University of KwaZulu-Natal

First integrals for spherically symmetric shear-free perfect fluid distributions

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by

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Abstract

In this dissertation we study spherically symmetric shear-free spacetimes. In particular we analyse the integrability of and find exact solutions to the Emden-Fowler equation $y_{xx} = f(x)y^2$, which is the master equation governing the behaviour of shear-free neutral perfect fluid distributions. We first review the study of Maharaj $et\ al$ (1996) by finding a first integral to this master equation. This first integral is subject to the integrability condition which we use to find restrictions on the function $f(x)$. We show that this first integral is a generalisation of particular solutions obtained by Stephani (1983) and Srivastava (1987). Furthermore, we use a similar method to obtain a new first integral of the master equation. This is achieved by multiplying the Emden-Fowler equation by an integrating factor. We then study the integrability condition, which is an integral equation, related to the new first integral. We find that the integrability condition can be written as a third order differential equation whose solution can be expressed in terms of elementary functions and elliptic integrals. In general the solution of the integrability condition is given parametrically. We believe that this is a new result. A particular form of $f(x)$ is identified which corresponds to repeated roots of a cubic equation giving an explicit solution.

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Contents

Chapter 1

Introduction

Before Einstein's theory of general relativity, which was proposed in 1916, Newtonian theory had been mostly used as the tool to describe the gravitational interactions between bodies. In Newtonian theory, the mass density is the source of the gravitational field. In the theory of general relativity the gravitational field of a body can be described by the curvature of spacetime. The curvature of spacetime is described by the Riemann tensor. Spacetime is taken to be a four-dimensional differentiable manifold endowed with a symmetric, non-degenerate metric tensor field. The geometry of spacetime is described using the Einstein tensor. The Einstein tensor is defined in terms of the metric tensor, the Ricci tensor as well as the Ricci scalar. The Einstein tensor is used to generate the nonlinear Einstein field equations, which are used to relate the matter content to the curvature of spacetime. The matter content is expressed in terms of the energy momentum tensor, which is a tensorial quantity, that is used to describe the density, flux of energy as well as momentum in spacetime. The Einstein field equations can be extended to the Einstein-Maxwell equations in the presence of the electromagnetic field.

Seeking exact solutions to the Einstein field equations has been the subject of study in many astrophysical and cosmological models. Exact solutions to the field equations have been used to investigate physical properties of observable phenomena such as relativistic stars as pointed out by Shapiro and Teukolsky (1983). In stellar models it is important to include effects due to heat flux, shear viscosity, bulk viscosity, the electromagnetic field and a superposition of different types of relativistic fluids. However, a number of exact solutions that are known have little physical importance. It is therefore important to find new exact solutions with desirable physical features. Delgaty and Lake (1998) provide the criteria for a class of exact solutions to have physical importance. Only a few models are known which satisfy all criteria for physical acceptability.

Exact solutions to the field equations for spherically symmetric spacetimes have been used to model many physical applications in astrophysics and cosmology. Some of the well known spherically symmetric solutions to the Einstein field equations are the Schwarzschild exterior solution (Schwarzschild 1916a), the Schwarzschild interior solution (Schwarzschild 1916b), the Reissner-Nordström solution (Nordström 1918), the Vaidya solution (Vaidya 1951) as well as the Kerr solution (Kerr, 1963). The Schwarzschild exterior solution describes the gravitational field outside the spherical body assuming that the electric charge of the body, the angular momentum as well as the cosmological constant are all equal to zero. This solution is often used to describe astronomical objects such as stars and the motion of planets. The Schwarzschild interior solution on the other hand describes the gravitational field in the interior of a spherical body which is nonrotating, has constant density and zero pressure at the surface. This solution can be used to model relativistic stars which have small fluctuations in energy. The Reissner-Nordström solution is used to describe the spacetime geometry of a nonrotating charged spherical body. The Reissner-Nordström solution is not very relevant in physical situations in cosmology since the universe at large scales appears to be neutral. It is sometimes used in modelling localised matter distributions. Some of the relatively recent exact solutions to the Einstein-Maxwell field equations for charged models were obtained by Komathiraj and Maharaj (2007a), Hansraj and Maharaj (2006) and Thirukkanesh and Maharaj (2006, 2009). The Vaidya solution is used to describe a nonempty external spacetime of a spherically symmetric, non-rotating body. The Vaidya solution presumes that the body either emits or absorbs null dust. This solution can be used to model the behaviour of non-adiabatic gravitational collapse for massive stars. The Kerr solution gives a description of the geometry of the empty spacetime of a rotating body. This solution is a generalisation of the Schwarzschild exterior solution. The Kerr solution arises from nonlinear differential equations for which exact solutions are not easy to find. Stephani (2004) and others have pointed out that an interior solution that matches to the Kerr line element is yet to be found. In cosmology, Krasinski (1997) used spherically symmetric spacetimes to model the gravitational behaviour and evolution of the early universe. Spherically symmetric models are used to generalise cosmological models which are both homogeneous and isotropic.

The general spherically symmetric spacetimes are expanding, accelerating and shearing. The absence of shear in spherically symmetric spacetimes simplifies the field equations. Shear-free solutions are given in isotropic and comoving coordinates. Most of the known exact solutions to the Einstein field equations are not shearing, as pointed out by Stephani $et\ al$ (2009) who provide a list and categories for a number of shear-free spherically symmetric solutions. Some of the classes of shear-free solutions include those obtained by Stephani (1983), Srivastava (1987), Sussman (1988a, 1988b) and Maharaj $et\ al$ (1996). Shear-free solutions have been used to model many physical applications in astrophysics and cosmology. In a recent treatment Brassel $et\ al$ (2015) found gravitational potentials for shear-free heat conducting fluids in terms of elementary functions.

Most of the solutions to the field equations are not shearing. However solutions with shear are difficult to find because of the nonlinearity of the field equations.

There exist some examples of spherically symmetric solutions which are shearing. These examples include solutions obtained by Vaidya (1953), Wesson (1978) and McVittie and Wiltshire (1977). Other solutions were found by Maharaj $et\ al$ (1993) and Naidu $et\ al$ (2006) in terms of potentials. Wiltshire (2006) derived exact solutions to the Einstein equations with shear in a comprehensive treatment using the Lie method of infinitesimal point symmetries. Exact solutions with shearing spacetimes may be applied in cosmological applications with no heat flux. Exact solutions with heat flux may be used to model radiating spheres in general relativity.

In this study, we seek exact solutions for spherically symmetric shear-free neutral perfect fluid distributions. An extensive review of known solutions is given by Stephani $et\ al$ (2009). There have also been substantial studies where the electromagnetic field is incorporated, in which case the Einstein field equations are supplemented by Maxwell's equations, and the solutions to the resulting Einstein-Maxwell equations need to be solved. Studies of charged shear-free fluids include solutions found by Ivanov (2002), Sharma $et\ al$ (2001) and Kweyama $et\ al$ (2012).

For spherically symmetric shear-free spacetimes with neutral matter, the Einstein field equations reduce to a system of nonlinear partial differential equations, which can be further reduced to a single Emden-Fowler equation $y_{xx} = f(x)y^2$ under a specific transformation. This equation was first introduced by Emden (1907) and was further studied by Fowler (1914). The Emden-Fowler equation is useful in finding solutions to the field equations with spherically symmetric shear-free matter. It also governs the behaviour of other physical systems as shown in Leach and Maharaj (1992). Various methods have been used to solve this equation. The more general approach is to apply the group theoretic technique of Sophus Lie.

This is a systematic geometric approach which uses the symmetries of a differential equation to reduce the order of the differential equation. This method was first introduced by Lie (1891) and has been widely applied. The first general solution to the Emden-Fowler equation in general relativity was found by Kustaanheimo and Qvist (1948). Other solutions were later found by Stephani (1983) and Srivastava (1987). Sussman (1987), Maartens and Maharaj (1990) and Maharaj et al (1991) also found classes of solutions to the Emden-Fowler equation under the assumption that the spacetime is invariant under a conformal Killing vector. A recent treatment of this problem is given by Maharaj et al (1996). In our study, we adopt an approach that is similar to the approach used by Maharaj et al (1996). We need to find exact solutions to the Emden-Fowler equation without specifying the function $f(x)$. We show that it is possible to obtain a new first integral to $y_{xx} = f(x)y^2$.

This dissertation is organised as follows:

Chapter 1 is this general introduction where we provide a background in general relativity, and the Einstein field equations. In particular we consider shearing and shear-free spherically symmetric solutions and their applications. We also refer to the literature where spherically symmetric models are studied.

In Chapter 2 we provide the basic theory in the spacetime geometry that is required for this dissertation. We discuss concepts such as manifolds, the metric tensor field, the Ricci tensor, the Einstein tensor and the energy momentum tensor. We use these concepts to generate the Einstein field equations for spacetimes which are static, shear-free and shearing.

In Chapter 3 we provide a review of the results of Maharaj $et\ al$ (1996). We generate a first integral for perfect fluid distributions which was obtained by Maharaj $et\ al$ (1996). We provide the first integral for the master equation $y_{xx} = f(x)y^2$ and study its integrability conditions. From the integrability conditions, we find restrictions for the function $f(x)$.

In Chapter 4 we follow the approach of Chapter 3 to obtain a new first integral for the equation $y_{xx} = f(x)y^2$. We also study the integrability condition for our new first integral and show that it can be solved. We use the integrability condition to find restrictions on the form of the function $f(x)$. An explicit form for $f(x)$ is obtained in a special case.

Chapter 5 is the conclusion of the dissertation. Here we summarize the main results obtained in our study.

Chapter 2

Spherically symmetric spacetimes

2.1 Introduction

Einstein's theory of general relativity is often used to describe gravitating distributions of matter with a specified spacetime geometry. Applications in relativistic astrophysics and cosmology often require a model with spherically symmetric geometry. Hence in this chapter we provide the basic theory of differential geometry, and give the Einstein field equations for spherically symmetric spacetimes. We provide only those details required for this dissertation. For more details on the spacetime geometry and the formulation of the Einstein field equations, the reader may consult Bishop and Goldberg (1968), Stephani (2004) and Wald (1984), amongst others. In section 2.2 we define and discuss the concepts of the metric tensor field, Christoffel symbols, Ricci and Einstein tensors, and the energy momentum tensor. This leads to the Einstein field equations on a manifold in section 2.3. In section 2.4 we consider spacetimes which are static. Nonstatic shear-free spacetimes are discussed in section 2.5. Nonstatic shearing spacetimes are discussed in section 2.6. In all three cases we generate the Ricci and Einstein tensors; the field equations are given explicitly.

2.2 Spacetime geometry

A differential manifold, one of the most fundamental structures in mathematics, is used in many physical applications, as pointed out by Carroll (1997). An

$n-$dimensional manifold is essentially a topological space which is locally similar to Euclidean space $\mathbb{R}^n$. Globally the manifold is different from $\mathbb{R}^n$ in general. The difference arises because of the appearance of spacetime curvature. In general relativity, we assume that the spacetime $\boldsymbol{M}$ is a four-dimensional, pseudo-Riemannian manifold endowed with a metric tensor field $\boldsymbol{g}$. The metric tensor field $\boldsymbol{g}$ is a non-degenerate and symmetric type $(0, 2)$ tensor field. It has signature $(- + ++)$ and represents gravitational potentials in general relativity. We use real coordinates $(x^a) = (x^0, x^1, x^2, x^3)$ to uniquely label the points in a manifold. The coordinate x^0 is timelike and x^1, x^2, x^3 are spacelike. Note that $x^0 = ct$ (c is the speed of light in vacuum). In this dissertation we use units where c is unity.

