Exploring the challenges of learning transformations of trigonometric functions by senior phase mathematics preservice teachers at a KwaZulu-Natal university

by

Francis Youbi

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Supervisor: Prof. Sarah Bansilal

DECLARATION

I, Mr Francis Youbi, declare as follows:

1. That the work described in this thesis has not been submitted to UKZN or other tertiary institution for purposes of obtaining an academic qualification, whether by myself or any other party. Where a colleague has indeed prepared a thesis based on related work essentially derived from the same project, this must be stated here, accompanied by the name, the degree for which submitted, the University, the year submitted (or in preparation) and a concise description of the work covered by that thesis such that the examiner can be assured that a single body of work is not being used to justify more than one degree.

2. That my contribution to the project was as follows: This is followed by a concise description of the candidate's personal involvement in and contribution to the project, in sufficient detail that the examiner is in no doubt as to the extent of their contribution.

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Signed [redacted] Date 30 January 2025

DEDICATION

To my wife, Emily Youbi: your unwavering support and encouragement have been the cornerstone of my dreams. Your patience and endless belief in my abilities are truly appreciated. To my late father, Fidele Ndongo, and my late mother, Odile Kiene, thank you for instilling in me a passion for learning and perseverance. My gratitude also extends to my mother-in-law, Linah Mafokwane, for being a constant source of motivation and joy. To my children, Yann, Fidel, Aden, Catherine and Pascale, your presence has been a significant inspiration in completing this Ph.D. To all my siblings, Clementine Manga, Dieudonne Tonda, Donatien Brice Ndongo, and Jean Luc Ndongo, your distant love and support have meant so much to me. This thesis is a testament to the strength of the community and love that surrounds me.

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LIST OF ACRONYMS/ABBREVIATIONS

NQF:	National Qualifications Framework
SAQA:	South African Qualifications Authority
GET:	General Education and Training
FET:	Further Education and Training
HG:	Higher Education
DoBE:	Department of Basic Education
NSC:	National Senior Certificate
NDRLP:	National Diagnostic Report on Learner Performance
PSTs:	Preservice Teachers
NCTM:	National Council of Teachers of Mathematics

ABSTRACT

This study investigates the conceptual and procedural challenges that senior phase mathematics preservice teachers (PSTs) encounter in understanding and applying the transformations of trigonometric functions, within the context of a higher education institution in KwaZulu-Natal, South Africa. Grounded in a qualitative research design, the study involved 125 first-year PSTs who participated in a diagnostic assessment consisting of Eight analytical tasks focused on trigonometric graphs and functions. In addition, semi-structured interviews were conducted with a purposive sample of five participants to gain deeper insights into their reasoning processes and conceptual misunderstandings.

Findings reveal a consistent pattern of difficulties and misconceptions among the PSTs in relation to core concepts such as amplitude, period, phase shift, and vertical shift. Many participants exhibited significant struggles in identifying and interpreting these features within graphical representations, particularly when asked to apply transformations such as vertical and horizontal shifts, reflections, and scaling. Furthermore, a prevalent issue was the misapplication of trigonometric identities, which resulted in errors when simplifying expressions or solving equations. A lack of fluency in transitioning between different representations, such as moving from algebraic equations to their corresponding graphs, or vice versa, further highlighted conceptual fragmentation in the PSTs' understanding.

Additional challenges included confusion in converting between radians and degrees, errors in determining the correct direction and magnitude of phase shifts, and difficulties in simplifying complex trigonometric expressions. These issues suggest that many PSTs are still operating at a procedural rather than a conceptual level, relying heavily on rote memorization without fully grasping the underlying principles of function behaviour and transformation.

The implications of the study point to a need for significant pedagogical reform in the way trigonometric concepts are taught in teacher preparation programmes. Specifically, there is a need for instructional strategies that are both visual and interactive, incorporating tools such as dynamic graphing software, manipulatives, and contextualized problems that bridge abstract mathematics with real-world

applications. Emphasizing multiple representations and encouraging metacognitive reflection may also foster deeper learning and retention.

Recommendations for future research include exploring the impact of targeted instructional interventions, such as inquiry-based learning, collaborative problem-solving, and the integration of educational technology, on improving PSTs' conceptual mastery of trigonometric transformations. By equipping future teachers with stronger foundational understanding and pedagogical content knowledge, this research seeks to contribute to broader efforts in revitalizing mathematics education and enhancing student outcomes at the senior phase level and beyond.

CHAPTER 1
INTRODUCTION AND RESEARCH PROBLEM

The aim of this research was to investigate the challenges faced by senior - phase mathematics preservice teachers at a university in KwaZulu-Natal when trying to understand and learn about trigonometric function transformations. This chapter will primarily focus on crafting and articulating the problem statement of the study, which involves providing a detailed overview and background information about the study, explaining the reasons for conducting it, and explicitly stating the specific issue being investigated. Additionally, the research questions that guided this study are outlined, along with a comprehensive structure of the thesis.

1.1 BACKGROUND OF THE STUDY

The South African education system is structured into three primary bands in alignment with the National Qualifications Framework (NQF), which was instituted in the early 1990s as part of post-apartheid educational reforms. These three bands, General Education and Training (GET), Further Education and Training (FET), and Higher Education (HE), form the foundation upon which the nation's educational aspirations are built. At the core of each of these bands is mathematics, a subject universally recognized for its significance in the development of critical thinking, logical reasoning, and problem-solving abilities. The importance of mathematics transcends academic boundaries, positioning it as a gateway subject to science, technology, engineering, and mathematical (STEM) fields. Consequently, mathematics is a compulsory component of the South African curriculum from the early grades through to secondary schooling, and is also pivotal within tertiary education programs, particularly in teacher training.

Despite curriculum transformations and government interventions aimed at improving mathematics education, performance outcomes in mathematics remain troubling, especially at higher school levels. Data from the Department of Basic Education (DoBE) indicate that learner performance in mathematics, particularly in the National Senior Certificate (NSC) examinations, continues to decline both in terms of achievement and participation. The 2019 NSC examination report, for example, revealed a worrying trend: fewer learners were electing to write mathematics, and among those who did, performance levels were generally poor. Specifically, the number of learners who took mathematics dropped from 233,858 in 2018 to 222,034 in 2019, representing a decline of 11,824 candidates in just one academic year. Over a broader timeframe, from 2016 to 2019, there was a sharp decrease of 43,878 learners participating in mathematics within the NSC framework.

Such a decline not only signals disinterest but also reflects deeper systemic issues. In 2019, the NSC mathematics pass rate was 54.6%, a decrease of 3.4% compared to the 2018 results. According to the National Diagnostic Report on Learner Performance (NDRLP), most learners who passed mathematics in 2019 did so with marginal scores: 34.4% achieved between 30–49.9%, 9.6% between 50–59.9%, and only 10.6% between 60–100%. These figures suggest that only 12,845 matriculants achieved scores that might qualify them for tertiary studies in mathematics, intensive disciplines, assuming all other entry criteria were also met. This dwindling pool of mathematically capable learners has serious implications for the STEM pipeline in South Africa, including the supply of future mathematics teachers.

In recognition of these challenges, South Africa's National Development Plan (NDP) has set an ambitious target: by 2030, at least 450,000 learners should be eligible to pursue bachelor's degrees in mathematics and science (NDP, 2012, p. 305). However, the trajectory of current data makes the achievement of this target increasingly unlikely. In the 2024 NSC examination, approximately 251,488 learners wrote the mathematics paper. Of these, 17.4% (about 44,636 students) achieved a score of 60% or higher, thereby meeting the university entrance benchmark for mathematics-intensive programs (Moonstone Information Refinery, 2025). This marks a noticeable improvement in eligibility compared to previous years. The overall mathematics pass rate also increased significantly, rising to 69.1% in 2024, up from 63.5% in 2023 (DoBE, 2024). While the enhanced pass rate is encouraging, the proportion of candidates meeting the 60% standard remains insufficient to substantially boost the pipeline of learners qualified for advanced science, engineering, and mathematics studies. These statistics represent a pressing national crisis that not only affects the future workforce in science and technology fields but also impinges on the number and quality of preservice mathematics teachers entering the education system.

This situation is further complicated by learner performance in specific areas of mathematics, most notably geometry and trigonometry, which form a significant portion of Paper Two of the NSC mathematics exam. Trends over the years show that learners often find these sections particularly challenging. With trigonometry, in particular, many students struggle with key conceptual areas, such as the transformation of trigonometric functions, identity manipulation, and problem-solving involving real-life applications. The ability to translate a real-world or verbal problem into symbolic mathematical expressions, a crucial skill in understanding and applying trigonometric principles, is notably weak. These challenges are exacerbated by unequal resource distribution and inconsistent teaching quality across South African schools. Learners in well-resourced, urban schools often outperform those in under-resourced, rural or township schools due to disparities in access to qualified teachers, teaching materials, and technological support (DoBE National Report on Grade 12 performance, 2019, 2020, 2021, & 2022).

Trigonometry, as a domain within mathematics, explores the relationships between angles and sides in triangles and is integral to many scientific and engineering applications (Orhun, 2010). It necessitates the integration of algebraic manipulation, geometric understanding, and graphical analysis. Its utility extends beyond education; for instance, trigonometric functions are foundational to astronomy, geography, and navigation systems. Despite its importance, South African learners repeatedly exhibit poor performance in this area, as reflected in multiple National Diagnostic Reports from 2019 to 2024. These reports consistently highlight a lack of conceptual understanding, a finding echoed by numerous researchers (e.g., Ngcobo et al., 2019; Walsh et al., 2017). Students often fail to develop the necessary mental schemas to solve trigonometric problems and display significant gaps in prerequisite knowledge, such as a lack of familiarity with radian measure, which is critical for a full understanding of trigonometric function behaviour.

In particular, the transformation of trigonometric functions, a topic that requires abstract reasoning and visual-spatial understanding, remains a persistent stumbling block. Transformations involve changes in the amplitude, period, phase shift, and vertical displacement of trigonometric graphs. Mastery of this topic requires students to interpret and switch between multiple representations (graphical, algebraic, and contextual), which many struggle to do. According to cognitive learning theories, such as APOS theory (Action–Process–Object–Schema), understanding transformations entails progressing through various stages of conceptual development (Ngcobo et al., 2019). Unfortunately, many learners and preservice teachers alike remain at the procedural “action” stage and do not reach the “schema” stage of full conceptual integration.

This challenge is not confined to secondary school learners. Preservice teachers (PSTs), who are supposed to be the future bearers of quality mathematics instruction, frequently display the same misunderstandings and learning difficulties they are meant to remediate in their future classrooms. Entrance into a Bachelor of Education (Bed) program for teaching mathematics at the senior phase (Grades 7–9) or further education and training phase (Grades 10–12) requires a minimum score of 60% in mathematics on the NSC. However, achieving this score does not guarantee deep conceptual understanding. An analysis of the 2019 first-year preservice teachers' exam scripts at the university where this study was conducted revealed widespread difficulties in dealing with transformations of trigonometric functions, mirroring the challenges documented in the NSC Diagnostic Reports. This parallel suggests that the conceptual deficits begin in basic education and persist into higher education, raising critical questions about the effectiveness of mathematics instruction and the preparedness of incoming university students.

Given that preservice teachers are tasked with teaching trigonometry in the senior phase, their difficulties in grasping transformations signal a potential crisis in mathematics teacher education. If these teachers enter classrooms without a robust understanding of the subject, they risk perpetuating cycles of underachievement and misconception. Furthermore, their pedagogical content knowledge (PCK), the ability to teach mathematical concepts effectively, is likely compromised by their own conceptual gaps. Effective teaching of transformations requires not only an understanding of the mathematical content but also the ability to anticipate student errors, present concepts in multiple ways, and use technology tools like graphing calculators or dynamic geometry software (e.g., GeoGebra) to enhance visualization. Without these skills, instruction may remain procedural, fragmented, and ineffective.

The educational context of KwaZulu-Natal, where this study was based, adds another layer of complexity. The province is characterized by a mix of urban, peri-urban, and rural schools, many of which operate under difficult conditions. Teacher education institutions in this region admit students from diverse socio-economic and educational backgrounds. Some of these students have attended schools with limited mathematics teaching, minimal access to laboratories, libraries, or ICT resources, and where English, the language of instruction, is not their home language. These factors contribute significantly to the development of alternative conceptions and misinterpretations of core mathematical concepts, including those in trigonometry.

Additionally, language itself can be a barrier. The terminology used in trigonometry, such as “amplitude,” “frequency,” “shift,” and “reflection”, often carries meanings in everyday usage that differ from their mathematical definitions. This semantic conflict can further confuse learners who are not operating in their first language. Moreover, without an adequate grounding in function concepts and algebraic manipulation, the added abstraction of function transformation can become overwhelming (Pillay, 2017).

This study, therefore, is driven by a clear need to understand the challenges that senior phase and FET preservice mathematics teachers (PSTs) face in learning and internalizing the transformation of trigonometric functions. As future educators, these individuals must be equipped with not only procedural fluency but also conceptual depth and pedagogical flexibility. By examining their learning processes, difficulties, misconceptions, and instructional backgrounds, this research seeks to illuminate some reasons for the persistent underperformance in this vital topic. The goal is to generate insights that can inform teacher education curricula, intervention programs, and policy decisions, thereby contributing to improved mathematics teaching and learning outcomes in South Africa.

1.2 Conceptual and historical context of trigonometry

Trigonometry is a foundational branch of mathematics that focuses on the study of relationships between angles and sides in triangles, particularly right-angled triangles. At its core, trigonometry involves six essential functions, sine, cosine, tangent, cosecant, secant, and cotangent, which serve as tools to describe periodic phenomena and model relationships in both pure and applied mathematics. These trigonometric functions are indispensable in various fields of science and engineering, including physics, astronomy, architecture, geography, and increasingly, computer science and machine learning (Oliveira & Costa, 2021). Their relevance extends to everyday applications such as satellite navigation, acoustics, and even medical imaging technologies.

As Boyer (1991) notes, while trigonometric functions are deeply embedded in contemporary mathematical systems and applied sciences, this does not render trigonometry immune to critical epistemological inquiry. The development of trigonometry must be understood not only as a set of computational tools but also as a field underpinned by logical and conceptual foundations that continue to require formal and dialectical scrutiny. This viewpoint is particularly relevant in mathematics education, where learners and preservice teachers often accept rules and formulas procedurally, without engaging with their underlying principles.

Historically, trigonometry emerged from practical necessities. Its early development was intimately tied to astronomical observations and land surveying practices in ancient civilizations such as Egypt, Babylon, India, and Greece. The Greeks, particularly Hipparchus and Ptolemy, formalized early trigonometric ideas through the construction of chord tables. Later, Indian mathematicians like Aryabhata and Bhaskara contributed to sine and cosine concepts, which were further refined by Islamic scholars during the Golden Age, who introduced systematic algebraic and geometric methods. European mathematicians during the Renaissance, such as Regiomontanus, transformed trigonometry into an independent field by separating it from astronomy and geometry (Katz & Parshall, 2021).

Modern trigonometry, as it is taught today, includes not only its classical triangle-based foundations but also unit circle representations, function transformations, and graphical analysis. The introduction of radians, periodicity, amplitude, and phase shifts mark the evolution of trigonometry into a more abstract and generalized framework that requires multi-representational understanding (Castro Superfine & Li, 2020). Nevertheless, despite its rich historical development and wide application, trigonometry remains a challenging topic for both learners and educators at various levels of the education system.

A significant body of literature over the past two decades has consistently pointed to trigonometry as a topic that poses deep conceptual challenges for learners. These difficulties are not limited to the memorization of formulas but are often tied to the abstract nature of trigonometric functions, the need for multi-modal representation (algebraic, graphical, and contextual), and the complex language used in mathematical descriptions (Li & Schoenfeld, 2021).

One major issue is that learners often approach trigonometry procedurally, relying on rote memorization of identities and formulas rather than developing conceptual understanding. This is compounded by difficulties in grasping the unit circle and the radian system, both of which are crucial for understanding the periodic and transformational properties of trigonometric functions (Karadag & Cakiroglu, 2022). In South Africa, multiple National Diagnostic Reports (DoBE, 2020–2024) have revealed that students struggle particularly with transformation of trigonometric functions, which includes understanding how changes to the equation of a sine or cosine function affect its graph (e.g., amplitude, period, phase shift, and vertical shift).

Moreover, students tend to have trouble when moving between different representations of the same concept. For instance, a student might be able to plot a sine graph based on a table of values but fail to interpret its equation in terms of transformations. This lack of representational fluency hinders deeper engagement with the content and affects their ability to solve real-world problems that involve trigonometric modelling (Özdemir & Ünlü, 2021).

Teachers, too, face several challenges in teaching trigonometry. Studies have highlighted that many teachers themselves harbour misconceptions or incomplete conceptual frameworks about the subject, particularly regarding function transformations (Cetin & Dubinsky, 2020). Furthermore, the abstract nature of trigonometric transformations requires high levels of PCK, including the ability to explain complex ideas in accessible ways, anticipate learner errors, and use technological tools for visualization. In under resourced contexts such as many South African schools, teachers often lack access to graphing tools or dynamic software (e.g., GeoGebra), making the teaching of transformations especially challenging.

Additionally, the linguistic complexity of trigonometry further contributes to student misunderstandings. Words like “amplitude,” “phase,” “shift,” and “period” have everyday meanings that can conflict with their mathematical definitions. For learners studying in a second or third language, such as many preservice teachers in KwaZulu-Natal, these conflicts exacerbate the difficulty in mastering trigonometric concepts (Makgato & Ramaboka, 2022).

At the University of KwaZulu-Natal (UKZN), the Bachelor of Education (B.Ed.) degree program specializing in Senior Phase Mathematics and Technology Education or FET Mathematics Education offers a structured approach to mathematics teacher preparation. In the first year of study, PSTs are introduced to both foundational and specialized content, with an emphasis on developing a deep understanding of key mathematical topics relevant to the senior phase (Grades 7–9) and FET phase (Grades 10–12).

Within the second semester, students are enrolled in a module titled “Mathematics Education: Senior Phase and FET 1 (SP1)”, which covers both measurement and trigonometry. This course is designed to bridge the gap between high school mathematics and university-level mathematical thinking. Key topics in the trigonometry component include:

- Basic trigonometric ratios (sine, cosine, and tangent)
- Application of trigonometric ratios in right-angled triangles
- Introduction to the unit circle
- Use of degrees and radians
- Definition of sine, cosine, and tangent as functions
- Graphs of trigonometric functions
- Transformation of trigonometric functions (amplitude, period, phase shift, and vertical shift)
- Solving trigonometric equations
- Real-world applications of trigonometry

The module emphasizes conceptual understanding and the ability to represent and interpret trigonometric functions in various forms. Students are required to demonstrate both procedural fluency and a conceptual grasp of how trigonometric functions behave under transformation. Additionally, PSTs are expected to engage with teaching strategies, design lesson plans, and reflect on common learner errors, thereby linking content mastery with pedagogical development (College Handbook, 2024).

Despite the intentions of the curriculum, evidence from student performance assessments and lecturer observations indicates that many PSTs struggle with trigonometric transformations, mirroring the performance trends observed among Grade 12 learners nationally (DoBE, 2024; Bowie & Frith, 2021). Common challenges include:

- Confusion between amplitude and period changes (Mason, 2010)
- Misidentification of phase shifts (Kastberg et al., 2020)
- Inability to interpret vertical translations in graphical form (Güven, 2006)
- Weak algebraic manipulation skills needed to rearrange trigonometric equations (Adas & Ubuz, 2013)
- A superficial understanding of the unit circle and radians (Trigueros & Ursini, 2003)

These challenges suggest that the conceptual difficulties learners face in high school are not being adequately addressed before or during their entry into teacher education programs. Instead, these issues are often carried forward, becoming entrenched in PSTs' own mathematical thinking and limiting their future effectiveness in the classroom. The result is a cyclical problem where teachers teach as they were taught, often relying on rote instruction and surface-level understanding.

1.3 Statement of the study

Mathematics, which is fundamental to scientific and technological knowledge, plays an important role in the progress of a nation's socio-economic development. The importance of mathematics has led researchers in education to explore the challenges faced by students and teachers worldwide in this field (Bansilal & Ubah, 2020). Bansilal and Ubah (2020) argue that a significant number of future teachers have inadequate mathematical skills, lacking comprehension in their chosen areas of study. The overall opinion of preservice teachers (PSTs) about trigonometry can differ. Some may perceive it as difficult and intricate, while others may consider it a significant mathematical concept that is crucial for understanding practical applications in the real world. Ultimately, individuals' prior knowledge, interest, and ease with mathematics play a role in shaping their perspective. Trigonometry is commonly seen as a challenging area of mathematics because it is perceived as inflexible, theoretical, difficult, and uninteresting to learn (Nabie et al., 2018).

Furthermore, studies on PSTs specializing in mathematics within the country reveal a concerning trend - a large majority lack a strong foundation in basic mathematical principles (Bansilal & Ubah, 2020). Preservice teachers who lack a solid understanding of mathematics will not be able to offer high-quality instruction that promotes a thorough comprehension of mathematical concepts. It is crucial for them to possess knowledge about various ways in which these concepts can be represented and the ability to establish connections between them. One such example is trigonometry, where problem-solving often necessitates the coordination of multiple representations. If confined solely to activities within one system, it may be impossible to reach a solution. These forms of representation are referred to as semiotic representations, involving the use of signs and symbols (Duval, 2006). In algebra, knowing the rules for adding variables allows you to solve any addition problem. However, in trigonometry, if you only know the formula $\tan(x) = \frac{\sin(x)}{\cos(x)}$ and are unaware that $\frac{1}{\cot(x)}$ is equivalent to $\tan(x)$, you won't be able to demonstrate that $\tan(x) + \frac{1}{\cot(x)} = 2\tan(x)$. This lack of understanding in trigonometry can affect how well learners and students perform when answering trigonometry questions.

The 2019 NDRLP provides an example of poor student performance on a trigonometry question in paper 2 of the NSC exam. The average score on this question was only 30%. According to the report, this may be due to students relying too heavily on past exam papers.

To improve performance, the report suggests focusing on strengthening content knowledge in trigonometry and exposing learners to complex problem-solving questions from an early stage in their education. Additionally, the report found that candidates struggled with understanding basic concepts related to graphs in trigonometry. It recommends that teachers expand their teaching of graphs beyond just sketching and instead provide exercises that require interpreting and extracting solutions from graphs.

Based on my extensive eight years of experience in the field of mathematics, specifically teaching first-year university students, we have noticed a recurring issue with their comprehension of trigonometric functions and their transformations. Despite being widely regarded as one of the most challenging subjects within the realm of mathematics, trigonometry possesses an abundance of practical applications across various disciplines. For instance, in the field of astronomy, trigonometry has long been used to accurately measure the distances between different stars within our galaxy. Similarly, in geography, this mathematical concept serves as a valuable tool for calculating the distances between landmarks. Moreover, trigonometry finds significant utility in satellite navigation systems, financial markets, electronics, medical imaging techniques such as ultrasound and CAT scans, and numerous other domains.

The wide-ranging applicability of trigonometry motivated me to delve deeper into its instruction and conduct research specifically focused on cosine and sine function transformations. These transformations are important in modelling real-life phenomena such as radio waves, tides, musical tones, and electrical currents. Recognizing that teachers' knowledge is frequently cited as a contributing factor to poor performance in mathematics by many researchers and educational stakeholders in South Africa (Bansilal et al., 2014; Brijlall et al., 2014; CED, 2011; Spaull, 2011; Taylor, 2011), we believe it is imperative to examine the challenges faced by senior-phase mathematics education students, who will eventually become future teachers within the FET band, when it comes to understanding trigonometric functions.

By exploring into these challenges faced by students aspiring to become mathematics teachers themselves, this study aims to provide a comprehensive understanding not only for them but also for teachers already working in the field and researchers investigating this topic. Furthermore, this research endeavour offers an opportunity for these prospective educators to address and rectify common mistakes made by learners, ultimately enhancing the overall quality of mathematics instruction.

1.4 Importance of the study

This study is important for several reasons. First, it addresses a significant gap in mathematics education in South Africa by focusing on the conceptual understanding of trigonometric function transformations among future mathematics teachers. Given the increasing demand for STEM-related skills and the widespread application of trigonometry in real-world contexts, strengthening the mathematical foundation of preservice teachers is not only an educational priority but also a national imperative.

The transformation of trigonometric functions, particularly, sine, cosine and tangent function, plays a critical role in mathematical modelling. Real-life phenomena such as sound waves, ocean tides, electrical currents, and radio signals all rely on these mathematical functions for accurate representation and analysis. A solid understanding of these concepts enables teachers to bridge the gap between abstract mathematics and practical applications, making the subject more meaningful and accessible to learners (Castro Superfine & Li, 2020; Oliveira & Costa, 2021).

Furthermore, the study has the potential to influence how mathematics teacher education is structured, particularly within the context of South African universities. By identifying specific conceptual difficulties and representational obstacles experienced by first-year PSTs, the research can inform targeted interventions, improved curriculum design, and enhanced teaching strategies in initial teacher education programs. Findings may support the integration of multi-representational teaching approaches, use of technology-enhanced visual tools (e.g., GeoGebra), and conceptual assessments that go beyond procedural proficiency.

This research also holds significance for educational policy and practice. As Bansilal and Ubah (2020) and Makgato and Ramaboka (2022) emphasize, the quality of teacher content knowledge directly impacts learner achievement. By strengthening PSTs' understanding of trigonometric transformations, we can indirectly improve teaching quality in the Senior Phase and the Further Education and Training (FET) band. This, in turn, can enhance national performance in mathematics and address ongoing equity and quality issues in South African education.

Finally, the study contributes to the body of literature in mathematics education research. While international studies have examined conceptual and representational challenges in trigonometry, there remains a paucity of research specifically focusing on South African PSTs and their engagement with function transformations. This study, therefore, not only addresses local educational needs but also adds a valuable perspective to global discussions about how best to prepare mathematics teachers in the 21st century.

1.5 Aim and Objectives of the study

The purpose of this study is to work diligently and make a strong effort to achieve a predetermined set of objectives that have been carefully planned and established. These objectives serve as the main goals for conducting this study, providing guidance to ensure that all efforts are concentrated on reaching these specific targets, which are:

- To determine the typical errors committed by prospective mathematics teachers while engaging with trigonometric function-related tasks.
- To use Duval's theory to explain why the preservice teachers make mistakes when working with problems related to trigonometric functions.

Knowing these mistakes and their potential origins in this research could assist educators in instructing students in a manner that enables them to avoid these errors or misunderstandings when faced with assessments.

1.6 Research questions

The research questions guided and controlled the study, acting as a compass to navigate through a vast amount of knowledge and information. The questions provided guidance and direction to achieve specific objectives. The questions that came up were:

1) *What errors do preservice mathematics teachers commonly make when working on tasks involving trigonometric functions?*
2) *How can Duval theory be used to explain these errors?*

Duval's theory of semiotic representations posits that mathematical understanding depends on the ability to interpret, transform, and coordinate between different semiotic representations, such as verbal, symbolic, graphical, and figural forms, rather than on manipulating symbols alone (Duval, 2006).

1.7 Methodological considerations

This study investigates the challenges encountered by first-year Senior Phase and FET mathematics PSTs in comprehending trigonometric functions, with a particular emphasis on the sine and cosine functions and their transformations. A qualitative case study methodology was adopted to explore these difficulties in depth, as this approach is well-suited for capturing complex cognitive and conceptual phenomena within specific educational contexts (Creswell & Poth, 2018; Yin, 2020).

Data were collected during the COVID-19 pandemic, necessitating the use of online activities and virtual semi-structured interviews as the primary modes of data gathering. These methods allowed for flexible, context-sensitive exploration of PSTs' reasoning processes and conceptual challenges in trigonometry, while accommodating health and safety restrictions. A sample of 125 PSTs from the University of KwaZulu-Natal was purposively selected for participation. Interviews were conducted with a subset of students who demonstrated low to moderate performance on diagnostic trigonometry tasks, enabling a focused investigation into common misconceptions and errors.

The study is grounded in Duval's (1983; 1993; 2006) theory of semiotic representation**,** which links mathematics learning with cognitive anthropology. This theoretical framework categorizes cognitive processing into three main representational registers: iconic (visual), enactive (action-based), and symbolic (algebraic/symbolic). According to Duval, mathematical comprehension is not limited to manipulating symbols but also involves coordinating between multiple representations to facilitate deep understanding. In the context of trigonometry, these representational systems are essential when exploring function transformations, graph interpretations, and symbolic manipulations (Brown, 2006; Demir, 2012; Martín-Fernández et al., 2022).

Trigonometric function transformations often require the learner to navigate across different representational systems, such as from symbolic to graphical forms. Limited fluency in these transitions can result in persistent misconceptions. Duval's framework was used deductively to analyse students' responses to activity tasks involving evaluation, simplification**,** and transformation of trigonometric expressions and functions. Interview questions were designed to probe these cognitive processes more deeply and were aligned with students' performance on the diagnostic activities.

The data analysis process involved both qualitative and basic quantitative techniques**.** Thematic analysis was used to identify recurring patterns, errors, and representational difficulties in student responses. Themes were then interpreted through the lens of Duval's semiotic theory and triangulated with relevant literature to enhance trustworthiness (Nowell et al., 2017). A deductive coding framework guided by theoretical constructs was employed to classify students' errors as either representational or procedural, (Cetin & Dubinsky, 2020).

This methodological approach provides a comprehensive view of the learning obstacles that PSTs face when engaging with trigonometric functions and their transformations. It further allows for an informed reflection on instructional practices that may support more effective representational reasoning in mathematics teacher education.

1.8 Ethical consideration of the study

To conduct the study, we had to first obtain ethical clearance from the Human and Social Sciences Research Ethics Committee at the University of KwaZulu-Natal (UKZN), under the protocol reference number HSSREC/00002036/2020. This official ethical approval confirms that the study fulfilled all the criteria stipulated by the university's policies for responsible conduct of research involving human participants.

Given the constraints imposed by the COVID-19 pandemic, particularly restrictions on physical gatherings, travel, and the enforcement of social distancing protocols, all data collection was conducted remotely. Online tools such as Zoom, Skype, and phone call were employed to carry out semi-structured interviews and administer diagnostic tasks related to trigonometric functions.

Prior to participation, all prospective participants received a detailed information sheet and consent form electronically.

Confidentiality and anonymity were maintained throughout the research process. Data were securely stored on password-protected devices and institutional cloud systems, accessible only to the researcher and supervisor.

The ethical design of the study ensured respect for participants' rights, minimized potential harm, and upheld the principles of beneficence, respect for persons, and justice, as outlined in the Belmont Report (National Commission for the Protection of Human Subjects of Biomedical and Behavioural Research, 1979) and the University of KwaZulu-Natal's research ethics framework (University of KwaZulu-Natal, 2023). The decision to conduct all interviews and tasks online not only complied with health guidelines but also ensured participants' safety and comfort during the research process.

1.9 Limitation of the study

While this study provides valuable insights into the challenges faced by first-year senior phase and FET PSTs in understanding the transformation of trigonometric functions, particularly sine and cosine, several limitations must be acknowledged.

Firstly, due to the COVID-19 pandemic and the resulting lockdown restrictions, all data collection was conducted online. While this ensured adherence to safety protocols and maintained participation continuity, the online format limited the researcher's ability to observe important non-verbal cues, spontaneous problem-solving behaviour, and in-person interaction, which are often significant in mathematics education research. In addition, challenges related to internet connectivity, unequal access to digital devices, and home learning environments may have influenced the quality of student engagement and the depth of their responses. A further limitation related to the online nature of the data collection process was that students may have shared their responses with their peers before submitting the tasks. This duplication may have resulted in the frequency of certain errors being amplified or reduced in the report.

Secondly, the study focused exclusively on first-year PSTs at the University of KwaZulu-Natal (UKZN). Although the sample size of 125 students provided a solid foundation for meaningful qualitative analysis, the findings may not be generalizable to other universities in South Africa or beyond, due to potential variations in student demographics, prior educational exposure, and institutional support structures. The specific context and selection criteria of the institution thus limit the broader applicability of the findings.

Another limitation arises from the study's narrow focus on the transformations of sine and cosine functions within trigonometry. While these are fundamental components of the topic, the research did not explore other crucial trigonometric concepts such as the unit circle, identities, solving trigonometric equations, or real-life applications. Consequently, the study offers a partial view of PSTs' overall understanding of trigonometry.

Lastly, the study employed a cross-sectional design, examining student understanding at a single point in their academic journey. As a result, the research does not provide insights into how PSTs' conceptual grasp of trigonometry may develop over time or in response to instructional interventions. A longitudinal study might have offered a more comprehensive perspective on the evolution of their mathematical understanding throughout their teacher education program.

Despite these limitations, the study contributes significantly to the field by identifying specific conceptual difficulties in trigonometric function transformations among future mathematics educators. It also highlights areas where targeted instructional strategies and curriculum development can improve the mathematical preparedness of PSTs, ultimately supporting the broader goals of improving mathematics education in South Africa.

1.10 Structure of the thesis

This thesis has of seven chapters. **Chapter 1** discusses the research background, with sections covering the overview and rationale for the study. It includes a review of previous research on mathematics performance and NSC Examination reports in South Africa to contextualize the problem. This chapter sets the stage for the study by outlining the research problem, purpose, and questions, as well as clarifying aims, objectives, and concepts.

The chapter begins by highlighting the critical role mathematics plays in the academic development and future opportunities of learners, particularly in gateway subjects such as mathematics, which directly influence access to higher education and employment. It draws attention to persistent national concerns over poor learner performance in mathematics, as evidenced in annual NSC reports, and identifies trigonometry, especially the understanding of trigonometric functions and their transformations, as a consistent area of underperformance. These issues are further examined through existing literature, shedding light on the nature and causes of students' misconceptions.

Furthermore, the chapter justifies the study's focus on senior phase PSTs, emphasizing their influence as future educators. It also explains the significance of exploring how these PSTs engage with foundational concepts like function transformations. This foundational chapter is essential in framing the theoretical, conceptual, and practical underpinnings of the study.

Chapter 2 presents a comprehensive review of the literature that forms the theoretical and empirical foundation of this study. It begins with an exploration of the history and definition of trigonometry, tracing the subject's origins from ancient civilizations such as the Babylonians, Egyptians, and Greeks to its development in Indian, Islamic, and European mathematics. The historical review underscores the role of trigonometry in solving practical problems in astronomy, geography, and architecture, while also examining how early definitions of trigonometric ratios and functions have evolved into the formal mathematical constructs used today. This historical grounding is essential, as it illustrates the progression of thought and notation that informs current mathematical practices and instructional approaches.

The chapter then moves into a detailed discussion of the evolution of trigonometry, emphasizing key conceptual shifts, such as the transition from geometric to analytic approaches and the formalization of functions. This evolution has influenced how trigonometric ideas are presented in modern education, and understanding these developments helps to clarify persistent student misconceptions. Following this, the chapter highlights the importance of teaching the history of mathematical concepts. It argues that historical context can help students make sense of abstract ideas, support conceptual change, and promote deeper engagement with mathematical content. Learning about the origins and evolution of mathematical ideas can demystify complex topics and show students that mathematics is a dynamic human endeavour rather than a static collection of rules.

A significant portion of the chapter is devoted to the concept of function, which is central to understanding transformations of trigonometric graphs. This discussion is organized into three key subsections. The first subsection, "Understanding the Concept of Functions: Historical Origins, Representations, and Educational Perspectives," explores the origins of the function concept, the various ways functions can be represented (e.g., algebraically, graphically, numerically), and the implications these representations have for teaching and learning. The second subsection, "Function in Algebra," discusses how functions are typically introduced and developed in algebra courses, emphasizing operations, composition, and transformations. The third subsection, "Function in Trigonometry," addresses how trigonometric functions are interpreted as mappings from angle measures to real numbers, and how these are graphed, analysed, and transformed in secondary and tertiary mathematics.

The literature review continues by examining common difficulties students experience in representing graphs of functions, particularly when shifting between different representations. Students often struggle with translating symbolic expressions into graphical or tabular formats, which affects their ability to interpret and analyse functions effectively. This challenge is especially pronounced in trigonometry, where understanding transformations, such as shifts, stretches, and reflections, requires fluency across multiple representational forms.

Further attention is given to the difficulties students face in comprehending trigonometric functions. Many students display limited conceptual understanding and rely heavily on procedural memorization. The chapter explores specific obstacles such as misconceptions about amplitude, period, and phase shift, as well as challenges in interpreting angle measures in degrees and radians. The confusion around radian measure is identified as a recurring barrier to deeper understanding.

The chapter also differentiates between the challenges faced by school learners and those encountered by university students and in-service teachers. While some misconceptions are common across levels, research indicates that even advanced learners and educators may struggle with foundational ideas in trigonometry, pointing to gaps in initial teacher education and curriculum design.

In conclusion, the chapter examines how students acquire trigonometric knowledge, and the role technology plays in this process. Dynamic software tools, graphing calculators, and visual aids have the potential to enhance understanding but may also introduce new misconceptions if not used thoughtfully. Finally, the chapter reviews various instructional approaches for teaching trigonometric concepts, including traditional methods, inquiry-based learning, visual strategies, and the use of real-world contexts, highlighting their strengths and limitations in addressing student difficulties.

Chapter 3 delves into the conceptual framework that underpins the analysis of data collected in this study. This framework provides a theoretical lens through which students' learning processes, errors, and conceptual challenges are interpreted. The chapter begins by examining how individuals acquire knowledge and develop cognitive abilities, particularly in the context of mathematics learning. It considers learning as an active, constructive process where learners build understanding by engaging with content through mental operations, experiences, and social interactions. Cognitive development is viewed as a gradual process influenced by both internal mental structures and external representations, such as symbols, diagrams, and language.

Central to this framework is Raymond Duval's theory of semiotic representation, which is used to explain how learners interact with and make sense of mathematical ideas. Duval argues that mathematical thinking is inherently dependent on the use of semiotic representations, such as algebraic symbols, graphs, tables, and verbal descriptions. According to Duval, understanding a mathematical concept involves more than just knowing its definition or computational procedures; it requires the ability to coordinate and convert between different types of representations. The failure to make these conversions effectively is often at the root of many conceptual misunderstandings in mathematics.

Duval's perspective on learning emphasizes the distinction between two key cognitive processes: *treatment* and *conversion*. Treatment refers to the manipulation of representations within the same register, such as simplifying an algebraic expression or shifting a graph along the x-axis. Conversion, on the other hand, involves translating information from one register of representation to another, for example, from a symbolic equation to a graphical representation. Duval argues that while treatments are often emphasized in traditional mathematics instruction, conversions are more cognitively demanding and are essential for genuine conceptual understanding. The inability to convert between representations can result in fragmented knowledge and surface-level comprehension.

The chapter further elaborates on the notion of registers of representations, which are systems or modes used to convey mathematical meaning. These include visual registers (like graphs and diagrams), symbolic registers (like algebraic notation), verbal or linguistic registers (spoken or written explanations), and numerical registers (tables or sequences of values). Each register offers specific affordances and constraints in expressing mathematical ideas. Duval contends that no single register is sufficient on its own to capture the full meaning of a mathematical concept; thus, learners must be able to navigate across multiple registers to achieve a deeper understanding.

In the final part of the chapter, Duval's theory is applied specifically to the study of trigonometric function transformations. Transforming trigonometric functions involves manipulating parameters such as amplitude, period, phase shift, and vertical shift, often represented algebraically, graphically, and descriptively. By using Duval's framework, the study aims to uncover the cognitive difficulties preservice teachers face when required to interpret, represent, and transform trigonometric functions across different semiotic registers. This theoretical perspective provides a structured approach to analysing errors and understanding the representational challenges that hinder students' mastery of trigonometric concepts. Ultimately, the framework supports a deeper, more nuanced interpretation of students' thinking and the learning barriers they encounter.

Chapter 4 presents a comprehensive overview of the research methodology employed in this study. It begins with an introduction that articulates the primary aim of the research: to investigate and understand the challenges encountered by first-year senior phase and FET PSTs in working with complex concepts related to trigonometric functions, particularly cosine, sine, and tangent. These trigonometric concepts are fundamental in senior phase and FET mathematics curricula, yet many learners and prospective educators continue to struggle with understanding their properties, applications, and transformations. This study seeks to uncover the roots of these challenges, focusing specifically on identifying both procedural errors and conceptual misunderstandings.

To achieve this objective, a qualitative research approach is adopted. This approach is well-suited to exploring the depth and complexity of participants' thinking, as it allows for a rich, contextualized analysis of how individuals interpret and respond to mathematical tasks. The study is positioned within an interpretive research paradigm, which emphasizes the importance of understanding how people make meaning of their experiences. In this context, the research aims not only to document what errors preservice teachers make but to gain insights into why these errors occur and how they reflect underlying cognitive and representational difficulties.

The chapter proceeds by outlining the overall research design and methodological orientation of the study. An exploratory case study design is employed to investigate the experiences of a specific group of PSTs within their authentic learning environment. This design is appropriate for uncovering detailed information about the specific educational context, participant thinking processes, and the teaching and learning dynamics at play.

The research sample is then described, including the selection criteria and demographic profile of the participants. This includes details about their academic backgrounds, prior exposure to trigonometry, and status within the university's mathematics education program. The chapter also outlines the various methods and tools used to collect data, including a combination of written tasks and interviews.

The data collection process is presented in two main components. First, the chapter discusses the administration of two mathematical activities, Activity 1 and Activity 2, designed to assess students' understanding of trigonometric graphs, parameters, and representations. Activity 1 focused on identifying parameters and interpreting graphs expressed in radians and degrees, while Activity 2 required graphical representation and verbal explanation of trigonometric functions. These tasks were crafted to reveal both procedural fluency and conceptual understanding.

Second, semi-structured interviews were conducted with selected participants to further probe their reasoning processes and clarify written responses. These interviews allowed the researcher to explore participants' thought patterns in depth and provided valuable qualitative data for analysis.

The chapter then details the data analysis procedures used, which were guided by Duval's theory of semiotic representation. This framework allowed for the identification of representational challenges and analysis of students' cognitive transitions across different mathematical registers.

Following the analysis section, the chapter addresses the issues of validity and reliability. Strategies to ensure trustworthiness include triangulation of data sources, member checking, and detailed documentation of procedures. The chapter concludes with a discussion of the ethical considerations observed throughout the study, including informed consent, confidentiality, voluntary participation, and the ethical handling of data.

Chapter 5 provides a detailed analysis and commentary on the data gathered during the pilot study. This preliminary investigation was conducted with the full approval of the Faculty of Education and aimed to explore the nature and frequency of errors made by first-year senior phase and FET PSTs at the University of KwaZulu-Natal. The data analysed in this chapter were obtained from a review of 85 examination answer books. These scripts were collected from a previous mathematics education exam, which included various topics, among them a range of trigonometry-related questions.

For the purposes of the pilot study, two specific trigonometric questions were selected from the broader exam paper. The first question focused on the identification and interpretation of vertical and horizontal shifts in the graphs of trigonometric functions, specifically cosine and sine. The second question required students to sketch the graphs of these functions based on their algebraic equations. These two questions were deliberately chosen because of their conceptual complexity and their alignment with the aims of the broader study, namely, to explore PSTs' understanding of transformations in trigonometric functions and to uncover common misconceptions that could inform both theory and practice in mathematics teacher education.

The chapter begins by presenting a brief overview of the pilot study, including its objectives, methods of data collection, and relevance to the larger research project. This section sets the context for the more detailed analysis that follows. The core of the chapter is divided into two major analytical sections. The first addresses how students identified vertical and horizontal shifts in trigonometric functions. It focuses on their responses to questions involving transformations of cosine and sine graphs and highlights various levels of understanding and error patterns, particularly in relation to the effects of coefficients and constants in the function expressions.

The second analytical section turns to students' attempts to sketch the graphs of trigonometric functions from their algebraic representations. This part of the analysis examines not only whether students could reproduce the correct shapes of sine and cosine graphs but also whether they could accurately apply amplitude changes, reflections, vertical and horizontal shifts. The analysis identifies a range of errors, from minor procedural slips to deeper conceptual misconceptions, and provides commentary on what these errors suggest about the students' grasp of function transformations.

The chapter concludes by discussing the foundational role played by the pilot study in shaping the main study. The insights gained from the pilot helped to refine the research focus, improve the clarity and structure of the main research instruments, and confirm the relevance of using trigonometric transformations as a lens for exploring PSTs' understanding. In sum, Chapter 5 not only documents the performance of PSTs on selected trigonometry tasks but also establishes a clear trajectory for the development and direction of the main study.

Chapters 6 and 7 present a detailed analysis of the data collected in this research study, with findings drawn from the participants' responses to the two assessment activities, Activity 1 and Activity 2, as well as from follow-up interviews. The analysis was structured to align closely with the objectives of the study, which aimed to identify and understand the nature of the conceptual and procedural challenges faced by first-year senior phase and FET PSTs when working with trigonometric functions.

In Chapter 6, the focus is on the data obtained from Activity 1. This activity was designed to assess participants' understanding of key features of trigonometric graphs, including the identification and interpretation of parameters such as amplitude, period, phase shift, and vertical displacement. The chapter begins by presenting a thematic analysis of the written responses provided by participants. These responses were then triangulated with insights gathered from the semi-structured interviews, which were conducted to gain a deeper understanding of participants' reasoning and to clarify any ambiguities present in their written work. Interview responses relating to specific questions from Activity 1 offered additional insight into participants' conceptual frameworks and highlighted the thought processes behind their approaches to solving the problems.

Chapter 7 continues the analysis by turning attention to Activity 2. This second activity required participants to graphically represent and interpret trigonometric functions, with a particular emphasis on function transformations. As with Chapter 6, written responses were analysed first, followed by a comparison with the interview data. The interviews in this phase were particularly useful for exploring how participants translated between different mathematical representations, symbolic, graphical, and verbal, and how they reasoned through the process of constructing and interpreting graphs of trigonometric functions.

