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Solutions of the Volterra Integro-Differential Equation

By

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Solutions of the Volterra Integro-Differential Equation

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Declaration

I, **Yusuf Ali**, declare that

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Mr Yusuf Ali

Date

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Abstract

Integro-differential equations has found extensive applications in the field of engineering, sciences and mathematical modelling of various physical and biological phenomena. In this thesis we focus on the Volterra type integro-differential equation which has been used to model biological species co-existing, heat diffusion, electromagnetic theory etc. In recent years much research has focused on finding approximate solutions of the integro-differential equation by polynomial methods, specifically focusing on the Lagrange collocation and piecewise cubic Hermite collocation methods. A further aspect to the thesis will be on analytical methods, mainly the applications of Lie group theory to the Volterra type equation. Lie group theory is one of the most powerful methods applied to obtain solutions of differential equations. We will present the linear independent symmetries of the Volterra type equation of the first and second kind. In addition, we shall apply the Laplace transform and it's inverse to determine general solutions for selected forms of kernel, specifically those with convolution integrands.

Notation

\mathbb{L}	Linear operator
\mathbf{R}	Residual
K	Kernel
y	Exact Solution of differential or integro-differential equation
p_n	Approximating polynomial of order n
y^i	Piecewise cubic approximating polynomial
ϕ_i	General basis function
M_i	Monomial basis function
\mathcal{L}_i	Lagrange basis function
\mathcal{H}_i	Hermite basis function
ψ_i	Nodal basis function
sup	Supremum
T_i	Chebyshev polynomial of the first kind of degree i
E_p	Discrete error with p collocation points
\mathbf{G}	Lie group
ϵ	Lie group parameter
Φ	Law of group composition
\mathcal{O}	Big O notation
x_1, y_1	Group of transformations of x, y
$y_1^{[n]}$	Extended group of transformations
ξ, η	Infinitesimals of \mathbf{G}

$\eta^{[n]}$	Extended infinitesimal of \mathbf{G}
X	Infinitesimal generator
$X^{[n]}$	Extended infinitesimal generator
X_i	Linearly independent symmetry
\mathcal{L}	Laplace transform
\mathcal{L}^{-1}	Inverse Laplace transform

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Chapter 1

Introduction

1.1 Integro-differential equations

Generally, a differential equation is defined as an equation that relates some function with its independent variables, constants and derivatives [7]. Conversely, an integral equation is one that relates some unknown function and integrals of the unknown function, [16].

Integro-Differential Equations (IDEs) are defined as equations in which some unknown function appears along with its derivatives and integrals. It can be seen as some hybrid of both differential and integral equations. The work covered by this thesis focuses on solving the case of first-order Volterra type linear integro-differential equations, discussed in more detail as follows, [25]. All numerical solutions are generated using the software packages Matlab and Python.

1.1.1 Differential equations

Ordinary differential equations

An equation containing some function $y(x)$ with a single independent variable as well as one or more of its derivatives $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^ny}{dx^n}$ is called an ordinary differential equation. That is, an n^{th} -order differential equation is defined in general as follows [27]

$$F\left(x, y, \frac{dy}{dx}, \dots, \frac{d^ny}{dx^n}\right) = 0, \quad (1.1)$$

where F is a function of the independent variable x , the dependent variable y and the derivatives of y up to order n .

Partial differential equations

An equation containing some function $y(x_1, x_2, \dots, x_n)$ with n independent variables as well as partial derivatives of two or more independent variables is referred to as a partial differential equation. Such equations take the general form as follows [27]

$$F \left(x_1, x_2, \dots, x_n, y, \frac{\partial y}{\partial x_1}, \dots, \frac{\partial y}{\partial x_n}, \frac{\partial^2 y}{\partial x_1 \partial x_1}, \dots, \frac{\partial^2 y}{\partial x_1 \partial x_n}, \dots \right) = 0, \quad (1.2)$$

where F is a function of the independent variables x_1, x_2, \dots, x_n , the independent variable y and the partial derivatives of y .

Initial conditions

When solving a differential equation, we are generally seeking a particular solution in order for our result to be of practical use in science, finance etc. Particular solutions can only be obtained if we have a set of known initial conditions sufficient for the order of the differential equation, such a problem is referred to as an initial value problem and is defined as follows. [7]

Definition 1.1 (Initial Value Problems). In order to obtain the particular solution for the differential equation given by

$$F \left(x, y, \frac{dy}{dx}, \dots, \frac{d^n y}{dx^n} \right) = 0,$$

we require the following initial conditions

$$y(x_0) = y_0, \quad \frac{dy}{dx} \Big|_{x=x_0} = y_1, \quad \dots, \quad \frac{d^{n-1}y}{dx^{n-1}} \Big|_{x=x_0} = y_{n-1},$$

where y_0, y_1, \dots, y_{n-1} are known constants.

Remarks

In this thesis, ordinary differential equations are of specific concern to us as we are studying ordinary integro-differential equations, that is integro-differential equations involving one independent variable only as opposed to partial integro-differential equations where multiple independent variables are involved.

1.1.2 Integral equations

Reiterating, an integral equation involves an unknown function say $y(x)$ and the integral $\int K(t, s, y(s)) ds$, where $K(t, s, y(s))$ is a known integral kernel. Suppose the unknown function $y(x)$ appears only as part of the integrand, such an equation is of the first kind whereas if the unknown function appears both in the integrand and outside, the equation is of the second kind. [25]

The above explanation of integral equations is too general in that it allows for many distinct forms to arise, therefore classical mathematics defines two distinct types of integral equations. Namely, the Fredholm type and the Volterra type.

The Fredholm type integral equation

In Fredholm type integral equations, the limits of integration are constant and take the following form (in the case of an equation of the second kind)

$$y(x) = g(x) + \int_a^b K(x, s, y(s)) ds. \quad (1.3)$$

The Volterra type integral equation

Fredholm type integral equations are special cases of Volterra type equations with the upper limit of integration a variable. In fact, Volterra type integral equations can be seen as a generalization of Fredholm type integral equations with respect to initial value problems [16]. The Volterra equation of the second kind takes the form

$$y(x) = g(x) + \int_{x_0}^x K(x, s, y(s)) ds, \quad (1.4)$$

where x_0 is a constant, y is unknown and we assume the kernel $K(x, s, y)$ and the term $g(x)$ are given.

In the linear case, the form of the Volterra type integral equation is simplified as here $K(x, s, y(s)) = K(x, s)y(s)$ and hence results in

$$y(x) = g(x) + \int_{x_0}^x K(x, s)y(s) ds. \quad (1.5)$$

1.1.3 Volterra type integro-differential equations

During his study of population growth, Volterra came upon a new type of equation. An equation in which both the derivatives of an unknown function and the integral terms appeared together [26]. In general, Volterra Type Integro-Differential Equations take the form

$$\frac{d^n y}{dx^n} = g(x) + \int_{x_0}^x K(x, s, y(s)) ds, \quad (1.6)$$

where n is an integer, $y(x)$ is the unknown function, $F(x)$ and $K(x, s, y)$ are given. As a result of (1.6) involving $y(x)$ and its derivatives, in order to determine a solution for (1.6), we require the initial conditions $y(0), \frac{dy}{dx}|_{x=0}, \dots, \frac{d^{n-1}y}{dx^{n-1}}|_{x=0}$ for reasons described in Definition 1.1.

Hence, the form of the First-Order Volterra Type Linear Integro-Differential Equation, the primary problem relevant to the subsequent chapters of this thesis is given in the following definition.

Definition 1.2 (First-Order Volterra Type Linear Integro-Differential Equation). An Integro-Differential equation of the form

$$\frac{dy}{dx} = g(x) + \int_{t_0}^x K(x, t)y(t) dt, \quad y(t_0) = y_0, \quad (1.7)$$

where $t_0 \leq t \leq T$ for finite constants t_0 and T , $y(x)$ the unknown solution of the IDE, $F(x)$ and $K(x, t)$ the known functions.

1.1.4 Applications

The study of Integro-Differential Equations is of great relevance as they frequently arise in modern science, that being in the fields of pure and applied mathematics and beyond. Integro-differential equations play an important role in heat and mass diffusion processes, biological predator and prey interaction, electromagnetic theory, ocean circulations amongst many others [23].

Next we give a brief outline of some recent examples of integro-differential equations in use, namely in the fields of financial mathematics, mathematical biology and electrical engineering respectively.

Option prices in exponential Lévy models

Large and sudden price movements in the financial markets where traditional diffusion equations often fall short has resulted in the development of newer pricing models incorporating jump

processes. Where large price movements are represented by discontinuous trajectories, thus providing a model with a more realistic representation of real world price fluctuation than diffusion equations could provide.

Such a model incorporates integro-differential equations, as defined by the Lévy-Khintchine representation given in the following definition [24].

Definition 1.3 (Lévy-Khintchine Representation). Define a Lévy process $(X_t)_{t \geq 0}$ on \mathbf{R}^n with Lévy-Khintchine characteristic triplet (A, ν, γ) , then its characteristic function is defined by

$$\begin{aligned} \phi_X(z)(t) &= E(e^{iz \cdot X_t}) = e^{t\psi(z)}, \quad \text{with} \\ \psi(z) &= -\frac{1}{2}z \cdot Az + i\gamma \cdot z + \int_{\mathbf{R}^n} (e^{iz \cdot x} - 1 - iz \cdot x 1_{|x| \leq 1}) \nu \, dx, \end{aligned}$$

for all $z \in \mathbf{R}^n$, where $1_{|x| \leq 1}$ is the indicator function.

Spread of viruses in epidemiology models

Traditionally, spatial processes such as the spread of viruses are modelled with diffusion equations. However, they only capture local contact between organisms and therefore underestimate the rates at which viruses spread. Integro-differential equations on the other hand can model non-local interactions as well, providing a more accurate prediction of these spatial processes. [17]

The speed of the spatial propagation of an epidemic, for example in the spread of spores in an environment, is modelled in [21] and is given in the following definition.

Definition 1.4 (Spatially Spread Epidemic). Let the density of susceptible individuals at time t and position x be denoted by $S(x, t) \geq 0$ and the density of already infected individuals at time t and position x be $I(x, t) \geq 0$, then

$$\frac{\partial I}{\partial t} = \beta \left(\int_{\Omega} K(x, y) I(y, t) \, dy \right) S(x, t),$$

where β is the rate of infection and kernel $K(x, y) \geq 0$ is the contact distribution.

The case where contact is local but infected individuals change location in a spatial domain Ω is modelled in [17], given in the following definition.

Definition 1.5 (Spatially Spread Infectives). Let the density of susceptible individuals at time t and position x be denoted by $S(x, t) \geq 0$, the density of infectives at time t and position x be $I(x, t) \geq 0$ and the constant dispersal rate of infectives be $\alpha \geq 0$, then

$$\frac{\partial I}{\partial t} = \beta I(x, t)S(x, t) - \alpha I(x, t) + \alpha \int_{\Omega} K(x, y)I(y, t) dy,$$

where β is the rate of infection and kernel $K(x, y) \geq 0$ is the contact distribution.

Impressed voltage in circuit analysis

Kirchhoff's second law, as a result of the law of conservation of energy, states that in a closed circuit the total voltage drop across an inductor, resistor and capacitor given by

$$L \frac{dI}{dt}, \quad RI(t) \quad \text{and} \quad \frac{1}{C} \int_0^t I(\tau) d\tau$$

respectively, is equivalent to $E(t)$ the impressed voltage at t . Where the current at t is given by $I(t)$ and L , R and C are the constants of inductance, resistance and capacitance respectively.

Hence, the voltage impressed $E(t)$ at time t in a closed circuit is given by the integro-differential equation [27]

$$E(t) = L \frac{dI}{dt} + RI(t) + \frac{1}{C} \int_0^t I(\tau) d\tau, \tag{1.8}$$

with initial condition $I(0) = I_0$.

1.2 The numerical approach

In the first part of this thesis we look at the methods of Lagrange collocation on a global domain as well as piecewise cubic Hermite collocation as tools to numerically approximate the solution of integro-differential equations.

The Lagrange polynomial, although named after Joseph Louis Lagrange, was first published in 1779 by Edward Waring. It is an elegant representation of Newton's Interpolation Formula which does not necessitate the computation of divided differences and therefore very useful in interpolation theory. [18]

In the piecewise cubic Hermite collocation method, the domain is divided into subintervals and an approximating polynomial is defined on each subinterval. The adjacent approximating

polynomials are then pieced together to form an approximation over the entire domain. The advantage of using Hermite polynomials as the basis is that, at the boundaries of each subinterval, they and their first derivatives are continuous thus reducing the number of equations that have to be satisfied. [8]

1.3 The analytical approach

In the second part of this thesis we present the method of Lie symmetry analysis, an area of mathematics first pioneered in the 19th century by the Norwegian mathematician Marius Sophus Lie. Under a Lie group of point transformations, if a system of differential equations is invariant, particular solutions called invariant solutions can be found. These solutions, which are invariant under some subgroup of the Lie group admitted by the system, result from solving a reduced system with less independent variables. The Lie symmetry method can be applied to virtually any system of differential equation and furthermore is highly algorithmic and thus amenable to computer programming. [5]

Lie symmetry analysis has been applied extensively in the study of ordinary differential equations as well as partial differential equations, developing into a useful tool in the classification and finding of solutions of differential equations [3], [11], [12]. We determine new symmetries for the general first-order Volterra integro-differential equation of the first and second kind.

Additionally, we use the Laplace transform and its inverse to determine exact solutions for certain forms of integro-differential equations. Also, we find exact solutions to integro-differential equations given various forms of the kernel, including degenerate forms, using ordinary methods.

Part I

The numerical approach

Chapter 2

Application of Lagrange collocation to solving IDEs

2.1 Polynomial interpolation - theoretical basis

Polynomial interpolation is a method of numerical analysis, whereby values between a known set of discrete data points is approximated using the lowest order polynomial that passes through the points of this dataset.

As outlined in [9], given $N + 1$ unique interpolation points x_i (for $i = 0, 1, \dots, N$), to approximate the function $f(x)$ defined and continuous on $[x_0, x_N]$, we seek a polynomial $p_N(x)$ such that

$$p_N(x_i) = f(x_i), \quad \text{for all } i = 0, 1, \dots, N. \quad (2.1)$$

2.1.1 Constructing the interpolating polynomial

As outlined in [2], the interpolating polynomial of order n can be written as

$$p_N(x) = \sum_{i=0}^N a_i \phi_i(x), \quad (2.2)$$

where a_i , for $i = 0, 1, \dots, N$ are the unknown parameters and $\phi_i(x)$, for $i = 0, 1, \dots, N$ the given basis functions.

In order to determine the interpolating polynomial through a set of unique nodes x_0, x_1, \dots, x_N for a given function $f(x)$ continuous on $[x_0, x_N]$, we must solve for the unknown parameters a_0, a_1, \dots, a_N . Hence, the resulting linear system is as follows

$$\begin{bmatrix} \phi_0(x_0) & \phi_1(x_0) & \phi_2(x_0) & \dots & \phi_N(x_0) \\ \phi_0(x_1) & \phi_1(x_1) & \phi_2(x_1) & \dots & \phi_N(x_1) \\ \phi_0(x_2) & \phi_1(x_2) & \phi_2(x_2) & \dots & \phi_N(x_2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \phi_0(x_N) & \phi_1(x_N) & \phi_2(x_N) & \dots & \phi_N(x_N) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \\ \vdots \\ f(x_N) \end{bmatrix}. \quad (2.3)$$

Monomial basis

For a polynomial interpolation, the most natural basis comprises of monomials $M_i(x)$ given by

$$M_i(x) = x^i, \quad \text{for } i = 0, 1, \dots, N, \quad (2.4)$$

which results in an interpolant of the form

$$p_N(x) = \sum_{i=0}^N a_i x^i. \quad (2.5)$$

The associated linear system $V\mathbf{a} = \mathbf{f}$ is given by

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \dots & x_0^N \\ 1 & x_1 & x_1^2 & \dots & x_1^N \\ 1 & x_2 & x_2^2 & \dots & x_2^N \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_N & x_N^2 & \dots & x_N^N \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \\ \vdots \\ f(x_N) \end{bmatrix}, \quad (2.6)$$

where the matrix V is known as the Vandermonde matrix. This matrix has the property

$$\det(V) = \prod_{i=0}^{N-1} \prod_{j=i+1}^N (x_j - x_i), \quad (2.7)$$

which implies that matrix V is non-singular since $x_i \neq x_j$, $i \neq j$. This property leads to the following theorem.

Theorem 2.1 (Unique Existence of a Polynomial Interpolant). *For a set of unique nodes x_0, x_1, \dots, x_N and corresponding set of points y_0, y_1, \dots, y_N there exists a unique polynomial $p(x)$ with a maximum order N , such that*

$$p(x_i) = y_i, \quad \text{for all } i = 0, 1, \dots, N.$$

Example 2.1 (Monomial Basis). Estimate $f(x) = \cos(5x)$, between $[-1, 1]$ using 8 equally spaced points with a monomial basis.

Plotting the monomial basis polynomial for this problem yields

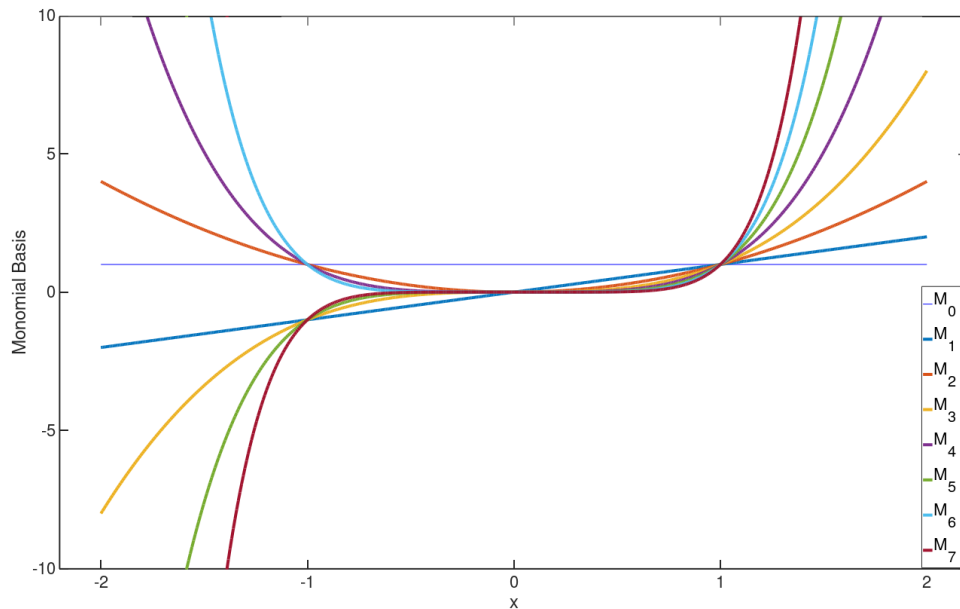


Figure 2.1: The Monomial basis polynomials $M_i(x) = x^i$, $i = 0, 1, \dots, 7$.

Hence, solving for the interpolant gives

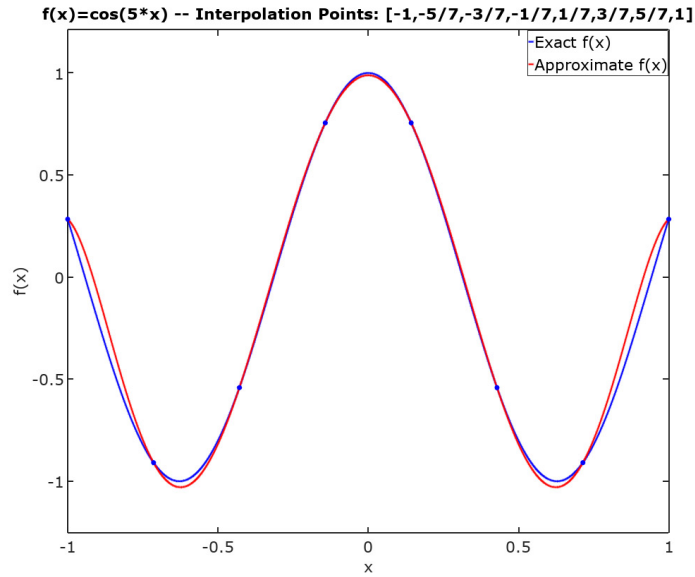


Figure 2.2: The 7th order polynomial interpolant approximating $\cos(5x)$.

Lagrange basis

As described in [13], instead of solving the system (2.6), it would be convenient if there was a basis $\phi_i(x)$ in (2.2), such that for each x_i , $a_i = f(x_i)$ for $i = 0, 1, \dots, N$. Such a basis is the Lagrange polynomial, $\mathcal{L}_i(x)$ which satisfies

$$\mathcal{L}_i(x_j) = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j, \end{cases} \quad (2.8)$$

where δ_{ij} is the Kronecker delta function.

