ND: Which are the two numbers that give 1 when multiplied?

Class: 1 and 1.

ND: Which are the two numbers that give 6 when multiplied?

Class: 2 and 3.

ND: After finding the factors we now cross multiply a factor of last term with a factor of the number that was a coefficient of x ; we find the sum of their product if they give us a middle term, we then take them as arranged. First set as first bracket and last set as last bracket,


$$\begin{array}{l} (1 \times 3) \\ (1 \times 2) \\ \hline +3 + 2 = +5 \end{array}$$

Hence the answer is $(x + 3)(x + 2)$.

ND: Did you understand what we did in the first approach and in second approach?

Class: Yes.

ND: Let us go to the example $x^2 + 5x - 6$ that has different signs and see how to approach it. Let us first approach this one. What are the two numbers that we multiply to give the product of x^2 ?

Class: x and x .

ND: Two numbers that we multiply to get 6?

Class: 2 and 3.

ND: We need the sum of 5 from 3 and 2, so what is the nature of their sign?

Class: Negative and positive.

ND: $(x - 3)(x + 2)$, we have to check if our factors -3 and 2, if they are added they give -1 which is not our middle term. Note that when doing the factors, check if they give you a product

of the original expression with a correct middle term. Then we can say that 3 and 2 are not the factors of this expression. What do we put there as our second terms of the brackets?

Class: 6 and 1.

ND: What should our signs be for us to get a positive 5?

Class: Positive 6 and negative 1.

She further factorised the expression to get an answer of $(x + 6)(x - 1)$ as the final answer, and together with the class agreed that this solution was correct.

ND: Let us use the second method and please note that we are factorising using this method, we look at the coefficients not variables. Which two numbers can be multiplied to give 1? Which two numbers can be multiplied to give 6?

Class: 1 and 1, 6 and 1, 3 and 2.

ND: One factor of 1 is multiplied with a factor of 6 ... we do the same thing to the two other remaining factors ... after multiplying them, their products need to be added so that they give a middle term. Note that the first sign goes with the first bracket and second sign goes with the second bracket.

Handwritten mathematical work on a chalkboard showing the factorization of the quadratic expression $x^2 + 5x - 6$. The work includes the expression $x^2 + 5x - 6$, its factorization $(x+6)(x-1)$, and a diagram illustrating the cross-multiplication method. The diagram shows two columns of numbers: 1 and 6 in the first column, and 3 and 2 in the second column. Arrows indicate the multiplication of 1 by 6 and 3 by 2, with the results 6 and 6 being added to get 12. The final result is $(x+6)(x-1)$.

She further explained that even though 2 and 3 are factors, when they are multiplied with factors of 1, the sum of their products need to give the middle term ... however, they are not the factors since the sum of their product does not give 5. Miss Ndulini further emphasised the importance of checking as a way of ensuring that the solution is correct.

Another example $6p^2 + 24p - 30$ was given.

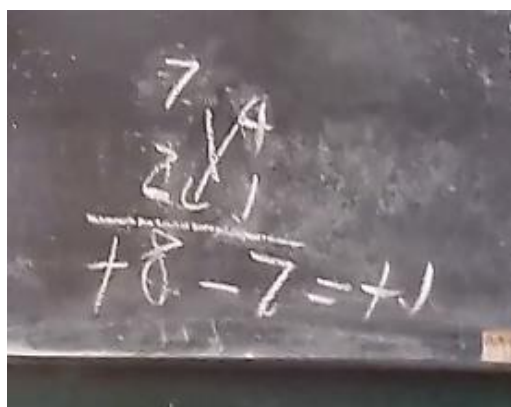
ND: Once the coefficient of p^2 is not equal to 1, the first thing you have to do is check if that number is not a common factor of the whole expression. If it is a common factor, then we take it out, hence $6(p^2 + 4p - 5)$, and now we can simply factorise the quadratic inside brackets, and get $6(p + 5)(p - 1)$.

Another example of $14a^2 + a - 4$ was also given. She led learners by emphasising that the first method only worked if the coefficient of x^2 is one, while the second method work well every time. She explained to learners that in this example they would use the second method to find the solution.

ND: Is there any common factor in $14a^2 + a - 4$?

Class: No.

ND: We write down possible factors, and we use the trial and error method to find the pair that can give us the middle term when added, which is positive 1... we have to play around with these factors to get the correct pair by using the second method.



After using the trial and error approach, teacher Ndulini was able to get the correct factors of $14a^2 + a - 4$ as $(7a + 4)(2a - 1)$.

She then gave learners a short verbal summary and wrote the classwork exercises on the chalkboard. While learners were busy copying down the classwork, she went desk to desk to check if they had been able to do the class activities successfully.

Classwork:

1. $a^2 + 7a + 12$
2. $y^2 - 6y + 8$
3. $2a^2 - 24a + 70$
4. $3x^2 + 20x - 32$

5.4 Case study three: Teacher Hlophe (Teacher HP).

5.4.1 The Biography of Teacher Hlophe.

Teacher Hlophe is a qualified mathematics teacher with a Bachelor of Education degree majoring in mathematics and physical sciences for the FET phase. He has five years teaching experience in mathematics at secondary school level, precisely the R-School featured in this study. Currently, he teaches Grade 10 and Grade 12 mathematics. He shares the Grade 10 teaching load with one other teacher and they are each responsible for a separate grade section. When preparing for his lessons for Grade 10 algebraic quadratic expressions, teacher Hlophe sometimes writes his own notes for his learners. He strongly recommends the use of different textbooks when teaching mathematics and says it exposes the teacher and learners to different methods of problem-solving as well as different problems. Teacher Hlophe has never been exposed to any form of continuous professional development, particularly on the topic of algebraic quadratic expressions. He is able to track the mathematics curriculum coverage through the use of various practices and documents. He makes use of the CAPS guidelines; the KZN annual teaching plan (ATP), and his work is continuously quality-checked by the school's departmental head (DH). He seldom uses the PILO Jika Imfundo trackers for his classwork. Considering the dynamics and the workload of teaching Grade 10 mathematics, he feels it is impossible to complete all the topics in the curriculum. According to teacher Hlophe, mathematics teachers at his school are supportive of one another in that all Grade 10 teachers do the same assessment for all Grade 10 mathematics classes. However, the teamwork is compromised in that Grade 10 teachers do not teach according to the same common teaching plan. He is therefore solely responsible for what he teaches in his class and it is interesting to note that he is unhappy with the pace of his teaching the Grade 10 mathematics curriculum. Speaking from his experience, he is of the opinion that the Grade 10 mathematics curriculum

can be improved by ensuring the use of practical activities throughout the term. Teacher Hlophe strongly believes that all content on algebraic expressions need to be covered in term one in order to consolidate the knowledge base of learners.

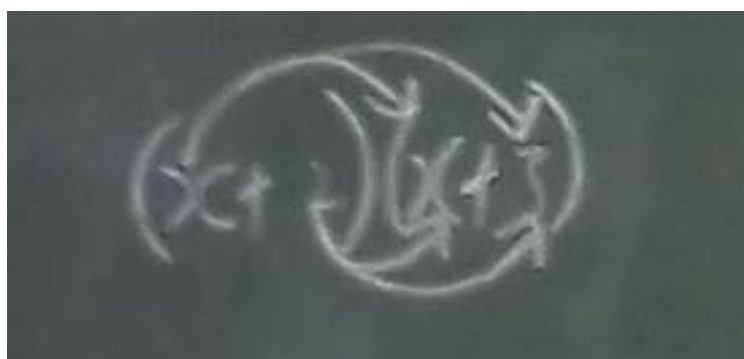
Teacher Hlophe agreed to be observed while delivering lessons to his Grade 10 class. The date and the period was agreed to both by the participant and myself as being accessible and convenient. The classroom lesson observations lasted an hour each. These sessions were video recorded as per agreement in the informed consent documentation.

5.4.2 Description of Teacher Hlophe's Lessons.

5.4.2.1 Lesson one of Teacher Hlophe: Product of binomial and binomials.

In teacher Hlophe's first lesson, I was introduced to the class while learners were being prepped for the lesson. Teacher Hlophe began by writing the title of the topic on the board alerting learners to the content that would be covered. He presented an example on the chalkboard $(x + 2)(x + 3)$ for the class to simplify. His aim was to reinforce the notion that the reverse of the multiplication of terms, is the factorisation.

Teacher Hlophe explained that products are answers obtained from multiplication. He went on to say that in multiplying the binomials each set in the bracket multiplies the set on the other side in brackets, the answer of which is called a product. Teacher Hlophe emphasised that the answer is called a product because it comes from multiplication. He told learners that the first term of the first bracket multiplies each term of the second bracket, and the second term of the first bracket multiplies each term of the second bracket.



HP: Now we are going to say x multiplied by x and what do we get?

Class: x^2

HP: What is the value of $x \times 3$?

Class: $3x$.

HP: We get a positive $3x$ because x and 3 are positive. Are we done with the x of the first bracket?

Class: Yes.

HP: Now, what is value of $2 \times x$?

Class: $2x$.

HP: What is the value of 2×3 ?

Class: 6 .

Teacher Hlophe told learners that it was important that during the multiplication of terms, learners should make sure they also multiplied the signs of the terms.

HP: Now, this is the product $x^2 + 3x + 2x + 6$ but we can still simplify our product by collecting like terms. So if we look at x^2 , does x^2 have any like term or something that looks similar to it?

Class: No.

HP: Therefore, we just have to write it as x^2 ... now we look at $3x$... does $3x$ have any like term?

Learner: Yes, $2x$.

HP: Then that means we can simply add the like terms to get $3x + 2x = 5x$...now we are only left with 6 which does not have any like term so our solution will be $x^2 + 5x + 6$

$$\begin{array}{l} (x+2)(x+3) \\ x^2 + 3x + 2x + 6 \\ \hline x^2 + 5x + 6 \end{array}$$

He then gave a second example of $(3a - 7)(2a + 1)$ for learners to work through. Teacher Hlophe together with learners engaged in the discussion.

HP: What is the value of $3a \times 2a$?

Class: $6a^2$.

HP: $3a \times 1$?

Class: Positive $3a$.

HP: $-7 \times 2a$?

Class: $-14a$.

HP: -7×1 ?

Class: -7 .

HP: Now we can start collecting the like terms.... does $6a^2$ have any like terms?

Class: No.

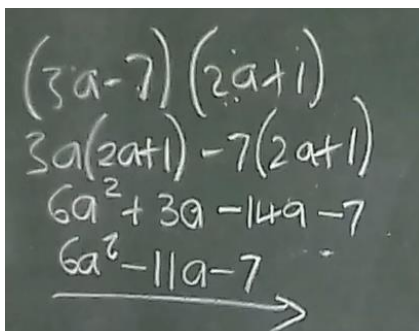
HP: So it will remain $6a^2$ does $3a$, have a like term?

Class: Yes, $-14a$...

HP: What is the value of $3a - 14a$?

Class: $-11a$.

HP: Now this is left as $6a^2 - 11a - 7$. This is how we multiply the binomials.



The image shows a chalkboard with the following handwritten work:

$$\begin{aligned} &(3a-7)(2a+1) \\ &3a(2a+1) - 7(2a+1) \\ &6a^2 + 3a - 14a - 7 \\ &6a^2 - 11a - 7 \end{aligned}$$

An arrow points to the right from the final result.

Teacher Hlophe then wrote the heading “squaring binominals” on the chalkboard to indicate he was about to start doing examples based on this subtopic. He presented the example $(x - 5)^2$ for the class to work through.

HP: What does $(x - 5)^2$ mean?

Class: No answer.

HP: This means we have two brackets of $(x - 5)$ which can be rewritten as $(x - 5)(x - 5)$. I hope everyone can see that this looks the same as the previous example. Now we apply the same procedure.... x multiplied by x ?

Class: x^2 .

HP: What is the value of $x \times -5$?

Class: $-5x$.

HP: $-5 \times x$?

Class: $-5x$.

H: What is -5×-5 ?

Class: 25.

HP: Finally we have $x^2 - 5x - 5x + 25$... now we can collect the like terms does x^2 have like terms?

Class: No.

HP: So, does $-5x$ have like terms?

Class: Yes, it is another $-5x$.

HP: What is the value of $-5x - 5x$?

Class: $-10x$.

HP: Does 25 have like terms?

Class: No.

HP: Finally we get $x^2 - 10x + 25$.

Handwritten algebraic expansion of $(x-5)^2$ on a chalkboard:

$$\begin{aligned} &(x-5)^2 \\ &(x-5)(x-5) \\ &x^2 - 5x - 5x + 25 \\ &x^2 - 10x + 25 \end{aligned}$$

HP: Let us do the last example together, by simplifying $(4p - 3q)^2$... so we've got $4p - 3q$...how many of them do we have?

Class: 2.

HP: So this means it is $(4p - 3q)(4p - 3q)$, so it is going to be the multiplication of the first bracket terms to second bracket terms and we have $16p^2 - 12pq - 12pq + 9q^2$, note that when writing a term with more than one variable, make sure you arrange the alphabets in an ascending order. Together with learners, with teacher Hlophe leading the discussion, they arrived at the solution of $16p^2 - 24pq + 9q^2$ classwork was then given to the class.

Classwork:

1. Find the product

1.1 $(a - 2)(a + 3)$

1.2 $(3a - 7)(2a + 1)$

1.3 $(3a + 2b)(2a + 11b)$

2. Squaring binomials

2.1 $(x + 3)^2$

2.2 $(2x - 3y)^2$

Teacher Hlophe checked how learners managed the classwork and noticed that some struggled to identify the like terms. He went to the chalkboard and asked questions that probed learners understanding on the identification of like terms. He wrote x on the chalkboard and began to ask questions.

HP: Is there any number that looks like x ?

Class: No.

HP: Therefore we take it as it is now let us consider an example of $3x$, ... what do call 3 in this term $+3x$?

Learner: It's a coefficient of x .

HP: And what do we call x ?

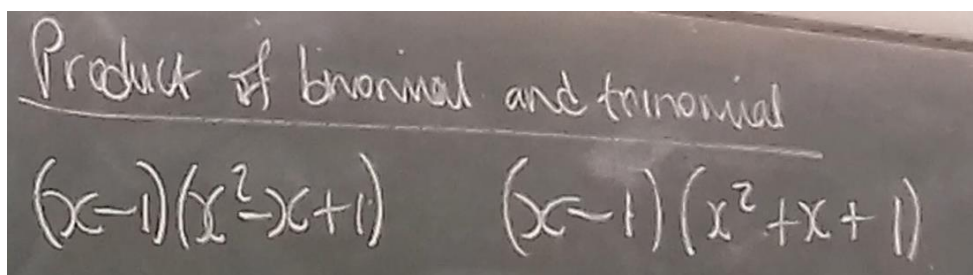
Class: Variable.

HP: When we have $+3x + 2x$, we add or subtract like terms; we do not look at a variable ...we look at their coefficient variables are there for us to notice the like terms. Hence we add $3x + 2x$ and get $+5x$. Let us go to this one, $6a^2 + 3a - 14a - 7$, we said like terms were $3a - 14a$ and they both have variables, we subtract the coefficients which then gives us $-11a$. When adding or subtracting numbers, note that the answer will take the sign of the bigger number. If you have the same signs, you just write the sign down and add the number like in $-5x - 5x$ to give $-10x$.

Teacher Hlophe then summarised the session by emphasising the important points to consider when working with the product of binomials. Due to the limited time, the uncompleted sums were converted to homework.

5.4.2.2 Lesson two of Teacher Hlophe: Products of binomials and trinomials.

On the second visit, teacher Hlophe did corrections for the work that was given the day before. After observing learners performing diligently, he proceeded to introduce the new topic. He wrote the title on the chalkboard: "Products of binomial and trinomial", and presented the class with the following example, $(x - 1)(x^2 - x + 1)$. Mr Hlophe began leading the discussion by providing the explanation on how to find the product of a binomial and trinomial.



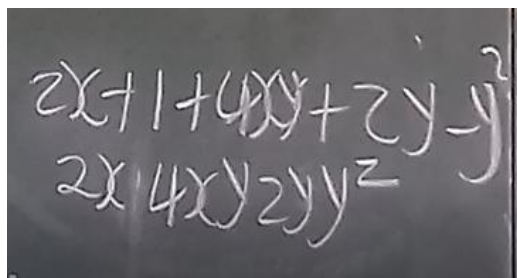
HP: Which ones in these brackets are binomials and which ones are trinomials?

Learner: First bracket is binomial and then second bracket is trinomial

HP: How many terms do we have here? $2x + 1 + 4xy + 2y - y^2$?

Class: 5.

HP: How many terms do we have here? $2x + 4xy + 2y - y^2$?



Class: 1.

HP: If we do not have plus or minus in the expression, it is an indication that it is one term. In this $(x - 1)(x^2 - x + 1)$ case we will do it the same way as we did when we found the products of binomials in the previous lesson. Now we have to multiply the first term of the first bracket with all the terms found in second bracket respectively, and do the same thing for the second term of the first bracket and reach $x^3 - x^2 + x - x^2 + x - 1 \dots$ now we have to collect the like terms.

HP: What are our like terms in this product $x^3 - x^2 + x - x^2 + x - 1$?

Learner: $-x^2$ and $-x^2$, x and x .

HP: Then we have to add or subtract the like terms, which means in this case, we will end up with, $x^3 - 2x^2 + 2x - 1$ as our answer.

$$(x-1)(x^2+x+1)$$

$$x^3 - x^2 + x - x^2 + x - 1$$

$$x^3 - 2x^2 + 2x - 1 \quad \checkmark$$

Example 2: $(4p^2 + 14p + 49)(2p - 7)$.

HP: In this problem, we will also use the same procedure even though we can see our first bracket is a trinomial. If you listened well you would remember that the first term of the first bracket must always multiply all terms of the second bracket ... and here we have two terms in the second bracket, which is $2p$ and -7 . Then we get

$$8p^3 - 28p^2 + 28p^2 - 98p + 98p - 343.$$

HP: Do we have any like terms?

Class: Yes, $-28p^2$ and $+28p^2$, $-98p$ and $+98p$.

HP: You can see that the signs of the identified pairs of like terms are different respectively... so that means they will cancel each other, and we end up with $8p^3 - 343$ as our product.

$$(4p^2 + 14p + 49)(2p - 7)$$

$$8p^3 - 28p^2 + 28p^2 - 98p + 98p - 343$$

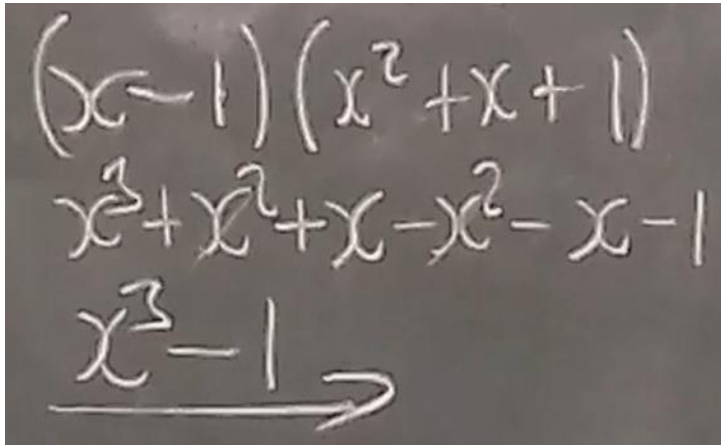
$$8p^3 - 343 \quad \checkmark$$

Example 3: $(x - 1)(x^2 + x + 1)$.

HP: What do we need to do here?

Learner: We will multiply all terms in the second bracket with x from the first bracket, and multiply all the terms of the second bracket with -1 from first bracket.

Teacher Hlophe wrote on the chalkboard what he was told by a learner, and got $x^3 + x^2 + x - x^2 - x - 1$. Together with the learners they worked out the like terms and ended up with $x^3 - 1$.



The image shows a chalkboard with handwritten algebraic work. The first line shows the product of two binomials: $(x-1)(x^2+x+1)$. The second line shows the result of distributing: $x^3 + x^2 + x - x^2 - x - 1$. The third line shows the simplified result after combining like terms: $x^3 - 1$, with an arrow pointing to the right.

HP: If I can be specific with you on what happened here in the last two examples, we had products, which resulted in the expressions called the difference of two cubes.

HP: What do we mean by the difference?

Class: No clear answer.

HP: Difference is an answer that we get when we subtract items from other items. You will understand better when we start factorisation. Why do we call the last two examples cubes? Do take note of such things; it is very important going forward.

Classwork was given to learners to solve to demonstrate their understanding of this concept.

Classwork: Simplify the following

1. $(5a^2 - 2b)(2a^2 + 4ab - 7b^2)$
2. $(2x - 5y)(3x^2 - 2xy + 7y^2)$

Learners were encouraged to do the activity individually and the teacher went desk-to-desk to check the work of each learner. All learners were able to complete the task in the given period, and were given a chance to write the corrections on the chalkboard.

HP: What type of expression is in the first bracket?

Class: Binomial.

HP: This means; we are multiplying a binomial with a trinomial. What type of expression is this $(5a^2 - 2b)(2a^2 + 4ab - 7b^2)$?

Class: A product of a binomial and a trinomial.

HP: We got this term $(5a^2 - 2b)$ and we multiplied it with this $(2a^2 + 4ab - 7b^2)$ because of the brackets. How many terms do we have here?

Class: No answer.

HP: Let's say we have $(2)(x)$, how many terms do I have?

Class: Two.

HP: What is it that is separating the two items?

Class: Brackets.

HP: In between these two terms $(5a^2 - 2b)(2a^2 + 4ab - 7b^2)$, do we have a plus or minus?

Class: No.

Mr Hlophe led the discussion until the outcome was reached $10a^4 + 20a^3b - 35a^2b^2 - 4a^2b - 8ab^2 + 14b^3$... the rest of the sums were converted to homework and the lesson was brought to an end.

5.5 Case study four: Teacher Mzimela (Teacher MZ).

5.5.1 The Biography of Teacher Mzimela.

Teacher Mzimela is a qualified mathematics teacher with a Bachelor of Education degree majoring in mathematics and physical sciences for the FET phase. He has eleven years of experience in teaching mathematics at secondary school level at the T-school of this study. Presently, he teaches Grade 10 and Grade 12 mathematics. He shares the Grade 10 teaching

with two other teachers, with each teacher teaching their own grade section. When preparing for his lessons teaching Grade 10 algebraic quadratic expressions, teacher Mzimela sometimes writes his own notes for his learners. He emphasised that it is important to use different textbooks as he felt that textbooks offer different approaches that can be taught to learners. Teacher Mzimela has received continual professional development on a number of occasions. These occasions include school mathematics meetings with the Departmental Head (DH); cluster meetings; workshops organised by the subject adviser; and mentoring visits from a colleague or the SMT. Teacher Mzimela is able to track using various practices and documents. He uses the CAPS guidelines; the KZN annual teaching plan (ATP), and his work is continuously quality-checked by the Departmental Head (DH) of the school. Teacher Mzimela does not use the PILO Jika Imfundo trackers in his class.

Considering the dynamics and the workload involved in teaching mathematics at Grade 10 level, he feels it is impossible to complete all the topics in the Grade 10 mathematics curriculum. According to teacher Mzimela, mathematics teachers at his school are supportive of one another in that the entire Grade 10 teachers do the same assessments for all Grade 10 mathematics classes. There is evidence of teamwork in the mathematics department even though the Grade 10 teachers do not teach according to a common teaching plan. Again, he is solely responsible for what he teaches in his class, and it is noted that he is not satisfied with the pace of his teaching the Grade 10 mathematics curriculum. Teacher Mzimela agreed to be observed while delivering lessons to his Grade 10 class, and the date and the period was agreed to by both the participant and myself as being accessible and convenient. The classroom lesson observation lasted for about an hour. This session was video recorded as per agreement in the informed consent documentation.

5.5.2 Description of Teacher Mzimela's Lessons.

5.5.2.1 Lesson one of Teacher Mzimela: Factorisation of quadratics of the difference of two squares.

Teacher Mzimela introduced me to his class as a person interested in the education of secondary school mathematics in the district. Teacher Mzimela had to mark the homework that he had given learners the day before our visit. The homework focused on the expansion of brackets if

given the product of expressions. Teacher Mzimela randomly checked if learners had done their homework by perusing their exercise books. Learners were then offered the opportunity to present the corrections on the chalkboard. Together with learners, teacher Mzimela verified that all the sums were done correctly before starting the new lesson. He introduced the new lesson on factorisation. His approach was to present factorisation as the reverse process of finding products. So, he started off by presenting learners with the expression $a(b + c)$ to find the products by removing the brackets. Students confidently achieved $ab + ac$. He then asked them to do the reverse, by taking out the common factor in the expressions $4a + 8$ and $12x^2 + 5x^3$. He led the discussion with students randomly uttering responses until they achieved $4(a + 2)$ and $x^2(12 + 5x)$ respectively.

The confidence displayed by learners while doing these examples made me think that a similar exercise might have been done in this class before my visit. Students were further given the example $2a(x + y) - 3b(x + y)$ to simplify. Again learners were quick to provide the answer of $(x + y)(2a - 3b)$. Learners were given another example $2x(m - 3n) - 5y(m - 3n)$ to do as a classwork exercise. One learner was able to demonstrate on the board and gave the answer $(m - 3n)(2x - 5y)$.

MZ: Is she correct?

Class: Yes.

Teacher Mzimela gave another example $2a(3x - 2y) + 5b(2y - 3x)$

MZ: How many terms do we have here?

Class: 2 terms.

MZ: Do we have a common factor?

Class: Yes, and No.

After receiving uncertain responses from learners, teacher Mzimela decided to assist learners by clarifying the differences between term one and term two.

MZ: In term one we have $3x - 2y$ and in term two, $2y - 3x$. These items look similar but the difference is in their signs, in that 3 is positive in the first bracket and negative in the second bracket; while 2 being negative in the first bracket and positive in the second bracket. We need

to change the sign in one of the brackets so that what is inside the brackets look similar in both terms. How do we do that?

Learner: We multiply the number inside the bracket with minus 1.

MZ: This view is partially correct. Do we have somebody with an alternate response or view?

Class: No response.

Teacher Mzimela then provided clarity by explaining that one way of addressing the situation is to take out the common factor minus 1 in one of the terms. He opted to take minus 1 as common factor on the second term, and the first term became $2a(3x - 2y)$. For the second term, he first wrote $5b$, which was followed by minus 1 as the common factor of $2y - 3x$ and items changed to $-2y + 3x$, then the whole expression became:

$$2a(3x - 2y) + 5b(-1)(-2y + 3x)$$

$$2a(3x - 2y) - 5b(-2y + 3x), \text{ for } -2y + 3x \text{ is the same as } 3x - 2y$$

$$(3x - 2y)(2a - 5b).$$

Teacher Mzimela gave learners a class exercise, and a learner demonstrated the answer on the chalkboard.

$$\text{Step 0: } 3a(5x + y) - (-5x - y)$$

$$\text{Step 1: } 3a(5x + y) - (5x + y)$$

$$\text{Step 2: } (5x + y)3a$$

MZ: Is step 1 correct?

Class: No.

MZ: Can we find somebody to assist?

Another student went up to the chalkboard and provided the following response:

$$\text{Step 0: } 3a(5x + y) - (-5x - y)$$

$$\text{Step 1: } 3a(5x + y) + (5x + y)$$

Step 2: $(5x + y)(3a + 1)$.

MZ: is the sum correct?

Class: Yes.

Teacher Mzimela went on to introduce the method of factorisation of the quadratic expression in the special case of the difference of two squares. He began by writing the topic, the difference of two squares, on the board. He reminded learners about finding the product of two linear expressions, which he presented on the board $(a + b)(a - b)$. So together with learners they were able to work it out and found $a^2 - ab + ab - b^2$, which teacher Mzimela and learners agreed that $-ab + ab$ cancel each other, hence it resulted in $a^2 - b^2$. He summarised his lesson by emphasising to learners that multiplication or removal of brackets is one way to find the product while factorisation is the reverse process of finding products. He instructed learners that if they are asked, for example, to factorise $a^2 - b^2$ they need to know that the answer should be $(a + b)(a - b)$. He explained that if you have two squared terms that are separated by minus, we refer to them as the difference of two squares. If you want to factorise the difference of two squares, you just need to open two brackets. Find the factors for the first term as well as those of the second or last term. In this case, the factors of a^2 are a and a , and factors of b^2 are b and b ; simply put negative and positive signs in-between the items in the brackets i.e. $(a + b)(a - b)$.

He provided two examples for students to do as classwork

1. $x^2 - 49$
2. $169x^2 - 100$

Learners did the examples with confidence, showing that they understood the concept. Teacher Mzimela then gave learners homework:

1. $4x^3 - 6x^2$
2. $3x^2 - 9y^2 = +12xy$
3. $x^2 - 4$

4. $16a^2 - 121b^2$
5. $27a^3 - 3b^2$
6. $25p^4q - 100p^4q^3$

5.5.2.2 Lesson two of Teacher Mzimela: Factorisation of quadratic trinomials.

Teacher Mzimela began the lesson by revisiting the homework that was given to learners on the day before my visit. Learners were given an opportunity to demonstrate the solution on the chalkboard with the discussion being dominated by the whole class. Learners were then encouraged to do the correction in their exercise books. Teacher Mzimela started by providing learners with a generalised explanation of factorisation. He introduced the factorisation of quadratic trinomials by asking a learner to find the product of the two given linear expressions $(a + b)(a + b)$. He took learner through the process to find the product and as result, they found: $a^2 + ab + ab + b^2$ which then gives $a^2 + 2ab + b^2$

MZ: From $a^2 + 2ab + b^2$, how did we get a^2 ?

Learner 1: We multiplied the first value of the first bracket with first value of second bracket, that is $a \times a$.

MZ: From $a^2 + 2ab + b^2$, how did we get b^2 ?

Learner 2: We multiplied the last value of the first bracket with last value of second bracket, that is $b \times b$.

MZ: How did we get the middle term $2ab$?

Learner 3: We multiply the first term brackets with the last term of the second bracket and get ab , and again multiply the second term of the first brackets with the first term of the second bracket, and we got ab , then we add like terms and get $2ab$.

He gave another example of finding the product of $(x + 2)(x + 3)$

MZ: Find the value of $x \times x$.

Class: x^2 .

MZ Find the value $x \times 3$.

Class: $3x$.

MZ: Find the value $2 \times x$.

Class: $2x$.

MZ: Find the value 2×3 .

Class: 6.

MZ: Finally we get $x^2 + 5x + 6$. Generally, we use the FOIL (First terms, Outer terms, Inner terms and Last terms) method. Let us now try to check if we are able to work backward on the given example $x^2 + 3x + 2 = 0$.

MZ: How do we begin?

Learner 2: We open two brackets.

MZ: How do we get x^2 ?

Class: $x \times x$.

MZ: What do we need to do get 2?

Class: 1×2 .

Before giving the conclusion on the factors, teacher Mzimela emphasised to learners the need to consider the sign of the factors as they have compound impact on the determination of the middle term, hence the answer was $(x + 1)(x + 2)$. Together with learners, teacher Mzimela was able to check if the answer was correct by multiplying two linear expressions where $x \times x = x^2$, $x \times 2 = 2x$, $x \times 1 = x$, and $1 \times 2 = 2$ then $2x + x = 3x$ hence $x^2 + 3x + 2$.

Another example of $x^2 + 6x + 8$ was given to learners to factorise. He emphasised that the first thing to do is to check if there is a common factor.

MZ: What are the factors of x^2 ?

Class: $x \times x$.

MZ: What are the factors of 8?

Class: 2×4 and 1×8 .

MZ: Which are the correct factors to choose to the sum and why?

Learner 3: 2×4 , because $2 + 4 = 6$, which is the middle term.

Another example of $x^2 - 5x + 6$ was given to learners to factorise. He gave learners an opportunity to do this example on the board. The learner wrote $(x - 2)(x - 3)$, then teacher Mzimela went to the chalkboard to explain the example. He asked the student why he chose -2 and -3 .

Learner: If I add -2 and -3 they give -5 .

MZ: I can also get 5 by saying $6 - 1 = 5$, will I be wrong if I say 6 and 1?

Class: No.

MZ: If we multiply -6 and 1 are we going to get positive?

Class: No.

MZ: That is why these factors are wrong -6 and 1, that is why we choose -2 and -3 .

$x^2 + 6x + 8$
 $(x+2)(x+4)$

$x^2 - 5x + 6$
 $(x-2)(x-3)$

$2 \times 4 = 8$
 $1 \times 8 = 8$
 $2 \times 3 = 6$
 $6 \times 1 = 6$

Another example of $3a^2 - 21a - 24$ was given for learners to factorise.

MZ: How many terms do we have here $3a^2 - 21a - 24$?

Class: Three.

MZ: The first thing we do when we talk about factorisation is whether we have a common factor or not. In this expression, do we have common factor?

Class: Yes.

MZ: What is our common factor?

Class: 3.

MZ: So that means we have to take it out, and we are left with $3(a^2 - 7a - 8)$, we can still factorise the brackets further; how are we going to factorise?

Class: Open two set of brackets.

MZ: What are factors of a^2 ?

Class: $a \times a$.

MZ: What are the factors of 8?

Class: 8×1 and 4×1 .

MZ: How do we get negative -8 ?

Class: We multiply negative number with positive number.

MZ: That means one of the factors must be negative and the other must be positive, which factors give us -7 when added?

Class: -8 and 1 , because the middle is negative.

MZ: That means we get $3(a^2 - 7a - 8)$ to get $3(a + 1)(a - 8)$

Another example of $x^3 + 2x^2 - 8x$ was given to learners to factorise.

MZ: Is there anyone who wants to do this one?

Learner:

Step 0: $x^3 + 2x^2 - 8x$

Step 1: $x(x^2 + 2x - 8)$

Step 2: $x(x - 2)(x + 4)$

MZ: Is he correct?

Class: Yes.

After checking the example together with the class, classwork was given:

1. $x^2 - 7x + 12$
2. $x^2 - 5x - 84$
3. $a^2 - 2a - 48$
4. $2x^2 - 14x - 36$
5. $4a^2 - 12a - 40$
6. $90 + -45x + 5x^2$

5.6 Case study five: Teacher Kweyama (Teacher KW).

5.6.1 The Biography of Teacher Kweyama.

Teacher Kweyama is a qualified teacher with a Bachelor of Education degree majoring in mathematics and physical sciences for the FET phase. He has five years of experience in teaching mathematics at secondary school level, two years of which he has taught at the T-School in this study. Currently, he teaches mathematics to Grade 10 learners and physical sciences to Grade 12 learners. He began his career teaching mathematics and physical sciences at schools in deep rural areas before being transferred to this T-School where he shares the Grade 10 teaching with two other teachers. Each teacher teaches their own grade section. When preparing lessons to teach Grade 10 algebraic quadratic expressions, teacher Kweyama sometimes writes his own notes for his learners. Teacher Kweyama shares the section with other teachers. He has received professional development on a number of occasions and this is ongoing. These occasions include school mathematics meetings with the Departmental Head (DH); cluster meetings; the AMESA workshop; workshops organised by the subject adviser; and mentoring visits from a colleague or the SMT. Mr Kweyama is able to track using various practices and documents. He uses CAPS guidelines; the KZN annual teaching plan (ATP), and his work is continuously quality checked by the Departmental Head (DH) of the school. Teacher Kweyama seldom uses the PILO Jika Imfundo trackers in his class. Considering the dynamics and the Grade 10 mathematics teaching workload, he is of the opinion that it is sometimes not possible to complete all the topics covered by the curriculum. According to teacher Kweyama, mathematics teachers at his school are supportive of one another and all teachers do the same assessments for all Grade 10 mathematics classes. He believes Grade 10

teachers teach according to the same common teaching plan and is therefore solely responsible for what he teaches in his class. It is noted that he is pleased with his pace in teaching the Grade 10 mathematics curriculum. In his opinion, based on his teaching experience, algebraic expressions sequencing of topics in Grade 10 mathematics make sense, and are necessary.

Teacher Kweyama agreed to be observed while teaching his Grade 10 class and the date and the period was agreed to by both the participant and myself as being accessible and convenient. The classroom lesson observations lasted for about two hours (one hour each session). This session was video recorded as per agreement in the informed consent documentation.

5.6.2 Description of Teacher Kweyama's Lessons.

5.6.2.1 Lesson one of Teacher Kweyama: Factorisation of quadratics forming a difference of two squares.

I was introduced to teacher Kweyama's class as a person with an interest in mathematics education in the district. I was welcomed by learners who shouted "shine Mr Vilakazi.". Teacher Kweyama wrote the date and the title of the topic of the lesson "difference of two squares" on the chalkboard. He led the discussion by first requesting learners find the product of the two given linear expressions: $(a + b)(a - b)$. He reminded the class of the use of the FOIL method in removing the brackets and the discussion commenced.

KW: What are we going to do here?

Learner 1: We multiply the first terms; we multiply the outer terms; we multiply the inner terms; we multiply the last terms; we look for the like terms.

KW: Good ...then let us do it.

Learner 1: We open the two sets of brackets, then $a \times a = a^2$; $a \times -b = -ab$; $b \times a = ab$; $b \times b = b^2$ and this is represented as $a^2 - ab + ab - b^2 = a^2 - b^2$ because $-ab + ab$ cancel one another.

Teacher Kweyama then led the discussion by giving learners a full description of a quadratic expression of the difference of two squares. According to teacher Kweyama, an expression that

is made by two squared terms, yet separated by a minus sign, is referred to as a difference of two squares.

He presented another example $x^2 - 9$, and together with the class worked to factorise it. He explained to learners that, “here we have two terms and this is the difference of two squares in that x^2 is a square derived from $x \times x$...and 9 is also a square that is derived from 3×3 . In this case the way we factorise them, is by opening up two sets of brackets ... from those two brackets, now the first term was x^2 then we will need to determine the factors that will give us x^2 ... obviously seeing that x^2 is a variable we also need to use variables ... so if you multiply x and x you will get x^2 ... then again we also have 9 as a constant value ...again the two similar numbers that can be multiplied together to give 9, is 3×3 ... thereafter you put the minus sign and the positive sign respectively in separating the items within the brackets as follows $(x - 3)(x + 3)$... again the order is not important meaning that $(x + 3)(x - 3)$ is also correct. Again, the use of minus and plus is to ensure that the final answer has a middle term of zero, because the middle terms will cancel one another”.

Learners were given classwork to do on factorisation

1. $x^2 - 16$
2. $4x^2 - 1$
3. $9x^2 - 25$

The teacher went around the class to mark learners’ work. Learners were then offered the opportunity to present their work on the board. All three learners were able to provide the correct answers; however, they could hardly explain their thoughts. Teacher Kweyama then intervened by engaging the whole class.

KW: How did we factorise $x^2 - 16$ to get $(x - 4)(x + 4)$?

Learner: We opened two sets of brackets; we put x and x the square root of x^2 ; we put 4 and 4 the square root of 16, we put plus and minus in between.

KW: Why 4 and 4 not 16 and 1 or 8 and 2?

Class: It is only -4 and 4 that can be added to give zero of the middle term.

KW: How did we factorise $4x^2 - 1$ to get $(2x - 1)(2x + 1)$?

Learner 3: We opened two sets of brackets; we put $2x$ *and* $2x$ the square root of $4x^2$; we put 1 *and* 1 the square root of 1 , we put plus and minus in between.

KW: How did we factorise $9x^2 - 25$ to get $(3x - 5)(3x + 5)$?

Learner 4: We opened two sets of brackets; we put $3x$ *and* $3x$ the square root of $9x^2$; we put 5 *and* 5 the square root of 25 , we put plus and minus signs in between.

KW: Good.

Teacher Kweyama went on to introduce another type of quadratic expression. He wrote “factorisation by the removal of the common factor” on the chalkboard. He presented the example $x^2 - 4x$. He stressed to learners the characteristics of this expression and led learners on its factorisation.

KW: In $x^2 - 4x$ we look for the common factor in this case which will be x ... we then say $\frac{x^2}{x} = x$ and $\frac{4x}{x} = 4$; we keep the sign in the middle; then we factorise to get $x(x - 4)$... Another example $2x^2 + 6x$... we look for the common factor which is $2x$ in this case; $\frac{2x^2}{2x} = x$; $\frac{6x}{2x} = 3$; we keep the sign in between; then we factorise to get $2x(x + 3)$ Did you understand?

Class: Yes.

More classwork on the factorisation of quadratic expressions was given to the class, of which, due to limited time, was then converted to homework.

Classwork

1. $x^2 + 4x$
2. $2x^2 - 4x$
3. $8x^2 + 2x$
4. $10k^2 + 4k$
5. $8p^2 - 24p$

5.6.2.2 Lesson two of Teacher Kweyama: Factorisation of quadratic trinomial expression.

Teacher Kweyama began this lesson by allowing learners the opportunity to mark the work given the day before. He presented the example of $x^2 - 2x - 15$ for the class to factorise.

KW: Here we use the method of factorisation by grouping. We consider each and every quadratic expression as represented by the general quadratic expression $ax^2 + bx + c$, where in this case $a = 1$; $b = -2$ and $c = -15$.

Step 1: We find $a \times c = 1 \times -15 = -15$.

Step 2: We use trial and error to search for the factors of -15 which when added will give -2 . After the trial of factors we found -5 and 3 as suitable factors.

Step 3: We then rewrite the given expression as the form of its factors

$$x^2 - 2x - 15$$

$$x^2 - 5x + 3x - 15.$$

Step 4: We group and then factorise

$$(x^2 - 5x) + (3x - 15)$$

$$x(x - 5) + 3(x - 5).$$

Step 5: Open the two sets of brackets and take out the common factor to achieve the following factors: $(x - 5)(x + 3)$.

Teacher Kweyama introduced another example of $6x^2 - 11x - 10$ to strengthen his approach to the factorisation of quadratics by grouping.

KW: Again, we consider each and every quadratic expression as represented by the general quadratic expression $ax^2 + bx + c$, where in this case $a = 6$; $b = -11$ and $c = -10$.

Step 1: We find $a \times c = 6 \times -10 = -60$.

Step 2: We use trial and error to search for the factors of -60 which when added will give -11 . After the trial of factors, we found -15 and 4 as suitable factors.

Step 3: We then rewrite the given expression as the form of its factors:

$$6x^2 - 11x - 10$$

$$6x^2 - 15x + 4x - 10$$

Step 4: We group and then factorise:

$$(6x^2 - 15x) + (4x - 10)$$

$$x(6x - 15) + 2(2x - 5)$$

Since we do not have factors that are the same in both brackets, we rearrange our expression in Step 4, and factorise:

$$(6x^2 + 4x) + (-15x - 10)$$

$$2x(3x + 2) - 5(3x + 2)$$

Step 5: Open the two sets of brackets and take out the common factor to achieve the following factors: $(2x - 5)(3x + 2)$.

Teacher Kweyama introduced another example of $x^2 - 4$ for the class to factorise.

KW: In factorising $x^2 - 4$, the first thing you need to do is to consider the given expression as the general quadratic expression $ax^2 + bx + c$, in this case $a = 1$, $b = 0$, and $c = -4$. We use the ac method ... in other words we multiply the coefficient of the first term with that of the last term $ac = (1)(-4)$... you will need to know about bx but in this case you can simply see that $bx = 0$ because we don't have the middle term ... we have to find factors of -4 which when added will give us b or will give us zero ... which are -2 and $+2$... then we substitute bx with $-2x$ and $+2x$ and have $x^2 - 2x + 2x - 4$ and then introduce the grouping

$(x^2 - 2x) + (2x - 4)$... look for the common factor in each group respectively

$x(x - 2) + 2(x - 2)$... further factorise and get $(x - 2)(x + 2)$.

Learners were given classwork which required them to factorise $x^2 - x - 6$ using the grouping method.

KW: Here we need to factorise ... I think the best way to explain these to learners is by looking at the last value ... in this case it is -6 ... you also look at the middle coefficient, in this case it is -1 ... here the coefficient of x^2 is 1, so now the best way to find factors here is to multiply 1 and -6 then I will get -6 ... from there, I just need to determine the factors of 6 which are 1 and 6; 3 and 2 ... to get middle term you need to subtract from these factors, if I take 6 and 1 ... $1 - 6 = -5$ and when I interchange factors $6 - 1 = 1$ then these factors are incorrect ... then I move to second factors $2 - 3 = -1$... then I get the answer which is -1 which is exactly the coefficient of the middle term ... I then factorise by grouping ... I rewrite $x^2 - x - 6$ as $x^2 + 2x - 3x - 6$... then group the terms $(x^2 + 2x) + (-3x - 6)$... then take out the common factor in both sets respectively $x(x + 2) - 3(x + 2)$... take out the common factor again $(x + 2)(x - 3)$... even though this could be seen as a very long method, however, it will benefit learners as they will be treating the quadratic of the coefficient of x^2 that is not equal to 1.

5.7 Case study six: Teacher Gwamanda (Teacher GS).

5.7.1 The Biography of Teacher Gwamanda.

Teacher Gwamanda is a qualified mathematics teacher with a Bachelor of Education degree majoring in mathematics and physical sciences for the FET phase. He has four years of experience in teaching mathematics at secondary school level at the T-School in this study. Presently, he teaches Grade 10 mathematics and Grade 12 Physical Sciences. Currently, he shares Grade 10 mathematics with two other teachers, each of whom teaches his grade section. When preparing lessons, teacher Gwamanda strongly advocates the use of a single textbook. Teacher Gwamanda sometimes writes his own notes for his learners to meet the standard maths needed for Grade 10 learners. Teacher Gwamanda has received continual professional development at the school mathematics meetings with the Departmental Head (DH); and cluster meetings. He is able to check the mathematics curriculum coverage using different practices and documents. He uses CAPS guidelines; the KZN annual teaching plan (ATP), and his work is regularly quality-checked by the Departmental Head (DH) of the school. Teacher Gwamanda does not use PILO Jika Imfundo trackers. Considering the dynamics and the workload involved in teaching mathematics at Grade 10 level, he feels it is not possible to complete all the topics required by the curriculum. Teacher Gwamanda believes that the time set by the curriculum to teach Grade 10 mathematics does not match the real teaching time.

According to teacher Gwamanda, mathematics teachers at his school are supportive of one another in that all Grade 10 teachers do the same assessment for all Grade 10 mathematics classes. There is evidence of teamwork in the mathematics department even though Grade 10 teachers do not teach according to a common teaching plan. Again, he is solely responsible for what he teaches in his class, and it is noted that he is unhappy with the pace of his teaching the Grade 10 mathematics curriculum. Teacher Gwamanda strongly believes that most topics required by Grade 10 mathematics curriculum are necessary and suitable for Grade 10 learners. He believes that sequencing of topics according to CAPS makes sense as well as the planning guidelines provided to teachers.

Teacher Gwamanda agreed to be observed while delivering a lesson to his Grade 10 class, and the date and period was agreed to by both the participant and myself as being accessible and convenient. The classroom lesson observation lasted for about an hour. This session was video recorded as per agreement in the informed consent documentation.

5.7.2 The description of Teacher Gwamanda's lessons.

5.7.2.1 Lesson one of Teacher Gwamanda: addition and subtraction of algebraic fractions.

I was introduced to the class and learners were then prepared for the lesson. Teacher Gwamanda began by writing the title of the topic on the board while alerting the learners to the content they would be covering. He presented the example: $\frac{x-1}{2} + \frac{x-2}{3} - \frac{x-3}{4}$ to the class and began his lesson by explaining the nature of the sum. He said, "Here we have an expression in a form of fractions, and these three fractions are in the form $\frac{a}{b}$ and are separated by positive and negative signs". Teacher Gwamanda then asked how this particular problem could be solved, however, there was no heading giving the instruction on what to do with the sum in question. He told learners that the first thing they had to do was to find the Lowest Common Denominator (LCD), for all of the three denominators 2; 3 and 4.

GS: How are we going to find the LCD for this sum?

Learner: We look for the smallest number that can be divisible by all the three denominators 2; 3 and 4 without leaving any remainder.

GS: When looking at 4 and 2 can we take 4 as the LCD?

Class: Yes.

GS: When looking at 4 and 3, can we take 4 as the LCD?

Class: No.

GS: Why?

Learner: This is because when 4 is divided by 3, there is a remainder of 1, hence it is not a factor.

Teacher Gwamanda led the discussion by showing learners how to explore using the trial and error method if certain numbers are LCDs by using 6, 8 and 10 as possible LCDs for a given sum, which later proved that the correct LCD is 12.

GS: Now we have transformed all the denominators in the sum to one common denominator which is 12. This can only be achieved if each term is multiplied by a fraction of one as follows:

$$\frac{x-1}{2} \times \frac{6}{6} + \frac{x-2}{3} \times \frac{4}{4} - \frac{x-3}{4} \times \frac{3}{3}$$
$$\frac{6(x-1)}{12} + \frac{4(x-2)}{12} - \frac{3(x-3)}{12}$$

We need to note that if we have a fraction of $\frac{a}{b} + \frac{c}{b}$, then we end up having a LCD that is b , then we look at how many time does b go into b and find it goes once, and $1 \times a = a$ and then apply the same procedure to c and get c . The final answer will be $\frac{a+c}{b}$, if we have $\frac{a}{b} + \frac{c}{b}$, since the denominators are the same we can simply take one denominator and add numerators. So, in this example we can write it as: $\frac{6(x-1)+4(x-2)-3(x-3)}{12}$. Then you can simplify the task by removing the brackets in each term of the numerator to reach $\frac{6x-6+4x-8-3x+9}{12}$, and after collecting the like terms we achieve $\frac{7x-5}{12}$.

Teacher Gwamanda then used another approach using the same example, $\frac{x-1}{2} + \frac{x-2}{3} - \frac{x-3}{4}$ to simplify:

Step 1: $\frac{12}{2} = 6$, then $6(x - 1)$

Step 2: $\frac{12}{3} = 4$, then $3(x - 2)$

Step 3: $\frac{12}{4} = 3$, then $3(x - 3)$

After dividing the LCD with each denominator, as a learner you need to write your LCD as the denominator and take all answers achieved in each step as the numerator. This leads us to achieve $\frac{6(x-1)+4(x-2)-3(x-3)}{12}$ and then we follow the similar approach used in the previous approach and arrive at the same step.

Teacher Gwamanda then gave learners a class activity.

1. $\frac{x-3}{3} - \frac{x-2}{2} + \frac{x+1}{6}$

After a few minutes he gave learners an opportunity to demonstrate the solution on the chalkboard. He asked two learners to write the same sum but demonstrate different approaches. Both learners arrived at the same answer which showed an understanding of both approaches teacher Gwamanda had taught. He then gave learners homework.

5.7.2.2 Lesson two of Teacher Gwamanda: Simplification of algebraically expression.

On the second visit, teacher Gwamanda allowed learners to do corrections on work that was given the previous day. After observing learners performing diligently, he proceeded to introduce the new topic. He wrote the title on the chalkboard “simplification of algebraic expressions”, and presented the class with the following example $\frac{x^2-12x+27}{4x^2-12x}$. Teacher Gwamanda began leading the discussion by providing the explanation on how to deal with this given problem.