Across both chapters, a rigorous qualitative coding process was applied to the data, leading to the development of categories for the different types of errors made by the participants. These categories emerged inductively from the data, based on patterns and commonalities observed in the participants' responses. The error types were further interpreted through the lens of Duval's theory of semiotic representation, which provided a framework for understanding the cognitive and representational difficulties underlying each category of error.

The analysis revealed a variety of challenges, including misinterpretations of parameter changes, confusion between degree and radian measures, difficulties in performing transformations, and struggles in shifting between symbolic and graphical representations. These findings were systematically organized into themes, each supported by excerpts from participants' work and corresponding interview responses. This dual-layered approach allowed for a nuanced understanding of how and why certain errors occurred, rather than merely documenting that they occurred.

Ultimately, Chapters 6 and 7 not only present the data and findings but also form the empirical backbone of the study. They illuminate the complex interplay between conceptual understanding, representational fluency, and pedagogical preparedness in the context of trigonometric functions, offering valuable insights for both mathematics education researchers and teacher educators

Chapter 8 presents a comprehensive analysis of the research results, focusing on participants' performance in the assessment activities and insights gathered from semi-structured interviews with PSTs. The chapter begins by detailing the findings of the study, carefully examining them through the dual lenses of existing literature and the study's guiding theoretical framework, particularly Duval's theory of semiotic representation. This analytical approach allows for a thorough interpretation of the results, enabling the identification of areas where the findings align with or diverge from those of previous research in mathematics education.

By situating the data within the context of related studies, the chapter provides a richer understanding of the challenges faced by PSTs when working with trigonometric functions and their transformations. It explores how errors and misconceptions observed in this study reflect broader trends in mathematics learning and instruction. The use of a theoretical lens ensures that the discussion is not limited to surface-level observations but instead delves into the underlying cognitive and representational issues that may contribute to the difficulties encountered by participants.

As the chapter progresses, it directly addresses the research questions posed at the outset of the study. Drawing conclusions from the analysed data, the discussion connects the empirical findings to the study's overall objectives. This section critically examines the specific types of errors made by participants, the nature of their misunderstandings, and how these relate to both procedural knowledge and conceptual understanding. The chapter highlights the importance of representational fluency, conversion between registers, and the role of semiotic reasoning in mastering trigonometric concepts.

The conclusion of Chapter 8 reiterates the key findings, emphasizing their significance in the broader context of mathematics education. It discusses how the identified challenges, such as difficulty interpreting parameter changes, confusion between radians and degrees, and struggles with graph-based reasoning, affect not only learners but also the development of future educators. These insights are positioned as important contributions to the field, especially in understanding the gaps in teacher preparation programs related to the teaching of trigonometric functions.

Importantly, the chapter acknowledges the limitations of the study. These include constraints related to sample size, contextual boundaries, and potential biases in data interpretation. By openly discussing these limitations, the chapter enhances the credibility of the research and allows readers to interpret the results with a clear understanding of the study’s scope and constraints.

In addition to interpreting the findings, the chapter explores their practical applications. It offers specific suggestions for improving the teaching and learning of trigonometric transformations, particularly in preservice teacher education. These include incorporating targeted activities that promote representational flexibility, using dynamic graphing tools, and embedding theoretical knowledge into practical teaching strategies.

Finally, the chapter outlines directions for future research. It identifies remaining gaps, such as the need for longitudinal studies to track the development of representational skills over time and investigations into how instructional interventions can address persistent misconceptions. By proposing avenues for continued inquiry, Chapter 8 contributes to the ongoing effort to enhance mathematics education and better prepare teachers to support student learning effectively.

CHAPTER 2
LITTERATURE REVIEW

Trigonometry is a branch of mathematics that deals with the relationships between angles and sides of triangles. It has proven to be extremely valuable in a wide range of disciplines, including Newtonian physics, astronomy, architecture, geography, financial market analysis, electronics, probability theory, medical imaging, and engineering. By utilizing principles and formulas from trigonometry, experts in these fields can gain deep insights into their respective subjects and understand even the most complex aspects within them. Trigonometry plays a crucial role in various applications such as researching celestial mechanics and movements, designing safe structures, and analysing complicated market trends. Its versatility allows for universally applicable fundamentals and accurate predictions across different phenomena. This makes it an essential tool for researchers and professionals alike who strive to uncover patterns and explore new dimensions in their fields. No other mathematical tool can match the current relevance and effectiveness of trigonometry when applied appropriately.

The focus of this chapter is a comprehensive examination of past research relating to trigonometric functions. Various topics are explored, including the nature and significance of mathematics, common errors and misconceptions in mathematics education, strategies for analysing mistakes made by learners during instruction, as well as challenges associated with teaching and learning mathematical concepts specifically related to trigonometry. The chapter also addresses gaps that exist within the current literature surrounding these areas.

2.1 History and definition of trigonometry

Trigonometry, a fundamental branch of mathematics, derives its name from the Greek words "trigonon" meaning triangle and "metron" meaning measure. Its origins are deeply embedded in the practical and observational needs of ancient civilizations, evolving over thousands of years from empirical problem-solving tools to an abstract mathematical discipline central to modern science, engineering, and education (Steckroth, 2007; Witzel & Van Brummelen, 2022).

The earliest concepts underpinning trigonometry can be traced back to the civilizations of ancient Egypt and Babylon. These cultures developed rudimentary trigonometric methods for architectural design, land measurement, and astronomical observation. In Egypt, for example, the construction of pyramids relied on practical understandings of angular inclination and shadow length, derived using a gnomon (See Figure 2.1), an upright stick whose shadow length varied with the position of the sun (Maor, 1998; Fauvel, 1991).

Figure 2. 1: Image of an ancient gnomon device

Similarly, the Babylonians made remarkable advances in astronomical calculations. They used a base-60 numeral system (sexagesimal), which enabled them to divide circles into 360 degrees, a system that remains in use today. Their innovations included the recording of arc and chord lengths to track celestial bodies, which played a critical role in predicting lunar eclipses and solstices (Van Brummelen, 2009; Witzel & Van Brummelen, 2022). These efforts signify the earliest forms of trigonometric thinking, deeply tied to practical needs and empirical observation.

The formal development of trigonometry, however, took a more systematic turn during the Greek period. The Greek astronomer Hipparchus (c. 190–120 BCE) is often credited as the "father of trigonometry" for compiling the first known trigonometric table based on chords in a circle. He introduced the idea of a circle divided into 360 degrees and used this framework to develop a method of calculating angles and arcs (Maor, 1998; Adamek et al., 2005). Hipparchus's primary motivation was astronomical: he sought to measure the angular positions of stars and planets with accuracy. His work also introduced the use of a coordinate system for celestial bodies, establishing a foundational link between trigonometry and astronomy (Sozio, 2005; Van Brummelen, 2009).

Subsequent Greek mathematicians, particularly Ptolemy, expanded on Hipparchus's work. Ptolemy's Almagest (circa 150 CE) contains a comprehensive set of chord tables and developed the basis for many classical trigonometric identities. These developments reflect a significant shift from purely empirical trigonometric uses to a more deductive, systematized science, blending geometry with numerical computation (Duval, 2006; Witzel & Van Brummelen, 2022).

Following the fall of the Greco-Roman world, the Islamic Golden Age (8th–14th century) became the centre of trigonometric advancement. Arabic mathematicians translated Greek texts and added substantial contributions of their own. Al-Battani, Al-Tusi, and Al-Khwarizmi introduced sine, cosine, and tangent functions and developed spherical trigonometry, essential for Islamic astronomy and navigation (Chevallard, 1992; Fauvel, 1991). They also developed algebraic techniques to work with trigonometric expressions, which later influenced European scholars during the Renaissance.

Trigonometry continued to evolve in the Indian subcontinent, where mathematicians such as Aryabhata and Bhaskara developed early versions of sine (jya), cosine (kojya), and versine (utkrama-jya) functions. Their approaches reflected a high level of abstraction and precision, contributing significantly to both astronomical models and mathematical theory (De Villiers, 2010; Joseph, 2020).

The term "trigonometry" itself first appeared in Latin as "trigonometria" in the 16th century, introduced by German mathematician Bartholomaeus Pitiscus. This was a period of intense mathematical development in Europe, aided by the introduction of the printing press, which allowed for wider dissemination of trigonometric tables and texts (Maor, 1998). European scholars such as Regiomontanus and Copernicus continued the legacy of their predecessors by applying trigonometry to new astronomical models that challenged Ptolemaic geocentrism.

During the 17th and 18th centuries, trigonometry began to detach from its purely astronomical roots and found increasing relevance in calculus and analytical geometry. Isaac Newton and Gottfried Wilhelm Leibniz incorporated trigonometric functions into their formulations of calculus, where they became essential tools for modelling oscillatory motion, waves, and harmonic analysis (Goldin, 2002; Eisenberg & Dreyfus, 1994).

In contemporary mathematics, trigonometry remains vital across multiple domains. It plays a central role in Fourier analysis, signal processing, electromagnetism, quantum mechanics, and GPS technology. Moreover, the transition from two-dimensional planar trigonometry to multi-dimensional vector and matrix-based approaches (such as linear algebra and spherical trigonometry) has expanded its applicability in computer graphics, robotics, and machine learning (Byers, 2010; Casula, Rangarajan, & Shields, 2021).

Recent scholarship also highlights the significance of integrating historical and cultural dimensions of trigonometry into educational settings. Using a historical approach not only improves conceptual understanding but also humanizes mathematics by showing its evolution across civilizations (Fauvel, 1991; Joseph, 2020; Eisner, 2017). This perspective is particularly valuable in multicultural classrooms where diverse learners can connect with the mathematical contributions of their ancestors.

Despite these advancements, the teaching and learning of trigonometry continue to present challenges. Research by Čižmešija and Šipuš (2013) and more recently by Chigonga (2016) and Demir (2011) highlights persistent student difficulties in understanding abstract concepts such as radian measure, function transformations, and inverse trigonometric functions. Many of these issues stem from inadequate representational fluency, the ability to move flexibly among symbolic, graphical, and numerical representations (Duval, 2006; Elia et al., 2007; Gould & Schmidt, 2010). As Duval's theory of semiotic representation suggests, students must coordinate between enactive (physical), iconic (visual), and symbolic (algebraic) forms of knowledge to master trigonometric reasoning.

Contemporary scholars like Witzel and Van Brummelen (2022) call for renewed emphasis on the historical and epistemological foundations of trigonometry to address pedagogical gaps. They argue that understanding how trigonometric ideas evolved can demystify abstract concepts for learners and provide a more coherent cognitive map. Moreover, advances in digital technology offer opportunities to simulate real-world applications of trigonometry in dynamic and interactive formats, enhancing student engagement and comprehension (Gould & Schmidt, 2010; Garofalo et al., 2000).

The South African educational context reflects similar challenges and opportunities. According to the Department of Basic Education (DoBE, 2022), many learners in the Further Education and Training (FET) phase struggle with trigonometry, especially in applying it to unfamiliar contexts. These challenges often persist into university-level teacher education, as preservice teachers (PSTs) enter with fragile conceptual foundations and limited exposure to multi-representational problem solving. Researchers such as De Villiers (2010) and Brijlall et al. (2012) emphasize the need for teacher training programs to integrate conceptual history and semiotic representation theory to address these gaps.

Moreover, curriculum designers are increasingly encouraged to draw from both indigenous knowledge systems and global mathematical traditions to make trigonometry more culturally responsive and contextually relevant (Choi-Koh, 2003; Joseph, 2020). For instance, using African cultural practices such as traditional surveying techniques or architectural forms may provide meaningful entry points into trigonometric thinking for local learners.

2.2 The evolution of trigonometry

The evolution of trigonometry is deeply rooted in human civilization’s need to solve both practical and abstract problems. From ancient times to modern applications, trigonometry has developed in response to challenges in architecture, astronomy, navigation, and more recently, in engineering, physics, and digital technologies. Its transformation over centuries exemplifies how mathematics evolves from real-world needs into abstract conceptual frameworks (Kennedy, 1991; Fuhrer, 1987; Van Brummelen, 2021). This historical and conceptual progression reveals the resilience, adaptability, and interdisciplinary value of trigonometric thinking.

Historically, the earliest roots of trigonometry were utilitarian. Ancient Egyptian and Mesopotamian civilizations employed geometric reasoning to survey land, construct pyramids and ziggurats, and align their architectural structures with celestial bodies. While these ancient cultures did not formalize trigonometry as a separate mathematical branch, they relied on implicit trigonometric concepts for measurement and design (Maor, 1998; Adamek et al., 2005). For example, the use of Pythagorean triples in building right angles reflects a primitive understanding of side ratios in right-angled triangles. In Mesopotamia, the Babylonians developed early tables of numbers that suggest familiarity with ratio-based computations.

The formal development of trigonometry began during the Hellenistic period with Greek mathematicians, especially in the field of astronomy. Hipparchus (c. 190–120 BCE), often referred to as the "father of trigonometry," introduced the chord function to measure angular distances between stars. This function, defined geometrically on a circle, was a precursor to the modern sine function (Kennedy, 1991). Ptolemy, in his seminal work *Almagest*, built upon Hipparchus's ideas and constructed extensive chord tables, which allowed for accurate astronomical predictions. His work emphasized the link between celestial mechanics and trigonometry, firmly establishing the discipline's importance in scientific inquiry.

Trigonometric knowledge spread across cultures, evolving through a process of translation, adaptation, and refinement. Indian scholars such as Aryabhata (c. 500 CE) and Bhaskara II (c. 1114–1185 CE) transitioned from chords to the more intuitive sine (jya) and cosine (kojya) functions. Their notational and conceptual innovations marked a significant shift, moving away from geometric dependence and toward algebraic formulation. These scholars also introduced fundamental trigonometric identities, including early versions of the Pythagorean identity, which were foundational for later mathematical advancements (Van Brummelen, 2009; Joseph, 2011). Importantly, these developments contributed to the abstraction of trigonometric concepts, paving the way for symbolic manipulation and broader applications.

he Islamic Golden Age (8th to 15th centuries) played a crucial role in preserving and expanding upon the Greek and Indian contributions. Scholars such as Al-Battani, Al-Khwarizmi, and Al-Tusi made systematic improvements in both theoretical and applied trigonometry. Al-Battani refined the sine and tangent functions and compiled highly accurate trigonometric tables, while Al-Tusi's *Treatise on the Quadrilateral* provided the first complete trigonometric formulations independent of Euclidean geometry (Kennedy, 1991). These advancements had a lasting influence, especially as Islamic mathematical texts were translated into Latin during the European Renaissance. The sophistication of Islamic trigonometry, including its use in spherical astronomy and cartography, was far ahead of its time and essential to the subsequent developments in Europe.

In medieval Europe, the reception of Islamic mathematical knowledge led to a renewed interest in trigonometry. The 12th-century translations of Arabic texts into Latin sparked intellectual revival, enabling scholars such as Regiomontanus (1436–1476) to compile the first European treatise on modern trigonometry. His work included sine and tangent functions as ratios rather than chords, standardizing their use in mathematical problem-solving. This shift was crucial for the integration of trigonometry into fields like navigation, engineering, and mapmaking. During this period, trigonometry became an essential tool in determining distances and angles across long journeys, a necessity for the Age of Exploration.

By the 17th and 18th centuries, trigonometry was fully integrated into the scientific revolution. Mathematicians like Isaac Newton applied trigonometric methods to develop theories of motion, gravitation, and optics. Leonhard Euler made one of the most significant contributions during the 18th century by connecting trigonometric functions with exponential expressions. Euler's famous identity $e^{ix} \cos(x) + \sin(x)$ bridged complex analysis and trigonometry, forming the basis for Fourier analysis, wave mechanics, and modern signal processing (Brannan et al., 2019). This marked a turning point where trigonometry transitioned from a largely geometric subject to an essential component of mathematical analysis and theoretical physics.

Despite its growing sophistication, the evolution of trigonometry was not always linear. Fuhrer (1987) and Van Brummelen (2021) both point out that knowledge was frequently lost, misinterpreted, or delayed during transmission between cultures and generations. Certain concepts had to be rediscovered or reformulated, highlighting the discontinuities in mathematical history. These interruptions, however, did not diminish the utility of trigonometry; rather, they underscore its adaptability across diverse intellectual traditions and practical needs.

In the 20th and 21st centuries, trigonometry has continued to evolve through its integration into digital technology, education, and interdisciplinary research. The advent of digital tools, such as graphing calculators, dynamic geometry software (e.g., GeoGebra), and computer algebra systems, has transformed the way trigonometry is taught and learned. These technologies allow for visual and interactive exploration of trigonometric concepts, helping students grasp ideas like amplitude, frequency, and phase shift with greater intuition (Thomas, 2020; Stylianides & Stylianides, 2021). Ferri and Ferrentino (2022) note that trigonometry is fundamental in modern computational tasks such as 3D modelling, image compression, and real-time rendering in computer graphics.

Real-world applications of trigonometry now extend far beyond astronomy and engineering. It is integral to fields such as signal processing, climate modelling, architecture, robotics, and machine learning. For instance, Fourier analysis, a powerful tool based on trigonometric decomposition, is used in sound engineering, medical imaging (like MRI), and telecommunications. These applications emphasize that trigonometric functions are not only mathematical abstractions but also essential tools in interpreting, modelling, and manipulating the world around us.

Educationally, the evolution of trigonometry has influenced pedagogical strategies. Traditional approaches based on memorization of identities and procedural manipulation are increasingly replaced by conceptually focused teaching that emphasizes understanding transformations, multiple representations, and contextual applications. Researchers advocate for instructional designs that use the unit circle, dynamic graphs, and real-life scenarios to enhance conceptual learning (Duval, 2006; Elia et al., 2007; Čižmešija & Šipuš, 2013). Brown (2006) stresses the importance of helping students perceive trigonometric functions as dynamic rather than static relationships, while more recent work suggests that real-world modelling can make learning more engaging and meaningful (Stylianides & Stylianides, 2021).

This shift has implications for curriculum design, particularly in preparing students for STEM-related fields. The Department of Basic Education (2022) in South Africa emphasizes a competency-based curriculum that incorporates trigonometric modelling, reasoning, and technological integration. Trigonometry now serves not only as a foundational mathematical topic but also as a bridge between arithmetic, geometry, and higher algebra, encouraging students to develop robust mathematical thinking skills.

In conclusion, the evolution of trigonometry is characterized by its responsiveness to human needs, its intercultural development, and its integration into both theoretical and applied domains. From the geometric insights of ancient civilizations to the symbolic formalism of modern mathematics, trigonometry has continually adapted to new challenges and technologies. Its enduring significance is evident in its educational value, practical utility, and capacity to evolve alongside scientific progress. As both a historical artifact and a living discipline, trigonometry offers rich opportunities for teaching, learning, and discovery in the contemporary world.

2.3 Importance of learning history of mathematics concepts

The history of mathematics enriches our understanding of its deep connection to social policy, cultural development, and educational practices, providing a valuable framework for teaching and learning (Fauvel, 1991). It highlights that mathematics is not a static collection of formulas and procedures, but a human endeavour shaped by the cultural, intellectual, and technological needs of its time. Understanding this historical trajectory allows learners to appreciate the subject as a living discipline that continues to evolve in response to changing societal needs.

Integrating the historical development of mathematics into teaching practices has been shown to enhance conceptual understanding and promote more meaningful engagement with the subject. Polya (1981) and Hull (1969) argue that teaching mathematical ideas through their historical development provides learners with richer insights into the logic and motivation behind mathematical discoveries. This approach also nurtures critical thinking, problem-solving, and an appreciation for the creativity involved in mathematical work. By contextualizing abstract ideas within a narrative of human discovery, educators can bridge the gap between procedural fluency and deeper conceptual understanding.

De Villiers (2008) provides practical frameworks and examples for incorporating history into mathematics education. His work demonstrates how historical anecdotes, primary texts, and contextual discussions can deepen students' understanding and inspire curiosity.

Teaching mathematics in this way promotes an appreciation not only of the logical structure of mathematics but also of its aesthetic and cultural dimensions. History humanizes the subject and opens possibilities for students to see themselves as part of the mathematical story.

Although current mathematics curricula often prioritize procedural fluency and content mastery, there is growing recognition of the need to integrate historical perspectives. The National Council of Teachers of Mathematics (NCTM, 1989) has long advocated for instruction that reflects the nature of mathematics as an evolving and human-centred discipline. They recommend the use of historical context to foster a more holistic understanding of mathematical concepts and to support the development of positive mathematical identities among students. This orientation encourages learners to move beyond rote memorization, developing instead a sense of the “why” behind the “how.”

Recent studies reinforce the pedagogical value of this approach. For instance, Tzanakis and Arcavi (2021) emphasize the importance of using history to promote meta-cognitive awareness and epistemological understanding in students. They suggest that historical narratives provide fertile ground for reflective thinking, allowing learners to question how knowledge is constructed, validated, and revised over time. This perspective is especially important in cultivating mathematically literate citizens who can critically engage with data and quantitative information in an increasingly complex world.

Incorporating history also aligns with contemporary educational goals related to equity, diversity, and inclusion. According to Swetz (2020), highlighting contributions from non-Western cultures—such as those of ancient Egypt, India, China, and the Islamic world—can help diversify the mathematical narrative and provide students from varied backgrounds with a stronger sense of connection and relevance. This broader historical lens counters the dominant Eurocentric portrayal of mathematics and underscores its truly global development. It affirms that mathematics is not the product of a single culture but a shared human endeavour with roots across civilizations.

From a cognitive standpoint, historical approaches have been found to enhance retention and conceptual transfer. Alpaslan and Yildiz (2022) report that learners exposed to historically contextualized mathematics instruction demonstrate stronger conceptual understanding and are more capable of applying mathematical ideas across contexts. Their research also suggests that historical integration can improve learners' attitudes toward mathematics, reducing anxiety and increasing motivation. Historical narratives seem to make abstract content more tangible and relatable, thus supporting long-term learning.

Furthermore, the use of historical problems and methods in instruction provides opportunities for developing problem-solving skills. For example, exploring how ancient mathematicians calculated areas, angles, or ratios without modern tools encourages learners to think creatively and reason logically. Liu (2020) argues that working with historical problems fosters mathematical reasoning and connects students with the epistemological roots of the discipline. This backward-looking exploration can stimulate forward-thinking insights by highlighting the trial-and-error processes behind mathematical advancement.

Digital technologies have also facilitated new ways of teaching the history of mathematics. Online platforms, simulations, and digital archives make it easier than ever to access historical texts, visualize ancient mathematical methods, and engage students through interactive storytelling (Clark-Wilson et al., 2020). These tools can help overcome the time and content constraints often cited by educators as barriers to integrating history into their lessons. Augmented and virtual reality applications now enable students to explore historical environments where mathematics evolved—bringing to life the contexts in which thinkers like Euclid, Al-Khwarizmi, or Aryabhata operated.

The accessibility of digital resources also contributes to differentiated instruction. Students with different learning preferences can engage with historical content through videos, interactive diagrams, or reconstructed ancient tools. As Arzarello and Robutti (2023) point out, technology-supported historical tasks support both visual and logical learners and enhance classroom discourse around historical mathematical practices.

Despite these benefits, challenges remain. Many teachers report limited training in the historical foundations of mathematics and often lack the confidence or resources to implement historically informed pedagogy effectively. As noted by Fauvel (1991), teacher education programs must prioritize this area to empower educators with both content knowledge and pedagogical strategies. More recent policy reports echo this call, advocating for professional development that equips teachers to use history not as an add-on, but as an integral part of their instructional design (Zengin, 2021).

Professional development initiatives such as the European History in Mathematics Education (HiME) project and national efforts in Turkey, South Africa, and China have begun to address this gap. For instance, Zengin and Aydın (2023) note that history-based teacher workshops led to significant improvements in preservice teachers' confidence and their ability to integrate historical material into classroom activities. Such training also fostered deeper pedagogical content knowledge, especially in complex areas like trigonometry and calculus.

Moreover, integrating the history of mathematics has implications beyond classroom learning. It supports the cultivation of mathematically literate citizens capable of understanding the social and ethical dimensions of mathematical applications. According to Jankvist (2015), historical inquiry helps students develop a sense of mathematics as a discipline that both reflects and influences broader societal changes. For example, learning about the development of logarithms in response to navigational challenges or the emergence of statistics during industrialization highlights mathematics' role in human problem-solving and innovation.

The COVID-19 pandemic underscored the importance of this broader understanding. As governments and health organizations relied on mathematical models to inform policy, public trust and comprehension became critical. Teaching the historical foundations of such models, how they evolved, were challenged, and revised, can equip students with the tools to critically engage with similar issues in the future. History, in this context, becomes not merely a pedagogical enhancement but a civic necessity.

The integration of history also promotes cross-disciplinary learning. Collaborations between mathematics and humanities departments, or the inclusion of historical texts in mathematics instruction, foster a more integrated and reflective curriculum. This aligns with global educational frameworks such as UNESCO's Education for Sustainable Development Goals, which encourage curriculum that connects knowledge, critical thinking, and cultural awareness.

In conclusion, embedding the history of mathematics into teaching practices is a powerful strategy for enriching students' understanding, promoting cultural appreciation, supporting conceptual development, and fostering critical engagement with mathematical ideas. While challenges related to teacher preparedness, curriculum time, and assessment frameworks persist, a growing body of research affirms the value of this approach. As mathematics education evolves to meet the demands of a complex and interconnected world, drawing on its rich historical legacy can help ensure that the subject remains meaningful, relevant, and human-centred.

2.4 Functions and definition

2.4.1 Understanding the Concept of Functions: Historical Origins, Representations, and Educational Perspectives

In mathematics, a function is a fundamental construct that assigns each element from one set (the domain) to exactly one element in another set (the range). Formally, if f is a function, and x is an element of its domain, then f(x) denotes the unique output associated with x. This mapping emphasizes a “one input, one output” relationship, a key tenet distinguishing functions from more general relations. Functions serve as an essential tool in abstraction: they model dependencies, between numbers, geometric constructs, or real-world phenomena. They can be described in multiple ways: algebraically (e.g. via formulae), graphically (e.g. plots), numerically (tables), and verbally (descriptions). Flexibly interpreting across these representations—symbolic, visual, discrete, contextual, is crucial for developing deep mathematic understanding (Duval, 2006; Elia et al., 2007).
According to De Villiers (2010), functions can also be represented using tables and graphs. However, these methods were not available during the time of the Greeks due to the absence of a coordinate system. This does not mean that the concept of a function did not exist at that time. Although it may not have been explicitly defined or formulated, there was certainly an intuitive understanding of it. Tall (1989) distinguishes between the concept image and concept definition, stating that Newton and Leibniz, for example, may not have had a formal definition for limits and functions but they still had a good understanding of them. Trigonometry was seen as part of geometry by Euclid in 300 BC. From the 1600s onwards, people struggled with reconciling trigonometric and algebraic functions. The late development of the coordinate system suggests that it was not an easy idea to conceive (De Villiers, 2010). The coordinate system originated from physics, mechanics, and astronomy.

As more problems involving periodic motion arose, the need for functions and further concepts in trigonometry became apparent.

According to De Villiers (2010), the idea of a function and its definition came about later, as people started applying mathematics and science to phenomena involving periodic functions. This indicates that the development of the function concept was driven by practical considerations. The need to formally define what a function has arisen from the increasing use of mathematical functions and calculus in scientific problems related to motion and forces, starting from the 1600s (De Villiers, 2010). This was made possible by advancements in algebraic symbolism and the Cartesian coordinate system, which simplified previous methods used by ancient civilizations like the Hindus, Greeks, and Arabs. However, this late formalization suggests that the concept of a function may be complex and profound. While we may have many characteristics associated with a particular concept, a definition only includes a small subset of those characteristics that are necessary and sufficient for defining the concept (De Villiers, 1984).

Functions are a fundamental concept in mathematics, playing a crucial role not only in education but also in various fields. They serve as a bridge connecting different areas of mathematical studies. In basic and secondary education, students are introduced to the foundational aspects of functions, beginning with linear and quadratic functions (Akkus et al., 2008; Elia et al., 2007). This early exposure helps build the necessary skills for more advanced studies in high school, where they encounter a variety of functions such as polynomial, rational, trigonometric, exponential, and logarithmic (Rocha, 2020).

In middle school, students start exploring functions through different representations, including numerical, graphical, and algebraic forms. These explorations often involve situations that depict linear or quadratic relationships. However, challenges arise when students must connect these diverse representations. They often struggle with interpreting graphs, manipulating symbols, and understanding the linkage between tables of discrete values and continuous graphs (Elia et al., 2007; Leinhardt et al., 1990; Viseu et al., 2022).

High school education aims to deepen students' comprehension of functions, enhancing their ability to work with diverse types of functions and understand their characteristics. At this stage, students learn to think about functions using different approaches and to describe their behaviour systematically. Furthermore, they are expected to grasp the terminology associated with functions, such as 'image', 'inverse image’, 'domain', and 'range'.

Understanding these terms allows students to conceptualize functions not just as processes, but as mathematical objects with distinct properties (Steketee & Scher, 2012).

Despite these educational efforts, some students find it difficult to interpret and apply the concepts related to functions. The transition from seeing functions as mere equations to appreciating them as dynamic relationships remains a critical challenge. Educators must focus on developing strategies that facilitate the integration of these numerous representations and terminologies, thus enabling students to make meaningful connections and gain a deeper understanding of mathematical functions (Markovits et al., 1988).

2.4.2 Function in algebra

In the domain of algebra, the concept of a function is foundational. A function is commonly defined as a rule that assigns exactly one output to each input, typically denoted as $f(x)$. It represents a formalized way to describe relationships between variables and plays a central role in expressing mathematical ideas, solving equations, modelling real-world phenomena, and connecting various mathematical domains such as geometry, calculus, and statistics. In school mathematics, students encounter functions through different representations: symbolic (e.g., $f(x) = 3x + 5$, graphical (plots on the Cartesian plane), numerical (tables of values), and contextual (situational or real-world problems). These representations offer multiple entry points for understanding and reasoning about functions. However, becoming proficient in interpreting and transitioning between these representations, a skill referred to as representational fluency, remains a persistent challenge for many learners.

The concept of function is often introduced in the intermediate grades, yet functional reasoning is a key component of secondary and post-secondary mathematics. Despite its critical importance, numerous studies have highlighted that students often struggle with functional thinking. Functional thinking goes beyond merely calculating function values; it includes recognizing patterns, understanding variable relationships, interpreting graphs and expressions, and making predictions. According to Ncube and Luneta (2024), learners tend to exhibit a fragmented understanding of function. While they may perform algorithmic procedures correctly, their conceptual grasp remains shallow. This gap results in difficulties applying function knowledge flexibly across novel problems or unfamiliar contexts.

One recurring issue is the over-reliance on symbolic manipulation without conceptual grounding. Students may learn to substitute and simplify expressions mechanically without understanding the underlying mathematical relationship. Stemele and Asvat (2024) found that many high school learners misinterpret symbolic function expressions, often confusing the independent and dependent variables or applying incorrect transformation rules. Their study suggests that deep-seated misconceptions, such as believing that every algebraic expression defines a function or misreading linear equations, stem from insufficient scaffolding and limited engagement with visual or tactile learning tools.

The challenges learners face with functions are not new. Seminal studies by Leinhardt, Zaslavsky, and Stein (1990) and Elia et al. (2007) emphasized the cognitive difficulty involved in transitioning between different representations of functions. For instance, students may understand a function algebraically but fail to connect it to its graphical representation or to interpret its behavior in a contextual setting. This disconnection is often exacerbated when representations are taught in isolation, without making their interconnectedness explicit.

One approach to mitigating these difficulties is the use of multiple representations in instruction. Research has consistently shown that engaging students with coordinated representations promotes deeper understanding. Duval's (2006) theory of semiotic representation stresses that true conceptual learning in mathematics requires the ability to convert meaningfully between different representational registers. This is particularly crucial in algebra, where moving from a symbolic form to a graphical one, or interpreting a table of values in relation to a function rule, is necessary for full comprehension.

To support learners in developing representational fluency, several instructional strategies have been proposed. One effective approach involves the use of visual and manipulative tools. For instance, algebra tiles help concretize abstract concepts like variables, expressions, and equations, enabling students to visualize operations and transformations. Dynamic software tools such as GeoGebra, Desmos, or AlgebraLab allow students to instantly see the effects of changing coefficients or constants in function expressions on their graphs. These tools provide immediate feedback and promote active exploration, which can strengthen both conceptual understanding and student engagement (Stemele & Asvat, 2024; Drijvers et al., 2019).

In addition to tools, task design plays a critical role in how students learn functions. Recent research by Mellor et al. (2024) underscores the impact of well-structured textbooks on student understanding. Their study found that resources integrating varied function representations, reasoning-focused tasks, and authentic real-world applications help bridge the gap between abstract reasoning and practical application. For example, embedding linear or quadratic functions in relatable contexts such as economics, biology, or architecture allows students to see the relevance of algebra and encourages transfer of learning to different domains.

Textbooks that fail to offer diverse representations or rely heavily on procedural drills may inadvertently contribute to misconceptions. Often, students are taught to "match the formula" without understanding why it works or what it represents. As pointed out by Stylianou and Silver (2004), procedural tasks alone are insufficient to develop higher order thinking or mathematical reasoning. Instead, instructional materials should foster opportunities for students to explain, justify, and critique mathematical ideas in multiple formats.

Curricula and teacher preparation programs must also emphasize the development of a conceptual foundation for functions. This includes teaching the formal definition of a function using appropriate mathematical language (domain, range, image, inverse) and ensuring that students understand both the formal and intuitive meanings of these terms. Ball, Thames, and Phelps (2008) argue that teachers need specialized content knowledge to explain such ideas clearly and accurately while making connections across topics. Teachers must also be trained to diagnose misconceptions, ask probing questions, and design learning sequences that gradually build students' understanding.

Furthermore, assessments should align with instructional goals that emphasize both procedural and conceptual understanding. Traditional tests that focus only on symbolic manipulation may reinforce narrow views of algebra. By contrast, assessment items that require students to interpret graphs, justify transformations, or model real-world situations using functions can reveal deeper levels of understanding. According to National Research Council (2001), such balanced assessment approaches are key to promoting meaningful learning and guiding instructional improvement.

Incorporating real-world contexts is another essential strategy for enhancing function learning. When students recognize how functions describe real-life phenomena, such as speed over time, growth rates, or cost-revenue relationships, they are more likely to engage with the content meaningfully. Research by White and Mitchelmore (2017) found that context-based instruction helps students develop a more flexible and intuitive understanding of function. It also supports the development of modelling skills, which are increasingly emphasized in modern curricula.

Language also plays a significant role in function learning. Many students struggle with the specialized vocabulary of functions, terms like “dependent,” “independent,” “slope,” “intercept,” and “rate of change.” Teachers must pay careful attention to these linguistic demands and support students in using precise mathematical language. According to Adler and Ronda (2015), language-sensitive instruction, particularly in multilingual contexts like South Africa, is crucial for equity and accessibility in mathematics learning.

Despite growing awareness of these pedagogical strategies, challenges remain. Time constraints, high-stakes testing, and rigid curriculum structures can make it difficult for teachers to implement deep and exploratory approaches. Moreover, many teachers report lacking confidence in their own understanding of functions, particularly in navigating between representations or applying functions in applied contexts (Ncube & Luneta, 2024). To overcome these barriers, professional development should include practical training on how to use manipulatives, interpret dynamic graphs, and construct meaning across representations. Programs that provide exemplar tasks, model lessons, and opportunities for collaborative planning can empower teachers to shift from a procedural to a conceptual emphasis in their instruction (Kazima & Adler, 2006; Drijvers et al., 2019). Continued research into classroom practice and curriculum innovation is needed to further refine strategies for supporting function learning.

The function concept in algebra is both foundational and complex. While students are typically introduced to functions early in their mathematical education, developing a robust and transferable understanding requires targeted instructional strategies. Research consistently shows that learners benefit from exposure to multiple representations, the use of visual and manipulative tools, meaningful real-world contexts, and language-rich environments. Teachers and curriculum designers must work together to ensure that function instruction balances procedural fluency with conceptual depth. By doing so, we can help students not only perform calculations but also appreciate the power of functions in describing and understanding the world around them.

2.4.3 Function in trigonometry

Trigonometric functions, namely sine, cosine, and tangent, are foundational concepts in secondary mathematics. Initially introduced through right-angled triangle ratios, these functions model relationships between the angles and sides of triangles. As students’ progress, the conceptualization of trigonometric functions expands through the unit circle model, allowing their domains to extend from acute angles in triangles to all real numbers on the Cartesian plane. This expansion marks a significant epistemological shift: from viewing trigonometric ratios as static geometric relationships to recognizing them as dynamic, continuous functions mapping real numbers to real numbers. Mastery of this transition is critical, not only for solving trigonometric equations and graphing periodic functions, but also for applying these concepts to real-world phenomena in physics, engineering, biology, and other sciences.

Trigonometric functions are particularly useful for modelling periodic behaviour, patterns that repeat over time or space. Examples include the motion of pendulums, the propagation of sound waves, the orbit of celestial bodies, and alternating electrical currents. In these contexts, functions like $\varphi(x) = \sin(x)$, $g(x) = \cos(x)$, or $k(x) = \tan(x)$ serve as abstract mathematical models describing real-world processes. Developing a coherent understanding of such functions requires students to appreciate their properties, transformations, and connections across multiple representations, including symbolic, graphical, numerical, and contextual.

However, a robust body of literature has shown that students face significant conceptual hurdles when learning trigonometric functions, particularly in transitioning from a geometric to an analytical understanding. Ncube and Luneta (2024) emphasize that many students persist in viewing sine, cosine, and tangent merely as fixed ratios within a triangle, rather than as dynamic functions whose outputs vary continuously with the input angle. This narrow perspective undermines their ability to work fluently with transformations such as phase shifts, amplitude changes, vertical translations, and period modifications, concepts essential for interpreting and graphing trigonometric functions accurately.

This misconception is often rooted in how trigonometry is introduced and sequenced within the curriculum. When learners first encounter trigonometry through right-angle triangles, they may develop a rigid association between the sine function and the opposite-over-hypotenuse definition, without recognizing that the same function can be extended to negative angles, angles greater than 90, or even radians on the unit circle. The shift from this static triangle-based perspective to the unit circle model is not intuitive and requires deliberate instructional support. As Elia et al. (2007) and Leinhardt, Zaslavsky, and Stein (1990) point out, learners often struggle to interpret or construct accurate graphs of trigonometric functions, in part because they fail to visualize how the rotation around the unit circle generates wave-like behavior in the graphs of sine and cosine.

The disconnect between geometric intuition and functional analysis is further exacerbated by how textbooks present the content. Mellor et al. (2024) found that many textbooks fail to adequately link graphical and algebraic representations of trigonometric functions. Instead of integrating explanations and activities that bridge symbolic expressions (e.g., $y = 2\sin(x - 90°)$ with corresponding transformations on a graph, textbooks often treat these as separate topics. As a result, students may learn the mechanics of function transformations procedurally, knowing how to shift or stretch a graph, without fully understanding why or how those changes occur in relation to the unit circle or real-world context.

Moreover, some learners find it difficult to reconcile their procedural fluency with conceptual understanding. For instance, they might apply transformation rules mechanically but lack the interpretive skills to analyse how these transformations affect the function's behaviour over its domain. Ncube and Luneta (2024) observed that even students who could correctly apply sine and cosine transformations on paper struggled to explain the impact of these changes on amplitude, frequency, or phase in real-life phenomena like waves and oscillations. This gap highlights the need for representational fluency, students' ability to translate and connect information across symbolic, graphical, and contextual forms.

Pedagogical strategies aimed at enhancing this fluency emphasize the use of dynamic visualizations, interactive tools, and contextual applications. Dynamic geometry software such as GeoGebra, Desmos, and PhET simulations enable students to manipulate function parameters in real time and observe immediate changes in the corresponding graph. This kind of visual feedback fosters a deeper conceptual understanding of function transformations and can correct misunderstandings more effectively than static diagrams (Clark-Wilson et al., 2020; Drijvers et al., 2019). For example, sliding the coefficient in front of the sine function visually demonstrates how amplitude increases or decreases, making abstract concepts more tangible.

Additionally, real-world contextual problems have been shown to improve student engagement and conceptual comprehension of trigonometric functions. Modelling periodic phenomena such as tides, heartbeat rhythms, sound waves, and seasonal temperature fluctuations allows students to see how mathematical functions describe and predict patterns in the natural world. According to Kerslake and Aizikovitsh-Udi (2022), integrating such contextual examples not only improves comprehension but also enhances student motivation and interest, making abstract mathematical ideas feel relevant and applicable.

Recent curriculum reforms advocate introducing the functional perspective of trigonometric concepts earlier in the learning sequence. Instead of delaying function-based interpretations until after students master triangle ratios, educators are encouraged to present the unit circle, radian measure, and periodic function behaviour alongside or even before geometric definitions. This approach promotes continuity between different branches of mathematics and supports students in developing a unified view of functions. According to Toh et al. (2021), students who learn trigonometric functions first as mappings of real numbers before delving into triangle-based ratios, exhibit stronger reasoning skills and are better able to interpret graphs and apply transformations.

Further support for this curriculum shift comes from a conceptual change framework, which suggests that students need to reorganize their mental models to accommodate new and more complex interpretations of familiar ideas (Vosniadou, 2013). Teaching strategies that confront and replace misconceptions, such as contrasting static triangle ratios with dynamic functional mappings, are more likely to foster long-term conceptual change. Teachers can support this process by using language that highlights the dynamic nature of functions (e.g., "input-output relationships" or "changing values across a domain") and by scaffolding tasks that require students to coordinate geometric and functional perspectives.

Despite these promising approaches, significant challenges remain. One major barrier is teacher preparedness. As highlighted by Zengin (2021), many mathematics teachers have not received sufficient training in how to teach trigonometric functions from a functional perspective. Their own learning may have been grounded in rote memorization of triangle ratios, leaving them ill-equipped to guide students through the conceptual shift to dynamic functions. Teacher education programs must therefore prioritize the development of pedagogical content knowledge (PCK) related to functions, particularly the use of representations, technological tools, and applications in real-world contexts (Ball et al., 2008).

Moreover, assessments need to reflect the multifaceted nature of trigonometric function understanding. Traditional tests that focus on solving equations or applying identities do not necessarily capture students' representational or conceptual understanding. Assessment tasks should include graph interpretation, transformation analysis, and modelling scenarios, ensuring alignment with instructional goals. As National Research Council (2001) recommends, assessments that emphasize understanding, reasoning, and application provide more accurate insights into students' learning and better guide future instruction.

Mastering trigonometric functions as mathematical functions is a complex yet essential goal in secondary mathematics education. While the traditional geometric approach provides a useful entry point, it is insufficient for developing the functional reasoning necessary for higher-level mathematics and real-world problem-solving. By emphasizing representational fluency, integrating dynamic technology, incorporating contextual problems, and revising curricular sequencing, educators can support learners in making the critical transition from static triangle-based thinking to dynamic functional understanding. Continued efforts in teacher professional development, curriculum design, and assessment reform are needed to ensure that trigonometric functions are taught not only as useful mathematical tools but as windows into the deep and elegant structure of mathematical relationships.

2.5 Difficulties of representing a graph of a function

Mathematical representation involves changing a problem into different forms such as graphs or models. This helps students effectively communicate their ideas or solutions (NCTM, 2000). Representation ability is crucial for several reasons. Firstly, it forms the foundational skill necessary for constructing mathematical concepts and fostering mathematical thinking. Secondly, it is essential for achieving a deep understanding of concepts and serves as a valuable tool in problem-solving. Representation involves various configurations like forms, characters, symbols, or objects, which can effectively describe or symbolize other forms (Goldin, 2002). Graciella and Suwangsih (2016) identified three types of representation in mathematics: symbolic, visual, and verbal. These forms can be demonstrated through actions, graphs, words, and language. Similarly, Alhadad (2010) discussed the ability to use and translate between these various mathematical forms to explain ideas and interpret phenomena, utilizing visual, symbolic, and verbal means.

Martin et al. (2023) explored the challenges faced by grade 10 students when studying functions, particularly injective, surjective, and bijective ones. In a study involving 27 students, it was observed that while they could easily connect independent and dependent variables using arrow diagrams, they found it difficult to derive necessary information from graphs to establish these connections. This difficulty aligns with the broader challenge learners face with abstract graphical analysis. Furthermore, the study highlighted that students often misinterpret algebraic expressions, leading to incorrect domain identification. Interestingly, some students managed to overcome these issues when functions were visually represented. However, graph transformations presented additional struggles, as students found it hard to interpret and manipulate information to solve these tasks. Describing changes in variable behaviour and understanding graph variations (horizontal and vertical shifts) were particularly tough areas. The findings indicate that difficulties in understanding functions vary with representation type. Teachers should note these differences when designing educational tasks. Although students showed a preference for graphical representations, the study found them more challenging than arrow diagrams, emphasizing the necessity of focusing on enhancing students' functional thinking and their ability to work with different representations.

For students to effectively understand functions, it is essential that they engage with problems requiring transitions between algebraic, numeric, and graphic representations. Cunningham (2005) highlights that students often struggle with these transfer problems, possibly due to current instructional practices. From a survey conducted among 28 algebra teachers, it was found that minimal class time is dedicated to graphic-to-numeric transfer problems compared to other types. Additionally, these types of problems are less frequently included in assessments. Cunningham (2005) pointed out that these tasks are precisely the ones students find most challenging. It is believed that this imbalance might stem from teachers' assumptions about their own comfort levels with these problems, coupled with a belief in student proficiency. Teachers might overestimate both their own and their students' abilities to handle these tasks, leading to less emphasis in the classroom. By increasing focus on these transfer problems, educators could significantly enhance students' comprehensive understanding of functions. Addressing this gap through balanced instructional practices could help students become more adept at handling varying representations of functions, thereby improving their overall competency in algebra (Cunningham, 2005).

