EXPLORING GRADE 11 MATHEMATICS TEACHERS' PEDAGOGICAL CONTENT KNOWLEDGE WHEN TEACHING EUCLIDEAN GEOMETRY IN THE UMLAZI DISTRICT

BY

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DEDICATION

This work is dedicated

To:

My wife **Pearl**

My daughters **Mangaliso** and **Okuhle,**

AND

All my family members,

who are the original sources of my inspiration.

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All those whose names I might have omitted.

DECLARATION

I, Abednigo Sibusiso Nojiyeza, declare that this is my own and original work, submitted in partial fulfilment of the degree of Master of Education in Curriculum Studies, entitled Exploring grade 11 mathematics teachers' pedagogical content knowledge when teaching Euclidean geometry.

Abednigo Sibusiso Nojiyeza

Date

As the candidate's supervisor, I agree to the submission of this thesis/dissertation for submission.

Prof Jayaluxmi Naidoo

Date

CONTENTS

ABBREVIATIONS AND ACRONYMS

Abbreviation	**Description**
CAPS	Curriculum and Assessment Policy Statement
CK	Content knowledge
DBE	Department of Basic Education
DoE	Department of Education
EG	Euclidian Geometry
FET	Further Education and Training
GCK	Geometrical content knowledge
GET	General Education and Training
GPCK	Geometrical pedagogical content knowledge
HDE	Higher Diploma in Education
NCS	National Curriculum Statement
NPDE	National Professional Diploma in Education
NSC	National Senior Certificate
OBE	Outcomes Based Education
PCK	Pedagogical content knowledge
PGCE	Post Graduate Certificate in Education
PK	Pedagogical knowledge
RNCS	Revised National Curriculum Statement
SMK	Subject matter knowledge
STD	Secondary Teachers Diploma
TCK	Teacher content knowledge
TK	Teacher knowledge
TPCK	Teacher pedagogical content knowledge

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ABSTRACT

Teaching Euclidean geometry is a complex activity. From my personal teaching experiences, it appears that most grade 11 learners perform poorly in geometry, especially in Euclidean geometry. Challenges encountered by learners in learning Euclidean geometry affect their overall performance in mathematics. Often the mathematics teacher is regarded as the main contributing factor in mathematics learner performance. The knowledge that mathematics teachers possess and use in the classrooms when teaching Euclidean geometry is the instrument of change in learners' learning. Since the process of learning Euclidean geometry is mainly influenced by the teacher, it is therefore important to understand how teachers explain or demonstrate ways of solving Euclidean geometry problems to learners to help them understand.

This research study focused on exploring grade 11 in-service mathematics teachers' pedagogical content knowledge when teaching Euclidean geometry. It was intended to investigate the methods used by mathematics teachers in teaching Euclidean geometry. This study used the van Hiele levels of geometrical thinking as a theoretical framework to understand and explore teachers' pedagogical content knowledge of geometry. Purposive sampling was used to select three grade 11 mathematics educators from three conveniently sampled South African secondary schools within the Umlazi district in KwaZulu-Natal, South Africa. A qualitative research approach was used in this study. Data was collected through open and closed-ended questionnaires, semi-structured face-to-face interviews as well as lesson observations to gather descriptive data from the three participants, and it was coded with set categories.

This research is important within the South African context where the teaching and learning of geometry, especially Euclidean geometry, is still a challenge. Euclidean geometry is still difficult for many teachers to teach and for many learners to learn. The findings of this study may be useful to all secondary school mathematics teachers in improving the teachers' skill in teaching Euclidean geometry. This is because teachers would become aware about the factors that teaching methods can bring about, thus influencing learners' participation in lessons to improve their performance.

Keywords: Euclidean geometry, knowledge, learning, learner performance, mathematics, teaching.

CHAPTER 1

INTRODUCTIONAND BACKGROUND TO THE STUDY

1.1 Introduction

This introductory chapter presents the background of this study, problem statement and its context and objectives followed by the research questions that guided this study. It further highlights the rationale, significance, scope, limitations, key concepts, and layout of this study. It also highlights an overview of Euclidean geometry and South African curriculum and ends with a conclusion.

1.2 Background to the study

The decreasing number of learners choosing mathematics in secondary schools (Kennedy, Lyons, & Quinn, 2014) and the poor quality passes in secondary school mathematics over the years (Adler & Pillay, 2016) provided a justifiable rationale to conduct this study. According to Venkat and Graven (2017), most learners are hesitant to take mathematics or do not take mathematics in secondary schools because of their lack of confidence in their ability in doing mathematics, and lack of motivation (Zámková, Prokop, & Stolín, 2016). Over the last decade, ongoing low-learner performance in mathematics has led to an increasing interest in understanding how teacher pedagogical practices and content knowledge may figure within these patterns of poor performance (Taylor and Taylor, 2013).

Poor and outdated teaching methods (Khumalo, Molepo, & Mji, 2016), lack of learner motivation and interest (Cook, 2017) and lack of basic content knowledge (CK) are some of the factors associated with learners' poor performance in mathematics (Makgato & Mji, 2006; Mamba, 2012; Tachie & Chireshe, 2013). According to Jones (2000), the successful teaching of geometry depends on teachers' content knowledge of geometry as well as how to teach it effectively. A teacher's knowledge of how to teach content is referred to as pedagogical content knowledge (PCK) (Rollnick & Mavhunga, 2017). Teachers' knowledge and skills affects their interaction in the classroom (Fung et al., 2017). The knowledge that teachers possess and use in their classrooms is the instrument of change in students' learning (Gess-Newsome et al., 2016). To effectively deal with learners' challenges, teachers should have appropriate content knowledge as well as pedagogical content knowledge (Rollnick & Mavhunga, 2017). Teachers require a deep and broad knowledge of mathematics to be effective in their teaching (Hill, 2010).

Mathematics results in all grades in the schooling system indicate that the problems facing the teaching and learning of mathematics have not yet been properly addressed (Chikiwa, 2017). According to Okitowamba, Julie, and Mbekwa (2018), in South Africa the national mathematics average achievement at each grade across the schooling system ranges between 30% to 40%, or even lower. This has been a serious problem in the changing South African basic education system since democracy in 1994 (Pournara, Hodgen, Adler, & Pillay, 2015) and has been linked to many factors such as poor teaching facilities, learners' negative attitude towards mathematics, shortage of qualified mathematics teachers, and poor teaching methods (Dube, 2016). Despite extensive intervention programmes made by the department of education for both teachers and learners, mathematics learners' performance is still poor (Biyela, Sibaya, & Sibaya, 2016).

According to Bansilal (2017), the 2017 NSC diagnostic report shows that only 35.1 % learners passed with 40% or above, which means that less than 35.1 % learners qualified for entries into science and engineering related careers since the minimum required pass rate in mathematics to these fields is 50 % or more. This was a similar case to 2014 when Curriculum and Assessment Policy Statement (CAPS) was implemented in grade 12 for the first time. These results show that there has been little or no sign of improvement in Grade 12 mathematics learner performance for the past five years.

Euclidean geometry (EG) has been cited as one of the topics that is problematic for the majority of the learners, and teachers are said to be contributing factors in the poor performance of mathematics (Novak & Tassell, 2017). Euclidean geometry was removed from the syllabus in 2006 but has been compulsory in the CAPS since 2012. According to Steyn (2016), NSC mathematics diagnostic reports have indicated low learners' scores in the geometry questions, and Euclidean geometry forms one third of the mathematics paper two examination.

CAPS' focus is more on content and each topic contributes towards the acquisition of specific skills and Euclidean geometry is one of the ten main content topics taught in mathematics in the Further Education and Training phase (DBE, 2011). It is taught in term three in grades 10 and 11 for about three weeks. As stated in the CAPS document, Euclidean geometry forms part of mathematics paper two and weighs about 20 % in grade 10 and 33 % in grades 11 and 12 (DBE, 2011). Table 1.1 shows an overview of grade 10 to 12 Euclidean geometry.

Table 1.1 An overview of grade 10 to 12 Euclidean geometry.

Grade	Content overview
10	(i) Revise basic results established in earlier grades regarding lines, angles and triangles, especially the similarity and congruence of triangles. (ii) Investigate line segments joining the midpoint of two sides of a triangle. (iii) Properties of special quadrilaterals: the kite, parallelogram, rectangle, rhombus, square and trapezium. (iv) Investigate and make conjectures about properties of the sides, angles, diagonals and areas of these quadrilaterals. (v) Prove these conjectures. (vi) Solve problems and prove riders using the properties of parallel lines, triangles and quadrilaterals.
11	(i) Investigate and prove theorems of the geometry of circles assuming results for earlier grades as axioms, together with one other result concerning tangents and radii of circles. • The line drawn from the centre of a circle perpendicular to a chord bisects the chord; • The perpendicular bisector of a chord passes through the centre of the circle; • The angle subtended by an arc at the centre of a circle is double the size of the angle subtended by the same arc at the circle (on the same side of the chord as the centre); • Angles subtended by a chord of the circle, on the same side of the chord, are equal; • The opposite angles of a cyclic quadrilateral are supplementary; • Two tangents drawn to a circle from the same point outside the circle are equal in length; • The angle between the tangent to a circle and the chord drawn from the point of contact is equal to the angle in the alternate segment. (ii) Solve circle geometry problems, providing reasons for statements when required. (iii) Prove riders.

12	(i) Revise earlier (Grade 9) work on the necessary and sufficient conditions for polygons to be similar. (ii) Prove (accepting results established in earlier grades) • that a line drawn parallel to one side of a triangle divides the other two sides proportionally (and the mid-point theorem as a special case of this theorem); • that equiangular triangles are similar; • that triangles with sides in proportion are similar; • the Pythagorean theorem by similar triangles. (iii) Prove riders.

Adapted from *CAPS for the FET phase Grades 10 -12, p.14, 25-48*

1.2.1 An overview of South African curriculum changes

Many changes have been made within the South African education system since the birth of democracy in 1994. The first democratic curriculum review was Curriculum 2005 (C2005), which was driven by outcomes-based education (Legault, Green-Demers, & Pelletier, 2006). This curriculum was launched in 1997 and was to be implemented in 1998 from grade 1 to 12 by the year 2005. Its main purpose was to completely change the apartheid education system. According to Khuzwayo and Mncube (2017), this marked a remarkable shift from content-based teaching and learning to a learner-centred approach. Outcomes Based Education (OBE), as part of Curriculum 2005, moved learners away from rote learning or memorising to a system that taught them to be critical thinkers.

The second review led to the National Curriculum Statement (NCS). According to Goos, Vale, and Stillman (2017) this started as the Revised National Curriculum Statement (RNCS) Grades R to 9, launched in 2004, as a second approach to address the problems of C2005. Mathematics became compulsory from grade 10, and later mathematical literacy was introduced in grade 10 where learners had to choose between mathematics and mathematical literacy (Dube, 2016). Geometry was not compulsory in mathematics in the FET phase as it was included in the optional paper three.

Another review was conducted in 2009, which led to the Curriculum Assessment Policy Statement (CAPS), the third curriculum approach, which has been actively implemented in schools since 2012 in grade 10, 2013 in grade 11 and 2014 in grade 12 (Goos et al., 2017). Under CAPS, mathematics content is emphasised more than learning outcomes and all optional topics including geometry become compulsory (Dube, 2016). According to Weber (1990), these three curricula reformations brought in changes in content topics, policy instructions and theories and teachers were expected make pedagogical practical changes to conform to the current policy. But it remains questionable whether the implementation of any change in curriculum is effective or not. Do teachers, as drivers of curriculum implementation, get prepared enough to successfully implement curriculum changes?

1.3 Problem statement

The poor quality of mathematics achievement by South African learners is well known (Pournara et al., 2015). It is documented that South Africa has been participating in various local and international mathematics assessment studies over the years to evaluate and monitor the quality of its education system, and according to these studies, South African learner performance in mathematics, throughout all grades, is not adequate (Chikiwa, 2017). The teaching and learning of mathematics is faced with serious challenges in certain content topics, especially Euclidean geometry in the Further Education and Training (FET) phase, and this causes mathematics learner performance to be poor (Herbst, Fujita, Halverscheid, & Weiss, 2017). It is well known that many secondary school learners perceive mathematics, especially Euclidean geometry, as a difficult and demanding subject to learn and understand (Schultze & Nilsson, 2018). According to Gresham (2018), the way Euclidean geometry is communicated by mathematics teachers in the classroom is one of many reasons for this.

According to Tsao (2017), learners' geometry achievement has always been lower than other areas of mathematics and it has also been noted that many learners shy away from learning geometry because of the negative attitude and their poor performance in geometry. This inadequate learner attainment is claimed to be caused by teachers' poor knowledge of mathematics content and poor teaching methods (Pournara et al., 2015). According to Chikiwa (2017), this continuous low-learner performance in mathematics has attracted a number of researchers seeking to investigate and understand how mathematics teachers' pedagogical practices and content knowledge contribute to these poor patterns oflearner performance in mathematics.

1.4 Research objectives

This study was intended to attain the following objectives:

- To explore pedagogical content knowledge of grade 11 mathematics teachers when teaching Euclidean geometry.
- To explore how grade 11 mathematics teachers apply their pedagogical content knowledge of Euclidean geometry when teaching this section.
- To explore why grade 11 mathematics teachers use their pedagogical content knowledge for teaching Euclidean geometry in the way that they do.

1.5 Research questions

Data generation in this study is aimed at responding to the following three critical research questions:

1. What is grade 11 mathematics teachers' pedagogical content knowledge of Euclidean geometry?

2. How do grade 11 mathematics teachers use their pedagogical content knowledge when teaching Euclidean geometry?

3. Why do grade 11 mathematics teachers' use their pedagogical content knowledge for teaching Euclidean geometry in the way that they do?

1.6 Significance of the study

The findings from this study seek to contribute to possible mathematics intervention strategies through investigating the teachers' pedagogical content knowledge of mathematics when teaching grade 11 Euclidean geometry in their classrooms. Mathematics teachers, student teachers of mathematics, mathematics teacher trainers and, to some extent, mathematicians and scholars, are likely to benefit from this research study.

1.7 Scope and limitations of the study

The study was conducted within the Umlazi District in the Phumelela Circuit in KwaZulu-Natal. Three in-service teachers were selected from three schools, school A, school B and school C. Each teacher selected for this study was required to be teaching grade 11mathematics and have a teaching qualification to teach mathematics in the Further Education and Training (FET) phase.

1.8 Key concepts discussed in this study

1.8.1 Mathematics teachers

Mathematics teachers in this study are full-time teachers teaching Grade 11 mathematics and in possession of a mathematics teaching qualification in the Further Education and Training (FET) phase. According to Ghousseini and Herbst (2016), teaching goes beyond reciting or reading to learners in a classroom situation. It involves incorporation of instructions, content understanding and a great deal of pedagogical knowledge, and it is basically a thinking practice. They continue that it does not only require pedagogical changes but rather the integration of policies, theories, and reasoning by teachers. Effective teachers have the knowledge necessary for good teaching and have a clear content knowledge of the subject they teach (Miller, 2017).

1.8.2 Geometry versus Euclidean geometry

Geometry is defined as the mathematical study of space. "Euclidean geometry is a mathematical system associated with Euclid, which he described in his geometry textbook called '*The Elements*'"(Steyn, 2016, p. 7). According to Yixuan, Lei, Peng, and Jinhong (2016), Euclidean geometry is the learning of formal axiomatic systems where learners write proofs and solve riders.

1.8.3 The teaching and learning of Euclidean geometry

According to Sulistiowati, Herman, and Jupri (2019) the teaching and learning of Euclidean geometry in South Africa has been identified as one of the topics that is a challenge for both teachers and learners, and this observation suggests the need for an urgent intervention to seek an alternative approach to the teaching and learning of Euclidean geometry (Jojo, 2015). Furthermore, teaching and learning of Euclidean geometry is in many ways similar to the teaching and learning of a new language, introducing learners to new words in unfamiliar and familiar contexts (Gresham, 2018). According to Dube (2016), the NCS era tried to address this problem by pushing for geometry to be written optionally by the learners in the FET phase and most teachers stopped teaching any geometry, since it was optional in grade 12. This implied that engineering learners and other science field learners lacked knowledge of geometry at tertiary institutions, and tertiary institutions had to introduce this concept from scratch (Jojo, 2015). CAPS corrected this mistake by reinstating the teaching and learning of geometry in 2011 and this again was a huge challenge for both teachers and learners (Dube, 2016).

1.8.4 Content knowledge (CK)

In this study, CK refers to the teachers‘ knowledge about specific content and this includes knowledge of correct interpretation and application of mathematical concepts, facts, procedures and principles, ideas and theories (Ramatlapana & Berger, 2018).

1.8.5 Geometrical content knowledge (GCK)

In this study GCK refers to the teacher‘s ability to relate to geometrical diagrams with their properties, and to prove and apply theorems when solving geometrical problems (Alex & Mammen, 2018).

1.8.6 Teachers’ pedagogical content knowledge (PCK)

Pedagogical content knowledge was developed on the basis of Shulman’s ideas and it refers to the teachers’ deep knowledge of the specific content to be taught, *how* it is learned and *how* best it can be taught (Tambara, 2015). This knowledge goes far beyond content knowledge (Chidziva, 2017), and according toRamatlapana and Berger (2018), PCK includes knowledge about methods and techniques used in the classroom, knowledge of the learners and knowledge of strategies used to evaluate their understanding. In this study, I refer to this as the teachers’ geometrical pedagogical content knowledge (GPCK).

1.8.7 The theoretical framework

This study uses the van Hiele Theory of Levels of Thought in Geometry as the theoretical framework for framing the study. The van Hiele theory is more pedagogical than psychological, and it has contributed largely in the learning and teaching of geometry (Van Putten, Howie, & Stols, 2010).

1.9 The structure of the study

This study consists of six chapters structured as follows:

Table 1.2 The structure of the study.

Chapter	
1	Provides the introduction and background to the study
2	Reviews the literature
3	Reviews the theoretical framework used as a lens for this research study
4	Describes the research design and methodological approach adopted in my study
5	Presents raw data and various methods used to analyse and interpret data
6	Concludes the research study by discussing the findings and possible recommendations of the study

1.10 Conclusion

In this introductory chapter, I provided an overview to the background of the study about poor learner performance in mathematics, objectives, problem statement, research questions that guided this study, rationale, significance of the study, scope and limitations, explained key terms and the structure of this study was also highlighted. In the next chapter, the literature review will be presented and discussed.

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

The teaching of mathematics, especially geometry, has proved to be a serious challenge to most in-service teachers where a large percentage of teachers have minimal geometry pedagogical knowledge (Sinclair & Bruce, 2015). The previous chapter outlined the background history to the study, showing an overview of Euclidean geometry in the Further Education and Training (FET) phase. This chapter further discusses the background history of geometry and also presents curriculum changes in the South African education system. Mathematics teachers' knowledge and the van Hiele theory of learning geometry are also discussed in this chapter.

2.2 South African curriculum reviews

The need to provide basic education for all is one of the main objectives in the transformation of the South African education system (Ramnarain, 2016). This objective is expressed in the current South African Curriculum and Assessment Policy Statement (CAPS) document where the sixth specific mathematics aim states that: "the purpose of mathematics is to promote accessibility of mathematical content to all learners" (DoE, 2011c, p. 7), and one of the ten main content areas of mathematics in the FET curriculum is Euclidean geometry. To address this objective, based on continuous curriculum reviews in South Africa, the Department of Basic Education has re-introduced the teaching and learning of Euclidean geometry. This section was examined in mathematics paper two as of 2012.

The implementation of the CAPS in 2012 brought about changes in mathematics curriculum. Topics that were excluded in the Outcomes Based Education (OBE) curriculum, such as Euclidean geometry and probability, were now reintroduced in the CAPS (Herbst et al., 2017). Thus, since 2012, the teaching and learning of Euclidean geometry (with its formal proofs) became compulsory in mathematics in the Further Education and Training (FET) phase as indicated in the CAPS document (Brijlall, 2017). CAPS is more prescriptive and structured, and has replaced OBE. The CAPS focuses on subject knowledge and excellence and also demands higher standards of learner performance (Maddock & Maroun, 2018).