GS: Step 1 we have to first factorise the numerator and then factorise the denominator, our numerator we have is $x^2 - 12x + 27$ and our denominator is $(4x^2 - 12x)$. Once we have

factorised, we need to observe that they are items that cancel one another. We cancel those values and end up with the final answer. He led learners in finding the factors of the numerator. In the numerator, we need to simplify this algebraically expression, the first thing you have to do is see how many terms there are here ($x^2 - 12x + 27$), we see there are three terms.

GS: What are the factors of x^2 ?

Class: x and x .

GS: What are the factors of 27? (Those factors when added should give -12)

Class: -9 and -3 .

However, I noted that the teacher was giving learners clues that led to the anticipated factors. Eventually they arrived at $(x - 9)(x - 3)$. Mr Gwamanda, together with learners, quickly began to check if the factors were correct $(x - 9)(x - 3)$ becomes $x^2 - 9x - 3x + 27$ which finally becomes $x^2 - 12x + 27$.

GS: In the first terms of brackets you need to make sure that they give you the first term of this expression ($x^2 - 12x + 27$) if multiplied, the last terms in the bracket, if multiplied, have to give you this value which is 27, and if you add or minus they give what we call the middle term. So what are the values that we can put which will give us 27, make sure that those values, if you add them or subtract them, give -12 . Learners engaged and gave -9 and -3 . Let's check $-9 \times -3 = 27$, and the middle term $-9 \times x = -9x$ and $x \times -3 = -3x$, so if we say $-9x - 3x$ we get $-12x$, which shows our factors are correct because they give us the last term and second or middle term.

GS: Let's go to the denominator. How are we going to factorise this value ($4x^2 - 12x$). Is there any common factor?

Class: $4x$.

While writing the learners response he said, "...this means we take out $4x$ and now we have $4x(x - 3)$ "

GS: Let's check. Is this ($4x(x - 3)$) going to give us back our original expression; $4x \times x = 4x^2$ and $4x \times -3 = 12x$.

Teacher Gwamanda went on further to simplify the algebraic expression saying: "The first thing you are going to do is to check how many terms we have, then after that you do the

factorisation, you do part of the numerator then part of the denominator, then what we have to do is just look for factors that are the same, then you cancel it out”.

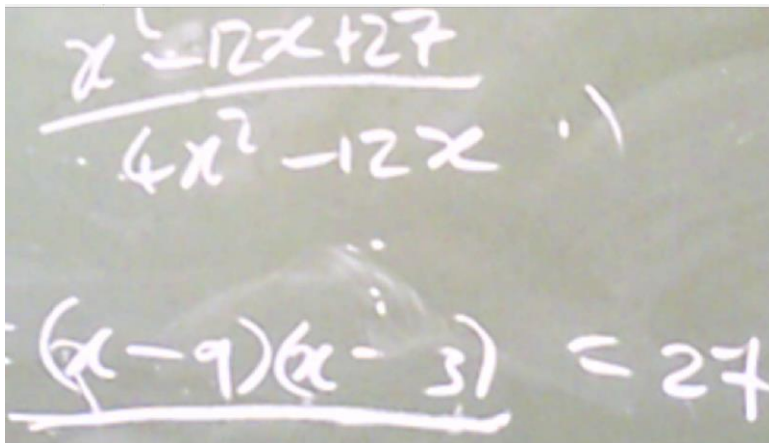
GS: Is there any $x - 9$ in the denominator?

Class: No.

GS: Is there any $x - 3$ in the denominator?

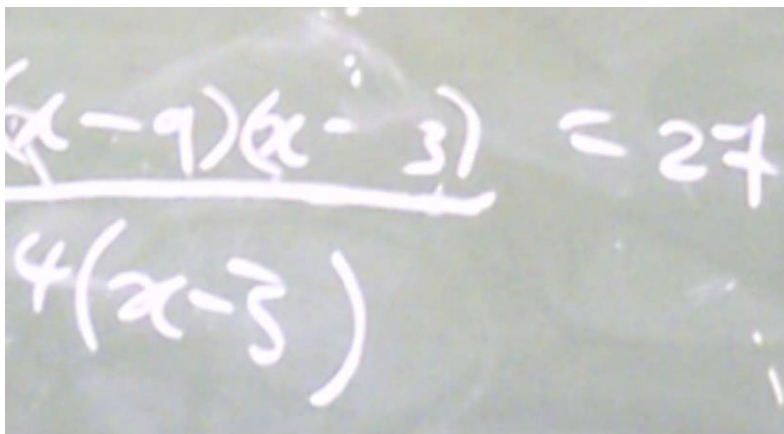
Class: Yes.

GS: Then $x - 3$ cancel $x - 3$ and you remain with $\frac{x-9}{4x}$ so there is no further possible simplification.



$$\frac{x^2 - 12x + 27}{4x^2 - 12x}$$

$$\underline{(x-9)(x-3) = 27}$$



$$\underline{(x-9)(x-3) = 27}$$

$$4(x-3)$$

Example two:

$$\frac{x-2}{x^2+4x-5} \div \frac{x^2+5x}{x^2-4}$$

Teacher Gwamanda explained the nature of this problem by emphasising that it is the division of two algebraic fractions. So, he indicated to learners that the first thing to do was to change the division sign between the two fractions to multiplication. Once the division sign is changed to multiplication the fraction on the right changes, its completion in that the numerator becomes the denominator and the denominator becomes the numerator. Then the expression becomes $\frac{x-2}{x^2+4x-5} \times \frac{x^2-4}{x^2+5x}$ which is followed by searching for factors.

GS: Can we factorise $x - 2$?

Class: No.

GS: Can we factorise $x^2 + 4x - 5$?

Class: Yes, $(x + 5)(x - 1)$.

Together with learners, teacher Gwamanda checked if the given factors were correct by performing the product rule and found $x^2 + 5x - x - 5 = x^2 + 4x - 5$

GS: Can we factorise $x^2 - 4$?

Class: No answer.

He then led learners by explaining to them this is a difference of two squares, then learners quickly recalled the difference of two squares and offered up the answer $(x - 2)(x + 2)$.

GS: Can we factorise $x^2 + 5x$?

Class: No answer.

GS: How many do we need to have?

Class: Two.

GS: Are we sure that we have two brackets?

Following this question, the class became undecided, some said one bracket and others said there were two sets of brackets. Teacher Gwamanda was obliged to break the deadlock. According to teacher Gwamanda the correct answer was one set of brackets based on the fact that when we rewrite the expression $x^2 + 5x$ as $x^2 + 5x + c$, and $c = 0$ hence, one set of brackets. He then led them by writing down a set of brackets giving them verbal clues on following the method of taking out the common factor and found $x(x + 5)$, and a new

expression $\frac{x-2}{(x+5)(x-1)} \times \frac{(x-2)(x+2)}{x(x+5)}$ was achieved. Teacher Gwamanda further advised students to observe if there were any items that cancelled one another.

Teacher Gwamanda took a few minutes to observe the expression and realised there were no items that could divide each other hence there was no way to further the simplification. This observation indicated that teacher Gwamanda did not have enough time to prepare for this lesson. He might have followed the teaching plan accordingly but without putting much focus on the lesson preparation. Again, adding to this observation, teacher Gwamanda was taking activities directly from the textbook, which is a clear indication that the teacher relied more on his experience while engaging with this exercise. He persuaded and convinced learners that the sum was written incorrectly and then decided to rewrite it as $\frac{x-2}{x^2+4x-5} \div \frac{x^2-4}{x^2+5x}$. Together with learners, they followed the procedure similar to the one they used in the incorrect sum and achieved the following: $\frac{x-2}{(x+5)(x-1)} \times \frac{x(x+5)}{(x-2)(x+2)}$.

The image shows a chalkboard with handwritten mathematical work. On the left, the expression $\frac{x-2}{x^2+4x-5}$ is written. To its right is a division symbol $\div$ followed by the expression $\frac{x^2-4}{x^2+5x}$. Further right, the division is converted to multiplication by the reciprocal, showing $\frac{a}{b} = \frac{b}{a}$ as a general rule, and then the final expression $\frac{x-2}{x^2+4x-5} \times \frac{x(x+5)}{(x-2)(x+2)}$.

GS: Do we have brackets that are similar?

Class: Yes.

GS: Which means they will be able to cancel each other hence $x + 5$ is cancelled and $x - 2$ is also cancelled, and finally we are left with $\frac{x}{(x+1)(x+2)}$.

The teacher opened the session for questions but there were no questions from the class. The teacher then summarised the session by emphasising the important points to consider when working on simplifying algebraic expressions. He advised learners to look for quadratic items. He further emphasised that learners should seek to find items that cancel each other.

Learners were then given classwork:

1. $\frac{x^2-5x+6}{2x-6}$
2. $\frac{x^2+x-6}{2x-8} \div \frac{x^2-2x}{x^2-16}$

Learners were encouraged to do the activity individually and the teacher went desk-to-desk to mark the work of each learner. There were a few learners who were able to complete the sum in the given period. Due to the limited time, the uncompleted sums were converted to homework.

5.8 Summary on teachers' lesson observations.

A summary on teachers' lessons observations is provided Table 5.1 that follows.

Table 5.1 Summary on teachers' lessons observations.

Educator	Lesson1	Lesson2
Rakhulu	Traditional; difference of two squares; examples; difference of two squares technique; trial & error; cross method	Traditional; Example; Grouping method
Ndulini	Traditional; example; greatest common factor technique	Traditional; Trial & error; cross method
Hlophe	Traditional; revisited the product of binomial & binomial	Traditional; Trial & error; cross method
Mzimela	Traditional; revisited the product of monomial & binomial difference of two squares; examples; trial and error	Traditional; revisited the product of binomial & binomial; trial & error
Kweyama	Traditional; revisited the product of binomial & binomial; difference of two squares; difference of two squares technique	Traditional; Example; difference of two squares; Grouping method
Gwamanda	Traditional; Trial & error; cross method; difference of two squares; difference of two squares technique	Traditional; Trial & error; cross method

5.9 Conclusion.

This chapter explored teachers in the execution of their lessons the focus of which was on teaching the factorisation of quadratic expressions. Each teacher presented two lessons, in total six lessons per school selected for this focus study. I made the following observations: Firstly, in both schools (R- School and T-School), teachers who teach Grade 10 mathematics, also teach other higher grades: some teach mathematics and others physical sciences. This approach has the advantage of exposing learners to a myriad of experiences. However, it also has its disadvantages in that there is an imbalance in the provision of knowledge as observed in the T-School where all three teachers taught different subtopics; this indicates that Grade 10 teachers are not planning together. Secondly, in both schools, teachers strived to engage learners even though their approaches were teacher-centred, in that only the correct responses were encouraged throughout the lessons. Thirdly, in most lessons, there was no evidence of teachers administering the variety of tasks to cater for different capabilities among their students except for one class in the R-School, and even in that case this approach was necessitated by the fact that this class was a combination of two sections, the commerce and sciences. Fourthly, in both schools, learners were always encouraged and offered opportunities to display their understanding on the chalkboard, however, less stress was placed on allowing them to speak their minds. Basically learners go to the chalkboard with their exercise books in their hands, which seems to defeat the purpose of peer sharing and learning.

CHAPTER 6

LEARNERS' NARRATIVES ABOUT GENERAL PRACTICES IN THE GRADE 10 MATHEMATICS CLASSROOM - RESPONSES TO THE INTERVIEWS

6.1 Introduction.

This chapter presents an analysis of the responses from learners' interviews. These interviews were semi-structured in that the researcher was allowed the flexibility of posing questions that were outside the scope of the scheduled interview questions, for clarity purposes. The scheduled learners' interview questions appear in Appendix I of this thesis document. The learners' interview questions were categorised into two parts, where the focus of the first part was on the general practices learners experience as their teachers teach Grade 10 mathematics, while the focus of the second part was on the activity on quadratic expressions, and is presented in Chapter 7 of this thesis. The results from the interview were arranged according to the issues that were raised by the researcher. The responses of the 12 learners were considered according to each question, and patterns were identified using a grounded approach for each of the responses. The heading of each subsection is described according to their perceptions of the particular issue.

6.2 On the understanding of the concept "algebra".

The researcher posed the following question to the interviewees: "What do you understand about the term algebra? What comes to mind whenever you come across this term? Learners shared their understanding of the concept algebra in a variety of ways. For example, Learner SZ said: *"Algebra is when we talk about solving ... particularly when the sum has an equal sign"*. This notion of solving was also shared by learner BT who said: *"Here we talk of the variables ... solve for x and y ... equations as well as expressions"*. When learner XD was asked if he had ever come across the term algebra, he shared similar feelings as the other two learners SZ and BT as he said: *"Yes, I did come across with it"*.

R: What comes to your mind when you think about this term?

XD: *"I think of multiplication; brackets off; factorise; equations....."*

Again, the very same notion of solving was observed in the explanation made by learner DB as she said: *"All I can say is...I think of the algebraic expressions...also solving"*

equations....group your expression into two...like $(x - 2)(x + 2)$ ". Learner MW supported the notion as she said: *"Okay, it is where you are expected to factorize; or simplify"*. Offering a similar explanation to that of learner MW, learner GE said: *"Yes, the things that cross my mind are factorising; finding common factors; dropping same set of pairs of factors"*. Learner SB offered this response: *"I think of the things like equations ... and we need to solve....simplify"*.

However, two learners indicated to have no understanding of the concept of algebra. For example when learner TM was asked what comes to her mind if people talk about algebra, she said: *"I am unable to explain the meaning of term algebra"*.

R: Do you understand the meaning of the term algebra?

TM: *"Yes"*.

Learner LH boldly responded by saying: *"No, it is for the first time I come across such a term"*.

6.3 On the definition of the concept "factorisation".

Learners offered their definitions on the concept of factorisation in a variety of ways. Speaking on conception of quadratic equations, Kabar (2018, p. 127) argue that knowing and understanding the formal definition of a quadratic equation is important for conceptually understanding the relationships beyond symbolic computations in solving quadratic equations. In this case, I strongly believe that students, when factoring quadratics, shall be guided by the formal definition, hence being able to argue why they factor quadratic expressions in ways they are factoring.

6.3.1 Generalised views about factoring.

When learner NQ was asked what factorisation means to her, she said: *"It is to search for the factors of that number or the factors of that variable.....in fact it is to break the expression to its factors.....but you need to be able to bring everything back to what was given"*. A similar notion of factors was also shared by learner GE: *"It is whereby we look at the number ... we want the factors of a number"*. Learner BT supported the descriptions were made by learners

NQ and GE. In her definition of the term factorisation she said: *“I can say is to take the big number or the complex expression and simplify it to its constituent factors.....taking out the common factor on those given numbers.....is to take out the factors such that when you multiply them back you achieve what was given initially”*. Learner XD supported this notion of factors as he focused on the construction of the expression. Learner XD said: *“Factorisation is where you show how what you are given, like the expression, was constructed”*.

6.3.2 Using examples to define factorisation.

Learner GD resorted to using an example to describe the term factorisation. She said: *“When you factorise you shrink a bigger expression make it into simplest form”*.

R: How?

GD: *“We factorise the expressions.....expressions are made of variables...let me make an example about $6x + 12$...to factorise it you must know the highest common factors of both $6x$ and that of $+12$... then you look for what is common”*. The emphasis on the use of the example was also followed by learner TM who said: *“It is where we are given maybe “4”, then you are asked to break it like $2 \times 2 = 4$ ”*.

The use of an example in defining the term factorisation was also noted in the description made by learner SK who said: *“The first thing that am going to emphasise is that when we factorise, you have the expression and equation...in an equation is when we’ve solving and in expression is when we are simplifying. If you simplify you are able to factorise and also if you solve you are able to factorise. In this case you have been able to factorise, you are also able to reverse the sum into what it was before factorisation - to its originality. If you are given the trinomial, the middle is positive and the two factors the bigger number determines what should be in the middle – how it is supposed to be like in terms of signs”*. From his descriptions it is clear that Sthembiso (learner SK), had established his ways of simplifying the methods of getting the factorisation done. Learner SZ also followed the route taken by learners SK, TM and GD saying: *“First of all I could say by factorisation you need to see if the sum is the difference of two squares; or you need to take out the common factor before you are able to reverse to reach the original sum ... Perhaps you are expected to expand something. Or you need to show how the sum before the current state was – show all your workings step by step.....”*

6.3.3 Vague descriptions.

There were learners who showed little understanding of the concept of factorisation in their descriptions. For example, learner SB in defining factorisation said: *“By factorisation I can say it is where you change maybe the expression to be viewed as simple”*. Another learner who offered a very thin description on how to define factorisation was learner LH who said: *“It is where we break the expression”*. Table 6.1 details the summary on how learners defined the concept “factorisation”.

Table 6.1 Summary on how learners defined the concept “factorisation”.

Perspective	Description on defining the concept “factorisation”
Generalised views(factors)	NQ: To break the expression to its factors; GE: we want the factors of a number. BT: Complex expression and simplify it to its constituent factors.
Use of examples	GD: To factorise it you must know the highest common factors of both $6x$ and that of $+12...$ then you look for what is common SK: The trinomial, the middle is positive and the two factors the bigger number determines what should be in the middle – how it is supposed to be like in terms of signs. SZ: You need to see if the sum is the difference of two squares; to take out the common factor; show how was the sum before the current state. TM: It is where we are given maybe “4”, then you are asked to dissociate it like $2 \times 2 = 4$.
Vague description	SB: By factorisation I can say it is where you change maybe the expression to be viewed as simple. LH: It is where we break the expression. XD: Like the expression, was constructed. DB: Is when you break up the expression. MW: Factorisation means breaking.

6.4 On the relation between the amounts of work required for the factorisation of quadratics in Grade 10 mathematics, and the amount of time for teaching.

Learners were asked the question, R: Do you see any correlation between the amount of work expected to be learnt in Grade 10 mathematics in the factorisation of quadratics, and the amount of time given to teachers for teaching? With regard to this question, it appeared from the learners’ responses that they did not share the same beliefs. On one hand, learners were of the view that the time allocated for teaching the concept of factorisation of quadratics in Grade 10 was too little when compared to the amount of work that needed to be taught. For example, learner SZ said: *“I think the time is too little”*. A similar view was shared by learner BT who said: *“There is huge amount of work in a less amount of time.....you are supposed learn a number of methods on factorisation like the method you taught us where you (researcher) multiply a and c and get the factors of the product thereof....by this time our maths teacher*

was long passed that portion of factorisation....we were dealing with the factorisation of fractional algebraic expression ...". Learner GD supported the views of learners SZ and BT that there was a great deal of work to get through in a short space of time. She explained: *"The work is greater than time....we have a very short amount of time"*.

On the other hand, some learners were of the opinion that sufficient time was given for the teaching of factorisation of quadratics in Grade 10 mathematics, despite the amount of work teachers were required to get through. For example learner XD said: *"Yes ... there is the correlation ... the work is equal to the time"*. A similar view was also shared by learner DB who said: *"Yes, it is right ... they are equivalent"*. Learner SK also supported the notion of correlation between the amount of work and the amount of time. Learner SK said: *"The way I see it, I think it correlates in that the pace of instruction is equal to the volume of work in the factorisation of quadratics at Grade 10 mathematics"*. Learner LH also supported this view, saying: *"No, it is related"*. Learner NQ held a similar perspective. She said: *"Yes, it is right ... the amount of work in factorisation in relation to time of doing it, they are equivalent"*.

6.5 On whether the teachers of mathematics in their previous grades prepared them for Grade 10 mathematics.

I posed this question to the interviewees. R: Do you think your mathematics teachers in Grade 8 and in Grade 9, prepared you enough to face Grade 10 mathematics? All learners seemed to value the mathematical support they received from teachers in their previous grades in preparation for Grade 10 mathematics. Learner SK said: *"I can agree, Yes they did prepare me.....in fact mathematics is the same in all grades...in fact as you migrate from Grade 9 to Grade 10, you just add a bit of effort from what you did in Grade 9.....as you enter Grade 10 you come across most of the things you did in Grade 9"*. Learner LH also acknowledged the positive contribution that was made by his teachers in Grades 8 and 9 saying: *"My mathematics teachers of Grades 8 and 9 did help me understand mathematics better ... When I asked questions, they responded earnestly.....there are a lot of things I understand now just because of them, like solving for x , I was helped by them"*. Learner BT as agreed with learners LH and SK saying: *"Yes... my previous mathematics teachers had a role in shaping me mathematically ... for an example, my Grade 9 mathematics teacher used to explain thoroughly when dealing with the topic.....she couldn't just pass without having made an assurance that all learners*

understood....apart from focusing on the formal tests, she could also make use of the informal assessment as a way of getting a clear picture on whether we understood or not”.

R: How did your previous mathematics teachers help you understand algebra better?

BT: *“They taught the variety of methods of solving problems....the Grade 8 mathematics teacher used to even code switch to IsiZulu for us to understand better which I believe was the right to do”.* Supporting this view, learner XD said: *“There is only one teacher I could remember.... that Grade 8 mathematics teacher was of help to me in that he helped in nurturing the love of mathematics in me ... he could explain ... leaving us all clear with understanding.... he could change a very difficult concept and made it simple”.*

Learner DB had this to share: *“I can say yes there was that Grade 9 mathematics teacher who was of assistance to me....I had a challenge when it comes to simplification of algebraic expressions, particularly in the collection of the like terms”.* Again, learner GD applauded the work that was done by her previous mathematics teachers in cultivating her knowledge of factorisation and said: *“Yes...they did help me a lot”.*

R: Can you single out any teacher?

GD: *“Yes...my Grade 9 mathematics teacher”.*

In responding to the question, learner MW also acknowledged the groundwork made by her previous mathematics teachers in shaping her mathematically. She said: *“I began understanding mathematics when I was in Grade 9 ... I was being taught by the one and the very same teacher who taught me in Grade 8.....I don’t know what happened but in Grade 9 there was a total shift in my mindset about mathematics”.*

R: Do you maintain that if your Grade 9 mathematics teacher was never in your life, you would still be struggling with mathematics?

MW: *“It is difficult to say so...maybe the change has been in me...I don’t know”.* Learner GE was very emotional when she said: *“What I can say is that ... my Grade 9 teacher was just so good on teaching mathematics ... she helped me learn algebra better ... even now I wish I can swap them; take my current teacher to Grade 9....in Grade 9 everybody would get level 5 and 4, I mean not just 30% the teacher who will work hard to bring us the best,.....another thing is she believed in all of us she unleashed the potential in us....She was the best as compared*

to the current one”. Learner SB also acknowledged the groundwork made by the mathematics teachers in his previous grades who nurtured in him a love for mathematics. Learner SB shared: *“My teachers of mathematics in most previous grades were knowledgeable...they did understand that most learners perceive mathematics as difficult....they loved their job and they were talented”*.

R: Do you have that special teacher?

SB: *“My Grade 6 mathematics teacher....one thing about my Grade 8 teacher is...he was knowledgeable but the problem was on imparting knowledge to learners”*.

Learner TM referred back to her primary school education as having shaped her with regard to mathematics, explaining: *“Yes....in the school where I did Grade 5 the best performance used to be awarded with certificate....that was motivating ... that contributed in my mathematics body of knowledge ... that cultivated the love of mathematics in me”*. Learner NQ was even more specific in acknowledging the work of her Grade 9 teacher in the factorisation of quadratics, saying: *“Yes ... my teachers of mathematics in the previous grades contributed in my mathematics knowledge ... they did prepare me enough to understand the factorisation of quadratics ... I entered Grade 10 already understanding some factorisation procedures which I learned with understanding at Grade 9”*.

6.6 On whether learners solicit support from sources other than their teachers.

I posed this question to participants “Do you have, other than your mathematics teacher, anything to point to as having assisted you in understanding mathematics better”? Learners highlighted different sources where they solicit support. For example, learner LH pointed at the use of the calculator and said: *“Yes I do get the assistance from the calculator that I am using...it makes it easy for me if I solve problems...my teacher stressed the need for us to use the calculators”*. Another source always at the disposal of learners was pointed out by learner SK who said: *“Oh yes....books...I know how to utilise books even at the absence of the teacher...this is because I do understand that a teacher is a person who gives you guideline; how to study...the person who has to study is you (pointing at himself)....so a person who is willing to learn can make use of the textbook with success...learning can also be made by the use of the previous examination papers and it is a good approach of learning how questions*

are structured for the future examinations”. Learner BT also pointed to the use of previous examination papers as the factor that helped her understand mathematics better. Learner BT said: *“I make use of the past examination papers.....I regularly visit the library and borrow those books that are not available at my school.....I watch mind-set maths Grade 10 as well as the programme of mathematics in 319 learning channel”*. The use of previous question papers as being a stimulating factor in understanding mathematics was also raised by learner GD who said: *“I ask around some past paper in mathematics ... and I practice all of them ... and If I come across through difficult question, I have a cousin who is doing Grade 11 ... and I go and ask her some questions that I don’t understand ... and if it’s like I don’t understand and I have it in the textbook, I go through the textbook over and over again until I understand it better”*.

Learner NQ attends Saturday classes as part of a local project. She views her commitment as the other form of support that aids her learning of mathematics with understanding. She had this to say: *“I am attending the project called PROTEC.....my brother is also helping me a lot in teaching me mathematics with understanding.....other mathematics teachers at school are also helping a lot”*. Learner TM also supported the view of using textbooks and explained: *“I make use of the text book... that is what is making me understand mathematics better ... I am able to catch and understand by just looking at the examples done by the teacher.....examples done, in particular, are so helpful in making me understand mathematics”*.

In responding to the question, learner DB again acknowledged the use of peers as the external support always at her disposal for learning mathematics. She said: *“I make use of my peer learners who have understood the concept better than I could do....those learners are very supportive when consulted”*. Learner GE pointed directly to her intrinsic motivation as one guiding principle that makes her understand mathematics better. She had this to offer: *“What I can say is that I’m not that good when it comes to maths, but what’s helps me is that when I get home I become restless looking for assistance..... I want myself, indeed as a learner to understand my subject.....subject can be understood by different people in different ways – some will be experts and others naïve. ...I for one, if I do not understand something, I worry myself a lot until I understand it”*.

R: Basically ... do you go an extra mile?

GE: “Yes”.

Other than the use of his mathematics teacher, learner SB pointed to his talent as having assisted him in understanding mathematics better. He had this to say: *“With maths...I don’t know how I can explain it....all I could say I do not know why people are complaining about maths....maybe I have a God’s given talent because I am able to grasp the concept even if I am taught for the first time....I can see things happening automatically”*. Learner MW supported the view offered by learner GE. Learner GE spoke about practice as the other form of support that aided in her learning and understanding mathematics. She said: *“All you need to do in order to do better in mathematics ... you have to love it....you have to practice it every time everyday...then you will understand it”*.

6.7 On whether the content on the factorisation of quadratics is appropriate for Grade 10 learners.

The majority of learners were of the opinion that the content on the factorisation of quadratics that is delivered at Grade 10 level is appropriate for learners. For example, learner SZ said: *“It is appropriate”*. His view was shared by learner LH who said: *“Yes, it is relevant for us.... it is neither difficult nor easier... it is relevant for the Grade 10”*. Learner XD also supported the presentations made by learners SZ and LH and further blamed himself for his inability to work on the activity, saying: *“I think it is appropriate....the fact is it is simple it is just that we do not want to practice...maybe if I had given myself time to practice I would have been able to do all the problems in the given task”*. A similar view of the grade content on factorisation of quadratics being appropriate for Grade 10 learners was shared by learner DB who said: *“I think it is appropriate ... it is neither difficult nor easy”*. Learner MW shared a similar perspective saying: *“For Grade 10 learners, I will not say it is not relevant to them.....it is not difficult, it is just that algebra is posing a challenge for the many.....many one can say trigonometry is one of those topic understood by the majority...”*. Learner GE supported MW’s view and said: *“I would not say the mathematics content in Grade 10 is easy ... but then I would not say it is difficult because we are trying to do it better at all times”*.

R: Is it appropriate?

GE: “Yes, it is appropriate”.

R: Is the content not overloading you?

GE: *“Some are overloading, sometimes it can be difficult and you may find that it is easy in class ... come out of the classroom ... yoooo it’s something else”*. Learner TM also suggested that the chapters in Grade 10 are not of the same difficulty, yet are appropriate for Grade 10 learners. Learner TM suggested: *“There are a variety of chapters....the chapters done in Grade 10 are not the same.....there are those that are difficult, but they are appropriate for the Grade 10 learners”*. Learner SB was of the opinion that Grade 10 mathematics is doable; hence it is appropriate for Grade 10 learners. However, he was also of the view that mathematics can be simple in the classroom, yet pose some challenges at examination time. He had this to offer: *“In mathematics....for me...for the fact that there are people who are able to understand it in the classroom, I can say it is simple....I think in the level at which it is in at the moment is an appropriate level.....the difficult side of mathematics is the examination time...it is where I can see the difficult side of mathematics because you can find that in the classroom you are an expert of mathematics but come the examination I will be the last to submit my exam script ... but the mathematics we learn in grade 10 is appropriate”*.

6.8 On whether every learner can understand mathematics.

The question was posed to learners: “Do you think the factorisation of quadratic expressions can be done by all learners or is it for the privileged few? Learners offered different views. For example learner SK said: *“Yes, I think it can be done by everybody”*. Learner GD shared a similar sentiment saying: *“Yes...I think everybody can learn mathematics”*. This view was also shared by learner SZ who said: *“It can be done by all learners”*.

R: If that is the case, why are learners not performing well in Grade 10?

SZ: *“It is just that some learners fail to try....it is already embedded in their minds that mathematics is difficult.... you need to love the subject and also love to read ... in order for you to master the subject, you have to love it.... the thing is the person is having a problem in understanding the concept...and on top of that if they do not make an extra effort ...then it becomes a problem”*.

Learner LH spoke about the level of concentration in his remarks as he said: *“Factorisation of quadratics in Grade 10 can be learnt by every learner.....as long as you are focused ... you can do everything”*.

R: Why is there such poor performance in Grade 10 mathematics?

LH: *“Learners do not concentrate”*.

Learner BT supported this view and added the notion of determination claiming: *“Factorisation can be done by everyone who is determined.....diligence is paramount in this case.....anyway it demands ones determination to do anything”*.

Learner SB was of the view that mathematics is doable, but added the following condition: *“I think every learner can understand mathematics....for one to understand, they are compelled to put more focus on it....you need to spare most of your time on it....practising...be attentive. mathematics is for everybody”*.

R: If that is the case, why then is there such poor performance in mathematics in our schools?

SB: *“I think most learners in my class do not give themselves time for mathematics....learners are not putting their attention on the lesson as the teacher teaches....they have told themselves that mathematics is difficult and this is a challenge on the side of the teacher...no matter how thorough the teacher may explain, learners will never understand”*.

R: Where does this perception of “mathematics is difficult” emanate?

SB: *“It begins from the primary schools....even myself.... I recall saying the same thing when I was doing like Grade 4.....but when I reached Grade 6, it was then I began to realise that ...no....mathematics is not difficult.....in Grade 8, I was cruising”*.

The view that different grades posed various levels of difficulties was raised by learner MW who added: *“I do not believe in that claim ... and I won’t believe in it”*.

R: So, do you believe everybody can do mathematics?

MW: *“Yes, I once told myself that mathematics is difficult...however, as times went by ... from grades to grades ... I realised that mathematics is doable”*.

Learner TM held a similar view to learner SB as she said: *“No ... I do not think that every learner can do mathematics”*.

R: Why do you say so?

TM: *“Most learners do not like mathematics....and they are not prepared to learn it ...mathematics demands time for you to understand.... mathematics can be for everybody if, and only if, all learners could give themselves time and space of willing to learn”*.

At first, learner GE felt mathematics is doable by everybody. However, as the interview progressed there was a change of tune. Here is how she responded: *“In my thoughts ... mathematics is for all learners, everybody can learn mathematics and what I can say nothing beats pushing too hard, nothing can beat hard work, you can do it, just as long as you give it focus on it, ...tell yourself that no I am going to do this ... even if it is hard ...even if they say ‘mathematics never loved all of us’, but then I won’t quit”*.

R: Do you think mathematics is only doable for people who love mathematics?

GE: *“I am the one who is supposed to love mathematics ... but then you can love it, put all your energies and only to find the negative response.....because things do not always go as you anticipated”*.

R: Which goes back to say mathematics is not for everyone?

GE: *“Yes ... mathematics is not for everyone”*.

R: So, are you changing the statement now?

GE: *“Actually ... this question is difficult”*.

6.9 On whether their teachers give them enough opportunities to explore mathematics.

Learners were divided into two groups of thinking as they responded to how they viewed their current mathematics teachers in light of the above question. On one hand, almost 50% of the learners were of the view that their teachers did not offer them enough opportunities to explore mathematics. For example learner SK said: *“The manner at which the clarities have been made at this interview in factoring $x^2 - 4$, I do not think they are giving us ample opportunities to explore mathematics”*. Learner SZ’s response was similar: I posed the question: *“How many methods did your teacher show you on the factorisation of the expression like $6x^2 + 7x - 20$?”*

SZ: *“Only one method of trial and error”*.

This response by learner SZ is evidence with regard to the factorisation of this type of quadratic, learners were not given an opportunity to explore. Similarly, learner BT’s response also clearly indicated that learners are sometimes not given opportunities to explore. She said: *“No.....most of the teachers are imparting concepts on the last minutes of the term ... they are expecting us*

to grasp everything in a short space of time.....everything is done on the rush during that period”.

R: Do you think they prepare their lessons?

BT: *“Yes, they do prepare....but lessons are conducted in a quick and rushing mood”.*

Learner NQ supported the claim made by learner BT. She said: *“No ... they do not give us enough time to explore mathematics”.*

R: Do you think your teachers should give you time to explore?

NQ: *“Yes, so that we will have the variety of methods to solve problems ... so that we are able to choose what is right for us so that we are able to choose what is easy for us”.*

In contrast, 50% of the learners held the view that their teachers did provide them with enough opportunities to explore mathematics, even though they were not satisfied with the number of methods they were given to tackle the concept.. For example learner LH said: *“They do give us time because we are able to learn....we do get the class activities.....it will be good if teachers provide the variety of methods for learners to choose the best possible and easy-to-use method”.* Learner XD supported the remark made by learner LH and said: *“Yes.... they do give us enough time to explore mathematics I can say that because I always use the methods taught by him in my mathematics....he also encourages us to use the variety of methods in solving mathematical problems...he could say you can use any method in mathematics as long as it will bring you to the desired destination”.*

R: In your view, do your teachers prepare their lessons?

XD: *“Yes...he does prepare his lessons”.*

R: What practice work does your teacher offer?

XD: *“He gives us extra work for practicing....if the majority of us was unable to do it, he will spend much of his time doing the clarifications....ultimately we all understand....he could even sacrifice the time that was intended for the new topic in ensuring that we understand better”.* A similar claim was echoed by learner DB who said: *“Yes....I can say the way he teaches us is it does afford us the opportunity to explore mathematics he does more examples for us to understand....he also gives us time to do corrections”.* Learner GD switched the focus from

that of the teacher and redirected it to peer learning. She said: *“Yes ... our teachers of mathematics do give us enough opportunity to explore mathematics ... because sometimes in the class, when we do corrections you find that there are some learners who did it in the other way when we do corrections ... our teacher would do both of those examples and say... if you think that one is better or you understand it better, you can use it, if you want to”*.

6.10 On the role they expect from their mathematics teachers in order for them to learn mathematics better.

Learners gave positive contributions which best describe the role of the mathematics teacher.

6.10.1 Show high command of the content.

Learner LH said: *“What my teacher is doing currently is acceptable ... but what I would prefer the most is my teacher to extend the lessons even on extra hours.....I expect him to clarify the concept till I understand.....he must provide the clear response if and ever I pose a question”*.

Learner DB was of a similar opinion, saying: *“For me to learn mathematics better ... my teacher should be able to respond in every aspect I am not sure about....he must be in class at all times....he must ensure that we understood everything and anything....always do the consolidation work as a way to check that we understood”*.

6.10.2 Provide multi-methods when addressing the concept.

The notion of the use of multi-methods was submitted by learner BT. In her response to the question of what teachers need to do so that learners are left convinced that enough time had been given to a concept, she said: *“I expect him.....not to teach on the last minutes....to dwell in the topic and be emphatic before he could proceed to the next topic ... teachers should expose us to all different methods.....they need not come to class with the pre-determined method to infuse in us....”*. This notion was supported by learner MW who added: *“The current teacher ... he is ok in the way he teaches us....I do understand and if I am not clear, I do ask questions....he exposes us to a variety of approaches.....”*. Learner TM affirmed this claim and said: *“If the teacher is using multi-methods I am able to catch things when he is teaching....I quickly understand the concept that is being taught....I am a fast learner”*.

R: I am talking about the actions of the teacher.

TM: *“Even while the teacher is explaining the concept on the chalkboard, I am able to catch and understand....the teacher does not have to make a thorough explanation”.*

6.10.3 To be practical in the approach and offer practice exercises.

Learner GD raised the notion of mathematics teachers being practical in the execution of mathematical concepts. She said: *“My teacher of mathematics should do ... like maybe ... make mathematics to be practical and simple ... work with examples that we can see around.... and be able to explain same things again and again.... if we don’t understand, be able to be patient and understand.”* This notion was supported by learner SB in her remarks: *“Our teacher is just a good teacher.... he teaches us a lot..... other grades are complaining about their teachers, but our teacher is the best teacher..... and I think I don’t need to go to other lessons to learn mathematics at my time, so I just need to focus on him a lot..... he has done a lot everything that came in exams we’ve made it.....If he explains thoroughly I quickly understand.....I need to know and understand the topic by making use of examples of what we are dealing with.....he should refrain from dealing with complex exercises....he must engage us with what is easy”.* This view was supported by learner XD who submitted his contribution by saying: *“I think my teacher is supposed to leave me with clear understanding on the particular concept my teacher need not be on the hurry when dealing with concept....he must make sure that we are together need not to teach more than one concept in a single period because as learners we are getting confused always ask if we are still on track..... always give us some work to do at home to foster practicing”.* Learner NQ was also emphatic regarding the role practice played in learning. According to her, the role of the teacher should be to enforce practice. She explained: *“Firstly, the teacher has to provide examples for us to understand better....he has to provide some activities for us to practice.....he has to provide the homework exercises as well”.*

6.10.4 Promote individualisation; differentiated teaching.

According to learner SB, the ideal teacher should be able to read the dynamics of the class with regard to learners’ capabilities. Such understanding can only be made if the teacher is aware of the capabilities of each individual learner. Learner SB had this to offer: *“The best teacher is*

the one who takes care of everybody in the class.....he will focus on those that are struggling by ensuring that they understand the concept that is being taught". The practice of differentiated teaching was also advocated by learner GE who said: "What I expect at the moment from my teacher: he need not be too fast for a sake of completing the syllabus....as a class we need to come with different views because as a learner you are able to understand another learner better because you speak the same language he must do the corrections or provide the platform to a learner who has done much better, for that learner to bring some clarity to the whole class.....this is because teachers may think they are clarifying challenges but only to find that they are confusing us as learners and we are far behind like they want us to do maths in their way.....it becomes difficult to grasp the teachers methods if you are failing to understand".

R: I heard you talking about covering the curriculum. Don't you think teachers also operate within the constraints of time, since the curriculum has to be taught within a specified period?

GE: *"Yes. It does help to cover the topics, however covering the topic without understanding is futile when it comes to the examinations".*

6.11 On motivating a learner who is experiencing some challenges regarding the concept of factorisation of quadratics.

All learners contributed and submitted innovative and positive responses on the matter.

6.11.1 The emphasis on practice as a practice.

Learner SK echoed his opinion and stressed the use of practice as a practice. He said: *"It is important for a learner to practice mathematics day-by-day...you cannot go to sleep without having done some mathematics... in most cases in maths we factorise and then simplify.....the way I look at maths it is easy to solve than to simplify....finally you have to check whether you are correct.....if you are incorrect you have to start afresh...people say maths is difficult... I think you have to know yourself and always believe in you that you can do it. You need to be attentive; work hard and be visionary".* This view of practice was also stressed by learner DB who said: *"In motivating the Grade 10 learner who is perceived to be struggling in mathematics ... I could say....practice is very important....consultation with those learners who seem to*

understand the concept is also important.....make use of the teacher all the times". Learner GD also supported the view of practice as held by learners SK and DB saying: *"In guiding a learner who is struggling in mathematics I might tell her to try to learn more regular things like square numbers, cube numbers... and to do more of practicing...practice makes perfect"*. The notion of practice as a learner practice also surfaced in the response offered by learner NQ who said: *"My advice will be ... since you have chosen to do mathematics, you do not have to be the free man...you have to study..... you have to practice.....you need to give yourself ample time to spend on your work"*.

6.11.2 Teacher consultation; the use of technology.

According to learner LH, it is critical to create a good relationship with your mathematics teacher in order to perform well in mathematics. He said: *"The key thing is to always raise your hand and ask for assistance from the teacher if you are unclear"*. Learner BT supported this notion and went on to stress the use of technology in assisting learners to perform well in mathematics. She said: *" if you come across a challenge, do not hesitate to consult the teacher for help and guidance.....approach another learner for help and guidance.....watch the TV programmes on mathematics.....these learners have cell phones, they can download the mind-set activities and engage on the practice"*.

6.11.3 Associating with diligent peers.

According to learner SB, it is paramount to associate with people who share your vision.. He explained: *"I can say....firstly....do not perceive mathematics as difficult.....refrain from listening to people who have already failed in mathematics because they will demotivate you.....learn to love mathematics for mathematics to love you back..... associate yourself with people offering the extra tuition on mathematics.... have time for your mathematics ... practice all the times"*. This opinion of learning from your peers was also shared and stressed by learner TM who said: *"I could say....practice is very important....consultation with those learners who seem to understand the concept is also important.....make use of the teacher all the times"*.

6.11.4 Intrinsic motivation.

Learner MW spoke about the role intrinsic motivation plays in helping learners learn successfully. She had this to offer: *“You can only learn mathematics with understanding if you are eager to learn....if you consult for assistance ... intrinsic motivation is paramount....she needs to love maths.....practice makes perfect”*. This view of self-dedication was shared by learner GE who said: *“What I can say mathematics needs someone who is dedicated and it needs passion, ...and it needs someone who can boldly say ‘I love mathematics’.....Mathematics needs practice and sharing of knowledge and opinions with your classmates....you need to believe in yourself”*.

6.12 Conclusion.

In this Chapter we were able to gather learners’ narratives based on how they view mathematical practices in their respective classrooms. In Chapter 7 we will be looking at how learners engaged with the mathematics activities.

CHAPTER 7

LEARNERS' INTERVIEW RESPONSES TO THE ACTIVITY OF FACTORING QUADRATICS, AND THE ANALYSIS OF THEIR PERFORMANCE IN THE MARCH 2019 GRADE 10 COMMON TEST.

7.1 Introduction.

This chapter presents an analysis of responses from learners' interviews on the mathematical activity. The interview questions were aimed at gaining a better understanding on how learners engaged with the activity on the factorisation of quadratics. Items of the activity on the factorisation of quadratics were adapted from the previous years' Grade 10 mathematics common paper. The activity on the factorisation of quadratics comprised of tasks that required learners to engage, firstly on quadratic expressions that were factorable, and secondly, on those that were not factorable. All twelve learners interviewed did the activity on the factorisation of quadratics. Two of the seven learners from the R-School participated at the pilot stage of this study. This means that there are only five learners who participated in the main research project.

7.2 Learners responding to items on factoring quadratics.

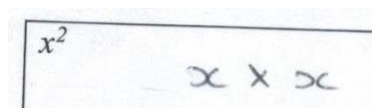
7.2.1 Task 1: Factorisation of x^2 .

For this task, all learners confidently arrived at the correct solution. However, I found the following explanations during the interviews:

7.2.1.1 Learners' use of the exponential laws.

Two learners (MW and SK) explained their procedures by making use of the exponential law. For example, MW offered this explanation:

MW provided the following working,



A photograph of a piece of paper with handwritten mathematical work. On the left, there is a box containing the expression x^2 . To the right of the box, the expression $x \times x$ is written.

and said “in factorising x^2 you get $x \cdot x$ and if you check $x \cdot x$ is the same as $x^1 \cdot x^1 = x^{1+1} = x^2$ ”.

On the factorisation of x^2 , Sthembiso arrived at the solution of $(x \cdot x)$ following the similar approach as learner MW. When asked why, Sthembiso responded by this way,

Handwritten work for factorising x^2 . The work shows x^2 in a box, then $x \cdot x$, then $x \cdot x^1$, and finally $x^{1+1} x^2$.

and said “it is easy just because when you factorise...you factorise what you are able to reverse....so I noticed that I have x^2 I then separated it to be $x \cdot x$ because there is a law on exponents which states that if you multiply the bases that are the same you simply add the exponents, that is $x^1 \cdot x^1 = x^{1+1} = x^2$ ”.

7.2.1.2 Learners’ use of the trial and error approach.

Another approach followed by learner (GD) was the use of the trial and error. Here is learner GD workings,

Handwritten work for factorising x^2 using trial and error. The work shows x^2 , then $x(x+0)$, then $x(x+1)$ and x^2+x , and finally $x(x+0)$ and x^2 .

coupled by the following explanation, “at first... I tried looking for the highest common factor in x^2 which is x I took it aside and opened brackets and I thought of putting $x(x - 1)$ and then when I checked my answer if it’s right, I distributed x inside and got $x^2 + x$ Then, I checked $x(x - 1)$ doesn’t give x^2 so then I tried using number less than 1 then it was $x(x - 0)$ hence $x(x)$ ”.

A summary of learners’ interviews in terms of correct/incorrect; valid/invalid reasons when responding to task 1 of the activity of factoring quadratics is presented in Table 7.1

Table 7.1: Consolidated summary of interview responses on Task 1 of learners' activity on factorisation of quadratics in terms of correct/incorrect; valid/invalid reasons.

x^2	No Response	Incorrect Response	$x \cdot x$ Without Reason	$x \cdot x$ With Invalid Reason	$x \cdot x$ With Valid Reason
Learners					SB; GE; BT; SK; SZ; LH; XD; DB; GD; MW; TM; NQ
Number of learners	0	0	0	0	12
%	0%	0%	0%	0%	100%
12 learners responded with reasons	8 learners spoke about the removal of the common factor; 3 learners spoke about the exponential laws; 1 learner spoke about the trial and error of factors				

Data collected from the learners' activity, and presented in Table 7.1, shows that all 12 learners were able to factorise x^2 to achieve $x \cdot x$ following different routes. Nine learners (75%) spoke about the removal of the common factor, with two learners speaking about the exponential laws. For this exercise alone, the use of exponents is permitted, since both routes lead to the desired destination.

7.2.2 Task 2: Factorisation of $x^2 - 4$.

7.2.2.1 Correct responses following the rule of factoring the type of difference of two squares.

Most learners were comfortable working with the given expression, with some even commenting that it was the difference of two squares. Some even went on to check without having being asked to do so. Here is the explanation from learner DB “*same thing applies here,*

$$\begin{array}{l} x^2 - 4 \\ (x-2)(x+2) \end{array} \quad \begin{array}{l} x^2 + 2x - 2x - 4 \\ x^2 - 4 \end{array}$$

...I found the factors of x^2 which are x and x ... I also found the factors of -4 which are -2 and $+2$ I opened the brackets and found $(x-2)(x+2)$ and I checked $(x-2)(x+2)$ becomes $x^2 + 2x - 2x - 4$ and $+2x - 2x$ cancel one another and leave $x^2 - 4$ ". From this discussion it is evident that learners have developed their own rules of working confidently with the quadratics of the difference of two squares.

7.2.2.2 Incorrect responses

However, one learner (LH) portrayed some degree of misconception in his explanation that when he said “*And in here I took out, x ,*

$$x^2 - 4$$

$$x(x) - 4$$

showing that it is $x \cdot x - 4$... here x will multiply only x not 4... hence $x \cdot x - 4$ ”.

R: Ok, have you ever come across something called the “difference of two squares”?

LH: “*Yes*”

R: If so, how do you perform the factorisation in the difference of two squares?

LH: “*I do not recall exactly*”

A summary of learners’ interviews in terms of correct/incorrect; valid/invalid reasons when they responded to task 2 of the activity of factoring quadratics is presented in Table 7.2 that follows.

Table 7.2: Consolidated summary of interview responses on Task 2 of learners’ activity on factorisation of quadratics in terms of correct/incorrect; valid/invalid reasons.

$x^2 - 4$	No Response	Incorrect Response	$(x - 2)(x + 2)$ Without Reason	$(x - 2)(x + 2)$ With Invalid Reason	$(x - 2)(x + 2)$ With Valid Reason
Learners		LH			SB; BT; SK; SZ; XD; DB; GD; TM; NQ; MW; GE
Number of learners	0	1	0	0	11
%	0%	8%	0%	0%	92%
11 learners with reasons	4 learners spoke about the Difference of Two squares; 7 learners spoke about the trial and error of factors				

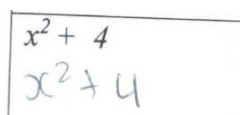
Data captured from the activity, and presented in Table 7.2, reveals that out of eleven learners that provided the correct response by factoring $x^2 - 4$ to get $(x - 2)(x + 2)$, four learners spoke about the “difference of two squares”, with seven learners speaking about the trial and error of factors.

7.2.3 Task 3: Factorisation of $x^2 + 4$.

7.2.3.1 Correct responses

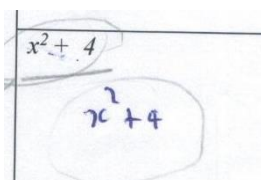
7.2.3.1.1 Learners stated “not factorisable” and followed the technique of difference of two squares type.

Learners BT and MW shared similar sentiments. Here is an example of the response from BT when she said “*I have never come across the addition of two squares*”


$$\begin{array}{l} x^2 + 4 \\ x^2 + 4 \end{array}$$

...I just left it as it is”.

Learner (SK) provided the following mathematical argument,


$$\begin{array}{l} x^2 + 4 \\ x^2 + 4 \end{array}$$

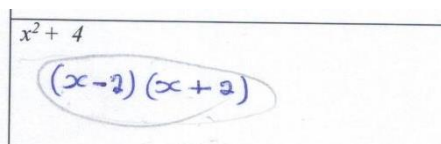
when he said “*The way I look it, it is different from $x^2 - 4$ because you can factorise it, but once when factorised, when you try to reverse to its originality it becomes difficult....this thing of $x^2 + 4$ has a trick because you are unable to reverse it to the original expression”.*

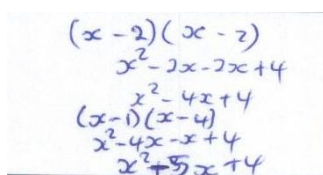
R: Do you agree that $x^2 + 4$ is a quadratic? ...and any quadratic can be factorised?

