MATRIX MODELS OF POPULATION THEORY.

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Contents

Abstract

Non-negative matrices arise naturally in population models. In this thesis, we first study Perron-Frobenius theory of non-negative irreducible matrices. We use this theory to investigate the asymptotic behaviour of discrete time linear autonomous models. Then we discuss an application for this in age structured population. Furthermore, we study Liapunov stability of a general non-linear autonomous model. We consider a general nonlinear autonomous model that arises in structured population. We assume that the associated nonlinear matrix of this model is non-increasing at all density levels. Then, we show the existence of global extinction. In addition, we show the stability condition of the extinction equilibrium of the this model in the Liapunov sense.

Declaration

I declare that the contents of this dissertation are original except where due reference has been made. It has not been submitted before for any degree to any other institution.

Suliman Jamiel

January 2013

I, Suliman Jamiel, declare that

1. The research reported in this thesis, except where otherwise indicated, is my original research.

2. This thesis has not been submitted for any degree or examination at any other university.

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Signed:

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Dedication

I dedicate this thesis to everyone who encourages, inspires and believed in me, especially to my teachers, friends and family members.

Chapter 1

Introduction

Let $x(t)$ be the density or the number of individuals of a population at time t. Assume that a population with density $x(t)$ has m different classes (age classes, size classes, developmental stages, spatial locations, etc). Then $x(t)$ can be written as follows

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_m(t) \end{pmatrix}, \tag{1.1}$$

or, sometimes, $x(t) = [x_i(t)]_{1 \leq i \leq m}$, where $x_i(t)$ represents the density or the number of individuals of the i-th class.

Over a unit of time, individuals move from class j to i at a rate p_{ij}, where $p_{ij} \geq 0$. The equation showing the rate at which the individuals move from one class to another is

$$x_i(t+1) = \sum_{i=1}^{m} p_{ij} x_j(t) \qquad \forall \ 1 \leq j \leq m. \tag{1.2}$$

The problem (1.2) can be written in a more compact form below:

$$\begin{aligned} x(t+1) &= Px(t) \\ x(0) &= x^0 \end{aligned} \tag{1.3}$$

where $P = [p_{ij}]_{1 \leq i,j \leq m}$ and $x^0 = [x(0)]_{1 \leq i \leq m}$. Here we assume that P is a constant matrix. The system (1.3) is an example of a linear autonomous model. We consider a general nonlinear autonomous system (1.4), but first of all, we introduce necessary definitions and notations. We denote the set of real numbers by $\mathbb{R}$ and the set of complex numbers by $\mathbb{C}$. We denote the $m-$fold cartesian product $\mathbb{C} \times \cdots \times \mathbb{C}$ by $\mathbb{C}^m$ and $\mathbb{R} \times \cdots \times \mathbb{R}$ by $\mathbb{R}^m$. If we define the set of non-negative real numbers by $\mathbb{R}_+ = \{x \in \mathbb{R} \text{ such that } x \geq 0\}$, then $\mathbb{R}_+^m$ denotes the $m-$fold cartesian product, $\mathbb{R}_+ \times \cdots \times \mathbb{R}_+$. We denote the set of non-negative integer numbers $\{k, k+1, k+2, \dots\}$ by $I[k, +\infty)$. Let $x = [x_i]_{1 \leq i \leq m}$. We say that $x > 0$ if $x_i > 0$ and we say that $x \geq 0$ if $x_i \geq 0$ for all $i = 1, 2, \dots, m$. We say that $x = 0$ if $x_j = 0$ for all $j = 1, 2, \dots, m$ and we say that $x \neq 0$ if $x_j \neq 0$ for some $j = 1, 2, \dots, m$. We denote $0 \neq x \geq 0$ if $x_i \geq 0$ for $i = 1, 2, \dots, m$ and $x_j \neq 0$ for some $j = 1, 2, \dots, m$.

If X is a topological space, we define a neighbourhood of $x \in X$ to be a subset H of X that includes an open set U containing x,

$$x \in U \subseteq H.$$

If H is open, then H is called an open neighbourhood. We define the boundary of a subset S of X to be the set of points $x \in X$ such that every neighbourhood of x contains at least one point of S and one point not in S. The union of the set S and its boundary is denoted by $\overline{S}$ and called the closure of S.

We now consider a general nonlinear autonomous system

$$\begin{aligned} x(t+1) &= f(x(t)), \\ x(0) &= x^0, \end{aligned} \tag{1.4}$$

where $f : K \to \mathbb{R}^m, K \subset \mathbb{R}^m$, is continuously differentiable in a appropriate open subset of K.

An equilibrium point $x_e \in \mathbb{R}^m$ of (1.4) is defined to be a solution of the equation

$$x_e = f(x_e). \tag{1.5}$$

If $x_e = 0$, then we say that x_e is an extinction equilibrium. If $x_e > 0$, then we say x_e is a positive equilibrium. An equilibrium x_e is called stable if for every $\epsilon > 0$, we can find a $\delta = \delta(\epsilon) > 0$ such that $\|x(0) - x_e\| < \delta$ implies $\|x(t) - x_e\| < \epsilon$ for all $t \in I\,[0, +\infty)$. If x_e is not stable, then x_e is said to be unstable. An equilibrium x_e is called an attractor if a $\delta > 0$ can be found such that $\|x(0) - x_e\| < \delta$ implies $\lim_{t \to +\infty} \|x(t) - x_e\| = 0$. A stable and attracting equilibrium is said to be locally asymptotically stable equilibrium.

Now, from (1.4), we have $f : K \to \mathbb{R}^m, K \subset \mathbb{R}^m$, we assume that f is continuously differentiable in particular equilibrium point $x_e \in \mathbb{R}^m$. That is, $\frac{\partial f}{\partial x_i}$ at x_e exists and is continuous on an open neighborhood of x_e for $1 \leq i \leq m$. Let us write $f = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{pmatrix} = \begin{pmatrix} f_1(x_1, x_2, \ldots, x_m) \\ f_2(x_1, x_2, \ldots, x_m) \\ \vdots \\ f_m(x_1, x_2, \ldots, x_m) \end{pmatrix}$. The Jacobian matrix of f, sometimes called the Jacobian, is defined by

$$Df = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_m} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_m} \end{pmatrix}.$$

The Jacobian of f at an equilibrium x_e, $Df(x_e)$, is essential to the stability properties of the equilibrium x_e. The Jacobian matrix $Df(x_e)$ is called hyperbolic if none of its eigenvalues λ satisfies $|\lambda| = 1$. Otherwise, $Df(x_e)$ is called non-hyperbolic matrix. If the Jacobian matrix $Df(x_e)$ is hyperbolic, then the point x_e is called hyperbolic equilibrium.

We now show that we can assume that the equilibrium x_e is equal to 0. Letting

$$y(t) = x(t) - x_e \tag{1.6}$$

in (1.4) yields

$$y(t+1) = f(y(t) + x_e) - x_e = F(y(t)). \tag{1.7}$$

Then we see that $F(0) = 0$ so that 0 is an equilibrium for (1.7). Note that this can be done only locally as (1.4) may have several equilibria. From now on, unless specifically stated, we will assume that our equilibrium is 0.

The linearization of $F(y)$ about $y = 0$ is

$$F(y) = F(0) + DF(0)(y - 0) + g(y)$$

with

$$\lim_{\|y\| \to 0} \frac{\|g(y)\|}{\|y\|} = \lim_{\|y\| \to 0} \frac{\|F(y) - F(0) - DF(0)y\|}{\|y\|} = 0.$$

Thus (1.7) becomes

$$\begin{aligned} y(t+1) &= Py(t) + g(y(t)), \\ y(0) &= y^0, \end{aligned} \tag{1.8}$$

where $P = DF(0)$. At $t = 0$, (1.8) becomes

$$y(1) = Py^0 + g(y^0) = Py(0) + g(y(0)).$$

At any $t \in I[1, +\infty)$, we have

$$\begin{aligned} y(t+1) &= P^{t+1}y^0 + \sum_{i=0}^{t} P^{t-i} g(y(i)), \\ &= P\left[P^t y^0 + \sum_{i=0}^{t-1} P^{t-i-1} g(y(i))\right] + g(y(t)), \\ &= Py(t) + g(y(t)). \end{aligned}$$

Thus, the initial value problem (1.8) has a unique forward solution given by

$$y(t) = \begin{cases} P^t y^0 + \sum_{i=0}^{t-1} P^{t-i-1} g(y(i)) & t \in I[1, +\infty), \\ y^0 & t = 0. \end{cases} \tag{1.9}$$

In Chapter 2, we describe the notation that is used in this thesis and introduce some definitions. Then we discuss some relevant preliminary results and theorems from spectral theory about general matrices. In Chapter 3, we study Perron-Frobenius theorems for non-negative irreducible matrices. We use these results to study the long time behaviour of the linear autonomous models (1.3), in Chapter 4, following [3] and [9]. Then we give a complete discussion of the long time behaviour of Leslie matrix models. In Chapter 5, we discuss Liapunov stability of a general nonlinear autonomous system (1.4), following [4]. Furthermore, we consider (1.3) when the matrix P depends explicitly on $x(t)$, that is,

$$\begin{aligned} x(t+1) &= P(x(t))x(t), \\ x(0) &= x^0. \end{aligned} \tag{1.10}$$

We assume that $P(x)$ has non-negative entries for $x \geq 0$ and $P(x)$ is continuously differentiable in x. Then, we show the existence of global extinction of (1.10) under assumptions that $P(x) \leq P(0)$ for all $x \in \mathbb{R}^m_+$. We conclude our thesis by providing the stability condition of a hyperbolic extinction equilibrium of (1.10).

Chapter 2

Preliminaries

In this chapter, we recall the standard definitions and introduce terms and notation that will be used in the thesis. Then we discuss the relevant theorems.

2.1 Basic definitions and notations

Let $z \in \mathbb{C}$, then $z = re^{i\theta}$, where $r \in \mathbb{R}$ and $\theta \in [0, 2\pi)$. We define $|z| = r$ and we denote the complex conjugate of z by z^*. We denote the real part of z is by $\Re(z)$ and the imaginary part by $\Im(z)$.

Let $x = [x_i]_{1 \leq i \leq m}$ be a vector. Unless stated otherwise, in this thesis by a vector we mean a column vector. We define $|x|$ by $[|x_i|]_{1 \leq i \leq m}$ and we denote the transposed vector of x by x^T. Let $y = [y_i]_{1 \leq i \leq m}$ be a vector, then we define inner product of x and y by

$$\langle x, y \rangle = x^T y = \sum_{i=1}^{m} x_i y_i.$$

We recall the definition of a vector norm [6], page 259, then the $p-$norm of x is defined by

$$\|x\|_p = \left(\sum_{i=1}^{m} |x_i|^p \right)^{\frac{1}{p}}$$

for $1 \leq p < \infty$. We have that $\|x\|_p = \||x|\|_p$. Thus, in order to find $\|x\|_p$ it is sufficient to find $\||x|\|_p$. We define the $\infty-$norm of x, $\|x\|_\infty$, by $\lim_{p \to \infty} \|x\|_p$. To find the formula for $\|x\|_\infty$, we proceed following [9], page 275, as follows. If $x = 0$, then

$$\|x\|_\infty = \lim_{p \to \infty} \|x\|_p = \lim_{p \to \infty} 0 = 0.$$

If $x \neq 0$, relabel the entries of $|x|$ by setting $\tilde{x}_1 = \max_{1\le i\le m}|x_i|$ and, if there are k entries with same magnitude as $\tilde{x}_1$, we label them $\tilde{x}_2, \dots, \tilde{x}_k$. Label any remaining coordinates as $\tilde{x}_{k+1}, \dots, \tilde{x}_m$. Consequently, $\frac{\tilde{x}_i}{\tilde{x}_1} < 1$ for $i = k+1, \dots, m$, so,

$$\begin{aligned}
\|x\|_\infty &= \lim_{p\to\infty}\|x\|_p = \lim_{p\to\infty}\left(\sum_{i=1}^{m}|x_i|^p\right)^{\frac{1}{p}} \\
&= \lim_{p\to\infty}\left(|\tilde{x}_1|^p + \dots + |\tilde{x}_k|^p + \left|\tilde{x}_{k+1}\right|^p + \dots + \left|\tilde{x}_m\right|^p\right)^{\frac{1}{p}} \\
&= \lim_{p\to\infty}\left(k|\tilde{x}_1|^p + \left|\tilde{x}_{k+1}\right|^p + \dots + \left|\tilde{x}_m\right|^p\right)^{\frac{1}{p}} \qquad (2.1)\\
&= \lim_{p\to\infty}\left(k|\tilde{x}_1|^p\left(1 + \frac{1}{k}\left|\frac{\tilde{x}_{k+1}}{\tilde{x}_1}\right|^p + \dots + \frac{1}{k}\left|\frac{\tilde{x}_m}{\tilde{x}_1}\right|^p\right)\right)^{\frac{1}{p}} \\
&= |\tilde{x}_1|\lim_{p\to\infty}k^{\frac{1}{p}}\lim_{p\to\infty}\left(1 + \frac{1}{k}\left|\frac{\tilde{x}_{k+1}}{\tilde{x}_1}\right|^p + \dots + \frac{1}{k}\left|\frac{\tilde{x}_m}{\tilde{x}_1}\right|^p\right)^{\frac{1}{p}} = |\tilde{x}_1| = \max_{1\le i\le m}|x_i|.
\end{aligned}$$

Let $(\mathbb{R}^m, \|\cdot\|)$ be a normed vector space. Let $\|\cdot\|$ and $\|\cdot\|_0$ be any two norms on $\mathbb{R}^m$. We define an equivalence relation between $\|\cdot\|$ and $\|\cdot\|_0$ if there exists two real numbers $a > 0$ and $b > 0$ such that

$$a\|x\| \le \|x\|_0 \le b\|x\|. \qquad (2.2)$$

Let $\{x_t\}$ be a sequence in $\mathbb{R}^m$; that is, $x_t \in \mathbb{R}^m$ for $t = 0, 1, \dots$ and let $\|\cdot\|$ be any norm defined on $\mathbb{R}^m$. We say that $\{x_t\}$ is bounded if there is an $M \in \mathbb{R}$ such that $\|x_t\| \le M$ for all $t \in I[0, +\infty)$. We say $x_t \to l$ in $(\mathbb{R}^m, \|\cdot\|)$ as $t \to \infty$, if for any $\epsilon > 0$ there exists $T = T(\epsilon) > 0$ such that for all $t > T$ we have $\|x_t - l\| < \epsilon$. The vector l is called the limit point of $\{x_t\}$. A set $\{x : x \in \mathbb{R}^m\}$ is said to be bounded, respectively closed, if every sequence in the set is bounded and respectively the set contains all its limit points.

Lemma 2.1.1 ([1], page 128)**.** *If $x_t \to l$ in $(\mathbb{R}^m, \|\cdot\|)$ as $t \to \infty$, then $\|x_t\| \to \|l\|$ in $\mathbb{R}_+$ as $t \to \infty$.*

Proof. If $x_t \to l$ in $(\mathbb{R}^m, \|\cdot\|)$ as $t \to \infty$, then for any $\epsilon > 0$ there exists $T = T(\epsilon) > 0$ such that for all $t > T$ we have $\|x_t - l\| < \epsilon$. If we recall the Backward Triangle Inequality ([9], page 273), then

$$|\|x_t\| - \|l\|| \le \|x_t - l\| < \epsilon.$$

Since $\|x_t - l\| \to 0$ as $t \to \infty$, then $\|x_t\| - \|l\| \to 0$. Thus $\|x_t\| \to \|l\|$ as $t \to \infty$. □

Lemma 2.1.2 ([7], page 72)**.** *Let $\{x_1, x_2, \dots, x_m\}$ be a linearly independent set of vectors in $\mathbb{R}^m$. Then there is a number $c > 0$ such that for every choice of scalars $\alpha_1, \alpha_2, \dots, \alpha_m$, we have*

$$\|\alpha_1 x_1 + \dots + \alpha_m x_m\| \ge c\left(|\alpha_1| + \dots + |\alpha_m|\right) \qquad (2.3)$$

Proof. We write $s = |\alpha_1| + \cdots + |\alpha_m|$. If $s = 0$, all α_j for $j = 1, 2, \ldots, m$ are zero, so that (2.3) holds for any c. Let $s > 0$. Then (2.3) is equivalent to the inequality which we obtain from (2.3) by dividing by s and writing $\beta_j = \frac{\alpha_j}{s}$, that is,

$$\|\beta_1 x_1 + \cdots + \beta_m x_m\| \geq c. \tag{2.4}$$

Hence it suffices to prove the existence of a $c > 0$ such that (2.4) holds for scalars $\beta_1, \beta_2, \cdots, \beta_m$ with $\sum_{j=1}^{m} |\beta_j| = 1$. Suppose that this is false. Then there exists a sequence $\{y_t\}$ in $\mathbb{R}^m$ with $y_t = \beta_1^{(t)} x_1 + \cdots + \beta_m^{(t)} x_m$ and $\sum_{j=1}^{m} |\beta_j^{(t)}| = 1$ such that $\|y_t\| \to 0$ as $t \to \infty$. Since $\sum_{j=1}^{m} |\beta_j^{(t)}| = 1$, we have $|\beta_j^{(t)}| \leq 1$ for each $1 \leq j \leq m$. Hence, for each fixed j, the sequence $\{\beta_j^{(t)}\} = \beta_j^{(1)}, \beta_j^{(2)}, \ldots,$ is bounded. Consequently, by the Bolzano-Weierstrass theorem, $\{\beta_1^{(t)}\}$ has a convergent subsequence. Let β_1 denote the limit of the subsequence and let $\{y_{1,t}\}$ denote the corresponding subsequence of $\{y_t\}$. By the same argument, $\{y_{1,t}\}$ has a subsequence $\{y_{2,t}\}$ for which the corresponding subsequence of scalars $\beta_2^{(t)}$ converges; let β_2 denote the limit. Continuing in this way, after m steps we obtain a subsequence $y_{m,t} = y_{m,1}, y_{m,2}, \ldots$ of $\{y_m\}$ whose terms are of the form

$$y_{m,t} = \sum_{j=1}^{m} \gamma_j^{(t)} x_j,$$

where $\sum_{j=1}^{m} |\gamma_j^{(t)}| = 1$ with scalars $\gamma_j^{(t)}$ satisfying $\gamma_j^{(t)} \to \beta_j$ as $t \to \infty$. Hence, as $t \to \infty$, $y_{m,t} \to y = \sum_{j=1}^{m} \beta_j x_j$ where $\sum_{j=1}^{m} |\beta_j| = 1$, so not all β_j can be zero. Since $\{x_1, \cdots, x_m\}$ is a linearly independent set, we have $y \neq 0$. By Lemma 2.1.1, $y_{m,t} \to y$ as $t \to \infty$ implies $\|y_{m,t}\| \to \|y\|$. Since $\|y_{m,t}\| \to 0$, $\|y\| = 0$. Thus $y = 0$. This contradicts $y \neq 0$, and the lemma is proved. □

Theorem 2.1.3 ([7], page 75)**.** *Let $\|\cdot\|$ and $\|\cdot\|_0$ be two norms in $\mathbb{R}^m$. Then $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent.*

Proof. Let $\{e_1, e_2, \ldots, e_m\}$ be a basis of $\mathbb{R}^m$. Then every $x \in \mathbb{R}^m$ has a unique representation $x = \alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_m e_m$. By Lemma 2.1.2, there is a positive constant c such that

$$\|\alpha_1 e_1 + \cdots + \alpha_m e_m\| \geq c\left(|\alpha_1| + \cdots + |\alpha_m|\right).$$

On the other hand, the triangle inequality gives

$$\|x\|_0 \leq \|\sum_{j=1}^{m} |\alpha_j| \|e_j\|_0 \leq k \sum_{j=1}^{m} |\alpha_j|,$$

where $k = \max_{1 \leq j \leq m} \|e_j\|_0$. Together, $a\|x\|_0 \leq \|x\|$ where $a = \frac{c}{k}$. The other inequality in (2.2) is now obtained by interchanging the roles $\|\cdot\|$ and $\|\cdot\|_0$ in the preceding argument. □

Since norms are equivalent in $\mathbb{R}^m$, in particular applications we shall use the most convenient norm.

Definition 2.1.4. Let $y \in \mathbb{R}^m$, we define

$$B(y, r) = \{x \in \mathbb{R}^m : \|x - y\| < r \text{ for some } 0 < r \in \mathbb{R}\},$$

and

$$\overline{B(y, r)} = \{x \in \mathbb{R}^m : \|x - y\| \leq r \text{ for some } 0 < r \in \mathbb{R}\}.$$

If $y = 0$, we denote $B(y, r)$ by

$$B(r), \tag{2.5}$$

and $\overline{B(y, r)}$ by

$$\overline{B(r)}. \tag{2.6}$$

Let $P = [p_{ij}]_{1 \leq i,j \leq m}$, sometimes written $P = [p_{ij}]_{m \times m}$, be a real valued square matrix of order $m \times m$. That is, $P \in \mathbb{R}^{m^2}$. Unless stated, otherwise, in this thesis any matrix is considered to be a real valued square matrix of order $m \times m$. A matrix P is said to be non-negative ($P \geq 0$), respectively positive ($P > 0$), if $p_{ij} \geq 0$ and $p_{ij} > 0$, respectively, for all $1 \leq i, j \leq m$. If $P \in \mathbb{C}^{m^2}$, then we define $|P|$ to be $[|p_{ij}|]_{1 \leq i,j \leq m}$. We denote the transposed matrix of P by P^T. The null space of P, $N(P)$, is defined by $N(P) = \{x \in \mathbb{R}^m : Px = 0\}$ and the range space $R(P) = \{y \in \mathbb{R}^m : Px = y \text{ for some } x \in \mathbb{R}^m\}$. The dimension of $N(P)$, $\dim(N(P))$, is defined to be the cardinality of $N(P)$. We say that P is a nilpotent matrix of index k if $P^k = 0$ for some k and $P^{k-j} \neq 0$ for all $j = 1, \cdots, k$, where P^0 denotes the identity matrix. If P is a singular matrix, we define index(P) to be the smallest integer t so that $R(P^t) \cap N(P^t) = 0$. If P is a non-singular matrix, then index(P) $= 0$. If λ is an eigenvalue of P (Definition 2.1.5), then the index of λ is defined to be the index of $(P - \lambda I)$.

If we regard a matrix P as element of $\mathbb{R}^{m^2}$, we can consider the set of all $m \times m$ matrices as normed space $(\mathbb{R}^{m^2}, \|\cdot\|)$ where, by Theorem 2.1.3, $\|\cdot\|$ can be any norm on $\mathbb{R}^{m^2}$. For applications we use a specific class of norms, called matrix norms, and defined by

$$\begin{aligned} \|P\| &= \max\left\{\frac{\|Px\|}{\|x\|} : x \in \mathbb{R}^m \text{ with } x \neq 0\right\} \\ &= \max\left\{\|Py\| : y \in \mathbb{R}^m \text{ with } \|y\| = \|\frac{x}{\|x\|}\| = 1\right\}. \end{aligned}$$

In particular, we define the $p-$norm of P by

$$\begin{aligned} \|P\|_p &= \max_{x \neq 0} \|P\frac{x}{\|x\|_p}\|_p \\ &= \max_{\|y\|_p = 1} \|Py\|_p, \end{aligned} \tag{2.7}$$

and the $\infty-$norm by

$$\|P\|_\infty = \max_{\|x\|=1} \|Px\|_\infty. \tag{2.8}$$

By the Bolzano-Weierstrass theorem, as we proved in Lemma 2.1.1, every sequence in a set $\{x \in \mathbb{R}^m : \|x\|_p = 1\}$ has a subsequence converging to a point in the set. Therefore, we have that this set is closed and bounded. Since $x \to \|Px\|_p$ is a continuous function on $\mathbb{R}^m$, by the Extreme Value Theorem, the maximum value $\|P\|_p$ exists.