In the manifold $\boldsymbol{M}$, the line element which measures the invariant distance between any two neighbouring points is given in terms of the metric tensor field $\boldsymbol{g}$ defined by

$$ds^2 = g_{ab}dx^a dx^b, \tag{2.2.1}$$

where g_{ab} is a function of the coordinates (x^a). This generalises the line element in the Cartesian plane which is given by

$$ds^2 = dx^2 + dy^2 + dz^2.$$

The metric tensor field components and its derivatives generate the connection coefficients, which are given by

$$\Gamma^a{}_{bc} = \frac{1}{2}g^{ad}(g_{cd,b} + g_{db,c} - g_{bc,d}). \tag{2.2.2}$$

The connection coefficients $\Gamma^a{}_{bc}$ are also referred to as the Christoffel symbols of the second kind. The commas in the Christoffel symbols represent partial derivatives. The connection coefficients do not transform tensorially. However, they are very useful in the construction of the type $(1,3)$ Riemann (curvature) tensor $\boldsymbol{R}$.

The curvature tensor $\boldsymbol{R}$ is very important in describing the geometry of spacetime, and is given by

$$R^d{}_{abc} = \Gamma^d{}_{ac,b} - \Gamma^d{}_{ab,c} + \Gamma^e{}_{ac}\Gamma^d{}_{eb} - \Gamma^e{}_{ab}\Gamma^d{}_{ec}. \tag{2.2.3}$$

The curvature tensor satisfies the following symmetries

$$R_{abcd} = -R_{bacd}, \tag{2.2.4a}$$

$$R_{abcd} = -R_{abdc}, \tag{2.2.4b}$$

$$R_{abcd} = R_{cdab}, \tag{2.2.4c}$$

$$R_{abcd} + R_{adbc} + R_{acdb} = 0. \tag{2.2.4d}$$

The Riemann tensor also satisfies the property

$$R^a{}_{abcd} = 0. \tag{2.2.5}$$

In a four-dimensional manifold, the Riemann tensor $\boldsymbol{R}$ has a total of 20 independent components. Furthermore, the Riemann tensor satisfies the Bianchi identity given by

$$R^a{}_{bcd;e} + R^a{}_{bec;d} + R^a{}_{bde;c} = 0. \tag{2.2.6}$$

The semi-colons in (2.2.6) represent covariant derivatives. By contraction on

(2.2.3) we obtain the Ricci tensor given by

$$R_{ab} = R^c{}_{acb} \tag{2.2.7}$$

$$= \Gamma^c{}_{ab,c} - \Gamma^c{}_{ac,b} + \Gamma^c{}_{dc}\Gamma^d{}_{ab} - \Gamma^c{}_{db}\Gamma^d{}_{ac}.$$

The Ricci tensor is symmetric. By contraction on (2.2.7), we obtain the Ricci (curvature) scalar

$$R = g^{ab}R_{ab}$$

$$= R^a{}_a.$$

The Ricci tensor and the Ricci scalar generate the Einstein tensor $\boldsymbol{G}$. This tensor is given by

$$G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab}. \tag{2.2.8}$$

The Einstein tensor is also symmetric and it can be shown that it has zero divergence, that is

$$G^{ab}{}_{;b} = 0. \tag{2.2.9}$$

The Riemann tensor is useful for describing the curvature of a pseudo-Riemannian manifold. In general relativity, the Einstein tensor $\boldsymbol{G}$ is used to derive the Einstein field equations for gravitation.

2.3 Field equations

The Einstein field equations are used to relate the curvature of spacetime to the matter content represented by the energy and momentum. The Einstein field equations are given by

$$G_{ab} = kT_{ab}, \tag{2.3.1}$$

where $\boldsymbol{G}$ is the Einstein tensor and $\boldsymbol{T}$ is the energy momentum tensor. The parameter $k = \frac{8\pi G}{c^4}$ is the coupling constant. We will take k to be unity in this dissertation so that the Einstein field equations become

$$G_{ab} = T_{ab}. \tag{2.3.2}$$

Using (2.3.2) and (2.2.9), we obtain

$$T^{ab}{}_{;b} = 0. \tag{2.3.3}$$

Equation (2.3.3) is the law of conservation of matter.

The energy momentum tensor is given by

$$T_{ab} = (\mu + p)u_a u_b + p g_{ab} + q_a u_b + q_b u_a + \pi_{ab}, \tag{2.3.4}$$

where μ is the energy density, p is the kinetic pressure, q_a is the heat flux vector $(q_a u^a = 0)$ and π_{ab} is the anisotropic pressure (stress) tensor $(\pi_{ab}u^a = 0 = \pi^a_a)$. These quantities are measured relative to the comoving fluid four-velocity $\boldsymbol{u}$ which is unit and timelike so that

$$u^a u_a = -1.$$

For isotropic heat conducting fluids, the energy momentum tensor (2.3.4) becomes

$$T_{ab} = (\mu + p)u_a u_b + p g_{ab} + q_a u_b + q_b u_a. \tag{2.3.5}$$

Perfect fluids have the property that $\pi_{ab} = q_a = 0$, so that (2.3.4) reduces to

$$T_{ab} = (\mu + p)u_a u_b + p g_{ab}. \tag{2.3.6}$$

In many applications in cosmology and relativistic astrophysics we assume that

$$p = p(\mu),$$

which is a barotropic equation of state.

In this dissertation, we study the behaviour of the gravitational field in spherically symmetric spacetimes. The physically relevant spherically symmetric spacetimes are static, shear-free or shearing. These spacetimes have been used to model applications in stellar structures, radiating stars and gravitational collapse. Highly compact static stars have been studied by Mafa Takisa $et\ al$ (2017), Ngubelanga $et\ al$ (2015) and Sunzu $et\ al$ (2014). Radiating stars with outgoing heat flow across the boundary of the star have been investigated by Govender $et\ al$ (2010), Naidu $et\ al$ (2006) and Reddy $et\ al$ (2015). Gravitational collapse of spherically symmetric spacetimes have received attention in the recent works of Brassel $et\ al$ (2017), Kumar and Srivastava (2018) and Sharma $et\ al$ (2015). We therefore focus our attention on spherically symmetric spacetimes and generate the corresponding field equations.

2.4 Static spacetimes

If the spacetime is static and spherically symmetric then the metric can be written in terms of the comoving coordinates $(x^a) = (t, r, \theta, \phi)$. The line element is of the form

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{2.4.1}$$

where the arbitrary functions $\nu(r)$ and $\lambda(r)$ relate to the gravitational potentials. The line element (2.4.1), which may be matched to the exterior Schwarzschild metric, is often used to model the behaviour of relativistic compact objects such as neutron stars. Some examples are discussed by Komathiraj and Maharaj (2007b) and Thirukkanesh and Maharaj (2008).

The nonzero connection coefficients, for the line element (2.4.1), are

$$\Gamma^0{}_{01} = \nu' \qquad \Gamma^1{}_{00} = \nu' e^{2(\nu-\lambda)}$$

$$\Gamma^1{}_{11} = \lambda' \qquad \Gamma^1{}_{22} = -re^{-2\lambda}$$

$$\Gamma^1{}_{33} = -re^{-2\lambda}\sin^2\theta \qquad \Gamma^2{}_{12} = \frac{1}{r}$$

$$\Gamma^2{}_{33} = -\sin\theta\cos\theta \qquad \Gamma^3{}_{13} = \frac{1}{r}$$

$$\Gamma^3{}_{23} = \cot\theta,$$

where the primes represent partial differentiation with respect to r. The nonvanishing Ricci tensor components (2.2.7) become

$$R_{00} = e^{2(\nu-\lambda)}\left(\nu'' + \nu'^2 - \nu'\lambda' + \frac{2\nu'}{r}\right), \tag{2.4.2a}$$

$$R_{11} = -\left(\nu'' + \nu'^2 - \nu'\lambda' - \frac{2\lambda}{r}\right), \tag{2.4.2b}$$

$$R_{22} = 1 - \frac{1}{e^{2\lambda}}\left[1 + r(\nu' - \lambda')\right], \tag{2.4.2c}$$

$$R_{33} = \sin^2\theta R_{22}. \tag{2.4.2d}$$

The Ricci (curvature) scalar (2.2.8) is then given by

$$R = 2\left[\frac{1}{r^2} - \left(\nu'' + \nu'^2 - \nu'\lambda' + \frac{2\nu'}{r} - \frac{2\lambda'}{r} + \frac{1}{r^2}\right)\frac{1}{e^{2\lambda}}\right]. \tag{2.4.3}$$

The nonzero Einstein tensor components have the form

$$G_{00} = \frac{e^{2\nu}}{r^2}\left[r\left(1 - \frac{1}{e^{2\lambda}}\right)\right]', \tag{2.4.4a}$$

$$G_{11} = -\frac{1}{r^2}\left(e^{2\lambda} - 1\right) + \frac{2\nu'}{r}, \tag{2.4.4b}$$

$$G_{22} = \frac{r^2}{e^{2\lambda}}\left(\nu'' + \nu'^2 + \frac{v'}{r} - v'\lambda' - \frac{\lambda'}{r}\right), \tag{2.4.4c}$$

$$G_{33} = \sin^2\theta G_{22}. \tag{2.4.4d}$$

for the spacetime metric (2.4.1).

The fluid-four velocity $\boldsymbol{u}$ can be written in the form

$$u^a = (e^{-\nu}, 0, 0, 0). \tag{2.4.5}$$

For this four-velocity the nonvanishing components of the energy momentum tensor $\boldsymbol{T}$ in (2.3.6) are given by

$$T_{00} = \mu e^{2\nu}, \tag{2.4.6a}$$

$$T_{11} = p e^{2\lambda}, \tag{2.4.6b}$$

$$T_{22} = p r^2, \tag{2.4.6c}$$

$$T_{33} = \sin^2\theta T_{22}. \tag{2.4.6d}$$

Then the Einstein field equations for static spherically symmetric spacetimes can be generated from (2.4.4) and (2.4.6). They are of the form

$$\mu = \frac{1}{r^2}\left[r\left(1 - \frac{1}{e^{2\lambda}}\right)\right]', \tag{2.4.7a}$$

$$p = \frac{2v'}{re^{2\lambda}} - \frac{1}{r^2}\left(1 - \frac{1}{e^{2\lambda}}\right), \tag{2.4.7b}$$

$$p = \frac{1}{e^{2\lambda}}\left(\nu'' + \nu'^2 + \frac{\nu'}{r} - \nu'\lambda' - \frac{\lambda'}{r}\right). \tag{2.4.7c}$$

Using the law of conservation of matter (2.3.3), we obtain

$$\frac{dp}{dr} = -(\mu + p)\frac{d\nu}{dr}, \tag{2.4.8}$$

which is also implied by the system of equations (2.4.7).

Equating (2.4.7b) and (2.4.7c) we obtain

$$\nu'' + \nu'^2 - \nu'\lambda' - \frac{(\nu' + \lambda')}{r} + \frac{(e^{2\lambda} - 1)}{r^2} = 0, \tag{2.4.9}$$

which is called the condition of pressure isotropy. This condition can also be written in the form

$$\frac{d}{dr}\left(\frac{e^{-2\lambda} - 1}{r^2}\right) + \frac{d}{dr}\left(\frac{e^{-2\lambda}}{r}\right) + e^{-2\nu-2\lambda}\frac{d}{dr}\left(\frac{e^{2\nu}\nu'}{r}\right) = 0. \tag{2.4.10}$$

Equation (2.4.10) is very useful in studying the behaviour of static stellar models. The first solutions to (2.4.10) were presented by Tolman (1939).