The Lagrange polynomial is therefore constructed as follows, where the interpolation points x_0, x_1, \dots, x_N are unique

$$\mathcal{L}_i(x) = \frac{(x - x_0)(x - x_1) \dots (x - x_{i-1})(x - x_{i+1}) \dots (x - x_N)}{(x_i - x_0)(x_i - x_1) \dots (x_i - x_{i-1})(x_i - x_{i+1}) \dots (x_i - x_N)} \quad (2.9)$$

$$= \prod_{\substack{j=0 \\ j \neq i}}^N \frac{(x - x_j)}{(x_i - x_j)}. \quad (2.10)$$

The resulting interpolating polynomial is given by

$$p_N(x) = \sum_{i=0}^N a_i \mathcal{L}_i(x). \quad (2.11)$$

As outlined in [9], due to the Kronecker delta property of Lagrange polynomials i.e. $\mathcal{L}_i(x_i) = 1$ and zero elsewhere. The linear system reduces to

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \\ \vdots \\ f(x_N) \end{bmatrix}. \quad (2.12)$$

Hence, the Lagrange interpolating polynomial for a known function $f(x)$ is given by

$$p_N(x) = \sum_{i=0}^N f(x_i) \mathcal{L}_i(x). \quad (2.13)$$

Example 2.2 (Lagrange Basis). Approximate $f(x) = \cos(5x)$, between $[-1, 1]$ using 8 equally spaced points with a Lagrange basis.

Plotting the Lagrange basis polynomial for this problem yields

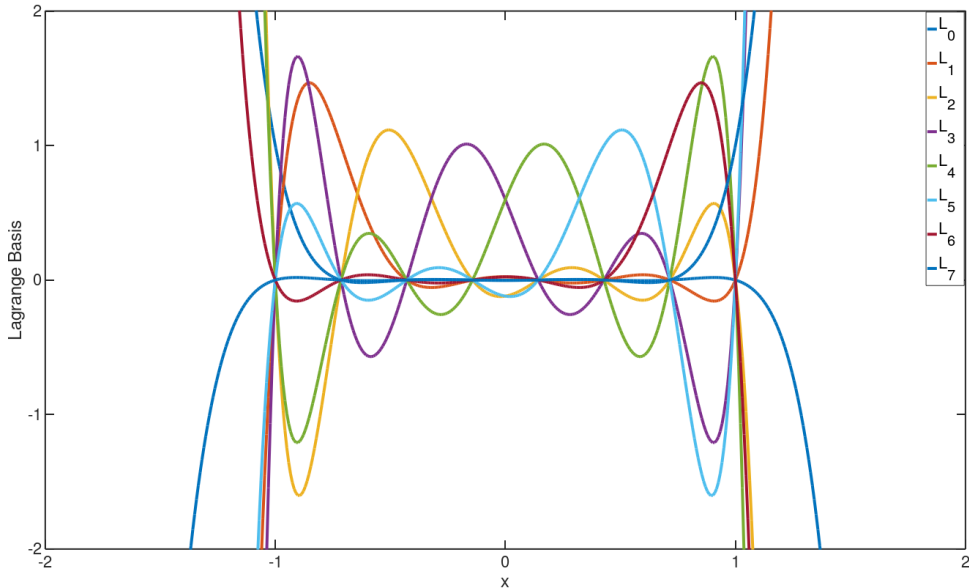


Figure 2.3: The Lagrange basis polynomials $\mathcal{L}_i(x) = \prod_{j=0, j \neq i}^7 \frac{(x-x_j)}{(x_i-x_j)}$, $i = 0, 1, \dots, 7$.

Hence, solving the interpolant and plotting

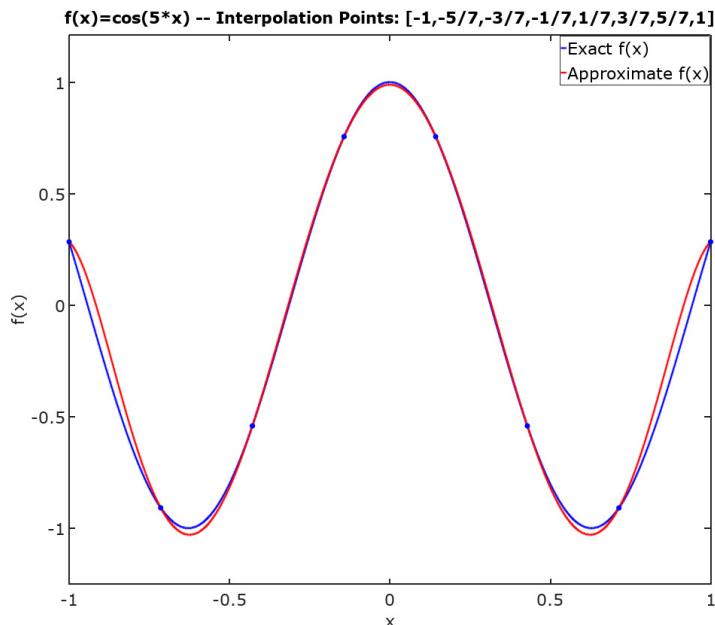


Figure 2.4: The 7th order polynomial interpolant versus the exact function $f(x)$.

2.1.2 Interpolation error at equidistant points

According to the Weierstrass approximation theorem, on a closed interval $[a, b]$ any continuous function $f(x)$ can be approximated as closely as desired with an interpolating polynomial $p_N(x)$ of sufficiently high degree [9], that is

$$\lim_{n \rightarrow \infty} \sup_{a \leq x \leq b} |f(x) - p_N(x)| = 0. \quad (2.14)$$

Example 2.3. Suppose we wish to approximate the continuous function $f(x) = \sin(5x)$ on $[-1, 1]$ with $N = 11$ and $N = 21$ equally spaced interpolation points. The approximation and absolute errors are illustrated in the following figures.

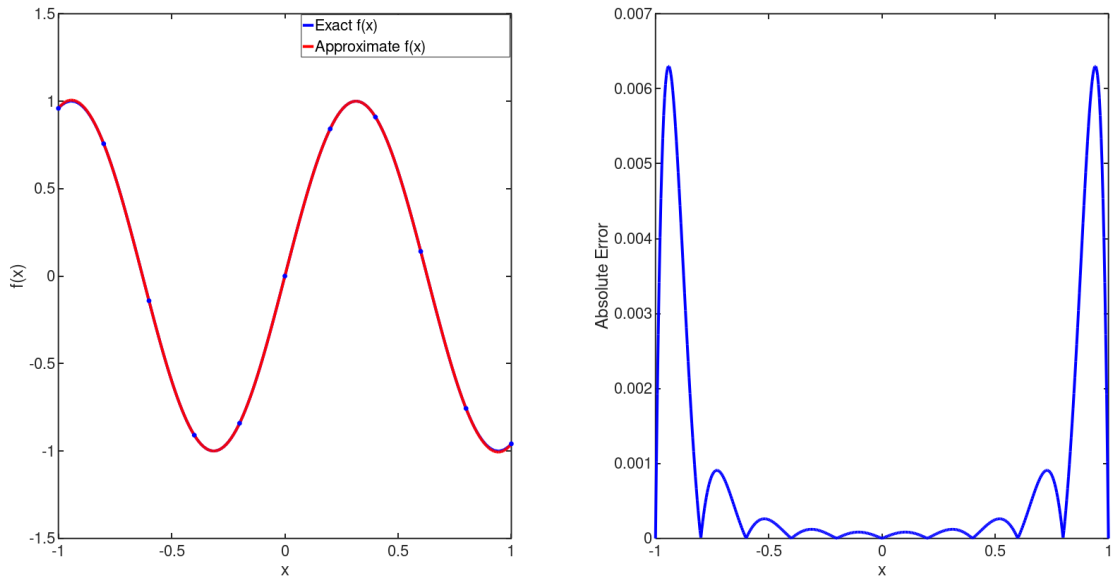


Figure 2.5: Error using p_{10} to approximate $\sin(5x)$.

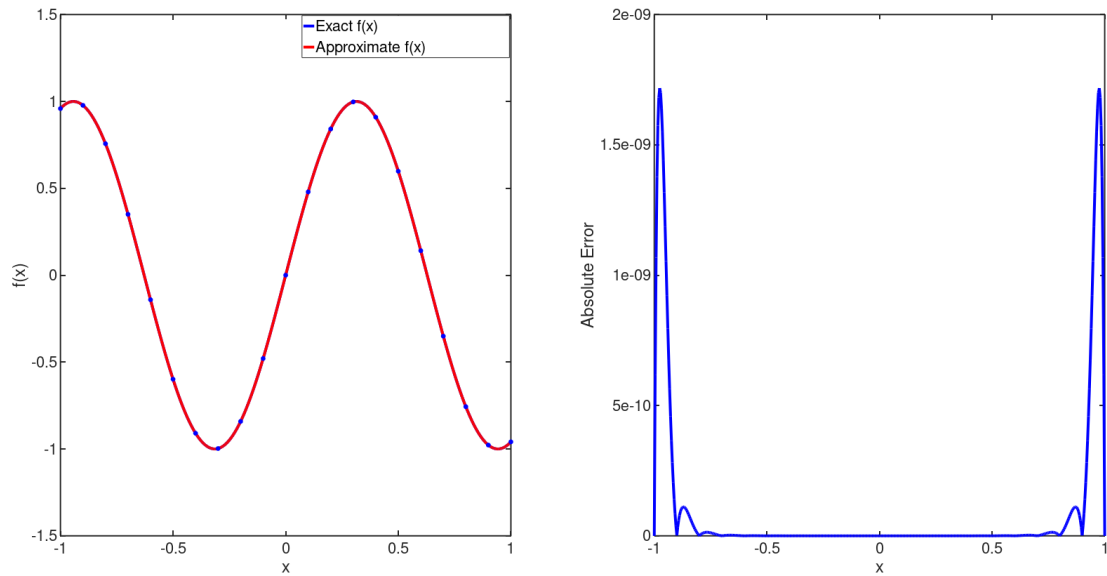


Figure 2.6: Error using p_{20} to approximate $\sin(5x)$.

From Example 2.3, as the order of the interpolating polynomial increases, the error approaches zero in absolute terms. This is as expected from the Weierstrass theorem as described in (2.14).

However Runge's phenomenon demonstrates that interpolation can result in errors that do not converge (in fact, diverge) as the degree of the interpolating polynomial increases as demonstrated in the next example. [6]

Example 2.4. Suppose we wish to approximate Runge's function $f(x) = \frac{1}{1+x^2}$ on $[-5, 5]$ with n equally spaced interpolation points.

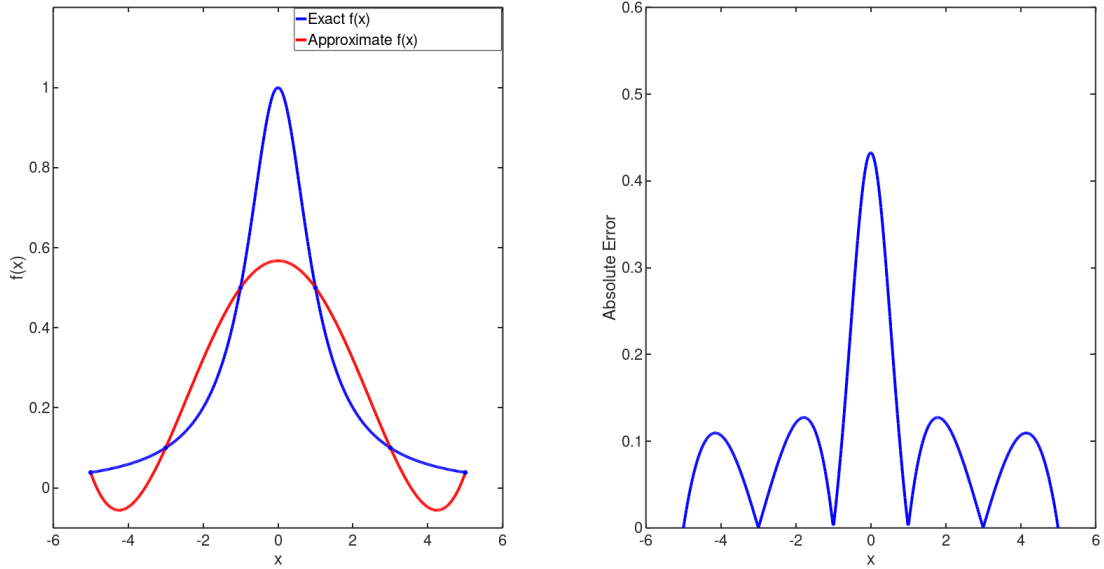


Figure 2.7: The 5th degree polynomial interpolant versus the exact function $f(x)$.

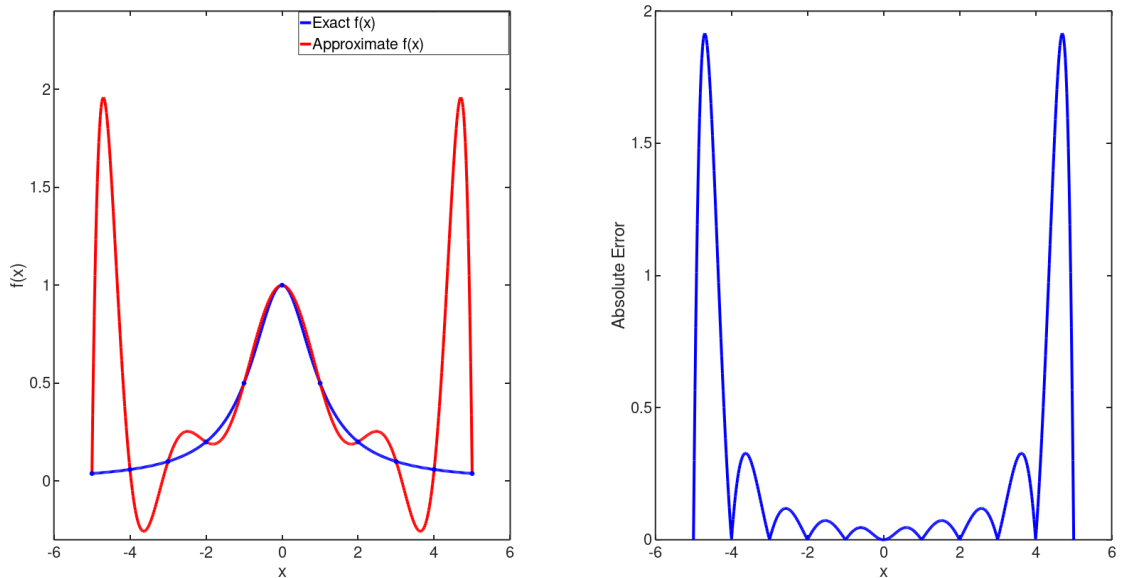


Figure 2.8: The 10th degree polynomial interpolant versus the exact function $f(x)$.

Example 2.4 shows that interpolating the Runge's function with polynomials using equally spaced interpolation points results in diverging error as the degree of the interpolant approaches infinity, that is

$$\lim_{n \rightarrow \infty} \sup_{-5 \leq x \leq 5} |f(x) - p_N(x)| = \infty. \quad (2.15)$$

To explore why this phenomenon occurs, we need a uniform approach for finding the error of a polynomial interpolant of degree N . The general form of such an error is derived in [6], and given in the following theorem.

Theorem 2.2 (Polynomial Interpolation Error). *Suppose the N^{th} degree polynomial $p_N(x)$ interpolates, at $N + 1$ distinct points x_0, x_1, \dots, x_N , the function $f(x)$ continuous on $[a, b]$ and if the $(N + 1)^{\text{th}}$ derivative $f^{(N+1)}(\xi)$ exists and is also continuous on $[a, b]$, then there exists $\xi \in (a, b)$ such that*

$$f(x) - p_N(x) = \frac{f^{(N+1)}(\xi)}{(N + 1)!} \prod_{i=0}^N (x - x_i). \quad (2.16)$$

Furthermore, in the context of Theorem 2.2 as outlined in [2], the error bound for $a \leq x \leq b$ can be given in the form

$$\sup_{a \leq x \leq b} |f(x) - p_N(x)| = \frac{1}{(N + 1)!} \sup_{a \leq \xi \leq b} |f^{(N+1)}(\xi)| \sup_{a \leq x \leq b} \prod_{i=0}^N (x - x_i). \quad (2.17)$$

As the degree of the interpolating polynomial (2.17), N approaches $+\infty$, the maximum of the absolute error $|f(x) - p_N(x)|$ may approach zero as the denominator $(N + 1)! \rightarrow +\infty$. However, as in the case of Runge's phenomenon, if the $(N + 1)^{\text{th}}$ derivative $f^{(N+1)}$ grows unbounded as $N \rightarrow \infty$, the maximum of the absolute error $|f(x) - p_N(x)|$ may not converge. Hence, the interpolation error (2.16) could grow unbounded.

The following figure shows graphically that for Runge's function $f(x) = \frac{1}{1+x^2}$, the supremum of the derivatives $f^{(N)}(x)$ increases exponentially as N increases, the possible cause of the error of divergence of polynomial interpolation in this case.

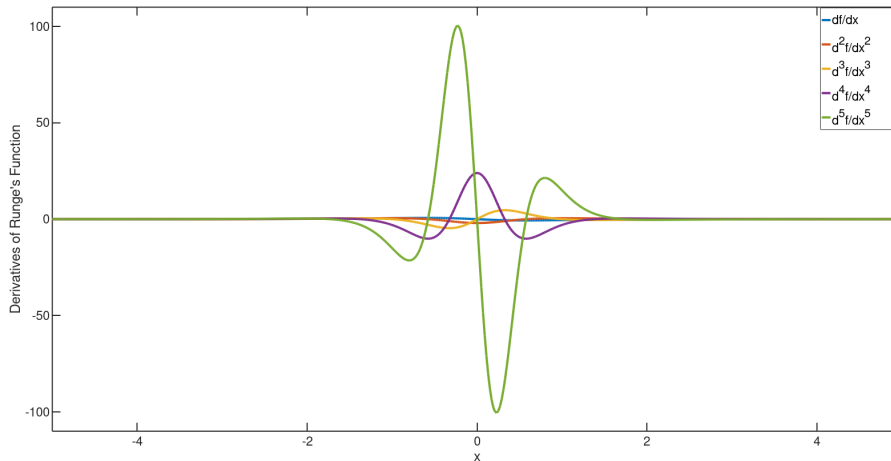


Figure 2.9: The first five derivatives of Runge's Function.

2.1.3 Advantages of polynomial interpolation

Some of the advantages of interpolating continuous functions using polynomials [4]

- They approximate continuous functions uniformly. According to the Weierstrass Approximation (Theorem 2.3), there exists a polynomial that is within a bounded error to any continuous function bounded on an interval, but this is not necessarily an interpolating polynomial.
- Their derivatives and anti-derivatives are also polynomials and therefore easy to determine and manipulate.

Theorem 2.3 (Weierstrass Approximation). *Suppose that a function $f(x)$ is continuous on $[a, b]$. For some small $\epsilon > 0$, there exists a polynomial of the form*

$$p_N(x) = a_N x^N + a_{N-1} x^{N-1} + \cdots + a_1 x + a_0,$$

such that

$$|f(x) - p_N(x)| < \epsilon, \quad \text{for all } x \in [a, b].$$

Theorem 2.3 is proved in [13].

2.1.4 Disadvantages of polynomial interpolation

There are cases where a polynomial approximation may fall short as a functional solution, some of the disadvantages of a polynomial interpolation are [6]

- Suppose we had to fit a polynomial to a large dataset, this could mean that the order of the polynomial may be prohibitively high to serve as a practical solution.
- The increasingly oscillatory nature of polynomials as it's order increases could result in the approximate solution oscillating unreasonably between data points. This was seen in the Runge's Phenomenon as observed in Example 2.4.

2.2 The theory of collocation

Consider a linear integro-differential equation of the form [10]

$$\mathbb{L}y = F(x), \quad (2.18)$$

where \mathbb{L} is a linear operator defined by

$$\mathbb{L}[\] = \frac{d}{dx} - \int_{t_0}^x K(x, t)[\] dt. \quad (2.19)$$

If $\tilde{y}(x)$ is an approximation to $y(x)$, then

$$\mathbb{L}\tilde{y} \approx F(x), \quad (2.20)$$

and the residual is given by

$$R(x) = \mathbb{L}\tilde{y} - F(x). \quad (2.21)$$

At collocation points x_i , $i = 0, 1, 2, \dots, N$ we force the residual to equal zero, that is

$$R(x_i) = (\mathbb{L}\tilde{y})(x_i) - F(x_i) = 0. \quad (2.22)$$

This implies that

$$(\mathbb{L}\tilde{y})(x_i) = F(x_i), \quad (2.23)$$

which is equivalent to interpolating $F(x)$ by $(\mathbb{L}\tilde{y})(x)$. Therefore, in order to numerically solve the integro-differential equation $\mathbb{L}y = F(x)$, we interpolate $F(x)$ by $(\mathbb{L}\tilde{y})(x)$. This is called collocation.

2.3 Methodology

As defined in Definition 1.2, we wish to find a numerical solution to the problem of the form

$$y'(x) - \int_{t_0}^x K(x, t)y(t) dt = F(x), \quad \text{with } y(t_0) = y_0. \quad (2.24)$$

2.3.1 Constructing the numerical solution

Let the approximate solution $p_N(x)$ of integro-differential equation (2.24), and its first derivative $y'(x)$ be defined, respectively, by the following series

$$p_N(x) = \sum_{i=0}^N a_i \mathcal{L}_i(x) \quad \text{and} \quad (2.25)$$

$$p'_N(x) = \sum_{i=0}^N a_i \mathcal{L}'_i(x), \quad (2.26)$$

where a_i for $i = 0, 1, \dots, N$ are the unknown parameters, \mathcal{L}_i for $i = 0, 1, \dots, N$ is the Lagrange basis for each i as defined by (2.10) and $N + 1$ is the number of interpolating points.