Tall and Bakar (1992) investigated how English A-level mathematics students understand the concept of functions, revealing significant gaps between practical application and theoretical understanding. Their study found that students often rely on familiar examples like $p(x) = x^2$ or $k(x) = \sin(x)$ as prototypes, which reflect everyday ideas rather than formal definitions. This reliance can lead to misconceptions, as students often overlook the rigorous definitions presented in educational curricula. For instance, the study revealed that 75% of incoming university mathematics students did not recognize a constant as a function and erroneously identified a circle as a function. These errors highlight a significant disconnect between what educators intend to teach and what students learn. The research suggests that students' understanding is largely grounded in the properties of common prototypes rather than theoretical definitions. This highlights the importance of curricula addressing these gaps and emphasizing a more thorough comprehension of mathematical concepts. By bridging the divide between practical examples and formal definitions, educators can help students develop a more accurate and comprehensive understanding of functions.

Mudaly and Rampersad (2010)'s study with Grade 11 learners in South Africa highlighted significant gaps in students' comprehension of graphical representations of functions. The research demonstrated that students leaned heavily on procedural knowledge when explaining fundamental concepts, revealing their reliance on classroom diagrams without a deeper understanding of the visuals. This dependency indicated a deficiency in visualization skills, which are crucial for grasping more complex mathematical ideas. The study underscored that without strong visualization abilities, students struggled to develop a profound conceptual understanding of functions, relying instead on rote learning and procedural steps that lacked deeper insight

Given these findings, Mudaly and Rampersad (2010) recommended that educators should place greater emphasis on enhancing visualization techniques in their instruction. By doing so, teachers can help students move beyond mere procedural knowledge to a more holistic understanding of functions. Integrating more dynamic and interactive methods for teaching graphical representations can engage students visually and conceptually, allowing them to make connections between abstract concepts and their graphical counterparts. This approach not only aids in grasping the current curriculum but also equips students with improved problem-solving skills applicable to real-world situations. Prioritizing visualization in education is thus essential for nurturing students’ comprehensive understanding and appreciation of mathematics.

Therefore, mathematical representation is important for understanding and problem-solving as it transforms problems into graphs or models. Key representation types include symbolic, visual, and verbal. These studies reveal challenges students face in switching between these forms, particularly with functions. Misinterpretations often arise in graphical analysis and algebraic expressions, highlighting gaps in educational practices. This research suggests that minimal focus on graphic-to-numeric tasks in teaching contributes to these issues. Reliance on procedural knowledge further impedes comprehension. Emphasizing visualization techniques and balanced instructional practices can boost students' functional understanding and problem-solving skills, bridging gaps between practical and theoretical knowledge.

2.6 Challenges in working with various representations of functions

Teachers should equip themselves with diverse methods of representation, as no single approach is universally effective in addressing the varied learning needs of students (Shulman, 1986). According to Stylianou (2010), a representation is a configuration that stands for something else, while Goldin (2002) warns that representations need to be viewed as parts of broader representational systems. Stylianou (2010) further explains that representation goes beyond mere imitation; it involves the innovation or adaptation of conventions within a representational system tailored to the specific task or purpose. Moore-Russo and Viglietti (2012) highlight that understanding a concept involves the interaction between personal mental representations and external physical representations. They argue that having a solid grasp of a concept requires access to diverse semiotic resources. This variety enables individuals to select the most suitable representation for a given context, effectively illustrating specific properties. Such an approach ensures a deeper comprehension and flexible application of concepts across different situations, demonstrating a well-rounded understanding.

The study by Brijlall et al. (2012) highlights the multifaceted understanding teachers have regarding mathematical representations. In the study, teachers recognized that these representations aid in clarifying concepts, expressing mathematical ideas, and depicting real-life situations. They also acknowledge that using various representations enhances comprehension of interrelated concepts. However, the study points out that recognizing the importance of diverse representations is only part of the solution. For teachers to effectively support students' access to various mathematical representations, they must themselves be fluent in these representations. Their ability to seamlessly navigate and integrate multiple forms of representation in teaching is crucial for fostering a deeper understanding among learners. Hence, the emphasis is not just on awareness but also on the practical application and fluency of teachers in using these diverse mathematical representations to enrich the learning experience.

Working with a variety of representations in mathematics opens doors to more complex and enriching activities across different systems. This is especially true in fields like trigonometry and geometry, where solving problems often necessitates coordinating multiple representational formats. Sticking to just one system may limit the ability to find a solution. While changing representations isn't always mandatory, engaging in activities that require shifts between them fosters a deeper understanding and aids in the conceptualization of mathematical ideas (Ubah & Bansilal, 2018). By moving between different forms, learners can gain a more comprehensive perspective. This ability to transition from one representation to another is frequently a crucial step in advancing mathematical understanding. Through experience in translating between various representational systems, students build a stronger, more flexible grasp of concepts, enabling them to tackle complex problems with greater confidence and skill (Duval, 2006).

Bansilal and Naidoo (2012) study highlights the challenges faced by Grade 12 students when working on transformation geometry problems, particularly emphasizing the disconnect between analytic and visual strategies. The students were proficient in utilizing algebraic rules and calculations in the analytic mode, often bypassing visualization altogether. This reliance on algebraic methods seemed to hinder their ability to effectively visualize transformations, making it difficult for them to grasp which properties of figures were preserved during these transformations. This difficulty underscores the importance of incorporating visualization skills in geometry education, as visualization plays a crucial role in understanding geometric transformations. The authors argue that interacting with different representations of a geometric object can enhance understanding by emphasizing various properties and relationships inherent in the object. This approach aligns with the belief that exposure to multiple representations enriches learners' conceptual knowledge and aids in deeper comprehension.

Further expanding on representation, Bansilal (2012) explored teachers' ability to solve problems related to the normal distribution curve, drawing on Duval's theory of transformations within and between semiotic representation systems. The research revealed that teachers were more successful in solving problems when a single representational system was used compared to tasks requiring the coordination of multiple systems. This finding suggests that the complexity of moving between different representational modes can be a barrier to problem-solving proficiency. Together, these studies underscore the critical role of representation and visualization in mathematical learning and teaching. They highlight the need for educational strategies that integrate both analytic and visual approaches, enabling students and teachers to navigate and understand complex mathematical concepts more effectively. Encouraging the development of skills in both areas could foster more comprehensive mathematical literacy, ensuring that learners not only apply algebraic techniques but also appreciate the geometric relationships they represent.

Mkwanazi et al. (2023) examined the conceptual challenges faced by South African high school mathematics teachers in trigonometry and coordinate geometry, highlighting difficulties in working with dual representations. The study involved 236 in-service teachers enrolled in a university program. Results showed that teachers performed well on trigonometric rules but struggled significantly with sketching and interpreting trigonometric graphs. The task of coordinating graphical and algebraic representations to find the equation of a line proved particularly challenging. The primary issue appears to be the teachers' difficulty in transitioning smoothly between algebraic and graphical representations. While memorizing rules is key, the real challenge lies in understanding the underlying concepts that connect these representations. For example, recognizing that parallel lines in a graph correlate with equal slopes in an algebraic formula goes beyond simple memorization. It requires an understanding of the patterns and relationships that transcend the specific representations.

Duval 's (2006) work emphasizes the importance of effectively converting between representations and choosing the right mode for problem-solving. Developing this skill is essential for both teachers and students. If teachers struggle, it reveals a gap in conceptual understanding rather than procedural ability. This gap can be identified in the study by Bansilal and Naidoo (2012), where learners had difficulty recognizing invariance under geometric transformations like reflections and rotations. This is often due to a focus on procedural knowledge over an intuitive grasp of concepts. A central pedagogical need emerges, encouraging the ability to translate between different representations and understand their connections. Visualization, geometric transformations, and algebraic contexts can be integrated to bridge this gap. Teachers could benefit from exploring concepts through varied perspectives-algebraic, graphical, and geometrical.

Improving mathematical instruction might require a dual approach: enhancing teachers' understanding of conversion skills and designing student learning experiences that challenge them to engage with multiple representations. By developing these cognitive skills, educators can foster a deeper and more flexible understanding of mathematics, transcending mere procedural fluency. Such an understanding empowers both teachers and students to tackle complex problems and identify connections across different representations (Bansilal and Naidoo, 2012).

A holistic approach can significantly contribute to more effective mathematics education and improve problem-solving abilities for both students and educators. By focusing on conceptual understanding, the ability to adeptly switch between representations is enhanced, ultimately leading to a more comprehensive grasp of mathematics and its applications. This effort will not only address current challenges but also pave the way for sustained improvement in mathematical competency.

2.7 Student’s comprehension of trigonometric functions

Varied methods of teaching trigonometry are found in different countries, including Canada, the United States, Australia, the United Kingdom, and South Africa (De Kee et al., 1996; Satty, 1976; Willis, 1966; Collins, 1973). Historically, trigonometry was introduced using ratios and right-angled triangles. However, in the 1960s, with the advent of "new Mathematics," there was a shift towards a functional approach with emphasis on the unit circle (Jugmohan, 2004). This approach became dominant to the point where it is often referred to as the "unit circle approach" rather than the functional approach (Pournara, 2001). Initially, trigonometric functions were taught as ratios of sides in a right triangle. Later (around the early 1960s), a modern method was introduced that defined trigonometric functions as x and y coordinates on a unit circle (Trende, 1962; Willis, 1966). Some textbooks focus exclusively on one method, while others try to incorporate both methods

Figure 2.2 encapsulates the core ideas and extended concepts in trigonometry, providing a comprehensive overview of the subject. At its core, trigonometry revolves around the study of triangles, particularly right-angled triangles, and the relationships between their angles and sides. These fundamental relationships are defined through trigonometric ratios: sine (sin), cosine (cos), and tangent (tan). Each ratio represents a specific relationship between the sides of a right-angled triangle: sine is the ratio of the opposite side to the hypotenuse, cosine is the ratio of the adjacent side to the hypotenuse, and tangent is the ratio of the opposite side to the adjacent side.

Extending beyond right-angled triangles, trigonometry delves into unit circles, where the radius is one unit, allowing for a seamless transition to the study of angles in any quadrant. This extension introduces the concept of radians, an alternative to degrees for measuring angles, thereby linking trigonometry more closely to calculus and advanced mathematics.

Furthermore, Figure 2.2 touches upon the inverse trigonometric functions, which are used to find angles when the values of the trigonometric ratios are known. These are essential for solving a variety of real-world problems. Additionally, the figure illustrates the trigonometric identities, such as the Pythagorean identity, angle sum and difference identities, and double-angle formulas, which are indispensable tools for simplifying and solving trigonometric equations.

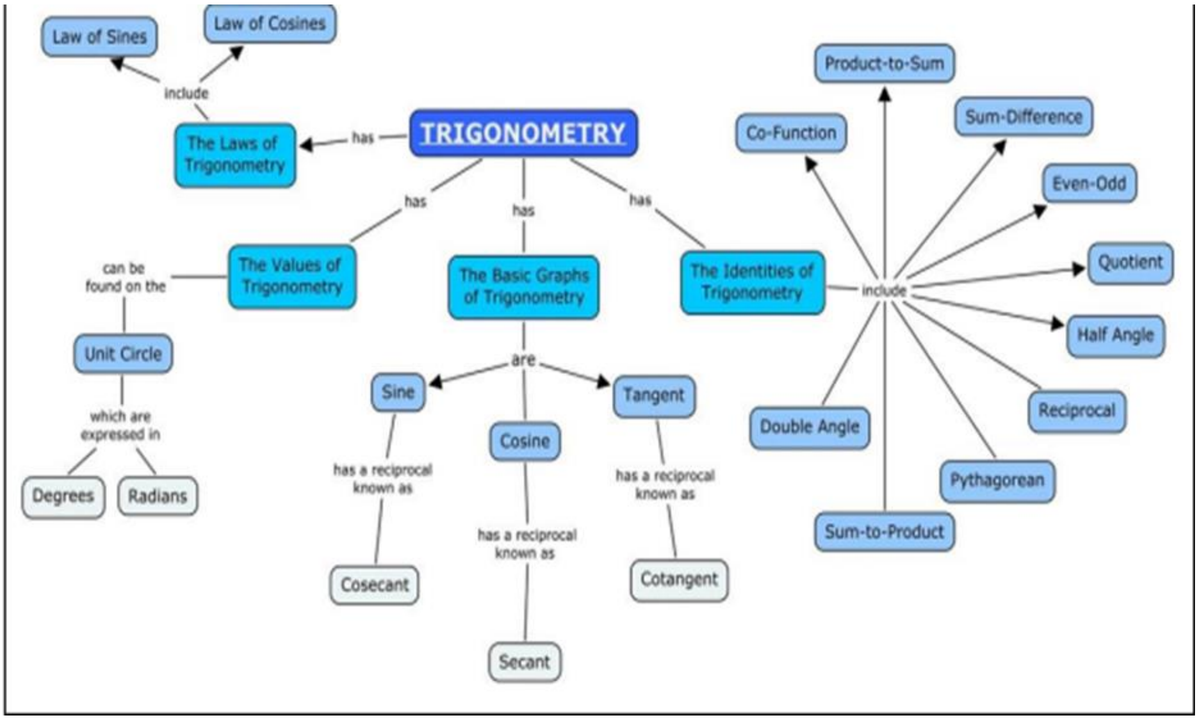


Figure 2. 2: A mind map for trigonometric concepts (Chigonga, 2016)

Weber (2005) conducted a study to understand how well students comprehended trigonometric functions. He compared two different teaching methods: experimental instruction and lecture-based instruction. The experimental instruction was founded on Gray and Tall's (1994) concept of procept, which involves using processes, mathematical objects, and symbols to represent them. This approach aimed to engage students in activities such as constructing the unit circle and drawing angles with their corresponding trigonometric values. In contrast, the lecture-based approach focused more on teaching procedural skills through explanations and examples from textbooks. After completing the procedures, all students were asked questions that required them to compare trigonometric values or explain certain concepts related to them. The questions included:

- Which is bigger sin 37° or sin 23°? Explain why.
- Without measuring, estimate the sin of 170°.
- Is sin 145° positive and why?
- Explain why sin θ can never equal 2.

The study's findings revealed that the traditional teaching method did not effectively help students grasp trigonometric functions. None of the students were able to explain why sin(θ) could never be 2, and they struggled to understand different properties of trigonometric functions or make reasonable estimates of their values. Despite being reminded of the definition of functions, none of the students could provide a satisfactory explanation for why sine is considered a function. Similar results were reported in other studies by Challenger (2009), Gür (2009), and Marchi (2012), which found that high school students taught through traditional lecture-based courses had a fragmented understanding of trigonometric functions.

In Weber's (2005) study, it was observed that the students in the experimental group were able to approximate values of basic trigonometric expressions and analyse properties of trigonometric functions. Specifically, around 30 out of 40 students could explain the sine function as a process involving an input and output quantity. They could also provide explanations for why these functions possessed certain characteristics. Weber further discovered that the experimental group understood the process of drawing angles and defining their sine values, recognizing that each angle corresponded to a unique point of intersection with the circle. As a result, he concluded that these students had a solid grasp of trigonometric functions due to their ability to visualize geometric processes associated with obtaining these values in relation to the unit circle. Like Moor's findings in 2010, Weber highlighted the significance of the unit circle representations in enhancing students' comprehension of trigonometric expressions. However, it is worth noting that these results contradict Kendal and Stacey's study from 1997.

Kendal and Stacey (1997) conducted a study with 178 high school students to evaluate their understanding of trigonometric functions using two different methods: the ratio method and the unit circle method. The findings revealed that students had a better grasp of trigonometric functions when they were taught using ratios of sides in right-angled triangles, such as the commonly used memory aid SOHCAHTOA, rather than through the unit circle method. These results align with Palmer's (1980) study, where students were randomly assigned either ratio or unit circle instruction, and the ratio group demonstrated superior comprehension of trigonometric functions compared to the unit circle group. Burch (1981) also supported these conclusions by noting that students struggled with interpreting trigonometric functions using the unit circle, as it involves associating x and y coordinates on the unit circle with cosine and sine values of corresponding angles, whereas other trigonometric functions can be expressed in terms of right triangles.

According to Brown (2005), Demir (2011), and Marchi (2012), students would have a better understanding of trigonometric functions if they had more opportunities to learn using both the unit circle and right triangles, rather than focusing on only one method. To investigate high school students' comprehension of trigonometric functions, Brown (2005) developed a model that incorporated both the geometry of triangles and angles in degrees, as well as the context of the unit circle. In this model, the sine and cosine of an angle could be defined in three ways: as ratios, distances, or coordinates. After studying 120 high school students, Brown (2005) discovered that those who could utilize and connect all three interpretations were able to define trigonometric functions effectively and solve problems proficiently. However, it was noted that most students had difficulty making connections between all three representations and instead relied on just one perspective. Therefore, Brown concluded that many students possessed an incomplete understanding of trigonometric functions.

Demir (2011) made modifications to Brown's learning model for trigonometric functions due to students' lack of understanding. He introduced a new learning trajectory involving three different representations: the right triangle, the unit circle, and the graph. Demir conducted a study with 24 students aged 16-17 and found that this new approach supported students' comprehension in various aspects. They were able to understand the coordinate definitions of sine and cosine in the context of the unit circle, make connections between the unit circle and triangle contexts, and interpret trigonometric graphs accurately. However, despite these improvements, some students still struggled to explain why sine and cosine are considered functions, which contrasts with findings from Weber (2005) and Challenge's (2009) studies that highlighted similar issues among students.

Few studies have been conducted on students' understanding of transformations of sinusoidal functions. Masomeh (2017) conducted a study using two tasks to investigate this topic. The tasks involved recognizing sinusoidal functions and assigning coordinates to these functions. The study was carried out with a sample of seven undergraduate students enrolled in Calculus I and Calculus II at a North American university. Three of these students had passed the Calculus I course, which covered transformations of trigonometric functions. Semi-structured interviews were used to collect data, and all tasks were presented using the Dynamic Geometry software, Sketchpad. The researcher discovered that students had different ways of recognizing periods and transformations. When the coefficients of x in the sinusoidal function were whole numbers, students were able to match algebraic representations with graphical representations. However, they struggled to perform the same task when the coefficients of x were fractions.

In Africa, as well as in other parts of the world, trigonometry has proven to be challenging for students at all levels. A study conducted by Nabie et al. (2018) focused on 119 PSTs from two Colleges of Education in Ghana's Northern Region. The researchers used the Trigonometry Perception Questionnaire (TPQ) and the Trigonometry Assessment Test (TAT) to collect data. The TPQ tasks revealed that over 80% of the participants found trigonometry concepts to be abstract, rigid, and uninteresting. Results from the TAT tasks showed that more than 50% of the PSTs could not complete the questions. Additionally, none of the students could explain why the trigonometric identity $cos^2x + sin^2x = 1$ holds true, and 81% did not understand the sign conventions of trigonometric ratios on the unit circle. Furthermore, most participants struggled with graphing a sine function - over 80% could not state its domain and range, while 91% could not find the line of symmetry, maximum or minimum points, or determine intervals of increase or decrease necessary for graphing. In conclusion, the researchers found that most PSTs in their study experienced conceptual difficulties with trigonometric relations, preventing them from effectively explaining or applying these relationships effectively.

In South Africa, there is concern expressed about students finding trigonometric functions difficult to learn. Numerous studies have been conducted on teaching and learning trigonometric functions in the country, mostly focusing on high school learners, with little to no research on PSTs. One study by Majengwa (2010) examined Grade 11 learners' understanding of the cosine function using Sketchpad at a school in KwaZulu-Natal. The data was collected from six randomly selected learners out of 123 enrolled in the grade, all studying Mathematics and Computer Application Technology (CAT). Through task-based interviews, it was found that learners had limited comprehension of trigonometric notations, struggled with calculator usage, algebraic manipulations, and understanding angle-to-ratio conversions. However, engaging in Sketchpad activities helped improve their understanding of the cosine function. Another study conducted by Ngcobo et al. (2019) focused on Grade 12 learners in the same province. The researchers explored how these learners mentally approached solving unknown properties of triangles in trigonometry. Data was gathered from written responses and semi-structured interviews. Findings revealed that 67% of participants could only identify the formula needed for the given tasks but were unable to proceed further in finding solutions.

2.8 Challenges with angle measurement in trigonometric functions

To address these conceptual difficulties, some researchers (e.g., Martinez-Sierra, 2008; Tuna, 2013) have advocated for the use of radians over degrees when teaching trigonometric functions. Tuna (2013) argued that radians provide a more mathematically coherent approach, as they relate angle measure to arc length on the unit circle. This approach helps students connect geometry and algebra in meaningful ways. According to Tuna, a radian is defined as the ratio of the arc length to the radius of a circle, a conceptual definition that aligns with real-number representation of angles through the wrapping function. Akkoç and Gül (2010) added that this interpretation underpins how mathematicians typically express trigonometric functions for real numbers. Degrees, by contrast, are arbitrarily based on dividing a circle into 360 parts and do not integrate as seamlessly into algebraic representations of trigonometric functions.

Despite the theoretical benefits of radians, research has revealed that students still experience significant challenges with this concept. Fi (2003), in a study involving 14 undergraduate students, found that while most students could perform mechanical conversions between degrees and radians, they were unable to define a radian or explain its meaning. This conceptual gap was echoed in studies by Orhun (2001), Steckroth (2007), Akkoç (2008), and Tuna (2013), where students struggled not only with conversions but also with the idea of radians as real-number angle representations. Akkoç (2008), for instance, noted that the majority of 42 undergraduate students could not provide a correct definition of a radian, although they could recall conversion formulas. Tuna (2013) similarly found that while undergraduate students were familiar with the numerical value of π in relation to degrees (e.g., 2π = 360°), many failed to visualize or define a radian as the arc-to-radius ratio on a unit circle.

These difficulties are not limited to university students. High school learners also show poor conceptual understanding of radians. Orhun (2001), Akkoc and Gül (2010), and Tuna (2013) each found that grade 10 students struggled to compute angles from arc lengths or to visualize the meaning of radian measure. For example, Orhun's (2001) study revealed that only 19.5% of students could correctly calculate angles based on given arc lengths, and many had misconceptions about the unit π, often mistaking it as a symbol rather than a number. Similarly, Fi (2003) and Akkoç (2008) reported that undergraduate students frequently misunderstood π, treating it as a unit of angle measurement rather than recognizing it as a real number constant used to define radian measure. These misunderstandings can seriously hinder students' abilities to interpret trigonometric functions correctly and flexibly.

More recent studies continue to confirm and expand upon these earlier findings. Maknun et al. (2022) found that many students rigidly associate trigonometric values with specific "well-known" angles, such as 30°, 45°, and 60°, and have difficulty generalizing these values across broader angular domains, especially when expressed in radians. Students also demonstrated a lack of awareness of how π operates beyond geometric formulas, failing to appreciate its role in defining periodic behaviour and function symmetry. These misconceptions can lead to systematic errors in graphing and solving trigonometric problems and indicate a disconnection between numerical understanding and geometric reasoning.

Further, Abdullah and Haron (2023) emphasize that students often lack exposure to radian measure at earlier stages of education, which creates barriers when they encounter it in higher-level mathematics. Their study involving high school learners in Malaysia revealed that over 60% of students preferred degrees because they had more experience using them. When introduced to radians, students found the numbers unfamiliar and confusing, particularly when working with π-based values like $\frac{\pi}{3}$ or $\frac{3\pi}{2}$. They tended to treat π as an unknown symbol rather than a fixed mathematical constant. Similarly, Shariff and Rohana (2021) observed that students often failed to understand the unit circle in terms of radians, leading to misinterpretation of trigonometric graphs and poor performance on periodicity questions.

One reason these issues persist, as reported by Bütüner and Ural (2020), is that many textbooks and teachers emphasize degrees in early instruction and delay the introduction of radians until much later. This sequencing may unintentionally reinforce the misconception that degrees are the “real” angle unit, while radians are merely an advanced topic. Their study recommended earlier and more integrated instruction of radian measure using visual and contextual examples that relate arc length to angles, especially within the context of circular motion and periodic phenomena.

Furthermore, research by Özdemir and Ünlü (2021) highlighted the importance of instructional strategies in addressing these difficulties. Their study found that when teachers used dynamic software to illustrate the wrapping function and unit circle in radians, students demonstrated improved understanding and retention of radian concepts. Interactive visualizations helped learners perceive π not just as a constant but as a measure of rotation and periodicity.

In conclusion, a strong conceptual grasp of angle measurement, particularly in radians, is essential for students to fully understand and apply trigonometric functions. While degrees are more familiar and often easier for students initially, they can constrain the broader, more generalizable understanding required in advanced mathematics and science. Radian measure, when properly introduced and contextualized, provides a more coherent and powerful foundation for learning trigonometry. However, the evidence from both historical and recent studies shows that students at both secondary and tertiary levels continue to face substantial challenges in conceptualizing radian measures and the meaning of π. These challenges are compounded by limited instructional time, textbook bias toward degrees, and insufficient use of visual and technological aids. Addressing these issues requires a deliberate pedagogical shift that emphasizes conceptual understanding, the use of dynamic representations, and early integration of radian concepts into the mathematics curriculum.

2.9 The challenges with graphs of trigonometric functions

2.9.1 Difficulties of school learners in understanding trigonometric functions and graphs

Trigonometric functions and their graphical representations remain one of the most conceptually demanding areas of school mathematics, presenting significant difficulties for learners globally. Despite being introduced in the senior phase, many learners fail to fully grasp the dynamic nature and interrelated representations of these functions, often perceiving them as abstract and disconnected from real-life applications. One of the primary challenges arises from learners' difficulties in interpreting the graphical nature of trigonometric functions, such as sine, cosine, and tangent. Learners often memorize the shapes of these graphs without understanding the underlying reasons for their periodic nature or how the graphs are derived from the unit circle (Naidoo & Kapofu, 2021). As highlighted by Mhlolo (2023), the reliance on rote learning inhibits students' ability to engage in deeper conceptual thinking, particularly when transitioning between graphical and algebraic representations. Learners tend to see trigonometric functions merely as ratios within right-angled triangles and struggle to understand them as continuous functions defined over the real numbers. This problem is compounded by a lack of integration between the unit circle definition of trigonometric functions and their graphical representations. Studies have shown that learners are often unable to link a rotation on the unit circle with a corresponding point on a sine or cosine graph, which is essential for conceptual understanding (Mabuza & Dlamini, 2020).

Another critical area of difficulty is learners' misunderstanding of the domain and range of trigonometric functions. According to Dube and Makwakwa (2022), many learners believe that trigonometric functions are only defined for acute angles, primarily due to their early exposure to right-angle triangle definitions. This misconception persists even after formal instruction on the unit circle and the Cartesian plane. Furthermore, learners frequently misinterpret the concept of periodicity, failing to recognize that trigonometric graphs extend infinitely in both directions. Instead, they tend to treat these graphs as finite diagrams to be memorized, rather than understanding the cyclical behaviour of the functions (Naidoo, 2022). Learners also find it challenging to determine the amplitude, period, and phase shift from an equation, which hinders their ability to sketch or interpret transformed graphs. For example, the transformation $f(x) = 2\sin(x - 180°) + 1$ is often not understood as involving vertical stretching, horizontal shifting, and vertical translation, but rather interpreted in piecemeal or incorrect ways (Mhlongo & Ramdhany, 2023). These difficulties reveal a lack of coherence in learners' mental models of trigonometric functions and their transformations.

A recurring theme in recent literature is the difficulty learners face when required to use multiple representations of trigonometric functions. According to Mthembu and Hlatshwayo (2020), learners often struggle to translate between algebraic expressions, graphical representations, and verbal descriptions of trigonometric phenomena. They may correctly identify the amplitude and period from a given equation but fail to reflect these characteristics accurately on a graph. Conversely, when presented with a graph, they may be unable to deduce the correct equation. This disconnect suggests that learners treat these representations in isolation rather than as different views of the same mathematical object. The work of Letsoalo and Setati (2021) supports this finding, noting that learners often fail to see the significance of phase shifts and how they affect the horizontal displacement of a trigonometric graph. This lack of representational fluency significantly affects their ability to solve trigonometric equations or model periodic phenomena.

Conceptual misunderstandings are further exacerbated by the complexity of radian measure. Although the curriculum introduces radians as the standard unit of angular measure in the senior phase, learners struggle to make sense of radians as real numbers representing arc length ratios. According to Mofokeng and Sibanda (2023), many learners continue to rely exclusively on degrees, resisting the use of radians even when solving problems that require this understanding. This is often because they fail to grasp the definition of a radian as the length of the arc on a unit circle subtended by a central angle. Instead, learners view radians as merely another way to "label" angles without connecting them to real number measures. This limits their ability to understand why trigonometric functions are defined for all real numbers, especially in calculus contexts where radians are essential. In many cases, learners interpret π radians simply as 180°, without any conceptual grounding in arc length or the radian's derivation from circle geometry (Mhlongo, 2022). These misunderstandings are symptomatic of a broader problem: learners often fail to see trigonometry as a system of interconnected ideas and instead focus on fragmented procedures and disconnected facts.

Graphing calculators and dynamic geometry software, which are designed to support visual learning, are not always utilized effectively to mitigate these challenges. Although studies show that technology can significantly enhance understanding when properly integrated (Mthembu and Mathonsi, 2024), it is also evident that learners may become overly reliant on visual aids without internalizing the underlying concepts. For instance, learners may use graphing tools to sketch sine and cosine functions but are unable to explain the behaviour of the graphs or predict the effects of altering parameters such as amplitude and frequency. According to Naidoo and Ramdhany (2023), many teachers also lack the pedagogical content knowledge needed to use these tools effectively, resulting in technology being used as a demonstration device rather than as a means for exploration and discovery. This has implications for both initial teacher education and professional development programs, which must equip teachers with the skills to help learners transition from visual observations to analytical reasoning.

The difficulties learners experience is not solely cognitive but also affective. Anxiety and negative attitudes towards trigonometry often hinder engagement and persistence. Many learners view trigonometry as an intimidating subject filled with strange symbols and unfamiliar rules (Gumede & Dlamini, 2021). This perception is reinforced by the cumulative nature of trigonometric learning, which builds on prior knowledge of algebra, geometry, and number systems. When foundational concepts are weak, learners struggle to keep pace, leading to frustration and disengagement. According to Nkosi and Ndlovu (2020), learners often express feelings of helplessness when asked to sketch or interpret graphs of trigonometric functions, citing confusion over where to begin or how to proceed. This emotional response can lead to surface learning strategies, such as memorizing graph shapes or using procedural shortcuts, which further inhibit deep understanding. Teachers must therefore be attentive not only to learners' conceptual development but also to their emotional responses to the material, creating classroom environments that foster confidence and curiosity.

Another important aspect of learners' difficulties involves language and notation. Trigonometric functions are often introduced using symbolic language that learners find unfamiliar, such as$\sin(x)$, $\cos(x)$, and $\tan(x)$. Learners may not immediately recognize these as function names and instead interpret them as separate variables or operations. As argued by Mlambo and Mavuso (2022), such misinterpretations are particularly prevalent in multilingual classrooms where English is not the home language. These challenges are compounded when learners are expected to interpret symbolic expressions that combine multiple transformations, such as $f(x) = -3\cos(2x + 180°) + 1$. In such cases, learners often misapply transformation rules or fail to recognize the composite nature of the function. Language barriers also affect learners' ability to interpret word problems involving trigonometric contexts, such as modelling tides, sound waves, or oscillating motion. If the linguistic demands of a problem are too great, learners may fail to recognize the relevance of trigonometric functions to the solution.

Lastly, the way trigonometry is assessed also contributes to learners' difficulties. Traditional assessments often emphasize procedural fluency over conceptual understanding, rewarding correct answers even when derived through memorized steps (Mofokeng & Nkosi, 2024). Learners may therefore pass assessments without developing a robust understanding of trigonometric functions and their properties. Furthermore, many assessments fail to incorporate tasks that require learners to make connections between multiple representations or to apply their knowledge to unfamiliar contexts. Without such opportunities, learners are unlikely to develop the flexibility needed for genuine mathematical reasoning. As a result, even when learners appear to perform adequately in class or on exams, they may lack the conceptual depth needed to succeed in higher-level mathematics. Teachers and curriculum developers must therefore strive to design assessments that probe learners' understanding more holistically, incorporating graphical, algebraic, and verbal elements in meaningful ways.

In conclusion, learners' difficulties with trigonometric functions and their graphs are multifaceted, involving conceptual, procedural, linguistic, and affective dimensions. These challenges are not isolated, but interconnected, and are often rooted in the fragmented ways trigonometry is taught and assessed in schools. Addressing these issues requires a shift towards more integrated, conceptually driven instruction that emphasizes connections between representations, fosters positive learner dispositions, and leverages technological tools meaningfully. Furthermore, teacher professional development must focus on strengthening pedagogical content knowledge related to trigonometric functions, including strategies for supporting learners in developing a unified and intuitive understanding of these important mathematical constructs. Only through such comprehensive efforts can we begin to bridge the gap between learners' experiences and the rich, interconnected world of trigonometric functions and graphs.

2.9.2 Difficulties of university students and schoolteachers in understanding trigonometric functions and graphs

The understanding of graphs associated with trigonometric functions is crucial for students to fully grasp the concept. Demir (2011), Breslich (1928), and Orhun (2001) have all highlighted the importance of using graphs in teaching trigonometric functions. Without utilizing graphs, students may only focus on the ratio aspect of trigonometric functions and overlook the function aspect. This means that they would only see sine and cosine as ratios without understanding them as functions of real numbers. However, incorporating graphing into the teaching of trigonometric functions can help students conceptualize these functions better. Brown (2006) and Demir (2011) suggest that when students are exposed to graphs, they are more likely to understand sine and cosine as functions. Moreover, by using graphs, students can explain why sine and cosine graphs represent functions. They can do this through either a formal definition like "There is only one y for every x", or a process definition based on an input-output mechanism. By understanding that there is a unique output for each input, students can grasp the concept of trigonometric functions represented by their respective graphs.

Many studies (Vinner,1983; Demir, 2011; Baki and Kutluca, 2009; and Rose, Bruce and Sibbald, 2011) have shown that students struggle with understanding trigonometric functions, particularly when it comes to graphs of these functions. Researchers have identified graphs of trigonometric functions as one of the most challenging topics for students in mathematics classrooms. Surveys conducted among grade 10 students and mathematics teachers have consistently found that trigonometric functions and their graphs are considered the most difficult topics in trigonometry (Baki and Kutluca, 2009). Additional studies (Adamek, Penkalski, and Valentine, 2005; and Tatars, Okur, and Tuna ,2008) further support the notion that this topic is complex and poses significant difficulties for students.

Students often struggle to understand graphs of trigonometric functions because they have difficulty connecting algebraic and graphical representations (Lambertus, 2007). Additionally, they find it challenging to transition between the unit circle and the graphs, which is essential for comprehending trigonometric functions (Rose et al., 2011; Demir, 2011; Brown 2006). This difficulty in making connections and transitions is not unique to trigonometric functions but also exists with other mathematical functions such as algebraic functions (Gagatsis et al., 2003).

In studies involving undergraduate students, researchers (Leinhardt, Zaslavsky and Stein, 1993; Yerushalmy and Schwartz, 1993; and Knuth, 2000) found that many of them relied heavily on algebraic representations when solving mathematics problems. They would use equations or graphs to complete tasks but would often choose the algebraic approach even when a graphical one was more appropriate. This limited their ability to effectively use both algebraic and graphical representations and understand the connections between equations and their graphs (Knuth, 2000). Similar difficulties were observed in high school students when it came to trigonometric functions. They struggled to find the coordinates where a given graph crossed the y-axis, instead focusing solely on finding points where the equation equalled zero. These students had trouble linking graphical and algebraic representations in the context of trigonometry (Challenger, 2009). Another study (Marchi, 2012) with high school students found that none of them used a graphical representation to solve trigonometric equations unless specifically instructed to do so. This indicated a lack of understanding about using graphs as a method for finding solutions in trigonometry.

As stated previously, several research studies focused on students' abilities to connect the unit circle with graphs of trigonometric functions. Brown (2005) found that some high achievers high school students were unable to make connections between a rotation on the unit circle and points on a graph of sine or cosine. Only a few students were able to conceptualize trigonometric graphs through arc lengths on the unit circle and corresponding horizontal positions. This finding was consistent with Marchi (2012)'s study, which also found that high school students struggled to recall information and make correct connections between the graph for sin (x) and the unit circle. Another study (Demir, 2011) observed that while some high school students had success, others still struggled to explain coordinates on graphs as arc lengths and vertical positions on the unit circle, or mark points on the graph corresponding to positions on the unit circle.

Trigonometry is a challenging subject for both students and teachers alike. While there have been numerous research studies focused on students' difficulties in understanding trigonometric concepts, there is a limited amount of research that explores teachers' understandings of these concepts. Interestingly, the difficulties faced by teachers are often like those faced by their students. One concept that proves challenging for both students and teachers is radians. In a study conducted by Topçu et al. (2006) involving 14 high school mathematics teachers, it was discovered that most of the teachers had a lack of understanding when it came to radians. For example, in the following question:

> $f\colon \to \mathbb{R} \to \mathbb{R}$ and $f(x) = x\sin(x)$. Plot the following points on the Cartesian plane:
> a) $\left(30°; f(30°)\right)$, b) $(2\pi; f(2\pi))$, c) $(6\pi; f(60))$, d) $(2; f(3\pi))$.

According to Topçu et al. (2006), a small percentage of teachers, specifically 3.9%, thought about "sin 30" in radians while the majority, which is 90.2%, considered it in degrees. This suggests that most teachers did not view radians as a real number, despite being explicitly defined as such. These findings align with another study conducted by Cizmesija and Siqus (2013), where participant teachers referred to pi as the unit for measuring radians, failing to recognize that real numbers not in the form q π (q, π), q $\in\mathbb{Q}$ could also be radian measures of an angle. Both studies concluded that teachers tended to prefer answering trigonometric tasks in degrees rather than in radians due to their lack of understanding of radians as a ratio between two lengths: the length of an arc subtending a central angle and the radius of the circle. These misunderstandings among teachers mirror those observed among students, as reported in studies by Moore (2010), Demir (2011), and Moor, LaForest and Kim (2012). These studies highlight common misconceptions held by students regarding radian measurements.

Fi's (2006) study also shed light on another area where teachers struggle with inverse trigonometric functions. Despite recognizing that given trigonometric functions were inverse functions, teachers often discussed them in terms of reciprocal functions. Fi attributed this confusion to the teachers' familiarity with the written form of real number reciprocals. While a non-zero real number x's multiplicative inverse is typically written as x^{-1} or $\frac{1}{x}$, this is not the case for trigonometric functions. Furthermore, Fi found that none of the participating teachers had a conceptual understanding of co-concepts such as sine-cosine, tangent-cotangent, and secant-cosecant. Teachers frequently treated inverse, reciprocal, and co-functions as interchangeable ideas when answering interview questions.

2.10 The acquisition of trigonometric knowledge by students and the impact of technology on this process.

According to Bruna (1996), one way to make learning easier is by utilizing the strengths of the learners. In today's world, many learners are familiar with using cell phones and computers, so incorporating computer use into mathematics education could help address difficulties in understanding certain concepts. Bruna also emphasizes that active participation, rather than passive observation, is crucial for learning. Learners should engage as much as possible, such as working on problems independently to discover effective problem-solving strategies. In recent years, with the rise of digital tools and online learning environments, this principle has gained new relevance. The integration of interactive technology in mathematics classrooms has provided platforms for students to explore concepts in a dynamic and personalized way (Utami, Susilo, & Subanji, 2020).

There have been a limited number of research studies conducted on how technology affects the learning of trigonometric concepts. For example, Blackett and Tall (1991), Yenitepe (2002), and Choi-Koh (2003) have all explored this topic. In one study by Robison (1996), 99 college students taking a trigonometry course were divided into two groups: one group used Mathematica, a computational software program, while the other group relied on static images and their own imaginations to solve trigonometric problems. Surprisingly, the results showed no significant difference in achievement between the two groups. However, these findings contradict those of Blackett and Tall (1991), whose study demonstrated that using computer software improved students' understanding of triangle trigonometry. Similarly, Yenitepe (2002) found that high school students who received technology-based instruction performed better on exams compared to those who did not use technology.

The use of technology in teaching trigonometric graphs has been found to enhance students' understanding and conceptual development (Choi-Koh, 2003; Army, 1991; Stacey and Ball, 2001). Traditional teaching methods relied on static media, such as tables and formulas, which provided only discrete data and limited graphical variations. In contrast, technology allows students to manipulate graphic objects freely, providing a more dynamic and interactive learning experience. Choi-Koh (2003) emphasized the importance of technology in graphing trigonometric functions, stating that it serves as a means for displaying representations. With traditional methods, teachers were limited to using static media, making it difficult to convey the continuous nature of graphs. Technology enables teachers to demonstrate graphical variations in real-time, helping students visualize how changes in variables affect the shape of the graph. Moreover, technology eliminates the need for students to manually sketch graphs by hand, saving time and mental energy. Kissane et al. (2018) noted that this frees up students' cognitive resources for better engagement with conceptual development. By automating the process of graphing trigonometric functions, technology allows students to focus on understanding the underlying concepts rather than getting caught up in tedious calculations.

Several research studies have supported the benefits of using technology in teaching trigonometric graphs. Army (1991) found that integrating computer-based graphing tools improved students' ability to understand and analyse trigonometric functions. Stacey and Ball (2001) also reported positive outcomes when incorporating technology into their teaching of trigonometry. In more recent literature, Sulistyo, Widodo, and Zubainur (2021) demonstrated that using dynamic graphing tools such as GeoGebra improved students' ability to understand amplitude, period, and phase shift in trigonometric graphs. Similarly, Kurniawan and Herayanti (2022) observed that students developed stronger graph interpretation skills when engaged with dynamic software compared to those using conventional methods.

In the use of technology-based learning environments in teaching trigonometric functions, Choi-Koh (2003)'s study was conducted by inviting a 10th-grade student to work on seven tasks, including creating graphs of trigonometric functions using a graphing calculator. The results showed that the student successfully utilized the calculator as a tool to graph sine functions and identify changes in the graphs with varying coefficients. Choi-Koh (2003) concluded, in line with previous research by Yerushalmy (1988), Army (1991), Stacey and Ball (2001), and Lambertus (2007), that technology-based learning environments provide an opportunity for students to strengthen their understanding of the connection between graphical and symbolic representations. This connection is often a challenge when graphing trigonometric functions. However, these findings contrast with another study conducted by Colgan (1992) involving eight students in grades 11 and 12. In this study, the students used a computer-based graphing tool called Zap-a-Graph to study trigonometric function graphs. Colgan observed that while the technical tool allowed students to plot graphs, it did not help them make connections between algebraic and graphic representations of trigonometric functions. In some cases, it even led to misunderstandings. The students reported that Zap-a-Graph was difficult to use because it could only display black and white images and did not allow for different coloured graphs.

While Choi-Koh's study suggests that technology-based learning environments can enhance students' understanding of graphical representations, Colgan's study raises concerns about potential limitations and drawbacks associated with certain graphing tools like Zap-a-Graph. This highlights the importance of selecting appropriate and user-friendly tools. Recent research has echoed this view. For example, Purnomo and Sari (2023) reported that learners using interactive tools that allowed for real-time manipulation of function parameters were significantly more likely to understand domain, range, and transformation of trigonometric functions. However, they also cautioned that without sufficient guidance, students might misuse the technology or become overly reliant on visual patterns without conceptual grounding.

Garofalo, Drier, Harper, Timmerman, and Shockey (2000), and Demir (2011), argue that technical tools like GeoGebra can help students understand the connections between different representational systems in trigonometry. They specifically highlight the use of technology in mathematics classrooms to help students establish connections between the unit circle and graphs, which is often a challenging concept for trigonometry students. Kissane and Kemp (2009), as well as Zengin, Furkan, and Kutluca (2012), also support the idea that Dynamic Geometry software can assist students in forming relationships between graphical representations of trigonometric functions and positions of points on the unit circle. These findings align with Tall's (1986) study, where 'A' level students successfully linked the unit circle and trigonometric graphs through computer graphics used as part of a teaching strategy. More recently, Maulana, Sugiman, and Pramudya (2020) found that GeoGebra-assisted instruction helped Indonesian students build deeper understanding of the sine and cosine wave formation through unit circle motion. They concluded that visualization from dynamic software significantly reduced misconceptions about amplitude and period.

These studies suggest that incorporating technology into mathematics education can be beneficial for understanding complex concepts like trigonometry. The use of tools like GeoGebra or Dynamic Geometry software helps students make connections between different representations of trigonometric functions and enhances their comprehension. By using these cognitive tools, students can visualize how the unit circle relates to graphs, particularly when graphing trigonometric functions. In recent comparative studies, Pratama and Rahmawati (2022) showed that students exposed to visual technology in learning trigonometric functions scored significantly higher on conceptual assessments than those in traditional classrooms. Similarly, Ngwenya and Mthethwa (2023) conducted an experimental study in South Africa and reported improved student engagement and retention when trigonometric concepts were taught using mobile applications and video demonstrations.

Furthermore, the integration of technology into trigonometry education can support inclusive education strategies. According to Mavhunga and Maharaj (2021), digital tools can help bridge the achievement gap for learners with diverse learning needs by allowing selfpaced exploration and immediate feedback. The interactive and multimodal nature of educational software is especially helpful for visual and kinesthetic learners who struggle with abstract representations in traditional instruction.

Although early studies such as Robison (1996) suggested that technology may not always produce measurable gains in student achievement, the overwhelming trend in contemporary research supports the effectiveness of technology-enhanced learning in trigonometry. Studies conducted between 2020 and 2024 affirm that when used appropriately, technology facilitates better conceptual understanding, fosters connections across representations, and enhances engagement. However, the success of such interventions depends on the careful integration of tools, proper scaffolding by teachers, and the selection of software that is pedagogically sound and user-friendly. As educational technology continues to evolve, its role in supporting mathematics instruction, particularly in complex topics like trigonometric functions, will likely become increasingly central.

2.11 Challenges faced by mathematics teachers when teaching trigonometric concepts

Trigonometry, with its complex blend of algebraic manipulation, geometric reasoning, and functional thinking, continues to pose significant learning challenges for students at secondary and tertiary levels. Despite extensive research into pedagogical strategies, there is still no consensus on a single instructional approach that guarantees success in learning trigonometry. Studies suggest that the effectiveness of instructional methods is often context-dependent and shaped by students' prior knowledge, representational fluency, and engagement levels. Weber (2005) criticizes the traditional textbook-driven method of instruction, asserting that it limits conceptual development and instead recommends that students be given opportunities to reason with numerical values in connection with geometric processes, such as the unit circle. He contends that using multiple representations and emphasizing their interrelations helps students develop a more profound and transferable understanding of trigonometric concepts. Marchi (2012) supports this view and adds that exposing learners to various ways of expressing mathematical ideas enhances their ability to understand and apply trigonometric principles in different contexts.

