EXPLORING PRE-SERVICE TEACHERS' UNDERSTANDING OF SIMILARITY AND PROOFS IN EUCLIDEAN GEOMETRY

BY

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A dissertation submitted to the University of KwaZulu-Natal (UKZN), Edgewood Campus, School of Education, in fulfilment of the academic requirement for the Degree of Master of Education in Mathematics

SUPERVISOR

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DECLARATION

I, **MDUDUZI MHLENGI MBATHA (214523136)**, declare that this dissertation entitled ***'Exploring pre-service teachers' understanding of similarity and proofs in Euclidean geometry'*** submitted for the qualification of Masters' Degree in Mathematics Education at the University of KwaZulu-Natal is my original work, and:

i. This dissertation has not been previously submitted to any tertiary institution for a degree or diploma.
ii. This dissertation does not contain other persons' tables, pictures, graphs, or other information unless specifically acknowledged as being sourced from others.
iii. Where the work of others has been used, it has been cited and referenced.

This study commenced from February 2020 to February 2022 under supervision of Prof Sarah Bansilal from the School of Education, Mathematics and Computer Science Education Cluster, University of KwaZulu-Natal, Edgewood Campus.

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As the candidate's supervisor, I agree to the submission of this dissertation.

Supervisor: Prof Sarah Bansilal

Signature: [redacted] Date: 02 February 2022

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DEDICATION

I dedicate this dissertation to my parents:

- My mother, Nomusa Buyenzeni Mbatha, raising a family of six siblings while not working – you have shown me how to sow the seeds of success. I love you always!

- My late father, Mfunelwa Petros Mbatha (1958-2001); a man who was an inspiration to everyone who knew him. Daddy, you are in my heart and thoughts always.

TABLE OF CONTENTS

LIST OF ABBREVIATIONS

CAPS: Curriculum and Assessment Policy Statement

DBE: Department of Basic Education

FET: Further Education and Training

KZN: KwaZulu-Natal

PCK: Pedagogical Content Knowledge

PSTs: Pre-Service Teachers

SA: South Africa

SCK: Subject Content Knowledge

VHL: Van Hiele Level

LIST OF TABLES

LIST OF FIGURES

ABSTRACT

This qualitative study explores pre-service teachers' understanding of similarity and proofs in Euclidean geometry in one South African university from KwaZulu-Natal (KZN) province. Such insight is vital for addressing pre-service teachers' geometric knowledge, which has been found lacking. The research participants were 34 pre-service teachers (PSTs) in their first year of study towards a Bachelor of Education degree specialising in mathematics at the FET phase. A pen and paper test and semi-structured interviews were employed in gathering the required data for this study. The Van Hiele levels of geometric thought were used as a theoretical framework, which formed the basis for the analysis and discussion of findings.

The findings indicated that most pre-service teachers performed adequately on familiar items but struggled with those unfamiliar, which were not typical grade 12 examinable questions. A follow-up of semi-structured interviews was conducted with seven PSTs of mixed abilities to probe the originality of their written responses. Although all interviewed PSTs indicated an improvement when responding to research items verbally than in writing, they did not reach the expected acquisition necessary to teach geometry effectively. Overall, this study found that many PSTs displayed poor levels of understanding similarity and proofs, including (1) limited understanding of the definition of similarity to triangles; (2) poor understanding of how to prove two figures are similar; (3) the haphazard use of geometric theorems in devising proofs; (4) a display of higher Van Hiele levels of understanding for familiar items but lower levels of understanding for unfamiliar items. These findings raised concerns about this group of PSTs teaching geometry, especially if certain concepts require more complex skills that are slightly beyond the secondary school curriculum.

It is recommended that professional teacher education training offered to pre-service teachers should include aspects such as (1) Improving PSTs' geometry content knowledge, (2) Teaching geometry for understanding and (3) Improving PSTs' written mathematical responses. These factors may be pivotal in improving pre-service teachers' geometric knowledge beyond the scope of the secondary school curriculum.

CHAPTER 1

INTRODUCTION TO THE STUDY

1.1 Introduction

Ball et al. (2005) argued that the quality of teachers' knowledge influences the quality of their teaching since the way the content is learned chiefly depends on the depth of the teacher's knowledge. It is incontestable that teachers who do not know a subject well might not help learners understand it. Having the required understanding of subject knowledge allows the teacher to assist learners in relaying ideas and addressing misconceptions. Hence this study explores pre-service teachers' content knowledge of similarity and proofs in Euclidean geometry since it has not been extensively researched. This chapter provides the problem statement, research questions, rationale of the study, and the organisation of chapters.

1.2 Problem statement

In South Africa (SA), the mathematics curriculum designers were challenged in deciding on poor performance in Euclidean geometry versus its importance. From 2008 to 2013, Euclidean geometry was made an optional section of mathematics examined as paper three in the grade 12 national examinations. It was also discovered that very few learners did mathematics paper three. Table 1.1 below compares the number of learners who wrote the grade 12 mathematics examinations versus those who did mathematics paper three, which included Euclidean geometry (Department of Basic Education, 2013, 2011). The fourth column presents the percentage of learners who did paper three.

Table 1.1: Number of learners who wrote the grade 12 mathematics examination versus mathematics paper three from 2009 to 2013 (DBE, 2013, 2011)

Year	Number of learners wrote mathematics	Number of learners wrote mathematics paper 3	Percentage of learners wrote mathematics paper 3
2009	290 407	11 755	4, 05%
2010	263 034	9 454	3,59%
2011	224 635	8 871	3,94%
2012	225 874	8 878	3,93%
2013	241 509	9 302	3,85%

Table 1.1 above shows that from 2009 to 2013, many teachers and learners did not take up mathematics paper three, suggesting that most teachers and learners did not feel adequately prepared for Euclidean geometry and other optional sections. Ndlovu (2011), Atebe and Schäfer (2010), and Bowie (2009) noted that teachers avoided the teaching of Euclidean geometry in schools because of poor mastery of subject content. Furthermore, a common perception was that Euclidean geometry was complex for both teachers and learners, and the mathematics pass rate for most schools was below average. Therefore, most schools chose to opt out of it to have enough time to focus on compulsory papers to boost their pass rate (Bowie, 2009). However, with the new Curriculum Assessment and Policy Statement (CAPS), Euclidean geometry has been reinstated as a compulsory examinable section forming part of mathematics paper two from 2012 in grade 10, 2013 in grade 11, and 2014 in grade 12. When Euclidean geometry was brought back into the core mathematics curriculum, most teachers did not feel confident about the strand since it had not been taught for long (Bowie, 2009).

Many researchers (Ngirishi & Bansilal, 2019; Bowie, 2009; Lee & Ginsburg, 2009; Mthembu, 2007; Singh, 2006) have expressed their concerns regarding underachievement in Euclidean geometry. According to Patkin and Lavenberg (2012, p.14), "Euclidean geometry is experienced as the most challenging section of the curriculum which is structured in an "unusual" manner which does not connect to reality". It may be why Euclidean geometry has posed a serious challenge to most teachers and learners in South African schools. According to Ndlovu (2012), this lack of understanding of geometric concepts results from students' poor geometry knowledge in secondary school. Ngirishi and Bansilal's (2019) study with grade 10 and 11 learners in SA also showed that they generally encountered difficulties in Euclidean geometry and performed poorly. Learners struggled with geometric proof questions that required two or more steps. The authors found that many learners operate at Van Hiele level 1, visualisation (Ngirishi & Bansilal, 2019). The Department of Basic Education (2020) has also commented on learners' lack of understanding geometry, showing the following errors made by learners: *a) "When proving theorems, statements are not written in a logical sequence b) Learners often base their responses on a question on the visual appearance of a given diagram, resulting in learners making assumptions not directly related to the given diagram"* (p.204). Hence, as a strategy for future improvement, it was recommended that significant time be devoted to teaching geometry to gain the skills required to succeed in geometry (DBE, 2020).

When Euclidean geometry was still optional, teachers decided whether they would teach it or not, depending on their confidence and knowledge of the subject content. Now that it is compulsory, teachers must teach Euclidean geometry regardless of whether they know the content well enough or not. This may be one of the reasons for continuous underachievement in Euclidean geometry. For example, the diagnostic report of 2016, 2017, 2018 and 2019 showed that, in mathematics paper two, Euclidean geometry was amongst poorly answered questions (DBE, 2020, p.192; DBE, 2019, p.175; DBE, 2018, p.187; DBE, 2017, p.180). This continuing trend of poor performance in Euclidean geometry led to the following questions: How well do pre-service teachers know the content knowledge of similarity and proofs in Euclidean geometry? What kind of content knowledge do pre-service teachers' need in similarity and proofs? What could be possible reasons for low levels of understanding similarity and proofs in Euclidean geometry? The current study explores these issues in detail. These questions are essential in restructuring mathematics teachers' education programs to respond to the current needs of pre-service teachers.

1.3 Research questions

In gaining insight into the issues mentioned above, this study conducted in one South African university in KwaZulu-Natal province explored the following research questions:

1. How do pre-service teachers perform on tasks based on similarity and proofs?
2. What are the misconceptions pre-service teachers have about similarity and proofs in Euclidean geometry?
3. What are the Van Hiele levels of geometric thought of pre-service teachers in similarity and proofs?

1.4 The rationale of the study

The rationale of this study comprises two parts: (1) addressing the gap in the field of study of similarity and proofs and (2) the researcher's personal experience of teaching similarity and proofs.

1.4.1 Addressing the gap in the study of similarity and proofs

Many scholars (Dhlamini, 2012; Van Putten, Stols & Howie, 2010; Bowie, 2009; Mthembu, 2007; de Villiers, 1997) have explored various topics in Euclidean geometry. However, research on similarity and proofs is restricted, especially in South Africa (Ubah & Bansilal, 2019).

Even on an international level, very little research on similarity and proofs exists on pre-service teachers' knowledge (Clark-Wilson & Hoyles, 2017; Seago et al., 2014). Thus, these reasons motivated the pursuit of this study to contribute to the body of knowledge in the field. This study explored the understanding of similarity and proofs by PSTs who matriculated when Euclidean geometry was reinstated as a compulsory mathematics section at secondary school. This investigation seems crucial because even when the minimum of 60% pass in mathematics in matric examination is placed as the prerequisite for PSTs to undertake mathematics FET courses, looking at the proportions of topics weighting, PSTs could obtain +60% without having mastered Euclidean geometry content. The previous statement supports Van Putten et al. (2010) that many pre-service teachers in SA leave matric with a poor understanding of geometry, even their marks seem to suggest otherwise. Therefore, it is essential that PSTs' understanding of Euclidean geometry should be monitored and supported during their professional teacher education training period.

1.4.2 Personal experience of teaching similarity and proofs

To a large extent, the rationale for this study is personally and professionally motivated. The researcher was a tutor for two geometry modules while studying at university and currently a secondary school mathematics teacher for the past four years. During these years of teaching mathematics, the researcher noticed that most students struggled to conceptualise geometric concepts.

According to the South African education system, mathematics is compulsory for all learners up to grade 9; thereafter, learners can either continue with mathematics or take up mathematical literacy. From the researcher's anecdotal observation, many grade-10 learners shy away from pure mathematics and prefer mathematical literacy, which they assume to be easier than pure mathematics. Furthermore, it was noticed that learners with mathematics fear Euclidean geometry most compared to other mathematics sections. It is no secret that our current teaching methods do little to equip learners with the required skills to succeed in geometry, as indicated by the low performance and a high number of learners giving up the subject area. Table 1.2 below compares the number of grade 12 learners who wrote mathematics national examinations versus those who wrote mathematical literacy from 2016 to 2020 (Department of Basic Education, 2020).

Table 1.2: Number of learners who wrote mathematics versus mathematical literacy

Year	Number of learners who wrote mathematics	Number of learners who wrote mathematical literacy
2016	265 810	361 865
2017	245 103	313 030
2018	233 858	294 204
2019	222 034	298 607
2020	233 315	341 363

Table 1.2 shows that from 2016 to 2020, more learners wrote mathematical literacy than pure mathematics, suggesting that more learners opt for mathematical literacy in grade 10. Therefore, research in mathematics education should address these challenges in geometry. Hence, it was necessary to unpack pre-service teachers' content knowledge of similarity and proofs before their full-time teaching career. By identifying their content knowledge, the mathematics community (parents, teachers, examiners, lecturers, subject advisors) can be better informed about the problems related to poor teaching and learning of similarity and proofs in Euclidean geometry.

1.5 Organisation of chapters

This dissertation is structured as follows:

Chapter 1

This chapter provides the reasons for advancing this study by outlining the problem statement, the research questions, and the rationale.

Chapter 2

This chapter discusses the literature review and theoretical considerations of the study.

Chapter 3

This chapter gives the research procedure employed in the current study, including paradigm, research approach, sampling procedures, data collection and analysis procedures, and ethical procedures.

Chapter 4

This chapter presents the findings and analysis of the data collected from the field.

Chapter 5

This chapter discusses the findings of this study related to the research questions, recommendations, limitations and suggestions for further research.

CHAPTER 2

LITERATURE REVIEW AND THEORETICAL CONSIDERATION

2.1 Literature review

Since the current study explored pre-service teachers' understanding of similarity and proofs in Euclidean geometry, the chapter discusses the importance of Euclidean geometry followed by the Euclidean geometry curriculum in South Africa. Later, the conceptual and procedural understanding in mathematics is explored. In conclusion, the similarity and proofs in Euclidean geometry followed by related misconceptions are discussed.

2.1.1 Importance of Euclidean geometry

Euclidean geometry is integral for analysing and describing the space in which one lives. It improves the ability of cognition of spatial items, their images and analysis (Battista, 2007). According to Bassarear (2012), Euclidean geometry involves computing the properties of shapes such as lengths, angles, area and perimeter to determine relationships and properties of shapes. These properties of shapes play a significant role in science and technology applications, including the construction industry, design, and architecture (Knight, 2006). According to Chambers (2008), geometry has links with the real world; for example, geometrical concepts such as circles, triangles, squares and rectangles are used as traffic road signs meant for danger warning signs, information and control (Siyepu & Mtonjeni, 2014). Euclidean geometry improves thinking and reasoning through the solution of riders and the writing of proofs to represent and make sense of the world (Ozerem, 2012; de Villiers, 1997). These skills are essential for technologically related careers such as planning and building quality houses and bridges in the 21st century. For this reason, in Nigeria, Adegun and Adegun (2013) emphasised the need to make Euclidean geometry a compulsory section of mathematics at schools if scientists, technologists, and engineers are to be produced in universities. Such views justify the importance of Euclidean geometry in our lives.

2.1.2 The Euclidean geometry curriculum in South Africa

In South Africa, Euclidean geometry was a compulsory section of mathematics before 2008, but after it became an optional section examined as paper three in the grade 12 national examinations (Alex & Mammen, 2014). Later, a new curriculum called Curriculum and Assessment Policy Statement (CAPS), first implemented in the FET phase in grade 10 in 2012,

brought back Euclidean geometry as a compulsory section of mathematics paper two after it was moved to optional paper three (DBE, 2011). The general aims of CAPS (2011) grades (R-12) include but are not limited to: a) promote "an active and critical approach to learning, rather than rote and uncritical learning of given truths" (p.3), b) produce learners that "collect, analyse, and organise quantitative data to evaluate and critique conclusions" (p.3). Moreover, the CAPS curriculum lists the following specific aims peculiar to mathematics (CAPS, 2011):

> a) Mathematical modelling is an essential focal point of the curriculum. Real-life problems should be incorporated into all sections whenever appropriate.
>
> b) To develop problem-solving and cognitive skills. Teaching should not be limited to "how" but instead feature the "when" and "why" of problem types. (p.8)

The sequence of geometry content in the CAPS curriculum suggests hierarchy building upon the work done in previous grades. For example, in grade 10, unique properties of quadrilaterals are explored using concepts introduced in the previous grades, such as triangles, parallel lines and angles. Likewise, in grade 11, circle theorems and grade 12 triangle theorems are explored using concepts learnt in previous grades.

When teaching Euclidean geometry and its relationship with the CAPS requirement, it can be assumed that learners must be offered appropriate opportunities to develop the skill of hypothesising, discovery, and experimentation. However, some scholars noted that the pedagogical approach of most textbooks and teachers is that of giving rules, proofs and procedures to learners without leaving an opportunity for learners to experiment, discover and critically conclude (de Villiers, 2012). The following example indicates how textbooks generally present theorems as a finished product to learners. Figure 2.1 shows how the textbook presents the tan-chord theorem to learners.

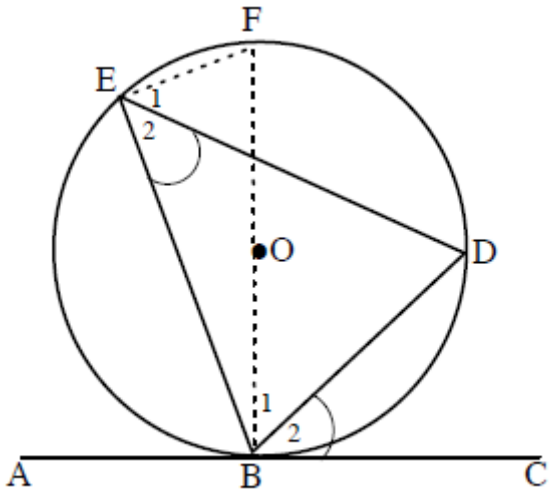


Figure 2.1: Diagram showing tan-chord theorem

The angle between a tangent to a circle and chord equals the angle the chord subtends at the alternate segment.

ABC is a tangent, BD is a chord, and E is a point on the major arc.

Required to prove (RTP): $C\hat{B}D = B\hat{E}D$

Proof:

Construction: Draw diameter BF and join EF.

Table 2.1: Proof of the tan-chord theorem

Statement	***Reason***
$\hat{B}1 + \hat{B}2 = 90°$	Tan ⊥ radius
$\hat{E}1 + \hat{E}2 = 90°$	∠ in semi-circle
$\hat{B}1 + \hat{B}2 = \hat{E}1 + \hat{E}2$	Both = 90°
$\hat{B}1 = \hat{E}1$	∠ in the same segment
$\Longrightarrow B2 = \hat{E}2$	
$\therefore C\hat{B}D = B\hat{E}D$	

When using this theorem to solve riders, the reason is ∠ between tangent and chord.

In the previous example, teachers strictly follow textbooks; hence, learners are deprived of critical learning and analysing the theorem's logic. If things are to be fixed, teachers and textbooks must offer learners the opportunity to discover and experiment for themselves instead of presenting them with a set of steps. Furthermore, there needs to be a well-sequenced and coherent geometry curriculum from primary to secondary school, allowing learners to develop skills as they move along (de Villers, 1997). Some authors (Siyepu, 2005; de Villers, 1997) attributed failure in geometry as being loaded with formal geometry in secondary school while very little is covered informal in primary school. Hence, learners entered secondary school without sufficient experience to cope with the demand of secondary school geometry.

2.1.3 Conceptual and procedural understanding in Mathematics

According to Stylianides and Stylianides (2007), learning with understanding has been a challenge in mathematics. This view is supported by Ndlovu (2012), who found that the university students emerging from the schooling system do not have a thorough understanding of geometric concepts. Instead, they have a collection of rules and algorithms that sometimes become barriers to their learning. They encounter difficulties sifting through the cluttered memorised rules to select the appropriate method for a particular problem. Instead of learning for understanding, students devise coping skills such as manipulative approaches and drills to pass through the examination without deeply engaging in understanding problems (Jojo, 2010).

To succeed in mathematics, conceptual understanding, which allows connections between concepts, is necessary. Hiebert and Carpenter (1992) defined conceptual understanding as organised knowledge showing through understanding the relationship between mathematics concepts, the ability to link the facts in mathematics, knowing different methods that could be useful to solve a particular problem and concept boundaries. Conceptual understanding requires more profound knowledge than memorising formulas and rules. When students produce correct answers to problems, it is assumed that they understand the concept, but research has shown that it is not always the case. Hasenbank (2006) argued that students with poor conceptual understanding could perform well in problem testing procedural understanding, but their knowledge is soon forgotten. Biggs and Tang (2007) refer to this as surface learning instead of deep learning.

Another critical part of mathematics is procedural understanding. Hiebert and Carpenter (1992) also define procedural understanding as a sequence of actions that allow the completion of mathematical tasks efficiently. It includes recalling facts and performing a skill in the same context learned.

Hiebert and Lefevre (1996) emphasised that procedures learned with meaning are connected to conceptual understanding, which means that knowing procedures for only familiar context is not enough to succeed in mathematics. For example, Brijlall and Ndlovu (2013) found matric learners who could use derivatives to find the maximum or minimum value of the cubic function but failed to provide the same application in a quadratic function when it was needed. It indicated that learners memorised the procedure for finding the maximum or minimum value of the cubic function but lacked conceptual understanding and appropriate use of function derivatives. It may suggest that some schools emphasise finding answers quickly without connecting them to conceptual understanding. Procedural and conceptual understanding tend to develop on separate tracks when this occurs. Learning with understanding becomes affected because procedures learned without a conceptual understanding may be forgotten quickly or cannot be applied to unfamiliar contexts.

In geometry, conceptual and procedural understanding are vital for distinctions between concept definition and a concept image. Vinner and Hershkowitz (1980) proposed concept definition and concept image to describe how students make sense of mathematical concepts. A concept definition is a set of words that explains a concept. Meanwhile, a concept image "is a set of cognitive structures involving mental images related to a concept" (p.177). Cunningham and Roberts (2010) found that when students are asked to state the definition of a concept, they depend on their prior experiences and examples; hence, they usually state the concept image, which sometimes is limited to prototype diagrams. The concept definition is knowing general definitions for non-prototype examples and concept boundaries. Conceptual understanding of geometrical concepts is broader than knowing concepts for specific figures and having "skills required to manipulate geometric shapes" (Luneta, 2015, p.1). Cunningham and Roberts (2010) found that pre-service elementary teachers lack conceptual understanding of geometric concepts required to teach effectively. Therefore, mathematics education must improve conceptual understanding to promote meaningful mathematics learning.

2.1.4 Similarity in Euclidean geometry

Many South African mathematics textbooks define similarity as two polygons that satisfy the two conditions: (1) they have equal corresponding angles, and (2) their corresponding sides are proportional. Therefore, similarity requires the understanding that the size of angles of a rectilinear figure could be conserved, although the length of its sides is altered according to the same ratio. Understanding similarity also requires mastering concepts like ratio and proportion as these concepts form the foundation. On the one hand, a ratio is the comparison of the two quantities with the same units in the form $\frac{a}{b} = k$ (where k is the constant). On the other hand, proportion is the comparison of two or more equal ratios such that $\frac{a}{b} = \frac{c}{d} = k$ (where k is the constant). Understanding these concepts is vital for identifying similar figures to locate equal corresponding angles and corresponding sides in proportion. The notation for similarity is (///); when naming similar figures, the order of naming figures is important because it shows equal corresponding angles and corresponding sides proportional, which both can help compute missing values.

The concept of similarity is a daily life experience; for example, the object and its image are an example of similar figures. Even "young children have experiences from daily life that contribute to their intuitions about scale, including playing with miniature toys, building scale models, enlarging or shrinking images" (Lehrer et al., 2002 cited by Cox, 2013, p.4). These rich experiences can improve the understanding of similarity relationships. While similarity is the general concept applicable to all rectilinear figures, some studies found that many textbooks focus on the similarity of triangles. For example, Amaral and Hollebrands (2017) examined context-based similarity tasks in Brazil and United States books. They found six different books, three from each country focusing more on the similarity of triangles. In addition, the context-based similarity task made up less than 30% of the similarity task, which was also of low cognitive demand. The few existing research about similarity also focused on the similarity of triangles (Haj-Yahya 2021; Ubah & Bansilal, 2019), although narrowing this concept to triangles has not improved its understanding to students. The study by Haj-Yahya (2021), which explored student conceptions of congruent and similar triangles, found a poor understanding of geometric definitions and theorems. Learners did not accept similar triangle theorems as formal definitions of similarity because of the concern for one uniformity concept definition.

Hence, de Villiers (2004) emphasised that definitions are essential for geometrical knowledge and the deductive structure of geometry; they are essential in understanding the meanings of the concepts and are used as building blocks for the construction of geometrical theorems. Turning definitions into theorems and theorems into definitions may be necessary to develop a complete understanding of geometric concepts. The theorems are rephrased to state the conditions necessary and sufficient to define the concepts.

In Ubah and Bansilal's (2019) exploration of pre-service teachers' reasoning about similar triangles, the findings showed that some participants struggled to identify similar triangles in different diagrams. Some pre-service teachers were challenged with basic skills like identifying equal angles in the triangles, which helps in naming similar triangles; they had no means to solve the problem. Ubah and Bansilal support Gal and Linchevski's (2010) findings that students struggle to differentiate between various configurations present in geometric diagrams, which assist in successfully solving the problem. Gal and Linchevski (2010) revealed that students found it easier to identify similar triangles in figures such as Shape 1 ($\Delta AB'C'$ and ΔABC, where $\angle AB'C'$ and $\angle ABC$ are right angles) than that of Shape 2 (ΔKLM and ΔKNL, where $\angle KLM$ and $\angle KNL$ are right angles). Shape 2 requires a deep understanding of geometry to identify equal angles, which can assist to name similar triangles within the two triangles.

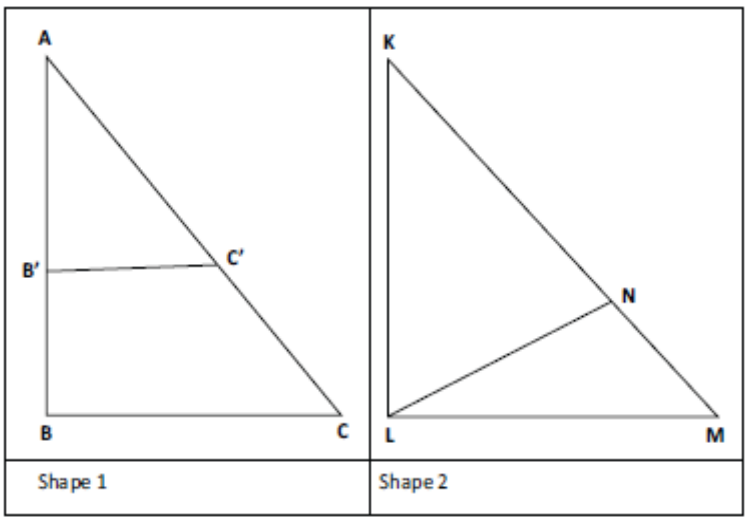
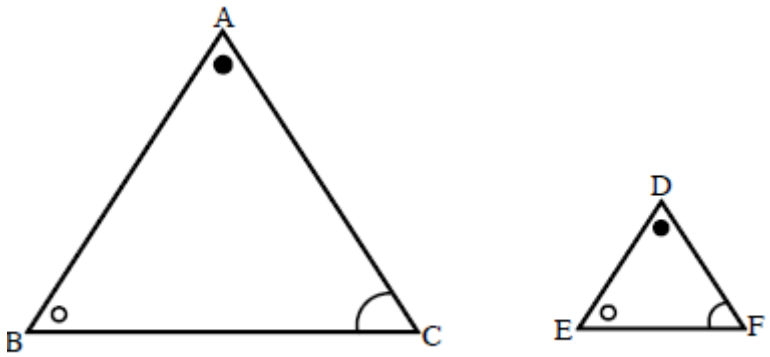


Figure 2.2: Shapes requiring different mental transformations (Gal & Linchevski, 2010)

2.1.5 Proof in Euclidean geometry

According to Otton (2007), proof has been the central focus of mathematics curricula in many countries. Mudaly and de Villiers (2004) justified the inclusion of formal geometric proofs in mathematic curricula as a medium for teaching and learning deductive reasoning. To emphasise the importance of proof, Hanna (2000) stated that learners who had not learned proofs had not learned mathematics. A proof is a logical argument statement of complete thinking procedures in a step-by-step manner explaining why these steps are achievable until a new valid conclusion is reached (Ngirishi & Bansilal, 2019). Proof uses definitions, axioms and previously proved theorems to arrive at a new conclusion. Proof can lead to further discoveries demonstrating the need for better definitions or yield a useful algorithm. For example, below is the proof showing that if two triangles are equiangular, their corresponding sides are in proportion and are similar. Thus, it demonstrates how the definition of similar triangles can be refined to yield the valuable algorithm that equiangular triangles are similar without establishing that the corresponding sides need to be proportional. Therefore, for two triangles to be similar, it is sufficient that they are equiangular. Figure 2.3 shows equiangular triangles.

Given: $\Delta ABC \; and \; \Delta DEF, with \; \hat{A} = \hat{D}, \hat{B} = \hat{E} \; and \; \hat{C} = \hat{F}.$



Required to proof: $\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$

Proof: On AB construct AG such that AG = DE.

On AC construct AH, such that AH =DF.

Join GH.