SK: “*Yes, it is a quadratic. In case we are asked to factorise....when you factorise ...is either we take common factor or you open two brackets, so I realised that in this case we will never be able to factorise....if I make use of 2 and 2, I get this $(x + 2)(x + 2)$, this thing will no way return to the original expression as it becomes $x^2 + 4x + 4$, that is why I left it unanswered”.*

7.2.3.1.2 Learners stated “not factorisable” and followed the trial and error approach.

Learner (GD) was also able to share her journey of understanding by writing the following,


$$\begin{array}{l} x^2 + 4 \\ (x-2)(x+2) \end{array}$$


$$\begin{array}{l} (x-2)(x-2) \\ x^2 - 2x - 2x + 4 \\ x^2 - 4x + 4 \\ (x-1)(x-4) \\ x^2 - 4x - x + 4 \\ x^2 + 5x + 4 \end{array}$$

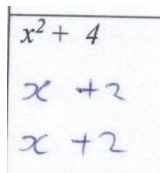
And said “I am not sure if this is right because there are no numbers that give me a positive 4 and when add those number give me zero. So, I then thought of putting negative 4 and 1 thought do not give me positive 4..... I was not able to find correct answer”.

7.2.3.2 Learners’ incorrect responses.

Five types of learners’ misconceptions were identified

7.2.3.2.1 Learners treated $x^2 + 4$ as $x^2 + 4x + 4$.

Learner SB wrote,



$$\begin{array}{l} x^2 + 4 \\ x + 2 \\ x + 2 \end{array}$$

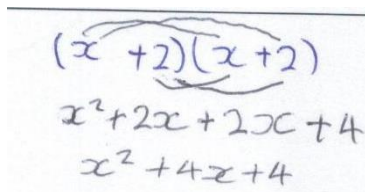
and said “I said $x^2 + 4 = (x + 2)(x + 2)$ you need to observe that x^2 easily breaks to x and x ...then the factors of 4 are 2×2 ; -2×-2 ”.

R: Why did you choose 2×2 ?

SB: “I chose 2×2 because the positive numbers are numbers that are easy to work with are easily understood”.

R: Now let us check if $(x + 2)(x + 2) = x^2 + 4$

SB wrote,



$$\begin{array}{l} (x + 2)(x + 2) \\ x^2 + 2x + 2x + 4 \\ x^2 + 4x + 4 \end{array}$$

and said “ $(x + 2)(x + 2)$ becomes $x^2 + 2x + 2x + 4$...this becomes $x^2 + 4x + 4$ mhhhh.... I have gone wrong somewhere”.

R: Do you still maintain that $(x + 2)(x + 2)$ are the factors?

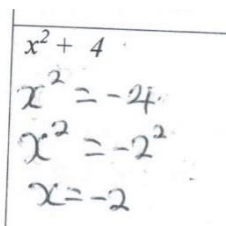
SB: “No”.

R: Why?

SB: *"Because when I simplify I do not get the original expression".*

7.2.3.3.2 Learners followed the exponential route to prove that $x^2 + 4$ can be factorised.

Learner SZ wrote,


$$\begin{array}{l} x^2 + 4 \\ x^2 = -4 \\ x^2 = -2^2 \\ x = -2 \end{array}$$

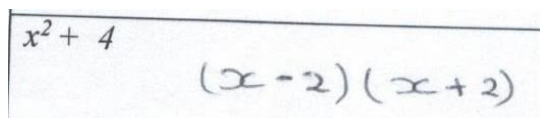
and said *"What I did first was to transpose 4 and achieved $x^2 = -4$. then what can we do to make the base the same as well as the exponents be the same. $x^2 = -2^2$, 4 can be 2^2 , because the exponents are the same, we will ignore the minus sign in 2".*

R: In the original expression there was no equal sign, where does it come from now?

SZ: *"I transposed 4 because it was going to be difficult to factorise if in the given expression: $x^2 + 4$ ".*

7.2.3.3.3 Learner treated $x^2 + 4$ as $x^2 - 4$.

Learner MW wrote,


$$\begin{array}{l} x^2 + 4 \\ (x-2)(x+2) \end{array}$$

and said *"I committed the similar mistake...I swopped the signs"*

R: Do you think $x^2 + 4$ is factorisable?

MW: *"Yes"*

7.2.3.3.4 Learner treated $x^2 + 4$ as $x^2 - 4x + 4$.

Learner DB wrote,

$$\begin{array}{l} x^2 + 4 \\ (x-2)(x-2) \end{array}$$

and said “Same thing here as well....I collected the factors of x^2 as well those of +4 and found the answer $(x-2)(x-2)$ ”.

R: Can you check if $(x-2)(x-2)$ are the real factors?

DB responded,

$$\begin{array}{l} x^2 - 2x - 2x + 4 \\ x^2 \end{array}$$

and said “ $(x-2)(x-2)$ becomes $x^2 - 2x - 2x + 4$ which finally is $x^2 - 4x + 4$ eish....I didn't get it right”.

7.2.3.3.5 Learner showing a shallow conceptual understanding of factorisation.

Learner LH wrote,

$$\begin{array}{l} x^2 + 4 \\ x(x) + 4 \end{array}$$

and echoed “I did exactly as in $x^2 - 4$... the difference is +4”.

A summary of learners' interviews in terms of correct/incorrect; valid/invalid reasons when they responded to task 3 of the activity of factoring quadratics is presented in Table 7.3

Table 7.3: Consolidated summary of interview responses on Task 3 of learners' activity on factorisation of quadratics in terms of correct/incorrect; valid/invalid reasons.

$x^2 + 4$	No Response	Incorrect Response	Correct Response Without Reason	Correct Response With Invalid Reason	Correct Response With Valid Reason
Learners		SB; GE; LH; XD; DB; MW; TM; NQ			BT; SK;SZ; GD
Number of learners	0	8	0	0	4
%	0%	67%	0%	0%	33%
4 learners with reasons	2 learners spoke about the difference of two squares; 2 learners spoke about the trial and error of factors				

Data presented in Table 7.3, shows that, out of the four learners who responded correctly, two learners spoke about the difference of two squares, with two learners speaking about the trial and error of factors. In arguing why $x^2 + 4$ cannot be factorised, learner (SK) spoke about the trial and error of factors approach, yet embedding the fundamental principles.

7.2.4 Task 4: Factorisation of $x^2 - 4x$.

In engaging with task 4, ten learners were able to arrive at the correct solution. During the interviews the following trends emerged, and were discussed as follows:

7.2.4.1 Learners' correct responses

7.2.4.1.1 Learners taking out the common factor.

Nine learners used the approach of taking out the common factor. For example, learner (LH) offered this response,



$$x^2 - 4x$$

$$\underline{x(x - 4)}$$

and said “Here there appears x in all terms...meaning I will take out x and put everything in brackets, hence $x(x - 4)$,....because I want $4x$, I will search for the number that must be multiplied by x to get $4x$, the number is 4”.

Three learners (TM, SB and BT) offered the following explanation:

TM wrote,

$$\begin{array}{l} x^2 - 4x \\ x(x-4) \end{array}$$

and said “In $x^2 - 4x$ there is a common factor x which will be taken out ... x^2 divided by x is x ... $4x$ divided by x is 4 ...and become $x(x - 4)$ ”.

Learner (SK) went even further to offer the extra example of his own. Here is how he explained his mathematical moves,

$$\begin{array}{l} x^2 - 4x \\ x(x-4) \end{array} \quad \begin{array}{l} x^2 - 6x \\ x(x-6) \end{array}$$

and said “This one is very easy – as I indicated earlier on – before you factorise, you first check if there exist a common factor or not...in this case we have common factor which is x then $x(x - 4)$, for an example if we have $x^2 - 6x$ it will become $x(x - 6)$. This is because if you multiply this you will get what was given before... it is one and the very same thing because you going to take out x as common factor”.

Learner (MW) went further on to talk about checking whether the solution is correct. Here is her mathematical presentation,

$$\begin{array}{l} x^2 - 4x \\ (x)(x-4) \quad x(x-4) \end{array}$$

and said “I took out the common factor x and got $x(x - 4)$...if you check you get $x^2 - 4x$ ”.

7.2.4.1.2 Learner using the fundamental approach of factoring the quadratic.

One learner (GD) used the fundamental approach when arguing how she arrived at the solution. Her explanation proceeded as follows,

$$x^2 - 4x$$

$$(x-4)(x-0)$$

$$(x-4)(x)$$

and offered this narrative “For this one, I went to look for two numbers that will give me -4 when added them together.... but then give me a product of zero, so I went for -4 and 0 ... I put $(x - 4)(x - 0)$ ”.

7.2.4.2 Learners’ incorrect responses.

7.2.4.2.1 Learners portraying shallow conceptual understanding of factorisation.

However, there were only two learners (XD and NQ), who portrayed some misconceptions. Surprisingly though, when both learners were offered the opportunity to check their responses, they respectively did, and boldly took their stands that they were incorrect. Learner NQ provided the following answer,

$$x^2 - 4x$$

$$(x-2x)(x+2x)$$

R: Can you check if $(x - 2x)(x + 2x)$, are the correct factors?

Learner NQ provided this working,

$$x^2 + 2x^2 - 2x^2 - 4x$$

$$x^2 - 4$$

as her answer. When she was asked to check she got $x^2 - 2x^2 - 2x^2 - 4x$, and she quickly realised that she was incorrect. She said: “*hhayi siwrong (meaning no, it is incorrect) ...it was supposed to be $-4x^2$ instead of $-4x$* ”

R: Don’t you have another approach for solving $x^2 - 4x$?

Learner NQ wrote,

$$x^2 - 4x$$

$$x(x-4)$$

$$x^2 - 4x$$

and said “There it is: $x(x - 4)$ ”.

R: So what was the problem initially?

NQ: “I forgot to be attentive here...really I did not focus”.

When asked how he factorised $x^2 - 4x$, learner XD offered the following response,

$$\begin{array}{l}
 x^2 - 4x \\
 (\cancel{x-2x})(x+2) \\
 (\cancel{x-2x})(\cancel{2x}+2) \\
 (x-2)(x+2x)
 \end{array}$$

and said “I said my answer here is $(x - 2)(x + 2x)$ ”.

R: Can you check if $(x - 2)(x + 2x)$ are the real factors?

Learner XD wrote,

$$\begin{array}{l}
 = x^2 + 2x^2 - 2x - 4x \\
 = 4x^2 - 6x
 \end{array}$$

and commented “ $(x - 2)(x + 2x)$ becomes $x^2 + 2x^2 - 2x - 4x$ which then becomes $4x^2 - 6x$...it is not correct”

Summary of learners’ interviews in terms of correct/incorrect; valid/invalid reasons when responding to task 4 of the activity of factoring quadratics is presented in Table 7.4

Table 7.4: Consolidated summary of interview responses on Task 4 of learners’ activity on factorisation of quadratics in terms of correct/incorrect; valid/invalid reasons.

$x^2 - 4x$	No Response	Incorrect Response	$x(x - 4)$ Without Reason	$x(x - 4)$ With Invalid Reason	$x(x - 4)$ With Valid Reason
Learners		XD; NQ	DB		SB; GE; BT; SK; SZ; LH; GD; MW; TM
Number of learners	0	2	1	0	9
%	0	17%	8%	0	75%
9 learners who responded with reasons	8 learners spoke about the removal of the common factor; 1 learner spoke about the trial and error of factors				

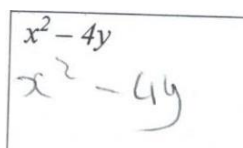
Data from the learners' activity presented in Table 7.4, reveals that 10 learners were able to factorise $x^2 - 4x$, to reach $x(x - 4)$. Seven of the ten learners followed the generalisation of “taking out the common factor”. One learner, (GD), was spotted using the trial and error method by first perceiving all types of quadratic expressions as $ax^2 + bx + c$. For the purpose of this thesis, I termed this kind of working with quadratic expressions “the fundamental consideration in factoring quadratics”, and discussed in detail in section 1.7.4.1 of this thesis.

7.2.5 Task 5: Factorisation of $x^2 - 4y$.

7.2.5.1 Learners' correct response of “not factorisable”

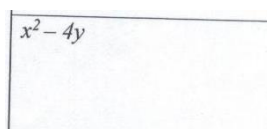
7.2.5.1.1 Learners following a technique of greatest common factor.

Two learners, BT and MW shared similar sentiments in stating why $x^2 - 4y$ is not factorisable. Here is an example of the response from both learners respectively. Learner BT wrote,



and said “*Here there is no common factor...I left it just as it is*”.

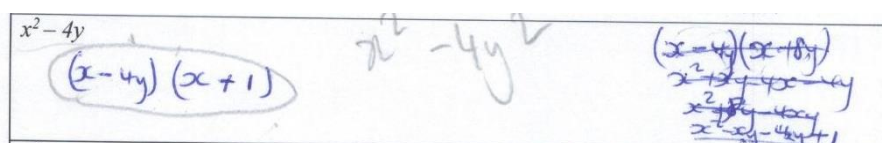
Learner MW wrote,



and said “*There is no way you can add or subtract the unlike terms.....again there is no common factor*”.

7.2.5.1.2 Learner following a trial and error approach.

Another mathematically argued response was offered by learner GD. Here follows her writings,



and explained “For this $x^2 - 4y$... I tried to look for factors that when add together will give $-4y$ so I wasn’t able to find the two numbers that will give sum of $-4y$... so I was having a bit of difficult in this question”.

7.2.5.1.3 Learner following a technique of difference of two squares.

Again, learner SK brought some clarity on the matter by following a technique of difference of two squares, when he wrote,

and said “This one the way I look at it...I see it the same as $x^2 - 4x$, if you look at $x^2 - 4x$ you could see that there is a common factor – you take out the common factor. Then here in $x^2 - 4y$ there is no common factor furthermore, there is no way you can open two brackets...let us say now you are closing those brackets, you will not be able to return to the original...the problem is there is no way to split this y in $4y$ I see in this case y is not squared... a squared is not the same as stand-alone x . the x^2 can be factorised like this $x^2 = x \cdot x$ yet in this case y is stand alone and is without a square – hence can never be factorised”.

7.2.5.2 Learners’ incorrect responses.

Four learners were identified as portraying shallow conceptual understanding of factorisation.

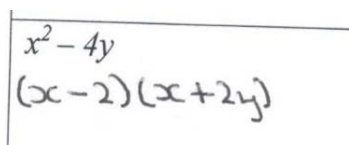
7.2.5.2.1 Learners factored $x^2 - 4y$ to be $x(x) - 4(y)$.

Learner SB wrote,

and said “This one was difficult.... I tried with my knowledge because I don’t want to fail.... then I first simplified x^2 to become $(x)(x)$..and then $-4y$ to become $(-4)(y)$...which I’m not sure if it is correct”.

7.2.5.2.2 Learners factored $x^2 - 4y$ to be $(x - 2)(x + 2y)$.

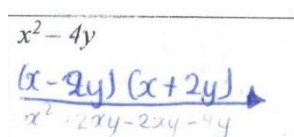
Learner XD wrote,


$$\begin{array}{l} x^2 - 4y \\ (x-2)(x+2y) \end{array}$$

and said “Eish....the solution here is the same as in $x^2 - 4x$ ”.

7.2.5.2.3 Learners factored $x^2 - 4y$ to be $(x - 2y)(x + 2y)$.

Learner NQ wrote,


$$\begin{array}{l} x^2 - 4y \\ (x-2y)(x+2y) \rightarrow \\ x^2 + 2xy - 2xy - 4y \end{array}$$

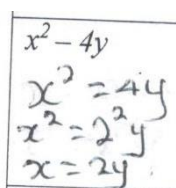
and said “Even though this solution is incorrect $(x - 2y)(x + 2y)$ ”.

R: What do you think needs to be done?

NQ: “I have forgotten”.

7.2.5.2.4 Learner followed the exponential route to show that $x^2 - 4y$ can be factorised.

Learner SZ followed the exponential route, where he wrote,


$$\begin{array}{l} x^2 - 4y \\ x^2 = 4y \\ x^2 = 2^2 y \\ x = 2y \end{array}$$

and said “Same thing here, I transposed $-4y$ (same as in $x^2 + 4$) I said $x^2 = 4y$ then changed 4 to be 2^2 , then $x^2 = 2^2 y$, 2 and 2 (referring to exponents) cancelled one another and I was left with $x = 2y$ ”.

A summary of learners’ interviews in terms of correct/incorrect; valid/invalid reasons when responding to task 5 of the activity of factoring quadratics is presented in Table 7.5.

Table 7.5: Consolidated summary of interview responses on Task 5 of learners' activity on factorisation of quadratics in terms of correct/incorrect; valid/invalid reasons.

$x^2 - 4y$	No Response	Incorrect Response	Correct Response Without Reason	Correct Response With Invalid Reason	Correct Response With Valid Reason
Learners		SB; GE; SZ; LH; XD; DB; TM; NQ			MW; GD; CF; BT
Number of learners	0	8	0	0	4
%	0	67%	0	0	33%
4 learners responded with reasons	3 learners spoke about the non-existence of the common factor; 1 learner spoke about the trial and error of factors				

Data presented in Table 7.5 shows that, of the four learners who responded with reasons three learners spoke about the non-existence of the common factor, while one learner, GD, spoke about the use of trial and error of factors in arguing why $x^2 - 4y$ cannot be factorised.

7.2.6 Task 6: Factorisation of $x^2 - x - 6$.

For this task, all twelve learners produced the correct factorisation. In the interviews, many interesting trends emerged and were discussed as follows:

7.2.6.1 Learners ignored the role of x^2 in task 6.

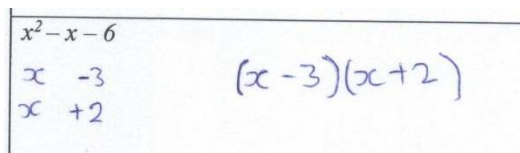
Four of the twelve learners focused on getting the factors of -6 and ignored the factors of x^2 in their discussion. For example, R-Learner (LH) offered this working,

$$\begin{array}{l} x^2 - x - 6 \\ (x - 3)(x + 2) \end{array}$$

and explained “Here I searched for two numbers which when are multiplied and give me -6 , when added give -1 ... I got -3 and 2 ...hence $(x - 3)(x + 2)$ ”. As is evident, although LH has identified the two factors, he did not consider it important to discuss the factors of x^2 . This practice among the learners is common if the given quadratic is of $a = 1$.

7.2.6.2 Learners recognised the role of x^2 in task 6.

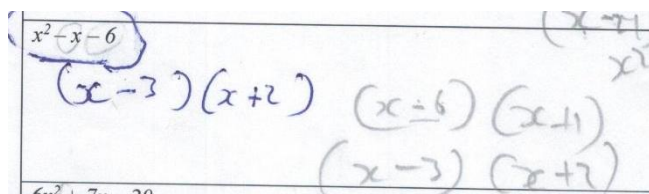
There were 7 students who made explicit mention of the factors of x^2 , as well as those of -6 . For example SB wrote,


$$\begin{array}{l} x^2 - x - 6 \\ x - 3 \\ x + 2 \end{array} \quad (x - 3)(x + 2)$$

and explained “*This was one of the easiest.....I found the factors x^2 which are x and x I then looked for the factors of -6 which when added will give -1 , and it was -3 and 2 and opened the brackets...hence $(x - 3)(x + 2)$ ”.*

7.2.6.3 Learners followed the trial and error approach.

A single learner (SK), went further to discuss how he engaged in the trial and error approach as a way of arriving at the solution, when he wrote,


$$\begin{array}{l} x^2 - x - 6 \\ (x - 3)(x + 2) \\ (x - 6)(x + 1) \\ (x - 3)(x + 2) \end{array}$$

and gave his explanation as follows “*This is a trinomial, it is easy as I indicated earlier on that if we factorise make sure that in your two factors the sign of the bigger factor is determined by the sign of the middle term. We use the trial and error e.g. if I use $(x - 6)(x + 1)$, when I multiply it will give x^2 and -6 but unable to give (-1) of the middle term... this means I had to look for other factors which when added will give me the middle term of (-1) – the terms are $(x - 3)(x + 2)$ ”.*

Summary of learners’ interviews in terms of correct/incorrect; valid/invalid reasons when responding to task 6 of the activity of factoring quadratics is presented in Table 7.6 that follows.

Table 7.6: Consolidated summary of interview responses on Task 6 of learners' activity on factorisation of quadratics in terms of correct/incorrect; valid/invalid reasons

$x^2 - x - 6$	No Response	Incorrect Response	$(x - 3)(x + 2)$ Without Reason	$(x - 3)(x + 2)$ With Invalid Reason	$(x - 3)(x + 2)$ With Valid Reason
Learners		DB			SB; GE; BT; SK; SZ; LH; XD; GD; MW; TM; NQ
Number of learners	0	1	0	0	11
%	0%	8%	0%	0%	92%
All learners	All 12 learners spoke about the trial and error factors, and this includes learner DB whose mistake was to swap the signs				

Data extracted from the learners' activity presented in Table 7.6, shows that all twelve learners adopted the use of the trial and error method to factoring a trinomial quadratic of $a = 1$, as they were presented with $x^2 - x - 6$, to factorise. This includes learner DB, the only learner who provided the incorrect response, and whose mistake was swapping the signs in her factors.

7.2.7 Task 7: Factorisation of $6x^2 + 7x - 20$.

7.2.7.1 Learners' correct responses

For task 7, four learners produced the correct factorisation, with correct explanations on how they arrived at their solutions. In the interviews, two interesting methods of dealing with factorisation of quadratics emerged, and are discussed as follows:

7.2.7.1.1 Learners used the Autrey's and Austin's method (A-A method).

Here is how learner (SB) explained his approach, as he wrote,

Handwritten work for $6x^2 + 7x - 20$:

Left side: $6x^2 + 7x - 20$, $6 \times 20 = 120$, Factors of 120: $+15 \quad -8$. The work shows the expression $\frac{(6x+15)(6x-8)}{6}$ and $\frac{3(2x+5)2(3x-4)}{6}$.

Right side: $\frac{6(2x+5)(3x-4)}{6}$, then $\frac{(2x+5)(3x-4)}{1}$, followed by $6x^2 - 8x + 15x - 20$, and finally $6x^2 + 7x - 20 = 0$.

and said “I first multiplied 6 and -20 and got -120 , I then looked for the two factors of -120 which when added will give $+7$ and I got $+15$ and -8 , I opened the brackets and divided all by 6 which was in first term $\frac{(6x+15)(6x-8)}{6}$ I then looked for the highest common factors in the numerator and got 3 and 2 respectively $\frac{3(2x+5)2(3x-5)}{6}$ I then simplified by dividing 6 and 6 and was left with $(2x + 5)(3x - 4)$ ”.

R: Can you check if these factors are correct?

Learner SB responded,

$$\begin{array}{r} (2x+5)(3x-4) \\ 6x^2 - 8x + 15x - 20 \\ \hline 6x^2 + 7x - 20 \end{array}$$

and explained “Yes $(2x + 5)(3x - 4)$ becomes $6x^2 + 15x - 8x - 20$ which becomes $6x^2 + 7x - 20$ ”.

Another learner MW produced the following workings,

$$\begin{array}{l} x^2 - x - 6 \quad (6x+15)(6x-8) \quad = \frac{6(2x+5)(3x-4)}{6} \\ 6x^2 + 7x - 20 \quad \quad \quad 6 \quad \quad \quad = (2x+5)(3x-4) \\ \quad \quad \quad = \frac{3(2x+5)2(3x-4)}{6} \end{array}$$

and she said “you first multiply first term with the last term and get answer A, ...look for the co-factors of answer A which when added will give the middle term.....I proceeded and got $(2x + 5)(3x - 4)$ ”.

7.2.7.1.2 Learners used the method of trial and error.

Even though learner GD was incorrect when inserting the signs within the factors, she was however clear in her use of the trial and error approach when dealing with the quadratics. Here is how she explained her approach, by writing,

Handwritten work for factoring $6x^2 + 7x - 20$. The correct factorization is $(2x - 5)(3x + 4)$. Several trial-and-error attempts are shown, including $(6x - 10)(x + 2)$ and $(6x - 8)(x + 15)$.

and say “For this one I tried looking for the combination of co-factors of 6 and 20... when add together they give the sum of 7... I tried 3 and 2 with 5 and 4, they give me sum product of 10 and 12...I swapped them then gave me 15 and 8... then I used $(2x - 5)$ and $(3x + 4)$ ”.

R: How did you know that you needed to put a minus sign between the brackets?

GD: “I first do trial and error method”.

7.2.7.2 Learners’ incorrect responses

7.2.7.2.1 Learners who were unable to reach the final stage of factorization.

Five learners were unable to provide the correct solution with a correct explanation. However, it is paramount to disclose that the discussion revealed that they were aware of the use of the approaches mentioned in 6.2.7.2 and 6.2.7.3. For example, here is what learner NQ said during the interview, when she wrote,

Handwritten work for factoring $6x^2 + 7x - 20$. The correct factorization is $(6x + 12)(6x + 4)$. Several trial-and-error attempts are shown, including $(6x - 20)$, $(6x - 14)$, and $(6x - 10)$.

and said “I began by saying: $6 \times -20 = -120$, I then failed to get the factors of -120 that will give me $+7$ when added. ...find the common factors in the numerator which when multiplied will give 6 and then divide the denominator of 6. You will be left with the factors”.

Also learner (LH) agreed when he provided his response,

Handwritten work for factoring $6x^2 + 7x - 20$. The correct factorization is $(3x - 4)(2x + 3)$.

and explained “Here I searched for two numbers which when multiplied will give $6x^2$, I put them in, I went on to look for two numbers which when added will give 7, which are -4 and 3, hence $(3x - 4)(2x + 3)$ ”.

7.2.7.2.2 Learners ignored the role played by $6x^2$.

Two learners (XD and BT) ignored the presence of $6x^2$ in their explanations, respectively.

Learner XD responded,

$$\begin{array}{l} 6x^2 + 7x - 20 \\ (6x + 5)(x - 4) \end{array}$$

and said “I failed this one....I first looked for the factors of 20....I enlisted them all...I tried to add them to get this one $+7x$ and found that no it is impossible”.

Also learner BT wrote,

$$\begin{array}{l} 6x^2 + 7x - 20 \\ 6x^2 + 7x - 20 \end{array} \quad 6x - 20 = -120$$

120	-1
-120	1
2	-60
-2	60
-10	12

Total and error

and said “Here I did not get the factors that could be multiplied and give -20 , again when added and $+7$ I then decided to leave it as it is”

R: Noting that there was a “6” as the coefficient in $6x^2 + 7x - 20$. Were you going to use the same approach of factorising as you did in $x^2 - x - 6$?

BT: “No,.....it is not the same.....Yes....No....it is the same...I simply look at the second term and the last term in both expressions.....(In deep thought..recalling something)...I also tried the method that you (researcher)....I multiplied 6 and -20 and got -120 I tried in vain to get the factors of -120 which when added give $+7$ ”.

A summary of learners’ interviews in terms of correct/incorrect; valid/invalid reasons when responding to task 7 of the activity of factoring quadratics is presented in Table 7.7

Table 7.7: Consolidated summary of interview responses on Task 7 of learners' activity on factorisation of quadratics in terms of correct/incorrect; valid/invalid reasons.

$6x^2 + 7x - 20$	No Response	Incorrect Response	$(3x - 4)(2x + 5)$ Without Reason	$(3x - 4)(2x + 5)$ With Invalid Reason	$(3x - 4)(2x + 5)$ With Valid Reason
Learners		GE; BT; LH; XD; NQ; GD; SZ; DB			SB; SK; MW; TM
Number of learners	0	8	0	0	4
%	0%	67%	0%	0%	33%
From the 12 learners	2 learners used the A-A method; 9 learners followed the trial and error of factors				

Data presented in Table 7.7 shows that only four learners (33%) of the 12 learners, provided the correct responses, with reasons for their answers. It was noted that learners followed the trial and error method in factoring a trinomial quadratic of $a \neq 1$, when they were asked to factorise $6x^2 + 7x - 20$. Nine learners of this study followed the trial and error of factors, but only two learners (TM and SK) made it to the correct solution. The activity also revealed that only two learners (SB and MW) of the study followed the Autrey's and Austin's method (A-A method) in factoring $6x^2 + 7x - 20$, and both completed the task successfully.

7.3 Summary of responses across the items.

7.3.1 Summary of methods used as learners engaged in the activity of factoring quadratics.

The consolidated summaries of interview responses according to what learners were doing in the activity on the factorisation of quadratics are presented in the table that follows. After engaging with the data collected from the learners' interviews and written work, I noted the methods of factoring quadratics used by learners, the results of which are provided in Table 7.8 that follows.

Table 7.8 Methods of factoring quadratic expressions used by learners (learners' activity).

Quadratic	Factoring Method								
	Trial & Error	Cross	Group ing	AA	Greatest common factor	Difference of two squares	Exponents	Incorrect answer	Correct answer with no reason
x^2	GD; TM (2)				SB; GE; BT; SZ; LH; XD; DB; (7)		SK; MW; NQ (3)		
$x^2 - 4$	SB; GE; DB; GD; MW; TM; NQ (7)					BT; SK; SZ; XD (4)		LH (1)	
$x^2 + 4$	GD; SZ (2)					SK; BT (2)		NQ; TM; MW; DB; XD; LH; GE; SB (8)	
$x^2 - 4x$	GD (1)				SB; GE; BT; SK; SZ; LH; MW; TM (8)			XD; NQ (2)	DB (1)
$x^2 - 4y$	GD (1)				WM; SK; BT (3)			NQ; TM;DB; XD; LH; SZ; GE; SB (8)	
$x^2 - x - 6$	SB; GE; BT; SK; SZ; LH; XD; DB; GD; MW; TM; NQ (12)							DB	
$6x^2 + 7x - 20$	SK; NQ; TM; DB; XD; LH; SZ; BT; GE (9)	GD (1)		MW; SB (2)				NQ; GD; DB; XD; LH; SZ; GE; GD (8)	

7.3.2 Summary according to APOS genetic decomposition.

From the learners' activity, I categorised their responses along the APOS levels. Guided by the genetic decomposition of the current study, as well as Table 3.1 of the APOS reflective indicators, and Table 4.2 of ranking the tasks in the activity of factoring quadratics according to APOS, I was able to gain a better understanding of what they could and could not do. Table 7.9 and Table 7.10 provide a summary of how learners are categorised in terms of APOS

placement. Table 7.9 is for learners from the township school (T-school), while Table 7.10 is for learners from the school in the rural setting (R-school).

Table 7.9 Learners' placement along APOS indicators (learners from the T-School)

Learner	A P O S level				Indicators
	A	P	O	S	
Bonke (SB)		X			- His definition of factorisation was vague "<i>By factorisation I can say it is where you change may be the expression to be viewed as simple</i>" - performed correctly on all five factorable quadratic expressions - unable to argue the non-factorability of the two expressions
Enhle (GE)	X				- Her definition of factorisation was specific to numbers "<i>It's whereby we look at the number; we want the factors of a number</i>" - performed correctly on only the first four factorable quadratics; unable to work with trinomial quadratic of $a \neq 1$ - unable to argue the non-factorability of the two expressions
Nomfundo (NQ)	X				- Her definition of factorisation was to the point "<i>Is to search for the factors of that number or the factors of that variable</i>" - performed correctly on only three factorable quadratics; unable to work with trinomial quadratic of $a \neq 1$ - unable to argue the non-factorability of the two expressions
Ntombizodwa (MW)		X			- Her definition of factorisation was vague "<i>Factorisation means breaking</i>" - performed correctly on all five factorable quadratic expressions - was able to argue the non-factorability of the one expression
Thembani (TM)		X			- Her definition of factorisation was partially correct "<i>it is where we are given maybe "4", then you are asked to break it like $2 \times 2 = 4$</i>" - performed correctly on all five factorable quadratic expressions - unable to argue the non-factorability of the two expressions

Table 7.10: Learners' placement along APOS indicators (learners from the R-School).

Learner	A	P	O	S	Indicators
Sandile (LH)	X				- His definition of factorisation was vague "<i>It is where we break the expression</i>". - performed correctly on only three factorable quadratics; unable to work with trinomial quadratic of $a \neq 1$ - unable to argue the non-factorability of the two expressions
Skhumbuzo (XD)	X				- His definition of factorisation was vague "<i>factorisation is where you show how what you are given was constructed</i>". - performed correctly on only three factorable quadratics; unable to work with trinomial quadratic of $a \neq 1$ - unable to argue the non-factorability of the two expressions
Bongekile (BT)			X		- Her definition of factorisation was to the point "<i>I can say is to take the big number or the complex expression and simplify it to its constituent factors.....</i>" - performed correctly on only the first four factorable quadratics; unable to work with trinomial quadratic of $a \neq 1$, but there are workings on trial and error - was able to argue the non-factorability of the two expressions
Sthembiso (SK) Pilot			X		- His definition of factorisation was vague "<i>is when we're solving and in expression is when we are simplifying</i>" - performed correctly on all five factorable quadratic expressions - was able to argue the non-factorability of the two expressions
Nobuhle (SZ) Pilot		X			- Her definition of factorisation was vague "<i>you need to show how was the sum before the current state</i>" - performed correctly on only the first four factorable quadratics; unable to work with trinomial quadratic of $a \neq 1$, but only swapped the signs - was able to argue the non-factorability of only one expression
Bayanda (DB)	X				- Her definition of factorisation was partially to the point "<i>group your expression into two...like $(x - 2)(x + 2)$</i>" - performed correctly on only three factorable quadratics; unable to work with trinomial quadratic of $a \neq 1$ - unable to argue the non-factorability of the two expressions
Gugulethu (GD)			X		- Her definition of factorisation was to the point "<i>let me make an example about $6x + 12$...to factorise it you must know the highest common factors of both $6x$ and that of $+12$... then you look for what is common</i>". - performed correctly on only the first four factorable quadratics; unable to work with trinomial quadratic of $a \neq 1$, but only swapped the signs - was able to argue the non-factorability of only one expression

7.4 On items that were challenging in the activity of factoring quadratics.

Learners respectively pointed out different areas where they encountered challenges.

7.4.1 Learners were unable to factorise $x^2 + 4$.

When asked about the area in the factorisation of quadratics that they found challenging, learners provided a myriad of responses. For example, learner SK had this to say: *“Yes there are aspects that are tricky....it just that some kind of experience derived from practices is needed to able to deal with instances like e.g. one person may look at $x^2 + 4$ as similar to $x^2 - 4$ yet there is difference...then and begin to try using $(x + 2)(x - 2)$...with no solution to found when trying to reverse”*. Learner SZ shared a similar view to learner SK, she said: *“One part of factorisation I could say was difficult was to factorise $x^2 + 4$ ”*. Learner GD highlighted the mathematics of working with cube numbers as a portion of the concept of factorisation that she regards as difficult. She had this to say: *“Working with cube numbers and perfect squares”*.

R: What do you think causes the challenge?

GD: *“It is because somehow I cannot be able to identify the cubes numbers and the perfect squares... I don't really understand them...I do not know the square numbers”*.

7.4.2 Learners were unable to factorise the trinomial quadratics of $a \neq 1$.

A greater portion of learners pointed to factorisation of the trinomial quadratics as always posing a challenge. For example, learner GE said *“Working with trinomials was one aspect of factorisation that I found challenging”*.

R: What were you doing on the trinomials?

GE: *“Factorisation; simplifying; and also exponential expressions”*.

R: What challenges do you face when dealing with the factorisation of quadratics?

GE: *“Algebra in general ... I could say on one day is easy ...on the other day is difficult.....the thing is the first obstacle we faced this year during Grade 10 it is our sir student-teacher who taught us, I even told our subject teacher....what I can say is he failed to teach us algebra the*

good way because we didn't understand the way in which he was expressing.....he confused us and we were left confused".

This view was also shared by learner LH who said: *"I encountered the challenges on the task on factorising $6x^2 + 7x - 20$... it was never easy....it was difficult to get the factors of $6x^2$ as well as of -20 in $x^2 - x - 6$ you are able to get the factors of -6 which when added give -1 ".*

R: Can you point the real challenge in factoring $6x^2 + 7x - 20$?

LH: *"You can get the factors of 20, the real challenge is on getting the middle term $+7x$ ".*

Learner BT supported the other learners as she said: *"Even though I can say I do not have challenges of dealing with the factorisation of quadratics ... but this part of factorising the trinomials like $6x^2 + 7x - 20$ is very difficult ... in other instances you are given big numbers which poses a challenge of getting the factors...the trial of getting the factors becomes a problem on its own".*

R: Can you share some experiences regarding the factorisation?

BT: *"Something needs to be done.....the activities in the book and in class are easy ...examination is different".*

Learner XD also supported the view held by other learners, that working with trinomial quadratics is challenging. He said: *"The area in factorisation that I perceive as difficult is that of the sum of two squares.....ei...I cannot say it is difficult.... It is just that I do not remember myself being taught that concept.....again, this part of factorising for an example $6x^2 + 7x - 20$ the part of finding the factors that correspond is very difficult".*

7.4.3 Learners were unable to work with the difference of two squares.

Learner DB pointed out that the challenge of working with quadratics of difference of two squares was one portion of factorisation at Grade 10 level that she perceived as difficult. She said: *"Yes.....the difference of two squares can get me confused sometimes."*

7.4.4 Learners were unable to work with factorisation embedded in fractions.

Learner MW said: *“Factorisation is simple...the challenge begins when you encounter an expression or equation with a fraction....it is very difficult to factorise the problem with a fraction”*.

R: In which area of the problem are you facing the challenge?

MW: *“All areas...I don't know where to begin”*.

7.4.5 Learners were unable to identify their inadequacy.

Learner NQ was contradictory in her response. When asked about the part of factorisation of quadratics that poses a challenge to her, she responded by saying *“None yet”*. The contradiction was observed in the following dialogue.

R: Can you identify the quadratics?

NQ: *“Yes”*.

R: Is $6x^2 + 7x - 20$ a quadratic?

NQ: *“No ... it is a trinomial”*.

R: Are the following expressions, $x^2 - x - 6$; $x^2 - 4y$; $x^2 - 4x$; $x^2 - 4$ the quadratics?

NQ: *“No ... these are: trinomial, binomial, binomial, difference of two squares”*.

R: Can you write down an example of a quadratics for me?

Learner NQ wrote, $x + 2x + 4y + 2y$

R: Is $x + 2x + 4y + 2y$ a quadratic?

NQ: *“yes ... it has 4 terms”*

7.5 What learners found easy to grasp in the factorisation of quadratics.

Learners respectively pointed out different areas they were comfortable in when dealing with the factorisation of quadratics.

7.5.1 Learners are able to factorise the trinomial quadratics of $a = 1$.

Learner DB was confident in saying: “Yes... *there is a part in the factorisation of quadratics at grade 10 that I perceive as easy ... there is $x^2 - x - 6$... I can say I understood when you (the researcher) taught us those methods of tabulating the co-factors of the last term....the use of the variety of approaches has been very helpful.*” Two learners also expressed a similar sentiment as learner DB. Learner NQ said: “Factorisation of trinomials like $x^2 - x - 6$ is very much easy”. Learner LH said: “The factorisation of $x^2 - x - 6$ and that of $x^2 - 4x$ are the portions of factorisation that I found easy to deal with.”

7.5.2 Learners are able to factorise the difference of two squares.

Three learners were comfortable in using the rule of working with the difference of two squares. For example, learner SK said: “It is for an example $x^2 - 4$... *the difference of two squares ... this is simple because there is only one thing that you do...open two sets of brackets... put inside the brackets the opposite signs....you put the negative sign in one bracket and put the positive sign on the other....then you factorise..... I can call it a rule*”. Learner SZ agreed with learner SK and said: “Yes, the one on the difference of two squares ... *is the section of the factorisation of quadratics that I can say is easy for me*”.

R: Why do you say it is easy?

SZ: “It is because I know that in the difference of two squares ... *in between the terms you have to have the negative; again, at the end the last term there has to be a number with 2 as an exponent as well as the base*”.

Learner XD also shared similar sentiments to learners SZ and SK as he said: “Yes....*difference of two square is the portion in the factorisation that I regard as the easiest here it is understood very easy....for an example if you are given $x^2 - 4$ you simply open two sets of brackets..... find the square root of x^2 which is x ...and the square root of 4 is 2 ...you drop the minus sign in the first bracket and positive sign in the second bracket.... $(x - 2)(x + 2)$ simple*”. Again, learner BT was comfortable working with rules as she spoke about the part of factorisation of quadratics that she recognised as simple. She said: “It is the difference of two squares; the addition or subtraction of two cubes.”

R: .Why are these areas easy for you?

BT: *“It is because you are given the cubes....even the calculator can help you get the cube roots and then follow the instructions from the teacher”.*

From this discussion it is evident that learners have developed their own rules of working confidently with the quadratics of difference of two squares.

7.6 Performance of learners in Grade 10 common test (algebra).

7.6.1 Overall summary of the test.

All the learners in Grade 10 in both schools wrote the common test. I looked at the performance in the algebra items, which were Questions 1, 2, 3 and 4 with a total of 46 marks. Here are descriptions of the questions:

Table 7.11: Description of questions in the March 2019 Grade 10 common test.

Question	Description	No of sub questions
1	Factorisation and simplification	5
2	Solving equations	4
3	Number system	4
4	Inequalities and word problem	5

It is of interest to note that learners were required to work with quadratic expressions in at least six of the sub questions accounting for 20 out of the total of 46 marks, which indicates how central the concept of quadratic expressions are to algebra. In Table 7.12, I present the average marks for each of the sub questions:

Table 7.12: Marks per sub question in the March 2019 Grade 10 common test.

School		1.1(2)	1.1(3)	1.2(3)	1.2(2)	1.2(4)	2.1(2)	2.2(3)	2.3(5)	2.4(2)	3.1(1)	3.2(2)	3.2(2)	3.3(1)	4.1(5)	4.2(3)	4.2(1)	4.2(3)	4.3(4)
I	Mean	0,36	0,51	1,38	0,93	0,51	1,05	0,40	0,44	0,28	0,78	0,00	0,18	0,06	1,44	0,99	0,31	0,06	0,00
	N	124	122	129	122	116	128	117	111	90	123	104	102	81	126	114	96	66	58
	Std. Deviation	0,747	0,911	1,347	0,946	1,009	0,987	0,938	0,849	0,654	0,416	0,000	0,408	0,242	2,022	1,320	0,529	0,240	0,000
K	Mean	0,32	0,44	1,11	1,02	0,55	0,66	0,31	0,27	0,25	0,69	0,00	0,00	0,00	0,67	1,00	0,25	0,08	0,00
	N	81	82	85	83	78	85	78	78	64	81	70	70	56	85	81	75	49	44
	Std. Deviation	0,722	0,848	1,310	0,950	1,224	0,907	0,887	1,113	0,713	0,605	0,000	0,000	0,000	1,340	1,313	0,468	0,277	0,000
Total	Mean	0,35	0,48	1,27	0,97	0,53	0,89	0,36	0,37	0,27	0,75	0,00	0,10	0,04	1,13	0,99	0,29	0,07	0,00
	N	205	204	214	205	194	213	195	189	154	204	174	172	137	211	195	171	115	102
	Std. Deviation	0,736	0,885	1,336	0,947	1,097	0,973	0,917	0,968	0,677	0,500	0,000	0,325	0,188	1,815	1,314	0,503	0,256	0,000

As is evident, the learners did very poorly in all these items. In terms of trying to compare which items were most difficult, the following graph illustrates the average percentage per sub question.

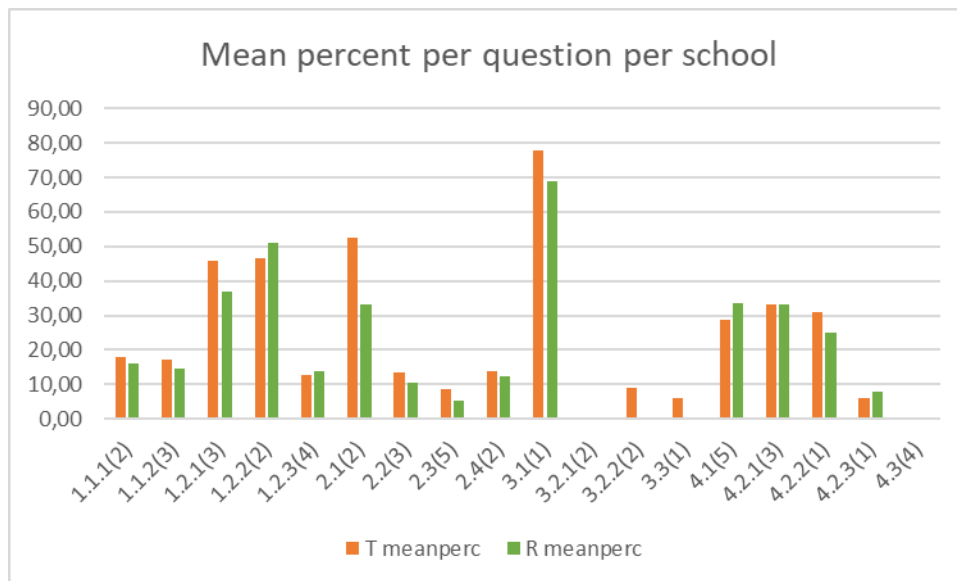


Figure 7.1: Mean percent per sub question per school.

The items experienced as most difficult were Questions 4.3; 3.2.1; 3.2.2 and 3.3. As seen in Figure 7.1, the mean of Question 3.2.1 was zero for both schools, and none of the learners received any marks for this question. This question asked them to find two integers between which a given expression, P , was for $P(7)$. In fact, there were only zeros in this question. The

same was true for Question 4.3 which was a problem-solving task. For Q3.2.2, and 3.3 the mean for the R-School was zero. The mean for the T-School for Q3.2.2 was 8.8% while for Question 3.3 it was 6.2%

The question which was experienced as the easiest was Question 3.1 where the mean in the T-School was 78.5% and in the R-School 74.5%. The question asked them to express the decimal 0.45 as a common fraction. Questions 1.2.1 and 1.2.2, which required simplification of algebraic expressions, were experienced as not as difficult by both schools. Question 2.1, which required the solution of an equation which was already given in factorised form, seemed to be less challenging for the T-School.

In terms of the median, this was zero across both schools for each of the sub questions, except 1.2.1; 1.2.2 and 3.1. These zero medians are a big concern because they reveal that there is an overwhelming amount of non-participation by these Grade 10 learners. For each question, most responses were zero or blank. To try to represent this situation of concern I present a graph of zeros for each school. I used a line graph to show the trends although the data is discrete, but a line graph allows us to see the patterns in the data.

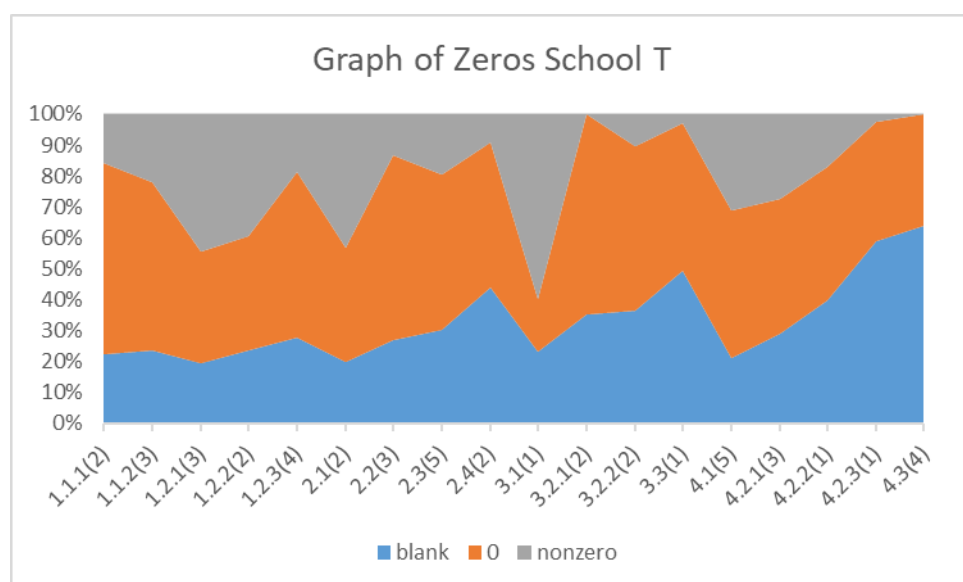


Figure 7.2: Graph of Zeros for the T-School.

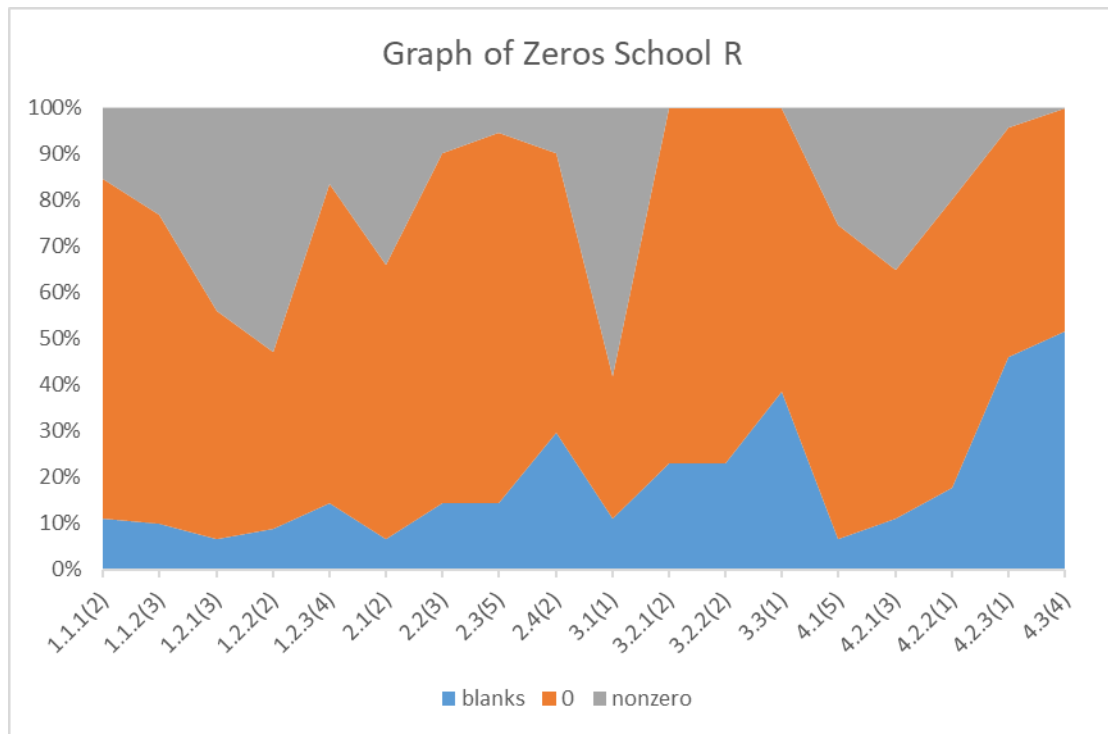


Figure 7.3: Graph of Zeros for the R-School.

It is concerning that because of the large numbers of zeros and blanks, the graphs shown in Figures 7.2 and 7.3 make more sense of the data than one using averages or medians. Figures 7.2 and 7.3 confirm that most learners scored zeros or blanks for all questions except for Questions 1.2.1; 1.2.2; and 3.1.