Let $x = [x_i]_{1\leq i\leq m} \in \mathbb{R}^m$ where $\|x\|_\infty = 1$. Then $|x_i| \leq 1$ for $1 \leq i \leq m$ and

$$\|P\|_\infty = \|Px\|_\infty = \max_{1\leq i\leq m} |\sum_{j=1}^m p_{ij}x_j| \leq \max_{1\leq i\leq m} \sum_{j=1}^m |p_{ij}||x_j| \leq \max_{1\leq i\leq m} \sum_{j=1}^m |p_{ij}|.$$

Thus

$$\|P\|_\infty \leq \max_{1\leq i\leq m} \sum_{j=1}^m |p_{ij}|. \tag{2.9}$$

We will prove that for some $x \in \mathbb{R}^m$, where $\|x\|_\infty = 1$, we have

$$\|P\|_\infty \geq \max_{1\leq i\leq m} \sum_{j=1}^m |p_{ij}|.$$

Find a number k for which $\sum_{j=1}^m |p_{kj}| = \max_{1\leq i\leq m} \sum_{j=1}^m |p_{ij}|$. Then let

$$\tilde{x}_j = \begin{cases} 1 & \text{if } p_{kj} \geq 0, \\ -1 & \text{if } p_{kj} < 0 \end{cases}$$

for $j = 1, 2, \ldots, m$. Hence for $x = [\tilde{x}_j]_{1\leq j\leq m}$ we have

$$\|P\|_\infty = \|Px\|_\infty = \max_{1\leq i\leq m} |\sum_{j=1}^m p_{ij}\tilde{x_j}| \geq |\sum_{j=1}^m p_{kj}\tilde{x_j}| = \sum_{j=1}^m |p_{kj}| = \max_{1\leq i\leq m} \sum_{j=1}^m |p_{ij}|.$$

Thus

$$\|P\|_\infty \geq \max_{1\leq i\leq m} \sum_{j=1}^m |p_{ij}|. \tag{2.10}$$

From (2.9) and (2.10) we have

$$\|P\|_\infty = \max_{1\leq i\leq m} \sum_{j=1}^m |p_{ij}|. \tag{2.11}$$

Let $P^t = [p_{ij}^{(t)}]_{1\leq i,j\leq m}$ for any $t = 1, 2, \ldots$ and $L = [l_{ij}]_{1\leq i,j\leq m}$. By Theorem 2.1.3, $P^t \to L$ in $(\mathbb{R}^{m^2}, \|\cdot\|)$ as $t \to \infty$, if and only if $p_{ij}^{(t)} \to l_{ij}$ as $t \to \infty$ for all $1 \leq i, j \leq m$.

Definition 2.1.5. The eigenvalues $\lambda \in \mathbb{C}$ of P are the the solutions of the characteristic equation of P or, equivalently, the roots of the characteristic polynomial of P.

Let $\sigma(P) = (\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_s)$ be the set of distinct eigenvalues of P which is called the spectrum of P. We define the spectral radius of P to be

$$\rho(P) = \max_{\lambda_i \in \sigma(P)} |\lambda_i| \text{ for } i = 1, 2, \dots, s \tag{2.12}$$

and the spectral circle of P to be the set

$$\{\lambda \in \sigma(P) : |\lambda| = 1\}.$$

The characteristic polynomial $q(\lambda)$ can be written in the factorized form

$$q(\lambda) = (\lambda_1 - \lambda)^{m_1}(\lambda_2 - \lambda)^{m_2} \cdots (\lambda_s - \lambda)^{m_s} \tag{2.13}$$

with $m_1 + m_2 + \cdots + m_s = m$. The exponent m_i, corresponding to each eigenvalue λ_i, is called the algebraic multiplicity of λ_i and $\dim(N(P - \lambda_i I))$ is called the geometric multiplicity of λ_i. If $\dim(N(P - \lambda_i I)) = m_i$, then λ_i is said to be semisimple eigenvalue. If $m_i = 1$, then λ_i is said to be simple eigenvalue.

Let $\lambda \in \sigma(P)$ be an eigenvalue of P and $v \neq 0$ be a vector satisfying $Pv = \lambda v$. Then, v is called an eigenvector of P corresponding to the eigenvalue λ. Sometimes v is also called a right eigenvector of P. The pair (λ, v) is called an eigenpair. A vector $\omega \neq 0$ which satisfies $\omega^T P = \lambda \omega^T$ or, equivalently, $P^T \omega = \lambda \omega$ is called a left eigenvector of P. If a vector $0 \neq v^i$ is found as a solution to the equation $(P - \lambda_i I)^t v^i = 0$ for some $1 < t \leq m_i$ and $(P - \lambda_i I)^{t-1} v^i \neq 0$, then v^i is called an associated or generalized eigenvector corresponding to the eigenvalue λ_i.

2.2 Similarity

Definition 2.2.1. Let P_1 and P_2 be two matrices. We say that P_1 and P_2 are similar, if there exists a non-singular matrix Q such that $P_2 = QP_1Q^{-1}$. The product QP_1Q^{-1} is called similarity transformation of P_1.

Theorem 2.2.2 ([9], page 508)**.** *Similar matrices have the same characteristic polynomial.*

Proof. Let P_1 and P_2 be similar matrices and let $\lambda \in \mathbb{C}$, then there exists a non-singular matrix Q such that

$$\begin{aligned}\det(P_1 - \lambda I) &= \det(Q^{-1}P_2Q - \lambda I) = \det(Q^{-1}(P_2 - \lambda I)Q) \\ &= \det(Q)^{-1}\det(P_2 - \lambda I)\det(Q) \\ &= \frac{1}{\det(Q)}\det(P_2 - \lambda I)\det(Q) \\ &= \det(P_2 - \lambda I).\end{aligned}$$

□

Two important corollaries of the above theorem are as follows:

Corollary 2.2.3. *If P_1 and P_2 are similar matrices, then $\lambda \in \sigma(P_1)$ if and only if $\lambda \in \sigma(P_2)$.*

Corollary 2.2.4. *If P_1 and P_2 are similar matrices, then $\lambda \in \sigma(P_1)$ is a simple eigenvalue if and only if $\lambda \in \sigma(P_2)$ is a simple eigenvalue.*

Theorem 2.2.5. *If P_1 and P_2 are similar matrices, then*

$$P_2^t = QP_1^tQ^{-1}$$

for any $t \in I[1, +\infty)$.

Proof. Similarity of P_1 and P_2 implies the existence a non-singular matrix Q such that $P_2 = QP_1Q^{-1}$ for any $t \in I[1, +\infty)$. Then

$$P_2^t = (QP_1Q^{-1})^t = (QP_1Q^{-1})(QP_1Q^{-1})\dots(QP_1Q^{-1}),$$

where QP_1Q^{-1} is repeated t times, then

$$\begin{aligned}P_2^t &= QP_1Q^{-1}QP_1Q^{-1}\dots QP_1Q^{-1} \\ &= QP_1^tQ^{-1}.\end{aligned}$$

□

2.3 Jordan forms

We say that a matrix is diagonalizable if it is similar to a diagonal matrix. A matrix P is diagonalizable if and only if P possess a complete set of eigenvectors([9], page 509). While it is not always possible to diagonalize any matrix with similarity transformation, Theorem 7.8.4 in [9], page 590, says that every square matrix P with distinct eigenvalues $\sigma(P) = \{\lambda_1, \lambda_2, \cdots, \lambda_s\}$ is similar to a block-diagonal matrix $J = \mathrm{diag}(J(\lambda_1), J(\lambda_2), \ldots, J(\lambda_s))$, called the Jordan form of P. That is, there is a non-singular matrix Q such that

$$P = QJQ^{-1} = Q \begin{pmatrix} J(\lambda_1) & 0 & \cdots & 0 \\ 0 & J(\lambda_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J(\lambda_s) \end{pmatrix} Q^{-1}. \tag{2.14}$$

Equation (2.14) is called the Jordan decomposition of P. Denote m_j to the algebraic multiplicity of λ_j. Each segment $J(\lambda_j)$ in J is an $m_j \times m_j$ block diagonal matrix made up of $d_j = \dim N(P - \lambda_j I)$ Jordan blocks $B_i(\lambda_j)$, where $1 \le i \le d_j$, as described below:

$$\begin{aligned} J(\lambda_j) &= \begin{pmatrix} B_1(\lambda_j) & 0 & \cdots & 0 \\ 0 & B_2(\lambda_j) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_{d_j}(\lambda_j) \end{pmatrix} \\ &= \mathrm{diag}(B_1(\lambda_j), \ldots, B_{d_j}(\lambda_j)). \end{aligned} \tag{2.15}$$

Thus

$$\begin{aligned} J &= \mathrm{diag}(J(\lambda_1), J(\lambda_2), \ldots, J(\lambda_s)) \\ &= \mathrm{diag}(B_1(\lambda_1), \ldots, B_{d_1}(\lambda_1), \ldots, B_1(\lambda_s), \ldots, B_{d_s}(\lambda_s)). \end{aligned}$$

In short,

$$J = \mathrm{diag}(B_1(\lambda_1), \ldots, B_k(\lambda_s)) = \begin{pmatrix} \ddots & & \\ & B_i(\lambda_j) & \\ & & \ddots \end{pmatrix}, \tag{2.16}$$

with $1 \le i \le k$ and $1 \le j \le s$. Each Jordan block $B_i(\lambda_j)$ in J is an $r_i \times r_i$ matrix defined by

$$B_i(\lambda_j) = \begin{pmatrix} \lambda_j & 1 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & & 1 \\ 0 & \cdots & 0 & \lambda_j \end{pmatrix} = D_i(\lambda_j) + N_i(\lambda_j).$$

where $D_i(\lambda_j) = \begin{pmatrix} \lambda_j & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & & 0 \\ 0 & \cdots & 0 & \lambda_j \end{pmatrix}$ and $N_i(\lambda_j) = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & & 1 \\ 0 & \cdots & 0 & 0 \end{pmatrix}$. The matrix $N_i(\lambda_j)$ is nilpotent of index r_i since $N_i^{r_i} = 0$ and $N_i^{r_i - j} \neq 0$ for all $j = 1, 2, \ldots, r_i$.

Theorem 2.3.1 ([9], page 593)**.** *Let $J = Q^{-1}PQ$ be the Jordan form of P defined by* (2.16)*. Then the columns of Q are the eigenvectors and generalized eigenvectors of P.*

Proof. We have $PQ = QJ$ and $J = \operatorname{diag}(B_1(\lambda_1), \ldots, B_k(\lambda_s))$, where each $B_i(\lambda_j)$ is an $r_i \times r_i$ Jordan block. We partition Q such that $Q = \begin{pmatrix} Q_1 & | & \cdots & | & Q_k \end{pmatrix}$ where $Q_i = \begin{pmatrix} x_1^i & x_2^i & \cdots & x_{r_i}^i \end{pmatrix}$ is an $m \times r_i$ matrix for all $i = 1, \cdots, k$ and x_j^i is an $m \times 1$ vector for $j = 1, 2, \ldots, r_i$. Thus

$$\begin{aligned} PQ &= P \begin{pmatrix} Q_1 & | & \cdots & | & Q_k \end{pmatrix} \\ &= \begin{pmatrix} PQ_1 & | & \cdots & | & PQ_k \end{pmatrix} \\ &= \begin{pmatrix} Px_1^1 & \cdots & Px_{r_1}^1 & | & \cdots & | & Px_1^k & \cdots & Px_{r_k}^k \end{pmatrix}. \end{aligned}$$

On the other hand

$$\begin{aligned} QJ &= \begin{pmatrix} Q_1B_1(\lambda_1) & | & \cdots & | & Q_kB_k(\lambda_k) \end{pmatrix} \\ &= \left(\begin{pmatrix} x_1^1 & \cdots & x_{r_1}^1 \end{pmatrix} B_1(\lambda_1) \quad | \quad \cdots \quad | \quad \begin{pmatrix} x_1^k & \cdots & x_{r_k}^k \end{pmatrix} B_k(\lambda_k) \right). \end{aligned}$$

But for each $\lambda_j \in \sigma(P)$ we have

$$Q_iB_i(\lambda_j) = \begin{pmatrix} x_1^i & x_2^i & \cdots & x_{r_i}^i \end{pmatrix} \begin{pmatrix} \lambda_j & 1 & 0 & \cdots & 0 \\ 0 & \lambda_j & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ & & & \ddots & 1 \\ 0 & \cdots & & 0 & \lambda_j \end{pmatrix}$$
$$= \begin{pmatrix} \lambda_j x_1^i & x_1^i + \lambda_j x_2^i & \cdots & x_{r_{i-1}}^i + \lambda_j x_{r_i}^i \end{pmatrix}$$

where $1 \le i \le k$. Therefore $PQ_i = Q_iB_i(\lambda_j)$ implies

$Px_1^i = \lambda_j x_1^i,$ that is, (λ_j, x_1^i) is an eigenpair;

$Px_2^i = x_1^i + \lambda_j x_2^i,$ that is, $(P - \lambda_j I)x_2^i = x_1^i \neq 0$ and $(P - \lambda_j I)^2 x_2^i = 0$;

$Px_3^i = x_2^i + \lambda_j x_3^i,$ that is, $(P - \lambda_j I)x_3^i = x_2^i \neq 0,$ and by induction $(P - \lambda_j I)^3 x_3^i = 0$;

$\vdots$

$Px_{r_i}^i = x_{r_{i-1}}^i + \lambda_j x_{r_i}^i,$ that is, $(P - \lambda_j I)x_{r_i}^i = x_{r_{i-1}}^i \neq 0,$ and by induction $(P - \lambda_j I)^{r_i} x_{r_i}^i = 0.$

Hence the matrix Q_i for each $1 \le i \le k$ (and thus Q) is the matrix whose columns are the eigenvectors and the generalized eigenvectors of P. □

The largest Jordan block $B_i(\lambda_j)$ in $J(\lambda_j)$ in (2.15) is an $r_j \times r_j$ matrix where $r_j =$ index(λ_j). Moreover, index$(\lambda_j) = 1$ if and only if every Jordan block $B_i(\lambda_j)$ is 1×1 which happens if and only if the number of eigenvectors associated with λ_j in Q such that $Q^{-1}PQ = J$ is the same as the number of Jordan blocks $B_i(\lambda_j)$. Since each $J(\lambda_i)$ is made up of $d_j = \dim(N(P - \lambda_j I)$, we have " the algebraic multiplicity equals to the geometric multiplicity d_j " This is just another way of saying that algebraic multiplicity and geometric multiplicity of λ_j are the same which is the definition of λ_j being semisimple. This can be summarised as follows:

Corollary 2.3.2 ([10], page 151)**.** *Let $\lambda \in \sigma(P)$. Then every Jordan block associated with λ is a 1×1 matrix if and only if λ is semisimple.*

Let $f(z) : \mathbb{C} \to \mathbb{C}$ be arbitrary function that has a series expansion about $\lambda_j \in \sigma(P)$ for some $1 \le j \le s$; that is

$$f(z) = f(\lambda_j) + f'(\lambda_j)(z - \lambda_j) + \frac{f''(\lambda_j)}{2!}(z - \lambda_j)^2 + \cdots$$

for $|z-\lambda_j| < r$ and some $r > 0$, then for any Jordan block $B_i(\lambda_j)$ in J defined in (2.16), we define the matrix function $f(B_i(\lambda_j))$ by

$$f(B_i(\lambda_j)) = f(\lambda_j)I + f'(\lambda_j)(B_i(\lambda_j) - \lambda_j I) + \frac{f''(\lambda_j)}{2!}(B_i(\lambda_j) - \lambda_j I)^2 + \cdots .$$

Since $N_i = B_i(\lambda_j) - D_i = B_i(\lambda_j) - \lambda_j I$ is nilpotent of index r_i, the series $f(B_i(\lambda_j)$ is finite:

$$f(B_i(\lambda_j)) = \sum_{i=0}^{r_i-1} \frac{f^{(i)}(\lambda_j)}{i!} N^{(i)} = \begin{pmatrix} f(\lambda_j) & 0 & 0 & \cdots & 0 \\ 0 & f(\lambda_j) & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ & & & & 0 \\ 0 & \cdots & & 0 & f(\lambda_j) \end{pmatrix} + \begin{pmatrix} 0 & f'(\lambda_j) & 0 & \cdots & 0 \\ 0 & 0 & f'(\lambda_j) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ & & & & f'(\lambda_j) \\ 0 & \cdots & & 0 & 0 \end{pmatrix} + \cdots$$

$$+ \begin{pmatrix} 0 & 0 & 0 & \cdots & \frac{f^{(r_i-1)}(\lambda_j)}{(r_i-1)!} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ & & & & \\ & & & & 0 \\ 0 & & \cdots & 0 & 0 \end{pmatrix} = \begin{pmatrix} f(\lambda_j) & f'(\lambda_j) & \frac{f''(\lambda_j)}{2!} & \cdots & \frac{f^{(r_i-1)}(\lambda_j)}{(r_i-1)!} \\ 0 & f(\lambda_j) & f'(\lambda_j) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \frac{f''(\lambda_j)}{2!} \\ & & & & f'(\lambda_j) \\ 0 & \cdots & & 0 & f(\lambda_j) \end{pmatrix}. \tag{2.17}$$

We define $f(J)$ by

$$f(J) = \operatorname{diag}(f(B_1(\lambda_1)), \ldots, f(B_k(\lambda_s))) = \begin{pmatrix} \ddots & & \\ & f(B_i(\lambda_j)) & \\ & & \ddots \end{pmatrix}. \tag{2.18}$$

For an arbitrary matrix P we define

$$f(P) = Qf(J)Q^{-1} = Q \begin{pmatrix} \ddots & & \\ & f(B_i(\lambda_j)) & \\ & & \ddots \end{pmatrix} Q^{-1} \tag{2.19}$$

where J is the Jordan form of P.

From (2.14), by similarity, we have $\lambda \in \sigma(J)$ if and only if $\lambda \in \sigma(P)$. From (2.17), if $\lambda \in \sigma(B_i(\lambda))$ then $f(\lambda) \in \sigma(f(B_i(\lambda))$, hence $\lambda \in \sigma(J)$ implies that $f(\lambda) \in \sigma(f(J))$. From (2.19), by similarity, we have $f(\lambda) \in \sigma(f(J))$ if and only if $f(\lambda) \in \sigma(f(P))$. Thus we have the following corollary

Corollary 2.3.3. *If $\lambda \in \sigma(P)$, then $f(\lambda) \in \sigma(f(P))$.*

Another result that follows from the Jordan form of a matrix can be written as follows.

Corollary 2.3.4. *If $f(\lambda)$ is a simple eigenvalue of $f(P)$, λ is a simple eigenvalue of P.*

Proof. We use the contrapositive argument. Let $\lambda_j \in \sigma(P)$ with algebraic multiplicity $m_j > 1$. Then, the Jordan form of P,

$$J = Q^{-1}PQ = \begin{pmatrix} J(\lambda_1) & 0 & \cdots & 0 \\ 0 & J(\lambda_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J(\lambda_s) \end{pmatrix},$$

where $J(\lambda_j)$ is an $m_j \times m_j$ matrix. We have

$$J(\lambda_j) = \begin{pmatrix} B_1(\lambda_j) & 0 & \cdots & 0 \\ 0 & B_2(\lambda_j) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_{d_j}(\lambda_j) \end{pmatrix},$$

where $d_j = \dim(N(P - \lambda_j I))$ and each Jordan block $B_i(\lambda_j)$ is an $r_i \times r_i$ block diagonal matrix for $i = 1, 2, \ldots, d_j$. By definition, $f(B_i(\lambda_j))$ is an $r_i \times r_i$ upper triangular matrix, given by (2.17), that has only $f(\lambda_j)$ in the diagonal, and

$$f(J(\lambda_j)) = \operatorname{diag}(f(B_1(\lambda_j)), f(B_2(\lambda_j)), \ldots f(B_{d_j}(\lambda_j)))$$

is an $m_j \times m_j$ upper triangular matrix. Thus, $f(\lambda_j) \in \sigma(f(J(\lambda_j)))$ is repeated m_j times. We have

$$f(J) = \operatorname{diag}(f(J(\lambda_1)), \ldots, f(J(\lambda_s))) = \begin{pmatrix} \ddots & & \\ & f(J(\lambda_j)) & \\ & & \ddots \end{pmatrix}.$$

Thus, $f(\lambda_j)$ is an eigenvalue of $f(J)$ that is repeated at least m_j times. Therefore $f(\lambda_j) \in \sigma(P)$ is repeated at least m_j times and the Corollary is proved. □

If we partition Q and Q^{-1} as $Q = (Q_1|\cdots|Q_s)$ and $Q^{-1} = \begin{pmatrix} \Psi_1 \\ \vdots \\ \Psi_s \end{pmatrix}$ so that (2.19) implies that

$$f(P) = Qf(J)Q^{-1} = Q \begin{pmatrix} \ddots & & \\ & f(J(\lambda_j)) & \\ & & \ddots \end{pmatrix} Q^{-1} = \sum_{j=1}^{s} Q_j f(J(\lambda_j))\Psi_j. \tag{2.20}$$

Since $Q^{-1}Q = I$,

$$\Psi_i Q_j = \begin{cases} I & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}. \tag{2.21}$$

Definition 2.3.5. A matrix G is said to be projector if and only if

$$G^2 = G.$$

Let

$$G_j = Q_j \Psi_j. \tag{2.22}$$

Then

$$G_j^2 = Q_j \Psi_j Q_j \Psi_j = Q_j I \Psi_j = Q_j \Psi_j = G_j.$$

Thus G_j is a projector. If G_j is the projector onto the generalized eigenspace $N(P - \lambda_j I)^{r_j}$ along $R(P - \lambda_j I)^{r_j}$, that is, $R(G_j) = N(P - \lambda_j I)^{r_j}$ and $N(G_j) = R(P - \lambda_j I)^{r_j}$ for $1 \leq j \leq s$, we define the spectral resolution of $f(P)$, [9], page 603, by

$$f(P) = \sum_{j=1}^{s} \sum_{i=0}^{r_j - 1} \frac{f^{(i)}(\lambda_j)}{i!}(P - \lambda_j I)^i G_j \tag{2.23}$$

where r_j = index(λ_j).