The derivative of the Lagrange basis

The Lagrange basis in the form (2.9), can be differentiated to obtain

$$\mathcal{L}'_i(x) = \sum_{\substack{j=0 \\ j \neq i}}^N \frac{\mathcal{L}_i(x)}{(x - x_j)}, \quad \text{for } x \neq x_j. \quad (2.27)$$

Evaluating at x_i , taking into account the Kronecker Delta property (2.8), the derivative of the Lagrange basis takes the form given in the following proposition.

Proposition 2.1 (First Derivative of the Lagrange Basis I). The first derivative of the Lagrange basis $\mathcal{L}'_i(x_i)$ is defined as

$$\mathcal{L}'_i(x_i) = \sum_{\substack{j=0 \\ j \neq i}}^n \frac{1}{(x_i - x_j)},$$

where x_i for $i = 0, 1, \dots, N$ are the interpolation points.

The Lagrange Basis (2.10), can easily be written in terms of the nodal basis function $\psi(x)$, that is

$$\mathcal{L}_i(x) = \frac{\psi(x)}{(x - x_i)\psi'(x_i)}, \quad (2.28)$$

where the nodal basis, $\psi(x)$ and its first derivative at x_i , $\psi'(x_i)$ is defined as

$$\psi(x) = \prod_{j=0}^N (x - x_j) \quad \text{and} \quad \psi'(x_i) = \prod_{\substack{j=0 \\ j \neq i}}^N (x_i - x_j), \quad (2.29)$$

respectively, where x_j for $j = 0, 1, \dots, N$ are the interpolating points.

Differentiating the Lagrange basis in the form (2.28), we obtain

$$\mathcal{L}_i(x)(x - x_i) + \mathcal{L}_i(x) = \frac{\psi'(x)}{\psi'(x_i)}, \quad (2.30)$$

evaluating at x_j , we can write

$$\mathcal{L}_i'(x_j) = \frac{\psi'(x_j)}{(x_j - x_i)\psi'(x_i)} - \frac{\mathcal{L}_i(x_j)}{x_j - x_i}. \quad (2.31)$$

Finally, applying the Kronecker Delta property (2.8), the derivative of the Lagrange basis is given by the following proposition.

Proposition 2.2 (First Derivative of the Lagrange Basis II). The first derivative of the Lagrange basis $\mathcal{L}_i'(x_j)$ for $i \neq j$, is defined as

$$\mathcal{L}_i'(x_j) = \frac{\psi'(x_j)}{(x_j - x_i)\psi'(x_i)}, \quad (2.32)$$

where x_i for $i = 0, 1, \dots, N$ are the interpolation points.

2.3.2 Solving the IDE

Substituting (2.25) and (2.26) into the problem (2.24), we obtain

$$\sum_{i=0}^N a_i \mathcal{L}'_i(x) - \sum_{i=0}^N \int_{t_0}^x K(x, t) a_i \mathcal{L}_i(t) dt = F(x), \quad (2.33)$$

rearranging, we have the system of equations for each interpolation point x_j at $j = 0, 1, \dots, N$

$$\sum_{i=0}^N \left\{ a_i \left(\mathcal{L}'_i(x_j) - \int_{t_0}^{x_j} K(x, t) \mathcal{L}_i(t) dt \right) \right\} = F(x_j). \quad (2.34)$$

Applying the initial condition $y(x_0) = 1$ yields

$$p_N(x_0) = \sum_{i=0}^n a_i \mathcal{L}_i(x_0), \quad (2.35)$$

it follows from the Kronecker delta property (2.8) that

$$p_N(x_0) = a_0 = 1. \quad (2.36)$$

Hence, the resulting system of equations can be expressed as the following matrix problem

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ L_0^1 & L_1^1 & L_2^1 & L_3^1 & \dots & L_N^1 \\ L_0^2 & L_1^2 & L_2^2 & L_3^2 & \dots & L_N^2 \\ L_0^3 & L_1^3 & L_2^3 & L_3^3 & \dots & L_N^3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ L_0^N & L_1^N & L_2^N & L_3^N & \dots & L_N^N \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} 1 \\ F(x_1) \\ F(x_2) \\ F(x_3) \\ \vdots \\ F(x_N) \end{bmatrix}, \quad (2.37)$$

where

$$L_i^j = \mathcal{L}'_i(x_j) - \int_{x_0}^{x_j} K(x_j, t) \mathcal{L}_i(t) dt. \quad (2.38)$$

2.3.3 Error calculation

The method used to determine the discrete error of our approximate solution versus the exact solution is defined as follows

$$E_{N+1} = \sqrt{\int_{x_0}^{x_N} (y(x) - p_N(x))^2 dx}, \quad (2.39)$$

where $y(x)$ is the exact solution, $p_N(x)$ is the N^{th} -degree polynomial approximate solution and $N + 1$ is the number of collocation points. That is, we define the discrete error as the square root of the integral square error over the domain $x \in [x_0, x_N]$.

2.4 The Application

Suppose we had the integro-differential equation of the form

$$y'(x) + \frac{101}{10}e^{-x} \sin(10x) + e^{-x} \cos(10x) = \int_0^x e^{t-x} y(t) dt, \quad (2.40)$$

subject to initial condition, $y(0) = 1$.

The exact solution to this problem is given by $y(x) = e^{-x} \cos(10x)$. The numerical solution using Lagrange collocation is presented below.

2.4.1 34 equally spaced points

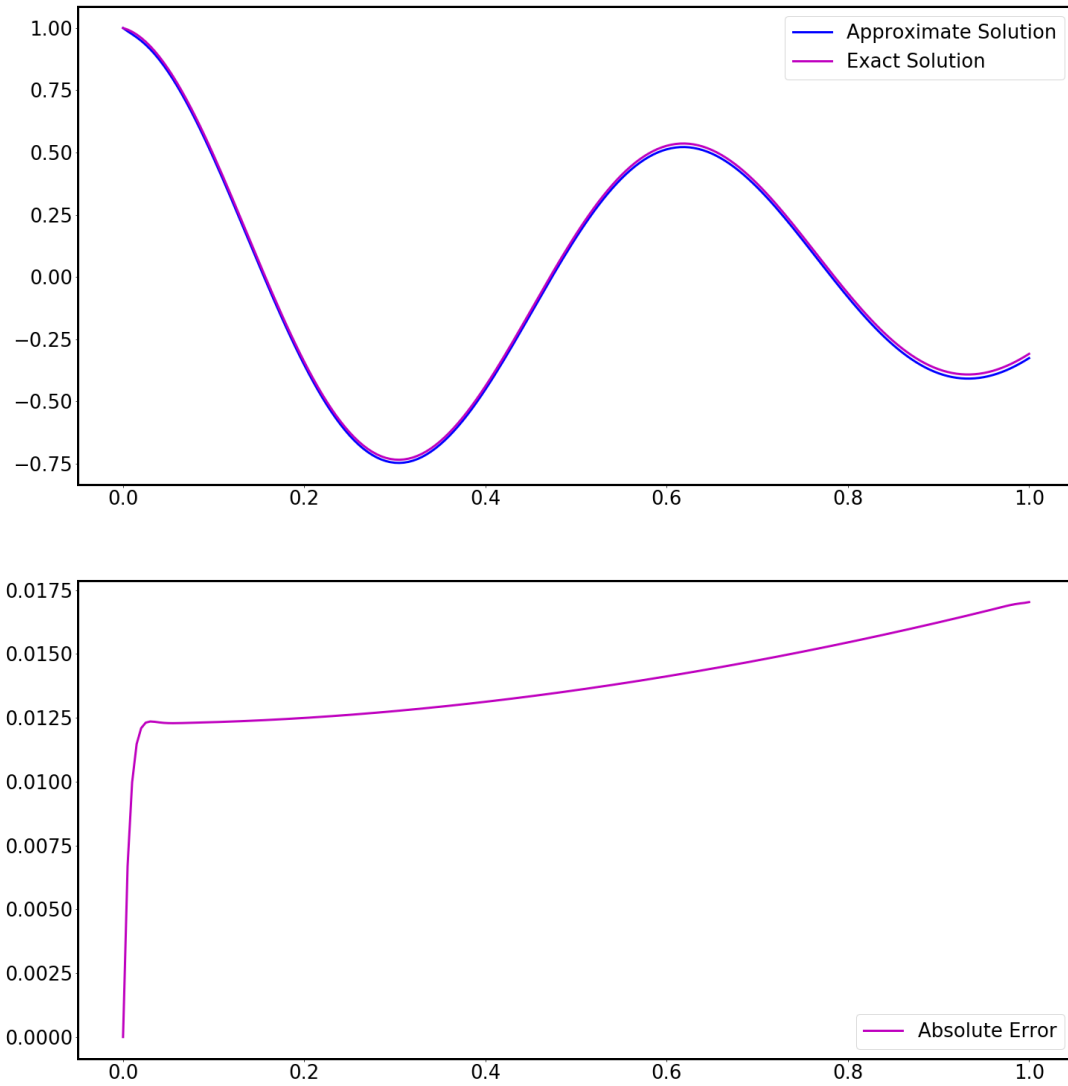


Figure 2.10: Lagrange Collocation using 34 equally spaced collocation points.

$$E_{34} = 0.013973603997490907$$

2.4.2 46 equally spaced points

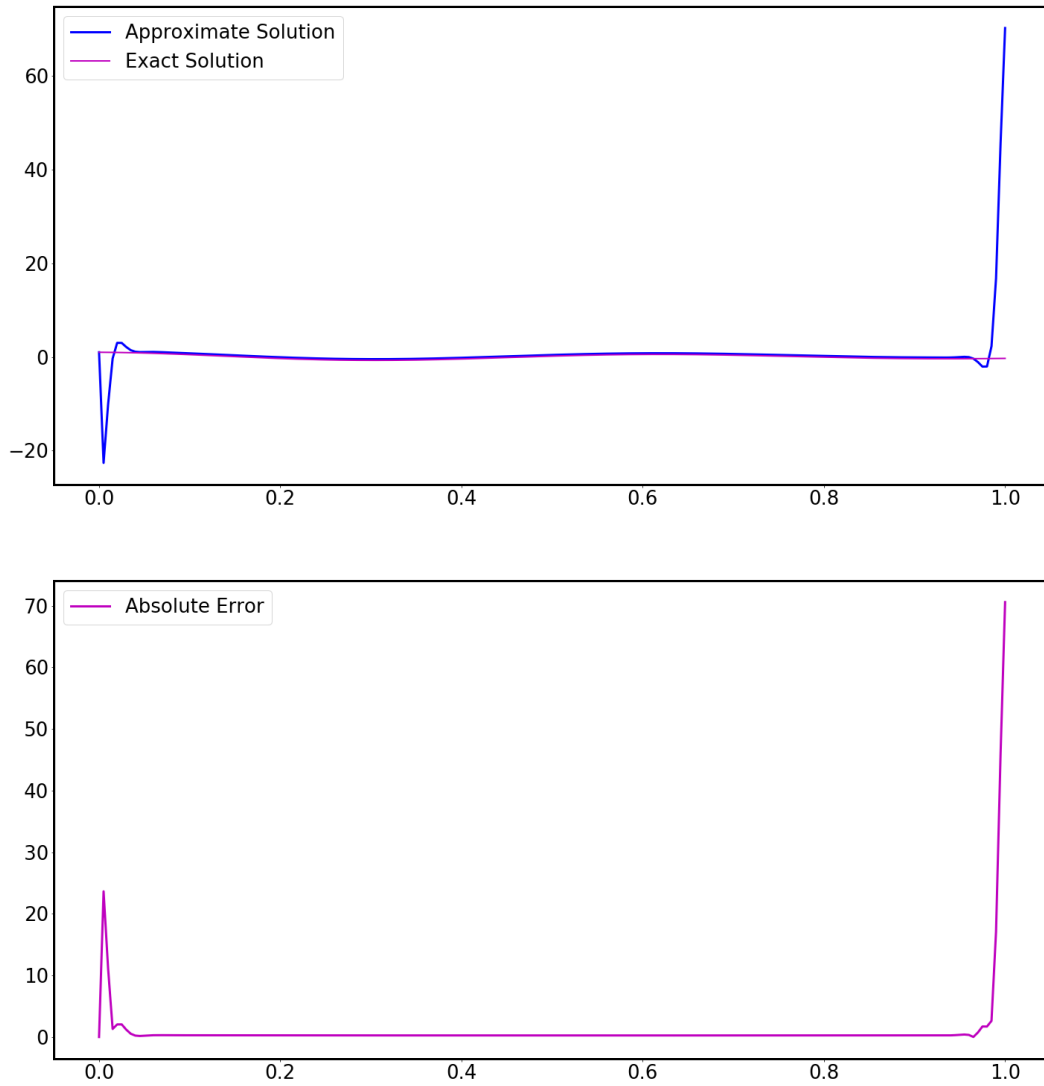


Figure 2.11: Lagrange Collocation using 46 equally spaced collocation points.

$$E_{46} = 5.31439610371601$$

Chapter 3

Application of cubic Hermite collocation to solving IDEs

3.1 Cubic Hermite collocation - theoretical basis

The interval $x \in [x_1, x_{N+1}]$, is partitioned into N equally spaced subintervals such that $x_1 < x_2 < \dots < x_N < x_{N+1}$ and $h = x_{i+1} - x_i$. We establish a new variable $u = \frac{x-x_i}{h}$ where $u \in [0, 1]$ such that as x varies in subinterval i given by $[x_i, x_{i+1}]$, u varies proportionally in the unit interval $[0, 1]$. [20]

3.1.1 The Hermite basis functions

The four cubic Hermite basis polynomials $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ and \mathcal{H}_4 are defined on the unit intervals, with the property that \mathcal{H}_1 on the i^{th} interval and \mathcal{H}_3 on the $(i+1)^{st}$ interval are continuous and have continuous first derivatives at the node x_{i+1} . A similar property is satisfied by \mathcal{H}_2 and \mathcal{H}_4 . That is, the cubic Hermite basis functions have the following properties

$$\begin{aligned}\mathcal{H}_1(0) &= 1, & \mathcal{H}_1(1) &= 0, \\ \mathcal{H}'_1(0) &= 0, & \mathcal{H}'_1(1) &= 0,\end{aligned}\tag{3.1}$$

$$\begin{aligned}\mathcal{H}_3(0) &= 0, & \mathcal{H}_3(1) &= 1, \\ \mathcal{H}'_3(0) &= 0, & \mathcal{H}'_3(1) &= 0,\end{aligned}\tag{3.2}$$

$$\begin{aligned}\mathcal{H}_2(0) &= 0, & \mathcal{H}_2(1) &= 0, \\ \mathcal{H}'_2(0) &= 1, & \mathcal{H}'_2(1) &= 0,\end{aligned}\tag{3.3}$$

$$\begin{aligned}\mathcal{H}_4(0) &= 0, & \mathcal{H}_4(1) &= 0, \\ \mathcal{H}'_4(0) &= 0, & \mathcal{H}'_4(1) &= 1.\end{aligned}\tag{3.4}$$

Solving for the basis function $\mathcal{H}_1(x)$ we see that we have a double root at $x = 1$, thus we must solve the equation

$$\mathcal{H}_1(x) = (x - 1)^2(ax + b),\tag{3.5}$$

subject to the conditions in (3.1). This yields $a = 2$ and $b = 1$, therefore

$$\mathcal{H}_1(x) = (x - 1)^2(2x + 1).\tag{3.6}$$

Similarly, solving for the remaining cubic polynomials $\mathcal{H}_2, \mathcal{H}_3$ and \mathcal{H}_4 which satisfy (3.3), (3.2) and (3.4) respectively, give the following Hermite basis functions

$$\mathcal{H}_2(x) = (x - 1)^2x,\tag{3.7}$$

$$\mathcal{H}_3(x) = x^2(3 - 2x),\tag{3.8}$$

$$\mathcal{H}_4(x) = x^2(x - 1).\tag{3.9}$$

The continuity of the Hermite basis functions and their first derivatives on adjacent subintervals can be seen graphically in the following figure.

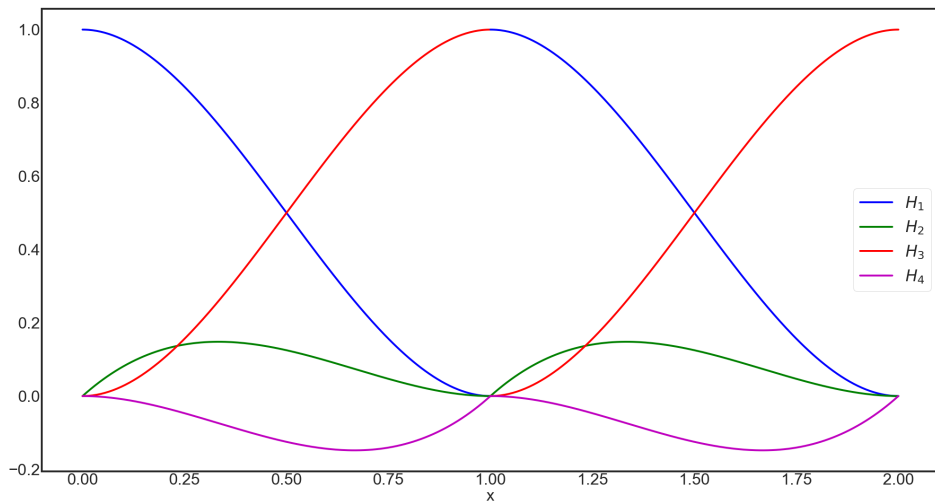


Figure 3.1: The Cubic Hermite Basis functions on two subintervals.

Linear independence of the Hermite basis functions

Consider $\sum_{k=1}^4 \alpha_k \mathcal{H}_k(x) = 0$, substituting

$$\begin{aligned} x = 0, & \quad \text{gives} \quad \alpha_1 = 0, \\ x = 1, & \quad \text{gives} \quad \alpha_3 = 0. \end{aligned}$$

Differentiating, we have $\sum_{k=1}^4 \alpha_k \mathcal{H}'_k(x) = 0$, substituting

$$\begin{aligned} x = 0, & \quad \text{gives} \quad \alpha_2 = 0, \\ x = 1, & \quad \text{gives} \quad \alpha_4 = 0, \end{aligned}$$

which proves that the Hermite basis functions are linearly independent.

3.1.2 Defining the piecewise approximating polynomials

For each subinterval i , defined on $[x_i, x_{i+1}]$, the general approximation of a given function $f(x)$ between x_i and x_{i+1} is given by

$$y^i(u) = \sum_{k=1}^4 a_k^i \mathcal{H}_k(u), \quad (3.10)$$

where a_k^i are unknown constants.

Adjacent approximations must be continuous at adjacent nodes, that is $y^i(1) = y^{i+1}(0)$, which implies that

$$y^i(1) = \sum_{k=1}^4 a_k^i \mathcal{H}_k(1) = \sum_{k=1}^4 a_k^{i+1} \mathcal{H}_k(0) = y^{i+1}(0), \quad (3.11)$$

from the defining properties of Hermite basis functions (3.1) and (3.2) above, it follows that

$$a_3^i = a_1^{i+1}. \quad (3.12)$$

The gradient of adjacent approximations must also be continuous at the nodes, that is $\frac{dy^i}{du} \Big|_{u=1} = \frac{dy^{i+1}}{du} \Big|_{u=0}$, which implies that

$$\frac{dy^i}{du} \Big|_{u=1} = \sum_{k=1}^4 a_k^i \mathcal{H}'_k(1) = \sum_{k=1}^4 a_k^{i+1} \mathcal{H}'_k(0) = \frac{dy^{i+1}}{du} \Big|_{u=0}, \quad (3.13)$$

from properties (3.3) and (3.4) above, it follows that

$$a_4^i = a_2^{i+1}. \quad (3.14)$$

Expanding the general approximation $y^i(x)$ for subintervals $i = 1, 2, \dots, n$ yields

$$\begin{aligned} y^1(u) &= a_1^1 \mathcal{H}_1(u) + a_2^1 \mathcal{H}_2(u) + a_3^1 \mathcal{H}_3(u) + a_4^1 \mathcal{H}_4(u), \\ y^2(u) &= a_1^2 \mathcal{H}_1(u) + a_2^2 \mathcal{H}_2(u) + a_3^2 \mathcal{H}_3(u) + a_4^2 \mathcal{H}_4(u), \\ y^3(u) &= a_1^3 \mathcal{H}_1(u) + a_2^3 \mathcal{H}_2(u) + a_3^3 \mathcal{H}_3(u) + a_4^3 \mathcal{H}_4(u), \\ &\vdots \\ y^N(u) &= a_1^N \mathcal{H}_1(u) + a_2^N \mathcal{H}_2(u) + a_3^N \mathcal{H}_3(u) + a_4^N \mathcal{H}_4(u), \end{aligned} \quad (3.15)$$

substituting the relations (3.12) and (3.14), yields

$$\begin{aligned} y^1(u) &= a_1^1 \mathcal{H}_1(u) + a_2^1 \mathcal{H}_2(u) + a_3^1 \mathcal{H}_3(u) + a_4^1 \mathcal{H}_4(u), \\ y^2(u) &= a_3^1 \mathcal{H}_1(u) + a_4^1 \mathcal{H}_2(u) + a_3^2 \mathcal{H}_3(u) + a_4^2 \mathcal{H}_4(u), \\ y^3(u) &= a_3^2 \mathcal{H}_1(u) + a_4^2 \mathcal{H}_2(u) + a_3^3 \mathcal{H}_3(u) + a_4^3 \mathcal{H}_4(u), \\ &\vdots \\ y^N(u) &= a_3^{N-1} \mathcal{H}_1(u) + a_4^{N-1} \mathcal{H}_2(u) + a_3^N \mathcal{H}_3(u) + a_4^N \mathcal{H}_4(u). \end{aligned} \quad (3.16)$$

This result enables us to simplify the notation for the general approximation by avoiding duplicate calculations at the nodes. Consequently, the total number of unknowns is reduced to $2n + 2$. These results are summarised in the following definition.