7.6.2 A closer look at the items based on quadratic factorization.

We now look more carefully at Question 1.1.1 and 2.3 which were direct questions involving the factorisation of quadratic expressions. Figure 7.4 shows Question 1.1.1

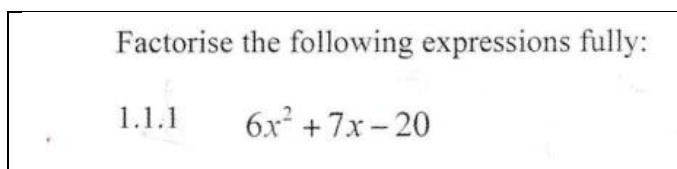


Figure 7.4: Question 1.1.1 of the March 2019 Grade 10 common test.

The breakdown of marks for Question 1.1.1 appears in Table 7.13, and shows that there were 32 students who were able to get full marks, which means that only 13 % of the learners were able to factorise the quadratic expression, $ax^2 + bx + c$.

Table 7.13: Distribution of marks for Question 1.1 1 of the March 2019 Grade 10 common test.

		Frequency
Valid	0	166
	1	7
	2	32
	Total	205
Missing	System	46
Total		251

Figure 7.5 presents a copy of Question 2.3 which asked learners to solve the given equation.

2.3 $\frac{4x^2-3x-1}{4x+1} + \frac{x^3+1}{x^2-x+1} = 2$ -

Figure 7.5: Question 2.3 of the March 2019 Grade 10 common test.

The detailed breakdown of marks for the question appears on Table 7.14 that follows:

Table 7.14: Distribution of marks for Question 2.3 of the March 2019 Grade 10 common test.

		Frequency
Valid	0	153
	1	19
	2	10
	3	2
	5	5
	Total	189
Missing	System	62
Total		251

Table 7.14 shows that there were only 5 learners who got the question completely correct. That is, only 2% of the group was able to correctly apply factorisation of quadratics to simplify an algebraic fraction.

7.7 Detailed qualitative analysis of the responses of the 10 interviewed participants to Question 1.1.1 and Question 2.3

I followed up on how the ten Grade 10 students who were interviewed performed in the two tasks that appeared in the March 2019 mathematics common test. For the purpose of this study, question 1.1.1 becomes task A and question 2.3 becomes task B.

Task A: 1.1 Factorise the following expressions fully:

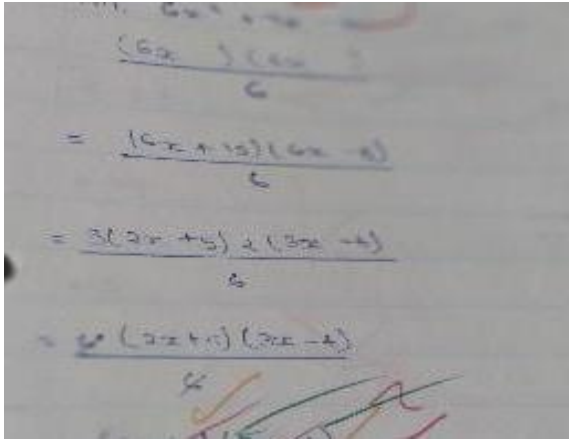
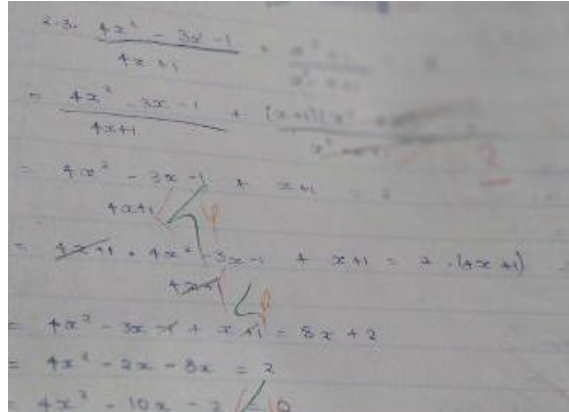
$$1.1.1 \quad 6x^2 + 7x - 20 \quad \quad \quad 2 \text{ marks}$$

Task B: Determine, without the use of a calculator, the value of x in each of the following:

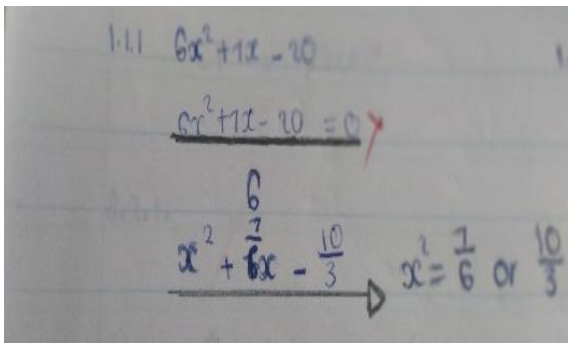
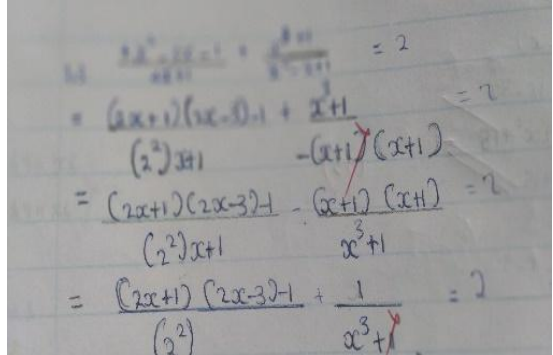
$$2.3 \quad \frac{4x^2-3x-1}{4x+1} + \frac{x^3+1}{x^2-x+1} = 2 \quad \quad \quad 5 \text{ marks}$$

7.7.1 How the interviewed learners from the T-School performed in Question 1.1.1 and Question 2. 3.

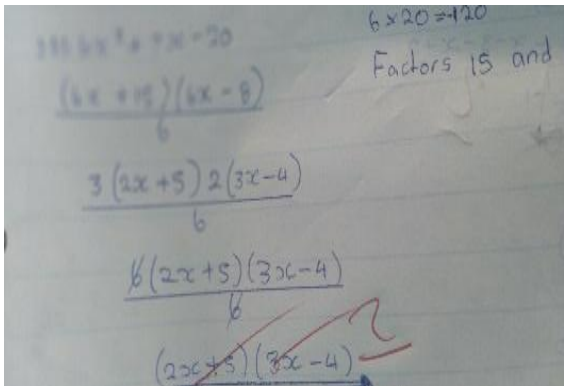
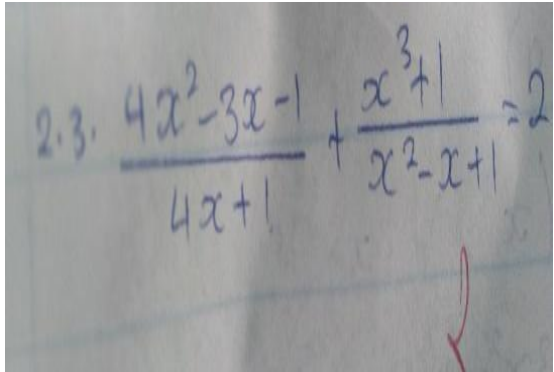
- Ntombizodwa (learner MW)

Task A	Task B
	

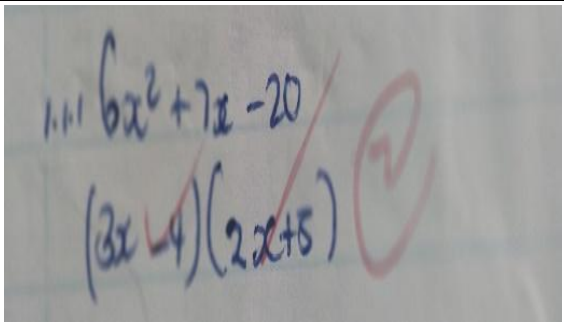
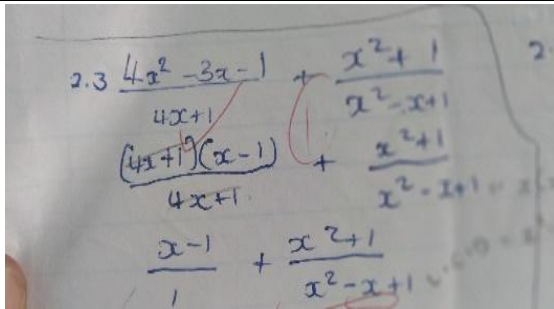
- Enhle (learner GE)

Task A	Task B
	

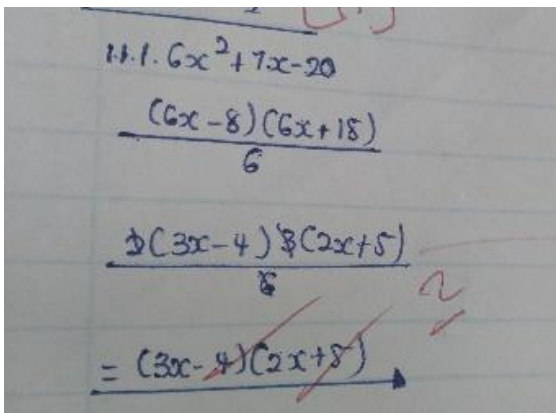
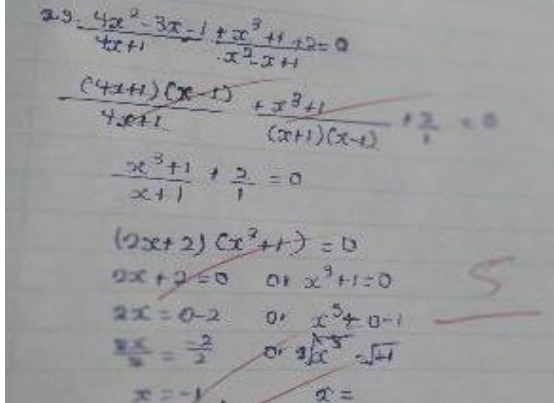
- Bonke (learner SB)

Task A	Task B
	

- Thembani (learner TM)

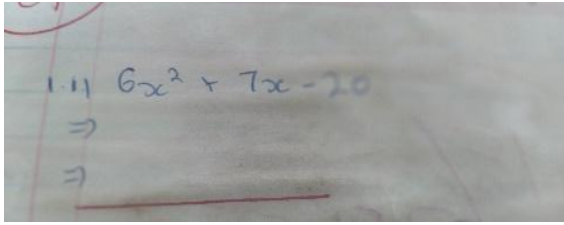
Task A	Task B
	

- Nomfundo (learner NQ)

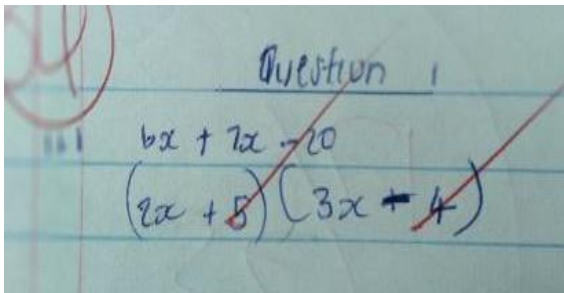
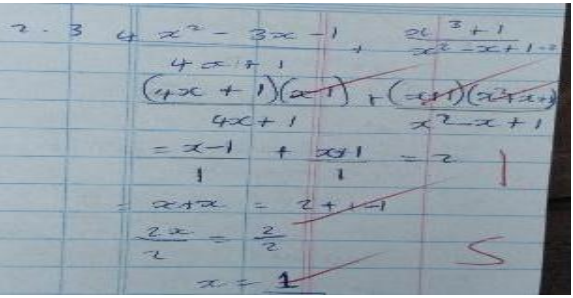
Task A	Task B
	

7.7.2 How the interviewed learners from the R-School performed in Question 1.1.1 and Question 2. 3.

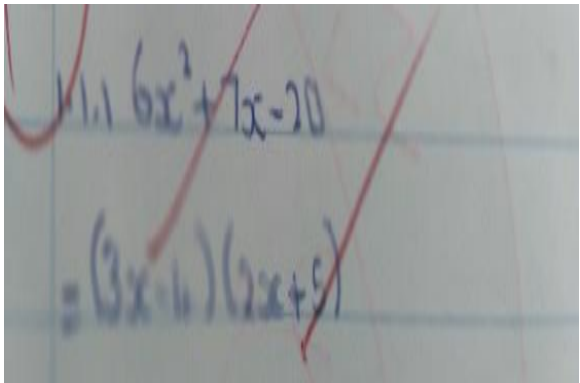
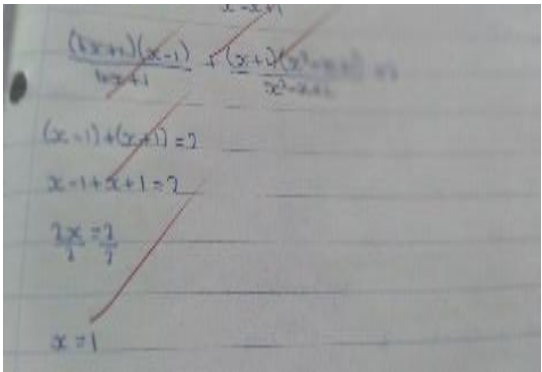
- Sandile (learner LH)

Task A	Task B
	

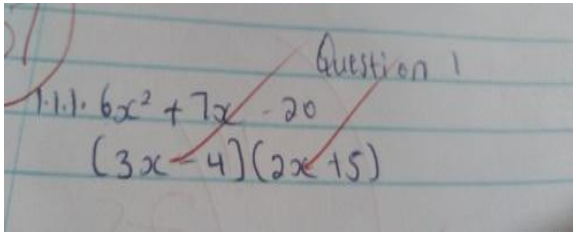
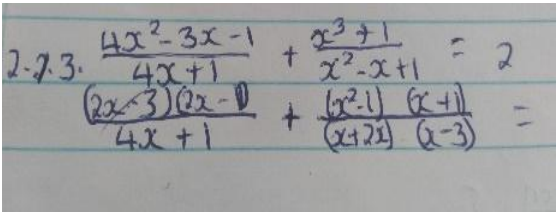
- Bongekile (learner BT)

Task A	Task B
	

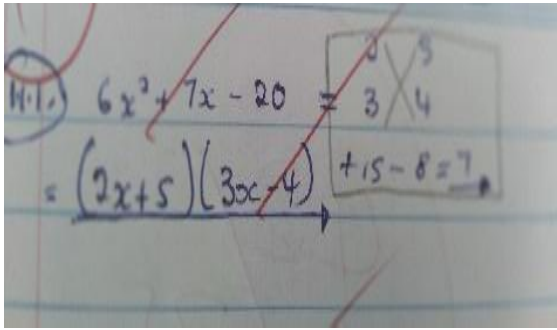
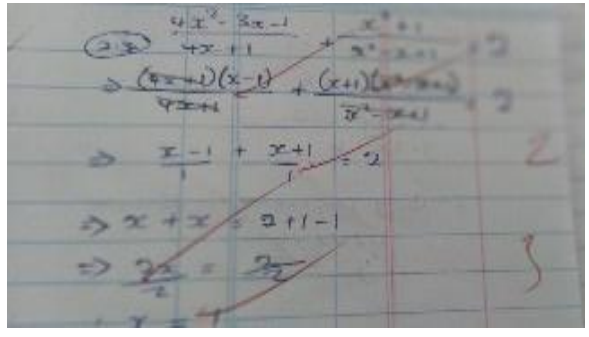
- Skhumbuzo (learner XD)

Task A	Task B
	

- Bayanda (learner DB)

Task A	Task B
	

- Gugulethu (learner GD)

Task A	Task B
	

7.7.3 Summary on how the ten interviewed learners performed in Question 1.1.1 and Question 2. 3; and their location according to APOS levels.

Being guided by Table 1.2 and the Table 4.1 of this thesis, I was able to locate the two given tasks according to APOS levels, where task A is at a Process level while task B is at an Object level. The summaries in terms of items being correct/incorrect/partially correct are provided in Table 7.15 and Table 7.16.

Table 7.15 Performance of interviewed learners from the T-School in Question 1.1.1 and Question 2. 3 and their location according to APOS levels.

Learner	Task A (Process)	Task B (Object)	Learners' APOS level
MW	Fully correct solution	Incorrect solution	Process
GE	Incorrect solution (attempted the A-A method)	Incorrect solution	Action
SB	Fully correct solution	Incorrect solution (not attempted)	Process
TM	Fully correct solution	Fully correct solution	Object
NQ	Fully correct solution	Fully correct solution	Object

Table 7.16 Performance of interviewed learners from the R-School in Question 1.1.1 and Question 2. 3 and their location according to APOS levels.

Learner	Task A (Process)	Task B (Object)	
LH	Incorrect solution (not attempted)	Incorrect solution (not attempted)	Action
BT	Fully correct solution	Fully correct solution	Object
XD	Fully correct solution	Fully correct solution	Object
DB	Fully correct solution	Incorrect solution	Process
GD	Fully correct solution	Fully correct solution	Object

For task A, which is located at the Process level, eight learners produced the correct factors, and only two learners could not enter the Process stage. These results indicate that the greater percentage of learners operate at the Process level. For task B, which is located at the Object level, only five learners (GD; XD; BT; NQ & TM) were able to pass the Process level and reach the object level. There were three learners (DB; SB & MW) who passed the Process stage but could not make it at the Object level. The two learners (LH & GE) who could not enter the Process level also did not make it at an Object level, indicating that they are trapped at an Action level. At least learner GE from the T-School attempted, learner LH from the R-School did not even make an attempt.

7.8 Conclusion.

This chapter gathered data from the learner interview responses as well as their written work, and categorised their responses along the APOS levels. The details of the analysis of this data is presented in Chapter 10.

CHAPTER 8

TEACHERS' VIEWS ON GENERAL PRACTICES IN THE GRADE 10 MATHEMATICS CLASSROOM - RESPONSES TO THE QUESTIONNAIRE

8.1 Introduction.

This chapter presents the results and the responses from the teacher questionnaires. The questionnaires were administered to 25 Grade 10 mathematics teachers in the Ilembe District. Six of the 25 teachers were from the two schools that participated in this study. Of the 25 teachers, 13 were males and 12 were females. The teacher questionnaire used in this study appears in Appendix J of this thesis. The questionnaire used intended to elicit responses on the perceptions of teachers regarding the use of mathematical knowledge in addressing activities on the manipulation of algebraic quadratic expressions. The information gained from the questionnaire is supplemented by teacher interviews, which are presented in Chapter 9 of this thesis.

8.2 Biographical information of teachers.

Section A of the teacher questionnaire required teachers to provide their teaching qualifications with a focus on Mathematics FET. The information is provided in table 8.1.

Table 8.1: Teachers' teaching qualifications.

3-year Diploma in Mathematics FET	PGCE in Mathematics FET	Bachelors' Degree in Mathematics FET	Higher than a Bachelor's degree
0	7	16	2

Teachers were also asked to provide the number of years they have taught Grade 10 mathematics. The information is provided in table 8.2.

Table 8.2: Number of years teaching Grade 10 mathematics.

< 5 years	5 – 10 years	11 – 15 years	16 – 25 years	> 25 years
6	6	5	7	1

The information gathered from Table 8.1 of the questionnaire revealed that teachers were qualified to teach mathematics at the FET phase. From Table 8.1 it is impressive to learn that the majority of teachers that participated in this study hold a Bachelors degree to teach mathematics at the FET level. This information was important in that, according to norms and standards, a teacher is qualified to teach if he/she holds a four-year degree. This information provided me with the grounds that the participants of this study were indeed relevant and suited to this study.

8.3 The use of mathematics textbooks in schools.

Teachers were asked to respond to the given statement by indicating either always; sometimes; or never. A summary of their responses follows:

Table 8.3: The use of mathematics textbooks in schools.

STATEMENT	always	sometimes	never
All Grade 10 teachers use one mathematics textbook for teaching algebraic expressions and equations.	13	8	4
I write my own notes for my learners when teaching Grade10 algebraic expressions and equations.	8	17	0

Teachers were asked for general comments about the use of the textbook on algebraic expressions and/or equations. There is a view that all textbooks are CAPS compliant. WMAN1 commented as follows: *“All textbooks are relevant to CAPS, and most of them makes mathematics so simply in algebraic expressions”*. GS said: *“I think all textbooks are very good in mathematics for the Grade10 learners”*. In seconding the motion, WMAN6 said: *“All textbooks that I use are good on algebraic equations/expressions in Grade 10”*. This view was also shared by WMAN4 when he said: *“All textbooks cover the topics adequately and all cognitive levels are there”*. Other teachers have their own preferences regarding the textbook they use. For example, WMAN2 said: *“Classroom mathematics textbook is as one of the most valuable book for Grade 10 for the algebra expressions and equations”*. In addition, WMAN3 had her preference saying: *“Platinum textbook is good but there are more clarity on the classroom mathematics textbook”*. WMAN10 supported this view explaining: *“The activities*

in the classroom mathematics textbook are simple, clear and learners are able to understand the examples done in the textbook. The class activities are graded according to levels i.e. simply to abstract”.

Other teachers held different views, for example, WMAN5 said: *“Some books are good and some are shallow in exercises, they do not expose learners to high level thinking e.g. $x - 1 = (x^{\frac{1}{2}} - 1)(x^{\frac{1}{2}} + 1)$ ”*. This view was supported by UDU1 who said: *“There are few examples and activities that are not related to examination questions, which are tested at the end of the year”*. UDU3 said: *“The textbooks always give the simple examples and there is no section on enrichment question examples which make the learners to not adapt to the examination question paper”*. MPQ4 had a similar view, saying: *“Textbooks that are used are relevant but it would have been good if they have included examples that are relevant when learners are assessed”*. In support of this notion, MPQ2 said: *“The use of textbook is beneficial notes including algebraic terminology or language - word sums help learners learn the language hence the questions and instructions during assessments. Tasks only cover lower order questions”*. Other teachers adopted the use of more than one textbook, for example, HP said: *“The use of different textbooks is very important as it exposes the teachers and learners on different methods of solving problems. Consulting different textbooks ensures exposure to different problems as well”*. MPQ1 added: *“There is no best textbook for teaching any topic ... there is always shortcomings, therefore I do not use one textbook so that they can complement each other”*. TR said: *“The textbook that I use is good, but sometimes I supplement it with other textbooks”*. Another reason for using different textbooks was shared by MZ who explained: *“I use different textbooks because some aspects are not treated the same way”*.

8.4 Teachers’ Continuous Professional Development (CPD) activities.

Teachers were further asked to indicate the number of times they had attended any professional developments on algebraic expressions and or equations, particularly quadratic expressions, during the last 18 months. They were expected to rate at a scale of 1 to 5, the activities they had attended where 1 is the least, and 5 is the most useful. The results are presented in Table 8.4.

Table 8.4: Teachers' Continuous Professional Development (CPD) activities.

Statement	Number of times							Ratings					
	0	1	2	3	4	5	6	0	1	2	3	4	5
School mathematics meetings on algebraic expressions/equations with HOD and teachers	6	6	8	3		1	1	6		2	2	8	7
Cluster meetings with other schools on algebraic expressions/equations	3	7	5	6	2	1		3	1	3	4	6	8
Workshops on algebraic expressions /equations by the subject advisor	9	4	6	2	3	1		9		2	5	5	4
A visit at your school from your subject advisor to monitor your progress on algebraic expressions /equations	11	9	3		2			11	1	1	6	3	3
Workshops on algebraic expressions /equations by your union	22	3						22			1		2
Mentoring visit from a colleague or School Management Team (SMT) on algebraic expressions /equations	9	9	3	4				9	1	1	3	7	4
AMESA workshop on algebraic manipulations	24	1						24		1			

The data gathered in Table 8.4 reveals that teachers are not adequately supported when it comes to the manipulation of algebraic expressions. About 80% of teachers indicated that they had held school mathematics meetings on algebraic expressions and equations with their DH and their teachers twice at most, with 24% of those teachers indicating that such meetings were not empowering. All teachers, with the exception of three teachers, held cluster meetings with other schools on algebraic expressions and equations. These meetings were rated beneficial for the majority of teachers. Nine teachers expressed the view that no workshops on algebraic expressions and equations had been held by the subject advisor. This number grew to eleven teachers who expressed the fact that they had never been visited at their schools by the subject advisor in order to monitor their progress on algebraic expressions and equations. Here again, the data shows that there is very little done by teacher unions to capacitate teachers by holding workshops on algebraic expressions and equations. There is also very little effort done on the

mentoring of teachers as the data reveals that their colleagues or School Management Team (SMT) do not hold classroom visits, resulting in the majority of teachers remaining less knowledgeable regarding the manipulation of algebraic expressions and equations. Finally, the data shows that mathematics teachers do not affiliate with mathematics formations like AMESA. AMESA, as the Association of Mathematics Education in South Africa holds its conferences annually, and extends invitations to teachers to participate by presenting papers, or by learning from other mathematics educators or participating in featured algebra activities.. Teachers were also asked to indicate whether there were any additional professional development activities in which they engaged, particularly regarding the manipulation of algebraic expressions. However, none of the participants of this study responded to this item.

8.5 Tracking the mathematics curriculum coverage.

Teachers were urged to comment on their efforts in tracking the mathematics curriculum coverage with special focus on the manipulation of algebraic expressions and equations. Their responses are presented in Table 8.5.

Table 8.5: Tracking the mathematics curriculum coverage.

STATEMENT	always	sometimes	never
In my school, the DH or senior teacher checks on my mathematics curriculum coverage.	20	5	0
I use the CAPS guidelines.	20	5	0
I use the KZN annual teaching plan (ATP)	23	2	0
I use the PILO Jika Imfundo trackers	8	8	9
I think it is not possible to complete all the topics in the Grade 10 mathematics curriculum	0	15	10

The data presented in Table 8.5 shows that there is continuous checking of mathematics curriculum coverage by the DHs and senior mathematics teachers. As indicated, teachers use the CAPS guidelines, and the KZN annual teaching plan (ATP) documents to track their work. Very few teachers of this study used the PILO Jika Imfundo trackers, showing that teachers largely follow the DBE policies. WMAN10 said: “*Educators should use the ATPs and CAPS*

guidelines and other textbooks (not one) when teaching mathematics". From this statement it is clear that teachers focus on departmental policies and nothing else.

Fifteen teachers of this study agreed that it was not possible to complete all the topics in the Grade 10 mathematics curriculum. GS said: *"I think the department must review how to set dates for the ATP in relation to the starting dates of the examinations in schools because it seems not corresponding"*. UDU1 said: *"The background on the topic from the previous grades is important in assisting with the clarification of concepts. Background like HCF; factors; squares; cubes; square roots; cube roots; etc, and to me this background is lacking at grade 10"*. This view was also held by MPQ1 who said: *"Mathematics curriculum is not that long, but the problem is that learners in Grade 10 have little knowledge of Grade 8 and 9 knowledge ...if you happen to be lucky enough to get one with it... So teachers spend a lot of time re-teaching Grade 8 and 9 work instead of Grade 10... in Grade 10 it is as if you are teaching three grades in one year"*.

8.6 Mathematics teachers' school environment.

Teachers were asked to comment on the state of support they received as teachers of Grade 10 mathematics. They were urged to use the scale of 1 to 5 indicating the degree to which they agreed with the given statement. Table 8.6 presents the distribution of the number of teachers as they responded to the statements where they were asked to indicate one of the following:

(1) Strongly Disagree; (2) Disagree; (3) Neutral; (4) Agree; (5) Strongly Agree; where necessary:

Table 8.6: Mathematics teachers' school environment.

Statement	Strongly Disagree	Disagree	Neutral	Agree	Strongly Agree
I am solely responsible for what I teach and assess in my Grade 10 mathematics class.	2	1	6	4	12
All Grade 10 teachers, teach according to the same common plan.	1	1	6	5	12
All the Grade 10 teachers do the same assessments for all Grade 10 classes.	1	4	5	7	8
My DH checks that I am on track with my Grade 10 mathematics teaching.	0	0	2	6	17
If I have a problem with mathematics, I go to my DH who helps me.	0	1	5	13	6
My mathematics colleagues are supportive of the work I do.	1	0	6	12	6
My Grade 10 learners appreciate my work.	1	0	7	6	11
Parents are supportive of our work.	4	6	10	4	1
Our school's mathematics department works together as a team.	0	3	3	11	8
Our mathematics subject head ensures that everybody does their share of work.	0	0	3	18	4

The majority of teachers agreed that their respective schools' mathematics departments, work together as a team. Firstly, their mathematics subject heads ensure that everybody does their share of the work; and their mathematics colleagues are supportive of the work that they do. This level of support is also observed by the fact that their DHs check that they are on track with their Grade 10 mathematics teaching. As a result, if they have problems with mathematics, they consult their DHs who help them. Secondly, all Grade 10 teachers teach according to the same common plan, hence, they do the same assessments for all Grade 10 classes in their respective schools. Consequently, their work is appreciated by Grade 10 learners. However, eventually they are solely responsible for what they teach and assess in their Grade 10 mathematics classes, and this is further compromised by the fact that parents are less supportive of their work as teachers.

8.7 Grade 10 mathematics curriculum.

Teachers were asked to comment on the Grade 10 mathematics curriculum by using the scale of 1 to 5 indicating the degree to which they agreed with the given statement. All the statements were focused on the manipulation of algebraic expressions and equations. Table 8.7 presents the distribution of the number of teachers as they responded to the statements which required them to indicate one of the following:

(1) Strongly Disagree; (2) Disagree; (3) Neutral; (4) Agree; (5) Strongly Agree; where necessary:

Table 8.7: Grade 10 mathematics curriculum.

Statement	Strongly disagree	Disagree	Neutral	Agree	Strongly agree
I am pleased with the pace of the Grade10 mathematics curriculum.	1	2	5	14	3
All the topics in the Grade 10 mathematics curriculum are necessary.	0	0	4	9	12
The sequencing of the topics according to CAPS, makes sense.	0	1	1	9	14
We stick rigidly to the departmental planning guidelines.	0	0	4	5	16

The data from Table 8.7 indicates that about 68% of the teachers were pleased with the pace they worked through the Grade10 mathematics curriculum, at their respective schools. Again, teachers agree that all the topics in the Grade 10 mathematics curriculum are necessary, and their sequencing of the topics according to CAPS, makes sense. Furthermore, all the teachers stick rigidly to the departmental planning guidelines.

Teachers commented on how management of the Grade10 mathematics curriculum in algebraic manipulation can be improved. WMAN7 felt that the status quo has to remain saying: *“For me, I think all the topics currently they are arranged accordingly”*. This view was shared by MPQ4 who said: *“Even though the ATP is relevant, but it is too long of which sometimes it is uncovered”*. Even though the curriculum is relevant, MPQ3 felt that the curriculum should focus on algebra more than any other topics because it has more percentage in common tasks.

A similar view was shared by GS when he said: *“Not that much, but it must be deeply based on the factorisation such that the factorisation is taught as revision in the upper grades”*. Taking a different angle, WMAN3 said: *“Time allocation is a challenge, we request more days for us to deliver the curriculum”*. WMAN5 pointed a finger at the lower grades saying: *“The problem lies with the lower grades where Grade 8 and Grade 9 learners are not prepared adequately”*. In agreement with this view, UDU2 said: *“They need to give the lower grades the guiding books which are linked with previous question papers, because even though the Grade 10 mathematics curriculum is ok ... but most of the work done there is the repetition of the work that is supposed to be done at Grade 8 and 9”*. Commenting on improving the curriculum coverage, WMAN10 said: *“More workshops should be done, especially for the newly-appointed educators”*. In addition to this, HP expressed the opinion that teachers should ensure that they practice as it was only through practicing with learners that could they be certain of grasping the content.

Teachers were urged to voice their comments to the Department of Education regarding the teaching of factorisation of algebraic expressions and the solution of quadratic equations in Grade 10 mathematics. WMAN2 said: *“Algebraic expressions form the base for any mathematical problem in Grade 11 and 12... more workshops should be done to equip teachers with different methods of dealing with it”*. These workshops should equip teachers with ways to deal with transition as WMAN9 explained: *“About quadratic equations, it should be made simple because learners starts with for an example $\blacksquare + 2 = 8$ and then are introduced to the unknown x and made to work with $x + 2 = 8$, but learners hate x . Let us introduce another unknown like b, m, k etc. not x ”*. MPQ2 agreed, saying: *“Factorisation planning should not assume that learners have information, rather begin from the basics”*. MPQ4 said: *“More workshops and seminars must be conducted to help educators on their teaching skills”*. HP added further: *“The DoE has to always keep algebraic expressions and equations in first term since it forms basis for solving even with other chapters”*. However, according to GS, *“the DoE must increase curriculum covered time so that learners will get more time of factorisation. Secondly, they must introduce the quadratic formula at Grade 10”*.

There is a view that the manipulation of algebraic expression is not adequately assessed, as WMAN3 commented: *“The allocation of marks must be checked, and more percentage of it*

must be assessed”. UDU1 agreed, saying: “*They have to ensure a balance in the examination questions because more work is done but less marks are given*”. According to WMAN4, Grade 10 should be provided with study guides for all sections where examination type questions appear in topics, as is done in the Grade 12 intervention programme.

8.8 Teachers’ responses to the activity on factoring quadratics that appeared in the questionnaire.

Section A Question 3 of the teacher questionnaire required teachers to elaborate on their teaching methodology as they dealt with the factorisation of algebraic quadratics at Grade 10 level. The questions for this activity were extracted from the March 2018 Grade 10 mathematics common paper, and appear in Appendix H of this thesis. The algebraic quadratics in this teacher questionnaire were categorised into:

- (i) Factorising quadratic expressions that were factorisable
- (ii) Factorising quadratic expressions that were not factorisable

8.8.1 Task 1: Factorisation of x^2 .

The teachers provided a myriad of responses.

8.8.1.1 Correct responses with correct explanations.

Twelve teachers produced the correct factorisation, with correct explanations. For example teacher MPQ1 responded: “*What was x^2 before it became x^2 ? Learners should be able to say $(x)(x)$. Explain that x^2 is a product of x and x are factors*”

In providing explanations, some teachers resorted to using examples in the form of numbers. For example, teacher WMAN4 said: “ *$8 = 2.4 \quad 8 = 1.8$* ” in emphasising how the factors are formed. Other teachers used variables, for example UDU2 said: “*Explain the concept of factors and square numbers, then give examples like $a^2 = a.a$* ”.

8.8.1.2 Correct responses without explanations.

Five teachers WMAN2; TR; MZ; ND; WMAN7 responded by saying: $x^2 = (x)(x)$ without providing an explanation.

8.8.1.3 Incorrect responses stating “no solution” with explanations.

Five teachers provided no solutions, but gave reasons why they responded that way. Here is how they explained their positions:

WMAN1 said: *“No solution. Expression with one term can never be factorised”*.

UDU3 said: *“The given problem is a quadratic equation and it has no x and y values is that why we can't factorise”*.

UDU4 said: *“The factor of x^2 is x, but x^2 cannot be factorised”*.

UDU5 said: *“No solution. Look for the square root of x^2 , or factor of x^2 - square is when you multiply a variable or a number by itself”*.

HP said: *“Monomial cannot be factorised”*.

8.8.1.4 Teachers who provided no response.

Three teachers WMAN5; WMAN6; UDU1; provided no responses. I am tempted to believe they were indicating that there is no solution.

8.8.2 Task 2: Factorisation of $x^2 - 4$.

8.8.2.1 Correct factorisation

Twenty-three teachers provided the correct response of $(x - 2)(x + 2)$

8.8.2.1.1 Correct factorisation with explanation using the law of exponents.

Teacher WMAN7 used the law of exponents and said: *“(1). Law of exponents $a^1 \times a^1 = a^{1+1} = a^2$ meaning $x^1 \times x^1 = x^{1+1} = x^2$ (reinforce law of exponents). (2). signs*

$(\times \div + -)$ (3). Brackets off (4). Bodmas Rule. All the above mentioned content; just reinforce it when in teaching the concept of factorisation”.

8.8.2.1.2 Correct factorisation with explanations about the difference of two squares.

Ten teachers announced prior to their solutions that this was the difference of two squares. They applied the rule of factorising the difference of two squares. For example, teacher MPQ3 said: *“Difference of two squares can be factorised as the product of the sum and difference of the two terms i.e, $x^2 - 2^2 = (x + 2)(x - 2)$ ”*. Teacher MZ noted another dimension in his explanation, saying: *“Difference of two squares, open up two brackets and determine the factors for the first term and the last term. Also make sure that the co-efficient of x is zero”*. Although teacher MZ spoke about the difference of two squares, he also stressed the fundamental approach to dealing with factorisation of quadratics by mentioning the aspect of ensuring the middle term is equal to zero. A similar approach of ensuring that the sum of the factors needs to be zero was also noted in the explanation provided by teacher KW who said: *“Find two numbers that have a product of -4 and a sum of zero. The numbers are -2 and $+2$ ”*.

8.8.2.1.3 Correct factorisation with explanations but which ignored the role of the middle term.

In contrast, teacher MPQ2 followed a similar approach as she spoke about the factors, but was however silent about the role of the middle term. Here is how teacher MPQ2 responded: *“Find the factors of 4 as well as those of x^2 ... the negative sign shows that one factor is negative and the other is positive”*. A similar response was also noted in the explanation provided by teacher GS who said: *“Again, here is to find the product of $x^2 - 4$ by opening two brackets. In the brackets the multiplication of first terms must give you the first term in equation, and the multiplication of the last terms must give you the last term”*.

8.8.2.1.4 Correct factorisation with explanations focused on the role played by the knowledge of finding products of binomials.

Teacher MPQ1 recognised the role played by prior knowledge in understanding new knowledge. In this case she felt it was important for learners to understand the process of determining the products of binomials as a stepping stone towards understanding the concept of factorisation, saying: *“From Grade 9 learners know that $a^2 - b^2 = (a - b)(a + b)$... the reverse process of this is factorisation OR to factorise difference of 2 squares one needs to open 2 brackets, find the square root of each term. Put inside the brackets, one set of square root must have positive and other the negative sign”*.

8.8.2.1.5 Correct responses without explanations.

Three teachers WMAN5; HP; ND provided the correct answer but without explanations

8.8.2.2 Correct explanations but no answers.

Two teachers provided no answers; however, they provided correct explanations. Teacher UDU3 said: *“This has factors so we solve by equating it to zero and get the values of x ”*. Teacher UDU5 said: *“Look for the square root of the first term and the second term, then if not possible find the factors of both terms”*

8.8.3 Task 3: Factorisation of $x^2 + 4$.

All 25 teachers agreed $x^2 + 4$ cannot be factorised and used different approaches to show why this quadratic cannot be factorised.

8.8.3.1 Use of the fundamental approach to factoring the quadratic.

Four teachers followed the fundamental approach to factorising the quadratic. WMAN7, in bringing forth his argument why $x^2 + 4$ cannot be factorised said: *“This expression it is not factorisable because we do not have two numbers that we can multiply and give us the last term ... and that if we add or subtract them and give us the middle term”*. This argument was also shared by teacher MPQ2 who said: *“Not factorisable; the positive sign shows that the*

factors have the same signs ... when the signs are the same it is impossible to get the middle term as zero". Teacher UDU5 said: *"It is not factorisable. First term has the factor and also the second term, but the sign between the terms restrict the factorisation because it gives the middle term whereas the original expression does not have it"*. Teacher UDU4 said: *"This expression can be solved using another method, because if we factorise it, it has no factors and it is impossible to return it to its original form if factorised"*.

8.8.3.2 Teachers equating the expression to zero.

Two teachers, WMAN2 and MZ equated the expression $x^2 + 4$ to zero and solved for x . The results they found made them conclude that $x^2 + 4$ cannot be factorised. Teacher WMAN2 said: *" $x^2 = -4$; $x = \sqrt{-4}$ can't be factorized because there is no square root for a negative number"*.

8.8.3.3 Teachers referring to the difference of two squares.

Eight teachers spoke about the notion of factorising the difference of two squares as they explained why $x^2 + 4$ cannot be factorised. For an example, teacher WMAN6 said: *"Not factorisable. Teach them that you can only factorise a difference of two squares"*. This explanation was also shared by teacher WMAN3 who said: *"Not difference of two squares ... $(x + 2)(x + 2)$ If we expand and use distributive law it becomes a trinomial"*.

8.8.3.4 Teachers referring to the sum of two squares.

Four teachers spoke about the sum of two squares. They all agreed that the sum of two squares cannot be factorised. For example, teacher UDU1 said: *"Sum of two squares cannot be factored because it is an irreducible polynomial"*. Teacher MPQ3 said: *"No factors ... the sum of two squares can never be factorised"*.

8.8.3.5 Teachers referring to the graphical solution.

Teacher MPQ1 said: *"Learners can see that there is no HCF, this is a sum of squares. Ask them to sketch the graph i.e. $f(x) = x^2 + 4$, they will see that it does not cut x-axis and therefore the*

values of x that will make this expression zero are not real. It is therefore not factorisable". Teacher WMAN5 supported the use of graphs saying: *"No factors because this is not a difference of two squares. When sketched it will not cut the x -axis"*.

8.8.4 Task 4: Factorisation of $x^2 - 4x$.

Twenty-four teachers said $x^2 - 4x$ can be factorised by means of taking out the common factor to become $x(x - 4)$, with the exception of teacher UDU5 who said *"it is not factorisable"*.

8.8.4.1 Teachers factoring by taking out the common factor – with no further explanation.

Without providing methods, 20 teachers, stated that the approach was to take out the common factor. WMAN3 said: *"Isolate what is common in the expression and then put it outside the brackets"*.

8.8.4.2 Teachers factoring by taking out the common factor and equating $x^2 - 4x$ to zero.

One teacher, UDU3, was noted to have correctly factorised $x^2 - 4x$ and then equated the results to zero. Here is what teacher UDU3 said: *"It is a must to equate zero in a given equation then find factors which is $x(x - 4) = 0$ then we find the values of x "*.

8.8.4.3 Teachers factoring by taking out the common factor – with sound explanation.

Teacher UDU1 clearly stated her procedure on how she arrived at the solution: *"Take out the highest common factor and place it in front of the brackets, then the expression inside the brackets is obtained by dividing each term by the highest common factor"*.

8.8.4.4 Teachers factoring by taking out the common factor – using examples to explain the method.

Two teachers, UDU2 and WMAN9 resorted to using examples in their explanations Teacher UDU2 said: *"Explain the common factors with examples e.g. $ab + bc = b(a + c)$ "*. WMAN9 said: *"Start using practical example: Got 2 bags with three items but one must be the*

same and tell that learner to look for the item that is similar and take it out. That's the introduction of the common factor".

8.8.5 Task 5: Factorisation of $x^2 - 4y$.

Twenty-three teachers said $x^2 - 4y$ cannot be factorised with the exception of two teachers who displayed some understanding of their engagement with the task.

8.8.5.1 Teachers treating $x^2 - 4y$ as $x^2 - 4y^2$.

Two teachers, WMAN1 and WMAN9 were noted to have treated $x^2 - 4y$ as $x^2 - 4y^2$ in their responses. Teacher WMAN1 said: " $(x - 2y)(x + 2y)$ difference of two squares is applicable". Teacher WMAN9 said: " $(x - 2y)(x + 2y)$ Two numbers that can make up two items that are not the same".

8.8.5.2 Teachers correct responses.

Twenty-three teachers provided and cited the following reasons why $x^2 - 4y$ cannot be factorised:

8.8.5.2.1 Teachers stated "not factorisable" – with no explanation but with some special reasons.

Nine teachers were unable to give reasons for saying $x^2 - 4y$ cannot be factorised. Two teachers, UDU3 and UDU4 had the following reasons to offer. Teacher UDU3 said: "*There is no such a given function that is why we can't find these values*". Teacher UDU4 said: "*Since we have two unknowns it will be impossible to factorise it. So it is undefined/unfactorised*".

8.8.5.2.2 Teachers stated "not factorisable" – no common factor.

Seven teachers said $x^2 - 4y$ cannot be factorised since no common factor existed in-between the two terms. WMAN3 said: "*No common factor ...you must check whether there is a common factor, hence you cannot factorise*". Teacher WMAN7 shared a similar opinion, saying: "*It is not factorisable. This binomial expression the way I use to teach my learners, I just do practical*

example of taking two pieces of paper, of different colours and the other of the same colours and tell them that we can only factorise the equation or expression with the like terms". Teacher MPQ2 said: "Not factorisable because the terms are not the same and there is no common factor". WMAN5 said: "No factors because there are unlike terms".

8.8.5.2.3 Teachers stated "not factorisable" – no difference of two squares.

One teacher, UDU1 found that $x^2 - 4y$ does not fit the characteristics of the difference of two squares, and concluded that it cannot be factorised. Teacher UDU1 said: "*It cannot be factored as the difference of two squares since y is not a square, therefore it cannot be factored*".

8.8.5.2.4 Teachers stated "not factorisable" – no common factor with no difference of two squares.

Teacher MPQ1 said: "Ask learners if there is an HCF, they will realise that there is no HCF. Ask if there is difference of squares. Is this quadratic equation? If the answer is no to all questions asked, then lead them to conclude that there are no factors and hence expression not factorisable". Teacher MPQ3 said: "No factors. No common factor and second term $4y$ is not a perfect square". Teacher WMAN4 wrote: " $x \cdot x - 4 \cdot y$ No common factors. Not difference of two squares". Teacher WMAN6 said: "Not factorisable. There is no common factor and y is not a square".

8.8.6 Task 6: Factorisation of $x^2 - x - 6$.

8.8.6.1 Teachers incorrect responses.

8.8.6.1.1 Teacher stated "no solution".

UDU5 said: "No solution – if factorisable, the first and last term has the factors which gives the middle. NB the sign of a middle term is determined by factors of the first and last term".

8.8.6.1.2 Teacher equated $x^2 - x - 6$ to zero.

UDU3 said: "To solve this problem, we firstly equate into zero then we find factors and give the value values of x ".

8.8.6.2 Teachers correct responses.

8.8.6.2.1 Teachers factorised correctly but provided no explanations.

Nine teachers managed to find the factors $(x - 3)(x + 2)$ but did not provide explanations on how they arrived at the solution. WMAN10 said: *“Trinomial expression ... Factorise by inspection ...middle term is negative and the third term is negative ... so you find factors”*. WMAN6 said: *“Factorise by opening two brackets”*. WMAN 8 said: *“Investigation. Question and answer method”*.

8.8.6.2.2 Teachers factorised correctly but ignored the role played by x^2 .

Thirteen teachers were noted to have ignored the role played by x^2 in their solutions. Teacher MPQ4: *“Factorising a trinomial, look for all possible factors of the last term and when multiplying those factors they must give the middle term”*. Teacher MPQ2 said: *“List all factors of 6... choose only those when subtracted will give a middle term ... which is equal to one”*. Teacher UDU2 said: *“Factorising a trinomial by opening two brackets, separate the square no., find the factors of the last no. or constant. Choose the factors that will give the difference equal to the middle term, the bigger factor will take the sign which is the same as that of the middle term”*. Teacher UDU1 said: *“Find two numbers to give the last term -6 when multiplied, but give middle term -1 when added or subtracted”*. Teacher WMAN3 said: *“You open two brackets then you check the sign of the last term if it is negative signs will not be the same within the brackets”*. Teacher UDU4 said: *“Open two brackets and use the smallest factors of 6 (3and2), our middle term is negative so make 3 negative”*.

8.8.6.2.3 Teachers factorised correctly and recognised the role played by x^2 .

Five teachers provided holistic explanations and spoke about the role played by x^2 . Teacher MPQ1 said: *“Ask for factors of the first term, and the factors of the third term. Learners are expected to be able to list i.e. $x \cdot x$ for x^2 ; -3 and 2; -2 & 3; -6 & 1; 6 & -1. Ask them to find those factors that will give a sum of -1 i.e. -3 & 2. Explain that the factors must give the middle term when added”*. Teacher MPQ3 said: *“Check for common factor-no common factor. Since trinomial, try grouping-not working. Open two sets of brackets and factorise the first term x^2 . Check for factors of -6 if multiplied ... and factors that give -1 if you add them”*. Teacher

WMAN2 said: *“Put square root of x^2 in opened brackets. Put factors of 6 in closed brackets that give a difference which is -1 . The bigger the factor goes along with a sign of a middle term”*.

8.8.7 Task 7: Factorisation of $6x^2 + 7x - 20$.

8.8.7.1 Teachers incorrect responses.

8.8.7.1.1 Teachers stated “no solution”.

Three teachers said there was no solution. Teacher UDU3 said: *“This equation is undefined because it has no factors as such values of x ”*. Teacher UDU5 said: *“No solution – if it is factorisable, there are factors of the first and the third term, which can be added together and give the middle term ... N.B. the signs”*. WMAN7 said: *“It is not factorisable”*.

8.8.7.1.2 Teachers provided nil responses.

Teacher MPQ1 said: *“Ask learners to list all factors of $6x^2$ and factors of -20 ”*. UDU4 said: *“No solution - Open two brackets and use the smallest factors of 20, then put the biggest factor with the smallest factor of 6”*. WMAN1 said: *“No given solution. The above method is applicable but firstly you need to find the factors of 6”*.

8.8.7.2 Teachers correct responses.

8.8.7.2.1 Teachers provided correct factors without explaining their procedures.

Six teachers managed to get $(2x + 5)(3x - 4)$ as factors but provided no explanations. WMAN10 said: *“Find the factors by inspection”*. WMAN8 said: *“Discussion. Factors of the first and last term. Question and answer method”*. WMAN6 said: *“Factorise by opening two brackets”*.

8.8.7.2.2 Teachers provided correct factors through the use of the quadratic formula.

Teacher WMAN5 said: *“Quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ should be used to assist learners not good”*. Teacher WMAN3 said: *“If you suspect that it is not easy to get the factors you simply use the quadratic equation”*. Teacher WMAN9 said: *“It can be factorised through the use of the quadratic expression”*.

8.8.7.2.3 Teachers provided correct factors using the grouping method.

Teacher UDU1 said: *“Multiply first and the last term $(6)(-20)=-120$. Find two numbers that multiplied to give -120, but added to give middle term +7. Replace the middle term with those numbers which are $-8x$ & $+15x$, then factorise by grouping”*.

8.8.7.2.4 Teachers provided correct factors using the trial and error method.

Teacher MPQ2 said: *“List factors of 20 and factors of $6x^2$; choose only those that give +7 as a middle term”*. Teacher MPQ4 said: *“The trinomial with the first term with coefficient greater than 1 must also have factors i.e. factors of $6x^2$ are $2x$ and $3x$ and last term is -ve, therefore factors are $(2x + 5)(3x - 4)$ ”*. Teacher MPQ3 said: *“No common factor. Check for factors of $6x^2$ put in brackets, now check for factors of (-20) . Multiply to check for the factors that can give us $+7x$ ”*. UDU2 said: *“Open two brackets, take the smallest factors of the constant, consider the co-efficient of x^2 , divide it. The bracket with the bigger coefficient of x will take the positive sign and the smaller factor of 20”*. Teacher GS said: *“Is to find the product of this equation by opening two brackets. Then first term in brackets must give the value of a and the last terms in the brackets must give you the value of c . Last term and first term must give the middle term value”*.

8.9 Summary of teachers’ responses to items on factoring quadratics that appeared in the teachers’ questionnaire.

The summary of the categories of teachers’ responses to the activity on factoring quadratics that appeared in the teachers’ questionnaire is presented in Appendix L.