Corollary 2.3.6 ([10], page 155)**.** *If (λ, x) is an eigenpair of P, then $(f(\lambda), x)$ is an eigenpair of $f(P)$.*

Proof. If $0 \neq x_h \in N(P - \lambda_h I)$ and if G_h is the projection onto $N(P - \lambda_h I)$ along $R(P - \lambda_h I)$ defined by (2.22), then $G_h x_h = x_h$. From (2.21) we have

$$G_j G_h = Q_j \Psi_j Q_h \Psi_h = \begin{cases} I & j = h \\ 0 & j \neq h \end{cases}$$

and

$$G_j x_h = G_j G_h x_h = \begin{cases} G_h^2 x_h = G_h x_h = x_h & \text{when } j = h \\ 0 & \text{when } j \neq h \end{cases}.$$

Thus, from (2.23), we have

$$\begin{aligned} f(P)x_h &= \sum_{j=1}^{s} \sum_{i=0}^{r_j-1} \frac{f^{(i)}(\lambda_j)}{i!}(P - \lambda_j I)^i G_j x_h \\ &= \sum_{i=0}^{r_j-1} \frac{f^{(i)}(\lambda_h)}{i!}(P - \lambda_h I)^i x_h \\ &= f(\lambda_h)x_h + \sum_{i=1}^{r_j-1} \frac{f^{(i)}(\lambda_h)}{i!}(P - \lambda_h I)^i x_h \\ &= f(\lambda_h)x_h + 0 \end{aligned}$$

because $(P-\lambda_h I)^i x_h = 0$ for all $i = 1, 2, \ldots$. Thus $f(P)x_h = f(\lambda_h)x_h$, hence $(f(\lambda_h), x_h)$ is an eigenpair of $f(P)$. □

Let $\{P^t\}$ be a sequence of matrices and let $G \in \mathbb{R}^{m^2}$. We say that P is convergent to G, if $\lim_{t\to\infty} P^t = G$. We say that P is Cesàro summable to G, if $\lim_{t\to\infty} \frac{I+P+P^2+\cdots+P^{t-1}}{t} = G$.

Proposition 2.3.7 ([9], page 630)**.** *Let $P \in \mathbb{R}^{m^2}$. Then P is convergent to 0 if and only if $\rho(P) < 1$. Furthermore, if $\rho(P) = 1$, then P is convergent to G, where G is the projector onto $N(I-P)$ along $R(I-P)$ if and only if $\rho(P)$ is a semisimple eigenvalue of P and $\rho(P)$ is the only eigenvalue in the unit circle.*

Proof. If $J = Q^{-1}PQ$ is the Jordan form for P, then from (2.19) and Theorem 2.2.5, for $t \in I[0, +\infty)$ we have

$$P^t = QJ^tQ^{-1} = Q \begin{pmatrix} \ddots & & \\ & B_i(\lambda_j)^t & \\ & & \ddots \end{pmatrix} Q^{-1}, \tag{2.24}$$

where $B_i(\lambda_j) = \begin{pmatrix} \lambda_j & 1 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & & 1 \\ 0 & \cdots & 0 & \lambda_j \end{pmatrix}$, $1 \leq j \leq s$ and $1 \leq i \leq k$.

Now we proceed as follows. From (2.24), we have

$$\lim_{t\to\infty} P^t = 0 \text{ if and only if } \lim_{t\to\infty} B_i(\lambda_j)^t = 0 \text{ for each } \lambda_j \in \sigma(P). \tag{2.25}$$

To prove $\lim_{t\to\infty} P^t = 0$, it is sufficient to show that $\lim_{t\to\infty} B_i(\lambda_j)^t = 0$ for each $\lambda_j \in \sigma(P)$.

Let $f(z) = z^t$. Then, by (2.17), we have

$$f(B_i(\lambda_j)) = B_i(\lambda_j)^t = \begin{pmatrix} \lambda_j^t & t\lambda_j^{t-1} & \frac{t(t-1)}{2!}\lambda_j^{t-2} & \cdots & \frac{t(t-1)\dots(t-r_i+1)}{(r_i-1)!}\lambda_j^{t-r_i+1} \\ 0 & \lambda_j^t & t\lambda_j^{t-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \frac{t(t-1)}{2!}\lambda_j^{t-2} \\ & & & \lambda_j^t & t\lambda_j^{t-1} \\ 0 & \cdots & & 0 & \lambda_j^t \end{pmatrix}. \tag{2.26}$$

If $B_i(\lambda_j)^t \to 0$ as $t \to \infty$ for each $\lambda_j \in \sigma(P)$, then from the diagonal entries we have that $\lambda_j^t \to 0$ as $t \to \infty$, so that $|\lambda_j| < 1$. Thus $\rho(P) < 1$.

On the other hand, if $\rho(P) < 1$, then for each $\lambda_j \in \sigma(P)$ we have $|\lambda_j| < 1$. This implies that for each $n = 1, \dots, r_i - 1$ we have

$$\lim_{t\to\infty} \left| \begin{pmatrix} t \\ n \end{pmatrix} \lambda_j^{t-n} \right| = \lim_{t\to\infty} \left| \frac{t(t-1)\cdots(t-n)}{n!}\lambda_j^{t-n} \right| \le \lim_{t\to\infty} \frac{t^n}{n!}|\lambda_j|^{t-n} \le \frac{|\lambda|^{-n}}{n!} \lim_{t\to\infty} t^n|\lambda_j|^t \tag{2.27}$$

We have $t^n|\lambda_j|^t = \dfrac{t^n}{\left(\frac{1}{|\lambda_j|}\right)^t} = \left(\dfrac{t}{c^t}\right)^n$ where $c = \left(\frac{1}{|\lambda_j|}\right)^{\frac{1}{n}}$. Since $|\lambda_j| < 1$, $c > 1$. Thus $c - 1 > 0$. Moreover

$$\begin{aligned} c^t &= (1 + c - 1)^t \\ &= 1 + t(c-1) + \frac{t(t-1)}{2}(c-1)^2 + \dots \\ &> \frac{t(t-1)}{2}(c-1)^2. \end{aligned}$$

Hence

$$\lim_{t\to\infty} t^n|\lambda_j| = \lim_{t\to\infty} \left(\frac{t}{c^t}\right)^n < \lim_{t\to\infty} \left(\frac{t}{\frac{t(t-1)}{2}(c-1)^2}\right)^n = 0.$$

Thus

$$\lim_{t\to\infty}\left|\binom{t}{n}\lambda_j^{t-n}\right| \leq \frac{|\lambda|^{-n}}{n!}\lim_{t\to\infty} t^n|\lambda_j|^t = 0 \tag{2.28}$$

for each $n = 1, 2, \ldots, r_i - 1$. Thus $B_i(\lambda_j) \to 0$ for each $\lambda_j \in \sigma(P)$. Hence $\lim_{t\to\infty} P^t = 0$.

Now suppose that $\lambda_j \in \sigma(P)$, $\rho(P) = |\lambda_j| = 1$ and that the limit of P^t as $t \to 0$ exists. If $\lambda_j \in \sigma(P)$ is not a semisimple, then from Corollary 2.3.2, there exists an $r_i \times r_i (r_i > 1)$ Jordan block $B_i(\lambda_j)$ of the form

$$B_i(\lambda_j) = B_i(\lambda_j) = \begin{pmatrix} \lambda_j & 1 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & & 1 \\ 0 & \cdots & 0 & \lambda_j \end{pmatrix}.$$

Then $B_i(\lambda_j))^t$ is given by (2.26). Moreover (2.27) becomes

$$\lim_{t\to\infty}\left|\binom{t}{n}\lambda_j^{t-n}\right| = \lim_{t\to\infty}\left|\frac{t(t-1)\cdots(t-n)}{n!}\lambda_j^{t-n}\right| \leq \lim_{t\to\infty}\frac{t^n}{n!}|\lambda_j|^{t-n} \leq \frac{|\lambda|^{-n}}{n!}\lim_{t\to\infty} t^n|\lambda_j|^t \leq \lim_{t\to\infty}\frac{1}{n!}t^n = \infty$$

because $|\lambda_j| = 1$. This implies that $\lim_{t\to\infty} B_i(\lambda_j)^t$ does not exist and hence $\lim_{t\to\infty} P^t$ does not exist either, which contradicts the assumption that $\lim_{t\to\infty} P^t$ exists. Therefore if $\rho(P) = 1$ and if $\lim_{t\to\infty} P^t$ exists, then $\rho(P)$ must be a semisimple eigenvalue. On the other hand, if there exists another eigenvalue of P different than $\rho(P) = 1$ on the unit circle; that is, there is $\lambda_j \in \sigma(P)$, where $\lambda_j \neq \rho(P)$ with $|\lambda_j| = 1$, then λ_j can be written as $\lambda_j = e^{i\theta},\ \theta \in (0, 2\pi)$, and the diagonal terms of $B_i(\lambda_j)^t$ are $\lambda_j^t = e^{it\theta}$. They oscillate as t changes which prevents $J(\lambda)^t$, and thus P^t, from having a limit.

Now suppose that $\rho(P) = 1$ and that it is a semisimple eigenvalue and $\rho(P)$ is the only eigenvalue on the unit circle. Let $\lambda_1 = \rho(P)$. Then, from the assumption λ_1, is the only eigenvalue on the unit circle; that is, $1 = \lambda_1 > |\lambda_2| \geq \cdots \geq |\lambda_s|$. From the spectral resolution formula of $f(P)$, (2.23), we have

$$\begin{aligned}\lim_{t\to\infty} f(P) = \lim_{t\to\infty} P^t = \lim_{t\to\infty}\sum_{j=1}^{s}\sum_{i=0}^{r_j-1}\binom{t}{i}\lambda_j^{t-i}(P-\lambda_j I)^i G_j \\ = G_1 + \lim_{t\to\infty}\sum_{j=2}^{s}\sum_{i=0}^{r_j-1}\binom{t}{i}\lambda_j^{t-i}(P-\lambda_j I)^i G_i \\ = G_1 + 0 = G_1\end{aligned}$$

because (2.28) implies that $\lim_{t\to\infty} \binom{t}{i} \lambda_j^{t-i} = 0$ for $j \geq 2$ and $i = 1, \ldots, r_j - 1$. Since λ_1 is a semisimple eigenvalue then, from Corollary 2.3.2, we have $\text{index}(\lambda_1) = 1$, thus from (2.23), the limit G_1 is the projector onto $N(I-P)$ along $R(I-P)$. □

Lemma 2.3.8 ([10], page 162)**.** *If P converges to G, then P is Cesàro summable to G.*

Proof. Let $\lim_{t\to\infty} P^t = G$ and define $S_t = \frac{\sum_{r=0}^{t-1} P^r}{t}$ for $t = 1, 2, \ldots$ where P^0 denotes the identity matrix. Then, for any $\epsilon > 0$, there exists $N = N(\epsilon)$ such that $\|P^t - G\| < \frac{\epsilon}{2}$ for any $\|\cdot\|$ and for all $t \geq N$. Furthermore, there exists a real number β such that $\|P^t - G\| < \beta$ for all t. Then, for any $t \geq N$, we have

$$\begin{aligned}\|S_t - G\| &= \|\frac{I + P + P^2 + \cdots + P^{t-1}}{t} - G\| \\ &\leq \frac{1}{t}\sum_{i=1}^{N} \|P^i - G\| + \frac{1}{t}\sum_{i=N+1}^{t} \|P^i - G\| \\ &\leq \frac{N\beta}{t} + \frac{t-N}{t}\frac{\epsilon}{2}.\end{aligned}$$

When t is sufficiently large, $\frac{N\beta}{t} \leq \frac{\epsilon}{2}$ so that $\|S_t - G\| < \epsilon$ and therefore $\lim_{t\to\infty} S_t = G$. □

Theorem 2.3.9 ([9], page 633)**.** *Let $P \in \mathbb{R}^{m^2}$. Then P is Cesàro summable to 0 if and only if $\rho(P) < 1$. Furthermore, P is Cesàro summable to G, where G is the projector onto $N(I-P)$ along $R(I-P)$, if and only if $\rho(P) = 1$ with each eigenvalue on the unit circle being semisimple.*

Proof. Let $J = Q^{-1}PQ$ be the Jordan form of P defined in (2.16). We have

$$\frac{I + P + \cdots + P^{t-1}}{t} = Q\left(\frac{I + J + \cdots + J^{t-1}}{t}\right)Q^{-1}.$$

Thus P is Cesàro summable if and only if J is Cesàro summable. This is equivalent to saying that each Jordan block $B_i(\lambda_j)$ in J is Cesàro summable. Consequently, P cannot be Cesàro summable if $\rho(P) > 1$, because if $B_i(\lambda_j)$ is a Jordan block in which $1 < |\lambda_j| \in \sigma(P)$ then, from (2.26), each diagonal entry of $\frac{I + B_i(\lambda_j) + \cdots + B_i(\lambda_j)^{t-1}}{t}$ is

$$\delta(\lambda_j, t) = \frac{1 + \lambda_j + \cdots + \lambda_j^{t-1}}{t} = \frac{1}{t}\left(\frac{1 - \lambda_j^t}{1 - \lambda_j}\right) = \frac{1}{1 - \lambda_j}\left(\frac{1}{t} - \frac{\lambda_j^t}{t}\right). \tag{2.29}$$

If $|\lambda_j| = 1 + x$ with $x > 0$, then

$$(1+x)^t \geq 1 + \binom{t}{1}x + \binom{t}{2}x^2 > \frac{t(t-1)}{2}x^2\,.$$

Thus

$$\left|\frac{\lambda_j^t}{t}\right| \geq \frac{t(t-1)(|\lambda_j|-1)^2}{2}\frac{1}{t} = \frac{(|\lambda_j|-1)^2}{2}(t-1).$$

Hence $\delta(\lambda_j, t)$ becomes unbounded as $t \to \infty$. In other words, it's necessary that $\rho(P) \leq 1$ for P to be Cesàro summable. From Proposition 2.3.7 and Lemma 2.3.8, we already know that P is convergent and hence Cesàro summable to 0 when $\rho(P) < 1$, therefore we only need to consider the case when P has eigenvalues on the unit circle.

If $\lambda_j \in \sigma(P)$ such that $|\lambda_j| = 1, \lambda_j \neq 1$, and if λ_j is not semisimple then, from Corollary 2.3.2, there is an associated Jordan block $B_i(\lambda_j) = \begin{pmatrix} \lambda_j & 1 & \\ & \ddots & \ddots \\ & & \lambda_j \end{pmatrix}$ that is larger than 1×1. From (2.26), each entry on the first superdiagonal of $\frac{I+B_i(\lambda_j)+B_i(\lambda_j)^2+\cdots+B_i(\lambda_j)^{t-1}}{t}$ is the derivative $\frac{\partial \delta}{\partial \lambda_j}$ of the expression (2.29) which oscillates indefinitely as $t \to \infty$. In other words, P cannot be Cesàro summable if there are eigenvalues $\lambda_j \neq 1$ on the unit circle such that λ_j is not semisimple. Similarly, if $\lambda_j = 1$ is not semisimple, then P cannot be Cesàro summable because each entry on the first superdiagonal of $\frac{I+B_i(\lambda_j)+\cdots+B_i(\lambda_j)^{t-1}}{t}$ is

$$\frac{1+2+\cdots+(t-1)}{t} = \frac{t(t-1)}{2t} = \frac{t-1}{2} \to \infty \text{ as } t \to \infty.$$

Therefore, if P is Cesàro summable and has eigenvalues λ_j such that $|\lambda_j| = 1$, then λ_j must be semisimple. On the other hand, if $\rho(P) = 1$ and each eigenvalue on the unit circle is semisimple, then P is Cesàro summable. This follows because, from Proposition 2.3.7, each Jordan block associated with an eigenvalue $\lambda_j \in \sigma(P)$ such that $|\lambda_j| < 1$ is convergent and hence, by Lemma 2.3.8, Cesàro summable to 0. For semisimple eigenvalues λ_j such that $|\lambda_j| = 1$, the associated Jordan blocks are 1×1 and hence Cesàro summable because (2.29) implies

$$\frac{1+\lambda_j+\cdots+\lambda_j^{t-1}}{t} = \begin{cases} \frac{1}{1-\lambda_j}\left(\frac{1}{t} - \frac{\lambda_j^t}{t}\right) \to 0 & \text{for } |\lambda_j| = 1, \lambda_j \neq 1, \\ 1 & \text{for } \lambda_j = 1. \end{cases}$$

Thus we have established that

$$\lim_{t\to\infty} \frac{I+B_i(\lambda_j)+\cdots+B_i(\lambda_j)^{t-1}}{t} = \begin{cases} [1]_{1\times1} & \text{if } \lambda_j = 1 \text{ and } \lambda_j \text{ is semisimple,} \\ [0]_{1\times1} & \text{if } |\lambda_j| = 1, \lambda_j \neq 1, \text{ and } \lambda_j \text{ is semisimple,} \\ 0 & \text{if } |\lambda_j| < 1. \end{cases}$$

Consequently, if P is Cesàro summable, then the Jordan form for P is $J = Q^{-1}PQ = \begin{pmatrix} I & 0 \\ 0 & C \end{pmatrix}$, where I is an $m_j \times m_j$ identity matrix, and the eigenvalues of C are such that $|\lambda_j| < 1$ or else $|\lambda_j| = 1, \lambda_j \neq 1$, where λ_j is semisimple. So C is Cesàro summable to 0, J is Cesàro summable to $\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ and

$$\begin{aligned} \lim_{t\to\infty} \left(\frac{I + P + \cdots + P^{t-1}}{t} \right) &= Q \lim_{t\to\infty} \left(\frac{I + J + \cdots + J^{t-1}}{t} \right) Q^{-1} \\ &= \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = Q_1 \Psi_1 = G. \end{aligned}$$

Since λ_j is a semisimple eigenvalue, then from Corollary 2.3.2 we have $\text{index}(\lambda_j) = 1$, thus from (2.23), the limit G is the projector onto $N(I - P)$ along $R(I - P)$. □

Chapter 3

Perron-Frobenius type theorems

The Perron-Frobenius type theorems describe the properties of the spectral radius and its corresponding eigenvectors. This result was first proved by Perron(1907) in the case of positive matrices. Frobenius(1912) then filled in all the details to identify the nature of all exceptions.

3.1 Non-negative matrices

Lemma 3.1.1 ([2], page 98). *Let* $P = [p_{ij}]_{1 \leqslant i,j \leqslant m}$ *and* $x = [x_i]_{1 \leqslant i \leqslant m}$. *The system*

$$x(t+1) = Px(t), \tag{3.1}$$

$$x(0) = x^0, \tag{3.2}$$

where $x^0 \geq 0$, *has a non-negative solution* $x(t) \geqslant 0$ *for* $t = [1, +\infty)$, *if and only if* P *is non-negative,* $P \geqslant 0$.

Proof. We have

$$x_i(t) = \sum_{j=1}^{m} p_{ij} x_j(t-1)$$

for $i = 1, 2, \ldots, m$. If $P \geq 0$, we want to show that given an initial condition $x^0 \geqslant 0$, we have $x(t) \geqslant 0$ for $t \in [1, +\infty)$. At $t = 1$, we have

$$x_i(1) = \sum_{j=1}^{m} p_{ij} x_j^0 \geqslant 0$$

for $i = 1, 2, ..., m$. Assuming that at $t = k$ that we have

$$x_i(k) = \sum_{j=1}^{m} p_{ij} x_j(k-1) \geqslant 0$$

for $i = 1, 2, ..., m$. Then at $t = k+1$, we have

$$x_i(k+1) = \sum_{j=1}^{m} p_{ij} x_j(k) \geqslant 0$$

for $i = 1, 2, ..., m$. This is true for all $t \geq 1$. Thus, by induction, the first part of the condition is proved. On the other hand, if $x(t) \geqslant 0$ for each $t = 1, 2, ...,$ then we show that $P \geqslant 0$. Let us assume the opposite, then there exists at least one element $p_{ij} < 0$ of $(0 \leqslant i, j \leqslant m)$. If we consider an initial condition $x^0 = e_j = \begin{pmatrix} 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \end{pmatrix}^T$ where 1 is the jth element of $x^0, 1 \leq j \leq m$. We have $x(1) = Px^0 = \begin{pmatrix} p_{1j} & p_{2j} & \cdots & p_{ij} & \cdots & p_{mj} \end{pmatrix}^T$. Then $x(1)$ is not non-negative because $p_{ij} < 0$ for some $1 \leq i \leq m$, but this contradicts the assumption that $x(t)$ is non-negative, $x(t) \geqslant 0$, for all $t = 0, 1, 2, ...$ Thus P must satisfy $P \geqslant 0$ and the second part of the condition is proved. □

We conclude from the above Lemma that in order to get non-negative solution $x(t)$, the matrix P must be non-negative. Therefore, from now on we only consider non-negative matrices.

Lemma 3.1.2 ([9]). *Let $0 \leq P \in \mathbb{R}^{m^2}$. Then*

$$\rho(P)^t = \rho(P^t)$$

for $t \in I[1, +\infty)$.

Proof. From Corollary 2.3.6, for any $t \in I[1, +\infty)$, we have that, since (λ, x) is an eigenpair of P, (λ^t, x) is an eigenpair of P^t. Thus, if $r \in \sigma(P)$ with $|r| = \rho(P)$, then $r^t \in \sigma(P^t)$, hence $|r|^t = |r^t| \leq \rho(P^t)$. Thus

$$|r|^t = \rho(P)^t \leq \rho(P^t). \tag{3.3}$$

On the other hand, we prove that if $\omega \in \sigma(P^t)$ with $\rho(P^t) = |\omega|$, then there exists $\alpha \in \sigma(P)$ with $\alpha^t = \omega$.

Let $\alpha_1, \ldots, \alpha_s$ be the roots (not necessarily distinct) of the polynomial $X^t - \omega$, so that

$$X^t - \omega = (X - \alpha_1) \cdots (X - \alpha_s).$$

Substituting matrix P into this equation yields

$$P^t - \omega I = (P - \alpha_1 I) \cdots (P - \alpha_s I). \tag{3.4}$$

Since $\omega \in \sigma(P^t)$, the left-hand side is singular, so at least one of the factors on the right must be singular. Hence at least one of $\alpha_1, \ldots, \alpha_s$ must be in $\sigma(P)$. Then for some $i \in (1, \ldots, s)$ we have

$$\omega = \alpha_i^t.$$

Since $\alpha_i \in \sigma(P)$, $|\alpha_i| \leq \rho(P)$ and $|\alpha_i|^t = |\alpha_i^t| \leq \rho(P)^t$. Hence $|\omega| = |\rho(P^t)| \leq \rho(P)^t$, thus

$$\rho(P^t) \leq \rho(P)^t \tag{3.5}$$

Therefore from (3.3) and (3.5) we have

$$\rho(P)^t = \rho(P^t). \tag{3.6}$$

□

Lemma 3.1.3. *Let $\|\cdot\|$ be a matrix norm. Then*

$$|\rho(P)| \leq \|P\|$$

and

$$|\rho(P)^t| \leq \|P^t\|.$$

Proof. For any eigenpair (λ, x), we have $|\lambda|\|x\| = \|\lambda x\| = \|Px\| \leq \|P\|\|x\|$. Thus $|\lambda| \leq \|P\|$ for all $\lambda \in \sigma(P)$. Hence $|\rho(P)| \leq \|P\|$. Similarly $|\rho(P^t)| \leq \|P^t\|$. Thus, from (3.6), we have

$$|\rho(P)^t| \leq \|P^t\|. \tag{3.7}$$

□

Lemma 3.1.4.

$$\rho\left(\frac{P}{\rho(P) + \epsilon}\right) < 1 \text{ for all } \epsilon > 0. \tag{3.8}$$

Proof. First we have that

$$\lambda \in \sigma(P) \text{ if and only if } c\lambda \in \sigma(cP),$$

where c is any non-zero constant. Next, we have $\rho(P) = \max\{|\lambda| : \lambda \in \sigma(P)\}$ so $\rho(cP) = |c|\rho(P)$.