Several other classes of exact solutions to both (2.4.9) and (2.4.10) are contained in the review of Delgaty and Lake (1998). Researchers transform (2.4.10) to equivalent forms, using a change of coordinates, which may lead to a new exact solution.

2.5 Shear-free spacetimes

If the spacetime is nonstatic and shear-free then the metric can be written in terms of coordinates which are both comoving and isotropic. In the coordinate system $(x^a) = (t, r, \theta, \phi)$, the line element takes the form

$$ds^2 = -A^2dt^2 + B^2[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)]. \tag{2.5.1}$$

where $A = A(t, r)$ and $B = B(t, r)$ are metric functions. The shear-free line element is used to model radiating stars in general relativity. Some models that have been studied include Tewari and Charan (2015) and Das *et al* (2016). For a recent treatment of the conformal symmetries of the spacetime (2.5.1) see the results of Moopanar and Maharaj (2013).

The nonvanishing connection coefficients (2.2.2), for the line element (2.5.1), are of the form

$$\Gamma^0{}_{00} = \frac{\dot{A}}{A} \qquad \Gamma^0{}_{01} = \frac{A'}{A}$$

$$\Gamma^0{}_{11} = \frac{B\dot{B}}{A^2} \qquad \Gamma^0{}_{22} = r^2\frac{B\dot{B}}{A^2}$$

$$\Gamma^0{}_{33} = r^2\sin^2\theta\frac{B\dot{B}}{A^2} \qquad \Gamma^1{}_{00} = \frac{AA'}{B^2}$$

$$\Gamma^1{}_{11} = \frac{B'}{B} \qquad \Gamma^1{}_{22} = -r^2\left(\frac{B'}{B}+\frac{1}{r}\right)$$

$$\Gamma^1{}_{33} = -r^2\sin^2\theta\left(\frac{B'}{B}+\frac{1}{r}\right) \qquad \Gamma^1{}_{01} = \frac{\dot{B}}{B}$$

$$\Gamma^2{}_{02} = \frac{\dot{B}}{B} \qquad \Gamma^3{}_{03} = \frac{\dot{B}}{B}$$

$$\Gamma^2{}_{12} = \frac{B'}{B}+\frac{1}{r} \qquad \Gamma^3{}_{13} = \frac{B'}{B}+\frac{1}{r}$$

$$\Gamma^2{}_{33} = -\sin\theta\cos\theta \qquad \Gamma^3_{23} = \cot\theta,$$

where dots and primes denote partial differentiation with respect to t and r, respectively. The nonvanishing components of the Ricci tensor (2.2.7) are given by

$$R_{00} = \frac{AA''}{B^2} + AA'\frac{B'}{B^3} - 3\frac{\ddot{B}}{B} + 3\frac{\dot{A}}{A}\frac{\dot{B}}{B} + \frac{2}{r}\frac{AA'}{B^2}, \tag{2.5.2a}$$

$$R_{01} = 2\frac{B'\dot{B}}{B^2} - 2\frac{\dot{B}'}{B} + \frac{A'\dot{A}}{A^2} + 3\frac{A'}{A}\frac{\dot{B}}{B} - \frac{1}{A^2}\left(\dot{A}^2 + A'^2\right), \tag{2.5.2b}$$

$$\begin{aligned} R_{11} = {} & 2\frac{\dot{B}^2}{A^2} + \frac{A'}{A}\frac{B'}{B} - \frac{2B'}{rB} - \frac{\dot{A}}{A^3}B\dot{B} - \frac{A''}{A} + \frac{B\ddot{B}}{A^2} \\ & + 2\frac{B'^2}{B^2} - 2\frac{B''}{B}, \end{aligned} \tag{2.5.2c}$$

$$\begin{aligned} R_{22} = {} & r^2\frac{B\ddot{B}}{A^2} - r^2\frac{\dot{A}\dot{B}B}{A^3} + 2r^2\frac{\dot{B}^2}{A^2} - r^2\frac{A'}{A}\frac{B'}{B} - r\frac{A'}{A} \\ & - 3r\frac{B'}{B} - r^2\frac{B''}{B}, \end{aligned} \tag{2.5.2d}$$

$$R_{33} = \sin^2\theta R_{22}. \tag{2.5.2e}$$

Using the above components, we obtain the Ricci scalar

$$\begin{aligned} R = {} & -2\frac{A''}{A}\frac{1}{B^2} - \frac{4}{r}\frac{A'}{A}\frac{1}{B^2} + \frac{6}{A^2}\frac{\dot{B}^2}{B^2} - \frac{8B'}{rB^2} + 2\frac{B'^2}{B^4} - 2\frac{A'}{A}\frac{B'}{B^3} \\ & - 4\frac{B''}{B^3} - 6\frac{\dot{A}}{A^3}\frac{\dot{B}}{B} + 6\frac{\ddot{B}}{B} \end{aligned} \tag{2.5.3}$$

for the line element (2.5.1). Substituting (2.5.2) and (2.5.3) in (2.2.8), we obtain the nonvanishing Einstein tensor components in the form

$$
\begin{aligned}
G_{00} &= 3\frac{\dot{B}^2}{B^2} - \frac{A^2}{B^2}\left(2\frac{B''}{B} - \frac{B'^2}{B^2} + \frac{4}{r}\frac{B'}{B}\right), && \text{(2.5.4a)}\\
G_{01} &= -\frac{2}{B^2}\left(B\dot{B}' - B'\dot{B} - \frac{A'}{A}B\dot{B}\right), && \text{(2.5.4b)}\\
G_{11} &= \frac{1}{A}\left(-2B\ddot{B} - \dot{B}^2 + 2\frac{\dot{A}}{A}B\dot{B}\right)\\
&\quad + \frac{1}{B^2}\left(B'^2 + 2\frac{A'}{A}BB' + \frac{2}{r}\frac{A'}{A}B^2 + \frac{2}{r}BB'\right), && \text{(2.5.4c)}\\
G_{22} &= -2r^2\frac{B\ddot{B}}{A^2} + 2r^2\frac{\dot{A}}{A^3}B\dot{B} - r^2\frac{\dot{B}^2}{A^2} + r\frac{A'}{A}\\
&\quad + r\frac{B'}{B} + r^2\frac{A''}{A} - r^2\frac{B'^2}{B^2} + r^2\frac{B''}{B}, && \text{(2.5.4d)}\\
G_{33} &= \sin^2\theta G_{22}, && \text{(2.5.4e)}
\end{aligned}
$$

for the metric (2.5.1).

The fluid four-velocity $\boldsymbol{u}$ is given by

$$
u^a = \left(\frac{1}{A}, 0, 0, 0\right),
$$

for shear-free spacetimes. Then the nonvanishing components of (2.3.5) are given by

$$T_{00} = \mu A^2, \tag{2.5.5a}$$

$$T_{01} = -qAB^2, \tag{2.5.5b}$$

$$T_{11} = pB^2, \tag{2.5.5c}$$

$$T_{22} = pB^2 r^2, \tag{2.5.5d}$$

$$T_{33} = \sin^2\theta T_{22}. \tag{2.5.5e}$$

If we substitute (2.5.5) and (2.5.4) in (2.3.2), then we obtain the field equations

$$\mu = \frac{3}{A^2}\frac{\dot{B}^2}{B^2} - \frac{1}{B^2}\left(2\frac{B''}{B} - \frac{B'^2}{B^2} + \frac{4}{r}\frac{B'}{B}\right), \tag{2.5.6a}$$

$$\begin{aligned} p = {} & \frac{1}{A^2}\left(-\frac{2\ddot{B}}{B} - \frac{\dot{B}^2}{B^2} + 2\frac{\dot{A}}{A}\frac{\dot{B}}{B}\right) \\ & + \frac{1}{B^2}\left(\frac{B'^2}{B^2} + 2\frac{A'}{A}\frac{B'}{B} + \frac{2}{r}\frac{A'}{A} + \frac{2}{r}\frac{B'}{B}\right), \end{aligned} \tag{2.5.6b}$$

$$\begin{aligned} p = {} & -2\frac{1}{A^2}\frac{\ddot{B}}{B} + 2\frac{\dot{A}}{A^3}\frac{\dot{B}}{B} - \frac{1}{A^2}\frac{\dot{B}^2}{B^2} + \frac{1}{r}\frac{A'}{A}\frac{1}{B^2} \\ & + \frac{1}{r}\frac{B'}{B^3} + \frac{A''}{A}\frac{1}{B^2} - \frac{B'^2}{B^4} + \frac{B''}{B^3}, \end{aligned} \tag{2.5.6c}$$

$$q = -\frac{2}{AB^2}\left(\frac{B'\dot{B}}{B^2} + \frac{A'}{A}\frac{\dot{B}}{B} - \frac{\dot{B}'}{B}\right). \tag{2.5.6d}$$

The nonlinear system of partial differential equations (2.5.6) gives the matter variables μ, p and q in terms of A and B. This system is used to describe the interior of the shear-free spherically symmetric radiating star.

Eliminating p in (2.5.6b) and (2.5.6c) gives the following partial differential equation

$$\frac{A''}{A}\frac{1}{B^2} + \frac{B''}{B^3} - 2\frac{B'}{B^3}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{1}{rB^2}\left(\frac{A'}{A} + \frac{B'}{B}\right) = 0. \tag{2.5.7}$$

This is the condition of pressure isotropy for shear-free fluids. Equation (2.5.7) can be equivalently written as

$$\frac{A''}{A} + \frac{B''}{B} = \left(2\frac{B'}{B} + \frac{1}{r}\right)\left(\frac{A'}{A} + \frac{B'}{B}\right). \tag{2.5.8}$$

If we define

$$x = r^2,$$

then (2.5.8) becomes the following differential equation

$$\left(\frac{A}{B}\right)_{xx} = 2A\left(\frac{1}{B}\right)_{xx}, \tag{2.5.9}$$

where the subscript represents partial differentiation with respect to x.

2.6 Shearing spacetimes

If the spacetime is nonstatic and shearing then the line element can be written in terms of coordinates which are comoving. With coordinates $(x^a) = (t, r, \theta, \phi)$, the metric becomes

$$ds^2 = -e^{2\nu(t,r)}dt^2 + e^{2\lambda(t,r)}dr^2 + Y^2(t,r)[d\theta^2 + \sin^2\theta d\phi^2], \tag{2.6.1}$$

where ν, λ and Y are functions of the spacelike and timelike coordinates r and $t,$ respectively. The shearing line element is utilized to model cosmological and astrophysical systems in general relativity. Some examples are given in the treatments of Kitamura (1994) and Ivanov (2016). For a recent investigation of the conformal geometry of the spacetime (2.6.1) see the results of Moopanar and Maharaj (2010).