In South Africa, research into the teaching and learning of geometry is ongoing (Roberts, 2002) as most teachers still find it difficult to teach this topic and learners find it difficult to learn geometry more so, in Euclidian geometry, where learners are expected to prove theorems and riders. This challenge experienced by learners may be due to ineffective teaching (Fosnot, 2013). According to Novak and Tassell (2017), the understanding of geometric concepts and geometric proofs is still problematic for many teachers and they conclude that both teachers and learners consider geometry to be a challenging topic in high school mathematics. In this regard, this study may be useful as it seeks to explore grade 11 mathematics in-service teachers' pedagogical content knowledge when teaching Euclidean geometry. Euclidean geometry is taught in grades 10, 11 and 12 and according to DoE (2011c), grade 11 learners are expected to: (a) investigate and prove theorems of the geometry of circles assuming results from earlier grades, together with one other result concerning tangents and radii of circles; and (b) solve circle geometry problems, providing reasons for statements when required. Some of these theorems will be discussed later on in this chapter.

2.3. Weighting of content within mathematics paper two

Weighting of content is an essential part of mathematics curriculum and assessment which must provide equal benefits to the learners regardless of their prior mathematics background, and this needs to be a motivating factor for the level of learner commitment (Franke, 2018). According to Butler (2018), content weightings forms part of teaching, learning and assessing, and mathematics content weightings within the curriculum may either be motivating or stressful for learners.

The weighting of mathematics content areas in the CAPS firstly gives teachers guidance on the amount of time needed to adequately address the content area, and secondly guidance on the spread of content in the examination, especially the end of year summative assessment (DoE, 2011c). This study focuses on grade 11 mathematics teachers' pedagogical content knowledge when teaching Euclidean geometry. Grade 11 Euclidean geometry weighs about 33% of the mathematics paper two content areas in the final examination, and 55% in the mid-year examination. Questions assess the performance of learners at different cognitive levels with an emphasis on critical thinking, process skills, strategies and scientific reasoning in solving and investigating problems in a variety of contexts (DoE, 2015). Table 2.1 shows the weightings of mathematics paper two content areas in the FET phase with a special focus on Grade 11 Euclidean geometry.

Table 2.1 Weighting of content areas in the end-of-year examination

Content areas	**Grade 10**	**Grade 11 Final examination**	**Grade 11 Mid-year examination**	**Grade 12**
Statistics	15 ± 3	20 ± 3	n/a	20 ± 3
Analytical geometry	15 ± 3	30 ± 3	45±3	40 ± 3
Trigonometry	40 ± 3	50 ± 3	n/a	40 ± 3
Euclidean geometry	**30 ± 3**	**50 ± 3**	**55±3**	**50 ± 3**
Total	100	150	100	150

Adapted from *(DoE, 2011c, p. 8)*

2.4 Background history of geometry

According to Kösa (2016) geometry is one of the mathematics topics that deals with points, lines, planes, space and spatial figures. Geometry is considered a rich topic in mathematics which helps learners make sense of the world and promotes learner critical thinking skills, deductive reasoning ability and logical argument (Jupri, 2017). Furthermore, Oflaz, Bulut, and Akcakin (2016) pointed out that geometry helps learners to efficiently solve problems they face in their daily lives. Thus, teachers' knowledge and understanding of geometry becomes crucial (Kovács, Recio, & Vélez, 2018).

The word geometry originally comes from the Greek word "geometrien" where "geo' means "place" and "metric" means" measure" (Hendricks & Adu, 2016, p. 1). This was the first field of science to study the sizes of objects (Zámková et al., 2016). The earliest geometers were Egyptian surveyors who built property boundaries and large buildings in shapes like squares, circles, triangles and rectangles which was not easy to do by eye (Van Manen, 2016, p. 11). McAndrew, Morris, and Fennell (2017) wrote that 'studying geometry focuses on the learning and comparing of different shapes.' Geometry has a long history starting from practical measurement of land in the old Egyptian days and studying properties of different geometric shapes in Greece.

The teaching and learning of geometry was only for high school learners until in the 1960s where it was introduced in primary schools (Fuller, Deshler, Darrah, Trujillo, & Wu, 2016). According to Fomunyam (2016, p. 3), geometry remains a prerequisite skill when studying fields such as geology, physics, architectural design, art, astronomy, mechanical drawing and in various areas involving construction. These fields are closely linked to learners' background of geometric understanding and are critical for any country to develop (Oflaz et al., 2016). Through geometric knowledge, learners are able to analyse, describe, and understand the world in which they live in as well as equip them with necessary tools which can be applied in other areas of mathematics (Kovács et al., 2018). It is, therefore, for this reason that learners study geometry although there are many challenges in its teaching and learning. According to Novak and Tassell (2017), geometry became an essential section of the mathematics curriculum in high schools although it is still regarded as problematic not only in South Africa but around the world and recent research has shown that the interest in the teaching and learning of geometry remains a topic of discussion.

2.5 Background history of Euclidean geometry

According to Hendricks and Adu (2016, p. 4), the most famous name associated with Greek geometry is that of Euclid who is believed to be the first mathematician who wrote geometry definitions, axioms and postulates based on points and lines, which laid the foundation of what we now call Euclidean geometry. As geometric knowledge grew, these mathematicians saw the need to approach geometry in a more systematic way, and this resulted in a breakthrough in Greece around 300 BC with the publication of Euclid's Elements (Fomunyam, 2016), which summarised and synthesised most of what was known about geometry in Greece at the time (Van Manen, 2016, p. 11).

According to Van Manen (2016, p. 12), Euclid's Elements consist of propositions and definitions in a book called "Elements" which is a good reference even up to this day. King (2018, p. 31), has highlighted some of Euclid's Elements from this book:

- Definition 1: "Equal circles have equal diameters or equal radii".
- Proposition 13: "If a straight line is vertically drawn to meet another straight line, then a right angle is formed or two right angles are formed".
- Definition 4: "Straight lines inside a circle are equally distant from the centre if perpendiculars drawn to them from the centre are equal".
- Proposition 6: "If two straight lines are perpendicular to the same plane, then the straight lines are parallel".

These elements are now a central pillar in geometry and are essential within geometry education (Clark & Worger, 2016, p. 2).

2.6 Basic education as a human right

"Access to a basic education is one of the fundamental human rights in the post-apartheid South Africa, included in the country's constitution" (Steffe et al., 2016, p. 9). According to Maharajh, Nkosi, and Mkhize (2016, p. 115) "this right is viewed as the most important right by which South Africans can be taken out of poverty, but this right can only be meaningful if all schools meet minimum standards of adequacy." However, the majority of South African schools are poorly resourced and their teachers, especially mathematics teachers, have poor teaching qualifications, most schools are largely dysfunctional and learner performance is generally poor (Ab Kadir, 2017). Based on research, better resourced schools which cater for the wealthy minority have good quality mathematics teachers with higher mathematics qualifications.

2.7 Curriculum changes in the post-apartheid South African education

Mathematics teaching and learning in post-apartheid South Africa has been faced with curricula reforms and reviews since the advent of democracy in 1994, thus exposing teachers and learners to numerous changes in content topics, policy instructions and theories and each of these curriculum reformations demanded changes in pedagogical practices from teachers (Cobbinah & Bayaga, 2017). Changes in curriculum involve either the removal of topic(s) or the addition of new topic(s) and changes in assessment criteria. In the case of new topic(s) being added, educators usually feel uncertain about their level of content knowledge (Phasha, 2016). Phasha (2016) suggests that within the South African education system there have been many unsuccessful changes, unsuccessful in the sense that new curricula are introduced and implemented in a short period of time while teachers, as drivers of curriculum change, are still trying to adapt to change.

NATED 550 curriculum (known as the old curriculum) was characterised by being educator-centred (Phasha, 2016). This curriculum saw teachers as the only source of knowledge and learners were recipients of knowledge or knowledge absorbers during the teaching and learning process (Cobbinah & Bayaga, 2017). According to Brinkmann (2014), curriculum was content led, biased, unrelated to learners' experiences, and classrooms were tense and strictly teacher-centred. This was subjected to criticism by many scholars and there was a need to transform the curriculum (Cobbinah & Bayaga, 2017).

This old curriculum was replaced by Curriculum 2005 with its Outcomes-Based Education (OBE) approach, the first version of the post-apartheid National Curriculum Statement (NCS) (Phasha, 2016). This was an important development in the post-apartheid era, where learners moved away from rote learning to a system that taught them to think critically (Borko & Livingston, 1989). The responsibility for learning resided increasingly on learners constructing their own understanding, because teachers as facilitators was emphasised as a means to minimise teacher inputs and maximise learner participation during teaching and learning (Cobbinah & Bayaga, 2017).

The recent curriculum change that is being implemented in South Africa is the Curriculum and Assessment Policy Statement (CAPS) (Phasha, 2016). Teaching under the CAPS seems not to differ from the NCS, it is an amendment of the NCS, and teaching is poised to deal with concerns that emanated from NATED 550 and NCS (Cobbinah & Bayaga, 2017). According to Hendricks and Adu (2016), in the CAPS the focus is on the role of the teacher as the primary curriculum implementer. Some topics that were removed previously have now been reinstated, and this puts teachers in the uncomfortable position of having to learn new topics (Phasha, 2016).

According toUgorji and Alfred (2017), these changes in the South African school curriculum brought many instructional challenges to teachers, especially secondary school mathematics teachers. They continue that in the NCS, some topics, including Euclidean geometry, were optional and the learners were required to write a separate examination paper (Mathematics paper 3) on the topics in addition to the compulsory paper 1 and paper 2. In the NCS, the majority of learners that opted for paper 3 between 2009 and 2013 did not do well in the examination, and as a result, most schools discouraged their learners from opting for paper 3 (Ugorji & Alfred, 2017). The implementation of CAPS in 2012 brought in sections which were not previously found in OBE curriculum such as Euclidean geometry and many of the under- or unqualified teachers in under-resourced schools had never studied these topics during their own schooling nor during their own teacher training (Herbst et al., 2017).

2.8 Mathematics and the mathematics curriculum within the South African schooling system

Mathematics is a language that makes use of symbols and notations for describing numerical, geometric and graphical relationships; it is a human activity that involves observing, representing and investigating patterns and quantitative relationships in physical and social phenomena and between mathematical objects themselves (Darling-Hammond et al., 2015). According to Chang and Sian (2016), mathematics helps develop mental processes that enhance logical and critical thinking, accuracy and problem-solving that contribute to decision-making.

According to Sharma (2016), mathematics education is one of the priorities highlighted in the South African National Development Plan (NDP) which indicates the demand for an increase in the number of learners achieving 50% and above in mathematics, although this faces many challenges. The secondary school mathematics curriculum prepares the learners' transition into higher education with minimum skills and knowledge required to function, as well as in professions such as engineering, medicine or business (Venkat & Graven, 2017).

Mathematics in the schooling system is a compulsory subject for all learners in South Africa in the General Education and Training (GET) band (Grades R-9) but in the Further Education and Training (FET) band (Grades 10-12), learners choose between mathematics and mathematical literacy (Steffe et al., 2016). According to Walshaw (2017) all learners entering grade 10 have to select which of these two mathematics curricula they will follow for grades 10-12. He also argued that most learners chose mathematical literacy instead of mathematics within the FET band and just prior to grade 12. Sharma (2016) has stated that teachers' lack of mathematical content knowledge and the skills required to apply what they know in the classroom may cause most learners to drop mathematics.

According to Maharajh et al. (2016), the mathematics curriculum may be seen as a set of legally prescribed standards that guides the teaching and learning of mathematics. In South Africa the National Curriculum Statement (Grade R-12) represents a policy statement for learning and teaching in schools and consists of the Curriculum and Assessment Policy for each subject (NCS CAPS FET Mathematics document). Curriculum may be differentiated into the intended and the enacted; the former refers to the curriculum structure and design, i.e., what the designers planned to accomplish, and the latter to how that design is implemented and its effects on learning and learners where teachers play an active role through their own pedagogical skills (Shay, Wolff, & Clarence-Fincham, 2016).

2.9 South African mathematics teachers

We need to explore why mathematics teachers in public schools who are better trained, better qualified, and equipped with better content knowledge, are not leading to better learning outcomes for learners (Tella, 2017). According to Barendsen and Henze-Rietveld (2017), it is what mathematics teachers '*believe and do*' in the classroom that has a huge impact on mathematics learner performance. Having only a good subject knowledge may not necessarily mean good teaching practice (Simsek & Boz, 2016).

Each country has differentminimum requirements that are needed to become a qualified secondary school mathematics teacher (Leikin, Zazkis, & Meller, 2018). According to Tarling and Ng'ambi (2016), in South Africa a high school mathematics teacher must hold a mathematics or any mathematically-related degree followed by one year of teacher education gained in a Post Graduate Certificate in Education (PGCE) course. Teacher research indicates a shortage of qualified and experienced mathematics teachers and in many schools mathematics is taught by teachers who struggle with both the content they teach and how learners learn (Dube, 2016). Most South African mathematics teachers still lack fundamental understanding of mathematics and pedagogical practices (Mamlok-Naaman, 2017).

Teacher qualification is a special skill or a special type of experience or a particular knowledge a teacher possesses in order to teach effectively (Santagata & Sandholtz, 2018). Crossfield and Bourne (2017), on the other hand, suggest that improved qualifications might increase teaching effectiveness. According to Tella (2017), teachers with higher teaching qualifications have an in-depth knowledge of content and are more likely to encourage and guide learners, as they know how to plan productive lessons and how to diagnose learner problems. They have significantly higher learner outcomes than those with lower or no teaching qualifications (Handal, Watson, & Maher, 2015).

According to Gokalp (2016), a teaching qualification in South Africa is a requirement in both primary and secondary education. To be a qualified secondary school teacher, a teacher may have any of these teaching qualifications but is not limited to Postgraduate Certificate in Education (PGCE), The National Professional Diploma in Education (NPDE), Higher Diploma in Education (HDE), Secondary Teachers Diploma (STD) and a Bachelor of Education (B.ED) (Tella, 2017). Furthermore, Handal et al. (2015) suggest that level of education, years of teaching experience and on-going professional development are some of the ways in which teacher qualification can be quantified.

Teachers with poor mathematics background and lack of mathematical pedagogical content knowledge may have challenges towards teaching mathematics at any grade. The quality of mathematics teachers and the quality of teaching are among the most important factors that shape the learning and growth of learners (Ajibade, 2016). Grobler, Moloi, and Thakhordas (2017, pp. 34-35) stated that "in South Africa, teacher's poor content knowledge and outdated teaching practices are some of the reasons for learners' poor performance in mathematics." This poor learner performance has been exacerbated by the large number of unqualified and inexperienced mathematics teachers (Hendricks, Botha, & Adu, 2016).

Teachers are key mathematics ''resources'' and mathematics agents in all schools, and particularly in ''schools for the poor'' and for working-class learners where teaching resources are still a challenge (Steffe et al., 2016, p. 16), and, therefore, it is obvious that mathematics teachers must possess a professional knowledge base and exhibit knowledge of the subject matter (Tella, 2017). According to Chan and Yung (2018), high-quality teachers can affect learner achievement positively. For mathematics teachers to be effective, they must have the knowledge necessary for good teaching and they must also know how to translate their subject knowledge into effective pedagogy and then apply this in the classroom (Bold, Filmer, Martin, Molinad, et al., 2017).

Liu and Li (2017) state that teachers also deal with different textbooks and learning materials that have already reviewed mathematical contents, and they must be able to see if these reviewed contents are appropriately transformed. To find a way of formulating a mathematics concept so that it is understood by the learners requires that teachers must understand the concept very well (Grobler et al., 2017). Successful mathematics teachers have a vast range of instructional strategies and techniques that reflect their knowledge of the subject (Tella, 2017). According to Marshall, Thomas, and Robinson (2016, p. 433), every learner learning mathematics must have a qualified and competent teacher in their respective mathematics classrooms for better performance.

According to Bietenbeck, Piopiunik, and Wiederhold (2017), teacher subject knowledge has a positive and significant impact on learner performance. Grobler et al. (2017) point to the problem of poor mathematics content knowledge of mathematics teachers in South Africa and state that most mathematics teachers struggle with the content they teach. Teacher knowledge of mathematics is necessary but it is not enough for effective and quality teaching, as a teacher needs to make his or her mathematics knowledge understandable for learners (Bold, Filmer, Martin, Molinad, et al., 2017). According to Tella (2017), teachers are those that consciously reflect upon, conceptualize, and apply understandings from one classroom experience to the next. There is a positive relationship between teacher knowledge and learner performance, and learners taught by teachers with higher content knowledge - specifically pedagogical content knowledge (PCK)- will always perform better than those taught by teachers who lack PCK (Arends, Winnaar, & Mosimege, 2017). Shuls and Trivitt (2015) note that good teachers have a thorough knowledge of the learners they teach, a sound content knowledge appropriate to the learners' level they teach, and rich knowledge of how learners learn mathematics. Teachers need to justify their practices in terms of improving learner performance (Goos, 2016).

2.10 The nature of a mathematics classroom

The nature of a mathematics classroom directly affects the nature and the level of learners' learning (Hiebert & Grouws, 2007). Teachers play a major role in creating an environment conducive for teaching and learning as proposed by Burden (2016). When the classroom environment and pedagogical practice are brought in line with the lived experiences of the learners, learners' performance is influenced positively (M. L. Franke, Kazemi, & Battey, 2007). According to Wilson-Patrick (2016, pp. 419-459), teacher knowledge "directly and positively affects classroom practice" and subsequently learner achievement. Mathematics teachers' limited content knowledge and ineffective teaching methods can affect learner performance (Kriek & Grayson, 2009), Reddy et al. (2016) further add to this list non-completion of the syllabus and lack of resources.

Effective classroom practice is critical for promoting learners' conceptual understanding of mathematics (Copur-Gencturk & Papakonstantinou, 2016). However, according to Ghousseini and Herbst (2016), secondary school mathematics classroom practice in the South African education system has a challenge of creating a mathematics learning environment where learners are actively engaged with mathematics and where the needs of diverse learners are met. Furthermore, as teachers' knowledge of mathematics increases, their classroom practice will change, and hence it will be more likely to create a more welcoming and a positive classroom environment (Goos, 2016). Instructional methods employed in mathematics classrooms need to meet the needs of all the learners, regardless of their background, gender or race, so that their performance may improve (Schettino, 2016).

2.11 Mathematics learner performance in South Africa

Poor learner performance in mathematics in South Africa is said to be influenced by many factors (Gafoor & Kurukkan, 2015). A wide-ranging body of assessment analyses points to poor mathematics learner performance across all levels of the South African schooling system (Venkat & Spaull, 2015), and these analyses rangefrom classroom observation and small-scale localised studies to national assessments of mathematics, such as TIMSS and SACMEQ (Kuhn, Alonzo, & Zlatkin-Troitschanskaia, 2016). According to Alfifi and Abed (2017), research has also indicated that mathematics teacher qualifications and teaching experience may be one of the factors causing learners to poorly perform in mathematics, especially in Euclidean geometry.

Hence, according to Harwell et al. (2015), this ongoing poor learner performance has led to increasing concern in understanding how teacher qualities, content knowledge and pedagogical practices may shape these patterns of poor learner achievement. Common findings across research studies relatethatmany South African mathematics teachers lack fundamental mathematics knowledge (Elizarov, Zhizhchenko, Zhil'tsov, Kirillovich, & Lipachev, 2016). According to Pournara and Barmby (2019), several attempts across the world have been made to improve mathematics teachers' content knowledge in order to increase learners' mathematical attainment. However, there is little evidence in South Africa that these interventions have had much impact on learners' mathematics performance, and the impact problem is usually linked to teachers' poor mathematical knowledge (Heeralal & Dhurumraj, 2016).