Let

$$c = \frac{1}{\rho(P) + \epsilon}.$$

Then

$$\rho(cP) = \frac{\rho(P)}{\rho(P) + \epsilon}.$$

Finally, for any $\epsilon > 0$ we have

$$\rho\left(\frac{P}{\rho(P) + \epsilon}\right) = \frac{\rho(P)}{\rho(P) + \epsilon} < 1.$$

□

Theorem 3.1.5 ([9], page 619)**.** *Let $\|\cdot\|$ be a matrix norm. Then,*

$$\rho(P) = \lim_{t\to\infty} \|P^t\|^{\frac{1}{t}}. \tag{3.9}$$

Proof. From Lemma 3.1.3 we have $\rho(P)^t = \rho(P^t) \leq \|P^t\|$ which implies $\rho(P) \leq \|P^t\|^{\frac{1}{t}}$. From Equation (3.8) and Proposition 2.3.7 we have

$$\lim_{t\to\infty} \left(\frac{P}{\rho(P) + \epsilon}\right)^t = 0$$

which implies

$$\lim_{t\to\infty} \frac{\|P^t\|}{(\rho(P) + \epsilon)^t} = 0.$$

It also follows from Proposition 2.3.7 that there is a positive integer T_ϵ such that $\frac{\|P^t\|}{(\rho(P)+\epsilon)^t} < 1$ for all $t \geq T_\epsilon$, so $\|P^t\|^{\frac{1}{t}} < \rho(P) + \epsilon$ for all $t \geq T_\epsilon$. Because this holds for each $\epsilon > 0$, it follows that $\lim_{t\to\infty} \|P^t\|^{\frac{1}{t}} = \rho(P)$. □

Corollary 3.1.6 ([9], page 619)**.** *Let $\tilde{P} \geq 0$ be a matrix such that $\tilde{P} \leq P$. Then*

$$\rho(\tilde{P}) \leq \rho(P). \tag{3.10}$$

Proof. First, $0 \leq \tilde{P} \leq P$ implies that $\tilde{P}^t \leq P^t$ for $t \in I[1, +\infty)$. This implies $\|\tilde{P}^t\|^{\frac{1}{t}} \leq \|P^t\|^{\frac{1}{t}}$ and $\lim_{t\to\infty} \|\tilde{P}^t\|^{\frac{1}{t}} \leq \lim_{t\to\infty} \|P^t\|^{\frac{1}{t}}$. From (3.9) we have

$$\rho(\tilde{P}) \leq \rho(P).$$

□

3.2 Positive matrices

In order to discuss the Perron-Frobenius theory for non-negative matrices, we first discuss Perron theory for positive matrices.

Lemma 3.2.1 ([9], page 661). *Let $0 < A \in \mathbb{R}^{m^2}$ be a matrix. Then $\rho(A) > 0$.*

Proof. We assume the contrary; that is, $\sigma(A) = \{0\}$. If $J = Q^{-1}AQ$ is the Jordan form of A, then

$$J = Q^{-1}AQ = \begin{pmatrix} \ddots & & \\ & B_i(\lambda_j) & \\ & & \ddots \end{pmatrix},$$

where $B_i(\lambda_j)$ is a Jordan block of the form

$$B_i(\lambda_j) = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Hence

$$A^m = QJ^mQ^{-1} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix}.$$

This implies that A is a nilpotent matrix satisfying $A^m = 0$ and $A^{m-i} \neq 0$ for $i = 1, 2, \ldots, m$. But $A > 0$ implies that $A^k > 0$, for any positive integer k. Thus, we have a contradiction. Hence $\sigma(A) \neq \{0\}$. Thus there exists $0 \neq \lambda \in \sigma(A)$ and hence $\rho(A) > 0$. □

From Lemma 3.2.1, for any $0 < A \in \mathbb{R}^{m^2}$ we have $\rho(A) > 0$. If

$$P = \frac{A}{\rho(A)}, \tag{3.11}$$

then $P > 0$ and $\rho(P) = 1$.

Theorem 3.2.2 ([9], page 664). *Let A be a matrix and let P be defined by (3.11). Then there exists $r \in \sigma(P)$ such that $r = \rho(P)$ and a unique, up to a constant multiplier, positive eigenvector associated with r.*

Proof. Let (λ, x) be an eigenpair of P such that $|\lambda| = 1$, then

$$|x| = |\lambda||x| = |\lambda x| = |Px| \leq |P||x| = P|x|. \tag{3.12}$$

If we let $z = P|x|$ and $y = z - |x|$, then we have $y \geq 0$. Suppose that $y \neq 0$. Then $Py > 0$, because $y \neq 0$ implies the existence of at least one element $y_j \neq 0$ for $1 \leq j \leq m$, such that if $y^T = \begin{pmatrix} 0 & \cdots & y_j & \cdots & 0 \end{pmatrix}$, then $Py = \begin{pmatrix} p_{1j}y_j & p_{2j}y_j & \cdots p_{mj}y_j \end{pmatrix}^T > 0$. Similarly, $|x| \neq 0$ implies that $z = P|x| > 0$. Hence there exists $\epsilon > 0$ such that $Py > \epsilon z$. That is, $P(z - |x|) > \epsilon z$. This implies $Pz > P|x| + \epsilon z = z + \epsilon z$. Thus

$$\frac{P}{1+\epsilon} z > z.$$

This can be written as $Bz > z$, where $B = \dfrac{P}{1+\epsilon}$. If we multiply both sides by B, we have $B^2 z > Bz > z$. Similarly, $B^3 z > B^2 z > Bz > z$. By induction we have that $B^t z > z$ for all $t = 1, 2, \ldots$. Since $\rho(B) = \rho(\dfrac{P}{1+\epsilon}) = \rho(\dfrac{P}{\rho(P)+\epsilon}) < 1$, then, from Lemma 2.3.7, we have $\lim_{t\to\infty} B^t = 0$. Thus, $\lim_{t\to\infty} B^t z > \lim_{t\to\infty} z$ implies that $0 > z$ which contradicts the fact that $z \geq 0$. This contradiction happened because we assumed that $y \neq 0$. Hence $0 = y = P|x| - |x|$. Thus $r = 1$ is an eigenvalue of P associated with the eigenvector $|x|$. Therefore $r \in \sigma(P)$ and

$$|x| = P|x| = z > 0. \tag{3.13}$$

We now show that if $Px = x$ and $Py = y$, then $x = \alpha y$ for some $\alpha \in \mathbb{C}$. Let $x = [x_i]_{1\leq i\leq m}$, $y = [y_i]_{1\leq i\leq m}$. Assume that $x \neq \alpha y$ for any $\alpha \in \mathbb{C}$ and set

$$\omega = x - \frac{x_k}{y_k} y \tag{3.14}$$

for some $1 \leq k \leq m$. Then

$$P\omega = Px - \frac{x_k}{y_k} Py = x - \frac{x_k}{y_k} y = \omega. \tag{3.15}$$

Equation (3.15) implies that $r = 1$ is an eigenvalue of P corresponding to ω. It follows from (3.13) that ω is an eigenvector corresponding to $r = 1$. That is,

$$\omega = P\omega > 0. \tag{3.16}$$

But (3.14) implies that

$$\omega_k = x_k - \frac{x_k}{y_k} y_k = 0$$

for some $1 \leq k \leq m$ which is a contradiction with (3.16). This contradiction occurred because we assumed that $x \neq \alpha y$ for any $\alpha \in \mathbb{C}$. Thus $x = \alpha y$ for some $\alpha \in \mathbb{C}$. $\square$

Next, we show that $r = 1$ is the only eigenvalue of P in the unit circle. To show this we first introduce the following lemma

Lemma 3.2.3 ([10], page 51)**.** *Let* $z = [z_j]_{1\le j\le m}$ *with* $0 \ne z_j \in \mathbb{C}$ *for all* $j = 1, 2, \dots, m$, *then*

$$|\sum_{j=1}^{m} z_j| = \sum_{j=1}^{m} |z_j| \tag{3.17}$$

if and only if $z_j = \alpha_j z_1$ *for some* $0 < \alpha_j \in \mathbb{R}$ *and* $j = 2, 3, \dots, m$.

To prove this lemma we need to use the case of equality in the Cauchy-Bunyakovskii-Schwarz(CBS) Inequality, which says that:

Theorem 3.2.4 ([9], page 271)**.** *For all* $x, y \in \mathbb{C}$, *we have*

$$\Re(x^*y) \le |x^*y| \le |x||y|. \tag{3.18}$$

Equality holds if and only if $y = \alpha x$ *for* $\alpha = \dfrac{\langle x^*, y\rangle}{\langle x^*, x\rangle}$.

Proof of Lemma 3.2.3. If $z_j = \alpha_j z_1$ for $\alpha_j > 0$ and $j = 2, 3, \dots, m$, then $|\sum_{j=1}^{m} z_j| = |z_1 + \alpha_2 z_1 + \cdots + \alpha_m z_1| = |(1 + \alpha_2 + \alpha_3 + \cdots + \alpha_m) z_1| = (1 + \alpha_2 + \alpha_3 + \cdots + \alpha_m)|z_1| = |z_1| + |z_2| + \cdots + |z_m| = \sum_{j}^{m} |z_j|$. Conversely, if

$$|\sum_{j=1}^{m} z_j| = \sum_{j=1}^{m} |z_j|, \tag{3.19}$$

then we need to show that $z_j = \alpha_j z_1$ for some $\alpha_j > 0$ and $j = 2, 3, \dots, m$.

From (3.19) we have

$$\left(|\sum_{j=1}^{m} z_j|\right)^2 = \left(\sum_{j=1}^{m} |z_j|\right)^2 \tag{3.20}$$

which implies that

$$\sum_{j=1}^{m} |z_j|^2 + \sum_{i=1}^{m} \sum_{i\ne j}\sum_{j=1}^{m} \Re(z_i^* z_j) = \sum_{j=1}^{m} |z_j|^2 + \sum_{i=1}^{m} \sum_{i\ne j}\sum_{j=1}^{m} |z_i||z_j|.$$

Thus

$$\sum_{i=1}^{m} \sum_{i\ne j}\sum_{j=1}^{m} \Re(z_i^* z_j) = \sum_{i=1}^{m} \sum_{i\ne j}\sum_{j=1}^{m} |z_i||z_j|.$$

Hence

$$\sum_{i=1}^{m} \sum_{i\ne j}\sum_{j=1}^{m} (|z_i||z_j| - \Re(z_i^* z_j)) = 0. \tag{3.21}$$

But from Theorem 3.2.4 we have

$$\Re(z_i^* z_j) \leq |z_i^* z_j| \leq |z_i||z_j|, \tag{3.22}$$

which implies

$$|z_i||z_j| - \Re(z_i^* z_j) \geq 0$$

for all $1 \leq i, j \leq m$ and $i \neq j$. Thus $|z_i||z_j| - \Re(z_i^* z_j) = 0$ in (3.21) and therefore (3.22) becomes

$$|z_i||z_j| = \Re(z_i^* z_j) \leq |z_i^* z_j| \leq |z_i||z_j|. \tag{3.23}$$

Hence

$$\Re(z_i^* z_j) = |z_i^* z_j| = |z_i||z_j| \tag{3.24}$$

for all $1 \leq i, j \leq m$ and $i \neq j$ which, from Theorem 3.2.4, implies that $z_j = \alpha_j z_i$ with $\alpha_j = \dfrac{z_j^* z_i}{|z_i|^2}$. If we choose $i = 1$, then

$$z_j = \alpha_j z_1 \tag{3.25}$$

for $j = 2, \ldots, m$. Thus, (3.24) becomes

$$\Re(\alpha_j z_1^* z_1) = |z_1|^2 \Re(\alpha_j) = |\alpha_j z_1^* z_1| = |z_1|^2 \sqrt{\Re(\alpha_j)^2 + \Im(\alpha_j)^2} = |z_1|^2 |\alpha_j|.$$

Hence $\Re(\alpha_j) = |\alpha_j|$, thus $\Im(\alpha_j) = 0$ and therefore

$$\alpha_j = \Re(\alpha_j) = |\alpha_j| \geq 0. \tag{3.26}$$

Since from (3.25) we have $0 \neq z_j = \alpha_j z_1$ for $j = 2, 3, \ldots, m$, it follows that $\alpha_j \neq 0$ for $j = 2, 3, \ldots, m$ and therefore $\alpha_j > 0$ for $j = 2, 3, \ldots, m$. □

Lemma 3.2.5 ([9], page 664)**.** *The spectral radius r is the only eigenvalue of P in the unit circle.*

Proof. From Theorem 3.2.2, if (λ, x) is an eigenpair of P such that $|\lambda| = 1$, then $0 < |x| = P|x|$ and $0 < |x_k| = (|x|)_k = |\sum_{j=1}^{m} p_{kj} x_j|$ for $1 \leq k \leq m$. From Lemma 3.2.3 we have

$$|\sum_{j=1}^{m} p_{kj} x_j| = \sum_{j=1}^{m} p_{kj} |x_j| = \sum_{j=1}^{m} |p_{kj} x_j|. \tag{3.27}$$

if and only if $p_{kj} x_j = \alpha_j (p_{k1} x_1)$ for some $\alpha_j > 0$, where $j = 1, 2, \ldots, m$.

Hence

$$x_j = \pi_j x_1 \text{ with } \pi_j = \frac{\alpha_j p_{k1}}{p_{kj}} > 0.$$

If $|\lambda| = 1$, then $x = x_1 u$ with

$$u = \begin{pmatrix} 1 & \pi_2 & \cdots & \pi_m \end{pmatrix}^T > 0, \tag{3.28}$$

so $\lambda x = Px$ implies $\lambda u = Pu = |Pu| = |\lambda u| = |\lambda| u = u$ which implies that $\lambda = 1$. Thus $\lambda = r = 1$ is the only eigenvalue of P in the unit circle. $\square$

We have $Pu = ru = u$ and $P\dfrac{u}{\|u\|_1} = r\dfrac{u}{\|u\|_1}$. We call $v = \dfrac{u}{\|u\|_1}$ the Perron vector and $r = 1$ the Perron root. Moreover, since Theorem 3.2.2 implies that $\dim N(P - I) = 1$, the Perron vector v of P is uniquely defined.

Theorem 3.2.1 ([9], page 665)**.** *The Perron root is a semisimple eigenvalue.*

Proof. If $r = 1$ is not semisimple then, from Corollary 2.3.2, there exists an $r_i \times r_i (r_i > 1)$ Jordan block $B_i(1)$ for some $1 \le i \le k$ such that

$$B_i(1)^t = \begin{pmatrix} 1 & t & \dfrac{t(t-1)}{2!} & \cdots & \dfrac{t(t-1)\dots(t-r_i)}{(r_i-1)!} \\ 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & & \dfrac{t(t-1)}{2!} \\ & & & & t \\ 0 & & \cdots & 0 & 1 \end{pmatrix}. \tag{3.29}$$

Since

$$\lim_{t\to\infty} t = \infty$$

then, from (2.11), we have

$$\|B_i(1)\|_\infty = \max_{\|x\|_\infty = 1} \|B_i(1)x\|_\infty \to \infty \text{ as } t \to \infty,$$

and thus $\|J^t\|_\infty \to \infty$ as $t \to \infty$. Moreover, since

$$\|J^t\|_\infty = \|Q^{-1}P^tQ\|_\infty \le \|Q^{-1}\|_\infty \|P^t\|_\infty \|Q\|_\infty,$$

then

$$\|P^t\|_\infty \ge \frac{\|J^t\|_\infty}{\|Q^{-1}\|_\infty \|Q\|_\infty} \to \infty \text{ as } t \to \infty.$$

Let $P^t = [p_{ij}^{(t)}]$ and let k denote the k-th row for which $\|P^t\|_\infty = \sum_{j=1}^m p_{kj}^{(t)}$ (see (2.11)). If $v > 0$ is the Perron vector of P satisfying $v = Pv$, then such a vector is a positive vector satisfying $\|v\|_1 = 1$ and hence $1 \geq \|v\|_\infty$. Thus

$$\begin{aligned} 1 \geq \|v\|_\infty \geq v_k = \sum_{j=1}^{m} p_{kj}^{(t)} v_j &\geq \left(\sum_{j=1}^{m} p_{kj}^{(t)}\right) \left(\min_{1\leq j\leq m} v_j\right) \\ &= \|P^t\|_\infty \left(\min_{1\leq j\leq m} v_j\right) \to \infty \end{aligned}$$

as $t \to \infty$. But this is impossible, thus the assumption that $r_i > 1$ must be false, hence $r_i = 1$. Therefore $r \in \sigma(P)$ is a semisimple eigenvalue. □

We have $P > 0$ if and only if $P^T > 0$ and $\det(P) = \det(P^T)$. Hence $\rho(P) = \rho(P^T)$. Thus, from Theorem 3.2.2, in addition to the Perron eigenpair (r, v), there exists a corresponding Perron vector for P^T satisfying $P^T\omega = r\omega$. Since $\omega^T P = r\omega^T$, the vector ω is called the left Perron vector for P.

Corollary 3.2.6 ([9], page 666)**.** *There are no other non-negative eigenvectors for P other than the Perron vector v and its positive multipliers.*

Proof. If (λ, y) is an eigenpair for P such that $y \geq 0$ and if $\omega > 0$ is the Perron vector for P^T, then $\langle \omega, y\rangle > 0$ so

$$r\omega^T = \omega^T P \text{ implies that } r\omega^T y = \omega^T P y = \lambda\langle \omega, y\rangle.$$

Thus $r = \lambda$. Hence from Theorem 3.2.2, we have $y = \alpha v$ for some $\alpha > 0$. □

3.3 Further non-negative matrices

Frobenius (1912) generalized the Perron theory for positive matrices for any non-negative square matrices. In this section we discuss Frobenius theory for non-negative matrices.

Theorem 3.3.1 ([9], page 670)**.** *Let $P \geq 0$ be a matrix. Then $r \in \sigma(P)$, where $r = \rho(P)$. Moreover, there exists $0 \neq v \geq 0$ such that $Pv = rv$.*

Proof. Let $P = [p_{ij}] \geq 0$, $P_t = [p_{ij} + \frac{1}{t}] > 0$, where $1 \leq i, j \leq m$ and $t \in I[1, +\infty)$. Let r_t be the Perron root and v_t be the Perron vector of P_t, then $v_t > 0$ and $\|v_t\|_1 = 1$. By Bolzano-Weierstrass theorem, as we proved in Lemma 2.1.2, every sequence $\{v_t\}$ in a set

$$B = \{v : v \in \mathbb{R}^m \text{ and } \|v\|_1 = 1\}$$

has a subsequence converging to a point in B. Let $\{v_{t_i}\}$ be such a subsequence, then we have $\lim_{i\to\infty} v_{t_i} = v \geq 0$ because $v_{t_i} > 0$ and $\|v_{t_i}\|_1 = 1$. Since $P_1 > P_2 > \cdots \geq P$, then, from Lemma 3.1.6, we have $r_1 \geq r_2 \geq \cdots \geq r$. Hence $\{r_t\}, t = 1, 2, \ldots$ is a non-increasing sequence bounded below by r and thus $\lim_{t\to\infty} r_t$ exists. Let $\lim_{t\to\infty} r_t = \rho \geq r$. If $\{r_{t_i}\}, i = 1, 2, \ldots$ is a subsequence of $\{r_t\}$, then $\lim_{i\to\infty} r_{t_i} = \rho$, where $\rho \geq r$. Since $\lim_{t\to\infty} P_t = P$ implies $\lim_{i\to\infty} P_{t_i} = P$ then

$$Pv = \lim_{i\to\infty} P_{t_i} v_{t_i} = \lim_{i\to\infty} r_{t_i} v_{t_i} = \lim_{i\to\infty} r_{t_i} \lim_{i\to\infty} v_{t_i} = \rho v.$$

This implies that $\rho \in \sigma(P)$. Thus $\rho \leq r$. Therefore $r = \rho \in \sigma(P)$. Moreover, $v \geq 0$ is an eigenvector corresponding to r. □

3.4 Irreducible matrices

Irreducible matrices are a especial class of non-negative matrices. We define irreducible matrices using the concept of permutation matrices. A matrix $\mathcal{I}$ is said to be a permutation matrix, if one entry in each row and column of $\mathcal{I}$ is 1 and all other entries are 0 (see [12]). Let $P = [p_{ij}]_{1\leq i,j\leq m}$. A matrix P is said to be reducible if there exists a permutation matrix $\mathcal{I}$ such that

$$\mathcal{I}^T P \mathcal{I} = \begin{pmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{pmatrix},$$

where the entries P_{11} and P_{22} are square matrices. If the above is not satisfied, then the matrix P is said to be irreducible matrix.

Irreducibility of P can be checked from the connectivity of the graph of P. To see this, let us first introduce these concepts. A graph is a set of points, or nodes, $\{N_1, N_2, \ldots, N_m\}$ together with a set of edges $\{E_1, E_2, \ldots, E_m\}$ between the nodes. If there is a sequence of edges linking between any pair of nodes, then the graph is called connected graph, and if there is a direction assigned to each edge, then the graph is called a directed graph. A graph of P, denoted by $G(P)$, is defined to be a directed graph of m nodes $\{N_1, N_2, \ldots, N_m\}$ in which there is a directed edge leading from N_i to N_j if and only if $p_{ij} \neq 0$.

Definition 3.4.1. The graph $G(P)$ is called strongly connected if for each pair of nodes (N_i, N_k) there is a sequence of directed edges leading from N_i to N_j.