Definition 3.1 (The Piecewise Hermite Approximation). The general approximation for a given function $f(x)$ on each subinterval i , given by $[x_i, x_{i+1}]$ can be written as

$$f(x) \approx y^i(u) = \sum_{k=1}^4 a_{k+2(i-1)} \mathcal{H}_k(u),$$

where a_j for $j = 1, 2, \dots, 2N + 2$ are the unknown constants and $u = \frac{x-x_i}{h} \in [0, 1]$.

3.1.3 Cubic Hermite interpolation

In order to approximate some function $f(x)$ on an interval $[x_1, x_N]$, we must determine the approximating polynomial $y^i(u)$ on each subinterval i , $i = 1, \dots, N$. In order to determine each $y^i(u)$ uniquely we need to solve for all unknown parameters $a_1, a_2, \dots, a_{2N+2}$, that is we require a system of $2N + 2$ equations.

Choosing interpolation points

Since we must have $2N + 2$ equations, we require that many interpolation points as well. Here we select three interpolation points in the first and last subintervals and two interpolation points in all other subintervals as illustrated in the figure below.

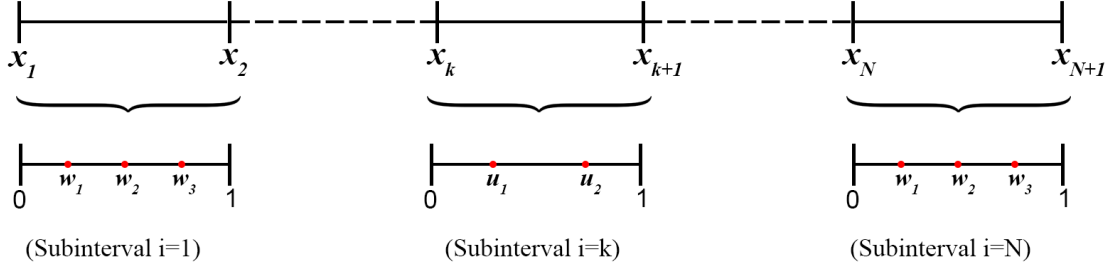


Figure 3.2: Interpolation points on each transformed subinterval.

In figure 3.2 above, the interpolation points are chosen such that $w_1, w_2, w_3 \in [0, 1]$, $w_1 \neq w_2 \neq w_3$ and $u_1, u_2 \in [0, 1]$, $u_1 \neq u_2$. In this thesis, we shall use roots of the Chebyshev polynomials $T_2 = 2x^2 - 1$ and $T_3 = 4x^3 - 3x$ shifted to $[0, 1]$. That is

$$T_2^* = \frac{1}{4}(8x^2 - 8x + 1) = 0, \quad \text{with roots} \quad (3.17)$$

$$u_1 = \frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right), \quad u_2 = \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right), \quad \text{and} \quad (3.18)$$

$$T_3^* = \frac{1}{8}(32x^3 - 48x^2 + 18x - 1) = 0, \quad \text{with roots} \quad (3.19)$$

$$w_1 = \frac{1}{2}\left(1 - \frac{\sqrt{3}}{2}\right), \quad w_2 = \frac{1}{2}, \quad w_3 = \frac{1}{2}\left(1 + \frac{\sqrt{3}}{2}\right). \quad (3.20)$$

The matrix system

Solving for unknowns $a_1, a_2, \dots, a_{2N+2}$ with interpolation points illustrated by Figure 3.2 leads to the following matrix problem

$$GA = F, \quad (3.21)$$

where

$$G = \begin{bmatrix} \mathcal{H}_1(w_1) & \mathcal{H}_2(w_1) & \mathcal{H}_3(w_1) & \mathcal{H}_4(w_1) & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathcal{H}_1(w_2) & \mathcal{H}_2(w_2) & \mathcal{H}_3(w_2) & \mathcal{H}_4(w_2) & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathcal{H}_1(w_3) & \mathcal{H}_2(w_3) & \mathcal{H}_3(w_3) & \mathcal{H}_4(w_3) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{H}_1(u_1) & \mathcal{H}_2(u_1) & \mathcal{H}_3(u_1) & \mathcal{H}_4(u_1) & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{H}_1(u_2) & \mathcal{H}_2(u_2) & \mathcal{H}_3(u_2) & \mathcal{H}_4(u_2) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots & \ddots & \ddots & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{H}_1(u_1) & \mathcal{H}_2(u_1) & \mathcal{H}_3(u_1) & \mathcal{H}_4(u_1) & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{H}_1(u_2) & \mathcal{H}_2(u_2) & \mathcal{H}_3(u_2) & \mathcal{H}_4(u_2) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{H}_1(w_1) & \mathcal{H}_2(w_1) & \mathcal{H}_3(w_1) & \mathcal{H}_4(w_1) \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{H}_1(w_2) & \mathcal{H}_2(w_2) & \mathcal{H}_3(w_2) & \mathcal{H}_4(w_2) \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{H}_1(w_3) & \mathcal{H}_2(w_3) & \mathcal{H}_3(w_3) & \mathcal{H}_4(w_3) \end{bmatrix},$$

$$A = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ \vdots \\ a_{2N-2} \\ a_{2N-1} \\ a_{2N} \\ a_{2N+1} \\ a_{2N+2} \end{bmatrix}, \quad F = \begin{bmatrix} f(x_1 + w_1h) \\ f(x_1 + w_2h) \\ f(x_1 + w_3h) \\ f(x_2 + u_1h) \\ f(x_2 + u_2h) \\ \vdots \\ f(x_{N-1} + u_1h) \\ f(x_{N-1} + u_2h) \\ f(x_N + w_1h) \\ f(x_N + w_2h) \\ f(x_N + w_3h) \end{bmatrix}.$$

Example 3.1. Approximate Runge's function $f(x) = \frac{1}{1+x^2}$ on $[-5, 5]$ using Cubic Hermite Interpolation.

We select the interpolation points described in Figure 3.2 using the roots of the shifted Chebyshev polynomials (3.18) and (3.20).

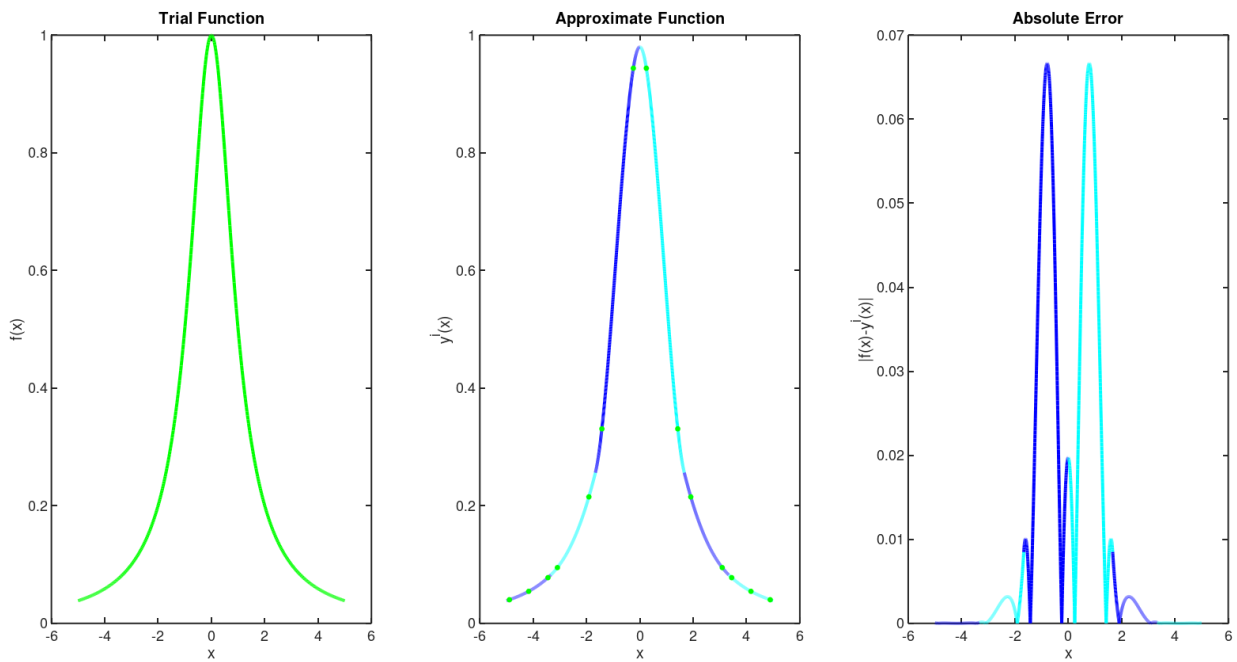


Figure 3.3: Cubic Hermite Interpolation applied to the Runge function with 6 subintervals or 14 interpolation Points.

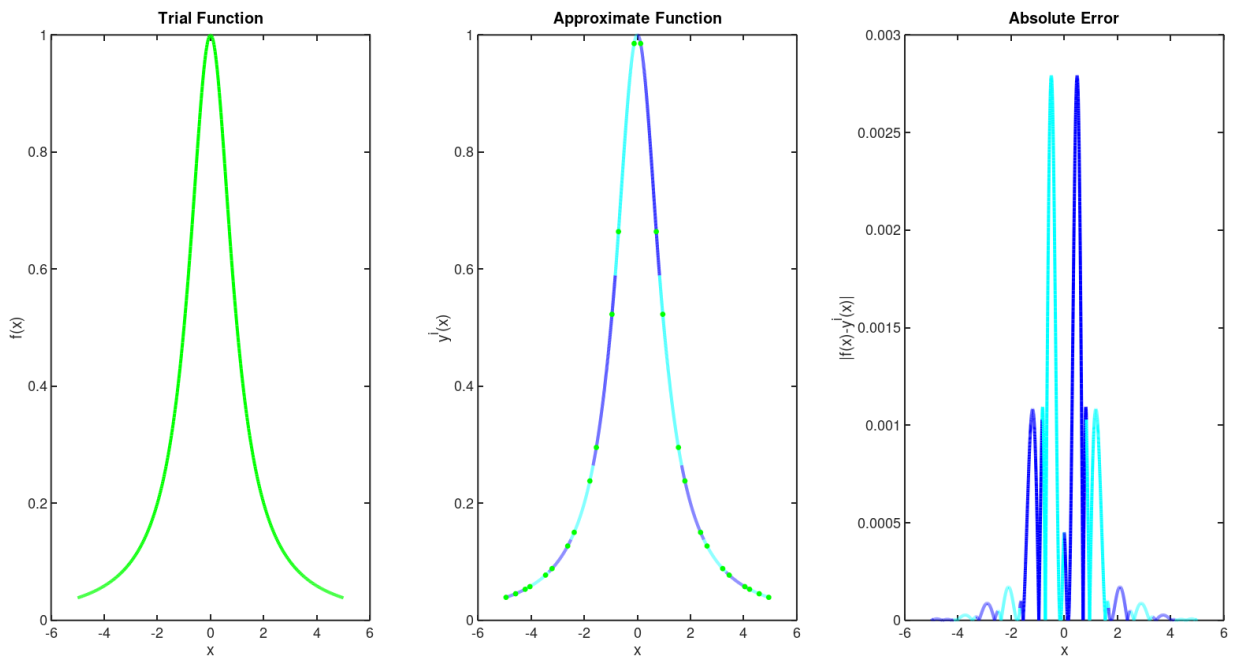


Figure 3.4: Cubic Hermite Interpolation applied to the Runge function with 12 subintervals or 26 interpolation Points.

3.2 Cubic Hermite collocation - methodology

We wish to find a numerical solution to the problem defined in Definition 1.2, that is

$$y'(x) - \int_{t_0=x_1}^x K(x,t)y(t) dt = F(x), \quad \text{with } y(t_0 = x_1) = y_0, \quad (3.22)$$

this time using the Hermite collocation method.

3.2.1 Constructing the numerical solution

Each piecewise approximate solution $y^i(x)$ of integro-differential equation (3.22), and its first derivative $\frac{dy^i}{du}$ is defined respectively as follows

$$y^i(u) = \sum_{k=1}^4 a_{k+2(i-1)} \mathcal{H}_k(u) \quad \text{and} \quad (3.23)$$

$$\frac{dy^i}{du}(u) = \sum_{k=1}^4 a_{k+2(i-1)} \mathcal{H}'_k(u), \quad (3.24)$$

in each subinterval i , where $\mathcal{H}'_k(u)$ denotes the first derivative of the k^{th} cubic Hermite basis defined earlier.

3.2.2 Transforming the IDE to each subinterval

Where $x \in [x_i, x_{i+1}]$ and $u = \frac{x-x_i}{h}$, the integral term in (3.22) is defined from x_1 to $x = x_i + uh$, therefore we must take into account each subinterval encompassed by this range, that is

$$\begin{aligned} \frac{dy^i}{du} \frac{du}{dx} - \left(\int_{x_1}^{x_2} K(x_2)y^1(t) dt + \int_{x_2}^{x_3} K(x_3)y^2(t) dt + \dots \right. \\ \left. + \int_{x_i}^{x_i+uh} K(x_i + uh, t)y^i(t) dt \right) = F(x_i + uh), \quad (3.25) \end{aligned}$$

since $\frac{du}{dx} = \frac{1}{h}$, the transformed IDE can be written in the following form

$$\frac{dy^i}{du} - h \left(\int_{x_1}^{x_2} K(x_2)y^1(t) dt + \int_{x_2}^{x_3} K(x_3)y^2(t) dt + \dots \right.$$

$$+ \int_{x_i}^{x_i+uh} K(x_i + uh, t) y^i(t) dt \Big) = hF(x_i + uh). \quad (3.26)$$

Substituting (3.23) and (3.24) into (3.26) yields the transformed IDE on subinterval i

$$\begin{aligned} \sum_{k=1}^4 a_{k+2(i-1)} H_k'(u) - h \left(\int_{x_1}^{x_2} K(x_2, t) \sum_{k=1}^4 a_k H_k(u) dt + \int_{x_2}^{x_3} K(x_3, t) \sum_{k=1}^4 a_{k+2} H_k(u) dt + \dots \right. \\ \left. + \int_{x_i}^{x_i+uh} K(x_i + uh, t) \sum_{k=1}^4 a_{k+2(i-1)} H_k(u) dt \right) = hF(x_i + uh). \quad (3.27) \end{aligned}$$

Choosing collocation points

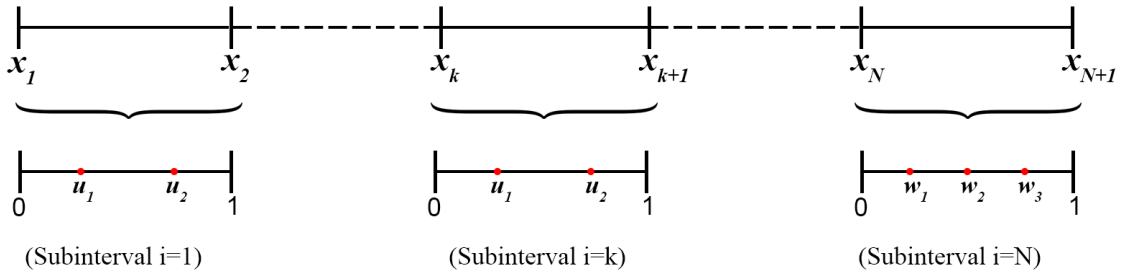


Figure 3.5: Collocation points on each transformed subinterval.

We have $2N + 2$ unknowns and one initial condition, hence we choose $2N + 1$ collocation points as the roots of the Chebyshev polynomials shifted to $[0, 1]$, as follows

$$u_1 = \frac{1}{2} \left(1 - \frac{1}{\sqrt{2}}\right), \quad u_2 = \frac{1}{2} \left(1 + \frac{1}{\sqrt{2}}\right), \quad (3.28)$$

$$w_1 = \frac{1}{2} \left(1 - \frac{\sqrt{3}}{2}\right), \quad w_2 = \frac{1}{2}, \quad w_3 = \frac{1}{2} \left(1 + \frac{\sqrt{3}}{2}\right). \quad (3.29)$$

The matrix system

Solving for unknowns $a_1, a_2, \dots, a_{2N+2}$ with collocation points illustrated by Figure 3.5 leads to the matrix system $G\mathbf{a} = \mathbf{F}$, with

(3.30)

$$\begin{aligned}
& \left[\begin{array}{cccccccc}
1 & & & & & & & \\
H_1^1(u_1) & H_2^1(u_1) & H_3^1(u_1) & H_4^1(u_1) & & & & \\
H_1^1(u_2) & H_2^1(u_2) & H_3^1(u_2) & H_4^1(u_2) & & & & \\
H_1^2(u_1) & H_2^2(u_1) & H_3^2(u_1) & H_4^2(u_1) & H_5^2(u_1) & H_6^2(u_1) & & \\
H_1^2(u_2) & H_2^2(u_2) & H_3^2(u_2) & H_4^2(u_2) & H_5^2(u_2) & H_6^2(u_2) & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
H_1^{N-1}(u_1) & H_2^{N-1}(u_1) & H_3^{N-1}(u_1) & H_4^{N-1}(u_1) & H_5^{N-1}(u_1) & \cdots & H_{2N-1}^{N-1}(u_1) & H_{2N}^{N-1}(u_1) \\
H_1^{N-1}(u_2) & H_2^{N-1}(u_2) & H_3^{N-1}(u_2) & H_4^{N-1}(u_2) & H_5^{N-1}(u_2) & \cdots & H_{2N-1}^{N-1}(u_2) & H_{2N}^{N-1}(u_2) \\
H_1^N(w_1) & H_2^N(w_1) & H_3^N(w_1) & H_4^N(w_1) & H_5^N(w_1) & \cdots & H_{2N-1}^N(w_1) & H_{2N}^N(w_1) \\
H_1^N(w_2) & H_2^N(w_2) & H_3^N(w_2) & H_4^N(w_2) & H_5^N(w_2) & \cdots & H_{2N-1}^N(w_2) & H_{2N}^N(w_2) \\
H_1^N(w_3) & H_2^N(w_3) & H_3^N(w_3) & H_4^N(w_3) & H_5^N(w_3) & \cdots & H_{2N-1}^N(w_3) & H_{2N}^N(w_3) \\
& & & & & & & H_{2N+1}^N(w_1) & H_{2N+2}^N(w_1) \\
& & & & & & & H_{2N+1}^N(w_2) & H_{2N+2}^N(w_2) \\
& & & & & & & H_{2N+1}^N(w_3) & H_{2N+2}^N(w_3)
\end{array} \right]
\end{aligned}$$

$$H_k^1(u) = \mathcal{H}'_k(u) - h \int_{x_1}^{x_1+uh} K(x_1 + uh, t) \mathcal{H}_k(t) dt, \quad (3.31)$$

$$H_k^i(u) = \begin{cases} -h \int_{x_1}^{x_2} K(x_2, t) \mathcal{H}_k(t) dt, & \text{for } k \leq 2 \\ -h \int_{x_\ell}^{x_{\ell+1}} K(x_{\ell+1}, t) \mathcal{H}_3(t) dt \\ \quad - h \int_{x_{\ell+1}}^{x_{\ell+2}} K(x_{\ell+2}, t) \mathcal{H}_1(t) dt, & \text{for } \begin{matrix} 2 < k \leq 2i - 2, \\ i > 2, \\ k \text{ is odd} \end{matrix} \\ -h \int_{x_\ell}^{x_{\ell+1}} K(x_{\ell+1}, t) \mathcal{H}_4(t) dt \\ \quad - h \int_{x_{\ell+1}}^{x_{\ell+2}} K(x_{\ell+2}, t) \mathcal{H}_2(t) dt, & \text{for } \begin{matrix} 2 < k \leq 2i - 2, \\ i > 2, \\ k \text{ is even} \end{matrix} \\ \mathcal{H}'_1(u) - h \int_{x_\ell}^{x_{\ell+1}} K(x_{\ell+1}, t) \mathcal{H}_3(t) dt \\ \quad - h \int_{x_{\ell+1}}^{x_{\ell+2}} K(x_{\ell+2}, t) \mathcal{H}_1(t) dt, & \text{for } \begin{matrix} 2i - 2 < k \leq 2i, \\ k \text{ is odd} \end{matrix} \\ \mathcal{H}'_2(u) - h \int_{x_\ell}^{x_{\ell+1}} K(x_{\ell+1}, t) \mathcal{H}_4(t) dt \\ \quad - h \int_{x_{\ell+1}}^{x_{\ell+2}} K(x_{\ell+2}, t) \mathcal{H}_2(t) dt, & \text{for } \begin{matrix} 2i - 2 < k \leq 2i, \\ k \text{ is even} \end{matrix} \\ \mathcal{H}'_3(u) - h \int_{x_\ell}^{x_\ell+uh} K(x_\ell + uh, t) \mathcal{H}_3(t) dt, & \text{for } \begin{matrix} 2i < k \leq 2i + 2, \\ k \text{ is odd} \end{matrix} \\ \mathcal{H}'_4(u) - h \int_{x_\ell}^{x_\ell+uh} K(x_\ell + uh, t) \mathcal{H}_4(t) dt, & \text{for } \begin{matrix} 2i < k \leq 2i + 2, \\ k \text{ is even} \end{matrix} \end{cases} \quad (3.32)$$

where $\ell = \lfloor \frac{k-1}{2} \rfloor$, the floor function of $\frac{k-1}{2}$, and

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ \vdots \\ a_{2N-2} \\ a_{2N-1} \\ a_{2N} \\ a_{2N+1} \\ a_{2N+2} \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} y_0 \\ F(x_1 + u_1h) \\ F(x_1 + u_2h) \\ F(x_2 + u_1h) \\ F(x_2 + u_2h) \\ \vdots \\ F(x_{N-1} + u_1h) \\ F(x_{N-1} + u_2h) \\ F(x_N + w_1h) \\ F(x_N + w_2h) \\ F(x_N + w_3h) \end{bmatrix}. \quad (3.33)$$

3.2.3 Error calculation

The method used to determine the discrete error of our approximate solution versus the exact solution is defined as follows

$$E_{2N+2} = \sqrt{\sum_{i=1}^N \int_{x_i}^{x_{i+1}} \left(y(x) - y^i \left(\frac{x - x_i}{h} \right) \right)^2 dx}, \quad (3.34)$$

where $y(x)$ is the exact solution, $y^i \left(\frac{x - x_i}{h} \right)$ is the piecewise approximating polynomial on subinterval i and N is the number of subintervals the domain is divided into. That is, we define the discrete error as the square root of the integral square error over the domain $x \in [x_1, x_{N+1}]$.