The National Senior Certificate in South Africa is a standardised assessment whose main function is to determine whether Grade 12 learners have mastered subject knowledge at the culmination of their secondary education and the condition in which the examination is administered is the same for all Grade 12 learners in South Africa (le Roux & Sebolai, 2017). This examination is a critical indicator of measuring quality basic education in the South African schooling system (NSC Examination Report: 2017, p. 3). According to Vadachalam and Chimbo (2017), very few learners enrol for mathematics at NSC level and many of these do not pass mathematics and these few passes are of poor quality.

It is well known that the majority of South African learners underperform in both national and international mathematics assessment instruments and many claim that one of the causes of this underperformance is teachers' weak knowledge of mathematics especially in geometry (Mogashoa, 2016). According to Beane (2016), South African learners across all grades continue to underperform in mathematics when compared to their counterparts globally and nationally. Trends in mathematics and science study (TIMSS) data continue to show very low mathematics performance among South African learners internationally (Hendricks et al., 2016).

Although the number of learners who wrote mathematics in grade 12 declined in 2017, learner performance in mathematics remains poor, and teachers are often reported to be part of this learner underperformance in mathematics (Voogt, Pieters, & Handelzalts, 2016). Table 2.2 shows the overall South African learner achievement in grade 12 mathematics for the years 2013-2017:

Table 2.2: South African grade 12 learners' performance in mathematics

Year	No. Wrote	% achieved at 30% and above	% achieved at 40% and above
2013	241 509	59.1	40.5
2014	**225 458**	**53.5**	**35.1**
2015	263903	49.1	31.9
2016	265810	51.1	33.5
2017	245103	51.9	35.1

Source*: 2013-2017 NSC Diagnostic reports (DBE, 2017, p. 153)*

Since 2014, when CAPS was introduced with Euclidean geometry in grade 12, mathematics results declined. In 2013, Euclidean geometry was an optional section in mathematics paper 3 and most schools opted not to teach this section. Furthermore, Umugiraneza, Bansilal, and North (2017) also indicated that in South Africa comments about poor results in mathematics naturally led to questions about whether mathematics teaching was as effective as it could be. There is still significant underperformance in mathematics teaching and learning, and many learners lack interest and proficiency in mathematics (Mohr-Schroeder et al., 2017).

According to Mamlok-Naaman (2017), this ongoing low-learner performance has led to increasing interest in understanding how teacher characteristics, pedagogical practices and content knowledge may figure within these patterns of poor learner performance. This poor learner performance may be caused by teachers' weak content knowledge of mathematics as well as poor pedagogical practices (Mabokela & Mlambo, 2017). They suggest that improving teachers' content knowledge and pedagogical practices would improve learner performance.

Despite many teacher professional development programmes designed to improve teacher's content knowledge and teacher's pedagogical practices (Mogashoa, 2016), learner performance in mathematics is still poor (Voogt et al., 2016). There is little evidence that mathematics professional development programmes in South Africa have an impact on learner attainment (Mogashoa, 2016).

2.12 Teaching and learning of Euclidean geometry in South Africa

The lack of attention paid to the teaching of Euclidean geometry is still a concern (Tutak & Adams, 2017). According to Wei, Darling-Hammond, Andree, Richardson, and Orphanos (2017) geometry is a complex topic and teaching Euclidean geometry is complex as it requires the application of higher cognitive functions. Teaching and learning Euclidean geometry requires abstract and critical thinking, ability to visualize abstract concepts, and logic (Seo, Hajishirzi, Farhadi, Etzioni, & Malcolm, 2015). Furthermore, Wei et al. (2017) stated that the reason why many learners fear mathematics was mainly due to their poor geometrical conception.

The teaching and learning of Euclidean geometry remains problematic across all grades, especially at secondary school level and this has been debated over many years (Clark & Worger, 2016). Wentzel (2016, p. 49) also confirmed that "Euclidean geometry education in South Africa is in a very poor state, and a great number of learners perform poorly in Euclidean geometry" and suggested that there is a need to seek an alternative approach to the study of Euclidean geometry. According to McAndrew et al. (2017), research has noted that Euclidean geometry is a difficult section to teach and to learn. There are some challenges experienced by teachers in presenting Euclidean geometry lessons, and as a result learners experience difficulties in learning this topic (Utami, Mardiyana, & Pramudya, 2017).

Several prevailing factors have been identified to explain why teaching and learning Euclidean geometry is considered a difficult task (Christie, 2016). Clark and Worger (2016) claimed that teachers' pedagogy and teaching methodologies are some of the critical factors that affect the teaching and learning of Euclidean geometry. According to Christie (2016, p. 795), "geometry language, visualization abilities and ineffective instructions are some other factors contributing to poor learner performance in Euclidean geometry". Furthermore, Wei et al. (2017) confirmed that teacher's pedagogy and teaching methods affect the teaching and learning of geometry.

According to Tsao (2017), research has shown that teacher's methods of Euclidean geometry teaching and their personalities greatly accounted for the learners' positive attitude towards geometry. He continues that teachers with a negative attitude toward Euclidean geometry are unlikely to cultivate a positive attitude in their own learners. The problem of teaching and learning of mathematics has been researched widely for decades and the teaching and learning of Euclidean Geometry to a lesser extent (Wei et al., 2017).

According to Ugorji and Alfred (2017), Euclidean geometry is among the new topics introduced into the school mathematics curriculum in South Africa. The teaching of the topic seems to be a challenge to most teachers and students find it difficult to grasp. Mathematics teachers still find Euclidean geometry difficult to teach in both primary and secondary schools, and this has led to many learners performing poorly in Euclidean geometry (Tutak & Adams, 2017).

Teaching Euclidean geometry involves an understanding and appreciation of the history and cultural context of geometry, knowing how to recognize interesting geometrical problems and theorems, and geometry content knowledge, competence and proficiency (Ramatlapana, 2017). The majority of mathematics teachers teaching Euclidean geometry are pedagogically ill-equipped to effectively teach the new topics as most of them were not taught the topic in their schooling (Ugorji & Alfred, 2017). According to Wei et al. (2017), the chances of learners understanding Euclidean geometry will be very little if they are taught by incompetent teachers.

2.13 Proofs and theorems of the geometry of circles

Proofs and theorems form part of Euclidean geometry and need to be understood well by teachers and learners. However, both teachers and learners encounter difficulties in the teaching and learning of proofs and theorems (Kolb, 2014). According to Brijlall (2017), a proof serves as a convincing demonstration that some mathematical statement is true in all cases without any exception. In Euclidean geometry, grade 11 learners are expected to know all theorems and proofs including theorems done in previous grades and must be familiar with acceptable reasons, and knowledge of all geometry concepts learned in previous grades is a prerequisite.

According to Kolb (2014), understanding and applying theorems is a highly ranked level of proof and reasoning ability. Teachers' content knowledge and their background learning experiences of proof and theorems may have an impact on the teaching and learning of proof and theorems which may affect learners' understanding in learning proof and theorems (Fung et al., 2017).

According to DoE (2011c, p. 14), grade 11 mathematics teachers are expected to teach learners to investigate and prove the following theorems, and be able to use them and their converses to solve riders, although converses are non-examinable:

1. The line drawn from the centre of a circle perpendicular to a chord bisects the chord;

2. The perpendicular bisector of a chord passes through the centre of the circle;

3. The angle subtended by an arc at the centre of a circle is double the size of the angle subtended by the same arc at the circle (on the same side of the chord as the centre);

4. Angles subtended by a chord at the circle, on the same side of the chord, are equal;

5. The opposite angles of a cyclic quadrilateral are supplementary;

6. Two tangents drawn to acircle from the same point outside the circle are equal in length;

7. The tangent-chord theorem.

Learners will also use concepts learned in earlier grades to prove circle geometry theorems such as:

1. Angles on a straight line add up to 180° (supplementary).

2. The angles in a triangle add up to 180°.

3. In an isosceles (two equal sides) triangle the two angles opposite the equal sides are themselves equal.

4. The exterior angle of a triangle is equal to the sum of interior opposite angles.

5. Congruency in proofs. Geometric figures that have the same shape and the same size are congruent.

2.14 The van Hiele theory of geometric thinking, teaching and learning

According to Piaget (2000), the process of teaching and learning geometry cannot be separated from the van Hiele theory. The van Hiele model of geometric thinking was developed in 1957 by two Dutch mathematics educators, P.M. van Hiele and Dina van Hiele (Mji & Makgato,

2006) to help learners learn geometry. These were experienced secondary mathematics educators who were concerned about difficulties learners faced when learning geometry in secondary schools (Eccles & Wang, 2016). Their intention was to explain and identify learners' difficulties in geometry (Chua, Tengah, Shahrill, Tan, & Leong, 2017). The van Hiele theory was mainly aimed at improving the teaching of geometry and helping learners improve their geometric understanding by organising activities that would take learners' thinking skills into account whilst new skills were being introduced (Alex & Mammen, 2016).

The theory describes five levels (starting from 1 to 5, where 5 is the highest level), that characterise the thinking of learners as they grow in their understanding of geometric relationships (Wilson-Patrick, 2016). According to this theory, learners progress through levels of thought in geometry (Senk, 1985), and these learners' levels of geometric thought are sequential and hierarchical, and are achieved largely as a result of effective geometric teaching (Troelstra & Van Dalen, 2014). Learners cannot bypass them and achieve understanding (Senk, 1985). These levels progress from a visual level to sophisticated levels of description, analysis, abstraction and proof (Senk, 1985). Teachers of Euclidean geometry need to be aware of these levels so they can select appropriate geometry problems and materials. According to Mji and Makgato (2006), teaching at a level higher than the level of learners can inhibit their learning and lead to frustration. Therefore, it is important that learners are taught at their level. According to Wadsworth (1996), the following Table explains van Hiele's levels:

Table 2.3 A summary of the van Hiele levels

Level	**Learners' thinking**
1	Visual or recognition level. According to McAndrew et al. (2017), learners can mention the name of an object based on its physical appearance (shape or form), they can only identify the shape but they cannot provide the properties. Shape is defined by its appearance. At this level, learners do not think about analysing and identifying properties of the figures, appearance is dominant and may overpower the properties of the figure (Mji & Makgato, 2006). According to Uygan and Kurtuluş (2016), mathematics, especially geometry, involves visualisation of shapes and objects.
2	Analysis or descriptive level. Learners start to identify and analyse particular properties of shapes but not in a logical manner (McAndrew et al., 2017). For example, a learner might think of a

	square as a shape that has four equal sides and four right angles; learners establish properties by observing, measuring, drawing and model thinking(Senk, 1985).
3	Informal deduction or abstract level. At this level, learners begin to recognise relationships between shapes (Mji & Makgato, 2006). They can distinguish the features of the figure, they will understand why each square is a square, but they cannot organise necessary lists in order to prove this observation. Learners see clearly, for example, why a square is a rectangle (Senk, 1985) because it has all the properties of a rectangle. They can compare shapes with their properties and provide precise definition and be able to relate the shape with other shapes (McAndrew et al., 2017). There is a logical order of properties of shapes at this level (Novak & Tassell, 2017).
4	Formal deduction level. Learners prove theorems deductively and establish interrelationships among system of theorems; they are able to explain the relationships between figures using theorems, definitions and axioms (McAndrew et al., 2017). They are now able to write proofs by themselves and Euclidean geometry requires learners to think on a formal deductive level (Novak & Tassell, 2017).
5	Rigour level. Learners establish theorems in different postulation systems and analyse these systems. They can compare and study different geometries in the absence of physical models (McAndrew et al., 2017).

The learners' inability to achieve level 1 (recognition) has resulted in researchers suggesting level 0, called pre-recognition level (Novak & Tassell, 2017). They continue to say that Euclidean geometry requires learners to think on level 4 of the van Hieles' theory (a formal deductive level), but this does not happen as most learners lack the prerequisite understanding of geometry.

Understanding these levels will help teachers identify the learners' learning needs and the level at which learners are operating (McAndrew et al., 2017). Teachers can design learning activities relevant to a particular level. The main reason for the failure of Euclidean geometry is that the topic is presented at a higher level than that of the learners (Kennedy et al., 2014). Hence the performance in Euclidean geometry tends to be much worse than performance in algebra, analytical geometry and trigonometry (Wei et al., 2017). Learners can only perform tasks at a certain level and are expected to perform all tasks below their level (Reece & Walker, 2016). According to Armah, Cofie, and Okpoti (2017), an in-depth conceptual understanding of geometrical properties at the primary level is not only important for the learners in learning about geometry, it will also determine whether they can cope with its continuation when they proceed to secondary level.

The van Hiele theory also emphasised the importance of language when teaching geometry and noted that failure in teaching geometry may be due to teachers teaching at a language higher than that of a learner (Abu, Ali, & Hock, 2012). According to Alex and Mammen (2016), teachers need to know that each level has its own language and teachers need to address learners in a language they understand at that level so as to inspire learners' confidence and assist them to better understand concepts.

The van Hieles' model of geometry thinking has been linked to Piaget's theory of five stages of child development and the role these play in learning geometry (McAndrew et al., 2017). Piaget wrote that learners' geometrical understanding develops with age and that for learners to create new ideas about shapes they need physical interaction with objects (McAndrew et al., 2017). According to Skilling, Bobis, Martin, Anderson, and Way (2016) Piaget was the first to study the foundations of geometrical thinking and to suggest that geometrical thinking develops sequentially in four stages.

2.15 Mathematics teachers' content knowledge (CK) for teaching.

Amongst many factors that shape the instructional practices of mathematics teachers is their level of content knowledge (Hughes, Swars Auslander, Stinson, & Fortner, 2019). According to Pournara et al. (2015), content knowledge is the knowledge of concepts, facts, and principles of the subject matter, it is more about knowing subject content, i.e., *what to teach*, for example, Euclidean geometry. This knowledge is closely parallel to what is taught in schools and it also respects the integrity of mathematics (Ramatlapana, 2017). It is sometimes referred to as "subject matter knowledge", which is not necessarily for teaching purposes and all mathematicians should have a certain level of this knowledge, but for teachers it needs to be relevant to the mathematics taught in schools. According to Akhtar, Shaheen, and Bibi (2016), subject matter or content knowledge is defined as knowledge of the actual subject matter that is to be taught, e.g. mathematics, physical science, or life science. Content knowledge forms the base for the development of PCK.

Shulman's (1987) definition refers to a teacher's subject knowledge and teachers who lack content knowledge can negatively affect not only the quality of the teaching itself but also learners' content skills and results (Lee, Capraro, & Capraro, 2018). They add that this knowledge develops from training colleges and universities and grows with teacher experience and also as teachers take more mathematics content courses. Mathematics teachers need to participate in advanced mathematics courses in order to improve their content knowledge. It is important for mathematics teachers to advance their content knowledge because content does not only change in volume but also changes in structure over time (Akhtar et al., 2016).

2.16 Mathematics teachers' pedagogical knowledge (PK) for teaching

According to Doody and Noonan (2013), pedagogical knowledge is the teachers' deep knowledge about the processes and practices or methods of teaching and learning, i.e., *how to teach it*. The source of pedagogical knowledge (*how*) is content knowledge (*what*), however, there is no guarantee that teachers with good content knowledge will also have good pedagogical knowledge (Novak & Tassell, 2017). According to Rosenkränzer, Kramer, Hörsch, Schuler, and Rieß (2016), pedagogical knowledge refers to non-content related knowledge of strategies and procedures for effective and well managed lessons and classrooms and this type of knowledge goes beyond subject matter knowledge.

2.17 Mathematics teachers' pedagogical content knowledge (PCK) for teaching

Pedagogical content knowledge was first introduced by Shulman in 1985 (Shulman, 1986) and has become internationally recognized as the most important part of the knowledge required for teaching (Alonzo & Kim, 2016). Shulman (1986, p. 7) stated that "PCK goes beyond knowing subject matter (CK) to the level of subject matter knowledge for teaching." This implies that having only a good content knowledge may not necessarily mean good teaching. Teachers need more than just subject content knowledge. He also argued that content knowledge and pedagogical knowledge have a special combination, and a teacher acts as a bridge, connecting these two (Simsek & Boz, 2016).

According to Steffe et al. (2016), pedagogical content knowledge (PCK) is knowledge which allows a teacher to convert his or her own knowledge to a form that learners understand. This implies that "PCK also includes an understanding of what makes the learning of specific topics easy or difficult" (Shulman, 1986, p. 9). For example, the learning of Euclidean geometry is regarded as difficult by most learners, and hence, according to Chan and Yung (2018), teachers need knowledge of fruitful strategies to improve the understanding of the learners.

PCK bridges subject matter content knowledge and the practice of teaching (Vermeulen & Meyer, 2017). PCK results from a combination of the teacher's content knowledge, knowledge of learners' understanding of the subject at different levels, as well as the different ways in which content knowledge can be used in teaching and learning in the classroom (Wei et al., 2017). According to Krzywacki, Kim, and Lavonen (2017), PCK is the synthesis of all knowledge needed for the teaching and learning of a certain topic, thus it is always related to subject matter knowledge. PCK may include how a particular topic is organised, represented and presented for instruction to meet the needs of the learners with diverse interests and abilities (Chan & Yung, 2018). PCK covers the core business of teaching, learning, assessment and reporting (Ramatlapana, 2017).

According to Lee et al. (2018), PCK is associated with content as well as practical experience, and it develops from coursework and mathematics education as well as from teacher professional workshops. Erwin and Rustaman (2017) argue that highly qualified and experienced teachers are those with substantial knowledge about varying teaching approaches relevant to the subject they teach, which includes classroom management, lesson design and implementation, understanding mathematics content and effectively engaging learners during teaching and learning. The teacher's experience has been highly recognised as a key factor in the development of pedagogical content knowledge (PCK) (Chan & Yung, 2018). Furthermore, as mathematics teachers gain experience in their teaching profession, their effectiveness tends to improve (Kini & Podolsky, 2016).

According to Nur and Nurvitasari (2017), the teachers' PCK of geometry is an important knowledge that is needed in order to make geometry interesting. Teaching is a complicated task and teachers need to have a variety of knowledge to be able to cope well (Masduki, Suwarsono, & Budiarto, 2017). Among the dominant categories of knowledge are content knowledge (CK), pedagogical knowledge (PK) and pedagogical content knowledge (PCK) and within mathematics knowledge, there is an interplay between mathematics content knowledge, pedagogical knowledge and pedagogical content knowledge (Ramatlapana, 2017). Content knowledge and pedagogical content knowledge are the most important components of teachers' knowledge that impacts learners' performance. According to Akhtar et al. (2016), the appropriate combination of CK, PK, and PCK can make teachers effective and competent. Effective and competent teachers have in-depth knowledge and understanding of content, curriculum, learner characteristics, teaching and learning methods, and good classroom management (Masduki et al., 2017).

2.18 Conclusion

This chapter presented an overview of literature by discussing the curriculum of the current South African education system. Since geometry is the mathematics content being explored, a broader overview of geometry was also discussed. A comprehensive discussion of South African mathematics curriculum, mathematics teachers, and mathematics learner performance was also presented. The next chapter presents the background of the teacher knowledge framework and the framework for understanding the process of learning geometry.

CHAPTER 3

THEORETICAL FRAMEWORK

3.1 Introduction

Over the years, much of the debate about learner performance in mathematics, especially geometry, has related to the issue of teachers' knowledge, and hence there is no doubt that teachers play a key role in the improvement of mathematics learner performance (Nilsson & Vikström, 2015). According to García-Santillán, Escalera-Chávez, Moreno-García, and Santana-Villegas (2016), the foundation and the understanding of basic geometry knowledge starts in primary school. How learners develop their understanding of geometry has been an area of research over the past 60 years (Watan, 2018). Defining teacher knowledge is important due to its impact on learner performance (Güler & Çelik, 2018). This chapter explores frameworks for teacher knowledge and for understanding the processes of learning in geometry.