Theorem 3.4.2 ([10], page 36)**.** *A matrix P is irreducible if and only if the graph of P is strongly connected.*

Lemma 3.4.3 ([5], page 51). *Let $P \geq 0$ be an irreducible matrix. Then*

$$(I+P)^{m-1} > 0. \tag{3.30}$$

Proof. Let $P = [p_{ij}]_{1\leq i,j\leq m}, y = [y_i]_{1\leq i\leq m}$ where $0 \neq y \geq 0$ and let $z = (I+P)y$. We claim that z must have strictly smaller number of zero coordinates than y. Let us assume the opposite, that is, z has greater or the same number of zero coordinates as y. Since $y_i \neq 0$ implies that $z_i = \left(y_i + \sum_{j=1}^{m} p_{ij}y_j\right) \neq 0$ for $i = 1, 2, \ldots, m$, then it is impossible for z to have greater number of zero coordinates than y. Moreover, the zero entries in y occur at the same places as in z. Thus we assume that y and z have the same zero coordinates. We have $z = (I+P)y$. Then there exists a permutation matrix $\mathcal{I}$ such that $\tilde{y} = \mathcal{I}y = \begin{pmatrix} u \\ 0 \end{pmatrix}$ with $u > 0$ and $\tilde{z} = \mathcal{I}z = \begin{pmatrix} v \\ 0 \end{pmatrix}$ with $v > 0$. Hence $\mathcal{I}z = \mathcal{I}(I+P)y$, which implies that $\mathcal{I}z = \mathcal{I}y + \mathcal{I}P\mathcal{I}^{-1}\mathcal{I}y$. Thus

$$\tilde{z} = \tilde{y} + \mathcal{I}P\mathcal{I}^{-1}\tilde{y}. \tag{3.31}$$

We take $P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$, $\mathcal{I} = \begin{pmatrix} \mathcal{I}_{11} & \mathcal{I}_{12} \\ \mathcal{I}_{21} & \mathcal{I}_{22} \end{pmatrix}$ and $\mathcal{I}^{-1} = \mathcal{Q} = \begin{pmatrix} \mathcal{Q}_{11} & \mathcal{Q}_{12} \\ \mathcal{Q}_{21} & \mathcal{Q}_{22} \end{pmatrix}$. Then

$$\mathcal{I}Q = \begin{pmatrix} \mathcal{I}_{11} & \mathcal{I}_{12} \\ \mathcal{I}_{21} & \mathcal{I}_{22} \end{pmatrix} \begin{pmatrix} \mathcal{Q}_{11} & \mathcal{Q}_{12} \\ \mathcal{Q}_{21} & \mathcal{Q}_{22} \end{pmatrix} = \begin{pmatrix} \mathcal{I}_{11}\mathcal{Q}_{11} & 0 \\ 0 & \mathcal{I}_{22}\mathcal{Q}_{22} \end{pmatrix} = I.$$

This implies that $\mathcal{I}_{11} \neq 0, \mathcal{I}_{22} \neq 0, \mathcal{Q}_{11} \neq 0$ and $\mathcal{Q}_{22} \neq 0$. Equation (3.31) implies that

$$\begin{pmatrix} v \\ 0 \end{pmatrix} = \begin{pmatrix} u \\ 0 \end{pmatrix} + \begin{pmatrix} (\mathcal{I}_{11}P_{11} + \mathcal{I}_{12}P_{21})\mathcal{Q}_{11}u + (\mathcal{I}_{11}P_{12} + \mathcal{I}_{12}P_{22})\mathcal{Q}_{21}u \\ (\mathcal{I}_{21}P_{11} + \mathcal{I}_{22}P_{21})\mathcal{Q}_{11}u + (\mathcal{I}_{21}P_{21} + \mathcal{I}_{22}P_{22})\mathcal{Q}_{21}u \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}$$

where $A = (\mathcal{I}_{11}P_{11} + \mathcal{I}_{12}P_{21})\mathcal{Q}_{11}u + (\mathcal{I}_{11}P_{12} + \mathcal{I}_{12}P_{22})\mathcal{Q}_{21}u + u$ and $B = (\mathcal{I}_{21}P_{11} + \mathcal{I}_{22}P_{21})\mathcal{Q}_{11}u + (\mathcal{I}_{21}P_{21} + \mathcal{I}_{22}P_{22})\mathcal{Q}_{21}u$. Hence $A = v$ and $B = 0$. But B is sum of non-negative values thus, in particular, $\mathcal{I}_{22}P_{21}\mathcal{Q}_{11}u = 0$. Since $u > 0, \mathcal{I}_{22} \neq 0$ and $Q_{11} \neq 0$, $P_{21} = 0$ which is a contradiction with P being irreducible. Therefore z cannot have the same zero coordinates as y. Hence z must have strictly smaller number of zero coordinates than y. Thus by recursion we have $(I+P)^{m-1}y > 0$. Since $0 \neq y \geq 0$ is an arbitrary, then $(I+P)^{m-1} > 0$. □

Theorem 3.4.4 ([9], page 673). *Let $P \geq 0$ be an irreducible matrix. Then there exists a positive simple eigenvalue $r \in \sigma(P)$, satisfying $r = \rho(P)$, with unique (up to a constant multiplier) corresponding positive left and positive right eigenvectors.*

Proof. From Theorem 3.3.1, we have $r = \rho(P) \in \sigma(P)$. Let $f(P) = (I + P)^{m-1}$. Then, from Lemma 3.4.3, we have $f(P) > 0$. From Corollary 2.3.3, $r \in \sigma(P)$ implies that $f(r) \in \sigma(f(P))$. Consequently, if $\mu = \rho(f(P))$, then

$$\mu = \max_{\lambda \in \sigma(P)} |1 + \lambda|^{m-1} = \left(\max_{\lambda \in \sigma(P)} |1 + \lambda| \right)^{m-1}. \tag{3.32}$$

To find

$$\max_{\lambda \in \sigma(P)} |1 + \lambda|$$

where

$$|\lambda| \leq r$$

for any $\lambda \in \sigma(P)$, we proceed as follows: Let $C = \{\lambda + 1 : \lambda \in \sigma(P) \text{ and } |\lambda + 1 - 1| \leq r\}$. Note that if $(\lambda + 1) \in C$, then $|\lambda + 1| - 1 \leq |\lambda + 1 - 1| \leq r$ which gives $|\lambda + 1| \leq r + 1$ for all $\lambda \in \sigma(P)$. Moreover $\lambda + 1 = r + 1 \in C$, thus

$$\max_{\lambda \in \sigma(P)} |1 + \lambda| = r + 1.$$

Hence

$$\mu = (1 + r)^{m-1}.$$

From Corollary 2.3.4, it follows that $r \in \sigma(P)$ is a simple eigenvalue, otherwise $\mu = \rho(f(P))$ is not a simple eigenvalue which is impossible because $f(P) > 0$. To see that P has a positive eigenvector associated with r, we recall from Theorem 3.3.1 that there exists a non-negative eigenvector $x \geq 0$ associated with r. From Corollary 2.3.6, (r, x) being an eigenpair of P implies that (μ, x) is an eigenpair of $f(P)$. Corollary 3.2.6 ensures that x must be a positive multiple of Perron vector of $f(P)$ and thus x must be in fact positive. Now $r > 0$, otherwise $Px = rx = 0$, which is impossible because $P \geq 0$ and $x > 0$ forces $Px > 0$. Moreover, we have $P \geq 0$ if and only if $P^T \geq 0$; then the existence of a positive simple eigenvalue and a unique positive eigenvector for P^T follows as for P. □

It follows from Theorem 3.4.4 that $\dim(N(P - I)) = 1$. The uniquely defined Perron vectors for P and P^T are, respectively, given by $v = \frac{x}{\|x\|_1}$ and $\omega = \frac{y}{\|y\|_1}$.

Corollary 3.4.5 ([9], page 674)**.** *Let $P \geq 0$ be an irreducible matrix. Then there are no other non-negative eigenvectors for P other than the Perron vector v and its positive multipliers.*

Proof. If (λ, y) is an eigenpair for P such that $y \geq 0$ and if $\omega > 0$ is the Perron vector for P^T, then $\omega^T y > 0$ so

$$r\omega^T = \omega^T P \text{ implies that } r\omega^T y = \omega^T P y = \lambda \omega^T y.$$

Thus $r = \lambda$. Hence, from Theorem 3.4.4, we have $y = \alpha v$ for some $\alpha > 0$. □

Lemma 3.4.6 ([9], page 674)**.** *Let $P \geq 0$ be irreducible matrix, $r = \rho(P)$ and assume that $rz \leq Pz$ for $z \geq 0$. Then $rz = Pz$ and $z > 0$.*

Proof. If $rz \leq Pz$ then, by using the Perron vector $\omega > 0$ for P^T, $(P - rI)z \geq 0$ implies that $\omega^T(P - rI)z > 0$, which is impossible because $\omega^T(P - rI) = 0$. Thus $rz = Pz$. From Corollary 3.4.5, z must be a multiple of the Perron vector for P. Thus $z > 0$. □

If $P \geq 0$ is irreducible matrix having only one eigenvalue $r = \rho(P)$ on the unit circle, then P is said to be primitive. If $P \geq 0$ is an irreducible matrix having $h > 1$ eigenvalues on the unit circle, then P is imprimitive.

Theorem 3.4.7 (Frobenius's Test for Primitivity, [9] page 678)**.** *A matrix $P \geq 0$ is primitive if and only if $P^t > 0$ for some $t > 0$.*

Theorem 3.4.8 ([9], page 674)**.** *Let $P \geq 0$ be irreducible and let $r = \rho(P)$. Then P is primitive if and only if $\lim_{k\to\infty} \left(\frac{P}{r}\right)^k$ exists, in which case*

$$\lim_{t\to\infty} \left(\frac{P}{r}\right)^t = G = \frac{v\omega^T}{\langle \omega, v \rangle}, \tag{3.33}$$

where G is the spectral projector onto $N(P - rI)$ along $R(P - rI)$, v and ω are, respectively, the Perron vectors of P and P^T.

Proof. Theorem 3.4.4 ensures that $1 = \rho\left(\frac{P}{r}\right)$ is a simple eigenvalue of $\frac{P}{r}$. We have that P is primitive if and only if $\frac{P}{r}$ is primitive. In other words, P is primitive if and only $1 = \rho\left(\frac{P}{r}\right)$ is the only eigenvalue on the unit circle which is equivalent, by Proposition 2.3.7, to saying that $\lim_{t\to\infty} \left(\frac{P}{r}\right)^t = G$, where G is the spectral projector onto $N(P - rI)$ along $R(P - rI)$. It remains to show that

$$G = \frac{v\omega^T}{\langle \omega, v \rangle},$$

where v and ω are the respective Perron vectors for P and P^T. First we have $\langle \omega, v\rangle > 0$ because v, the Perron vector for P, and ω, the Perron vector for P^T, are positive. Moreover

$$G^2 = \frac{v\langle \omega, v\rangle \omega^T}{\langle \omega, v\rangle\langle \omega, v\rangle} = \frac{v\omega^T}{\langle \omega, v\rangle} = G.$$

Thus G is a projector (Definition 2.3.5) . Since

$$(I-G)^2 = I - 2G + G^2 = I - 2G + G = I - G,$$

hence $I-G$ is also a projector. To determine $R(G)$, we observe that for any $z \in \mathbb{R}^m$, we have $Gz = \alpha v$ where $\alpha = \frac{\langle \omega, z\rangle}{\langle \omega, v\rangle}$. Thus, if $x \in R(G)$ then $x \in \operatorname{span}(v)$, hence $R(G) = \operatorname{span}(v) = N(P-rI)$. Define

$$R(P-rI)^{\perp} = \{x \in \mathbb{R}^m : \langle x, y\rangle = 0 \text{ for any vector } y \in R(P-rI)\}$$

and

$$N(P-rI)^{\perp} = \{x \in \mathbb{R}^m : \langle x, y\rangle = 0 \text{ for any vector } y \in N(P-rI)\}.$$

If $x \in R(P-rI)^{\perp}$, then for any $y \in R(P-rI)$ we have $\langle x, y\rangle = x^T(P-rI)z = 0$ for any $0 \neq z \in \mathbb{R}^m$. This implies that $x^T(P-rI) = 0$. Since $\dim N(P-rI)^T = 1$, $x^T = \omega^T$. From the definition of G, we have $x^T(I-G) = 0$, hence $x^T(I-G)u = 0$ for any $u \in \mathbb{R}^m$, in particular for $(I-G)u = u$. This implies that $u \in N(G)$, hence $x \in N(G)^{\perp}$. Thus

$$R(P-rI)^{\perp} \subseteq N(G)^{\perp}.$$

Thus

$$\left(N(G)^{\perp}\right)^{\perp} \subseteq \left(R(P-rI)^{\perp}\right)^{\perp}.$$

That is,

$$N(G) \subseteq R(P-rI).$$

If we recall that Rank Plus Nullity Theorem in [9], page 199 that $\dim N(G) = m - \dim R(G) = m - 1 = m - \dim N(P-rI) = \dim R(P-rI)$. Thus $N(G) = R(P-rI)$. □

Theorem 3.4.9 ([9], page 675). *Let $\tilde{P}$ be a matrix and let P be an irreducible matrix. If $|\tilde{P}| \leq P$, then $\rho(\tilde{P}) \leq \rho(P)$. If the equality holds, that is, if $\gamma = \rho(P)e^{i\phi} \in \sigma(\tilde{P})$ for some $0 < \phi < 2\pi$, then $\tilde{P} = e^{i\phi}DPD^{-1}$ for some*

$$D = \begin{pmatrix} e^{i\theta_1} & 0 & \cdots & 0 \\ 0 & e^{i\theta_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{i\theta_m} \end{pmatrix}, \tag{3.34}$$

where each $0 < \theta_i < 2\pi$ *for* $i = 1, 2, \ldots, m$*. On the other hand, if* $\tilde{P} = e^{i\phi} DPD^{-1}$ *for some* $0 < \phi < 2\pi$ *and for some* D *given by* (3.34)*, then* $\gamma = \rho(P)e^{i\phi} \in \sigma(\tilde{P})$*.*

Proof. We already know that $\rho(\tilde{P}) \le \rho(P)$ by Lemma 3.1.6. If $\rho(\tilde{P}) = r = \rho(P)$, and if (γ, x) is an eigenpair for $\tilde{P}$ such that $|\gamma| = r$, then

$$r|x| = |\gamma||x| = |\gamma x| = |\tilde{P}x| \le |\tilde{P}||x| \le P|x|.$$

This implies that $|\tilde{P}||x| = r|x|$ because Lemma 3.4.6 ensures that $P|x| = r|x|$ and $|x| > 0$. Consequently, $(P - |\tilde{P}|)|x| = 0$. But $P - |\tilde{P}| \ge 0$ and $|x| > 0$, so $P = |\tilde{P}|$. Since $\frac{x_k}{|x_k|}$ for $1 \le k \le m$ is on the unit circle and $\frac{x_k}{|x_k|} = e^{i\theta_k}$ for some $0 < \theta_k < 2\pi$. Set $D = \begin{pmatrix} e^{i\theta_1} & 0 & \cdots & 0 \\ 0 & e^{i\theta_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{i\theta_m} \end{pmatrix}$ and notice that $x = D|x|$.
Since $|\gamma| = r$, there is a $\phi \in \mathbb{R}$ such that $\gamma = re^{i\phi}$, and hence $\tilde{P}D|x| = Px = \gamma x = re^{i\phi}x = re^{i\phi}D|x|$ implies

$$e^{-i\phi}D^{-1}\tilde{P}D|x| = r|x| = P|x|. \tag{3.35}$$

Let $C = e^{-i\phi}D^{-1}\tilde{P}D$ and note that $|C| = |\tilde{P}| = P$ to write (3.35) as $0 = (|C| - C)\,|x|$. Considering only the real part yields $0 = (|C| - \Re(C))\,|x|$. But $|C| \ge \Re(C)$, and $|x| > 0$. Thus $\Re(C) = |C|$, and hence $\Re(c_{ij}) = |c_{ij}| = \sqrt{\Re(c_{ij})^2 + \Im(c_{ij})^2}$ implies that $\Im(c_{ij}) = 0$ which implies $\Im(C) = 0$. Therefore, $C = \Re(C) = |C| = P$, which implies $\tilde{P} = e^{i\phi}DPD^{-1}$. Conversely, if $\tilde{P} = e^{i\theta}DPD^{-1}$, then Corollary 2.2.3 ensures that $\rho(\tilde{P}) = \rho(e^{i\phi}P) = \rho(P)$. □

Lemma 3.4.10. *The* $h-$*th power of every element in a finite group of order* h *is the identity element of the group.*

Proof. The order of every element of a group is a divisor of h by Lagrange's Theorem [11]. So, if x is an element of our group of order k, then $h = k \cdot l$ for some l. Hence, we have

$$x^h = (x^k)^l = (e)^l = e.$$

where e is the identity element of the group. □

Theorem 3.4.11 ([9], page 676)**.** *Let* $P \ge 0$ *be an irreducible that has* $\{\lambda_1, \lambda_2, \ldots, \lambda_h\}$ *eigenvalues on its unit circle. Then each eigenvalue* λ_i *for* $i = 1, 2, \cdots, h$*, is a simple eigenvalue. Moreover,* $\{\lambda_1, \lambda_2, \ldots, \lambda_h\}$ *is the set of roots of* $r = \rho(P)$ *given by* $\{r, r\omega, r\omega^2, \ldots, r\omega^{h-1}\}$*, where* $\omega = e^{\frac{2\pi i}{h}}$*.*

Proof. Let $S = \{r, re^{i\theta_1}, \ldots, re^{i\theta_{h-1}}\}$ denote the eigenvalues on the unit circle of P. Applying Theorem 3.4.9 with $\tilde{P} = P$ and $\gamma = re^{i\theta_k}$ ensures the existence of a diagonal matrix D_k such that $P = e^{i\theta_k} D_k P D_k^{-1}$. Let $f(P) = e^{i\theta_k} P$. By Theorem 3.4.4, r is a simple eigenvalue of P. Corollary 2.3.4 implies that $f(r) = re^{i\theta_k}$ is also a simple eigenvalue of $f(P)$.

If we consider another eigenvalue $re^{i\theta_s} \in S$, then we can write $P = e^{i\theta_s} D_s P D_s^{-1}$ for some D_s, so

$$P = e^{i\theta_k} D_k P D_k^{-1} = e^{i\theta_k} D_k e^{i\theta_s} D_s P D_s^{-1} D_k^{-1} = e^{i(\theta_k+\theta_s)} D_k D_s P D_s^{-1} D_k^{-1} = e^{i(\theta_k+\theta_s)} (D_k D_s) P (D_k D_s)^{-1}$$

and, consequently, from Corollary 2.2.3, we have that $re^{i(\theta_k+\theta_s)}$ is also an eigenvalue on the unit circle of P. This means that $G = \{1, e^{i\theta_1}, \ldots, e^{i\theta_{h-1}}\}$ is closed under multiplication, and it follows that G is a finite commutative group of order h. By Lemma 3.4.10, we have $\left(e^{i\theta_k}\right)^h = 1$ for each $k = 0, 1, \ldots, h-1$, so G is the set of the h-th roots of unity $e^{\frac{2\pi ki}{h}}$, where $k = 0, 1, \ldots, h-1$, and thus S must be the h-th roots of r. $\square$

Theorem 3.4.12 ([9], page 677)**.** *Let P be imprimitive with h eigenvalues on its unit circle, then $\sigma(P)$ is invariant under rotation about the origin through an angle $\frac{2\pi}{h}$. No rotation less than $\frac{2\pi}{h}$ can preserve $\sigma(P)$.*

Proof. Since $\lambda \in \sigma(P)$ if and only if $\lambda e^{\frac{2\pi i}{h}} \in \sigma(P)$, it follows that $\sigma(e^{\frac{2\pi i}{h}} P)$ is $\rho(P)$ rotated through $\frac{2\pi}{h}$. But Theorem 3.4.9 and Theorem 3.4.11 ensure that P and $e^{\frac{2\pi i}{h}} P$ are similar and, consequently, $\sigma(P) = \sigma(e^{\frac{2\pi i}{h}} P)$. No rotation less than $\frac{2\pi}{h}$ can keep $\rho(P)$ invariant because Theorem 3.4.11 makes it clear that the eigenvalues on the unit circle will not go back into themselves for rotation less than $\frac{2\pi}{h}$. $\square$

Corollary 3.4.13 ([9], page 677)**.** *Let $P \geq 0$ be imprimitive and let $r = \rho(P)$. Then $\frac{P}{r}$ is summable to $\frac{v\omega^T}{\langle \omega, v\rangle}$ where v and ω are the respective Perron vectors for P and P^T.*

Proof. Being imprimitive means that P is non-negative and irreducible with more than one eigenvalue on the unit circle. However, Theorem 3.4.11 says that each eigenvalue on the unit circle is simple, so Theorem 2.3.9 can be applied to $\frac{P}{r}$ to conclude that $\frac{P}{r}$ is summable to G, that is,

$$\lim_{t\to\infty} \frac{I + \left(\frac{P}{r}\right) + \cdots + \left(\frac{P}{r}\right)^{t-1}}{t} = G,$$

where G is the spectral projector onto $N(\frac{P}{r} - I)$ along $R(\frac{P}{r} - I)$. We have

$$N\left(\frac{P}{r} - I\right) = \{x \in \mathbb{R}^m : \left(\frac{P}{r} - I\right) x = 0\} = \{x \in \mathbb{R}^m : (P - rI)x = 0\} = N(P - rI)$$

and

$$
\begin{aligned}
R\left(\frac{P}{r}-I\right) &= \{y \in \mathbb{R}^m : \left(\frac{P}{r}-I\right)x = y \text{ for some } x \in \mathbb{R}^m\} \\
&= \{y \in \mathbb{R}^m : (P-rI)x = ry \text{ for some } x \in \mathbb{R}^m\} \\
&= \{y \in \mathbb{R}^m : (P-rI)z = y \text{ for some } z = \frac{x}{r} \in \mathbb{R}^m\} \\
&= R(P-rI).
\end{aligned}
$$

From Theorem 3.4.8, the spectral projector onto $N(P-rI)$ along $R(P-rI)$ is given by $G = \frac{v\omega^T}{\langle \omega, v\rangle} > 0$ where v and ω are the respective Perron vectors for P and P^T. □

Corollary 3.4.14 ([9], page 678)**.** *Let $P \geq 0$ be irreducible. If P has at least one positive diagonal element, then P is primitive.*

Proof. Suppose there are $h > 1$ eigenvalues on the unit circle. We know from Theorem 3.4.12 that if $\lambda_0 \in \sigma(P)$, then $\lambda_k = \lambda_0 e^{\frac{2\pi i k}{h}} \in \sigma(P)$ for $k = 0, 1, \ldots, h-1$, so

$$
\sum_{k=0}^{h-1} \lambda_k = \lambda_0 \sum_{k=0}^{h-1} e^{\frac{2\pi i k}{h}} = \lambda_0 \frac{1-e^{2\pi i}}{1-e^{\frac{2\pi i}{h}}} = \lambda_0 \frac{1-1}{1-e^{\frac{2\pi i}{h}}} = 0.
$$

In other words, if P is imprimitive, then $\mathrm{trace}(P) = 0$. Therefore, if P has a positive diagonal entry, then P must be primitive. □

Chapter 4

Linear autonomous models

We consider the linear autonomous model introduced in (1.3),

$$x(t+1) = Px(t),$$
$$x(0) = x^0, \tag{4.1}$$

where $0 \neq x^0 \geq 0$. We assume that $P \geq 0$ is irreducible and primitive matrix with constant entries. We have $x(1) = Px(0) = Px^0$ and $x(2) = Px(1) = PPx^0 = P^2x^0$. Assuming that at $t-1$ we have $x(t-1) = P^{t-1}x^0$, then at t we have $x(t) = Px(t-1) = PP^{t-1}x^0 = P^tx^0$. This is correct for all $t > 1$. It follows by recursion that $x(t) = P^tx^0$ for all $t = 1, 2, \ldots$. Thus (4.1) have the following solution

$$x(t) = P^tx^0, \;\; t \in I[1, +\infty)$$
$$x(0) = x^0. \tag{4.2}$$

Let $r = \rho(P)$. Then, from Theorem 3.4.8, we have

$$\lim_{t\to\infty} \frac{x(t)}{r^t} = \lim_{t\to\infty} \frac{P^tx^0}{r^t} = \lim_{t\to\infty} \frac{P^t}{r^t}x^0 = v\frac{\langle \omega, x^0\rangle}{\langle \omega, v\rangle} \tag{4.3}$$

where v and ω are the Perron vectors for P and P^T respectively.