Contrasting these approaches, Kendal and Stacey (1997) champion the ratio-based method of teaching trigonometric functions, finding in their study that students exposed to this approach performed better on trigonometric tasks than those taught via the unit circle. This highlights that instructional strategies must be flexible and responsive to different learner profiles. Delice and Monaghan (2005) argue that students' struggles often stem from weak algebraic foundations, especially when simplifying trigonometric expressions. They stress the importance of strengthening these algebraic skills and embedding them within real-life contexts to increase relevance and student engagement. Their work suggests that contextual applications can bridge the gap between abstract content and students' everyday experiences. Taking this idea further, Gould and Schmidt (2010) conducted an innovative study in which high school students developed digital story problems rooted in real-life situations, then applied trigonometric functions to solve them. This non-traditional method increased student motivation and led to deeper learning, demonstrating that making the subject more relatable and creative can foster meaningful engagement.

Similarly, Barnes (1999) integrated creative writing into trigonometry lessons by assigning students word problems that they solved in groups and then explained in writing. This method helped students overcome anxiety around mathematical problems and improved their understanding by requiring them to verbalize and reflect on their mathematical reasoning. In another study, Gur (2009) found that many Grade 10 students heavily relied on memorized formulas from textbooks, such as those related to the unit circle, but could not justify their answers or explain concepts in their own words. Gur advocated for instructional strategies that prioritize students' ability to articulate definitions and concepts independently, emphasizing that this approach fosters deeper understanding and long-term retention.

Graphical representations also play a pivotal role in improving conceptual understanding. Orhun (2001), Topçu et al. (2006), and Akkoc (2008) argue for using graphs in teaching trigonometric functions and introducing core definitions before delving into angle-based representations. Orhun specifically cautions against using degree measures prematurely, as doing so can misalign with broader mathematical definitions and confuse learners. Byers (2010) and Pesek and Kirshner (2000) likewise advocate for a definition-first pedagogy where students construct meanings through multiple representations, including the unit circle, vectors, ratios, and functions, before applying formulas. Though effective, this method can be challenging for learners unfamiliar with abstract reasoning, requiring careful scaffolding by teachers to support representation-building processes.

Addressing difficulties with radians, Akkoc and Gül (2010) and Moor (2012) emphasize linking radian measurement with arc length to make the concept more intuitive. They suggest using digital tools to illustrate how arc length corresponds to angles in the unit circle, reinforcing the understanding that one radian subtends an arc equal in length to the radius of the circle. Moor's arc-based approach further clarifies the conceptual equivalence between degrees and radians, arguing that both measure the same quantity on different scales, degrees as fractions of 360 and radians as fractions of 2π. These insights highlight the pedagogical importance of grounding angle measurement in tangible, geometrically meaningful constructs.

In recent years, the integration of digital technologies into trigonometry instruction has shown promising results. Sulistyo et al. (2021) found that using GeoGebra to explore amplitude, period, and phase shift helped students grasp transformations in trigonometric graphs more effectively. The dynamic nature of these tools allows students to manipulate parameters in real time, enhancing their intuition about function behaviour. Pratama and Rahmawati (2022) reported similar findings, noting that learners who used mobile applications to explore trigonometric functions demonstrated superior conceptual understanding compared to those taught via traditional methods. In South Africa, Ngwenya and Mthethwa (2023) found that video-based mobile tutorials improved both comprehension and engagement among students, especially in under-resourced schools. These studies confirm that when integrated thoughtfully, technology can provide powerful visualizations and exploratory environments that foster deeper learning.

However, successful technology integration depends on more than just tool availability. Utami et al. (2020) stress the importance of balancing student agency with teacher guidance. While technology offers opportunities for discovery, unstructured exploration may lead to superficial understanding if not anchored in conceptual goals. Teachers must scaffold tasks to ensure learners move beyond experimentation to structured reasoning. Naidoo and Ramdhany (2023) found that teachers who received training in both trigonometric content and the use of graphing tools facilitated significantly better student outcomes. Conversely, Mavhunga and Maharaj (2021) highlighted that without pedagogical training, teachers may misuse technology, focusing on visual presentation rather than conceptual development. These findings underscore the necessity of targeted professional development that supports both content mastery and effective use of educational technology.

Incorporating new pedagogical models alongside technology, Kurniawan and Herayanti (2022) advocate for active learning approaches that emphasize collaborative problem solving and interactive visualization. They found that when students worked in pairs or groups using digital tools, they not only performed better on assessments but also demonstrated more positive attitudes toward mathematics. In line with this, Mthembu and Mathonsi (2024) argue for a multimodal instructional approach that blends visual, symbolic, and narrative modes of reasoning. Their research shows that students exposed to such integrative instruction are better able to connect concepts across representations and retain knowledge more effectively. This pluralistic pedagogy, which includes unit circles, function graphs, real-world applications, and verbal articulation, responds to the cognitive diversity present in most classrooms.

The convergence of historical and recent studies suggests a number of best practices for effective trigonometry instruction. First, teaching should begin with concept definitions, not rote procedures, to lay a strong foundational understanding. Second, multiple representations, algebraic, graphical, geometric, and contextual, should be used in tandem to reinforce learning. Third, technology should be embedded in a purposeful and pedagogically sound manner, enabling exploration, visualization, and reinforcement. Fourth, student reflection should be encouraged through writing, discussion, and creative expression. Fifth, real-life contexts should be employed to make abstract concepts tangible and relevant. Finally, teachers must receive ongoing professional development that covers both content and pedagogy, particularly regarding technology integration.

Therefore, while there is no single definitive method for teaching trigonometry, a growing body of evidence, both historical and contemporary, supports a flexible, multimodal, and conceptually grounded approach. Effective instruction blends definitions, real-world applications, technological tools, and reflective practices to help students internalize and apply trigonometric concepts. By moving beyond formulas and fostering genuine understanding, teachers can help learners develop not only proficiency in trigonometry but also a deeper appreciation for mathematics as a meaningful and powerful language. With sustained effort in curriculum design, professional development, and classroom innovation, the persistent difficulties students face in trigonometry can be substantially reduced.

This section provides a summary of several research studies that focus on the challenges faced by students when dealing with trigonometric concepts such as radian conception, inverse trigonometric functions, and co-functions. Many of the difficulties encountered by students in understanding the definition of radian and converting between radians and angles mirror the misconceptions held by teachers regarding trigonometric functions. Consequently, it is not surprising to observe students making similar mistakes while working with trigonometry due to their mathematics teachers' incomplete grasp on these complex ideas. However, what remains absent from all this existing literature is an in-depth investigation into the transformations of trigonometric functions. It is precisely this phenomenon that we are keen on further exploring.

CHAPTER 3
CONCEPTUAL FRAMEWORK USED TO ANALYSE DATA GATHERED

Duval's theory is a widely used framework for examining data collected in various research fields. By using Duval's theory, researchers can effectively organize and categorize their data, identify any anomalies or outliers, and ultimately extract valuable insights. Whether in the social sciences, natural sciences, or any other field involving data analysis, Duval's theory can serve as a tool to assist researchers in drawing evidence-based conclusions from their data. In this chapter, we will describe how Duval's theoretical framework is used in analysing the participants' responses to the tasks and interviews. In Chapter 8, we will then proceed to analyse and interpret the data guided by Duval's theoretical framework. Note that much of Duval's work (Duval, 1983; 1993; 1996a; 1996b; 1999) was published in French, which is my first language; hence, we were able to access and translate the work into English for this thesis.

3.1 The process of acquiring knowledge and developing cognitive abilities

The acquisition of new knowledge involves a process where resistance is encountered until it is fully understood and mastered. According to Chevallard (1992), the process of learning mathematics can be considered a part of cognitive anthropology, which is the study of how humans perceive and interact with various forms of knowledge. This means that understanding the teaching of mathematics cannot be separated from understanding how knowledge is developed in general. Chevallard (1992) argues that the study of didactics should not only focus on teaching methods and techniques but also consider the cultural, social, and historical contexts in which education takes place. By adopting an anthropological approach, Chevallard (1992) emphasizes the importance of understanding how educational practices are shaped by factors such as societal norms, beliefs, and values. This perspective provides a broader understanding of education and highlights the need for a more comprehensive approach to teaching and learning. For Schneuwly (2008), Chevallard (1992)'s argument highlights a dual effect of semiotic representation, which is the process of understanding and representing knowledge. When a teacher explains a notion, concept, or theory, they engage in semiotic representation by mentally visualizing the object of knowledge based on their expertise. However, the teaching materials themselves, such as texts, exercises, or worksheets, also reflect the semiotic representation process employed by their creators. Semiotic representation involves creating a modified version of an object or idea based on human perception. It refers to the process by which human actions alter the symbolic characteristics and significance of something (Maran, 2017). Each time a learner encounters different ways of interpreting something, it can result in the formation of a fresh mental representation of the subject being studied. This leads to multiple subsequent interpretations. As a result, the objects being learned acquire new significance for the learner that may vary over time. These meanings may differ from those of the teacher and what is presented in the study materials. This explains why learners may go through stages of advancement and decline as they engage in this process of interpretation before fully mastering the new concept (Schneuwly, 2008).

The National Council of Teachers of Mathematics (NCTM) (2000) emphasizes the importance of expressing mathematical ideas through writing. This is crucial for learning and creating mathematics. Students can use various representations to externalize their thoughts and make connections among them. NCTM suggests that students should be encouraged to represent their mathematical ideas in ways that are personally meaningful, even if these representations are initially unconventional. By utilizing instructional methods that promote connections between various types of representations, students can enhance their understanding of concepts, as well as improve their ability to think and communicate mathematically with others (Duval, 1995, 2003, 2017; Guerreiro et al., 2015).

Other theoretical perspectives also explore the concept of progress and regression in cognitive development. For instance, Carle (2009) and Mahy and Carle (2012) argue that there are universally nonlinear processes involving stages of progression and regression. Drawing on the socio-cognitive approach of Kuhn (1983) and Scharmer (2007), they illustrate how a U-shaped learning curve model is observed across different domains of knowledge. The U curve represents the phenomenon where an individual, who previously had a stable understanding of a concept, experiences instability when faced with new situations or interactions with others, such as a teacher. During this process of learning, previous knowledge resists before the emergence of new knowledge takes place.

3.2 Duval's perspective on learning and representations

Although there are various distinctions among their approaches, such as Chevallard (1992), Schneuwly (2008), and the previously mentioned U-curve theory, Duval emphasizes the importance of the cognitive aspect in learning. Consequently, Duval (2006b) asserts that it gives priority to the cognitive perspective rather than the mathematical perspective.

> 3) La prise en compte du point de vue cognitif met en évidence un autre objectif: comprendre non pas d'abord pour devenir capable de valider mais pour apprendre à comprendre, c'est-à-dire pour être ensuite capable de se poser de nouvelles questions, de trouver des moyens de les explorer et par suite de contrôler la pertinence de ses explorations et de ses interprétations. Ce qui est l'autonomie par excellence. (Duval, 2006b, p. 88-89)
> *"Taking the cognitive point of view into account highlights another objective: to understand not first to become able to validate but to learn to understand, to then be able to ask new questions, find ways to explore them and by to monitor the relevance of its explorations and interpretations. Which is autonomy par excellence. (Duval, 2006b, p. 88-89)"*

Figure 3. 1: A passage from Duval 2006b (p. 88-89.)

In Figure 3.1, Duval emphasizes in this passage the importance of recognizing the significance of interpretations, but he does not suggest that an interpretation can fully encompass a concept. Additionally, throughout his works, Duval focuses more on learners' "representations" of a concept. Duval (1993) highlights the importance of representation due to a cognitive contradiction in mathematical thinking: mathematical entities are neither physically present nor directly observable but can only be understood through representations. In further clarification, Duval (2006b) underscores this cognitive contradiction, where a single mathematical concept can be depicted in various forms, making it challenging to recognize that these different representations refer to the same object that is not visually observable. He adds that the unique nature of mathematics, compared to other fields of knowledge, highlights the significance of using representations and semiotics. Representations are necessary for understanding mathematical objects, and there is a cognitive challenge in transitioning from one representation of an object to another. Moreover, mathematical approaches inherently involve transforming semiotic representations. Duval (1993) investigates the use of signs, symbols, and gestures in various forms of theatre and art to convey meaning. The study emphasizes how performers apply semiotics to effectively communicate emotions, ideas, and narratives through physical movements, facial expressions, attire, and other visual elements. By examining this relationship between semiotics and performances, Duval (1993) provides valuable insights into the complex methods employed to convey messages on stage. This concept is part of Duval's differentiation between two processes: semiosis and noesis. Noesis refers to understanding an object as a concept, while semiosis involves perceiving certain properties of the object as representations. These representations serve as a means for humans to "see" characteristics of mathematical objects that cannot be directly perceived through the senses. For instance, a numerical representation in the form of a table can help visualize values in a function, while a graphical representation can indicate whether the function is continuous or not. Thus, mathematical learning always requires both noesis and semiosis.

In Duval's (1993) perspective, the process of learning mathematics involves connecting one's understanding and communication through two key factors. Firstly, since mathematics cannot be directly observed, it necessitates the use of representations to comprehend it. Hence, relying solely on one's understanding is insufficient; semiotic elements are essential. Secondly, while representations provide a glimpse into a concept or object, they do not encompass its entirety. Therefore, the cognitive activity of connecting understanding with communication, or semiosis, requires the involvement of noesis as well.

Additionally, Duval argues in his writings that meaning cannot be attributed to representations solely within a specific system. Instead, it is the system itself that enables individuals to link an object with its representation and identify the pertinent characteristics within that representation. In line with this viewpoint, Duval (2006b, p. 69) asserts that "representations are not primarily dependent on individuals but on systems that generate representations." This means that an individual cannot understand or interpret a representation without engaging with a particular system. For example, when presented with the visual depiction displayed in Figure 3.2, a person may interpret it using principles of physics and conclude that the objects depicted are two dice. However, in a mathematics context, that same individual would likely analyse it through a semiotic system and recognize it as a representation of an abstract mathematical concept-specifically, the sum of the numbers 1 and 3 or 5 and 1, which cannot be directly observed.

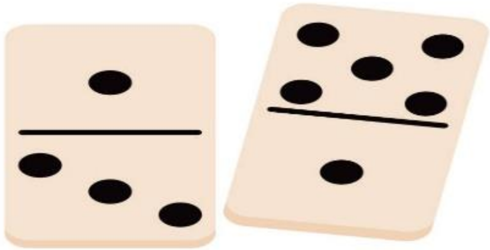


Figure 3. 2: Example demonstrating the importance of the system to associate a representation with an object. (Duval, 2006b, p.69)

Therefore, when attempting to understand a mathematical concept that lacks physical existence, a system of representation can be viewed as a form of a semiotic system. According to Duval (2006b, p. 81), decimal, binary, and other number representations are the most fundamental and effective semiotic systems available.

While recognizing the existence of other types of systems, our primary focus will now be on semiotic systems of representation. According to Duval (2006b, p. 74), in mathematics, a representation is only meaningful if it can be transformed into another form. By "transformation," he refers to two types of actions: processing (treatments) and conversion. These actions are essential in defining a register of representation. Additionally, Duval (1995) argues that the ability to vary the registers of semiotic representation is crucial for knowledge development.

3.3 Registers of representations

According to Duval (1993, p. 41), certain systems of representation can be considered registers of representation if they facilitate the three fundamental cognitive activities associated with semiosis. Duval (1998) identifies three primary types of representations: iconic, enactive, and symbolic.

Iconic representation comprises visual depictions that closely resemble the object or concept being represented. Enactive representation includes physical actions or manipulations that imitate the characteristics of the object. Symbolic representation relies on abstract symbols or signs that do not inherently resemble the object but are conventionally associated with it. Duval's classification provides a framework for understanding how different forms of representation can be applied in various cognitive tasks and fields. These three types hold special significance in the field of trigonometry. The enactive representation involves physical actions and manipulations, which in trigonometry could entail using physical objects or models that demonstrate properties of objects such as angles and triangles. The iconic representation includes visual aids such as diagrams and drawings, which are used to visually depict relationships and connections between angles and the sides of triangles. The graphical representation of a trigonometric function can also be regarded as an iconic representation. Finally, the symbolic representation consists of mathematical language and symbols. Trigonometry heavily relies on symbols like $\sin\theta$, $\cos\theta$, $\tan\theta$, and formulas to succinctly express relationships between angles and ratios.

Duval (1993, 1995) defines a register of representation as a semiotic system that enables three cognitive activities. The first activity involves accessing specific properties of an object through a representation. For example, when considering $\tan 40°$ in terms of this symbolic trigonometric representation, it can be seen as the value or output of the tangent function, with the input being 40 °. The second activity entails transforming this representation within the same system, using a rule to incorporate additional knowledge or data, referred to by Duval as treatment. For example, if one wishes to find cot 60°, the trigonometric relationship between the tangent and cotangent functions expressed in the rule $\cot\theta = \frac{1}{\tan\theta}$ can be employed. This allows finding the value of $\cot\theta$ as the reciprocal of $\tan\theta$ by using a treatment that relies on one register. Lastly, the third cognitive activity involves converting the initial representation into a second system to derive multiple meanings from the represented element-known as conversion. For example, one can interpret the expression $\tan 40°$ in terms of a right-angled triangle. In this right-angled triangle, where one angle is 40°, the ratio of the side opposite the angle to the side adjacent to the angle represents the value of tan 40°. That is, $\tan 40° = \frac{AB}{BC}$. See Figure 3.3.

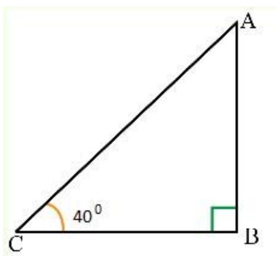


Figure 3. 3: Representation of tan 40° in terms of an iconic or geometric representation

Let us consider a problem that appeared in Activity 2 to explore how it could be solved using treatments and conversions. The students were asked to determine the value of $\tan(40°)$ in terms of a when $\cos(40°) = a$. Some students may choose to work with treatment transformations only. The task can be solved by staying strictly within the symbolic trigonometric register and using a treatment transformation. They can first find sin $40°$, by applying the trigonometric Pythagorean identity, $\sin^2 40° + \cos^2 40° = 1$, to get $\sin 40° = \sqrt{1 - cos^2 40°} = \sqrt{1 - a^2}$. Thereafter, they could perform a second action, which is also a treatment, which using the trigonometric relationship $\tan 40° = \frac{\sin 40°}{\cos 40°} = \frac{\sqrt{1-a^2}}{a}$. Alternatively, they could opt for a different approach to solving the problem using conversion transformations. In this approach, the student considers the trigonometric representation of $\cos(40°) = a$ and makes a conversion to an iconic representation on the Cartesian coordinate system, using a triangle embedded within a unit circle. The radius that forms the hypotenuse of the right- angled triangle is 1 unit, and the angle between the radius and the x-axis is $40°$. This is similar to Figure 3.3, except that the hypotenuse of the right-angled triangle is 1 unit, as illustrated in a student's response in Figure 3.4. Here, the student can execute a treatment within the iconic representation using the Pythagorean Theorem to calculate the value of all the sides of the triangle, thus involving a coordination between the trigonometric symbolic representation and the iconic representation. This coordination between the registers enables the student to use the properties of the triangle (iconic) to determine the lengths of the sides of the triangle and to then use the trigonometric symbolic register to derive the ratio $\tan 40°$. This is demonstrated in the student's work in Figure 3.4.

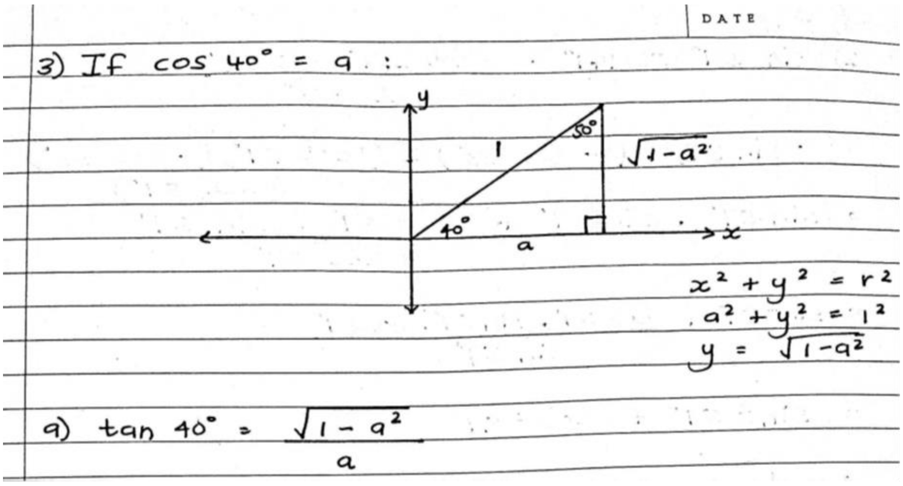


Figure 3. 4: Unit circle method used by a student

Therefore, the distinction between cognitive processing activity (treatment) and conversion lies in the fact that processing occurs within the same system, such as manipulating equations, whereas conversion involves a shift to a different system, like representing the relationships within a geometric figure as an algebraic expression. The student's conceptual understanding depends on the three cognitive activities mentioned above, as well as coordination, which is the ability to perform this conversion with speed and ease (Duval, 1993), and articulation, which is the ability to convert back and forth, for example, from a geometric figure to a numerical expression and vice versa (Duval, 2006b).

According to Duval (1993, 1995, & 1998), when learning mathematics, there is a need to convert between different forms of representation such as numbers, algebraic writings, and natural language. This can create difficulties in understanding the mathematical object. Even though representations like tables, graphs, and algebraic expressions belong to different forms of representation, they are not developed separately but rather interconnected. Duval (1988) argues that each form of representation has its properties, such as variables and symbols. To fully understand and coordinate these properties, they must be transformed into more meaningful units as we move from one form of representation to another. Therefore, the process of transitioning between different forms of representation should be undertaken to gain a clearer understanding of concepts such as functions.

However, according to Duval (1995) and Guzman R. (1998), it is emphasized that thought typically utilizes only one mode of representation at a given moment. The processes of conversion, coordination, and articulation are only employed when required by the specific situation that needs to be resolved. Difficulties in carrying out these conversions can potentially result in misunderstandings at a conceptual level.

3.4 Using Duval's theory to explain the underlying of transformation of trigonometric functions

From Duval (1993), a drawing can serve as a geometric depiction of a mathematical entity like a function. This offers a simplified means of illustrating the dynamics of the variables without explicitly providing all the components of the function, thanks to its visual nature. Another method of representing the function is through verbal representation, which Duval (1993) refers to as expressing the problem in words. In this form, the expression may include numerical or symbolic data that is always connected to a specific mathematical object, such as the functions we are currently interested in. Figure 3.5 shows an example of a function presented in the form of problems in words. By using the example in Figure 3.5, Duval (2005) states that the example serves as an illustration that exemplifies the process of constructing knowledge objects. Additionally, Duval explains that a statement expressed in everyday language can serve as a function because it represents various aspects, such as relationships, object definitions, and magnitude comparisons.

4) "Pour mesurer le volume d'un gaz, on a utilisé un ballon rempli d'oxygène, puis à l'aide des masses déposées par-dessus, on a exercé une pression sur le gaz. On a observé la relation entre la masse et le volume du gaz."

5) *To measure the volume of a gas, we used a balloon filled of oxygen, then using the masses placed on top, pressure was exerted on the gas. We observed the relationship between the mass and the volume of the gas.*

Figure 3. 5: Example of an implicitly expressed function as a word problem: volume of a gas as a function of mass. (Duval, 2005)

Duval's notion of processing (treatment) can be broken down into three different types: operative treatment, process-object treatment, and object-treatment relationship. These approaches enable students to interact with objects using various methods, both mentally and physically, which assists them in better understanding mathematical concepts. For trigonometric functions, operative treatment emphasizes utilizing the properties and relationships of trigonometric functions to solve mathematical problems. This approach often involves applying formulas, identities, and theorems from trigonometry to simplify expressions or find solutions. By leveraging trigonometric functions such as sine, cosine, tangent, and their inverses, the operative treatment seeks to provide efficient and accurate methods for tackling various mathematical scenarios related to angles, triangles, periodic phenomena, and more. For example, a crane is used in the construction of a shopping complex, where AC is the crossbeam supported by two metal stays, as shown in Figure 3.6. The length of AB is 32 m, and the length of BC is 15 m, and $\widehat{BCA} = 46°$, therefore, using the sine rule we can determine the size of the angle A. From

$$\frac{\sin \widehat{BAC}}{15m} = \frac{\sin 46°}{32m}$$

We have

$$\sin \widehat{BAC} = \frac{15m \sin 46°}{32m} = 0{,}3372$$

So,

$$\widehat{BAC} = 19{,}7°$$

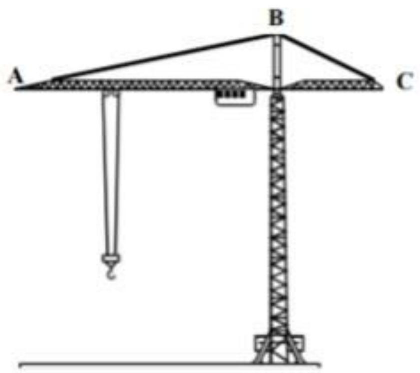


Figure 3. 6: Example of operative treatment problem

Process-object treatment refers to recognizing the processes involved in manipulating trigonometric functions while considering their underlying concepts. For example, transforming a sum or difference of two trigonometric functions into a product can simplify an equation significantly. By employing identities such as the sine or cosine addition formulas during this transformation process, students gain a deeper understanding of how trigonometric functions interact and can effectively solve complex equations more efficiently. Process-object treatment is another important approach. In this treatment, students focus on understanding the transformational processes involved in manipulating expressions containing trigonometric functions. They learn how to simplify complex expressions using various identities such as Pythagorean identities, double-angle identities, sum-to-product identities, etc. For instance, when solving a trigonometric equation like $\sin^2 x - \cos^2 x = 5$, students use the Pythagorean identity $\sin^2 x + \cos^2 x = 1$, to rewrite the original equation in term of a single function sine or cosine as $1 - \cos^2 x - cos^2 x \ = 5$. This will lead to a simplified equation $-2\cos^2 x = 4$, which simplifies to $\cos^2 x = -2$. From this simplified equation, the students can now easily identify that the solution will be found using the angles from the second and third quadrants. This treatment helps students develop problem-solving skills and strengthens their algebraic manipulation abilities. It also allows them to see connections between different trigonometric functions and explore the relationships among them.

Object-treatment relationship involves analysing the properties, characteristics, and relationships between different objects. In trigonometry, this treatment becomes relevant when considering the periodic behaviour of functions. For instance, students can explore the concepts of amplitude and period by studying sine or cosine graphs.

Understanding that changes in the amplitude impact how much a graph is stretched or compressed vertically, and changes in the period affect its horizontal lengthening or shrinking, allows students to connect function transformations with their graphical representations. For instance, when analysing the graphs of $y = 3\sin x$ and $y = \frac{1}{2}\sin x$ shown in Figure 3.7, we can observe that the amplitude of 3 in $y = 3\sin x$ causes the graph to extend higher and lower. On the other hand, the amplitude of $\frac{1}{2}$ in $y = \frac{1}{2}\sin x$ makes the graph shrink in value.

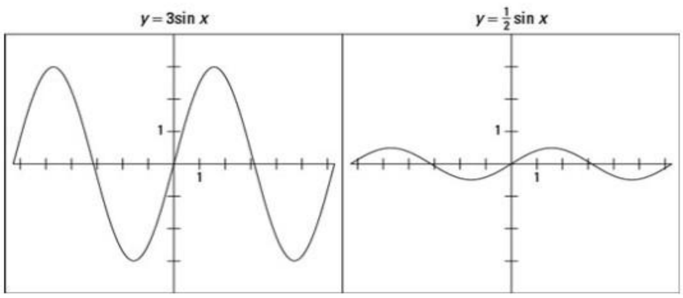


Figure 3. 7: Two graphs showing the change in amplitudes of a sine function

To examine the impact of a period of $y = 3\sin\frac{1}{2}x$, as depicted in Figure 3.8, it is evident that the function has twice the period of $y = \sin x$. Consequently, this causes the graph of $y = 3\sin\frac{1}{2}x$ to stretch horizontally by a factor of 2.

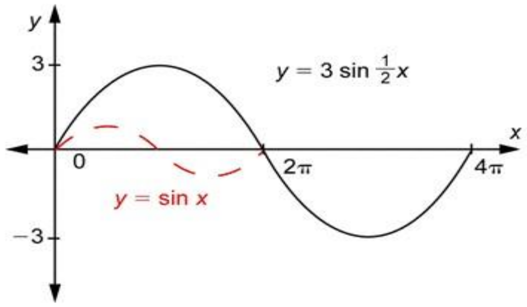


Figure 3. 8: A graph showing the change in period of a sine function

In contrast, conversion occurs when we change a geometric shape into a mathematical expression. This requires translating the visual aspects of the shape into symbols and equations that accurately depict its characteristics. Through this process, we transition from observing physical attributes such as shapes, angles, and sizes to representing them using variables, coefficients, operators, and constants. This transformation enables us to work with abstract ideas and manipulate them through algebraic operations, providing opportunities for deeper analysis and problem-solving in mathematics. We can find the equation of a sine graph, as shown in Figure 3.9, by using all relevant properties of the sine function equation. Hence, the Figure 4.9 represents the function $y = -3\sin x + 2$.

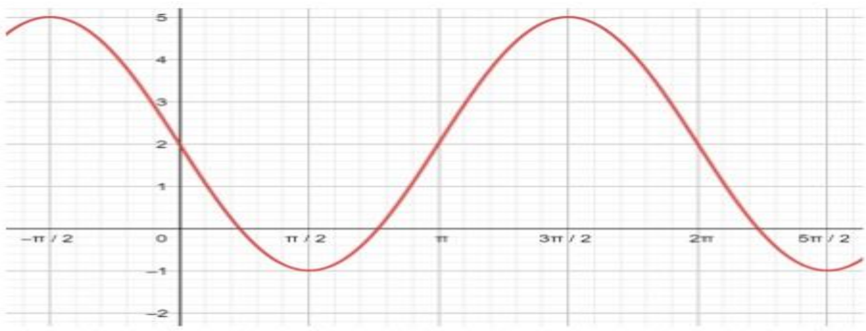


Figure 3. 9: Graph of $y = -3\sin x + 2$

Overall, the concept of using treatment and conversion in the transformation of trigonometric functions offers valuable insights into how these functions can be modified according to specific needs or problem-solving requirements.

CHAPTER 4
METHODOLOGY

4.1 Introduction

The primary aim of this study is to investigate and understand the challenges faced by first-year senior phase and FET PSTs in grappling with complex concepts related to trigonometric functions, specifically sine, cosine, and tangent. These concepts are foundational in senior phase and FET mathematics, yet they continue to pose considerable difficulties for many learners and prospective educators. As part of a broader effort to uncover the roots of these challenges, this study adopts a qualitative research approach aimed at unpacking both procedural errors and conceptual misunderstandings.

According to McMillan and Schumacher (2010), research methods refer to systematic strategies employed for the collection and rigorous analysis of data. Guided by this framework, the study commenced with a review of previous examination results pertaining to cosine and sine functions. Pilot studies are critical in educational research as they help to test research instruments, clarify procedures, and uncover potential challenges that might arise during the actual study. By doing so, researchers are better positioned to improve the reliability and validity of their tools, refine interview questions, and confirm the suitability of the data collection process. Ultimately, conducting a pilot study enhances the overall robustness of the research and increases the likelihood of producing meaningful and applicable results in the main study. Subsequently, a series of assessment activities, namely Activity 1 and Activity 2, were administered. These tasks were designed to evaluate PSTs' procedural knowledge and their ability to apply trigonometric principles in practical contexts.

In addition to written tasks, semi-structured interviews were conducted to gain deeper insights into the reasoning processes and misconceptions held by the participants. Due to the global Covid-19 pandemic, which significantly disrupted the educational landscape for students, teachers, and researchers alike, all research activities were carried out in a remote format. The assessments were submitted digitally, and interviews were conducted via video conferencing platforms, ensuring adherence to public health protocols and institutional safety guidelines. Despite the limitations of remote engagement, this mode of data collection provided an opportunity to continue meaningful academic inquiry during a period of widespread disruption. The overarching aim of the study is to address the following research questions:

1) *What errors do preservice mathematics teachers commonly make when working on tasks involving trigonometric functions?*
2) *How can Duval theory be used to explain these errors?*

4.2 Pilot study

This preliminary review was crucial in identifying prevalent trends, recurring errors, and deep-rooted conceptual difficulties experienced by PSTs, particularly regarding trigonometric functions. By systematically analysing past examination responses, the study established a foundational understanding of where PSTs struggle the most, thereby shaping the direction of the main research. This step served as a diagnostic tool, offering valuable insight into student thinking, error patterns, and misconceptions, and ultimately guiding the refinement of both the research design and focus areas for deeper investigation.

To strengthen the reliability and effectiveness of the study, a pilot phase was conducted prior to the main data collection. The aim of this pilot study was to test the research instruments, validate the methodology, and assess the feasibility of the planned procedures. The pilot focused specifically on the exam scripts of first-year senior phase and FET mathematics education PSTs from a KwaZulu-Natal university. With formal permission obtained from the university's Faculty of Education, a total of 85 examination scripts from the 2019 academic year were reviewed. These scripts were selected because they represented the most recent, accessible set of summative assessments aligned with the course's trigonometry content.

During the review process, special attention was paid to questions related to trigonometric functions, as these align directly with the objectives of the main study. The purpose was to explore students' responses to trigonometric tasks and assess how well they demonstrated conceptual understanding, procedural fluency, and representational flexibility. Two trigonometry questions were selected for deeper analysis based on their relevance and the frequency with which students attempted them. These questions were systematically analysed to identify patterns of success, types of errors, and areas of misunderstanding.

To maintain ethical standards and ensure participant anonymity, each script was assigned a unique code ranging from T1 to T85. These codes allowed for the accurate tracking and analysis of responses without revealing the identity of any student. This anonymization process was essential not only for maintaining confidentiality but also for promoting impartiality in the evaluation of students' work.

The analysis of the pilot data yielded rich information that informed the development of the main study's data collection instruments, including assessment activities and interview protocols. It also helped to fine-tune the categories for coding errors, misconceptions, and strategies evident in the students' responses. Furthermore, the pilot study confirmed the relevance of trigonometric transformation tasks in revealing underlying cognitive and representational challenges faced by PSTs. Overall, the pilot study served as a valuable preparatory step, helping to refine the study design, enhance the validity of the research instruments, and ensure a focused and ethical approach to data collection in the main study. The first question focused on the graphs of $f(x) = \cos(x) + q$ and $g(x) = \sin(x + p)$ which were drawn on the same system of axes for $-240° \leq x \leq 240°$ and students were asked to identify the values of q and p. The second question asked them to draw the graphs of $f(x) = 1 - 2\cos x$ and $g(x) = 3\sin(x + 30°)$ within the interval $-360° \leq x \leq 360°$.

4.3 Approach and design of the study

This study employed a qualitative research approach and utilized a case study design to explore the conceptual and procedural challenges that first-year senior phase and FET PSTs face in understanding and transforming trigonometric functions. The primary rationale for adopting a qualitative approach lies in its strength to provide a rich, in-depth exploration of learners' misconceptions and cognitive processes, particularly when working with complex mathematical concepts such as sine, cosine, and tangent functions. According to Kumar (2019), qualitative research is particularly well-suited for uncovering underlying understandings and misconceptions, especially when data are collected through open-ended questions and interactive interviews. By focusing on the descriptive and interpretive aspects of learner responses, this approach allows for a nuanced understanding of where and how learning breakdowns occur.

The research adopted a case study design, which is ideal for investigating a specific issue within its real-life context. Yin (2018) emphasizes that a case study is most effective when the boundaries between the phenomenon and the context are not clear and when researchers aim to answer "how" and "why" questions. Given that this study sought to explore how students conceptualize trigonometric transformations and why certain misconceptions persist, the case study design was highly appropriate. Furthermore, Seaton and Schwier (2014) argue that an exploratory case study is useful when the phenomenon under investigation involves multiple outcomes and when both "what" and "why" questions are central to the research inquiry. The present study aligns with this rationale as it explores the diversity of students' responses and errors in trigonometric problem-solving, which stem from various cognitive and instructional factors.

The participants in this research were 125 first-year senior phase and FET PSTs enrolled in a mathematics education module at the University of KwaZulu-Natal. These students were purposefully selected due to their exposure to trigonometric content in both high school and tertiary education contexts. Data were collected using multiple methods: two diagnostic assessment activities (Activity 1 and Activity 2) focused on trigonometric graphs and identities, and semi-structured interviews with a stratified sample of students representing different performance levels. The dual use of assessment tasks and interviews provided triangulation, thereby enhancing the validity of the study (Creswell & Poth, 2018). The assessments were designed to elicit specific errors in conceptual understanding and procedural fluency, such as incorrect identification of amplitude and period, misuse of trigonometric identities, and misinterpretation of phase shifts and domain restrictions.

In addition to the newly designed diagnostic tasks, permission was obtained from the university registrar to access previous examination scripts from 2019 on topics related to trigonometry and measurement. These historical scripts were used as part of a pilot study to identify recurring difficulties among PSTs. Analysis of the exam responses informed the design of Activity 1 and Activity 2, ensuring that the tasks addressed common conceptual gaps that had already been observed in the student population. This helped strengthen the reliability and relevance of the research instruments.

Although some descriptive quantitative data were employed to categorize students based on performance (i.e., low, average, and high), the primary focus of the study remained qualitative. The purpose was not to generalize findings statistically but to understand and interpret the types and sources of errors in-depth. As Marshall and Rossman (2016) argue, qualitative studies often incorporate limited numerical data to support categorization and sampling, but the emphasis lies in describing patterns and constructing meaning from the participants' perspectives.

Semi-structured interviews were conducted with 5 students selected across three performance bands, ensuring a diverse range of perspectives. Notably, the study also included high-performing students to gain a broader understanding of whether some misconceptions persist despite overall academic success. This strategy allowed the researcher to uncover subtle conceptual challenges that might not significantly impact performance scores but still reveal cognitive fragility (diSessa, 2018).

A critical feature of the data collection process was the use of open-ended questions during the interviews. These questions enabled the researcher to probe beyond surface-level responses and explore participants' reasoning, strategies, and decision-making processes. Open-ended questioning, as noted by Patton (2015), encourages participants to express their understanding in their own words, which is essential in educational research where learners often employ varied personal interpretations of mathematical concepts. Moreover, it allowed the interviewer to follow up on ambiguities, request clarifications, and delve into areas of confusion that might otherwise go unnoticed in closed-question formats.

The flexibility afforded by open-ended questions was particularly valuable in identifying representational and cognitive difficulties that are not always apparent from written assessments. For example, students were asked to explain why they chose a particular transformation rule or why they interpreted a graph in a certain way. Their explanations often revealed deeper misconceptions, such as conflating amplitude with vertical shifts or misunderstanding the role of radians versus degrees. These insights were only accessible because participants were encouraged to elaborate, justify, and even reconsider their reasoning during the interview process.

Furthermore, open-ended interviews created a dialogic interaction between the researcher and the participant. As argued by Brinkmann and Kvale (2015), the conversational nature of qualitative interviews allows researchers to co-construct meaning with participants. This form of engagement is especially useful in mathematics education, where learners' conceptions are often fragmented or incomplete. By guiding students through a reflective dialogue, the interviewer could uncover both the implicit and explicit structures of their mathematical thinking.

The use of video conferencing tools for conducting interviews, due to the Covid-19 pandemic, did not significantly hinder the depth of interaction. In fact, the online format provided a more comfortable setting for some participants, enabling them to articulate their thoughts more freely. The pandemic, while posing logistical challenges, also necessitated innovations in research methodology. As noted by Johnson et al. (2021), remote qualitative research has increasingly demonstrated its feasibility and effectiveness when appropriately planned and executed.

Ethical protocols were strictly adhered to throughout the study. Informed consent was obtained from all participants, and anonymity was preserved by assigning pseudonyms and student codes. All data, including video recordings and written assessments, were securely stored and used solely for research purposes. Ethical clearance was obtained from the University's Research Ethics Committee.

4.4 Research sample

This study employed a non-probability sampling method to collect data, allowing for the intentional selection of participants who could provide rich, relevant insights into the research problem. According to Creswell and Poth (2018), non-probability sampling is commonly used in qualitative research to identify individuals who are especially knowledgeable about the phenomenon under investigation. In this study, both convenience and purposive sampling techniques were employed. A total of 125 first-year PSTs were purposefully selected to participate, as they had already been introduced to trigonometric functions in their academic curriculum and were therefore capable of providing informed responses relevant to the study’s objectives.

Before data collection commenced, ethical clearance was obtained from the Department of Mathematics Education and the Humanities and Social Sciences Research Ethics Committee (HSSREC). Consent was granted to use students’ responses from Activity 1 and Activity 2, which were administered online. Only responses directly relevant to the aims of the study, specifically, those that demonstrated students' engagement with trigonometric functions and transformations, were selected for analysis. These written responses served as the primary data source for identifying common errors and misconceptions among the participants.

In selecting students for follow-up interviews, the researcher purposefully excluded those who achieved exceptionally high performance (i.e., an average score of 85% or above across both activities). This decision assumed that such students might not offer detailed insights into conceptual errors, which were central to the focus of this study. Instead, a performance-stratified sampling strategy was used to ensure a diverse representation of understanding levels among the interviewees. Specifically, five students were selected: one top performer (average scores between 70% and 84%), two middle performers (scores ranging from 50% to 69%), and two low performers (scores below 50%). This stratified approach allowed for a more comprehensive exploration of the various cognitive challenges students faced when engaging with trigonometric content.

All scripts from the selected participants were included in the final analysis, and additional permission was obtained where necessary to ensure compliance with ethical standards. The use of random purposive sampling, as advocated by Miles and Huberman (1994), was integral in enhancing the credibility and trustworthiness of the sample selection process. By purposefully selecting cases within defined performance brackets and introducing an element of randomization, the researcher was able to reduce bias and ensure that the interview data reflected a range of learner experiences and levels of understanding.

In sum, the strategic use of non-probability, purposive, and convenience sampling methods was appropriate for this qualitative, exploratory study. These techniques ensured that participants had relevant prior knowledge, allowed for focused data collection, and provided a well-rounded view of the learning difficulties encountered by PSTs in mastering trigonometric functions.

4.5 Methods and tools used to gather information, and the procedure followed in collecting data

4.5.1 Data collection process

During the second semester of the 2020 academic year, a total of 125 first-year senior phase and FET students enrolled in a mathematics education programme participated in a structured assessment process focused on trigonometric concepts. The assessment tasks were designed to evaluate students’ understanding across a range of foundational trigonometric skills. Specifically, the activities required students to define and identify key parameters of trigonometric graphs, compare trigonometric functions expressed in both radians and degrees, apply fundamental trigonometric identities to rewrite expressions in terms of variables, evaluate functions without the use of calculators, and simplify algebraic trigonometric expressions.

To ensure alignment with the curriculum, the assessment questions were developed by the researcher and provided to the lecturer responsible for teaching trigonometry and measurement within the module. These questions were subsequently integrated into two formal assessments administered to the students: the Tutorial Test and Sitting Test 1. These assessments were made available online on two different dates: the Tutorial Test was administered two weeks prior to Sitting Test 1. For this study, these assessments were referred to as Activity 1 and Activity 2, respectively.

Following the administration of the tests, which took place online due to Covid-19 restrictions, the module lecturer was responsible for grading all student submissions. Upon completion of the marking process, the researcher obtained access to the students completed assessments by downloading them from the university's online learning platform. The items specifically contributed by the researcher were extracted from the marked assessments and consolidated under the labels of Activity 1 and Activity 2 for analysis in the study.

After the assessment phase, individual interviews were conducted via Zoom with five selected students based on their performance levels in the two activities. These interviews were audio-recorded, transcribed verbatim, and converted into textual data for in-depth qualitative analysis. To facilitate accuracy and clarity during the interviews, the researcher employed screen-sharing to display each participant's own written responses to the relevant activity questions. This visual aid enabled students to reflect on their original answers, articulate their reasoning processes, and elaborate on the conceptual difficulties they encountered during the tests. This multimodal data collection process, combining online written assessments and interactive, screen-based interviews, enabled a deeper exploration of students' understanding and misconceptions related to trigonometric functions, thus enriching the study's findings.

4.5.2 Questions of Activity 1 and 2

Activity 1: Investigating Graphical Interpretation and Unit Conversion in Trigonometric Functions

The first of these diagnostic activities, referred to as Activity 1, was administered through an online learning platform. This task was accessible to students for a duration of eight hours. The relatively long availability window was a deliberate choice, rooted in the research design's goal of reducing performance anxiety and limiting the influence of time pressure. By ensuring that students had ample time to reflect on the tasks, the researchers aimed to isolate genuine conceptual and representational misunderstandings from simple mistakes that might arise under constrained test-taking conditions. The online mode of delivery also provided flexibility for students to engage with the material in a more self-directed and less stressful environment, potentially increasing the authenticity of their responses.

The primary goal of Activity 1 was to assess students' knowledge and comprehension of a few fundamental concepts related to trigonometric functions, with a particular emphasis on graphical representation. The task was specifically constructed to probe students' ability to identify, describe, and manipulate various parameters associated with trigonometric graphs. These parameters included amplitude, period, phase shift (horizontal translation), and vertical shift, each of which plays a significant role in the transformation and interpretation of sine, cosine, and tangent functions. Students were expected to demonstrate their understanding by recognizing how changes in these parameters affect the overall shape and position of trigonometric graphs.