8.9.1 Summary according to correct/incorrect and valid/invalid reasons when teachers responded to items of the activity.

The Tables in Section 8.9.1 provide consolidated summaries of interview responses pertaining to the activity on factoring quadratics that appeared in the teachers’ questionnaire.

8.9.1.1 Task 1: Factorisation of x^2 .

This item is located at an Action level. Six teachers (24%) were able to provide correct solutions with valid reasons. Twelve teachers (48%) provided correct answers without reasons, and seven teachers (28%) provided incorrect answers. Table 8.8 provides comprehensive details.

Table 8.8: Consolidated summary of questionnaire responses on Task 1 of teachers' activity on factorisation of quadratics in terms of correct/incorrect; valid/invalid reasons.

x^2	No Response	Incorrect Response	$x \cdot x$ Without Reason	$x \cdot x$ With Invalid Reason	$x \cdot x$ With Valid Reason
Number of teachers (25)	2	5	5	7	6
%	8%	20%	20%	28%	24%
11 teachers responded with reasons	6 teachers spoke about factors; 3 teachers used examples to illustrate their approaches				

Data collected from the teachers' questionnaire and presented in section 8.8.1 shows that 18 teachers (72%) were able to factorise x^2 to achieve, by following different approaches. Six teachers spoke about factors, while three teachers used examples to illustrate their approach. Of the seven teachers who were unable to respond correctly, two teachers did not respond at all. Teacher (UDU4) who, out of the seven teachers responded incorrectly said: "*The factor of x^2 is x , but x^2 cannot be factorised*". Unfortunately, I was unable to further engage UDU4 because he was not part of the interviews. However, I was able to engage with one of the teachers, teacher HP who was one of the seven teachers who responded to the questionnaire by saying: "*Monomial cannot be factorised*". The details of the discussion appear in Chapter 9 of this thesis.

8.9.1.2 Task 2: Factorisation of $x^2 - 4$.

This item is located at an Process level. Ten teachers (40%) were able to provide correct solutions with valid reasons. Thirteen teachers (52%) provided correct answers without reasons, and two teachers (8%) provided incorrect answers. Table 8.9 provides comprehensive details.

Table 8.9: Consolidated summary of questionnaire responses on Task 2 of teachers' activity on factorisation of quadratics in terms of correct/incorrect; valid/invalid reasons.

$x^2 - 4$	No Response	Incorrect Response	$(x - 2)(x + 2)$ Without Reason	$(x - 2)(x + 2)$ With Invalid Reason	$(x - 2)(x + 2)$ With Valid Reason
Number of teachers (25)	2	0	3	10	10
%	8%	0%	12%	40%	40%
20 teachers with reasons	10 teachers spoke about the difference of two squares; 15 teachers spoke about the trial and error of factors; 8 teachers spoke about the combination of both				

Data collected from the teachers' questionnaire and presented in section 8.8.2 shows that 23 teachers (92%) responded correctly in factorising $x^2 - 4$, to produce the factors of $(x - 2)(x + 2)$ as the solution. Amongst the teachers who provided reasons in support of their solutions, there were 10 who followed a short-cut by using "the difference of two squares". Of the ten teachers, five submitted incorrect responses but showed that they recognised the type of quadratic expression they were dealing with by using "the difference of two squares". For example, teacher WMAN10 said: "*Difference of two squares; one factor must be positive and another be negative*". This indicates that in as much as she was unable to provide a clear explanation for her solution, she was aware of the type of quadratic she was dealing with.

A total of 15 teachers spoke about the trial and error of factors as one approach to deal with this task. Lastly, a total of eight of the 20 teachers followed the use of both approaches; that is the trial and error of factors; and the specialised approach to deal with the difference of two squares type. This was noted in the explanation provided as a breakthrough by teacher MZ who said: "*Difference of two squares, open up two brackets and determine the factors for the first term and the last term. Also make sure that the co-efficient of x is zero*". Although teacher MZ spoke about the difference of two squares, he also stressed the fundamental approach in dealing with factorisation of quadratics by mentioning the aspect of ensuring the middle term is equal to zero. The emphasis made by teacher MZ of the middle term being zero, is very important in that, as observed from the teachers' questionnaire, teacher MPQ2 followed the trial and error approach as she spoke about the factors, but was however silent about the role of the middle term.

8.9.1.3 Task 3: Factorisation of $x^2 + 4$.

This item is located at an Object level. Twenty-four teachers (96%) were able to provide correct solutions with valid reasons. One teacher (4%) provided a correct answer without reasons. Table 8.10 provides comprehensive details.

Table 8.10: Consolidated summary of questionnaire responses on Task 3 of teachers' activity on factorisation of quadratics in terms of correct/incorrect; valid/invalid reasons.

$x^2 + 4$	No Response	Incorrect Response	Correct Response Without Reason	Correct Response With Invalid Reason	Correct Response With Valid Reason
Number of teachers (25)	0	0	1	0	24
%	0%	0%	4%	0%	96%
24 teachers with reasons	14 teachers spoke about the difference of two squares; 10 teachers spoke about the trial and error of factors; 2 teachers spoke about both the difference of two squares and the trial and error of factors; 1 teacher spoke about the difference of two squares no common factor				

Teachers were tested to see if they could argue the non-factorability of $x^2 + 4$. Data collected from the questionnaire and presented in section 8.8.3 shows that 24 teachers responded correctly with reasons on why $x^2 + 4$ can never be factorised. Of the 24 teachers who responded correctly with reasons, 14 teachers spoke about the difference of two squares; 10 teachers spoke about the trial and error of factors; and one teacher spoke about the difference of two squares no common factor. Data captured from the questionnaire indicated that 14 teachers spoke about the difference of two squares. This shows that a large number of teachers rely on the characteristics and properties of a particular prototype of quadratics to substantiate their arguments. This is mathematically incorrect if one views the approach taken by four teachers who followed the fundamental approach to factorising the quadratic expressions to argue why $x^2 + 4$ cannot be factorised.

8.9.1.4 Task 4: Factorisation of $x^2 - 4x$.

This item is located at a Process level. Eighteen teachers (72%) were able to provide correct solutions with valid reasons. Seven teachers (28%) provided correct answers without reasons. Table 8.11 provides comprehensive details.

Table 8.11: Consolidated summary of questionnaire responses on Task 4 of teachers' activity on factorisation of quadratics in terms of correct/incorrect; valid/invalid reasons.

$x^2 - 4x$	No Response	Incorrect Response	$x(x - 4)$ Without Reason	$x(x - 4)$ With Invalid Reason	$x(x - 4)$ With Valid Reason
Number of teachers (25)	0	0	5	2	18
%	0%	0%	20%	8%	72%
18 teachers with valid reasons	18 teachers spoke about the removal of the common factor				

The data from the teachers' questionnaire presented in section 8.8.4, shows that in factoring $x^2 - 4x$, all 25 teachers were able to reach the solution of $x(x - 4)$. Of the 18 teachers who were able to provide an explanation on how they went about reaching their solutions, 15 teachers (60%), opted for the approach of "taking out the common factor", while the remaining five opted to make use of examples in order to explain how they reached the solution.

8.9.1.5 Task 5: Factorisation of $x^2 - 4y$.

This item is located at an Object level. Eighteen teachers (72%) were able to provide correct solutions with valid reasons. Four teachers (16%) provided correct answers without reasons, and three teachers (12%) provided an incorrect answer. Table 8.12 provides comprehensive details.

Table 8.12: Consolidated summary of questionnaire responses on Task 5 of teachers' activity on factorisation of quadratics in terms of correct/incorrect; valid/invalid reasons.

$x^2 - 4y$	No Response	Incorrect Response	Correct Response Without Reason	Correct Response With Invalid Reason	Correct Response With Valid Reason
Number of teachers (25)	1	2	3	1	18
%	4%	8%	12%	4%	72%
18 teachers with reasons	15 teachers indicated no common factors; 3 teachers spoke about the difference of two squares; 2 teachers spoke about both the common factor and the difference of two squares				

Teachers were given a non-factorable algebraic expression, $x^2 - 4y$, to see if they were able to apply the knowledge gained in working with this type of quadratic to argue why other quadratics could not be factored. Data collected from the teachers' questionnaire and presented in section 8.8.5 shows that 22 teachers responded correctly with reasons on why $x^2 - 4y$ can never be factorised. Of the 18 teachers who responded with reasons, 13 teachers indicated the non-existence of a common factor; five teachers spoke about the difference of two squares; and four teachers spoke about both the common factor and the difference of two squares. These responses are indicators that show that teachers are comfortable using the properties of particular prototypes of quadratics to argue general quadratics.

8.9.1.6 Task 6: Factorisation of $x^2 - x - 6$.

This item is located at an Action level. Twelve teachers (48%) were able to provide correct solutions with valid reasons. Eleven teachers (44%) provided correct answers without reasons, and two teachers (8%) provided incorrect answers. Table 8.13 provides comprehensive details.

Table 8.13: Consolidated summary of questionnaire responses on Task 6 of teachers' activity on factorisation of quadratics in terms of correct/incorrect; valid/invalid reasons.

$x^2 - x - 6$	No Response	Incorrect Response	$(x - 3)(x + 2)$ Without Reason	$(x - 3)(x + 2)$ With Invalid Reason	$(x - 3)(x + 2)$ With Valid Reason
Number of teachers (25)	2	0	6	5	12
%	8%	0%	24%	20%	48%
	12 teachers trial and error of factors				

Data captured from the teachers' questionnaire and presented in section 8.8.6 shows that 23 teachers responded correctly in factoring $x^2 - x - 6$ to give the factors $(x - 3)(x + 2)$. Twelve teachers adopted the trial and error approach, with 11 teachers providing no explanation on how they went about reaching their factors.

8.9.1.7 Task 7: Factorisation of $6x^2 + 7x - 20$.

This item is located at a Process level. Five teachers (20%) were able to provide correct solutions with valid reasons. Seventeen teachers (68%) provided correct answers without reasons, and three teachers (12%) provided incorrect answers. Table 8.14 provides comprehensive details.

Table 8.14: Consolidated summary of questionnaire responses on Task 7 of teachers' activity on factorisation of quadratics in terms of correct/incorrect; valid/invalid reasons.

$6x^2 + 7x - 20$	No Response	Incorrect Response	$(3x - 4)(2x + 5)$ Without Reason	$(3x - 4)(2x + 5)$ With Invalid Reason	$(3x - 4)(2x + 5)$ With Valid Reason
Number of teachers (25)	3	0	5	12	5
%	12%	0%	20%	48%	20%
5 teachers with valid reasons	2 grouping method; 3 trial and error of factors				

Data captured from the teachers' questionnaires and presented in section 8.8.7 shows that 22 teachers responded correctly in factoring $6x^2 + 7x - 20$ to produce the factors $(3x - 4)(2x + 5)$. Of the 22 teachers who responded correctly, only five teachers were able to provide valid reasons. The data further shows that teachers adopt the use of the trial and error approach when factoring the trinomial quadratic expression of $a \neq 1$. Of the five teachers, three teachers adopted the use of the trial and error of factors, while the remaining two teachers followed the grouping method of factoring a quadratic. Teacher UDU1 used the grouping method when factoring $6x^2 + 7x - 20$ saying: "*Multiply first and the last term $(6)(-20)=-120$. Find two numbers that multiplied to give -120, but added to give middle term +7. Replace the middle term with those numbers which are -8x & +15x, then factorise by grouping*".

8.9.2 Summary according to methods followed by teachers in factoring quadratics.

The data extracted from the teachers' responses on the activity, and presented in Table 8.15 shows that teachers use a variety of methods in factoring quadratic expressions.

Table 8.15 Summary according to methods followed by teachers in factoring quadratics.

	Trial & Error	Equate to zero	Grouping	No solution	Greatest common factor technique	Difference of two squares technique	Correct no reason	graphing	Quadratic formula
x^2	10			7			8		
$x^2 - 4$	10			2		10	3		
$x^2 + 4$	10	2				11	1	1	
$x^2 - 4x$					18		7		
$x^2 - 4y$				5	15	5			
$x^2 - x - 6$	12			2			11		
$6x^2 + 7x - 20$	6		2	3			11		3

8.9.3 Summary of how teachers' responses to activity of factoring quadratics are located according to APOS.

In Table 4.1 of this thesis, all seven items of this activity of factoring quadratics were ranked in accordance to APOS levels. This section attempts to locate teachers' responses to the activity of factoring quadratics according to APOS. The activity required teachers to provide the full account of what they did to reach their solutions. For the purpose of this research, a teacher who is unable to provide reasons for how they reached their solution is considered incompetent,

even if their solution was correct. The summary on how teachers are located according to APOS is discussed in detail in section 10.3.3 of this thesis.

8.10 Conclusion.

This chapter has managed to navigate the responses from the teacher questionnaire, and present the results. The analysis of these results are presented in Chapter 10. In chapter 9, I will be looking at how teachers engaged with the mathematics activity on the factorisation of quadratics.

CHAPTER 9

TEACHERS' NARRATIVES ON GENERAL PRACTICES IN THE GRADE 10 MATHEMATICS CLASSROOM - RESPONSES TO THE ACTIVITY ON FACTORING QUADRATICS - TEACHERS' INTERVIEWS - THE RESULTS

9.1 Introduction.

This chapter presents an analysis of the responses from teacher interviews. Six teachers from the two participating schools were invited to participate in the interviews. The participating schools were categorised into two, one was the T-School (township school), and the other the R-School (school from a rural setting). Three teachers were from the R-School, while the other three were from the T-School. The interviews were semi-structured in that I was allowed the flexibility of posing questions that were outside the scope of the scheduled interview questions, for clarity purposes. The scheduled teacher interview questions appear in Appendix K of this thesis document.

The teacher interview questions were categorised into two parts. The focus of the first part was on general practices of Grade 10 mathematics teachers, while the second part focused on the activity on quadratic expressions. The teacher interview questions on the mathematical activity were based on questions that appeared in the teachers' questionnaire. All six teachers interviewed were also part of the cohort of teachers who participated in the questionnaire.

9.2 Teacher narratives on the general practices of Grade 10 mathematics teachers.

9.2.1 On the understanding of the concept “algebra”.

RT: *“Yah... I think algebra is about simplification of the expressions; solving for x ; it is only about that, when we speak of algebra”.*

MZ: *“Well, obvious I will start by telling the learner that algebra is under mathematics ... okay algebra as a term it is broad, but then mathematically algebra has to do with calculations ... we talk of the variables; formulas; expressions; equations; solve; simplify ... yes something like that”.*

ND: *“Algebra, it is about the equations and the expressions...here is where we look at factorisation, addition and subtractions of fractions...yah”.*

HP: *“Algebra is one piece of mathematics or some section in mathematics whereby we seek ways of solving ... it helps us in establishing these ways of solving problems ... it equips us with basics of dealing with problem-solving”.*

KW: *“I can say that ... algebra it is just a study of mathematical signs and rules that are used to manipulate these signs and symbols ... which can involve multiplications sign, division sign, subtraction sign as well and addition ... but you must also know the rules if you multiplying which rule you must use, if you adding which rules you must use, I can simple that the way that I understand it”.*

GS: *“Algebra ... eh hh ...it is one of the parts in mathematics where we use letters ... maybe the symbols representing certain information ... that information can be in the form of equation ... or in the form of formulas ... I think I can say that those letters and symbols they stand or they represent numbers maybe which are unknowns needed to be found ... and also in algebra I can say that it is a learner may deal with concepts like expressions of various forms ... binomial, trinomial; common factor; difference of two squares; sum and difference of two cubes”.*

9.2.2 On the definition of the concept “factorisation.

Teachers were asked to define the concept “factorisation”. In one way or another, they all spoke about the factors, yet provided different descriptions. Their responses follow:

Teacher Rakhulu (teacher RT) resorted to using the example to provide her explanation saying: *“It is to find factors, in other words to find, if it’s a number, the numbers when multiplied give you the same number, for example here I have x^2 it means when we speak about factors of x^2 , we are looking for two numbers when multiplied can give x^2 . It may be the similar numbers or it may be the different numbers e.g. if I got six, the factors of six can be two and three”.* Teacher Mzimela (teacher MZ) followed a similar approach, also using the example to define the concept factorisation by explaining: *“I think the simplest way to explain ... maybe I can choose to make an example of a number like 16 ... first of all I think a learner have to understand the meaning of the factors. I think in the lower grades learners used to determine multiples of certain numbers ... maybe if you are given like 16 then you need to check the numbers that you can multiply to get 16 ... it could be 4×4 , those are multiples of 16 ... you can also choose*

2 and 8, they will also give you 16 ... then therefore this 2 and 8 I have chosen I can also call them factors of 16 and again 4 and 4 will also be the factors of 16. Basically when you factorise you have to look for numbers that you can multiply to get a certain value ... we can therefore refer to them as factors of that number ... So this means that in the case of variables you must look at the variables that you must multiply to give the variable that was given ...generally factorisation is about determining the factors”.

Teacher Ndulini (teacher ND), spoke about the identical items within factorisation, and further used the notion of the concept of exponents. Here follows her description: *“When we are factorising...it is where we look, for instance, if we can make use of x^2 ..it is where we check what similar items that we can multiply and give us in return what was given before...we usually use that language for our learners to understand...for an example x^2 is made by x multiplied by x ...the similar thing if we have x^4 we get x^2 times x^2 ...it is the matter of keeping the base and adding the exponents”.* This notion of identical items also surfaced in the description provided by teacher Hlophe (teacher HP). He said: *“Factorisation is where we expose the basic building blocks of the expression ...the similar blocks that are between the two terms or more or something that is related between the given terms, or some relation in-between those terms ... so that is what we do when we factorise”*

In consolidating his description, Teacher Kweyama (teacher KW) swiftly made use of the checking stage. He described the concept of factorisation as follows: *“If we talk about factorisation, the first thing that you must know, is that ... if I can break this term ... it involves factors; the product of the factors must give you, your last term of which is c ... and if you add them, they must give you middle term of which is b ...all I can say is it is based on the factors”.* This explanation of using the checking stage as a means of bolstering the description of the concept of factorisation was also followed by Teacher Gwamanda (teacher GS) who said: *“Let me respond this way ... I think factorisation is associated with expanded expressions ... to expand the expression ... I think there is that relation between expanding expression and factorisation ...I think in factorisation is where we look at the equations or the expressions with an intention to simplify or breaking it down ... but the way we breaking it down should be such that it is able to reverse back to the original information ... for an example, maybe if you have an equation or the expression you look for terms or values that are common ... so if you*

see something common ... make sure that you are able to multiply it and reverse to the original expression ... so it just that we look for the factors that could be multiplied and return back to the original expression”.

9.2.3 On the role of the teaching experience in teaching factorisation of algebraic quadratic expressions.

As the researcher (R), I asked the participants: “In your view, does the teaching experience play any role in teaching factorisation of algebraic expressions?”

Teacher RT responded: “Yah, it has, because as you grow on, as you accumulate the experience, you get a lot of different methods of factorisation, so you find that if the teacher is new in the system, she/he may be not in a possession of the different methods of factorising; furthermore, it becomes more simpler to teach as you accumulate more experience in teaching”.

Teacher MZ shared a similar view to other teachers saying: “In my case I would say the experience counts a lot ... in fact all the school subjects not only in mathematics and then even at algebra as whole because for an example on your first year of teaching you might get 1 or 2 textbooks ...and obvious everything that you will be teaching is going to be mainly based on those textbooks ... the more the years in your teaching, you come across different textbooks ... then you realise that okay there is a better way of explaining this thing as compared to what I read from first textbook ... and then when you interact with other educators from other schools they will also give you their methods of teaching particular concepts ... now you have a variety of methods to choose from as engage your own learners ... on that note I would say the experience counts a lot when it comes to teaching”.

Teacher ND agreed, her views are narrated in her response: “ Yes, let us say maybe it is your first year in the teaching field you experience some problems, which may be you last did at the tertiary institution....in the year that follows you become aware on how to tackle some similar problems....and you are able to explain explicitly to the learners of that grade...I think it is very important to stay in the grade for some time, for example take three years teaching Grade

10 in order to accumulate the experience....when you are offered to teach the grade that follows like Grade 11, you are now more experienced...and you know what to wait for in the examination”.

Again, the notion was seconded by teacher HP who said: *“Yes that is the case ... the reason is if you are teaching something with experience you are in a better position of understanding how learners for an example receive some forms of approaches when dealing with particular concepts ...when you approach learners you do that with necessary tools in your on how are you going deal with dynamics as they manifest in the classroom environment ... learners usually are operating at different levels, so knowledge acquisition among learners will differ ... the teacher with experience will know how to handle those situations...”*

R: Are you implying that with a mathematics teacher of little experience there is a likelihood of not teaching with success?

HP: *“Yes he/she is unlikely to teach with no success ... he will have a challenge of getting to all sides of the problem ... the experience is the best teacher ... he may use the techniques in solving a particular problem that he transferred from the problem situation he encountered ten years ago ... he will also use the experience to gauge learners’ levels of concentrations and be quick to slow or up his pace”.*

Teacher KW agreed on the notion, saying: *“I can say that the teaching experience has a role to play when it comes to Grade 10 mathematics, because you have to be able to identify the problems that learners have mostly ... so it is of fundamental importance to have experience about mathematics and then you have to deal with problems ... how you solve problems step wise ... which steps that are important when you are doing ... as you know, that in most times the problems on learners begin at step number 3 whereby you have to factorise and give factor, that is where the main problem start ... so I maintain that the mathematics teaching experience does have a role to play in this regard”.*

And finally, teacher GS offered a similar perspective, saying: *“I think in this question of experience it is not the same with the question of the number of years for the teaching service”?*

R: Yes, it is about the experience.

GS: *“Experience ... I can say the person with experience has a role to play ... he has that advantage and he has an important role on that area which may be specifically algebra from Grade 10 ... because he has been exposed on different scenarios of algebra from different textbook, different resources ... maybe he has conducted some content workshop and everything ... he has grown knowledge wise since the teacher is a lifelong learner ... so if that person is teaching first year then second”.*

9.2.4 On the area in the factorisation of quadratic expressions that is challenging to teach at a Grade 10 level.

Emanating from the teachers’ responses, it was evident that most teachers found it challenging to teach the factorisation of a trinomial quadratic with $a \neq 1$ at Grade 10 level.

For example, teacher RT had this to say: *“It is when we have mixture of variables with constant on both term like e.g. $6y^2$, or when you have more than one variable on an expression, so for them it becomes difficult to find factors, or they don’t factorise at all, but those top high flyers can factorise; once you have very big number and worse if the number is negative sign, it better if it is plus, and it is better if negative is in the middle term, once on the last term becomes a problem”.* This difficulty in teaching factorisation of this quadratic was also expressed by teacher ND who said: *“Once x^2 has a coefficient other than one, it becomes a problem to make learners understand it is easy if x^2 has a coefficient of one”.* Teacher GS shared a similar opinion saying: *“I think to factorise expression with three terms ... it is more challenging particularly if the coefficient of x^2 is not equal to 1 ... you end up as a teacher applying many methods of working out the problem ... that poses a challenge to our learners ...”.* Teacher MZ supported teachers RT, ND, and GS on the matter saying: *“I can say most of the time learners begin to struggle when it comes to factorising trinomials with a coefficient of x^2 being greater than one; as much as we try by all means to use different approaches to teaching them, but it is always a challenge”.*

However, he blamed mathematics teachers themselves for contributing to the challenges faced by learners.

R: So what is it that you feel is posing a challenge?

MZ: *“Maybe it is the way we (teachers) break it down to them (learners) ... I don’t know because I know that we as teachers, we are the one who teach them to factorise a trinomial of the coefficient of x^2 being one, so then again that means we are the one who should also teach them to be able to deal with that of the coefficient of x^2 being greater than one ... so the problem lies with us as teachers”.*

Teacher HP pointed to the simultaneous use of different approaches in factorising the quadratics as one contributor to learners not understanding the concept of factorisation. He said: *“One of the challenges may be when learners mix up these different ways of factorising these algebraic quadratic expressions ... if today learner is taught to factorise by taking out common factor ... learner understands ... tomorrow a learner is taught the rule of dealing with difference of two squares ... it seems much better ... now on the other day you teach the learner the third method ... now that is where the confusion begin to manifest ... I don’t know maybe learners do not give themselves time to practice to know the difference between these rules ... in class it is as if they are clear but the problem starts when they are working alone”.*

Teacher KW on the other hand, pointed to the inability of teachers to manage the shift in learners’ practices as the main contributor in learners not understanding the concept of factorisation. He said: *“Factorisation begins at Grade 8 ... at Grade 8 we are not using the same methods as we are treating them at Grade 10 ... I am not the teacher of mathematics at Grade 8 ... at Grade 8 learners use the table method, and the theory of which is something I do not understand ... to me this table does no logic ... now at Grade 10 I begin to introduce my method ... the method that I understand ... they will persist using their table method which then becomes difficult for them to master my method”.*

R: Is there no way you can develop to try and go back to understand this table method in order to manage this shift?

KW: *“I think in that case it is difficult because these learners have been on this practice for two academic years ... and at the same time the ATP does not allow us enough time ... each and every section has its time frame ... so that is why I find it difficult ... the only solution maybe is to embark on the team teaching ... when it comes the beginning of the first term in Grade 10 is the very busy period since there are so many concepts to deal with as per the ATP”.*

9.2.5 On the area that is easy to teach in the factorisation of quadratic expressions at Grade 10 level.

Four teachers found it easy to teach the difference of two squares. Teacher ND, for example, made use of the example to stress her opinion saying: “Yes, this one $x^2 - 4$ just because here we speak of the items that are identical...for an example if given 4 we then multiply 2 and 2... if given 16, we then multiply 4 and 4”. Teacher MZ supported the view, adding: “I think it is the one where we deal with difference of two squares ... that one is always easy to deal with because some of learners in their earlier grade the use to transpose one term ... the one that does not have an x to the other side then use square root sign; to factorise that one or to get factors, so I think that one is the easiest”. Teacher KW shared a similar sentiment saying: “Personally, I can say it is the difference of two squares ... because for the difference of two squares you can be able even to plot a graph so that they to understand without looking at the factors directly ... but they can plot graph and observe that the graph cuts the $x -$ axis so there are factors ... I see it is much more easy to teach”. Even though teacher RT thought the difference of two squares was easy to teach, she also added that the simplicity of dealing with factorisation of the quadratic of $a = 1$ was also easy to teach. She said: “The one where x^2 have coefficient of one, that is the easiest, or if all the signs are positive or the last term is positive; difference of two square is easy...just because you only apply the rule”.

Two teachers, HP and GS, expressed their adequacy in dealing with the quadratic where the method is to take out the common factor. Teacher HP said: “Taking out the common factor is easy for the learners because, through observation, one can identify items that are common between the terms”. This notion was supported by teacher GS who said: “I think it is the one whereby the learner has to factorise by taking out the common factor”.

R: What is it that you find easy in dealing with that one?

GS: I think in that case ... I think it become easy for a learner to see the value or the term that is common in any two terms ... this approach is grounded at their earlier grades”.

9.2.6 On how they prepare their lessons on the factorisation of quadratic expressions for their diversely-talented Grade 10 learners.

Teachers shared their views on how they prepared lessons on the factorisation of quadratic expressions for their diversely-talented Grade 10 learners as follows:

Teacher RT said she differentiated approaches for learners according to their abilities. She explained: *“It depends on what I am teaching at that instance e.g. if I teach the difference of two squares, ... for slow learners I just give them the rule, and for those who are highflyers we try to expose them to the variety of methods. If you give the slow learners multi-methods, there is a likelihood for them to come with alternate incorrect methods so stick to the rule for them not to be confused”*.

Teacher ND said she administered the same activity to all learners but ensured that the activities were graded from easy to difficult. She explained: *“Usually I begin the introduction of the concept by engaging learners on problems that are less challenging ...I slowly and gradually introduce those problems that are challenging as the lessons progressed... I also use the highflyers as tutors to further clarify issues and matters of concern in the small groups ... I then consolidate for the whole class”*. Teacher HP supported teacher ND’s presentation saying: *“To be honest, I cannot say there is a special tactic that I use ... knowing that learners are at differing levels I clarify my all the steps that I do ... I try to be simpler and move in slow pace ... I teach the concept step-by-step ... I spend much of my time clarifying why we teaching the particular step”*. Teacher MZ said he followed a similar route to that taken by teacher ND and teacher HP, adding: *“Well, I guess this is the reason why the textbook activities are designed in that order ... textbooks always make sure that whenever they deal with the chapter they start with what is perceived to be simple for everyone to grasp... and then the more you carry on with the chapter, you see that things are getting complicated ... so obviously I will always start with what is simple to everyone, what I think could be understood by everyone ...then in a case where they do not understand, something I asked them if they do get what you explaining, then slow learners I know some of them do not understand, sometimes we have to repeat ourselves unfortunately, we cannot run away from the fact that smart ones, in the case you are busy repeating the simplest things they are discouraged ... some of them are getting bored but I guess it is a protocol we cannot leave the ones that are struggling, so we need to attend to them as well”*. Teacher KW shared a similar view to the other teachers saying: *“If I am*

preparing ...let us say maybe the new topic ... in my class I begin by administering the activity of easy problems ... this I do with an intention that everybody need to score marks ... I begin the individualisation exercise where I observe how learners are performing individually ... I give basic skills to those who are struggling ... I refrain from giving the challenging activities because they may end up being discouraged ... I provide the individualised activities for different learners based on the speeds at which they are responding”.

Even though teacher GS shared similar views to the other teachers, he stressed the notion of assessment. He explained: “*Learners are not the same when it comes to the rate of acquiring knowledge ... some took long to absorb the knowledge ... others are quick to absorb ... I prepare my lessons to cater for those differences such that lessons suit both of those learners*”.

R: How do you then structure your activities?

GS: *I think in terms of delivering the lesson across the classroom ... it is the same ... the difference is when it comes to assessing knowledge ... it also differs in terms of high order and low order questions ... the activities administered to the learners are not the same.*

R: Do you prepare your activities so that they cater for these differences?

GS: Yes.

9.2.7 On how they manage their Grade 10 work while also teaching higher grades.

R: In most instances, we teachers teach both Grade 10 and Grade 12. How do you strike the balance in terms of lesson preparations for the two grades?

RT: *“Yah, indeed the efforts can never be the same, but during teaching hours it is the same ... but for Grade 10 there is that feeling that I am still going to be with them at Grade 11 ... the difference is noted when we engage learners in extra- classes ... there is no way I can spend the same amount of in both grades ... the founding reason is the Grade 10 writes the internal examination while the Grade 12 writes the external paper, but I put more energy at Grade 12 and I don’t want to lie But during teaching hours the effort is the same for all grades.....but during the extra classes I put more effort on Grade 12.....reason behind the injection of lesser energy at Grade 10 is I am still going to be with them at Grade 11 - simplification and solving of equation are there at Grade 11, so it means that if there exist some gaps, I can fill the gaps*

in them at Grade 11 and still move with them to Grade 12, so it is just that there is still a long road ahead of us (me and my learners) all”. The notion of the imbalance was also expressed by teacher ND who said: “No, there is no balance ...I concentrate more on Grade 11 and 12 ...I don’t know how it just happens ...another point is we are being judged by means of the NSC results in Grade 12 ...so much of time will be allocated to Grade 12 ...however as you progress to other grades with your learners you begin to fill up the gaps ... it is therefore advisable for the teacher to move with his/her learners from grade to grade”.

Teacher HP jumped up to state the reason why there is no balance in the teaching of Grade 10s compared to Grade 12s, saying: “We get judged by the matric results ... the focus is more on matric ... where you are judged more is where you put most of your energy ... really the smaller grades get affected ... there is no balance in terms of injecting energy”.

Teacher MZ was honest in his response saying: “When it comes to preparing ... I’m not going to lie ... I spend too little time when it comes to preparing for the Grade 10 ... well basically one of the reason why I do that ... it is because sometimes I take things for granted ...I have that perception which says to me almost 80% of the things which are done in Grade 10 are things that I know by heart ...maybe the chances are very slim that I might come across the question in Grade 10 which will challenge me to an extent that I am required to go and seek for assistance from other teachers ... however, in Grade 12 there are cases where you get into class and you come across the problem that is challenging ... so basically that will mean before you hand out class work or homework activities for Grade 12 you must have first done them ...so that should the questions arise you are able to deal with them ... in Grade 10 almost everything that is done there I can be able to answer any question”. This honesty was also expressed by teacher GS who said: “No ... to be honest it does happen for the Grade 10 teacher to realise that he is behind ... sometimes the alarm will be rung by the availability of the examination time table ... then you now call the learners to the Saturday class ...you now need to push in order to cover the scope since learners are writing the external paper ... there is no balance at all”.

It was only teacher KW who offered an alternative response saying: *“If you can look at my timetable ... it is flexible because most of the times I see Grade 12 on period 3 before the break ...whereas the Grade 10 periods are after the break ... so I have the ample time to prepare ... as a result the focus becomes the same ... however, Grade 10 does not get the enough time for you to do the revision ... maybe the adjustments has to be done in the our ATP ... the ATP is failing us ... you find that you have to deal with the concept in a given period of time ... when you finish that topic you find that you only have two days to revise”*.

9.2.8 On the extent at which they incorporate the context-based activities in their teaching of factorisation of quadratics in Grade 10.

Teachers expressed reasons why they are unable to incorporate the context-based activities in their teaching of factorisation of quadratics in Grade 10. Teacher RT gave the following reasons: *“I don’t know at Grades 8 and 9 but at Grade 10 there is nothing to bring to class that is talking to factorisation in context...it is difficult to make factorisation to be practical”*. Teacher HP supported the notion saying: *“This part is difficult to achieve ... even though we may love to make factorisation practical, but it is difficult”*. Teacher MZ shared similar sentiments of finding it difficult to incorporate the context-based activities in teaching factorisation of quadratics to his Grade 10 class. *“In Grade 10 we hardly go to that ... I think most of the times we tend to focus on things like variables where we normally deal with x and y ”*.

Teacher KW cited the Annual Teaching Plan (ATP) as the one Department of Basic Education (DBE) document that hinders him in exploring context-based activities, saying: *“The problem here we work according to the ATP ... in the past examination papers there are no such activities ... the textbooks do not provide such activities ... your class work and homework should be in line and according to the ATP... if you give something outside of the ATP the departmental officials will view you as a person who is not working according to expectations”*. Again, this notion on the use of the ATP as hindering teachers from exploring the use of context-based activities was also stressed by teacher GS who said: *“No ... I only follow the activities on the textbook ... I follow the CAPS”*.

9.2.9 On the extent at which technological advances can assist in enabling Grade 10 learners perform successfully when dealing with the factorisation of quadratics.

Five teachers seemed to agree that technology can benefit the education system in ensuring Grade 10 learners perform successfully when dealing with the factorisation of quadratics. The teachers offered different reasons in substantiating their claims. For example, teacher RT said: *“I don’t know about the technology ... I am not used to it; I am a BBT (Born Before Technology) ... however, I have noted the technology making things easier in other areas of mathematics, like graphs and geometry...so I don’t if there is anything it could do in the factorisation ... I have only observed the technology playing the role on graphs; it also helps in geometry; so I’m not aware how it can assist in the factorisation – I have not seen it as yet”*.

Teacher ND supported teacher RT’s claim saying: *“I think technology can assist in the number of ways ...learners can make use of the internet and search how certain mathematical problems are solved ... by such engagement learners will get extra practise”*. Teacher MZ shared a similar opinion, saying: *“Well most of the things I guess they depend to the subject teacher and the learner ... because we have got something like Google, where you can just type question and then Google will explain to you how that problem is solved ...so I personally think Google is helpful because you are able to get access to YouTube ... a learner may then come to the class with very few clarity seeking questions ...again, in YouTube learners are presented with the multi-methods of dealing with a problem ... the work of the teacher becomes easy ... internet can also assist on issues regarding the concept of factorisation ... so for me technology can assist the learner a lot ...”*.

Teacher KW praised the important role technology can play in addressing issues of methodology, saying: *“Yes, we can use the technology ... even I have learnt a lot through technology ... I have searched through internet and learned other methods of factorisation ...I have able to marry the approaches I get from the textbooks and those I have learnt from the internet ...it is now easy to introduce these methods to my learners”*.

Teacher GS supported other teachers by adding: *“That question depends on the availability of technology... time is also needed to fine tune the minds of learners when it comes to the use of*

technology ... but once the technology is mastered then it will be much better when it comes to explaining the concept to the learner ... everything will be clear”.

There was only one teacher, HP, who did not appear to know what technology had to offer education. He said: *“Currently, I have nothing that I know that uses technology”.*

9.2.10 On the extent to which English, as the language of instruction, can impact on the success of Grade 10 learners when they deal with the factorisation of quadratics.

There was a balance of views as teachers responded to this question. On one hand, three teachers felt English, as the language of instruction, could impact on the success of Grade 10 learners when learning the factorisation of quadratics. Teacher MZ explained: *“Yes, I think it has an effect ...the way I always observe in our classrooms ... on certain occasions we as teachers try to explain some concept to them (learners) in English if we see that they do not understand all that I am saying then you try to break it down in IsiZulu so that they will be able to understand whereas the challenge will arise when the tests and examinations are written ...I will not be there to break the questions down to IsiZulu ... they will have to understand the statement and the information which is being said, on their own”.* This view was supported by teacher GS who said: *“ I think it may have the effect because the examination paper they are writing sometimes it needs to be interpreted ...even though at factorisation there is little of English, but learners are always terrified since English is not their mother language”.* Teacher ND shared a similar opinion, saying: *“Yes, because other learners are finding hard to understand the question and they do not know what is expected from them and what does the question want ... factorisation is better, however you do get some phrases like “hence” and those phrases are posing some challenges for our learners.*

On the other hand, three teachers were adamant that English as the language of instruction had no impact on the success of Grade 10 learners as they learnt the factorisation of quadratics. Teacher RT remarked: *“I am not sure, I don’t know because even I was taught in English but there is that addition of IsiZulu, I am not talking in terms of factorisation but with other parts of mathematics there is an effect. But with factorisation I don’t know because there is the use of simple English, “factorise; find factors” but I don’t know how isiZulu can come on board. I*

don't see any effect on the use of English on factorisation except for other chapters". Teacher HP held a similar view, saying: "With a big no ... in algebraic expressions I don't see any effect ... even if you can translate the name factorisation into IsiZulu language, factorisation will still be factorisation ...". Teacher KW shared his view on why English would have no impact on learners' success in mastering the factorisation of quadratics saying: "No ...I don't think there is a problem of language when it comes to factorisation ... this is because you are able to see the problem, and you are able to follow the steps as you factorise ... and if you are given any statement ... it is clear ...factorise this ... you see ... language has no effect at all ... maybe the problem will be on financial mathematics".

9.2.11 Dialogue on matters within the factorisation curriculum - the Grade 10 content factorisation of quadratics.

Three questions on the matters surrounding the Grade 10 content factorisation of quadratics, were put to the participants.

9.2.11.1 Question 1: Do you think the number of concepts on factorisation of quadratics in Grade 10 is adequate for Grade 10 learners to progress the FET band?

All teachers agreed that the knowledge of factorisation of quadratics Grade 10 learners gained was sufficient for them to progress the FET band. Teachers offered a number of reasons in support of their stands.. Teacher RT said: *"Yah, Grade 10 has it all, because as I look in there: we factorise binomials; trinomials; even the factorisation by grouping, yah it is complete, may be if the factorisation by using the quadratic formula could be added, because the majority of learners enter Grade 11 with no skills of dealing with factorisation only to be rescued by the use of the quadratic formula. Exactly, I don't know what could assist, but the factorisation in grade 10 is complete and implicit".* Teacher ND agreed with teacher RT saying: *"Yes, I think it is adequate ... this is because even at matric they will do just one and the same thing ...even though there are some additions, but the knowledge is the same".* Teacher HP offered an explanation that echoed the opinions of other teachers. He said: *"Even though I cannot say it is enough, but I can say it is sufficient because it form the basic for what they are about to get moving forward with the grades to Grade 12 ... most of the Grade 12 work is covered at Grade 10".* Teacher MZ agreed, saying: *"I will say it is right for algebra to be dealt thoroughly at*

Grade 10 ... as we said earlier on almost all the other chapters do rely on it ... simplify or solve ... you need to apply the knowledge you have acquired in the Grade 10 algebra throughout the school mathematics ...”.

Teacher KW firmly believed the knowledge of factorisation of quadratics learners in Grade 10 gained was enough to prepare them to progress the FET band. He said: *“I can say it does prepare them adequately ... this is because most of the secondary school mathematics concepts are dealt with at Grade 10 ... at Grade 11 they are more at the enhancing and exploring the other methods of factorisation; completing the square; use the quadratic formula ... ”*

Teacher GS supported the other teachers saying: *“I can say it is adequate and relevant”.*

R: Do you think the knowledge they acquire at Grade 10 can deliver them to matric?

GS: Yes.

9.2.11.2 Question 2: Do you see the Grade 10 content of factorisation of quadratics being appropriate for Grade 10 learners?

Teachers shared different views on the question of the Grade 10 content of factorisation of quadratics being appropriate for Grade 10 learners. Teacher RT raised concern about learners being afforded the space to practice. She said: *“I, speaking from my experience, I see it being appropriate, its more appropriate it is just a matter of affording learners a space to practise for them to be perfect”.*

She was supported by Teacher ND who said: *“Yes it is appropriate even though it is bit easy”.*

R: Why are learners performing poorly if the content is easy?

ND: *“Our learners are not dedicated to learning ...if you can compare the current maths of the Grade 10 and the maths that we did ten years ago you can see that we were overloaded with content ... we were dedicated that is why we were successful”.* In support of the other teachers, teacher KW said: *“I could say it is appropriate ... even though we find it difficult to complete the curriculum in that specified period of time”.*

Teacher HP mentioned the need for learners to practice, expressing his opinion thus: *“It varies, it is not too easy and also I cannot say it is too hard ... but it is right for our learners ... this content get learners equipped for the mathematics in the upper grades ... learners need to practice all the times otherwise the content is appropriate for them to grow mathematically”*.

Teacher MZ felt some content had to be added, saying: *“Well I think the only one aspect that I feel should be taught in Grade 10 the quadratic formula to determine the factors ... in case where you have to factorise the quadratic because again it is still quadratic ... we always given expressions that are factorisable ... what do we do in the cases where we do not get the factors? ... which approach can we use? ... I think if we already started with it in Grade 10 and being infused at that level it will be much easier for them to use at Grade 11 ... this is because in Grade 10 if a learner comes to you with a quadratic that you cannot factorise it becomes a challenge because some learners are smart ...”*.

R: Don't you think that perhaps CAPS wants learners to get a feel for mathematics at a young age before being exposed to ready-made formulae?

MZ: *Maybe we can view it at that angle ... but then again factorisation of quadratics begins at Grade 8; the more they enter the upper grades the more it becomes tense ... that is why I am saying they should start with it in Grade 10 because we say from grade 10 to 12 is a senior phase (FET) where they must start getting a feel of the real mathematics”*.

9.2.11.3 Question 3: What is your view regarding the amount of content required for Grade 10 factorisation of quadratics in relation to the expected pace of instruction?

All teachers remained resolute on this question stating categorically that the amount of content required for Grade 10 factorisation of quadratics had no positive relation to the expected pace of instruction. Teacher RT said: *“There is no correlation, there is a lot that need to be done in the factorisation but you find that the time is too little for the teacher to teach, that is why learners enter Grade 11 with little knowledge on factorisation, and only to be rescued by the quadratic formula, it's because time is too little”*. Teacher RT was supported by teacher ND who said: *“The time allocated to us for teaching the concepts in Grade 10 is too little ... you prepare the lesson of 55 minutes and find that learners are not grasping anything ... it is therefore difficult to take the time allocated for the other day because it has some work*

allocated to it ...you cannot keep teaching one and the very same concept – the ATP dictates your movements". Teacher HP favoured the views of the other teachers saying: *"I think the work schedules need to be revised ...there is this feeling that most of the concepts are expected to be dealt with in a short space of time ...the time is too short for the teacher of Grade 10 to deal with the factorisation of quadratics as per the expectations of CAPS ... teaching more topics in a short space of time brings confusion to our learners"*. Teacher MZ was short and precise in his answer, saying: *"No ... the amount of time I think it is very little, there is too much work"*. Teacher KW supported the views of the other teachers saying: *"Some of the topics are given a very little time ... the time is not enough because we go pass those concepts by compliance and with high speed assuming that learners understood ... justice with time is not done ... the assumption is that some of the concepts have been visited at Grade 9 ... there is no relation at all"*. Teacher GS hammered the final nail in by saying: *"There is no correlation ... the time is too little"*.

9.2.12 On their views on the use of short-cuts by teachers in solving problems in mathematics.

The following question was put to the interviewees.

R: In problem-solving exercises, teachers tend to teach short-cuts rather than the fully-fledged approaches. Do you agree? What is your position? What do you think might be their reasons?

In responding to the question, teachers offered reasons why they taught short-cuts. Teacher RT said: *"Yes ... I strongly agree. Reason is it is because the scope is too wide, it is too long, so if you try to avoid short-cuts you will end up not completing the ATP we are being pushed by the ATP....ATP is very long and we have to make sure that we complete this and we move to the next one before the given specified period of time lapses It is the time frame that guides us"*.

R: Doesn't over-reliance on short-cuts lead to confusion for our learners because of the many rules"?

RT: *"Yes...because of many rules, our learners become confused...that is why learners end up with misconceptions....learners end up e.g. not knowing when to take a common factor; when to think of difference of two squares; when to factorise by trial and error.... because of too*

many rules...this I think is the reason why mathematics is poorly performed even at Grades 11 and 12....too many rules...too many formulae.....so short-cuts lead to confusion”.

Teacher ND offered a similar response responding as follows:

ND: *“Yes, I do agree”.*

R: Are you also doing it? What do you think are the reasons?

ND: *“I personally do it ...I think the textbooks we are using are fostering the use of short-cuts ...it is again the ways we were taught in schools and even at the tertiary institutions”.*

R: Don't you think the use of many rules of factoring the quadratics may confuse learners?

ND: *“Yes ... learners get confused... they do not know which rule to apply and where to apply it”.*

R: Is there any ruling in CAPS which says when we deal with the factorisation of quadratics in Grade 10 we must start with the monomial; followed by the binomial; then the trinomial?

ND: *“No”.*

R: Can you follow any order?

ND: *“Taking from our (researcher and Ms Ndulini) earlier discussion...yes”.*

Teacher HP agreed saying: *“I agree ... for me that is just the case ... under normal circumstances when you are to deliver the lesson for an example on the factorisation of the difference of two squares, there are less chances of first exposing learners to a version of a complete quadratic by emphasising that there is zero middle term ... we do not speak about the trinomial here ... we only push them by going straight to the answer ...”.*

R: Why?

HP: *“I don't have the clear answer but you end up teaching learners some other methods of being able to recognise patterns for them to progress even though by telling learners it is like you skip some steps somehow it is like no complete learning took place ... however the focus is primarily on the answer ...”.*

R: If our focus is on the answer, are we not depriving learners the opportunity to observe the beauty of mathematics?

HP: *“Even though we may love to see that happening, however, there are some hindrances. Firstly, the time allocated to teach is very little when related to the amount of content to be taught. Secondly, the learner fail rate is too high ... we are not intending to deprive them but short-cuts are the solution to this predicament ... we want marks and nothing else”.*

R: Do you teach the short-cuts?

HP: *“Yes ... I do teach the short-cuts ... for an example I come to class and teach the rule to follow when dealing with expressions of difference of two squares ... how do we recognise it? ... what short routes to take to reach the solution ... this we do for them to get marks”.*

When teacher MZ was asked if he taught short-cuts, he admitted he did, saying:

MZ: *“Yah, no I do. I definitely do”.*

R: What do you think might be the reasons?

MZ: *“Well, I guess our priority is the issue of the time frame because some of the things in other times one will go to class with intents of allowing learners an opportunity to explore on their own and realise that with time constraints on our side one will land nowhere. Maybe 2 or 3 learners or may be 10% of learners will be able to achieve and see that about 90% of the class is not going anywhere. Again, I have the content to cover. Because of time I will rush them and drive them to conclusions”.*

R: Now, if we reflect – does this approach not cripple our learners in terms of mastering the content? For example, one learner when asked to factorise $x^2 + 4$, she responded by saying that she cannot factorise the difference of two squares that is separated by the plus sign.

MZ: *“Yeah well (smiling), in a case like that one it is fact that we cannot run away from the notion that some of the short-cuts that we infuse to our learners do in the long run confuse our learners in terms of choosing the solving approaches”.*

Teacher KW was another teacher who admitted to teaching short-cuts. He said: *“Yes, I agree ... let me respond this way ... sometimes we come to class with imbedded assumptions ... long methods will discourage learners ... we are then tempted to tell learners that for an example $x^2 - 4$ is the difference of two squares ... we do sometimes do that to be honest ... we strive to*

make mathematics easier by doing so ... however, I personally always try to make sure that learners are at least given more than one approach to the problem”.

R: Do you agree with the claim?

Teacher KW contradicted himself when he said: “*No ... personally I disagree with the claim ... as I have indicated earlier on ... that I am always giving more than one approaches to the solution”.*

R: What do you think might be the reasons for teachers using shortcuts?

KW: “*This is possible in the cases where teachers are without other approaches of dealing with that problem ... they only rely on the singular approach that they have ... if the short-cut is administered without other approaches it promotes cramming”.*

Teacher GS showed some support for teacher KW, saying: “*I disagree because what we give learners basically... we attended the tertiary institutions to learn that information ... we do have enough methods to help these kids to solve any kind of expression that they will face across been learner”.*

R: Do you use the short-cuts?

GS: “*I agree ... I do make use of the short-cuts ... when the going gets tough for the learners, we end up trying to make things easier for them to understand”.*

R: Does the CAPS prescribe approaches for the teachers to use in the factorisation of quadratics at Grade 10 level?

GS: “*No ... it is us teachers who introduces the approaches that we feel will be easier understood by our learners”.*

R: Don’t you think calling $x^2 - 4$ the difference of two squares may lead to confusion in the mind of the learner thinking that this is not a quadratic?