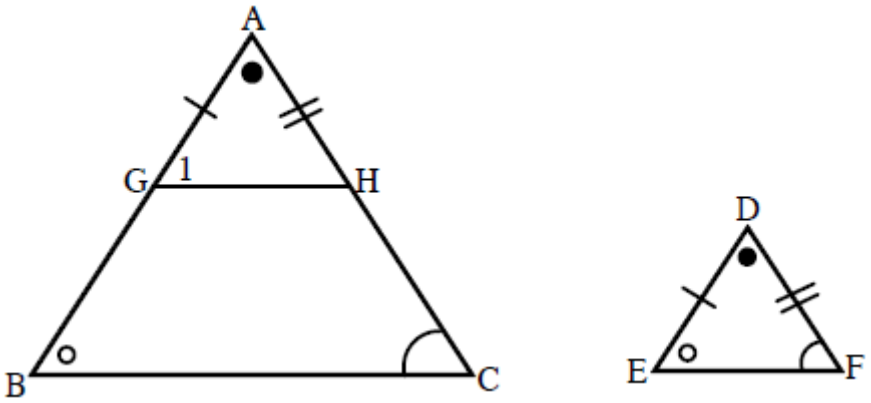


Figure 2.3: Equiangular triangles

Table 2.2: Proof showing equiangular triangles are similar

Statement	***Reason***
1. AG =DE	Construction
2. $\hat{A} = \hat{D}$	Given
3. AH = DF	Construction
$\therefore \Delta AGH \equiv \Delta DEF$	Side, Angle, Side
$\Rightarrow \hat{G}1 = \hat{E}$	$\Delta AGH \equiv \Delta DEF$
But $\hat{B} = \hat{E}$	Given
$\Rightarrow \hat{G}1 = \hat{B}$	Both equal to $\hat{E}$
Therefore, GH//BC	Corresponding angles equal ($\hat{G}1 = \hat{B}$)
$\therefore \frac{AB}{AG} = \frac{AC}{AH}$	Line // to one side of triangle divide two sides proportionally
$\Rightarrow \frac{AB}{DE} = \frac{AC}{DF}$	$AG = DE \text{ and } AH = DF$
Similarly, by constructing BG and BH on AB and BC respectively, it can be proved that: $\frac{AB}{DE} = \frac{BC}{EF}$	
$\therefore \frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$ Therefore, triangles are similar.	Equal corresponding angles and corresponding sides are in proportion.

It is evident from the above proof that although similarity is defined as figures with equal corresponding angles and corresponding sides are proportional, for triangles, equal corresponding angles imply corresponding sides are proportional; hence triangles are similar.

In this way, the proof leads to the discovery of special conditions for the similarity of triangles. Furthermore, it can be concluded that in two triangles, if two pairs of corresponding angles are equal, the remaining third angles will also be equal since the angles of a triangle add up to 180°. Thus, triangles with two pairs of equal corresponding angles can be classified as similar. These are rich experiences which proofs make necessary to learn. However, understanding learners' cognitive thinking is important to avoid memorisation when teaching proofs (de Villiers, 2004).

Despite the fruitfulness of proofs, they are problematic to most learners and university students (Thompson et al., 2012; de Villiers, 1997; Senk, 1989). Ndlovu (2012) showed pre-service teachers' poor understanding of geometric proofs due to poor knowledge of geometry which the pre-service teachers received during the study of geometry at high school. The study by Cassim (2006, p.22) concurred with these and illustrated learners' negative attitude with proofs as follows:

"Had to prove theorems all year long."

"I passed geometry by memorising proofs."

The comments above showed the narrow underlying understanding of the importance of proofs held by learners. First, the issue is to teach the roles, functions, and importance of proof to learners. According to Webber (2003), several proof functions include discovery, explanation, communication, justification, intuition development, and autonomy. Therefore, proofs improve reasoning ability and provide a method of evaluating arguments presented.

The other reason for teaching proofs is to show the universality of mathematical statements (Gfeller, 2010). However, learners have another challenge with proofs: lacking understanding of geometric concepts and beginning and monitoring proof development (Thompson et al., 2012). To make proofs more meaningful and appreciated by learners, Chinnappan et al. (2012) suggested that using geometric knowledge during the construction of proofs should be based on problem-solving incorporating reasoning.

2.1.6 Misconceptions and errors in Euclidean geometry

Many misconceptions that hinder a thorough understanding of geometry concepts have been identified in the research. Michael (2001) defines misconception as an inability to properly understand the concept, which can cause a poor understanding of the subject matter.

In most cases, misconceptions are a result of over or under- generalisation of a concept, rule, or presenting a different conception of the situation (Hansen et al., 2017). Many

misconceptions stem from communication gaps between teachers and learners during the teaching and learning process. Luneta (2008) mentioned factors like different levels of reasoning, lack of concentration during instruction, limited understanding of the situation and language barriers as other causes of misconceptions. Misconceptions can be valuable resources for effective learning if they are adequately addressed; however, they can result in severe problems if not appropriately addressed. For example, misconceptions can result in mathematics being viewed as complex, setting emotions of fear, anxiety, and frustration, often threatening performance and mathematics participation (Makhubele, 2014).

Related to misconception is an error. Harper cited by Makhubele (2014, p.44) defined error "as a deviation from accuracy or correctness". Understanding underlying causes of errors is critical when teachers use errors for remedial procedures. Although Luneta and Makonyane (2010) indicated that misconceptions and errors are different, Confrey (1990) and Micawber (2005) point out that misconception leads to error. According to Confrey (1990, p.33), misconceptions are "a line of thinking that causes a series of errors resulting from incorrect underlying premises". Micawber (2005) suggested that misconceptions give rise to a pattern of errors resulting from learners' prior knowledge and resistance to change. In healthy terminology, misconceptions may be considered a disease, whereas error may be the symptoms of that disease. This research argues that consistently repeated errors are the signs of misconception. Hence in this study, errors and misconceptions are treated more the same.

Below is a discussion of some commonly identified students' misconceptions and errors in Euclidean geometry.

2.1.6.1 Hierarchical classification of quadrilaterals

The hierarchical classification of quadrilaterals is vital in establishing their relations, solving problems and developing geometric reasoning skills (Turnuklu et al., 2012; Fujita & Jones, 2007; NCTM, 2000). However, research shows that learners experience difficulty understanding class inclusion of quadrilaterals. Figure 2.4 shows the hierarchical class inclusion of quadrilaterals.

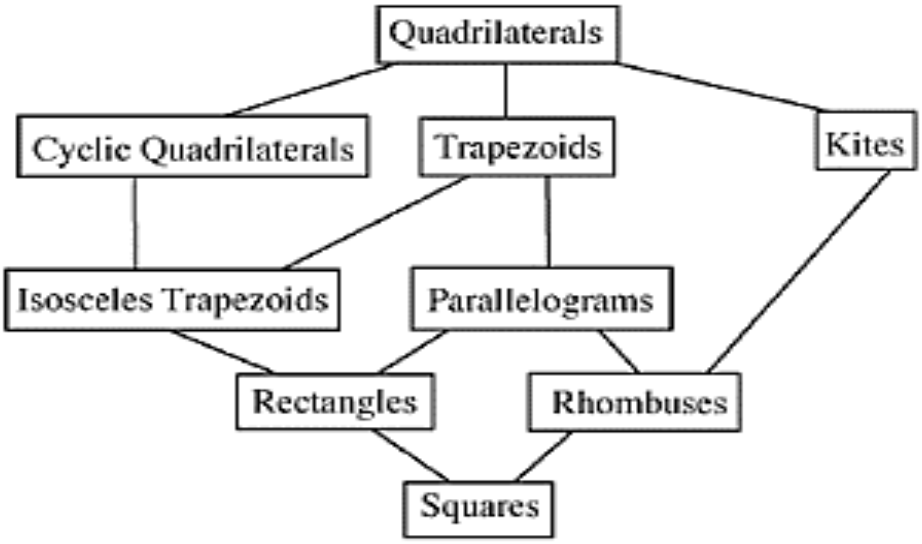


Figure 2.4: Hierarchical class inclusion of quadrilaterals (Usiskin et al., 2008**)**

Several studies on the hierarchical classification of quadrilaterals indicated that many students experience difficulties identifying simple shapes (Fujita, 2012; Marchis, 2008; Fujita & Jones, 2007; Feza & Webb, 2005; Siyepu, 2005; Roux, 2003; de Villers, 1998). Roux (2003) found that secondary school learners' performance in the USA is poor regarding items involving understanding properties of shapes. In the same vein, Siyepu (2005) highlighted difficulties South African secondary school learners face in naming and identifying simple quadrilaterals like the rhombus, kite, trapezium, and parallelogram. Another study conducted by Feza and Webb (2005) showed that learners had challenges understanding the hierarchy classification of quadrilaterals. For example, they did not perceive a square as a rectangle. These findings were consistent with the research done by Marchis (2008), which found that rectangles, squares, and rhombus, were all excluded from the class of parallelograms by all learners who also failed to perceive squares to be rectangles (or rhombus). Likewise, Atebe's (2008) study of grouping quadrilaterals according to special properties conducted with 36 students (18 Nigerians and 18 South Africans) found some misconceptions. For example, four learners (3 Nigerians and 1 South African) reasoned that all four-sided shapes were parallelograms.

Furthermore, the study showed that many learners perceive shapes in their totality without understanding their properties. For example, although some learners could name rhombus, they could not state its properties. These results show the challenge in understanding the properties of shapes, which is essential in reasoning about the relationship between geometric shapes and their properties.

2.1.6.2 Recognition of figures in their standard orientation

Marchis (2008) also showed other challenges learners experience when identifying shapes not positioned in their standard orientation. The author found that some learners could not identify the square if the base were not horizontal. An example of such a misconception is illustrated in figure 2.5 below. Some learners did not recognise the second shape as the square but as a kite in this diagram.

Figure 2.5: Different orientations of a square

Hershkowitz (1990) found that prototypical images were considered more important than attributes and definitions of geometric figures. These prototypical images are often identified correctly but are not recognised in different orientations (Fujita, 2012; Fujita & Jones, 2007). To address this challenge, teachers should try to put shapes in different orientations because learners may develop misconceptions that specific shapes must look a certain way rather than using characteristics of the shapes to identify them. It may also improve learners' understanding of the properties of shapes.

2.1.6.3 The use of visualisation and misapplication of theorems in presenting proofs

Research into geometric proof shows that students generally find the rigour of proofs difficult, especially when deep abstract thinking is required (de Villiers, 1997) or many steps are required (Ngirishi & Bansilal, 2019). Cassim's (2006) findings corroborate with those of Ngirishi and Bansilal, indicating that students use visualisation as a method of proving. Furthermore, many students made faulty assumptions based on the visual appearance of a given diagram suggesting that learners are struggling to extract valuable information from the diagram that can be combined to devise logical and coherent proof. It shows students' poor understanding of proofs and the inability to present convincing arguments justifying relationships between statements logically. Weber's (2001) research with university students found a lack of understanding proof.

Students lacked the core meaning of what constitutes proof; they lacked "understanding of a theorem and systematically misapply it" (Weber 2001, p.201). Similarly, Cassim's (2006) study showed learners using theorems haphazardly; for example, many learners employed cyclic quadrilateral theorems to solve some tasks despite no cyclic quadrilateral being present in the diagram.

2.1.7 Causes of misconception

The following subsections discuss the potential causes of misconceptions in geometry.

2.1.7.1 Negative attitude and mathematical anxiety

A positive attitude is the first requirement in accomplishing every individual's task. According to Nkwe (1985), a negative attitude towards geometry may trigger misconceptions and contribute to learners' poor performance. Tella (2007) argued that the cause of learners' negative attitude towards learning mathematics might be a result of teachers' methods of teaching, which sometimes does not connect with learners' daily experiences. Similarly, Rossnan (2006) argued that mathematical anxiety could develop due to the learner's prior negative experiences of learning mathematics. According to Hlalele (2012), mathematical anxiety is one of the sources of misconceptions. Anxiety is the feeling of fear which is caused by learners' continuous failure to complete a mathematics task successfully (Cavanagh & Sparrow, 2011). Moreover, it frustrates the learner and inhibits them from unleashing their innate mathematical abilities contributing to poor performance in the subject.

2.1.7.2 Language

Whether language proficiency influences academic achievement and can be a potential cause of misconception has been discussed for many years. Secada (1992) highlighted that there seems to be a correlation between language proficiency and mathematics achievement. This view was supported by Roux's (2003) study, which showed a significant difference between the performance of the more proficient language learners and the less proficient language learners concerning each of the first three Van Hiele levels. The author found reading comprehension and vocabulary as solid predictors for geometric thinking on the first three Van Hiele levels. Likewise, the study undertaken by Adler (2004) found that learners in Africa performed poorly in mathematics and science, where the language of teaching and learning is not their mother tongue compared to those taught using their mother tongue. Buchanan (2007) concluded that those who clearly understand the language used in mathematics can achieve better in this discipline.

Therefore, language has been identified as one of the causes of misconception, especially for second language speakers (Dikgomo, 1994). In support, Douglas et al. (2006) identified language as a means of gaining critical reasoning and understanding for success in geometry. The authors assert that improving learners' understanding of geometry is related to the accuracy of the language used. Learners must clearly understand the language used in mathematics to effectively understand the problem, organise their thinking and communicate their thoughts. Therefore, teachers are expected to assist learners in gaining the required understanding of mathematical concepts and communicating their thoughts because it will enable the teacher to assess students in terms of mathematical understanding.

2.1.7.3 Teachers' subject and pedagogical content knowledge of Euclidean geometry

According to Ndlovu et al. (2013), the most valuable resource for effective teaching of geometry is the teachers' subject content knowledge (SCK), which some teachers seem to lack. Brijlall (2011) defines subject content knowledge as the disciplinary knowledge of a subject, including mathematics concepts, rules, and associated procedures for problem-solving. A study undertaken by Bansilal et al. (2014), which investigated grade 12 teachers' content knowledge of mathematics, showed that they are performing unsatisfactorily. Although their study tested teachers' SCK for other topics but not geometry, it is disappointing that teachers struggle with the content they teach their grade 12 learners. Bowie (2009) asserts that a significant factor contributing to learners' poor achievement in geometry is teachers' lack of understanding of geometry. This view was supported by Turnuklu et al. (2012), who found that the quadrilaterals whose attributes were known most by the teachers were the square and rectangle. However, they had problems with their diagonal properties. Similarly, some researchers (Zeybek, 2018; Ozdemir, Erdogan & Dur, 2014) found that pre-service primary mathematics teachers' understanding of hierarchical classifications of quadrilaterals is poor. They were not able to establish relationships among quadrilaterals. Others (Zilkova, 2015; Türnüklü, 2014; Fujita, 2012) found pre-service teachers who possessed formal definitions of quadrilaterals but recognised them through their prototypical examples, making it difficult for them to understand their relationships.

Moreover, teachers need to develop pedagogical content knowledge (PCK) in addition to SCK. Shulman (1987) defines PCK as "making sense of student work and choosing powerful ways of demonstrating the subject so that it is comprehensible to students" (p. 9). Effective instruction requires teachers to develop knowledge of valuable resources and design activities that improve learners' understanding of geometry concepts (Ding & Jones, 2006).

According to the Van Hiele model, teachers' instructions must be aligned to learners' cognitive level of thought to make content accessible. Some research has shown that teachers' pedagogical approach to geometry may contribute to failure. For example, de Villiers (2004) found that in SA, teaching methods of Euclidean geometry do not promote a robust understanding of geometric concepts but instead focus on memorising theorems and application in solving riders without the understanding of the underlying problem. A method is a traditional form of teaching that allows the mathematics teacher to present examples and give learners similar exercises to demonstrate procedural understanding of learnt work. Dubinsky (1997) points out that this does not account for meaningful teaching and learning; it ignores the learners' prior knowledge and experiences. Alex and Mammen (2014), Siyepu and Mtonjeni (2014) add that the teaching methods used in South Africa do not promote problem-solving skills for competency in the real world but rather perpetuate learning by memorisation and rote.

2.2 Theoretical consideration

Due to a poor understanding of geometry, there is a need to prioritise the theories explaining learners' difficulties and means of assisting them in learning geometry effectively. At the centre of this challenge is how can it be done? According to Jaime and Gutierrez (1995), the Van Hiele model is a well-established model that explains students' struggle with understanding geometric reasoning and assists in planning effective instructions that can develop learners' thinking in geometry. Such sentiment can justify the importance of the Van Hiele model in geometry education.

The Van Hiele (1986) model pioneers were a Dutch couple, husband Pierre Van Hiele and his wife Dina Van Hiele-Geldof, who were teachers experiencing challenges in their geometry classes. These challenges underpinned the focus for their respective doctoral dissertations at the University of Utrecht, Netherlands. They intended to categorise learners' cognitive thinking in geometry by levels and explain how learners could be assisted to reach a higher level of geometric thinking. Pierre Van Hiele was accountable for describing levels of geometric thinking, which learners progress through in learning geometry. In contrast, Dina Van Hiele-Geldof focused on the teaching practice of assisting learners in making progress with the levels. Below is the discussion of the Van Hiele levels in detail.

2.2.1 The Van Hiele levels of geometric thought

Pierre Van Hiele identified five hierarchical and discrete levels of geometric thought which learners must progress through to reach the stage for accessing the critical reasoning for formal proof-writing. These levels include visualisation, analysis, informal deduction, formal deduction, and rigour. Although these levels are specific to geometry, Van Hiele (1986, p. 41) concedes that they are stages of development since "they are levels situated, not in the subject matter but in the thinking of the man". The levels are summarised as follows:

- **Level 1 (Visualisation):** This is the elementary level where figures are recognised as a whole, disregarding their constituent properties. According to Feza and Webb (2005), a person at this level could name and reproduce shapes like a rectangle but cannot identify essential properties that make a specific shape a rectangle.
- **Level 2 (Analysis):** At this level, the person starts showing mastery of the properties of shapes. However, when mentioning properties to illustrate the shapes, the person cannot understand the interrelationship between necessary and sufficient properties; they mention all characteristics of the figure they know (de Villiers, 1997).
- **Level 3 (Informal deduction)**: A person can construct significant definitions and provide casual arguments to substantiate their reasoning. Definitions and relations between figures begin to make sense for a person. For instance, persons can accept that a square is a rectangle because it meets all the properties of one. However, the function and implications of formal deductive reasoning are not understood (Crowley, 1987).
- **Level 4 (Formal deduction)**: At this level, a person acknowledges the importance of definitions in devising formal deductive proofs and can use present proofs on demands without memorisation. Furthermore, "the interaction of necessary and sufficient conditions is understood; distinctions between a statement and its converse can be made" (Crowley, 1987, p.3).
- **Level 5 (Rigour)**: This is the most advanced level of geometric thought where a person can go beyond Euclidean geometry and study non-Euclidean systems such as spherical geometry. However, this last level is less practical, and little is known about it; even Van Hiele himself has doubted its existence and considered it of no practical value.

According to the Van Hiele model, developments along these levels are based on teaching instruction and experimental opportunities rather than age. The student growth in age does not automatically imply growth in geometric thinking; however, the carefully planned instruction

is responsible for moving learners from one level to the next (Jaime & Gutierrez, 1995). Also, for students to have a thorough understanding of geometric concepts, it is impossible to skip levels; however, when teachers use rote and memorisation learning, they deprive learners of the opportunity to perform at the expected level. Hence, they will not understand the content taught but rather memorise the concepts (Clements & Battista, 1992). Fuys et al. (1985) gave an example of this whereby students "tried to recall (rather than think out) what their teacher had told them…thus, when geometry was taught, it appeared to be mainly at a recall or knowledge level" (p. 155).

Although the model has been found useful for learning geometry at all levels, some critiques exist about the levels being discrete. Some studies like that of Burger and Shaughnessy (1986) challenged the levels' discretion showing students oscillating between levels on the same concept. Clements et al. (2001) further corroborated Burger and Shaughnessy's results, finding it challenging to classify students' thinking according to neatly discrete levels. Therefore, Van Hiele's claim of levels being discrete is questionable.

2.2.2 Teaching implications of the Van Hiele levels of geometric thought

The Van Hiele model has effectively promoted classroom instructions that encourage geometric understanding. The model suggests that students should be assisted to pass through five levels of geometric thought which prepares them for advanced reasoning required in Euclidean geometry. Below is the discussion of the implications of the model for classroom teaching.

2.2.2.1 Teachers' geometry content knowledge and pedagogical knowledge

For teachers to assist students in progressing from a basic level to a more advanced level of thinking in geometry, they need to possess an adequate understanding of geometry content knowledge. A thorough understanding of geometry content knowledge will allow teachers to own the material they teach and identify content gaps reported in textbooks. Some researchers found that some textbooks have limited definitions of concepts and examples to illustrate exceptional cases of concepts (Kajander & Lovric, 2009).

Some textbooks present only prototypical examples, which may hinder conceptual understanding (Hershkowitz, 1987; Vinner & Hershkowitz, 1980). Thus, it makes teachers' geometry content knowledge essential, so they do not depend much on textbooks. In addition, it applies to pre-service teachers since it has been reported that their mathematical

understanding necessary to teach school mathematics is unsatisfactory (Brown et al., 1990). Some researchers (Gutierrez & Jaime, 1999; Hershkowitz, 1987; Mayberry, 1981) that found poor knowledge of geometric concepts by pre-service teachers concur with the previous findings. The poor knowledge of geometric concepts includes what Vinner and Hershkowitz (1980) classified as a gap between formal concept definitions and concept images. The authors found that the concept image held by some students can be limited to a single prototypical image. They added that over-dependence on it further contributed to a lack of understanding of geometric concepts.

Moreover, teachers choose a teaching method beyond their content knowledge in their classes. According to Claxton (1997), the teaching method can be either traditional or discovery learning. Traditional learning focuses on giving learners already made information that they must assimilate and reproduce on demand. On the other hand, discovery learning engages those being taught and focuses on making new knowledge collaboratively. Even though traditional learning allows the teacher to present an accurate concept definition and solving procedures, researchers found that material learnt without understanding may quickly be forgotten and irrelevant to learners (Gutierrez & Jaime, 1999; Vinner & Hershkowitz, 1980). When considering which learning method is relevant, discovery learning is arguably better than traditional learning because it engages and promotes understanding. According to de Villiers (1997), geometry has failed because it is imposed as ready-made pieces of knowledge on the learners. Hence teachers must utilise discovery learning to provide learners with opportunities to discover and explore geometry facts necessary for real-life applications of geometry skills and enhancing learners' understanding of geometry concepts (Jones, 2002).

Therefore, teachers need to possess a deep mathematical understanding to see relationships among concepts to enhance the transfer of new knowledge and influence positive beliefs about mathematics (Ma, 1999; Hiebert & Carpenter, 1992). These abilities are vital in the scaffolding process as they assist students in mastering the Van Hiele levels of geometric thought. However, Telima (2011) showed that limited resources and time frames influenced teachers' decision to opt for traditional teaching methods, which do not promote understanding of geometry concepts.

2.2.2.2 The use of appropriate geometric terms and activities

According to the Van Hiele model, terms used during instruction are critical in developing concepts from concrete to abstract levels. Progression from elementary to a more advanced

level is best facilitated through strategically planned instruction using relevant terms (Clements, 2003). Van Hiele emphasised that in most cases, a lack of understanding in geometry results from communication gaps between teacher and learner, where the teacher used terms above learners' level of understanding. The terms which match learners' level should clarify their thoughts rather than confuse them. For example, a teacher's use of special terms in mathematics can confuse learners. For instance, when a teacher uses the term parm (short for parallelogram), the learner may be confused not thinking of parallelogram. Thus, these special mathematics terms can cause misunderstanding, leading to misconceptions. Mudaly (2010) asserts that these special terms confuse learners invoking fear in mathematically challenged learners. As cited by Feza and Webb (2005, p. 37), de Villiers stated that mastery of relevant terms is critical for effective geometry learning. The proper geometric terms assist learners to communicate their ideas about concepts in geometry with understanding (Atebe & Schäfer, 2010).

Furthermore, realising the critical role of terms in mathematics classes should influence teachers in considering their usage in mathematics teaching. Genz (2006) found evidence that geometric concepts at the primary school level are not taught meaningfully to promote robust understanding required for success in geometry but are taught for memorisation. Therefore, Clark (2012) suggests that to promote a thorough understanding of mathematics concepts, teachers should be aware that concepts learned in the previous grade may not be understood. Thus, discussion and emphasis of concept definitions encourage advanced reasoning of geometric knowledge and understanding.

Moreover, Van Hiele (1986) believed that the way geometry is taught at the primary school level does little to equip learners with knowledge and skills to cope with the demand of thinking required at the secondary school level. Therefore, teachers should design appropriate activities matching their students' levels of reasoning and challenge them with questions that aim to stretch their reasoning to the next level of geometric thought (Lim, 2011; Van de Walle, 2004). Thus, it suggests that learners experience a range of activities that challenge them to think critically until they reach an advanced level of geometric thinking. However, as indicated by studies like Dhlamini's (2012), some teachers teach at a basic level, emphasising the procedural understanding of recurring problems depriving learners of the opportunity to engage with higher-order reasoning activities. Consequently, it contributes to learners' low achievement in Euclidean geometry.

2.3 Conclusion

The first section of this chapter discussed the literature review, which showed vast challenges in teaching and learning geometry. The second section explored the VHLs of geometric thought and its teaching implication to promote understanding in the geometry classroom. In a nutshell, teachers' geometric knowledge and how they design and select activities appropriate for students' level of thinking that ultimately improve their understanding of geometry was considered paramount. In doing so, geometry can be meaningful to the students. Ignoring this results in learners using rote memory learning and applying rules and theorems without understanding.

CHAPTER 3

RESEARCH METHODOLOGY

3.1 Introduction

This chapter discusses the research methodology utilised in this study. Cohen, Manion and Morrison (2007) stated that the purpose of research methodology is to clarify the process of scientific inquiry in the broadest possible terms and not its products. Hence, this chapter aimed to discuss the research procedure employed in this study. The chapter presents the research paradigm, research approach and style of research adopted in the current study. Thereafter, the data collection process, instruments used, participants, and data analysis procedures are discussed. Lastly, the ethical considerations and trustworthiness of this study are provided.

3.2 Research paradigm

Huitt (2011) defined a research paradigm as one's way of seeing reality. Correspondingly, Bertram and Christiansen (2014) defined research paradigm as "a particular worldview that defines, for the researchers who hold this view, what is acceptable to research and how this should be done" (p. 24). Moreover, Kivunja and Kuyini (2017) stipulate that "this worldview is the perspective, or thinking, or school of thought, or set of shared beliefs, that informs the meaning or interpretation of research data" (p. 26). The most used paradigms, as indicated by Kumar (2011), are: (a) postpositivist, where the researcher assumes that there is only one truth about social events which cannot be known completely; (b) interpretivism, where the researcher assumes that there is more than one truth based on many interpretations of events about the social world; and (c) critical paradigm, where a researcher sees the reality as shaped by political, economic, social, and other dynamics which exhibit unequal power relations.

The primary aim of this current study was to explore pre-service teachers' understanding of similarity and proofs. Therefore, the interpretive paradigm was considered appropriate because it is used to study the subjective world of human experiences and reasons for their actions (Cohen et al., 2011). In addition, Lewis (2015) argued that the interpretive paradigm aims to gain a deep understanding of how humans' interpretation of their reality is based on the contexts in which they find themselves.

The interpretive paradigm suited this study because the researcher aimed to understand the phenomenon investigated from participants' perspectives and experiences. Scotland (2012) asserts that interpretivists believe there is no one right or exact approach to knowledge. Hence,

in this study, participants' responses were assumed to represent their feelings, attitudes, and understanding of the situation they found themselves.

Furthermore, the interpretivist paradigm is flexible since it allows the participants to account for their actions; thus, it produces rich information to understand the phenomenon investigated in this study. This paradigm also matched this study's data collection, which included written documents and interviews to improve understanding of investigated phenomena. Thorne (2016) suggested that these methods provide a more significant opportunity for discussion between the researcher and participant to construct a meaningful reality in the participants' context. Hence, this study's findings reflect participants' understanding of similarity and proofs based on their experiences.

3.3 Research approach

According to Durrheim (2002), a research approach refers to collecting and analysing data required to respond to the research questions. According to Creswell (2014), the research approach adopted by a researcher depends on both the research paradigm and the critical questions the study seeks to answer. According to Williams (2007), researchers commonly utilise three main research approaches. Firstly, researchers can choose a quantitative approach grounded within the positivism or postpositivism paradigm. This research approach mainly uses numeric and statistical descriptions to test or confirm assumptions and theories (Creswell, 2009). Secondly, researchers can utilise a qualitative approach grounded within the interpretivism paradigm. Kumar (2011) outlines that "the main focus in qualitative research is to understand, explore, discover and clarify situations, feelings, perceptions, attitudes, values, beliefs and experiences of an individual or group of people" (p.104). Thirdly, Williams (2007) posits that a mixed methods research approach may be used for research questions requiring textual and numerical data.

As this study aimed to explore PSTs' understanding of similarity and proofs in their natural setting, the qualitative approach was considered appropriate for collecting and analysing data for the present study. The rationale for choosing a qualitative approach was that it allows understanding of the participants' feelings, attitudes, perceptions, and beliefs about the phenomena investigated.

Furthermore, the qualitative approach allows multiple understanding of the inherent complexity and variability of human behaviour and experience (Neuman, 2013).

The research questions of this study aimed to answer questions like what, how and why, which characterise qualitative understanding, rather than test hypothesis which characterises a quantitative approach. Qualitative researchers undertake an in-depth investigation of a small group of first-year pre-service teachers at one university in the KwaZulu-Natal province. According to Taylor et al. (2016), qualitative researchers desire to step beyond the known and enter participants' worlds to make discoveries that contribute to the development of empirical knowledge. For example, in using a qualitative approach for this study, underlying factors which affected PSTs' understanding of similarity and proofs can be revealed. According to McMillan and Schumacher (2001), "qualitative research is inquiry in which researchers collect data in face-to-face situations by interacting with selected participants in their settings" (p. 393). It suggests that the researcher is in a favourable position to understand how social, cultural, and environmental factors influence human behaviour and interaction.

3.4 Research style

Bertram and Christiansen (2014) asserted that various research styles could be used when conducting research. The authors argued that the research style to be used when conducting any research is usually influenced by research questions, methods of data collection and the research paradigm a researcher is working within. According to Rule and John (2011), a case study is an in-depth systematic study about a specific case in its setting where the case may be a person (a learner, teacher, or parent), a group of people (a class of learners or community). Case studies are useful where the researcher explores a single case within its real-life context, bound by time and activity. It deals with contemporary events and is concerned with how and why things happen. In other words, the researcher takes an in-depth look at the case study to capture the reality of the participants lived experiences and thoughts about a situation (Cohen et al., 2011). Cohen et al. (2007) suggest that "case studies can penetrate situations in ways that are not always susceptible to numerical analysis" (p. 253). It serves as an advantage of using the case study as it focuses on gaining a comprehensive understanding of a phenomenon, which is keeping in line with the qualitative research.