The nonvanishing connection (2.2.2) coefficients for the line element (2.6.1) take the form

$$\Gamma^0{}_{00} = \dot{\nu} \qquad \Gamma^0{}_{01} = \nu'$$

$$\Gamma^0{}_{11} = e^{2(\lambda-\nu)}\dot{\lambda} \qquad \Gamma^0{}_{22} = e^{-2\nu}Y\dot{Y}$$

$$\Gamma^0{}_{33} = e^{-2\nu}Y\dot{Y}\sin^2\theta \qquad \Gamma^1{}_{00} = e^{2(\nu-\lambda)}v'$$

$$\Gamma^1{}_{01} = \dot{\lambda} \qquad \Gamma^1{}_{11} = \lambda'$$

$$\Gamma^1{}_{22} = -e^{2\nu}Y\dot{Y} \qquad \Gamma^1{}_{33} = -e^{-2\lambda}YY'\sin^2\theta$$

$$\Gamma^2{}_{02} = \frac{\dot{Y}}{Y} \qquad \Gamma^2{}_{12} = \frac{Y'}{Y}$$

$$\Gamma^2{}_{33} = -\sin\theta\cos\theta \qquad \Gamma^3{}_{03} = \frac{\dot{Y}}{Y}$$

$$\Gamma^3{}_{13} = \frac{Y'}{Y} \qquad \Gamma^3{}_{23} = \cot\theta.$$

The nonvanishing Ricci tensor components for the line element (2.6.1) are then given by

$$\begin{aligned} R_{00} &= -\ddot{\lambda} - \dot{\lambda}^2 + \dot{\lambda}\dot{\nu} + 2\dot{\nu}\frac{\dot{Y}}{Y} \\ &\quad + e^{2(\nu-\lambda)}\left(\nu'' + \nu'^2 - \nu'\lambda' + 2\nu'\frac{Y'}{Y}\right), \end{aligned} \tag{2.6.2a}$$

$$R_{01} = 2\left(\dot{\lambda}\frac{Y'}{Y} + \nu'\frac{\dot{Y}}{Y} - \frac{\dot{Y}'}{Y}\right), \tag{2.6.2b}$$

$$\begin{aligned} R_{11} &= -\nu'' - \nu'^2 + \lambda'\nu' + 2\lambda'\frac{Y'}{Y} - 2\frac{Y''}{Y} \\ &\quad + e^{2(\lambda-\nu)}\left(\ddot{\lambda} + \dot{\lambda}^2 - \dot{\lambda}\dot{\nu} + 2\dot{\lambda}\frac{\dot{Y}}{Y}\right), \end{aligned} \tag{2.6.2c}$$

$$\begin{aligned} R_{22} &= \frac{1}{e^{2\nu}}Y\dot{Y}\left(\dot{\lambda} - \dot{\nu} + \frac{\dot{Y}}{Y} + \frac{\ddot{Y}}{\dot{Y}}\right) + 1 \\ &\quad + \frac{1}{e^{2\lambda}}YY'\left(\lambda' - \nu' - \frac{Y'}{Y} - \frac{Y''}{Y'}\right), \end{aligned} \tag{2.6.2d}$$

$$R_{33} = \sin^2\theta R_{22}. \tag{2.6.2e}$$

The Ricci scalar (2.2.8) then becomes

$$\begin{aligned} R &= \frac{2}{e^{2\nu}}\left(\ddot{\lambda} + \dot{\lambda}^2 - \dot{\lambda}\dot{\nu} + 2\dot{\lambda}\frac{\dot{Y}}{Y} - 2\dot{\nu}\frac{\dot{Y}}{Y} + \frac{\dot{Y}^2}{Y^2} + \frac{\ddot{Y}}{Y}\right) + \frac{2}{Y^2} \\ &\quad - \frac{2}{e^{2\lambda}}\left(\nu'' + \nu'^2 - \nu'\lambda' - 2\lambda'\frac{Y'}{Y} + 2\nu'\frac{Y'}{Y}\right. \\ &\quad \left. + \frac{Y'^2}{Y^2} + 2\frac{Y''}{Y}\right). \end{aligned} \tag{2.6.3}$$

The nonzero components of the Einstein tensor for the line element (2.6.1) take the form

$$
\begin{aligned}
G_{00} &= \frac{e^{2\nu}}{Y^2} + \frac{\dot{Y}^2}{Y^2} + 2\dot{\lambda}\frac{\dot{Y}}{Y} - e^{2(\nu-\lambda)}\left(2\frac{Y''}{Y} + \frac{Y'^2}{Y^2} - 2\lambda\frac{Y'}{Y}\right), \quad \text{(2.6.4a)}\\
G_{01} &= 2\dot{\lambda}\frac{Y'}{Y} + 2\nu'\frac{\dot{Y}}{Y} - 2\frac{\dot{Y}'}{Y}, \quad \text{(2.6.4b)}\\
G_{11} &= 2\nu'\frac{Y'}{Y} + \frac{Y'^2}{Y^2} - \frac{e^{2\lambda}}{Y^2} + e^{2(\lambda-\nu)}\left(2\dot{v}\frac{\dot{Y}}{Y} - \frac{\dot{Y}^2}{Y} - 2\frac{\ddot{Y}}{Y}\right), \quad \text{(2.6.4c)}\\
G_{22} &= \frac{1}{e^{2\lambda}}\left[\left(\nu'' + \nu'^2 - \nu'\lambda'\right)Y^2 + \left(\nu'Y' - \lambda'Y' + Y''\right)Y\right]\\
&\quad - \frac{1}{e^{2\nu}}\left[\left(\ddot{\lambda} + \dot{\lambda}^2 - \dot{\lambda}\dot{\nu}\right)Y^2 + \left(\dot{\lambda}\dot{Y} - \dot{\nu}\dot{Y} + \ddot{Y}\right)Y\right], \quad \text{(2.6.4d)}\\
G_{33} &= \sin^2\theta G_{22}, \quad \text{(2.6.4e)}
\end{aligned}
$$

for the metric (2.6.1).

For shearing spacetimes the fluid-four velocity $\boldsymbol{u}$ is comoving so that

$$
u^a = \left(e^{-\nu}, 0, 0, 0\right). \quad \text{(2.6.5)}
$$

Then the nonzero components of the energy momentum tensor $\boldsymbol{T}$ in (2.3.6) are

$$T_{00} = \mu e^{2\nu}, \tag{2.6.6a}$$

$$T_{11} = pe^{2\lambda}, \tag{2.6.6b}$$

$$T_{22} = pY^2, \tag{2.6.6c}$$

$$T_{33} = \sin^2\theta T_{22}. \tag{2.6.6d}$$

The Einstein field equations for shearing spacetimes with line element (2.6.1) are obtained by using (2.6.6) and (2.6.4). We get the field equations

$$\begin{aligned}
\mu &= \frac{1}{Y^2} - \frac{2}{Y}\frac{1}{e^{2\lambda}}\left(Y'' - \lambda' Y' + \frac{Y'^2}{2Y}\right) \\
&\quad + \frac{2}{Y}\frac{1}{e^{2\nu}}\left(\dot{\lambda}\dot{Y} + \frac{\dot{Y}^2}{2Y}\right),
\end{aligned} \tag{2.6.7a}$$

$$\begin{aligned}
p &= -\frac{1}{Y^2} + \frac{1}{Y}\frac{1}{e^{2\lambda}}\left(\nu' Y' + \frac{Y'^2}{2Y}\right) \\
&\quad - \frac{2}{Y}\frac{1}{e^{2\nu}}\left(\ddot{Y} - \dot{\nu}\dot{Y} + \frac{\dot{Y}^2}{2Y}\right),
\end{aligned} \tag{2.6.7b}$$

$$\begin{aligned}
p &= \frac{1}{e^{2\lambda}}\left[\left(\nu'' + \nu'^2 - \nu'\lambda'\right) + \frac{1}{Y}\left(\nu' Y' - \lambda' Y' + Y''\right)\right] \\
&\quad - \frac{1}{e^{2\nu}}\left[\left(\ddot{\lambda} + \dot{\lambda}^2 - \dot{\lambda}\dot{\nu}\right) + \frac{1}{Y}\left(\dot{\lambda}\dot{Y} - \dot{\nu}\dot{Y} + \ddot{Y}\right)\right],
\end{aligned} \tag{2.6.7c}$$

$$0 = \dot{Y}' - \dot{Y}\nu' - Y'\dot{\lambda}, \tag{2.6.7d}$$

which is a nonlinear system.

From (2.6.7b) and (2.6.7c) we obtain

$$\begin{aligned}
&\frac{1}{e^{2\nu}}\left[\left(\ddot{\lambda}+\dot{\lambda}^2-\dot{\lambda}\dot{\nu}\right)+\frac{1}{Y}\left(\dot{\lambda}\dot{Y}-\dot{\nu}\dot{Y}+\ddot{Y}\right)\right.\\
&\left.-\frac{2}{Y}\left(\ddot{Y}-\dot{\nu}\dot{Y}+\frac{\dot{Y}^2}{2Y}\right)\right]-\frac{1}{e^{2\lambda}}\left[\left(\nu''+\nu'^2-\nu'\lambda'\right)\right.\\
&\left.+\frac{1}{Y}\left(\nu'Y'-\lambda'Y'+Y''\right)+\frac{2}{Y}\left(\nu'Y'+\frac{Y'^2}{2Y}\right)\right]-\frac{1}{Y^2}=0, \quad (2.6.8)
\end{aligned}$$

which is the condition of pressure isotropy. We observe that this condition is more difficult to solve than the corresponding equations for static and shear-free spacetimes.

Chapter 3

A first integral for perfect fluid distributions

3.1 Introduction

When seeking exact solutions to the Einstein field equations for neutral and shear-free matter, spherical symmetry is usually assumed. This assumption leads to the nonlinear partial differential equation $y_{xx} = f(x)y^2$. This equation is of Emden-Fowler form and arises in many other physical applications. For a geometric analysis of a generalized Emden-Fowler equation see the analysis of Leach and Maharaj (1992). We will show in this chapter that its integrability is related to a third order ordinary differential equation. An unusual approach to solving $y_{xx} = f(x)y^2$ was followed by Maharaj $et\ al$ (1996). We review this approach and show that the first integral obtained is subject to an integral equation. In Section 3.2 we show how the Einstein field equations for the nonstatic shear-free metric reduce to $y_{xx} = f(x)y^2$. In Section 3.3 we review a first integral of $y_{xx} = f(x)y^2$ obtained by Maharaj $et\ al$ (1996) and study its integrability conditions. We conclude this chapter with Section 3.4 where we compare the first integral discussed in 3.3 to the first integrals obtained by Stephani (1983) and Srivastava (1987).

3.2 Field equations

We consider the shear-free spacetime (2.5.1). We let $A = e^{\nu}$ and $B = e^{\lambda}$. Then the metric becomes

$$ds^2 = -e^{2\nu(t,r)}dt^2 + e^{2\lambda(t,r)}\left[dr^2 + r^2(d\theta^2 + \sin^2\theta\phi^2)\right], \tag{3.2.1}$$

for a shear-free, perfect fluid in the comoving and isotropic coordinate system $(x^a) = (t, r, \theta, \phi)$. For our application we set $q = 0$ in (2.5.6) so that the fluid is not heat conducting. Then the Einstein field equations for the line element (3.2.1) take the form

$$\mu = 3\frac{{\lambda_t}^2}{e^{2\nu}} - \frac{1}{e^{2\lambda}}\left(2\lambda_{rr} + \lambda_r^2 + \frac{4\lambda_r}{r}\right), \tag{3.2.2a}$$

$$\begin{aligned} p = {} & \frac{1}{e^{2\nu}}\left(-2\lambda_{tt} - 3\lambda_t^2 + 2\nu_t\lambda_t\right) \\ & + \frac{1}{e^{2\lambda}}\left(\lambda_r^2 + 2\nu_r\lambda_r + \frac{2\nu_r}{r} + \frac{2\lambda_r}{r}\right), \end{aligned} \tag{3.2.2b}$$

$$\begin{aligned} p = {} & \frac{1}{e^{2\nu}}\left(-2\lambda_{tt} - 3\lambda_t^2 + 2\nu_t\lambda_t\right) \\ & + \frac{1}{e^{2\lambda}}\left(\nu_{rr} + \nu_r^2 + \frac{\nu_r}{r} + \frac{\lambda_{rr}}{r} + \lambda_{rr}\right), \end{aligned} \tag{3.2.2c}$$

$$0 = \nu_r\lambda_t - \lambda_{tr}. \tag{3.2.2d}$$

The subscripts r and t in equations (3.2.2) above represent partial derivatives with respect to r and t, respectively.