4.1 Long time behaviour of population models

Define the distribution of the total population at time t, by

$$p(t) = \|x(t)\|_1 = \sum_{i=1}^{m} x_i(t). \tag{4.4}$$

Then, from Theorem 3.4.8, we have

$$\lim_{t\to\infty}\frac{p(t)}{r^t}=\lim_{t\to\infty}\|\frac{x(t)}{r^t}\|_1=\lim_{t\to\infty}\|\frac{P^tx^0}{r^t}\|_1=\|\lim_{t\to\infty}\frac{P^tx^0}{r^t}\|_1=\|v\frac{\langle\omega,x^0\rangle}{\langle\omega,v\rangle}\|_1=\frac{\langle\omega,x^0\rangle}{\langle\omega,v\rangle}>0, \tag{4.5}$$

because v, the Perron vector of P, satisfies $\|v\|_1=1$. Thus, from (4.3) and (4.5), we have

$$\lim_{t\to\infty}\frac{x(t)}{p(t)}=\lim_{t\to\infty}\frac{\frac{x(t)}{r^t}}{\frac{p(t)}{r^t}}=\frac{\lim_{t\to\infty}\frac{x(t)}{r^t}}{\lim_{t\to\infty}\frac{p(t)}{r^t}}=v\frac{\frac{\langle\omega,x^0\rangle}{\langle\omega,v\rangle}}{\frac{\langle\omega,x^0\rangle}{\langle\omega,v\rangle}}=v. \tag{4.6}$$

The dynamics of the total population $p(t)$ can be described by

$$p(t+1)=\|\ x(t+1)\ \|_1=\|\ Px(t)\ \|_1=\frac{\|\ Px(t)\ \|_1}{p(t)}p(t)=\left\|\frac{Px(t)}{p(t)}\right\|_1 p(t) \tag{4.7}$$

because $p(t)\geq 0$ for all $t\in I[0,+\infty)$.

Let

$$c(t)=\left\|\frac{Px(t)}{p(t)}\right\|_1. \tag{4.8}$$

Then (4.7) becomes

$$p(t+1)=c(t)p(t). \tag{4.9}$$

The recursion formula of $p(t)$ for all $t>T$ can be written as

$$p(t)=p(T)\prod_{i=T}^{t}c(i). \tag{4.10}$$

Lemma 4.1.1 ([3], page 167)**.** *Let $c(t)$ be defined by* (4.11)*. If $c(0)\neq 0$, then $c(t)\neq 0$ for all $t\in I[1,+\infty)$.*

Proof. Let $c(0)\neq 0$. If $c(t_0)=0$ for some $t_0\in I[1,+\infty)$ then, from (4.10), we have

$$p(t)=p(t_0)\prod_{i=t_0}^{t}c(i)=0$$

for all $t\geq t_0$. Thus, from (4.4), we have

$$p(t)=\|x(t)\|_1=\|P^tx(0)\|=0$$

for all $t\geq t_0$. Since $x(0)\geq 0$ and $P\geq 0$, then

$$P^tx(0)=0$$

for all $t \geq t_0$. Therefore

$$\lim_{t\to\infty} \frac{P^t x(0)}{r^t} = \lim_{t\to\infty} \frac{P^t x^0}{r^t} = Pv = 0v = 0.$$

Thus 0 is an eigenvalue corresponding to the eigenvector v. From Theorem 3.4.4, v is an eigenvector corresponding to $r > 0$ which is a contradiction. Thus $c(t) \neq 0$ for all $t \in I[1, +\infty)$. □

From (4.8) and (4.7) we have

$$\lim_{t\to\infty} c(t) = \lim_{t\to\infty} \left\| \frac{Px(t)}{p(t)} \right\|_1 = \left\| P \lim_{t\to\infty} \frac{x(t)}{p(t)} \right\|_1 = \|Pv\|_1 = \|rv\|_1 = r, \tag{4.11}$$

where $r = \rho(P)$.

Corollary 4.1.2 ([3], page 7)**.** *Let $r = \rho(P)$ and let $p(t)$ defined by* (4.10)*. Then*

(a) If $r < 1$, then $\lim_{t\to\infty} p(t) = 0$.

(b) If $r > 1$, then $\lim_{t\to\infty} p(t) = +\infty$.

Proof. (a) Let $r < 1$. From (4.11) we have $\lim_{t\to\infty} c(t) = r$. Then, given any $\epsilon > 0$, in particular $0 < \epsilon < 1 - r$ or equivalently $r < r + \epsilon < 1$, there exists $T = T(\epsilon)$ such that for all $t > T$, we have $c(t) \in (r - \epsilon, r + \epsilon)$. Since $0 < c(t) < r + \epsilon$ for all $t > T$, we have $\prod_{i=T}^{t} c(i) < (r + \epsilon)^{t-T+1}$. Then $p(T) \prod_{i=T}^{t} c(i) < (r + \epsilon)^{t-T+1} p(T)$, this, in addition to (4.10), implies $p(t) < (r + \epsilon)^{t-T+1} p(T)$. Let $p(T) < M$ for some $M \in \mathbb{R}$. Since $(r + \epsilon) < 1$, then $\lim_{t\to\infty} p(t) = 0$.

(b) Let $r > 1$. Since $\lim_{t\to\infty} c(t) = r$, given any $\epsilon > 0$, in particular $0 < \epsilon < r - 1$ or equivalently $1 < r - \epsilon$, there exists $T = T(\epsilon)$ such that for all $t > T$ we have $c(t) \in (r - \epsilon, r + \epsilon)$. Thus for all $t > T$ we have $1 < r - \epsilon < c(t)$ which implies $(r - \epsilon)^{t-T+1} < \prod_{i=T}^{t} c(i)$. Hence

$$p(T)\,(r - \epsilon)^{t-T+1} < p(T) \prod_{i=T}^{t} c(i). \tag{4.12}$$

From (4.10) we have $p(t) = p(T) \prod_{i=T}^{t} c(i)$. Hence, (4.12) becomes

$$p(t) > p(T)\,(r - \epsilon)^{t-T+1}$$

for all $t > T$. Since $1 < r - \epsilon$, $(r - \epsilon)^{t-T+1} \to \infty$ as $t \to \infty$, and hence $\lim_{t\to\infty} p(t) = +\infty$.

□

Example 4.1.3 (Leslie age distribution model)**.** The Leslie Model is used to describe the changes in a population of individuals over a unit time which is the reproduction period. We divide a population into m age classes,

$$x_1(t), x_2(t), \ldots, x_m(t),$$

where each $x_i(t)$ represents the density or the number of individuals of the $i-$th age class at time t. Thus a population at time t has a class distribution vector

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_m(t) \end{pmatrix}, \tag{4.13}$$

If $l_j \neq 0$ denotes the probability of survival till the j-th age class, then the conditional probability of survival to the $j+1$st age class if one survived till jth age from birth $0 < s_{j+1,j} = \frac{l_{j+1}}{l_j} \leq 1$. Then the transition matrix T can be written as

$$T = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ s_{21} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & s_{m,m-1} & 0 \end{pmatrix}. \tag{4.14}$$

The matrix T has only $s_{j+1,j}$ entries because it is assumed that an individual in the age class j can only move to the next age class $j+1$ in a unit of time with probability $s_{j+1,j}$. If we hypothetically assume that an individual in the age class j can move to any age class i in a unit of time with probability s_{ij}, then these events must be mutually exclusive. That is, if an individual in the age class j in a unit of time can move to the age class 1 or 2 or $\cdots$ or $m-1$ or m with probability $s_{1j}, s_{2j}, \cdots, s_{m-1,j}, s_{m,j}$, respectively, then the probability that an individual can move from age class j to any age class i will be the sum of the probabilities of the mutually exclusive events. That is, we must have

$$0 \leq \sum_{i=1}^{m} s_{ij} \leq 1 \tag{4.15}$$

for $j = 1, 2, \ldots, m$.

Let F be the fertility matrix given by

$$F = \begin{pmatrix} f_{11} & f_{12} & \cdots & f_{1,m-1} & f_{1m} \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \tag{4.16}$$

where, $f_{1j} = \beta_{j+1}s_{j+1,j} \geq 0$ for $j = 1, 2, \ldots, m$ are the number of offspring produced by an age i individual. The matrix F has only entries in the first row because it is biologically meaningful that a new born individual is born as a juvenile, that is, in the first age class.

The dynamics of such population is described by

$$\begin{aligned} x(t+1) &= Fx(t) + Tx(t) \\ &= \begin{pmatrix} f_{11} & f_{12} & \cdots & f_{1,m-1} & f_{1m} \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_m(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ s_{21} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & s_{m,m-1} & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_m(t) \end{pmatrix}. \end{aligned} \tag{4.17}$$

Thus

$$x(t+1) = \begin{pmatrix} f_{11} & f_{12} & \cdots & f_{1,m-1} & f_{1m} \\ s_{21} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & s_{m,m-1} & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_m(t) \end{pmatrix}. \tag{4.18}$$

In short

$$x(t+1) = Px(t), \tag{4.19}$$

where

$$P = \begin{pmatrix} f_{11} & f_{12} & \cdots & f_{1,m-1} & f_{1m} \\ s_{21} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & s_{m,m-1} & 0 \end{pmatrix}. \tag{4.20}$$

The matrix P is called a projection matrix or map.

The time t in (4.19) is assumed to be increasing one unit at a time, that is, $t = t_0, t_0 + 1, t_0 + 2, \ldots,$ where t_0 represents the initial time. The number of individuals for each age class i is assumed to be given at the initial time t_0, that is, $x_i(t_0) = x_i^0$ is initially known. Equation (4.19) is known as the Leslie Model and the matrix P is called Leslie matrix(see [8]). When some individuals remain in the same age class with probability s_{ii}, then model (4.19) is known as Usher's Model, see [3], and the transition matrix T given in (4.14) becomes

$$T = \begin{pmatrix} s_{11} & 0 & \cdots & 0 & 0 \\ s_{21} & s_{22} & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & s_{m,m-1} & s_{mm} \end{pmatrix}. \tag{4.21}$$

Then the population dynamics is described by

$$x(t+1) = \begin{pmatrix} f_{11} + s_{11} & f_{12} & \cdots & f_{1,m-1} & f_{1m} \\ s_{21} & s_{22} & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & s_{m,m-1} & s_{mm} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_m(t) \end{pmatrix}. \tag{4.22}$$

Equation (4.22) can be written as

$$x(t+1) = Px(t), \tag{4.23}$$

where

$$P = \begin{pmatrix} f_{11} + s_{11} & f_{12} & \cdots & f_{1,m-1} & f_{1m} \\ s_{21} & s_{22} & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & s_{m,m-1} & s_{mm} \end{pmatrix} \tag{4.24}$$

is a projection matrix.

The projection matrix P given in (4.20) and (4.24) is non-negative. Let $\{N_1, N_2, \ldots, N_m\}$ be the set of nodes of a graph of $P, G(P)$. There is a directed edge $E_{i,i+1}$ leading from N_i to N_{i+1} because $s_{i+1,i} > 0$ for $i = 1, 2, \ldots, m-1$. Furthermore, if $f_{1m} > 0$, there is an edge E_{m1} leading from N_m to N_1. Hence, there is a directed path any node to any other, which is the definition of $G(P)$ being strongly connected (Definition 3.4.1). Thus, from Theorem 3.4.2, P is irreducible. If in addition to $f_{1m} > 0$, $f_{1j} > 0$ for $j = 1, \ldots, m-1$; that is, any age group is capable of reproduction, then from Corollary 3.4.14, the matrix P is primitive. From (4.3), we have that

$$\lim_{t\to\infty} \frac{x(t)}{r^t} = \lim_{t\to\infty} \frac{P^t x^0}{r^t} = \lim_{t\to\infty} \frac{P^t}{r^t} x^0 = v\frac{\langle \omega, x^0 \rangle}{\langle \omega, v \rangle}, \quad x_0 \in \mathbb{R}^m.$$

This means that a proportion of individuals in each age class becomes stable as time increases, while from Corollary 4.1.2, the total population may increase or decrease depending on the value of $r = \rho(P)$. This result assumed that young individuals reproduce. It is suffices to assume that $f_{1,m-1} > 0$ and $f_{1m} > 0$. If $f_{1,m-1} > 0$, then there is an edge $E_{m-1,1}$ leading from N_{m-1} to N_1. If we start from N_1 to return to N_1 either by following $E_{12}E_{23}\ldots E_{m-2,m-1}E_{m-1,1}$ which gives a cycle of length $m-1$ or by following $E_{12}E_{23}\cdots E_{m-1,m}E_{m1}$ which gives a cycle of length m. Thus $[P^{m-1}]_{11} > 0$ and $[P^m]_{11} > 0$. To pass from N_j to N_i with $j \geq i$, we follow $E_{j,j+1}E_{j+1,j+2}\cdots E_{m-1,m}E_{m1}E_{12}E_{23}\cdots E_{i-1,i}$. We take $m-j$ steps to reach N_m, one step to go to N_1 and $i-1$ steps to reach N_i from N_1, thus we reach N_i from N_j in $m-j+i$ steps. If $i > j$, we follow $E_{j,j+1}\cdots E_{i-1,i}$, hence we reach N_i from N_j in $i-j$ steps but we can add a cycle of length m, thus in both cases we reach N_j from N_i in $m-j+i$ steps. We cycle at N_1 appropriate number of k steps to eliminate the dependence on j and i as shown below. Since $1 \leq i, j \leq m$, then $-m \leq -j \leq -1$ and

$$-m+1 \leq -j+i \leq m-1. \tag{4.25}$$

We observe that $[P^k]_{11} > 0$ for any

$$k = \alpha(m-1) + \beta m \tag{4.26}$$

with α and β being natural numbers or 0, because $[P^m]_{11} > 0$, $[P^{m-1}]_{11} > 0$ and $[P^0]_{11} = 1 > 0$ (where P^0 denotes the identity matrix). If $\alpha = \left[\frac{k}{m-1}\right] m - k$ and $\beta = k - \left[\frac{k}{m-1}\right](m-1)$, where $[\cdot]$ denotes the integer part of a number, then (4.26) is satisfied. We write

$$k = r(m-1) + s \tag{4.27}$$

with integer r and integer s satisfying $0 \leq s \leq m-1$ so that $\left[\frac{k}{m-1}\right] = r$. Thus, we must have

$$r \geq r\left(1 - \frac{1}{m}\right) + \frac{s}{m}$$

for any s as above; that is,

$$rm \geq r\,(m-1) + s$$

so that we get $r \geq s$, that is, $r \geq m-1$. Hence, (4.26) is valid for any $k \geq (m-1)^2$. Indeed, in such case $k = (m-1)^2 + l$ for some $l \geq 0$ and

$$\frac{k}{(m-1)} = \frac{(m-1)^2 + l}{(m-1)},$$

thus

$$r = \left[\frac{k}{m-1}\right] = \left[m - 1 + \frac{l}{(m-1)}\right] \geq m - 1.$$

Then, we write

$$m - j + i + k = m - j + i + (m-1)^2 + l = m - j + i + m^2 - 2m + 1 + l = m^2 - m - j + i + l_1$$

where $l_1 = l + 1$. We can take

$$m - j + i + k = m^2 \tag{4.28}$$

for some k expressible through (4.26), because (4.28) implies that

$$k = m^2 - m + j - i = m^2 - 2m + m + j - i,$$

but $m + j - i \geq 1$, from (4.25), then

$$k = m^2 - 2m + m + j - i \geq m^2 - 2m + 1 = (m-1)^2.$$

We conclude that, we can reach any N_i from any N_j in m^2 steps: $m - j$ steps from N_j to N_m, 1 step to state N_1, 'cycling' for k times around N_1 and then going from N_1 to N_i in $i - 1$ steps. Hence, $P^{m^2} > 0$. Therefore, from Theorem 3.4.7, we conclude that P is primitive. By (4.3), we have that

$$\lim_{t\to\infty} \frac{x(t)}{r^t} = \lim_{t\to\infty} \frac{P^t x^0}{r^t} = \lim_{t\to\infty} \frac{P^t}{r^t} x^0 = v\frac{\langle \omega, x^0\rangle}{\langle \omega, v\rangle}, \quad x_0 \in \mathbb{R}^m.$$

Let us now consider a more complicated case where the fertility is restricted to some interval $[m_1, m_2]$, that is, $f_{1j} > 0$ for $j \in [m_1, m_2]$. If $m_2 < m$, then the graph $G(P)$ is not connected because there is no edge connecting N_m with N_1. Thus the matrix P is not irreducible, however the model shows constant long time behaviour under certain conditions. Let $m_1 < m_2$ and divide the population into productive age with $f_{1j} \geq 0$ for $j \leq m_2$ and post-productive population with $f_{1j} = 0$ for $j > m_2$. We assume that $m_1 < m_2$ and we introduce the matrix restricted to the productive ages

$$\tilde{P} = \begin{pmatrix} f_{11} & f_{12} & \cdots & f_{1,m_2-1} & f_{1m_2} \\ s_{21} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & s_{m_2,m_2-1} & 0 \end{pmatrix},$$

and the matrix providing the link from productive to post-reproductive ages,

$$\mathcal{R} = \begin{pmatrix} 0 & \cdots & s_{m_2+1,m_2} & 0 & \cdots & 0 & 0 \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & s_{m,m-1} & 0 \end{pmatrix}.$$

If $f_{1m_2} > 0$ and $f_{1,m_2-1} > 0$, we can apply the previous considerations; therefore $\tilde{P}$ is irreducible and primitive. Thus there exists a left eigenvector of $\tilde{P}$, $\omega \in \mathbb{R}_+^{m_2}$ and a right eigenvector $v \in \mathbb{R}_+^{m_2}$ such that

$$\lim_{t\to\infty} \rho(\tilde{P})^{-t}\tilde{P}^t x^0 = v\frac{\langle \omega, x^0\rangle}{\langle \omega, v\rangle}, \quad x^0 \in \mathbb{R}^{m_2}.$$

For $m_2 \leq j < m, t \geq 0$, we have

$$x_{j+1}(t+1) = s_{j+1,j}x_j(t).$$

Hence, starting from $x_{m_2}(t)$, we get

$$x_{m_2+i}(t+i) = c_i x_{m_2}(t)$$

where $c_i = s_{m_2+i,m_2+i-1} \cdot \ldots \cdot s_{m_2+1,m_2}$ as long as $i \leq m - m_2$. So

$$\lim_{t\to\infty} \rho(\tilde{P})^{-t}x_{m_2+i}(t+i) = c_i \lim_{t\to\infty} \rho(\tilde{P})^{-t}x_{m_2}(t) = c_i v_{m_2}\frac{\langle \omega^T, x^0\rangle}{\langle \omega^T, v\rangle}$$

If we change $t+i$ to t, then

$$\lim_{t\to\infty} \rho(\tilde{P})^{-t}x_{m_2+i}(t) = c_i\rho(\tilde{P})^{-i}v_{m_2}\frac{\langle \omega, x^0\rangle}{\langle \omega, v\rangle}$$

for $i = 1, \ldots, m - m_2$. Hence, if we take $x^0 \in \mathbb{R}^m$,

$$v = \begin{pmatrix} v_1 & v_2 & \cdots & v_{m_2} & c_1\rho(\tilde{P})^{-1}v_{m_2} & \cdots & c_{m-m_2}\rho(\tilde{P})^{-(m-m_2)}v_{m_2}\end{pmatrix}^T$$

and

$$\omega = \begin{pmatrix} \omega_1 & \omega_2 & \cdots & \omega_{m_2} & 0 & \cdots & 0\end{pmatrix}^T,$$

we have

$$\lim_{t\to\infty} \frac{x(t)}{\rho(\tilde{P})^t} = \lim_{t\to\infty} \frac{P^t x^0}{\rho(\tilde{P})^t} = v\frac{\langle \omega, x^0\rangle}{\langle \omega, v\rangle}. \tag{4.29}$$

Chapter 5

Nonlinear autonomous models

We consider a general nonlinear autonomous model introduced in (1.4),

$$\begin{aligned} x(t+1) &= f(x(t)), \\ x(0) &= x^0, \end{aligned} \tag{5.1}$$

where $f : K \to \mathbb{R}^m$ is continuously differentiable at an equilibrium point $x_e \in \mathbb{R}^m$. According to the introduction, substitution $y(t) = x(t) - x_e$ transforms (5.1) to an equivalent problem with 0 an equilibrium point of (5.1). Close to the extinction equilibrium $y = 0$, the problem can be written as

$$\begin{aligned} y(t+1) &= Py(t) + g(y(t)), \\ y(0) &= y^0, \end{aligned} \tag{5.2}$$

where

$$\lim_{\|y\| \to 0} \frac{\|g(y)\|}{\|y\|} = 0.$$

Thus the linearization about the extinction equilibrium is

$$\begin{aligned} y(t+1) &= Py(t) \\ y(0) &= y^0. \end{aligned} \tag{5.3}$$

The initial value problem (5.2) has a unique forward solution

$$y(t) = \begin{cases} P^t y^0 + \sum_{i=0}^{t-1} P^{t-i-1} g(y(i)) & t \in I[1, +\infty), \\ y^0 & t = 0. \end{cases} \tag{5.4}$$

5.1 Liapunov stability

Let $V : \mathbb{R}^m \to \mathbb{R}$. Then we define a variation of (5.1) by $\Delta V(x) = V(f(x)) - V(x)$ and

$$\Delta V(x(t)) = V(f(x(t)) - V(x(t)) = V(x(t+1)) - V(x(t)).$$

If $\Delta V(x) \leq 0$, then V is non-increasing along solutions of (5.1). The function V is said to be a Liapunov function on $H \subset \mathbb{R}^m$ if:

(i) V is continuous on H, and

(ii) $\Delta V(x) \leq 0$, whenever x and $f(x)$ belong to H.

We say that the function V is positive definite at the extinction equilibrium of (5.1), if $V(0) = 0$ and $V(x) > 0$ for all $0 \neq x \in B(\gamma)$. If $V(0) = 0$ and $V(x) \geq 0$ for all $0 \neq x \in B(\gamma)$, then V is said to be a positive semidefinite.