In this research, a case study was used to explore PSTs' understanding of similarity and proofs in Euclidean geometry. The case in this setting was a group of first-year mathematics PSTs studying towards a Bachelor of Education degree in one university from KZN. The rationale for using a case study comes from the purpose of this study.

It was necessary to collect data from a person's own written and spoken words to explore PSTs understanding of similarity and proofs in Euclidean geometry. According to Guthrie (2010), case studies do not represent the entire population, but findings could be used in other settings if appropriately selected.

3.5 Sampling procedure and participants

According to Cohen et al. (2007), sampling is a process of decision-making about the population, settings, events, or deeds to observe. Patton (2002) accentuates that there are no standard rules for the sample size of a qualitative study; it is somewhat informed by the purpose of the inquiry, what is at stake, what is useful, what will have credibility, and what can be done with available time and resources. However, the researcher must choose a sample size and characteristic of the population in which conclusions will be drawn out. For this study, purposive sampling was used. According to Etikan and Bala (2017), purposive sampling is "based on the judgement of the researcher as to who will provide the best information to succeed for the objectives of the study" (p. 215). Purposive sampling involves hand-picking certain groups or individuals to include in the sample based on their knowledge and relevance to the problem under study (Cresswell, 2003).

The targeted population for this study was all first-year PSTs studying for a Bachelor of Education degree majoring in mathematics at the FET phase at the university where the study was conducted. Only PSTs who completed grade 12 when Euclidean geometry was reinstated as a compulsory section of the mathematics curriculum participated. These specific criteria were placed to ensure the purposive selection of participants relevant to the phenomena investigated. Finally, 34 PSTs of mixed abilities and mixed gender who met the requirement voluntarily participated in a pen and paper test, and seven PSTs further participated in semi-structured interviews. It ensured that participants willing to give free and honest responses to questions participated.

3.6 Data collection methods and instruments

According to Kabir (2018), data collection "is the process of gathering and measuring information on variables of interest, in an established systematic fashion that enables one to answer stated research questions, test hypotheses, and evaluate outcomes" (p.202). The scholar further emphasises that for all data collection, the goal is obtaining quality evidence which is then translated into rich information and produced as new knowledge by providing credible answers to the research question(s).

The data collected in this study was in two phases, the pen and paper test and the semi-structured interviews.

3.6.1 Pen and paper test

Parahoo (2006) stated that one way of exploring individuals understanding of the phenomena is to analyse documents they produce. The document can be in the form of a questionnaire, where participants respond to items to extract research-related information (Kamgar & Navvabpour, 2017). In this study, the purpose of letting pre-service teachers complete the test was twofold. Firstly, to investigate the degree of success of PSTs in solving similarity and proof riders to place them into various VHLs according to their performance. Secondly, to identify the misconceptions, if any, and use the pre-service teachers' written responses to develop semi-structured interview questions for each interviewed participant to further probe for the originality of written responses.

A research task was designed to gain insight into pre-service teachers' knowledge of similarity and proofs. The task was administered online to all 34 PSTs who agreed to participate in this study. The test only accessed what is prescribed in the secondary school curriculum. The test consisted of riders, which PSTs should have encountered in secondary school when studying Euclidean geometry. The test was designed with the following characteristics:

• The duration for completion was one hour.
• The test consisted of three main questions, and marks were allocated for each question (See Appendix A).

Below is the summary of the content accessed in the test.

QUESTION 1 (14 marks) initial consisted of seven sub-questions (1.1 to 1.7); however, 1.6 was cancelled due to misprint. These questions required PSTs to show their understanding of the definition of similarity in mathematics, the minimum conditions required for figures to be similar and how to prove certain figures are similar.

QUESTION 2 (5 marks) had five sub-questions (2.1 to 2.5), which tested PSTs' knowledge of identifying and naming similar triangles in various geometric figures in the correct order.

QUESTION 3 (8 marks) has three sub-questions (3.1 to 3.3) which explored PSTs ability to construct geometric proofs about similar triangles.

3.6.2 Semi-structured interviews

According to Cohen et al. (2007), an interview is a direct face-to-face conversation between interviewer and interviewee to get research information such as beliefs, views, perceptions, and opinions. In this study, semi-structured interviews captured the respondents' perceptions and experiences, knowledge, attitudes and beliefs about similarity and proofs. The semi-structured interviews mainly probed PSTs' originality of their written responses for items in the test. In support, Sorsa et al. (2015) highlighted that interviews are a good data collection tool for discovering what a person knows. This study used semi-structured interviews with pre-service teachers' written responses from the pen and paper test.

Haahr et al. (2014) stated that interviews allow insight into another person's perspective, as they provide the opportunity for the researcher to probe and gather data, which could not have been obtained in other ways (Galvin, 2015). It lets the researcher explore misconceptions PSTs have about similarity and proofs and extract information on how knowledgeable they are. The advantages of interviews are that they are flexible and provide direct human interaction that allows for probing and clarification of answers with the respondents, follow-up leads, elaborating responses and obtaining more data in detail providing clarity. Suppose the researcher makes the respondents comfortable and secure; they are likely to reveal supplementary information that could not be revealed under any other circumstances that might be of use to the researcher at the data analysis stage. Lastly, the researcher can immediately validate the data when sensing that the respondent gives false information through facial expressions and tone of voice. Table 3.1 summarises the data collection methods used in this study.

Table 3.1: Data collection procedure and participants

Research question	Participants	Research instruments
1. How do pre-service teachers perform on tasks based on similarity and proofs?	Pre-service teachers	• Pen and paper test
2. What misconceptions do pre-service teachers have of similarity and proofs in Euclidean geometry?	Pre-service teachers	• Pen and paper test • Semi-structured interviews
3. What are the Van Hiele levels of geometric thought of pre-service teachers in similarity and proofs?	Pre-service teachers	• Pen and paper test

3.7 Data Analysis

According to Henning (2004), data analysis is "a relatively systematic process of coding, categorising, summarising and interpreting data to provide explanation of a single phenomenon of interest" (p. 33). The analysis intends to ease data to an interpretable and understandable form so that research problems can be studied and tested, and conclusions drawn (De Vos, 2002). In this study, the data were collected from PSTs' written and spoken words. Below is a detailed discussion of how data collected was analysed to respond to the research questions of this study.

3.7.1 Pen and paper test analysis

The researcher marked and analysed the 34 scripts of PSTs who participated in this study to assess PSTs' performance in similarity and proofs in Euclidean geometry. Pre-service teachers were then coded from PST1 to PST34 in ascending order of performance. PST refers to a pre-service teacher, and numbers 1 to 34 refer to the ascending order of performance in the test. Each response was coded during the marking process (C1 to C4), where C1 represents no responses, C2 represents incorrect responses, C3 represents correct but incomplete responses, and C4 represents complete, correct responses (see table 3.2).

Some advantage of using scoring rubrics in analysing task includes systematisation and consistency in judging responses (Becker, 2016).

Table 3.2: The scoring rubric for analysis of the behaviour

Category	Description of category
C1	Not answered
C2	Answered incorrect
C3	Some correct statements but incomplete
C4	Answered completely correct

The above rubric for PSTs written responses was used to determine the number of C1 to C4 responses for each item. Then the percentage of PSTs who completed each item correctly (C4) was calculated by taking the number of C4 responses for each item divided by the total number of participants (34) multiplied by 100. Thereafter, items were classified as not challenging or challenging based on the percentage of PSTs who achieved complete, correct responses for each item. Because the research accessed secondary school content, items with +80% complete, correct (C4) responses were classified as not challenging. The items with -60% of complete correct (C4) responses were classified as challenging. The poorly achieved items were studied to investigate the nature of knowledge resulting in an incorrect response, enabling the researcher to identify the misconceptions PSTs have about similarity and proofs in Euclidean geometry. Finally, each PSTs' responses were analysed according to the Van Hiele level of the item required. Below is the detailed description of analysing PSTs VHL for each item.

Table 3.3: Explanation of Van Hiele levels for each research item

Item	Van Hiele level	Explanation
1.1	Level 2: Analysis	This item required PSTs to deduce a general definition of similarity from analysing conditions of similar figures, which required reasoning at VHL2. PSTs who achieved the full two marks showed that they knew the proper definition of similarity in mathematics.
1.2	Level 2: Analysis	This item required PSTs to mention two minimum conditions required for the figures to be similar. PSTs who achieved the full two marks showed that they knew the minimum

Item	Van Hiele level	Explanation
		conditions for the figures to be similar, which is an expectation of VHL2.
1.3	Level 2: Analysis	This item required PSTs to decide if two arbitrary squares are similar by analysing properties of squares and conditions of similarity. PSTs who achieved the full two marks showed engagement in reasoning suitable for VHL2.
1.4	Level 2: Analysis	This item required PSTs to decide if two arbitrary rectangles are similar by analysing the properties of rectangles and confirming if the similarity conditions are satisfied for VHL2. PSTs who achieved the full three marks showed that they could use reasoning suitable for VHL2.
1.5	Level 3: Informal Deduction	This item required PSTs to calculate ratios of corresponding sides to decide if given triangles are similar. PSTs who achieved the full three marks demonstrated they could use reasoning suitable for VHL3.
1.7	Level 3: Informal Deduction	This item required PSTs to calculate the third angle in each triangle to decide if the given triangles were similar. PSTs who achieved the full two marks showed they could engage in informal deduction, a VHL3 skill.
2.1	Level 3: Informal deduction	This item required PSTs to name similar triangles correctly using circle theorems. PSTs who achieved the mark showed that they could engage in informal deduction, a VHL3 skill.
2.2	Level 3: Informal deduction	This item required PSTs to name similar triangles correctly using circle theorems. PSTs who achieved the full mark showed that they could engage in informal deduction using circle theorems, a VHL3 skill.
2.3	Level 3: Informal deduction	This item required PSTs to name similar triangles correctly using angles' parallel lines theorems. PSTs who achieved the full mark showed that they could engage in informal deduction using theorems about parallel lines, a VHL3 skill.
2.4/2.5	Level 3: Informal deduction	These two items required PSTs to name similar triangles in the correct order using the right angle and sum of triangle

Item	Van Hiele level	Explanation
		angles. PSTs who achieved the full mark showed that they could engage in an informal deduction based on triangle properties, a VHL3 skill.
3.1	Level 4: Formal deduction	This item required PSTs to prove AE = BE using applicable geometric facts. PSTs who achieved the full three marks showed that they could derive proof from scratch using formal deductive reasoning, a VHL4 skill.
3.2	Level 4: Informal deduction	This item required PSTs to prove that $\Delta AED /// \Delta CEB$ using the result that angles in the same segment are equal. PSTs who achieved the full three marks showed that they could use reasoning suitable for VHL3.
3.3	Level 3: Formal deduction	This item required PSTs to show that $AE^2 = BE.CE$ using the results from Q3.1 and Q3.2. PSTs who achieved the full two marks showed that they could engage in reasoning suitable for VHL4.

Table 3.4: Summary of Van Hiele levels of research items

ITEMS	VHL	NUMBER OF VHL
1.1, 1.2, 1.3 and 1.4	VHL2 Analysis	4
1.5, 1.7, 2.1, 2.2, 2.3, 2.4, 2.5 and 3.2	VHL3 Informal deduction	8
3.1 and 3.3	VHL4 Formal deduction	2

After analysing the response of each PST to all the items according to VHL, the PST was placed in one overall VHL based on the competency displayed for each VHL. PSTs were considered having achieved VHL accessed if they got +75% of items in that level. Therefore, for PSTs to be classified as operating at the VHL4, they needed to correctly obtain all two VHL4 items. PSTs who achieved one item were identified as beginning to show competence at VHL4. However, for PSTs to be classified as operating at VHL4, they would have to meet all the requirements for the lower levels. Otherwise, they would be placed at a lower level, and their ability to achieve VHL4 would then be attributed to external factors rather than their ability.

For PSTs to be classified as operating at the VHL3, they needed to get six or more of the eight VHL3 items. PSTs who achieved four or five of VHL3 items were identified as just starting to show competence at VHL3.

However, for PSTs to be classified as operating at VHL3, they would have to meet all the requirements for the lower levels. Otherwise, they would be placed at a lower level, and their ability to achieve VHL3 would then be attributed to external factors rather than their ability. For PSTs to be classified as operating at the VHL2, they must get three or more of the four VHL2 items. Those who achieved two or fewer VHL2 questions were placed at a lower level even if they managed to get some marks in the items that needed thinking at higher levels.

3.7.2 Semi-structured interview analysis

After analysing the pen and paper test result, in-depth semi-structured interviews of about 45-50 minutes were conducted with seven participants to verify what transpired in the pre-service teachers' written responses. It gave PSTs another chance to reflect on their responses and check if their understanding was still the same or improved. During the interviews, PSTs were requested to explain how best to solve the question to extract information on how knowledgeable they are with similarity and proofs. In ensuring that every aspect of the semi-structured interview was captured, they were audio-recorded before their responses were transcribed verbatim. The PSTs' explanations expressed in their home language were translated to English. Probing was extensively used to elicit more information to ascertain PSTs understanding of similarity and proofs. Talking with PSTs made it possible to delve more deeply into their thinking and determine their knowledge of similarity and proofs. Finally, the PSTs' verbal and written responses were analysed to thoroughly understand their knowledge of similarity and proofs (see chapter 4).

Table 3.5: List of PSTs who participated in the interviews

Participants	Marks achieved (out of 27)	Percentage Achieved	Gender
PST1	07	25.53	Male
PST7	16	59.26	Male
PTS8	16	59.26	Male
PST9	17	62.96	Female
PST14	19	70.73	Male
PST32	24	88.89	Male
PST34	25	92.59	Male

3.8 Ethical considerations

According to Mthethwa (2015), research ethics are essential for protecting participants from unfair criticism from participating in the research. Ethical issues concerning the research were observed through an application for ethical clearance from the University of KwaZulu-Natal Research Office seeking permission to conduct the study (see Appendix D). The researcher explained the purpose of the study and procedures during the research process to the participants and provided relevant contact details of personnel at the university. All participants were provided with an informed consent letter explaining the right to withdraw from the study, anonymity and confidentiality. They were ensured that the data collected would only be used for the study and not benefit assessment. Codes were used instead of participants names to ensure anonymity, and data collected would be kept safely in the university storeroom and destroyed by shredding after five years. Digitally recorded data will also be deleted after five years. If participants wanted to read the information before it became public, they were informed of their freedom to access the information. Each participant was required to provide their signed consent form. Finally, the researcher negotiated appointment dates with the participants for individual interviews on the agreed dates; the interview was audiotaped.

3.9 Trustworthiness of the study

The trustworthiness of a qualitative study can be ensured by addressing credibility, transferability, dependability, and conformability. These are described below in depth.

- ***Credibility***

Guba and Lincoln (2005) define credibility as the confidence placed in the data and analysis process. Credibility ensures that the research measures or tests what it intends to measure. Choosing participants with various experiences increases the possibility of shedding light on the research question from various aspects (Lewis, 2015). In this study, the effort was exerted in ensuring PSTs' written responses and spoken words were authentically presented without additions or deletions throughout the analysis. The presentation of some of the PSTs' written work and spoken words inserted, including direct quotes in the data analysis section, helped ensure credibility.

- ***Transferability***

Koul (2008) defines transferability as the evidence supporting the generalisation of the findings to other contexts involving different participants, groups and situations. This study described the settings, including the targeted population, to allow other researchers to apply the findings to similar settings.

- ***Dependability***

Dependability considers whether the research was transparent and written clearly. Howitt (2007) posits that it must be possible to trace the data, and the findings must be consistent. The pre-service teachers' original written responses can be used to confirm the dependability of this study.

- ***Conformability***

According to Howitt (2007), conformability is objectivity, neutrality, and control of research bias. In this study, conformability was achieved through the data transcripts and data analysis and the description of the findings in connection to the literature.

3.10 Conclusion

This chapter presented the research procedures employed in this study. Through an interpretivist paradigm, a qualitative case study approach was used to gather the authentic perceptions of pre-service teachers' understanding of similarity and proofs in Euclidean geometry. A purposive sampling procedure was used to select the study participants, and semi-structured interviews and a pen and paper test were applied in collecting data. The data was further analysed through verbatim transcription of interviews and applying the Van Heile levels of geometric thought. Lastly, ethical procedures and the trustworthiness of the study were discussed. The next chapter presents the findings of the study.

CHAPTER 4

RESEARCH FINDINGS

4.1 Introduction

This chapter presents the findings of this study. It begins with an overall summary of the results for the items in the questionnaire, followed by a presentation of PSTs' general interview responses that capture their experiences and feelings towards similarity and proofs. Thereafter, a detailed discussion of responses to each item and an analysis of students' performance in terms of the Van Hiele levels of geometric thought is provided.

4.2 Summary of results from the questionnaire

The pen and paper test comprised 13 questions, as indicated in chapter 3. Table 4.1 below shows the number of responses to each question according to each item's various performance levels. The various performance levels, categories C1, C2, C3 and C4, described in Chapter 3, represent not answered, answered incorrectly, some correct but incomplete and answered completely correct.

Table 4.1: Overall analysis for each item

QUESTION	C1	C2	C3	C4	Percentage of correctly answered (n=34)
	Not answered	Answered incorrectly	Some correct but incomplete	Answered completely correct	
1.1	0	2	3	29	85.29
1.2	0	2	2	30	88.24
1.3	4	2	2	28	82.35
1.4	3	27	4	0	00.00
1.5	0	3	10	21	61.76
1.7	0	8	0	26	76.47
2.1	0	13	0	21	61.76
2.2	0	10	0	24	70.59
2.3	0	9	0	25	73.53
2.4	0	15	0	19	55.88
2.5	0	16	0	18	52.94

3.1	0	13	1	20	58.82
3.2	0	2	1	31	91.18
3.3	1	0	0	33	97.06

Table 4.1 above indicated that most pen and paper test questions were answered well, except question 1.4, which was poorly answered. Below is the summary of seven PSTs' general interview responses selected to capture their experiences and feelings regarding similarity and proofs in Euclidean geometry. Later, their written responses supported by relevant interview responses are provided.

1. **Can you tell me when first you were introduced to similarity in Euclidean geometry? Did you understand it?**

   PST1: *I was introduced in grade 10, if I am not mistaken, and were well understood.*

   PST7: *It was in grade 12. When it was first introduced, I had some challenges here and there, but later I understand.*

   PST8: *I have been introduced to similarity in Euclidean geometry in grade 10, and I did understand it.*

   PST10: *I think I was in grade 10 or 11; I cannot remember. How much I understand it out of 10, I will give myself 7.*

   PST14: *Similarity ngaqala uku-introduswa kuyona ka-Grade 10 but uyi-understander ngaqala kahle ka-grade 12. (I was first introduced to similarity in grade 10, but I started to understand it clearly in grade 12).*

   PST24: *I was introduced to similarity in grade 9 but I did not understand it clearly in grade 9. I understood it clearly when I was doing grade 10.*

   PST34: *Yes, I did understand them well. I started to learn about similarity in grade 10.*

According to CAPS, similarity should be first encountered in grade 7, especially on the similarity of 2D shapes. Many PSTs indicated they were introduced to similarity in the FET (10-12) phase, except one participant mentioned they were first introduced in grade 9. Nevertheless, many also claimed they understood the concept of similarity.

2. **Can you tell me when first you were introduced to formal proofs in Euclidean geometry? Did you understand them?**

   PST1: *We were introduced in formal proofs in grade 11 and were well understood because our teacher did explain it better; she was so patient.*

PST7: *Yes, I did understand them, and we did them in grade 11. They were not difficult to understand because our teacher knows the subject well. He knows what he was doing.*

PST8: *I was introduced in grade 11, if I am not mistaken. I did understand, but not all of them.*

PST10: *I was introduced to them in grade 11, and I did understand them.*

PST14: *Ka-grade 11(in grade 11), and I understand them.*

PST24: *I was introduced to formal proofs in grade 11; yes, I understood them.*

PST34: *Yes, I did understand them; I learned them in grade 11.*

All participants claimed they first learnt about formal proofs in grade 11. According to CAPS, formal proofs should be first encountered in grade 10, where learners use previous geometric concepts to prove special quadrilaterals. Also, the participants claimed they understood them.

3. At this stage (PST), are you confident and comfortable solving and teaching similarity and proofs in Euclidean geometry?

PST1: *Yes, I am ready, and I am proud.*

PST7: *Eish, I am not confident and comfortable because of the pressure I will have standing in front of people for the first time, hoping to produce correct and accurate answers all the time. The knowledge for the topics is there, but the problem is confidence….you know how first times are.*

PST8: *Eeeeee, I can say I am comfortable but not kahle (not properly); like, there are proofs I am still struggling. So I still need umuntu ozongichazela kulezizinto engazi-understandi (a person who will explain those things that I do not understand).*

PST10: *Yeah, especially when I prepare first, not just anytime.*

Researcher: *What if a learner comes to you randomly asking a question you did not prepare?*

PST10: *I will send the learner back and tell him/her I will come back with an answer.*

PST14: *Yes, I am comfortable because we did EDMA160 last semester, ngathola okuningi khona as si-prover ama theorems i-understanding ngayithola sekungabalula ukuthi ngichazela izingane ayikho inkinga (I gain a lot of understanding there as we prove theorems. It will be easy to explain to learners there is no problem).*

PST24: *Yes, I am very comfortable because last year, in the class EDMA160, I learned new things that I did not understand clearly at high school, but here at university, it was clear. Now I can explain it to other people; I am confidence and comfortable to solve and teach similarity and proofs.*

PST34: *Yes, I am confident; I had a courage to do all the questions. I had no difficulty.*

Most PSTs indicated confidence in teaching similarity and proofs to learners; some further acknowledged that the university module they did equipped them with skills and knowledge, which boosted their understanding and confidence in teaching this section of mathematics.

4. **What is your general attitude/feeling towards teaching and learning Euclidean geometry?**

   PST1: *I can say my attitude are feeling very enthusiastic by this topic because it is very understood by me, but some learners find it so difficult because they need enough time to understand it, but I do not know why because I feel it is easy.*

   PST7: *It is my favourite chapter in mathematics. I feel it is something we need to teach because it exercises your mind.…as a maths teacher, I feel like when I am relaxing too much, there is something wrong. I just take few questions to do so that I can maintain standard of thinking.*

   PST8: *I feel happy about it and ngikulangazelele ukuyifundisa (I am eager to teach it), I want ukuchazela ingane zibone i-easy (to explain to the learners so that they can see it is easy) zingayiphasa if ngizama uku-understander every concept (they can pass if they can try to understand every concept).*

   PST10: *I have a positive attitude; although it is a bit confusing topic, but it is very interesting. It should be taught in the way that learners understand and enjoy it.*

   PST14: *Engakusho lana ukuthi lechapter iright noma beyithatha ngenjechapter enzima. (I can say here that this chapter is good, although they perceive it as challenging). But into edingekalayo ukuthi uyi introducer ngendlela youkuthi ibe lula kuma learners (what is needed is to introduce it in a simple way to learners) you introduce them step by step so that they will be able to do and understand even difficult questions.*

   PST24: *First of all, I love to teach geometry because, during high school, I feel like other teachers were teaching us maths, but they did not have enough knowledge of Euclidean geometry because when you ask some questions, they say they will answer on another day. They left us unanswered. Also, when they teach, they will just browse and stick on these topics that they love, calculus, algebra etc. I love to teach geometry because many people find it difficult. I believe geometry is based on researching and finding more, then you get clear and be able to teach it to learners. I have patience for teaching it because I believe it requires most patience.*

   PST34: *The content of geometry is good; as I explain early that as you learn this topic, you eventually develop mentally; learners should learn it.*

Many participants showed a very positive attitude towards Euclidean geometry, showing a willingness to teach it. Interestingly, one participant claimed their love and willingness to teach geometry was inspired by their former teacher's lack of geometry content knowledge. Finally, all the participants showed a positive attitude towards teaching similarity and proofs in

Euclidean geometry. Although challenging, they were willing to teach it to secondary school learners. The following is a detailed discussion of PSTs' written responses for each item in the questionnaire, supplemented by verbal responses from the interviews where possible.

4.3 Presentation of findings

4.3.1 Item 1.1

Item 1.1 was: ***Define the meaning of similarity in mathematics as best you can.*** **(2)**

The above question was set at the VHL2 analysis level. It required PSTs to use conditions for similar figures to deduce a general working definition of similarity in mathematics. The expected definition was figures with equal corresponding angles and corresponding sides in the same proportion. As indicated in table 4.1, out of 34 PSTs who participated, 2 PSTs answered incorrect (C2), 3 PSTs answered correct but with an incomplete response (C3), while the remaining 29 (85.29%) PSTs answered completely correct (C4). Figure 4.1 below shows the response of PST1, which was categorised as C2.

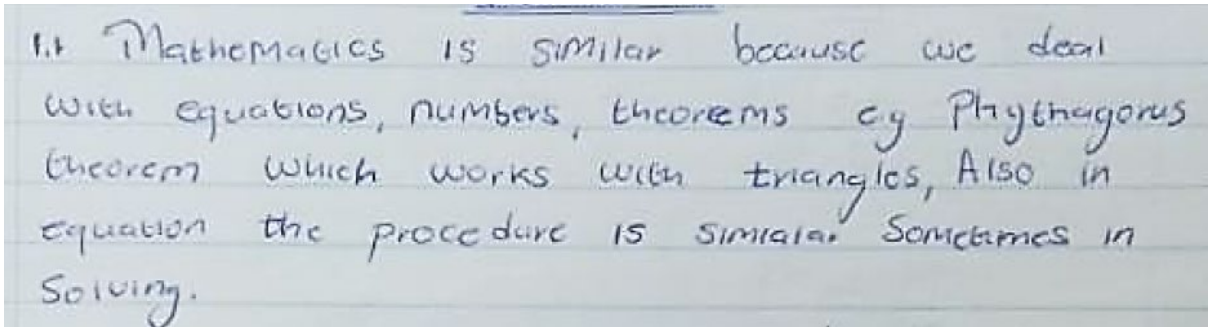


Figure 4.1: Response by PST1 to Item 1.1 classified as C2

The response by PST1 was considered incorrect (C2) because it was difficult to comprehend the intended meaning. However, during the interview, PST1 showed an understanding of the term, which was not evident in their initial written response. When probed about their response, participant PST1 explained that they did not understand the written instruction entirely at the time but could later explain similarity more clearly, as shown in the following excerpts.

> ***PST1:*** *I did not understand the question very well. I thought I am giving the similarity between the topics of mathematics. I did not explain the similarity as it is in mathematics.*
>
> ***Researcher:*** *Can you now give the best definition of similarity as it was supposed to be?*
>
> ***PST1:*** *It means objects with same shape but not necessarily same size.*

Figure 4.2 below shows PST9's response classified as C2.

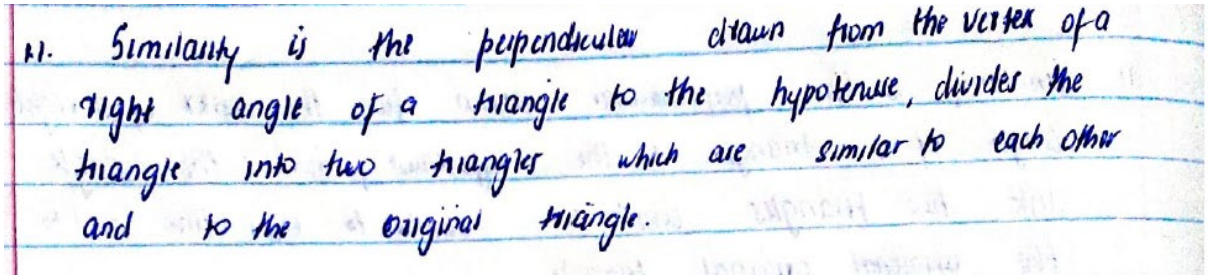


Figure 4.2: Response by PST9 to item 1.1 classified as C2

One of the participants (PST9) gave a particular case of similar triangles in the pen and paper test instead of defining similarity in mathematics. However, during the interview, PST9 was able to provide a complete explanation when probed as follows:

> ***Researcher:*** *Is this a general definition of similarity in mathematics?*
>
> ***PST9:*** *Yes, sir.*
>
> ***Researcher:*** *Does similarity apply only in the case of a right-angle triangle only?*
>
> ***PST9:*** *This definition is not the correct one; it is an example of similar triangles.*
>
> ***Researcher:*** *What is the correct general definition of similarity in mathematics?*
>
> ***PST9:*** *Figures with the same shape but not necessarily same in terms of size.*

Figure 4.3 below shows the response by PST2, which was classified as C3.

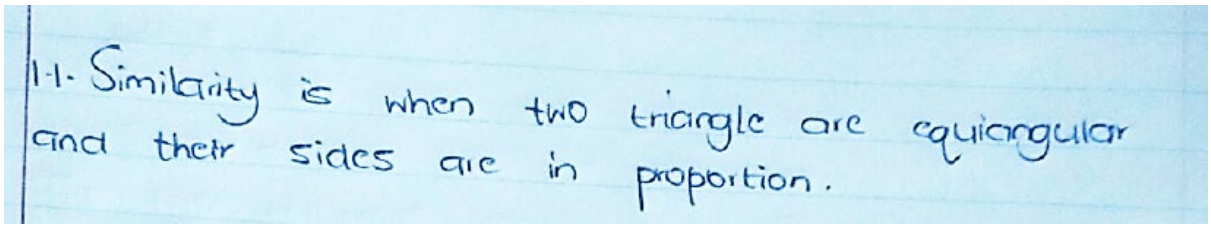


Figure 4.3: Response by PST3 to Item 1.1 classified as C3

The response by PST3 was considered correct but incomplete (C3) because they only explained similarity for triangles ignoring other figures that might be similar.

4.3.2 Item 1.2

The question for Item 1.2 was: ***What are the minimum conditions required to show that two polygons are similar? (2)***

This question was set at the VHL2 analysis level since it required PSTs to recall the definition and explicitly mention sufficient conditions for two figures to be similar. For any figures, there are two minimum conditions:

a. All pairs of corresponding angles are equal, and,
b. All pairs of corresponding sides are in proportion.