GS: “*It does lead to confusion ...the headings in the textbooks are presented as if difference of two squares are isolated from the quadratics”.*

9.2.13 On their views on whether teacher knowledge impacts on learner performances.

All six teachers agreed that teacher knowledge is an important contribution to learner performance. Teacher RT said: *“Yes, I agree because the teacher teaches those areas of comfortability – if the teacher is not comfortable about the concept, he/she will never treat that concept especially in the grades like Grade 10. You find that the teacher will only do the factorisation of a trinomial where the coefficient of x^2 is one. The teacher will then skip those trinomials where the last term is a variable and where the coefficient of x^2 is not equal to 1. In this case the teachers’ knowledge has an influence on the performance of learners. That unattended body of knowledge will remain a gap throughout the schooling of the learners. To ensure that there are no gaps in learners, teachers should make it a point that everything on the ATP is taught”*. Teacher ND supported teacher RT’s views, saying: *“Yes, because even if I don’t know how to deal with algebraic expressions, how I can expect my learners to understand all I will be saying ... and how can I explain the different approaches if I know nothing?”*

Teacher HP agreed with the other teachers, adding: *“I would say that is true, reason being, how vast a teachers knowledge is, would also assist in learners becoming vast knowledge-wise in that manner ... for example, you find that there are learners who are able express their thoughts so much in that they like to interact and discuss robustly ... a teacher with little knowledge will find it hard to handle such class of learners ...the teacher has to be knowledgeable so as to able to lead some discussions, otherwise he will feel inferior and uncomfortable to entertain the foreign ideas”*. Teacher MZ was also in agreement, saying: *“On this one, I can say I agree because you were to the best way is to look at it by looking at two cups – if my cup is full of water and then we do understand that the learners come to class knowing that we are the ones who have to teach them new things; so it will depend on the amount of knowledge that I instil to them – if I inject the little knowledge, it is obvious that they will give little in return. So if I know more that will definitely benefit my learners”*. Teacher KW was vocal in his response saying: *“I strongly agree ... let me say ... as a teacher of mathematics you have to be the first to understand and master the concept ... you need to master the strategies on how the steps are followed when we are solving ... it becomes more easier when you are confronted with the task of teaching”*.

Teacher GS was asked the following question: R: Do you think the teacher's knowledge makes a contribution to the learner's performance?

GS: *"Yes ... it does"*.

R: Does that imply that the teacher contributed to the best performing learner?

GS: *"Yes"*.

R: Does that imply that the teacher contributed to the poor performing learner?

GS: *"Ha ha ha ... kind of ... maybe"*.

9.2.14 On whether their teaching practices offer learners opportunities to explore mathematics.

R: One of the learners interviewed expressed concern that teachers do not offer them enough opportunities to explore – teachers teach according to their plans. Do you agree with your learner's allegation? What might be the reason behind such practice?

Once again, teachers responded to this question by giving reasons why learners are not afforded opportunities to explore. Teacher RT said: *"Yes, their allegations are true, we don't have time, you see if you recall the times of OBE it was said learners should discover on own on their pace, it was never achieved, it is very difficult to allow learners to discover something at a given time of 45 minutes or 1 hour – it is not happening"*.

Teacher ND agreed, saying: *"Yes that is true ... the problem here is ...in most instances our learners are not challenging ... they will just take whatever you feed them ...it is much better if you have learners who go an extra mile to seek for knowledge; questions; other methods and bring it back in class for further discussion...if learners come to class without the challenging questions, obviously they will only rely on the method and the approach you are feeding them as a teacher"*.

Teacher HP had this explanation to offer: *"Even though to a very less extent, I personally do offer learners opportunities to explore ... I encourage learners to engage with learners of other schools with regard to the various ways at which particular topics are treated"*.

R: Does CAPS prescribe how the sequencing of Grade 10 subtopics in the factorisation of quadratics should be followed?

HP: *“No ... we teachers use the fact that we know our learners’ capabilities ... the build-up of subtopics rests in the shoulders of the subject teacher”.*

R: If given the opportunity to advise other teachers of mathematics, how would you advise them to sequence the subtopics in the factorisation of quadratics in Grade 10?

HP: *“I think they can follow this order:*

- (i) x^2 ;*
- (ii) $x^2 - 4x$ taking out the common factor ... this I do in view of generalisations;*
- (iii) $x^2 - 4y$ to make the good contrast of (ii);*
- (iv) $x^2 - 4$ the difference of two squares*
- (v) $x^2 + 4$ to make the good contrast of (iv);*
- (vi) $x^2 - x - 6$ coefficient of x^2 being equal to 1*
- (vii) $6x^2 + 7x + 20$ coefficient of x^2 being unequal to 1”*

Although teacher MZ held similar views to the other teachers, he did offer an alternative approach regarding the sequencing of the subtopics in the factorisation of quadratics. *“Well, again it goes back to what I said earlier on – if I’m using classroom mathematics and then it has one method of factorising so obviously in class I will also use it, so then again that will deprive my learners because it is a bit hard to allow them to explore if even myself I do not have enough information – so it goes back to the previous question on the teacher knowledge – if I know 2 methods I will teach 2 methods but I know 5 methods my learners will be.... have an upper hand when it comes to that because I definitely will teach 5 methods and they will get the chance to choose one that suit them”.*

R: If given the opportunity to advise other teachers of mathematics, how would you advise them to sequence the subtopics in the factorisation of quadratics in Grade 10?

MZ: *“I think they can follow this order:*

Start by looking at these two types simultaneously

(i) $x^2 - x - 6$ coefficient of x^2 being equal to 1

(ii) $6x^2 + 7x + 20$ coefficient of x^2 being unequal to 1

Begin to eliminate the last term

(iii) $x^2 - 4x$ taking out the common factor ... this I do in view of generalisations;

(iv) $x^2 - 4y$ to make the good contrast of (ii);

Eliminate the middle term

(v) $x^2 - 4$ the difference of two squares

(vi) $x^2 + 4$ to make the good contrast of (iv);

Eliminate the second term and the last term

(vii) x^2 ”

Teacher KW pointed a finger at the DBE saying: *“In this case ... mhhh ... I point the finger at the government in this case because he is the one who compiled the ATP ... we are expected to do things at the high pace and at the given space of time ... otherwise if you give time to explore you will never finish the curriculum ...”*.

Only one teacher GS, held an alternative view, saying: *“On that one ... I’m quite not sure ... but if I can personal speak about me ... I can say I do give them an opportunity to explore mathematics ... to understand how they interpret particular scenarios with what is in their minds ... so not only that if you came with a plan then you just teach ... maybe there is a learner who will approach this in different way we need to give them opportunities to find out how other learners approach the same problem from different perspectives ... we do offer such opportunities”*.

R: What do you think might be the reasons for teachers not offering learners such an opportunity?

GS: *“It may happen that the teacher concerned is treating learners as empty vessels ... he only believes that he is the source of knowledge”*.

9.3 Teachers' responses to the activity on the factorisation of quadratics.

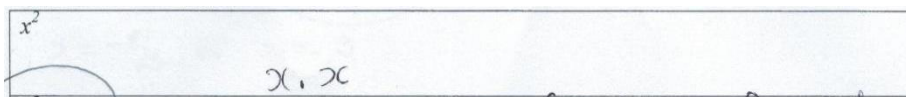
The second section of the teachers' interviews focussed on the activity on quadratic expressions. I used the teacher interviews to gain a better understanding on what transpired as teachers engaged in the mathematical activity that appeared in the teacher questionnaire.

9.3.1 Task 1: Factorisation of x^2 .

9.3.1.1 Teachers' correct responses.

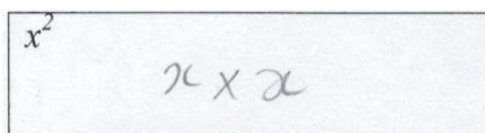
Out of the six teachers who were interviewed, four teachers from the onset were in agreement that $x^2 = x \cdot x$ while two teachers, Hlophe (teacher HP), and Ndulini (teacher ND), had alternative views.

Teacher RT was one of the teachers who was in agreement,



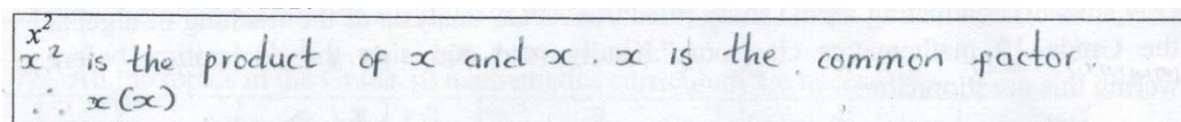
and responded by saying: “I have looked for two numbers that I can multiply and give me x^2 and found that it is $(x)(x)$ ”.

Teacher MZ shared a similar opinion,



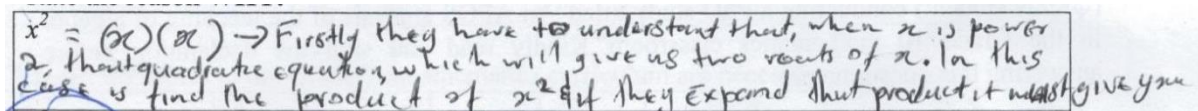
saying: “First of all, when it comes to factorising, there are many things that we need to look into ... for instance we have to check if we do have common factor or not from our terms ... and also check how many terms do we have ... well in this case we were given x^2 ... from my definition when we factorise x^2 ... since it is a variable I will need to determine variables that we will multiply to give me x^2 ... those variables are x and x ... in this case, if I multiply x and x I will get x^2 ”.

Teacher KW agreed,



saying: “In accompanying my learners I will start by saying we need to find the factors of x^2 which are x and x ... and another example maybe I can give ... that is all”.

Teacher GS agreed,



$x^2 = (x)(x) \rightarrow$ Firstly they have to understand that, when x is power 2, that quadratic equation, which will give us two roots of x . In this case is find the product of x^2 if they expand that product it will give you

saying: “Here there are factors ... what I always check is whether the variable x has a power of one or the power of two ... if you have x^2 ... you open two brackets ... look for factors of that could be multiplied to give x^2 ... the factors are x and x ”.

However, it took two teachers, HP and ND, some time to shift from their initial thinking to agree that if you factorise x^2 , you get $(x)(x)$. Their initial thoughts are presented, respectively, as follows:

9.3.1.2 Monomial cannot be factorised.

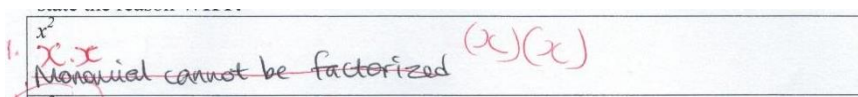
When teacher HP was asked how he factorised x^2 , he responded by saying “*monomial cannot be factorized*”.

R: Why?

HP: “As I have pointed my reason when I was defining factorisation ... I said we expose the basic building blocks that connects two terms or more ... so, in this case we were given one term ... so the factors of x^2 are $(x)(x)$ but now I need to compare this x^2 with something ... since I found no other term or terms to make the comparison I then concluded by saying *monomial cannot be factorised*”.

After a thorough discussion between myself and Mr Hlople (teacher HP), an alternative explanation was then given.

R: Do you still maintain your initial thoughts?



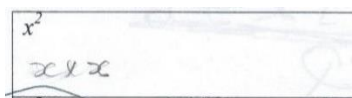
x^2
Monomial cannot be factorized $(x)(x)$

HP: “Ok you can factorise of x^2 to get $(x)(x)$... from my understanding I thought I need to compare of x^2 with something, for example, in $x^2 - 4$ we have of x^2 and 4 so I need to take out something common that relates these two ... allow to fix this by deleting this statement (monomial cannot be factorised)”.

During our engagement in the interview session, it was clear from his responses that even though teacher HP accepted that x^2 can be factored to get $(x)(x)$ it appeared that he was not willing to unlearn that factoring can only happen on a polynomial.

9.3.1.3 Factorisation is about breaking the expression into identical items.

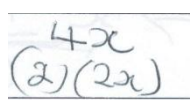
Another example is the response by teacher ND who wrote,



A handwritten note on a piece of paper. At the top left, it says x^2 . Below it, there are three 'x' characters written in a row: $x \ x \ x$. There is a faint, larger 'x' written to the right of the three 'x's.

and said: “When we factorising...it is where we look for instance if we can make use of x^2 , ..it is where we check what similar items that we can multiply and give us in return what was given before...we usually use that language for our learners to understand...for an example x^2 is made by x multiplied by x ...the similar thing if we have x^4 we get x^2 times x^2 ...it is the matter of keeping the base and adding the exponents”.

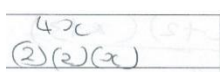
R: Speaking of identical things, let us say I had $4x$ and factorised it to $(2)(2x)$...



A handwritten note on a piece of paper. At the top, it says $4x$. Below it, it says $(2)(2x)$. There is a horizontal line drawn below the expression $(2)(2x)$.

, does this action hold the definition of factorisation?

ND: “No...we need something that is common...like in the binomial”.

R: Can we say we are factorising in  $(2)(2)(x)$?

ND: “Mhhhhh....yes....because when we multiply everything we get $4x$ ”.

9.3.2 Task 2: Factorisation of $x^2 - 4$.

Even though all teachers responded correctly by giving $(x + 2)(x - 2)$, their reasons were diverse. Their responses were as follows:

9.3.2.1 Teachers used the rule of dealing with difference of two squares, then the trial and error.

The rule of working with the factoring of the difference of two squares is articulated in the response of teacher RT when she was asked how she factorised $x^2 - 4$.

RT wrote,

Handwritten note by teacher RT: $x^2 - 4$ is circled. The text says: "Find the factors of first term, a factor of the last term, then put different signs." Below this, the factored form $(x-2)(x+2)$ is written. To the right, there is a small diagram showing a bracketed 'x' multiplied by a bracketed '2'.

and said “For this one, you just look for the two factors of the first term x^2 which is x and x , and for the last term which is 2 and 2 , then for the signs because I’m looking for factors of -4 , it can be -2 and 2 , in other words we look for factors of the term and those factors of the last term, which will give $-2x$ and $2x$ ”.

R: Why -2 and 2 ?

RT: “Yah, it can be 4 and 1 , but these are different of 2 squares, so 4 and 1 cannot give the original; once you have identified the factors you have to multiply them back to give you the original, so if it is -4 and 1 or 4 and -1 they can’t give you the original expression”.

R: What is your advice for the teacher who speaks of the difference of two squares as she/he teaches factorisation of the expression like $x^2 - 4$ for the first time?

RT: “Indeed this is the difference of two squares, so I think factorisation, is built on trial and error, you can start by asking your learners to give factors, and then ask them to multiply them back, to check if they do reach the original expression. So you cannot just start by saying these are difference of two squares, you can look for different factors of -4 , write those factors ask them to multiply them back, to check if they give original expression back”.

Teacher HP shared a similar understanding by writing,

Handwritten note by teacher HP: $x^2 - 4$ is circled in red. Below it, the factored form $(x-2)(x+2)$ is written. To the right, the expression $x^2 + 0x - 4$ is written, followed by $(x-2)(x+2)$ with a red arrow pointing to it.

and said: “I would say here in $x^2 - 4$... since this is the difference of two squares, usually the heading says factorise the difference of two squares ... then what we look for in both these two terms are called binomial ... because we have got two terms and one of the terms has to be the perfect square ... actually both terms must be perfect squares ... secondly, the difference of two squares is denoted by the presence of the negative sign between the two terms ... we open the two sets of brackets ... we break x^2 to get $(x)(x)$... we insert the minus and the plus signs in the sets brackets respectively ... we break 4 to 2 and 2 hence the solution is $(x - 2)(x + 2)$ ”.

9.3.2.2 Teachers solved the equation $x^2 - 4 = 0$ and used roots to identify factors.

R: How did you factorise $x^2 - 4$?

ND wrote,

$$\begin{array}{l} x^2 - 4 \\ (x-2)(x+2) \end{array}$$

and said “First of all I look at the sign between the terms...if it is negative, we emphasise that they need to check that because if it is negative it means factorisation is possible...there and then you now look your two terms...you check if they are square numbers...and then look for their square roots... the square root of x^2 is x and the square root of 4 is 2...we then open the two sets of brackets and drop positive and negative signs in both brackets respectively, hence the solution is $(x + 2)(x - 2)$ ”.

R: Why +2 and - 2?

ND explained by writing,

$$\begin{array}{l} x^2 - 4 = 0 \\ x^2 = 4 \\ x = \pm 2 \end{array}$$

and said “It is easy if you have the equation...let us say we have $x^2 - 4 = 0$...we then say $x^2 = 4$...then $x = +2$ or $x = -2$...since we introduce the square root both sides”.

R: Is it mathematically correct to introduce the equal sign if it was not given in the initial expression?

ND: “No ...we do not use the equal sign anyhow...however, we usually use the language of square root to simplify matters for our learners”.

R: Is the use of the square root a rule?

ND: *"I cannot say it is a rule, but it is one way of making it simpler for our learners".*

Teacher ND has introduced a new short cut, which is commonly observed followed by learners through the use of calculators.

9.3.2.3 Teachers used the rule of dealing with the difference of two squares, then the fundamental approach to factorising the quadratics.

R: How did you factorise $x^2 - 4$?

MZ wrote,

Handwritten work showing the factorization of $x^2 - 4$. The student writes $x^2 - 4$, then $x^2 + 0x - 4$, and finally $(x-2)(x+2)$. A note explains the difference of two squares method: "difference of two squares open up two brackets and determine the factors for the first term and the last term. Also make sure that the coefficient of x is 0."

and said *"Here we have two terms and I think this is the difference of two squares ... in this case the way we factorise them, we factorise by opening up two sets of brackets ... from those two brackets, now the first term was x^2 then we will need to determine the factors that will give us x^2 ... obviously seeing that x^2 is a variable we also need to use variables ... so if you multiply x and x you will get x^2 ... then again we also have -4 as constant value ...but 4 is negative ...let me start with the numbers ... we got 4 , the factors of which, will be 4 and 1 ; 2 and 2 ... now since 4 is negative in this case that will mean the signs if you multiply $+$ with $+$ you will get an answer which is positive ... to get a negative product I will need to multiply a negative with a positive ... in my case one sign should be negative and the other be positive because the last value here is negative ... hence the solution is $(x - 2)(x + 2)$ ".*

R: Why -2 and 2 not -4 and 1 ?

MZ: *"I can choose to expand $x^2 - 4$ and rewrite as $x^2 + 0x - 4$ and use the normal factorisation".*

R: How?

MZ: *"Open up two sets of brackets ... look for factors of x^2 which are x and x ... then look for factors of -4 which when added give zero and they are -2 and 2 ".*

Teacher GS had similar view writing,

$x^2 - 4 = (x-2)(x+2)$ → Again here is to find the product of $x^2 - 4$ by opening two brackets. In the brackets, the first term multiplication of first terms must give you the first term in equation and the multiplication of the last terms must give you the last term.

, and echoed “Since it is x^2 ... so you open two sets of ... from those two brackets, now the first term was x^2 then we will need to determine the factors that will give us x^2 which are x and x ... we have got 4, the factors of which, will be 4 and 1; 2 and 2 ... one factor must be negative and the other be positive e ... hence the solution is $(x - 2)(x + 2)$ ”.

R: Why did you choose -2 and 2 not -4 and 1 ?

GS: “Our expression here is $ax^2 + c$ so -4 and 1 will not give us zero when added ...wherein -2 and 2 does give us zero”.

9.3.2.4 Teachers used the fundamental approach to factorise the quadratics.

R: How did you factorise $x^2 - 4$?

KW wrote,

$x^2 - 4$
find two numbers that have a product of -4 and a sum of 0 . is -2×2 ()

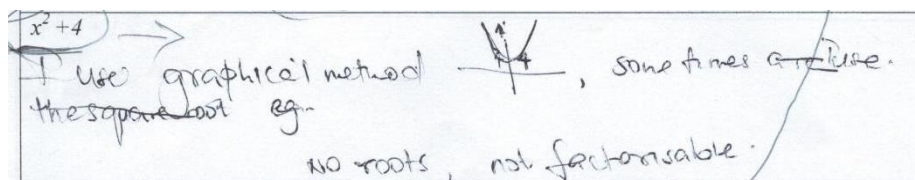
and said “In factorising $x^2 - 4$, the first thing that I can do in this problem ...because I want to factorise this ... in general I know this is like $ax^2 + bx + c$, so I can tell the learner to use the ac method ... in other words we multiply the coefficient of the first term with that of the last term $ac = (1)(-4)$... learners will need to know about bx but in this case you can simply see that $bx = 0$ because we don't have the middle term ... we have to find factors of -4 which when added will give us b or will give us zero ...which are -2 and $+2$...then we substitute bx with $-2x$ and $+2x$ and have $x^2 - 2x + 2x - 4$ and then introduce the grouping $(x^2 - 2x) + (2x - 4)$... look for the common factor in each group respectively $x(x - 2) + 2(x - 2)$... further factorise and get $(x - 2)(x + 2)$ ”.

9.3.3 Task 3: Factorisation of $x^2 + 4$.

All six teachers agreed that $x^2 + 4$ cannot be factorised, however, their explanations differed.

9.3.3.1 Teachers used the graphical representation.

In her explanation, teacher RT supported the use of the graphical method,



and said “For this one, you can’t have the factors of 4, the factors of positive 4 can be 2 and 2, so when you multiply them back they can’t give you $x^2 + 4$ so it means that there are no factors, we cannot factorise this $x^2 + 4$ because even if you can take -2 and -2, as I have said before when you find factors you have to check if those factors are the correct one by multiplying them back, if they don’t give the original, then it means factors are incorrect”.

9.3.3.2 Teachers used the rule of the sum of two squares.

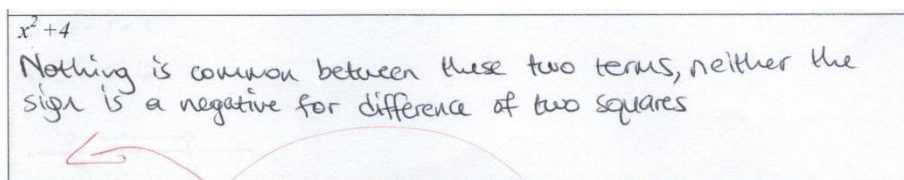
For this rule, if there are two squares separated by “plus”, then there are no factors. This rule has been developed to counter the rule of the difference of two squares.

In her explanation teacher ND provided a narrative that supported the rule of the sum of two squares saying: “Since there is a positive sign means we do not have factors ... we cannot open two sets of brackets with two different signs because when we are to multiply our factors back, we must be able to reverse back to the original expression”.

9.3.3.3 Teachers stated “no common factor”; “not difference of two squares”.

R: How did you factorise $x^2 + 4$?

HP responded,

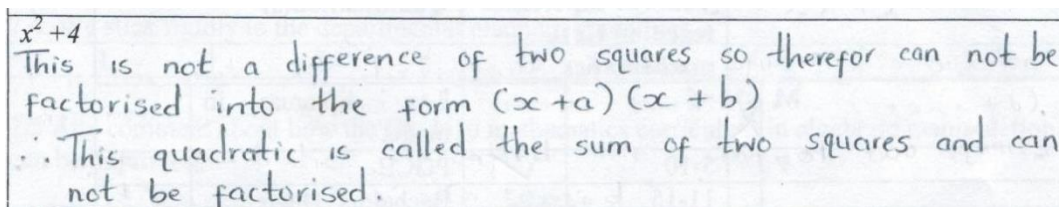


and explained “Is this the difference of two squares?... no ... the only difference between $x^2 - 4$ and $x^2 + 4$ is the sign only ... so, kay it is still right to use the square roots since both terms are perfect squares, but right now the problem is this positive sign ... I then said nothing is common between these two terms neither the sign is negative for difference of two squares ... this expression does not fit this category of difference of two squares ...”

R: So is there no mathematical approach in place to convince learners that $x^2 + 4$ can never be factorised?

HP: “I think I can follow the trial and error approach by showing that $(x - 2)(x + 2)$ does not work ... again $(x + 2)(x + 2)$ does not work as well I can then conclude by indicating to learners that the trialling of all the factors of 4 does not bring the expression back to $x^2 + 4$ ”.

Teacher KW shared a similar view,



saying: “The first thing that you must note in this problem is that this is not the difference of two squares ...but the sum of two squares ... in this case we look how can we get common factor and we don’t have the common factor ... we can’t be able to factorise this one according to my understanding, because we can’t be able to write it in a form of $(x + a)(x + b)$... we won’t able to find middle term”.

9.3.3.4 Teachers solved the equation $x^2 + 4 = 0$ and used the results to declare that $x^2 + 4$ cannot be factorised.

In his response teacher MZ wrote,

and said: “ $x^2 = -4$... well I said this one is undefined ... meaning there are no factors”. Teacher KW supported the notion saying “... let $x^2 + 4 = 0$... because I want to try to solve for x in this problem ... if I can’t get the value of x or if it is undefined, it will simply mean that I can’t factorise it ... $x^2 = -4$... introducing the square root sign on both sides ... we can get the square root of -4 ... therefore x is undefined ... in that way we can conclude that, this problem is not factorisable”.

R: How do you help your learners to better understand why there are no factors?

MZ: “I will use the trial and error method by trialling all the factors of 4 with a view of getting back to the original expression ... factors of 4 are 4 and 1; 2 and 2 ... there are no factors of 4 that give zero when added”.

9.3.3.5 Teachers followed the fundamental approach to factorising the quadratic expression.

Teacher GS offered an alternative view,

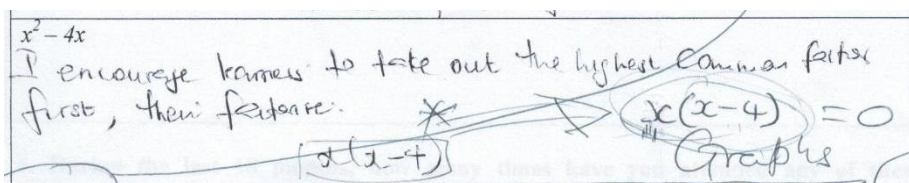
and said: “ $x^2 + 4$... even here the learner needs to open the two sets of brackets ... we look for factors that will give us x^2 which are x and x ... we look for the factors of 4 which are 4 and 1; 2 and 2 ... none of these factors are giving zero when added ... hence there are no factors”.

9.3.4 Task 4: Factorisation of $x^2 - 4x$.

All teachers found $x(x - 4)$ as the factors of this activity, offering only two different reasons.

9.3.4.1 Teachers taking out the common factor.

Five teachers said that they used the method of taking out the common factor. For example teacher RT said,



and explained “I use to tell them that, before you factorise any expression check for common factors... so here there is a common factor which is x , which can be taken out so as a result of it you end up having $x(x - 4)$, so that is where it begins”.

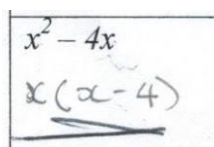
R: Is checking the common factor a rule?

RT: “Yes...it’s my rule.... it’s not my rule... yes it’s a rule but in every expression, check common factor: to take out the common factor makes easy to simplify, because it might happen that the expression looks complicated but once the common factor has been removed it becomes more simple to work it”.

R: Now in this case what happens on functions and graphs, where it takes very long for learners to notice that they have to find two possible values of x ?

RT: “In fact, this x is part of the factors, we are not saying we get rid of x but we are taking it out because we are factorising...you see... x is still part of the expression”.

Teacher ND offered a similar response,



and said: “Here we are looking for the common factor which is x , then the solution is $x(x - 4)$ ”.

Teacher HP agreed, saying:

$$x^2 - 4x$$

$$x(x-4)$$

$$x^2 - 4x + 0$$

$$(x-4)(x+0)$$

$$(x-4)(x)$$

$$ax^3by^4 + x^2b$$

and explained “Since we are factorising here ... earlier on we said when we factorise we look for something that brings relationship in these two terms of $x^2 - 4x$... there is x in x^2 and there is x in $-4x$... we take x as the common factor by saying $x^2 \div x = x$ and $-4x \div x = -4$... then we open the brackets and get $x(x - 4)$ ”.

A similar sentiment was also shared by teacher MZ who said:

$$x^2 - 4x$$

$$x(x-4)$$

factorize by taking out a common factor

and explained “Again we are given two terms ... the most important thing that we teach learners when it comes to factorising is that they must always check if there exist a common factor or not. So in this case we have two terms, if we check here since we have common factor ... our common factor is x ... obviously you need to take it out ... you will be left with x on the first term and you will be left with -4 on the second term. This is a solution $x(x - 4)$ ”.

9.3.4.2 Teachers take out the common factor, and then use the fundamental approach to factorising quadratics

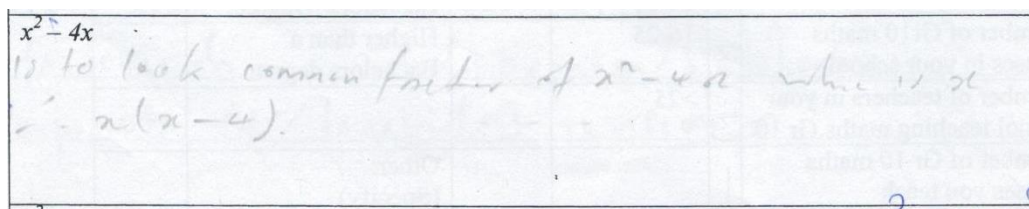
Even though teacher KW spoke about using the method of taking out the common factor, he also mentioned the use of the fundamental approach to factorising quadratics. The fundamental approach to factoring quadratic expressions is detailed in section 1.7.4.1 of this thesis. Teacher KW wrote,

$$x^2 - 4x$$

Find the common factor of x^2 and $-4x$ which is $+x$ and open the brackets inside the bracket will be $x-4$. $x(x-4)$

and said: “In $x^2 - 4x$ we can look at the common factor which will be x and factorise to get $x(x - 4)$... but you can also use the method of ac because you can see that $c = 0$ ”.

Using the method of ac , teacher KW refers to the method of grouping. A similar explanation was also offered by teacher GS who wrote,



and said: “In this case we have the common factor, which is x ... then you take out common factor from x^2 and we are left with x ... and take out x from $-4x$ and we are left with -4 ... hence the solution is $x(x - 4)$ ”

9.3.5 Task 5: Factorisation of $x^2 - 4y$.

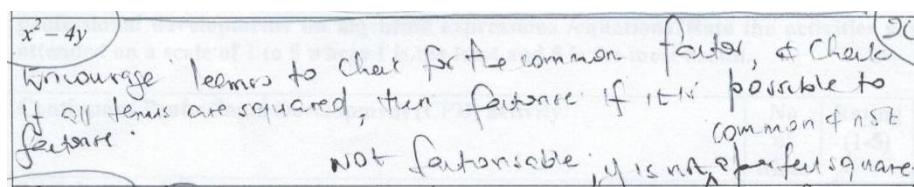
All six teachers shared the same understanding in that $x^2 - 4y$ cannot be factorised, yet provided different reasons in their explanations.

9.3.5.1 Teachers stated “no common factor”; “not difference of two squares”.

Four teachers, RT, ND, HP and GS were of the opinion that $x^2 - 4y$ needed to either have the common factor or behave as the difference of two squares for it to be factorised.

R: How did you factorise $x^2 - 4y$?

RT wrote,



and said “For this one, ... as I have said before you have to look for the common factor ... here there is no common factor ... and not all terms are squared ... it is not factorisable because there are no factors that you can multiply by each other to get the original. You can have the

factors of x^2 which are x and x and the factors of $-4y$ can be -4 and y but when you multiply them back you can't have original back".

R: Let say we have $x^2 - 4y^2$.

RT: "Yes, we can factor that".

R: How do you assist your learners in establishing and identifying the differences when factorising the following: $x^2 - 4$, $x^2 - 4y$ and $x^2 - 4y^2$?

RT: "I think the key in this case is at y , because y as to the power one, when I open two brackets where am I going to put y to the power one you see, this will bring a challenge such that you will not be able to go back to the original expression. What will make it easier is the availability of two y s. This is because, as I begin to factorise this binomial, I can have two brackets and have two y but if one y , ok I can put it anywhere, but when I'm checking, multiplying back I will not be able to reach the original, as I have said that finding factors is about going forward and moving back to the original. For my learners, every time once the factors are found, what I emphasise to them is that they should multiply them back to check their answer".

Turning to teacher Rakhulu, I drew her attention to the fact that somewhere in the country, one teacher factorised $x^2 - 4y$ to reach the results $(x + 2y)(x - 2y)$, and asked for her thoughts about the action to be taken to modify this misconception. Teacher RT said: "I think it might have been an error, I don't know but maybe it was an error; you can't say y as y to the power one and say it is y multiplied by y . I do not think there will be a misconception of this margin for the teacher who is teaching mathematics Grade 12 – maybe that teacher thought you forgot to put the square on y ".

R: With your experience, what is your take on the teacher who at the onset says to learners: 'this is difference of two squares - you open two brackets - you put negative and plus – then put the square roots of the original?'

RT: "Yes, it helps them to identify the difference of two squares. This means you (teacher) have to make sure that all the terms in the bracket are written in a simplest form - if it is 4, it can be 2^2 if different because there is $(-)$ in between the two terms. For those learners who are slow you can say open two bracket, the first bracket you can put the factors of the first term and the other factors, of the second term, and then you change the sign".

R: So do you apply the rules?

RT: “Yes, I apply the rules, but they (learners) have to make sure that they write all the terms in a square form”.

The notion of checking whether $x^2 - 4y$ qualified to have either the common factor, or being the difference of two squares, was also shared by teacher ND who responded,



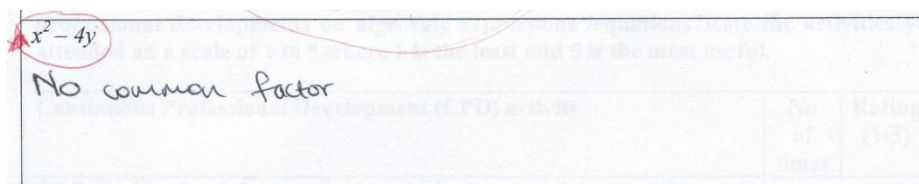
A handwritten note on a piece of paper. The expression $x^2 - 4y$ is written in blue ink and is circled with a blue pen. To the right of the circle, there are some faint, illegible handwritten notes.

and explained: “With this one our terms are not alike...we have unlike terms ...the expression is not factorisable. Here we do have two things that we can multiply and get x^2 which are x and x ... we have two things that we can multiply and get 4 which are 2 and 2... but we do not have two things that we can multiply and get y ...we must have two identical factors ...there is no number that can be multiplied to each other and give y ”.

R: Let us say we had to factorise $x^2 - 4y^2$?

ND: “Here we do have the answer because we have two things that we can multiply and get x^2 ... we have two things that we can multiply and get 4... we have two things that we can multiply and get y^2 ...”

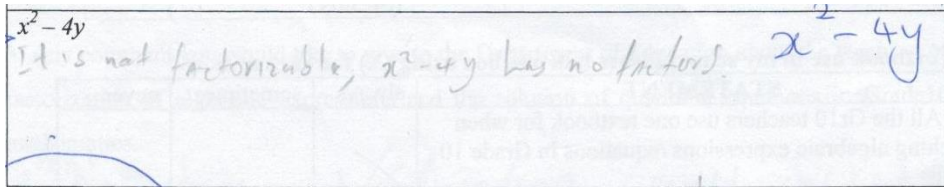
Teacher HP agreed,



A handwritten note on a piece of paper. The expression $x^2 - 4y$ is written in red ink and is circled with a red pen. Below the circle, the words "No common factor" are written in red ink. The background of the paper shows some faint, illegible text from another page.

saying: “Here we do not have the common factor ... once again between these two terms we don’t have something common ... there is nothing related ... the perfect square x^2 is there ... the minus sign is there ... but there is no perfect square in $4y$ so the rule for working the difference of two squares is not possible ... it therefore cannot factorise”.

Teacher GS supported the view,



responding by saying: “No ... here I just said it is not factorisable”.

R: Is $x^2 - 4y^2$ factorisable?

GS: “Yes”

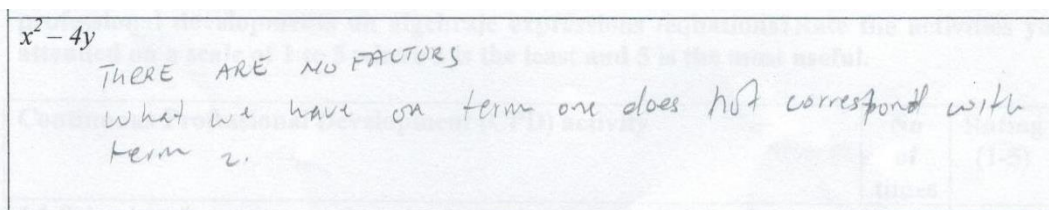
R: Why do we say $x^2 - 4y^2$ and $x^2 - 4$ are factorisable, while $x^2 - 4y$ is not factorisable?

GS: “The first two expressions are the difference of two squares and are factorisable ...for the third one ... the first term; the minus sign; the 4 are all qualifying it to be the difference of two squares but the problem is with y of 4y, which is not a square ... so the expression then does not qualify to be the difference of two squares ... hence it is not factorisable”.

9.3.5.2 Teachers stated, “no common factor”; “expression cannot be addressed through the use of fundamental approach to quadratics”.

R: How did you factorise $x^2 - 4y$?

MZ responded as follows:

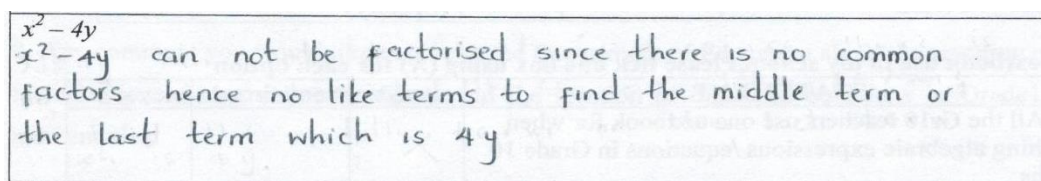


and said “In this case, there is no common factor and again yes, we can get the factors of x^2 but for 4y there will be no factors of which, that may give me 0xy when added ... in that way I said there are no factors”.

9.3.5.3 Teachers stated “no common factor”; “expression cannot be addressed through the use of fundamental approach to quadratics”; “expression cannot be represented graphically”.

R: How did you factorise $x^2 - 4y$?

KW wrote:



$x^2 - 4y$ can not be factorised since there is no common factors hence no like terms to find the middle term or the last term which is $4y$

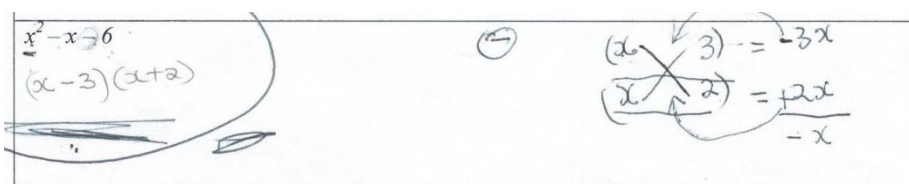
and said “For this one we can’t factorise it ... again we have a variable y ... in Grade 8 we used to say -4 is a constant but in this case is not a constant and we have two variables ... on $x^2 - 4$ we can plot the graph ... and on $x^2 - 4y$ we can’t plot the graph.

9.3.6 Task 6: Factorisation of $x^2 - x - 6$.

All six teachers provided $(x - 3)(x + 2)$ as the factors of this activity. Their reasons were as follows:

9.3.6.1 Teachers used the trial and error method – cross multiplication.

Teacher ND provided the following response:



Handwritten work showing the factorisation of $x^2 - x - 6$. On the left, the expression $x^2 - x - 6$ is written above the factors $(x - 3)(x + 2)$. On the right, a cross-multiplication diagram is shown. It consists of two columns of terms: $(x \quad 3)$ and $(x \quad 2)$. Arrows indicate the multiplication of terms across the columns: $x \times 2 = 2x$ and $3 \times x = 3x$. These two products are then subtracted from each other, resulting in $-x$, which matches the middle term of the original expression.

and explained “For this one, what I usually say to learners ...they need to look at the sign of the middle term ...which two numbers to multiply and get x^2 ...the numbers are x and x ”

R: What about $-x$ and $-x$? Why did you opt to use x and x ?

ND: “We do not worry ourselves much about $-x$ and $-x$ because ultimately we get x^2 ...we then look for the factors of -6 ...we need to bear in our minds that those two factors must give the coefficient of the middle term when added...those two factors need to have the opposing

signs in this case ...other learners may come with 1 and 6 but their difference can never be -1 ...the correct factors are 3 and 2 ...hence the solution is $(x - 3)(x + 2)$ ".

9.3.6.2 Teachers used the trial and error method – inspection.

Teacher TR said: "I usually ask learners to look for the factors of the first and last term; check the signs and then add or subtract the product to check if they give a middle term". The idea of using the trial and error method was also used by teacher KW. He responded by saying:

$x^2 - x - 6$
 Find two numbers that have a product of -6 and a sum of -1 . These are $2 \times (-3)$ will also give a sum of -1
 $\therefore (x+2)(x-3)$

and said "Find two numbers that have a product of -6 and a sum of -1 . These are $2(-3)$ will also give a sum of -1 . Therefore $(x + 2)(x - 3)$ ". When the three teachers ND; TR and KW were using the trial and error method in factoring the quadratic expression $a = 1$, it was noted that they ignored the role played by the value of a in reaching the solution. It was the two teachers, HP and GS, who recognised the role played by the value of a in reaching the solution. Teacher HP provided this response,

$x^2 - x - 6$
 $(x-3)(x+2)$

and said "this one is a quadratic since we have x^2 and it is a trinomial we will open two brackets which will make it to result in two roots ... in this case I would use one method ... which two factors do I need to get x^2 ? ... it is x and x ... we come to the last term and we look for two numbers which when multiplied give -6 and when added give -1 ... the numbers are -3 and 2 ... hence the answer is $(x - 3)(x + 2)$ ". Teacher GS supported the use of the method of trial and error in his response,

$x^2 - x - 6$
 they have to understand this trinomial, so they have to open two brackets. After that is to find factors of $x^2 - x - 6$ which is $(x-3)(x+2)$

as he said "in this case we have $ax^2 + bx + c$..."

R: Does the Grade 10 curriculum speak of $x^2 + bx + c$?

GS: “No ... here, a learner will open two sets brackets ... then the first terms in the two brackets will be x and x because it is these factors that could be multiplied to give x^2 ... we also look for the factors of -6 that which when added will give -1 and those factors are -3 and $+2$... hence the answer is $(x - 3)(x + 2)$ ”.

9.3.6.3 Teachers followed the method of grouping to factorise the quadratics.

Teacher MZ said,

$x^2 - x - 6$
 $(x - 3)(x + 2)$

- determine the factors for the first term and (coefficient of x^2) and the factors for the last term, constant value
- They give you the coefficient of x when you add them

and explained: “Here we need to factorise ... I think the best way to explain these to learners is by looking at the last value ... in this case it is -6 ... you also look at the middle coefficient, in this case it is -1 ... here the coefficient of x^2 is 1 , so now the best way to find factors here is to multiply 1 and -6 then I will get -6 ... from there, I just need to determine the factors of 6 which are 1 and 6 ; 3 and 2 ... to get middle term you need to subtract from these factors, if I take 6 and 1 ... $1 - 6 = -5$ and when I interchange factors $6 - 1 = 5$ then these factors are incorrect ... then I move to second factors $2 - 3 = -1$... then I get the answer which is -1 which is exactly the coefficient of the middle term ... I then factorise by grouping ... I rewrite $x^2 - x - 6$ as $x^2 + 2x - 3x - 6$... then by group the terms $(x^2 + 2x) + (-3x - 6)$... then take out the common factor in both sets respectively $x(x + 2) - 3(x + 2)$... take out the common factor again $(x + 2)(x - 3)$... even though this could be seen as a very long method, however, it will benefit learners as they will be treating the quadratic of the coefficient of x^2 that is not equal to 1 ”.

9.3.7 Task 7: Factorisation of $6x^2 + 7x - 20$.

All teachers worked confidently to achieve $(3x - 4)(2x + 5)$, using different approaches to factorise the quadratics.

9.3.7.1 Teachers followed the trial and error method – inspection.

Teacher ND said “ We use the different approach if we have the coefficient of x^2 that is not equal to one. We combine the relationship between the factors of the first term as well as the factors of the last term such that they produce the coefficient of the middle term”.

9.3.7.2 Teachers followed the trial and error method – cross multiplication.

Two teachers used the trial and error method of cross multiplication, for example teacher HP wrote,

A handwritten note showing the cross-multiplication method for factorising $6x^2 + 7x - 20$. The equation is written at the top. Below it, the factors $(3x-4)(2x+5)$ are written. To the right, a cross-multiplication diagram is shown: $(3x-4)$ and $(2x+5)$ are written in a box, with arrows indicating the cross-multiplication process. The result $-8 + 15 = 7$ is written below the diagram.

and said “Since the coefficient of the first term x^2 is not equal to 1, we are forced to use the other method ... we now look for the two numbers that can multiplied and give 6 which are 3 and 2 ... we use the method which I call “the cross multiplication method”... we look for two numbers which when multiplied give 20 which are 4 and 5 ... then we cross multiply them and get the factors $(3x - 4)(2x + 5)$ ”.

The use of cross multiplication as an approach was also recognised by teacher GS as he wrote,

A handwritten note explaining the cross-multiplication method for factorising $6x^2 + 7x - 20$. The equation is written at the top. Below it, the text reads: “Is to find product of this equation by opening two brackets. Then first term in bracket must give the value of $6x^2$. Last term in the ~~pro~~ brackets must give you the value of -20 . Last term and first term must the middle term (value of $7x$).

and said “In that case where the coefficient of x^2 is not equal to 1 ... learners should now also consider the factors of $6x^2$ which could be $6x$ and x ; $3x$ and $2x$... and also look for the

factors of -20 ... then cross multiply the factors until the correct sets are achieved ... hence the answer is $(3x - 5)(2x + 4)$... we use the trial and error in finding the factors”.

9.3.7.3 Teachers factorised quadratics using the method of grouping.

Teacher MZ was more comfortable to use the method of grouping,

$$6x^2 + 7x - 20$$

$$(3x-4)(2x+5)$$

and he said “I used the similar approach ... the factorisation by grouping and got $(3x - 4)(2x + 5)$ ”. Teacher KW also supported the use of the grouping method,

$$6x^2 + 7x - 20$$

In the form $ax^2 + bx + c$. multiply ac which in this problem is $(6)(-20) = -120$. Find two numbers whose product is $b = +7$ product to be negative, one number is $+$ and $-$ these are (-8) (15) the sum is $+7$. $(6x^2 - 8x) + (15x - 20) = 2x(3x-4) + 5(3x-4)$. $\therefore (2x+5)(3x-4)$

saying: “I used the similar approach ... the factorisation by grouping and got $(3x - 4)(2x + 5)$ ”.

What can be deduced from these discussions is that the advantage of using the grouping method lies in the fact that it can be used to solve different types of quadratic expressions. In comparison to their colleagues who were observed using different approaches to factor specific types of quadratic expressions, Teachers MZ and KW did not shift between approaches.

9.4 Summary of various strategies used by teachers in factoring quadratics.

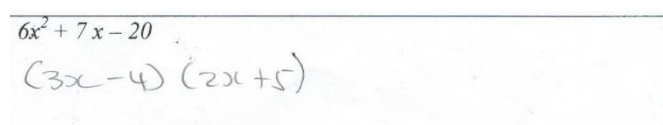
Data captured from the teacher interviews and presented in Table 9.1 show that different methods of factoring quadratic expressions were used by teachers when they engaged with the activity in the teachers’ questionnaire.

Table 9.1 Summary of various strategies used by teachers in factoring quadratics.

	Factoring methods							
	Trial & Error	Cross	Graphing	Difference of 2 squares	Common Factor	Com Fact + Dif of 2 sq	Grouping	Equate to zero
x^2	GS; MZ; RT; KW							
$x^2 - 4$	MZ; GS			RT; HP; ND			KW	
$x^2 + 4$	GS		RT	ND		HP; KW		MZ
$x^2 - 4x$	KW				RT; ND; HP; MZ; GS			
$x^2 - 4y$	MZ		KW			RT; ND; HP; GS		
$x^2 - x - 6$	GS; HP; RT	ND					MZ; KW	
$6x^2 + 7x - 20$	ND	GS; HP; RT					MZ; KW	

9.4.1 Teachers interviewed in this study believe that different prototypes of quadratic expressions can be factored by different approaches.

I have discussed in various sections of Chapter 8 and Chapter 9 of this thesis, how teachers develop short cuts when factoring the quadratics of the difference of two squares and those quadratics that require the greatest common factor technique. In this section I present data that shows that teachers treat the trinomial quadratics of $a = 1$ as different to the trinomial quadratics of $a \neq 1$, in that they perceive them as requiring different methods when being factorised. For example, in factoring $6x^2 + 7x - 20$ two teachers ND and HP said a different approach was inevitable, other than the one used in factoring $x^2 - x - 6$. Teacher ND provided the following response,



$$6x^2 + 7x - 20$$

$$(3x - 4)(2x + 5)$$

and said “We use the different approach if we have the coefficient of x^2 that is not equal to one. We combine the relationship between the factors of the first term as well as the factors of the last term such that they produce the coefficient of the middle term”.

Teacher HP provided this response,

$$6x^2 + 7x - 20$$

$$(3x-4)(2x+5)$$

$$\begin{array}{r} (3x) \times (4) = 12x \\ (3x) \times (5) = 15x \\ (-4) \times (2x) = -8x \\ \hline 12x - 8x + 15x = 7x \end{array}$$

and agreed when he said “Since the coefficient of the first term x^2 is not equal to 1, we are forced to use the other method ... we now look for the two numbers that can be multiplied and give 6 which are 3 and 2 ... we use the method which I call “the cross multiplication method” ... we look for two numbers which when multiplied give 20 which are 4 and 5 ... then we cross multiply them and get the factors $(3x - 4)(2x + 5)$ ”. Teacher GS also supported the notion in his response,

$$6x^2 + 7x - 20$$

Is to find product of this equation by opening two brackets. Then first term in bracket must give the value of a . Last term in the bracket must give the value of c . Last term in first term must the middle term (value of b).

when he said “In that case, where the coefficient of x^2 is not equal to 1 ... learners should now also consider the factors of $6x^2$ which could be $6x$ and x ; $3x$ and $2x$... and also look for the factors of -20 ... then cross multiply the factors until the correct sets are achieved ... hence the answer is $(3x - 5)(2x + 4)$... we use the trial and error in finding the factors”.

9.5 Conclusion.

This chapter was able to present responses drawn from the teacher interviews. The details of the analysis are presented in Chapter 10.

CHAPTER 10

FINDINGS AND DISCUSSIONS

10.1 Introduction.

This chapter presents (1) the findings and conclusions of this study, (2) recommendations, (3) limitations of this study, and (4) themes for further research. In the main, the findings and conclusions are in response to the four research questions as they appear in Chapter 1, and are presented in a structured discussion under the following heading: Teachers' and learners' understanding of the factorisation of quadratic expressions.

10.2 Answers to RQ 1: What does the analysis of a common test reveal about the Grade 10 learners' understanding of factorisation of quadratic expressions?

In this research question I consider the analysis of the provincial algebra common test written by Grade 10 learners, with a particular focus on those items that required knowledge of factorisation of quadratic expressions. A copy of the test which appears in Appendix G. Table 7.6 presents the results for the items on algebra. The results show that most learners performed very poorly on many of the items. The medians for each of the sub questions, except 1.2.1; 1.2.2 and 3.1, were zero across both schools. These zero medians are of concern because they reveal an overwhelming amount of non-participation by these Grade 10 learners. For each question, most responses were zero or blank.

The items that were experienced as most difficult were Questions 4.3; 3.2.1; 3.2.2 and 3.3. As seen in Figure 7.1, the mean of Question 3.2.1 was zero for both schools, and no learners scored any marks for this question. The question asked them to find two integers between which a given expression, P , was for $P(7)$. The same was true for Question 4.3 which was a problem-solving task. For Q3.2.2, and 3.3 the mean for the R-School was zero. The mean for the T-School for Q3.2.2 was 8.8% and 6.2% for Question 3.3.