3.2 Exploring the teacher knowledge framework

Mathematics teachers are critical role players in the learning of mathematics. They need to have strategies to help learners develop and understand the content knowledge (CK) of mathematics so that the learners' achievement is high (Ijeh, 2018). According to Bokosmaty, Mavilidi, and Paas (2017) the interaction between the mathematics teacher and the learners is fundamental for effective instruction of Euclidean geometry. To effectively deal with learners' difficulties in mathematics, especially in Euclidean geometry, teachers need to develop and maintain sufficient knowledge and skills specifically for this task, and they need to be knowledgeable in the content of mathematics as well as pedagogical content knowledge (PCK), including methods of recognising and addressing learners' misconceptions, misunderstandings and misinterpretations of concepts (Lee et al., 2018).

Shulman (1986) claims that to know a certain concept does not necessarily mean that one is able to teach that concept. He claims that teachers need to know far beyond content knowledge (subject matter per se) to the level of subject matter knowledge for teaching. They need to be knowledgeable about Euclidean geometry and pedagogy. Teachers need to have strategies that will make the learning of a particular topic easy and interesting to the learners, they need to be experts of the content (subject matter) and they need to master the science of teaching (pedagogy) (Erwin & Rustaman, 2017). For example, the learning of Euclidean geometry (EG) is known to be difficult (Lee et al., 2018), hence the most important characteristic of this study was to investigate and understand mathematics teachers' pedagogical content knowledge (PCK) when teaching Euclidean geometry content in grade 11 and it is hoped that this will improve the teaching and learning of geometry in mathematics.

Content knowledge (CK), general pedagogical knowledge (PK), knowledge of classroom management, pedagogical content knowledge (PCK), curriculum knowledge, learner knowledge and their characteristics, knowledge of educational contexts, and knowledge of educational outcomes and aims all provide a framework for teacher knowledge (Ijeh, 2018). These are crucial in teaching and in enhancing learners' understanding of any concept. It is important to have all these discussed knowledge components since they all affect instructional practice and learners' mathematics learning. However, this study will focus more on CK, PK and PCK.

According to Lee et al. (2018), teacher knowledge (TK) is the knowledge and understanding of facts and concepts, as well as knowledge about the subject, and the ability to effectively translate this knowledge to promote learning. This forms the basis for teachers' instructional classroom practices. The classroom practice as well as learners' achievements are both directly and positively affected by the teacher's knowledge (Campbell et al., 2014). Ottmar, Rimm-Kaufman, Larsen, and Berry (2015) said that for teachers to be effective teaching practitioners they must have the ability to translate mathematics knowledge in their classrooms to promote learning and improve learner attainment.

The commonly used definition of teacher knowledge was developed by Shulman (1986) who claimed that teacher knowledge consisted of content or subject matter knowledge (SMK) (referring to the content a teacher teaches, such as mathematics), pedagogical knowledge (PK), pedagogical content knowledge (PCK) and curriculum knowledge (CK) (Tutak & Adams, 2017). According to Lee et al. (2018) insufficient TK among mathematics teachers may lead learners to develop misunderstandings, misconceptions, and misinterpretations regarding the content taught. He adds that TK does not only affect teachers' classroom practice but also affects their PCK, as teachers who have higher CK have more positive attitudes in their teaching than those with limited CK.

3.2.1 Content knowledge (CK)

Content knowledge (CK) may be defined as the actual subject matter knowledge (SMK) of a teacher (Ramatlapana, 2017), for example, the teacher's knowledge of geometry. The CK construct in this study is conceptualised as the knowledge of circle geometry concepts, theorems, proofs and solving riders. Teachers with high CK will perform better in their classroom practice. Geometry connects mathematics curriculum, and a good understanding of geometry will help learners gain increased access to abstract mathematical concepts (Tsao, 2018). According to Campbell et al. (2014), teachers need this knowledge, including knowledge of mathematical facts, concepts and procedures, so that they are able to make correct statements and differentiate an incorrect from a correct statement and recognise learners' incorrect statements (Ottmar et al., 2015).

Shulman (1986) introduced content knowledge as the knowledge in the mind of a teacher about a specific topic. For instance, according to DoE (2011c, p. 34), content knowledge of one of the grade 11 Euclidean geometry theorems states that: "a line segment drawn from the centre of the circle to the midpoint of the chord is perpendicular to the chord". To be able to teach this theorem, teachers need to have a good understanding of "congruency". This is the knowledge of the depth and breadth of a particular topic (Mishra & Koehler, 2006), and Shulman (1986) earlier said that teachers need not only understand concepts, they need to further understand why they are so, and which concepts are most relevant to a particular topic.

3.2.2 Pedagogical knowledge (PK)

Ijeh (2018) defined pedagogical knowledge as a special kind of information necessary for teachers to use when performing their everyday teaching tasks. This knowledge relates to the ways and processes of teaching which include the knowledge and understanding of content to be taught, knowledge of learners targeted, effective lesson preparation, developing good quality learning activities, teaching styles (instructional skills) and strategies, strategies for evaluating learner understanding, classroom management skills, and sequencing of the learning outcomes and assessment (Erwin & Rustaman, 2017; Shulman, 1986). According to Tutak and Adams (2017), teachers who lack content knowledge have poor pedagogical knowledge.

3.2.3 Pedagogical content knowledge (PCK) – the heart of teaching

Teachers need to have an adequate knowledge of geometry in order to effectively teach it; they need to be able to translate their personal understanding of geometry into their pedagogy in ways that make it accessible to their learners (Correa-Bautista, 2017). Shulman (1986) referred to this translation process as pedagogical content knowledge (PCK) (Sunzuma & Maharaj, 2019). Mulhall, Berry, and Loughran (2003) defined PCK as a unique extraordinary knowledge that teachers have about how a particular content topic is addressed to the learners in a way that promotes and motivates learning. According to Correa-Bautista (2017), teaching mathematics, especially geometry, is not simple but it is a complex process where a mathematics teacher requires specialised knowledge (known as PCK) to be able to transmit his/her personal knowledge of geometry to the learners.

PCK was conceptualised in 1986 by Shulman as a knowledge base for teaching which includes knowledge of content and curriculum, and in 1987 Shulman developed four other knowledge bases such as "pedagogical knowledge (PK), knowledge of learners and their characteristics, knowledge of educational contexts, and knowledge of educational ends and purposes"(Ward & Ayvazo, 2016, p. 194). The foundation of PCK is the intersection of both content knowledge (CK) (content competence) and pedagogical knowledge (PK) (pedagogic competence) (Erwin & Rustaman, 2017).

According to Loewenberg Ball, Thames, and Phelps (2008), rather than just knowing geometry for example, the teacher has to know exactly how to get geometry concepts across to learners, i.e., how the teacher uses examples to provide explanations and demonstrations of the concept to the learners. Shulman (1986, p. 9) also emphasised that "PCK goes beyond knowledge of subject matter per se to the dimensions of subject matter knowledge for teaching" and this type of knowledge distinguishes between a teacher and a mathematician (Tutak & Adams, 2017, p. 302).

Therefore, teachers' PCK is considered to be the core knowledge required when it comes to supporting, promoting and developing learners' geometrical knowledge (Taşdan & Çelik, 2016). Shulman (1986) embraced the idea that effective and competent teachers have a very special understanding of content knowledge (CK) and pedagogical knowledge (PK) when teaching a particular content (Ijeh, 2018). PCK concept, according to Shulman (1986), was a missing connection of knowledge between teachers' subject content knowledge and their pedagogical knowledge (Mecoli, 2013).

According to Mosia (2016), teachers' knowledge, content knowledge and pedagogical knowledge should not to be considered in isolation as there is an interrelationship between them, i.e. successful teachers confront both content and pedagogy simultaneously. He continues that PCK blends CK and PK into knowledge of understanding how a particular concept is taught in a way that learners can easily understand. Included in the PCK is the understanding and knowledge of learner ideas and what makes a particular topic easy or difficult for the learners, including common learner misconceptions and strategies for recognizing and addressing them (Shulman, 1986).

3.3 Exploring the van Hiele model of learning geometry

Beyond the understanding of content knowledge (CK), pedagogical knowledge (PK) and pedagogical content knowledge (PCK) when teaching Euclidean geometry, teachers need to have a good understanding of theories relevant to the teaching and learning of geometry (Subbotin & Voskoglou, 2017). One of the theories specifically about the teaching and learning of geometry included, among others, is the van Hiele model of geometrical thinking (Sinclair et al., 2016). According to Stumbles (2018), van Hiele presented five sequential teaching levels of geometry that represent a framework for understanding the progress of learners from one geometrical level to the next.

This theory suggests that learners' competencies in mathematics have been linked closely to their levels of thinking in geometry (Armah, Cofie, & Okpoti, 2018) and it has contributed positively in teaching geometry to the learners (Al-ebous, 2016).Yunus, Suraya, Ayub, and Fauzi (2019) state that the van Hiele theory suggests that learners progress through numerous levels of geometric thinking, from recognising geometry shapes to constructing formal geometry proofs. The theory offers teachers a teaching model to apply and practice in order to improve their learners' levels of geometric thinking, and it can be used to explain why many learners encounter difficulties in geometry (Seah & Horne, 2019).

According to Alex and Mammen (2018), the van Hiele model emphasised the existence of hierarchical levels learners go through when learning geometry, and how they move sequentially from one level to the next. This framework is common to all studies of geometry and can be used in geometry classes to measure learners' and/or teachers' geometrical thought (McIntyre, 2017). A good understanding of this theory will enable teachers of geometry to identify the direction of learners' learning and the level at which they are operating (Luneta, 2015), and teachers can use this framework to predict learners' present and future performance in geometry (Subbotin & Voskoglou, 2017).

The process of learning and teaching geometry in secondary schools can hardly be separated from the van Hiele theory which was developed by Pierre and Dina van Hiele in the 1950s (Nur & Nurvitasari, 2017), and this theory states that learners progress through several levels or stages of reasoning during their learning process in geometry (McIntyre, 2017). According to Alex and Mammen (2018), this is known as a fixed sequence of levels, where learners cannot be at level n without having attained level $n-1$, i.e., learners cannot move from level 0 straight to level 2 or level 3 without having mastered level 1 first. Furthermore, the advancement from level $n-1$ to level n largely depends on instruction and not on age (George, 2017).

Jones (1998) describes this model as the development of reasoning and thinking process in the learning of geometry and further tells us that it is an important approach to the teaching of geometry, especially in secondary schools. Therefore, this theory of levels can be used in measuring geometrical thought in learners as well as in teachers (McIntyre, 2017). For example, if a teacher and a learner are functioning at different levels, then they may differ in communication, causing difficulty in the teaching and learning. One of the main characteristics of this theory is that instruction at a level higher than that of a learner may hamper the learning process and may lead to frustration. Therefore, teaching learners at their level is important.

This theory is very relevant in South African mathematics education, where geometry education is still a problematic area (Alex & Mammen, 2016).

The following discussion in Table 3.1 below about ideas and definitions of the five levels of van Hiele theory is taken from Subbotin and Voskoglou (2017, pp. 1-2):

Table 3.1 Exploring the van Hiele levels of geometric thinking

Level	**Description**	
Level 1	*Basic level of visualisation or recognition*	Learners can recognise, identify, name and compare geometric figures such as rectangle, square and triangle by their shapes, i.e., by their physical appearance; they can group shapes or objects which are similar. They may not recognise properties such as that which says that opposite sides are parallel. All learners in primary schools are assumed to be at this level
Level 2	*Analysis*	Learners can analyse geometric figures in terms of their properties, for example, all angles of a square are right angles, and a parallelogram has equal opposite angles. Although learners have an idea of the properties, however, they cannot compare properties between different geometric shapes, e.g. a square and a rectangle both have four right angles
Level 3	*Informal deduction*	Learners can logically inter-relate or inter-link properties by giving informal arguments, e.g. if opposite sides of a quadrilateral are parallel, then its opposite angles are equal or they can deduce that a rectangle is made up of two squares. Definitions are now meaningful to the learners and they can follow informal arguments. Learners begin to see proofs and can follow them without necessarily writing and structuring them. By the time learners finish grade 9, they should have achieved this level since formal proofs begin from the grade 10 curriculum.
Level 4	*Formal deduction*	Learners can construct and prove theorems deductively. It is the beginning stage of formal deductions, and learners now understand the importance of deductions and the roles of theorems, postulates and proofs, i.e. they can now structure and write proofs with understanding. By the time they finish grade 12, they should have attained this level. This level is appropriate for a typical geometry class at a high school, especially grades 10, 11 and 12.

Level 5	*Rigor*	This is the last stage, known as post-grade 12 level, where learners see geometry in the abstract. They can now analyse and compare theorems. Van Hiele acknowledged his interest in the first three levels since the majority of high school geometry is taught at level 3 and it is very rare for this level to appear in secondary schools.

Adapted from*Subbotin and Voskoglou (2017, pp. 1-2)*

According to Armah et al. (2018), in this theory, geometric thinking levels are emphasised as one of its main objectives. These levels are known to be sequential and were originally labelled from level 0 to 4, but were later changed to start from level 1 to 5 to make provision for a new level called 'pre-recognition level' which is called level 0 (Stols, Long, & Dunne, 2015). This model provides the teachers with clear explanations on how they can proceed to help and guide learners move from one level to the next, and by designing geometry activities which are relevant to the level of the learners (Armah et al., 2018).

The development of procedural geometrical fluency starts from level 0 to level 3, whilst from level 4 to 5 the development of conceptual understanding is displayed (Luneta, 2015). Luneta (2015) adds that each level must be fully developed by the learner before moving to the next level. Table 3.2 below shows an overview of Euclidean geometry curriculum as stated in the CAPS document from the Intermediate phase to the Further Education and Training phase.

Table 3.2: The overview of Euclidian geometry

Intermediate Phase	**Senior Phase**	**Further Education and Training Phase**
(Grades 4, 5 and 6)	**(Grades 7,8 and 9)**	**(Grades 10,11 and 12)**

Learners: 1. Recognise, visualise, name, describe, sort, and compare geometric 2D shapes as well as 3D objects. 2. Know the characteristics of shapes and objects and how to make models. 3. Recognise and name angles.	Learners: 1. Recognise, describe, sort, name and compare triangles and quadrilaterals. 2. Solve geometric problems using known properties. 3. Build and construct 3D models of geometric solids. 4. Define line segment, ray, straight line, parallel lines and perpendicular lines. 5. Measure, construct and classify angles. 6. Bisect lines and angles 7. Investigate the angles and sides in triangles and quadrilaterals.	Learners: 1. Investigate line segments joining the midpoints of two sides of a triangle and properties of special quadrilaterals. 2. Solve problems involving volume and surface area of solids 3. Investigate and prove theorems of the geometry of circles. 4. Solve circle geometry problems and prove riders.

Adapted from DBE ***(2011a, pp. 1-2; 2011b, pp. 27-28; 2011c, p. 14)***

3.3.1 Visualisation/recognition level –recognising and naming figures

According to Kemp (2018), visualisation occurs when learners are able to recognise figures and relate them to objects they know, they are not focused on individual parts, but on the overall appearance of a figure. Visualisation is a required skill in geometry which helps learners recognise geometric shapes or objects experimentally and it is an important factor which may affect the choice of strategy when learners undertake geometric activities (Gunhan, 2014).Visualization in geometry is essential as it facilitates learners' thinking in geometric concepts (Arıcı & Aslan-Tutak, 2015).

According to Seah and Horne (2019), visualisation is an important element in the process of learning geometry, as learners reason about basic geometric shapes without explicit regard to properties of its components. McAndrew et al. (2017) indicated that learners can mention the name of an object based on its physical appearance (shape or form); they can only identify the shape but they cannot provide the properties. Shape is defined by its appearance. At this visualisation level, learners do not think about analysing and identifying properties of the figures as appearance is dominant and may override the properties of the figure (Seah & Horne, 2019).

3.3.2 Analysis or descriptive level

At this level, learners start to identify and analyse particular properties of shapes but not in a logical manner (McAndrew et al., 2017). They begin to identify properties of geometric shapes and learn to use appropriate vocabulary related to properties (Watan, 2018). For example, a learner might think of a square as a shape that has four equal sides and four right angles. Learners establish properties by observing, measuring, drawing and making models (Seah & Horne, 2019). According to Kemp (2018), learners can identify the properties of figures, but do not know which properties are explicit enough to define the figure itself; they are aware of properties, but do not comprehend the significance of adequacy of conditions.

In both the intermediate and senior phases, a learner's experience of geometry moves from recognition and simple description to classification and more detailed description of characteristics and properties of geometric shapes and objects, and they develop an appreciation for the use of constructions to investigate the properties of geometric shapes and objects (DoE, 2011b). They develop clearer and more precise descriptions and classification categories of geometric shapes and objects (DoE, 2011a).

3.3.3 Informal deduction or abstract level

At this level, learners begin to recognise relationships between shapes (Yudianto, Sugiarti, & Trapsilasiwi, 2018). They can distinguish the features of the figure and they will understand why each square is a square, but they cannot organise necessary lists in order to prove this observation (Seah & Horne, 2019). According to Sulistiowati et al. (2019), learners see clearly, for example, why a square is a rectangle, because it has all the properties of a rectangle. They can compare shapes with their properties and provide precise definitions and are able to relate the shape with other shapes (McAndrew et al., 2017). There is a logical order of properties of shapes at this level (Novak & Tassell, 2017). Learners are now able to classify figures and use basic logic to justify their reasoning (Kemp, 2018).

3.3.4 Formal deduction level

According to Kemp (2018), learners are now able to write simple proofs and understand axiomatic systems. Learners prove theorems deductively and establish interrelationships among system of theorems; they are able to explain the relationships between figures using theorems, definitions and axioms (McAndrew et al., 2017). They are now able to write proofs by themselves and Euclidean geometry requires learners to think on a formal deductive level (Novak & Tassell, 2017). However, Rosyidi and Kohar (2018) explored numerous challenges that learners experienced when making the transition from the first three levels to this formal level of constructing proofs. For instance, learners need to understand and apply the theorem, and to make connections between mathematical objects.

According to Miyazaki et al. (2019), teaching and learning proofs is internationally recognized as a key component of mathematics curriculum; however, learners still experience difficulties in learning proofs. Currently, South African learners learn formal proofs as early as grades 8 and 9. According to DoE (2011b), progression in geometry in the Senior phase is achieved primarily by investigating new properties of shapes and objects, developing from informal descriptions of geometric figures to more formal definitions and classification of shapes and objects, solving more complex geometric problems using known properties of geometric figures, and developing from inductive reasoning to deductive reasoning. This lays the foundation for geometry in the Further Education and Training phase.

3.4 Constructing proofs – formal deduction level of the van Hiele model

Geometrical proofs help learners distinguish the difference between seeing and deducing (Kemp, 2018). Although learners are expected to construct and prove theorems deductively at this stage, the construction of geometrical proofs is said to be a major hurdle for both learners and teachers in the FET phase (Kempen & Biehler, 2015). Many high school learners perform below this level of formal deduction (Bowie, 2015). As discussed earlier, proofs and theorems form part of Euclidean geometry and need to be thoroughly understood by teachers and learners. However, both teachers and learners encounter difficulties in the teaching and learning of proofs and theorems (Kolb, 2014). In the grade 11 Euclidean geometry, learners are expected to know all theorems and proofs as stated in the Curriculum and Assessment Policy (CAPS), including theorems done in previous grades and must be familiar with acceptable reasons (DoE, 2011c).

All geometry concepts learned in previous grades are a prerequisite in the introduction of any new concept in geometry; however, in geometry classes, learners operate at different levels to that of their teachers, and, hence, they may not understand each other (Subbotin & Voskoglou, 2017). For example, a learner reasoning at level n will fail to understand a teacher who is reasoning at level $n+1$ (Al-ebous, 2016).

3.5 Conclusion

This chapter discussed and presented teacher knowledge framework as the best framework for this study, showing that content knowledge (CK), general pedagogical knowledge (PK), knowledge of classroom management, pedagogical content knowledge (PCK), curriculum knowledge, learner knowledge and characteristics, knowledge of educational contexts, and knowledge of educational outcomes and aims, all provide a framework for teacher knowledge. It also emphasised that these are crucial in teaching and in enhancing learners' understanding of any concept. Strengthening teachers' pedagogical content knowledge (PCK) is important in order to improve the teaching and learning of geometry (Appova & Taylor, 2019).