However, for triangles, it turns out that any one of the above conditions is sufficient. According to the South African curriculum, the theorems proving the sufficiency are studied in Grade 12. As indicated in Table 4.1, out of 34 PSTs who participated, 2 PSTs answered incorrectly (C2), another 2 PSTs provided incomplete responses. However, it contained some correct statements (C3) and the majority of 30 (88.24%) PSTs answered completely correctly (C4). Figure 4.4 below shows the responses of PST14, which was classified as C2.

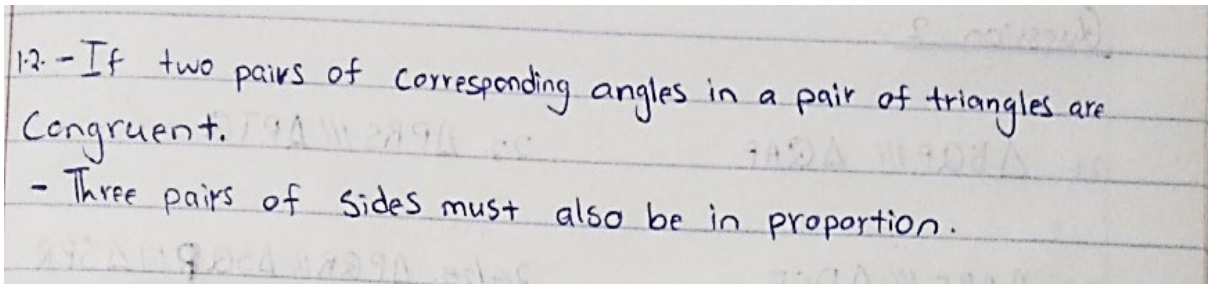


Figure 4.4: Response by PST14 to Item 1.2 classified as C2

The response by PST14 was considered incorrect (C2) because it limited the conditions of similarity to triangles ignoring other figures which might be similar. In trying to explore PST14's understanding of minimum conditions of similarity, the interview went as follows:

> ***Researcher:*** *Does similarity apply in triangles only, or can other figures like squares and rectangles be similar?*
>
> ***PST14:*** *Similarity can be applied to other figures like squares and rectangles, not only in triangles.*
>
> ***Researcher:*** *Then why it seems your conditions of similarity are limited to triangles?*
>
> ***PST14:*** *Yes, I see it now.*
>
> ***Researcher:*** *What should be general conditions for figures to be similar?*

PST14: *-All pairs of corresponding angles are congruent.*

-*All pairs of corresponding sides are in proportion.*

Figure 4.5 below shows the response from PST1, which was classified as C3.

Figure 4.5: Response by PST1 to Item 1.2 classified as C3

The response from PST1 was unclear, consisting of phrases without any connecting explanations. It was classified as C3 because the condition of equal angles was given. While PST1 could correct their original explanation by stating the correct conditions of similarity during the interview, they could not explain the reason for mentioning common angles/sides. Moreover, the verbal response indicated their limited understanding of applying conditions of similarity with varying figures. The interview was as follows:

Researcher: *Can you please elaborate on what you mean by "common angle/sides"?*

PST1: *I did not explain in a way that I'm briefing it out. I was supposed to say there are two common angles that need to be equal and one side of that polygon.*

Researcher: *Does similarity only occur when two polygons have two common angles equal and one side equal?*

PST1: *No, they must have corresponding sides in proportion and corresponding angles equal.*

Researcher: *Do these two conditions need to be both satisfied, or one condition is enough to declare any two figures similar?*

PST1: *It is the one that says corresponding angles equal, then corresponding sides will be automatically in proportion.*

Figure 4.6 below shows the response from PST8, which was classified as C3.

We need to show that two conditions are true:

a) All pairs of corresponding sides are in a same proportion

b) All pairs of corresponding sides are in the same proportion.

Figure 4.6: Response by PST8 to item 1.2 classified as C3

The response from PST8 was classified as C3 because it stated only one correct condition twice. However, during the interview, It was evident that PST8 knew the second condition but made a slip. Moreover, PST8 showed a sound understanding of applying conditions of similarity in varying figures. The interview was as follows:

Researcher: *Did you notice that numbers (a) and (b) are the same?*

PST8: *I realised my mistake when my script come back.*

Researcher: *What other condition do you want to include, if there is any?*

PST8: *I should have written:*

a. *All pairs of corresponding sides are in a same proportion.*

b. *All pairs of corresponding angles are congruent.*

Researcher: *Both conditions need to be satisfied, or one is enough to show that two polygons are similar?*

PST8: *Both should be satisfied because it might happen that their corresponding angles are congruent, but their corresponding sides are not in same proportion.*

Figure 4.7 below shows the response from PST32, which was classified as C4.

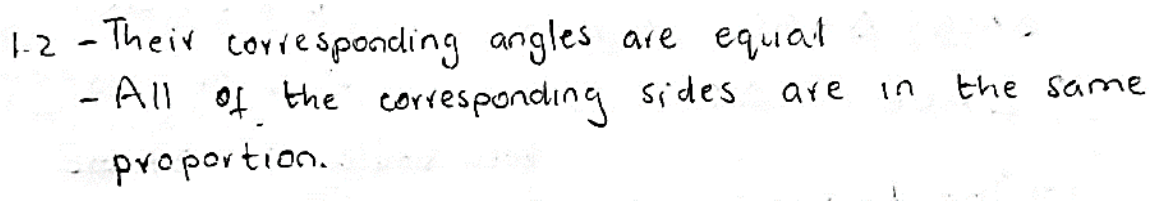


Figure 4.7: Response by PST32 to Item 1.2 classified as C4

Although the response from PST32 was completely correct (C4), the interview probing his understanding of these conditions with varying figures revealed some deficit. The participant (PST32) claimed that any figures with only equal corresponding angles would be similar.

It may not necessarily be valid as figures with equal corresponding angles are not always similar because their corresponding sides may not be proportionate. The interview was as follows:

> ***Researcher:*** *Both conditions need to be satisfied, or one is enough to show that two polygons are similar?*
>
> ***PST32:*** *I can say only one condition needs to be satisfied, which is corresponding angles equal. If I can prove all angles of two polygons are equal. Then I have proved that those two polygons are similar without mentioning that their corresponding sides are in proportion.*

4.3.3 Item 1.3

The question for item 1.3 was: ***Given any two arbitrary squares of different sizes, are they similar? Explain (2)***

This question was set at the VHL2 analysis level since it required PSTs to first analyse properties of squares then decide if the conditions were satisfied for the figures to be similar. Although the similarity of squares is not studied in detail in secondary school, any two arbitrary squares have all angles equal to 90° and have all sides equal (therefore proportional). Hence by understanding the properties of squares and conditions of similarity, it turns out that any two squares satisfy both conditions of similarity; hence, they will always be similar. As indicated in table 4.1, out of 34 PSTs who participated, four PSTs did not respond to this question (C1), two PSTs answered incorrect (C2), another two PSTs answered correct, but with incomplete responses (C3), while the remaining majority of 28 (82.35%) PSTs answered completely correct (C4). Furthermore, what was interesting is that 13 out of 28 PSTs who answered correct (C4) used both minimum conditions of similarity to conclude that two arbitrary squares are similar. The other 15 PSTs used one condition: corresponding angles equal.

Below is the interview with PST1, classified as C1 because they did not respond to this question. When probed about the challenges experienced with this question, PST1 was in a rush when he worked on the task but answered this question correctly when responding verbally. The interview was as follows:

> ***Researcher:*** *Was this question difficult for you since you did not respond to it?*
>
> ***PST1:*** *I was at rush when doing this work, but I was supposed to say they are similar because they have equal angles, which are equal to* 90°.

Figure 4.8 below shows the response from PST6 classified as C2.

Rhombus and Parallelogram, yes, they are similar. They have all the sides equal leading to the rule of SSS.

Figure 4.8: Response by PST6 to Item 1.3 classified as C2

The response of PST6 was considered incorrect (C2) because it was unclear how they included rhombus and parallelogram in their written response.

Figure 4.9 below shows the response from PST9, also classified as C2.

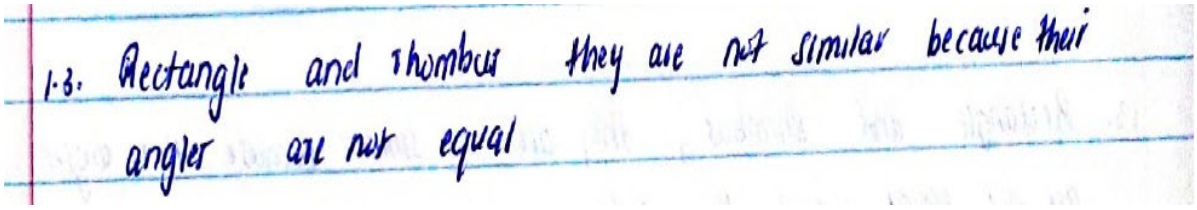


Figure 4.9: Response by PST9 to Item 1.3 classified as C2

The response of PST9 was considered incorrect (C2) because it was unclear how a rectangle and rhombus were included on the question based on squares. It may be that PST9 considered a rectangle and rhombus as being included in the set of squares or did not realise that the question was restricted to squares only. Although PST9 could not elaborate on why they included rectangles and rhombus in their response, they could correct their written response. The interview was as follows:

> ***Researcher:*** *Please explain why you included rectangle and rhombus in this question which was asking about similarity of two arbitrary squares?*
>
> ***PST9:*** *I do not know why. But I think two squares are similar because they have corresponding angles that are equal and corresponding sides in proportion.*

Figure 4.10 below shows the response from PST2 classified as C3.

Yes, they are similar. Because they both have four 90-degree angles & four equal sides even if the lengths of their sides are different. (2)

Figure 4.10: Response by PST2 to Item 1.3 classified as C3

Although the response from PST2 was correct, that two arbitrary squares are similar, their reasoning and mathematical language were not precise; hence the response was grouped in C3.

4.3.4 Item 1.4

The question for item 1.4 was: ***Given any two arbitrary rectangles of different sizes, are they similar? Explain. (3)***

This question was set at the VHL2 analysis level since it required PSTs to analyse properties of rectangles then decide if the conditions were satisfied or not for the figures to be similar. Although the similarity of rectangles is not studied in detail in secondary school, the basic understanding of the properties of rectangles and conditions of similarity will assist in deciding whether they are or are not similar. All angles in a rectangle are 90° but, the ratio of corresponding sides in two rectangles may be different, so it turns out that two arbitrary rectangles may not necessarily be similar depending on the ratios of their corresponding sides. Hence, for rectangles, it is important to check if the corresponding sides are in proportion. As indicated in Table 4.1, out of 34 PSTs: three PSTs did not respond to this question (C1), most PSTs (27) answered incorrectly (C2), stating that arbitrary rectangles are similar; meanwhile, the remaining four PSTs answered correctly, but their response was incomplete (C3).

Figure 4.11 below shows the response from PST6 classified as C2.

Kite and Trapezoid, they are not similar. They both have at least one set of parallel sides. Trapezoids have one set of parallel sides, but kites do not have any parallel sides.

Figure 4.11: Response by PST6 to Item 1.4 classified as C2

The response of PST6 was considered incorrect (C2) because it was unclear how PST6 included a kite and trapezoid in their written response. It may be that PST6 assumed that kites and trapezoids were examples of rectangles and was pointing out that a kite was not similar to a trapezoid.

Figure 4.12 below shows the response of PST9 classified as C2.

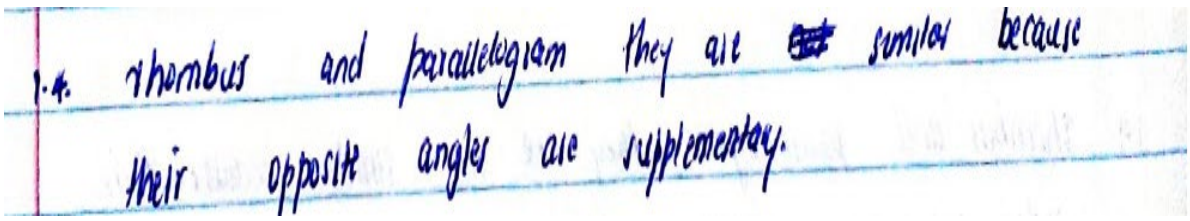


Figure 4.12: Response by PST9 to Item 1.4 classified as C2

The response of PST9 was incorrect (C2) because it was unclear how a rhombus and parallelogram were included on the question based on rectangles. Interestingly, PST9's response to this question is similar to their response about whether any two squares are similar (Item 1.3). For item 1.3, when asked about the similarity of squares, PST9 responded that a rectangle and rhombus were not similar. It seems they provided a counterexample to show that not all squares are similar. However, the error is that a rectangle and rhombus are not included in the class of squares. For Item 1.4, concerning the question of whether any two rectangles are similar, PST9 seemed to use a similar incorrect approach that a rhombus and a parallelogram belong to the class of rectangles by referring to them when asked about rectangles. It seems that their misconception is about the concept of class inclusion of quadrilateral. During the interview, PST9 admitted to misunderstanding the questions and could correct their written response verbally. The interview was as follows:

> ***Researcher:*** *Please explain why you included rhombus and parallelogram in this question which was asking about similarity of two arbitrary rectangles?*
>
> ***PST9:*** *I think it was a misunderstanding of the question. I do not think rectangles, they will always be similar because their corresponding sides are not always in proportion.*

Figure 4.13 below shows the response from PST7 classified as C3.

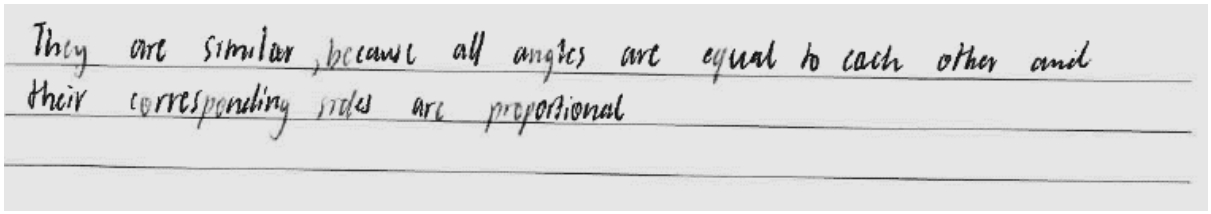


Figure 4.13: Response by PST7 to Item 1.4 classified as C3

The response of PST7 showed they did not carefully think about the possible ratios of corresponding sides of rectangles which sometimes may not be in the same proportion. The interview was as follows:

> ***Researcher:*** *Will it be always the case that two arbitrary rectangles are similar?*
>
> ***PST7:*** *Yes, Sir, because their corresponding angles are equal, and their corresponding sides are proportional.*
>
> ***Researcher:*** *Will it be always the case that rectangles with corresponding sides are in proportion?*
>
> ***PST7:*** *Yes, Sir.*

Researcher: *Consider the typical example of two rectangles; the first is 16 by 4 units, and the second is 8 by 3 units. Are their corresponding sides proportional?*

PST7: *The ratio of corresponding sides will not be same then these rectangles are not similar. It is clear now how these conditions of similarity can apply (laughing).*

As seen in the interview above, PST7 firmly believed that the corresponding sides would always be in proportion for any two rectangles. However, when PST7 was presented with a counterexample of two rectangles whose corresponding sides were not in proportion, they were convinced that any two rectangles are not necessarily similar.

Figure 4.14 below shows the response from PST32 classified as C3.

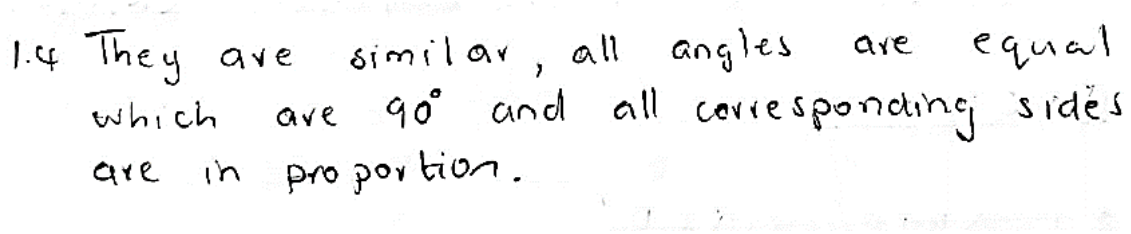


Figure 4.14: Response by PST32 to Item 1.4 classified as C3

The response of PST32 shows similar reasoning to PST7 that the two similarity conditions were satisfied for rectangles and not carefully thinking about the possible ratios of corresponding sides of rectangles which sometimes may not be proportional. During the interview, PST32 corrected their written response and gave examples of rectangles that might not be similar. The interview was as follows:

Researcher: *Will it always be a case that two arbitrary rectangles are similar?*

PST32: *Sir, I can see that I made a mistake because when we discuss about a square, I understood a rectangle is not like a square, If I can make the first rectangle with longest sides four and shortest sides two and the second rectangle with longest sides nine and shortest sides three. When doing ratios, I can see that their ratios are not the same; therefore, their corresponding sides are not in proportion.*

Figure 4.15 below shows the response from PST29, also classified as C3.

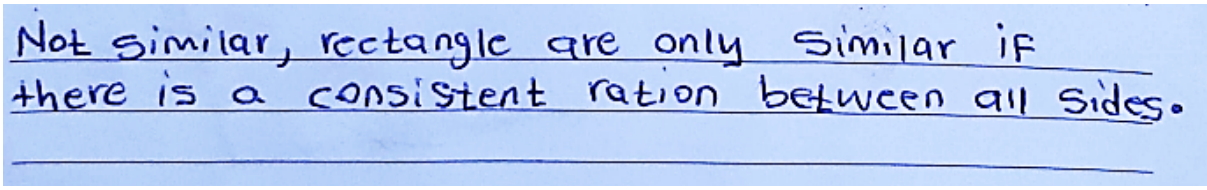
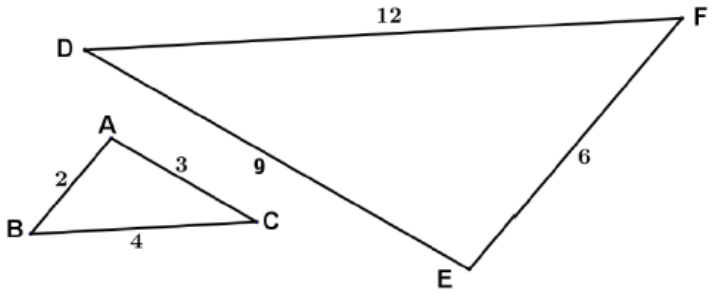


Figure 4.15: Response by PST29 to Item 1.4 classified as C3

The response of PST29 was almost correct (C3). PST29 indicated "not similar", implying that all rectangles are not similar. Although they included that rectangles are only similar if there is a consistent ratio between all the sides, the first part of the response was not completely correct. It would have been completely correct if they had said "not necessarily" or "not always".

4.3.5 Item 1.5

The question for item 1.5 was: ***The sides in the diagram have lengths as indicated. Are these two triangles similar, Yes or No? Explain why you say so. (3)***



This question was set at the VHL3 informal deduction since it required PSTs to use any one of the two minimum conditions of similarity to check if these given triangles are similar or not. Since the lengths of all sides were given, the convenient way was to check if the corresponding sides were proportional. However, another method of calculating interior angles in each triangle using the cosine rule could have been used. As indicated in table 4.1, out of 34 PSTs who participated, 3 PSTs answered incorrect (C2), 10 PSTs answered correct but with an incomplete response (C3), while 21 (61.76%) PSTs answered correctly (C4).

Figure 4.16 below shows the response from PST2, which was classified as C2.

$\frac{AB}{ED} = \frac{2}{9}, \frac{BC}{DF} = \frac{4}{12} = \frac{1}{3}, \frac{AC}{EF} = \frac{3}{6} = \frac{1}{2}$

No, they are not similar. The sides are not proportional, so they are not similar.

Figure 4.16: Response by PST2 to Item 1.5 classified as C2

The response of PST2 was considered incorrect (C2) because when calculating ratios, PST2 was not able to accurately pick corresponding sides, which would have been in the same proportion.

Figure 4.17 below shows the response of PST16 classified as C2.

No, because sides are unequal, no corresponding angles.

Figure 4.17: Response by PST16 to Item 1.5 classified as C2

The response of PST16 was incorrect (C2). The participant indicated that the given triangles were not similar because the sides were unequal; it is not required for figures to be similar. They confused the meaning of corresponding sides in proportion to mean that sides are equal.

Figures 4.18 below shows the response from PST1 classified as C3.

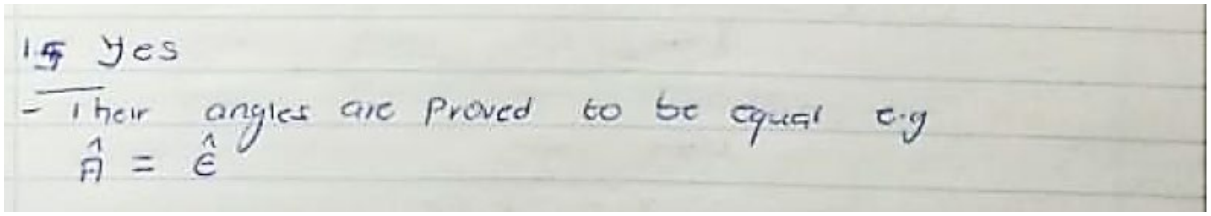


Figure 4.18: Response by PST1 to Item 1.5 classified as C3

The response of PST1 was correct but incomplete (C3) due to reasoning that is not convincing since only one pair of angles was compared as equal, and no mathematical proof was provided that indeed these angles are equal. Probing their response further confirmed the shortfall as PST1 was using visualisation to conclude that the pair of angles were equal. However, they were able to correct their response; the interview went as follows:

Researcher: *Please explain your approach to this question.*

PST1: *$\angle A$ and $\angle E$ seem to be equal; that is why I say these triangles are similar.*

Researcher: *In geometry, can we use our sight to conclude that angles are equal?*

PST1: *No, I think if you can use the cosine rule to prove that $\angle A = \angle E$ and calculate the other pair of angles, then the remaining one will be equal because of the sum of angles of a triangle. Then you proved they are similar.*

Researcher: *There is no possible convenient way to prove similarity if we have the length of sides?*

PST1: *Oh yes, you can calculate ratios and conclude they are similar.*

Figure 4.19 below shows the response of PST32 classified as C4.

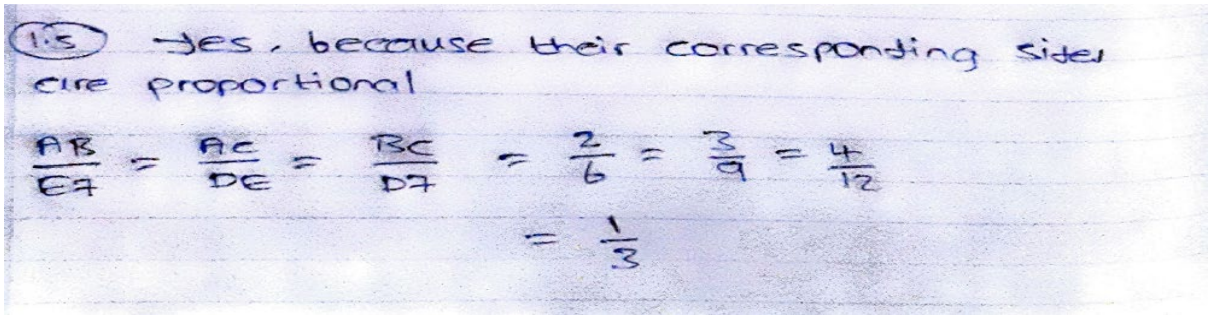


Figure 4.19: Response by PST32 to Item 1.5 classified as C4

Although the response of PST32 was correct (C4), probing their understanding on how to appropriately select sides in triangles that might give the same ratio revealed that they used a trial-and-error approach to test the ratio of different sides. They did not realise that corresponding sides would be the pairs of the longest sides, middle side, and shortest sides in each triangle, which might give the same ratio. The interview was as follows:

Researcher: *Do you have a hint of which side to divide when checking for the proportionality of triangles?*

PST34: *No, I tried to use different sides to see if they will give me the same ratio throughout and eventually, I found those sides which will give the same ratio.*

The response from PST34 shows they did not have a systematic method based on a strong conceptual understanding of how to solve this problem immediately without trial and error.

Figure 4.20 below shows the response from PST33, also classified as C4.

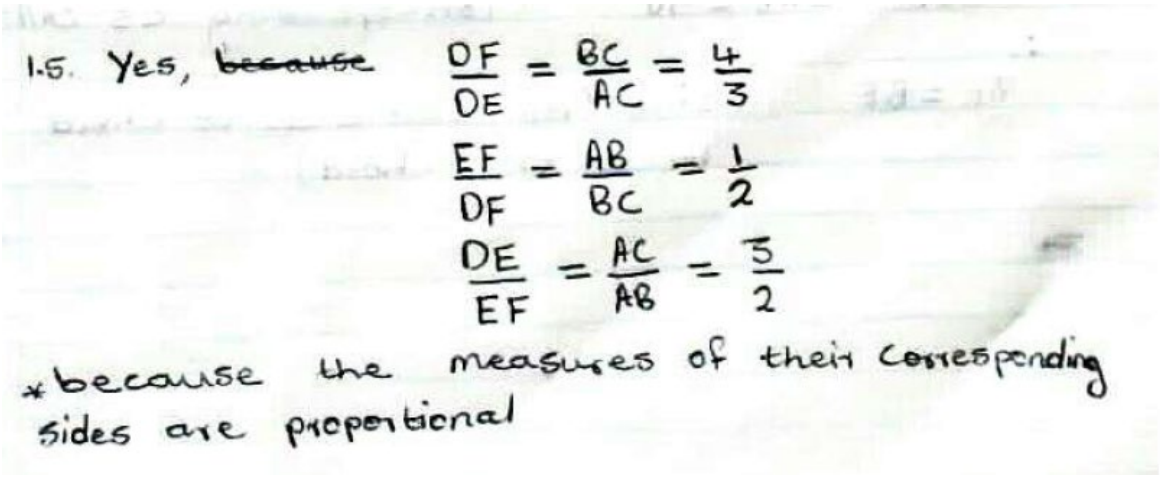
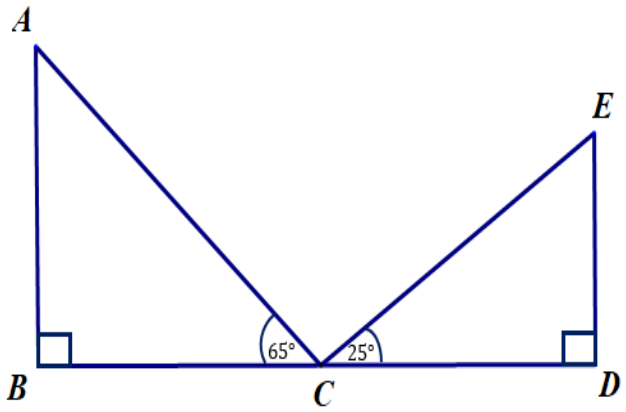


Figure 4.20: Response by PST33 to Item 1.5 classified as C4

The response from PST33 was correct. He used an unusual method of taking ratios of sides within each triangle to prove the proportionality of two triangles.

4.3.6 Item 1.7

The question for item 1.7 was: ***State with reason whether or not the following triangles are similar. (2)***



The question was set at the VHL3 informal deduction. It required PSTs to first calculate unknown angles in each triangle using the theorem that the sum of the interior angles of triangles is supplementary. Then, the PSTs could check if corresponding angles are equal, proving similarity. As indicated in table 4.1, out of 34 PSTs who participated, 8 PSTs answered incorrect (C2) while the remaining 26 PSTs answered correct (C4).

Figure 4.21 below shows the response from PST29 classified as C2.

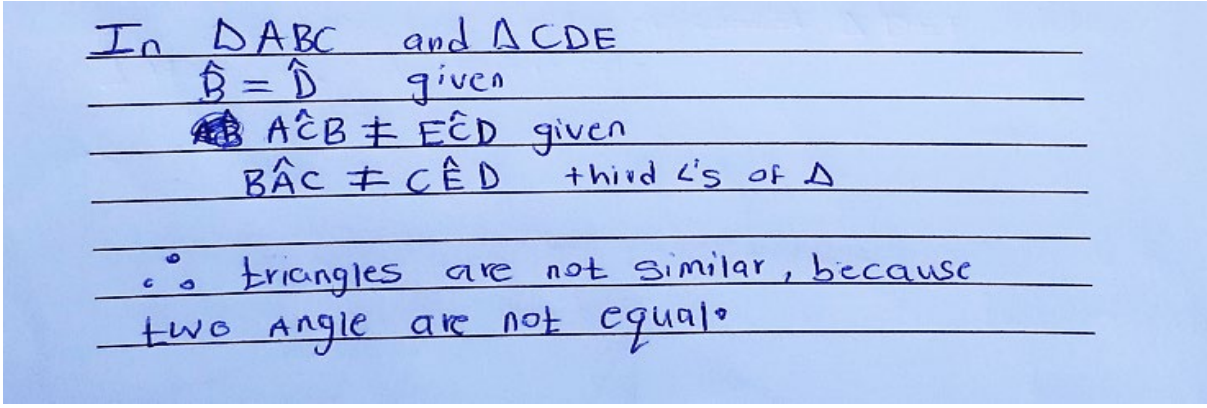


Figure 4.21: Response by PST29 to Item 1.7 classified as C2

The response from PST29 was considered incorrect (C2) due to comparing unknown angles and claiming they are not equal without calculating their actual values or providing any sound mathematical justification. It seems PST29 compared these angles using visualisation.

Figure 4.22 below shows the response from PST1, also classified as C2.