Equation (3.2.2d) can also be written as

$$\nu_r = (\ln \lambda_t)_r \, . \tag{3.2.3}$$

Then from (3.2.2b) and (3.2.2c) we obtain the equation

$$\left[e^{\lambda} \left(\lambda_{rr} - \lambda_r^2 - \frac{\lambda_r}{r} \right) \right]_t = 0,$$

where the variable ν has been eliminated.

The system of equations (3.2.2) can then be written in the form

$$\mu = 3e^{2h} - e^{-2\lambda} \left(2\lambda_{rr} + \lambda_r^2 + \frac{4\lambda_r}{r} \right), \tag{3.2.4a}$$

$$p = \frac{1}{\lambda_t e^{3\lambda}} \left[e^{\lambda} \left(\lambda_r^2 + \frac{2\lambda_r}{r} \right) - e^{3\lambda + 2h} \right]_t, \tag{3.2.4b}$$

$$e^{\nu} = \lambda_t e^{-h}, \tag{3.2.4c}$$

$$e^{\lambda} \left(\lambda_{rr} - \lambda_r^2 - \frac{\lambda_r}{r} \right) = -g(r), \tag{3.2.4d}$$

where $h = h(t)$ and $g = g(r)$ are arbitrary functions of integration. The functions h and g need to be specified in order to find exact solutions for the field equations. The metric function λ is obtained from the condition of pressure isotropy (3.2.4d). The remaining metric function ν then follows from (3.2.3) or (3.2.4c). The energy density μ and the isotropic pressure p can be calculated using equations (3.2.4a) and (3.2.4b). Using the transformation

$$\begin{aligned} x &= r^2, \\ y(x,t) &= e^{-\lambda}, \\ f(x) &= \frac{g}{4r^2}, \end{aligned}$$

(3.2.4d) reduces to

$$y_{xx} = f(x)y^2, \tag{3.2.5}$$

as first shown by Kustaanheimo and Qvist (1948). Equation (3.2.5) is the master equation governing the gravitational dynamics of a shear-free fluid in general relativity.

There have been a number of studies seeking solutions of the field equation (3.2.5). However, the solution is known for only a few forms of $f(x)$. The solution with

$$f(x) = \left(a + bx + cx^2\right)^{-5/2},$$

was given by Kustaanheimo and Qvist (1948). Solutions with

$$f(x) = x^{-20/7}, x^{-15/7}, e^x,$$

were found by Stephani (1983). General analyses of the equation (3.2.5) were completed by Wafo Soh and Mahomed (1999), Maharaj $et\ al$ (1996) and Stephani $et\ al$ (2009). A charged generalization was studied by Kweyama $et\ al$ (2012).

3.3 A known first integral

It is desirable to find a general class of solutions to (3.2.5) by restricting the functional form of $f(x)$. An unusual approach was followed by Maharaj $et\ al$ (1996) which leads to a first integral. We review their result, and extend their approach in the next chapter.

Integrating (3.2.5), we obtain

$$y_x = \int f(x)y^2 dx - \phi_0(t), \tag{3.3.1}$$

where $\phi_0(t)$ is a function of integration. The integral on the right hand side of (3.3.1) can be written as

$$\int f(x)y^2 dx = f_I y^2 - 2\int f_I y y_x dx. \tag{3.3.2}$$

For convenience, in (3.3.2) we have used the notation

$$\int f(x)dx = f_I.$$

Using integration by parts, the integral of $f_I y y_x$ is given by

$$\int f_I y y_x dx = f_{II} y y_x - \int f_{II} y_x^2 dx - \int f_{II} y y_{xx} dx. \tag{3.3.3}$$

Then using (3.2.5), we can write (3.3.3) as

$$\int f_I y y_x dx = f_{II} y y_x - \int f_{II} y_x^2 dx - \int f f_{II} y^3 dx. \tag{3.3.4}$$

Similarly, we can evaluate the integrals on the right hand side of (3.3.4), and substituting in (3.3.2) gives

$$\begin{aligned}\int f(x)y^2 dx &= f_I y^2 - 2f_{II} y y_x + 2f_{III} y_x^2 + 2(f f_{II})_I y^3 \\ &\quad -2\left[\int [2f f_{III} + 3(f f_{II})_I] y^2 y_x dx\right].\end{aligned} \tag{3.3.5}$$

The integral on the right hand side of (3.3.5) can be evaluated if

$$2ff_{III} + 3(ff_{II})_I = K_0, \tag{3.3.6}$$

where K_0 is an arbitrary constant. Note that the condition (3.3.6) is an integral equation in $f(x)$.

Substituting (3.3.5) in (3.3.1), we obtain the result

$$\phi_0(t) = -y_x + f_I y^2 - 2f_{II} y y_x + 2f_{III} y_x^2 + 2\left[(ff_{II})_I - \frac{1}{3}K_0\right] y^3. \tag{3.3.7}$$

We then observe that (3.3.7) is the first integral of (3.2.5) provided that condition (3.3.6) is satisfied. To complete the analysis we need to indicate the form of the function $f(x)$. In an attempt to find the form of the function f, the integral equation (3.3.6) can be transformed into an ordinary differential equation which is easy to solve. Differentiating (3.3.6) gives

$$2f_x f_{III} + 3ff_{II} = 0. \tag{3.3.8}$$

Using the transformation

$$L \equiv f_{III}, \tag{3.3.9}$$

equation (3.3.8) becomes a fourth order ordinary differential equation which is given by

$$2LL_{xxxx} + 5L_x L_{xxx} = 0. \tag{3.3.10}$$

We can integrate (3.3.10) to obtain a third order differential equation. We obtain

$$L_{xxx} = K_1 L^{-5/2}, \tag{3.3.11}$$

where K_1 is a constant of integration. We observe that the third order ordinary differential equation (3.3.10), together with the transformation (3.3.9), is equivalent to the integrability condition (3.3.6). A solution of (3.3.11) gives L and then $f(x)$ can be found using (3.3.9). Below we indicate how (3.3.11) can be integrated, giving L.

The nonlinear differential equation (3.3.11) may be written as

$$\begin{aligned} LL_{xxx} &= K_1 L^{-3/2}, \\ (LL_{xx}) - L_x L_{xx} &= K_1 L^{-3/2}, \\ (LL_{xx})_x - \frac{1}{2}(L_x^2)_x &= K_1 L^{-3/2}. \end{aligned}$$

Integration gives

$$LL_{xx} - \frac{1}{2}L_x^2 = -2K_2 + K_1 \left(\int L^{-3/2} dx \right),$$

where $-2K_2$ is a constant. The second order equation above may be written as

$$2(L^{1/2})_{xx} = -2K_2 L^{-3/2} + K_1 L^{-3/2} \left(\int L^{-3/2} dx \right),$$

whose integration yields

$$2(L^{1/2})_x = -K_3 - 2K_2 \left(\int L^{-3/2} dx \right) + \frac{1}{2}K_1 \left(\int L^{-3/2} dx \right)^2,$$

where $-K_3$ is a constant. For convenience, we write the first order differential equation above in the form

$$-L^{-2}L_x = K_3 L^{-3/2} + 2K_2 L^{-3/2} \left(\int L^{-3/2} dx \right) - \frac{1}{2}K_1 L^{-3/2} \left(\int L^{-3/2} dx \right)^2,$$

whose integration yields the result

$$L^{-1} = K_4 + K_3\left(\int L^{-3/2}dx\right) + K_2\left(\int L^{-3/2}dx\right)^2 - \frac{1}{6}K_1\left(\int L^{-3/2}dx\right)^3, \tag{3.3.12}$$

where K_4 is a constant. Therefore the third order equation (3.3.11) has been fully integrated. We can write the solution parametrically.

For convenience, we let

$$u = \int L^{-3/2}dx$$

so that

$$u_x = (L^{-1})^{3/2}. \tag{3.3.13}$$

By substituting (3.3.13) in (3.3.12), we obtain the following result

$$x - x_0 = \int \frac{du}{(K_4 + K_3u + K_2u^2 - (1/6)K_1u^3)^{3/2}}, \tag{3.3.14}$$

where x_0 is constant. We observe that the differential equation (3.3.10) can be reduced to the quadrature (3.3.14) and this can generally be evaluated in terms of elliptic integrals. Once the integral in (3.3.14) is evaluated, (3.3.13) can be used to find $L = u_x^{-(2/3)}$; the function $f(x)$ can then be found using (3.3.9). In order to find $f(x)$ satisfying the integrability condition (3.3.6), it is convenient to express the solution to (3.3.6) in the parametric form

$$\begin{aligned} f(x) &= L_{xxx}, \\ u_x &= L^{-3/2} = [g'(u)]^{-1}, \\ x - x_0 &= g(u). \end{aligned}$$

In the above we have defined

$$g(u) = \int \frac{du}{(K_4 + K_3u + K_2u^2 - (1/6)K_1u^3)^{3/2}}. \qquad (3.3.15)$$

The evaluation of the integral depends on the values of the constants K_1, K_2, K_3 and K_4.

The above solution has five cases depending on the nature of the factors of the polynomial $K_4 + K_3u + K_2u^2 - \frac{1}{6}K_1u^3$. The five cases are:

Case I: One order-three factor

Case II: One order-one factor and one order-two factor

Case III: Three order-one (non-repeated) factors

Case IV: One linear factor and one quadratic factor

Case V: No factors

Maharaj $et\ al$ (1996) studied only the first three cases. For the new first integral in the next chapter, we will also consider the last two cases: Case IV and Case V.

Case I: One order-three factor

If $K_4 + K_3u + K_2u^2 - \frac{1}{6}K_1u^3$ has one factor repeated three times then we can write

$$K_4 + K_3u + K_2u^2 - \frac{1}{6}K_1u^3 = (a + bu)^3,$$

with $b \neq 0$. In this case, the integral in (3.3.14) can be evaluated to obtain

$$g(u) = -\frac{2}{7b}(a + bu)^{-7/2}.$$

In this case it is possible to take this form of $g(u)$ and (3.3.14) to find $u(x)$, which can then be used to obtain

$$f(x) = \frac{48}{343}\left(-\frac{7b}{2}\right)^{6/7}(x-x_0)^{-15/7}.$$

After reparametrisation, $f(x)$ can be written as

$$f(x) = x^{-15/7}, \tag{3.3.16}$$

which is the same as the solution of Stephani (1983).