3.3 The application

As in the previous chapter, suppose we had the integro-differential equation of the form

$$y'(x) + \frac{101}{10}e^{-x} \sin(10x) + e^{-x} \cos(10x) = \int_0^x e^{t-x} y(t) dt, \quad (3.35)$$

subject to initial condition, $y(0) = 1$.

The exact solution to this problem is given by $y(x) = e^{-x} \cos(10x)$. The numerical solution using cubic Hermite collocation on finite elements is presented below.

3.3.1 34 collocation points

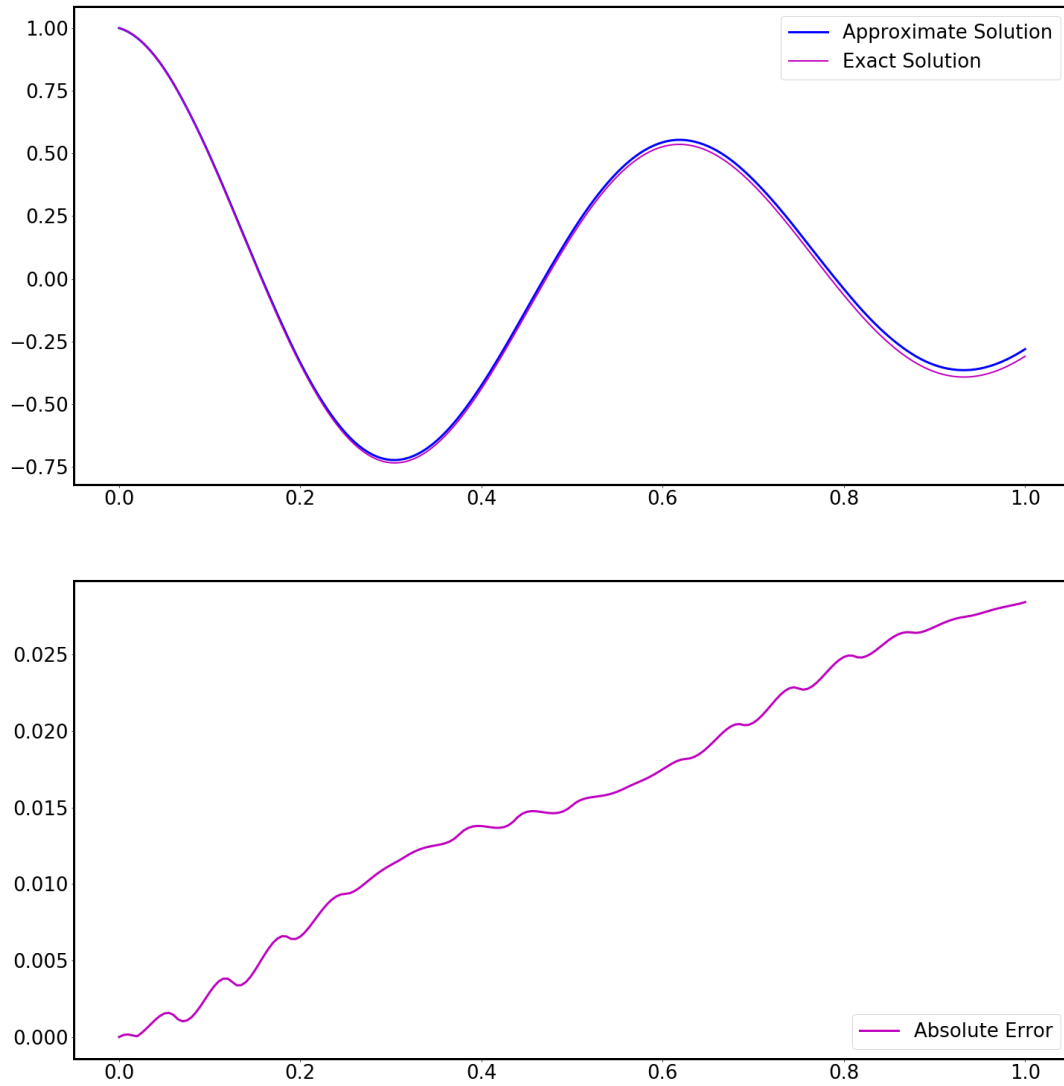


Figure 3.6: Cubic Hermite Collocation using 34 collocation points.

$$E_{34} = 0.01749210157742673$$

3.3.2 46 collocation points

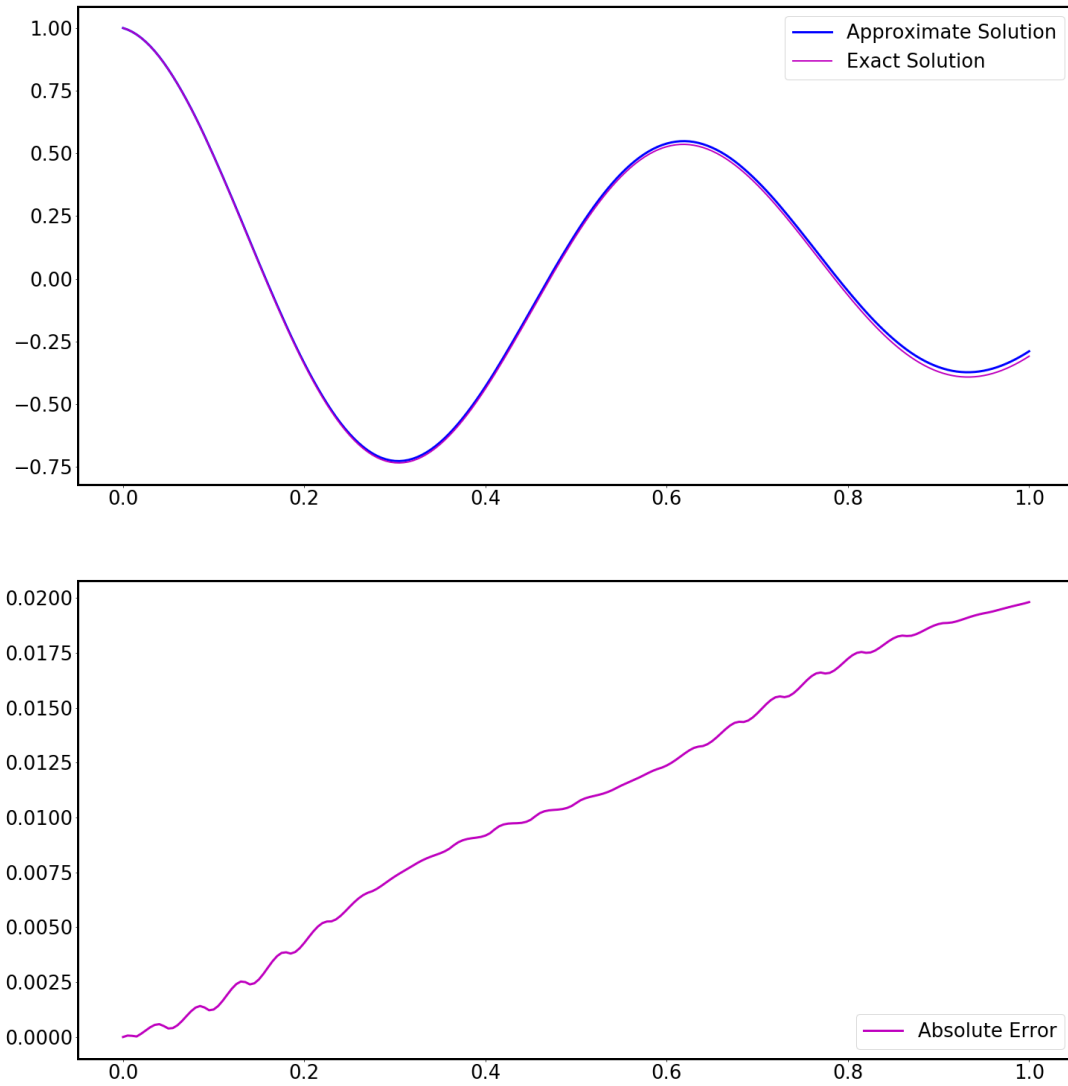


Figure 3.7: Cubic Hermite Collocation using 46 collocation points.

$$E_{46} = 0.012198741786086155$$

3.3.3 512 collocation points

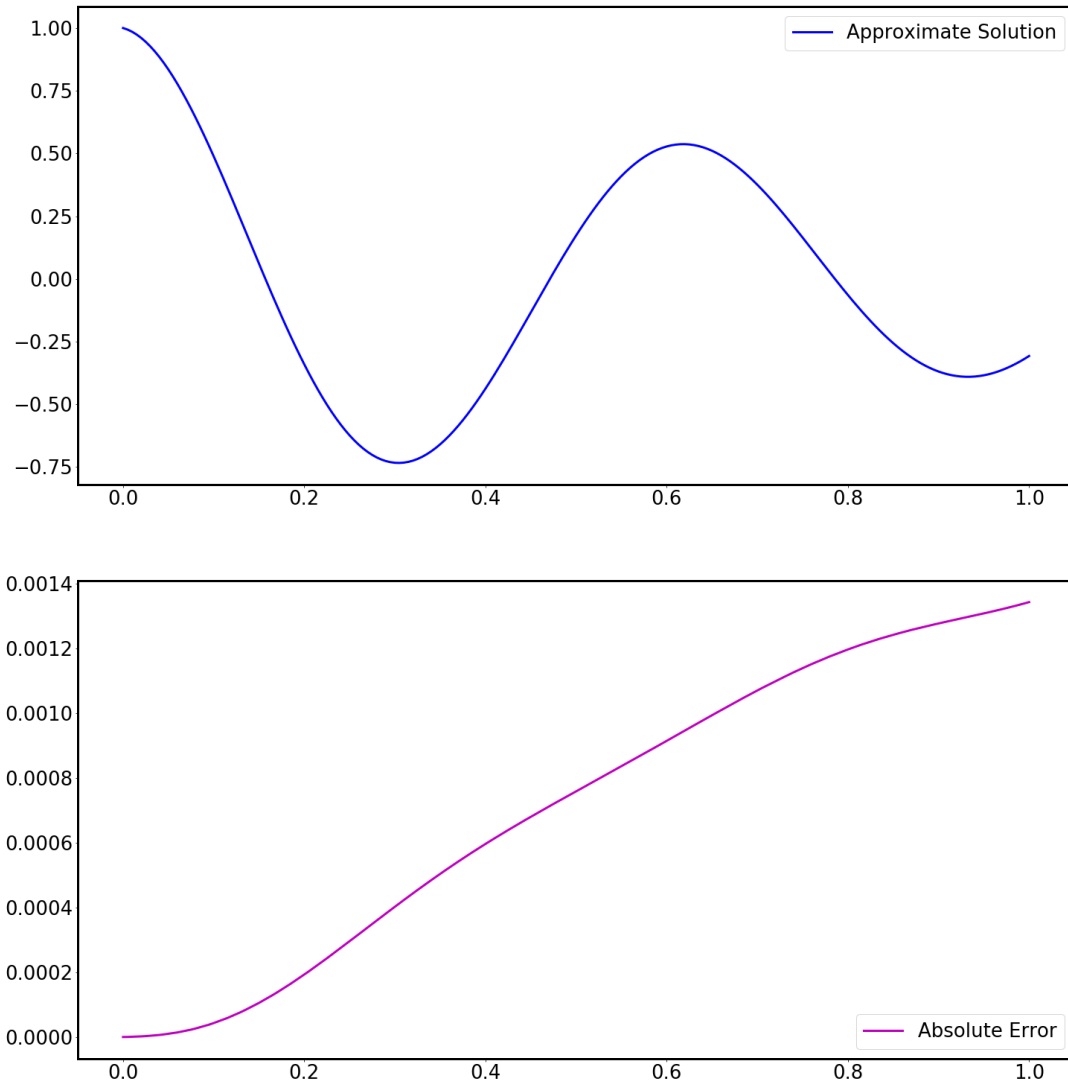


Figure 3.8: Cubic Hermite Collocation using 512 collocation points.

$$E_{512} = 0.0008406754614169248$$

3.3.4 1024 collocation points

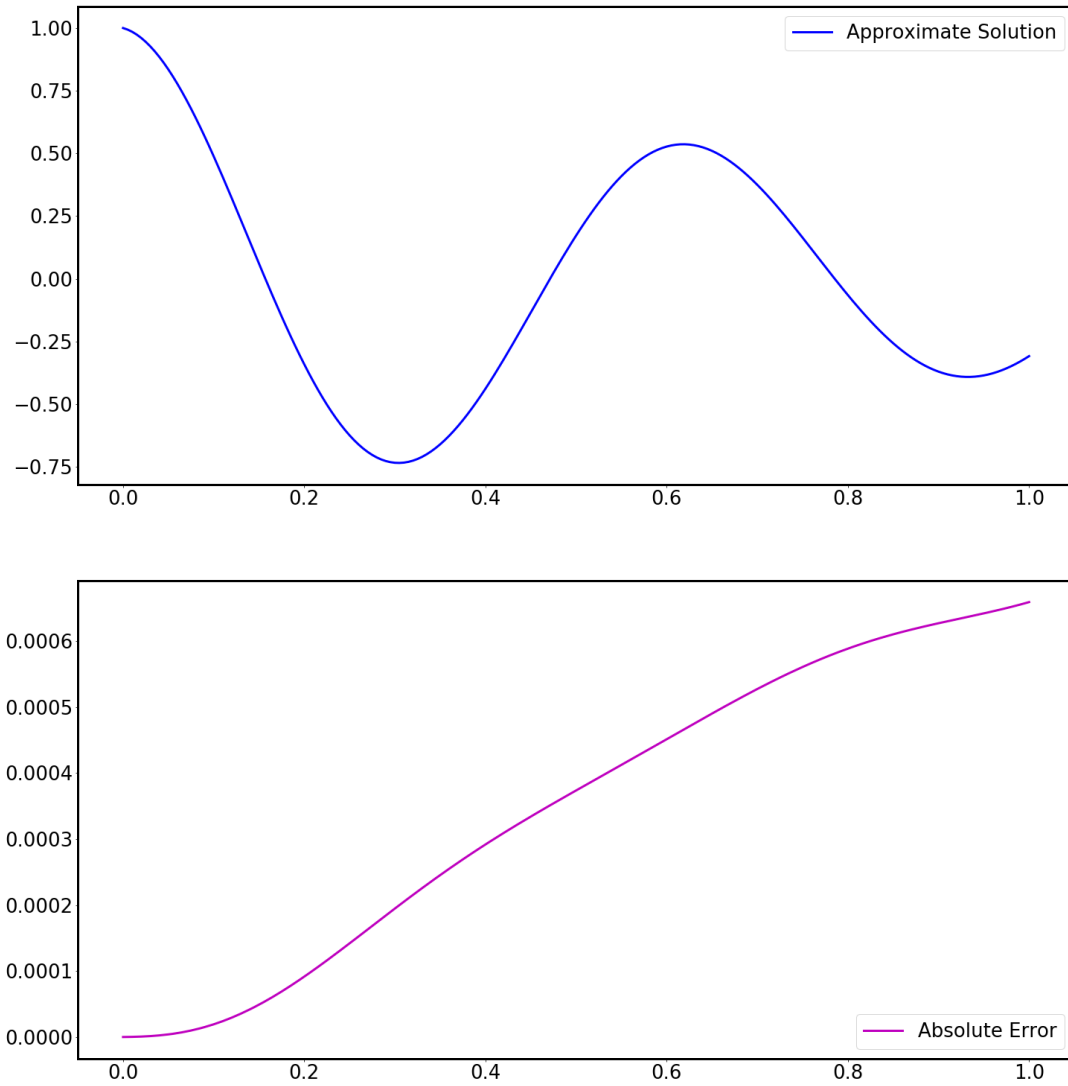


Figure 3.9: Cubic Hermite Collocation using 1024 collocation points.

$$E_{1024} = 0.00041307847311208916$$

Part II

The analytical approach

Chapter 4

Lie symmetry analysis

4.1 Lie symmetry analysis - theoretical basis

4.1.1 Symmetries

Definition 4.1 (Symmetry). A transformation that leaves a mathematical object unchanged is called a symmetry. The object is then said to be invariant under the given transformation. According to [11], a transformation of an object is a symmetry if the following conditions are satisfied.

1. The transformation preserves the structure of the object.
2. The transformation is a diffeomorphism.
3. The transformation results in the planar object in the (x, y) plane being indistinguishable to its image in the (x_1, y_1) plane.

The following are some examples of transformations which satisfy the conditions to be a symmetry.

Consider the parabola $y = x^2$. Its reflection about the y -axis is an example of a symmetry transformation, and is given by

$$x_1 = -x, \tag{4.1}$$

$$y_1 = y. \tag{4.2}$$

The following figure illustrates this symmetry.

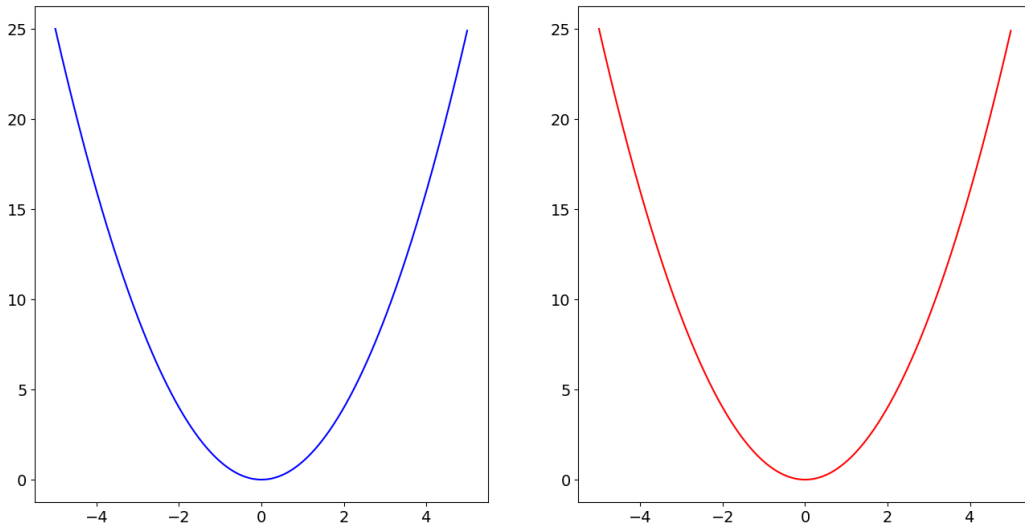


Figure 4.1: Reflection of $y = x^2$ about the y-axis.

The rotation of a disk through angle ϵ is also an example of a symmetry transformation. Assuming points (x, y) and (x_1, y_1) lie on the boundary of a circle of radius r , in terms of a reference angle θ and angle $\theta + \epsilon$ after rotation, we have

$$x = r \cos \theta, \quad x_1 = r \cos(\theta + \epsilon) = x \cos \epsilon - y \sin \epsilon, \quad (4.3)$$

$$y = r \sin \theta, \quad y_1 = r \sin(\theta + \epsilon) = y \cos \epsilon + x \sin \epsilon. \quad (4.4)$$

The following figure illustrates this symmetry.