Furthermore, the van Hiele theory of geometrical thinking was described as an important approach to the teaching and learning of geometry, especially in secondary schools, and teachers need to link this theory when teaching geometry. The geometry teaching sequence presented by the van Hiele framework has a positive impact on the teaching and learning of geometry and may promote the understanding and development of higher-order geometry skills. Teachers should be aware of the levels at which their learners are thinking when they arrive at their classroom, and they need to apply different dynamic strategies of teaching in order to help learners understand geometry, and to address the many challenges learners encounter when learning geometry.

CHAPTER 4

RESEARCH METHODOLOGY

4.1 Introduction

The main goal of this chapter is to describe and discuss research design and methods employed in this study, i.e., how the research was conducted or carried out from data collection to data processing and analysis procedures. "Methodology is the strategy or a plan of action lying behind the choice of using particular methods to the desired outcomes, it is the way in which we approach problems and seek answers, i.e., how the research is conducted" (Creswell & Creswell, 2017; pp. 3-7). According to Walliman (2017) research methods are a range of techniques used when doing research, and they provide a researcher with ways to collect, sort and analyse data so that the researcher can come to some conclusions.

The focus of this study was on Grade 11 mathematics teachers, and the research was planned to respond to the following questions:

1. What are Grade 11 mathematics teachers' pedagogical content knowledge of Euclidean geometry?

2. How do Grade 11 mathematics teachers use their pedagogical content knowledge when teaching Euclidean geometry?

3. Why do Grade 11 mathematics teachers use their pedagogical content knowledge for teaching Euclidean geometry in the way that they do?

An outline of the descriptions and justifications of the research design and approaches are articulated in this chapter. It also describes the methodology followed with respect to data collection, recording, summarising, analysing and interpretation. Participants, sampling procedures and ethical considerations that were used are also articulated in this chapter.

4.2 Research paradigm

According to O'Donoghue (2018, pp. 23-24)

> "Research paradigm is a framework or a pattern that guides a scientific study, and it functions as a map which clearly defines acceptable theories and methods to solve defined problems."

The research paradigm helps the researcher to describe reality, knowledge and truth (Rahi, 2017). Furthermore, the research paradigm influences the way researchers think about the topic (Kamal, 2019). The purpose of conducting research is mainly to solve problems and address knowledge gaps, as well as to influence policy (Kassa, 2017). A research paradigm helps the researcher define a valid research (Ridder, 2017) since paradigms represent researchers' thoughts, values and beliefs about the world (Kamal, 2019).

Positivism, interpretivism, critical and postmodernism are four major research paradigms used to guide research methods (Ryan, 2018) and each paradigm is based upon different assumptions about how knowledge is generated and accepted as valid (O'Donoghue, 2018). Ryan (2018) concludes by saying that being able to justify the decision to reject or adopt a paradigm forms the basis of any research, and therefore it is important to understand these paradigms and their origins, and to decide which is most appropriate for a particular study.

4.2.1 Interpretivism

The interpretivist paradigm was considered to be the most relevant paradigm in this study, because it emphasises the importance of social interaction as the basis for knowledge where the researcher uses his or her skills as a social being to try to understand how others understand their world (O'Donoghue, 2018). Therefore, knowledge is constructed by mutual negotiation and is specific to the situation being investigated.

Interpretivism argues that truth and knowledge are culturally and historically situated; they are based on people's experiences and their understanding (Ryan, 2018). Ryan (2018) further argues that one can never separate researchers from their own values and beliefs, and these will inform the way in which data is collected, interpreted and analysed. According to Archibald (2016) interpretivism is particularly useful when research is conducted in natural settings, such as the classroom. Interpretivists believe in deep understanding and interpretation of a concept and they explore the understanding of the world in which they live by developing subjective meanings of their experiences (Rahi, 2017).

The purpose of using the interpretivist paradigm is to provide information that will allow the researcher to "make sense" of the world from the perspective of participants (Bygstad, Munkvold, & Volkoff, 2016, pp. 83-96). Munkvold and Bygstad (2016, pp. 83-96) continue that the aim of interpretive research is to "understand phenomena through accessing the meaning that participants assign to them." According to Baker and Young (2016, p.26) an interpretivist research is required when one is studying phenomena that cannot be reduced or divided into discrete variables. Baker and Young (2016, p.26) also propose that "the purpose is to fully describe and understand a phenomenon in its wholeness and within a real world or natural context that can possibly inform other research settings." Interpretivism's main tenet is that research can never be objectively observed from the outside but must rather be observed from the inside through the direct experience of the people (Rahi, 2017).

Researchers working within this paradigm seek to understand rather than to explain (Cohen, Manion, & Morrison, 2013). According to Rahi (2017), interpretive researchers place strong emphasis on the better understanding of the world through first-hand experience, truthful reporting and quotations of actual conversation from insiders' perspectives. They employ data gathering methods that are sensitive to context, and which enable rich and detailed, or thick descriptions of social phenomena by encouraging participants to speak freely (Le Grange, 2018). Researchers who follow interpretive paradigm view the world through the experiences and perceptions of the participants, and they use those experiences to interpret and construct their understanding from collected data (Rahi, 2017). The researcher in this study followed the interpretive paradigm since this study explored teachers' pedagogical content knowledge when teaching Euclidean geometry. Each participant was given a questionnaire and was interviewed. Gathered data was used to construct and interpret participants' understanding.

4.3 Research approach

This research study focussed on exploring Grade 11 mathematics teachers' pedagogical content knowledge when teaching Euclidean geometry, making this study qualitative in nature (Taylor, Bogdan, & DeVault, 2015). In a qualitative study, a researcher works towards gaining insights into participants' thoughts and feelings about a particular problem, and this may provide the basis for future qualitative studies (Sutton & Austin, 2015). According to Rahi (2017), quantitative and qualitative research methods are the most dominant and commonly used research methods.

Qualitative methods are used to collect in-depth details on a particular subject, whereas a quantitative method is a scientific method which focuses on fresh data collection relating to the problem from a large population and analysis of the data but ignores an individual's emotions and feelings or environmental context (Kumar, 2019). Qualitative research tends to be field focussed, for example, going out to schools, visiting classrooms, observing teachers and analysing textbooks used (Eisner, 2017). According to Kamal (2019), qualitative researchers explore, and are more concerned with, how individuals or groups of people describe their own worlds.

In qualitative research, participants are studied in great detail through in-depth questionnaires, interviews and classroom observations. According to Ary, Jacobs, Irvine, and Walker (2018), qualitative research seeks a deeper understanding and discovering of the situation by focusing on the total picture rather than analysing it numerically. Jing and Huang (2015) maintain that qualitative research is a comprehensive exploration process which focuses more on understanding and interpreting the social phenomena in natural settings that people give meaning to. Qualitative researchers use non-quantitative research methods in providing descriptions of current situations, and through the qualitative research approach, a researcher is able to analyse various conditions which may cause people to behave in a particular manner (Kapur, 2018). This research approach is generally appropriate if a researcher wants to understand participants' perspectives, explore the meanings they give to phenomena, or observe a process in depth (Green & Thorogood, 2018). Qualitative researchers explore, investigate, describe, explain, and learn about social phenomenon (Leavy, 2017).

According to Kumar (2019), useful information cannot be reduced to numbers. People's ideas, feelings, emotions, beliefs, etc. can only be recorded as words. Taylor et al. (2015, p. 7) refer to "qualitative research as the kind of research which produces language data, i.e., peoples' own written responses or spoken words, and observing their actions". Qualitative researchers do not base their research on what they can measure or count; they are more concerned with the meaning of how people see things from their own real life experiences (Berger, 2015). According to Holloway and Galvin (2016), researchers using a qualitative approach immerse themselves in a culture or group by observing its people and their interactions, often by participating in activities, interviewing key people, taking life histories, constructing case studies, and analysing existing documents or other cultural artefacts. The qualitative researcher's goal is to ascertain an insider's view of the group under study (Taylor et al., 2015). According to Lewis (2015), the methods or procedures of qualitative research are usually seen as inductive and emerging, and are also shaped by the researcher's own subjective experience and meaning in collecting and analysing the data. He continues that research questions may also change in the middle of the study to better reflect the types of questions needed to understand the research problem and the planned data collection strategy will need to be modified to accommodate the new questions.

4.4 Research design

Taylor et al. (2015, p. 3) state that "research design refers to the ways or methods of approaching problems and seeking solutions" or a framework created to find solutions to research questions. It is a guideline or an instrument to be followed when trying to address the research problem (Leavy, 2017). Furthermore, it expresses the procedures for conducting the study, including aspects such as from whom, when, and under what conditions will the data be obtained (Creswell & Poth, 2017). According to Walliman (2017), there are different types of research designs that are suitable for different types of researches, and choosing which design to use depends on the nature of the research problem. He continues that each research design applies a range of research methods in collecting and analysing data.

4.4.1 Descriptive research design

According to Walliman (2017), descriptive research design mostly depends on observation as a means of data collection, where all observations are written and recorded, in order to analyse data. This qualitative study applied descriptive research design as all three participants were observed during teaching Grade 11 Euclidean geometry. Observations were written down in Appendix 4, and they were also recorded.

4.5 Case studies

According to Gustafsson (2017) case studies can be defined as intensive studies about a person or a group of people to explore and understand a setting. This study explored grade 11 mathematics teachers' PCK when teaching Euclidean geometry. Participants' lessons were observed when teaching. "Case studies allow a phenomenon to be studied in its real-world context" (Bass, Beecham, & Noll, 2018, p. 13). According to Yin (1981, p. 98), "The need to use case studies arises whenever an empirical inquiry must examine a contemporary phenomenon in its real-life context and especially when the boundaries between phenomenon and context are not clearly evident".Yazan (2015) described case study research as one of the most commonly used qualitative research strategies in educational research. Furthermore, case studies provide deeper understanding of a particular phenomenon studied, and because of their explorative nature and by using various methods, they are generally used to explore phenomenon within its natural context (Nielsen, Kines, Pedersen, Andersen, & Andersen, 2015).

Ary et al. (2018), stated that case studies investigate actions and attitudes of individuals, groups or institutions by applying a variety of data sources such as document analysis, interviews and observations, and the more all these methods are used in one study, the stronger the case study data will be. Case study researchers may gain a deeper understanding of why something happens or happened, and what might be important to investigate in future research (Bass et al., 2018). In this study, questionnaires, interviews and lesson observations were methods used to gather data.

4.5.1 Multiple-case study

This study builds on a multiple-case study in which the researcher interacted with three Grade 11 mathematics teachers in the Umlazi district, Phumelela Circuit in KwaZulu-Natal. Single-case design and multiple-case design are two basic types of case study designs. Single-case design is when, for example, a single school is selected as the subject of a case study, and a multiple-case design is when conclusions are drawn from different schools (Nielsen et al., 2015).“The rationale behind using a multiple-case study over a single-case study is to obtain a more comprehensive understanding of the subjects studied” (Wahyuni, 2012, p. 72). According to Gustafsson (2017), a multiple-case study allows wider exploring of research questions, and hence creates findings that are stronger and more valuable.

4.6 Ethical considerations

Permission to conduct the study was sought from the Principals of three secondary schools in the Umlazi District, Phumelela circuit. They were given consent letters to sign (Appendix 5). Subsequently, informed consent letters were given to the participants (three Grade 11 mathematics educators) and the purpose of the study and participants’ rights such as withdrawal from participation should they wish to do so without being compelled to give an explanation were explained (Appendix 6). In this study, parents and learners were also given informed consent letters to sign (Appendix 7 and 8 respectively) since participants were going to be observed during lesson observations.

According to Walliman (2017), working with human participants in a research will always raise ethical issues about how well they are treated, and the question of honesty in the way data is collected, analysed and interpreted. He adds that participants should be treated with respect throughout the research process. In this study, all participants remained anonymous and were all treated with respect. Each participant signed a consent form, which explained the purpose of the study and their rights such as withdrawal from participating should they wish to do so without being compelled to give an explanation.

4.7 Population

The research population is generally a substantial collection of those individuals with similar characteristics, which serves as the central point of a scientific enquiry (Archibald, 2016). Due to cases of large population sizes, researchers may select a subsection of the population, known as the study population. This research study population comprised of three in-service grade 11 mathematics teachers in the Umlazi district within the Phumelela circuit.

4.8 Sample method

Due to this large population size, the researcher selected a subsection of the population referred to as the study population. It is therefore from this population that the research sample was drawn. According to Walliman (2017) sampling is the process of selecting a small group of cases from a large group.

4.9 Sample size

In this study, the sample consisted of three high schools, and only 1 Grade 11 mathematics teacher per school was selected. Sampling is central to the practice of qualitative research methods (Robinson, 2014). Convenience and purposive sampling were used in this research study. According to Valerio et al. (2016), the rationale behind using convenience and purposive sampling is that it is impossible to use all mathematics teachers as their numbers are almost finite. Participants were selected based on the fact that they were easily and conveniently available, and were willing to provide data by virtue of their experience and knowledge (Robinson, 2014). The purpose of using convenience sampling is also based on affordability and accessibility (Etikan, Musa, & Alkassim, 2016). Relevant participants were purposively identified due to the qualities they possessed (Orcher, 2016).

4.10 Coding of the three participants

Codes were used in this study to maintain the anonymity of the three sampled schools and to protect the identity of the three sampled participants (Rahim, Allana, Steinke, Ali, & Khan, 2017). According to Coffelt (2017), maintaining the anonymity of the study participants is an ethical practice designed to protect their privacy while collecting, analysing, and interpreting data.

Table 4.1 shows codes used to identify both the three participants and their respective schools.

Table 4.1 Coding of the three participants

School	Teacher	Code	Gender
School A	Teacher A	TA	Male
School B	Teacher B	TB	Male
School C	Teacher C	TC	Female

4.11 Research instruments, data collection methods and procedures

This section describes the instruments that were used in collecting data. According to O'Donoghue (2018), a method is a program or a set of procedures for designing, conducting and reporting research; it is the technique or process used to collect and analyse data, e.g., questionnaires, interviews and observations.

An open-ended questionnaire, face-to-face interviews and lesson observations were the data collection methods used in this study. The interview schedule and questionnaires were designed according to the objectives of the study. An open-and closed-ended questionnaire (Appendix 2) was given to the three participants during the first meeting; thereafter semi-structured face-to-face interviews were conducted during the second meeting using Appendix 3, and, lastly, lesson observations were conducted during the last meeting using Appendix 4.All the meetings were conducted at the schools where the three participants teach, and field notes were taken in all three conducted meetings.

Semi-structured face-to-face interviews and lesson observations lasted about 30 minutes each and these were audio and video recorded and were then transcribed prior to data analysis stage. A folder of field notes was also maintained to complement the audio and video-taped recordings. This allowed the researcher to comment on environmental contexts and other behaviours, and would act as a reminder of other factors that could be important when analysing data. These notes were safely kept as they were relevant to the study and contained information that may be sensitive. According to Sutton and Austin (2015), the process of transcribing is done to convert the participant's spoken words to written words so as to facilitate the analysis of data. Coding began immediately after the transcription of all audio and video recordings, and was done by hand on a transcript copy. Compiled field notes were also useful in facilitating the coding process. Codes were then drawn together from the transcripts to form themes, and each theme became a section heading in the report.

4.11.1 Open- and closed-ended questionnaire

This study used both open- and closed-ended questions (Appendix 2). Part A of Appendix 2consisted of closed-ended questions focussing on teacher qualifications and experience in mathematics teaching. This included secondary education, professional development and mathematics teaching experience, especially of grade 11, as this study focused in grade 11. From Part A, the three participants were expected to individually respond about their background content knowledge from secondary education to professional development.

Part B of Appendix 2consisted of open-ended questions, focussing on teacher content knowledge. In Part B, all three participants were expected to individually indicate whether they studied Euclidean geometry in grade 12, and to give their background details on how they were taught Euclidean geometry in school as well as how they taught learners the Euclidean geometry theorem which states that: "The angle between the tangent and chord at the point of contact is equal to the angle in the alternate segment."

According to Archibald (2016), questionnaires may be described as a set of questions used in gaining knowledge and are often used in research for collecting data related to experiences, opinions and interests which cannot be easily gained through simple observation. Questionnaires are a useful research method for gathering original data about people's behaviour, their experiences, social interactions, opinions and attitudes, and awareness of events (Walliman, 2017).

A closed-ended questionnaire was administered to establish the participants' qualifications and experience in mathematics teaching (Part A of Appendix 1), and Part B was open-ended and the participants were required to give their background on how they were taught Euclidean geometry from their secondary school level to their post-secondary school qualifications, and also to provide a brief explanation on how they currently taught grade 11 learners Euclidean geometry. Each participant was given a questionnaire to complete in the absence of the researcher and prior to completing the questionnaire, the scope and purpose was explained well to participants and they were assured that anonymity would be maintained.

According to Popping (2015), open-ended questions are questions in which possible answers are not necessarily suggested, and the respondents answer in their own words, while closed questions seek quantitative data from the respondents. Using open-ended questions enables the researcher to explore in-depth by posing complex questions that can reveal people's practical experiences, interpretations and understandings of particular settings as well as their reactions to them (McGuirk & O'Neill, 2016). Open-ended questions allow participants to respond in their own words.

4.11.2 Semi-structured qualitative face-to-face interviews

In the semi-structured face-to-face interviews of this study, the three participants, TA, TB and TC, were free to respond to the open-ended questions as they wished. According to McGuirk and O'Neill (2016), semi-structured face-to-face interviews are commonly used since they provide a flexible technique for small scale qualitative data. From interview questions (Appendix 2), participants were expected to respond to the causes of learner poor mathematics performance, teaching methods or techniques they used when teaching Euclidean geometry (teacher pedagogical knowledge), learning theory/theories they incorporated in teaching Euclidean geometry and any challenges they had, especially teacher content knowledge, in teaching Euclidean geometry. There were ten semi-structured face-to-face interview questions which were conducted immediately after each lesson observation in this study.

These interview questions were developed based on the teaching and learning of Euclidean geometry and were used to collect qualitative data regarding mathematics teachers' views on poor learner performance in Euclidean geometry and the skills they used to effectively teach Euclidean geometry. Additional questions were also asked as follow-ups and probes on the participants' responses when needed. Field notes were made and interview sessions were also audio-recorded and analysed.

Semi-structured interviews were used as a means to explore participants' written responses. They were preferred because of their flexible approach to gaining insight into participants' thinking and to tap into their Euclidean geometry pedagogical content knowledge. The participants, who were grade 11 mathematics educators, were individually interviewed after completing the questionnaire and before the classroom observation session. The interview focused on the participants' explanations about their own experiences in the teaching of Euclidean geometry, and to encourage them to generate narratives on their experiences relating to the teaching of Euclidean geometry, the methods they used, and theories they incorporated. The semi-structured interviews allowed the participant and the researcher to connect in a dialogue allowing for questioning of responses by the interviewer. These one-on-one interviews were noted and also audio recorded for the purpose of storing the data in its original form for analysing at a later stage.

The researcher collected data by interviewing the participants (Seidman, 2013) using a face-to-face semi-structured interview to gather descriptive data in the respondent's own words (Brinkmann, 2014). According to Brinkmann (2014) face-to-face, semi-structured interviews are selected because they are well suited for the exploration of the perceptions and opinions of respondents regarding complex and sometimes sensitive issues and enable probing for more information and clarification of answers. Ribbins (2007) further states that" we interview people to explore their views in ways that cannot be achieved by other forms of research and report our findings as near as we reasonably can in their words". The advantage of a semi-structured interview is that it is not as rigid as a structured interview as it can be modified based upon the interviewer's perception of what seems most appropriate (O'Keeffe, Buytaert, Mijic, Brozović, & Sinha, 2016). During the interview session, Appendix 3 was used to record participants' responses to interview questions. Interviews were also audio-recorded and later transcribed for analysis (Doody & Noonan, 2013).