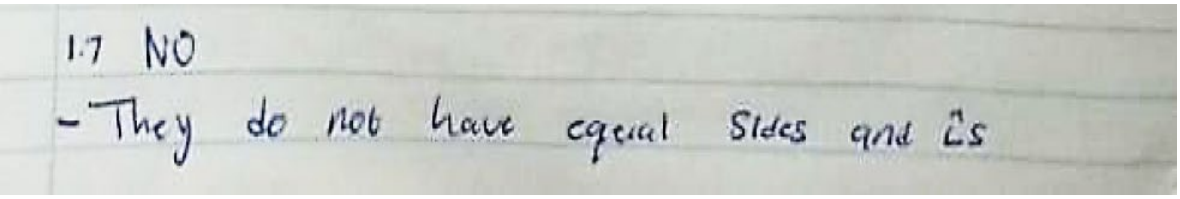


Figure 4.22: Response by PST1 to Item 1.7 classified as C2

The response from PST1 was incorrect (C2) due to comparing unknown sides and angles, claiming they are not equal without any sound mathematical justification. Furthermore, the interview confirmed that PST1 used visualisation to conclude that these triangles do not have equal sides and angles. During the interview, after being probed further, PST1 was able to correct their response verbally as follows:

> ***Researcher:*** *How did you know that these triangles do not have equal sides and angles?*
>
> ***PST1:*** *I was looking at them.*
>
> ***Researcher:*** *In geometry, can we use our sight to conclude if sides and angles are equal?*
>
> ***PST1:*** *No, but I think they are similar because if you can calculate* $\angle A = 25°$ *and* $\angle E = 65°$*, they are similar because the corresponding angles are equal.*

As seen in the excerpt above, PST1 thought some more about the size of the angles and realised that the corresponding angles were equal.

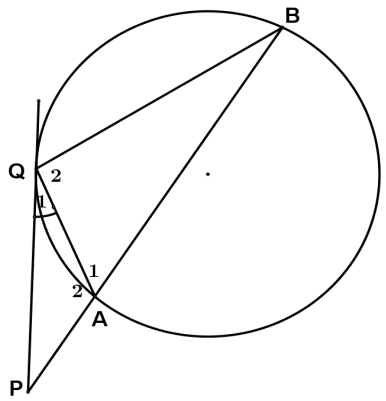
QUESTION 2

All sub-questions in this section were based on the question: ***For each diagram below, write down a triangle similar to the given triangle. Naming must be in the correct order.***

All the sub-questions below were set at the VHL3 informal deduction. It required PSTs to use geometry theorems to determine equal angles, which would help determine a triangle similar to the given one in the correct order.

4.3.7 Item 2.1

In the diagram below, PQ is a tangent to the circle at Q.



ΔBQP /// Δ____

As indicated in table 4.1, out of 34 PSTs who participated, 13 answered incorrectly (C2), while the remaining 21 PSTs answered this question correctly (C4). Figure 4.23 below shows the response from PST7 classified as C2.

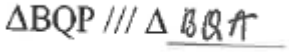


Figure 4.23: Response by PST7 to Item 2.1 classified as C2

The response of PST7 was incorrect (C2) due to the incorrect order of naming similar triangles. During the interview, after probing further, PST7 was able to correct their written response as follows:

> ***Researcher:*** *Please explain how you concluded these triangles are similar?*
>
> ***PST7:*** *Eish my order was wrong ….*
>
> ***Researcher:*** *Can you now please try to give the correct order of naming?*

PST7: *BQP/// QAP because:*

∠B = ∠A tan chord theorem

∠P is common

∠Q1 + ∠Q2 = ∠A third angle of a tringle

Then triangle BQP is similar to triangle QAP, corresponding angles equal.

The interview response from PST7 shows that as soon as they looked at their original answer, they realised that the order of the naming was not correct. They were able to correct the answer.

Figure 4.24 below shows the response by PST34 classified as C4.

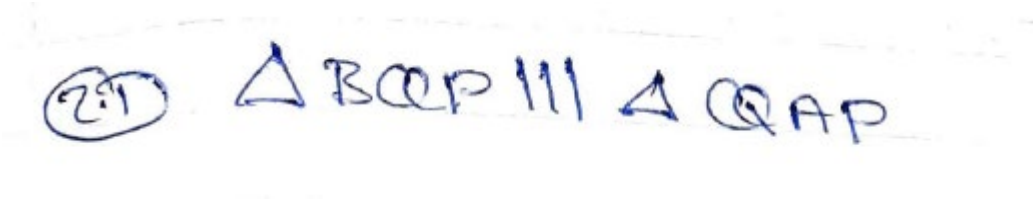


Figure 4.24: Response by PST34 to Item 2.1 classified as C4

The written response from PST34 was correct (C4). During the interview, PST34 was probed further to understand the reasoning used in answering this question.

Researcher: *Please explain how you concluded these triangles as similar?*

PST34: *The statement on top says PQ is a tangent, and if I see the word tangent, I know exactly there are theorems associated with tangent.*

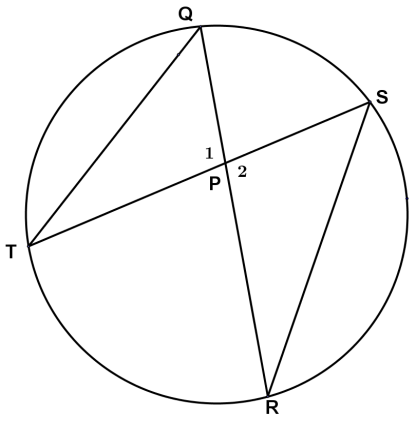
Then I say ∠Q1= ∠B……..tan-chord theorem

∠P is common from both triangle

Then ∠Q = ∠A, the reason sum of angles of a triangle.

Therefore, they are similar because of corresponding angles equal.

4.3.8 Item 2.2



ΔPRS /// Δ____

As indicated in table 4.1, out of the 34 PSTs who participated, 10 answered incorrect (C2), while the remaining 24 PSTs answered this question correct (C4). The ten incorrect responses were due to the incorrect order of naming a triangle similar to the given one.
Figure 4.25 shows the response from PST8, which was classified as C2.

ΔPRS /// ΔTQP

Figure 4.25: Response by PST8 to Item 2.2 classified as C2

The written response from PST8 was considered incorrect (C2) due to the incorrect order of naming similar figures. Probing was used to gain insight into PST8's understanding of the question. The interview was as follows:

Researcher: *Please explain how you concluded these triangles similar?*

PST8: *My naming was wrong.*

Researcher: *Can you now give the correct order*

PST8: *PRS///PTQ*

∠P1 = ∠P2 vertical opposite

∠T =∠R subtended by the same arch

∠Q = ∠S same reason subtended by the same arch, or I should say remaining angle.

Hence, the findings from the interview showed that PST8 was able to correct their answer once they reflected on it, as was evident with many of the interviews in this study.

Figure 4.26 below shows the response from PST14 classified as C4.

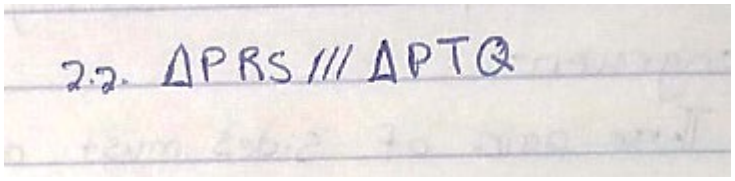


Figure 4.26: Response by PST14 to Item 2.2 classified as C4

The written response from PST14 was considered correct (C4). The interview was as follows:

Researcher: *Please explain how you concluded these triangles similar?*

PST14: *I think it will be simple, in ΔPRS and ΔPTQ:*

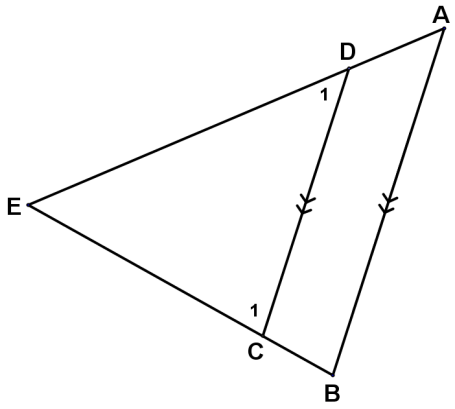
∠P2 = ∠P1 it is vertically oppositive angles

∠R= ∠T it is angles on the same segment

∠S = ∠Q it is angles on the same segment

Therefore, they are similar because the corresponding angles are equal.

4.3.9 Item 2.3



ΔABE /// Δ____

As indicated in table 4.1, out of 34 PSTs who participated, 9 PSTs answered this question incorrectly (C2), while the remaining 25 PSTs answered correctly (C4).

Figure 4.27 shows the response from PST20 classified as C2.

Figure 4.27: Response by PST20 to Item 2.3 classified as C2

The response by PST20 was considered incorrect (C2) due to the incorrect order of naming similar triangles.

Figure 4.28 below is the response from PST32 classified as C4.

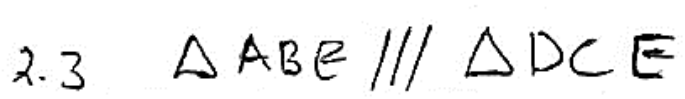


Figure 4.28: Response by PST32 to Item 2.3 classified as C4

The response from PST32 was correct (C4). The interview was as follows:

Researcher: *Please explain how you concluded these triangles similar?*

PST32: *We have bigger ΔABC and small ΔDCE*

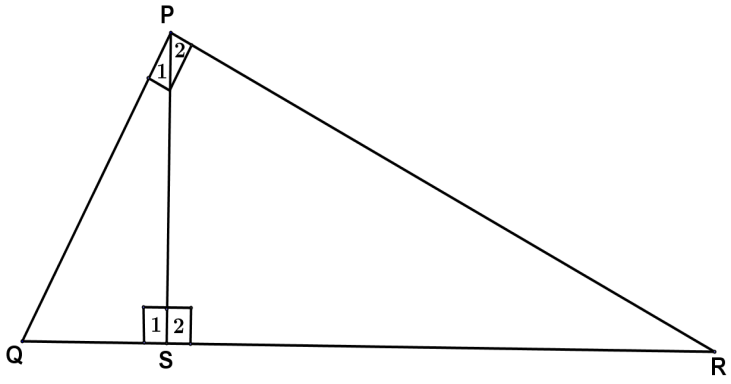
∠A= ∠D1 corresponding ∠'s DC//AB

∠B= ∠C1 corresponding ∠'s DC//AB

∠E is common on both triangles. Therefore, they are similar.

4.3.10 Item 2.4/2.5

In the following diagram, $Q\widehat{P}R = 90^0$ and PS⊥QR.



ΔPQR /// Δ____ ///Δ____

As indicated in table 4.1, out of the 34 PSTs who participated, for question 2.4, 15 PSTs answered incorrect (C2), while the remaining 19 PSTs answered correct (C4). For question 2.5, 16 PSTs answered incorrect (C2), while the remaining 18 PSTs answered correct (C4).

Figure 4.29 below is the response from PST11 classified as C2.

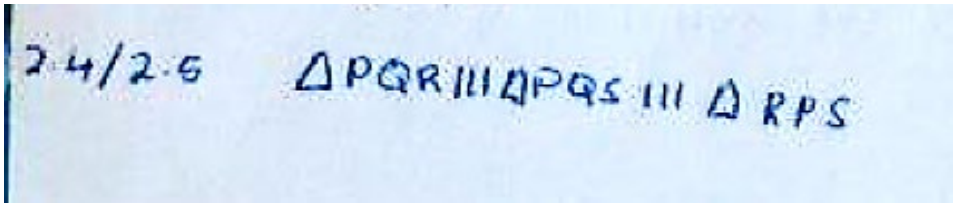
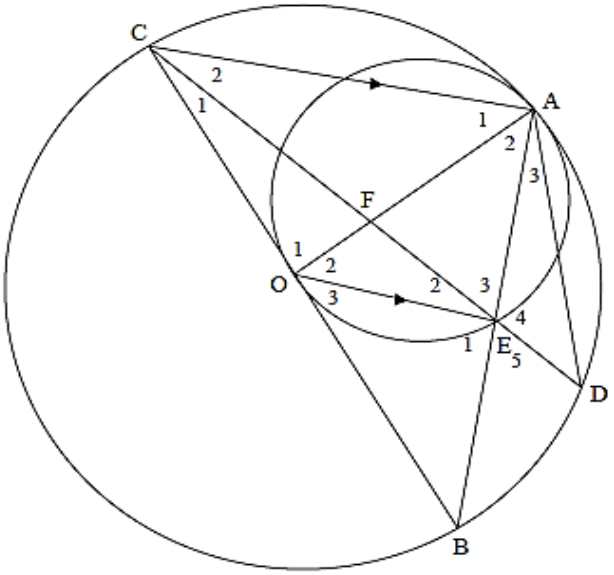


Figure 4.29: Response by PST11 to Item 2.4, classified as C2

The response from PST11 was incorrect (C2) due to the incorrect naming of similar triangles.

QUESTION 3

Two circles touch each other at point A, the smaller circle passes through O at the centre of the larger circle, and point E is on the circumference of the smaller circle. A, D, B and C are points on the circumference of the larger circle, OE//CA.



4.3.11 Item 3.1

The question for Item 3.1 was: **Prove, with reasons that AE=BE (3).**

This question was set at the VHL4 formal deduction since it required PSTs to construct proof by putting together theorems like an angle subtended by diameter, angles of parallel lines and line from centre perpendicular to the chord. As indicated in table 4.1, out of the 34 PSTs who participated, 13 answered incorrect (C2), 1 PST answered correct but incomplete (C3), while the remaining 20 PSTs answered correct (C4).

Figure 4.30 below shows the response by PST7, classified as C2.

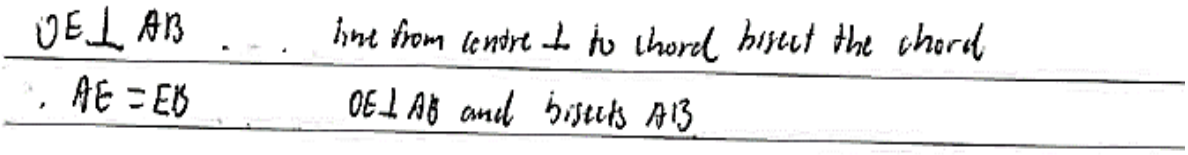


Figure 4.30: Response by PST7 to Item 3.1, classified as C2

The written response of PST7 was considered incorrect (C2) because he claimed OE⊥AB with the reason that line from centre⊥ to chord bisects the chord when this was not given. It means PST7 needed to prove that it was true before using it as fact. Interestingly, when probed to gain insight on their understanding of the above question, PST7, who had initially assumed the theorem to use, was later able to discover more methods of proving that line AE = BE. The interview was as follows:

> ***Researcher:*** *Why in the first step you use a theorem that states a line from the centre perpendicular to the chord bisects the chord?*
>
> ***PST7:*** *I was using theorem 1*
>
> ***Researcher:*** *But here, a line from centre OE is not given that it is perpendicular to chord AB?*
>
> ***PST7:*** *BC is a diameter, ∠A1 + ∠A2 = 90° ....∠ subtended by diameter*
>
> *∠E2 + ∠E3 = 90° corresponding angles*
>
> *Then now, I can use the theorem, which says the line from the centre perpendicular to the chord bisects the chord, then AE= EB.*
>
> *Also, the other method I could use is that ∠A1 + ∠A2 = ∠E1 = 90°*
>
> *∠E1 = ∠E2 + ∠E3 = 90°*
>
> *Then I prove that ΔOBE ≡ ΔOEA (RHS)*
>
> *Then conclude that AE = EB*
>
> *(Laughing) These answers just come by now.*

The findings from the interview show that when challenged, PST7 was able to fill in gaps in their reasoning and realised that there was more than one way to get the answer.

Figure 4.31 below shows the response from PST14, which was also classified as C2.

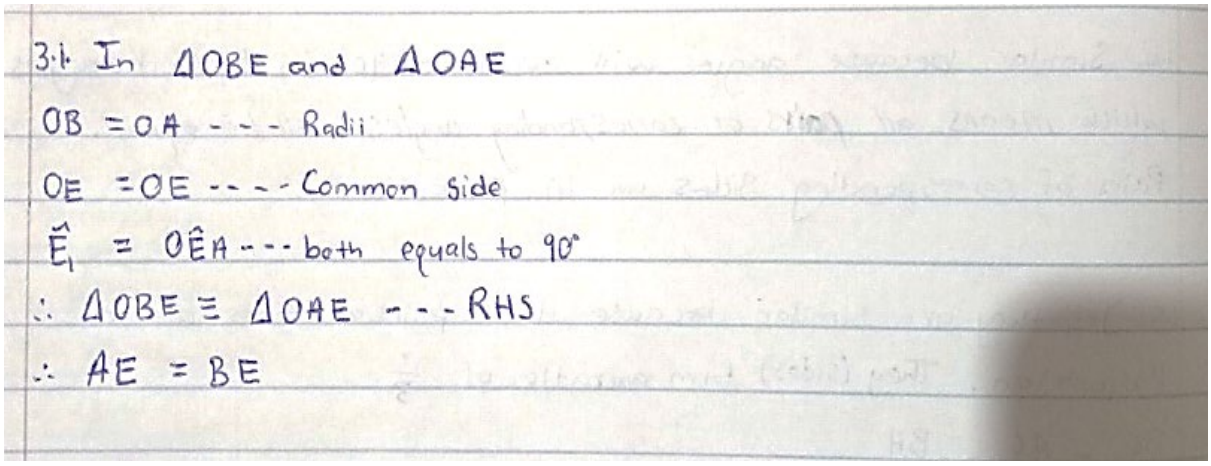


Figure 4.31: Response by PST14 to Item 3.1, classified as C2

The written response by PST14 was considered incorrect (C2) because they claimed that $\angle E1 = \angle OAE$ both equals 90°. The interview was as follows:

> ***Researcher:*** *Please explain step 3 of your solution why* $\angle E1 = \angle OEA = 90°$*?*
>
> ***PST14:*** *From theorem 1, which says the line from the centre perpendicular to the chord bisects the chord.*
>
> ***Researcher:*** *But it was not given that the lines are perpendicular. How do you conclude that they are perpendicular?*
>
> ***PST14:*** *Oh, okay, I see the mistake. In this case, I must first get:*
>
> $\angle CAB = 90°$ *...*$\angle's$ *angles on the semi-circle*
>
> $\angle CAB$ *is corresponding with* $\angle E1$*; they are equal to* $90°$.
>
> *Then conclude AE= BE using theorem 1 because a line from the centre is perpendicular to the chord.*

4.3.12 Item 3.2

The question for Item 3.2 was: ***Prove that*** $\Delta AED /// \Delta CEB$ ***(3)***

This question was set at the VHL3 informal deduction since it required PSTs to construct a simple proof that $\Delta AED /// \Delta CEB$ using theorems of circle geometry whereby angles are equal. It requires one to prove corresponding angles equal, which is one of the conditions of similarity. As indicated in table 4.1, out of the 34 PSTs who participated, two answered incorrect (C2), one PST answered correct but incomplete (C3), while the remaining 31 (88,23%) answered correct (C4).

Figure 4.32 below shows the response from PST2 which was classified as C2.

$\hat{C}_2 \cong \hat{A}$ (corresponding $\angle s$)

$\hat{B} \cong \hat{D}$ (corresponding $\angle s$)

$\therefore \Delta AED /// \Delta CEB$ (AA)

Figure 4.32: Response by PST2 to Item 3.2, classified as C2

The above response from PST2 shows a lack of understanding of corresponding angles. In many cases, equal corresponding angles are found on a parallel line. The term corresponding angles are also used to indicate the two angles within figures of the same shape, which are in the same position relative to the two shapes, respectively.

Figure 4.33 shows the response from PST22, which was classified as C3.

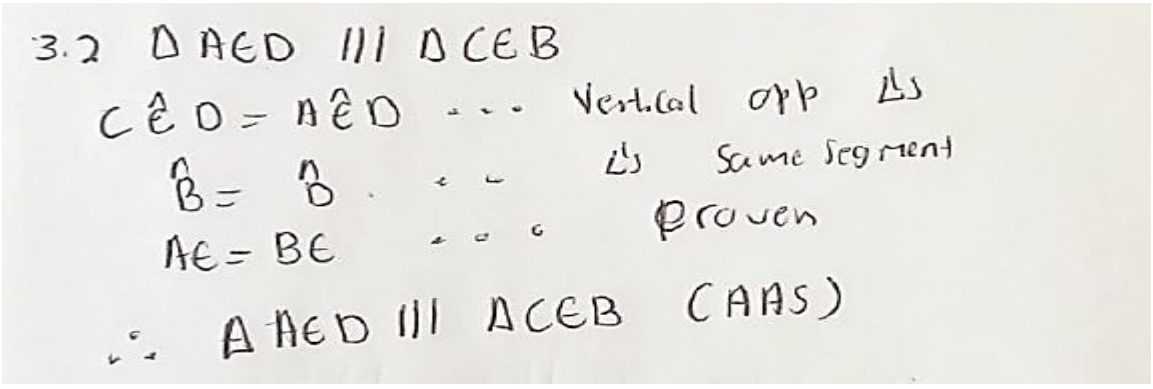


Figure 4.33: Response by PST22 to Item 3.2, classified as C3

PST22 chose the approach used to show that these triangles are congruent by picking out two sets of equal angles and a pair of equal sides, whilst the question required proof of similarity. Notably, the triangles are not congruent.

4.3.13 Item 3.3

The question for Item 3.3 was: ***Hence or otherwise, show that*** $\mathbf{AE^2 = DE.CE}$ ***(2)***

This question was set at the VHL4 of formal deduction since PSTs were required to express these sets of equal ratios arising from the fact that $\Delta \boldsymbol{AED} /// \Delta \boldsymbol{CEB}$, proved in Item 3.2. Then, they needed to draw on the fact proved in Item 3.1 that AE = BE. By substituting and manipulating expressions, PSTs could show the required expression and identify which triangles needed to be proved similar. They were further required to prove that similarity,

identify the sided relationship, and set up the sides in the same proportion using triangles that proved similar. Additionally, noticing that AE is squared and there was no BE, the relationship of AE and BE being equal must be utilised. As indicated in table 4.1, out of 34 PSTs who participated, one PST did not respond to this question (C1), while the remaining 33 (97,06%) PSTs answered correct (C4).

Figure 4.34 shows the response from PST3, classified as C4.

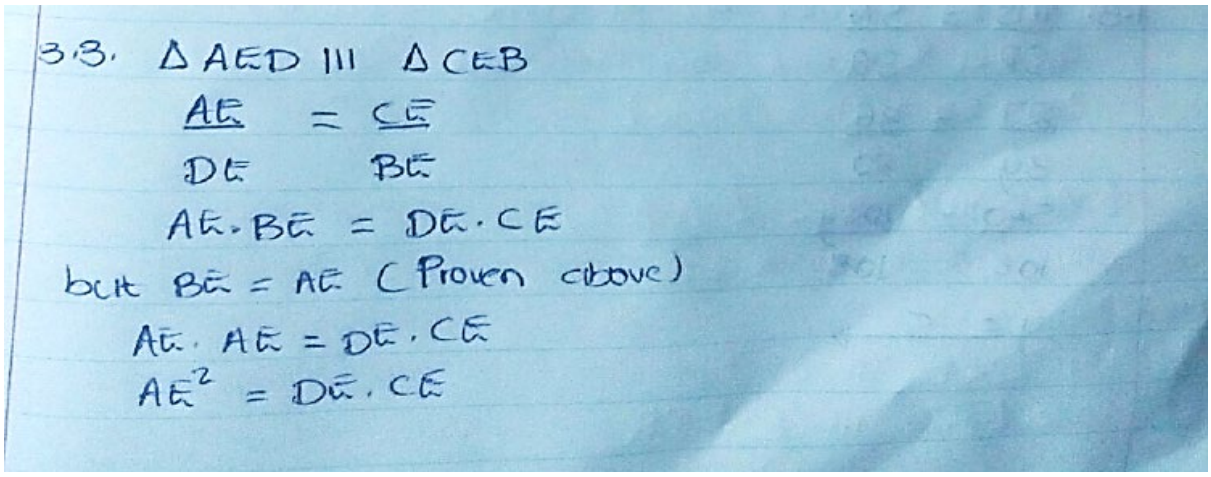


Figure 4.34: Response by PST3 to item 3.3 classified as C4

Figure 4.35 below shows the response by PST32, which was classified as C4.

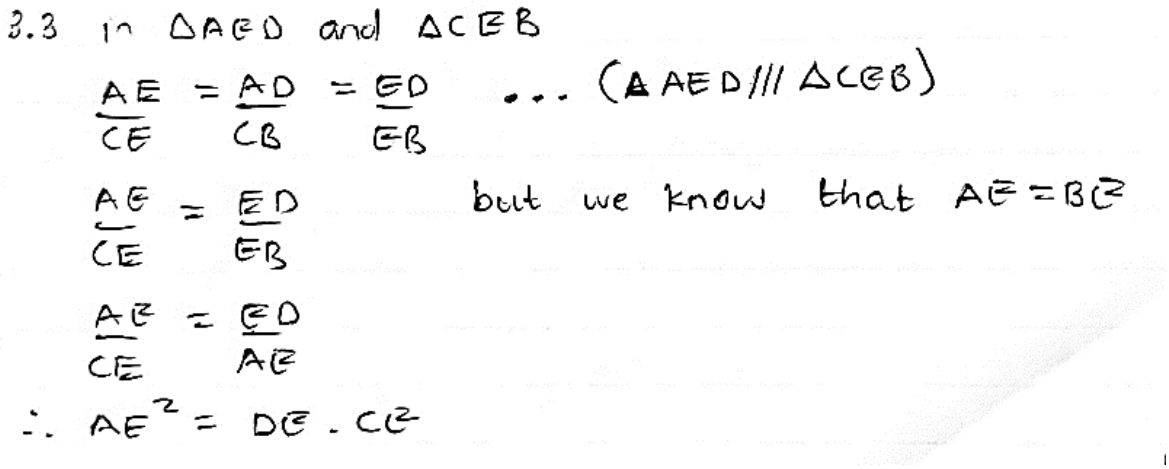


Figure 4.35: Response by PST32 to item 3.3, classified as C4

4.4 Pre-service teachers' Van Hiele levels of geometric thought

The raw data for classifying PSTs into different VHLs using criteria in chapter 3 (table 3.2 and table 3.3) can be found in appendix C. The table below summarises the VHLs for PSTs.

Table 4.2: Pre-service teachers' Van Hiele levels

Van Hiele Levels	Number of PSTs working on a certain level	Percentage of PSTs working on a certain level (n=34)
Below Analysis, VHL2	13	38.24%
Analysis, VHL2	9	26.47%
Informal deduction, VHL3	2	5.88%
Formal deduction, VHL4	10	29.41%

The results show that 13 PSTs (38.24%) involved in this study were still operating below VHL2, and 9 PSTs (26.47%) were still operating at the VHL2. Furthermore, 2 PSTs (5.88%) were operating at the VHL3. Only 10 PSTs (29.41%) showed competence at the VHL4.

4.5 Conclusion

This chapter presented the findings of research instruments. The next chapter discusses the results of this study in relation to this study's research questions.

CHAPTER 5

DISCUSSION AND CONCLUSION

5.1 Introduction

The previous chapter presented the findings of this study. This chapter discusses the current study's findings in relation to the research questions. The focus of the current study was exploring pre-service teachers' understanding of similarity and proofs in Euclidean geometry. In gathering the required data, the following research questions were posed:

1. How do PSTs perform on tasks based on similarity and proofs?
2. What misconceptions do PSTs have about similarity and proofs?
3. What is the VHLs of geometric thought for this group of PSTs in similarity and proofs?

The chapter further provides recommendations, limitations of the study, suggestions for further study and a conclusion.

5.2 Discussion of findings

This section presents the key findings related to each research question posed in this study.

5.2.1 Performance of pre-service teachers on tasks based on similarity and proofs

The pen and paper test and the semi-structured interviews were used to discern PSTs' performance on similarity and proofs in Euclidean geometry. The summary of the findings for all research items (see table 4.1 in chapter 4) reveals that on all research items except item 1.4, most PSTs provided correct answers, as more than half (50%) of the participants got each item completely correct. Since the research task examined secondary school content and all participants matriculated after the reintroduction of Euclidean geometry as a compulsory section of mathematics and have completed a university geometry module, they were expected to show a sound understanding of the material they learnt and will teach. Based on each item's performance, they were classified as challenging or not challenging depending on their number of C4 responses.

5.2.1.1 Least challenging items

Item 3.3, located at VHL4, was experienced as the least challenging since 97% of the PSTs got the item completely correct.

Such performance was unexpected for this item which was classified as a VHL4 item and followed VHL4 item 3.1 and VHL3 item 3.2 that required proofs. This item (3.3) required

PSTs to show that $AE^2 = DE.CE$, having shown AE=BE (Item 3.1) and $\Delta AED /// \Delta CEB$ (Item 3.2). In classifying Item 3.3 as VHL4, it was assumed that PSTs would need to set out a deductive proof using the coordination of the results for items 3.1 and 3.2. The satisfactory performance may be that PSTs have routinised the application of proportional relations within similar triangles. This type of question likely follows any question that requires showing two similar triangles. Furthermore, the correct order of naming the triangles was provided in the previous item 3.2. Thus, they were not required to start the proof but to derive the set of equal ratios from the statement of similarity and provide a substitution for AE = BE.

Item 3.2, located at VHL3, was also experienced as very easy, having 91% of C4 responses. This item (3.2) required PSTs to prove that $\Delta AED /// \Delta CEB$. The satisfactory performance of PSTs may be attributed to the question being frequently encountered in the secondary school content; hence they could have recalled the procedure to solve the question. This item was straightforward because showing the two triangles are similar requires using theorem angles subtended by the same chord or arc, which are equal. As seen in Table 4.1, amongst the different configurations for similar triangles, 70% of the students were able to correctly identify and name in the correct order the two similar triangles that had the same configuration as Δs AED and CEB and required the same reasoning item 2.2. Ubah and Bansilal (2019) also noted that the students in their study found it relatively easy to show similarity in the same situation.