Case II: One order-one factor and one order-two factor

If $K_4+K_3u+K_2u^2-\frac{1}{6}K_1u^3$ has one factor repeated two times then we can write

$$K_4+K_3u+K_2u^2-\frac{1}{6}K_1u^3 = (a+bu)(u+c)^2,$$

with $b \neq 0$. In this case, the integral in (3.3.14) can be evaluated to give

$$\begin{aligned} g(u) \quad = \quad & \left(\frac{15b^2}{4(a-bc)^3} + \frac{5b}{4u(a-bc)^2} - \frac{1}{2u^2(a-bc)}\right)\frac{1}{\sqrt{a+bu-bc}} \\ & + \frac{15b^2}{8(a-bc)^3}\int \frac{du}{u\sqrt{a+bu-bc}}, \end{aligned}$$

where the integral on the right can be expressed in terms of elementary functions depending on the sign of $a-bc$. In this case it is not possible to obtain $u(x)$ explicitly. Hence the solution can only be expressed parametrically.

Case III: Three order-one (non-repeated) factors

If $K_4+K_3u+K_2u^2-\frac{1}{6}K_1u^3$ has three non-repeated factors then we can write

$$K_4+K_3u+K_2u^2-\frac{1}{6}K_1u^3 = d(a-u)(b-u)(c-u),$$

with $d \neq 0$. In this case, the integral in (3.3.14) can be evaluated in terms of elliptic integrals to give

$$\begin{aligned} g(u) \quad = \quad & \frac{2[c(a-c)+b(a-b)-u(2a-c-b)]}{d^{3/2}(a-b)(a-c)(b-c)^2\sqrt{(a-u)(b-u)(c-u)}} \\ & +\frac{2(b-c)(a+b-2c)F(\alpha,\beta)}{d^{3/2}(a-b)^2(b-c)^2\sqrt{(a-c)^3}} \\ & -\frac{2(a^2+b^2+c^2-ab-ac-bc)E(\alpha,\beta)}{d^{3/2}(a-b)^2(b-c)^2\sqrt{(a-c)^3}}, \end{aligned}$$

where we have set

$$\alpha = \arcsin\sqrt{\frac{a-c}{a-u}}$$

and

$$\beta = \sqrt{\frac{a-b}{a-c}}.$$

In the solution above, $F(\alpha,\beta)$ and $E(\alpha,\beta)$ are the elliptic integrals of the first and second kind, respectively. In this case, we also cannot obtain $u(x)$ explicitly, and hence the solution can only be expressed in parametric form.

In **Case IV** and **Case V**, the integral (3.3.14) can only be expressed using elliptic integrals. These cases were not considered by Maharaj *et al* (1996). In these two cases, we also cannot obtain $u(x)$ explicitly. Therefore we can only express $u(x)$, and hence $f(x)$, explicitly only in the case where the polynomial $K_4 + K_3u + K_2u^2 - \frac{1}{6}K_1u^3$ has one order-three factor.

3.4 The Stephani and Srivastava solutions

Earlier results and particular first integrals are contained in the results of this section. As much as the first integral (3.3.7) of the equation (3.2.5) was obtained without a choice of a specific function $f(x)$, any choice for the function $f(x)$ that satisfies (3.3.6) will yield a first integral of the form (3.3.7). With the choice

$$f(x) = (ax+b)^n\,,$$

equation (3.3.7) becomes

$$\begin{aligned}
\psi_0(t) &= \frac{1}{6}\phi_0(t) \\
&= -6y_x - \frac{21}{4a}(ax+b)^{-8/7}y^2 - \frac{3}{2}\left(\frac{7}{a}\right)^2(ax+b)^{-1/7}yy_x \\
&\quad +\frac{1}{4}\left(\frac{7}{a}\right)^3(ax+b)^{6/7}y_x^2 - \frac{1}{6}\left(\frac{7}{a}\right)^3(ax+b)^{-9/7}y^3, \qquad (3.4.1)
\end{aligned}$$

where $n = -\frac{15}{7}$. If $a = 1$ and $b = 0$ then equation (3.4.1) gives

$$\begin{aligned}
\chi_0(t) &= \frac{1}{6}\phi_0(t) \\
&= -6y_x - \frac{21}{4}x^{-8/7}y^2 - \frac{3}{2}7^2x^{-1/7}yy_x \\
&\quad +\frac{1}{4}7^3x^{6/7}y_x^2 - \frac{1}{6}7^3x^{-9/7}y^3. \qquad (3.4.2)
\end{aligned}$$

The first integral (3.4.1) was obtained by Srivastava (1987). This solution is a special case of (3.3.7) and it satisfies (3.3.6) with $K_0 = 0$. The first integral (3.4.2) was found by Stephani (1983). We have shown that the first integrals (3.4.1) and (3.4.2) can be regained as special cases of (3.3.7).

Note that there exist solutions which do not satisfy (3.3.6). For example the choice $f(x) = e^x$ reported in Stephani (1983) does not satisfy the integrability condition (3.3.6). Also the Kustaanheimo and Qvist (1948) solution
$f(x) = (a + bx + cx^2)^{-5/2}$ does not satisfy (3.3.6).

Chapter 4

A new first integral for perfect fluid distributions

4.1 Introduction

In Chapter 3 we discussed the Emden-Fowler equation. As it is an important equation in mathematical physics, and represents spherically symmetric fields in general relativity, it is desirable to find new exact solutions. Therefore in this chapter we find a new class of solutions to the field equation $y_{xx} = f(x)y^2$ using an approach that is similar but different from the approach used in Chapter 3. In Section 4.2 we use the technique of integration by parts to obtain a new first integral of $y_{xx} = f(x)y^2$. This involves multiplying the master field equation by an integrating factor. The structure of the equation is changed but it remains integrable. This first integral is subject to an integral equation. In Section 4.3 we use this integrability condition to find the functional form of $f(x)$, adopting an approach that is similar to that used in Chapter 3.

4.2 A first integral

Another general class of solutions to (3.2.5) can be found with a different restriction on a functional form of $f(x)$, when compared to Chapter 3. The procedure is similar to that of Maharaj $et\ al$ (1996) but the differential equation is different. This unusual approach in integrating a differential equation leads to a new first integral. We believe that this result has not been obtained previously.

When (3.2.5) is multiplied by x, it becomes

$$xy_{xx} = xf(x)y^2. \tag{4.2.1}$$

The left hand side of equation (4.2.1) can be written as a total derivative as follows

$$(xy_x - y)_x = xf(x)y^2.$$

Integrating (4.2.1) gives

$$xy_x - y = \int xfy^2dx - \phi_1(t), \tag{4.2.2}$$

where $\phi_1(t)$ is a function of integration. The integral on the right hand side of this equation may be evaluated by parts to give

$$xy_x - y = \bar{f}_I y^2 - 2\int \bar{f}_I yy_x dx - \phi_1(t), \tag{4.2.3}$$

where for convenience, we have used $xf(x) = \bar{f}$ and $\int \bar{f}dx = \bar{f}_I$.

Integrating $\bar{f}_I yy_x$ by parts we obtain

$$\int \bar{f}_I yy_x dx = \bar{f}_{II} yy_x - \int \bar{f}_{II} y_x^2 dx - \int \bar{f}_{II} yy_{xx} dx. \tag{4.2.4}$$

Using (3.2.5), (4.2.4) may be written as

$$\int \bar{f}_I yy_x dx = \bar{f}_{II} yy_x - \int \bar{f}_{II} y_x^2 dx - \int f\bar{f}_{II} y^3 dx. \tag{4.2.5}$$

Substitution of (4.2.5) in (4.2.3) yields

$$xy_x - y = \bar{f}_I y^2 - 2\bar{f}_{II} yy_x + 2\int \bar{f}_{II} y_x^2 dx + 2\int f\bar{f}_{II} y^3 dx - \phi_1(t). \tag{4.2.6}$$

Integrating $f\bar{f}_{II}y^3$ and $\bar{f}_{II}y_x^2$, by parts again, we obtain

$$\int f\bar{f}_{II}y^3 dx = (f\bar{f}_{II})_I y^3 - 3\int (f\bar{f}_{II})_I y^2 y_x dx, \tag{4.2.7}$$

and

$$\begin{aligned}\int \bar{f}_{II}y_x^2 dx &= \bar{f}_{III}y_x^2 - 2\int \bar{f}_{III}y_x y_{xx} dx \\ &= \bar{f}_{III}y_x^2 - 2\int f\bar{f}_{III}y^2 y_x dx.\end{aligned} \tag{4.2.8}$$

Substituting (4.2.7) and (4.2.8) in (4.2.6) gives

$$\begin{aligned}xy_x - y &= \bar{f}_I y^2 - 2\bar{f}_{II}yy_x + 2\bar{f}_{III}y_x^2 + 2(f\bar{f}_{II})_I y^3 \\ &\quad -2\int 2f\bar{f}_{III}y^2 y_x dx - 2\int 3(f\bar{f}_{II})_I y^2 y_x dx - \phi_1(t),\end{aligned}$$

which can be written in the form

$$\begin{aligned}xy_x - y &= \bar{f}_I y^2 - 2\bar{f}_{II}yy_x + 2\bar{f}_{III}y_x^2 + 2(f\bar{f}_{II})_I y^3 \\ &\quad -2\left[\int [2f\bar{f}_{III} + 3(f\bar{f}_{II})_I]y^2 y_x dx\right] - \phi_1(t).\end{aligned} \tag{4.2.9}$$

The integral on the right hand side of (4.2.9) can be evaluated if

$$2f\bar{f}_{III} + 3(f\bar{f}_{II})_I = K_5, \tag{4.2.10}$$

where K_5 is a constant such that

$$\begin{aligned}\int [2f\bar{f}_{III} + 3(f\bar{f}_{II})_I]y^2 y_x dx &= \int K_5 y^2 y_x dx \\ &= \frac{K_5}{3}y^3.\end{aligned}$$

This yields the result

$$\begin{aligned}\phi_1(t) &= y - xy_x + \bar{f}_I y^2 - 2\bar{f}_{II} y y_x + 2\bar{f}_{III} y_x^2 \\ &\quad + 2\left[(f\bar{f}_{II})_I - \frac{1}{3}K_5\right] y^3,\end{aligned} \tag{4.2.11}$$

where $\phi_1(t)$ is an arbitrary function of integration. We then observe that (4.2.11) is another first integral of (3.2.5) provided that condition (4.2.10) is satisfied. It is important to note that the first integral $\phi_1(t)$ is linearly independent of the first integral $\phi_0(t)$ in (3.3.7). Hence (4.2.11) is a new result and $f(x)$ is constrained by the condition (4.2.10) which is an integral equation.