4.11.3 Classroom or lesson observations

The third data collection instrument used was a lesson observation schedule. According to Creswell (2013), an observation is a significant tool to generate data in qualitative research. Walliman (2017) also advises that some research questions can be answered best through observing how participants act and how things look. Participants were observed when teaching Euclidean geometry in grade 11 using the lesson observation tool. This tool appears in Appendix 3. Lesson observations were used to answer research question 2 of this study: "How do grade 11 mathematics teachers use their pedagogical content knowledge when teaching Euclidean geometry?" The observation for each participant lasted for about fifty minutes. Lesson observation was for the full duration of the lesson, and Appendix 3 was used during lesson observation. Lessons were videotaped to allow there searcher to replay them while transcribing and analysing the data; this assisted in avoiding misunderstanding and distortion of the authenticity of the data (Walliman, 2017).

4.12 Field notes

In this study, data collected through questionnaire, interviews and lesson observation was recorded in Appendix 1, Appendix 2 and Appendix 3 respectively. Furthermore, data collected through interviews and lesson observation was also recorded through field notes for subsequent analysis. Field notes were taken to support the analysis of interviews and classroom observations.

4.13 Thematic analysis

The collected data was analysed and explained in response to the three research questions stated in Chapter 1 of this study through thematic analysis. Thematic analysis is a method known for identifying, analysing, and interpreting patterns of meanings or themes within qualitative data (Glisczinski, 2018). In this study, thematic analysis was used in identifying and analysing small and large patterns within and across collected data. According to Nowell, Norris, White, and Moules (2017) in qualitative data, thematic analysis provides systematic and accessible procedures when generating codes and themes and it focuses on the content of statements from the participants. Questionnaires, interviews and lesson observations helped in identifying common themes about teachers' PCK when teaching grade 11 Euclidean geometry. Themes that emerged from the thematic analyses are discussed in the next chapter:

4.14 Pilot study

According to Kumar (2019), a pilot study may be conducted in preparation for the major study to give advance warning about where the main research project could fail, where research protocols may not have been followed, or whether proposed methods or instruments are inappropriate or too complicated. Piloting in this study was conducted in the researcher's previous school. According to Coughlan & Cronin (2016), a pilot study can be described as field-testing an instrument to determine how well it works with a smaller sample of the population; difficulties with sample selection and sample participation will be diagnosed and corrected at this stage. Piloting of the data generating tools affords the researcher an opportunity to refine the tools in order to generate rich data that respond to the research questions (Yin, 2009).

In this study, interview questions, questionnaire and lessons observation tools were piloted. It was important to determine how the design could be improved and to identify flaws in the measuring instrument, and whether phrasing of words in questions required cross-checking

(Kumar, 2019). In this study, the tools were revised after the pilot study and the changes were effected in Appendix 1, Part B, question 2 and in Appendix 2, questions 5, 6, 9 and 10.

4.15 Data generation plan

Table 4.2 illustrates the process that was followed when generating data from the three participants through the three research questions of this study.

Table 4.2 Data generation plan

Research questions	Method used to generate data
1. What is Grade 11 mathematics teachers' pedagogical content knowledge of Euclidean geometry?	• Open-ended questionnaire. • Semi-structured individual face-to-face interviews.
2. How do Grade 11 mathematics teachers use their pedagogical content knowledge when teaching Euclidean geometry?	• Open-ended questionnaire. • Semi-structured individual face-to-face interviews. • Lesson observations.
3. Why do Grade 11 mathematics teachers use their pedagogical content knowledge for teaching Euclidean Geometry in the way that they do?	• Semi-structured individual face-to-face interviews.

4.16 Methodological norms

4.16.1 Trustworthiness

According to Sutton and Austin (2015, p. 229), "trustworthiness aims to evaluate the following from the research:

1. Credibility (believing in the truth from the findings);

2. Transferability (the findings of the research may apply in any other context);

3. Dependability (research findings show consistency and the findings may be repeated); and

4. Confirmability (no researcher influence on the research findings and the findings are only shaped by respondents)."

This study used qualitative approaches, interviews, questionnaires and classroom lesson observations. Qualitative data was drawn from participants (Grade 11 mathematics teachers) at different times and in different schools.

4.16.2 Credibility

Believable and trusted data and data analysis in a qualitative study is known as credibility (Smith, 2015). According to Creswell and Creswell (2017), data is credible when there is a match between research findings and reality; i.e., when the data is free from error or bias and distortion. According to Scott (1990) the question of credibility should concern the extent to which the researcher is sincere in his/her opinion and in their attempt to record an accurate account from that chosen standpoint. In this study, all participants completed questionnaires, were interviewed and were observed whilst teaching in class. This was done to avoid misleading and inaccurate conclusions (Walliman, 2017).

4.16.3 Validity of research study

An important aspect of any research instrument is its validity, which is its ability to measure what it is supposed to measure (Coughlan & Cronin, 2016).Validity refers to the extent to which inferences made based on numerical scores are appropriate, meaningful and useful to the sample (Merriam & Tisdell, 2015). Validity also checks whether the instruments provide an adequate sample of items that represent that concept (Corbin, Strauss, & Strauss, 2014). In this study, both construct and content validity were used to check if the questionnaires and interview questions really measured the concepts the study assumed it measured.

4.16.4 Reliability of research study

The reliability of a measurement procedure is the stability or consistency of the measurement (Merriam & Tisdell, 2015). This means that if the same variable is measured under the same conditions, a reliable measurement procedure will produce identical measurements (Creswell & Creswell, 2017). In other words, it refers to a measuring instrument's ability to yield consistent numerical results each time it is applied; it does not fluctuate unless there are variations in the variable being measured (Meyers, Gamst, & Guarino, 2016).

The reliability of a test or instrument refers to the extent to which it consistently measures what it is supposed to measure (Lewis, 2015). A test is reliable to the degree that it measures accurately and consistently, yielding comparable results when administered a number of times (Smith, 2015). To ensure reliability of the data collected in this study, the contents of the questionnaires and interviews were verified by an independent body (a colleague) who is knowledgeable in the line of mathematics education to ascertain the degree to which the contents of the test items and interview were in harmony with the intended purpose. The initial suggestions and input from the verification exercise from my professional colleagues led me to reframe, add and delete some existing questions.

4.17 Data analysis

Collected data from open-ended questionnaires, semi-structured face-to-face interviews and lesson observations were analysed using qualitative techniques. According to Kamal (2019) the main instrument for data collection and analysis in a qualitative study is the researcher whose aim is to explore and understand people's experiences. Hence, the data was inductively analysed: i.e., the researcher generated explanations from the collected data in the form of concepts, theories or hypotheses. Finally, rich descriptions were produced from multiple sources of data such as interviews, documents and field notes.

Qualitative data analysis is the process of reducing and transforming collected data with the goal of discovering clear, useful, manageable, understandable, insightful and trustworthy information (Walliman, 2017). Furthermore, in qualitative data analysis; identification, evaluation, and interpretation of themes and patterns in written data help answer research questions at hand. Analysing qualitative data involves these three steps: organising, description and interpretation of data (Shotter, 2017).

The open-ended questionnaire, face-to-face interviews and lesson observations all produced a wealth of narrative data or text in this study. Questionnaires generated words, phrases and paragraphs; participants' word-for-word interviews were audio-recorded and transcribed; and lesson observations were video-recorded and transcribed in the researcher's field notes. Collected qualitative data was first grouped into themes and patterns in order for the data to be analysable, and these were organised into categories to bring meaning to the words. This is the core process of qualitative data analysis and it is generally conducted in two ways: content and thematic analysis. A study may use one type or both.

4.18 Data interpretation

According to Sutton and Austin (2015), data interpretation is when meaning and significance is attached to data analysis. This is the stage where the researcher moves from a mass of words to a final report, i.e., 'cutting and sorting process'(Le Grange, 2018). In this study, themes and patterns were used to explain the findings.

4.19 Conclusion

In conclusion, this chapter focused on the research design and methods employed to collect and analyse qualitative data. A questionnaire, interviews and lesson observations were data collection methods used in this study. The next chapter presents the analysis and interpretation of the data that was generated for this study.

CHAPTER 5

PRESENTATION, ANALYSIS AND INTERPRETATION OF DATA

5.1 Introduction

The previous chapter described and explained in detail the qualitative research design and methods employed in this study. The purpose of this study was to explore grade 11 mathematics teachers' pedagogical content knowledge when teaching Euclidean geometry. It sought to explore how grade 11 mathematics teachers use their pedagogical content knowledge when teaching Euclidean geometry.

This chapter presents, analyses and interprets the data generated from this qualitative study. The researcher was able to collect and analyse the data qualitatively by means of an open- and closed-ended questionnaire, individual semi-structured face-to-face interviews and lesson observations.

5.2 Themes

An open- and closed-ended questionnaire a semi-structured face-to-face interview and lesson observations from the three participants assisted the researcher in this study to identify common themes focussing on Euclidean geometry. Three themes that emerged from data are as follows:

- Exploring the teaching of Euclidean geometry;
- Methods of teaching that emerged from this study; and
- Exploring teachers' pedagogical content knowledge when teaching Euclidean geometry.

The following section discusses these three themes in detail.

5.3 Exploring the teaching of Euclidean geometry

5.3.1 Perceptions of participants regarding possible causes of learners' poor performance in Euclidean geometry

This study focuses on grade 11 mathematics teachers' pedagogical content knowledge (PCK) when teaching Euclidean geometry. The study focussed mainly on what grade 11 mathematics teachers do in their classrooms when teaching Euclidean geometry.

The view of participant Teacher A (TA) regarding possible causes of learner poor performance in mathematics is as follows:

> **TA:** "*…It is at times lack of content knowledge from teachers…*"

Teacher content knowledge is an important element of what is to be taught and learned. It is obvious that when mathematics teachers lack content knowledge, the quality of mathematics teaching and learning will be compromised, and hence, mathematics learners' performance will be at risk. Mathematics teachers who lack content knowledge may lose interest in what is to be taught and may end up teaching less or not teaching the topic at all (Good & Lavigne, 2017). This will negatively affect learners' performance as they will be progressing with content gaps. Therefore, mathematics teachers having content knowledge should not be a matter of chance if the quality of mathematics teaching and learning is to improve.

Finding possible causes of poor learner performance in mathematics has been ongoing in research in mathematics education (García-Santillán et al., 2016). However, according to Vidermanova and Vallo (2015), the nature of teachers' knowledge of geometry and methods used by teachers to teach geometry are implicated as one of the factors responsible for poor learner achievement in geometry. Research conducted in many South African schools has shown that as well asthe lack of resources, many South African school teachers still lack basic content knowledge, and this has contributed to the poor mathematics performance of the learners (Heeralal & Dhurumraj, 2016). Ottmar et al. (2015) also confirm that poor learner performance in mathematics may be due to teachers lacking content knowledge, and some other unpleasant experiences including among others, poor teaching strategies by some teachers (Sithole et al., 2017).

5.3.2 Teachers' background knowledge of Euclidean geometry

Based on the presented data in Tables 5.1, 5.2 and 5.3, it was noted through a closed-ended questionnaire that all three participants possessed a mathematics teaching qualification and a range of mathematics teaching experience as they had been teaching mathematics for an average of more than twenty-five years. All three participants were taught Euclidean geometry during their secondary education. Hence, this study assumed that these participants did not lack content knowledge. Teacher qualifications and experience play an important role in the teaching of mathematics. Qualified and experienced mathematics teachers are more confident in the classroom than unqualified and less experienced mathematics teachers (Tella, 2017). Tables 5.1, 5.2 and 5.3 that follow present the three participants' qualifications and experience in teaching grade 11mathematics.

Table 5.1: Teacher A's qualifications and experience in the teaching of grade 11 mathematics.

Teacher A	**Yes/ No**	**Year**	**Maths symbol**
Matriculated	Yes	1986	D (50-59%)
Post-Matric qualification(s)	**Institution**	**Year**	**Major subjects**
One-year N3 Electrical certificate	Technical College	1988	Mathematics
Three-year STD	Teacher College	1992	Mathematics
Higher Diploma in Education	Teacher College	1999	Mathematics
Teaching experience in mathematics	**Grade 10**	**Grade11**	**Grade12**

	26 years	26 years	26 years
Grade(s) currently teaching	✓	✓	✓
Number of learners in your class	35	30	25

Table 5.2: Teacher B's qualifications and experience in the teaching of grade 11 mathematics.

Teacher B	**Yes/ No**	**Year**	**Maths symbol**
Matriculated	Yes	1983	B (70-79%)
Post-Matric qualification(s)	**Institution**	**Year**	**Major subjects**
BPAED Arts Degree	University	1988	Mathematics
Teaching experience in mathematics	**Grade 10**	**Grade11**	**Grade12**
	29 years	29 years	29 years
Grade(s) currently teaching	✓	✓	✓
Number of learners in your class	40	35	30

Table 5.3: Teacher C's qualifications and experience in the teaching of grade 11 mathematics.

Teacher C	**Yes/ No**	**Year**	**Maths symbol**
Matriculated	Yes	1974	C (60-69%)
Post-Matric qualification(s)	**Institution**	**Year**	**Major subjects**
Four-year Bachelor of Arts degree	University	1980	Mathematics
Teaching experience in mathematics	**Grade 10**	**Grade11**	**Grade12**
	30 years	30 years	30 years
Grade(s) currently teaching	✓	✓	✓
Number of learners in your class	45	40	35

According to Aslam, Rehman, Imran, and Muqadas (2016), teacher qualification may be defined as a special teaching skill or subject matter knowledge or type of teaching experience that makes a teacher suitable to teach. Such teaching skills include among others, formal education, subject matter knowledge and teaching experience (Ho, Lee, & Teng, 2016). Furthermore, according to Yates and Johnston (2018), teacher formal education is crucial, and formal education means various academic diplomas or degrees a teacher may possess in order for him or her to teach.

Teachers with formal education are significantly more effective than those without(Kola & Sunday, 2015). According to Aslam et al. (2016), learners taught by better qualified and experienced mathematics teachers perform better in mathematics than those taught by less qualified and inexperienced mathematics teachers. As mathematics teachers gain experience in their teaching profession, their effectiveness tends to improve (Kini & Podolsky, 2016). Mathematics teachers bring their beliefs, perceptions, and their learning experiences into their classrooms (Tutak & Adams, 2017). Classroom practice style depends on the teacher's earlier experiences with the learners and may even be greatly influenced by the teacher's own time as a learner (Dean, 2019).

Subject matter knowledge is further acquired while the teacher is in pre-service teacher training, which prepares pre-service teachers to become qualified teachers(Kola & Sunday, 2015). According to Bietenbeck et al. (2017), teacher subject matter knowledge has a positive and significant impact on learner performance. Teacher subject matter knowledge is necessary but not enough for effective and quality teaching, as a teacher needs to make his or her mathematics knowledge understandable to the learners that he or she teaches (Bold, Filmer, Martin, Molinad, et al., 2017). Hence, improving teaching methods used by teachers in their classrooms may yield positive results in learner performance.

Although all three participants acquired subject matter knowledge during their pre-service training, Teacher B (TB) indicated that:

> "*…at the University we were not provided with much practical content training in Euclidean geometry…* "

In this study, two participants out of three were trained at the University, and hence two participants were not provided with much practical training of Euclidean geometry, and as a result, these teachers may not be able to expose the learners to the practical learning of Euclidean geometry. According to Biyela et al. (2016), pre-service teacher training programs constitute the foundation for competent mathematics teaching, and are therefore expected to produce competent and effective mathematics teachers who are ready to teach mathematics effectively.

Mathematics teachers should, at the most basic level, have background knowledge of mathematics content for the grade they are teaching, for the earlier grades, as well as for the several grades beyond that which they are expected to teach in order for them to deal adequately with learners' difficulties in Euclidean geometry (Venkat & Spaull, 2015).

The following are the responses of the three participants from the individual semi-structured face-to-face interviews responding to how they were taught Euclidean geometry in secondary school, especially in grade 11.

> **TA:**…"*I was taught about circles, parallel lines, cyclic quadrilaterals, how to prove the theorems*". TA further stated that teachers explained theorems on the chalkboard from textbooks, and all exercises were taken from the textbook"…*we did not investigate theorems, and we were taught to follow textbooks' methods"*

> **TB:**…"*We were first taught all Euclidean geometry theorems, and after each theorem, we were given exercises to work on. A lot of work was self-study, meaning we were given exercises to practice on our own. At University, we were not provided with much practical or content training in geometry.*"

> **TC:**…"*Teachers first taught us to prove the theorems. We were then given application problems to solve from textbooks*."

All three participants in this study agreed that mathematics teachers taught Euclidean geometry from textbooks and that all exercises were taken from textbooks. TA mentioned that teachers were simply following textbooks when teaching Euclidean geometry theorems and they were explained on the chalkboard. TB and TC also confirmed that teachers first explained Euclidean geometry theorems from textbooks, and after each theorem, application exercises were also given from textbooks.

According to Lepik, Grevholm, and Viholainen (2015), mathematics classroom teaching, in many cases, is still generally organised around and delivered through the use of mathematics textbooks. When learners are taught according to what is written in the textbooks, they remain passive listeners as textbooks may not meet the needs of all the learners (Vidermanova & Vallo, 2015). According to Dean (2019), mathematics teachers need to be able to combine various teaching methods to give different styles of teaching.

The role of mathematics teachers is to facilitate learners' thinking and learning, and should attempt to motivate learners to learn (Kösa, 2016). According to Abiam, Abonyi, Ugama, and Okafor (2016), Euclidean geometry continues to be regarded by learners as an area of greatest challenge to learn and is difficult for teachers to teach. Various strategies of teaching Euclidean geometry where learners are active constructors of knowledge may develop their thinking abilities and reasoning skills when solving Euclidean geometry problems (Archibald, 2016).

5.4 Methods of teaching that emerged from this study

This section explores the methods of teaching Euclidean geometry used by the three participants, TA, TB and TC respectively. These methods originated from the three participants themselves and they were based on the open-ended questionnaire, semi-structured face-to-face interview and lesson observation.

When the participants were asked how they teach Euclidean geometry and a theorem which states that: **"The angle between the tangent to a circle and the chord drawn from the point of contact is equal to the angle in the alternate segment"**, the views of the three participants coded as TA, TB and TC were as follows:

5.4.1 Exploring Teacher A's (TA) teaching methods

> **TA:**…"*I teach the proof in a formal way*". *I start by basic information, e.g. corresponding angles, alternate angles, triangles and parts of the circle, then explain how to prove theorems and give reasons. I explain the theorem from different positions, as well as emphasise that the other angle must be at the circumference. I then do simple application problems.*"

5.4.2 Exploring Teacher B's (TB) teaching methods

> **TB:**…"*Foundation is important. I explain basic concepts, e.g. points, lines, triangles, etc, and learners measure and discover the size of angles and lines. I do many examples, and give enough for learners to practice. I walk around, and assist learners. I then explain what a tangent and a chord are, and then ask the learners to locate them in the diagram. I explain the proof, then show them how to prove the theorem using* x *and* y.

Thereafter we go through the examples that require the application of the theorem. If learners experience some difficulty, we construct a circle; draw a tangent to the circle and a chord from the point of contact of the tangent. Using a protractor, we measure the angle formed between the angle and the chord as well as the angle subtended in the alternate segment. We then arrive at the conclusion that the angle between a tangent and a chord is equal to the angle in the alternate segment. Thereafter, we start with numerical riders, proceeding to the proof. Learners study the diagrams and identify the theorems presented."

5.4.3 Exploring Teacher C's (TC) teaching methods

TC:…"*For learners to learn better, I teach from the known to the unknown. There are ten main theorems excluding their converses. I teach theorem by theorem, and its converse, and allow learners to apply. I explain the proof using prior knowledge. Grade 11 requires knowledge from previous grades. I revise grade 10 geometry with the learners prior to grade 11geometry, and always highlight key concepts. I give learners a circle with a tangent, and chords in a circle are joined. Learners are asked to measure the angle between a tangent and a chord, and the angle in alternate segment. Learners are then given simple calculation problems and later given proofs of problems to solve.*"

Based on the responses from the three participants, proving Euclidean geometry lessons start by explaining the theorem and then giving learners application problems. In TA, TB and TC's classes, learners do not experience the discovery of geometric theorems, nor invent any knowledge, but they are told definitions, theorems are explained and geometric problems and proofs are assigned. This was also noted when TA, TB and TC were observed when teaching, where previous work of three geometric diagrams was discussed on the board; thereafter, the theorem was explained to the learners on the board step by step.