In addition to testing students' facility with graphical transformations, Activity 1 also required them to engage with trigonometric functions expressed in two different measurement systems: degrees and radians. This dual representation was intentionally incorporated to assess whether students could fluently navigate between these units and understand the implications of each within the context of graphing and solving problems. Although most high school curricula in South Africa introduce trigonometric angles in degrees, more advanced mathematical contexts, particularly those in university-level mathematics, use radians extensively. For this reason, a fundamental aspect of trigonometric literacy involves the ability to interpret and convert between these systems seamlessly. Activity 1 thus served as a mechanism to uncover any gaps in students' conceptual grasp of angle measurement and to determine whether students' difficulties stemmed from unit confusion, representational shortcomings, or underlying conceptual weaknesses.

As students worked through the items in Activity 1, their responses were recorded digitally. This allowed for a subsequent detailed analysis of error patterns and solution strategies. In particular, the design of the activity enabled the identification of several categories of student misconceptions, such as the misinterpretation of the effects of a negative coefficient on graph orientation, incorrect assumptions about the relationship between angle measure and wave frequency, or confusion arising from switching between degree and radian modes on digital calculators. By targeting both parameter identification and unit conversion within a trigonometric context, Activity 1 functioned as a diagnostic tool capable of revealing nuanced aspects of students' conceptual frameworks.

Activity 2: Probing Algebraic Manipulation and Evaluation of Trigonometric Expressions

To complement the graphical and unit-based focus of Activity 1, a second assessment, Activity 2, was developed and implemented. This activity was made accessible to students online for a period of twelve hours. The extended timeframe was again chosen to ensure that students had sufficient opportunity to work through the problems methodically, without the added pressure of timed testing. The shift in focus from graphical interpretation to algebraic manipulation and evaluation allowed for a broader understanding of students' trigonometric competencies and a deeper exploration of their conceptual challenges.

Activity 2 was designed around three central learning objectives, each corresponding to a critical component of trigonometric proficiency. The first objective concerned the application of fundamental trigonometric identities. Students were required to demonstrate their ability to recognize and apply identities such as the Pythagorean identities (e.g., $\sin^2(\theta) + \cos^2(\theta) = 1$), quotient identities, and co-function identities to express trigonometric functions in terms of other variables. This aspect of the task was included to determine whether students possessed the algebraic dexterity and identity recognition skills necessary to manipulate expressions in a meaningful way. It also provided insight into students' ability to reason deductively within the trigonometric framework, an essential skill for success in advanced mathematics.

The second objective of Activity 2 focused on the evaluation of trigonometric functions without the use of calculators. This requirement served a dual purpose. On one hand, it assessed students' memorization and conceptual understanding of key trigonometric values at standard angles, such as 0°, 30°, 45°, 60°, and 90° (and their radian equivalents). On the other hand, it tested students' capacity to apply this knowledge in unfamiliar or semi-structured contexts. The expectation was that students would be able to evaluate expressions such as $\sin(30°)$, $\cos\left(\frac{\pi}{4}\right)$, or $\tan(60°)$ based on their understanding of the unit circle or reference triangles. This skill is often underdeveloped among learners who have grown accustomed to relying on calculators for such tasks, and the activity was intentionally structured to assess students' mental engagement with foundational concepts.

The third and final objective of Activity 2 involved the simplification of complex trigonometric expressions. In this portion of the assessment, students were presented with algebraically intricate expressions involving multiple trigonometric terms, often requiring the use of identities, factoring techniques, and logical reasoning to simplify. The aim here was to evaluate students' procedural fluency and their ability to make mathematically justified transformations. For example, a typical item might ask students to simplify an expression like $\frac{(1-\cos^2(\theta))}{\sin(\theta)}$ or to prove that a given identity holds true for all permissible values of θ. By analyzing students' attempts to simplify these expressions, the researchers could identify specific procedural errors, such as misapplication of identities, arithmetic miscalculations, or a failure to recognize structurally equivalent forms.

Overall, the design of Activity 2 provided a rich source of data on students' algebraic reasoning, identity use, and symbolic manipulation within trigonometry. It also offered insight into their problem-solving strategies and highlighted common points of failure that might not be immediately apparent through graphical analysis alone.

Purpose and Integration of Assessment Activities

Both Activity 1 and Activity 2 were integral to the research methodology of this study. Together, they offered a comprehensive framework for assessing the multifaceted nature of students' understanding of trigonometric functions. Whereas Activity 1 emphasized interpretive and representational aspects, focusing on graph parameters and unit conversion, Activity 2 targeted symbolic manipulation and analytical reasoning through identities, evaluation, and simplification.

The combined use of these two assessment tools allowed for a triangulated approach to data collection, enhancing the robustness of the findings and enabling the researcher to draw more nuanced conclusions about students' learning difficulties. Furthermore, by offering the assessments online and allowing extended access periods (eight hours for Activity 1 and twelve hours for Activity 2), the study attempted to minimize the impact of extraneous variables such as test anxiety, time pressure, or technological limitations. This design decision was grounded in the belief that conceptual understanding can only be accurately diagnosed in conditions that support thoughtful engagement, reflection, and multiple attempts.

Another significant feature of the assessments was their diagnostic orientation. Rather than serving as summative evaluations of performance, the tasks were designed primarily to reveal how students think and where their reasoning processes might go awry. This orientation aligns with the broader theoretical framework underpinning the study, which views errors not as mere failures but as indicators of students' current cognitive schemas and areas for instructional intervention.

The responses collected from both activities were subsequently subjected to a detailed error analysis, using both qualitative coding schemes and descriptive statistics to identify recurrent patterns and critical areas of misunderstanding. These findings then informed the discussion and interpretation of results within the broader context of PSTs education, with a particular focus on the development of mathematical proficiency in trigonometric contexts.

The online diagnostic assessments in this study played a crucial role in uncovering the intricacies of students' understanding and misunderstanding of trigonometry. Through a combination of graph interpretation, unit conversion, identity application, non-calculator evaluation, and expression simplification, these tasks provided a rich dataset from which meaningful educational insights could be drawn. The intentional design, scope, and flexibility of the assessments ensured that they functioned as authentic measures of conceptual comprehension rather than superficial checks of rote knowledge or procedural memorization.

As stated earlier, only the responses to questions directly relevant to the objectives of this study were selected for analysis. The questions that comprised Activity 1 are presented below:

1) *Sketch a graph of any trigonometric function and a graph of any non-trigonometric function.*
2) *Find the amplitude and the period of the functions* $f(x) = 2\,sin\left(\frac{\pi x}{4} + 3\right) + 4$ *and* $g(x) = -6\,cos\left(\frac{x}{3} - \frac{\pi}{4}\right) + 1$ *on the interval* $[0\,;2\pi]$.
3) *Sketch the graph of* $f(x) = cos\,x$ *and* $h(x) = 2\,cos\left(\frac{x}{2} - 30°\right) + 1$ *on the same set of axes using the interval* $[0°; 360°]$.
4) *Does* $\sin(45°)$ *and* $\sin\left(\frac{\pi}{4}\right)$ *have the same value, and can you provide a reason to support your answer?*
5) *Write an equation for a cosine function with an amplitude of 3, a period of π, a phase shift of* $\frac{\pi}{4}$ *to the left, and translated 1 unit up. Additionally, write an equation for a sine function with an amplitude of 3, a period of 5π, a phase shift of* $\frac{3\pi}{4}$ *to the right, and a vertical translation of 6 units down.*

Lastly, the questions given for Activity 2 were:

1) *Use the fundamental identities of trigonometric functions to determine the value of* $tan(40°)$ *and* $sec^2(130°)$ *when* $cos(40°) = a$.
2) *Calculate the value of* $sin\left(\frac{13\pi}{6}\right)$ *and* $2\,sin\,150°$ *without the use of a calculator.*
3) *simplify the expressions* $tan\,x\,csc\,x$ *and* $\frac{(cos^2 x)-4}{(cos\,x)-2}$ *for* $(cos\,x) - 2 \neq 0$.

4.6 Semi structured interviews

Yin (2013) emphasized that interviews are a valuable method for collecting data in case study research, defining them as two-way conversations in which the interviewer poses questions to obtain information and gain insight into the thoughts, opinions, ideas, and perspectives of the interviewee regarding the topic under investigation. Similarly, Warren and Karner (2015) highlighted that interviews function as social interactions grounded in conversation. Creswell (2014), Merriam and Tisdell (2016), and Punch and Oancea (2014) further classified interviews into three commonly used types: structured, semi-structured, and unstructured.

In this study, individual interviews were conducted using the Zoom video conferencing platform, following a semi-structured format. This format was selected due to the exploratory nature of the research, which necessitated the use of follow-up questions, probing, and prompts based on participant responses. The semi-structured approach allowed for flexibility, enabling the interviewer to adapt the sequence and phrasing of questions according to each student's responses while still ensuring that all key themes were explored.

All 125 participants in this study were assigned identification codes to maintain anonymity and ensure confidentiality. Students who participated in Activity 1 were labelled using the codes S1 through S125, while those involved in Activity 2 were coded from P1 through P125. These codes were used to identify students during the interview selection and data analysis phases. Interviewees were selected based on their performance in Activities 1 and 2 to provide a balanced representation of high-, middle-, and low-performing students.

Each interview session was scheduled for approximately 30 minutes, depending on student availability. The primary objective of these interviews was to investigate the underlying reasons for errors and misconceptions related to trigonometric function transformations among first-year mathematics education students. The interview questions were designed to be flexible and adapted to each student's specific activity responses. This allowed the researcher to delve deeper into the cognitive and conceptual difficulties faced by participants and to identify recurring patterns and categories of errors.

Following the guidance of Cohen and Arieli (2011), the interview protocol was carefully constructed to ensure that all relevant dimensions of the study were addressed. Open-ended questions formed the foundation of the interviews, providing PSTs with the opportunity to articulate their reasoning behind specific errors observed in their written responses. This open-ended structure allowed students to elaborate on their thought processes, enabling the researcher to capture detailed and nuanced insights into their understanding of trigonometric functions and their transformations.

The use of open-ended questions also fostered a dialogic interaction between the interviewer and the interviewees, facilitating a more natural and in-depth exploration of mathematical thinking. According to Kallio et al. (2016), a well-designed semi-structured interview enhances the credibility and reliability of qualitative research, as it balances consistency in questioning with the flexibility to explore emergent themes. In alignment with this recommendation, the interviews in this study followed a logical structure, yet the sequence and focus of questions were guided by participants' responses to previous prompts.

Although the interviews primarily concentrated on students' understanding of trigonometric function transformations, the organic flow of conversation allowed for the identification of other relevant mathematical challenges and misconceptions. Additional follow-up questions were asked where necessary to clarify students' explanations or to explore their reasoning more deeply.

All interviews were audio-recorded with the consent of the participants and later transcribed verbatim for data analysis. The verbatim transcripts served as a crucial source of qualitative data, complementing the written activity responses and enabling triangulation of data sources. This process ensured a more comprehensive understanding of students' difficulties and enriched the interpretation of findings within the broader context of the study.

4.7 Data analysis

Qualitative data analysis is a structured and interpretive process that involves the systematic organization, categorization, and interpretation of non-numerical data to uncover deeper meanings and insights. It places emphasis on the contextual relevance and significance of the data, aiming to enhance understanding of participants' perspectives, lived experiences, and interactions within a particular setting (Cohen & Arieli, 2011). The process typically involves identifying recurring patterns, thematic consistencies, meaningful categories, and regularities that emerge from the data (Creswell & Poth, 2018; Hesse-Biber & Leavy, 2011). Such analysis goes beyond mere description, seeking to interpret and explain underlying meanings and the dynamics at play in the data set.

In this study, a deductive approach was employed to guide the analysis of data derived from student activities and semi-structured interviews. The nature of the research, which is exploratory and interpretive, influenced the selection of this analytical approach. As argued by Casula et al. (2021) and Pearse (2019), deductive reasoning is particularly suitable for exploratory studies because it allows for the application of pre-established theoretical constructs while still accommodating the nuanced understanding of emerging phenomena. This approach enabled us to align our analysis with a conceptual framework informed by existing literature, providing a structured lens through which to interpret the data.

The analysis of the students' activities incorporated both quantitative and qualitative elements. Quantitative analysis involved basic descriptive statistics that helped to outline observable trends and frequencies in student responses. This provided a foundational overview of students' performance across the tasks. In parallel, qualitative content analysis was conducted to explore the depth and quality of students' reasoning and engagement. This was undertaken with reference to the components delineated in our conceptual framework, allowing us to identify the presence of specific cognitive, procedural, or representational challenges in their responses.

With respect to the interview data, the analysis was purely qualitative in nature. Since the data consisted of verbatim transcriptions of participants' responses, a content analysis approach was applied to interpret their explanations, reasoning processes, and reflections. This analysis aimed to uncover how participants conceptualized and articulated their understanding of trigonometric functions and their transformations, providing insight into their thought processes and potential misconceptions.

The theoretical lens underpinning our analysis was Duval's theory of semiotic representation, which offers a robust framework for examining the cognitive processes involved in understanding mathematical concepts. This theory, which will be elaborated on in the following chapter, provided the basis for interpreting the various semiotic transformations (e.g., conversions and treatments) evident in the participants' work. By examining students' responses through this lens, we were able to assess the extent to which they were able to navigate between different representations and correctly interpret transformations of trigonometric functions.

Furthermore, each participant's performance was evaluated across both data sources, the activity responses and interview transcripts. This dual analysis strategy allowed us to triangulate data and draw more comprehensive conclusions about the depth and consistency of participants' understanding. Details on the analytical procedures, coding schemes, and interpretation strategies are fully described in Chapters 6 and 7, where the findings are also presented and commented in relation to the study's objectives and theoretical framework.

4.8 Validity and reliability of the analysis

To ensure the reliability and integrity of data analysis in this study, a comprehensive set of evaluation procedures was adopted. These procedures were designed to strengthen the study's trustworthiness by incorporating multiple strategies across data collection, documentation, and interpretation stages. A key strategy employed was the use of multiple data collection methods, a practice which enhances the credibility and richness of qualitative research (Nieuwenhuis, 2020). Data were collected through online trigonometric function activities and virtual interviews with participants. The activity-based tasks assessed PSTs conceptual and procedural knowledge, while the interviews were used to delve deeper into the errors and misconceptions evident in their responses. This triangulation of data sources allowed for the validation of findings and the reduction of potential researcher bias.

The combination of activity responses and interview data provided complementary insights into participants' understanding of trigonometric concepts and transformations. Interviews were particularly effective in revealing the reasoning behind students' choices, clarifying ambiguities in written responses, and exploring the cognitive processes underlying their errors. As such, this mixed-method approach facilitated a more holistic and nuanced understanding of the learning challenges faced by PSTs in this mathematical domain.

In addition to methodological triangulation, the study implemented several measures to ensure the trustworthiness and dependability of the research process. These included strict adherence to research timelines, systematic data documentation through audit trails, and the consistent availability of original data for verification (Eisner, 2017). A detailed and transparent research process was maintained to allow for the replication of methods and the confirmation of findings. Each stage of data collection and analysis was meticulously documented in real-time, consistent with the audit trail principles outlined by Leedy and Ormrod (2019). This documentation included records of recruitment procedures, informed consent, interview protocols, and analysis strategies.

The researcher's role in facilitating the trigonometry and measurement module also contributed to the reliability of the research. Serving as a course tutor provided a unique opportunity to manage participant engagement, ensure timely communication, and observe learning processes throughout the academic term. Regular debriefing sessions with the research supervisor further enhanced the quality and consistency of the study. These consultations allowed for the discussion of emerging insights, alignment of procedures with ethical standards, and reflective decision-making throughout the research cycle.

Participant selection was guided by purposeful sampling, focusing specifically on first-year senior phase and FET PSTs. This choice was informed by an analysis of past examination performance data, which consistently highlighted challenges in the understanding and application of trigonometric functions. The targeted nature of the sample ensured relevance to the research problem and supported the generalizability of findings within similar educational contexts.

Moreover, the research process emphasized detailed descriptions during both data collection and data analysis phases, enabling transferability of findings. By thoroughly contextualizing the study environment, participants, and analytical procedures, other researchers can assess the applicability of the findings to comparable settings. The conclusions drawn were grounded in the participants' actual responses, thereby minimizing researcher bias and reinforcing the objectivity of the interpretations.

Collectively, these methodological strategies contributed to a robust and credible analysis process. The integration of diverse data sources, transparent documentation, adherence to ethical standards, and sustained engagement with participants and supervisory support all worked in concert to ensure the reliability and integrity of the study's findings and conclusions.

4.9 Ethical considerations

This study was conducted with a rigorous commitment to ethical standards, strictly adhering to South African research regulations to protect participants' rights, dignity, and welfare throughout the research process. Ethical approval was secured from the Human and Social Sciences Research Ethics Committee (HSSREC) at the University of KwaZulu-Natal. This approval affirmed that all research procedures met the ethical standards required for studies involving human participants and that adequate safeguards were in place to minimize any potential risk.

Prior to data collection, participants were provided with comprehensive information about the purpose, objectives, and procedures of the study. This information enabled them to make informed decisions regarding their participation, thereby reinforcing the principle of informed consent. Both verbal and written consent were obtained to ensure participants fully understood their rights and the voluntary nature of their involvement. Emphasis was placed on participants' autonomy, with assurances that they could withdraw from the study at any point without fear of penalty or negative consequences.

In line with ethical best practices, the confidentiality and anonymity of participants were rigorously upheld. All personal identifiers were removed or anonymized to ensure that participants' identities remained protected. The data collected were stored securely and accessed only by the research team, ensuring that responses could not be traced back to individual participants. These measures aimed to foster a secure environment in which participants felt comfortable sharing their thoughts, experiences, and reflections.

Recognizing the potential ethical complexities arising from the dual role of the researcher as both tutor and investigator, the study proactively implemented strategies to address potential power dynamics. This included maintaining a transparent and open dialogue with participants, clarifying that their participation, or decision not to participate, would not affect their academic standing or relationship with the tutor. By creating an atmosphere of mutual respect and trust, the study encouraged open and honest engagement from participants.

Additionally, this relational approach promoted reflective conversations around the value and significance of the research. These interactions helped align the academic interests of both the researcher and participants and emphasized the reciprocal benefits of the study. Participants were made aware of how their contributions would inform and potentially improve future mathematics education, particularly within the methodology course. This acknowledgment of mutual value served to reinforce the ethical principle of beneficence, ensuring that participants not only contributed to the study but also gained insight into their own learning and teaching practices.

Ultimately, the ethical framework guiding this research was grounded in the principles of autonomy, beneficence, confidentiality, and justice. Through transparent communication, consent procedures, and strategies to mitigate researcher–participant power imbalances, the study upheld its ethical responsibility to protect and respect the participants while generating valuable contributions to the field of mathematics teacher education.

CHAPTER 5
PILOT STUDY DATA ANALYSIS AND COMMENTS

This chapter provides a detailed analysis and commentary on the pilot study conducted using written responses from a cohort of 85 first-year senior phase and FET PSTs. These responses were collected from two selected trigonometry questions that formed part of a previous formal examination administered in 2019. The exam, which was a traditional sit-down paper, had a total duration of three hours and was part of the end-of-semester assessment for mathematics education students. The aim of this pilot analysis was to assess common trends, errors, and conceptual misunderstandings in students' solutions, thereby laying the groundwork for the main study. To uphold ethical standards and ensure the anonymity of participants, each of the 85 examination scripts was assigned a unique alphanumeric code ranging from T1 to T85. This coding system allowed for the systematic tracking and referencing of individual responses without revealing the identity of any student. By analysing the written work in this structured and confidential manner, the study was able to develop preliminary insights into the nature of errors preservice teachers make when engaging with trigonometric concepts. These findings played a crucial role in refining the research instruments and focus of the broader investigation that followed.

5.1 Identification of the vertical and horizontal shifts in cosine and sine graphs

In the first question identified for the pilot study, students were presented with a task (illustrated in Figure 1) that required them to determine the vertical shift of the cosine graph and the horizontal shift of the sine graph. The question is stated as follows:

In the diagram below, the graphs of $f(x) = \cos(x) + q$ and $g(x) = \sin(x + p)$ are drawn on the same system of axes for $-240° \leq x \leq 240°$. The graphs intersect at $\left(0°; \frac{1}{2}\right)$, $(-120°; -1)$ and $(240°; -1)$. Find the value of q and p.

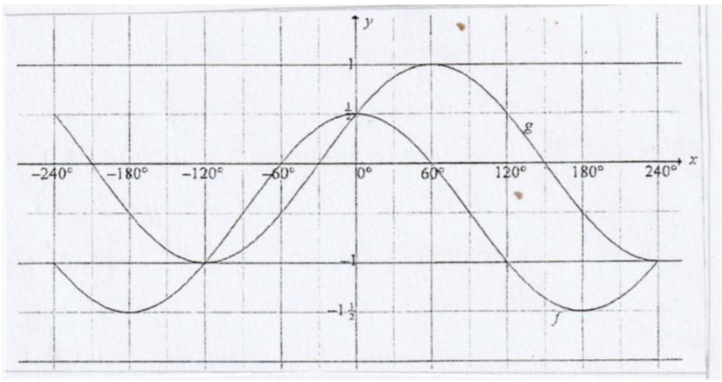


Figure 5.1: Question 1 of pilot study

This problem is a well-structured test of understanding trigonometric graph transformations, specifically vertical shifts (via q) and horizontal phase shifts (via p). The use of intersection points allows students to reverse-engineer the transformations by equating the values of the functions at specific points. It encourages conceptual reasoning, linking the graphical and algebraic forms of sine and cosine functions.

Moreover, the inclusion of symmetrical points like $(-120°)$ and $240°$, both mapping to $y = -1$, reinforces understanding of periodicity and function behaviour beyond the standard 0° to $360°$ interval. The problem is pedagogically effective in diagnosing common student difficulties with shifts and identifying function parameters from graphs.

We begin by discussing the responses related to the vertical shift in the cosine function.

5.1.1 Results for the item about the vertical shift (q) of the cosine graph

In terms of identifying the vertical shift in the graph of $f(x) = \cos(x) + q$, most of the students (all 85 participants) were able to arrive at the correct value, namely $q = \frac{1}{2}$. However, it is important to note that 47 of these students did not provide any written explanation or justification for their answer. While it is possible that they intuitively arrived at the correct value by comparing the standard cosine graph $y = \cos(x)$ to the transformed version $f(x) = \cos(x) + q$, the lack of reasoning makes it difficult to determine whether their responses reflected a deep conceptual understanding or mere guesswork.

Among those who did show their working, various strategies were employed. One of the most common methods used to determine the value of q was through substitution into the function at specific points of intersection that were clearly identified in the question. This approach was adopted by 33 students, who made use of the fact that the graph of $f(x) = \cos(x) + q$ intersected the sine graph $y = \sin(x + p)$ at three known points: $(-120°;\ -1)$,

$\left(0°;\ \frac{1}{2}\right)$, and $(240°; -1)$. By substituting these coordinates into the cosine equation and solving for q, students demonstrated a more structured and mathematical method for identifying the vertical shift.

A portion of this group, 25 students, chose to substitute the point $\left(0°;\ \frac{1}{2}\right)$, into the equation $f(x) = \cos(x) + q$, likely because the cosine of 0 degrees is a well-known value: $\cos(0°) = 1$. Using this information, they set up the equation $\frac{1}{2} = 1 + q$, and correctly solved for $q = -\frac{1}{2}$. They used the following steps:

at $\left(0^{\circ}, \frac{1}{2}\right)$, we have

$$f(0°) = \cos 0° + q$$

$$\frac{1}{2} = 1 + q$$

$$q = -\frac{1}{2}$$

This demonstrates that these students were not only capable of recalling trigonometric identities but were also able to apply them in a problem-solving context, using logical reasoning to determine a transformation parameter from a graphical representation.

Nine students applied a visual comparison method, noting that the maximum point of the standard cosine graph is 1, and observing that the maximum point in the graph provided in the question was located at $y = \frac{1}{2}$. From this comparison, they deduced that the entire graph must have shifted downward by $\frac{1}{2}$ units. Although this approach is less algebraic, it reflects a correct conceptual understanding of the impact of the constant q on the vertical translation of a function.

Despite the high number of correct answers, the lack of explanation by over half the students raises concerns about whether the understanding was procedural or conceptual. This observation suggests the need for encouraging students to always justify their answers, especially in diagnostic assessments, to better gauge their depth of understanding and identify potential misconceptions that may otherwise remain hidden.

There were six students who opted for the substitution at the point $(-120°, -1)$, and two students used the point $(240°, -1)$**.** Eight students used substitution with the point $(60°, 0)$. One student found the average of the maximum and minimum point as the vertical shift. They used the following:

$$q = \frac{max + min}{2}$$

We have, $$q = \frac{\frac{1}{2} + (-1)}{2}$$

$$q = -\frac{1}{2}$$

There were nine students who provided incorrect answers when attempting to determine the vertical shift q in the equation $f(x) = \cos(x) + q$. Unfortunately, none of these students included explanatory notes or reasoning for their responses, making it difficult to fully understand the nature of their misconceptions. However, an analysis of their answers does offer some insight into the types of errors made and their possible cognitive origins.

Two of the students submitted $q = 30°$ as their answer. This response appears to reflect a mix-up between the parameters p and q in the given question. In the context of the problem, p represents the horizontal shift (or phase shift) of the sine function and is measured in degrees, while q represents the vertical shift of the cosine function and is a numerical value, not an angle. The incorrect use of an angular value for a vertical translation indicates that these students may have had difficulty distinguishing between different transformation parameters in trigonometric equations. It also suggests a lack of familiarity with the roles that constants play in function transformations, particularly when multiple transformations are involved simultaneously.

The remaining seven students incorrectly identified $q = \frac{1}{2}$ as the vertical shift. This error appears to stem from a misunderstanding of the basic properties of the cosine function. Specifically, these students seem to have assumed that the cosine function starts at zero, i.e., that $\cos(0°) = 0$, which is incorrect. The correct value is $\cos(0°) = 1$. Based on this flawed assumption, they may have reasoned that if the graph passed through the point $\left(0°; \frac{1}{2}\right)$, then the shift from an assumed baseline of zero must be $\frac{1}{2}$. In other words, they substituted $\cos(0°) = 0$ into the expression $\cos(0°) + q = \frac{1}{2}$, leading to the conclusion that $q = \frac{1}{2}$.

This error highlights a significant conceptual gap in their understanding of the cosine function. It reflects not only a misremembering of trigonometric values but also a weak grasp of how vertical shifts affect function graphs. The confusion between the amplitude and the vertical shift may also be a contributing factor. Some students may have misinterpreted the vertical position of the graph's peak as the result of a vertical shift, when in fact it is a combination of the function's standard behaviour and the vertical transformation.

These errors suggest that while some students can apply formulae and transformations correctly, others may rely too heavily on visual intuition or memorized values without a firm conceptual foundation. To address such misconceptions, instruction should focus on reinforcing the key properties of trigonometric functions, such as standard values at specific angles, amplitude, period, and transformations. Additionally, encouraging students to explain their reasoning in written form, even in multiple-choice or numerical answer contexts, can help educators identify and remediate underlying misconceptions before they become entrenched.

*5.1.2 Responses to the item about the horizontal shift (*p*) of the sine graph*

With respect to identifying the horizontal shift p in the sine graph $g(x) = \sin(x + p)$, a total of 79 students provided correct responses. Among these, 40 students included written working or calculations to support their answers. The analysis of their strategies reveals a strong reliance on substitution techniques, particularly using known intersection points that were clearly stated in the question. This suggests that many students were comfortable applying algebraic reasoning when given specific coordinates, a positive indicator of procedural fluency.

Of the 40 students who showed their work, 32 used substitutions as their primary method to determine the value of p. These students substituted one of the three points of intersection, namely $(-120°;\ -1)$, $\left(0°;\ \frac{1}{2}\right)$, or $(240°; -1)$ into the function $g(x) = \sin(x + p)$ and solved for p. Among this group, 23 students chose the point $\left(0°;\ \frac{1}{2}\right)$, which was also commonly used in calculating the vertical shift q for the cosine graph. This pattern indicates that students recognized this point as particularly useful, likely because both sine and cosine values are straightforward to compute at 0°, and the function values align neatly with familiar trigonometric identities.

Another subset of seven students opted to use the point $(120°;\ -1)$, again reflecting a consistent strategy of selecting coordinates that had been helpful in the earlier task of determining q. This reuse of the same point across both functions suggests an efficient and structured approach to problem-solving. However, it may also reflect a reliance on memorization or pattern recognition rather than a deeper understanding of the distinct roles that horizontal and vertical shifts play in graph transformations.

Two students selected the point $(150°;\ 0)$, a less commonly used coordinate, but one that is still valid. Their successful use of this point to determine p reflects a broader understanding of sine function behaviour, particularly its zero-crossings and the implications of phase shifts on its periodic structure.

Additionally, ten students adopted a more visual approach. They identified the coordinate $(60°;\ 1)$ directly from the graph and substituted it into the equation $g(x) = \sin(x + p)$ to solve for p. Their ability to read this point accurately from the graph and apply it successfully in a substitution demonstrates both graphical literacy and algebraic competency. This approach, while dependent on visual accuracy, reflects a solid understanding of the relationship between graphical features and algebraic representations of trigonometric functions.

Overall, the high success rate in determining the horizontal shift and the diversity of methods used suggest that most students were confident in applying function transformations to sine graphs. Nevertheless, the variation in strategies used, ranging from algebraic substitution to visual analysis, highlights the importance of offering multiple representations and solution paths in instruction. Encouraging students to explain their reasoning, even when answers are correct, remains critical for reinforcing conceptual understanding and identifying any lingering misconceptions.

Among the students who answered incorrectly regarding the horizontal shift of the sine function $g(x) = \sin(x + p)$, four students provided the answer $p = 60°$. These students correctly understood that the question required them to determine a horizontal shift, which indicates a partial conceptual grasp of the transformation being assessed. However, they mistakenly based their calculations on the x −intercept of the cosine function, $f(x) = \cos(x) + q$, rather than analyzing the sine function g, as was required. This confusion likely stems from the graphs of both functions being presented on the same set of axes, which may have led these students to misidentify key features and attribute them to the wrong function. By focusing on the x −intercept of $f(x)$, these students misapplied a graphical characteristic of one function in the context of another. While their identification of a shift is encouraging in that it demonstrates some understanding of graph transformations, it also suggests an underlying difficulty in distinguishing between multiple functions when represented simultaneously. This highlights the importance of emphasizing function-specific features in teaching, especially in scenarios where students must interpret composite graphs or compare multiple functions.

In another type of error, two students provided the answer $x = 30°$ instead of identifying the correct value of p. The written work of one such student is shown below. In this student's working, the equation $x + p = 0$ was initially stated, indicating an awareness that the phase shift involves a transformation of the input variable. However, the student then manipulated the equation incorrectly:

$$x + p = 0$$
$$x - 30° = 0$$
$$x = 30°$$

This solution reveals a conceptual misunderstanding in how the value of p should be isolated and interpreted. Rather than solving for p, the student substituted $p = 30°$ into the equation and treated x as the unknown. The error here is subtle but important: the student appears to understand that the sine graph has been shifted horizontally, but they lack fluency in applying algebraic reasoning to determine the magnitude and direction of this shift. It also suggests some confusion about the relationship between the functional transformation and its impact on the graph's coordinates.

These errors reflect broader difficulties students face when working with horizontal transformations. Unlike vertical shifts, which tend to be more straightforward and intuitive, phase shifts require students to understand the manipulation of the input variable and how it alters the graph's position along the $x - axis$. Moreover, the fact that these students recognized the shift but failed to derive it correctly shows a gap between conceptual awareness and procedural accuracy.

To address these issues, it is important that educators reinforce not only how transformations are represented algebraically but also how they connect to specific features of graphs. Scaffolded practice that isolates each function and transformation, along with reflective tasks where students explain their reasoning, can help bridge the gap between recognition and proper execution.

5.1.3 Concluding comments for the items on vertical and horizontal shift

The analysis of students' responses reveals several key errors and misconceptions in determining the vertical shift q and horizontal shift p of trigonometric functions. For q, some students (5) confused it with the horizontal shift and incorrectly gave $q = 30°$, indicating a mix-up between parameters. Others (7) assumed $\cos(0°) = 0$ instead of 1, leading them to conclude $q = \frac{1}{2}$ by misinterpreting the point $\left(0°; \frac{1}{2}\right)$. These misconceptions reflect misunderstandings of cosine's standard values and the nature of vertical transformations.

In determining p, a larger number of students (18) erred. Five repeated the mix-up and gave $p = \frac{1}{2}$, likely reversing the roles of p and q. Four students used the x-intercept of the cosine graph instead of analysing the sine graph, showing confusion between the two functions. Two students gave $x = 30°$, incorrectly manipulating the equation $x + p = 0$, substituting p instead of solving for it. These patterns suggest challenges in distinguishing between function behaviours and correctly interpreting algebraic transformations, particularly with horizontal shifts. Targeted instruction on function roles and transformations is needed to address these gaps.

5.2 Sketching graphs of trigonometric function from the algebraic equations

Sketching the graphs of trigonometric functions from their algebraic equations is a crucial skill in mathematics education. It enables learners to connect symbolic representations with visual understanding and is fundamental in exploring the behaviour of periodic phenomena. Trigonometric functions such as sine, cosine, and tangent are defined algebraically, but their graphical representations reveal essential characteristics such as amplitude, period, phase shift, and vertical shift.

The second selected question for this pilot study was to sketch the graphs of cosine and sine functions from their algebraic equations. The question stated to draw the graphs of $f(x) = 1 - 2\cos x$ and $g(x) = 3\sin(x + 30°)$ within the interval $-360° \leq x \leq 360°$, and show all important points.

A total of 48 students successfully sketched the graphs of both functions $f(x) = 1 - 2\cos x$ and $g(x) = 3\sin(x + 30°)$ over the specified interval $-360° \leq x \leq 360°$. These students accurately represented all the key features of the graphs, including amplitude, phase shift, vertical shift, and period. Their responses are illustrated in Figure 1, which clearly displays the correct graphical representations of both trigonometric functions within the given domain. In Figure 5.2, the graph of function f and g are shown. The function f exhibits a vertical shift of +1 and an amplitude of 2, resulting in a wave that oscillates between a maximum of 3 and a minimum of (-1). The cosine nature of the function means it begins at a maximum point when $x = 0°$ and continues its periodic pattern at regular intervals of $360°$. The function g represents the sine function with an amplitude of 3 and a phase shift of 30° to the left. This shift is evident in the placement of its maximum and minimum points within the interval. The function oscillates between (-3) and (+3) and completes a full cycle every 360°.

These 60 students demonstrated a solid understanding of trigonometric transformations and accurately applied their knowledge to graphically represent the given functions over the specified interval.

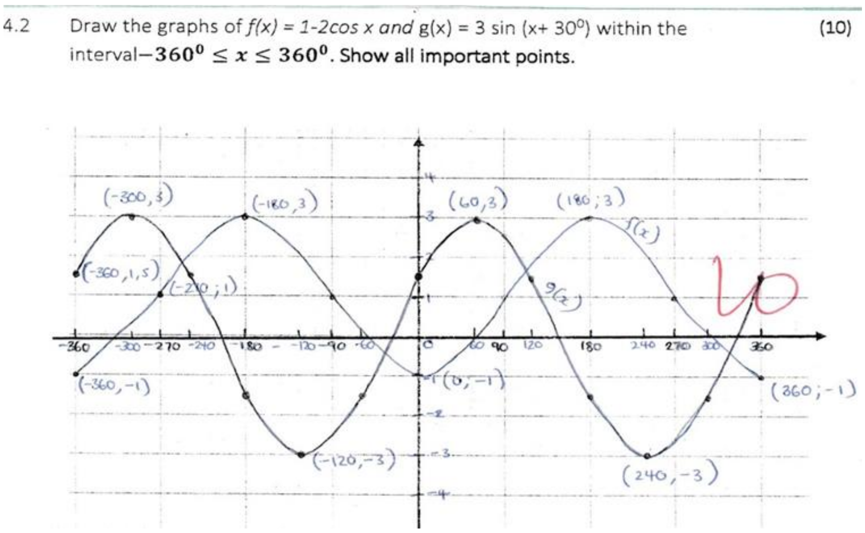


Figure 5.2: Correct drawing of the graphs of *f* and *g* within the given interval provided by T43

One student did not answer the question as shown by Figure 5.3. This indicates a conceptual gap in recognizing the fundamental shapes and properties of sine and cosine graphs. It also shows a lack of understanding of how to apply transformations (amplitude, phase shift, and vertical shift) to standard trigonometric functions. Corrective feedback should focus on revisiting the basic shapes and transformations of trigonometric functions and practicing how to sketch them accurately from their algebraic forms.

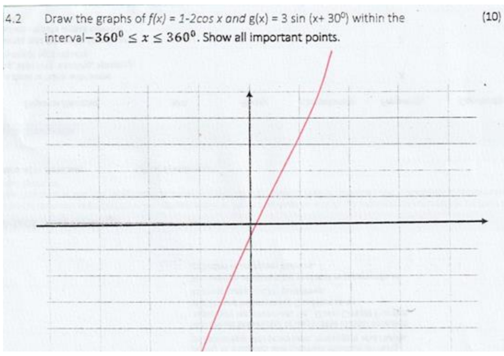


Figure 5.3: No answer was given by T27

A total of 36 students were unable to correctly draw the graphs of functions f and g. Of these, six students made errors in sketching the graph of f, while 30 students struggled with the graph of g.

The responses provided indicate various conceptual misunderstandings, particularly regarding transformations such as reflections, phase shifts, and amplitude changes. These difficulties suggest gaps in students' knowledge of trigonometric graph behaviour. In the sections that follow, we provide a detailed analysis of the specific errors observed in the students' representations of each individual function.

Among the six students who failed to correctly draw the graph of the function $f(x) = 1 - 2\cos(x)$, four made a significant conceptual error by not recognizing that the graph reflects the standard cosine graph about the x-axis. This misconception reveals a lack of knowledge regarding the influence of negative coefficients in trigonometric equations. In the function $f(x) = 1 - 2\cos(x)$, the coefficient of (-2) indicates two key transformations: first, a vertical stretch of the cosine graph by a factor of 2, and second, a reflection across the x-axis. These transformations mean that instead of starting at the maximum point, as is typical for the basic cosine function $\cos(x)$, the graph begins at its minimum point. From there, it rises to a maximum and then returns to the minimum over the course of one full period. This behaviour is characteristic of a negative cosine function and must be clearly understood to graph it accurately.

However, the four students in question failed to apply the reflection and instead drew the graph as if it were a positive cosine function. This resulted in a graph that started at a maximum point and displayed the shape of a standard $\cos(x)$ graph, but this was fundamentally incorrect given the negative coefficient. Figure 5.4 illustrates one such incorrect response. This figure clearly shows that the student began the graph at the wrong point and did not reflect the graph as required. Such an error demonstrates that these students may memorize the basic shape of trigonometric graphs without fully understanding how various parameters, such as the sign and magnitude of coefficients, transform the graph.

In addition to missing the reflection, all six students, not just the four mentioned above, failed to apply the correct vertical shift of the graph. The function $f(x) = 1 - 2\cos(x)$ includes a constant term of $(+1)$, which means that the entire graph must be shifted one unit upward. This transformation changes the central axis of the graph from $y = 0$ (as in the standard cosine function) to $y = 1$. Consequently, instead of oscillating between (-2) and 2 (as would happen with $(-2\cos(x))$), the graph of $f(x)$ oscillates between (-1) and 3. The amplitude remains 2, but the vertical shift repositions the graph entirely above and below the line $y = 1$. However, none of the six students reflected this transformation in their graphs. Figure 5.5 displays one such incorrect attempt, where the graph is drawn without any visible vertical shift, staying centred around the x-axis as if the "+1" term were not present in the function at all.

This pattern of error suggests that students either overlooked or misunderstood the role of the constant "+1" in the equation. In many cases, students treat the constant term as part of the amplitude or ignore it entirely. This may stem from limited exposure to transformed trigonometric functions in classroom practice or a lack of experience in combining multiple transformations (reflection, vertical shift, and amplitude change) in a single function. Moreover, it appears that these students may not systematically analyse the components of a trigonometric equation. Instead of deconstructing the function into its individual transformation parts (amplitude, period, phase shift, and vertical shift), they may rely on superficial characteristics or pattern-matching strategies that break down when faced with more complex transformations.

Another aspect of the error is the failure to adjust the range of the function accordingly. The standard cosine function $\cos(x)$ has a range of $[-1; 1]$, but when it is multiplied by (-2), its range becomes $[-2; 2]$, and after applying the vertical shift of $(+1)$, the correct range of the function $f(x) = 1 - 2\cos(x)$ becomes $[-1; 3]$. None of the student graphs reflected this adjusted range. Instead, the graphs remained confined to the default cosine range, reinforcing the observation that these students did not process how the vertical shift and amplitude affect the output values.

In addition to misapplying the transformation, another issue observed was lack of labelling of critical points. Important features of the graph such as maximum and minimum values, intercepts, and the midline were either incorrectly placed or entirely omitted in most of the incorrect responses. Accurate graphing of trigonometric functions requires not only plotting the general wave shape but also identifying and marking these key points, which help verify whether the graph reflects the equation accurately. The absence of these points suggests that students may not fully understand how to calculate or interpret them within the context of transformed functions.

Furthermore, these mistakes point to a broader pedagogical issue: students may not be taught to visualize how each part of a trigonometric function contributes to the overall graph. Rather than thinking of the function as the result of a sequence of transformations applied to a basic graph, they may treat each new function as a separate, unrelated case. This lack of relational understanding leads to confusion when multiple transformations are present, as was the case in this question.

To support students in overcoming these misconceptions, instruction should emphasize the process of analysing and sketching trigonometric functions step by step. Teachers should guide students in identifying amplitude, vertical shifts, reflections, and horizontal shifts separately before combining them. Providing structured exercises where students are explicitly asked to describe the transformation effects of each parameter could help solidify their understanding. Visual aids, such as dynamic graphing tools or step-by-step transformations using graph overlays, can also help students see how changes in the equation result in visible changes to the graph. Additionally, students should be given practice opportunities to compare original and transformed graphs and to explain the reasoning behind each transformation.

The errors made by these six students in graphing the function $f(x) = 1 - 2\cos(x)$ highlight a consistent lack of understanding regarding reflections and vertical shifts. These are critical components of trigonometric graphing, and their omission leads to inaccurate representations. By focusing on a detailed analysis of each transformation and emphasizing reasoning over memorization, educators can help students develop a stronger conceptual foundation and greater confidence in graphing transformed trigonometric functions.

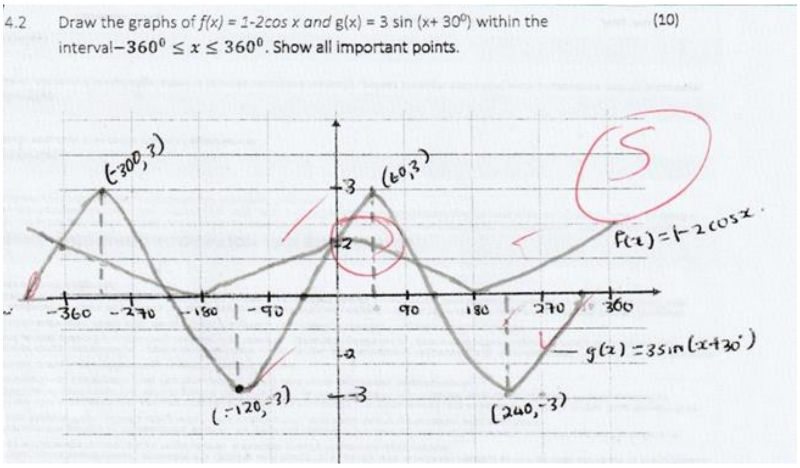


Figure 5.4: Incorrect reflection of f about the x-axis from T10

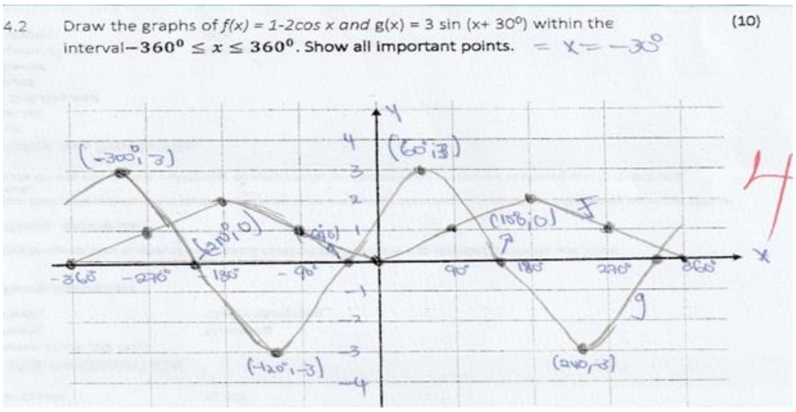


Figure 5.5: Incorrect vertical upward shift of the function *f* from T32

Among the students who failed to correctly draw the graph of the function $g(x) = 3\sin(x + 30°)$, 28 out of 30 did not correctly represent the graph's amplitude and phase shift, two of the most critical components in understanding and sketching transformed sine functions. Only two students managed to draw the graph with the correct amplitude, but even they misapplied the phase shift, thereby producing a graph that was still inaccurate. These consistent errors point to deep-rooted misconceptions regarding how transformations affect the sine function, especially when multiple transformations are involved simultaneously.