Item 1.2, located at VHL2, lies third as the most achieved item having 88% of C4 responses. This item questioned the minimum conditions required for any figures to be similar. Item 1.1, located at VHL2, lies forth as the most achieved item having 85.07% of C4 responses. This item questioned the definition of similarity in mathematics. Item 1.3, located at VHL2, lies fifth as the most achieved item having 82.35% of C4 responses. This item questioned if two arbitrary squares are similar or not, implying that most PSTs can interrogate the minimum conditions of similarity and properties of squares to conclude that any two arbitrary squares are similar. However, it was noticeable that many PSTs used only one condition as a reason why squares are similar.

All items experienced as the least challenging required either procedural or basic understanding of similarity in mathematics, including knowing the definition of similarity, minimum conditions required for figures to be similar, and proving triangles and squares are similar. Although many PSTs demonstrated adequate understanding of these items, it was noticed that few PSTs' understanding of similarity is limited to triangles. It was revealed by some

responses, which (i) indicated a particular case of similar triangles instead of a general definition; (ii) gave a definition and minimum conditions of similarity only applicable to triangles; and (iii) many PSTs conceived that one condition of similarity especially equal corresponding angles is enough to declare any two rectilinear figures similar since it is the case with triangles (see figure 4.2, 4.3 and 4.4 in chapter 4). These few mistakes indicated that PSTs considered similarity in familiar situations they studied in schools, which is the similarity of triangles. Hence, they did not show conceptual understanding of similarity. Likewise, research by Ndlovu (2012), which investigated students' understanding of Euclidean geometry, revealed that many students preferred questions that required procedural rather than conceptual understanding. In addition, the same study revealed that many students mastered procedures of getting answers by following rules and algorithms, which sometimes become a barrier to their learning.

5.2.1.2 Most challenging items

Item 1.4, located at VHL2, was the most challenging item for all PSTs since none of them could provide a completely correct response. This item questioned if any two arbitrary rectangles are similar. The poor results of this item were unexpected, indicating that more than 88% of PSTs could answer a similar question for squares (Item 1.3). Being successful with the first three items indicated that many PSTs can: (i) correctly define similarity in mathematics, (ii) state minimum conditions required for figures to be similar and (iii) apply similarity to figures like squares. Many PSTs incorrectly assumed that equal corresponding angles are sufficient to prove that two figures are similar or that corresponding sides will be proportional. In an interview with PST32, he asserted that to prove any figures are similar, "only one condition needs to be satisfied, which is corresponding angles must be equal" (see Section 4.3.2). This faulty reasoning led many PSTs to get the correct answer for Item 1.3. Still, it did not work for Item 1.4 since exploring the similarity of two rectangles requires checking if the two minimum conditions of similarity are satisfied. Even corresponding angles will always be equal (all 90°) for rectangles, though the ratios of corresponding sides might not always be in the same proportion.

Hence, for Item 1.4, PSTs were required to recognise that one condition is insufficient, whereas, for Item 1.3, one condition may be sufficient since the other will automatically hold. This unsatisfying performance suggests that many PSTs used their procedural knowledge to prove figures are similar; one condition is enough; nevertheless, they have listed two minimum

similarity conditions. This procedural knowledge affected PSTs' conceptual understanding of similarity to recall that ratios of corresponding sides of rectangles might not be in the same portion; hence, rectangles do not always satisfy both similarity conditions. These findings seem to support Hasenbank (2006), who claimed that students might perform well on procedural questions; however, their conceptual knowledge is poor when deeply probed. Biggs and Tang (2007) refer to this as surface learning instead of deep learning because it means passing the examination without understanding what is learned. Jojo (2010) also outlined this issue, showing that in most cases, students use manipulative approaches and drills to pass through the examination without engaging in problems that involve insight and understanding.

This result may support the claim that PSTs apply their knowledge as their teachers taught them. Dhlamini (2012) found that teachers emphasise procedural knowledge to solve routine problems, not allowing their learners to engage in questions that needed high cognitive analysis. Many studies have shown that a task requiring students to show a statement is false is of a higher cognitive demand than one requiring them to identify (in the same setting) it is true (Mutambara & Bansilal, 2019; Bansilal, 2015; Zaslavsky & Ron, 1998). Bansilal et al. (2017), in their study about functions, noted that students found it easier to identify that a function satisfied certain conditions rather than showing that a function does not satisfy a condition. The authors (Bansilal et al., 2017) recommended that students be given more opportunities to construct arguments for why an object does not satisfy a condition. Similarly, PSTs needed more opportunities to argue why polygons do not satisfy both similarity conditions in this study.

Moreover, Items 2.4/2.5 located at VHL3 appear second as challenging items achieving 55% and 52% C4 responses. These items required PSTs to name similar triangles correctly from geometric configurations in the same way as in items 2.1, 2.2 and 2.3; however, more cognitive reasoning was required in this item since three triangles were to be compared. The diagram was a special case of similar right-angles where one right-angled triangle is divided into two right-angled triangles, which are similar and are similar to the original triangle. This case of similarity is studied in the school curriculum; it was what PST9 provided as a definition of similarity in this study (see figure 4.2 in chapter 4). Despite the case being familiar to PSTs, many PSTs did not answer this question as expected. The challenge was related to identifying equal angles, which would help name similar triangles correctly.

Likewise, Ubah and Bansilal (2019) found that students performed poorly in a similar situation. The authors interviewed two students; one struggled to identify equal angles in the triangles, which would help name similar triangles but had no means of solving the problem. The second student was only successful in identifying similar triangles after creating physical triangles that could be manipulated to locate equal angles. The study suggested that students were challenged in identifying relevant geometric characteristics in given diagrams, which would assist in successfully solving the problem. Although challenging, in instances where increases in geometry required extensive cognitive reasoning, Ubah and Bansilal (2019) revealed that skilful conversion movements between the visual and symbolic registers led to success in solving Euclidean geometry problems.

Item 3.1, located at VHL4, appeared third as a challenging item achieving 58% of C4 responses. This item required PSTs to prove that AE = BE from figure 4.3.13 using geometric facts such as angles on a parallel line and a line from the centre perpendicular to the chord. Although such questions are frequently encountered in the school syllabus, more than 40% of PSTs could not provide logical and coherent proof in geometry. A logically coherent proof in geometry is a step-by-step deductive analysis where each step is derived from the results of the previous one until the unknown is justified (Ngirishi & Bansilal, 2019). In this study, it was evident that sometimes PSTs did not consider the conditions of geometric theorems; theorems were haphazardly used (see figure 4.30 and 4.31 in chapter 4). These findings support the DBE (2019) assertation that the organised thinking and reasoning required in Euclidean geometry and the ability to extract valuable properties from given figures to allow successful problem solving further complicate this strand of mathematics for learners.

5.2.2 Pre-service teachers' misconceptions about similarity and proofs

This study aimed to identify further PSTs' misconceptions evident in their responses than only acknowledging their responses as correct or incorrect. Guided by the belief that identifying misconceptions PSTs have would lead to proffering remedial recommendations, teachers can improve their teaching strategies by turning those misconceptions into helpful teaching resources.

Similarly, Chauraya and Brodie's (2018) study showed the importance of teachers understanding the nature of errors made by their learners as it improves their mathematical knowledge and teaching methods. Fang (2010) also supports that accessing students should

incorporate studying the nature and source of errors to provide them with the necessary support. Below is a discussion of the common misconceptions found in this study.

5.2.2.1 The misconception that similarity applies to triangles only

This study found many PSTs indicating that similarity applies to triangles only. This misconception was identified when PSTs: (i) gave a special case of similar triangles instead of the general definition of similarity in mathematics; (ii) gave definitions and minimum conditions of similarity which is only applicable to triangles; and (iii) incorrectly assumed that all figures with equal corresponding angles would have corresponding proportional sides like triangles. The above reasoning suggested a narrow understanding of the similarity concept limited to triangles. The lack of understanding may be attributed to some books considering similarity focusing mainly on triangles. For example, the study by Amaral and Hollebrands (2017) that examined context-based similarity task in Brazil and United States books found that six different books, three from each country, focus more on the similarity of triangles and context-based similarity tasks encountered for less than 30% of the similarity tasks. Many of those contextual tasks were of low cognitive demand.

To some extent, it may be argued that this focus on the similarity of triangles has created a misconception that similarity applies to triangles only. Kajander and Lovric (2009) revealed that inadequate understanding of geometric concepts might result from some textbooks and teachers' presentations. According to Hershkowitz (1987), some textbooks and teachers present limited definitions and standard orientation of figures to illustrate the concept. This limited presentation of geometric concepts further creates a gap in students' understanding of proper concept definition as they end up having mastered only concept images (Vinner & Hershkowitz, 1980). As a result, when students have to show understanding of a concept in different situations, they only have a concept image, which is their prior experience with diagrams, attributes, and examples associated with the concept limited to a single prototype image, dependent on a proper understanding of the concepts. From the findings of this study, it was clear that many PSTs seem to have a prototypical image of similar figures referring to similar triangles only. They understand the similarity of triangles they are familiar with rather than a broader conception of the similarity of figures in general.

5.2.2.2 The misconception that rectangles are always similar

In this study, many PSTs incorrectly assumed that all rectangles are similar because they have corresponding angles equal to 90°, which satisfies one similarity condition. They further

asserted that one similarity condition is enough to conclude any two figures are similar without checking whether the second condition of corresponding proportional sides is satisfied. This understanding of similarity resulted in a misconception that all rectangles are similar. PSTs did not recognise that for rectangles, it is not always the case that if corresponding angles are equal, then corresponding sides are proportional, implying that one condition may not be necessarily sufficient. This poor result about the similarity of rectangles can be attributed to a lack of general understanding of similarity definition in mathematics and the application of minimum conditions of similarity with varying properties of figures. Also, this indicates that stating the concept definition and its necessary required conditions do not guarantee the conceptual understanding of the concept amongst PSTs.

This finding supports de Villiers (1997), who revealed that some teachers focus on procedural fluency at the expense of conceptual understanding, teaching how to find answers quickly. Similarly, Luneta (2015) argued that some teachers do not teach mathematical concepts to empower students' understanding, negatively affecting learners' mathematical reasoning and performance. According to Tall (1988), the limited experience with prototypes, examples, and non-examples are attributed to robust concept understanding. Mason and Spence (1999) concluded that challenges in correctly understanding the concepts are due to compartmentalising rather than forming connections, which is needed for the highest level of knowledge referred to as *know-to act*. Item 1.4 required PSTs to form connections of conditions required for figures to be similar and the properties of rectangles. Still, many could not apply concepts outside of those involving prototypical examples.

5.2.2.3 The misconception about naming similar figures in order

Many incorrect responses for items 2.1 to 2.5 were due to incorrect order of naming similar triangles regardless of whether the instruction emphasised such aspects. PSTs needed to understand that mathematics is a symbolic language where symbols denote specific meaning shared by the mathematics community. For example, in this context, the symbol for similarity (///) is a specialised symbol that goes beyond the identification of similar triangles but indicates the order of equal angles and corresponding proportional sides.

For example, ΔABC/// ΔDEF indicate that $\hat{A} = \hat{D}, \hat{B} = \hat{E}\ and\ \hat{C} = \hat{F}$ and sides AB/DE = BC/EF = AC/DF. This matching of angles and sides is useful in setting proportional sides. Therefore, PSTs should learn this special implication of the symbol to justify teaching mathematics as a subject. This finding was similar to that of Ubah and Bansilal (2019), who

explored pre-service teachers' semiotic representing of similar triangles and found that many participants did not pay special attention to the key feature of similarity (///) notation. It is because they did not consider the order in naming similar triangles.

5.2.2.4 The misconception about the class inclusion of quadrilaterals

In this study, a few items required an understanding of quadrilaterals (items 1.3 and 1.4). This study found PST6 and PST9 showing consistent misconceptions about the class inclusion of quadrilaterals. PST6 included rhombus and parallelogram when responding to the possibility of similarity of squares (item 1.3), including kite and trapezoid when responding to the possibility of the similarity of rectangles (item 1.4). PST9 included rectangles and rhombus when responding to the possibility of similarity of squares (item 1.3), including rhombus and parallelogram when responding to the possibility of the similarity of rectangles (item 1.4). The findings suggested that the two PSTs were trying to consider all quadrilaterals, which belong to a family of squares and rectangles, but their knowledge of class inclusion was poor. Many studies (Ngirishi & Bansilal, 2019; Marchis, 2008; Feza & Webb, 2005; Siyepu, 2005; Roux, 2003) have identified several misconceptions in learners' understanding of class inclusion of quadrilaterals, which include a lack of understanding of interrelationships between different quadrilaterals. For example, pupils might think that a square is not a rectangle. The class inclusion of shapes is vital in geometry because it enables one to reason about the relationship between geometric shapes and their properties.

5.2.2.5 The misconception about using visualisation in presenting geometric proofs

In this study, presenting sound geometric proofs based on geometric facts was challenging to some PSTs. They opted to utilise their visualisation or use theorems haphazardly where their conditions were not applicable. A few PSTs were challenged with presenting sound geometry proof for item 3.1, demonstrating that AE = BE.

This result supported the Department of Education's (2020) assertion that the structured requirements of providing appropriate reasons for statements make this subject section seem complicated.

Furthermore, the result is consistent with Ngirishi and Bansilal's (2019) findings which revealed that grade 10 and 11 learners struggled to answer proof questions, especially when more than two steps were required. After spending years at secondary school and a semester at university, the expectation was that PSTs would have developed the understanding of the nature of geometric proofs, accepting that figures are not necessarily drawn into scale. Hence,

this makes one accept visualisation as insufficient to substantiate mathematics proofs. Some PSTs in this study showed a deficit in understanding geometric proofs when applying visualisation to conclude that angles and sides are either equal or not equal. The PSTs considered the necessary intertwining of the visual and analytic representations challenging since one representation supports and underpins the other.

5.2.3 Pre-service teachers' Van Hiele levels of geometric thought

The other interest of this study was to explore the VHLs of geometric thought of PSTs about similarity and proofs in Euclidean geometry. A summary of pre-service teachers' VHLs according to written responses from the pen and paper test (see Table 4.2) shows that pre-service teachers' VHLs ranged from VHL2 to VHL4, with some PSTs operating at VHL4. A particular finding in this study was that some items targeted at VHL3 and VHL4 were not necessarily the most challenging. Item 3.2, located at VHL3, was experienced as very easy, having 91% C4 responses. This item required PSTs to prove $\Delta AED /// \Delta CEB$ in figure 4.3.13. The satisfactory response could be that this was a proof question that needed only reasons for equal corresponding angles to substantiate the similarity of the two given triangles. Also, item 3.3 located at VHL4 was easy since 97% of the PSTs got the item entirely correct. This item required PSTs to show that $AE^2 = DE.CE$ in figure 4.3.13.

The satisfactory performance may be that PSTs have routinised the application of proportional relations within similar triangles. This type of question likely follows any question that requires one to show that two triangles are similar. Furthermore, the correct order of naming the triangles was provided in the previous items 3.2, so they did not have to start the proof but only needed to derive the set of equal ratios from the statement of similarity and then do a substitution for AE = BE. Nevertheless, it was not reassuring to note that a total of 22 PSTs (+64%) were operating at VHL2 or below. This finding raises concerns about how those PSTs will assist learners in acquiring a high level (VHL3 or VHL4) of geometric thinking that they did not achieve.

To cope with the demands of the axiomatic system and proofs as required by the curriculum, learners need to be on the formal deduction of the Van Hiele levels (DBE, 2011). Teachers must assist learners in navigating the five Van Hiele levels from visualisation to the formal deduction level. Thus, it is a concern if teachers themselves lack the levels expected to assist their learners.

The findings of this study were almost similar to some studies, which commonly found many PSTs working at the bottom of the VHLs (Van Putten et al., 2010) except for 10 PSTs operating at VHL4. Some scholars like Burger and Shaughnessy (1986) have noted that a student's level may vary across topics and tasks in geometry, showing that students can appear to be at a higher Van Hiele level of geometric thought for concepts learned through memorising.

5.3 Recommendations

This study identified several misconceptions regarding PSTs' understanding of similarity and proofs in Euclidean geometry. Therefore, teachers' professional development courses are considered vital in assisting PSTs in overcoming the identified misconceptions and equipping them with the necessary skills for effective geometry teaching. The following recommendations are suggested for teachers' professional development to assist PSTs: (1) Improving PSTs' geometry subject content knowledge, (2) Teaching Euclidean geometry for understanding, and (3) Improving PSTs' presentation of the written mathematics responses.

5.3.1 Improving PSTs' geometry subject content knowledge

This study found a lack of geometry subject content knowledge in some PSTs, including limited understanding of the definition of similarity, the application of minimum conditions required to prove figures are similar, and the haphazard use of geometric theorems.
Therefore, it is recommended that teachers' professional development improve PSTs' geometry subject content knowledge. Some scholars (Ball et al., 2005; Bischoff et al., 1999) argued that understanding subject content knowledge is the first requisite for proper teaching. It gives teachers the authority to own the material they are teaching, and help learners relate concepts, make sense of mathematics problems and address misconceptions. These abilities allow individuals to decide which method is suitable and possible ways of deriving the answer.

The above mentioned coincides with the studies undertaken by Ball et al. (2001) and The National Research Council (2001), which identified teachers' subject content knowledge as a crucial factor for effective education. It is incontestable that it is impossible to teach something one does not clearly understand effectively. However, that does not mean that teachers must be perfect; teachers' lack of geometry content knowledge may cause them to skip geometry, as geometry was optional in South Africa.

Teachers with poor subject content knowledge will struggle to provide learners with relevant mathematical skills to become effective thinkers. For example, Lubinski et al. (1998) found

that a novice teacher with a weaker background in mathematics was better at listening to learners and shaping their lessons to take advantage of what the learners were learning. The novice teacher, however, often presented incorrect mathematics to her learners. The study showed that a robust pedagogical background does not always lead to effective mathematics learning when the teacher's subject content knowledge is deficient. Also, the study by Van der Sandt and Niewoudt (2005) indicated that elementary pre-service teachers lacked adequate control of the material they teach their learners. Hence, the current study recommends improving PSTs subject content knowledge because it is unfair to attribute failure in geometry to learners whilst teachers themselves lack adequate content knowledge to teach the subject area.

5.3.2 Teaching Euclidean geometry for understanding

This study also showed that many PSTs performed satisfactorily on familiar items but struggled with those unfamiliar that required conceptual understanding. This finding supported Ndlovu's (2012) study, which explored PSTs' geometry content knowledge and found that many acquired rules and procedures to solve problems without proper understanding of the mathematics involved. Therefore, it is recommended that teachers' professional development should include Euclidean geometry to improve PSTs' understanding since most geometry questions do not have standard procedures on how to solve them but require conceptual understanding. Many studies have highlighted poor performance in geometry as the result of teachers' methods of teaching (de Villiers, 1997; Van Hiele, 1986). Sani and Salahudeen (2016) found that some teachers tend to teach mathematics in abstract terms, depriving learners of the connection between real-life problems and mathematics.

Likewise, Luneta (2015) found that some teachers do not explain geometric concepts well to empower their learners with the conceptual understanding of geometric concepts. Many studies recommended that geometry be taught in a stimulating and engaging way, leading to understanding and success in mathematics rather than collecting rules and solving routine problems. Although some basic concepts in geometry need to be memorised to facilitate understanding, memory can eliminate delays caused by searching for what can be compared to missing pieces of information. However, understanding is more important than memory; it helps one construct meaning in each problem situation using memorised facts. Euclidean geometry problems require abstract thinking; conceptual understanding becomes more relevant than memorised facts. Therefore, it is crucial to adopt teaching methods that encourage

understanding mathematics concepts. Knowing theorems and using them to solve simple geometric riders is not enough; learners need to see their application to other concepts and in real life. Furthermore, it requires those being taught to develop knowledge to analyse critically, question what is discussed, and explore its relevance to their lives. In support, Wasserman's (2016) view suggests that pre-service teachers need to start to engage with abstract geometry to develop a deep sense of the nature and role of definitions in mathematics.

Teachers must also improve textbook activities to teach geometry for understanding by supplementing content gaps in textbooks with additional examples to challenge student growth in reasoning ability (Fuys et al., 1985). De Villiers (1997) found that in South Africa, geometry teaching has mainly been text-book dependent and teacher-centred. It relates to teachers encouraging rote memorisation when their level of reasoning is higher than the students' (Clements & Battista, 1992). There is no understanding in teaching using this method; instead, the skill of applying rules and procedures is emphasised. Effective teaching must include discussing whether solutions to a problem are reasonable and improving understanding of mathematics in the problem; only finding answers is not primarily significant for real-life problems. Teachers may find different strategies to convert textbook activities to high cognitive demand.

5.3.3 Improving on their presentation of written mathematical response

One of the unexpected findings of this study was a mismatch between the PSTs' poorly written responses versus improved verbal responses when probed about the originality of their thinking. In most cases where the PST's presented incorrect written responses, they would correct their mistakes during interviews. To some extent, it made it challenging to have a complete picture of the PSTs' understanding of the task, especially for those who did not participate in the interviews since they might have also shown improvement when responding verbally. For this reason, it is recommended that teachers' professional development encourage PSTs to improve their written mathematical responses such that it reveals how best they know about the question. PSTs also need to improve on presenting good and convincing geometric proof. This study also discovered that when PSTs present a proof, statements are written randomly, without considering the coherence of deductive reasoning. Sometimes theorems are used haphazardly, their conditions not being sufficiently met.

5.4 Limitations and suggestions for further exploration

This study was a small-scale study with 34 PSTs from one university. The use of such a small sample could raise concerns about the external validity and reliability of the findings. It is not certain that all PSTs in South African universities have the same ability as those who participated in this study. It would have been appropriate to explore PSTs' understanding of similarity and proofs in Euclidean geometry in all South African universities; however, finance, time frame and other logistics limited this study. Secondly, this study utilised a qualitative paradigm, employing a test and semi-structured interview as data collection methods. Marshall and Rossman (1995) argued that qualitative research has a weakness regarding the transferability of results, as each qualitative research approach has its unique features.

Furthermore, the pen and paper tests were administered online, and no measure was taken to avoid copying. Pre-service teachers might have tried to get as much information as possible to produce correct answers. Also, the semi-structured interview took place online via zoom; challenges were experienced, like poor connectivity. In addition, most of the interview questions dwelt more on the aspects where the PSTs showed a lack of understanding. In retrospect, it might have provided a complete picture about the PSTs' understanding of similarity and proofs in Euclidean geometry if interviews balanced between poorly and successfully answered questions. Lastly, the researcher is a mathematics teacher, and subjectivity could have affected the interview process, for instance, asking leading questions; however, the researcher attempted to be as 'objective' as possible.

Considering the limitations mentioned above, for further exploration, researchers should:

- Increase the sample size and recruit pre-service teachers from different universities.
- Explore pre-service teachers understanding of similarity and proofs using other frameworks and using more non-routine questions.

5.5 Conclusion

This study explored pre-service teachers' understanding of similarity and proofs in Euclidean geometry in one South African university. Overall, it emerged that most PSTs who participated in this study had a substantial understanding of similarity and proofs on familiar items but struggled with unfamiliar and non-examinable grade 12 questions, predominantly those that were not typical textbook exercises but falls within the parameters of the secondary school curriculum. The poor performance of PSTs in unfamiliar items raised concerns about their

competency in mathematics when some concepts goes slightly beyond the secondary school curriculum content. This study pointed to the weak conceptual understanding of geometric definitions amongst PSTs, which suggested that knowing concept definition does not guarantee success if only a prototype is associated with the concept. This study showed the challenge which might contribute to continuous low achievement in geometry if future mathematics teachers have a shortfall in understanding geometric concepts. Therefore, future mathematics teachers should possess a deep understanding of the material they will teach beyond the school curriculum as a critical element in providing learners with quality geometric knowledge. Their deep understanding of geometric concepts will allow them to see textbook content limitations and have the power to design relevant activities to promote students' development of the Van Hiele levels of geometric thought.

REFERENCES

Adegun, B.O., & Adegun A.I. (2013). Students and teachers views of difficult areas in mathematics syllabus. Basic requirement for science and engineering education. *Journal of Education and Practice, 4(12)*, 235-243.

Adler, J. (2004). Research inside teacher education: The QUANTUM project, its context, some results and implications. Paper presented at the AERA conference in San Diego, April 2004.

Alex, J. K., & Mammen, K.J. (2014). An assessment of the readiness of grade 10 students for geometry in the context of curriculum and assessment policy statement (CAPS) expectation. *International Journal of Educational Sciences*, *7*(1), 29 -39.

Amaral, R.B., & Hollebrands, K. (2017). An analysis of context-based similarity tasks in textbooks from Brazil and the United States. *International Journal of Mathematical Education in Science and Technology, 48*(8), 1166-1184, DOI: 10.1080/0020739X.2017.1315188.

Atebe, H.U. (2008). Students' Van Hiele levels of geometric thought and conception in plane geometry: A collective case study of Nigeria and South Africa. Unpublished dissertation. Rhodes University, South Africa.

Atebe, H.U., & Schäfer, M. (2010). The face of geometry instruction and learning opportunities in selected Nigerian and South African high schools. In M. Schäfer & C. McNamara (Eds.), Proceedings of the 17th Annual Conference of the Southern African Association for Research in Mathematics, Science and Technology Education (pp. 7–21). Grahamstown: SAARMSTE.

Ball, D. L., Hill, H. H., & Bass, H. (2005). Knowing mathematics for teaching: Who knows mathematics well enough to teach third grade, and how can we decide? *American Educator*, pp. 14-46.

Ball, D.L., Lubienski, S.T., & Mewborn, D.S. (2001). Research on teaching mathematics: The unsolved problem of teachers' mathematical knowledge. In: Richardson V (ed). *Handbook of research on teaching*. 4th edition. New York: Macmillian.

Bansilal, S. (2015). An exploration of students' conversions from a symbolic to a verbal representation. *Journal of Communication, 6*(1), 38–47. https://doi.org/10.1080/0976691X.2015.1188484

Bansilal, S., Mkhwanazi, T., & Brijlall, D. (2014). An exploration of the common content knowledge of high school mathematics teachers. *Perspectives in Education, 32*(1), 34-50.

Bansilal, S; Brijlall, D & Trigueros, M. (2017). An APOS study on pre-service teachers' understanding of injections and surjections. *Journal of mathematical behaviour*, 48, 22-37. http://dx.doi.org/10.1016/j.jmathb.2017.08.002

Bassarear, T. (2012). *Mathematics for Elementary School Teachers*. 5th Edition. Brooks /Cole.

Battista, M.T. (2007). The development of geometric and spatial thinking. In F.K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 843–908). Information Age.

Becker, A. (2016). Student-generated scoring rubrics: Examining their formative value for improving ESL students' writing performance. *Assessing Writing*, 29, 15-24.

Bertram, C., & Christiansen, I. (2014*). Understanding research: An introduction to reading research*. Van Schaik Publishers.

Biggs, J., & Tang, C. (2007). Applying constructive alignment to outcomes-based teaching and learning, pp. 1-8.

Bischoff, P.J., Hatch, D.D., & Watford, L.J. (1999). The state of readiness of initial level preservice middle grades science and mathematics teachers and its implications on teacher education programs. *School Science and Mathematics, 99*(7), 394-399.

Bowie, L. (2009). What is Mathematics Paper 3 for? Marang Centre for Mathematics and Science Education, Marang News, Issue 5.

Brijlall, D. (2011). A case study of learner's views on content knowledge and practice. Proceedings of the nineteenth annual meeting of the South African Association for Research in Mathematics, Science and Technology Education (SAARMSTE).

Brijlall, D., & Ndlovu, A.Z. (2013). High school learners' mental construction during solving optimisation problems in calculus: a South African case study. *South African Journal of Education, 33*(2), 180-197.

Brown, S.I., Cooney, T.J., & Jones, D. (1990). Mathematics teacher education. In W.R. Houston (Ed.). Handbook of research on teacher education (pp.639-656). NY: Macmillan.

Buchanan, T. (2007). The importance of teaching students how to read to comprehend mathematical language. Action Research Projects. Paper 5. http://digitalcommons. unl.edu/mathmidaction research/5.

Burger, W.F., & Shaughnessy, J.M. (1986). Characterizing the Van Hiele levels of development in geometry. *Journal for Research in Mathematics Education*, *17*, 131–148. https://doi.org/10.2307/749317

Cassim, I. (2006). An exploratory study into grade 12 learners' understanding of Euclidean Geometry with special emphasis on cyclic quadrilateral and tangent theorems. Unpublished Master's thesis. Johannesburg: University of the Witwatersrand.

Cavanagh, R. & Sparrow, L. (2011). Mathematics anxiety: Scaffolding a new construct model. Mathematics: Traditions and [New] Practices. In Proceedings of the Annual Conference of Mathematics Education Research Group of Australasia, Alice Springs.

Chambers, P. (2008). *Teaching mathematics: Developing as a reflective secondary teacher*. Sage Publications.

Chauraya, M., & Brodie, K. (2018). Conversations in a professional learning community: An analysis of teacher learning opportunities in mathematics. *Pythagoras, 39*(1), a363.https://doi.org/10.4102/Pythagoras. V39i1.363

Chinnappan, M., Ekanayake, M.B., & Brown, C. (2012). Knowledge use in the construction of geometry proof by Sri Lankan students. *International Journal of Science and Mathematics Education, 10(4):* 865-887.

Clark, D.M. (2012). *Euclidean Geometry: A guided inquiry approach*. Mathematical Sciences Research Institute.

Claxton, G.L. (1997) *Hare Brain, Tortoise Mind: why intelligence increases when you think less*. London, Fourth Estate.

Clements, D. H. (2003). "Teaching and Learning Geometry." In *A Research Companion to Principles and Standards for School Mathematics*, edited by Jeremy Kilpatrick, W.