Theorem 5.1.1 ([4], page 205)**.** *Let V be a Liapunov function for* (5.1) *in a neighbourhood H of the extinction equilibrium and let V be positive definite with respect to the extinction equilibrium. Then the extinction equilibrium is stable. If, in addition, $\Delta V(x) < 0$ whenever $x \in H, f(x) \in H$ and $x \neq 0$, then the extinction equilibrium is asymptotically stable.*

Proof. By assumption, we have $f : K \to \mathbb{R}^m, K \subset \mathbb{R}^m$, is continuously differentiable at the extinction equilibrium. Let $\alpha_1 > 0$ such that $B(\alpha_1) \subset K \cap H$ (where $B(\alpha_1)$ denotes an open ball about the extinction equilibrium with radius α_1, Definition 2.1.4 and Equation (2.5)). By the continuity of f, there exists $\alpha_2 > 0$ such that if $x \in B(\alpha_2)$, then $f(x) \in B(\alpha_1)$. Let $0 < \epsilon \leq \alpha_2$ and define

$$\psi(\epsilon) = \min\{V(x) : \epsilon \leq \|x\| \leq \alpha_1\} > 0.$$

We have $V(0) = 0$. By the continuity of V, for any $\eta > 0$ there exists a δ such that $\|x\| < \delta$ implies that $0 < V(x) < \eta$. We choose $\eta = \psi(\epsilon)$, then there exists $0 < \delta < \epsilon$ such that $0 < V(x) < \psi(\epsilon)$ whenever $\|x\| < \delta$. If $x^0 \in B(\delta)$, then $x(t) \in B(\epsilon)$ for all $t \geq 0$. Otherwise, there exists $x^0 \in B(\delta)$ and a positive integer T such that $x(t) \in B(\epsilon)$ for $1 \leq t \leq T$ and $x(T+1) \notin B(\epsilon)$. Since $x(T) \in B(\epsilon) \subset B(\alpha_2)$ then, by the continuity of f, it follows that $x(T+1) = f(x(T)) \in B(\alpha_1)$. Consequently, $V(x(T+1)) \geq \psi(\epsilon)$. However, $V(x(T+1)) \leq \cdots \leq V(x^0) < \psi(\epsilon)$ and thus we have a contradiction. This establishes stability. To prove asymptotic stability, assume that $x^0 \in B(\delta)$, then $x(t) \in B(\epsilon)$ holds true for all $t \geq 0$. If $\{x(t)\}$ does not converge to 0, then it has a subsequence $\{x(t_i)\}$ that converges to $0 \neq l \in \mathbb{R}^m$. Let $E \subset B(\alpha_2)$ be an open neighbourhood of l with $0 \notin \overline{E}$. We define the function $h(x) = \frac{V(f(x))}{V(x)}$, $x \in E$. Then $h(x)$ is well defined and continuous. Moreover, $h(x) < 1$ for all $x \in E$, because $V(f(x)) < V(x)$. Now, by the continuity of $h(x)$, if $\eta \in (h(l), 1)$, there exists $\alpha > 0$ such that $x \in B(l, \alpha)$ implies $h(x) \leq \eta$, that is, $V(f(x)) \leq \eta V(x)$. Since $\Delta V(x) < 0$, then V is monotone decreasing. By the continuity of V, $x(t_i) \to l \neq 0$ as $i \to \infty$ implies that

$$\lim_{i\to\infty} V(x(t_i)) = V(l) \neq 0. \tag{5.5}$$

Therefore, by monotonicity of V,

$$\lim_{t\to\infty} V(x(t)) = V(l) \neq 0. \tag{5.6}$$

Now, for any $\epsilon > 0$, find $x(t_i) \in B(l, \alpha)$ such that $V(x(t_i)) < V(l) + \epsilon$. Then

$$V(x(t_i + 1)) = V(f(x(t_i)) \leq \eta V(x(t_i)) < \eta V(l) + \epsilon < V(l)$$

for $\epsilon < (1 - \eta)V(l)$ which is a contradiction with the monotonicity of V, thus $V(l) = 0$. Consequently $l = 0$. □

Theorem 5.1.2 ([4], page 213). *If $\Delta V(x) < 0$ for $0 \neq x \in B(\epsilon)$ for some $\epsilon > 0$, $V(0) = 0$ and there exists a sequence $a_i \to 0$ with $V(a_i) < 0$, then the extinction equilibrium of* (5.1) *is unstable.*

Proof. We recall the definition of instability which says that the extinction equilibrium is unstable if there exists an $\epsilon > 0$ such that for any $\delta > 0$, there is $x^0 \in \mathbb{R}^m$ such that $\|x^0\| < \delta$ and $\|x(t)\| \geq \epsilon$ for some $t \in [0, \infty)$. Let $\epsilon > 0$, so that $\Delta V(x) < 0$ for $0 \neq x \in B(\epsilon)$ and $V(0) = 0$. Let $x^0 = a_j$ for some j with $V(x^0) < 0$. We claim that there is $0 < \eta < \epsilon$ such that the sequence $\{x(t)\}$ of iterations starting from x^0 satisfies $\|x(t)\| \geq \eta$. Otherwise, there exists a subsequence $\{x(t_i)\}$ such that $x(t_i) \to 0$ as $i \to \infty$, hence by the continuity of $V(x)$ and $V(0) = 0$, we have $V(x(t_i)) \to V(0) = 0$ which is a contradiction because $\Delta V(x) < 0$ for $0 \neq x \in B(\epsilon)$ implies $V(x(t_i)) < V(x(t_{i-1})) < \cdots < V(x(t_1)) < V(x(t_0)) = V(x^0) < 0$ for all $i = 1, 2, \ldots$. By continuity and $\Delta V(x) < 0$ we have

$$\sup_{\eta \leq \|x\| \leq \epsilon} \Delta V(x) = m < 0.$$

Hence

$$\Delta V(x^0) = V(x(1)) - V(x^0) \leq m < 0,$$

and

$$\begin{aligned}V(x(1)) &\leq m + V(x^0),\\ V(x(2)) &\leq 2m + V(x^0),\\ &\vdots \\ V(x(t)) &\leq tm + V(x^0)\\ &\vdots\end{aligned} \tag{5.7}$$

as long as $x(t) \in B(\epsilon)$. If there is some $t \in I[0, +\infty)$ for which $\|x(t)\| > \epsilon$ then, by the definition of instability, the extinction equilibrium is unstable. Otherwise, if for all $t \in I[0, +\infty)$, $\|x(t)\| \leq \epsilon$, then $\{x(t)\}$ is a bounded set and therefore, by the continuity of V, $V(x(t))$ is bounded. On the other hand, since $x(t) \in B(\epsilon)$ for all t, we can apply (5.7) and, since $V(x(t)) \leq tm + V(x^0)$ for all t, $V(x(t))$ is unbounded, which is a contradiction. Hence the assumption that $x(t) \in B(\epsilon)$ for all t is wrong. Therefore there is some $t \in I[0, +\infty)$ for which $\|x(t)\| > \epsilon$ and therefore the extinction equilibrium is unstable. □

Definition 5.1.3. Let $B = [b_{ij}]_{1 \leq i,j \leq m}$ be a real symmetric matrix and let

$$V(x) = \langle x, Bx \rangle = \sum_{i=1}^{m} \sum_{j=1}^{m} b_{ij} x_i x_j.$$

A matrix B is said to be positive definite, respectively positive semidefinite, if $V(x)$ is positive definite and positive semidefinite respectively.

Sylvester's criterion is a test for positive definiteness of symmetric matrices with real entries ([4], page 214). It says that a real symmetric matrix $B = [b_{ij}]_{1 \leq i,j \leq m}$ is positive definite if and only if the determinants of its leading principle minors are positive, that is, if and only if

$$b_{11} > 0, \quad \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} > 0, \quad \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix}, \quad \ldots, \quad \det(B) > 0.$$

Let B be a positive definite and let $\lambda \in \sigma(B)$. Then $Bx = \lambda x$ for some $0 \neq x \in \mathbb{R}^m$ implies that $V(x) = \langle x, Bx \rangle = \lambda \langle x, x \rangle = \lambda \|x\|_2^2 > 0$, hence $\lambda > 0$ for any $\lambda \in \sigma(B)$. We recall from, [9], page 549, that a matrix B is real symmetric if and only if B is orthogonally similar to a real diagonal matrix D, that is, $Q^T BQ = D$ for some orthogonal matrix Q.

Lemma 5.1.4 ([9], page 549)**.** *Let* $\lambda_{\max} \in \sigma(B)$, $\lambda_{\min} \in \sigma(B)$ *such that* $\lambda_{\min} \le \lambda \le \lambda_{\max}$ *for any* $\lambda \in \sigma(B)$. *Then*

$$\lambda_{\max} = \max_{\|x\|_2=1} \langle x, Bx\rangle$$

and

$$\lambda_{\min} = \min_{\|x\|_2=1} \langle x, Bx\rangle.$$

Proof. Since B is real symmetric, then there is an orthogonal matrix Q such that $Q^T BQ = D = \operatorname{diag}(\lambda_1, \ldots, \lambda_m)$, or equivalently, $B = QDQ^T$. Since $\|x\|_2 = 1$ if and only if $\|y\|_2 = 1$ for $y = [y_i]_{1\le i\le m} = Q^T x$,

$$\begin{aligned}\max_{\|x\|_2=1} \langle x, Bx\rangle &= \max_{\|x\|_2=1} \langle x, QDQ^T x\rangle = \max_{\|x\|_2=1} \langle Q^T x, DQ^T x\rangle = \max_{\|y\|_2=1} \langle y, Dy\rangle \\ &= \max_{\|y\|_2=1} \langle y, \lambda y\rangle = \lambda\langle y, y\rangle \le \lambda_{\max}\langle y, y\rangle = \lambda_{\max}\|y\|_2^2 = \lambda_{\max}.\end{aligned}$$

The above equality is attained, when x is a normalized eigenvector of B corresponding to $\lambda_{\max}$, because if u is an eigenvector of B corresponding $\lambda_{\max}$, then $\langle u, Bu\rangle = \lambda_{\max}\langle u, u\rangle = \|u\|_2^2$ which implies

$$\frac{\langle u, Bu\rangle}{\|u\|_2^2} = \lambda_{\max}.$$

Then

$$\lambda_{\max} = \langle \frac{u}{\|u\|_2}, B\frac{u}{\|u\|_2}\rangle = \langle x, Bx\rangle,$$

where $\|x\|_2 = \left\|\frac{u}{\|u\|_2}\right\|_2 = 1$. The expression for the smallest eigenvalue $\lambda_{\min}$ is obtained by writing

$$\begin{aligned}\min_{\|x\|_2=1} \langle x, Bx\rangle &= \min_{\|x\|_2=1} \langle x, QDQ^T x\rangle = \min_{\|x\|_2=1} \langle Q^T x, DQ^T x\rangle \\ &= \min_{\|y\|_2=1} \langle y, Dy\rangle = \min_{\|y\|_2=1} \lambda\langle y, y\rangle \ge \min_{\|y\|_2=1} \lambda_{\min}\|y\|_2^2 = \lambda_{\min}.\end{aligned}$$

The equality is attained when x is a normalized eigenvector of B corresponding to $\lambda_{\min}$. Therefore,

$$\lambda_{\min} = \min_{\|x\|_2=1} \langle x, Bx\rangle.$$

□

Note that the above characterizations can be written in the equivalent forms

$$\lambda_{\max} = \max_{x\neq 0} \frac{\langle x, Bx\rangle}{\langle x, x\rangle},$$

and

$$\lambda_{\min} = \min_{x \neq 0} \frac{\langle x, Bx \rangle}{\langle x, x \rangle}.$$

Consequently,

$$\lambda_{\min} \|x\|_2^2 \leq V(x) \leq \lambda_{\max} \|x\|_2^2 \tag{5.8}$$

for all $x \neq 0$. Let B is a positive definite matrix and let $V(y) = \langle y, By \rangle$ be a continuous function defined for all vectors $y \in \mathbb{R}^m$. Then V is positive definite at the extinction equilibrium (because $V(0) = 0$ and $V(y) > 0$ for all $y \neq 0$ by positive definiteness of B.) Then, relative to (5.3), we have

$$\Delta V(y(t)) = V(y(t+1)) - V(y(t)) = \langle y(t), P^T BPy(t) \rangle - \langle y(t), By(t) \rangle. \tag{5.9}$$

This can be written as

$$\Delta V = \langle y, (P^T BP - B)y \rangle.$$

Thus $\Delta V < 0$ if and only if

$$P^T BP - B = -C \tag{5.10}$$

for some positive definite matrix C. Equation (5.10) is called the Liapunov equation of the linear equation (5.3). If C is a positive definite symmetric matrix such that (5.10) has a solution B that is also symmetric and positive definite, then $\Delta V < 0$. We may consider V to be a Liapunov function of (5.3) (because V is continuous for $y \in \mathbb{R}^m$ and $\Delta V(y) < 0$ for all $y \in \mathbb{R}^m$ and $Py \in \mathbb{R}^m$.) Therefore, from Theorem 5.1.1, the equilibrium of (5.3) is asymptotically stable. On the other hand, if the equilibrium of (5.3) is asymptotically stable then, for every positive definite symmetric matrix C, (5.10) has a unique solution B that is also symmetric and positive definite as shown in the following theorem.

Theorem 5.1.5 ([4], page 216)**.** *If the equilibrium of* (5.3) *is asymptotically stable then, for every positive definite symmetric matrix* C, *Equation* (5.10) *has a unique solution* B *that is also symmetric and positive definite.*

Proof. Assume that the equilibrium of (5.3) is asymptotically stable. Let C be a positive definite symmetric matrix. We will show that the Liapunov equation (5.10) has a unique solution B. Multiply (5.10) from the left by $(P^T)^r$ and from the right by P^r to obtain

$$(P^T)^{r+1} BP^{r+1} - (P^T)^r BP^r = -(P^T)^r CP^r.$$

Hence

$$\lim_{t\to\infty}\sum_{r=0}^{t}\left[(P^T)^{r+1}BP^{r+1}-(P^T)^rBP^r\right]=-\lim_{t\to\infty}\sum_{r=0}^{t}(P^T)^rCP^r.$$

Thus

$$-B+\lim_{t\to\infty}(P^T)^{t+1}BP^{t+1}=-\lim_{t\to\infty}\sum_{r=0}^{t}(P^T)^rCP^r. \tag{5.11}$$

Since $x(t)$ is asymptotically stable, $\lim_{t\to\infty}x(t)=\lim_{t\to\infty}P^tx(0)=0$. Hence, from Proposition 2.3.7, $\rho(P)<1$. Consequently $\rho(P^T)<1$, because P and P^T have the same characteristic polynomial. This implies that $\lim_{t\to\infty}(P^T)^{t+1}BP^{t+1}=\left(\lim_{t\to\infty}(P^T)^{t+1}\right)B\left(\lim_{t\to\infty}P^{t+1}\right)=0$. Thus (5.11) yields

$$B=\sum_{r=0}^{\infty}(P^T)^rCP^r. \tag{5.12}$$

From Theorem 5.1.6 below we have that $\rho(P)<1$ implies the existence of a norm of P, $\|\cdot\|$, such that $\|P\|<1$. Now, from (5.12), we have

$$\|B\|=\|\sum_{r=0}^{\infty}(P^T)^rCP^r\|\le\sum_{r=0}^{\infty}\|(P^T)\|^r\|C\|\|P\|^r=\sum_{r=0}^{\infty}\|C\|\|P\|^{2r}=\|C\|+\|C\|\lim_{t\to\infty}\sum_{r=1}^{t}q^r,$$

where $q=\|P\|^2<1$. Thus the series in (5.12) is absolutely convergent and

$$\|B\|\le\|C\|+\|C\|\lim_{t\to\infty}q\frac{(1-q^t)}{1-q}=(1+\frac{q}{1-q})\|C\|=(\frac{1}{1-q})\|C\|=\frac{\|C\|}{1-\|P\|^2}.$$

Since C is a symmetric, for each $r\in I[0,\infty)$ we have that

$$\left((P^T)^rCP^r\right)^T=\left((P^r)^TC^T(P^T)^{r^T}\right)^T=(P^T)^rCP^r.$$

Thus the convergent series in (5.12) is a sum of symmetric matrices and hence B is symmetric. Moreover, since C is a positive definite matrix satisfying $\langle x,Cx\rangle>0$ for $x\neq0$, then for $x\neq0$ we have

$$\langle x,Bx\rangle=\sum_{r=0}^{\infty}\langle P^rx,CP^rx\rangle=\langle x,Cx\rangle+\sum_{r=1}^{\infty}\langle P^rx,CP^rx\rangle>0. \tag{5.13}$$

Thus B is positive definite. □

Theorem 5.1.6. *Let P be a matrix. If $\rho(P)<1$, then there exists a norm of $P,\|\cdot\|$, such that $\|P\|<1$.*

Proof. If J is the Jordan form of P, then there exists a matrix Q such that

$$QPQ^{-1}=J=\begin{pmatrix}\lambda_1 & 1 & \cdots & 0\\ 0 & \lambda_2 & \ddots & \vdots\\ \vdots & \ddots & \ddots & 1\\ 0 & \cdots & 0 & \lambda_k\end{pmatrix}$$

where λ_j for $j = 1, 2, \ldots, k$ are the eigenvalues of P (which might be repeated). For $0 < \delta \in \mathbb{R}$, denote $D_\delta = \mathrm{diag}(1, \delta, \ldots, \delta^{m-1})$, and put $V_\delta = QD_\delta$. Then

$$V_\delta^{-1} P V_\delta = \begin{pmatrix} \lambda_1 & \delta & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \delta \\ 0 & \cdots & 0 & \lambda_k \end{pmatrix}.$$

We fix δ such that $\delta < \varepsilon = \frac{1-\rho(P)}{2}$ and we define a norm of a matrix M by

$$N(M) = \max_{\|x\|=1} \|V_\delta^{-1} M V_\delta x\|.$$

Hence $N(P) = \rho(P) + \delta \leq \rho(P) + \varepsilon = \rho(P) + \frac{1-\rho(P)}{2} = \frac{\rho(P)}{2} + \frac{1}{2} < \frac{1}{2} + \frac{1}{2} = 1$. Thus the matrix P has a norm $\|\cdot\|$ such that $\|P\| = N(P) < 1$. $\square$

Corollary 5.1.7 ([4], page 216)**.** *Let P be a hyperbolic matrix. If $\rho(P) > 1$, then there exists a real symmetric matrix B that is not positive semidefinite, such that* (5.10) *holds for some symmetric positive definite matrix C.*

Proof. Let $J = QPQ^{-1} = \begin{pmatrix} J_{11} & 0 \\ 0 & J_{22} \end{pmatrix}$ be the Jordan form of P such that $|\lambda| > 1$ for $\lambda \in \sigma(J_{11})$ and $|\lambda| < 1$ for $\lambda \in \sigma(J_{22})$. Then

$$P^T B P - B = -C$$

becomes

$$(Q^{-1} J Q)^T B (Q^{-1} J Q) - B = -C$$

which implies that

$$Q^T J^T {Q^{-1}}^T B Q^{-1} J Q - B = -C.$$

Then

$$J^T {Q^{-1}}^T B Q^{-1} J - {Q^{-1}}^T B Q^{-1} = -{Q^{-1}}^T C Q^{-1},$$

and

$$J^T \tilde{B} J - \tilde{B} = -\tilde{C}, \tag{5.14}$$

where $\tilde{B} = Q^{-1^T} B Q^{-1}$ and $\tilde{C} = Q^{-1^T} C Q^{-1}$. Let $\tilde{B} = \begin{pmatrix} \tilde{B}_{11} & 0 \\ 0 & \tilde{B}_{22} \end{pmatrix}$ and $\tilde{C} = \begin{pmatrix} \tilde{C}_{11} & 0 \\ 0 & \tilde{C}_{22} \end{pmatrix}$, where $\tilde{C}_{11}$ and $\tilde{C}_{22}$ are positive definite symmetric matrices, so that

$$\begin{pmatrix} J_{11}^T & 0 \\ 0 & J_{22}^T \end{pmatrix} \begin{pmatrix} \tilde{B}_{11} & 0 \\ 0 & \tilde{B}_{22} \end{pmatrix} \begin{pmatrix} J_{11} & 0 \\ 0 & J_{22} \end{pmatrix} - \begin{pmatrix} \tilde{B}_{11} & 0 \\ 0 & \tilde{B}_{22} \end{pmatrix} = - \begin{pmatrix} \tilde{C}_{11} & 0 \\ 0 & \tilde{C}_{22} \end{pmatrix}.$$

Then

$$J_{11}^T \tilde{B}_{11} J_{11} - \tilde{B}_{11} = -\tilde{C}_{11}, \tag{5.15}$$

and

$$J_{22}^T \tilde{B}_{22} J_{22} - \tilde{B}_{22} = -\tilde{C}_{22}. \tag{5.16}$$

Since for any $\lambda \in \sigma(J_{11})$ we have $|\lambda| > 1$, then for any $\lambda \in \sigma(J_{11}^{-1})$ we have $|\lambda| < 1$. Note that for a positive definite symmetric matrix $\tilde{C}_{11}$, we have that $(J_{11}^{-1})^T \tilde{C}_{11} J_{11}^{-1}$ is positive definite and symmetric. From Theorem 5.1.6, since $\tilde{C}_{11}$ is a positive definite symmetric matrix, there exists a positive definite symmetric B_0 such that

$$(J_{11}^{-1})^T B_0 J_{11}^{-1} - B_0 = -(J_{11}^{-1})^T \tilde{C}_{11} J_{11}^{-1}.$$

Then

$$B_0 - J_{11}^T B_0 J_{11} = -\tilde{C}_{11},$$

hence

$$J_{11}^T (-B_0) J_{11} - (-B_0) = -\tilde{C}_{11}.$$

From (5.15), we can define $\tilde{B}_{11} = -B_0$. Since $\langle x, B_0 x \rangle > 0$ for any $x \neq 0$, then $\langle x, \tilde{B}_{11} x \rangle < 0$ for any $x \neq 0$. Moreover, since $\rho(J_{22}) < 1$ and $\tilde{C}_{22}$ is a positive definite matrix, from Theorem 5.1.6 we have that (5.16) has a unique solution $\tilde{B}_{22}$, a symmetric matrix, satisfying $\langle x, \tilde{B}_{22} x \rangle > 0$ for any $x \neq 0$. Now, we have

$$Q^{-1^T} B Q^{-1} = \tilde{B} = \begin{pmatrix} \tilde{B}_{11} & 0 \\ 0 & \tilde{B}_{22} \end{pmatrix}, \tag{5.17}$$

where $\langle x, \tilde{B}_{11} x \rangle < 0$ and $\langle x, \tilde{B}_{22} x \rangle > 0$ for any $x \neq 0$. Equation (5.17) can be written as

$$B = Q^T \tilde{B} Q.$$

Then

$$\langle x, Bx \rangle = x^T B x = x^T Q^T \tilde{B} Q x = \langle Qx, \tilde{B} Qx \rangle$$

for $x \in \mathbb{R}^m$. We choose x, so that $Qx = \begin{pmatrix} y & 0 \end{pmatrix}^T$. Then

$$\langle x, Bx \rangle = \langle Qx, \tilde{B}Qx \rangle = \begin{pmatrix} y & 0 \end{pmatrix} \begin{pmatrix} \tilde{B}_{11} & 0 \\ 0 & \tilde{B}_{22} \end{pmatrix} \begin{pmatrix} y \\ 0 \end{pmatrix} = \langle y, \tilde{B}_{11} y \rangle < 0$$

for any $y \neq 0$. Thus $\langle x, Bx \rangle < 0$ for some $x \neq 0$ which means that B is not positive definite. □

The above result can be strengthened with this example.

Example 5.1.8. The result of Theorem 5.1.6 above only works when the matrix P is hyperbolic. Let $P = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, a non-hyperbolic matrix. We want to solve for a real symmetric matrix B the Liapunov equation

$$P^T BP - B = -C,$$

where C is a positive definite matrix. Let $C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b \\ b & b_{22} \end{pmatrix}$. Then

$$P^T BP = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b_{11} & b \\ b & b_{22} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2b_{11} & 2b \\ b & b_{22} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4b_{11} & 2b \\ 2b & b_{22} \end{pmatrix},$$

and

$$P^T BP - B = \begin{pmatrix} 4b_{11} & 2b \\ 2b & b_{22} \end{pmatrix} - \begin{pmatrix} b_{11} & b \\ b & b_{22} \end{pmatrix} = \begin{pmatrix} 3b_{11} & b \\ b & 0 \end{pmatrix} = - \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}.$$

Hence $c_{22} = 0$ which, from Sylvester's criterion, contradicts C being a positive definite matrix.