The question which was experienced as easiest was Question 3.1 which asked them to express the decimal 0.45 as a common fraction. The mean for Question 3.1 in the T-School and the R-School was 78.5% and 74.5% respectively. Questions 1.2.1 and 1.2.2 which required

simplification of algebraic expressions were also experienced as not so difficult by both schools. Question 2.1, which required the solution of an equation which was already given in factorised form, seemed to be less challenging for the T-School as well.

For Question 1.1.1 (factorisation of a quadratic expression $6x^2 + 7x - 20$) as presented in Figure 7.4, the T-School average was 18% while the R-School average was 16%. It is important to note that the median was 0 for both schools. For Question 2.3, which was based on the solution of an equation made up of algebraic fractions, the average was 8.8% in the T-School and 5.4 % in the R-School. Note also that the median was also 0 for this question. What can be deduced from the data presented in Table 7.14 is that the learning and teaching of factorisation of quadratic expressions has been unsuccessful for 87% of the learners at the school considering less than 13% of the Grade 10 mathematics learners in the two schools were able to factorise a quadratic equation, and only 2% were able to apply factorisation of quadratics to the simplification of an algebraic expression.

Further analysis of the common test scripts from the ten learners who were interviewed, provided further insights into the understanding. Tables 7.15 and 7.16 show that on task A, eight students performed well in factoring the quadratic expression of $a \neq 1$. Four learners from the T-School showed how they used the Autrey's and Austin's (A-A) method to achieve the factors. Learner GD also showed that she used the cross method in achieving the factors. In addition, data presented in Tables 7.9 and 7.10 show that on task B, only four of the 10 learners did the sum correctly, but there were five in total who managed to pass the step of factoring $4x^2 - 3x - 1$ correctly. While one learner made no attempt, four learners did not manage to pass the stage of factoring $4x^2 - 3x - 1$.

10.3 Answers to RQ 2: What levels of APOS thinking do teachers engage in when dealing with items based on the concept of factorisation of quadratic expressions?

The teachers' questionnaire responses; teachers' interviews; and learners' interviews, were analysed using the Genetic Decomposition (G.D.) of Section 3.7 of this thesis. Teachers were given an activity on the factorisation of quadratic expressions to work on, and were asked to elaborate on how they went about reaching their solutions. Each task on the activity was

allocated to an APOS theory conception level, as presented in Table 4.1 of this thesis, in accordance with the genetic decomposition as discussed in section 3.7 of this thesis. In presenting this summary, I focus on the responses given by the 25 teachers to the seven items on the questionnaire. I first present a summary of the engagement of the teachers for each item using an APOS perspective, and then present a summary of their engagement using APOS levels across the seven items as a whole.

10.3.1 Summary of teachers' written responses per item.

10.3.1.1 Task 1: Teachers factoring the monomial quadratic expression of x^2 .

Data on how teachers responded to factoring a monomial quadratic expression are presented in Table 8.8, while the discussion around their responses are detailed in section 8.9.1.1 of this thesis. For teachers, this task was targeted at an Action level, since it required some mathematics knowledge about factorisation in order to find the answers, which in this case is to isolate the common factor. However, as presented in section 9.3.1.2 of the teachers' interviews, a few teachers misunderstood the question because they believed that a monomial is not a quadratic.

10.3.1.2 Task 2: Teachers understanding of the factorisation of a binomial quadratic expression with $b = 0$: $x^2 - 4$.

Data generated from Table 8.9 shows that 23 teachers (92%) responded correctly in factorising $x^2 - 4$, to get factors of $(x - 2)(x + 2)$ as the solution. This was reflected in the teachers' interviews where teachers performed excellently. In fact, some teachers (MZ and KW), developed a rule which is described in section 9.3.2 of this thesis. The use of the short-cut involves actions specific to this type of example (difference of two squares) with which teachers have become very familiar, and revealed that the teachers engaged with the task at a Process level according to the G.D. of Section 3.7 of this thesis. Two teachers did not provide their responses respectively.

10.3.1.3 Task 3: Teachers’ understanding of the non-factorisation of a binomial quadratic expression where $b = 0$: $x^2 + 4$.

This task was aimed at assessing whether teachers could argue the non-factorability of $x^2 + 4$. Data reported in Table 8.10 shows that 24 teachers responded correctly with reasons, on why $x^2 + 4$ can never be factorised. The responses to this task suggested that the majority of teachers were able to access Object level reasoning as required by the task. The majority of the teachers (14) used the reasoning that because the expression was not a “difference of two squares” it was not factorisable, while others used a combination of reasons.

10.3.1.4 Task 4: Teachers understanding in factoring a binomial quadratic expression where $c = 0$: $x^2 - 4x$.

This task aligned to the Process level according to the Genetic Decomposition in Section 3.7, with the emergence of the short-cut “take out the common factor”, since the solution could be reached in one single step. The data from the teachers’ questionnaire shows that in factoring $x^2 - 4x$, all 25 teachers were able to reach the solution of $x(x - 4)$, as presented in Table 8.11. Of the 18 teachers who provided an explanation on how they went about reaching their solutions, 15 teachers (60%), opted for the “take out the common factor” approach, while the remaining five chose to make use of examples to explain how they reached the solution.

10.3.1.5 Task 5: Teachers’ understanding of the non-factorisation of a binomial quadratic expression where $c = 0$: $x^2 - 4y$.

This task targeted the Object level according to the G.D. in Section 3.7 since it required an argument to show why the expression was not factorisable. Data captured in Table 8.12 shows that 19 teachers were able to argue why, $x^2 - 4y$ can never be factorised. An analysis of their responses reveals that most of the teachers (15), used the argument of the “non- existence” of a common factor, while three teachers reasoned that it did not satisfy the “difference of two squares” pattern, and two teachers used the argument about no common factors.

10.3.1.6 Task 6: Teachers understanding of factoring a trinomial quadratic expression where $a = 1 : x^2 - x - 6$.

Data presented in Table 8.13 shows that 23 teachers were able to respond by giving factors of $x^2 - x - 6$ to be $(x - 3)(x + 2)$. Table 8.13 further reveals that two teachers did not respond. This task is located at an Action level according to the G.D. Teachers appear to find such a task very easy because it is so familiar. However, only 12 teachers provided appropriate explanations, most likely because the others thought the task was too easy to explain.

10.3.1.7 Task 7: Teachers understanding on factorising a trinomial quadratic expression where $a \neq 1: 6x^2 + 7x - 20$.

Data presented in Table 8.14 shows that 22 teachers were able to factorise $6x^2 + 7x - 20$ correctly to produce $(3x - 4)(2x + 5)$. According to the G.D., this task is located at Process level. The analysis shows that only five teachers (20%) of this study worked confidently at this level by providing the correct responses with acceptable reasons. Most teachers did not explain how they approached the problem. These results imply that the secondary school mathematics teachers possessed good procedural knowledge regarding the factorisation of quadratic expression of $a \neq 1$, yet had difficulty in providing meaning behind the algorithms. Of the 22 teachers, five did not provide reasons, 12 provided incorrect reasons, and only five provided the correct responses. This suggests that the remaining 17 teachers (68%) were able to work with problems targeted at Process level, but were unable to explain how they went about reaching their solutions.

10.3.2 Summary of teachers' APOS mental constructions as revealed by the written responses for all items in the activity sheet.

In this paragraph, I look at the teachers' APOS reasoning across all seven items that appeared in the activity on factoring quadratics. For Tasks 1 and 6, which were classified as Action level, teachers appeared reluctant to provide explanations on how they went about factorising the expressions. My observation was that teachers were reluctant to explain things that they perceived as being easy, or where the answer was obvious. For tasks 2; 4 and 7, which belong to a Process level, teachers again provided correct answers but were still reluctant to provide explanations about their procedures. This attitude is of concern since learners often complain

about mathematics teachers who know the content very well, but are not able to help learners understand how to solve the problems themselves. For Tasks 3 and 5, which were targeted at the Object level, most teachers provided correct solutions with explanations. This finding supports the claim I made that teachers are only prepared to explain problems which they perceive to be complicated, and appear reluctant to explain problems they think are simple.

Overall, it is difficult to identify with certainty the APOS mental constructions developed by teachers based on their written responses only. It was observed that some teachers were not willing to provide reasons for tasks they perceived as being too easy. Hence one cannot deduce on the absence of a reason, that they did not develop a particular mental construction. By considering the two Object level tasks 3 and 5, one may deduce that most teachers were able to identify that they could not be factorised while also providing correct reasons. One teacher was unable to provide a correct response for Task 3, while three teachers failed to provide a correct response for Task 5. It is likely that all the teachers, apart from these three, have constructed the concept of factorisation of quadratic expressions as an Object.

10.4 Answers to RQ 3: What levels of APOS thinking do the learners engage in when dealing with items based on the concept of factorisation of quadratic expressions?

In providing answers to this research question, I first present a summary of the engagement with the 12 learners for each item using an APOS perspective. I then present a summary of their engagement using APOS levels across the seven items as a whole.

10.4.1 Summary of learners' performance per item.

10.4.1.1 Task 1: Learners' factorisation of a monomial quadratic expression of x^2 .

Considering Grade 10 learners, this task was targeted at the Action level, since the knowledge of working with monomials is dealt with at Grade Eight. All learners, presented in Table 7.1, like their teachers, were able to provide appropriate explanations for this task, with two learners using trial and error, while seven reasoned using the greatest common factor technique, and three reasoned using properties of exponents.

10.4.1.2 Task 2: Learners' understanding of factorisation of a binomial quadratic expression where $b = 0$: $x^2 - 4$.

Data presented in Table 7.2, shows that at the heart of factoring the quadratic expression of $b = 0$, with c being the square number, teachers and learners have adopted the special method that works well with this type of quadratic expression. The use of the short-cut in deriving the factors renders shows that it is a Process level, according to the G.D. of Section 3.7, although the ease of deriving the factors required actions specific to this type of example (difference of two squares) that the learners were familiar with. Only one learner was not able to reach the answer, while seven learners used trial and error techniques, and four used the difference of two squares technique.

10.4.1.3 Task 3: Learners' understanding of the non-factorisation of a binomial quadratic expression where $b = 0$: $x^2 + 4$.

This question was targeted at the Object level according to the G.D. of Section 3.7 of this thesis. Data presented in Table 7.3 shows a dominance of reasoning that because the expression is not a “difference of two squares”, it is not factorisable. A greater percentage of teachers (32%) compared to learners (17%) used this reasoning. Two learners used this technique while two others argued that it was not factorisable using trial and error techniques. None of the other learners provided correct answers, showing that learners, unlike their teachers, struggled to use Object level reasoning, as the task required.

10.4.1.4 Task 4: Learners' understanding in factoring a binomial quadratic expression where $c = 0$: $x^2 - 4x$.

This task targets the Process level, according to the G.D. in Section 3.7, since it requires learners to use factorisation content knowledge to find the answers, which is what learner GD did by using a trial and error approach. The emergence of the short-cut “take out the common factor”, allowed the learners to reach the solution in a single step. Data presented in Table 7.4 reveals that ten of the 12 learners were able to factorise $x^2 - 4x$, to reach $x(x - 4)$ using the short cut.

10.4.1.5 Task 5: Learners' understanding of the non-factorisation of a binomial quadratic expression where $c = 0$: $x^2 - 4y$.

This task was classified at the Object level according to the G.D. in Section 3.7. Data presented in Table 7.5 shows that only four of the 12 learners were able to argue why $x^2 - 4y$ can never be factorised. The analysis drawn from the pool of responses indicated that the answer “non-existence of a common factor” dominated, with three learners using this reasoning. Here too, a greater percentage of teachers used this reasoning in comparison to learners. One learner used a trial and error approach.

10.4.1.6 Task 6: Learners' understanding of the factorisation a trinomial quadratic expression where $a = 1$: $x^2 - x - 6$.

Data presented in Table 7.6 on the learners' written and interview responses, revealed that 11 of the 12 were able to carry out the Actions required in the factorisation of $x^2 - x - 6$. One student was unable to carry out these Actions.

10.4.1.7 Task 7: Learners' understanding in factoring a trinomial quadratic expression where $a \neq 1$: $6x^2 + 7x - 20$.

This task was classified as suitable for Process level. Data presented in Table 7.7 shows that, from the learners' activity, only four learners (33%) out of the 12, provided the correct responses as well as acceptable reasons in substantiating their answers.

10.4.2. Summary of learners' APOS mental constructions across the items.

Section 7.3.2 provides a detailed summary of the probable APOS constructions that particular learners have made judging from their written and interview responses. The responses of each of the learners from the T-School and R-School are summarised in Table 7.9 and Table 7.10, respectively, and are used to identify the APOS levels that they are most likely to work on. From the two tables, it can be seen that three learners (SB, BT, SK) used reasoning aligned to the Object level according to the G.D. It can be deduced that these three learners constructed the concept of factorisation of quadratic expressions as an Object. There are four learners (MW, TM, SZ, GD), whose reasoning suggest that they have constructed the concept of factorisation of quadratic expressions as a Process. The remaining learners show that they are still bound to

the step by step Actions described in the G.D., and consequently have not moved past an Action level of reasoning.

10.5 Further issues arising from the items analysed for RQ 2 and RQ 3.

There are four pertinent issues arising from the analysis of the teachers' and learners' responses to the items from the activity sheet which I would like to share.

10.5.1 Most teachers found it challenging to explain how the trinomial quadratic expression of $a \neq 1$, could be factorised.

Data presented in Table 8.14 shows that five of the 22 teachers responded correctly to the questionnaire, while only five teachers supported their answers with reasons. This shows that teachers find it difficult to explain the procedures they use, meaning they find it challenging to explain these procedures to their learners. This claim is supported through the data derived from the analysis of teachers' responses during the interviews, and reported in section 9.3.7 where it was evident that most teachers found it challenging to teach the factorisation of a trinomial quadratic expression with $a \neq 1$ at a Grade 10 level.

10.5.2 Teachers found it challenging to provide sound algebraic arguments why certain quadratics could not be factorised.

Table 8.9 shows that 11 of the 25 teachers used the argument that because the expression $x^2 + 4$ was not a difference of two squares then it meant that it could not be factorised. This argument is not a strong one, because there are many expressions which are not a difference of two squares, but can still be factorised e.g. $x^2 + 2x + 1$. Furthermore, Table 8.9 shows that 15 of the 25 teachers used the technique of the greatest common factor in arguing why $x^2 - 4y$ can never be factorised. It is my belief that teachers rely on the use of the prototypes to substantiate their arguments instead of following a generalised view that is applicable for every quadratic.

10.5.3 Most learners found it challenging to factorise the trinomial quadratic expression of $a \neq 1$.

Table 7.8 shows that eight (67%) of the 12 learners were unable to factorise the quadratic of $a \neq 1$. Again, during the learners' interviews as reported in section 7.4.2, most learners pointed to factorisation of the trinomial quadratic expressions as always posing a challenge.

10.5.4 Most learners found it challenging to provide sound algebraic arguments why certain quadratics could not be factorised.

Table 7.8 shows that eight (67%) of the twelve learners were unable to argue why $x^2 + 4$ can never be factorised. When learners were asked to factorise $x^2 + 4$ during the interviews, as reported in section 7.4.1, they responded by indicating that this is one area in the factorisation of quadratics they found challenging. Again, Table 7.8 shows that eight (67%) of the 12 learners were unable to argue why $x^2 - 4y$ can never be factorised. As discussed in 7.2.5.2, most learners responded incorrectly yet portrayed a shallow conceptual understanding of factorisation.

10.5.5 A research perspective on RQ 2 and RQ 3.

In a quest almost similar to that of this study, Didis and Erbas (2015), conducted a study on performance and difficulties of students in formulating and solving quadratic equations with one unknown. The data revealed that some students did not correctly judge whether the quadratic equation to be solved was factorable over some domain. Students' struggle to define the term "factorisation" is one reason contributing to their errors in understanding factorising quadratic expression (Yahya and Shahrill, 2015). In the study conducted on the strategies used in solving algebra by secondary school Year 11 repeating students in Brunei, Shahrill and colleague (2015) found that quite a number of the participants were unable to define the term factorisation when asked for its definition during the interviews. Students could not relate factorisation with distributive law, which is putting the common terms or linear expression in brackets. Data generated from the current study and presented in Table 7.1 shows that only three (25%) of the 12 learners were able to provide a clear description on the definition of the concept of "factorisation". Results discussed in section 9.1.1 show that teachers are clear in their definition and all speak about the factors. Teachers chose examples as a means of solidifying their descriptions regarding their definition of factorisation. The choice of

examples by teachers ND and HP, in the interviews, leads to the impression that factorisation leads to producing identical factors. This contradicted what was observed in the classroom when they were factoring the items which in most cases produced unidentical factors. I am strongly of the opinion that the definition should include general conditions that are not limited to specific examples (Zazkis and Leikin, 2008).

From the descriptions by both teachers and learners, it is evident that they have developed and established their own ways of defining the concept “factorisation” to improve their classroom practices when dealing with quadratic expressions. From my experience as a secondary school mathematics teacher, I have perused the secondary school mathematics textbooks and observed that “factorisation” as a concept is not distinctively defined. However, when this concept is introduced, the authors only show how the factorisation is done, thus prompting teachers to follow suit when introducing this concept. As a result, teachers tend to choose workable quadratic expressions, usually the binomials and trinomials, to illustrate the methods of getting the factorisation done. Learners’ future successes in solving problems based on factorisation may depend on the type/s of example/s their teacher chooses and it is my argument that this is likely to develop the entrenched misconceptions about the concept of factorisation. The misconception I refer to has been prevalent and observed among a few teachers in this study, where participants were asked to factorise x^2 , and described the process they took to reach their solutions.

Table 8.9 shows that 11 of the 25 teachers followed the difference of two squares technique in arguing why $x^2 + 4$ can never be factorised. Data captured in Table 7.8 shows that two of the four learners who responded correctly on why $x^2 + 4$ can never be factorised, used this technique. Table 8.9 shows that 15 of the 25 teachers used the greatest common factor technique to argue why $x^2 - 4y$ can never be factorised. Table 7.8 also shows that three of the four learners who responded correctly on why $x^2 - 4y$ can never be factorised, used the greatest common factor technique. The reasoning offered by both teachers and learners in determining whether the expressions were factorable, were only limited to the use of the properties of the prototypes of quadratics, namely, the difference of two squares and/or the greatest common factor techniques. This study shows that the use of short-cuts while solving problems in mathematics deprives learners of the opportunity to understand mathematics holistically. Chimhande et al., (2017) shared similar sentiments when they argued that learners

have a tendency to memorise and carry out procedures or algorithms without understanding them, which limits their use of these procedures since they can only perform them in one direction. I argue that the teaching and learning of mathematics should remain anchored in the bigger picture and, in that way, mathematics becomes meaningful, accessible, expandable and transferable (Maoto, Masha and Maphutha, 2016). Overreliance on the use of short-cuts by both teachers and learners, as reported in Table 8.9 and Table 7.8, resulted in them using the rule of dealing with the difference of two squares to explain why $x^2 + 4$ cannot be factorised, and the use of the greatest common factor technique, to argue why $x^2 - 4y$, cannot be factorised. Because of their limited understanding, teachers and learners were not able to recognise that the reason why the two expressions ($x^2 + 4$ and $x^2 + 4y$) were not factorisable was because they did not satisfy the definition of factorisation that says factoring refers to the decomposition of a polynomial into a product of its factors (Kotsopoulos, 2007). I support the argument presented by Wulan and El Milla (2020) that says the definition of a concept will always be a tool to identify the examples not belonging to the concept in question. In section 1.7.4.1, I asserted that in factorising any other quadratic, a similar approach to factorisation can also be used to factorise $x^2 + 4$. The problem-solver needs to rewrite $x^2 + 4$ as $x^2 + 0x + 4$, and then search for the factors of $+4$ that, when added, will give zero. Here are the sets of cofactors of $+4$: $(-2 \text{ \& } -2)$; $(+2 \text{ \& } +2)$; $(-4 \text{ \& } -1)$; $(+4 \text{ \& } +1)$. It is evident from the list of cofactors of $+4$, that there is no set that can produce zero when added. For this reason, it can be concluded that $x^2 + 4$ cannot be factorised. One teacher, MPQ2, used the fundamental approach to factoring quadratics. This approach was also observed and is reported in section 7.2.3.2 when learner GD explained why $x^2 + 4$ cannot be factorised.

In conclusion, teachers and learners interviewed indicated a full understanding of factoring the following quadratic expressions: x^2 ; $x^2 - 4$; $x^2 - 4x$; and $x^2 - x - 6$. Furthermore, not all participants interviewed displayed a full understanding of factoring $6x^2 + 7x - 20$. For this item, the majority of participants were unable to operate at the Object level of APOS, since they could not establish that the trinomial quadratic of $a \neq 1$ can be treated equally as the trinomial quadratic of $a = 1$. They have developed their own rules where they perceive a difference between the two types of trinomial quadratics. Again, very few participants were able to argue why $x^2 + 4$ and $x^2 - 4y$, respectively, can never be factorised. For these items, very few were able to apply the definition of factoring quadratic expressions as one guide to arguing why $x^2 + 4$ and $x^2 - 4y$, respectively, can never be factorised. I believe such

incompetency is necessitated by the reality that says not all the participants were able to provide a concise definition of factorisation. These findings indicate that most participants displayed inadequate understanding of the concept of factoring quadratics when confronted by higher order questions.

10.6 Answers to RQ 4(a): How do mathematics teachers approach the teaching of factorisation of quadratic expressions with their Grade 10 learners?

There are five main findings revealed in the data relating to how teachers approach the teaching of factorisation of quadratic expressions presented below:

10.6.1 Teachers follow the traditional approach to teaching factorisation of quadratic expressions – where they use examples as a means of teaching the concept.

My observations of the lessons which are presented in Table 5.1 reveal that all six teachers of this study followed the traditional approach to teaching factorisation of quadratic expressions. According to Allen (2011), traditional teaching is most prevalent at the secondary level, but found regularly in elementary classrooms as well; it looks something like this:

1. Teacher begins the lesson with a warm- up or other launch activity.
2. Class corrects homework from the previous lesson.
3. Teacher presents new material.
4. Class practices new idea or technique.
5. Teacher assigns homework for the next class.

In traditional lessons, the teacher talks more than the students and classroom discourse patterns feature many sequences of chorused answers, with teachers channelling students' thinking towards answers they want the students to give (Vaiyavutjamai et al., 2005). In the traditional teaching classroom, the teacher insists students learn the same way, work toward a single best solution, minimise interaction and teamwork, and focus on the mathematical ends or answers rather than the means or processes (Allen, 2011).

In the study conducted by Cabahug (2012), teachers were found presenting several examples as means for students to learn how to factor polynomials. Cabahug (2012) argues that in teaching mathematics, one of the common problems of teachers is how to teach its abstractness in such a way that the students would easily grasp the concepts. This leads the teachers to look for the appropriate pedagogy that would fit the students' learning styles to help them understand the concepts.

Several data sources show that teachers of this study make use of specific examples in teaching the factoring of quadratic expressions. Firstly, as noted in 9.2.2 on the definition of the concept "factorisation", two teachers RT and MZ opted to use examples to make their points heard when they were defining factorisation. Secondly, as observed in the lesson proceedings, most teachers used specific examples to develop learners conceptually on the concept of factorisation. For example, teacher ND presented the example $x^2 + 5x + 6$ for the class to factorise. Unlike other teachers, teacher MZ introduced the factorisation of quadratic trinomials by requiring learners to find the product of the two given linear expressions $(a + b)(a + b)$. He followed the journey of generating a generalised approach to factorisation instead of using the specific examples. This approach enhances conceptual development in learners as it exposes learners to broader thinking when dealing with factoring of quadratics.

I strongly agree with Cabahug's (2012, p. 87) argument that, "the use of examples forces students to memorise these formulas without understanding and, factor the polynomials based on these formulas without considering that there might be another way of finding the factors of the given polynomials". My mathematics teaching experience and exposure has shown that when teachers introduce the concept of factorisation of quadratics, they focus on binomials and trinomials, ignoring the monomials. The use of such examples creates the impression that the monomial can never be factorised. As discussed in 10.3.1 teacher UDU4 was quoted as saying "*the factor of x^2 is x , but x^2 cannot be factorised*". Teacher HP held a similar view saying: "*monomial cannot be factorised*". On our engagement during the interview session, when teacher HP was asked why he responded by saying, "monomial cannot be factorised", when factoring x^2 , he said: "*as I have pointed my reason when I was defining factorisation ... I said we expose the basic building blocks that connects two terms or more ... so, in this case we were given one term ... so the factors of x^2 are $(x)(x)$ but now I need to compare this x^2 with*

something ... since I found no other term or terms to make the comparison I then concluded by saying monomial cannot be factorised". I argue that this kind of thinking is necessitated by the choice of examples we make when introducing the concept of factoring quadratics. In addition, there is a belief that factorisation has to do with identical outputs. Teacher ND spoke about the identical items within the factorisation, in her description "When we are factorizing...it is where we look for instance if we can make use of x^2 ..it is where we check what similar items that we can multiply and give us in return what was given before...we usually use that language for our learners to understand...for an example x^2 is made by x multiplied by x ...the similar thing if we have x^4 we get x^2 times x^2 ...".

Allen (2011, p. 2) argues that among the challenges to any sort of instructional change in mathematics, is the longstanding tradition of a particular lesson structure commonly found in American classrooms. In opposing the traditional teaching, Allen proposed the introduction of the democratic mathematics education which subscribes and advocates the constructivist approach to teaching. Allen contends that, in a democratic mathematics classroom, teachers play the role of facilitator, refraining from providing direct instruction or telling students anything that they might be able to discover for themselves. A similar call was made by Chua (2017, p. 14) when he said: "traditionally, the teaching of expansion and factorisation is developed in a procedural way, often with too much reliance on teacher demonstration and explanation". According to Chua (2017), mathematical skills such as algebraic manipulation like expansion and factorisation tends to be taught in a procedural manner, placing little emphasis on student inquiry and explanation. This way of learning deprives students of an enriching learning experience in which they discover and create new knowledge for themselves, as well as engage in mathematical reasoning and justification.

10.6.2 Teachers opted to use short cuts when factoring the quadratic expressions.

Teachers advocated the use of short-cuts on two occasions. Firstly, teachers adopted the technique of factoring the difference of two squares when they were factoring the quadratic expression of $b = 0$. The overemphasis on the use of "difference of two squares" has been prevalent in this study, where teachers were persistent in referring to these special types of quadratics as difference of two squares. The difference of two squares is a special type of quadratic expression with the middle term of zero. Calling these quadratics, the difference of two squares, has been noted to drive and sway the thinking of teachers to use a short cut to

factorise them, rather than the fundamental approach of factorising the quadratics. Results from Table 5.1 and Table 8.9 indicate that teachers opted to follow the short-cut approach in classrooms where this type of quadratics was taught, as well as when they engaged with the questionnaire. Secondly, in factoring the quadratic expression where $c = 0$, teachers adopted the use of the greatest common factor technique. Table 8.15 shows that 18 teachers used the greatest common factor technique in factoring $x^2 - 4x$. Again, Table 9.1 shows that five of the six teachers interviewed used a similar technique in factoring $x^2 - 4x$. In affirming the use of the rule of the greatest common factor technique, teacher Rakhulu said: *“I used to tell them that, before you factorise any expression check for common factors... so here there is a common factor which is x , which can be taken out so as a result of it you end up having $x(x - 4)$, so that is where it begins”*.

Researcher: Is the checking of the common factor a rule?

Rakhulu: *Yes...it's my rule.... it's not my rule... yes it's a rule but in every expression, check common factor: to take out the common factor makes easy to simplify, because it might happen that the expression looks complicated but once the common factor has been removed it becomes more simply to work it.*

Most teachers find that quadratic factoring is difficult to teach (Kotsopoulos, 2007), and at the same time, most students feel that quadratic factoring is difficult to understand, therefore many students attempt to learn algebra by using memorisation, and many teachers use direct-instruction methods to encourage memorisation too (Cabahug, 2012; Leitze and Kitt, 2000). However, Watanabe (1991) issued a warning when she revealed that learners perform easy methods which are alternatives and end up making errors. Watanabe called these alternative methods ‘short-cuts’ and pointed out that the short-cuts are easier attempts by learners to solve the problem. A short-cut, as described by Watanabe, will not constitute a correct tool of solving the problem (Watanabe, 1991). Teachers of this study concurred that the use of shortcuts did confuse learners. While rules are aimed at alleviating problem-solving abilities, choosing them appropriately becomes a problem in itself, and R-School teacher RT was quoted as saying: *... “Yes...because of many rules, our learners become confused...that is why learners end up with misconceptions...learners end up e.g. not knowing when to take a common factor; when to think of difference of two squares; when to factorize by trial and error.... because of too many rules...this I think is the reason why mathematics is poorly performed even at Grades 11 and*

12...too many rules...too many formulae....so generalisations lead to confusion". Teacher HP held a similar view when he pointed to the simultaneous use of different approaches in factorising the quadratics as one contributor to learners not understanding the concept of factorisation. He said: "One of the challenges may be when learners mix up these different ways of factorising these algebraic quadratic expressions ... if today learner is taught to factorise by taking out common factor ... learner understands ... tomorrow a learner is taught the rule of dealing with difference of two squares ... it seems much better ... now on the other day you teach the learner the third method ... now that is where the confusion begin to manifest ... I don't know maybe learners do not give themselves time to practice to know the difference between these rules ... in class it is as if they are clear but the problem starts when they are working alone".

10.6.3 Teachers believe that different prototypes of quadratic expressions can be factored by different approaches.

In section 10.6.2 I discussed how teachers develop short-cuts when factoring the quadratics of difference of two squares, and those quadratics requiring the greatest common factor technique. In this section I present data that shows that teachers treat the trinomial quadratics of $a = 1$ as different to the trinomial quadratics of $a \neq 1$, in that they are perceived as requiring different methods when being factorised. A presentation made in section 9.3.7 reveals that three (ND, HP & GS) of the six teachers interviewed expressed the view that different prototypes of quadratics require different approaches to factorisation, and maintained that the method used in factoring $6x^2 + 7x - 20$ will differ from the method used in factoring $x^2 - x - 6$.

10.6.4 Teachers taught the factoring of quadratic expressions by building on factorisation as the reverse process of the expansion of products.

The data revealed that some teachers considered the use of prior learning when introducing the concept of factorisation. From the teachers' questionnaire, teacher MPQ1 recognised the role played by prior knowledge in understanding new knowledge. In this case, she felt it was important for the learner to understand the process of determining the products of binomials as a stepping stone towards understanding the concept of factorisation saying: "*From Grade 9 learners know that $a^2 - b^2 = (a - b)(a + b)$... the reverse process of this is factorisation OR to factorise difference of 2 squares one needs to open 2 brackets, find the square root of each*

term. Put inside the brackets, one set of square root must have positive and other the negative sign". Data captured from the classroom lesson observations and presented in Table 5.1 shows that three teachers (HP; MZ and KW) also revisited the knowledge of finding products of binomials prior to engaging learners in factorisation. This approach to teaching promotes learners' conceptual understanding. Kilpatrick et al., (2001, p. 116) describes conceptual understanding as a critical component of mathematical proficiency that "is necessary for anyone to learn mathematics successfully". Kilpatrick et al., (2001) argued that: "Students with a conceptual understanding know more than isolated facts and methods. They understand why a mathematical idea is important and the kinds of contexts in which it is useful. They have organised their knowledge into a coherent whole, which enables them to learn new ideas by connecting those ideas to what they already know" (Kilpatrick et al., 2001, p. 118).

The use of prior learning is considered by many researchers as benefiting the new concept to be taught. According to Chua (2017), as teachers make use of existing mathematical ideas in building a new concept, students begin to make sense of what they learn in mathematics. Chua further contends that, "thus students will have to realise that the procedure for expanding two linear binomials is simply an extension of what they had learnt about expanding algebraic expressions in Secondary One, and that factorising quadratic trinomials requires undoing the binomial expansion procedure" (Chua, 2017, p. 14). A similar contribution of considering prior knowledge in understanding the concept at hand is advocated by Chasanah, Zulkardi and Darmawijoyo (2015) who used the area model to support students to make sense during the transition from binomials multiplication to factorisation. Chasanah et al., (2015) argue that, "the students built up understanding and made sense of the principles of factoring quadratic equation by relating the problem with the principles of using the area model to multiply binomials. In other words, the area model supports the students in the context of understanding the principles of factorisation".

10.6.5 Teachers displayed knowledge of multiple approaches to teaching the factoring of quadratic expressions.

Prevalent in this study is the use of multiple approaches to factoring quadratic expressions. However, data shows that learners were only armed with singular approaches as per their mathematics teachers, and displayed only single methods. This is contradictory as different researchers argue the benefits of presenting multiple external representations to students in the

likelihood of stimulating in them the development of conceptual understanding of mathematics (Panasuk, 2010). Exposure to these multiple strategies of solving mathematical problems will likely help students understand how and why certain strategies work ((Naroth, 2010). Ibeawuchi, Ogbonnanya and Mogari (2015, p. 250), goes further to say, “secondary school mathematics teachers’ subject matter knowledge regarding the quadratic functions, teachers should be provided with opportunities to explore and discover different methods and strategies to solve problems. Teachers should be offered multi-representations for the problem situations, and also the inter-relatedness of the different representations should also be emphasised”. Across teachers and learners of this study, the following methods were observed: the trial and error method or the cross method; the grouping method; the Autrey’s and Austin’s method (A-A).

10.6.5.1 Teachers factoring of quadratic expressions by the Trial and Error approach and the Cross Method.

After having consulted with local mathematics teachers and Subject Advisors, I came to the conclusion that the cross method is in fact the trial and error approach. While the cross method is more visual, the trial and error happens in ones’ mind, otherwise the advocacy is the same.

There were 25 teachers who participated in the teachers’ questionnaire which had seven items. This means that there was a total of 25 teachers x 7 items = 175 responses. Table 8.2 shows that out of 175 responses, the trial and error approach was attempted 48 times (27%). Table 5.1 shows that out of the 12 lessons observed, the trial and error approach to factoring quadratics appeared seven times. Six teachers were interviewed on the seven items of the activity. This means there was a total of 6 teachers by seven items = 42 responses. Table 9.2 also shows that during the teachers’ interviews, out of 42 responses, the use of the trial and error approach appeared 14 times (33%). Table 5.1 shows that out of the 12 lessons observed, the trial and error approach to factoring quadratics appeared five times. Table 9.2 also shows that during the teachers’ interviews, out of 42 expected responses, the use of the cross method appeared 4 times (10%).

These findings indicate that both teachers and learners rely on the use of the trial and error or cross method more than any other approach to factoring the quadratic expressions. Similar

findings of learners relying too much on the use of the cross method, even if they taught other methods, were also noted in the study conducted in Darussalam, Brunei on the use of the concrete model to enhance conceptual knowledge of low ability students in factorising quadratic expression (Latiff et al., 2017). Latiff et al., (2017) reported that half the participants in their research study applied the cross-multiplication method as their approach to factorisation while 32% utilised algebra tiles and 18% did both ways.

Most teachers of this study were able to factorise the quadratic expressions using the trial and error approach. However, it was noted that they were unable to give detailed explanations on how they arrived at the solutions using the approach chose. For example in factoring $6x^2 + 7x - 20$ to reach $(2x + 5)(3x - 4)$, UDU2 said: *“Open two brackets, take the smallest factors of the of the constant, consider the co-efficient of x^2 , divide it. The bracket with the bigger coefficient of x will take the positive sign and the smaller factor of 20”*.

10.6.5.2 Teachers factoring quadratic expressions by the grouping method.

Table 5.1 shows that out of 12 lessons observed, the grouping method to factoring quadratics appeared twice, being used by KW and RT. Table 8.2 shows that out of 175 responses, the grouping method was attempted twice (1.1%). Table 9.2 also shows that during the teachers' interviews, out of 42 expected responses, the use of the grouping method appeared 5 times (12%). The use of the grouping method evolved around two teachers (MZ and KW), This indicates that teachers did not believe in using the grouping method to factorise quadratic expressions. However, where the grouping method was used, it always produced the correct solutions, and that is the reason why it is the most preferred method of factoring quadratic expressions (Dorado and Torres, 2019; Patrick, 2012).). Furthermore, those teachers who used the grouping method were able to narrate exactly how they reached their solutions. For example in factoring $6x^2 + 7x - 20$ to reach $2x + 5)(3x - 4)$, teacher KW said: *In the form $ax^2 + bx + c$, multiply ac which in this problem is $(6)(-20) = -120$. Find two numbers whose product is -120 and their sum is $+7$, the numbers are -8 & $+15$. Rewrite $6x^2 + 7x - 20$ as $6x^2 + 15x - 8x - 20$ then group the expression into two i.e. $(6x^2 + 15x)(-8x - 20)$. Factorise both sets and leave something identical on both sets i.e. $3x(2x + 5x) - 4(2x + 5)$ then $2x + 5$ is a common factor with $3x - 4$ left.*

Patrick (2012) argues that for the grouping method, students can apply the prior knowledge of algebra they have learnt in lower forms, which is factoring polynomials by using common factors and grouping of terms, exactly as narrated by teacher KW. He, Patrick (2012) further argues that working with the grouping method; students develop the relational (conceptual) understanding and the logical understanding, with the procedural understanding throughout the process. Teacher KW in his narrative shows how scaffolding of concepts can lead to a better understanding of the new concept.

10.7 Answers to RQ 4(b) about why teachers use the approaches that they do in RQ 4(a).

In section 10.5.2 we discussed the use of short-cuts by teachers in factoring the quadratics. Teachers provided the following reasons:

10.7.1 Teachers found the short-cuts easy to teach when factoring quadratic expressions.

During the interviews, teachers were asked which area in the factorisation of quadratic expressions at Grade 10 level was easy to teach. Firstly, four teachers interviewed found it easy to teach the difference of two squares. Data recorded in section 9.2.5 reveals that four teachers pointed to the factorisation of the difference of two squares as the easiest to teach. I argue that teachers are giving themselves unnecessary pressure by teaching learners methods that attend to prototypes of quadratic expression. We showed in Chapter 3 of this study that, the difference of two squares as a special type of quadratic can be factorised fully without using the short cut approach. Teacher MZ in his response, unlike his colleagues, followed the approach where you rewrite for example $x^2 - 4$ as $x^2 + 0x - 4$, then search for the factors of -4 that when added will give zero, and such factors are -2 and $+2$, hence the $x^2 - 4 = (x - 2)(x + 2)$.

Secondly, teachers adopted the technique of the greatest common factor when factoring the quadratic expression of $c = 0$. Data presented in Table 8.2 indicates that of the 25 teachers who participated in the questionnaire, 18 teachers (72%), followed the greatest common factor technique. Data captured in Table 7.3 indicates that of the 12 learners interviewed, eight learners (67%) followed the greatest common factor technique. Again, when teachers were asked which area was easy to teach, they pointed to this prototype quadratic expression of $c = 0$ and echoed their views. Two teachers, HP and GS, expressed their adequacy in dealing with

the quadratic where the method is to take out the common factor. Again, I object to the notion of teaching another short cut, as in section 1.7.4.1 of this thesis, we showed that the quadratic expression of $c = 0$ can be factored following the fundamental approach. When following the fundamental approach to factoring quadratics, we consider every quadratics as $ax^2 + bx + c$, and in this case of $x^2 - 4x$, $a = 1$; $b = -4$; and $c = 0$, hence $x^2 - 4x + 0$. The fundamental approach in this case is excellently narrated in the response provided by learner GD, and reported in section 7.2.4.1.2 of this thesis.

10.7.2 In CAPS, the focus when factoring quadratics is on the answer not on the pathway.

Teachers and learners had their reasons why they followed the use of the short-cuts. Teachers felt there was no need to follow the long method since the focus is only on the answer not on the pathway. On the extent at which teachers are using short-cuts, teacher HP agreed and said: *“I agree ... for me that is just the case ... under normal circumstances when you are to deliver the lesson for an example on the factorisation of the difference of two squares, there are less chances of first exposing learners to a version of a complete quadratic by emphasising that there is zero middle term ... we do not speak about the trinomial here ... we only push them by going straight to the answer ...”*. A similar response was also noted in the study conducted in Mauritius where teachers argued why they opted for the traditional approach in teaching quadratics. One respondent was quoted as saying, *“teaching approaches are not important, what really matters is that students are passing”* (Dhunney and Angateeah, 2019, p. 225).

R: Why?

HP: *I don't have the clear answer but you end up teaching learners some other methods of being able to recognise patterns for them to progress even though by telling learners it is like you skip some steps somehow it is like no complete learning took place ... however the focus is primarily on the answer ...”*.

R: Do you teach the short-cuts?

HP: *Yes ... I do teach the short-cuts ... for an example I come to class and teach the rule to follow when dealing with expressions of difference of two squares ... how do we recognise it? ... what short routes to take to reach the solution ... this we do for them to get marks”*.

10.7.3 The use of short-cuts is encouraged in the ATP.

In responding to the question, teachers offered reasons why they taught the generalisations. Teacher RT said: *“Yes ... I strongly agree. Reason is, it is because the scope is too wide, it is too long so if you try to avoid generalising you will end up not completing the ATP we are being pushed by the ATP....ATP is very long and we have to make sure that we complete this and we move to the next one before the given specified period of time lapses....It is the time frame that guides us”*. A similar argument was raised in the study conducted by Dhunny and Angateeah (2019) on the impact of spiral teaching on quadratics for Grade 11 students in Mauritius. Teachers complained about the bulk of the syllabus which they were required to complete within a limited time frame. As a result, teachers favoured the traditional teaching approach or strategy to teach the quadratics, and were reluctant to use other teaching strategies like guided discovery or differentiated instruction which might be risky and time-consuming.

When teacher MZ of this study was asked if he taught short-cuts, he admitted he did and said in response:

MZ: *Yah, no I do. I definitely do.*

R: What do you think might be the reasons?

MZ: *“Well, I guess our priority is the issue of the time frame because some of the things in other times one will go to class with intents of allowing learners an opportunity to explore on their own and realise that with time constraints on our side one will land nowhere. Maybe 2 or 3 learners or may be 10% of learners will be able to achieve and see that about 90% of the class is not going anywhere. Again, I have the content to cover. Because of time I will rush them and drive them to conclusions”*.

R: If our focus is on the answer, are we not depriving learners the opportunity to observe the beauty of mathematics?

HP: *“Even though we may love to see that happening, however, there are some hindrances. Firstly, the time allocated to teach is very little when related to the amount of content to be taught. Secondly, the learner failure rate is too high ... we are not intending to deprive them, but short-cuts are the solution to this predicament ... we want marks and nothing else”*. Bohlmann, Prince and Deacon (2017) share a similar view as they argue that teachers who are pressed for time, and teachers who need or want to ensure specific test results, tend to teach to

the test. Similar observations of teaching for the examinations were also noted in the study conducted by Chitera (2016) in Malawi where they oppose the notion of teaching something that will not be examined by the teachers, but support the idea of teaching what will benefit the students, i.e. passing of the examinations as this is the yardstick that they will be measured with. As observed in the secondary schools in Brunei Darussalam, students are usually taught using a “drilling and practice” method which focuses more on t preparation for a test or examination (Badarudin and Khalid, 2008). According to Badarudin and Khalid (2008), most teachers felt they were judged based on their students’ examination results, pressuring them to cover all syllabuses as quickly as possible to prepare students for tests and examinations. Due to this norm, teachers had limited time to teach students conceptual knowledge, despite agreeing to its importance.

10.7.4 The use of short-cuts is encouraged in the prescribed mathematics textbooks.

Teacher ND and teacher GS talked about the use of textbooks when responding on the extent at which teachers used short-cuts. They said:

ND: *“Yes, I do agree”.*

R: Are you also doing it? What do you think are the reason?

ND: *“I personally do it ...I think the textbooks we are using are fostering the use of generalisations ...it is again the ways we were taught in schools and even at the tertiary institutions”.* Teacher GS agreed saying: *“...the headings in the textbooks are presented as if difference of two squares are isolated from the quadratics”.* Teacher GS meant the activities on the use of the technique of difference of two squares are given the heading “difference of two squares” instead of factorisation of quadratic expression.

I concur with teacher ND and teacher GS’s claims that the use of short-cuts is encouraged in the textbooks. As an experienced mathematics teacher, I have perused most of the prescribed Grade 10 mathematics textbooks and found that short-cuts are being taught. In my engagements with teachers on the ground, I also noted that the study guides, mostly used for the matric interventions also advocate the use of short-cuts when dealing with the factorisation of quadratics.

10.8 Further issues arising from the study.

In this section I briefly discuss seven issues which emerged during the analysis but were not explicitly discussed in the answers to the research questions.

10.8.1 Learners use the same strategies as their teachers when factorising quadratics.

10.8.1.1 Learners found it easy to use the short-cuts in factoring quadratics of the difference of two squares type.

A discussion presented in section 10.7.1 paragraph one of this thesis shows that teachers adopted the use of the short-cut in factoring quadratics of the difference of two squares type. Similarly, Table 7.8 reveals that four learners followed a technique of applying the rule of factoring the difference of two squares type. During the interviews, these four learners, as reported in section 7.2.2.1, insisted they were comfortable using the rule of working with the difference of two squares. Learners claimed that it was easy to work with the difference of two squares technique.

10.8.1.2 Learners factorising quadratic expressions by trial and error method.

A discussion presented in section 10.6.5.1 of this thesis shows that on numerous occasions, teachers adopted the use of the trial and error approach in factoring quadratics. Similarly, data presented in Table 7.8 on the learners' interviews shows that the trial and error approach was attempted in every item of the learners' activity. Table 7.8 shows that the cross method was attempted only twice by learners as they engaged in the learners' activity. However, I believe learners have followed the cross method on numerous occasions because it was noted during the learner interviews they explained their workings using the cross method but did not write it together with their solutions.

While learners were competent in factoring other types of quadratic expressions using the trial and error approach, factoring the quadratic expression of $a \neq 1$ was a different case. Data presented in Table 7.8 shows that of the 10 learners who followed the trial and error approach to factoring the quadratic expression of $a \neq 1$, only one learner, SK, was successful in reaching the correct answer. This observation affirms claims made by different researchers when they

asserted that the cross method is destined to fail learners (Chua, 2017; Kotsopoulos, 2007; Patrick, 2012; Saputro et al., 2018b). Even learner GD, reported in section 7.2.7.1.2, who consistently used the trial and error approach successfully throughout the activity, was unable to produce the correct solution when using the cross method in factoring the quadratic expression of $a \neq 1$. The workings of learner GD shows that the only error she committed was the incorrect selection of multiplication factors in the factorisation of quadratic expressions (Yahya and Shahrill, 2015), otherwise her factors would have been correct. Kotsopoulos (2007) reported that quadratic relations are one of the most conceptually challenging aspects of the high school curriculum. She went on to say the reason was because many secondary students have difficulty with basic multiplication table fact retrieval.

10.8.2 The fifth operation “cancel” has emerged.

It was observed on a number of occasions that both teachers and learners use the ‘cancel’ operation with confidence. As noted from the lesson observations, teacher KW presented an example of $x^2 - 9$, and together with the class worked to factorise it. Teacher KW said: *“Again, the use of minus and plus is to ensure that the final answer has a middle term of zero, because the middle terms will cancel one another”*. In teacher MZ’s lesson, together with learners they were able to work it out and found $a^2 - ab + ab - b^2$, which teacher MZ and learners agreed that $-ab + ab$ cancels each other hence it resulted $a^2 - b^2$. Another observation was made in teacher GW’s lesson where they worked on the example: Simplify: $\frac{x-2}{x^2+4x-5} \div \frac{x^2+5x}{x^2-4}$. Teacher GS advised learners to look for quadratic items. He further emphasised that students should seek to find items that can cancel each other.

Learners were also found to use operation ‘cancel’ in a number of instances. The discussion between the researcher (R) and learner TM progressed as follows:

R: Coming to this one $\frac{4x^2-3x-1}{4x+1} + \frac{x^2+1}{x^2-x+1} = 2$

TM: I first factorised.....I dropped the rest as they were and tried to factorise...but I was not successful.

R: Here I see the marks of the pencil. What were you doing here?

Question 2

$$2. (x-3)(x+2)=0$$

$$x-3=0 / x+2=0$$

$$x=3 / x=-2$$

2.3 $\frac{4x^2 - 3x - 1}{4x+1}$

$$\begin{array}{r} 4x+1 \overline{) 4x^2 - 3x - 1} \\ \underline{4x^2 + x} \\ -4x - 1 \\ \underline{-4x - 1} \\ 0 \end{array}$$

$(4x+1)(x-1)$

$$\begin{array}{r} x^2 + 1 \\ + x^2 - x + 1 \\ \hline 2x^2 - x + 2 \end{array}$$

TM: “*I was cancelling*”.

R: Why were you cancelling here?

TM: “*I was cancelling because there exists a division sign*”.

R: Do you remember when we were dealing with $x^2 + 2x - 2x - 4$?

TM: “*Yes*”.

R: What happened to $+2x - 2x$?

TM: “*They cancelled one another because one was positive and the other was negative*”.

From the discussion, it does appear that the mathematical operation ‘cancel’ has emerged and was commonly used in this study in the following two instances: Firstly, it is used when two equal items divide each other. Secondly, it is also used when two items of additive inverses are added. The use of “cancel” was noted in the study conducted by Maharaj et al., (2015) who argue that the words cancel and divide should not be used interchangeably when the operation in question is division, and such misuse of mathematical terms or concepts could lead to mental conflicts.

10.8.3 Teachers do not focus on the work in Grade 10 as much as they do in Grade 12.

The biographies of all the teachers who were interviewed indicate that they all teach Grade 10 and Grade 12. On the question on how they manage their Grade 10 work while also teaching higher grades, as recorded in section 9.2.7, teachers agreed on the imbalances this brings to teaching and learning, responding as follows: Teacher RT said: “... *The founding reason is the*

Grade 10 writes the internal examination while the Grade 12 writes the external paper, but I put more energy at Grade 12 and I don't want to lie....”. Such notions of the imbalance was also expressed by teacher ND who said: “*...Another point is we are being judged by means of the NSC results in Grade 12 ...so much of time will be allocated to Grade 12 ...*”. Teacher HP said: “*We get judged by the matric results ... the focus is more on matric ... where you are judged more is where you put most of your energy ...*”. Emanating from this discussion, it is evident that teachers are unable to strike a balance of teaching Grade 10 efficiently, as the majority also teach other grades, like Grade 12 for example, where the full focus is on matric.

10.8.4 Teachers offer few opportunities for learners to explore mathematics.

Data reported in section 9.2.14 revealed that teachers agreed that their practices do not allow learners fair opportunities to explore mathematics. The following reasons were noted: Teacher RT said: “*...It is very difficult to allow learners to discover something at a given time of 45 minutes or 1 hour – it is not happening*”. Teacher ND was agreed, saying: “*...In most instances our learners are not challenging ... they will just take whatever you feed them ...*”. Teacher KW pointed a finger at the DBE saying: “*...the ATP ...we are expected to do things at the high pace and at the given space of time... otherwise if you give time to explore you will never finish the curriculum ...*”.