According to the DoE (2011c), learners are required to investigate, prove and apply the theorems when solving riders. Explaining theorems to the learners may not be effective if it is done in a way that imposes a heavy cognitive demand (Bokosmaty, Sweller, & Kalyuga, 2015). To deal adequately with learner difficulties regarding the teaching of Euclidean geometry theorems it is important for teachers to have deep levels of both content knowledge and pedagogical content knowledge (Depaepe et al., 2015).

According to Tella (2017), both Euclidean geometry content knowledge and pedagogical content knowledge of mathematics teachers is important, and mathematics teachers need to be fully equipped with necessary Euclidean geometry content and pedagogical content knowledge in teacher education programmes. For teachers to be effective in teaching, they must have a clear understanding of the subject they teach (Bold, Filmer, Martin, Molina, et al., 2017). According to Minor, Desimone, Lee, and Hochberg (2016), teacher subject knowledge and teacher learning experience play an important role in classroom practice, especially in explaining new concepts and in addressing learner misconceptions. Ghousseini and Herbst (2016) suggest that subject knowledge is essential not only for teaching purposes but also for the selection of relevant teaching aids and using technology. They also add that teachers with strong subject knowledge may teach in a more dynamic and interesting manner.

Effective classroom environment is critical for promoting learners' conceptual understanding of Euclidean geometry (Copur-Gencturk & Papakonstantinou, 2016). However, according to Ghousseini and Herbst (2016), high school mathematics classroom practice in the South African education system has a challenge of creating a mathematics learning environment where learners are actively engaged with mathematics and where the needs of diverse learners are met. This was confirmed in this study as all three participants had indicated the traditional way of teaching Euclidean geometry theorems, where geometry learning is not collaborative but more teacher-centred. Mathematics lessons need to be more learner-centred where learners are actively involved in the construction of knowledge.

5.5 Exploring teachers' pedagogical content knowledge when teaching Euclidean geometry

This study also looked at the participants' lesson preparation and presentation, teacher methods or techniques, and learning theories teachers incorporated in teaching Euclidean geometry. Lesson observations were conducted particularly to observe teacher's pedagogical content knowledge when teaching Euclidean geometry in grade 11. Three lessons were observed from three different schools, school A, school B and school C. Teacher A, teacher B and teacher C were observed teaching Euclidean geometry in grade 11. Each lesson observed lasted an hour with an average of 35 learners per class.

The lesson observation tool was used; this tool appears in Appendix 3. Lesson observations were used to respond to research question 2 of this study: ***How do grade 11 mathematics teachers use their pedagogical content knowledge when teaching Euclidean geometry?*** The observation for each participant lasted for about thirty minutes. All three participants had evidence of Euclidean geometry lesson preparation. Lessons taught focussed on analysing and proving Euclidean geometry theorems. Teacher and learner activities were stated. The lesson observation tool (Appendix 3) provided the description of a lesson, looking at the introduction, teaching resources, lesson presentation and teacher methodology, learner involvement, learner activities and learner assessment.

5.5.1 Exploring TA's pedagogical content knowledge when teaching Euclidean geometry:

5.5.1.1 Topic

Theorem 4 which states that: "Angles subtended by the same arc are equal."

5.5.1.2 Brief description of a lesson

Learners were expected to prove and apply the theorem which states that: "The angles subtended by the same arc are equal."

5.5.1.3 Lesson introduction

Previous work of three diagrams was discussed on the board; thereafter the theorem was explained to the learners on the board step by step.

5.5.1.4 Prior knowledge identified

Learners appeared to be aware of theorems previously discussed.

5.5.1.5 Presentation

Learners were seated and worked in pairs. Worksheets and notes were provided from different textbooks. TA discussed previous concepts on the board from homework exercises, and learners appeared to have the required prior knowledge. Diagrams drawn on the chalkboard were explained using different colours. Theorem 4, thereafter, was explained to the learners using the chalkboard. Common errors and misconceptions were identified and corrected during the lesson. The question and answer method was used. Although all learners had textbooks they also had Euclidean geometry booklets with all theorems, notes, and activities.

5.5.2 Exploring TB's pedagogical content knowledge when teaching Euclidean geometry:

5.5.2.1 Topic

Proving the theorem which states that: "The angle between a tangent and a chord is equal to the angle in the alternate segment".

5.5.2.2 Brief description of a lesson

Learners were expected to prove and apply the theorem which states that: "The angle between a tangent and a chord is equal to the angle in the alternate segment."

5.5.2.3 Lesson introduction

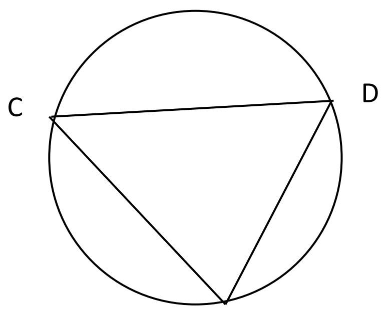
The teacher explained tangent and a chord using chalkboard illustrations, and different chalk colours were used.

5.5.2.4 Prior knowledge identified

Learners appeared to have done application exercises of this theorem; they were able to solve practical examples by applying this theorem.

5.5.2.5 Presentation

Learners were seated and worked in pairs. TB explained tangent and a chord using chalkboard illustrations, different chalk colours were used. The position of an angle between a tangent and a chord was illustrated on the chalkboard. Learners understood these concepts. Learners appeared to have done application exercises of this theorem; they were able to solve practical examples by applying this theorem. The most common misconception was addressed, i.e. the position of an angle in the alternate segment. Figure 5.1 illustrates this theorem.



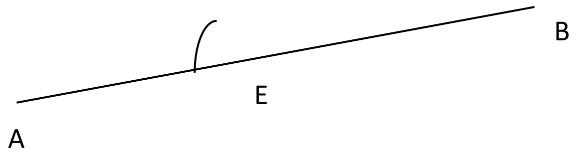


Figure 5.1 Tan-chord theorem

Some learners were not clear whether A$\hat{E}$C equals to E$\hat{C}$D or E$\hat{D}$C, but this was addressed. Chalkboard, textbook and worksheets were used in this lesson. This was mostly a teacher demonstration and explanation lesson, and the questions and answer method was used. TB was walking around checking if the learners were managing to prove the theorem and class exercises.

5.5.3 Exploring TC's pedagogical content knowledge when teaching Euclidean geometry:

5.5.3.1 Topic

Proving and application of the theorem which states that: "The angle subtended by a diameter at the circumference of a circle is a right angle."

5.5.3.2 Brief description of a lesson

Learners were expected to prove and apply the theorem which states that: "The angle subtended by a diameter at the circumference of a circle is a right angle."

5.5.3.3 Lesson introduction

Review of homework was done, and learners were asked to discuss and present their approaches. The theorem was then explained on the chalkboard where diagrams were drawn and explained to the learners.

5.5.3.4 Prior knowledge identified

Recap on theorem 2 which states that: "The angle which an arc of the circle subtends at the centre of the circle is twice the angle it subtends at any point on the circle." It was also emphasised that this theorem forms part of the examinable theorems. Previous grades concepts were also discussed and learners were given an opportunity to ask questions after every concept discussed.

5.5.3.5 Presentation

> Learners were seated and worked individually as well as in pairs. All had workbooks and worksheets. Recap and application of the theorem which states that: "The angle which an arc of a circle subtends at the centre of a circle is twice the angle it subtends at any point on the circle" was discussed. Review of homework was done, and learners were asked to discuss and present their approaches. The theorem was then explained on the chalkboard where diagrams were drawn and explained to the learners. Previous grades concepts were also discussed and learners were given an opportunity to ask questions after every concept discussed. Notes on circle geometry with activities were provided to the learners, and also included to the notes were converse theorems. The question and answer method was used in this lesson. It was also explained to the learners that this was a non-examinable theorem, and learners needed to apply the theorem.

Based on the evidence, only one kind of teaching method, known as the traditional teaching method was used in all three observed lessons. All three classes observed were managed by more teacher-centred techniques and the levels of teacher creativity were limited. It was also noted that teaching strategies used to promote learners' understanding of Euclidean geometry were limited. Not all learners participated in class discussions on the knowledge taught; the majority of the learners were passive listeners.

It was indicated earlier in this study that mathematics teachers' lacking of content knowledge may shape the patterns of poor mathematics learner achievement (Elizarov et al., 2016). However, the observation in this study supported that all three participants from the three different lessons observed were knowledgeable about Euclidean geometry content, as they were able to respond to questions posed by the learners. Therefore, teacher content knowledge may not necessarily be the cause of poor mathematics learner performance in this study. However, according to Wang, Wang, and An (2018), many learners still have difficulties in understanding the construction of geometric proofs and mathematics educators are failing to get learners to understand the construction of geometric proofs. It is therefore imperative to investigate what kinds of teaching methods teachers use when teaching Euclidean geometry.

Mathematics teachers need to be aware of different teaching methods in order to succeed in teaching Euclidean geometry (Bhagat & Chang, 2015). Euclidean geometry's fundamental skills such as visualisation, problem solving, construction, reasoning and proving can be fostered through proper teaching and learning of geometry (Arıcı & Aslan-Tutak, 2015). While the deductive method is central to geometry theorems, providing learners with meaningful and practical experiences of geometry theorems in the classroom is still a challenge for many mathematics educators.

Learners still fail to see a need for geometry proofs and get little meaningful mathematics out of the traditional, proof-oriented geometry (Fan, Qi, Liu, Wang, & Lin, 2017). Through observations of the three participants' lessons it was clear that the majority of the learners faced difficulties with proving Euclidean geometry theorems. According to Jones and Tzekaki (2016), teaching and learning of Euclidean geometry theorems have been considered as one of the most difficult mathematics content in secondary schools which causes learners' mathematics results to be poor.

5.6 Exploring the traditional method of teaching

It was observed during the lesson observations that TA, TB and TC all used the traditional methods of teaching, such as lecturing, when teaching Euclidean geometry. They focused more on teaching basic skills and computational procedures rather than concentrating on new Euclidean geometry concepts.

According to Jupri (2017), in a traditional geometry class, learners do not experience the discovery of geometric theorems, nor invent any knowledge, but they are told definitions and theorems and are assigned geometric problems and proofs. The learner-centred approach may be attained if teachers use various flexible teaching strategies (In'am & Hajar, 2017). To improve the teaching of mathematics, especially Euclidean geometry, mathematics teachers are encouraged to improve the quality of teaching methods and create meaningful classroom environments (Khalil, Khalil, & ul Haq, 2019). Mathematics teaching should not only encourage memorising of concepts but must create bridges between learners' experiences and mathematics curriculum goals (Kovács et al., 2018).

The traditional teaching method is known to be teacher-centred and a lecture method (Kösa, 2016). In a traditional teaching method, a teacher reviews previous homework, and then explains new concepts to the learners followed by learners imitating the teacher's demonstration (Lessani, Yun, & Bakar, 2017). The primary focus of this pedagogical approach is on the teacher transmitting knowledge to the learners, i.e., teaching by telling (Kösa, 2016).

Ina traditional geometry lesson, teachers tend to explain to the learners and thereafter require the learners to attempt given exercises, and few attempts are made to encourage the learners to explain their reasoning and to make logical connections (Ahamad, Li, Shahrill, & Prahmana, 2017). Mathematics teachers need to have a good understanding of mathematics content and knowledge of effective non-traditional teaching methods to effectively teach mathematics. Utilisation of non-traditional teaching methods in mathematics classrooms such as small groups, problem-solving strategies, playing games, and watching video lessons may help learners understand new concepts better (Gresham, 2018).

According to Parrot and Eu (2018), in order for mathematics teachers to meet the needs of all types of learners in the classroom, teachers need to be advised to use a variety of teaching methods and strategies to benefit all types of learners in the classroom. As indicated earlier in this study poor and outdated teaching methods (Khumalo et al., 2016), lack of learner motivation and interest (Cook, 2017) and lack of basic content knowledge (CK) are some of the factors associated with learners' poor performance in mathematics. Hence, effective classroom practice is critical and instructional methods employed in mathematics classrooms need to meet the needs of all the learners, regardless of their background, gender or race, so that their performance may improve (Schettino, 2016). According to Heeralal and Dhurumraj (2016), mathematics classrooms need to be learner-centred by allowing all learners to actively participate in mathematics lessons.

5.7 Exploring mathematics teachers' continuous professional development

Professional development may be defined as any activity aimed at improving classroom instruction or teachers' subject knowledge and skills (Early et al., 2016). All mathematics teachers need to be supported to continuously develop further in the mathematics they teach in order to have good mathematics knowledge and skills, and a good knowledge of how mathematics should be taught and learned in both primary and high schools (Junqueira & Nolan, 2016). According to Umugiraneza et al. (2017), mathematics teachers who continuously develop further in the subject they teach would be exposed to more diverse methods of teaching during their professional development programmes than those who do not continuously develop further. They further stated that not all experienced teachers are more effective, and not all inexperienced teachers are less effective. Having only a good subject knowledge may not necessarily mean good teaching (Simsek & Boz, 2016).

All three participants in this study are qualified and experienced mathematics teachers as they have been teaching grades 10, 11 and 12 mathematics for more than twenty years, and they studied mathematics in grade 12 and beyond. This may, therefore, imply that they are knowledgeable with secondary school geometry content. Being knowledgeable and confident in geometry is critical in shaping a learner's attitude towards learning Euclidean geometry (Gresham, 2018). However, according to Geldenhuys and Oosthuizen (2015), continuous teacher professional development programmes in the subject are necessary in order to improve the quality of classroom teaching and learning, and teachers need to be re-skilled in their areas of specialisation as well as in the pedagogical content knowledge.

Continuous professional development process requires constant and active learning for teachers throughout their professional life, where teachers continuously perform content activities aiming at increasing both their quality of classroom teaching as well as their professional development (Özdemir, 2019). According to Parrot and Eu (2018), these teacher professional development programmes need to be based on practical activities and not only on theoretical information where teachers are passive listeners with very low participation; they need to be relevant to their classroom practices. When these professional development programmes are content focused and use active learning where teachers are not passive listeners they may improve teacher content knowledge and teacher pedagogical content knowledge (Early et al., 2016). Some studies have indicated that teachers who participate in effective continuous professional development programmes which are content-specific are likely to attain better mathematics results for their learners (Sithole et al., 2017).

5.8 Conclusion

This chapter presented analysed and interpreted data that was generated from three participants of this study. Research questions, the data generation plan and themes that emerged from the collected data, were discussed in this chapter. The sources used to collect data were an open-ended questionnaire, semi-structured face-to-face interviews and lesson observations. Field notes were also used to collected data. Qualitative data was analysed through thematic analysis. The analysis of collected data demonstrated the ways in which the Grade 11 mathematics teachers used their PCK when teaching Euclidean geometry.

Mathematics teachers need to be aware of different teaching strategies in order to succeed in teaching mathematics, especially Euclidean geometry, so that learners may develop a solid understanding of Euclidean geometry (Kösa, 2016).Furthermore, mathematics teachers need to give learners enough confidence to discover Euclidean geometry problems and think critically in order to solve them (Bhagat & Chang, 2015). The discussion of the conclusion and recommendations will be presented in the next chapter.

CHAPTER 6

CONCLUSION AND RECOMMENDATIONS

6.1 Introduction

The purpose of this study was to explore grade 11 in-service mathematics teachers' pedagogical content knowledge (PCK) when teaching Euclidean geometry (EG). It was intended to investigate the methods used by mathematics teachers in teaching Euclidean geometry, and its results emphasise the need for seeking better methods of teaching grade 11 Euclidean geometry. The main findings of the current study are discussed in this chapter, and also a summary of how I, the researcher, achieved the objectives of the current study. This is clearly denoted in the way research questions were answered. Furthermore, some recommendations, suggestions for further research and limitations of the current study are highlighted in this chapter. I also discuss how this study was concluded.

6.2 The three critical research questions and the main findings

6.2.1 What are grade 11 mathematics teachers' pedagogical content knowledge of Euclidean geometry?

6.2.1.1Teacher qualifications and content knowledge

Based on evidence from the data, the questionnaire completed by the three participants indicated that all three participants were qualified mathematics teachers and had a range of teaching experience. It was evident that they gained knowledge of Euclidean geometry in their secondary education as well as at university. Hence, this study assumed that these three participants did not lack content knowledge. According to Bietenbeck et al. (2017), teacher subject knowledge has a positive and significant impact on learner performance.

Teacher qualifications and experience play an important role in the teaching of mathematics. According to Khumalo et al. (2016), mathematics teachers with good mathematics qualifications are content specialists in a school situation and have a huge advantage over those who do not. For example, a mathematics teacher in possession of a mathematics degree will find it easier to explain challenging mathematics problems to the learners (Tella, 2017).

6.2.1.2 Teacher pedagogical and pedagogical content knowledge

The evidence collected in this study indicated that one kind of teaching method, called the traditional teaching method, was used in all three observed lessons. What was evident was that all three classes observed were managed by more teacher-centred techniques and the levels of teacher creativity were limited. It was also noted that various teaching strategies used to promote learners' understanding of Euclidean geometry were limited. Not all learners participated in class discussions on the knowledge taught, and the majority of the learners were passive listeners. Mathematics teachers need to be aware of different teaching methods in order to succeed in teaching Euclidean geometry (Bhagat & Chang, 2015).

6.2.2 How do grade 11 mathematics teachers use their pedagogical content knowledge when teaching Euclidean geometry?

Based on the responses from the open-ended questionnaire, semi-structured face-to-face interviews and lesson observations, it was noted from the three participants that when proving grade 11 Euclidean geometry theorems, the participants started by explaining the theorem and then provided learners with application-based problems. In TA, TB and TC's classes, learners did not experience the discovery of Euclidean geometric theorems, nor invent any knowledge, but they were told definitions, theorems were explained and they were assigned Euclidean geometric problems and proofs. This was also noted when TA, TB and TC were observed when teaching grade 11 Euclidean geometry, where previous work of the three geometric diagrams was discussed on the chalkboard; thereafter the theorem was explained to the learners on the chalkboard step by step. Basically, there was a direct flow of information from teacher to learners.

All three participants used the traditional way of teaching grade 11 Euclidean geometry theorems, which mainly relied on textbooks. All three grade 11 Euclidean geometry lessons observed were not collaborative but were more teacher-centred. According to Mapesos (2017), in a traditional teacher-centred class learners remain passive and are merely listeners; they are deprived of their freedom of expression, group interactions and collaborative learning.

For effective teaching, a learner-centred approach may be used where learners are actively involved in the construction of knowledge, and learners' communication and critical thinking skills are developed. The learner-centred approach employs active learning to promote deep learning skills and enhances learner adaptability in problem-solving situations (Frame et al., 2015). This approach results in greater retention of knowledge, possibly because learners understand and make personal sense of the knowledge, rather than simply memorize and reproduce knowledge (Habók & Nagy, 2016). According to Kramarenko (2019), learners perform better in learner-centred teaching strategies than in traditional teaching strategies. Furthermore, the learner-centred approach also provides learners with greater learner-teacher engagement and a higher degree of learning than traditional teaching methods (Carter, Creedy, & Sidebotham, 2016). Teachers need to adopt a variety of teaching strategies and methods which consider learners' backgrounds, environment and their different learning abilities to ensure that all learners have equal learning opportunities.

6.2.3 Why do grade 11 mathematics teachers use their pedagogical content knowledge when teaching Euclidean geometry the way they do?

Based on the responses from the open-ended questionnaire, participant Teacher B (TB) indicated that they were not provided with much practical content training in Euclidean geometry. All three participants also indicated that at school, teachers were simply following textbooks, and lessons were all teacher-centred. This implies, according to this study, that all three participants were not provided with much practical training of Euclidean geometry and as a result, these teachers may not be able to expose the learners to the practical learning of Euclidean geometry and other various teaching strategies.