4.3 Integrability conditions

To complete the analysis we need to determine the form of the function $f(x)$ (or $\bar{f}(x)$). In an attempt to seek the form of the function f, the integral equation (4.2.10) can be transformed into an ordinary differential equation. It is easier to solve the differential equation rather than the integral equation. Differentiating (4.2.10), we obtain

$$2f_x\bar{f}_{III} + 5f\bar{f}_{II} = 0. \tag{4.3.1}$$

Multiplying (4.3.1) by x and using $(xf)_x = f + xf_x$, we have

$$2xf_x\bar{f}_{III} + 5xf\bar{f}_{II} = 0$$

$$2\left[(xf)_x - f\right]\bar{f}_{III} + 5\bar{f}\bar{f}_{II} = 0$$

$$2\left[\bar{f}_x - \frac{1}{x}\bar{f}\right]\bar{f}_{III} + 5\bar{f}\bar{f}_{II} = 0, \tag{4.3.2}$$

which contains $\bar{f}$ only. Note the following transformations

$$
\begin{aligned}
\bar{L} &\equiv \bar{f}_{III} \\
\bar{L}_x &\equiv \bar{f}_{II} \\
\bar{L}_{xx} &\equiv \bar{f}_I \\
\bar{L}_{xxx} &\equiv \bar{f} \\
\bar{L}_{xxxx} &\equiv \bar{f}_x.
\end{aligned}
\tag{4.3.3}
$$

This enables us to eliminate $\bar{f}$ in (4.3.2) which then becomes

$$2\left[\bar{L}_{xxxx} - \frac{1}{x}\bar{L}_{xxx}\right]\bar{L} + 5\bar{L}_x\bar{L}_{xxx} = 0. \tag{4.3.4}$$

This is a nonlinear equation and more complicated than its counterpart (3.3.10). However it is still possible to solve it. We can write (4.3.4) in the form

$$2\left[\frac{\bar{L}_{xxxx}}{\bar{L}_{xxx}} - \frac{1}{x}\right] + 5\frac{\bar{L}_x}{\bar{L}} = 0, \tag{4.3.5}$$

which is separable. Integrating (4.3.5) leads to the integral given by

$$
\begin{aligned}
2\ln\left(\frac{\bar{L}_{xxx}}{x}\right) + 5\ln\bar{L} &= constant \\
\bar{L}_{xxx} &= C_0 x \bar{L}^{-5/2},
\end{aligned}
\tag{4.3.6}
$$

where C_0 is a constant of integration. Note (4.3.6) is similar to (3.3.11) in Chapter 3; however it is a different differential equation with a new solution.

We observe that the third order ordinary differential equation (4.3.6) together with the transformations (4.3.3), is equivalent to the integrability condition (4.2.10). Integrating (4.3.6) repeatedly we find $\bar{L}$, and then $\bar{f}(x)$ can be found using (4.3.3). Below we outline how (4.3.6) can be integrated giving $\bar{L}$.

The nonlinear differential equation (4.3.6) may be written as

$$\bar{L}\bar{L}_{xxx} = C_0 x\bar{L}^{-3/2},$$

$$(\bar{L}\bar{L}_{xx})_x - \bar{L}_x\bar{L}_{xx} = C_0 x\bar{L}^{-3/2},$$

which can also be written as

$$(\bar{L}\bar{L}_{xx})_x - \frac{1}{2}(\bar{L}_x^2)_x = C_0 x\bar{L}^{-3/2}.$$

Integrating the equation above, we get

$$\bar{L}\bar{L}_{xx} - \frac{1}{2}\bar{L}_x^2 = C_1 + C_0\int x\bar{L}^{-3/2}dx,$$

where C_1 is constant. This second order equation can be written in the form

$$x\bar{L}^{-3/2}\left(\bar{L}\bar{L}_{xx} - \frac{1}{2}\bar{L}_x^2\right) = C_1 x\bar{L}^{-3/2} + C_0 x\bar{L}^{-3/2}\int x\bar{L}^{-3/2}dx.$$

The equation above can also be written in the form

$$x(\bar{L}^{1/2})_{xx} = C_1 x\bar{L}^{-3/2} + C_0 x\bar{L}^{-3/2}\int x\bar{L}^{-3/2}dx, \tag{4.3.7}$$

where we have absorbed the factor of $\frac{1}{2}$ in C_0 and C_1. Observe that (4.3.7) is difficult to solve. However, note that we can write the left hand side as

$$x(\bar{L}^{1/2})_{xx} = \left[x(\bar{L}^{1/2})_x\right]_x - (\bar{L}^{1/2})_x,$$

so that (4.3.7) can then be written in the form

$$\left[x(\bar{L}^{1/2})_x\right]_x - (\bar{L}^{1/2})_x = C_1 x\bar{L}^{-3/2} + C_0 x\bar{L}^{-3/2}\int x\bar{L}^{-3/2}dx,$$

whose integral is given by

$$x(\bar{L}^{1/2})_x - \bar{L}^{1/2} = C_2 + C_1\int x\bar{L}^{-3/2}dx + \frac{1}{2}C_0\left(\int x\bar{L}^{-3/2}dx\right)^2, \qquad (4.3.8)$$

where C_2 is a new constant. The equation above is not in standard form. However, it is still possible to make progress. When multiplied by a factor $x\bar{L}^{-3/2}$, equation (4.3.8) above can be written as

$$\begin{aligned}\frac{1}{2}x^2\bar{L}^{-2}\bar{L}_x - x\bar{L}^{-1} &= C_2 x\bar{L}^{-3/2} + C_1 x\bar{L}^{-3/2}\int x\bar{L}^{-3/2}dx \\ &\quad + \frac{1}{2}C_0 x\bar{L}^{-3/2}\left(\int x\bar{L}^{-3/2}dx\right)^2.\end{aligned}$$

The left hand side can be written as a total derivative, and we have

$$\begin{aligned}\left(-\frac{1}{2}x^2\bar{L}^{-1}\right)_x &= C_2 x\bar{L}^{-3/2} + C_1 x\bar{L}^{-3/2}\int x\bar{L}^{-3/2}dx \\ &\quad + \frac{1}{2}C_0 x\bar{L}^{-3/2}\left(\int x\bar{L}^{-3/2}dx\right)^2.\end{aligned}$$

The integral of the equation above is given by

$$\begin{aligned}-\frac{1}{2}x^2\bar{L}^{-1} &= C_3 + C_2\left(\int x\bar{L}^{-3/2}dx\right) + \frac{1}{2}C_1\left(\int x\bar{L}^{-3/2}dx\right)^2 \\ &\quad + \frac{1}{6}C_0\left(\int x\bar{L}^{-3/2}dx\right)^3,\end{aligned}$$

where C_3 is constant. This can be simplified to

$$
\begin{aligned}
x^2\bar{L}^{-1} &= -2C_3 - 2C_2\left(\int x\bar{L}^{-3/2}dx\right) - C_1\left(\int x\bar{L}^{-3/2}dx\right)^2 \\
&\quad -\frac{1}{3}C_0\left(\int x\bar{L}^{-3/2}dx\right)^3.
\end{aligned}
$$

Redefining the constants in the above equation, we can write it as

$$
\begin{aligned}
x^2\bar{L}^{-1} &= \bar{C}_3 + \bar{C}_2\left(\int x\bar{L}^{-3/2}dx\right) + \bar{C}_1\left(\int x\bar{L}^{-3/2}dx\right)^2 \\
&\quad +\bar{C}_0\left(\int x\bar{L}^{-3/2}dx\right)^3, \qquad (4.3.9)
\end{aligned}
$$

where $\bar{C}_3 = -2C_3$, $\bar{C}_2 = -2C_2$, $\bar{C}_1 = -C_1$ and $\bar{C}_0 = -\frac{C_0}{3}$. Therefore the third order equation (4.3.6) has been integrated. In general we can write the solution parametrically.

For convenience, we let

$$u = \int x\bar{L}^{-3/2}dx,$$

so that

$$\frac{u_x}{x} = \left(\bar{L}^{-1}\right)^{3/2}, \qquad (4.3.10)$$

or equivalently

$$\bar{L}^{-1} = \frac{u_x^{2/3}}{x^{2/3}}. \qquad (4.3.11)$$

Substituting (4.3.11) in (4.3.9), we obtain

$$x^2u_x = \left(\bar{C}_3 + \bar{C}_2u + \bar{C}_1u^2 + \bar{C}_0u^3\right)^{3/2}.$$

In the above equation the variables separate, and we can write

$$x_0 - \frac{1}{x} = \int \frac{du}{\left(\bar{C}_3 + \bar{C}_2 u + \bar{C}_1 u^2 + \bar{C}_0 u^3\right)^{3/2}}, \tag{4.3.12}$$

where x_0 is constant. Now the function $\bar{f}(x)$ must be found satisfying the integrability condition (4.2.10). In order to find $\bar{f}(x)$ satisfying this integrability condition, it is convenient to express the solution in the parametric form

$$\begin{aligned} \bar{f}(x) &= \bar{L}_{xxx}, \\ u_x &= x\bar{L}^{-3/2}, \\ x_0 - \frac{1}{x} &= g(u). \end{aligned}$$

In the above we have defined

$$g(u) = \int \frac{du}{\left(\bar{C}_3 + \bar{C}_2 u + \bar{C}_1 u^2 + \bar{C}_0 u^3\right)^{3/2}}.$$

The evaluation of the integral is determined by the values of the constants $\bar{C}_0, \bar{C}_1, \bar{C}_2$ and $\bar{C}_3$. The above solution has five cases depending on the nature of the factors of the polynomial $\bar{C}_3 + \bar{C}_2 u + \bar{C}_1 u^2 + \bar{C}_0 u^3$. This is similar to the solution discussed in Chapter 3. The five cases are

Case I: One order-three factor

Case II: One order-one factor and one order-two factor

Case III: Three order-one (non-repeated) factors

Case IV: One linear factor and one quadratic factor

Case V: No factors

Note that the resulting solutions will be different as the third order equation (4.3.6) is not the same as the corresponding equation (3.3.11) of Chapter 3.

Case I: One order-three factor

This is the simplest case as the factors are repeated. If $\bar{C}_3 + \bar{C}_2 u + \bar{C}_1 u^2 + \bar{C}_0 u^3$ has one factor repeated three times then we can write

$$\bar{C}_3 + \bar{C}_2 u + \bar{C}_1 u^2 + \bar{C}_0 u^3 = (A + Bu)^3,$$

with $B \neq 0$. In this case, the integral in (4.3.12) can be evaluated to give

$$x_0 - \frac{1}{x} = -\frac{2}{7B}(A + Bu)^{-7/2},$$

so that

$$u = -\frac{A}{B} + \frac{1}{B}\left(-\frac{2}{7B}\right)^{2/7}\left(x_0 - \frac{1}{x}\right)^{-2/7}.$$

Now we use (4.3.11) to find $\bar{L}$ from $u(x)$, and hence we find $\bar{f}(x)$. We find that

$$u_x = \frac{1}{x^2}\left(-\frac{2}{7B}\right)^{9/7}\left(x_0 - \frac{1}{x}\right)^{-9/7},$$

and

$$\bar{L} = x^{2/3} {u_x}^{-2/3} = x^2 \left(-\frac{2}{7B}\right)^{-6/7}\left(x_0 - \frac{1}{x}\right)^{6/7}.$$

Differentiating $\bar{L}$ three times we obtain

$$\begin{aligned}
\bar{L}_{xxx} &= \frac{12}{7x^2}\left(-\frac{2}{7B}\right)^{-6/7}\left(x_0-\frac{1}{x}\right)^{-1/7} - \frac{12}{7x^2}\left(-\frac{2}{7B}\right)^{-6/7}\left(x_0-\frac{1}{x}\right)^{-1/7} \\
&\quad -\frac{12}{49x^3}\left(-\frac{2}{7B}\right)^{-6/7}\left(x_0-\frac{1}{x}\right)^{-8/7} + \frac{12}{49x^3}\left(-\frac{2}{7B}\right)^{-6/7}\left(x_0-\frac{1}{x}\right)^{-8/7} \\
&\quad +\frac{48}{343x^4}\left(-\frac{2}{7B}\right)^{-6/7}\left(x_0-\frac{1}{x}\right)^{-15/7}, \\
&= \frac{48}{343x^4}\left(-\frac{2}{7B}\right)^{-6/7}\left(x_0-\frac{1}{x}\right)^{-15/7}.
\end{aligned}$$

Now using the relation $\bar{L}_{xxx} = \bar{f}(x)$ we have

$$\bar{f}(x) = \frac{48}{343x^4}\left(-\frac{2}{7B}\right)^{-6/7}\left(x_0-\frac{1}{x}\right)^{-15/7}, \tag{4.3.13}$$

and eliminating the bar gives

$$f(x) = \frac{48}{343x^5}\left(-\frac{2}{7B}\right)^{-6/7}\left(x_0-\frac{1}{x}\right)^{-15/7}. \tag{4.3.14}$$