In data reported in section 6.11, a question on whether teachers offer learners enough opportunities to explore trinomial quadratics, particularly those of $a \neq 1$, most learners agreed that their teachers did not offer them enough opportunities to explore mathematics. For example learner GD said: “*The manner at which the clarities have been made at this interview, I don't think they are giving us ample opportunities to explore.*” A similar view was evident in the response provided by learner SZ. The researcher posed the question: “How many methods did your teacher show you on the factorisation of the trinomial quadratic expression like $6x^2 + 7x - 20$?

SZ: “*Only one method of trial and error*”.

This response by learner SZ is evidence that on the factorisation of this type of quadratic, learners were not given an opportunity to explore. Supporting the view was learner LH who said: “*They do give us time because we are able to learn...we do get the class activities....but*

it will be good if teachers provide the variety of methods for learners to choose the best possible and easy to use method”.

Again, the response by learner BT clearly indicated that learners are sometimes not given opportunities to explore. She said: *“No...most of the teachers are imparting concepts on the last minutes of the term ... they are expecting us to grasp everything in a short space of time.....everything is done on the rush during that period”.*

R: Do you think your teachers should give you enough time to explore?

NQ: *“Yes, so that we will have the variety of methods to solve problems ... so that we are able to choose what is right for us ...so that we are able to choose what is easy for us”.*

10.8.5 Teachers and learners are driven by the “push to solve” goal.

When teachers and learners are confronted by a situation of not knowing what to do, they tend to modify the problem so that it fits a state they are able to manipulate. I showed in section 8.8.5.1 how two teachers WMAN1 and WMAN9 changed the expression $x^2 - 4y$ for it to be factorable. They did this to get the factors. It was difficult for them to accept that $x^2 - 4y$ can never be factorised. Similarly, learners also do everything to find a solution to the given problem. For example, when learners were asked to factorise $x^2 - 4y$, learner SB, presented in section 7.2.5.2.1, produced the following response *“This one was difficult.... I tried with my knowledge because I don’t want to fail....”*. This remark shows that learners are not aware that for certain problems there are no solutions.

10.8.6 There is limited attention given to checking whether the answers are correct.

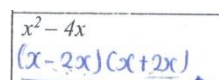
The work presented by learner DB showed that checking the solution in solving problems is covered by some schools, which is an important part of problem solving in mathematics. Here is the explanation from learner DB about checking whether his response was correct,

The image shows handwritten mathematical work. On the left, the expression $x^2 - 4$ is written above the factored form $(x-2)(x+2)$. On the right, the expression $x^2 + 2x - 2x - 4$ is written above $x^2 - 4$, demonstrating the expansion of the factored form to verify it.

and she further explained “Same thing applies here....I found the factors of x^2 which are x and x ... I also found the factors of -4 which are -2 and $+2$ I opened the brackets and found $(x-2)(x+2)$ and I checked $(x-2)(x+2)$ becomes $x^2 + 2x - 2x - 4$ and $+2x - 2x$ cancel one another and leave $x^2 - 4$ ”

However, throughout the study most learners seemed to be unaware that checking the solution is part of the problem-solving strategy. Most learners’ work presented in section 7.2.3.3 showed that learners did not check their solutions, as a result they were not able to identify errors in their answers.

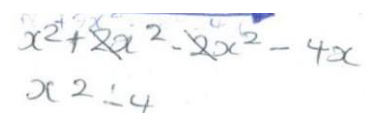
In another instance, where learners were asked to factorise $x^2 - 4x$, two learners (XD and NQ), displayed some misconceptions. Surprisingly though, when both learners were offered the opportunity to check their responses, they did and admitted that their responses were incorrect. Learner NQ provided the following answer:



$$\frac{x^2 - 4x}{(x-2x)(x+2x)}$$

R: Can you check if $(x-2x)(x+2x)$, are the correct factors?

NQ wrote:

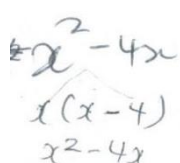


$$\begin{aligned} x^2 + 2x^2 - 2x^2 - 4x \\ x^2 - 4 \end{aligned}$$

When she was asked to check she got $x^2 - 2x^2 - 2x^2 - 4x$, and quickly realised that she was incorrect. She said “*hhayi siwrong (meaning no, it is incorrect)...it was supposed to be $-4x^2$ instead of $-4x$* ”.

R: Don’t you have another approach for solving $x^2 - 4x$?

NQ further wrote,



$$\begin{aligned} x^2 - 4x \\ x(x-4) \\ x^2 - 4x \end{aligned}$$

and said “There it is: $x(x - 4)$ ”.

R: So what was the problem initially?

NQ: “I forgot to be attentive here.....really I did not focus”.

Learner XD offered the following response when asked to factorise $x^2 - 4x$:

$$\begin{array}{l} x^2 - 4x \\ \hline \cancel{(x-2x)(x+2)} \\ \cancel{(x-2x)(2x+2)} \\ (x-2)(x+2x) \end{array}$$

and further explained “I said my answer here is $(x - 2)(x + 2x)$ ”.

R: Can you check if $(x - 2)(x + 2x)$ are the real factors?

XD wrote,

$$\begin{array}{l} = x^2 + 3x^2 - 2x - 4x \\ = 4x^2 - 6x \end{array}$$

and said “ $(x - 2)(x + 2x)$ becomes $x^2 + 2x^2 - 2x - 4x$ which then becomes $4x^2 - 6x$ it is not correct”.

These extracts from the interviews show that checking answers is an important part of solving the problem. Teachers need to emphasise this step with their learners as it is one of the four steps in problem-solving advocated by Polya (1962).

10.8.7 Learners factoring of quadratic expressions following the Autrey’s and Austin’s method (A-A method).

It was noted that the learners in this study were versatile in employing multiple approaches when factoring quadratics. Data presented in Table 7.8 shows that the Autrey’s and Austin’s method (A-A method) was used twice as learners (WM and SB) engaged with the activity on the factoring of quadratics. The A-A method is shown in Figure 2.7 of this thesis, and is known to work all the time. Table 7.8 shows that both learners who opted to use the A-A method

responded to the task correctly. Section 7.2.7.1.1 shows how the two learners, MW and SB, followed the A-A method in factoring quadratics.

10.8.8 Appropriate content in an inappropriate grade.

This study focused on Grade 10 mathematics, and the activities conducted within the study were designed to be responded to in accordance with Grade 10 mathematics requirements. It was, however, noted during the study that teachers made use of tactics that are appropriate for upper grades other than Grade 10 in responding to the tasks that were designed for Grade 10 level.

10.8.8.1 Teachers spoke about the graphical solution in factoring quadratics at Grade 10 level.

Teacher MPQ1 said: *“Learners can see that there is no HCF, this is a sum of squares. Ask them to sketch the graph i.e. $f(x) = x^2 + 4$, they will see that it does not cut x-axis and therefore the values of x that will make this expression zero are not real. It is therefore not factorisable”*. Teacher WMAN5 supported the use of graphs saying: *“No factors because this is not a difference of two squares. When sketched it will not cut the x-axis”*.

10.8.8.2 Teachers spoke about the use of the quadratic formula in factoring quadratics at Grade 10 level.

Few teachers held the view that the correct factors may be derived through the use of the quadratic formula. Teacher WMAN5 said: *“Quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ should be used to assist learners not good”*. Teacher WMAN3 said *“If you suspect that it is not easy to get the factors, you simple use the quadratic equation”*. Teacher WMAN9 said: *“It can be factorised through the use of the quadratic expression”*. Teacher MZ felt some content had to be added and said: *“Well I think the only one aspect that I feel should be taught in Grade 10 the quadratic formula to determine the factors ... in case where you have to factorise the quadratic because again it is still quadratic ...we always given expressions that are factorisable ...what do we do in the cases where we do not get the factors? ... which approach can we use? ... I think if we already started with it in Grade 10 and being infused at that level*

it will be much easier for them to use at Grade 11 ...this is because in Grade 10 if a learner comes to you with a quadratic that you cannot factorise it becomes a challenge because some learners are smart”.

R: Don't you think maybe CAPS wants learners to have a feel of mathematics at the very young age before being exposed to ready-made formulae?

MZ: *“Maybe we can view it at that angle ... but then again factorisation of quadratics begins at Grade 8; the more they enter the upper grades the more it becomes tense ... that is why I am saying they should start with it in Grade 10 because we say from grade 10 to 12 is a senior phase (FET) where they must start getting a feel of the real mathematics”.*

10.9 Implications and recommendations of the study.

This study has found that, firstly, teachers and learners tend to rely too much on the use of rules in factoring certain quadratics. In so doing, the prototype quadratics is perceived as isolated and disconnected concepts. As a result, most learners were not able to factor the trinomial quadratic of $a \neq 1$, since they perceived the factoring of $ax^2 + bx + c$ with $a = 1$ and that of $a \neq 1$ as two different procedures. Secondly, there are also students whose mental constructions (conception) are limited in certain stages of the APOS theory.

The study makes recommendations for learners, teachers, and subject advisors. The study recommends that teachers and learners should be able to consider quadratic expressions as one big idea and follow the fundamental considerations when factoring the quadratic expression. The fundamental considerations will minimise the use of many short cuts as all types of quadratic expressions are viewed as one, with one approach to factor them all successfully. Subject advisors and teachers of mathematics should play a central role in supporting the recommendation proposed by teacher MZ when he was asked in section 9.2.14 if, given the opportunity to advise other teachers of mathematics, on the sequencing of the subtopics in the factorisation of quadratics at Grade 10, he proposed that the teaching of quadratics should be such that teachers move from teaching the trinomial quadratic; progress to teaching the binomial quadratics; and finally look at the monomial.

This view is driven by the belief that using the fundamental considerations first, will give learners opportunities to engage and deduce the short-cuts themselves, rather than being taught the short cuts as separate and disconnected rules. Such exposure will help learners develop conceptually as they will begin to understand how the whole idea of factoring is interpreted in a particular short cut. Furthermore, this approach will help develop learners in developing conceptual change through the stages of APOS theory.

Secondly, the study strongly supports and subscribes to the notion of the use of multi methods in teaching the factorisation of quadratics. The use of multi-methods has been supported by different researchers (Jojo, 2015; Panasuk, 2010) in this study for a number of benefits. The use of multi-methods in factoring quadratics is encouraged and needed for students to better understand the connections between different methodologies for conceptual development.

10.10 Themes for further research.

This study has made findings about the APOS understanding of quadratic factoring by teachers and Grade 10 learners. According to the ACE cycle (Asiala et al., 1997), the theoretical analysis is followed by the instructional treatment which is then followed by the analysis of the data. In this project, I have undertaken a theoretical analysis, but, however the instructional treatment was not influenced by this analysis since teachers taught in the ways that they preferred to. The analysis has revealed many findings as reported in this chapter with respect to teachers' understanding and teaching approaches as well as learners' understanding and strategies. The recommendations of the study were that teachers should consider treating quadratic expressions as a big idea and as a totality instead of looking at them as disconnected sections, each with a different rule. Hence a suggestion for further research would be to implement such a teaching approach and to observe whether this results in better understanding of the concepts by learners. This would fulfil the three stages of the ACE cycle and make an important contribution to the field.

10.11 Conclusion.

This study has taken up the major part of my life for the past 5 years, during which time I have pondered and reflected on the collection of the data and the interpretations of the analysis. I am

a little pleased that this journey has resulted in this thesis which records my ideas and personal thoughts about the teaching of factoring quadratics. However it is by no means the end of this journey as the thesis has provided me with many more ideas and thoughts about how the teaching and learning of mathematics could be improved in my country.

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APPENDICES

APPENDIX A

UKZN ETHICAL CLEARANCE LETTER



22 March 2017

Mr Aubrey S Vilakazi 202521963
School of Education
Edgewood Campus

Dear Mr Vilakazi

Protocol reference number: HSS/0222/0170

Project title: An APOS analysis of the teaching of algebra in the Grade 10 mathematics classroom.

Expedited Approval

In response to your application dated 30 March 2017, the Humanities & Social Sciences Research Ethics Committee has considered the abovementioned application and the protocol have been granted **FULL APPROVAL**.

Any alteration/s to the approved research protocol i.e. Questionnaire/Interview Schedule, Informed Consent Form, Title of the Project, Location of the Study, Research Approach and Methods must be reviewed and approved through the amendment/modification prior to its implementation. In case you have further queries, please quote the above reference number. Please note: Research data should be securely stored in the discipline/department for a period of 5 years.

The ethical clearance certificate is only valid for a period of 3 years from the date of issue. Thereafter Recertification must be applied for on an annual basis.

I take this opportunity of wishing you everything of the best with your study.

Yours faithfully



Dr Shamila Naidoo (Deputy Chair)

/px

cc Supervisor: Dr Sarah Bansilal
cc Academic Leader Research: Dr SB Khoza
cc School Administrators: Ms T Khumalo, Ms M Ngcobo and Ms N Ncayiyana

Humanities & Social Sciences Research Ethics Committee

Dr Shemuka Singh (Chair)

Westville Campus, Govan Mbeki Building

Postal Address: Private Bag 354001, Durban 4000

Telephone: +27 (0) 31 260 3687/83504587 Facsimile: +27 (0) 31 260 4800 Email: vinke@ukzn.ac.za / research@ukzn.ac.za / ethics@ukzn.ac.za

Website: www.ukzn.ac.za

APPENDIX B

KZN DoE CLEARANCE LETTER FOR CONDUCTING RESEARCH IN SCHOOLS



education

Department:
Education
PROVINCE OF KWAZULU-NATAL

Enquiries: Phindile Duma

Tel: 033 392 1041

Ref:24/8/1150

Mr AS Vilekazi
PO Box 3942
124 Esther Roberts Road
Sundumbili
4491

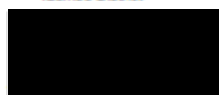
Dear Mr Vilekazi

PERMISSION TO CONDUCT RESEARCH IN THE KZN DoE INSTITUTIONS

Your application to conduct research entitled: "EXPLORING THE USE OF APOS IN THE SIMPLIFICATION OF ALGEBRATIC EXPRESSIONS IN THE GRADE 10 MATHEMATICS CLASSROOM", in the KwaZulu-Natal Department of Education Institutions has been approved. The conditions of the approval are as follows:

1. The researcher will make all the arrangements concerning the research and interviews.
2. The researcher must ensure that Educator and learning programmes are not interrupted.
3. Interviews are not conducted during the time of writing examinations in schools.
4. Learners, Educators, Schools and Institutions are not identifiable in any way from the results of the research.
5. A copy of this letter is submitted to District Managers, Principals and Heads of Institutions where the Intended research and interviews are to be conducted.
6. The period of investigation is limited to the period from 27 January 2017 to 07 June 2019.
7. Your research and interviews will be limited to the schools you have proposed and approved by the Head of Department. Please note that Principals, Educators, Departmental Officials and Learners are under no obligation to participate or assist you in your investigation.
8. Should you wish to extend the period of your survey at the school(s), please contact Miss Connie Kehlogile at the contact numbers below
9. Upon completion of the research, a brief summary of the findings, recommendations or a full report/dissertation/thesis must be submitted to the research office of the Department. Please address it to The Office of the HOD, Private Bag X9137, Pietermaritzburg, 3200.
10. Please note that your research and interviews will be limited to schools and institutions in KwaZulu-Natal Department of Education.

Lembe District



Dr. EV Msimba
Head of Department: Education
Date: 01 February 2017

..Championing Quality Education - Creating and Securing a Brighter Future

KWAZULU-NATAL DEPARTMENT OF EDUCATION
Postal Address: Private Bag X9137 - Pietermaritzburg - 3200 - Republic of South Africa
Physical Address: 247 Burger Street - Anton Lembede Building - Pietermaritzburg - 3201
Tel.: +27 33 392 1004/11 - Fax: +27 333 392 1200- Email: Kehlogile.Conn@kzndoe.gov.za/Phindile.Duma@kzndoe.gov.za - [Web:www.kzndoe.gov.za](http://www.kzndoe.gov.za)
Facebook: KZNDOE...Twitter: @DOE_KZN...Instagram: km_education...Youtube:kzndoe

APPENDIX C



School of Science, Mathematics and

Technology Education

Faculty of Education

Private Bag X03

Ashwood, 3605

27 JUNE 2017

P.O. Box 3942

Sundumbili

4491

INFORMED CONSENT - TEACHERS

Dear Mathematics Teacher

Title of the Project: An APOS analysis of the teaching of algebra in the Grade 10 mathematics classroom

I am a PhD (Mathematics) student at the University of KwaZulu-Natal. I am currently conducting research on an APOS analysis of the teaching of algebra in the Grade 10 mathematics classroom. Participation in this study is voluntary. This study involves the learner baseline assessment; the learner group activity; and the lesson observations. This will be followed by the completion of the questionnaire by the teachers. It also includes interviews with selected teachers and learners in order to clarify my understanding of the responses in the learner group activity, as well as in the questionnaires. These interviews will be tape-recorded. Confidentiality is assured because none of the names will be used in the analysis or the report. If you have any queries or concerns, kindly contact the following persons:

Researcher: Mr A.S. Vilakazi Cell: 082 227 8635 Email: aubreysifiso@hotmail.com	Supervisor: Prof. Sarah Bansilal Tel: 031-260 3451 Cell: 0832795916 Email: bansilals@ukzn.ac.za
HSSREC contact details: Tel: 031 260 3587 Fax 031 260 4809 Email: ximbap@ukzn.ac.za	

Declaration:

I..... (full names of participant) hereby confirm that I understand the contents of this document and the nature of the research project, and I consent to participating in the research project. I also consent to be audio recorded.

I understand that I am at liberty to withdraw from the project at any time, should I so desire.

SIGNATURE OF PARTICIPANT

DATE

.....

APPENDIX D



School of Science, Mathematics and Technology Education,

Faculty of Education

Private Bag X03, Ashwood, 3605

30 August 2017

P.O. Box 3942

Sundumbili, 4491

INFORMED CONSENT – PARENT

Dear Parent

Title of the Project: An APOS analysis of the teaching of algebra in the Grade 10 mathematics classroom

I am Mr. A.S. Vilakazi a PhD student at the University of KwaZulu-Natal. I am currently conducting research on An APOS analysis of the teaching of algebra in the Grade 10 mathematics classroom. I seek permission from yourself to analyse your child's/ward's responses to the March and June Examination and to conduct interviews, if necessary, in order to clarify my understanding of the responses, with him/her. Confidentiality is assured because no names will be used in the analysis or the reports. Participation is voluntary and your child/ward is free to withdraw from the study at any stage and for any reason.

If you have any queries or concerns, kindly contact the following persons:

Researcher: Mr A.S. Vilakazi Cell: 082 227 8635 Email: aubreysifiso@hotmail.com	Supervisor: Prof. Sarah Bansilal Tel: 031-260 3451 Cell: 0832795916 Email: bansilals@ukzn.ac.za
HSSRECcontact details: Tel: 031 260 3587 Fax 031 260 4809 Email:ximbap@ukzn.ac.za	

Declaration:

I.....(full names of parent / guardian)
of.....(full name of learner), hereby confirm that I understand the contents of
this document and the nature of the research project, and I consent to participating in the research project. I also consent my child to be audio
recorded. I understand that my child is at liberty to withdraw from the project at any time, should he/she so desire.

SIGNATURE OF PARTICIPANT

DATE

.....

.....

APPENDIX E



School of Science, Mathematics and

Technology Education

Faculty of Education

Private Bag X03

Ashwood, 3605

27 JUNE 2017

P.O. Box 3942

Sundumbili

4491

INFORMED CONSENT - LEARNERS

Dear Mathematics Learner

Title of the Project: An APOS analysis of the teaching of algebra in the Grade 10 mathematics classroom

I am a PhD (Mathematics) student at the University of KwaZulu-Natal. I am currently conducting research on an APOS analysis of the teaching of algebra in the Grade 10 mathematics classroom. Participation in this study is voluntary. This study involves the learner baseline assessment; the learner group activity; and the lesson observations. It also includes interviews with selected learners in order to clarify my understanding of the responses in the learner group activity. These interviews will be tape-recorded. Confidentiality is assured because none of the names will be used in the analysis or the report. If you have any queries or concerns, kindly contact the following persons:

Researcher: Mr A.S. Vilakazi Cell: 082 227 8635 Email: aubreysifiso@hotmail.com	Supervisor: Prof. Sarah Bansilal Tel: 031-260 3451 Cell: 0832795916 Email: bansilals@ukzn.ac.za
HSSREC contact details: Tel: 031 260 3587 Fax 031 260 4809 Email: ximbap@ukzn.ac.za	

Declaration:

I..... (full names of participant) hereby confirm that I understand the contents of this document and the nature of the research project, and I consent to participating in the research project. I also consent to be audio recorded.

I understand that I am at liberty to withdraw from the project at any time, should I so desire.

SIGNATURE OF PARTICIPANT

DATE

.....

APPENDIX F

CLASSROOM OBSERVATION SCHEDULE

The learning environment	Notes
Usable chalkboards?	
Sufficient seating or desks or writing surface(s) per learner?	
Sufficient space for the teacher to organize different activities or seating arrangements?	
Noise or outside distraction?	
Classroom organization	Notes
Are learners seated alone at individual desks?	
Are learners seated in pairs?	
Are learners seated in groups?	
During the course of the lesson: Does the teacher remain in one place? Does the teacher move around the class?	
Lesson topic	Notes
What is the Maths topic being addressed?	
Is this lesson: An introductory lesson? A continuation of a previous lesson? The end of a series of lessons?	
Lesson structure	Notes
Is the lesson structured to address the unforeseen misconceptions?	
How does the teacher pace the lesson in terms of available time? Very efficiently? Efficiently? Inefficiently?	
What is the focus of the lesson activities? Whole class teaching? Whole class discussion? Learners working in pairs? Learners working alone?	
Organisation of the activities	Notes
Does the teacher organize tasks so that learners work: Individually without the assistance from the teacher? Individually with assistance from the teacher? Together as a class with the teacher assisting the whole class? Together as a class with learners responding to one another? In pairs or small groups without assistance from the teacher? In pairs or small groups with assistance from the teacher? Teacher does not organize the activities	

Language(s) of learning and teaching	Notes																						
Activities are written in:																							
English/ the vernacular																							
Maths terminology/ numbers/ maths notation only																							
English / the vernacular but mainly English																							
English/ the vernacular but mainly the vernacular																							
Activities not used																							
Learners complete activities in:																							
English/ the vernacular																							
Maths terminology/ numbers/ maths notation only																							
English / the vernacular but mainly English																							
English/ the vernacular but mainly the vernacular																							
Activities not used																							
Teacher instructs in:																							
English/ the vernacular																							
Maths terminology/ numbers/ maths notation only																							
English / the vernacular but mainly English																							
English/ the vernacular but mainly the vernacular																							
In teacher-learner interactions, learners mainly use:																							
English/ the vernacular																							
Maths terminology/ numbers/ maths notation only																							
English / the vernacular but mainly English																							
English/ the vernacular but mainly the vernacular																							
In learner-learner interactions, learners mainly use:																							
English/ the vernacular																							
Maths terminology/ numbers/ maths notation only																							
English / the vernacular but mainly English																							
English/ the vernacular but mainly the vernacular																							
Learner participation and involvement																							
<table border="1"> <tr> <td>Not at all</td> <td>1</td> <td>occasionally</td> <td>2</td> <td>frequently</td> <td>3</td> <td>All the time</td> <td>4</td> </tr> </table> The scale above will be used to measure the extent of learner involvement during the course of the lesson <table border="1"> <tr> <td></td> <td>Scale (1 – 4)</td> </tr> <tr> <td>Listening to the teacher</td> <td></td> </tr> <tr> <td>Observing demonstrations</td> <td></td> </tr> <tr> <td>Copying down teachers' notes</td> <td></td> </tr> <tr> <td>Memorizing and or repeating words or Maths terms</td> <td></td> </tr> <tr> <td>Responding to teachers' questions</td> <td></td> </tr> <tr> <td>Asking questions</td> <td></td> </tr> </table>		Not at all	1	occasionally	2	frequently	3	All the time	4		Scale (1 – 4)	Listening to the teacher		Observing demonstrations		Copying down teachers' notes		Memorizing and or repeating words or Maths terms		Responding to teachers' questions		Asking questions	
Not at all	1	occasionally	2	frequently	3	All the time	4																
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Memorizing and or repeating words or Maths terms																							
Responding to teachers' questions																							
Asking questions																							

	Completing tasks/ activities in their exercise books	
	Reading textbooks	
	Discussing with their peers	
	Writing their own notes	
	Using calculators	
	Marking/ reviewing of own/ others' homework/ classwork	
Learner assessment		Notes
Are learners assessed on: Oral responses? Learners' written work?		
Are learners provided with feedback: Individually? As a class?		

APPENDIX G

Mathematics Common Test

Grade 10 (KZN)

March 2019

QUESTION 1

1.1 Factorise the following expressions fully:

1.1.1 $6x^2 + 7x - 20$ (2)

1.1.2 $x^3 + x^2 - x - 1$ (3)

1.2 Simplify the following expressions fully:

1.2.1 $(2x + 3)(5 - x)$ (3)

1.2.2 $(xy^3 - 3)(x^2y^6 + 3xy^3 + 9)$ (2)

1.2.3 $\frac{3^{2x-1} \cdot 5^{x-3}}{45^{x-2}}$ (4)

(14)

QUESTION 2

Determine, without the use of a calculator, the value of x in each of the following:

2.1 $(x - 3)(x + 2) = 0$ (2)

2.2 $11 \times 3^{2x+1} = 297$ (3)

2.3 $\frac{4x^2-3x-1}{4x+1} + \frac{x^3+1}{x^2-x+1} = 2$ (5)

2.4 $\pi x^2 h = V$ (2)

(12)

QUESTION 3

3.1 Rewrite 0,45 as a common fraction, in simplest form. (1)

3.2 Given: $P = \sqrt{\frac{-20x}{1-3x}}$; where $x \in R$

3.2.1 For which value(s) of x is P undefined? (2)

3.2.2 If $x = 7$, determine which two integers P lies. (2)

3.3 The larger of two consecutive integers is represented by $x + 3$.

Determine an expression which represents the smallest integer. (1)

(6)

QUESTION 4

4.1 Solve for x and y simultaneously:

$$4x = 10 - 3y$$

$$y + 2x = 6 \quad (5)$$

4.2 $-9 \leq 2x + 3 < 5$; where $x \in R$

4.2.1 Solve for x in the inequality (3)

4.2.2 Represent your answer to QUESTION 4.2.1 on a number line. (1)

4.2.3 Write your answer to QUESTION 4.2.1 in interval notation. (1)

4.3 The mean of three numbers is 25. The second number is four less than twice the first. The third number is two more than four times the first.

(Hint: Let the smallest number be x)

Calculate the numerical value of the smallest number. (4)

(14)

APPENDIX H

LEARNER ACTIVITY

SCHOOL:

NAME:

CLASS:

In the spaces provided, answer the following:

1. FACTORISE the following expressions

$$x^2$$

$$x^2 - 4$$

$$x^2 + 4$$

$$x^2 - 4x$$

$$x^2 - 4y$$

$$x^2 - x - 6$$

$$6x^2 + 7x - 20$$

APPENDIX I

INTERVIEW SCHEDULE – LEARNERS

Section A

1. Why did you chose Mathematics instead of Mathematical Literacy in your Grade 10?
2. Do you have somebody who inspired you to take Mathematics?
3. What role do you expect from your teacher of mathematics, for you to learn mathematics better?
4. Beside the role of the mathematics teacher, what other factors that makes you understand mathematics better?
5. How have your previous teachers helped you learn mathematics better?

Our focus will be on algebra

6. When you think about algebra, what is the first thing that comes to your mind? Tell me more about that.
7. How do you feel about the amount of factorisation of algebraic expressions material in grade 10 and the pace of instruction?
8. How have your teachers (current or previous) helped you learn the factorisation of algebraic expressions more easily?
9. What challenges do you face with the factorisation of algebraic expressions?
10. Do you think that all students should be required to learn the same maths?

On the Activity and On the 2019 March Common Test

11. Which area within the activity has been the easiest? What do you think made it easier?
12. Which area within the activity has been the most difficult? What do you think made it difficult?

General

13. How would you motivate your peers who seem to be so discouraged in doing Mathematics?
14. Is there anything else you would like to share with me about your experiences in your engagement with your Grade 10 Mathematics?

APPENDIX J

TEACHER QUESTIONNAIRE

Dear Participant

Thank you in advance for your participation in this questionnaire. I am Aubrey Sifiso Vilakazi (UKZN student) conducting a PhD study titled “An APOS analysis of the teaching of algebra in the Grade 10 mathematics classroom”. Kindly read and sign the declaration before answering this questionnaire.

This questionnaire consists of 6 pages and 8 questions.

SECTION A

1. Please tick using (X) or write a number in the relevant box

Gender /classes		Years of experience teaching Gr10 mathematics		Qualification(s)	
	M	<5		3-yr diploma in maths	
	F	5-10		PGCE	
		11-15		Bachelors' Degree	
Number of Gr 10 maths classes in your school		16-25		Higher than a Bachelors degree	
Number of teachers in your school teaching maths Gr 10		>25			
Number of Gr 10 maths classes you teach				Other: (Specify)	

2. Textbook use in my school. Please tick one box using (X) for each option

STATEMENT	always	sometimes	never
2.1 All Grade 10 teachers use one mathematics textbook for teaching algebraic expressions and equations.			
2.2 I write my own notes for my learners when teaching the Gr10 algebraic expressions /equations			
2.3 Any comment about the textbook use on algebraic expressions /equations:			

3. METHODOLOGY ON CONTENT MATTERS

In the spaces provided, briefly describe the methods you use to explain to your learners in the given situations:

3.1 **FACTORISE** the following algebraic expressions. If they are not factorizable, please state the reason **WHY**.

x^2
$x^2 - 4$
$x^2 + 4$
$x^2 - 4x$

$$x^2 - 4y$$

$$x^2 - x - 6$$

$$6x^2 + 7x - 20$$

4. During the last 18 months, how many times have you attended any of these professional developments on algebraic expressions /equations? Rate the activities you attended on a scale of 1 to 5 where 1 is the least and 5 is the most useful.

Continuous Professional Development (CPD) activity	No of times	Rating (1-5)
4.1 School mathematics meetings on algebraic expressions /equations with HOD and teachers		
4.2 Cluster meetings with other schools on algebraic expressions /equations		
4.3 Workshops on algebraic expressions /equations by the subject advisor		
4.4 A visit at your school from your subject advisor to monitor your progress on algebraic expressions /equations		
4.5 Workshops on algebraic expressions /equations by your union		
4.6 Mentoring visit from a colleague or School Management Team (SMT) on algebraic expressions /equations		
4.7 AMESA workshop on algebraic manipulations		

4.8 Other professional developments on algebraic expressions /equations (specify) • •		

5. Mathematics Curriculum tracking. Your responses should be strictly on algebraic expressions /equations. Please tick one box using (X) for each option

STATEMENT	always	sometimes	never
5.1 In my school, the HOD or senior teacher checks on my mathematics curriculum coverage.			
5.2 I use the CAPS guidelines.			
5.3 I use the KZN annual teaching plan (ATP)			
5.4 I use the PILO trackers			
5.5 I think it is not possible to complete all the topics in the Gr 10 mathematics curriculum			
5.6 Any other comment: 			

SECTION B.

Using the following 1-5 scale, please indicate the degree to which you agree with the statements (6.1 to 7.4) listed below by marking with an (X) the most correct response where necessary:

1	2	3	4	5
Strongly disagree	disagree	neutral	agree	Strongly agree

6. My School Environment.

6.1 I am solely responsible for what I teach and assess in my Gr 10 mathematics class.

1	2	3	4	5

6.2 All the Gr 10 teachers, teach according to the same common plan.

1	2	3	4	5

6.3 All the Gr 10 teachers do the same assessments for all the Gr 10 classes.

1	2	3	4	5

6.4 My HOD checks that I am on track with my Gr 10 mathematics teaching.

1	2	3	4	5

6.5 If I have a problem with the mathematics, I go to my HOD who helps me.

1	2	3	4	5

6.6 My mathematics colleagues are supportive of the work I do.

1	2	3	4	5

6.7 My Gr 10 learners appreciate my work.

1	2	3	4	5

6.8 Parents are supportive of our work.

1	2	3	4	5

6.9 Our school's mathematics department works together as a team.

1	2	3	4	5

6.10 Our mathematics head ensures that everybody does their share of work.

1	2	3	4	5

7. Grade 10 Mathematics Curriculum: All the statements are only focused on algebraic expressions /equations.

7.1. I am pleased with the pace of the Grade10 mathematics curriculum.

1	2	3	4	5

7.2. All the topics in the Grade 10 mathematics curriculum are necessary.

1	2	3	4	5

7.3. The sequencing of the topics according to CAPS, makes sense.

1	2	3	4	5

7.4. We stick rigidly to the departmental planning guidelines.

1	2	3	4	5

7.5 Any comment about how the Grade10 mathematics curriculum in algebraic manipulation can be improved?

8. Any comment you would like to give to the Department of Education about the teaching of factorization of algebraic expressions and the solution of quadratic equations in Grade10 mathematics.

THANK YOU FOR YOUR PARTICIPATION

APPENDIX K

INTERVIEW SCHEDULE – TEACHERS

Section A

The teacher will be expected to describe in detail to the researcher the methods and techniques in responding to the Question 3 of Section A of the Teacher Questionnaire of this study.

Section B

The researcher will pose to the teacher the questions based on the observed lessons.

Section C:

General Questions

1. When you think about algebra, what is the first thing that comes to your mind? Tell me more about that.
2. What is your view on the teaching experience in the teaching of Grade 10 factorisation of algebraic expressions Maths? Do you think algebraic teaching experience has a role to play in the teaching of factorisation of algebraic expressions in Grade 10?
3. What challenges do you as a teacher face with the teaching of factorisation of algebraic expressions?
4. Which area in the teaching of factorisation of algebraic expressions in Grade 10 has been the easiest? What do you think made it easier?
5. Which area in the teaching of factorisation of algebraic expressions in Grade 10 has been the most difficult? What do you think made it difficult?
6. How have your teachers/ lecturers (current or previous) helped you teach grade 10 the concept of the factorisation of algebraic expressions more easily?
7. Do you think that all learners should be required to learn the same maths?

Lesson Preparation on Algebra and algebraic expressions and equations

8. How do you prepare your lesson on the factorisation of algebraic expressions in Grade 10 for your diverse talented class?
9. To what extent do you incorporate the context-based activities in your teaching of factorisation of algebraic expressions?

10. To what extent do you think technological advances can assist in enabling our grade 10 learners to perform successfully when dealing with the factorisation of algebraic expressions?
11. Teaching of factorisation of algebraic expressions in Grade 10 class of Non-English speakers. Do you think it has an effect? why?

Curriculum

12. Secondary school mathematics is characterized by the extensive use of algebra. Do you think the amount of algebraic knowledge (algebraic expressions) in grade 10 is adequate for learners to progress the FET band?
13. Do you believe that the content of algebraic expressions in grade 10 mathematics curriculum is appropriate for grade 10 learners? Tell me more about why you feel that way.
14. How do you feel about the amount of the manipulation of algebraic expressions material in grade 10 and the pace of instruction?

Learning Environment

15. Tell me about some features of your math class that help you teach the factorisation of algebraic expressions better.
16. Looking at the poorly resourced class and the well-resourced class. Do you think such conditions may have an impact on the learner attainment, particularly in the teaching of the factorisation of algebraic expressions?

Research Claims

17. In problem-solving exercises, teachers tend to teach the generalisations (short-cuts) than the fully-fledged approaches. Do you agree? What is your position? What do you think might be their reasons?
18. The teacher knowledge has a profound contribution on the learner performance. Do you agree? What is your position?
19. One of your learners expressed with concern that teachers do not offer them the opportunity to explore. Teachers teach according to their plan. Do you agree with your learner's allegation? What might be the reason behind such practice?
20. Is there anything else you would like to share with me about your experiences in the teaching of the factorisation of algebraic expressions?

APPENDIX L

Data Captured from the Teachers' Questionnaire: The activity on factoring quadratics

x^2

no response	WMAN6; UDU1
no solution. Expression with one term can never be factorized	WMAN1
No solution. Look for the square root of x^2 , or factor of x^2 - square is when you multiply a variable or a number by itself	UDU5
The factor of x^2 is x , but x^2 cannot be factorised	UDU4
Monomial cannot be factorised	HP
The given problem is a quadratic equation and it has no x and y values is that why we can't factorise	UDU3
Answer = $x.x$	
no explanation	WMAN2; TR;MZ;ND WMAN7
Firstly they have to understand that when x is power 2 that is quadratic equation which will give us two roots of x . in this case is find the product of x^2 and if they expand that product it must give you	GS
What was x^2 before it became x^2 ?Learners should be able to say $(x)(x)$. Explain that x^2 is a product of x and x are factors	MPQ1
In x^2 , 2 tells us that x has multiplied itself twice i.e. $x.x$, to factorise we must find the square root of x^2	MPQ2
Factors are $x.x$ (factorisable)	MPQ4
Break up x^2 to $(x)(x)$	MPQ3
Explain the concept of factors and square numbers, then give examples like $a^2 = a.a$	UDU2
x^2 is the product of x and x . and x is the common factor	KW
e.g $8 = 2.4$ $8 = 1.8$	WMAN4
explain by how many x are there	WMAN3
question and answer method. Use the knowledge of exponential products	WMAN8
Start by using a number for instance 4, and ask what are two numbers that can make up 4 and they change that to variable	WMAN9
when finding factors in the expression, make sure that if you multiply you will get the same expression	WMAN10
Answer = 0	WMAN5

$x^2 - 4$

No response	
This has factors so we differentiate and equate it to zero and get values of x	UDU3
Look for the square root of the first term and the second term, then if not possible find the factors of both terms	UDU5
$(x - 2)(x + 2)$	
No explanation	WMAN5; HP;ND
Difference of two squares, open up two brackets and determine the factors for the first term and the last term. Also make sure that the co-efficient of x is zero	MZ
From gr9 learners know that $a^2 - b^2 = (a + b)(a - b)$ – the reverse process of this is factorization OR To factorise difference of 2 squares one needs to open 2 brackets, find the square root of each term. Put inside the brackets, one set of square root must have +ve and other -ve sign	MPQ1
Find the factors of 4 and x^2 , the negative sign shows that one factor is negative and the other is positive	MPQ2
Difference of 2 squares can be factorized as the product of the sum and difference of the 2 terms i.e. $x^2 - 2^2 = (x + 2)(x - 2)$	MPQ3
Difference of two squares (binomial) factors of the first term $x^2 = x.x$ and $(-)$ is made of $(+)$ and $(-)$ i.e. $+ - 2$	MPQ4
Open two brackets with different signs	UDU4
From the factors and squares further explain the difference of two squares	UDU2
Again, here is to find the product of $x^2 - 4$ by opening two brackets. In the brackets the multiplication of first terms must give you the first term in equation and the multiplication of the last terms must give you the last term	GS
Find two numbers that have a product of -4 and a sum of zero. The numbers are -2 and +2	KW

Teaching the difference of two squares	WMAN3
Find the square roots of the two terms that are perfect squares. Write the factorization as the sum and difference of the two square roots.	UDU1
Here we look at these two terms respectively. Ask ourselves what are two numbers that can make up each term and put them in brackets since factors comes through multiplication.	WMAN9
Question and answer method; investigate and self-discovery	WMAN8
For difference of two squares, put the half of next to opened bracket, then the square root of 4 next to closed brackets. Now you put opposite signs between the x-variable and the number.	WMAN2
$\sqrt{x^2} = x$ $\sqrt{4} = 2$	WMAN4
Find the factors of first term, & factors of the last term, then put different signs.	TR
Difference of two squares; one factor must be positive and another be negative.	WMAN10
Difference of two squares is applicable in this problem.	WMAN1
Factorize by opening two brackets.	WMAN6
1. Law of exponents $a^1 \times a^1 = a^{1+1} = a^2$ $x^1 \times x^1 = x^{1+1} = x^2$ LAW1 (reinforce law of exponents) 2. signs ($\times \div + -$) 3. Bracket 4. Bodmas Rule. All the above mentioned content; just reinforce when in teaching the concept of factorization.	WMAN7

$$x^2 + 4$$

No explanation	ND
Sum of two squares cannot be factored because it is an irreducible polynomial	UDU1
Learners can see that there is no HCF, this is a sum of squares. Ask them to sketch the graph i.e. $f(x) = x^2 + 4$, they will see that it does not cut x-axis and therefore the values of x that will make this expression zero are not real. It is therefore not factorisable	MPQ1
Not factorisable; the positive sign shows that the factors have the same signs; when the signs are the same it is impossible to get the middle term as zero	MPQ2
No factors. Sum of 2 squares can never be factorised	MPQ3
Sum of the two squares has no factors	MPQ4
This expression can be solved using another method, because if we factorise it, it has no factors and it is impossible to return it to its original form if factorised	UDU4
This is not factorisable because of the sign, even there is no common factor	UDU2
It is not factorisable. First term has the factor and also second term, but the sign between the terms restrict the factorization because its gives the middle term whereas the original expression does not have it	UDU5
This given problem is undefined because they is no factors of such a given problem because of absence of factors	UDU3
Using the methods of factorization cannot be factorized. It is the test of the difference of two squares.	WMAN8
$x^2 = -4$; $x = \sqrt{-4}$ can't be factorized because there is no square root for a negative number.	WMAN2
$x^2 = -4$ is undefined	MZ
This expression it is not factorisable because we do not have two numbers that we can multiply and give us the last term & that if we add or subtract and give us the middle term.	WMAN7
Not factorisable, because there no two numbers that can give $x^2 + 4$ through multiplication.	WMAN9
Not factorisable, because there no two numbers that can give $x^2 + 4$.	GS
Not difference of two squares. $(x + 2)(x + 2)$ If we expand and use distributive law its becomes a trinomial	WMAN3
This is not a difference of two squares so therefore can not be factorized into the form $(x + a)(x + b)$. this quadratic is called the sum of two squares and can not be factorised	KW
Not factorizable. Teach them that you can only Factorize a difference of two squares.	WMAN6
N/A Positive sign does not allow us to factorize this expression.	WMAN1
No factors. This is not a difference of two squares.	WMAN10
Nothing is common between the terms, neither the sign is a negative for the difference of two squares	HP
I use graphical method sometimes I use the square root. No roots, not factorisable.	TR
Cannot factorise – not difference of two squares.	WMAN4
No factors because this is not a difference of two squares. When sketched it will not cut the x-axis	WMAN5

$$x^2 - 4x$$

$x(x - 4)$	
No explanation.	WMAN5; WMAN7; HP;ND
It is not factorisable	UDU5
Take out the common factor.	WMAN1; UDU4; MPQ2;MPQ3
Explain the common factors with examples e.g. $ab + bc = b(a + c)$	UDU2
It is a must to equate zero in a given equation then find factors which is $x(x - 4) = 0$ then we find the values of x	UDU3
Take out the highest common factor and place it in front of the brackets, then the expression inside the brackets is obtained by dividing each term by the highest common factor.	UDU1
Factorise by taking out a common factor	MZ; KW; MPQ1;MPQ4
Take out the highest common factor.	WMAN10
Take out the highest common factor.	GS
I encourage learners to take out the highest common factor first, then factorise	TR
Here, you have to factor out "x" and divide each term by "x"	WMAN2
Isolate what is common in the equation then put it outside the brackets.	WMAN3
Take out the common factor.	WMAN4
Discovery method; question and answer method	WMAN8
Start using practical example: got 2 bags with three items but one must be the same and tell that learner to look for the item that is similar and take it out. That's the introduction of the common factor.	WMAN9
$x(x^2 - 4) \rightarrow x(x - 2)(x + 2)$ Factorise by taking out a common factor.	WMAN6

$$x^2 - 4y$$

No response	WMAN2
Not factorisable	UDU2; UDU5; MPQ4
There is no such a given function that is why we can't find these values	UDU3
Since we have two unknowns it will be impossible to factorise it. So it is undefined/ unfactorised	UDU4
No factors because are unlike terms.	WMAN5
Not factorisable because the terms are not the same and there is no common factor	MPQ2
Ask learners if there is an HCF, they will realise that there is no HCF. Ask if there is difference of squares. Is this quadratic equation? If the answer is no to all questions asked, then lead them to conclude that there are no factors and hence expression not factorisable	MPQ1
No factors. No common factor and second term 4y is not a perfect square	MPQ3
It cannot be factored as the difference of two squares since y is not a square, therefore it cannot be factored	UDU1
No factors. Nothing common	WMAN10
No common factor. You must check whether there is a common factor, hence you cannot factorise.	WMAN3
Investigation. No common factors.	WMAN8
No common factors.	HP
There are no factors. What we have on term one does not correspond with term2	MZ; KW
Encourage learners to check for the common factor and check if all terms are squared, then factorise if it is possible to factorise. Not factorisable y is not common and not a perfect square.	TR
$x \cdot x - 4 \cdot y$ No common factors. Not difference of two squares.	WMAN4
$(x - 2y)(x + 2y)$ difference of two squares is applicable.	WMAN1
$(x - 2y)(x + 2y)$ Two numbers that can make up two items that are not the same.	WMAN9
Not factorizable. There is no common factor and y is not a square	WMAN6
Not factorizable. There is no factors	GS
It is not factorizable. This binomial expression the way I use to teach my learners, I just do practical example of taking two piece of paper, different colours & the other of the same colours and tell them that we can only factorise the equation or expression with the like terms.	WMAN7

$$x^2 - x - 6$$

no response- to solve this problem we firstly equate into zero then we find factors and give the value values of x	UDU3
No solution – if factorisable, the first and last term has the factors which gives the middle. NB the sign of a middle term is determined by by factors of the first and last term	UDU5
$(x + 2)(x - 3)$	

No explanation.	WMAN4; WMAN5 WMAN7 WMAN9; HP;ND
Trinomial expression. Factorise by inspection (middle term is – and the third term is -1 so you find factors	WMAN10
Ask factors of the 1 st term, factors of the 3 rd term. Learners are expected to be able to list i.e. x.x for x ² ; -3 and 2; -2 & 3; -6 & 1; 6&-1. Ask them to find factors that will give a sum of -1 i.e. -3&2. Explain that the factors must give the middle term when added.	MPQ1
Factorizing a trinomial look for all possible factors of the last term and when multiplying those factors they must give the middle term	MPQ4
List all factors of 6; choose only those when subtracted will give a middle term equal to one	MPQ2
Check for common factor-no common factor. Since trinomial try grouping-not working. Open 2 brackets and factorise 1 st term x ² . Check for factors of -6 if multiplied & -1 if you add.	MPQ3
Factorizing a trinomial by opening two brackets, separate the square no., find the factors of the last no. or constant. Choose the factors that will give the difference equal to the middle term, the bigger factor will take the sign which is the same as that of the middle term.	UDU2
Find two numbers to give the last term -6 when multiplied, but give middle term -1 when added or subtracted	UDU1
They have to understand this trinomial, so they have to open two brackets. After that is to find factors.	GS
Investigation. Question and answer method.	WMAN8
Look for factors of 6 e.g. (6.1) (3.2) that when you add or subtract give you the middle term which is -1 in this case.	WMAN1; KW
Put half of x ² in opened brackets. Put factors of 6 in closed brackets that give a difference which is – 1. The bigger the factor goes along with a sign of a middle term	WMAN2
You open two brackets then you check the sign of the last term if it is negative signs will not be the same within the brackets.	WMAN3
Open two brackets and use the smallest factors of 6(3and2), our middle term is negative so make 3 negative	UDU4
Factorise by opening two brackets	WMAN6
I usually ask learners to look for the factors of the first and last term; check the signs & then add or subtract the product to check if they give a middle term	TR
Determine the factors for the first term and the co-efficient of x ² and the factors for the last term (constant value)-they give you the co-efficient of x when you add them.	MZ

$$6x^2 + 7x - 20$$

This equation is undefined because it has no factors as such values of x	UDU3
No solution – its factorisable, there are factors of first and second term, which can be added together and give the middle NB the signs	UDU5
Ask learners to list all factors of 6x ² and factors of –20	MPQ1
(2x + 5)(3x – 4)	
No explanation.	WMAN2;WMAN4; HP; MZ;ND
Find the factors by inspection	WMAN10
List factors of 20 and factors of 6x ² ; choose only those that give +7 as a middle term	MPQ2
The trinomial with the first term with coefficient greater than 1 must also have factors i.e. factors of 6x ² are 2x and 3x and last term is –ve, therefore factors are (2x + 5)(3x – 4)	MPQ4
No common factor. Check for factors of 6x ² put in brackets, now check for factors of (-20). Multiply to check for the factors that can give us +7x	MPQ3
In the form ax ² + b x + c, multiply ac which in this problem is (6)(-20)=-120. Find two numbers whose product is -120 and their sum is +7, the numbers are -8 & +15. Rewrite 6x ² + 7 x – 20 as 6x ² + 15 x – 8x – 20 then group the expression into two i.e. (6x ² + 15 x)(– 8x – 20). Factorise both sets and leave something identical on both sets i.e. 3x(2x+5x)-4(2x+5)then 2x+5 is a common factor with 3x-4 left.	KW
Open two brackets, take the smallest factors of the of the constant, consider the co-efficient of x ² , divide it. The bracket with the bigger coefficient of x will take the positive sign and the smaller factor of 20	UDU2
No solution - Open two brackets and use the smallest factors of 20, then put the biggest factor with the smallest factor of 6	UDU4
Multiply first and the last term (6)(-20)=-120. Find two numbers that multiplied to give -120, but added to give middle term +7. Replace the middle term with those numbers which are -8x & +15x, then factorise by grouping.	UDU1
Is to find the product of this equation by opening two brackets. Then first term in brackets must give the value of a and the last terms in the brackets must give you the value of c. last term and first term must the middle term value.	GS
Discussion. Factors of the first and last term. Question and answer method.	WMAN8
Quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ should be used to assist learners not good.	WMAN5
If you suspect that it is not easy to get the factors you simple use the Quadratic equation	WMAN3

<i>It can be factorized through the use of the Quadratic expression</i>	WMAN9
<i>No given solution. The above method is applicable but firstly you need to find the factors of 6.</i>	WMAN1
<i>Factorise by opening two brackets</i>	WMAN6
<i>Look for the factors of the first term, last term; check the signs</i>	TR
<i>It is not factorisable</i>	WMAN7

APPENDIX M

Editors' Certificate

06 December 2021

CONFIRMATION OF EDITING

This is to confirm that the PhD thesis by Aubrey Sifiso Vilakazi (202 521 961) was forwarded to me for editing and proofreading.

The 346-page document comprising 10 chapters and Referencing Index has been edited for English grammar, punctuation and clarity and proofread for consistency and readability.

Sincerely

Wanda Daly [English Editor]

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APPENDIX N

Turnitin Report

draft1			
ORIGINALITY REPORT			
15%	14%	6%	%
SIMILARITY INDEX	INTERNET SOURCES	PUBLICATIONS	STUDENT PAPERS
PRIMARY SOURCES			
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