Gary Martin, and Deborah Schifter, pp. 15–78. Reston, VA.: National Council of Teachers of Mathematics.

Clements, D.H., & Battista, M.T. (1992). Geometry and spatial reasoning. In D. A. Grouws (Ed.), *Handbook of research on mathematics and learning*. Macmillan.

Clements, D.H., Battista, M.T., & Sarama, J. (Eds.). (2001). *Journal for Research in Mathematics Education Monograph 10: Logo and Geometry*. National Council of Teachers of Mathematics.

Cohen, L., Manion, L. & Morrison, K. (2007). *Research methods in education* (6th ed.). Routledge.

Cohen, L., Manion, L., & Morrison, K. (2011). *Surveys, longitudinal, cross-sectional and trend studies: Research Methods in Education*, (7th ed.). Routledge, 261-264.

Confrey, J. (1990). A review of research on student conceptions in mathematics, science and programming. In C.B. Cazden (Ed.), *Review of research in education* (Vol. 16, pp. 3-56). American Educational Research Association.

Cox, D.C. (2013). Similarity in Middle School Mathematics: At the Crossroads of Geometry and Number. *Mathematical Thinking and Learning, 15*(1), 3-23, DOI: 10.1080/10986065.2013.738377

Creswell, J.W. (2009). *Research design: qualitative, quantitative and mixed method approaches*. Sage Publications, Inc.

Creswell, J.W. (2014). *Research Design: Qualitative, Quantitative and Mixed Methods Approaches* (4th ed.). Sage.

Crowley, M.L. (1987). The van Hiele model of the development of geometric thought. In M. M. Linquist, & A.P. Shalte (Eds), *Learning and Teaching Geometry, K-12,* (pp. 1-16). National Council of Teachers of Mathematics.

Cunningham, R.F., & Roberts, A. (2010). Reducing the mismatch of geometry concept definitions and concept images held by pre-service teachers. Implications for Mathematics Education Programs: The Journal. Retrieved from http://www.k-12prep.math.ttu.edu/journal/1.contentknowledge/cunningham01/article.pdf

Curriculum and Assessment Policy Statement (CAPS). (2011). KwaZulu-Natal Department of Education: Annual Report 2010/2011.

De Villers, M. (2012). An illustration of the explanation and discovery functions of proof. *Pythogoras, 33*(3), 1-8.

De Villiers, M. (1997). The future of secondary school geometry. *Pythagoras*, 37-54.

De Villiers, M. (1998). To teach definitions in geometry or teach to define? In A. Oliver and K. Newstead (Eds.), Proceedings of the 22nd Conference of the International Group for the Psychology of Mathematics Education 2, 248-255.

De Villiers, M. (2004). Using dynamic geometry to expand mathematics teachers' understanding of proof. *International Journal of Mathematical Education In Science and Technology, 35(5),* 703-724.

De Vos, A.S. (2002). *Research at grass roots for the social sciences and human service professions*. Van Schaik Publishers.

Department of Basic Education (DBE). (2011). 2010 *Examination Report*. Pretoria: Department of Basic Education.

Department of Basic Education (DBE). (2011). Curriculum and Assessment Policy Statement (CAPS) FET Band Mathematics Grades 10-12. Pretoria: Government Printers.

Department of Basic Education (DBE). (2013). 2012 *Examination Report.* Pretoria: Department of Basic Education.

Department of Basic Education (DBE). (2016). *2015 Examination Report.* Pretoria: Department of Basic Education.

Department of Basic Education (DBE). (2017). *2016 Examination Report.* Pretoria: Department of Basic Education.

Department of Basic Education (DBE). (2018). *2017 Examination Report.* Pretoria: Department of Basic Education.

Department of Basic Education (DBE). (2019). *2018 Examination Report.* Pretoria: Department of Basic Education.

Department of Basic Education (DBE). (2020). *2019 Examination Report.* Pretoria: Department of Basic Education.

Dhlamini, S.S. (2012). An investigation into grade 12 teachers' understanding of Euclidean Geometry. Unpublished Masters' dissertation. University of KwaZulu Natal.

Dikgomo, K.T. (1994). Misconception in the Learning of Linear Inequalities in Mathematics at Standard Eight: A Constructivist Perspective. M. Sc. Dissertation. Johannesburg: University of Witwatersrand.

Ding, L., & Jones, K. (2006). Teaching geometry in lower secondary school in Shangai, China. Proceedings of the British Society for Research into Learning Mathematics, *26*(1), 41–46. Available from http://www.bsrlm.org.uk/IPs/ip26-1/ BSRLM-IP-26-1-8.pdf

Douglas, R., Klentschy, M. P., Worth, K., & Binder, W. (Eds.). (2006). *Linking science and literacy in the K–8 classroom*. National Science Teachers Association.

Dubinsky, E.D. (1997). Some Thoughts on a First Course in Linear Algebra at the College Level. Purdue University.

Etikan, I., & Bala, K. (2017). Sampling and sampling methods. *Biometrics & Biostatistics International Journal, 5*(6), 215–217.

Fang, Y. (2010). The cultural pedagogy of errors: teacher Wang's homework practice in teaching geometric proofs. *Journal of Curriculum Studies, 42*(5), 597-19.

Feza, N., & Webb, P. (2005). Assessment standards, Van Heile levels and grade seven learners understanding of Geometry. *Pythagoras, 62,* 36-47.

Fujita, T. (2012). Learners' level of understanding of the inclusion relations of quadrilaterals and prototype phenomenon. *The Journal of Mathematical Behavior*, *3(1)*, 60–72.

Fujita, T., & Jones, K. (2006). Primary trainee teachers' understanding of basic geometrical figures in Scotland. Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education, 3, 14-21.

Fujita, T., & Jones, K. (2007). Learners' understanding of the definitions and hierarchical classification of quadrilaterals: towards a theoretical framing. *Research in Mathematics Education*, 9 (1&2), 3-20. http://dx.doi.org/10.1080/14794800008520167

Fuys, D., Geddes, D., & Tischler, R. (1985). An investigation of the van Hiele model of thinking in geometry among adolescents. Brooklyn, NY: Brooklyn College of the City University of New York, School of Education.

Gal, H., & Linchevski, L. (2010). To see or not to see: Analysing difficulties in geometry from the perspective of visual perception. *Educational Studies in Mathematics, 74*(2), 163–183. https://doi.org/10.1007/s10649-010-9232-y

Galvin, R. (2015). How many interviews are enough? Do qualitative interviews in building energy consumption research produce reliable knowledge? *Journal of Building Engineering, 1*, 2-12.

Genz, R. (2006). Determining high school geometry students' geometric understanding using Van Hiele levels: Is there a difference between standards- based curriculum students and non-standards-based curriculum students? Unpublished master's thesis, Department of Mathematics Education, Brigham Young University, Provo, Utah, United States.

Gfeller, M.K. (2010). A teacher's conception of communication in geometry proofs. *School Science and Mathematics, 110*(7): 341-51.

Guba, E.G., & Lincoln, Y.S. (2005). Paradigmatic controversies, contradictions and emerging confluences. In Denzin, N.K. & Lincoln, Y.s. (Eds.). *The SAGE Handbook of Qualitative Research* (3rd ed.), (pp. 191-215). SAGE Publications.

Guthrie, G. (2010). *Basic research methods: An entry to social science research*: SAGE Publications India.

Gutierrez, A. & Jaime, A. (1999). Pre-service Primary Teachers' Understanding of the Concept of Altitude of a Triangle. *Journal of Mathematics Teacher of Education, 2*(3), 253-275.

Haahr, A., Norlyk, A., & Hall, E.O. (2014). Ethical challenges embedded in qualitative research interviews with close relatives. *Nursing ethics, 21*(1), 6-15.

Haj-Yahya, A. (2021). Students' conceptions of the definitions of congruent and similar triangles. *International Journal of Mathematical Education in Science and Technology*, 1-25. DOI: 10.1080/0020739X.2021.1902008

Hanna, G. (2000). Proof, explanation and exploration: an overview. *Educational Studies in Mathematics,44*, 5-23.

Hansen, A., Drews, D., Dudgeon, J., Lawton, F., & Surtees, L. (2017). Children's errors in mathematics. Learning Matters: British library Cataloguing in Publication Data.

Hasenbank, J.F. (2006). The effects of a framework for procedural understanding on college algebra students' procedural skill and understanding. Unpublished Doctoral Dissertation, Montana State University, Bozeman.

Henning, E. (2004). *Finding your way in qualitative research.* W. Smith & B. Van Schaik Publishers.

Hershkowitz, R. (1987). The acquisition of concepts and misconceptions in basic geometry or when "a little learning is a dangerous thing." In J. Novak (Ed.), Proceedings of the 2nd International Seminar on Misconceptions and Educational Strategies in Science and Mathematics (Vol. III, pp. 238-251). Cornell University.

Hershkowitz, R. (1990). Psychological Aspects of Learning Geometry. In P Nesher & J Kilpatrick (Eds.), Mathematics and Cognition. (pp. 70-95). Cambridge: CUP. http://dx.doi.org/10.1017/CBO9781139013499.006

Hiebert, J., & Carpenter, T.P. (1992). Learning and teaching with understanding. In D.A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 65-97). Macmillan.

Hierbert, T. J., & Lefevre, P. (1996). Conceptual Knowledge in mathematics: The case of mathematics In T. Hierbert (Ed.), *Conceptual and Procedural Knowledge* (pp. 1-27). Hillsdale: Erlbaum.

Hlalele, D. (2012). Exploring rural high school learners' experience of mathematics anxiety in academic settings. *South African Journal of Education*, (2), 267-278.

Howitt, C. (2007). Quality Standards Associated with the Post-positivist and Interpretive Research Paradigms. SMEC, Curtin University of Technology, Perth.

Huitt, W. (2011). Analysing paradigms used in education and schooling. *Educational Psychology Interactive*. Valdosta State University. Retrieved 21/03/2015 from http://www.edpsycinteractive.org/topics/intro/paradigm.pdf

Jaime, A., & Gutierrez, A. (1995). Guidelines for teaching plane isometries in secondary school. *The Mathematics Teacher, 88*(7), 591-597.

Jojo, Z. (2010). Students' understanding of the Chain rule. Unpublished Masters' dissertation. University of KwaZulu Natal, Durban.

Jones, K. (2000), Critical Issues in the Design of the Geometry Curriculum. *In*: Bill Barton (Ed), *Readings in Mathematics Education*. University of Auckland, pp. 75-90.

Jones, K. (2002). Issues in the teaching and learning of geometry. In *Linda Haggarty (Ed) Aspects of teaching secondary mathematics: perspectives on practice.* Routledge Falmer.

Kabir, S.M. (2018). Methods of Data Collection. *ResearchGate*, 201-276.

Kajander, A. & Lovric, M. (2009). Mathematics textbooks and their potential role in supporting misconceptions. *International Journal of Mathematical Education in Science and Technology*, *40*(2), 173-181.

Kamgar, S., & Navvabpour, H. (2017). An efficient method for estimating population parameters using split questionnaire design. *Journal of Statistical Research of Iran JSRI*, *14*(1), 77-99.

Kivunja, C., & Kuyini, A. B. (2017). Understanding and Applying Research Paradigms in Educational Contexts. *International Journal of Higher Education, 6*(5), 26-41. doi:doi:10.5430/ijhe.v6n5p26

Knight, K.C. (2006). *An investigation into the change in the van Hiele level of understanding geometry of pre-service elementary and secondary mathematics teachers*. Unpublished master dissertation, University of Maine.

Koul, R.B. (2008). *Educational Research and Ensuring Quality Standards.* Retrieved from html:file://F:/educational%20RESEARCH%20AND%20ENSURINGQUALITYSTANDARDS

Kumar, R. (2011). *Research Methodology: A step-by-step guide for beginners* (3rd ed.). SAGE Publications Ltd.

Lee, J.S, & Ginsburg, H.P. (2009). Early childhood teachers' misconceptions about mathematics education for young children in the United States. *Australasian Journal of Early Childhood, 34* (4), 37–45.

Lewis, S. (2015). Qualitative inquiry and research design: Choosing among five approaches. *Health promotion practice*, *16*(4), 473-475.

Lim, S. (2011). Applying the Van Heile theory to the teaching of secondary school geometry. *Teaching and learning, 13(1)*, 32-40.

Lubinski, C.A., Otto, A. D., Rich, B.S., & Jaberg, P. A. (1998). An analysis of two novice K-8 teachers using a model of teaching-in-context. In S. Bereson, K. Dawkins, W. Coulombe, J. Kolb, K. Norwood, and L. Stiff (Eds.). *Proceedings of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 704-709). Columbus, OH: ERIC

Luneta, K. (2008). Error discourse in Fundamental Physics and Mathematics: Perspectives of students' misconceptions. *ICET International Year Book*, 385-400.

Luneta, K. (2015). Understanding student's misconceptions: An analysis of final Grade 12 examination questions in geometry. *Pythagoras: Journal of Association of Mathematics Education in South Africa, 36*(1), 261-272.

Luneta, K., & Makonye, P.J. (2010). Learner Errors and Misconceptions in Elementary Analysis: A Case Study of a Grade 12 Class in South Africa. *Acta Didactica Napocensia*, *3*(3), 35-46.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Lawrence Erlbaum.

Makhubele Y.E. (2014). Misconceptions and Resulting Errors displayed by grade 11 learners in the learning of geometry. Unpublished master's dissertation. University of Johannesburg, Johannesburg, South Africa.

Marchis, I. (2008). Geometry in primary school Mathematics. *Educatia, 21* (6), 131-139.

Marshall, C. and Rossman, G.B. (1995). *Designing qualitative research* (2nd ed.). SAGE Publications.

Mason, J. & Spence, M. (1999). Beyond Mere Knowledge of Mathematics: The Importance of Knowing-To Act in the Moment. *Educational Studies in Mathematics*, *38* (1-3), 135-161.

Mayberry, J. (1981). An investigation in the van Hiele levels of geometric thought in undergraduate pre-service teachers. *Dissertations Abstract International, 42*, 2008A. (UMI No. 80-23078).

McMillan, J.H. and Schumacher, S. (2001). *Research in education: a conceptual introduction,* (5th ed.). Addison Wesley Longman, Inc.

Micawber, M. (2005). What is the difference between to make an error and to make a mistake? Retrieved May 28th 2011. From html:file://F:Wha's%20the20%20dfference%20between%20'to%20make%20an%20erro…

Michael, J. (2001). The significance of misconceptions for teaching and learning. Paper Presented in the 2001- 2002 SOTL Schedule of Events, Indiana University. Retrieved April 12, 2011, from http://www.indiana.edu/sotl/eventsdescr.html.

Mthembu, S.G. (2007). Instructional approaches in the teaching of Euclidean geometry in grade 11. Unpublished master dissertation University of KwaZulu Natal, Durban, South Africa.

Mthethwa, M.Z. (2015). Application of GeoGebra on Euclidean geometry in rural high schools - grade 11 learners. *(Master's Thesis, University of Zululand)*. Retrieved from https://pdfs.semanticscholar.org/585f/e87c309485574efeec4b47d90184c042f274.pdf

Mudaly, V. (2010). Thinking with Diagrams Whilst Writing with Words. *Pythagoras, 71,* 65-75.

Mudaly, V., & De Villiers, M. (2004, July). Mathematical modelling and proof. Paper presented at the 10th Annual National Congress of the Association for Mathematics Education of South Africa, University of the North West, Potchefstroom.

Mutambara, L.H.N., & Bansilal, S. (2019). An exploratory study on the understanding of the vector subspace concept. *African Journal of Research in Mathematics, Science and Technology Education, 1*(20), 1–13. Available from https://doi.org/10.1080/18117295.2018.1564496

National Council of Teachers of Mathematics (NCTM). (2000). *Principles and standards for school mathematics.* Reston, VA: National Council of Teachers of Mathematics.

National Research Council. (2001). *Adding it up: Helping children learn mathematics.* J. Kilpatrick, J. Swafford, & B. Findell (Eds.), Mathematics Learning Study Committee, Center for Education, Division of Behavioral and Social Sciences and Education. Washington, DC: National Academy Press.

Ndlovu, B. (2012). Exploring Pre-Service Teacher's Knowledge of proof in Geometry. A dissertation submitted in partial fulfilment of the requirements for the Degree of Master of Education. Durban: University of KwaZulu-Natal.

Ndlovu, M., Wessels, D., & De Villiers, M. (2013). Competencies in using Sketchpad in geometry teaching and learning: Experiences of preservice teachers. *African Journal of Research in Mathematics, Science and Technology Education, 17(3)*, 231–243. https://doi.org/10.1080/10288457.2013.848536

Ndlovu, M.C. (2011). Re-envisioning the scholarship of engagement: lessons from a university school partnership project for mathematics and science teaching. *South African Journal of Higher Education, 25* (7), 1397-1415.

Neuman, W.L. (2013). *Social research methods: Qualitative and quantitative approaches:* Pearson education.

Ngirishi, H., & Bansilal, S. (2019). An exploration of high school learners' understanding of geometric concepts. *Problems of Education in the 21st Century, 77*(1), 82–96. https://doi.org/10.33225/pec/19.77.82

Nkwe, D.T. (1985): Teachers attitude, with special references to mathematics teaching (M.Ed. dissertation. University of Witwatersrand, Johannesburg)

Otton, S. (2007). Research and practice: proof in the geometry classroom. Available from http://www.msu.edu/~ottensam/Otten_842_PROOF.doc

Ozdemir Erdogan, E., & Dur, Z. (2014). Preservice Mathematics Teachers' Personal Figural Concepts and Classifications About Quadrilaterals. Australian Journal of Teacher Education, 39(6). http://dx.doi.org/10.14221/ajte.2014v39n6.1

Özerem, A. (2012). Misconceptions in geometry and suggested solutions for seventh grade students. *Procedia-Social and Behavioral Sciences*, *55*, 720-729.

Parahoo, K. (2006). *Nursing Research.* Process and Issues, (2nd ed.). Palgrave Macmillan, Basingstoke.

Patkin, D., & Levenberg, I. (2012). Geometry from the world around us. *Learning and Teaching Mathematics, 13*(1), 14–18. Available from http://www.amesa.org.za/ Patkin.pdf

Patton, M.Q. (2002). Qualitative evaluation and research methods (3rd ed.). Sage Publications, Inc.

Rossnan, S. (2006). Overcoming mathematics anxiety. *Mathitudes*, 1, 1-4.

Roux, A. (2003). *The impact of language proficiency on mathematical thinking.* Proceedings of the Annual Meeting of the Association for Mathematics Education of South Africa (AMESA), University of Cape Town, Vol. 2, pp 361-371.

Rule, P., & John, V. (2011*). Your guide to case study research.* Van Schaik.

Sani, S., & Salahudeen, B. (2016). Effects of Geoboard and geographical globe on senior secondary school students' performance in mathematics in Kuduna state ATBU. *Journal of Science, Technology & Education, 4(1),* 140-148.

Scotland, J. (2012). Exploring the philosophical underpinnings of research: Relating ontology and epistemology to the methodology and methods of the scientific, interpretive, and critical research paradigms. *English Language Teaching, 5(9),*9.

Seago, N.M., Jacobs, J.K., Heck, D.J., Nelson, C.L. & Malzahn, K.A. (2014) Impacting teachers' understanding of geometric similarity: results from field testing of the Learning and Teaching Geometry professional development materials, *Professional Development in Education*, *40*(4), 627-653. DOI: 10.1080/19415257.2013.830144

Secada, W.G. (1992). Race, ethnicity, social class, language and achievement in mathematics. In: Grouws DA (ed.). *Handbook of research on mathematics teaching and learning*. New York: Macmillan/NCTM.

Senk, S.L. (1989). Van Hiele levels and achievement in writing geometry proofs. *Journal for Research in Mathematics Education, 20*(3), 309-321.

Shulman, L.S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review*, 57, 1-22.

Singh, R.I. (2006). An investigation into learner understanding of the properties of selected quadrilaterals using manipulatives in grade eight mathematics classes. Masters' thesis. Durban, South Africa: University of KwaZulu-Natal.

Siyepu, S.W. (2005). The use of Van Hiele theory to explore problems encountered incircle geometry: A grade 11 case study. Unpublished masters' thesis. Grahamstown: Rhodes University.

Siyepu, S.W., & Mtonjeni, T. (2014). Geometrical concepts in real-life context: a case of South African traffic road signs. Proceedings of the 20th Annual National Congress of the Association for Mathematics of South Africa, Volume 1, 07 to 11 July 2014, Kimberley.

Sorsa, M. A., Kiikkala, I., & Åstedt-Kurki, P. (2015). Bracketing as a skill in conducting unstructured qualitative interviews. *Nurse researcher, 22*(4).

Stylianides, A.J., & Stylianides, G.J. (2007). Learning mathematics with understanding: A critical consideration of the learning principle in the Principles and Standards for School Mathematics. *The Montana Mathematics Enthusiast, 4*(1), 103-114. https://scholarworks.umt.edu/tme/vol4/iss1/8/

Tall, D. (1988). Concept Image and Concept Definition. In J. de Lange & M. Doorman (Ed.), *Senior Secondary Mathematics Education (*37-41). Utrecht: OW & OC.

Taylor, S., Bogdan, R., & De Vault, M.L. (2016). Introduction to qualitative research methods: A guidebook and resource (4th ed.). Wiley.

Telima, A. (2011). Problems of teaching and learning of geometry in secondary schools in River State, Nigeria. *Geometry Concepts Perceived Difficult To Learn By Senior Secondary School. International Journal of Emerging Science, 1*(2), 143-152.

Tella, A. (2007). The impact of motivation on student's academic achievement and learning outcomes in mathematics among secondary school students in Nigeria. *Eurasia Journal of Mathematics, Science & Technology Education*, *3*(2), 149-156.

Thompson, D.R., Senk, S.L. & Johnson, G.J. (2012). Opportunities to Learn Reasoning and Proof in High School Mathematics Textbooks. *Journal for Research in Mathematics Education, 43*(3). 253-295.

Thorne, S. (2016). *Interpretive description: Qualitative research for applied practice:* Routledge.

Turnuklu, E. (2014). Construction of inclusion relations of quadrilaterals: Analysis of pre-service elementary mathematics teachers' lesson plans. *Education and Science, 39*(173), 198–208.

Turnuklu, E., Akkas, E. N., & Gundogdu-Alayli, F. (2012). Mathematics teachers' perceptions of quadrilaterals and understanding the inclusion relations. Eighth Congress of European Research In Mathematics Education (CERME 8).

Ubah, I., & Bansilal, S. (2019). The use of semiotic representations in reasoning about similar triangles in Euclidean geometry. *Pythagoras*, *40*(1), a480. https://doi.org/10.4102/pythagoras.v40i1.480

Usiskin, Z., Griffin, J., Witonsky, D., & Willmore, E. (2008). The classification of quadrilaterals: A study in definition. Charlotte, NC: Information Age Publishing.

Van de Walle, J. A. (2004). *Elementary school mathematics: Teaching developmentally.* Longman.

Van de Walle, J. A., Karp, K. S., & Bay-Williams, J. M. (2012). Elementary and Middle School Mathematics: Teaching Developmentally (7th Edition), Translation Ed. Durmu,S., Nobel Akademik Yayoncolok, Ankara.

Van der Sandt, S. & Nieuwoudt, H.D. (2005). Geometry content knowledge: Is pre-service training making a difference? *African Journal of Research in SMT Education*, *9*(2), 109-120.

Van Hiele, P. (1986). *Structure and Insight: a theory of mathematics education.* Academic Press.

Van Putten, S.H. Stols and Howie (2010). Making Euclidean geometry compulsory: Are we prepared?. *Perspectives in Education, 28*(4), 22-31.

Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. In D. Tall, (Ed.), Advanced mathematical thinking (pp. 65-81). Dordrecht: Kluwer Academic.

Vinner, S., & Hershkowitz, R. (1980). Concept images and common cognitive paths in the development of some simple geometrical concepts. In R. Karplus (Ed.), *Proceedings of the fourth international conference for the psychology of mathematics education* (pp. 177–184). Berkeley: University of California, Lawrence Hall of Science.

Wasserman, N.H. (2016). Abstract algebra for algebra teaching: Influencing school mathematics instruction. *Canadian Journal of Science, Mathematics and Technology Education, 16*(1), 28-47.

Webber, K. (2003). *Students' difficulties with proof* (Research Sampler No. 8). Retrieved on 23 June, 2003, from, the Mathematical Association of America website: http://www.maa.org/t_and_l/sampler/rs_8.html

Weber, K. (2001). Student difficulty in constructing proofs: the need for strategic knowledge. *Educational Studies in Mathematics*, 48, 101-119. Kluwer Academic Publishers.

Williams, C. (2007). Research Methods. *Journal of Business & Economic Research, 5*(3), 65-72.

Zaslavsky, O., & Ron, G. (1998). Students' understandings of the role of counterexamples. In A. Olivier, K. Newstead, A. Olivier, & K. Newstead (Eds.), Proceedings of the 22nd conference of the international group for the psychology of mathematics education (Vol.4, pp.225-232). PME

Zeybek, Z. (2018). Understanding inclusion relations between quadrilaterals. *International Journal of Research in Education and Science (IJRES), 4*(2), 595-612. DOI:10.21890/ijres.428968

Zilkova, K. (2015). Misconceptions in pre-service primary education teachers about quadrilaterals. *Journal of Education, Psychology and Social Sciences*, *3*(1), 30–37.

APPENDICES

APPENDIX A

Ethical clearance certificate

UNIVERSITY OF KWAZULU-NATAL
INYUVESI YAKWAZULU-NATALI

22 September 2020

Mr Mduduzi Mhlengi Mbatha (214523136)
School Of Education
Edgewood Campus

Dear Mr Mbatha,

Protocol reference number: HSSREC/00001901/2020
Project title: Exploring pre-service teachers understanding of similarity and proofs in Euclidean Geometry
Degree: Masters

Approval Notification – Expedited Application

This letter serves to notify you that your application received on 27 August 2020 in connection with the above, was reviewed by the Humanities and Social Sciences Research Ethics Committee (HSSREC) and the protocol has been granted FULL APPROVAL.

Any alteration/s to the approved research protocol i.e. Questionnaire/Interview Schedule, Informed Consent Form, Title of the Project, Location of the Study, Research Approach and Methods must be reviewed and approved through the amendment/modification prior to its implementation. In case you have further queries, please quote the above reference number. PLEASE NOTE: Research data should be securely stored in the discipline/department for a period of 5 years.

This approval is valid until 22 September 2021.
To ensure uninterrupted approval of this study beyond the approval expiry date, a progress report must be submitted to the Research Office on the appropriate form 2 - 3 months before the expiry date. A close-out report to be submitted when study is finished.

All research conducted during the COVID-19 period must adhere to the national and UKZN guidelines.

HSSREC is registered with the South African National Research Ethics Council (REC-040414-040).

Yours sincerely,

[redacted]

Professor Dipane Hlalele (Chair)

/dd

Humanities & Social Sciences Research Ethics Committee
UKZN Research Ethics Office Westville Campus, Govan Mbeki Building
Postal Address: Private Bag X54001, Durban 4000
Tel: +27 31 260 8350 / 4557 / 3587
Website: http://research.ukzn.ac.za/Research-Ethics/

APPENDIX B

Pen and paper test and solutions

Student Code: _____

Mathematics Education for the FET phase Geometry and Trigonometry

EDMA160

Question set 1

30 Marks

QUESTION 1 (17 MARKS)

1.1 Define the meaning of similarity in mathematics as the best you can? (2)

Figures with the same shape but not necessarily the same size. ***(VHL2, Analysis)***

1.2 What are the minimum conditions or requirements to show that two polygons are similar? (2)

i. Equiangular and.

ii. Corresponding sides in proportion ***(VHL2, analysis)***

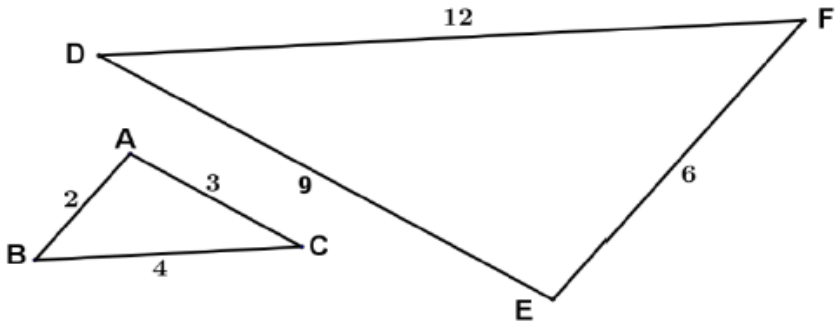
1.3 Given any two arbitrary squares of different sizes, are they similar or not? Explain (2)

Yes, they are similar because they will be always equiangular (all angles are 90 degrees) and their corresponding sides are in proportion. ***(VHL3, analysis)***

1.4 Given any two arbitrary rectangles of different sizes, are they similar or not? Explain. (3)

Not always similar, although they are equiangular (all angles are 90 degrees) but their corresponding sides will not always be in proportion. ***(VHL3, analysis)***

1.5 The sides in the diagram has lengths as indicated. Are these two triangles similar, Yes or No? Explain why you say so. (3)

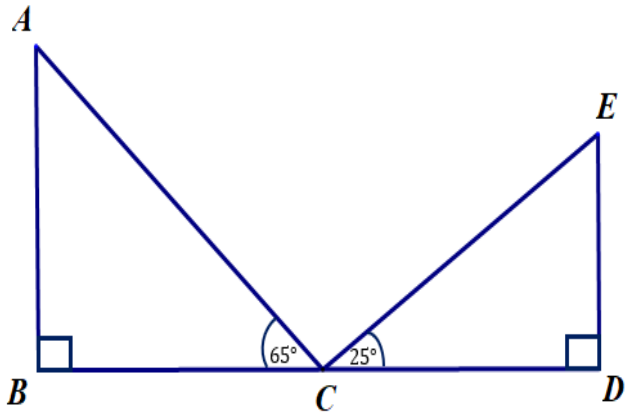


$$\frac{DF}{CB} = \frac{12}{4} = 3\ ,\ \frac{FE}{BA} = \frac{6}{2} = 3\ ;\ \frac{DE}{CA} = \frac{9}{3} = 3$$

Yes, they are similar because their corresponding sides are in proportion.