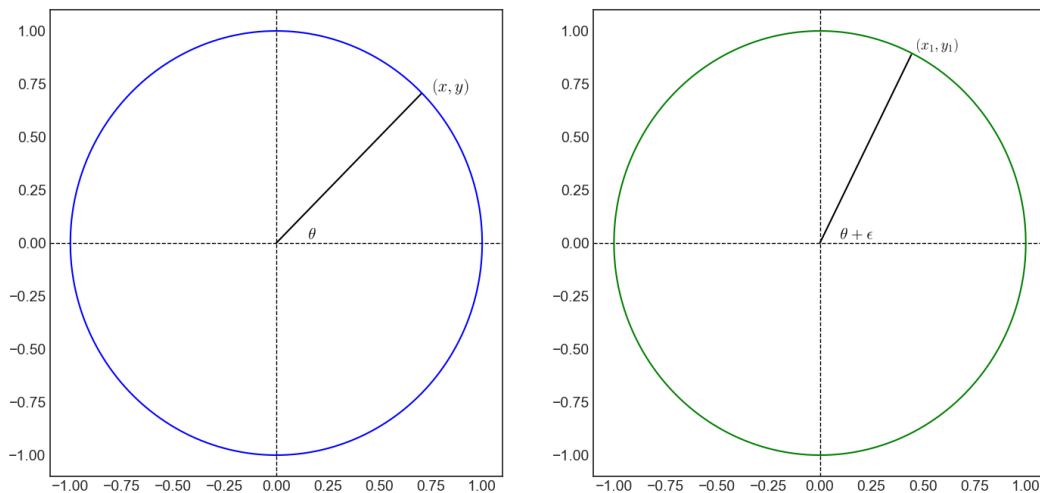


Figure 4.2: Rotation of $x^2 + y^2 = 1$ by ϵ degrees.

Another example of a symmetry is the scaling transformations, which is of the form

$$x_1 = e^\epsilon x, \quad (4.5)$$

$$y_1 = e^\epsilon y. \quad (4.6)$$

The following figure illustrates this symmetry.

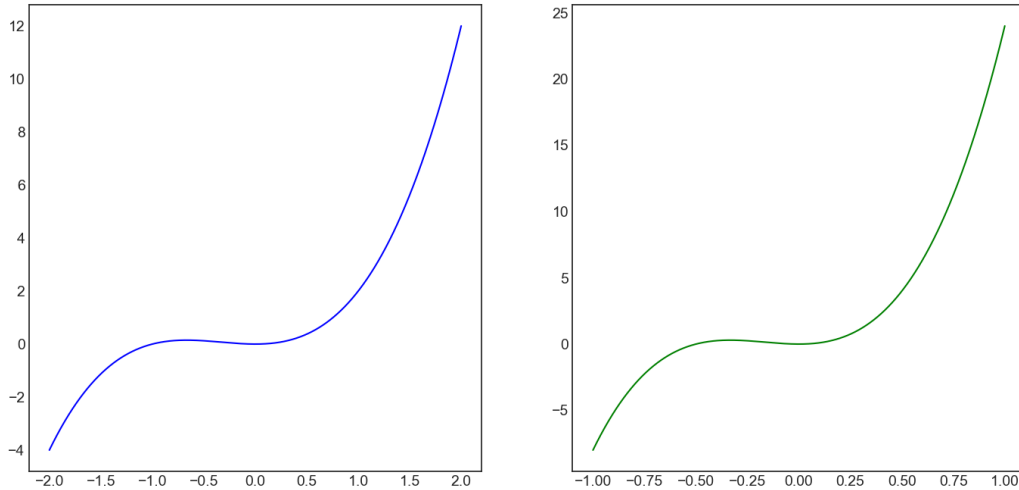


Figure 4.3: Scaling of $y = x^3 + x^2$ with $\epsilon = \ln(2)$.

4.1.2 One-parameter Lie groups

A one-parameter Lie group \mathbf{G} is a transformation of the form

$$T_\epsilon := \begin{cases} x_1 = f(x, y, \epsilon), \\ y_1 = g(x, y, \epsilon), \end{cases} \quad (4.7)$$

where ϵ is a scalar parameter, f and g are smooth functions that are expandable in a Taylor series about any value of ϵ and $f(x, y, \epsilon)$ and $g(x, y, \epsilon)$ comprise the global form of the group.

The Lie group \mathbf{G} satisfies the following axioms, where $\Phi(a, b)$ is the law of group composition which can be taken as the binary operation $\Phi(a, b)$, the reader can find similar treatments in the text [1].

1. Closure. If $a, b \in \mathbf{G}$, then $\Phi(a, b) \in \mathbf{G}$
2. Identity. If $a \in \mathbf{G}$, there is $e \in \mathbf{G}$ such that $\Phi(a, e) = \Phi(e, a) = a$ holds.

3. Inverse. If $a \in \mathbf{G}$, then a unique element $a^{-1} \in \mathbf{G}$ exists such that $\Phi(a, a^{-1}) = \Phi(a^{-1}, a) = e$. Hence, the associative property follows.
4. Associative. If $a, b, c \in \mathbf{G}$, then $\Phi(a, \Phi(b, c)) = \Phi(\phi(a, b), c)$.

4.1.3 The infinitesimal form

Expanding (4.7) in Taylor series about $(x, y, 0)$ gives

$$f(x, y, \epsilon) = f(x, y, 0) + \left. \frac{\partial f(x, y, \epsilon)}{\partial \epsilon} \right|_{\epsilon=0} \frac{\epsilon}{1!} + \left. \frac{\partial^2 f(x, y, \epsilon)}{\partial \epsilon^2} \right|_{\epsilon=0} \frac{\epsilon^2}{2!} + \dots, \quad (4.8)$$

$$g(x, y, \epsilon) = g(x, y, 0) + \left. \frac{\partial g(x, y, \epsilon)}{\partial \epsilon} \right|_{\epsilon=0} \frac{\epsilon}{1!} + \left. \frac{\partial^2 g(x, y, \epsilon)}{\partial \epsilon^2} \right|_{\epsilon=0} \frac{\epsilon^2}{2!} + \dots. \quad (4.9)$$

Applying the identity axiom of \mathbf{G} , the infinitesimal form of the Lie group of transformations are

$$x_1 = x + \epsilon \xi(x, y) + \mathcal{O}(\epsilon^2), \quad (4.10)$$

$$y_1 = y + \epsilon \eta(x, y) + \mathcal{O}(\epsilon^2), \quad (4.11)$$

where the functions $\xi(x, y)$ and $\eta(x, y)$ are known as the infinitesimals of \mathbf{G} and are defined by

$$\xi(x, y) = \left. \frac{\partial x_1}{\partial \epsilon} \right|_{\epsilon=0}, \quad \eta(x, y) = \left. \frac{\partial y_1}{\partial \epsilon} \right|_{\epsilon=0}. \quad (4.12)$$

Theorem 4.1 (First Fundamental Theorem of Lie). *A parametrization $\tau(\epsilon) = \int_0^\epsilon \Gamma(\mathcal{E}) d\mathcal{E}$ exists, such that the global form of the Lie group (4.7) are equivalent to the solution of an initial value problem of the system*

$$\frac{dx_1}{d\epsilon} = \xi(x_1, y_1), \quad \frac{dy_1}{d\epsilon} = \eta(x_1, y_1), \quad (4.13)$$

with $x_1 = x$ and $y_1 = y$ when $\epsilon = 0$, where $\Gamma(\epsilon) = \left. \frac{\partial \phi(\alpha, \beta)}{\partial \beta} \right|_{\alpha=\epsilon^{-1}, \beta=\epsilon}$ and $\Gamma(0) = 1$.

4.1.4 The infinitesimal generator

The one-parameter transformations (4.10) and (4.11) can be written as

$$x_1 = (1 + \epsilon \xi(x, y) \frac{\partial}{\partial x}) x, \quad (4.14)$$

$$y_1 = (1 + \epsilon \eta(x, y) \frac{\partial}{\partial y})y, \quad (4.15)$$

omitting $\mathcal{O}(\epsilon^2)$ as ϵ is an infinitesimal. The infinitesimal generator is defined as

$$X = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y}. \quad (4.16)$$

The infinitesimal form of the group of transformations (4.10) and (4.11) can be expressed in terms of the infinitesimal generator X

$$x_1 = (1 + \epsilon X)x, \quad (4.17)$$

$$y_1 = (1 + \epsilon X)y. \quad (4.18)$$

4.1.5 Lie symmetries of functions

Suppose we wish to find symmetries of a function $F(x, y)$, to begin we expand the function $F(x_1, y_1)$ in Taylor series about point $(x, y, 0)$, which yields

$$F(x_1, y_1) = F(x, y) + \epsilon \left(\frac{dF}{d\epsilon} \right) \Big|_{\epsilon=0} + \mathcal{O}(\epsilon^2) \quad (4.19)$$

$$= F(x, y) + \epsilon \left(\frac{\partial F}{\partial x_1} \frac{\partial x_1}{\partial \epsilon} + \frac{\partial F}{\partial y_1} \frac{\partial y_1}{\partial \epsilon} \right) \Big|_{\epsilon=0} + \mathcal{O}(\epsilon^2) \quad (4.20)$$

$$= F(x, y) + \epsilon \left(\frac{\partial F}{\partial x} \xi(x, y) + \frac{\partial F}{\partial y} \eta(x, y) \right) + \mathcal{O}(\epsilon^2) \quad (4.21)$$

$$= F(x, y) + \epsilon \left(\xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y} \right) F(x, y) + \mathcal{O}(\epsilon^2). \quad (4.22)$$

When higher order terms are omitted $\mathcal{O}(\epsilon^2)$, the function can be written in terms of the generator

$$F(x_1, y_1) = (1 + \epsilon X)F(x, y). \quad (4.23)$$

To admit X as a symmetry on $F(x, y)$, we require that the transformation $F(x_1, y_1)$ be invariant of $F(x, y)$, that is

$$(1 + \epsilon X)F(x, y) = F(x, y). \quad (4.24)$$

Therefore, if $X F(x, y) = 0$, then X is a symmetry of $F(x, y)$.

Theorem 4.2 (Invariant Functions). *$F(x, y)$ is invariant under the one-parameter Lie group of transformations (4.7) if and only if, [3]*

$$X F(x, y) \equiv 0. \quad (4.25)$$

4.1.6 The extended generator

Let x be the independent variable and y the dependent on x , then the function $F(x, y)$ can be seen as a function of $x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots$ as well. In this case, we need to determine how these derivatives transform under the generator (4.16).

Extending the group of transformations (4.7) to space $(x, y, y', \dots, y^{(n)})$, $n \geq 2$ gives the following Lie group of transformations in $(x, y, y', \dots, y^{(n)})$ -space

$$x_1 = f(x, y, \epsilon), \quad (4.26)$$

$$y_1 = g(x, y, \epsilon), \quad (4.27)$$

$$y_1^{[1]} = g_1(x, y, y', \epsilon), \quad (4.28)$$

\vdots

$$y_1^{[n]} = g_n(x, y, y', \dots, y^{(n)}, \epsilon). \quad (4.29)$$

Definition 4.2 (Total Derivative). The total derivative, with respect to x , is the operator defined by

$$D_x = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + y'' \frac{\partial}{\partial y'} + \dots + y^{(n+1)} \frac{\partial}{\partial y^{(n)}}, \quad (4.30)$$

where $y' = \frac{dy}{dx}$, $y'' = \frac{d^2y}{dx^2}$ and $y^{(k)} = \frac{d^k y}{dx^k}$ for $k \in \mathbb{N}$.

The extended form of the group (4.7) is defined as

$$y_1^{[k]} = \frac{D_x [g_{k-1}(x, y, y', \dots, y^{(k-1)}, \epsilon)]}{D_x [f(x, y, \epsilon)]}, \quad k = 1, \dots, n, \quad (4.31)$$

where $g_0 = g(x, y, \epsilon)$.

The extended group of transformations, truncating from $\mathcal{O}(\epsilon^2)$, can be written as follows in $(x, y, y', \dots, y^{(n)})$ -space

$$x_1 = x + \epsilon \xi(x, y), \quad (4.32)$$

$$y_1 = y + \epsilon \eta(x, y), \quad (4.33)$$

$$y_1^{[1]} = y' + \epsilon \eta^{[1]}(x, y, y'), \quad (4.34)$$

\vdots

$$y_1^{[n]} = y^{(n)} + \epsilon \eta^{[n]}(x, y, \dots, y^{(n)}). \quad (4.35)$$

We introduce the extended infinitesimal generator $X^{[n]}$, dealing with derivatives up to order n , which is given by

$$X^{[n]} = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y} + \eta^{[1]}(x, y, y') \frac{\partial}{\partial y'} + \cdots + \eta^{[n]}(x, y, y', \dots, y^{(n)}) \frac{\partial}{\partial y^{(n)}}, \quad (4.36)$$

where $n = 1, 2, \dots$ and $\eta^{[n]}$ are the coefficients defined in the extension of the transformation group. The reader can refer to texts [3] [11].

Substituting (4.35) into (4.31) and simplifying

$$y_1^{[k]} = \frac{D_x [g_{k-1}(x, y, y', \dots, y^{(k)}, \epsilon)]}{D_x [f(x, y, \epsilon)]} \quad (4.37)$$

$$= \frac{D_x [y^{(k-1)} + \epsilon \eta^{[k-1]}(x, y, \dots, y^{(k-1)})]}{D_x [x + \epsilon \xi]} \quad (4.38)$$

$$= \frac{D_x [y^{(k-1)}] + \epsilon D_x [\eta^{[k-1]}]}{1 + \epsilon D_x [\xi]} \quad (4.39)$$

$$= \frac{y^{(k)} + \epsilon D_x [\eta^{[k-1]}]}{1 + \epsilon D_x [\xi]} \quad (4.40)$$

$$= (y^{(k)} + \epsilon D_x [\eta^{[k-1]}])(1 - \epsilon D_x [\xi] + \epsilon^2 D_x [\xi]^2 - \dots) \quad (4.41)$$

$$= y^{(k)} + \epsilon (D_x [\eta^{[k-1]}] - y^{(k)} D_x [\xi]). \quad (4.42)$$

However, from (4.35) we know that $y_1^{[k]} = y^{(k)} + \epsilon \eta^{[k]}$, which leads to the following theorem.

Theorem 4.3 (Extended Infinitesimals). *The infinitesimals in $(x, y, y', \dots, y^{(k)})$ -space satisfy the relation*

$$\eta^{[k]}(x, y, y', \dots, y^{(k)}) = D_x [\eta^{[k-1]}] - y^{(k)} D_x [\xi], \quad (4.43)$$

where $k = 1, 2, \dots$ and $\eta^{[0]} = \eta(x, y)$. The reader can refer to text [3].

This theorem provides a convenient recursive method to obtain each $\eta^{[k]}(x, y, y', \dots, y^{(k)})$, $k = 1, 2, \dots, n$ in (4.36). Applying this method, we give the first two of these extended infinitesimals

$$\eta^{[1]} = D_x [\eta] - y' D_x [\xi] \quad (4.44)$$

$$= \eta_x + y' \eta_y - y' (\xi_x + y' \xi_y) \quad (4.45)$$

$$= \eta_x + (\eta_y - \xi_x) y' - \xi_y y'^2, \quad (4.46)$$

$$\eta^{[2]} = D_x [\eta^{(1)}] - y'' D_x [\xi] \quad (4.47)$$

$$= \eta_{xx} + (2\eta_{xy} - \xi_{xx}) y' + (\eta_{yy} - 2\xi_{xy}) y'^2 - \xi_{yy} y'^3 + (\eta_y - 2\xi_x - 3\xi_y y') y''. \quad (4.48)$$

4.1.7 Symmetries of ordinary differential equations

Suppose we have an ordinary differential equation of the form

$$\frac{d^n y}{dx^n} = f \left(x, y, \frac{dy}{dx}, \dots, \frac{d^{n-1}y}{dx^{n-1}} \right), \quad (4.49)$$

then the criteria for the invariance of an ordinary differential equation is given by the following theorem.

Theorem 4.4 (Invariance of an ODE). *Let X be the infinitesimal generator of the one-parameter Lie group of transformations, a symmetry is admitted by the ODE (4.49) if and only if*

$$X^{[n]} \left[\frac{d^n y}{dx^n} - f \left(x, y, \frac{dy}{dx}, \dots, \frac{d^{n-1}y}{dx^{n-1}} \right) \right] = 0, \quad (4.50)$$

when $\frac{d^n y}{dx^n} = f$.

Here, we present an illustrative example displaying the application of Lie's algorithm.

Example 4.1. Determine the symmetries of the ODE

$$y''(x) = \frac{1}{y(x)^4}. \quad (4.51)$$

Since this a second order ODE, we use the second extension of the generator and apply the invariance condition given in Theorem 4.4, that is

$$X^{[2]} \left(y''(x) - \frac{1}{y(x)^4} \right) \Big|_{y''=\frac{1}{y^4}} = 0, \quad (4.52)$$

which results in the following equation

$$y' \left[2\eta_{xy} - \xi_{xx} - \frac{3\xi_y}{y^4} \right] + y'^2 [\eta_{yy} - 2\xi_{xy}] - y'^3 \xi_{yy} + \eta_{xx} + \frac{\eta_y}{y^4} - \frac{2\xi_x}{y^4} + \frac{4\eta}{y^5} = 0. \quad (4.53)$$

Equating the powers of y' to zero, we obtain the following system of differential equations

$$y'^3 : \quad \xi_{yy} = 0, \quad (4.54)$$

$$y'^2 : \quad \eta_{yy} - 2\xi_{xy} = 0, \quad (4.55)$$

$$y'^1 : \quad 2y^4 \eta_{xy} - y^4 \xi_{xx} - 3\xi_y = 0, \quad (4.56)$$

$$y'^0 : \quad y^5 \eta_{xx} + 4\eta + y\eta_y - 2y\xi_x = 0. \quad (4.57)$$

We solve (4.54)-(4.57), obtaining

$$\xi(x, y) = \frac{5}{2}c_1x + c_2, \quad (4.58)$$

$$\eta(x, y) = c_1y, \quad (4.59)$$

which yields the following two symmetries

$$X_1 = \frac{5x}{2} \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \quad (4.60)$$

$$X_2 = \frac{\partial}{\partial x}. \quad (4.61)$$

4.2 Integro-differential equations of the first kind

In this section we apply Lie's algorithm to the general Volterra integro-differential equation of the first kind in order to determine its linear independent symmetries.

We consider the equation of the form

$$y'(x) = \int_a^x K(s, y(s))ds, \quad (4.62)$$

differentiating with respect to x yields the following second-order ODE

$$y''(x) - K(x, y(x)) = 0. \quad (4.63)$$

Since this is an ODE of order two, we require the extended generator of order two, that is

$$X^{[2]} = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y} + \eta^{(1)}(x, y, y') \frac{\partial}{\partial y'} + \eta^{(2)}(x, y, y', y'') \frac{\partial}{\partial y''} \quad (4.64)$$

$$= \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + [\eta_x + (\eta_y - \xi_x)y' - \xi_y y'^2] \frac{\partial}{\partial y'} + [\eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - \xi_{yy}y'^3 + (\eta_y - 2\xi_x - 3\xi_y y')y''] \frac{\partial}{\partial y''}. \quad (4.65)$$

In order to find the symmetries of (4.63), we must solve equation

$$X^{[2]}(y''(x) - K(x, y))|_{y''=K} = 0, \quad (4.66)$$

applying the generator (4.65), we have

$$-\xi K_x - \eta K_y + \eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - \xi_{yy}y'^3 + (\eta_y - 2\xi_x - 3\xi_y y')y'' = 0. \quad (4.67)$$

Grouping by powers of y' , we obtain the following determining equations

$$(y')^3 : \quad \xi_{yy} = 0, \quad (4.68)$$

$$(y')^2 : \quad \eta_{yy} - 2\xi_{xy} = 0 \quad (4.69)$$

$$(y')^1 : \quad 2\eta_{xy} - \xi_{xx} - 3\xi_y K = 0, \quad (4.70)$$

$$(y')^0 : \quad -\eta K_y - \xi K_x + (\eta_y - 2\xi_x)K + \eta_{xx} = 0. \quad (4.71)$$

From equations (4.68) and (4.69) we get

$$\xi(x, y) = a(x)y + b(x), \quad (4.72)$$

$$\eta(x, y) = a'(x)y^2 + c(x)y + d(x), \quad (4.73)$$

substituting this into equations (4.70) and (4.71) yields

$$3a''(x)y - 3Ka(x) + 2c'(x) - b''(x) = 0, \quad (4.74)$$

$$\begin{aligned} & - (a'(x)y^2 + c(x)y + d(x))K_y - (a(x)y + b(x))K_x + (2a'(x)y + c(x)) \\ & - 2a'(x)y - 2b'(x))K + a'''(x)y^2 + c''(x)y + d''(x) = 0. \end{aligned} \quad (4.75)$$

Grouping (4.74) by powers of y

$$y^1 : \quad a''(x) = 0, \quad (4.76)$$

$$y^0 : \quad 2c(x) - 3Ka(x) - b''(x) = 0. \quad (4.77)$$

From (4.76) it follows that $a(x) = c_1x + c_2$ where c_1 and c_2 are constants. Now grouping (4.75) by powers of y yields

$$y^2 : \quad c_1K_y = 0, \quad (4.78)$$

$$y^1 : \quad -c(x)K_y - (c_1x + c_2)K_x + c''(x) = 0, \quad (4.79)$$

$$y^0 : \quad -d(x)K_y - b(x)K_x + c(x)K - 2b'(x)K + d''(x) = 0. \quad (4.80)$$

In order to satisfy equation (4.78), there are two possible cases, $K = r(x)$ and $c_1 \neq 0$ or $c_1 = 0$ and $K = K(x, y)$.