One of the most recurring issues was the misinterpretation of the amplitude. Although the coefficient "3" in the given function clearly defines the amplitude of the sine wave, many students failed to acknowledge this. Instead, they incorrectly estimated the amplitude to be around 2.5 or 2.6, a value close to but not equal to the expected value. This error did not result from a misreading of the function but rather from a flawed calculation process. Many of these students continued to assume that the function's maximum point occurs at $x = 90°$, as it does in the standard sine function $y = \sin(x)$. They then substituted $x = 90°$ into the function $g(x) = 3\sin(x + 30°)$, leading them to calculate:

$$g(90°) = 3(90° + 30°) = 3\sin(120°) \approx 2{,}598$$

While this calculation is mathematically accurate, it reflects a fundamental misunderstanding: due to the phase shift of $(+30°)$, the location of the turning point (maximum) of the graph is no longer at $x = 90°$, but at $x = 60°$. Hence, although their calculation yielded a plausible value for the y-coordinate, it was based on a misaligned x-coordinate, showing that students were applying function values procedurally without fully grasping the structural transformation caused by the phase shift. This misalignment distorted their sketches, as they retained the standard positions of turning points rather than adjusting them according to the transformation.

As a result, 18 students produced what can be described as skewed graphs, where the turning points remained fixed at $x = -270°; -90°;\ 90°$, and so on, as though no horizontal shift had occurred. These graphs appear to have the correct shape and general oscillation, and notably, these students correctly placed the zeros of the function at $x = -120°;\ -30°; 150°; 330°$. This suggests that they were partially aware of the phase shift but applied it inconsistently, successfully to the zeros but not to the maxima and minima. Figure 5.6 illustrates one such case, where the graph appears structurally sound but contains misaligned turning points due to an incomplete understanding of the phase shift's effect.

Meanwhile, the remaining students made errors that were more pervasive. The final 12 students incorrectly placed both the zeros and the turning points, producing graphs that were wholly inconsistent with the expected wave pattern of $g(x) = 3\sin(x + 30°)$. These students seemed to disregard or misinterpret both the amplitude and the phase shift, as shown in Figure 5.7. Their graphs failed to demonstrate any consistent application of the sine function's periodic nature or the effect of transformations, indicating a more fundamental gap in conceptual understanding.

An overarching misconception observed in all 30 students was their inability to interpret the meaning of the "$+30°$" inside the sine function. The function $g(x) = 3\sin(x + 30°)$ contains a horizontal shift embedded in its argument. However, the students did not correctly understand this notation as indicating a leftward phase shift of 30°. In trigonometry, a function of the form $\sin(x + c)$ results in a horizontal translation of the graph c units to the left, while $\sin(x - c)$ results in a shift to the right. This principle seems to have been overlooked entirely. Students either ignored the transformation or treated it as a vertical shift or numerical constant.

To accurately find the phase shift, students should have solved for the horizontal translation by setting the expression inside the sine function to zero:

$$x + 30° = 0 \Rightarrow x = -30°$$

This result tells us that every point on the graph is shifted 30° to the left. Unfortunately, none of the students correctly demonstrated this process. Instead, they drew sine graphs starting at $x = 0°$, which is the standard starting point for $y = \sin(x)$, not for $y = \sin(x + 30°)$. This incorrect starting point compromised the placement of all key features in their graphs. Figure 5.7 shows a clear example of such a mistake, where the student failed to apply any horizontal shift and used an incorrect amplitude of approximately 2.5 instead of the correct value of 3.

These findings reveal a consistent pattern: while students may be able to recall the general shape of the sine function and even some of its key points, they often struggle to apply transformations, especially horizontal ones, in a precise and meaningful way. Phase shifts are particularly challenging because they require students to understand how changes inside the function's argument affect the graph's position along the x-axis, a concept that is less intuitive than vertical changes, which directly affect output values.

Furthermore, the students' inability to isolate and interpret each transformation separately suggests that they may not have been taught to analyse trigonometric functions systematically. Instead, they may rely on visual memorization of graph shapes without engaging with the mathematical reasoning behind each parameter in the function. This makes their understanding vulnerable to collapse when they encounter more complex or compounded transformations, as in the case of $g(x) = 3\sin(x + 30°)$.

To address these misconceptions, instruction must explicitly focus on decomposing transformed trigonometric functions into their constituent components: amplitude, period, phase shift, and vertical shift. Teachers should model how each component alters the base graph and provides structured exercises where students calculate and plot each transformation step by step. The use of dynamic graphing software could also help students visually link algebraic transformations with graphical behaviour. Additionally, students should be encouraged to justify their graphs using algebraic reasoning rather than visual estimation or memorized patterns.

The widespread errors made in sketching the graph of $g(x) = 3\sin(x + 30°)$ underscore students' difficulty in interpreting and applying amplitude and phase shift transformations correctly. Whether it was an underestimation of the amplitude or the omission of a horizontal shift, these mistakes point to a need for deeper, conceptually driven instruction on trigonometric graph transformations. Only through guided, deliberate practice will students develop the confidence and accuracy required to handle such functions proficiently.

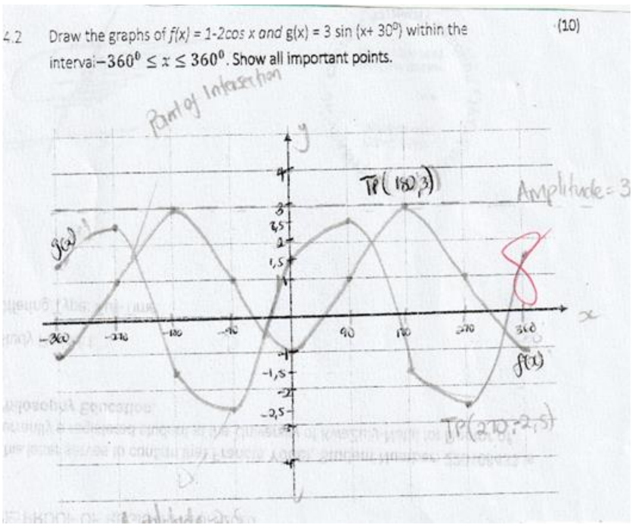


Figure 5.6: Incorrect amplitude of the function *f* from T66

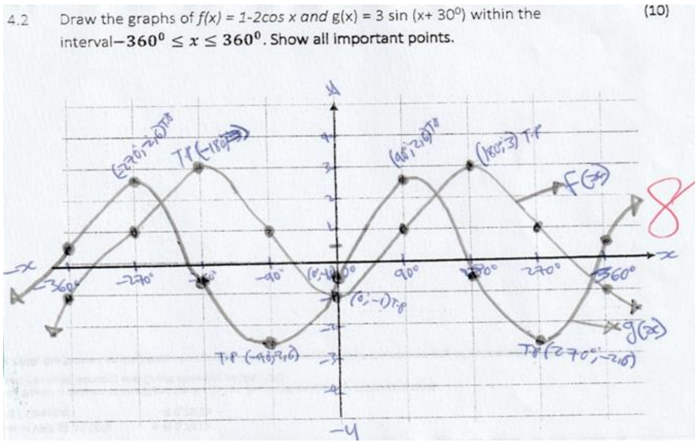


Figure 5.7: Incorrect phase shift of the function *f* from T76

5.3 Groundwork to the main study

The pilot study provided essential groundwork for the main study by highlighting key conceptual and procedural difficulties faced by PSTs in graphing and interpreting transformed trigonometric functions. By analysing the written responses of 85 first-year senior phase and FET mathematics PSTs to two carefully selected trigonometry questions from a previous exam, the pilot study offered an early indication of the depth and nature of student understanding. This preliminary investigation identified recurring errors and misconceptions, particularly around the vertical and horizontal transformations of sine and cosine functions. It laid the foundation for refining the research tools and deepening the inquiry into students' representational and reasoning strategies.

One of the most significant contributions of the pilot study was its exposure of students' misconception of trigonometric transformations, especially reflections, vertical shifts, and phase shifts. For example, many students failed to recognise that a negative coefficient in a cosine function reflects the graph across the x-axis, or that a positive constant outside the function causes a vertical shift. Similarly, many students either misinterpreted or completely omitted the horizontal shift resulting from the phase shift in the sine function. These findings provided critical insight into specific content areas where PSTs struggled, and they informed the formulation of the main study's research questions, which were aimed at investigating these conceptual gaps more systematically.

Furthermore, the pilot study allowed for a preliminary categorisation of the types of errors students made, procedural (incorrect substitution or manipulation), representational (inaccurate graphs), and conceptual (misunderstanding transformations). This classification was essential in shaping the error analysis framework for the main study. It also enabled the identification of themes related to the semiotic representations used by students, linking algebraic forms to graphical output, which was further explored in the main research using Duval's theory of semiotic representation.

Importantly, the pilot study also tested the suitability of data collection methods and materials. It confirmed that the chosen trigonometric tasks were both appropriately challenging and rich in revealing student thinking. Additionally, it provided an opportunity to assess ethical procedures, such as maintaining student anonymity and obtaining appropriate permissions. The pilot process also validated the coding scheme and analysis techniques to be used in the main study, ensuring they were robust and aligned with the research objectives.

In summary, the pilot study laid a strong empirical and methodological foundation for the main study. It helped refine the research focus, shaped the analytical framework, and confirmed the relevance of trigonometric transformations as a critical topic for deeper investigation. It ensured that the main study would be both informed and targeted, aiming to uncover not just what errors PSTs make, but why these errors persist and how they can be addressed in teacher education.

CHAPTER 6

ANALYSING DATA FROM ACTIVITY 1 QUESTIONS

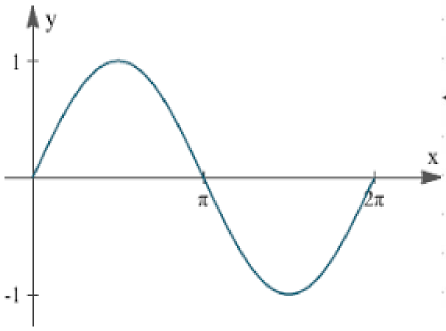
This chapter presents the analysis of written responses submitted by 107 students for the four trigonometry tasks administered during the study. While the task was made available online for a specified period, only 107 students successfully submitted their responses. An additional 18 students were unable to participate due to issues such as poor internet connectivity or health-related challenges. In addition to the written responses, the chapter also includes an analysis of interview data collected from five participants who were available and consented to take part in the interview process. For the purposes of anonymity, each of the 107 written response participants is referred to using a unique identifier ranging from S1 to S107. The ordering of these codes does not indicate any ranking or sequence of submission. The four tasks have been categorized based on the nature of their cognitive demands. The first category includes three tasks that focus on sketching trigonometric graphs and identifying their defining parameters. The second category consists of a task requiring students to compare trigonometric functions expressed in radians and degrees. The final category involves two tasks in which students are required to formulate the defining equations of trigonometric functions based on specific features. Each section of the analysis explores the patterns of responses, identifies common errors, and links the findings to conceptual and representational challenges in learning trigonometric transformations.

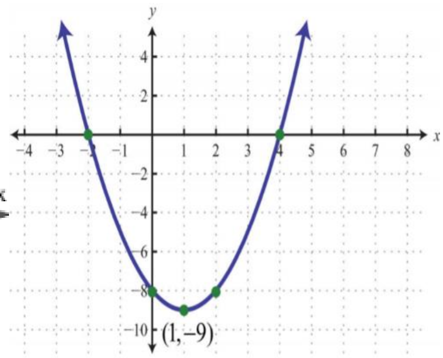
6.1 sketching and transforming a trigonometric function

6.1.1 The difference between a graph of a trigonometric function and a graph of a non-trigonometric function

Participants were asked to sketch a graph of any trigonometric function and a graph of any non-trigonometric function. The question and the answer appear in Figure 6.1.1 below.

Sketch a graph of any trigonometric function and a graph of any non-trigonometric function.





Graph 1: Graph of $\sin x$

Graph2: Graph of a parabola, $y = x^2 - 2x - 8$

Figure 6.1. 1: Graphs of a trigonometric function (Graph 1), and a non-trigonometric function (Graph 2)

Out of the 107 students who participated in the task, 81 students successfully sketched both a trigonometric function and a non-trigonometric function, indicating a sound understanding of the task requirements. A total of 16 students only managed to sketch one of the two required graphs: 11 of these provided only the graph of a trigonometric function, while 5 submitted only the graph of a non-trigonometric function. Notably, 10 students did not attempt the question at all.

Among the correctly and partially completed responses, several patterns emerged. For the trigonometric functions, the sine function was overwhelmingly the most frequently drawn, appearing in 80% of the responses that included a trigonometric graph. This was followed by the cosine function, represented in 18% of responses, while the tangent function appeared in only 2%. Regarding non-trigonometric functions, the straight-line graph dominated the responses, featuring in 75% of cases, while parabolic graphs appeared in 25% of the responses.

Figure 6.1.2 illustrates a typical correct representation of a non-trigonometric function: a straight-line graph drawn by student S3. This indicates familiarity with linear functions and the ability to translate algebraic form into graphical representation.

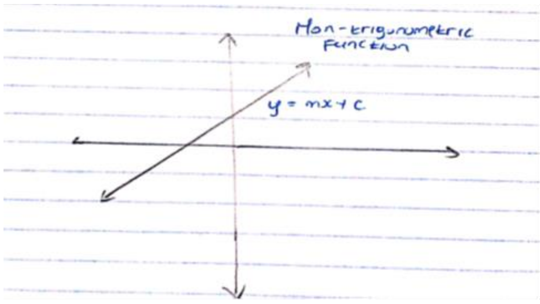


Figure 6.1. 2: Response from S3 showing a graph of a straight line

Figures 6.1.3 display responses from students S6 and S50, who correctly presented both a trigonometric and a non-trigonometric function. These examples exemplify an integrated understanding of different function types and their graphical representations, meeting the core objective of the task.

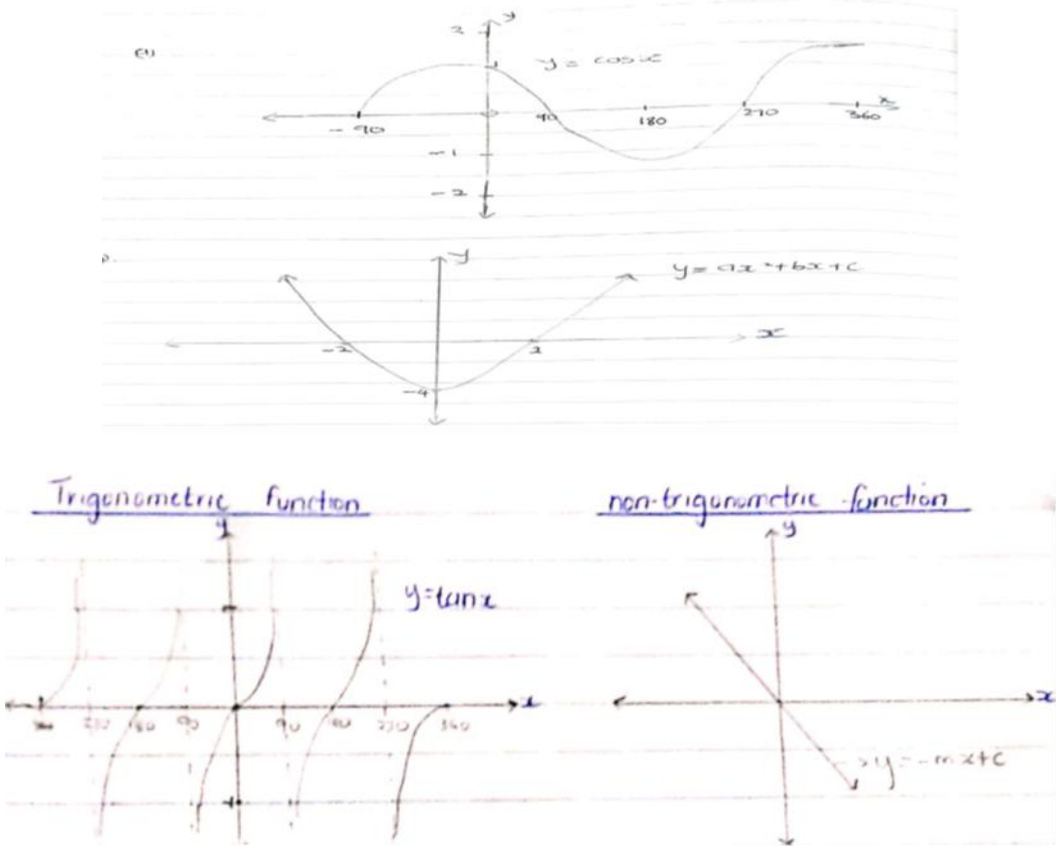


Figure 6.1. 3: Response from S6 and S50 showing the graph of a trigonometric function and the graph of a non-trigonometric function

Figure 6.1.4 shows the work of student S1, who sketched only a trigonometric function, omitting the non-trigonometric component. This response suggests either a lack of confidence or limited knowledge of non-trigonometric function types.

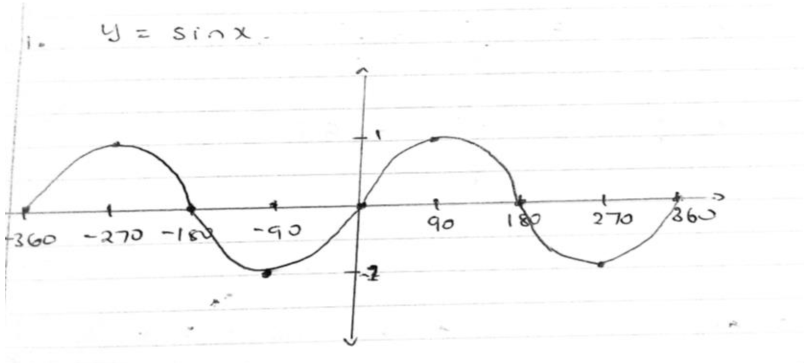


Figure 6.1. 4: Response from S1 showing only the graph of a trigonometric function

Conversely, Figure 6.1.5 highlights a response from student S93, who presented only a non-trigonometric function. This omission of the trigonometric graph could indicate difficulty in recalling the shapes or properties of trigonometric functions or confusion about their graphical features.

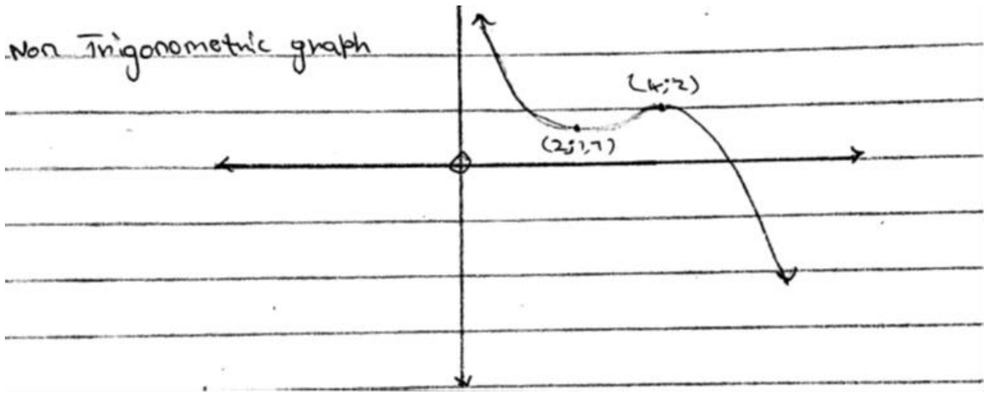


Figure 6.1. 5: Response S93 showing only the graph of a non-trigonometric function

Overall, while most students demonstrated an understanding of the graphical representation of both function types, the incomplete and incorrect responses highlight areas where misconceptions or knowledge gaps persist, particularly in the recognition and differentiation of trigonometric functions.

6.1.2 *Finding an amplitude and a period from an equation of a transformed trigonometric function*

6.1.2.1 Finding the amplitude

The students were asked to find the amplitude of $f(x) = 2\sin\left(\frac{\pi x}{4} + 3\right) + 4$ and $g(x) = -6\cos\left(\frac{x}{3} - \frac{\pi}{4}\right) + 1$ on the interval $[0\,;2\pi]$. To answer this question, the students should have used the general rule for each case as follows.

From the general form of the function $f(x) = A\sin(Bx - C) + D$ or $f(x) = A\cos(Bx - C) + D$,

$Amplitude = |A|$.

Hence, the amplitude of the function *f* is 2 since $|2| = 2$ and the amplitude of the function *g* is 6 since $|-6| = 6$. There were 47 students who correctly found the two amplitudes for both functions. An interview with one student who correctly found the amplitudes of f and g revealed that the student could not find these amplitudes without drawing the graphs.

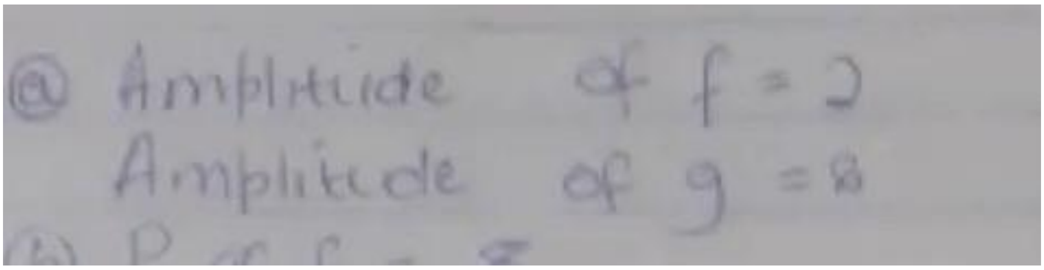


Figure 6.1. 6: Correct response provided by S85

Researcher: *What is an amplitude in trigonometry?*

Student S85: *According to my understanding……*

Student S85: *Amplitude is the …yoh,..yoh… is the length or distance between the X axis and the turning point, can you hear me?*

The student continued:

Student S85: *By drawing, the graphs because uh there's no other way that I know off, I just know that you draw the graph. I just drawn the graphs.*

Student S85: *Based on my drawings…that is how I got the answers.*

Researcher: *So, you are saying that if they gave you two trigonometry two functions like these two functions here, you can't find the amplitude without drawing the graphs first? That's what you are saying?*

Student S85: *No, I can't, yes.*

Nineteen students did not answer the question. These students did not know the amplitude of a cosine or sine function; hence, they could not provide any answers for either function. Forty-one students incorrectly found the amplitude of one or both functions. These incorrect answers are discussed as follows.

6.1.2.2 Wrong application of the amplitude formula

Thirty students provided the correct amplitude of the function f but failed to find the correct amplitude of the function g. They knew which value in each equation of f and g represented the amplitude, but they did not apply the rule of absolute value to those values. The interviews with S13 and S37, who had given answers to Figures 6.1.6 and 6.1.7, reveal that both students held the misconception that the amplitude is the turning point of a trigonometric graph.

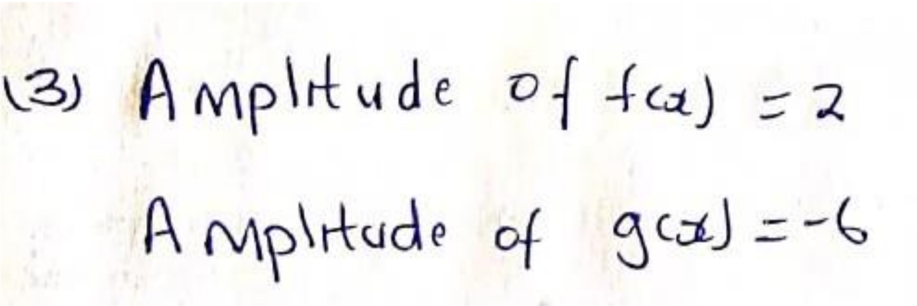


Figure 6.1. 7: Incorrect amplitude of the function g provided by S13

Researcher: *How did you arrive at your answer?*

Student S13: *I did some research on the internet. I found that the amplitude of the function is the first number behind cosine.*

Researcher: *The first number?*

Student S13: *Yes.*

Researcher: *What do you mean by behind cos?*

Student S13: *In the function of G of X, sir.*

Researcher: *Ok.*

Student S13: *Yes, because there's 6 cos and then the other values, negative 6 cos is the amplitude then for sine, 2 is the amplitude sir.*

When asked to explain what the amplitude was, the student responded.

Student S13: *Is, I think, where the graph of the function g of x or f of x turns.*

Researcher: *Ok.*

Student S13: *Yes, sir, I think so.*

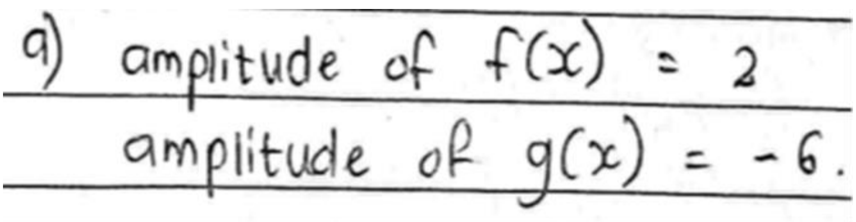


Figure 6.1. 8: Incorrect answer for the amplitude of g provided by S37

When asked about his answer, S37 responded:

Student S37: *I took my amplitudes in the formula of the function f of x and g of x, sir.*

Researcher: *So, you said your amplitude from the formula of the function f of x is 2?*

Student S37: *Yes.*

Researcher: *And then for the second function you said minus 6 is the amplitude?*

Student S37: *Yes.*

Researcher: *So, you just took it straight. There was nothing that you used? You just concluded that your amplitude from the given formula is minus 6.*

Student S37: *Uh that's what I was not sure sir, writing this.*

The student tried to explain what his idea of amplitude was.

Student S37: *Amplitude is where the graph will turn. Amplitude, that is where my graph would turn, uh---*

Researcher: *You said it's your turning point?*

Student S37: *It's where my graph will turn when I'm doing my graph it never goes up the amplitude.*

The interview responses from students S13 and S37 reveal significant misconceptions and procedural reasoning gaps related to the concept of amplitude in trigonometric functions. Both students identify the amplitude as the numerical coefficient in front of the trigonometric function (e.g., 6 in "$6\cos(x)$"), which aligns with a correct procedural interpretation. However, neither demonstrates a conceptual understanding of what amplitude represents in the context of a graph. For example, both students confuse amplitude with the "turning point" of a function, which more accurately refers to features of quadratic functions rather than sinusoidal ones.

Student S13 vaguely describes amplitude as "where the graph turns," suggesting a misunderstanding of periodic behaviour and peak values in sinusoidal functions. Similarly, S37 reiterates the idea that amplitude is "where the graph will turn," and expresses uncertainty about whether the sign of the coefficient (negative or positive) affects the amplitude, which it does not, it only influences the graph's orientation, not its magnitude.

6.1.2.3 Misconception of the meaning of the parameters A and D in the general functions, $\boldsymbol{f(x) = A\,sin(Bx - C) + D}$ *or* $\boldsymbol{f(x) = A\,cos(Bx - C) + D}$

From the general formula of cosine and sine functions, the value A is defined as the distance from the highest point of the curve to the horizontal axis, and the value of D is defined as the vertical shift of the graph. Five students had a misconception in which they found the amplitude of f and g by adding the values of A and D from each equation (See Figure 6.1.9).

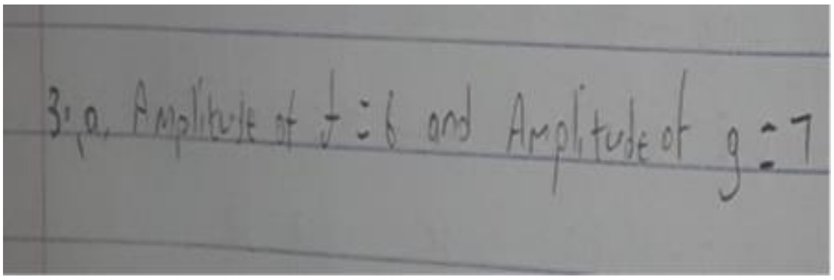


Figure 6.1. 9: One example of the incorrect response provided by S14

6.1.2.4 Incorrect use of the definition of an amplitude

The response provided by student S25 (See Figure 6.1.10) reveals a key misconception regarding the general form of trigonometric functions, specifically in relation to amplitude. The student incorrectly interpreted the value of *D* in the general equation of sine and cosine functions as representing the amplitude, rather than recognizing that amplitude is determined by the absolute value of *A*, the coefficient in front of the sine or cosine term. Furthermore, the student assumed that the sign of *A* affects the amplitude's value, indicating a lack of understanding that amplitude is a measure of vertical distance from the midline to the peak or trough of the wave and is therefore always a positive quantity. This misconception suggests that the student may be conflating different parameters in the trigonometric function formula and has not yet internalized the distinct roles of each component, *A* (amplitude), *B* (frequency), *C* (phase shift), and *D* (vertical shift).

This type of error highlights the need for instructional approaches that emphasize both the graphical meaning and algebraic structure of trigonometric functions, helping students to better connect the symbolic representation with its geometric and functional implications.

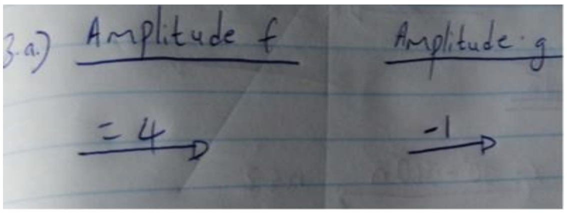


Figure 6.1. 10: Incorrect response provided by S25

6.1.2.5 Confusion about what the amplitude is

Three students incorrectly stated that the amplitude of both functions was 6, without showing any working or justification. As seen in Figure 6.1.11 (response by S4), it appears that these students may have fixated on the number 6, possibly the most visually prominent coefficient in the equation, regardless of its actual position or mathematical significance. This implies a surface-level engagement with the equation, where values are selected based on appearance rather than structural meaning. It also indicates a misconception of how the coefficient of the sine or cosine function (usually denoted as A in the standard form $f(x) = A\sin(Bx + C) + D$) determines amplitude through its absolute value.

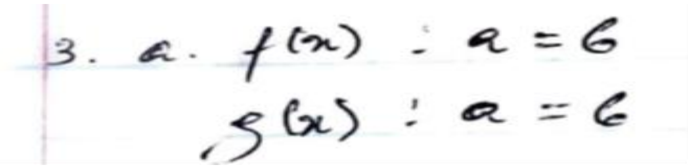


Figure 6.1. 11: Incorrect response provided by S4

Furthermore, the use of interval notation to express amplitude, as seen in the responses of two students including S22 in Figure 6.1.12, points to a more fundamental confusion. These students appeared to equate amplitude with the domain or range of the function rather than recognizing it as a scalar quantity representing the maximum vertical displacement from the midline of the graph. By writing expressions like [–6, 6] or [0, 6], the students demonstrated an attempt to express a range of values rather than a single value that quantifies the graph's height above and below the horizontal axis.

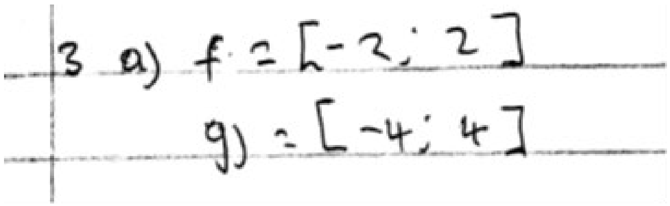


Figure 6.1. 12: Incorrect response provided by S22

This confusion might stem from prior exposure to range concepts in algebra or functions, leading them to apply familiar but inappropriate strategies in a trigonometric context**.**

6.1.2.6 Finding the period

For a trigonometric function, the length of one complete cycle is called the period of the graph. From the general form of the function $f(x) = A\sin(Bx - C) + D$ or $f(x) = A\cos(Bx - C) + D$, one can find the period by using the formula:

$$Period = \frac{2\pi}{|B|}.$$

The task given to the participants was to find the period of $f(x) = 2\sin\left(\frac{\pi x}{4} + 3\right) + 4$, and $g(x) = -6\cos\left(\frac{x}{3} - \frac{\pi}{4}\right) + 1$. Using the above formula, the period of *f* is 8 since $\frac{2\pi}{|B|} = \frac{2\pi}{\frac{\pi}{4}} = 2\pi \times \frac{4}{\pi} = 8$, and the period of g is 6π since $\frac{2\pi}{|B|} = \frac{2\pi}{\frac{1}{3}} = 2\pi \times 3 = 6\pi$. There were 13 students who correctly found the two periods of both functions (See Figure 6.13). An interview with S85, who correctly found both periods, shows that the student did use the formula of the function but worked with the graph of the functions to determine the period of each function.

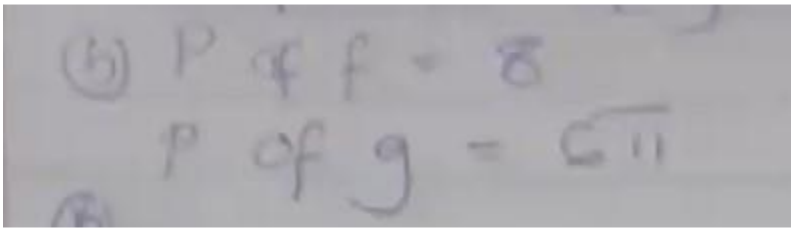


Figure 6.1. 13: An example of a correct response provided by S85

Student S85: *I don't know how to find the periods of trigonometric functions without drawing their graphs.*

Researcher: *You did not rely on the formula for a period?*

Student S85: *No.*

Three students found the correct period of f but did not provide any answer for finding the period of g. Thirty-seven did not provide any answers to the question. There were 52 students who incorrectly found the period of both functions, and two students who correctly found the correct period of g but incorrectly computed the period of f. These incorrect answers can be classified as follows:

6.1.2.6.1 Challenges with the radius

The responses of the seven students who converted 2π to 360° when applying the formula $p = \frac{2\pi}{|B|}$ demonstrate a significant conceptual misconception regarding the use of radians versus degrees in trigonometry. This mistake suggests a strong fixation on degrees as the default unit of angular measurement, likely stemming from prior exposure to trigonometry in high school, where angles are predominantly taught in degrees. As a result, students appear to have internalized the belief that the period of a trigonometric function must always be expressed in degrees, overlooking the fact that the standard mathematical expression of the period formula, especially in functions such as $f(x) = \sin(Bx)$, is based on radians. This is evident when, for example, a student incorrectly computes the period of $f(x) = \sin(3x)$ as $\frac{360°}{3} = 120°$ instead of the correct radian measure $\frac{2\pi}{3}$. Such confusion reflects a lack of flexibility in moving between angular units and a failure to recognize the context in which each unit is appropriate. The consequence is not only computational inaccuracy but also a weakened conceptual foundation, especially when transitioning to more advanced mathematical contexts like calculus, where radians are the default. Moreover, if these PSTs retain and carry this misconception into their teaching practice, they risk perpetuating the error in the classroom, potentially hindering learners' understanding of periodicity, unit circle properties, and function behaviour.

6.1.2.6.2 Misconception about the coefficient of x in the expression $\frac{\pi x}{4}$ and $\frac{x}{3}$

One student (S2), whose response appears in Figure 6.1.14, displayed basic algebraic misconceptions and, seemed to read $\frac{\pi}{4}$ as a variable. This student has two different misconceptions: firstly, he took B as 1 instead of $\frac{\pi}{4}$, in $f(x) = 2\,sin\left(\frac{\pi x}{4} + 3\right) + 4$. This is a well-known algebraic misconception where students ignore the letter, believing that the coefficient of x must be a number. The student took *B* as 3 in $g(x) = -6\cos\left(\frac{x}{3} - \frac{\pi}{4}\right) + 1$ instead of $\frac{1}{3}$.

To compute the period of *f*, the student considered the parameter B as equal to 1 to find $p = \frac{360°}{1} = 360°$. In computing the period of g, student S2 took *B* as 3, instead of $\frac{1}{3}$, and obtained the answer as $p = \frac{360°}{3} = 120°$.

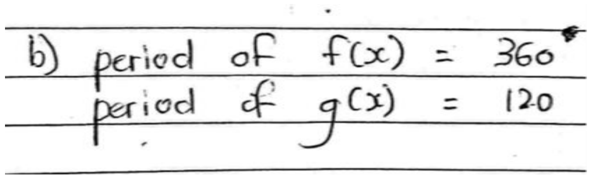


Figure 6.1. 14: An example of an incorrect response provided by S2

Again, thinking that π is a variable, two students took the coefficient of $\frac{\pi x}{4}$ as $\frac{1}{4}$, where the student has taken B as $\frac{1}{4}$ instead of $\frac{\pi}{4}$ (See Figure 6.1.15).

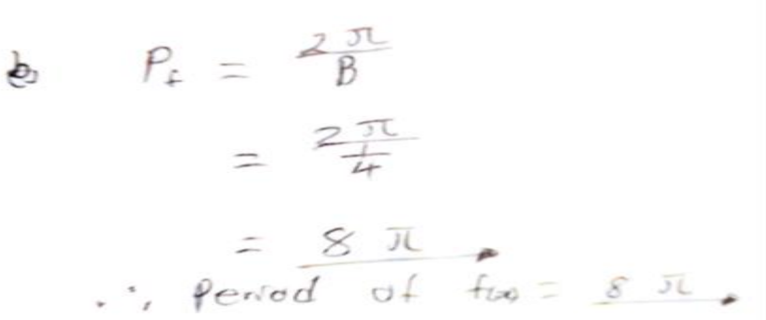


Figure 6.1. 15: An example of an incorrect response provided by S10

6.1.2.6.3 Misconception that a period of a trigonometric function is always 360°

Thirty-three students exhibited a misconception that the period of a trigonometric function is always equal to 360° or 2π. An example of this response is shown in Figure 6.1.16. An interview with S13, who provided a response from Figure 6.1.16, revealed that the students believed the period of cosine and sine functions is often equal to 360 degrees or 2π.

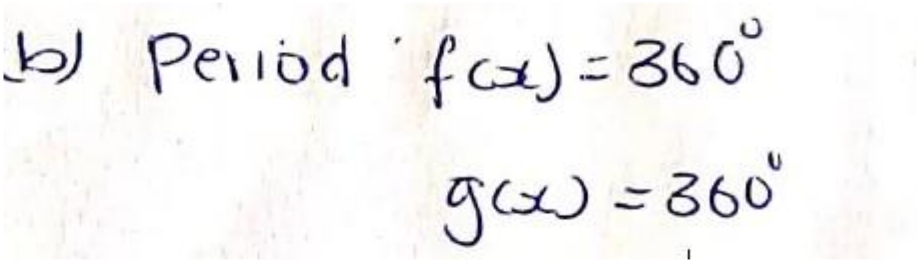


Figure 6.1. 16: An example of the misconception that the period is always 360° provided by S13

Student S13: *Uh the period sir, as I know that period of sine and period of cosine is equal to 360 degrees and only the period of tan is equal to 180 degrees so, I just looked at the 2 functions and I saw cosine and sine and I concluded that the 2 periods are 360 degrees, sir.*

Researcher: *So, in other words, you want to say that for any functions of sine or cosine their periods are always 360 degrees.*

Student S13: *No, it is not always 360 degrees, sir.*

Researcher: *Okay, so?*

Student S13*: But most of the time the period of cosine and sine is 360 degrees.*

Student S37 explained that it is difficult for her to find the period of the cosine or sine function without drawing their graphs. However, the student was aware of the formula but did not seem to realise that these parameters could change.

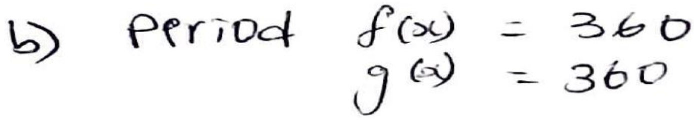


Figure 6.1. 17: An example of the misconception that the period is 360° by S37

Researcher: *Okay then, now we are going to the period. Again, the same functions. Then they asked you to find the period and then you said for f it's 360 degrees, for g it's 360 degrees. So how did you arrive at this conclusion of 360 degrees?*

Student S37: *Because I don't know what I was given but the period is where my graph will end when I'm drawing it.*

Researcher: *Okay, then, are you aware of the period's formula?*

Student S37: *Period formula? Which, it's 2 pi over absolute value of B.*

When asked about what the period meant, the student explained:

Student S37: *A period is a repeating of a pattern when I'm drawing a graph.*

Supervisor: *It's a repeat?*

Student S37: *It's repeating.*

Supervisor: *Okay.*

Student S37: *Yes, because you know the form of drawing a circle, let's speak about sine, you know the form of drawing a sine graph. Then it's repeating itself, it's a repeating period. Then if the original graph of sine is when it's repeating, repeating it's a rotating period.*

It may be that the student is confusing the idea of repeating cycles with the idea of rotation of angles within the unit circle.

6.1.2.6.3 Misconception that the period can be represented in an interval notation

One student did not understand the difference between a period and the interval of a trigonometric function (See Figure 6.1.18). The student failed to recognize that the period is the length of one complete cycle of a trigonometric function, which is a number, while an interval denotes a portion of the axis or number line.

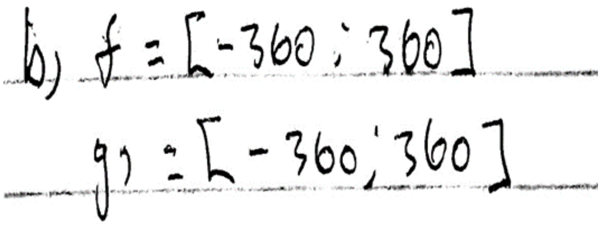


Figure 6.1. 18: Incorrect response provided by S22 for the period of both functions

<u>6.1.2.6.5 Misconception about taking the parameter B as the period of a function</u>

From the general form of the function $f(x) = A\sin(Bx - C) + D$ or $f(x) = A\cos(Bx - C) + D$, two students interpreted the period as B in both functions. S54 and S60 believed that the period of the function f was $\frac{\pi}{4}$ which they converted to 45˚. For the period of the function g, S60 considered the period to be $\frac{1}{3}$, while student S54 had a different approach. This student calculated the period to be $\frac{2\pi}{\frac{\pi}{4}}$, which he converted to $\frac{360°}{45°} = 8$. Here, the student performed an arbitrary operation. The student took the value of $C = \frac{\pi}{4}$ in $g(x) = -6\cos\left(\frac{x}{3} - \frac{\pi}{4}\right) + 1$. She converted the value of C into degrees to get $C = \frac{\pi}{4} = 45°$. Then, she divided 360° by 45° to obtain 8. The student could not define a period and did not understand why she used all those steps to find the period of each function (See Figure 6.1.19).

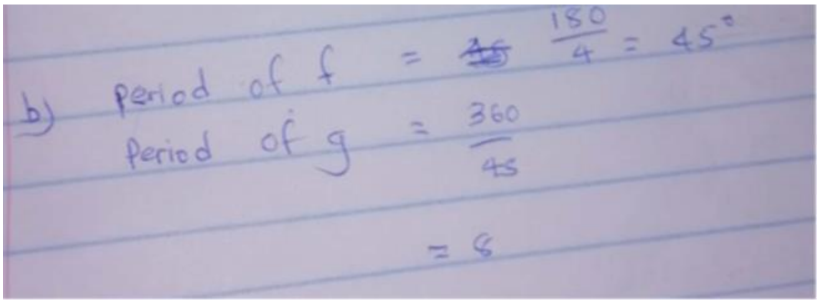


Figure 6.1. 19: Incorrect response of S54 to find the period of both functions

Researcher: *Okay then now take us through the period as well, how did you find these two periods?*

Student S54: *Okay since the function F of x is sine and I don't know if it is the standard or what but it's 180 over 4.*

Researcher: *Why are you using 180 over 4?*

Student S54: *Because sir I took…, okay the normal…, the standard is 360 for sine and cos and since the amplitude is 2 on the function F, then I took that, I said 360 divided by 2.*

Researcher: *Okay.*

Student S54: *Then I got, I got 360 divided by 2, it's 180, sir.*

Researcher: *Okay, then?*

Student S54: *Then why did I divide it by 4?*

Mr. Francis: *Yes.*

Student S54: *Oh, I can't remember sir, why I divided by 4.*

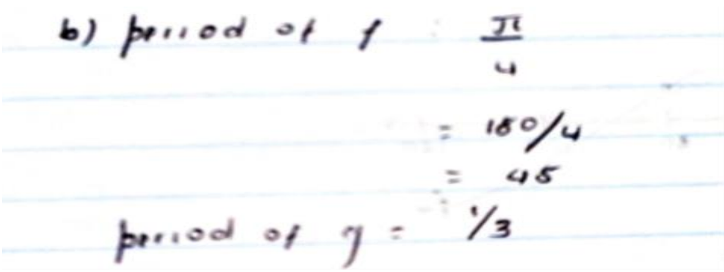


Figure 6.1. 20: Incorrect response provided by S60 for the period of both functions

Student S54 initially attempted to find the period by dividing 360 by 2, stating that the amplitude was 2, thereby demonstrating a critical conceptual error: conflating the amplitude with the frequency-modifying coefficient B. The student believed that the amplitude (which affects the height of the graph) was somehow related to the period (which affects the horizontal stretching or compression), which indicates a misunderstanding of the distinct roles of parameters in the general sine and cosine functions.

More telling, however, was her further division by 4 without being able to recall or justify the reason for this step, eventually admitting, "I can't remember why I divided by 4." This uncertainty highlights not just a gap in procedural knowledge but a lack of conceptual understanding of what the period represents, the length of one full cycle of a trigonometric function. The student's arbitrary calculations and inability to define or explain the reasoning behind her method suggest that her approach was based on memorised, unconnected procedures rather than logical or mathematical understanding.

Similarly, Figure 6.1.20 (from Student S60) showed another incorrect response where the value for the period appears to have been guessed or derived from unrelated numerical patterns. Such attempts may stem from PSTs' exposure to rote learning environments in secondary school, where procedures are often taught without sufficient emphasis on underlying concepts.

6.1.2.6.6 Confusion about what a period is

The responses illustrated in Figures 6.1.21 through 6.1.23 demonstrate a lack of understanding among four students regarding how to correctly determine the period of trigonometric functions. These students were unable to identify the correct parameter in the function equation that influences the period, and as a result, their answers appear either arbitrary or based on misconceptions. In Figure 6.1.21, one student gave a period of 1 for both functions, possibly mistaking the coefficient of x, or defaulting to a general value without computation or justification. This suggests either confusion with the concept of frequency or a misconception of how function transformations affect the graph.