After reparametrisation, $f(x)$ can be written as

$$f(x) = \frac{1}{x^5}\left(1-\frac{1}{x}\right)^{-15/7}. \tag{4.3.15}$$

This gives a new solution to the master equation (3.2.5) for a shear-free fluid which is different to the solution (3.3.16) found by Stephani (1983). Now using the relations (4.3.3) we can write the first integral (4.2.11) in terms of x. We obtain the result

$$\begin{aligned}
\phi_1(t) &= y - xy_x + 2\left(-\frac{2}{7B}\right)^{-6/7}\left(x_0 - \frac{1}{x}\right)^{6/7} y^2 \\
&\quad + \frac{12}{7x}\left(-\frac{2}{7B}\right)^{-6/7}\left(x_0 - \frac{1}{x}\right)^{-1/7} y^2 \\
&\quad - \frac{6}{49x^2}\left(-\frac{2}{7B}\right)^{-6/7}\left(x_0 - \frac{1}{x}\right)^{-8/7} y^2 \\
&\quad - 4x\left(-\frac{2}{7B}\right)^{-6/7}\left(x_0 - \frac{1}{x}\right)^{6/7} yy_x \\
&\quad - \frac{12}{7}\left(-\frac{2}{7B}\right)^{-6/7}\left(x_0 - \frac{1}{x}\right)^{-1/7} yy_x \\
&\quad + 2x^2\left(-\frac{2}{7B}\right)^{-6/7}\left(x_0 - \frac{1}{x}\right)^{6/7} y_x^2 \\
&\quad + \left[\frac{192}{343x^4}\left(-\frac{2}{7B}\right)^{-12/7}\left(x_0 - \frac{1}{x}\right)^{-9/7}\right]_I y^3 \\
&\quad + \left[\frac{576}{2401}\left(-\frac{2}{7B}\right)^{-12/7}\left(x_0 - \frac{1}{x}\right)^{-16/7}\right]_I y^3 \\
&\quad - \frac{2}{3}K_5 y^3. \qquad (4.3.16)
\end{aligned}$$

The subscripts I in the equation (4.3.16) denote a remaining integration. It can be observed that this first integral is different from the first integral (3.4.2).

Case II: One order-one factor and one order-two factor

If $\bar{C}_3 + \bar{C}_2 u + \bar{C}_1 u^2 + \bar{C}_0 u^3$ has one factor repeated three times then we can write

$$\bar{C}_3 + \bar{C}_2 u + \bar{C}_1 u^2 + \bar{C}_0 u^3 = (A + Bu)(u + C)^2,$$

with $B \neq 0$. In this case, the integral in (4.3.12) can be evaluated to obtain

$$
\begin{aligned}
& g(u) = \\
& \left(\frac{15B^2}{4(A-BC)^3} + \frac{5B}{4u(A-BC)^2} - \frac{1}{2u^2(A-BC)} \right) \frac{1}{\sqrt{A+Bu-BC}} \\
& + \frac{15B^2}{8(A-BC)^3} \int \frac{du}{u\sqrt{A+Bu-BC}},
\end{aligned}
$$

where the integral can be expressed in terms of elementary functions depending on the sign of $A - BC$. For this case it is not possible to obtain the function $u(x)$ explicitly. Therefore the solution can only be given parametrically.

Case III: Three order-one (non-repeated) factors

If $\bar{C}_3 + \bar{C}_2 u + \bar{C}_1 u^2 + \bar{C}_0 u^3$ has three non-repeated factors, then we can write

$$
\bar{C}_3 + \bar{C}_2 u + \bar{C}_1 u^2 + \bar{C}_0 u^3 = D(A-u)(B-u)(C-u),
$$

with $D \neq 0$. In this case, the integral in (4.3.12) can be written in terms of elliptic integrals to obtain

$$
\begin{aligned}
g(u) \quad = \quad & \frac{2[C(A-C) + B(A-B) - u(2A-C-B)]}{D^{3/2}(A-B)(A-C)(B-C)^2\sqrt{(A-u)(B-u)(C-u)}} \\
& + \frac{2[(B-C)(A+B-2C)F(\alpha,\beta)]}{D^{3/2}(A-B)^2(B-C)^2\sqrt{(A-C)^3}} \\
& - \frac{2[(A^2+B^2+C^2-AB-AB-BC)E(\alpha,\beta)]}{D^{3/2}(A-B)^2(B-C)^2\sqrt{(A-C)^3}},
\end{aligned}
$$

where we have set

$$
\alpha = \arcsin\sqrt{\frac{A-C}{A-u}}
$$

and

$$
\beta = \sqrt{\frac{A-B}{A-C}}.
$$

In this form of the solution, the quantities $F(\alpha, \beta)$ and $E(\alpha, \beta)$ are elliptic integrals of the first and second kind, respectively. In this case of non-repeated factors, we also cannot obtain $u(x)$ explicitly and hence the solution can only be given in parametric form.

Case IV: One linear factor and one quadratic factor

If $\bar{C}_3 + \bar{C}_2 u + \bar{C}_1 u^2 + \bar{C}_0 u^3$ has one linear and one quadratic (irreducible) factor, then we can write

$$\bar{C}_3 + \bar{C}_2 u + \bar{C}_1 u^2 + \bar{C}_0 u^3 = (a+u)(cu^2 + du + e),$$

where $cu^2 + du + e$ has no real roots. Therefore we can write

$$g(u) = \int \left[\frac{1}{(a+u)(cu^2+du+e)} \right]^{3/2} du.$$

Using partial fractions it is possible to show that

$$\begin{aligned} &\frac{1}{(a+u)(cu^2+du+e)} \\ &= \frac{1}{(a+u)(ca^2-ad+e)} - \frac{cu+d-ac}{(cu^2+du+e)(ca^2-ad+e)}, \end{aligned}$$

so that

$$\begin{aligned} &g(u) = \\ &\int \left[\frac{1}{(a+u)(ca^2-ad+e)} - \frac{cu+d-ac}{(cu^2+du+e)(ca^2-ad+e)} \right]^{3/2} du. \end{aligned}$$

Further simplification of this integrand is not possible to express the integral above in terms of elementary functions. The solution can only be written in parametric form.

In **Case V**, the integral (4.3.12) can only be expressed using elliptic integrals. For this case, we also cannot obtain $u(x)$ explicitly. Therefore we can only express

$u(x)$, and hence $\bar{f}(x)$ explicitly only in the case where the polynomial $\bar{C}_3 + \bar{C}_2 u + \bar{C}_1 u^2 + \bar{C}_0 u^3$ has one order-three factor. For this case the function $f(x)$ has the explicit form given in equation (4.3.15).

Chapter 5

Conclusion

The aim of this dissertation was to investigate the integrability of and find a new class of exact solutions to the Emden-Fowler equation

$$y_{xx} = f(x)y^2,$$

which governs the behaviour of spherically symmetric shear-free uncharged fluids. This Emden-Fowler equation has several applications in general relativity and other areas of mathematical physics.

We now outline the contents of this dissertation and highlight the results obtained. In Chapter 1 we provided a general background on the history of Einstein's theory of general relativity as well as the field equations. We discussed some examples of known solutions to the field equations focusing on spherically symmetric spacetimes. In particular we were concerned with both shear-free and shearing spacetimes. We gave a brief history of the Emden-Fowler equation and the subsequent research in general relativity that has been done to obtain its solutions.

In Chapter 2 we discussed fundamental concepts of spacetime geometry. We began by discussing the concepts of differential manifolds as well as the metric tensor field and its line element. We introduced the concepts of connection coefficients, the Riemann tensor, the Ricci and Einstein tensors as well as the Ricci scalar. This led to the formulation of the Einstein field equations. We then studied these quantities for line elements of static and nonstatic spherically symmetric spacetimes. We considered both the shear-free and shearing spacetimes. For each metric, we gave the system of field equations and the condition of pressure isotropy.

In Chapter 3 we studied the integrability of

$$y_{xx} = f(x)y^2.$$

This chapter included a review of the research by Maharaj $et\ al$ (1996). We began this chapter by showing how the field equations for the shear-free nonstatic line element reduce to the Emden-Fowler equation. We then used integration by parts to obtain a first integral of $y_{xx} = f(x)y^2$, which is given by

$$\phi_0(t) = -y_x + f_I y^2 - 2f_{II} y y_x + 2f_{III} y_x^2 + 2\left[(ff_{II})_I - \frac{1}{3}K_0\right] y^3,$$

subject to the integrability condition

$$2ff_{III} + 3(ff_{II})_I = K_0.$$

This is an integral equation. It is interesting to observe that the integral equation may be transformed to the third order differential equation

$$L_{xxx} = K_1 L^{-5/2},$$

where $L \equiv f_{III}$. We solved this third order equation and expressed its solution in parametric form. A particular solution has the functional form

$$f(x) = x^{-15/7},$$

after reparametrisation. The first integrals of Stephani (1983) and Srivastava (1987) were regained from the first integral.

In Chapter 4 we first multiplied the Emden-Fowler equation by x to give

$$xy_{xx} = xf(x)y^2,$$

and investigated its integrability. This enabled us to find a new first integral. We used integration by parts to obtain the first integral given by

$$\begin{aligned}\phi_1(t) &= y - xy_x + \bar{f}_I y^2 - 2\bar{f}_{II} y y_x + 2\bar{f}_{III} y_x^2 \\ &+2\left[(f\bar{f}_{II})_I - \frac{1}{3}K_5\right] y^3,\end{aligned}$$

where $\bar{f}(x) = xf(x)$. This first integral is subject to the condition

$$2f\bar{f}_{III} + 3(f\bar{f}_{II})_I = K_5.$$

This is an integral equation which is different from the corresponding condition in Chapter 3. This integral equation can be transformed to the third order ordinary differential equation

$$\bar{L}_{xxx} = C_0 x \bar{L}^{-5/2},$$

where $\bar{L}_{xxx} = \bar{f}$. Remarkably the third order equation can be integrated to give

$$\begin{aligned}x^2\bar{L}^{-1} &= \bar{C}_3 + \bar{C}_2\left(\int x\bar{L}^{-3/2}dx\right) + \bar{C}_1\left(\int x\bar{L}^{-3/2}dx\right)^2 \\ &+\bar{C}_0\left(\int x\bar{L}^{-3/2}dx\right)^3.\end{aligned}$$

Then the new solution can be written parametrically in the form

$$\begin{aligned}\bar{f}(x) &= \bar{L}_{xxx}, \\ u_x &= x\bar{L}^{-3/2}, \\ x_0 - \frac{1}{x} &= \int \frac{du}{\left(\bar{C}_3 + \bar{C}_2 u + \bar{C}_1 u^2 + \bar{C}_0 u^3\right)^{3/2}}.\end{aligned}$$

It is important to note that a particular solution has the functional form

$$f(x) = \frac{1}{x^5}\left(1 - \frac{1}{x}\right)^{-15/7}.$$

This form is different to any of the solutions found previously. Note that the solutions of Stephani (1983), Srivastava (1987) and Maharaj $et\ al$ (1996) are not regained. Therefore our new first integral has different properties to the first integral of Chapter 3. It is possible that the new first integral is related to the geometrical structure of the Emden-Fowler equation. This will be the basis for future research.

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