Case I : $c_1 \neq 0, K = r(x)$

We have the following three equations to solve

$$2c'(x) - b''(x) - 3r(x)(c_1x + c_2) = 0, \quad (4.81)$$

$$c''(x) - (c_1x + c_2)r'(x) = 0, \quad (4.82)$$

$$d''(x) - b(x)r'(x) + [c(x) - 2b'(x)]r(x) = 0. \quad (4.83)$$

Integrating (4.82), (4.81) and (4.83) respectively give the following general solutions

$$c(x) = c_1 \int_0^x \int_0^t sr'(s) ds dt + c_2 \int_0^x \int_0^t r'(s) ds dt + c_3x + c_4, \quad (4.84)$$

$$b(x) = c_1 \left[2 \int_0^x \int_0^t \int_0^s vr'(v) dv ds dt - 3 \int_0^x \int_0^t sr(s) ds dt \right] + c_2 \left[2 \int_0^x \int_0^t \int_0^s r'(v) dv ds dt - 3 \int_0^x \int_0^t r(s) ds dt \right] + c_3x^2 + c_5x + c_6, \quad (4.85)$$

$$d(x) = c_1 \left[2 \int_0^x \int_0^u r'(s) \int_0^s \int_0^t \int_0^w vr'(v) dv dw dt ds du + 3 \int_0^x \int_0^u r(s) \int_0^s \int_0^t vr'(v) dv dt ds du - 3 \int_0^x \int_0^u r'(s) \int_0^s \int_0^t vr(v) dv dt ds du - 6 \int_0^x \int_0^u r(s) \int_0^s vr(v) dv ds du \right] + c_2 \left[2 \int_0^x \int_0^u r'(s) \int_0^s \int_0^t \int_0^w r'(v) dv dw dt ds du + 3 \int_0^x \int_0^u r(s) \int_0^s \int_0^t r'(v) dv dt ds du - 3 \int_0^x \int_0^u r'(s) \int_0^s \int_0^t r(v) dv dt ds du - 6 \int_0^x \int_0^u r(s) \int_0^s r(v) dv ds du \right] + c_3 \left[\int_0^x \int_0^u s^2 r'(s) ds du + 3 \int_0^x \int_0^u sr(s) ds du \right] - c_4 \int_0^x \int_0^u r(s) ds du + c_5 \left[2 \int_0^x \int_0^u r(s) ds du + \int_0^x \int_0^u sr'(s) ds du \right] + c_6 \int_0^x \int_0^u r'(s) ds du + c_7x + c_8. \quad (4.86)$$

Thus from (4.72) and (4.73), we have

$$\xi(x, y) = c_1 \left[xy + 2 \int_0^x \int_0^t \int_0^s vr'(v) dv ds dt - 3 \int_0^x \int_0^t sr(s) ds dt \right] + c_2 \left[y + 2 \int_0^x \int_0^t \int_0^s r'(v) dv ds dt - 3 \int_0^x \int_0^t r(s) ds dt \right] + c_3x^2 + c_5x + c_6, \quad (4.87)$$

$$\begin{aligned}
\eta(x, y) = & c_1 \left[y^2 + y \int_0^x \int_0^t sr'(s) ds dt + 2 \int_0^x \int_0^u r'(s) \int_0^s \int_0^t \int_0^w vr'(v) dv dw dt ds du \right. \\
& + 3 \int_0^x \int_0^u r(s) \int_0^s \int_0^t vr'(v) dv dt ds du - 3 \int_0^x \int_0^u r'(s) \int_0^s \int_0^t vr(v) dv dt ds du \\
& \left. - 6 \int_0^x \int_0^u r(s) \int_0^s vr(v) dv ds du \right] + \\
& c_2 \left[y \int_0^x \int_0^t r'(s) ds dt + 2 \int_0^x \int_0^u r'(s) \int_0^s \int_0^t \int_0^w r'(v) dv dw dt ds du \right. \\
& + 3 \int_0^x \int_0^u r(s) \int_0^s \int_0^t r'(v) dv dt ds du - 3 \int_0^x \int_0^u r'(s) \int_0^s \int_0^t r(v) dv dt ds du \\
& \left. - 6 \int_0^x \int_0^u r(s) \int_0^s r(v) dv ds du \right] \tag{4.88}
\end{aligned}$$

$$\begin{aligned}
& + c_3 \left[x + \int_0^x \int_0^u s^2 r'(s) ds du + 3 \int_0^x \int_0^u sr(s) ds du \right] - \\
& c_4 \left[1 + \int_0^x \int_0^u r(s) ds du \right] + c_5 \left[2 \int_0^x \int_0^u r(s) ds du + \int_0^x \int_0^u sr'(s) ds du \right] + \tag{4.89} \\
& c_6 \int_0^x \int_0^u r'(s) ds du + c_7 x + c_8.
\end{aligned}$$

For the symmetry $X = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y}$, we have the following eight linearly independent symmetries listed

$$\begin{aligned}
X_1 = & [xy + 2P_2(x) - 3N_2(x)] \frac{\partial}{\partial x} + \left[y^2 + P_2(x)y + 2 \int_0^x \int_0^u r'(s)P_3(s) ds du \right. \\
& + 3 \int_0^x \int_0^u r(s)P_2(s) ds du - 3 \int_0^x \int_0^u r'(s)N_2(s) ds du \\
& \left. - 6 \int_0^x \int_0^u r(s)N_1(s) ds du \right] \frac{\partial}{\partial y} \tag{4.90}
\end{aligned}$$

$$\begin{aligned}
X_2 = & [y + 2M_3(x) - 3L_2(x)] \frac{\partial}{\partial x} + \left[M_2(x)y + 2 \int_0^x \int_0^u r'(s)M_3(s) ds du \right. \\
& + 3 \int_0^x \int_0^u r(s)M_2(s) ds du - 3 \int_0^x \int_0^u r'(s)L_2(s) ds du \\
& \left. - 6 \int_0^x \int_0^u r(s)L_1(s) ds du \right] \frac{\partial}{\partial y} \tag{4.91}
\end{aligned}$$

$$X_3 = x^2 \frac{\partial}{\partial x} + [x + Q_2(x) + 3N_2(x)] \frac{\partial}{\partial y} \tag{4.92}$$

$$X_4 = [1 + L_2(x)] \frac{\partial}{\partial y} \tag{4.93}$$

$$X_5 = x \frac{\partial}{\partial x} + [2L_2(x) + P_2(x)] \frac{\partial}{\partial y} \quad (4.94)$$

$$X_6 = \frac{\partial}{\partial x} + M_2(x) \frac{\partial}{\partial y} \quad (4.95)$$

$$X_7 = x \frac{\partial}{\partial y} \quad (4.96)$$

$$X_8 = \frac{\partial}{\partial y} \quad (4.97)$$

where the functions L_i , M_i , N_i , P_i and Q_j where $i = 1, 2, 3$ and $j = 1, 2$; are defined as

$$\begin{aligned} L_1(x) &= \int_0^x r(t)dt, & L_2(x) &= \int_0^x \int_0^t r(s)dsdt, & L_3(x) &= \int_0^x \int_0^t \int_0^s r(v)dvdsdt, \\ M_1(x) &= \int_0^x r'(t)dt, & M_2(x) &= \int_0^x \int_0^t r'(s)dsdt, & M_3(x) &= \int_0^x \int_0^t \int_0^s r'(v)dvdsdt, \\ N_1(x) &= \int_0^x tr(t)dt, & N_2(x) &= \int_0^x \int_0^t sr(s)dsdt, & N_3(x) &= \int_0^x \int_0^t \int_0^s vr(v)dvdsdt, \\ P_1(x) &= \int_0^x tr'(t)dt, & P_2(x) &= \int_0^x \int_0^t sr'(s)dsdt, & P_3(x) &= \int_0^x \int_0^t \int_0^s vr'(v)dvdsdt, \\ Q_1(x) &= \int_0^x t^2 r'(t)dt, & Q_2(x) &= \int_0^x \int_0^t s^2 r'(s)dsdt. \end{aligned}$$

Case II : $c_1 = 0$, $K = K(x, y)$

We have the following three equations to solve

$$c''(x) - c(x)K_y - c_2K_x = 0, \quad (4.98)$$

$$d''(x) - d(x)K_y - b(x)K_x + c(x)K - 2b'(x)K = 0, \quad (4.99)$$

$$2c'(x) - b''(x) - 3c_2K = 0. \quad (4.100)$$

Since $K_y(x, y)$ is a variable coefficient, this system of ODEs cannot be solved in general by ordinary methods. The handbooks of Polyanin [22] and Kamke [14] provides solutions for specific polynomial and exponential forms of the kernel.

4.3 Integro-differential equations of the second kind

In this section we apply Lie's algorithm to the general Volterra integro-differential equation of the second kind in order to determine its linear independent symmetries.

Consider the equation of the form

$$y'(x) = y(x) + \int_a^x K(s, y(s))ds, \quad (4.101)$$

differentiating with respect to x yields the following second-order ODE

$$y''(x) - y'(x) - K(x, y(x)) = 0. \quad (4.102)$$

Following a similar approach to that of the first kind equation, we obtain the following determining equations

$$\xi_{yy} = 0, \quad (4.103)$$

$$\eta_{yy} - 2\xi_y - 2\xi_{xy} = 0, \quad (4.104)$$

$$2\eta_{xy} - 3K\xi_y - \xi_x - \xi_{xx} = 0, \quad (4.105)$$

$$\eta_{xx} - \eta K_y - \xi K_x - \eta_x + (\eta_y - 2\xi_x)K = 0, \quad (4.106)$$

from (4.103) and (4.104) we get

$$\xi(x, y) = a(x)y + b(x), \quad (4.107)$$

$$\eta(x, y) = a(x)y^2 + a'(x)y^2 + c(x)y + d(x). \quad (4.108)$$

Hence, (4.105) and (4.106) yields the equations

$$-3a(x)K - b'(x) + 3a'(x)y + 2c'(x) - b''(x) + 3a''(x)y = 0, \quad (4.109)$$

$$\begin{aligned} &2a(x)Ky + c(x)K - 2b'(x)K - y^2a'(x) - d'(x) - c'(x)y \\ &+ d''(x) + c''(x)y + a'''(x)y^2 - a(x)K_y y^2 - d(x)K_y \\ &- c(x)K_y y - a'(x)K_y y^2 - b(x)K_x - a(x)K_x y = 0, \end{aligned} \quad (4.110)$$

grouping (4.109) by powers of y

$$3a'(x) + 3a''(x) = 0, \quad (4.111)$$

$$2c'(x) - 3a(x)K - b'(x) - b''(x) = 0. \quad (4.112)$$

Equation (4.111) implies that $a(x) = -c_1 e^{-x} + c_2$, now grouping (4.112) by powers of y gives

$$c_2 K_y = 0, \quad (4.113)$$

$$c''(x) + 2K(-c_1e^{-x} + c_2) - c'(x) - K_x(-c_1e^{-x} + c_2) - K_y c(x) = 0, \quad (4.114)$$

$$d''(x) + Kc(x) - 2Kb'(x) - d'(x) - K_x b(x) - K_y d(x) = 0, \quad (4.115)$$

In order to satisfy equation (4.113), there are two possible cases, $K = r(x)$ and $c_2 \neq 0$ or $c_2 = 0$ and $K = K(x, y)$.

Case I: $c_2 \neq 0, K = r(x)$

We have the following three equations to solve

$$-3r(x)(-c_1e^{-x} + c_2) + 2c'(x) - b''(x) - b'(x) = 0, \quad (4.116)$$

$$c''(x) + 2r(x)(-c_1e^{-x} + c_2) - c'(x) - r'(x)(-c_1e^{-x} + c_2) = 0, \quad (4.117)$$

$$d''(x) + r(x)c(x) - 2r(x)b'(x) - d'(x) - r'(x)b(x) = 0. \quad (4.118)$$

Equation (4.117) yields

$$c(x) = c_1[-B_2(x) + 2A_2(x)] - c_2[-B_1(x) + 2A_1(x)] + c_3[e^x - 1] + c_4, \quad (4.119)$$

substituting this into (4.116) gives

$$\begin{aligned} b(x) = & c_1[-2D_2(x) + 4C_2(x) + 3E_1(x)] - c_2[-2D_1(x) + 4C_1(x) + 3E_2(x)] \\ & - c_3[2 - e^{-x} - e^x] + c_5[1 - e^{-x}] + c_6. \end{aligned} \quad (4.120)$$

Finally, substituting (4.119) and (4.120) into (4.118) yields

$$\begin{aligned} d(x) = & c_1[4R_2(x) - 2S_2(x) - 4T_2(x) + U_2(x) + 3V_2(x) + 8W_2(x) - 2Y_2(x) + 6Z_2(x)] - \\ & c_2[4R_1(x) - 2S_1(x) - 4T_1(x) + U_1(x) + 3V_1(x) + 8W_1(x) - 2Y_1(x) + 6Z_1(x)] + \\ & c_3[-2B_1(x) + B_0(x) + B_2(x) - e^x A_1(x) + A_1(x) + 2A_0(x) - 2A_2(x)] - \\ & c_4 A_1(x) + c_5[B_1(x) - B_2(x) + 2A_2(x)] + c_6 B_1(x) + c_7[e^x - 1] + c_8, \end{aligned} \quad (4.121)$$

where the functions $A_i, B_i, C_j, D_j, E_j, R_j, S_j, T_j, U_j, V_j, W_j, Y_j$ and Z_j where $i = 0, 1, 2$ and $j = 1, 2$; are defined as

$$\begin{aligned} A_0(x) &= \int_0^x e^t \int_0^t r(s) ds dt, & B_0(x) &= \int_0^x e^t \int_0^t r'(s) ds dt, \\ A_1(x) &= \int_0^x e^t \int_0^t e^{-s} r(s) ds dt, & B_1(x) &= \int_0^x e^t \int_0^t e^{-s} r'(s) ds dt, \\ A_2(x) &= \int_0^x e^t \int_0^t e^{-2s} r(s) ds dt, & B_2(x) &= \int_0^x e^t \int_0^t e^{-2s} r'(s) ds dt, \\ C_1(x) &= \int_0^x e^{-w} \int_0^w e^{2v} \int_0^v e^{-s} r(s) ds dv dw, \end{aligned}$$

$$\begin{aligned}
C_2(x) &= \int_0^x e^{-w} \int_0^w e^{2v} \int_0^v e^{-2s} r(s) ds dv dw, \\
D_1(x) &= \int_0^x e^{-w} \int_0^w e^{2v} \int_0^v e^{-s} r'(s) ds dv dw, \\
D_2(x) &= \int_0^x e^{-w} \int_0^w e^{2v} \int_0^v e^{-2s} r'(s) ds dv dw, \\
E_1(x) &= \int_0^x e^{-w} \int_0^w e^{-v} r(v) dv dw, \quad E_2(x) = \int_0^x e^{-w} \int_0^w e^v r(v) dv dw, \\
R_1(x) &= \int_0^x e^q \int_0^q e^{-p} r'(p) \left(\int_0^p e^{-t} \left(\int_0^t e^{2v} \int_0^v e^{-s} r(s) ds dv \right) dt \right) dp dq, \\
R_2(x) &= \int_0^x e^q \int_0^q e^{-p} r'(p) \left(\int_0^p e^{-t} \left(\int_0^t e^{2v} \int_0^v e^{-2s} r(s) ds dv \right) dt \right) dp dq, \\
S_1(x) &= \int_0^x e^q \int_0^q e^{-p} r'(p) \left(\int_0^p e^{-t} \left(\int_0^t e^{2v} \int_0^v e^{-s} r'(s) ds dv \right) dt \right) dp dq, \\
S_2(x) &= \int_0^x e^q \int_0^q e^{-p} r'(p) \left(\int_0^p e^{-t} \left(\int_0^t e^{2v} \int_0^v e^{-2s} r'(s) ds dv \right) dt \right) dp dq, \\
T_1(x) &= \int_0^x e^q \int_0^q e^{-2p} r(p) \left(\int_0^p e^{2t} \int_0^t e^{-s} r'(s) ds dt \right) dp dq, \\
T_2(x) &= \int_0^x e^q \int_0^q e^{-2p} r(p) \left(\int_0^p e^{2t} \int_0^t e^{-2s} r'(s) ds dt \right) dp dq, \\
U_1(x) &= \int_0^x e^q \int_0^q e^{-p} r(p) dp \left(\int_0^x e^t \int_0^t e^{-s} r'(s) ds dt \right) dq, \\
U_2(x) &= \int_0^x e^q \int_0^q e^{-p} r(p) \left(\int_0^p e^t \int_0^t e^{-2s} r'(s) ds dt \right) dp dq, \\
V_1(x) &= \int_0^x e^q \int_0^q e^{-p} r'(p) \left(\int_0^p e^{-t} \left(\int_0^t e^v r(v) dv \right) dt \right) dp dq, \\
V_2(x) &= \int_0^x e^q \int_0^q e^{-p} r'(p) \left(\int_0^p e^{-t} \left(\int_0^t e^{-v} r(v) dv \right) dt \right) dp dq, \\
W_1(x) &= \int_0^x e^q \int_0^q e^{-2p} r(p) \left(\int_0^p e^{2t} \int_0^t e^{-s} r(s) ds dt \right) dp dq, \\
W_2(x) &= \int_0^x e^q \int_0^q e^{-2p} r(p) \left(\int_0^p e^{2t} \int_0^t e^{-2s} r(s) ds dt \right) dp dq, \\
Y_1(x) &= \int_0^x e^q \int_0^q e^{-p} r(p) dp \left(\int_0^x e^t \int_0^t e^{-s} r(s) ds dt \right) dq, \\
Y_2(x) &= \int_0^x e^q \int_0^q e^{-p} r(p) \left(\int_0^p e^t \int_0^t e^{-2s} r(s) ds dt \right) dp dq, \\
Z_1(x) &= \int_0^x e^q \int_0^q e^{-2p} r(p) \left(\int_0^p e^t r(t) dt \right) dp dq, \\
Z_2(x) &= \int_0^x e^q \int_0^q e^{-2p} r(p) \left(\int_0^p e^{-t} r(t) dt \right) dp dq.
\end{aligned}$$

Thus from (4.107) and (4.108), we have

$$\begin{aligned}\xi(x, y) = & c_1 [-e^{-x}y - 2D_2(x) + 4C_2(x) + 3E_1(x)] - c_2 [y - 2D_1(x) + 4C_1(x) + 3E_2(x)] \\ & - c_3 [2 - e^{-x} - e^x] + c_5 [1 - e^{-x}] + c_6,\end{aligned}\tag{4.122}$$

$$\begin{aligned}\eta(x, y) = & c_1 [-B_2(x)y + 2A_2(x)y + 4R_2(x) - 2S_2(x) - 4T_2(x) + U_2(x) \\ & + 3V_2(x) + 8W_2(x) - 2Y_2(x) + 6Z_2(x)] \\ & - c_2 [-B_1(x)y + 2A_1(x)y + y^2 + 4R_1(x) - 2S_1(x) - 4T_1(x) + U_1(x) \\ & + 3V_1(x) + 8W_1(x) - 2Y_1(x) + 6Z_1(x)] \\ & + c_3 [(e^x - 1)y - 2B_1(x) + B_0(x) + B_2(x) - e^x A_1(x) + A_1(x) + 2A_0(x) - 2A_2(x)] \\ & - c_4 [A_1(x) + y] + c_5 [B_1(x) - B_2(x) + 2A_2(x)] + c_6 B_1(x) + c_7 [e^x - 1] + c_8.\end{aligned}\tag{4.123}$$

For the symmetry $X = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y}$, we have the following eight linearly independent symmetries listed

$$\begin{aligned}X_1 = & [-e^{-x}y - 2D_2(x) + 4C_2(x) + 3E_1(x)] \frac{\partial}{\partial x} + [-B_2(x)y + 2A_2(x)y + 4R_2(x) - 2S_2(x) \\ & - 4T_2(x) + U_2(x) + 3V_2(x) + 8W_2(x) - 2Y_2(x) + 6Z_2(x)] \frac{\partial}{\partial y},\end{aligned}\tag{4.124}$$

$$\begin{aligned}X_2 = & [y - 2D_1(x) + 4C_1(x) + 3E_2(x)] \frac{\partial}{\partial x} + [-B_1(x)y + 2A_1(x)y + y^2 + 4R_1(x) \\ & - 2S_1(x) - 4T_1(x) + U_1(x) + 3V_1(x) + 8W_1(x) - 2Y_1(x) + 6Z_1(x)] \frac{\partial}{\partial y},\end{aligned}\tag{4.125}$$

$$\begin{aligned}X_3 = & [2 - e^{-x} - e^x] \frac{\partial}{\partial x} + [(e^x - 1)y - 2B_1(x) + B_0(x) + B_2(x) - e^x A_1(x) \\ & + A_1(x) + 2A_0(x) - 2A_2(x)] \frac{\partial}{\partial y},\end{aligned}\tag{4.126}$$

$$X_4 = [A_1(x) + y] \frac{\partial}{\partial y},\tag{4.127}$$

$$X_5 = [1 - e^{-x}] \frac{\partial}{\partial x} + [B_1(x) - B_2(x) + 2A_2(x)] \frac{\partial}{\partial y},\tag{4.128}$$

$$X_6 = \frac{\partial}{\partial x} + B_1(x) \frac{\partial}{\partial y},\tag{4.129}$$

$$X_7 = [e^x - 1] \frac{\partial}{\partial y},\tag{4.130}$$

$$X_8 = \frac{\partial}{\partial y}.\tag{4.131}$$

Case II : $c_2 = 0, K = K(x, y)$

We have the following three equations to solve

$$c''(x) - 2c_1Ke^{-x} - c'(x) + c_1K_xe^{-x} - K_y c(x) = 0, \quad (4.132)$$

$$d''(x) + Kc(x) - 2Kb'(x) - d'(x) - K_x b(x) - K_y d(x) = 0, \quad (4.133)$$

$$2c'(x) + 3c_1Ke^{-x} - b''(x) - b'(x) = 0. \quad (4.134)$$

This system of ODEs cannot be solved in general by ordinary methods. The handbooks of Polyanin [22] and Kamke [14] provides solutions for specific polynomial and exponential forms of the kernel.