In Figure 6.1.21, another student presented the answer $900°$ for both functions, an unusually high and clearly incorrect value. This could indicate a misapplication of prior knowledge or an attempt to guess the period based on irrelevant numerical patterns. It may also reflect confusion between angle measures, like thinking in terms of revolutions or using accumulated values rather than the specific length of one cycle.

Figures 6.1.22 and 6.1.23 further exemplify these misconceptions. One student (S25) again failed to correctly identify or apply the coefficient B in the standard period formula $\frac{360°}{|B|}$, and another (S92) provided a similarly incorrect and unexplained response. Across all these cases, it is evident that the students did not understand that the period of sine or cosine functions is derived from the horizontal stretching or compressing factor B, and that this parameter directly modifies the standard period of $360°$ (or 2π in radians).

These incorrect responses suggest that the students lacked not only procedural fluency but also a conceptual framework for interpreting trigonometric functions. The absence of consistent reasoning in their work, combined with their incorrect values, implies they may have been guessing or relying on rote recall without comprehension.

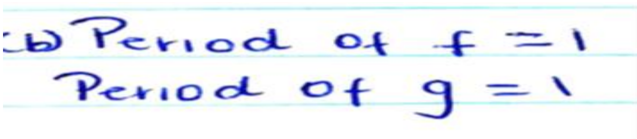


Figure 6.1. 21: Incorrect response provided by S15 for the period of both functions

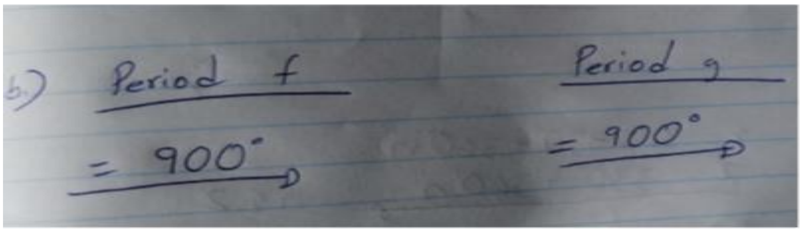


Figure 6.1. 22: Incorrect response provided by S25 for the period of both functions

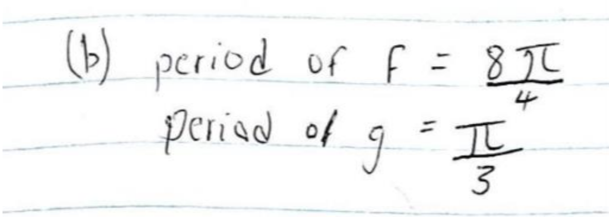


Figure 6.1. 23: One example of an incorrect response provided by S92 for the period of both functions

6.1.3 Sketching a graph of the original trigonometric function $\boldsymbol{f(x) = \cos x}$ *and its transformed function*

Participants were asked to sketch the graphs of $f(x) = \cos x$ and $h(x) = 2\cos\left(\frac{x}{2} - 30°\right) + 1$ on the same set of axes using the interval $[0°; 360°]$. In addition, they were asked to show all the important points. There were 43 students who correctly drew and represented all the important points of the graphs of f *and* h within the requested interval. Figure 6.1.24 represents the graphs of f *and* h within the interval $0° \leq x \leq 360°$. Ten students did not answer the question, and 54 students did not answer it correctly. The incorrect answers are grouped into four observations.

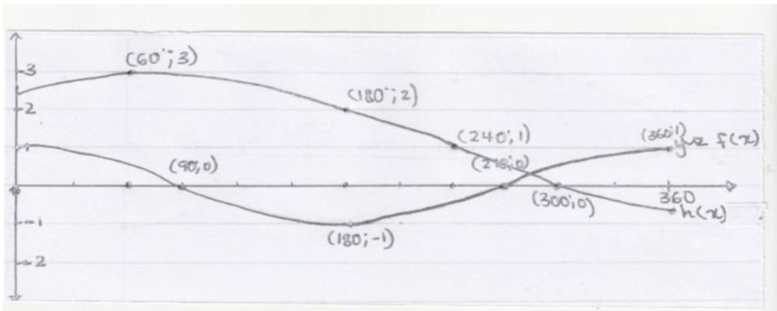


Figure 6.1. 24: An example of a correct response provided by S3

6.1.3.1 Misconception about the horizontal phase shift

The first group consists of 20 students who answered the question as represented in Figure 6.1.25. These students correctly drew the graph of the function f but did not correctly draw the graph of h. They cannot distinguish between the starting point of the function and the horizontal phase shift. They shifted the starting point by 30˚ to the right. They had misconceptions about the horizontal shift and the starting point. They assumed that since the function has a horizontal shift of 30˚ then the starting point of the function must also shift by 30˚ to the right. They correctly applied the horizontal shift since when $y = 0°$, the function crosses the $x - axis$ at $x = 300°$. S85 provided the same answer shown in Figure 6.1.25. She confirmed in her interview's responses that the function h of x has shifted to the right along the x-axis. The student, therefore, assumed that the starting point should have an x coordinate equals to 30 degrees. The following script reveals her misconception about the horizontal shift of a function.

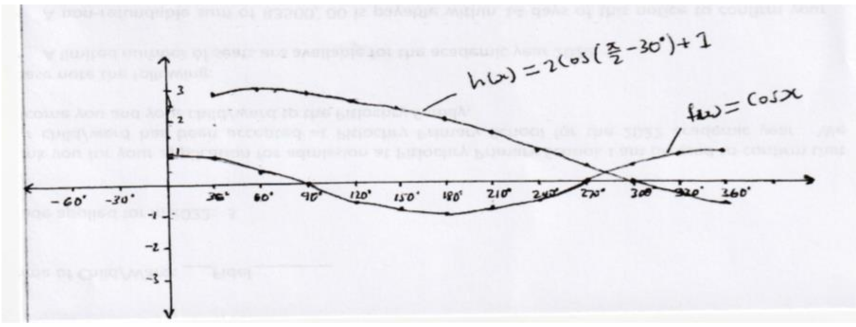


Figure 6.1. 25: An example of an incorrect response for h provided by S85

Researcher: *Why does the starting point of h of x have a coordinate x equal to 30 degrees?*

Student S85: *Because cosine is positive, and it has shifted by 30 degrees to the right.*

Researcher: *So, but now, what will be the value of h if x equals 0?*

Student S85: *Mmm, can I use the calculator, sir?*

Researcher: *Yes, you can.*

Student S85: *It is 2 comma seventy-three, sir.*

Researcher: *So, what is that value telling you?*

Student S85: *It's the value of the function h of 0.*

Researcher: *Is this not where your starting point was supposed to be?*

Student S85: *Uh…Yes, sir.*

Researcher: *Ok, so why did your starting point start at x equal to 30 degrees?*

Student S85: *I can see that I was wrong, sir.*

The responses of S85 reveal both a partial understanding and a conceptual gap in working with phase shifts and evaluating trigonometric functions. Initially, the student confidently claims that the graph of $h(x)$ starts at $x = 30°$ due to a perceived rightward shift in the cosine function. This suggests that the student is aware that a horizontal translation can occur in trigonometric functions, which is a good sign. However, the explanation is superficial, relying on memorised rules rather than a deeper understanding of what a phase shift means in relation to the function's behaviour.

When asked about the value of the function at $x = 0$, the student correctly uses a calculator to compute $h(0) \approx 2.73$, but initially does not connect this result to the function's graphical starting point. This indicates a disconnection between the algebraic form of the function and its graphical representation, a common issue among learners who struggle to integrate different semiotic registers (Duval, 2006).

The final exchange is particularly revealing once the student realises the contradiction between their graphical assumption and the actual output of the function at $x = 0$, they acknowledge their error. This moment illustrates the student's openness to reflection and the potential for conceptual growth when misconceptions are addressed through guided questioning

6.1.3.2 Wrong representation of all shifts and the domain of definition

In the second group, 28 students did not use the translation of one unit up of h (see Figure 6.1.26) and did not apply the phase shift of 30° to the right when drawing the function h. As shown in Figure 6.1.26, at $y = 0°$, the function h crosses the $x - axis$ at 240˚ instead of 300˚. It has also been observed that the students did not understand the difference between a closed and open interval representation on a graph. In Figure 6.1.26, the student represented the endpoint with an arrow, indicating that the function continues beyond that point, however, the question specified a closed interval for the domain of the function as the interval $[0°; 360°]$, meaning that the function was not defined beyond 360°.

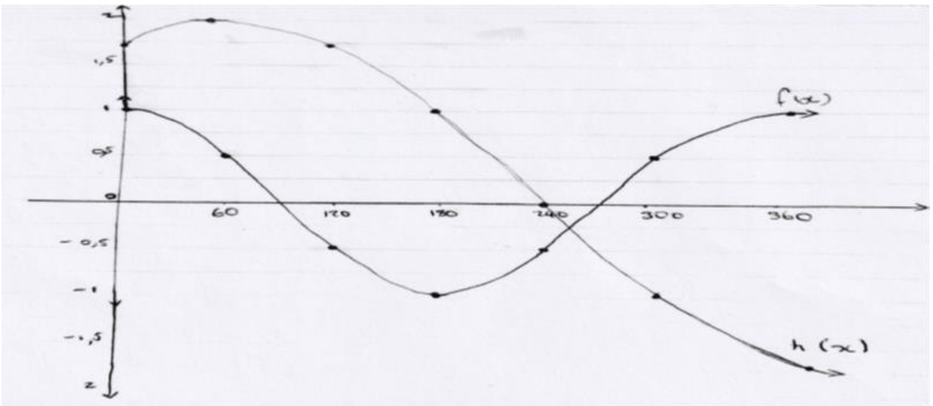


Figure 6.1. 26: One example of an incorrect response for h provided by S77

6.1.3.3 Wrong representation of the domain of definition

The third group consists of four students who answered the question, as indicated by Figure 6.1.27. These students did not use the given interval for both functions and drew the graphs on the interval [-360°, 360°]. They correctly drew both functions but failed to consider the given domain of definition. S54 justifies her answer in Figure 6.1.27 by stating that she used the calculator to plot the functions, and that the interval of a trigonometric function is determined by its period. The script below presents her account of her answer.

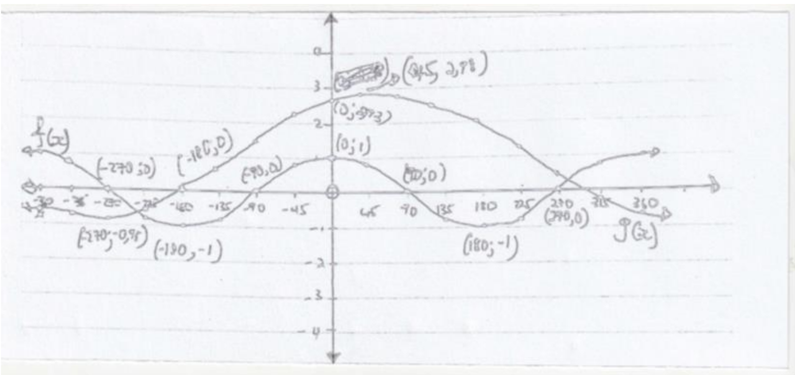


Figure 6.1. 27: One example of an incorrect domain of definition of both functions provided by S54

Student S54: *I don't know sir, since I used the calculator.*

Researcher: *Oh okay, so which element tells you where the function must start and where it must stop?*

Student S54: *Um I think its period.*

Researcher: *The period?*

Student S54: *Yes.*

Researcher: *So, what does the interval tell you?*

Student S54: *Interval?*

Researcher: *You have no idea?*

Student S54: *Yes, sir I just have no idea.*

The student seems to have a confusion between the shifting method and the period.

6.1.3.4 Incorrect representation of both functions

The fourth group consists of two students who answered the question as represented in Figure 6.1.28. The students assigned the incorrect amplitudes to both graphs. They applied a phase shift of 90° to the right on the graph of h, and they did not use the translation of one unit upward on the graph of h.

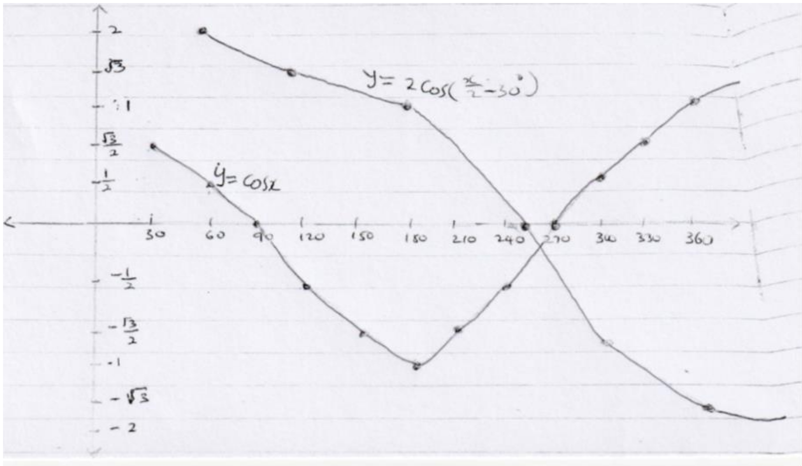


Figure 6.1. 28: An example of an incorrect response of both graphs provided by S102

6.2 Evaluating a trigonometric function

The task given to the participants was to determine if there was a difference between the values of $\sin(45°)$ and $\sin\left(\frac{\pi}{4}\right)$ and to justify their answers.

The expected response to this question was that there is no difference in the values between of $\sin\left(\frac{\pi}{4}\right)$ *and* $\sin(45°)$ since their outputs are identical. Thus, $\sin\left(\frac{\pi}{4}\right) = \sin(45°) = \frac{\sqrt{2}}{2}$. There were 82 students who correctly answered the question, and six students who simply stated that there was no difference between the two expressions without providing any explanation. A total of five students did not attempt the question, and 14 students incorrectly answered the question. The various incorrect responses are discussed below.

6.2.1 *Misunderstanding of the degree and radian value*

There were four students who said there was a difference but did not provide a reason. These students noticed that there were differences in the expressions but could not determine if the values were different. There were five students who pointed out that the argument for the sine function in the first case was in radians, while the second one was in degrees. Figure 6.2.1 presents the response of S52 as an example of such a response.

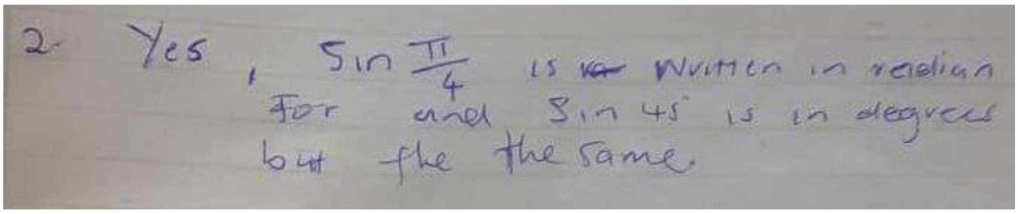


Figure 6.2. 1: Responses given by S52 to the question

6.2.2 *Errors with the use of a calculator*

Two students made errors with the calculator steps. They found different values for the expressions. One student, S8, did not provide his calculations but merely stated that the y-values were not the same. See Figure 6.2.2 below. Clearly, this student used the calculator to find the output of each function but did not change the mode when entering each expression.

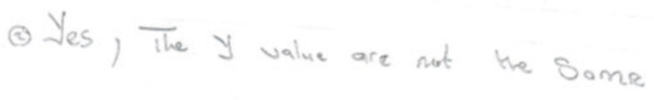


Figure 6.2. 2: Response given by S8 to the question

The second student (S19) stated that there is a difference because $\sin(45°) = \frac{\sqrt{2}}{2}$ whereas $\sin\left(\frac{\pi}{4}\right) = 0.01$. Figure 6.2.3 shows that this student performed the calculation for both expressions while the calculator was in degree mode, instead of adjusting the settings for the second expression to radians.

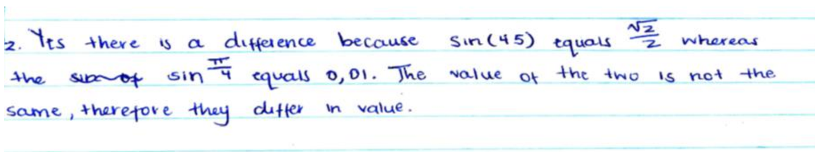


Figure 6.2. 3: Response given by S19 to the question

6.2.3 Error with notation

One student (S86) displayed a misconception about the notation as shown in Figure 6.2.4. From the figure, $\sin(45°) = \frac{\sqrt{2}}{2}$, but $\sin\left(\frac{\pi}{4}\right) = \frac{\pi}{4} \times \frac{180°}{\pi} = 45°$. The student recognised that 45° was equivalent to $\frac{\pi}{4}$. However, his response shows a disregard for the function operator $(\sin x)$.

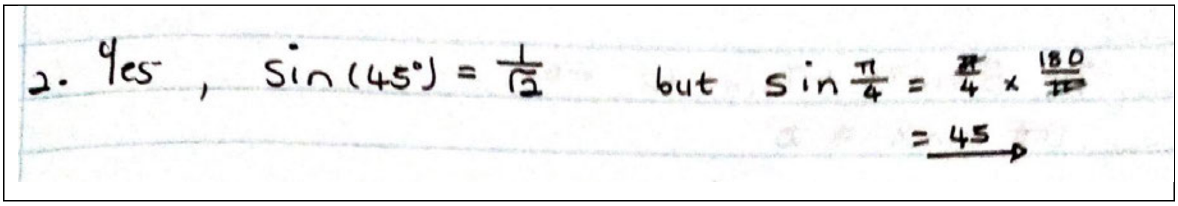


Figure 6.2. 4: Response given by S86 to the question

6.2.4 Misconception about the use of values

One student (S59) just wrote that they were different because $\pi = 180°$. Here, this student S59 demonstrated an understanding of the equivalence between radians and degrees but seemed not to have understood the question about the values of the expressions.

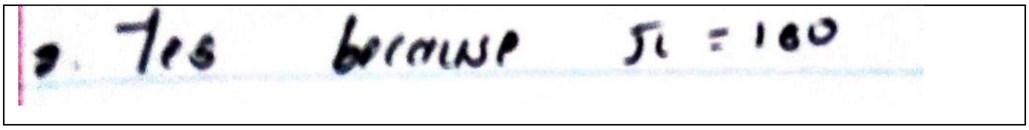


Figure 6.2. 5: Response given by S59 to the question

Another student (S101) stated that there was a difference but provided an irrelevant answer, claiming that $\mathbf{sin}\left(\frac{\pi}{4}\right)$ indicates how many rotations are needed to complete a cycle, while $\mathbf{sin(45°)}$ is a special angle. This student did not understand the distinction between the revolution of an angle concept, the period of a function, and the special angle concept.

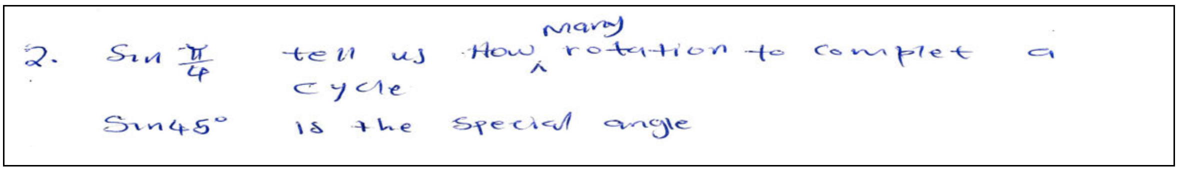


Figure 6.2. 6: Response given by S101 to the question

6.3 Creating a trigonometric function equation

In this task, participants were asked to write an equation for a cosine function with an amplitude of 3, a period of π, a phase shift of $\frac{\pi}{4}$ to the left, and translated 1 unit up. Additionally, they were asked to write an equation for a sine function with an amplitude of 3, a period of 5π, a phase shift of $\frac{3\pi}{4}$ to the right, and a vertical translation of 6 units down.

6.3.1 Writing an equation for a cosine and sine function

From the given question about writing an equation for cosine and sine functions, 16 students correctly found the equation $y = 3\cos\left(2x + \frac{\pi}{2}\right) + 1$, and two correctly found $y = 3\sin\left(\frac{2}{5}x - \frac{3\pi}{10}\right) - 6$**.** To achieve such a positive outcome, the following steps were used to write the equations for both functions. From the general forms, $y = A\cos(Bx + C) + D$, and $y = Asin(Bx + C) + D$, we have:

Step 1: Finding the values of A, B and D

- For the cosine function, the amplitude of the function is 3 hence, $A = |3| = 3,$ and the function is translated one unit up, therefore, $D = 1$. Additionally, the function has a period equal to π, which translates to $Period = \pi$, therefore $B = \frac{2\pi}{Period} = \frac{2\pi}{\pi} = 2$.
- For the sine function, the amplitude of the function is 3 hence, $A = |3| = 3,$ and the function is translated 6 units down, therefore, $D = -6$. Additionally, the function has a period equal to 5π, which translates to $Period = 5\pi$, therefore $B = \frac{2\pi}{Period} = \frac{2\pi}{5\pi} = \frac{2}{5}$.

Step 2: Finding the value of **C**

- It was given that the cosine function had a phase shift of $\frac{\pi}{4}$, therefore, the phase shift can be expressed as $-\frac{C}{B}$, leading to $-\frac{\pi}{4} = -\frac{C}{2}$. Thus, $C = \frac{2\pi}{4} = \frac{\pi}{2}$.
- It was given that the sine function had a phase shift of $\frac{3\pi}{4}$, therefore,

$Phase\ shift = -\frac{3\pi}{4} = \frac{C}{B}$

$\therefore C = -\frac{3\pi}{4} \times \frac{2}{5} = -\frac{3\pi}{10}$.

Step 3: Writing of the equation

By substituting the values of A, B, C, and D into the general form of the cosine and sine functions we obtain:

$$y = 3\cos\left(2x + \frac{\pi}{2}\right) + 1 \text{ } and \text{ } y = 3\sin\left(\frac{2}{5}x - \frac{3\pi}{10}\right) - 6.$$

Ten students did not answer both questions. They probably could not provide an equation for the requested function. Eighty-one students found an incorrect cosine equation, and 95 students incorrectly answered the question about finding the sine function's equation. All these students failed to find the value of C for the given phase shift. They did not know that the phase shift, θ, is equal to $\frac{C}{B}$.The value of θ is positive if the function is shifted to the right, and it is negative if the function is shifted to the left. Hence, from the general form

$y = A\cos(Bx + C) + D \text{ } or \text{ } y = Asin(Bx + C) + D$, we have; $C = \theta B.$

Among these incorrect answers, 20 students only failed to obtain the correct value of C in the general form of the cosine function equation (see Figure 6.3.1), and 43 students only failed to find the value of C in the sine function equation (see Figure 6.3.2). Sixty-one students gave an incorrect value and a sign of C, as shown in Figure 6.3.3, and 35 students had an incorrect value and sign of C for the sine function as shown in Figure 6.3.4. They all failed to calculate the correct value of C and did not understand the concept of a phase shift to the right or to the left

Figure 6.3. 1: Response given by S3 for the cosine equation

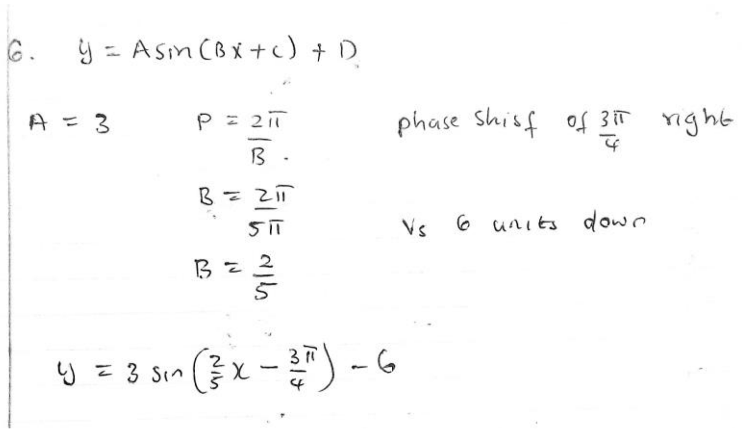


Figure 6.3. 2: Response given by S1 for the sine equation

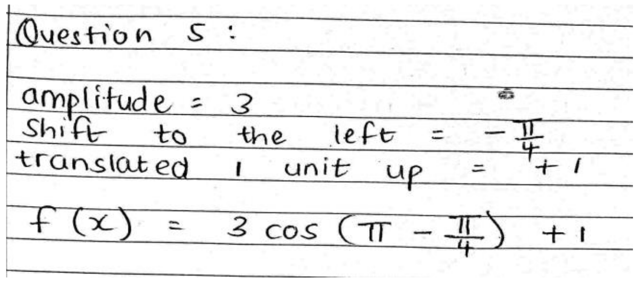


Figure 6.3. 3: An example of a response with a wrong sign and value of C given by S2

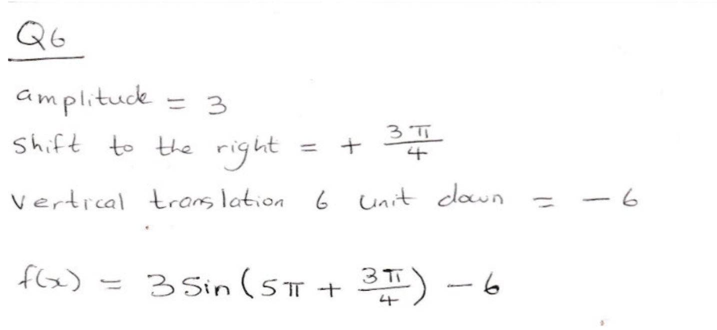


Figure 6.3. 4: An example of a response with a wrong sign and value of C given by S9

A total of 12 students had an incorrect notation for the cosine equation, and 20 students had an incorrect notation for the sine equation. They implied that the general form of the equation $y = A\cos(Bx + C) + D$ is the same as $y = A\cos B(x + C) + D$. These students did not understand the properties of a trigonometric function. They failed to see that, a trigonometric function, $y = A\cos(B(x + C)) + D$ does not yield the same outcome as the trigonometric function, $y = A\cos B(x + C) + D$. Here the misconception lies in the use of brackets in a trigonometric function. Figure 6.3.5 illustrates this misconception. The same analogy was applied to the sine function, as shown in Figure 6.3.6.

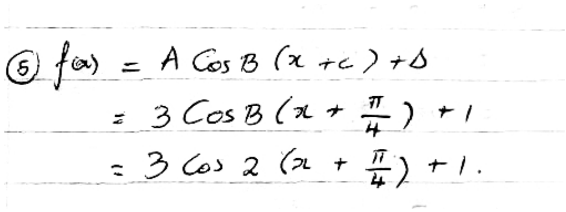


Figure 6.3. 5: Response given by S88 for cosine equation

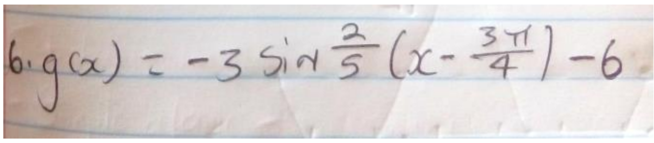


Figure 6.3. 6: Response given by S51 for sine equation

6.3.2 Misconception about the value of A, B, and D when writing a cosine and sine equation

For each problem, an incorrect value of A, B, or D was found. Forty-nine students had these incorrect values in the equation of the cosine function, and 32 students had these incorrect values in the equation of the sine function. These results are represented as follows:

6.3.2.1 Incorrect value of A (Amplitude)

Seven students provided the wrong answer for the value of the amplitude in the equation of the cosine function. Among these answers, six referenced Figure 6.3.7, while one student referred to Figure 6.3.8. In the first case, the students had a misconception about the values of A and B. From the general form of the equation, they considered that the value of A is equal to the value of B; in the second case, the student believed that the value of A was twice the value of B. Such responses indicate that this group of students did not understand that, in the general form of the cosine function, $y = A\cos(Bx + C) + D$, the value of A, representing the amplitude of the function, is not the same as the value of B.

Six students provided the wrong answer for the amplitude in the equation of the sine function. Of these answers, one referenced Figure 6.3.8, while five students provided an answer from Figure 6.3.9. S14 submitted an incorrect answer for A in the cosine function, finding it to be equal to 2; he made the same mistake in the equation of the sine function. The students who provided the answer in Figure 6.2.10 understood what represents the amplitude in the general form of a sine function. The misconception demonstrated by the students is that the vertical translation affects the sign of the amplitude in the equation.

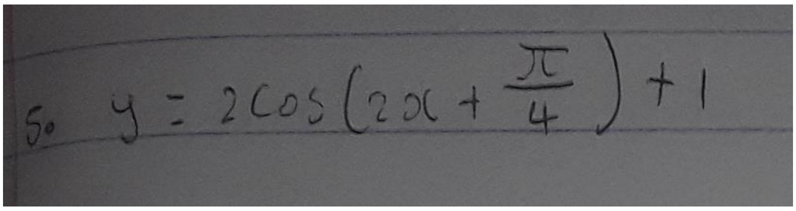


Figure 6.3. 7: An example of an incorrect value of A given by S14

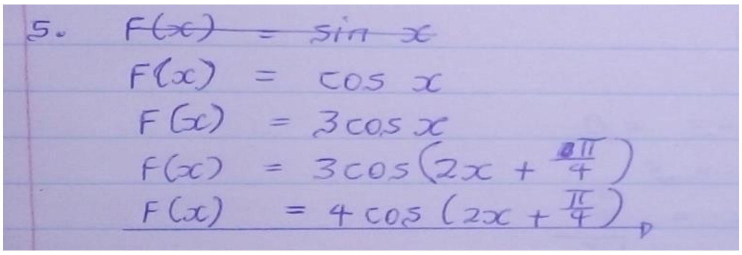


Figure 6.3. 8: A response of an incorrect wrong value of A given by S70

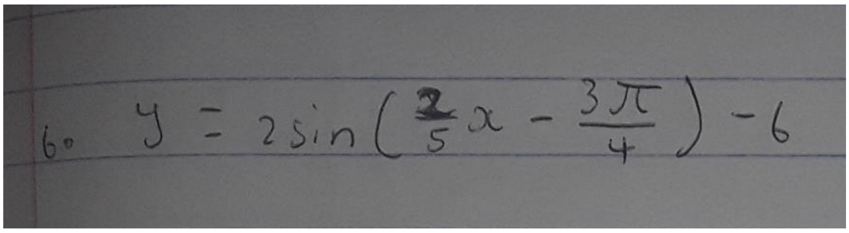


Figure 6.3. 9: A response of an incorrect value of A given by S14

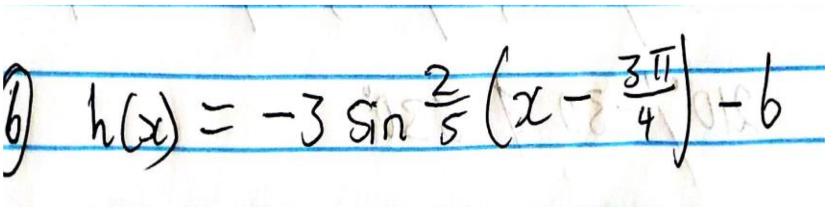


Figure 6.3. 10: A response of an incorrect value of A given by S72

6.3.2.2 Misconception about the period and the value of B

Nineteen students had a misconception about the value of B and the period. From these answers, ten students assigned π (six students referenced Figure 6.3.11, three students referenced Figure 6.3.12, and one student referenced Figure 6.3.13), and five students assigned $1/\pi$ as the value of B (see Figure 6.3.14). Three students assigned 1 as the value of B (see Figure 5.3.16), and one student stated that B was equal to x (see Figure 6.3.15). From the general form of the cosine function, $y = A\cos(Bx + C) + D$, all these students did not distinguish between the value of B in the general form and the period of the function. They did not know how to find the value of B when given the period of the cosine function. They all failed to use the formula $B = \frac{2\pi}{Period}$. In addition, some of the students did not know the difference between the commutative property and the distributive property of multiplication, as well as the use of brackets in a trigonometric function.

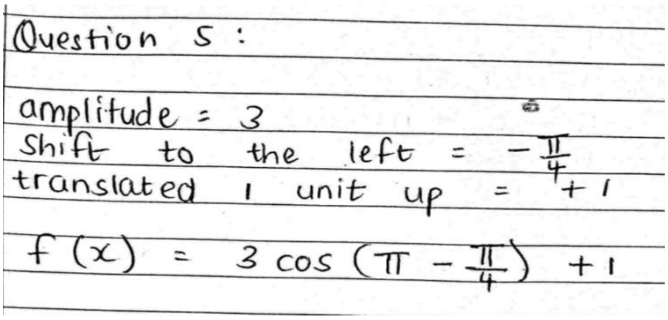


Figure 6.3. 11: A response of an incorrect value of B given by S2

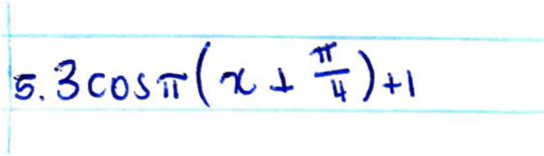


Figure 6.3. 12: A response of an incorrect value of B given by S18

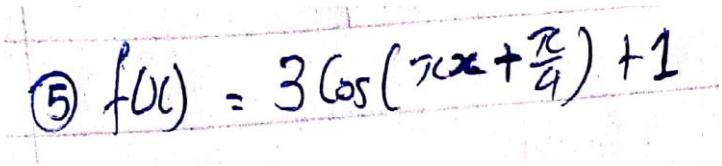


Figure 6.3. 13: A response of an incorrect value of B given by S45

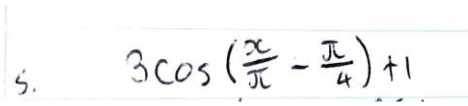


Figure 6.3. 14: A response of an incorrect value of B given by S81

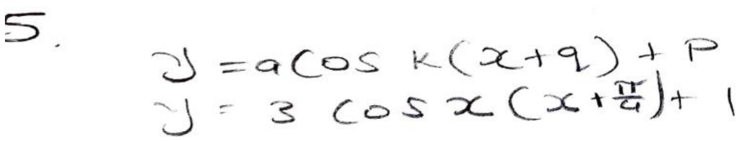


Figure 6.3. 15: A response of an incorrect value of B given by S95

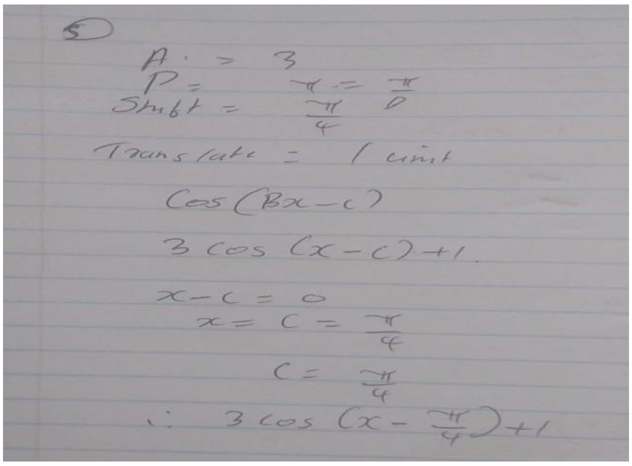


Figure 6.3. 16: A response of an incorrect value of B given by S102

6.3.2.3 Incorrect value of D

Twenty-three students provided an incorrect value of D. Among these responses, 20 students found $D = -1$ (see Figure 6.3.17), and three students found $D = 0$ (see Figure 6.3.18). These students did not understand the difference between the translation of one unit up and the translation of one unit down.

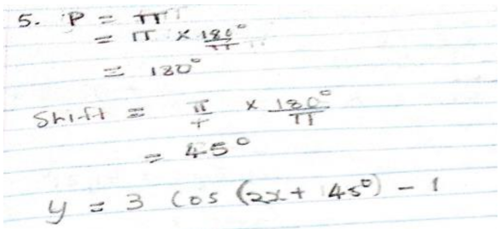


Figure 6.3. 17: A response of an incorrect value of D given by S64

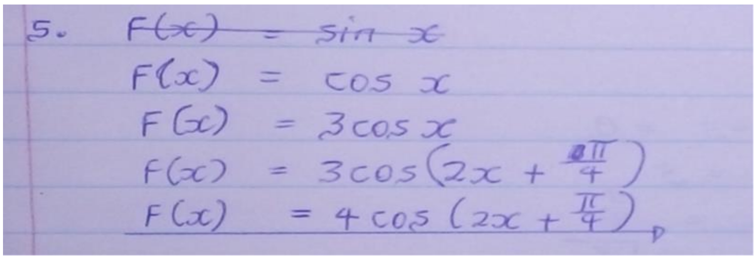


Figure 6.3. 18: A response of an incorrect value of D given by S70

For the equation of the sine function, the mistakes and misconceptions observed in finding the values of B and D were like those in finding these values from the cosine equation.

Twenty students had incorrect values of B, and six had wrong values for D. Some of those incorrect responses are shown in the figures below.

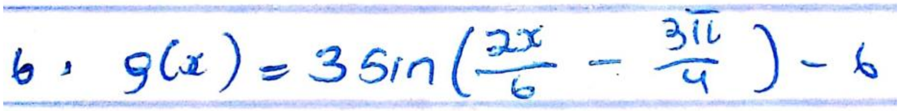


Figure 6.3. 19: A response of an incorrect value of B given by S3

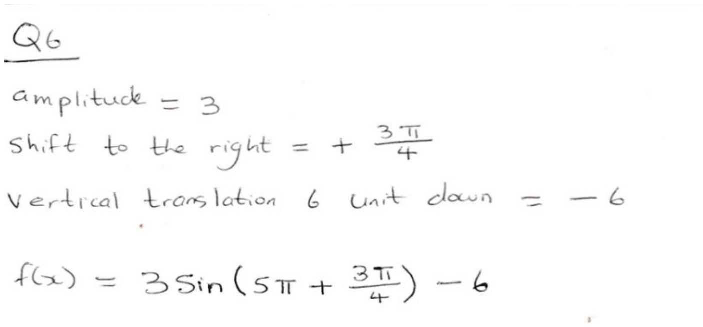


Figure 6.3. 20: A response of an incorrect value of B given by S9

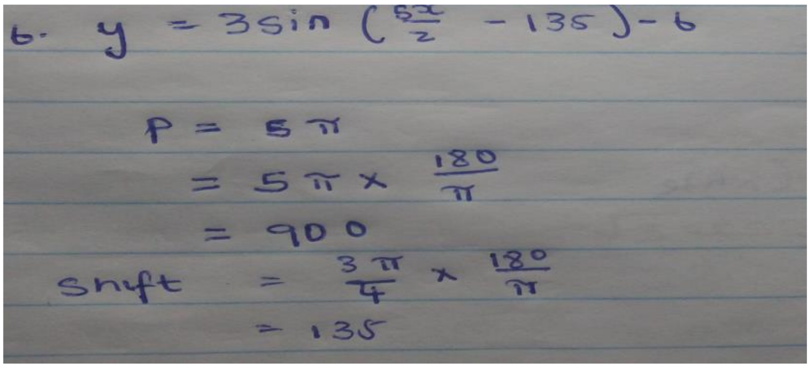


Figure 6.3. 21: A response of an incorrect value of B given by S5

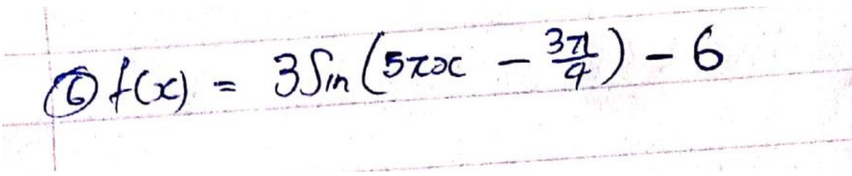


Figure 6.3. 22: A response of an incorrect value of B given by S45

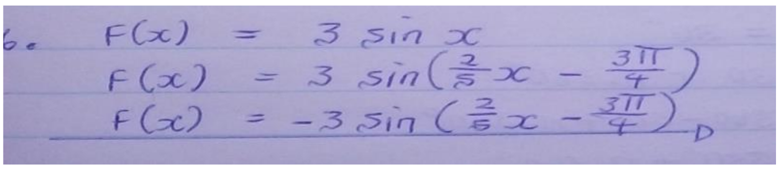


Figure 6.3. 23: A response of an incorrect value of D given by S70

CHAPTER 7
ANALYSING DATA GATHERED FROM ACTIVITY 2 QUESTIONS

In this chapter, we present the analysis of the written responses to the six tasks that the 125 students were required to complete. We analyse the interview responses of five participants who were available for the interview process. The participants are referred to as P1 to P125, where the order does not represent any significance. The tasks are arranged according to three types of demands. We first discuss the responses to four tasks based on using the fundamental identities of trigonometric functions to find an expression of a trigonometric function in terms of a variable. The next four tasks deal with evaluating a trigonometric function expressed in radians and degrees without the use of a calculator. Thirdly, we examine at three tasks that require students to simplify a trigonometric expression.

7.1 Using algebra and trigonometric relationships to represent trigonometric functions as an algebraic expression

This task aimed to identify prospective teachers' skills in finding the value of a trigonometric function with respect to a variable when the value of one trigonometric function was expressed as a variable. The participants were asked to use fundamental identities of trigonometric functions to determine the value of $\tan(40°)\ and\ \sec^2(130°)$ when $\cos(40°) = a$. Each function was determined as follows:

7.1.1 Expressing tan (40°) in terms of a

The students could have applied the Pythagorean identity method or the unit circle method, to find expressions for all the trigonometric functions of 40°. Using the unit circle, a right-angled triangle could be constructed where the hypotenuse is the radius of the circle, and the sine of an angle *t* equals the $y - value$ of the endpoint on the unit circle of an arc of length t, while the cosine of an angle *t* equals the $x - value$ of the endpoint. Of the 125 students who responded to the question. Among these students, six students did not provide any answer to the question while 18 just wrote the answer $\tan\frac{\sqrt{1-a^2}}{a}$ without providing any working details (see Figure 7.1). During the interview, student P14 was unable to clearly explain how he used the unit circle method. He failed to identify each ratio of sides of the right-angle triangle when an acute angle is given. This was due to not properly identifying the hypotenuse, the opposite side, and the adjacent angle.

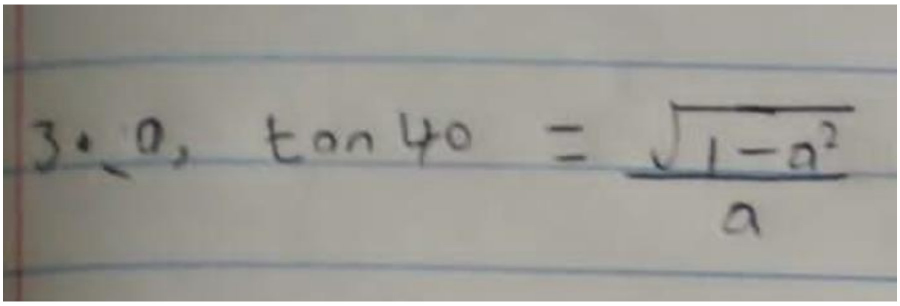


Figure 7. 1: Answer given by P14

Researcher: *Now, you said you used a right-angle triangle diagram, right?*

Student P14: *Yes.*

Researcher: *Do you feel like there was any other method that you could've used?*

Student P14: *I think there was sir.*

Researcher: *Which one?*

Student P14: *I think there was sir, there was another method I could use but I didn't understand it very well, so this is the only one that I understood well.*

Eighty-nine students answered the question correctly, all of whom applied the unit circle method as shown in Figure 7.2. No one used the Pythagorean identity, $\sin^2\theta + \cos^2\theta = 1$ to obtain,

$the\ \sin 40° = \sqrt{1 - \cos 40°} = \sqrt{1 - a^2}$ whereby $\tan 40° = \frac{\sin 40°}{\cos 40°} = \frac{\sqrt{1-a^2}}{a}$.

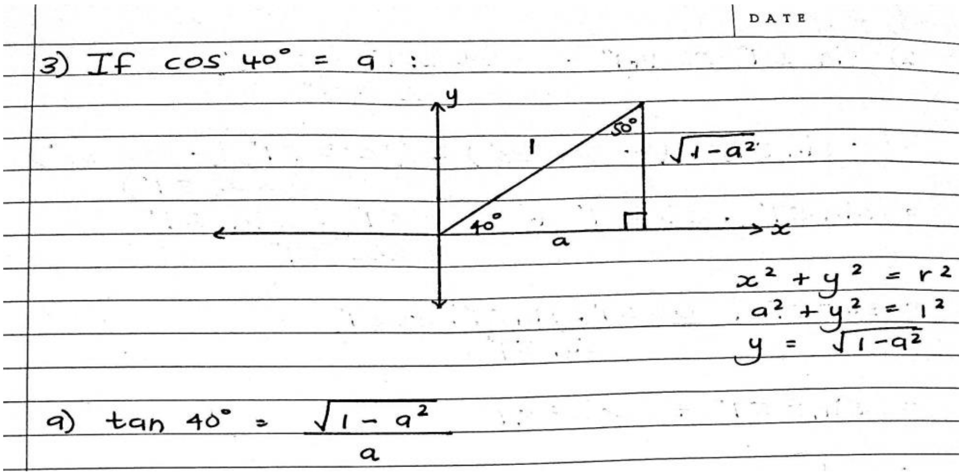


Figure 7. 2: Unit circle method used by P2

Twelve students provided an incorrect answer to the question. One student was able to correctly substitute the value of the radius and the $x - value$, but encountered difficulties in calculating the $y - value$. The student calculated the $y - value$ as $\sqrt{a^2 - 1}$ instead of $\sqrt{1 - a^2}$ as depicted in Figure 6.3.