Theorem 5.1.9 ([4], page 226)**.** *Let P be a hyperbolic matrix. If $\rho(P) > 1$, then the equilibrium of* (5.2) *is unstable.*

Proof. If $\rho(P) > 1$ then, from Corollary 5.1.7, there exists a real symmetric matrix B satisfying $\langle y, By \rangle < 0$ for some $0 \neq y$ such that

$$P^T BP - B = -C, \tag{5.18}$$

for some symmetric and positive definite matrix C. Let $V(y) = \langle y, By \rangle$. Then $V(0) = 0$. Relatively to (5.2), we have

$$\begin{aligned} \Delta V(y(t)) = V(y(t+1)) - V(y(t)) &= \langle y(t+1), By(t+1) \rangle - \langle y(t), By(t) \rangle \\ &= y(t+1)^T By(t+1) - y(t)^T By(t) \\ &= (Py(t) + g(y(t)))^T B (Py(t) + g(y(t))) - y(t)^T B(t) y(t). \end{aligned}$$

Thus

$$\Delta V(y) = \left(y^T P^T B + g(y)^T B\right)(Py + g(y)) - y^T B y$$

which implies that

$$\begin{aligned}
\Delta V(y) &= y^T P^T BPy + g(y)^T BPy + y^T P^T Bg(y) + g(y)^T Bg(y) - y^T By \\
&= y^T P^T BPy - y^T By + g(y)^T BPy + y^T P^T Bg(y) + g(y)^T Bg(y) \\
&= y^T \left(P^T BP - B\right) y + g(y)^T BPy + y^T P^T Bg(y) + g(y)^T Bg(y).
\end{aligned}$$

Substituting the value of C from (5.18) implies that

$$\Delta V(y) = -y^T Cy + g(y)^T BPy + y^T P^T Bg(y) + V(g(y)).$$

Now, from (5.8), we have

$$\lambda_{min} \|y\|_2^2 \leq y^T Cy$$

for all $y \neq 0$, where $0 < \lambda_{min} \in \sigma(C)$ such that $\lambda_{\min} \leq \lambda$ for any $\lambda \in \sigma(C)$. From Theorem 2.1.3, for any norm $\|\cdot\|$ there exists $\gamma > 0$ such that

$$4\gamma \|y\|^2 \leq \lambda_{min} \|y\|_2^2 \leq y^T Cy$$

for all $y \neq 0$. Hence

$$-y^T Cy \leq -4\gamma \|y\|^2$$

for all $y \neq 0$. Since

$$\lim_{\|y\| \to 0} \frac{\|g(y)\|}{\|y\|} = 0,$$

for any $\epsilon > 0$, there exists $\delta > 0$ such that $\|y\| < \delta$ implies that $\frac{\|g(y)\|}{\|y\|} < \epsilon$. Hence $\|g(y)\| < \epsilon \|y\|$. Thus

$$|g(y)^T BPy| \leq \|g(y)^T\| \|B\| \|P\| \|y\| < \epsilon \|y^T\| \|P\| \|B\| \|y\| \leq \gamma \|y\|^2, \tag{5.19}$$

for some $\epsilon = \epsilon_1 > 0$. This occurs because $\|P\|$ and $\|B\|$ are bounded, and $\|P\| \neq 0$ and $\|B\| \neq 0$. We see that $\|P\| \neq 0$ and $\|B\| \neq 0$ as follows. We have that $\|P\| = 0$ if and only if $P = 0$. Then, $\rho(P) > 1$ implies that $P \neq 0$. Similarly, $\langle x, Bx \rangle < 0$ for some $x \neq 0$ implies that $B \neq 0$. Next,

$$|y^T P^T Bg(y)| \leq \|y^T\| \|P^T\| \|B\| \|g(y)\| < \epsilon \|y^T\| \|P^T\| \|B\| \|y\| \leq \gamma \|y\|^2$$

for some $\epsilon = \epsilon_1 > 0$. Hence

$$|y^T P^T Bg(y)| < \gamma \|y\|^2,$$

which implies that

$$-\gamma\|y\|^2 < y^T P^T Bg(y) < \gamma\|y\|^2.$$

Similarly, from (5.19) we have

$$|y^T P^T Bg(y)| < \gamma\|y\|^2,$$

then

$$-\gamma\|y\|^2 < y^T P^T Bg(y) < \gamma\|y\|^2.$$

Moreover,

$$|V(g(y))| = |g(y)^T Bg(y)| \leq \|g(y)^T\|\|B\|\|g(y)\| < \epsilon^2\|y\|^2\|B\| \leq \gamma\|y\|^2$$

for some $\epsilon = \epsilon_2 > 0$. Hence

$$-\gamma\|y\|^2 < V(g(y)) < \gamma\|y\|^2.$$

Thus for $\epsilon = \min\{\epsilon_1, \epsilon_2\}$, we have that

$$\begin{aligned}\Delta V(y) &= -y^T Cy + g(y)^T BPy + y^T P^T Bg(y) + V(g(y)) \\ &< -4\gamma\|y\|^2 + 3\gamma\|y\|^2 = -\gamma\|y\|^2\end{aligned}$$

for all $\|y\| < \delta$ and some $\gamma > 0$. Therefore, $\Delta V(y) < 0$ for $0 \neq y \in B(\delta)$ for some $\delta > 0$ and $V(0) = 0$. Moreover, P is hyperbolic and $\rho(P) > 1$ implies that $V(y) = y^T By < 0$ for some vectors $y \neq 0$ (Corollary 5.1.7). Let $\{\epsilon_i\}$ be a sequence of scalars such that $\epsilon_i \to 0$ as $i \to \infty$. Define $a_i = \epsilon_i y$ such that $V(y) = y^T By < 0$, then $V(a_i) = V(\epsilon_i y) = \epsilon_i^2 y^T By = \epsilon_i^2 V(y) < 0$. Thus $\{a_i\}$ is a sequence of vectors with $V(a_i) < 0$ such that $a_i \to 0$ as $i \to \infty$. Then, from Theorem 5.1.2, the extinction equilibrium is unstable. □

Theorem 5.1.10 ([3], page 155)**.** *If* $\rho(P) < 1$, *then the equilibrium of* (5.2) *is asymptotically stable.*

Proof. From (5.4), we have that (5.2) has a unique forward solution

$$y(t) = \begin{cases} P^t y^0 + \sum_{i=0}^{t-1} P^{t-i-1} g(y(i)) & t \in I\,[1, +\infty)\,, \\ y^0 & t = 0. \end{cases}$$

Let $\rho(P) < \eta < 1$ and $A = \eta^{-1}P$, then $\rho(A) = \rho(\eta^{-1}P) = \max\{|\lambda| : \lambda \in \sigma(\eta^{-1}P)\}$ so $\rho(A) = \rho(\eta^{-1}P) = |\eta^{-1}|\rho(P) = \eta^{-1}\rho(P) < 1$. Then, from Proposition 2.3.7, $A^t \to 0$ as $t \to \infty$. By the definition of convergence in a normed space, we have that $A^t \to 0$ as $t \to \infty$ if and only if $\|A^t\| \to 0$. Hence $\|A^t\| < c$ for some $c \in \mathbb{R}$. Then $\|P^t\| < c\eta^t$. Then, for all $t \in I\,[T+1, +\infty)$, we have

$$\begin{aligned}\|y(t)\| &= \left\|P^t y^0 + \sum_{i=0}^{t-1} P^{t-i-1} g(y(i))\right\| \leq \left\|P^t y^0\right\| + \left\|\sum_{i=0}^{t-1} P^{t-i-1} g(y(i))\right\| \\ &\leq \left\|P^t\right\| \|y^0\| + \sum_{i=0}^{t-1} \|P^{t-i-1}\| \|g(y(i))\| \leq c\eta^t \|y^0\| + \sum_{i=0}^{t-1} c\eta^{t-i-1} \|g(y(i))\|.\end{aligned}$$

Let $c' = \max\{1, c\}$. We have

$$\lim_{\|y\| \to 0} \frac{\|g(y)\|}{\|y\|} = 0.$$

Let

$$\epsilon = \frac{1}{2c'}(1 - \eta). \tag{5.20}$$

We can find $\delta > 0$ such that for all $y \in B(\delta)$ we have

$$\|g(y)\| < \epsilon \|y\|.$$

So, as long as $y(t) \in B(\delta)$ for $t \in I[T+1, +\infty)$, we have

$$\|y(t)\| \leq c'\eta^t \|y^0\| + \sum_{i=0}^{t-1} c'\eta^{t-i-1} \epsilon \|y(i)\|,$$

which implies

$$\eta^{-t} \|y(t)\| \leq c' \|y^0\| + \sum_{i=0}^{t-1} c'\eta^{-1} \epsilon \eta^{-i} \|y(i)\|.$$

If we let $z(t) = \eta^{-t}\|y(t)\|$, then

$$z(t) \leq c'\|y^0\| + \sum_{i=0}^{t-1} c'\eta^{-1}\epsilon z(i).$$

If we denote $m = c'\|y^0\|, k(i) = c'\eta^{-1}\epsilon$ for any i then, from Lemma 5.1.11 below, we have

$$z(t) \leq m \prod_{i=0}^{t-1} (1 + k(i))$$

as long as $z(t) \leq \eta^{-t}\delta$. That is,

$$z(t) \leq c'\|y^0\| \prod_{i=0}^{t-1} \left(1 + c'\eta^{-1}\epsilon\right).$$

Since

$$\prod_{i=0}^{t-1} \left(1 + c'\eta^{-1}\epsilon\right) = \left(1 + c'\eta^{-1}\epsilon\right) \cdots \left(1 + c'\eta^{-1}\epsilon\right) = \left(1 + c'\eta^{-1}\epsilon\right)^t,$$

we have

$$z(t) \leq c'\|y^0\| \left(1 + c'\eta^{-1}\epsilon\right)^t .$$

Substituting $z(t) = \eta^{-t}\|y(t)\|$ back, we get

$$\eta^{-t}\|y(t)\| \leq c'\|y^0\| \left(1 + c'\eta^{-1}\epsilon\right)^t ,$$

which implies

$$\|y(t)\| \leq c'\|y^0\| \left(\eta + c'\epsilon\right)^t .$$

Substituting the value of ϵ from (5.20), we have

$$\left(\eta + c'\epsilon\right) = \eta + c'\frac{1}{2c'}(1-\eta) = \eta + \frac{1-\eta}{2} = \frac{2\eta + 1 - \eta}{2} = \frac{\eta+1}{2} < 1$$

because $\eta < 1$. Therefore,

$$\|y(t)\| \leq c'\|y^0\| \left(\frac{\eta+1}{2}\right)^t \tag{5.21}$$

as long as $y(t) \in B(\delta.)$ By choosing $\|y^0\| < \min\{\delta, \frac{\delta}{c'}\}$ we have, by induction, that $y(t) \in B(\delta)$ and hence (5.21) holds for all $t \in I\,[T+1, +\infty)$. It follows that the equilibrium of (5.2) is asymptotically stable. □

Lemma 5.1.11 ([3], page 154)**.** *Let* $0 \leq k(i) \in \mathbb{R}$ *for* $i = 0, 1, 2, \ldots,$ $z(0) \leq m$ *(with* $m \geq 0$*) and*

$$z(t) \leq m + \sum_{i=0}^{t-1} k(i)z(i), \qquad t \in I\,[1, +\infty)\,, \tag{5.22}$$

then

$$z(t) \leq m \prod_{i=0}^{t-1}(1 + k(i)) \qquad t \in I\,[1, +\infty)\,. \tag{5.23}$$

Proof. Let $0 \leq u(0) = m$ and let

$$0 \leq u(t) = m + \sum_{i=0}^{t-1} k(i)u(i), \qquad t \in I\,[1, +\infty)\,.$$

Then, for all $t \in I\,[0, +\infty)$ we have

$$u(t+1) = m + \sum_{i=0}^{t} k(i)u(i) = m + \sum_{i=0}^{t-1} k(i)u(i) + k(t)u(t) = u(t) + k(t)u(t) = (1 + k(t))u(t). \tag{5.24}$$

At $t = 0$

$$u(1) = (1 + k(0))u(0) = m(1 + k(0)).$$

At $t = 1$

$$u(2) = (1 + k(1))u(1) = (1 + k(1))(1 + k(0))m = m \prod_{i=0}^{1} (1 + k(i)).$$

If we assume that at t we have $u(t) = m \prod_{i=0}^{t-1} (1 + k(i))$, then at $t + 1$ we have

$$u(t + 1) = (1 + k(t))u(t) = m(1 + k(t)) \prod_{i=0}^{t-1} (1 + k(i)) = m \prod_{i=0}^{t} (1 + k(i)).$$

It follows by induction that for all $t \in I[1, +\infty)$ we have

$$u(t) = m \prod_{i=0}^{t-1} (1 + k(i)). \tag{5.25}$$

Now, we recall that $z(0) \leq m = u(0)$. From (5.22) and (5.25), at $t = 1$ we have that $z(1) = m \leq m = u(1)$. We assume that at t we have

$$z(t) \leq m + \sum_{i=0}^{t-1} k(i)z(i) \leq m + \sum_{i=0}^{t-1} k(i)u(i) = u(t).$$

Then at $t + 1$ we have

$$\begin{aligned} u(t+1) = (1 + k(t))\, u(t) \geq (1 + k(t)) \left(m + \sum_{i=0}^{t-1} k(i)z(i) \right) &= \left(m + mk(t) + \sum_{i=0}^{t-1} k(i)z(i) + k(t) \sum_{i=0}^{t-1} k(i)z(i) \right) \\ &= \left(m + \sum_{i=0}^{t-1} k(i)z(i) + k(t) \left(m + \sum_{i=0}^{t-1} k(i)z(i) \right) \right) \geq \left(m + \sum_{i=0}^{t-1} k(i)z(i) + k(t)z(t) \right) \\ &= m + \sum_{i=0}^{t} k(i)z(i) \geq z(t + 1). \end{aligned}$$

This is correct for all $t \in I[0, +\infty)$. Thus, (5.23) is obtained. □

5.2 Long time behaviour of population models

We recall the nonlinear autonomous model (1.4),

$$\begin{aligned} x(t+1) &= P(x(t))x(t), \\ x(0) &= x^0, \end{aligned} \tag{5.26}$$

where $x^0 = [x_i^0]_{1\le i\le m}$ and $P(x) = [p_{ij}(x)]_{1\le i,j\le m}$. Let

$$P(x)x = \begin{pmatrix} [P(x)x]_1 \\ [P(x)x]_2 \\ \vdots \\ [P(x)x]_m \end{pmatrix}.$$

The Jacobian of $P(x)x$ is

$$D\left(P(x)x\right) = \begin{pmatrix} \frac{\partial}{\partial x_1}[P(x)x]_1 & \frac{\partial}{\partial x_2}[P(x)x]_1 & \cdots & \frac{\partial}{\partial x_m}[P(x)x]_1 \\ \frac{\partial}{\partial x_1}[P(x)x]_2 & \frac{\partial}{\partial x_2}[P(x)x]_2 & \cdots & \frac{\partial}{\partial x_m}[P(x)x]_2 \\ \vdots & \vdots & & \vdots \\ \frac{\partial}{\partial x_1}[P(x)x]_m & \frac{\partial}{\partial x_2}[P(x)x]_m & \cdots & \frac{\partial}{\partial x_m}[P(x)x]_m \end{pmatrix}.$$

Hence

$$D\left(P(x)x\right) = \begin{pmatrix} \left[\left(\frac{\partial}{\partial x_1}P(x)\right)x + P(x)e_1\right]_1 & \left[\left(\frac{\partial}{\partial x_2}P(x)\right)x + P(x)e_2\right]_1 & \cdots & \left[\left(\frac{\partial}{\partial x_m}P(x)\right)x + P(x)e_m\right]_1 \\ \left[\left(\frac{\partial}{\partial x_1}P(x)\right)x + P(x)e_1\right]_2 & \left[\left(\frac{\partial}{\partial x_2}P(x)\right)x + P(x)e_2\right]_2 & \cdots & \left[\left(\frac{\partial}{\partial x_m}P(x)\right)x + P(x)e_m\right]_2 \\ \vdots & \vdots & & \vdots \\ \left[\left(\frac{\partial}{\partial x_1}P(x)\right)x + P(x)e_1\right]_m & \left[\left(\frac{\partial}{\partial x_2}P(x)\right)x + P(x)e_2\right]_m & \cdots & \left[\left(\frac{\partial}{\partial x_m}P(x)\right)x + P(x)e_m\right]_m \end{pmatrix},$$

where $e_1 = \begin{pmatrix}1 & 0 & \cdots & 0\end{pmatrix}^T, e_2 = \begin{pmatrix}0 & 1 & \cdots & 0\end{pmatrix}^T, \ldots, e_m = \begin{pmatrix}0 & 0 & \cdots & 1\end{pmatrix}^T$.

Thus

$$D\left(P(x)x\right) = \begin{pmatrix} \left[\left(\frac{\partial}{\partial x_1}P(x)\right)x\right]_1 & \left[\left(\frac{\partial}{\partial x_2}P(x)\right)x\right]_1 & \cdots & \left[\left(\frac{\partial}{\partial x_m}P(x)\right)x\right]_1 \\ \left[\left(\frac{\partial}{\partial x_1}P(x)\right)x\right]_2 & \left[\left(\frac{\partial}{\partial x_2}P(x)\right)x\right]_2 & \cdots & \left[\left(\frac{\partial}{\partial x_m}P(x)\right)x\right]_2 \\ \vdots & \vdots & & \vdots \\ \left[\left(\frac{\partial}{\partial x_1}P(x)\right)x\right]_m & \left[\left(\frac{\partial}{\partial x_2}P(x)\right)x\right]_m & \cdots & \left[\left(\frac{\partial}{\partial x_m}P(x)\right)x\right]_m \end{pmatrix} + \begin{pmatrix} [P(x)e_1]_1 & [P(x)e_2]_1 & \cdots & [P(x)e_m]_1 \\ [P(x)e_1]_2 & [P(x)e_2]_2 & \cdots & [P(x)e_m]_2 \\ \vdots & \vdots & & \vdots \\ [P(x)e_1]_m & [P(x)e_2]_m & \cdots & [P(x)e_m]_m \end{pmatrix}.$$

Let $P(x) = \begin{pmatrix} \vdots & \vdots & \cdots & \vdots \\ P^1(x) & P^2(x) & \cdots & P^m(x) \\ \vdots & \vdots & \cdots & \vdots \end{pmatrix}$, where $P^i(x)$ denotes the ith column of $P(x)$ for $i = 1, 2, \ldots, m$. Then $P(x)e_i = P^i(x)$ for $i = 1, 2, \ldots, m$, implies that

$$\begin{pmatrix} [P(x)e_1]_1 & [P(x)e_2]_1 & \cdots & [P(x)e_m]_1 \\ [P(x)e_1]_2 & [P(x)e_2]_2 & \cdots & [P(x)e_m]_2 \\ \vdots & \vdots & & \vdots \\ [P(x)e_1]_m & [P(x)e_2]_m & \cdots & [P(x)e_m]_m \end{pmatrix} = \begin{pmatrix} \left[P^1(x)\right]_1 & \left[P^2(x)\right]_1 & \cdots & [P^m(x)]_1 \\ \left[P^1(x)\right]_2 & \left[P^2(x)\right]_2 & \cdots & [P^m(x)]_2 \\ \vdots & \vdots & & \vdots \\ \left[P^1(x)\right]_m & \left[P^2(x)\right]_m & \cdots & [P^m(x)]_m \end{pmatrix} = P(x).$$

Therefore,

$$D\left(P(x)x\right) = P(x) + \left[\left(\frac{\partial}{\partial x_i}P(x)\right)x\right],$$

where $\left[\left(\frac{\partial}{\partial x_i}P(x)\right)x\right]$ is the matrix whose i^{th} column is $\left(\frac{\partial}{\partial x_i}P(x)\right)x$. Hence, the Jacobian $D\left(P(x)x\right)$ at $x = 0$ is $P(0)$. Thus, the linearization of $P(x)x$ about $x = 0$ is

$$P(x)x = P(0)x + g(x),$$

with

$$\lim_{\|x\|\to 0} \frac{\|g(x)\|}{\|x\|} = 0.$$

Therefore, the problem (5.26) close to the extinction equilibrium $x = 0$ can be written as

$$x(t) = P(0)x(t) + g(x(t)),$$
$$x(0) = x^0,$$

Denote $p(t) = \|x(t)\|_1$ to the total population at time t. We make a biologically meaningful assumption that the initial density of a population $x^0 \geq 0$. From Lemma (3.1.1), in order to get a non-negative solution $x(t) \geq 0$, we must have $P(x) \geq 0$. Under the assumption that the matrix $P(x)$ is deleterious at all density levels; that is, $0 \leq P(x) \leq P(0)$ for all $x \in \mathbb{R}_+^m$, it follows that $0 \leq x \leq y$ implies that $0 \leq P(x)x \leq P(0)x \leq P(0)y$. Then

$$0 \leq P(x(t))x(t) \leq P(0)x(t) \tag{5.27}$$

for all $t \in I\,[0, +\infty)$. We denote by $y(t)$ the solution of the initial value problem,

$$\begin{aligned} y(t+1) &= P(0)y(t), \\ y(0) &= x^0, \end{aligned} \tag{5.28}$$

for $t \in I\,[0, +\infty)$. Then at $t = 0$ we have that $y(1) = P(0)y(0) = P(0)x^0$. Hence, from (5.27), it follows that $x(1) = P(x^0)x^0 \leq P(0)x^0 = P(0)y(0) = y(1)$. Thus

$$x(1) \leq y(1). \tag{5.29}$$

At $t = 1$, we have $y(2) = P(0)y(1)$. Then from (5.27) and (5.29), it follows that $x(2) = P(x(1))x(1) \leq P(0)x(1) \leq P(0)y(1) = y(2)$. Thus

$$x(2) \leq y(2). \tag{5.30}$$

If we assume that at $t - 1$ we have

$$x(t-1) \leq y(t-1)$$

then, from (5.27), at t we have $y(t) = P(0)y(t-1) \geq P(0)x(t-1) \geq P(x(t-1))x(t-1) = x(t)$. Thus

$$x(t) \leq y(t). \tag{5.31}$$

Hence, by induction, we have that

$$x(t) \leq y(t) \tag{5.32}$$

for all $t \in I\,[0, +\infty)$. We make a mathematical assumption that $P(0)$ is an irreducible, primitive and hyperbolic matrix. Denote $r = \rho(P(0))$. If $r < 1$, then from Theorem 5.1.10 the equilibrium $x = 0$ of (5.26) is asymptotically stable. Moreover, from Theorem 4.1.2, we have that $\|y(t)\|_1$ tends exponentially to 0 as $t \to +\infty$. Consequently, from (5.32), $p(t) = \|x(t)\|_1 \to 0$ as $t \to \infty$. Thus, we have global extinction of the population. If $r > 1$, from Theorem 5.1.9, we have that $x = 0$ is unstable.

Conclusion

In this work we study matrix models of population theory. In order to fully understand the evolution of such models, we study the theory of general matrices from spectral theory point of view and provide an overview of their properties. Thus, we study the Jordan forms of general matrices and use them to show the limit behaviour of a matrix (Proposition 2.3.7 and Theorem 2.3.9). We study the Perron-Frobenius type theorems for both positive and irreducible matrices regarding their spectral properties and provide detail of their proofs. We use these theorems to investigate the asymptotic behaviour of linear autonomous models arising in structured population following [3]. They consider diagonalizable matrices and they find that while the total population may increase or decrease depending on the spectral radius of the associated matrix, there is a proportion of individuals in each class stabilizes as time increase. In this thesis, we generalize their work to any matrix whether it is diagonalizable or not. We study stability of a hyperbolic equilibrium of a general nonlinear autonomous model following [4]. They find that the stability conditions of such equilibrium are determined by the spectral radius of the associated nonlinear matrix. Finally, we consider a general nonlinear autonomous model that arises in structured population. We assume that the associated nonlinear matrix of this model depends explicitly on the population density and that this matrix is non-increasing at all density levels. We investigate the stability of a hyperbolic extinction equilibrium of this model.

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