Chapter 5

Selected forms of the kernel

In this chapter, we select forms of the kernel that enable us to find general solutions of the first and second kind Volterra integro-differential equations by standard techniques.

5.1 When the kernel has the form $K(x, y(x)) = f(x)$

5.1.1 IDE of the first kind

Suppose we have an integro-differential equation of the first kind, with a kernel of the form $K(x, f(x)) = f(x)$, that is we have

$$f'(x) = \int_a^x f(s) ds, \quad (5.1)$$

differentiating with respect to x , we have

$$f''(x) - f(x) = 0. \quad (5.2)$$

Let $f(x) = e^{rx}$, we have the characteristic equation

$$r^2 - 1 = 0, \quad (5.3)$$

which implies that $r = \pm 1$, therefore the general solution is

$$f(x) = c_1 e^x + c_2 e^{-x}. \quad (5.4)$$

5.1.2 IDE of the second kind

Suppose we have an integro-differential equation of the second kind, with a kernel of the form $K(x, f(x)) = f(x)$, that is we have

$$f'(x) = f(x) + \int_a^x f(s) \, ds, \quad (5.5)$$

differentiating with respect to x , we have

$$f''(x) - f'(x) - f(x) = 0. \quad (5.6)$$

Let $f(x) = e^{rx}$, we have the characteristic equation

$$r^2 - r - 1 = 0, \quad (5.7)$$

which implies that $r = \frac{1 \pm \sqrt{5}}{2}$, therefore the general solution is

$$f(x) = c_1 e^{\frac{1+\sqrt{5}}{2}x} + c_2 e^{\frac{1-\sqrt{5}}{2}x}. \quad (5.8)$$

5.2 When the kernel has the form $K(x, y(x)) = g(y)h(x)$

5.2.1 IDE of the first kind

Suppose we have an integro-differential equation of the first kind, with a kernel of the form $K = g(y)h(x)$, that is we have

$$f'(x) = \int_a^x g(y(s))h(s) \, ds, \quad (5.9)$$

differentiating with respect to x , we have

$$f''(x) - g(y)h(x) = 0. \quad (5.10)$$

It follows that the general solution is

$$f(x) = c_2 x + c_1 + \int_0^x \left(\int_0^t h(s)g(y(s)) \, ds \right) dt. \quad (5.11)$$

5.2.2 IDE of the second kind

Suppose we have an integro-differential equation of the first kind, with a kernel of the form $K = g(y)h(x)$, that is we have

$$f'(x) = f(x) + \int_a^x g(y(s))h(s) ds, \quad (5.12)$$

differentiating with respect to x , we have

$$f''(x) - f'(x) - g(y)h(x) = 0. \quad (5.13)$$

It follows that the general solution is

$$f(x) = \int_0^x \left(c_1 e^t + e^t \int_0^t e^{-s} h(s) g(y(s)) ds \right) dt + c_2. \quad (5.14)$$

5.3 When the kernel has the form $K(x, y(x)) = \frac{h(x)}{g(y)}$

5.3.1 IDE of the first kind

Suppose we have an integro-differential equation of the first kind, with a kernel of the form $K = \frac{h(x)}{g(y)}$, that is we have

$$f'(x) = \int_a^x \frac{h(s)}{g(y(s))} ds, \quad (5.15)$$

differentiating with respect to x , we have

$$f''(x) - \frac{h(x)}{g(y)} = 0. \quad (5.16)$$

it follows that the general solution is given by

$$f(x) = c_2 x + c_1 + \int_0^x \left(\int_0^t \frac{h(s)}{g(y(s))} ds \right) dt. \quad (5.17)$$

5.3.2 IDE of the second kind

Suppose we have an integro-differential equation of the first kind, with a kernel of the form $K = \frac{h(x)}{g(y)}$, that is we have

$$f'(x) = f(x) + \int_a^x \frac{h(s)}{g(y(s))} ds, \quad (5.18)$$

differentiating with respect to x , we have

$$f''(x) - f'(x) - \frac{h(x)}{g(y)} = 0. \quad (5.19)$$

it follows that the general solution is given by

$$f(x) = \int_0^x \left(c_1 e^t + e^t \int_0^t \frac{e^{-s} h(s)}{g(y(s))} ds \right) dt + c_2. \quad (5.20)$$

5.4 When the kernel has the form $K(x, y(x)) = g(y) + |f(x)|$

5.4.1 IDE of the first kind

Suppose we have an integro-differential equation of the first kind, with a kernel of the form $K = g(y) + |f(x)|$, that is we have

$$f'(x) = \int_a^x (g(y) + |f(s)|) ds, \quad (5.21)$$

differentiating with respect to x , we have

$$f''(x) - g(y) - |f(x)| = 0. \quad (5.22)$$

it follows that the general solution is given by

$$f(x) = \begin{cases} c_1 e^x + c_2 e^{-x} + e^x \int_0^x \frac{1}{2} e^{-s} g(y(s)) ds \\ \quad + e^{-x} \int_0^x -\frac{1}{2} e^s g(y(s)) ds, & \text{if } f(x) > 0, \\ c_2 \sin(x) + c_1 \cos(x) + \cos(x) \int_0^x \sin(s)(-g(y(s))) ds \\ \quad + \sin(x) \left(\int_0^x \cos(s) g(y(s)) ds \right), & \text{if } f(x) < 0. \end{cases} \quad (5.23)$$

5.4.2 IDE of the second kind

Suppose we have an integro-differential equation of the first kind, with a kernel of the form $K = g(y) + |f(x)|$, that is we have

$$f'(x) = f(x) + \int_a^x (g(y) + |f(s)|) ds, \quad (5.24)$$

differentiating with respect to x , we have

$$f''(x) - f'(x) - g(y) - |f(x)| = 0. \quad (5.25)$$

it follows that the general solution is given by

$$f(x) = \begin{cases} c_1 e^{\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)x} + c_2 e^{\left(\frac{\sqrt{5}+1}{2}\right)x} + e^{\left(\frac{\sqrt{5}+1}{2}\right)x} \int_0^x \frac{e^{\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)s-s} g(y(s))}{\sqrt{5}} ds \\ \quad + e^{\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)x} \int_0^x -\frac{e^{\left(\frac{\sqrt{5}+1}{2}\right)s-s} g(y(s))}{\sqrt{5}} ds, & \text{if } f(x) > 0, \\ c_2 e^{\frac{x}{2}} \sin\left(\frac{\sqrt{3}x}{2}\right) + c_1 e^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}x}{2}\right) \\ \quad + e^{x/2} \cos\left(\frac{\sqrt{3}x}{2}\right) \int_0^x -\frac{2e^{-\frac{s}{2}} \sin\left(\frac{\sqrt{3}s}{2}\right) g(y(s))}{\sqrt{3}} ds \\ \quad + e^{x/2} \sin\left(\frac{\sqrt{3}x}{2}\right) \left(\int_0^x \frac{2e^{-\frac{s}{2}} \cos\left(\frac{\sqrt{3}s}{2}\right) g(y(s))}{\sqrt{3}} ds \right), & \text{if } f(x) < 0. \end{cases} \quad (5.26)$$

Chapter 6

Laplace transforms

6.1 Theoretical basis

The Laplace transform is an integral transform that takes a function $f(t)$, where t is a real variable and transforms it into a function $F(z)$, where z is a complex variable. This powerful transform can transform convolutions into a product, ODE's into algebraic equations and therefore has many applications in science and technology.

6.1.1 The Laplace transform

Definition 6.1 (The Laplace Transform). Suppose $f(t)$ is a given exponentially-restricted function that is locally integrable on $[0, \infty)$ and t is a real valued variable. The Laplace transform of $f(t)$ is given by

$$\mathcal{L}[f(t)] = \int_0^{\infty} e^{-zt} f(t) dt, \quad (6.1)$$

if (6.1) converges for some value of z . For convenience, the Laplace transform can also be denoted as $F(z) = \mathcal{L}[f(t)]$. As outlined in [15].

The Laplace transform, like differentiation and integration, is a linear operator. The Laplace transform of $af(t) + bg(t)$, where a, b are constants and $f(t), g(t)$ are exponentially bounded functions, is given by

$$\mathcal{L}[af(t) + bg(t)] = \int_0^{\infty} e^{-zt}(af(t) + bg(t)) dt \quad (6.2)$$

$$= a \int_0^{\infty} e^{-zt} f(t) dt + b \int_0^{\infty} e^{-zt} g(t) dt \quad (6.3)$$

$$= a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)], \quad (6.4)$$

hence, the following theorem.

Theorem 6.1 (The Laplace Transform as a Linear Operation). *For any functions $f(t)$ and $g(t)$ whose transforms exist and any constants a and b , the Laplace transform of $af(t) + bg(t)$ is a linear operation given by*

$$\mathcal{L}[af(t) + bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)]. \quad (6.5)$$

The following examples highlight some useful Laplace transforms.

Example 6.1.

$$\mathcal{L}[f'(t)] = \int_0^{\infty} e^{-zt} f'(t) dt \quad (6.6)$$

$$= f(t)e^{-zt} \Big|_0^{\infty} + z \int_0^{\infty} f(t)e^{-zt} dt \quad (6.7)$$

$$= sF(z) - f(0). \quad (6.8)$$

Example 6.2.

$$\mathcal{L}[f''(t)] = \int_0^{\infty} e^{-zt} f''(t) dt \quad (6.9)$$

$$= f'(t)e^{-zt} \Big|_0^{\infty} + z \int_0^{\infty} f'(t)e^{-zt} dt \quad (6.10)$$

$$= -f'(0) + z(zF(z) - f(0)) \quad (6.11)$$

$$= s^2F(z) - zf(0) - f'(0). \quad (6.12)$$

Example 6.3.

$$\mathcal{L}[tf(t)] = \int_0^{\infty} te^{-zt} f(t) dt \quad (6.13)$$

$$= -\frac{d}{dz} \int_0^{\infty} e^{-zt} f(t) dt \quad (6.14)$$

$$= -\frac{d}{dz} \mathcal{L}[f(t)] \quad (6.15)$$

$$= -\frac{dF}{dz}. \quad (6.16)$$

Example 6.4.

$$\mathcal{L}[tf'(t)] = \int_0^{\infty} te^{-zt} f'(t) dt \quad (6.17)$$

$$= -\frac{d}{dz} \int_0^{\infty} e^{-zt} f'(t) dt \quad (6.18)$$

$$= -\frac{d}{dz} \mathcal{L}[f'(t)] \quad (6.19)$$

$$= -\frac{d}{dz}(zF(z) - f(0)) \quad (6.20)$$

$$= -z\frac{dF}{dz} - F(z). \quad (6.21)$$

Example 6.5.

$$\mathcal{L}[tf''(t)] = \int_0^\infty te^{-zt} f''(t) dt \quad (6.22)$$

$$= -\frac{d}{dz} \int_0^\infty e^{-zt} f''(t) dt \quad (6.23)$$

$$= -\frac{d}{dz} \mathcal{L}[f''(t)] \quad (6.24)$$

$$= -\frac{d}{dz}(z^2F(z) - zf(0) - f'(0)) \quad (6.25)$$

$$= -z^2\frac{dF}{dz} - 2zF(z) + f(0). \quad (6.26)$$

6.1.2 The inverse Laplace transform

The inverse Laplace transform is an integral transform that takes a function $F(z)$, where z is a complex variable and transforms it into a function $f(t)$, where t is a real variable, in other words it is the reverse of the Laplace transform outlined in the previous section. The following definition highlights this.

Definition 6.2. Suppose that $F(z)$ is the Laplace transform of some function $f(t)$, then $f(t)$ is called the inverse Laplace transform of $F(z)$. This is denoted by

$$f(t) = \mathcal{L}^{-1}[F(z)]. \quad (6.27)$$

In practice, in order to find the inverse transform we generally work backwards from known Laplace transforms. The following examples illustrate how this is done.

Example 6.6.

$$\mathcal{L}[e^{at}] = \int_0^\infty e^{-t(z-a)} dt \quad (6.28)$$

$$= \frac{1}{z-a}, \quad (6.29)$$

which implies that

$$\mathcal{L}^{-1}\left[\frac{1}{z-a}\right] = e^{at}. \quad (6.30)$$

Example 6.7.

$$\mathcal{L} \left[\int_0^t f(\tau) f(t - \tau) d\tau \right] = \int_0^\infty e^{-zt} \int_0^t f(\tau) f(t - \tau) d\tau dt, \quad (6.31)$$

changing the order of integration, we have

$$\mathcal{L} \left[\int_0^t f(\tau) f(t - \tau) d\tau \right] = \int_0^\infty f(\tau) \int_\tau^\infty e^{-zt} f(t - \tau) dt d\tau \quad (6.32)$$

$$= \int_0^\infty f(\tau) \int_0^\infty e^{-z(t+\tau)} f(t) dt d\tau \quad (6.33)$$

$$= F(z) \int_0^\infty e^{-z\tau} f(\tau) d\tau \quad (6.34)$$

$$= F(z)F(z). \quad (6.35)$$

This implies that

$$\mathcal{L}^{-1} [F(z)F(z)] = \int_0^t f(\tau) f(t - \tau) d\tau. \quad (6.36)$$

6.2 Selected forms of the IDE

The Laplace transform can be used to reduce certain forms of integro-differential equations, specifically those with a convolution-type integral, to ordinary differential equations. In this section we illustrate the Laplace transform (and inverse transform) being applied to some examples of these modified integro-differential equations. The reader can refer to [19] for similar treatments.

6.2.1 Modified Volterra equation of the first kind

Consider the integro-differential equation of the form

$$f'(t) - \int_0^1 f(t) f(t(1 - s)) ds = 0, \quad (6.37)$$

where $f(t)$ is exponentially-restricted. If we let $y = ts$, then

$$tf'(t) - \int_0^t f(y) f(t - y) dy = 0. \quad (6.38)$$

Applying Definition 6.1 and Example 6.4, letting $u(z) = F(z)$ for convenience, gives

$$zu'(z) + u(z) + u(z)^2 = 0. \quad (6.39)$$

Solving this ODE yields

$$\frac{du}{dz} = -\frac{u^2 + u}{z}, \quad (6.40)$$

$$\int \frac{du}{u^2(1 + \frac{1}{u})} = -\int \frac{dz}{z}, \quad (6.41)$$

$$\ln\left(\frac{u}{u+1}\right) = -\ln z + \ln A, \quad (6.42)$$

$$\frac{u}{u+1} = \frac{A}{z}, \quad (6.43)$$

$$u = \frac{A}{z-A}, \quad (6.44)$$

where $\ln A$ is a constant of integration. Applying the inverse Laplace transform results in the following solution

$$f(t) = \mathcal{L}^{-1}\left[\frac{A}{z-A}\right] = Ae^{At}. \quad (6.45)$$

6.2.2 Modified Volterra equation of the second kind

Consider the integro-differential equation of the form

$$f'(t) - f(t) - \int_0^1 f(t)f(t(1-s)) ds = 0, \quad (6.46)$$

where $f(t)$ is exponentially-restricted. If we let $y = ts$, then

$$tf'(t) - tf(t) - \int_0^t f(y)f(t-y) dy = 0. \quad (6.47)$$

Applying Definition 6.1, Example 6.3 and Example 6.4, letting $u(z) = F(z)$ gives us

$$(z+1)u'(z) + u(z) + u(z)^2 = 0. \quad (6.48)$$

Solving this ODE yields

$$\int \frac{du}{u+u^2} = - \int \frac{dz}{z+1}, \quad (6.49)$$

$$\ln \left(\frac{u}{u+1} \right) = - \ln(z+1) + \ln A, \quad (6.50)$$

$$\ln \left(\frac{u}{u+1} \right) = \ln \left(\frac{A}{z+1} \right), \quad (6.51)$$

$$\frac{u}{u+1} = \frac{A}{z+1}, \quad (6.52)$$

$$u = \frac{A}{z - (A-1)}, \quad (6.53)$$

where $\ln A$ is a constant of integration. Applying the inverse Laplace transform results in the following solution

$$f(t) = \mathcal{L}^{-1} \left[\frac{A}{z - (A-1)} \right] = Ae^{(A-1)t}. \quad (6.54)$$

6.2.3 Second-order modified integro-differential equation

Suppose we had an integro-differential equation of the form

$$f''(t) - f(t) - \int_0^1 f(t)f(t(1-s)) ds = 0, \quad (6.55)$$

where $f(t)$ is exponentially-restricted. Let $y = ts$, then

$$tf''(t) - tf(t) - \int_0^t f(y)f(t-y) dy = 0. \quad (6.56)$$

Applying Definition 6.1, Example 6.3 and Example 6.5, letting $u(z) = F(z)$ gives us a Riccati ODE of the form

$$(z^2 - 1)u'(z) + 2zu(z) + u(z)^2 = f(0). \quad (6.57)$$

Suppose $f(0) = 0$, letting $y = u^{-1}$ results in the following linear differential equation

$$y'(z) - \frac{2z}{z^2 - 1}y(z) = \frac{1}{z^2 - 1}, \quad (6.58)$$

solving this equation yields

$$y(z) = \frac{1}{4}(z^2 - 1) \left(4c_1 - \frac{2z}{z^2 - 1} - \ln(1 - z) + \ln(z + 1) \right), \quad (6.59)$$

hence, the solution of (6.57) is

$$u(z) = \frac{2}{2c_1 z^2 - 2c_1 - z^2 \tanh^{-1}(z) - z + \tanh^{-1}(z)} \quad (6.60)$$

A solution to equation (6.56) can be obtained by the inverse transform of equation (6.60).

Chapter 7

Conclusion

Our aim in this dissertation was to explore methods, both numerical and analytical, that could be used to successfully determine approximate solutions as well as general solutions to Volterra integro-differential equations. Specifically, the numerical methods applied were the Lagrange collocation method on a global domain and the piecewise cubic Hermite collocation method. The analytical methods utilized was Lie symmetry analysis, Laplace transforms as well as ordinary methods where possible.

In chapter 2, we applied Lagrange collocation as a tool to approximate the solution of integro-differential equations. From figure 2.4.1 we see that for lower order approximating polynomials, Lagrange collocation is a viable tool to numerically solve an integro-differential equation. However if the exact solution of the integro-differential equation happens to be highly oscillatory, a lower order polynomial may not be a sufficiently accurate approximation. This poses a problem as a high order approximating polynomial may exhibit Runge's phenomenon, as is the case in figure 2.4.2.

Chapter 3 was the application of cubic Hermite collocation as a tool to approximate the solution of integro-differential equations. Piecewise cubic Hermite collocation appears to avoid the problems that arise with Lagrange collocation of integro-differential equations as seen in chapter 2. The method proves to be stable in the 46 collocation points example of figure 3.3.2, where in the previous chapter Lagrange collocation exhibited Runge's phenomenon. Additionally, the method continues to be stable even for a very large number of collocation points as seen in figures 3.3.3 and 3.3.4, errors getting smaller as the number of collocation points used increases.

In chapter 4 we applied Lie's algorithm to the general Volterra integro-differential equation of the first and second kind. We determined a set of eight linearly independent symmetries for both the first and second kind general equations. In particular, we had discussed the case that $K(x, y) = r(x)$, while in other cases symmetries for specific forms of the kernel were obtained. Chapter 5 dealt with selected general forms of the kernel, where we presented general solutions via the aid of standard methods. In chapter 6, we applied Laplace transform to transform the integro-differential equation to a differential equation without quadrature, thus being able to integrate the differential equation. We obtained these solutions for various modified Volterra

equations, specifically those that could be arranged with a convolution-type integral.

The work of this dissertation largely dealt with first-order ordinary integro-differential equations, further research in the future could explore higher-order ordinary integro-differential equations as well as multivariable partial integro-differential equations.

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