(VHL3, informal deduction)

1.7 State with reason whether or not the following triangles are similar. (2)



In ΔABC

$$\hat{A} = 180 - (90 + 65) = 25$$

Therefore, ΔABC // /ΔECD

In ΔECD

$$\hat{E} = 180 - (90 + 25) = 65$$

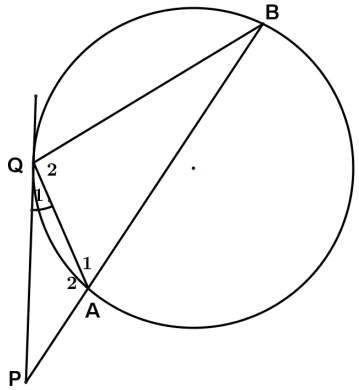
AAA

(VHL3, informal deduction)

QUESTION 2 (5 MARKS)

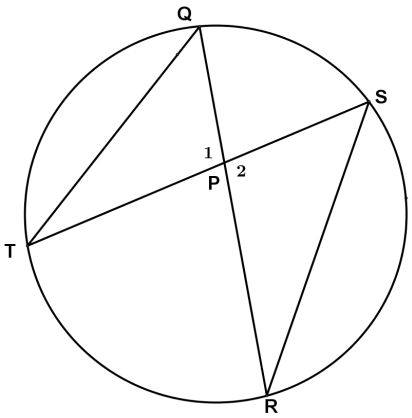
For each diagram below, write down a triangle similar to the given triangle. Naming must be in the correct order: ***(VHL3, informal deduction)***

2.1PQ is a tangent to the circle at Q.



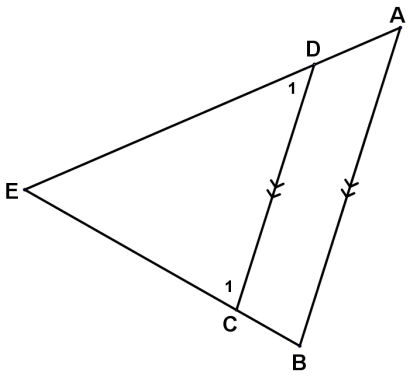
ΔBQP /// ΔQAP

2.2



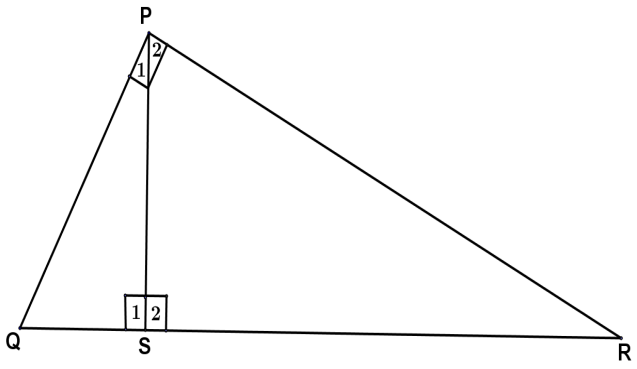
ΔPRS /// ΔPTQ

2.3



ΔABE /// ΔDCE

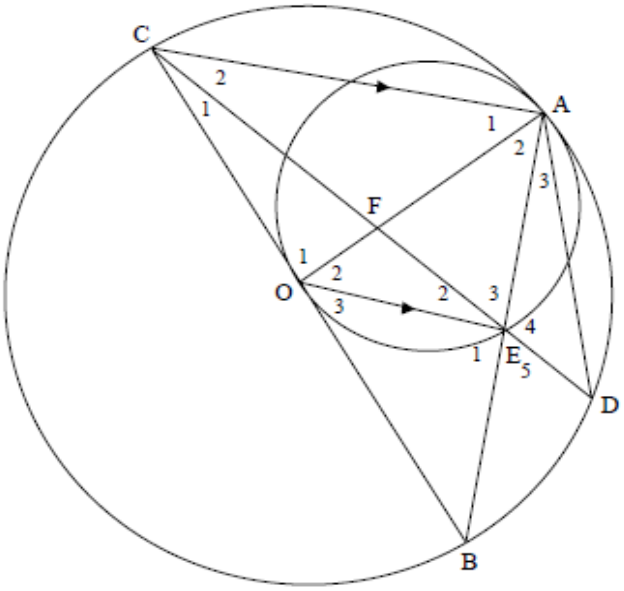
2.4/2.5 $\widehat{QPR} = 90^0$ and PS⊥QR.



ΔPQR /// ΔSPR ///ΔSQR

QUESTION 3 (8 MARKS)

Two circles touch each other at point A. The smaller circles pass through O, the centre of the larger circle. Point E is on the circumference of the smaller circle. A, D, B and C are points on the circumference of the larger circle. OE//CA.



3.1 Prove, with reasons that AE=BE (3)

***In* ΔABC**

- $\hat{E}1 = \hat{A} = 90$ ... ... $\angle$ ***SUBT. BY DIAMETER***
- $B\hat{E}O = A\hat{E}O = 90$ ... ... ***OB*** $\perp$ ***AB***

$\Rightarrow$ ***AE = BE*** ... ... ***line from the cenrtre chord, bisect the chord.***

(VHL4, formal deduction)

3.2 Prove that $\Delta AED /// \Delta CEB$ (3)

$\hat{A} = \hat{C}$ ... ... $\angle$ **subt. by the same arc BD**

$\hat{E}4 = \hat{E}1$ ... ... **VERT. OPP.** $\angle$**s**

$\hat{D} = \hat{B}$ ... ... **remaining** $\angle$ **OR** $\angle$**s subt. by the same chord AC**

Therefore, Δ**ABC // /**Δ**ECD** **AAA**

(VHL4, informal deduction)

3.3 Hence, or otherwise, show that $AE^2 = DE.CE$ (2)

$\frac{\mathbf{AE}}{\mathbf{CE}} = \frac{\mathbf{DE}}{\mathbf{BE}}$ **… … … … … … … … … … … … ΔABC // /ΔECD proved above**

$\Longrightarrow$ **AE. BE = DE. CE**

$\Longrightarrow$ **AE^2 = DE. CE … … … … … … … … … … … … ..BE=AE proved in question 3.1**

(VHL4, formal deduction)

THE END.

APPENDIX C

Semi-structured interview questions

1. Can you tell me when first you were introduced to similarity in Euclidean geometry? Did you understand it?

2. Can you tell me when first you were introduced to formal proofs in Euclidean geometry? Did you understand them?

3. At this stage (PST), are you confident and comfortable to solve and teach similarity and proofs in Euclidean geometry?

4. What is your general attitude/feeling towards teaching and learning of Euclidean geometry?

APPENDIX D

Individual performance in terms of category

PST	Q1.1		Q1.2		Q1.3		Q1.4		Q1.5		Q1.7		Q2.1		Q2.2		Q2.3		Q2.4		Q2.5		Q3.1		Q3.2		Q3.3		TOTAL
1	0	C2	1	C3	-	C1	0	C2	1	C3	0	C2	0	C2	0	C2	0	C2	0	C2	0	C2	0	C2	3	C4	2	C4	**7**
2	2	C4	2	C4	1	C3	0	C2	0	C2	0	C2	0	C2	1	C4	0	C2	0	C2	0	C2	0	C2	0	C2	2	C4	**8**
3	1	C3	2	C4	2	C4	0	C2	1	C3	2	C4	0	C2	1	C4	1	C4	1	C4	1	C4	0	C2	2	C2	2	C4	**14**
4	2	C4	2	C4	2	C4	0	C2	2	C3	0	C2	0	C2	1	C4	0	C2	0	C2	0	C2	0	C2	3	C4	2	C4	**14**
5	2	C4	2	C4	2	C4	0	C2	2	C3	0	C2	1	C4	0	C2	1	C4	0	C2	0	C2	0	C2	3	C4	2	C4	**15**
6	2	C4	2	C4	0	C2	0	C2	2	C3	2	C4	1	C4	0	C2	1	C4	0	C2	0	C2	0	C1	3	C4	2	C4	**15**
7	2	C4	2	C4	2	C4	0	C2	3	C4	2	C4	0	C2	0	C2	0	C2	0	C2	0	C2	0	C2	3	C4	2	C4	**16**
8	2	C4	1	C3	2	C4	0	C2	2	C3	0	C2	0	C2	0	C2	1	C4	0	C2	0	C2	3	C4	3	C4	2	C4	**16**
9	0	C2	2	C4	0	C2	0	C2	3	C4	2	C4	1	C4	1	C4	1	C4	1	C4	1	C4	0	C2	3	C4	2	C4	**17**
10	2	C4	2	C4	1	C3	0	C2	3	C4	2	C4	0	C2	1	C4	1	C4	1	C4	1	C4	0	C2	3	C4	-	C1	**17**
11	2	C4	2	C4	2	C4	0	C2	3	C4	2	C4	0	C2	0	C2	0	C2	0	C2	0	C2	2	C3	3	C4	2	C4	**18**
12	1	C3	2	C4	2	C4	0	C2	3	C4	2	C4	0	C2	0	C2	0	C2	0	C2	0	C2	3	C4	3	C4	2	C4	**18**
13	2	C4	2	C4	-	C1	-	C1	2	C4	2	C2	0	C2	1	C4	1	C4	1	C4	1	C4	3	C4	3	C4	2	C4	**18**
14	2	C4	0	C2	2	C4	0	C2	3	C4	2	C4	1	C4	1	C4	1	C4	1	C4	1	C4	0	C2	3	C4	2	C4	**19**
15	2	C4	2	C4	2	C4	0	C2	3	C4	2	C4	1	C4	1	C4	1	C4	0	C2	0	C2	0	C2	3	C4	2	C4	**19**

16	2	C4	2	C4	2	C4	0	C2	0	C2	2	C4	1	C4	1	C4	1	C4	0	C2	0	C2	3	C4	3	C4	2	C4	**19**
17	2	C4	2	C4	2	C4	0	C2	0	C2	2	C4	1	C4	1	C4	1	C4	0	C2	0	C2	3	C4	3	C4	2	C4	**19**
18	2	C4	2	C4	2	C4	0	C2	3	C4	2	C4	0	C2	0	C2	0	C2	0	C2	0	C2	3	C4	3	C4	2	C4	**19**
19	2	C4	2	C4	-	C1	-	C1	2	C3	2	C4	1	C4	1	C4	1	C4	1	C4	1	C4	3	C4	3	C4	2	C4	**21**
20	2	C4	2	C4	2	C4	0	C2	2	C3	2	C4	1	C4	1	C4	0	C2	1	C4	0	C2	3	C4	3	C4	2	C4	**21**
21	2	C4	2	C4	2	C4	2	C3	3	C4	2	C4	0	C2	0	C2	0	C2	0	C2	0	C2	3	C4	3	C4	2	C4	**21**
22	2	C4	2	C4	2	C4	0	C2	3	C4	2	C4	1	C4	1	C4	1	C4	1	C4	0	C2	3	C4	2	C3	2	C4	**22**
23	2	C4	2	C4	2	C4	0	C2	3	C4	2	C4	0	C2	0	C2	1	C4	1	C4	1	C4	3	C4	3	C4	2	C4	**22**
24	2	C4	2	C4	2	C4	0	C2	2	C3	2	C4	0	C2	1	C4	1	C4	1	C4	1	C4	3	C4	3	C4	2	C4	**22**
25	2	C4	0	C2	2	C4	0	C2	3	C4	2	C4	1	C4	1	C4	1	C4	1	C4	1	C4	3	C4	3	C4	2	C4	**22**
26	2	C4	2	C4	2	C4	0	C2	3	C4	0	C2	1	C4	1	C4	1	C4	1	C4	1	C4	3	C4	3	C4	2	C4	**22**
27	2	C4	2	C4	-	C1	-	C1	3	C4	2	C4	1	C4	1	C4	1	C4	1	C4	1	C4	3	C4	3	C4	2	C4	**22**
28	2	C4	2	C4	2	C4	2	C3	3	C3	2	C4	1	C4	1	C4	1	C4	1	C4	1	C4	0	C2	3	C4	2	C4	**22**
29	2	C4	2	C4	2	C4	2	C3	3	C4	0	C2	1	C4	1	C4	1	C4	0	C2	1	C4	3	C4	3	C4	2	C4	**23**
30	2	C4	2	C4	2	C4	0	C2	3	C4	2	C4	1	C4	1	C4	1	C4	1	C4	1	C4	3	C4	3	C4	2	C4	**24**
31	2	C4	2	C4	2	C4	0	C2	3	C4	2	C4	1	C4	1	C4	1	C4	1	C4	1	C4	3	C2	3	C4	2	C4	**24**
32	2	C4	2	C4	2	C4	0	C2	3	C3	2	C4	1	C4	1	C4	1	C4	1	C4	1	C4	3	C4	3	C4	2	C4	**24**
33	2	C4	2	C4	2	C4	0	C2	3	C4	2	C4	1	C4	1	C4	1	C4	1	C4	1	C4	3	C4	3	C4	2	C4	**24**
34	1	C3	2	C4	2	C4	2	C3	3	C4	2	C4	1	C4	1	C4	1	C4	1	C4	1	C4	3	C4	3	C4	2	C4	**25**

APPENDIX E

Individual performance in each test question with VHL

	QUESTIONS AND VAN HIELE LEVELS																														
PST	1.1	VHL	1.2	VHL	1.3	VHL	1.4	VHL	1.5	VHL	1.7	VHL	2.1	VHL	2.2	VHL	2.3	VHL	2.4	VHL	2.5	VHL	3.1	VHL	3.2	VHL	3.3	VHL	OVERAL
1	0	*l2*	1	*l2*	-	*l2*	0	*l2*	1	*l3*	0	*l3*	0	*l3*	0	*l3*	0	*l3*	0	*l3*	0	*l3*	0	*l4*	3	**L 3**	2	**L 4**	**B L2**
2	2	**L 2**	2	**L 2**	1	*l2*	0	*l2*	0	*l3*	0	*l3*	0	*l3*	1	**L 3**	0	*l3*	0	*l3*	0	*l3*	0	*l4*	0	*l3*	2	**L 4**	**B L2**
3	1	*l2*	2	**L 2**	2	**L 2**	0	*l2*	1	*l3*	2	**L 3**	0	*l3*	1	**L 3**	1	**L 3**	1	**L 3**	1	**L 3**	0	*l4*	2	*l3*	2	**L 4**	**B L2**
4	2	**L 2**	2	**L 2**	2	**L 2**	0	*l2*	2	*l3*	0	*l3*	0	*l3*	1	**L 3**	0	*l3*	0	*l3*	0	l3	0	*l4*	3	**L 3**	2	**L 4**	**L2**
5	2	**L 2**	2	**L 2**	2	**L 2**	0	*l2*	2	*l3*	0	*l3*	1	**L 3**	0	*l3*	1	**L 3**	0	*l3*	0	*l3*	0	*l4*	3	**L 3**	2	**L 4**	**L2**
6	2	**L 2**	2	**L 2**	0	*l2*	0	*l2*	2	*l3*	2	**L 3**	1	**L 3**	0	*l3*	1	**L 3**	0	*l3*	0	*l3*	0	*l4*	3	**L 3**	2	**L 4**	**B L2**
7	2	**L 2**	2	**L 2**	2	**L 2**	0	*l2*	3	**L 3**	2	**L 3**	0	*l3*	0	*l3*	0	*l3*	0	*l3*	0	*l3*	0	*l4*	3	**L 3**	2	**L 4**	**L2**
8	2	**L 2**	1	*l2*	2	**L 2**	0	*l2*	2	*l3*	0	*l3*	0	*l3*	0	*l3*	1	**L 3**	0	*l3*	0	*l3*	3	**L 4**	3	**L 3**	2	**L 4**	**B L2**
9	0	*l2*	2	**L 2**	00	*l2*	0	*l2*	3	**L 3**	2	**L 3**	1	**L 3**	1	**L 3**	1	**L 3**	1	**L 3**	1	**L 3**	0	*l4*	3	**L 3**	2	**L 4**	**B L2**

10	2	**L2**	2	**L2**	1	*l2*	0	*l2*	3	**L3**	2	**L3**	0	*l3*	1	**L3**	1	**L3**	1	**L3**	1	**L3**	0	l4	3	**L3**	-	*l4*	**BL2**
11	2	**L2**	2	**L2**	2	**L2**	0	*l2*	3	**L3**	2	**L3**	0	*l3*	0	*l3*	0	*l3*	0	*l3*	0	*l3*	2	*l4*	3	**L3**	2	**L4**	**L2**
12	1	*l2*	2	**L2**	2	**L2**	0	*l2*	3	**L3**	2	**L3**	0	*l3*	0	*l3*	0	*l3*	0	*l3*	0	*l3*	3	**L4**	3	**L3**	2	**L4**	**BL2**
13	2	**L2**	2	**L2**	-	*l2*	-	*l2*	2	*l3*	2	**L3**	0	*l3*	1	**L3**	1	**L3**	1	**L3**	1	**L3**	3	**L4**	3	**L3**	2	**L4**	**BL2**
14	2	**L2**	0	*l2*	2	**L2**	0	*l2*	3	**L3**	2	**L3**	1	**L3**	1	**L3**	1	**L3**	1	**L3**	1	**L3**	0	*l4*	3	**L3**	2	**L4**	**BL2**
15	2	**L2**	2	**L2**	2	**L2**	0	*l2*	3	**L3**	2	**L3**	1	**L3**	1	**L3**	1	**L3**	0	*l3*	0	*l3*	0	*l4*	3	**L3**	2	**L4**	**L3**
16	2	**L2**	2	**L2**	2	**L2**	0	*l2*	0	*l3*	2	**L3**	1	**L3**	1	**L3**	1	**L3**	0	*l3*	0	*l3*	3	**L4**	3	**L3**	2	**L4**	**L2**
17	2	**L2**	2	**L2**	2	**L2**	0	*l2*	0	*l3*	2	**L3**	1	**L3**	1	**L3**	1	**L3**	0	*l3*	0	*l3*	3	**L4**	3	**L3**	2	**L4**	**L2**
18	2	**L2**	2	**L2**	2	**L2**	0	*l2*	3	**L3**	2	**L3**	0	*l3*	0	*l3*	0	*l3*	0	*l3*	0	*l3*	3	**L4**	3	**L3**	2	**L4**	**L2**
19	2	**L2**	2	**L2**	-	*l2*	-	*l2*	2	*l3*	2	**L3**	1	**L3**	1	**L3**	1	**L3**	1	**L3**	1	**L3**	3	**L4**	3	**L3**	2	**L4**	**BL2**
20	2	**L2**	2	**L2**	2	**L2**	0	*l2*	2	*l3*	2	**L3**	1	**L3**	1	**L3**	0	*l3*	1	**L3**	0	*l3*	3	**L4**	3	**L3**	2	**L4**	**L2**
21	2	**L2**	2	**L2**	2	**L2**	2	**L2**	3	**L3**	2	**L3**	0	*l3*	0	*l3*	0	*l3*	0	*l3*	0	*l3*	3	**L4**	3	**L3**	2	**L4**	**L2**
22	2	**L2**	2	**L2**	2	**L2**	0	*l2*	3	**L3**	2	**L3**	1	**L3**	1	**L3**	1	**L3**	1	**L3**	0	*l3*	3	**L4**	2	*l3*	2	**L4**	**L4**
23	2	**L2**	2	**L2**	2	**L2**	0	*l2*	3	**L3**	2	**L3**	0	*l3*	0	*l3*	1	**L3**	1	**L3**	1	**L3**	3	**L4**	3	**L3**	2	**L4**	**L4**

24	2	**L2**	2	**L2**	2	**L2**	0	*l2*	2	*l3*	2	**L3**	0	*l3*	1	**L3**	1	**L3**	1	**L3**	1	**L3**	3	**L4**	3	**L3**	2	**L4**	**L4**
25	2	**L2**	0	*l2*	2	**L2**	0	*l2*	3	**L3**	2	**L3**	1	**L3**	1	**L3**	1	**L3**	1	**L3**	1	**L3**	3	**L4**	3	**L3**	2	**L4**	**B L2**
26	2	**L2**	2	**L2**	2	**L2**	0	*l2*	3	**L3**	0	*l3*	1	**L3**	1	**L3**	1	**L3**	1	**L3**	1	**L3**	3	**L4**	3	**L3**	2	**L4**	**L4**
27	2	**L2**	2	**L2**	-	*l2*	-	*l2*	3	**L3**	2	**L3**	1	**L3**	1	**L3**	1	**L3**	1	**L3**	1	**L3**	3	**L4**	3	**L3**	2	**L4**	**B L2**
28	2	**L2**	2	**L2**	2	**L2**	2	**L2**	3	**L3**	2	**L3**	1	**L3**	1	**L3**	1	**L3**	1	**L3**	1	**L3**	0	*l4*	3	**L3**	2	**L4**	**L3**
29	2	**L2**	2	**L2**	2	**L2**	2	**L2**	3	**L3**	0	*l3*	1	**L3**	1	**L3**	1	**L3**	0	**L3**	1	**L3**	3	**L4**	3	**L3**	2	**L4**	**L4**
30	2	**L2**	2	**L2**	2	**L2**	0	*l2*	3	**L3**	2	**L3**	1	**L3**	1	**L3**	1	**L3**	1	**L3**	1	**L3**	3	**L4**	3	**L3**	2	**L4**	**L4**
31	2	**L2**	2	**L2**	2	**L2**	0	*l2*	3	**L3**	2	**L3**	1	**L3**	1	**L3**	1	**L3**	1	**L3**	1	**L3**	3	**L4**	3	**L3**	2	**L4**	**L4**
32	2	**L2**	2	**L2**	2	**L2**	0	*l2*	3	**L3**	2	**L3**	1	**L3**	1	**L3**	1	**L3**	1	**L3**	1	**L3**	3	**L4**	3	**L3**	2	**L4**	**L4**
33	2	**L2**	2	**L2**	2	**L2**	0	*l2*	3	**L3**	2	**L3**	1	**L3**	1	**L3**	1	**L3**	1	**L3**	1	**L3**	3	**L4**	3	**L3**	2	**L4**	**L4**
34	1	*l2*	2	**L2**	2	**L2**	2	**L2**	3	**L3**	2	**L3**	1	**L3**	1	**L3**	1	**L3**	1	**L3**	1	**L3**	3	**L4**	3	**L3**	2	**L4**	**L4**

APPENDIX F

Consent form

Mathematics and Computer Sciences Cluster
University of KwaZulu-Natal,
School of Education,
Edgewood Campus
Date: 25 September 2020

Dear Student

My name is Mduduzi Mhlengi Mbatha, I am full-time educator who is currently studying towards a **Master of Education Degree** in Mathematics Education at the University of KwaZulu- Natal, Edgewood campus. I am conducting the research titled **'Exploring pre-service teachers' understanding of similarity and proofs in Euclidean Geometry'.**

Several studies indicated that Euclidean geometry is one of the sections considered most difficult for both teachers and learners in secondary school. Furthermore, the difficult of Euclidean geometry has contributed to the mass failure of grade 12 learners in Mathematics. In view of the foregoing, I intend to explore pre-service teachers' understanding of similarity and proofs in Euclidean geometry. **The objectives of the research are as follows:**

1. To explore pre-service teacher's understanding of similarity and proofs in Euclidean Geometry.
2. To explore the misconceptions pre-service teachers', have about similarity and proofs in Euclidean geometry.
3. To identify the Van Hiele levels of Geometric thinking that the group of pre-service teachers are operating at.

To gather the required data for the study, I kindly request you to participate in this study by reflecting critically on your understanding of similarity and proofs in Euclidean geometry. You are invited to participate in this study because you are a pre-service teacher who is studying the Mathematics module EDMA160 which deals with Euclidean geometry. The duration of the pen and paper test is 1 hour. In addition, I will also ask you to partake in a semi-structured interview which may last about 45-50 minutes. This study has been ethically reviewed and approved by the UKZN Humanities and Social Sciences Research Ethics Committee (approval number HSSREC/00001901/2020).

Please note that:

- Your participation is voluntary. If you do not participate you **will not be penalized** in anyway.
- You have a choice to participate, not participate or stop participating in the study. Meaning you are free to withdraw from the study at stage for any reason. You will not be penalized for taking such an action.
- Your confidentiality is guaranteed as your inputs will not be attributed to you in person but reported only as a population member opinion.
- Any information given by you cannot be used against you, and the collected data will be used for purposes of this research only.
- Data collected will be stored in secure storage and destroyed by shredding after 5 years. Digitally recorded data will be deleted after five years.
- Your involvement is purely for academic purposes only, and there are **no financial** benefits involved.

Thank you

Yours faithfully

Mduduzi Mhlengi Mbatha

My contact details are as follows:

Email: 214523136@stu.ukzn.ac.za

Cell phone number: 076 963 4305

My supervisor is prof Sarah Bansilal She is a Mathematics lecturer at Edgewood Campus, University of KwaZulu-Natal

My supervisor's contact details are:

Email: BansilalS@ukzn.ac.za

Cell phone number: 083 279 5916

You may also contact the Research Office at:

University of KwaZulu-Natal

Humanities and Social Sciences Research Ethics

Govan Mbeki Centre

Tel +27312604557

Email: HSSREC@ukzn.ac.za

Thank you for reading this document about this research.

DECLARATION OF CONSENT

I……

(Full names of participant) hereby confirm that I have been informed about the study entitled **'Exploring pre-service teachers' understanding of similarity and proofs in Euclidean Geometry'** by…….. (provide name of researcher). **I understand the contents of this document and the nature of the study, and I consent to participating in the research project.**

I understand the purpose and procedures of the study (add these again if appropriate).

I have been given an opportunity to answer questions about the study and have had answers to my satisfaction.

I declare that my participation in this study is entirely voluntary and that I may withdraw at any time without negative consequences.

I voluntarily give permission for the interviews to be audio-recorded.

My identity will not be disclosed, and pseudonyms will be used to protect my identity

If I have any further questions/concerns or queries related to the study or about my rights as a study participant I understand that I may contact the researcher at …….....(provide details).

HUMANITIES & SOCIAL SCIENCES RESEARCH ETHICS ADMINISTRATION

Research Office, Westville Campus

Govan Mbeki Building

Private Bag X 54001

Durban

4000

KwaZulu-Natal, SOUTH AFRICA

Email: HSSREC@ukzn.ac.za

Additional consent, where applicable

I am willing to be part of the paper and pen test and semi-structured interviews. I am also willing to allow recording by the following equipment, and the use of other data:

Digital audio recording of interv	Willing	Not willing
Semi-structured interview		

……...

Name of Participant

…… **……**

Signature of Participant **Date**

APPENDIX G

Gatekeeper's letter of permission to conduct research

UNIVERSITY OF KWAZULU-NATAL
INYUVESI YAKWAZULU-NATALI

13 September 2020

Mr Mduduzi Mbatha (SN 214523136)
School of Education
College of Humanities
Edgewood Campus
UKZN
Email: 214523136@stu.ukzn.ac.za

Dear Mr Mbatha

RE: PERMISSION TO CONDUCT RESEARCH

Gatekeeper's permission is hereby granted for you to conduct research at the University of KwaZulu-Natal (UKZN) towards your postgraduate studies, provided Ethical clearance has been obtained. We note the title of your research project is:

"Exploring pre-service teachers' understanding of similarity and proofs in Euclidean Geometry."

It is noted that you will be constituting your sample as follows:

- By conducting interviews with students on the Edgewood campus
- With a request for responses on the website. The questionnaire must be placed on the notice system http://notices.ukzn.ac.za. A copy of this letter (Gatekeeper's approval) must be simultaneously sent to (govenderkog@ukzn.ac.za) or (ramkissoonh@ukzn.ac.za).

Please ensure that the following appears on your questionnaire/attached to your notice:

- Ethical clearance approval letter;
- Research title and details of the research, the researcher and the supervisor;
- Consent form is attached to the notice/questionnaire and to be signed by user before he/she fills in questionnaire;
- gatekeepers approval by the Registrar.

You are not authorized to contact staff and students using the 'Microsoft Outlook' address book. Identity numbers and email addresses of individuals are not a matter of public record and are protected according to Section 14 of the South African Constitution, as well as the PAIA and POPI Act. For the release of such information over to yourself for research purposes, the University of KwaZulu-Natal will need express consent from the relevant data subjects. Data collected must be treated with due confidentiality and anonymity.

Yours sincerely

[redacted]

DR KE CLELAND: REGISTRAR (ACTING)

Office of the Registrar
Postal Address: Private Bag X54001, Durban, South Africa
Telephone: +27 (0) 31 260 8000/2206 Email: registrar@ukzn.ac.za
Website: www.ukzn.ac.za

1910 - 2010
100 YEARS OF ACADEMIC EXCELLENCE

Founding Campuses: Edgewood, Howard College, Medical School, Pietermaritzburg, Westville

INSPIRING GREATNESS

APPENDIX H

Editor's letter

15 December 2021

TO WHOM IT MAY CONCERN

This is to confirm that the dissertation written by Mduduzi Mhlengi Mbatha titled **'*Exploring pre-service teachers' understanding of similarity and proofs in Euclidean geometry'*** was copy edited for layout (including pagination, numbering, heading format, formatting of tables and figures), grammar, spelling, language type consistency, punctuation, and references by the undersigned. The document was subsequently proofread, and a number of additional corrections were advised.

Mrs. Barbara L. Mutula-Kabange

Copy Editor, Proofreader
BEd (UBotswana), BSSc Hons Psychology (UKZN),
MEd Educational Psychology (UKZN)

APPENDIX I

Turnitin report

Masters Thesis

ORIGINALITY REPORT

12%	11%	3%	%
SIMILARITY INDEX	INTERNET SOURCES	PUBLICATIONS	STUDENT PAPERS

PRIMARY SOURCES

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