According to Biyela et al. (2016), pre-service teacher training programs constitute the foundation for competent mathematics teaching, and are therefore expected to produce competent and effective mathematics teachers who are ready to teach mathematics effectively. Mathematics teachers are an important factor in the role of learner success in mathematics; therefore, pre-service teacher preparation programs should transform and be relevant should provide pre-service mathematics teachers with adequate pre-service trainings (Craig, Evans, & Stokes, 2017). Furthermore, according to Baran, Canbazoglu Bilici, Albayrak Sari, and Tondeur (2019), a range of mathematics teaching strategies needs to be integrated into pre-service teacher training programs to develop teachers' attitude, knowledge and skills for effective teaching of mathematics, especially Euclidean geometry. The most common strategies may include offering technology since all three participants in this study had never used technology in their lessons, and developing mathematics material is still a challenge to many mathematics teachers as many of them still rely only on textbooks.

6.3 Recommendations

Based on the findings of this study after exploring grade 11 mathematics teachers' pedagogical content knowledge (PCK) when teaching Euclidean geometry (EG), the following Euclidean geometry instructional approaches are suggested as recommendations:

6.3.1 The problem-solving teaching method

In comparison to a traditional method of teaching, problem solving allows learners to work collaboratively, constructing knowledge through social interaction (Ahamad et al., 2017).The problem solving method is commonly known to be teacher-learner-centred, and has long been taken to be the most important aspect in the teaching and learning of mathematics by many mathematics teachers (Bhagat & Chang, 2015). According to Liljedahl, Santos-Trigo, Malaspina, and Bruder (2016, pp. 12-14), there are four main steps involved in mathematics problem solving:

1. Understand the problem: you need to know what is known.

2. Devise a plan: you need to start connecting between the known and the unknown, e.g., you may apply the known theorem to find the unknown.
3. Carry out the plan: check whether steps done are correct.
4. Look back: examine the results obtained.

Problem solving is an essential aspect in the teaching and learning of Euclidean geometry (Seo et al., 2015). However, according to Wei et al. (2017), most researchers have found this does not occur within South African mathematics classrooms. According to Bhagat and Chang (2015), problem solving has always been seen as a central theme for mathematics, especially in geometry and it is considered an important tool for understanding mathematics. Learners need to understand the problem they are solving, they need to know what is the unknown, and what data are given (Liljedahl et al., 2016).

According to Wei et al. (2017), learners should know and understand the learned theorems in order to correctly apply them when solving Euclidean geometry problems since most geometric riders are about theorems, and it follows that if learners do not know the theorems, they cannot solve geometric riders. Problem solving in mathematics assists learners by increasing reasoning skills, and therefore problem solving is considered an essential aspect in order to increase the level of mathematics learner achievement and it can also be utilised to solve real life problems (Kösa, 2016).

6.3.2 The discovery learning method

This is a learner-centred approach which emphasises the importance of learners being actively involved in constructing the knowledge, and the teacher acts as a facilitator by involving all learners in different activities where learners discover and construct knowledge (Lessani et al., 2017). According to Kösa (2016), teachers do not simply pour out information to the learners, but they engage learners in dialogue and allow them to discuss amongst themselves. Furthermore, in discovery learning, learners are not passive recipients of knowledge but they actively participate in the construction of the knowledge (Lessani et al., 2017). The teacher creates a classroom environment where learners engage in the discovery of the knowledge, and this motivates them to think deeply (Leikin, 2015).

Furthermore, the following recommendations are also suggested to alleviate this poor and outdated mathematics teaching method:

1. The South African education system should encourage mathematics teachers to further their mathematics education. This can be supported by funding and granting study leave to teachers. This will have an overall positive effect on learners' performance in mathematics.

2. Re-training or refresher mathematics courses and mathematics workshops (that specifically address pedagogical content knowledge) should be on-going as this would help update mathematics teachers on innovative ways to teach mathematics. These ongoing professional development courses need to be pedagogically sound, content rich, curriculum relevant and quality assured.

3. The teaching and learning of geometry can be very challenging to many teachers and learners as it involves abstract and complex ideas; hence, teaching geometry using other resources such as laptops and projectors may help learners develop their understanding of abstract geometry concepts through visualisation and graphic representation (Parrot & Eu, 2018). Mathematics teachers should be given professional training on the active usage of technology in the teaching and learning process (Yildiz & Baltaci, 2016).

4. Mathematics learners should all be taught by the qualified and experienced mathematics teachers from primary school level through to secondary school level as this will no doubt enhance their academic achievement.

5. Both pre-and in-service mathematics teachers need to be trained to integrate technology in the teaching of mathematics, especially Euclidean geometry. Introducing technology in the classroom may change the traditional teaching strategies to a more learner-centred strategy where learners are actively engaged in the construction of knowledge (Giles & Kent, 2016).

At this point, knowledge of this study's findings may not be enough; however, according to Geldenhuys and Oosthuizen (2015), changes need to be implemented to mathematics teachers' beliefs, attitudes and practices through accredited continuous professional development programmes. Mathematics teachers need to be professionally developed in order for them to apply different teaching strategies in their mathematics classrooms for the benefit of all types of learners (Özdemir, 2019). Furthermore, these professional developments may enable mathematics teachers to acquire the necessary knowledge and skills to enhance their classroom mathematical abilities (Yow & Lotter, 2016). More research would be needed to investigate the effectiveness of these mathematics teachers' professional developments in improving mathematic teachers' classroom practices and hence learner performance in general (Gomez et al., 2015).

6.4 Suggestions for further research

Further studies could be extended to other aspects of Euclidean geometry and also to the other content areas of grade 11 mathematics namely: Algebra, Functions and graphs; Number Patterns, Finance, growth and decay; Probability; Measurement; Statistics; Analytical Geometry and Trigonometry. Further studies could well include a bigger sample space that will accommodate more South African secondary schools and even more districts and more provinces.

Further studies could also explore the possible influence of learner and parent variables such as: learners' interest and attitude, parental mathematics background and attitude, and peer influence, to mention just a few. This may prevent assumptions and generalisations that learner performance is only a result of applied teacher pedagogical content knowledge.

6.5 Study limitations

As is typical of every research study, this research study also acknowledges some limitations that might have affected its end results in one way or another. This research study was restricted to exploring grade 11 teachers' pedagogical content knowledge when teaching Euclidean geometry. Only three secondary schools were sampled in the Umlazi district, Phumelela circuit which comprises 160 secondary schools. This study was only conducted on the Euclidean geometry topic which is one of the ten content areas making up the grade 11 mathematics curriculum.

This current study had a very small sample of only three grade 11 mathematics educators as participants, and these were only chosen from one circuit (Phumelela) in one district, (Umlazi) in KwaZulu-Natal. Therefore, the results and findings were only obtained from the three participants. Hence, the results of the current study may not be generalised. Different results might possibly be obtained from various contexts with larger samples. Besides qualitative research, other research approaches and research instruments could have been employed.

6.6 Conclusion

The current study has contributed towards exploring methods used by grade 11 mathematics teachers when teaching Euclidean geometry. It has shown how grade 11 mathematics teachers use their pedagogical content knowledge when teaching Euclidean geometry and why they use their pedagogical content knowledge for teaching Euclidean geometry in the way they do. The current study may add value in the teaching and learning of mathematics, especially geometry education.

According to the analysis and findings, this study revealed that mathematics teachers still use the traditional lecture method of teaching when teaching Euclidean geometry. Teachers simply follow textbooks when teaching Euclidean geometry theorems and explain basic skills and computational procedures on the chalkboard. It was also confirmed that teachers first explain Euclidean geometry theorems, and after each theorem, application exercises are given to learners. Various strategies of teaching Euclidean geometry where learners are active constructors of knowledge may develop their thinking abilities and reasoning skills when solving Euclidean geometry problems (Archibald, 2016). As a recommendation, this ought to be researched further.

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APPENDIX 1

UNIVERSITY OF KWAZULU-NATAL
INYUVESI YAKWAZULU-NATALI

31 August 2017

Mr Abednigo Sibusiso Nojiyeza (954003825)
School of Education
Edgewood Campus

Dear Mr Nojiyeza,

Protocol reference number: HSS/1429/017M
Project title: Exploring Grade 11 Mathematics teachers' pedagogical content knowledge when teaching Euclidean Geometry in the Umlazi District

Approval Notification – Expedited Application

In response to your application received on 11 August 2017, the Humanities & Social Sciences Research Ethics Committee has considered the abovementioned application and the protocol has been granted **FULL APPROVAL.**

Any alteration/s to the approved research protocol i.e. Questionnaire/Interview Schedule, Informed Consent Form, Title of the Project, Location of the Study, Research Approach and Methods must be reviewed and approved through the amendment/modification prior to its implementation. In case you have further queries, please quote the above reference number.

PLEASE NOTE: Research data should be securely stored in the discipline/department for a period of 5 years.

The ethical clearance certificate is only valid for a period of 3 years from the date of issue. Thereafter Recertification must be applied for on an annual basis.

I take this opportunity of wishing you everything of the best with your study.

Yours faithfully

....

Dr Shenuka Singh (Chair)

/ms

Cc Supervisor: Dr Jayaluxmi Naidoo
Cc Academic Leader Research: Dr SB Khoza
Cc School Administrator: Ms Tyzer Khumalo

Humanities & Social Sciences Research Ethics Committee
Dr Shenuka Singh (Chair)
Westville Campus, Govan Mbeki Building
Postal Address: Private Bag X54001, Durban 4000
Telephone: +27 (0) 31 260 3587/8350/4557 **Facsimile:** +27 (0) 31 260 4609 **Email:** ximbap@ukzn.ac.za / snymanm@ukzn.ac.za / mohunp@ukzn.ac.za
Website: www.ukzn.ac.za

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APPENDIX 2

Questionnaire

Please fill in the information below.

PART A: Teacher qualifications and experience in Mathematics teaching.

	Yes/ No	Year	Mathematics symbol
Matriculated			
Post-Matric qualification(s)	**Institution**	**Year**	**Major subjects**
Teaching experience in mathematics	**Grade 10**	**Grade11**	**Grade12**
Grade(s) currently teaching			
Number of learners in your class			

PART B: Teacher pedagogical content knowledge.

1. Did you study Euclidean Geometry in Grade 12? Yes/No____________
2. Give background details on how you were taught Euclidean Geometry in school.

3. How do you teach the learners the following theorem which states that "The angle between the tangent and chord at the point of contact is equal to the angle in the alternate segment? (May use an additional writing material for this question)

Thank you for your time in completing this questionnaire

APPENDIX 3

INTERVIEW QUESTIONS
1. From your own experience, why do you think learners perform poor in mathematics?
2. Which method/technique do you apply when teaching Euclidean Geometry? And Why?
3. Do your Euclidean Geometry lessons use something more visual?
4. Do you incorporate learning theory/theories in teaching Euclidean Geometry?
5. If so, what learning theory/theories? And
6. How?
7. How do you evaluate yourself after each lesson taught?
8. Do learners enjoy your Euclidean Geometry lessons?
9. Are you having any challenge in teaching Euclidean Geometry?

APPENDIX 4

CLASS AND LESSON OBSERVATION

Date: ________________ Grade: ____________ Number of learners: ___________

Topic/Lesson: ____

Brief description of a lesson:

____
____
____

Learning environment:

____
____
____

Introduction of topic:

____
____
____
____

Prior knowledge identified:

____
____
____
____

Representations and Examples:

____
____
____
____

Identifies and addresses errors and misconceptions:

____
____
____
____

Teaching Resources:

Lesson presentation:

Learner involvement:

Learner activities:

Learner Assessment:

Feedback on learner assessment:

Teacher methodology (more than one method used?):

Evidence of Lesson planning/preparation:

General comments:

APPENDIX 5

School of Education, College of Humanities,
University of KwaZulu-Natal,

Edgewood Campus,

Dear Principal

INFORMED CONSENT LETTER

My name is Abednigo Sibusiso Nojiyeza. I am a Masters candidate studying at the University of KwaZulu-Natal, Edgewood campus, South Africa. I am interested in conducting a study in your school in Grade 11 mathematics classes. The title of my research topic is: **Exploring Grade 11 mathematics teachers' Pedagogical Content Knowledge when teaching Euclidean Geometry**.

To gather the information, I am interested in asking you some questions.

Please note that:

- Your confidentiality is guaranteed as your inputs will not be attributed to you in person, but reported only as a population member opinion.
- The interview may last for about 45 minutes to 1 hour.
- Any information given by you cannot be used against you, and the collected data will be used for purposes of this research only.
- Data will be stored in secure storage and destroyed after 5 years.
- You have a choice to participate, not participate or stop participating in the research. You will not be penalized for taking such an action.
- Your involvement is purely for academic purposes only, and there are no financial benefits involved.
- If you are willing to be interviewed, please indicate (by ticking as applicable) whether or not you are willing to allow the interview to be recorded by the following equipment:

Equipment	**Willing**	**Not willing**
Audio equipment		
Photographic equipment		
Video equipment		

I can be contacted at:

Email: snojiyeza11@gmail.com

Cell: 0833487481

My supervisor is Dr.Jayaluxmi Naidoo who is located at the School of Education, Edgewood campus of the University of KwaZulu-Natal.

Contact details: email: naidooj2@ukzn.ac.za Phone number: +27312601127.

You may also contact the Research Office through:

Ms P Ximba (HSSREC Research Office)

Tel: 031 260 3587

Email: ximbap@ukzn.ac.za)

Thank you for your contribution to this research.

DECLARATION

I……………………………………………………………………………(full names of participant) hereby confirm that I understand the contents of this document and the nature of the research project, and I consent to participating in the research project.

I understand that I am at liberty to withdraw from the project at any time, should I so desire.

SIGNATURE OF PARTICIPANT DATE

……………………………………… …………………

APPENDIX 6

School of Education, College of Humanities, University of KwaZulu-Natal,

Edgewood Campus,

Dear Grade 11 Mathematics Teacher

INFORMED CONSENT LETTER

My name is Abednigo Sibusiso Nojiyeza. I am a Masters candidate studying at the University of KwaZulu-Natal, Edgewood campus, South Africa. I am interested in conducting a study in your school in Grade 11 mathematics classes. The title of my research topic is: **Exploring Grade 11 mathematics teachers' Pedagogical Content Knowledge when teaching Euclidean Geometry**.

To gather the information, I am interested in asking you some questions.

Please note that:

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Thank you for your contribution to this research.

DECLARATION

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I understand that I am at liberty to withdraw from the project at any time, should I so desire.

SIGNATURE OF PARTICIPANT DATE

……………………………………… …………………

APPENDIX 7

School of Education, College of Humanities, University of KwaZulu-Natal,

Edgewood Campus,

Dear Parent

INFORMED CONSENT LETTER

My name is Abednigo Sibusiso Nojiyeza. I am a Masters candidate studying at the University of KwaZulu-Natal, Edgewood campus, South Africa. I am interested in conducting a study in your school in Grade 11 mathematics classes. The title of my research topic is: **Exploring Grade 11 mathematics teachers' Pedagogical Content Knowledge when teaching Euclidean Geometry**.

To gather the information, I am interested in asking you some questions.

Please note that:

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- The interview may last for about 45 minutes to 1 hour.
- Any information given by you cannot be used against you, and the collected data will be used for purposes of this research only.
- Data will be stored in secure storage and destroyed after 5 years.
- You have a choice to participate, not participate or stop participating in the research. You will not be penalized for taking such an action.
- Your involvement is purely for academic purposes only, and there are no financial benefits involved.
- If you are willing to be interviewed, please indicate (by ticking as applicable) whether or not you are willing to allow the interview to be recorded by the following equipment:

Equipment	**Willing**	**Not willing**
Audio equipment		
Photographic equipment		
Video equipment		

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Ms P Ximba (HSSREC Research Office)

Tel: 031 260 3587

Email: ximbap@ukzn.ac.za)

Thank you for your contribution to this research.

DECLARATION

I……………………………………………………………………………(full names of participant) hereby confirm that I understand the contents of this document and the nature of the research project, and I consent to participating in the research project.

I understand that I am at liberty to withdraw from the project at any time, should I so desire.

SIGNATURE OF PARENT **DATE**

………………………………………… …………………..

APPENDIX 8

School of Education, College of Humanities,
University of KwaZulu-Natal,

Edgewood Campus,

Dear Grade 11 Mathematics learner

INFORMED CONSENT LETTER

My name is Abednigo Sibusiso Nojiyeza. I am a Masters candidate studying at the University of KwaZulu-Natal, Edgewood campus, South Africa. I am interested in conducting a study in your school in Grade 11 mathematics classes. The title of my research topic is: **Exploring Grade 11 mathematics teachers' Pedagogical Content Knowledge when teaching Euclidean Geometry**.

To gather the information, I am interested in asking you some questions.

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- Your confidentiality is guaranteed as your inputs will not be attributed to you in person, but reported only as a population member opinion.
- The interview may last for about 45 minutes to 1 hour.
- Any information given by you cannot be used against you, and the collected data will be used for purposes of this research only.
- Data will be stored in secure storage and destroyed after 5 years.
- You have a choice to participate, not participate or stop participating in the research. You will not be penalized for taking such an action.
- Your involvement is purely for academic purposes only, and there are no financial benefits involved.
- If you are willing to be interviewed, please indicate (by ticking as applicable) whether or not you are willing to allow the interview to be recorded by the following equipment:

Equipment	**Willing**	**Not willing**
Audio equipment		
Photographic equipment		
Video equipment		

I can be contacted at:

Email: snojiyeza11@gmail.com

Cell: 0833487481

My supervisor is Dr.Jayaluxmi Naidoo who is located at the School of Education, Edgewood campus of the University of KwaZulu-Natal.

Contact details: email: naidooj2@ukzn.ac.za   Phone number: +27312601127.

You may also contact the Research Office through:

Ms P Ximba (HSSREC Research Office)

Tel: 031 260 3587

Email: ximbap@ukzn.ac.za)

Thank you for your contribution to this research.

DECLARATION

I………………………………………………………………………… (full names of participant) hereby confirm that I understand the contents of this document and the nature of the research project, and I consent to participating in the research project.

I understand that I am at liberty to withdraw from the project at any time, should I so desire.

SIGNATURE OF PARTICIPANT DATE

……………………………………… …………………

SIGNATURE OF PARENT (If participant is a minor) DATE

………………………………………… …………………..

APPENDIX 9

Exploring Grade 11 mathematics teachers' pedagogical content knowledge when teaching Euclidean Geometry

ORIGINALITY REPORT

15%	9%	7%	7%
SIMILARITY INDEX	INTERNET SOURCES	PUBLICATIONS	STUDENT PAPERS

PRIMARY SOURCES

1	Felix Schultze, Pernilla Nilsson. "Coteaching with senior students – a way to refine teachers' PCK for teaching chemical bonding in upper secondary school", International Journal of Science Education, 2018 Publication	<1%
2	www.ncca.ie Internet Source	<1%
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5	Submitted to Edith Cowan University Student Paper	<1%
	Submitted to Higher Education Commission	

APPENDIX 10

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25 November 2019

TO WHOM IT MAY CONCERN

TITLE OF PAPER: EXPLORING GRADE 11 MATHEMATICS TEACHERS' PEDAGOGICAL CONTENT KNOWLEDGE WHEN TEACHING EUCLIDIAN GEOMETRY IN THE UMLAZI DISTRICT

STUDENT/AUTHOR: ABEDNIGO SIBUSISO NOJIYEZA (954003825)

The above-mentioned coursework dissertation submitted for the Degree of Master of Education (Curriculum Studies) at the University of KwaZulu-Natal was proofread by me for English language, grammar, spelling, punctuation and formatting errors. I endeavoured throughout the process to retain the writing style of the student/author and to remain true to his research content and intentions.

Please note that Tables and References were not checked for accuracy although obvious errors, for example in headings or punctuation, were corrected.

My suggested changes may be accepted or not at the student's and your discretion.

M J Dalby