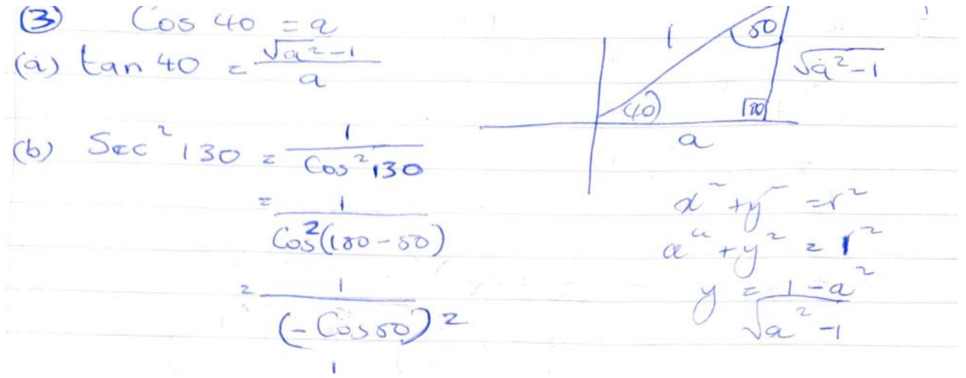


Figure 7. 3: Answer provided by P8

Three students seemed to be stuck after they substituted correctly for the horizontal side as seen in Figure 7.4. They could not find $\sin 40°$, by using the Pythagorean Theorem to determine the third side of the right-angled triangle within the unit circle. During an interview with one of them, student P79, the student was asked to explain his answer. The interview responses show that although the student had knowledge of the trigonometric ratios which he was supposed to use, but he could not find $\sin 40°$. He needed to calculate the opposite side by using the Pythagorean Theorem. Even when he acknowledged his mistake, he still could not explain how to apply the theorem. He then conceded by saying that the question was difficult, and he did not understand it. This indicates that these students faced challenges of using the unit circle method. The following interview responses outline these difficulties.

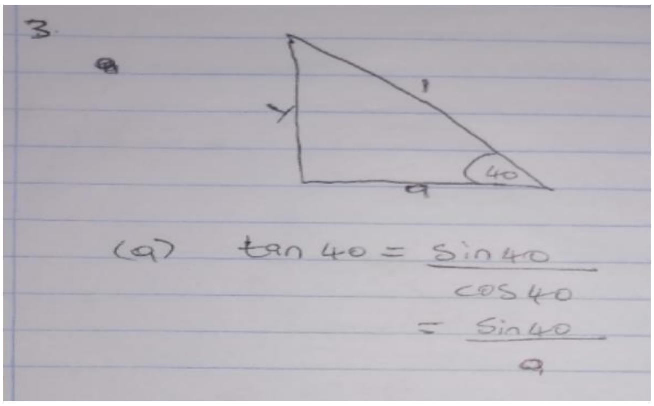


Figure 7. 4: Incorrect answer provided by P79

In summary, the student did not realise that he could use the Pythagorean Theorem to find the third side.

Researcher: *Why did you not complete your answer?*

Student P79: *I was panicking because I was trying to understand even myself whether, I understood clearly cause now that I see it, it is so clear if I'm asked to determine tan 40˚ in terms of a when cos 40˚ is equal to a. We know that tan is sine over cosine in terms of a, tan shouldn't be the answer I provided.*

Researcher: *What do you think you should have done?*

Student P79: *I think from this point now,*

Researcher: *Yes!*

Student P79*: I think, I should have, now that I was given that cos 40˚ is a, right? What I should have done after, we are writing tan 40˚, in terms of sine and cos 40˚ when sine is one of the reciprocate of…, I do believe that I should have given it, uh, another method using the identities that I know.*

Researcher: *Okay.*

Researcher: *So how are you going to achieve that? Is there a particular formula that you're going to use…, is there a particular property that you are going to use to come up with the correct answer?*

Student P79: *Yes, from my formula sheet that I have been given, using the tan formula, uh, let me see, if I can see the one that I can use, it won't take me long because I do have the identities with me, that I think, maybe, will come out to the answer , but if I were to say tan 40˚ is sine over cos.*

Researcher: *Okay.*

Student P79: *Then maybe I would have determined it in terms of a.*

Researcher: *So, you were going to start with tan 40˚ equals to sine over cos, that would be your first step?*

Student P79: *I think that would be my first step now.*

Researcher: *And from there?*

Student P79: *Then, I already have cos 40˚*

Researcher: *Okay.*

Student P79: *Because I have given that cos 40˚ is a, right!*

Researcher: *Okay.*

Student P79: *Then from there…, then I have a over sine 40˚, right?*

Researcher: *a over sine 40˚?*

Student P79: *Sorry, it is sine over cos, right? So, I will have sine 40˚ over a.*

Researcher: *Okay.*

Student P79: *Then knowing that sine 40˚ can be written as 90. Ay, I don't know if changing it to 90˚ minus 50˚ would make any difference?*

The student seems to be confused….

Researcher: *I don't know, what do you think?*

Student P79: *Eh, maybe saying..., no.., if 40˚..., if you say…, 180˚ minus…, eish…, I'm still trying to work it out.*

Researcher: *Okay.*

Student P79: *It's too difficult… (silent for one minute).*

Researcher: *Okay.*

Two students displayed some confusion between the x and y coordinates. They incorrectly took the $y - coordinate$ as a and then computed an incorrect value for the $x - coordinate$, as seen in Figure 7.5. The working details revealed confusion between the x and y coordinates. They correctly expressed $\tan\theta$ as the ratio of the $y - coordinate$ to the $x - coordinate$; however, due to the confusion, their answer was the value of $\cot\theta$.

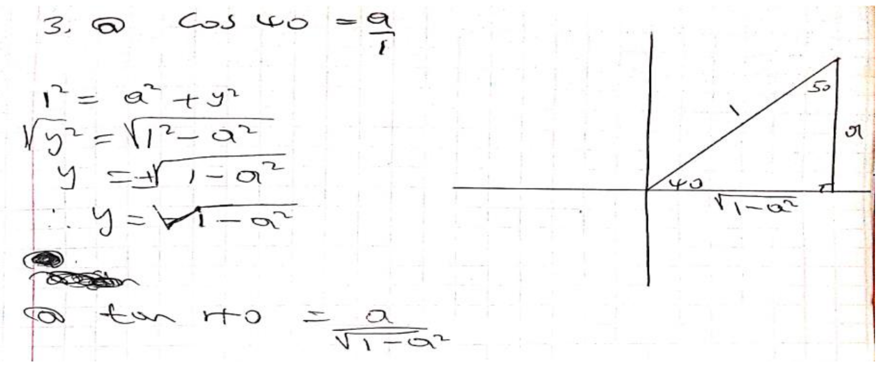


Figure 7. 5: Incorrect answer provided by P22

There were six students who made a correct substitution and computed all the sides of the right-angled triangle accurately but took the $y - coordinate$ as the value of $\tan\theta$, as shown in Figure 7.6. The students held a misconception that the values on the $y - axis$ represent the outcome of the trigonometric function $\tan\theta$.

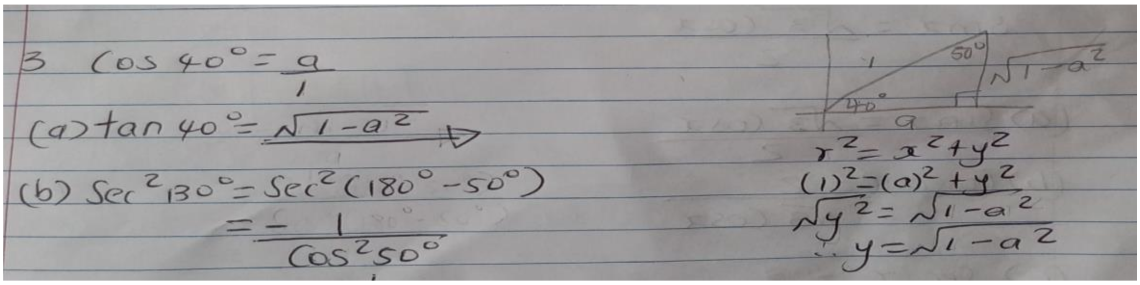


Figure 7. 6: Incorrect answer provided by P83

7.1.2 Finding the value sec² (130°) in terms of **cos (40°) = a**

This item required some manipulations based on the trigonometric relationships of reduction, co-functions for complementary angles, and the reciprocal relationship between cosine and secant functions. The students could have first simplified $\sec^2 130°$ to $\frac{1}{\cos^2 130°}$ and then used reduction and complementary angle rules to obtain $\frac{1}{\cos^2 50°} = \frac{1}{\sin^2 40°} = \frac{1}{1-a^2}$. Sixty-four students completed the question correctly. Among these correct answers, the students used three different approaches, which are illustrated in Figures 7.7, 7.8, and 7.9.

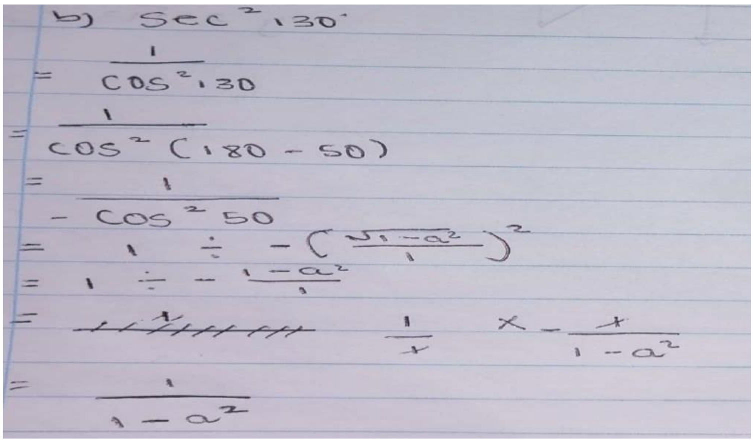


Figure 7. 7: A correct answer provided by P6

Twelve students obtained a correct answer by first translating the sec function to cos before opting for the reduction of $130° = 180° - 50°$ as shown in Figure 6.7. This then enabled them to express the equation in terms of a, by using the unit triangle from the previous sub-question.

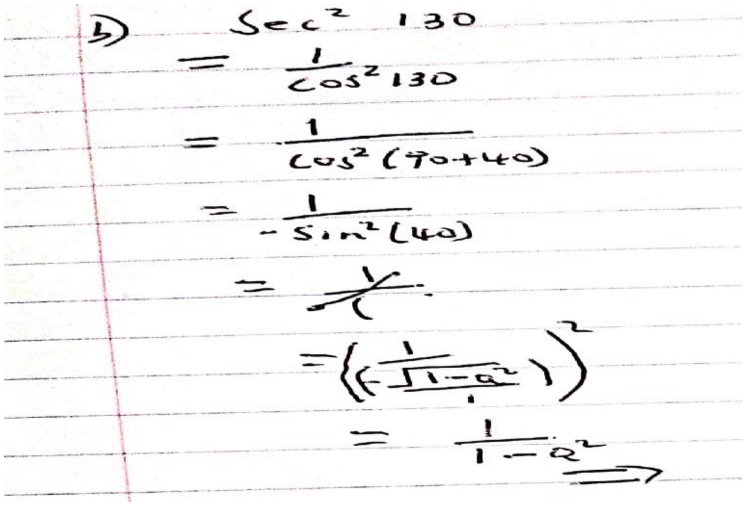


Figure 7. 8: A correct answer provided by P102

Sixteen students provided an answer represented in Figure 6.8. They first simplified $\sec^2 130°$ to $\frac{1}{\cos^2 130°}$, then they used cofunction identities to relate the different trigonometric functions and their complementary angles. Although the students had a misconception about the use of brackets in their third step by assuming that $\frac{1}{-\sin^2 40°}$ equals to $\frac{1}{(-\sin 40°)^2}$, they correctly used the unit triangle from the previous sub-question in the next steps to complete the question.

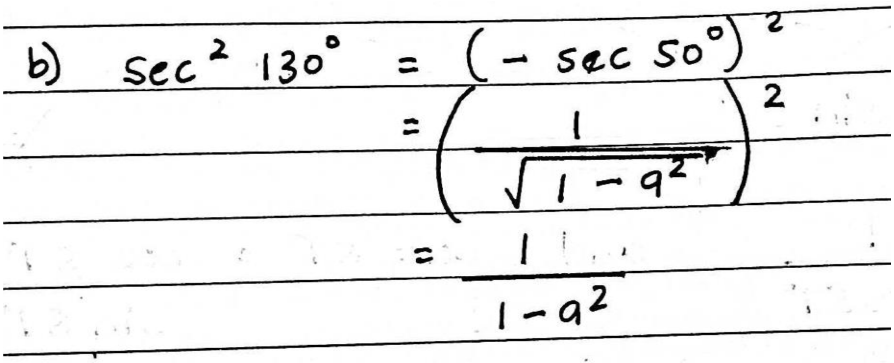


Figure 7. 9: A correct answer provided by P2

Thirty-six students provided an answer from Figure 6.9. They directly used the cofunction identity of the secant function and applied the unit triangle from the previous sub-question to complete the task.

Fifteen students did not submit any answer, and 51 students provided an incorrect answer. Four students presented an answer from Figure 7.10. Thes students begin by correctly rewriting $130°$ as $180° - 50°$. However, they then incorrectly simplify $\sec^2(130°)$ to

$\sec^2(50°)$, ignoring the fact that cosine is negative in the second quadrant, where 130° lies. Therefore, the correct relationship should include a negative sign: $\cos(130°) = -\cos(50°)$, which means $\sec(130°) = \frac{1}{-\cos(50°)}$, and squaring it still leads to a positive value. The omission of this sign consideration indicates a conceptual gap in understanding trigonometric function behaviour in different quadrants.

The students then attempt to evaluate $\sec^2(50°)$ by substituting $\cos(50°) = a$, though they appear to confuse secant with sine. They write $\sec(50°) = \frac{\sqrt{1-a^2}}{1}$, which is the expression for $\sin(50°)$, not secant. The correct identity is $\sec(50°) = \frac{1}{\cos(50°)} = \frac{1}{a}$, and hence $\sec^2(50°) = \frac{1}{a^2}$. The students then square $\sqrt{1-a^2}$ to get $1 - a^2$, which again reflects confusion between sine and secant. As a result, the final answer $\sec^2(130°) = 1 - a^2$ is mathematically and conceptually incorrect.

Overall, the students' response shows a fundamental misunderstanding of trigonometric identities and quadrant sign behaviour. There is clear confusion between the definitions of secant and sine, and the student incorrectly applies the identity for sine in place of secant. Additionally, they neglect the impact of angle location on the sign of trigonometric functions. To improve, the student needs to revisit the definitions of secant and cosine, pay closer attention to angle placement in the unit circle, and carefully apply identities rather than relying on incorrect substitutions.

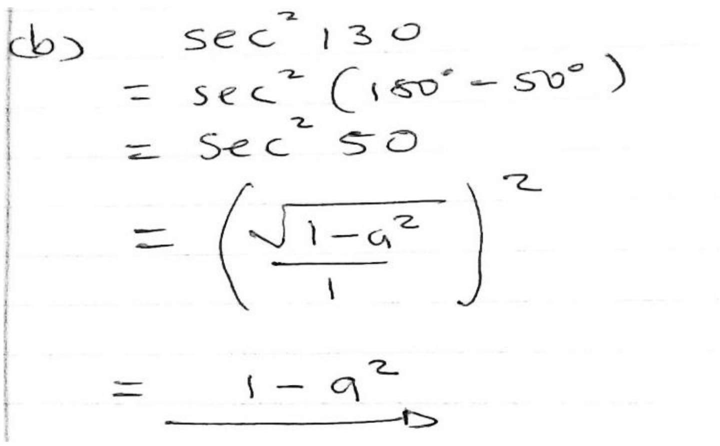


Figure 7. 10: Incorrect answer provided by P1

In Figure 6.11, P3 carried out most of the steps correctly but had a misconception about the square of a trigonometric function. The student assumed that $(-\csc x)^2 = \csc x$. This incorrect answer was provided by eighteen students.

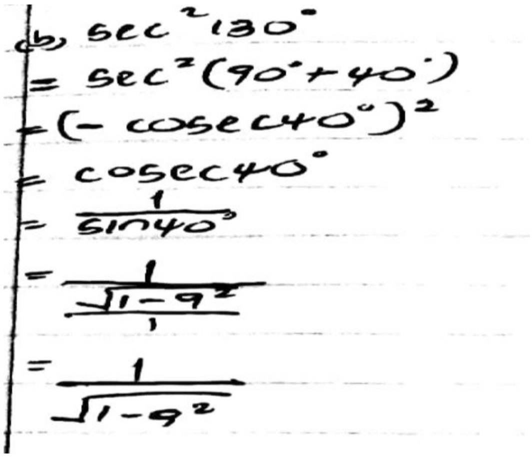


Figure 7. 11: Incorrect answer provided by P3

The answer in Figure 7.12 demonstrates the incorrect use of cofunction identities. P12 wrote $\sec^2(90° + 40°) = \sin^2 40°$ instead of $-\csc^2 40°$. This error was made by seven students. P79 was one of the students who referenced an answer from Figure 7.12. The interview with P79 revealed that the student could not correctly apply the trigonometric relationships of reduction to find co-functions. He also had a misconception about the reciprocal relationship between trigonometric functions, and he did not know how to properly use the trigonometric unit circle. In his answers, the student assumed that 130˚ is in the first quadrant of the unit circle and that sec (130˚ + 40˚) equals negative sec 40˚. The following transcript details his response to the question.

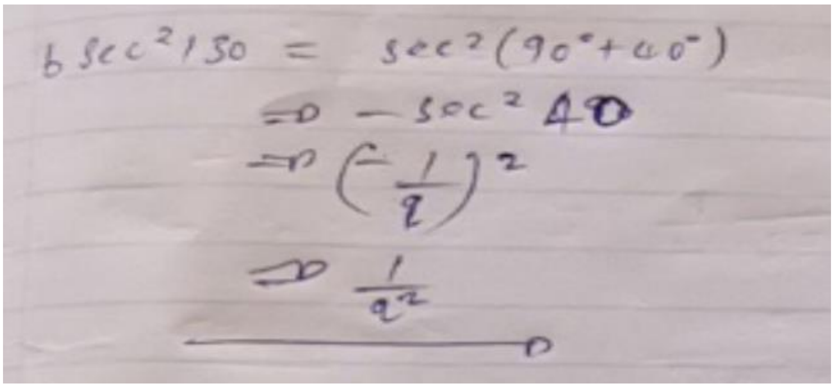


Figure 7. 12: Incorrect answer provided by P79

Researcher: *Can you please explain your answer?*

Student P79: *Alright, so I know that sec squared (130˚), if you, I don't, I think, I divided right, when I said sec squared. Sec squared of 130˚ is 90˚ + 40˚ cause 90˚ + 40˚ =130˚.*

Researcher: *Ok.*

Student P79: *Then knowing that if you are using the first quadrant then the identity changes its reciprocal form if I'm not mistaken.*

It seems that the student does not know the basic definition of a reciprocal function. He is confusing the use of trigonometric functions in the unit circle with the trigonometric reciprocal functions. A reciprocal function is the inverse of a function, so when a function changes quadrants, its reciprocal form will not change. The only change occurring in both the function and its reciprocal will be the signs of the functions.

Researcher: *Okay.*

Student P79: *So, deriving it from here when I have the -sec 40˚, I think given the amount of time I have now, I believe I have now, I would've gone the other way.*

Researcher: *Okay.*

Student P79: *But I do believe that I would've gone if I gave myself enough time to understand the question that was required of me, cause from here now if I can play it around using the identities that I do know, I would have gotten to the cos 40˚ again then expressed it in terms of a.*

Student P79: *That was my trouble because I know that if you checked all the scripts that were submitted, you'll realize that I didn't copy it from someone else. So, it was my own work. But knowing that I was not prepared.*

Researcher: *Oh, okay.*

Seventeen students provided an answer like Figure 7.13. The students correctly identified the cofunction. Their mistake was an algebraic one. They believed that $\frac{-1}{\left(\sqrt{1-a^2}\right)^2} = \frac{1}{1-a^2}$. An interview with P57 revealed that the student had a misconception which is not outlined in her answer to the question. During the interview, she mentioned that the reciprocal of sec was cosec, and that the reciprocal of cosec was sine. She doesn't know that there are three reciprocal trigonometric functions, totalling six, including cosine, sine, and tangent. Each function has its own reciprocal function. The discussion with the student also highlighted her error regarding the negative sign and her written algebraic expression. Her term "multiply" was incorrect.

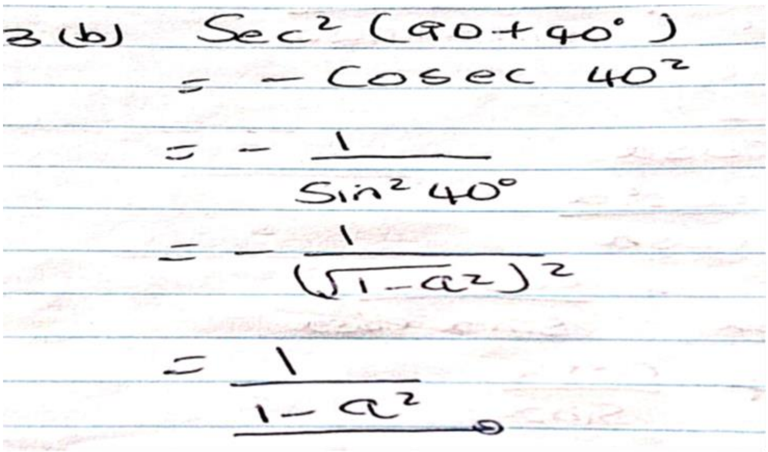


Figure 7. 13: Incorrect answer provided by P57

Researcher: *Can you take us through your answer?*

Student P57: *Okay sir, so I will start the question by putting sec squared 130 equals sec squared 90 °+ 40°.*

Researcher: *Okay.*

Student P57: *Then, what I know in trigonometry is that for each trigonometric ratio, when it has 90°, it takes the reciprocal.*

Here, she used the word reciprocal to mean cofunction. She has confusion about terminology…

Researcher: *Okay.*

Student P57: *Then the reciprocal of sec is cosec.*

Researcher: *Okay.*

Student P57: *Then I know it's cosec, what I must ask myself is which quadrant is sec, and is it positive or negative where there's 90 plus?*

Researcher: *Okay.*

Student P57: *Then I got it on a, it's on a second quadrant where it's 90 plus.*

Researcher: *Ok.*

Student P57: *And my sec with sec negative because it is on a second quadrant.*

Researcher: *Okay.*

Student P57: *Then the negative cosec is the reciprocal of sine.*

Researcher: *Okay.*

Student P57: *When said the reciprocal…, the reciprocal of cosec will be one over sine squared. Then I put squared because I have squared, I am dealing with trigonometric ratio with a square.*

Researcher: *Okay.*

Student P57: *Then I continue with negative one over sine squared 40, then equal to…, I'm going with my negative because they don't consider…I'll say negative one and that's the opposite side. We go to the definition of sine. Sine is equal to the opposite over the hypotenuse.*

Researcher: *Okay.*

Student P57: *Sine is equal to the opposite over hypotenuse. Remember my Pythagoras it was the square root of one minus a squared.*

Researcher: *Yes.*

Student P57: *And my hypotenuse was one---*

Researcher: *Yes.*

Student P57: *So, I will not put all over one. I know in my head that it is all over one.*

Researcher: *Okay.*

Student P57: *So, our negative one over the square root of one minus a squared only squared.*

Researcher: *Okay.*

Student P57: *Then this negative, oh the square root will cancel, all squared that I've written.*

Researcher: *Oh okay.*

Student P57: *Square will cancel all squares, then negative will cancel what? Negative I don't know like that, then our square root square cancels the square, my answer will be one over one minus a squared.*

Supervisor: *Sorry, sorry, sorry, can I just say something here, I want to find out from you. What happened to the minus sign there?*

Student P57: *Minus sign?*

Supervisor: *You got a minus in the first step, the second step, the third step, and the last step you got no minus.*

Student P57: *I don't remember why I left minus.*

Supervisor: *Um do you see that you put your square only by the 40? Perhaps you needed to put the bracket around.*

It seems the student had a problem with the notation of her expression. She did not place the bracket correctly. She was supposed to write $\sec^2(90° + 40°) = (-\text{cosec } 40°)^2$ instant of $\sec^2(90° + 40°) = -\text{cosec } 40^2$.

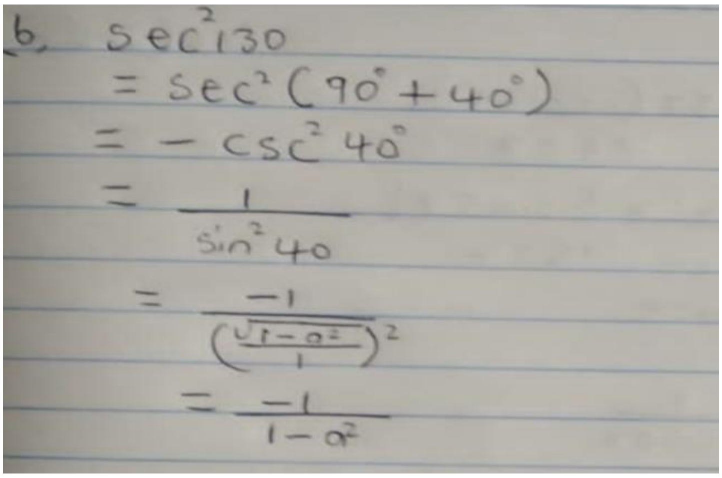


Figure 7. 14: Incorrect answer provided by P108

In Figure 7.14, P108 correctly finds the cofunction of $\sec^2 130°$ to be $-\csc^2 40°$. The student made an algebraic error in the third step. He wrote $-\csc^2 40°$ equals to $\frac{1}{\sin^2 40°}$ instead of $\left(\frac{1}{-\sin 40°}\right)^2$. This incorrect answer was provided by five students. The interview with P108 reveals that instead of writing $-\csc 40° = \frac{1}{-sin 40°}$ she wrote $-\csc^2 40° = \frac{-1}{\sin^2 40°}$. After a detailed discussion with the interviewer, she conceded that she made a mistake as shown by the following transcript:

Researcher: *Can you explain your answer?*

Student P108: *Oh, okay so* $\sec^2 130° = \sec^2(90° + 40°)$.

Researcher: *Okay.*

Student P108: *And 90˚... uh. Okay, so I used the Cartesian plane method.*

Researcher: *Okay.*

Student P108: *Where it says 90˚ plus…Yoh.*

Researcher: *Okay, so you can just take me through what you are seeing in your answer.*

Student P108: *Oh yes so 90˚, uh 90˚ changes the sec to be negative co-sec squared 40˚.*

Researcher: *Okay.*

Student P108: *So, we are going to use the previous right-angle triangle.*

Researcher: *Yes.*

Student P108: *To find negative co-sec squared 40˚.*

Researcher: *Okay.*

Student P108: *So, we are going to substitute that, our opposite was* $\sqrt{1 - a^2}$*, and our hypotenuse was-*

Student P108: *Okay and then we are going to have* $\frac{-1}{(\sqrt{1-a^2})}$.

Researcher: *Okay.*

Student P108: *So, our final answer is negative one over one minus a squared.*

Researcher: *Ok, how did get* $-\operatorname{cosec}^2 40° = \frac{1}{\sin^2 40°}$*?*

Student P108: *Uh…sorry, I think I was supposed to write* $\frac{1}{sin^2 40°}$.

Researcher: *Okay. Why did you write* $\frac{-1}{sin^2 40°}$ *instead* $\frac{1}{sin^2 40°}$*?*

Student P108: *Because minus multiply cosec squared 40 degrees and cosec squared 40 degrees equal to one over sine squared 40 degrees. So that is why I have a negative one over an open bracket square root of one minus a squared, closed bracket.*

Researcher: *So, don't you think that minus was supposed to multiply sine square 40 degrees?*

Student P108: *Um…I don't think so, sir.*

Researcher: *Why?*

Student P108: *Because minus multiplies the fraction, sir.*

Researcher: *So,* $\frac{1}{sin^2 40°}$ *is equal to* $-\operatorname{cosec} 40°$ *or it is equal to* $\operatorname{cosec} 40°$*?*

Student P108: *Uh…... I think it is equal to minus cosec 40 degrees.*

Researcher: *So, don't you think that it is the same with minus co-sec squared 40 degrees?*

Student P108: *Oh yes, sir.*

The student's response reveals multiple conceptual misunderstandings regarding trigonometric identities, angle transformations, and sign rules across quadrants. The problem asks for the evaluation of $\sec^2(130°)$ given that $\cos(40°) = a$. The student begins by rewriting $\sec^2(130°)$ as $\sec^2(90° + 40°)$, which is a non-standard and problematic approach. While technically $130° = 90° + 40°$, expressing angles in terms of 90° shifts requires caution because trigonometric functions transform into their co-functions (e.g., $\cos(90° + x) = -\sin(x)$), and secant does not directly convert into cosecant with a simple negative sign. The student incorrectly concludes that $\sec^2(90° + 40°) = -\csc^2(40°)$, which is not a valid trigonometric identity. This critical error misguides the entire solution. The student proceeds to evaluate $\csc^2(40°)$ using the identity $\csc(\theta)\,\frac{1}{\sin(\theta)}$, and attempts to find $\sin(40°)$ based on a previous triangle where $\cos(40°) = a$. Using the Pythagorean identity $\sin^2(40°) = 1 - \cos^2(40°) = 1 - a^2$, the student identifies $\sin(40°) = \sqrt{1 - a^2}$, which is a correct substitution. However, the calculation then becomes inconsistent when they write the expression as $-\frac{1}{\sin^2(40°)} = -\frac{1}{1-a^2}$. This expression assumes the negative sign applies to the whole fraction, reinforcing the earlier error from the incorrect identity. When questioned by the researcher, the student insists that the minus applies to the entire result due to the transformation from secant to negative cosecant but fails to justify this claim with correct reasoning or knowledge of quadrant rules.

Throughout the exchange, it becomes evident that the student lacks a firm understanding of function transformations across quadrants and the behaviour of trigonometric functions under angle shifts involving 90° and 180°. The student confuses secant with cosecant and does not apply the reciprocal or Pythagorean identities appropriately. Moreover, her justification for the negative sign appears memorized rather than reasoned, as she is unable to connect their steps to standard trigonometric principles or correct co-function identities. The dialogue shows that while the student attempts to defend her process, she is not confident in the foundational rules needed to construct or verify their solution. To improve, the student must revisit the fundamental identities of trigonometric functions, understand how angles relate on the unit circle, and develop a clearer grasp of how co-function and sign changes work in different quadrants.

7.2 How do PSTs calculate the value of a trigonometric function without the use of a calculator?

In this task, the students were asked to calculate the value of $\sin\left(\frac{13\pi}{6}\right)$ $and\ 2\sin 150°$ without the use of a calculator. The aim of the task was to examine the PSTs' knowledge in working with radians and degrees when evaluating a trigonometric function.

7.2.1 Using radian to evaluate a trigonometric function

The question required respondents' knowledge of using radians to evaluate a trigonometric function. The aim was to simplify the function of the given angle in terms of a reference angle. For the question $\sin\left(\frac{13\pi}{6}\right)$, it would be easier to first subtract the full rotations of 2π and express the argument as an angle between 0 and 2π. The results indicated that although 107 students provided the correct answer to the question, only 12 students provided the correct answer without converting radians to degrees (see Figure 7.15).

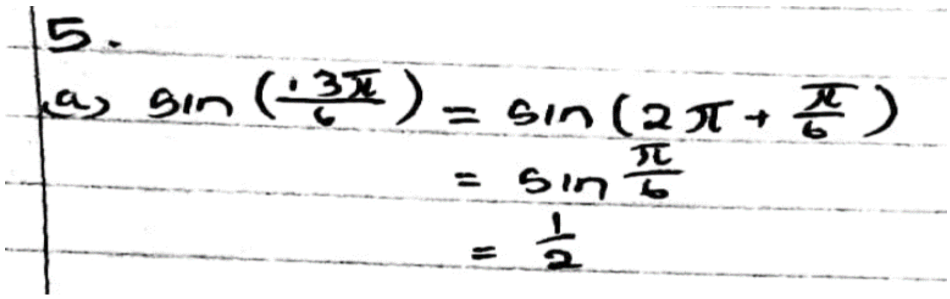


Figure 7. 15: Correct answer provided by P4 without changing the radian

Ninety-five students (as shown in Figure 6.16) converted radians to degrees before simplifying and then computing the value of sin (30°). It can be assumed that these students used calculators to answer the question since this is the usual method for determining the value of sin (30°), as seen in Figures 6.16. This shows that the participants' conceptual knowledge of radians was not as robust as their conceptual knowledge of degrees. In their interview responses, the students had different reasons for why they couldn't work with radians. Among the students who used the conversion method, 49 applied the cofunction method after converting radians to degrees, as shown in Figure 6.16. P108 argued that it is easier to work with degrees than with radians.

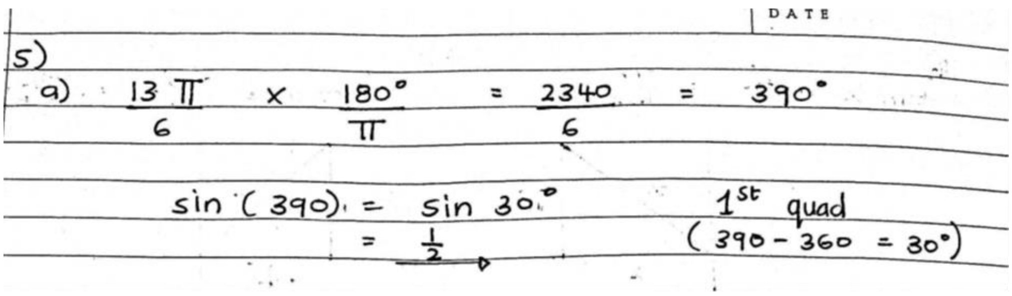


Figure 7. 16: Correct answer provided by P108 by changing the radian

Researcher*: Why did you convert radians to degrees, why did you change the radians to degrees?*

Student P108: *Because, um, degrees are easier to find answer without using a calculator.*

Forty-six students used the addition formula after converting radians to degrees as seen in Figure 7.17.

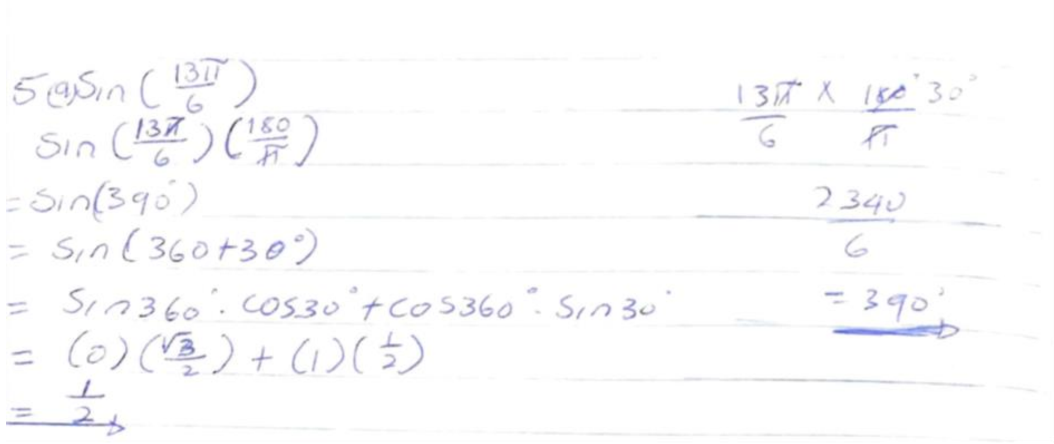


Figure 7. 17: Correct answer provided by P55 by changing the radian

P55 said that she was not going to find the correct answer if she had to use the radians.

Researcher: *Why did you convert radians to degrees?*

Student P55: *Um, I think if I didn't do that, I would just punch 13 pi over 6 into the calculator and get the answer and see what happened after that.*

Researcher: *Yes, but that question I think said, without the use of a calculator, let me go through, they said to evaluate the following without the use of a calculator. So that means, without the use of a calculator, how are you going to find an answer by using the radian only?*

Student P55: *Ay, there was no other way for me sir.*

Nine students did not answer the question, and nine students provided a wrong answer to it. Of the nine incorrect answers, two students made a conversion from radians to degrees and stopped at that point, as shown in Figure 6.18. They may have wanted to use the calculator but realized that its use was prohibited. Hence, the incomplete work presented in Figure 7.18.

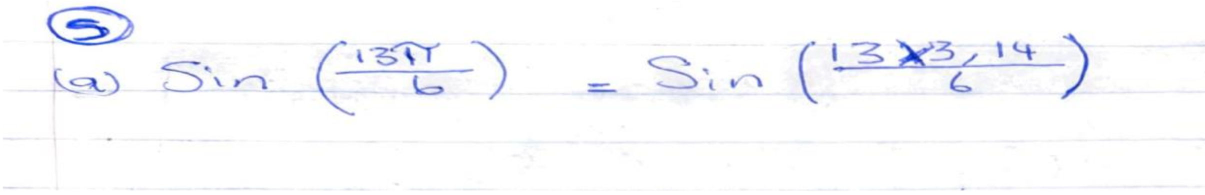


Figure 7. 18: Incorrect answer provided by P17

One student was confused by the conversion from radians to degrees, as shown in Figure 6.19. Here, the student did not understand the question, since he converted the radian to a degree and then reverted to expressing the argument in radians as a final answer.

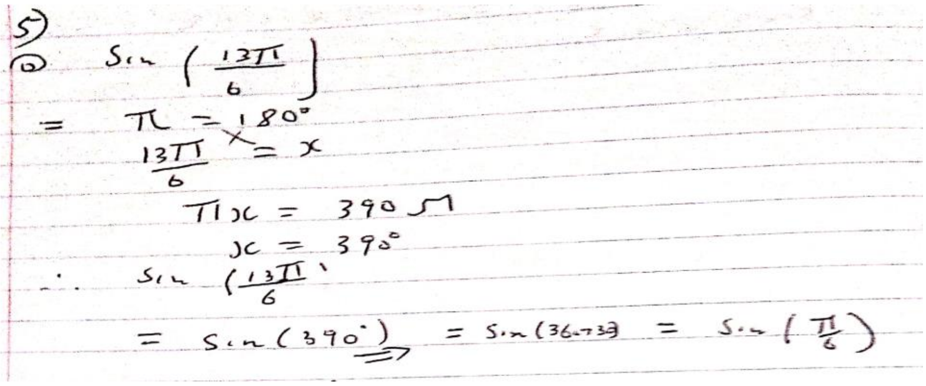


Figure 7. 19: Incorrect answer provided by P56

In Figures 7.20 and 7.21, P86 and P106 did not know how to find a reference angle in radians. In Figure 7.20, P86 incorrectly broke down $\frac{13\pi}{6}$ as the sum of $\frac{5\pi}{6} + \frac{3\pi}{4}$, and then applied the compound angle formula for $\sin(A + B)$ correctly to this incorrect result and then computed the answer based on this value.

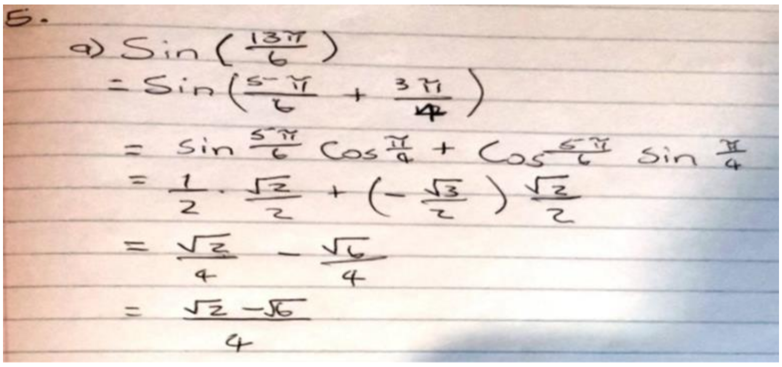


Figure 7. 20: Incorrect answer provided by P86

Figure 7.21 shows a misconception from P106 about the addition formula of a trigonometric function of three angles. His unique approach was to expand $\sin(A + B + C)$ as $\sin(A) + sin(B) + sin(C)$. It is as if the student 'multiplied' the 'sin' by the three given angles (Figure 7.20). That is, 'sin' was considered a variable that was then distributed over the sum of the three angles, and then the sin of each of the single angles was computed. Furthermore, the student couldn't correctly simplify fractions. In his responses from the interview, P79 acknowledged that he incorrectly used a method that he learned via the internet. He couldn't recall the method used to solve a similar problem. The following transcript displays his misuse of information he gathered from the internet.

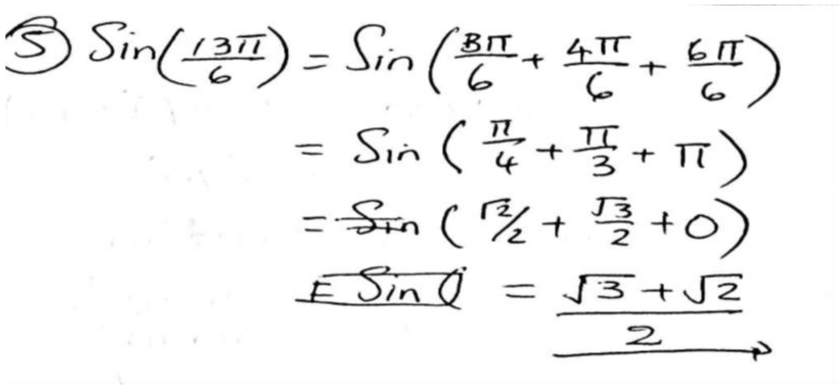


Figure 7. 21: Incorrect answer provided by P79

Researcher: *Please explain your answer*

Student P79: *What I did here, let me recall, this is 13pi over 6, so it is... Yes, so what I was trying to express here is…, because I saw while I was studying that you can add decompose such an expression…, that what I was trying here.*

Student P79: *If you can add the fractions that I have in the brackets, I think they still add up to 13 over 6.*

Referring to the fact that $\frac{13}{6} = \frac{3}{6} + \frac{4}{6} + \frac{6}{6}$.

Researcher: *So now, what were you aiming to achieve? Like which rule were you aiming to apply here? Is it an addition, a subtraction or which rules were you trying to apply?*

Student P79: *Yeah, I was trying to apply the rule that will simplify these fractions if they were decomposed. It's the trick that I took out, I remember I had to watch some tutorials, not from the ones you had to load on module but from the other educators on YouTube.*

Researcher: *Okay.*

Student P79: *YouTube.*

Researcher: *Yes but, you see, here you said sin 13 pi over 6 equals to sine 3 pi over 6, Right?*

Student P79: *Right.*

Researcher: *And then + 4 pi over 6 + 6 pi over 6. So, and then you said this equal to sine pi over 4 pi, pi over 3 + pi, all in closed brackets and then the next step the sine disappeared and then you said this equal to square root of 2 over 2 + square root of 3 over 2 + 0. Which rules are you using here from these steps? Which rules did you apply here?*

Student P79: *Yoh, I don't recall it since it's been a long time, but it was a very long time. I thought maybe it's something that is familiar to you since you have so much more mathematics than me. Because they called it; they used a very big term that I had to watch repeatedly to understand what they'd done. They used almost the same method.*

Researcher: *By using 3 terms in the brackets? Did they use 3 terms or 2?*

Student P79: *I think they had 2 or so. There was another problem that had 3 but the third one was like this one of 6 over 6, which is almost one. I will check after the meeting then I will send it to you so that you will know.*

Student P79: *Oh, so now the rules are there but I applied it wrongly, right?*

Student P79: *No, that's what I'm trying to understand. That if, that if I applied something that is not in mathematics terms, or do you know something that is close to what I've applied but used it wrongly?*

He asked the researcher to indicate where he went wrong or if what he wrote had mathematical significance.

The Figures 7.22 and 7.23 show that the students did know how to find a cofunction using degrees or radians. They incorrectly assigned 360˚ or $\frac{13\pi}{6}$ to the fourth quadrant. Consequently, they all found the cofunction of each case to be negative. P34 correctly simplified radians to degrees before using the cofunction rule. The student then used the incorrect cofunction identity $\sin(390°) = \sin(360° - 30°) = -\sin(30°)$. This identity represents an angle of a fourth quadrant. The correct identity should have been $\sin(390°) = \sin(360° + 30°) = \sin(30°)$ since the angle 390˚ lies in the first quadrant. The same error was made by P100 while working with radians. Three students provided an answer from Figure 6.22 and one student provided the answer in Figure 7.23.

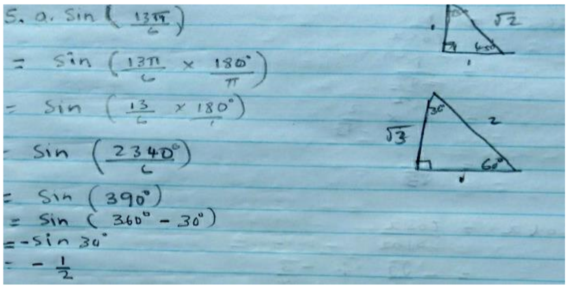


Figure 7. 22: Incorrect answer provided by P34

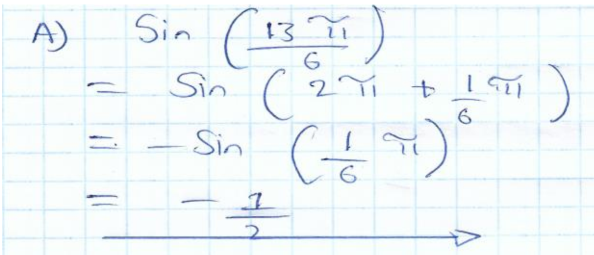


Figure 7. 23: Incorrect answer provided by P100

7.2.2 *Simplification of the expression a sin (b), where 0° < b > 180° and a is any real number*

The analysis of all given responses to the question shows that the prospective teachers had knowledge of how to evaluate a trigonometric function with a degree value. Considering this information, we can confirm that the prospective teachers were successful, especially in answering procedural knowledge questions that required basic trigonometric knowledge. Hence, 118 participants provided correct answers to the question, as seen in Figures 7.24 and 7.25; six participants did not answer the question, and one participant gave a wrong answer to the question, as seen in Figure 7.26.

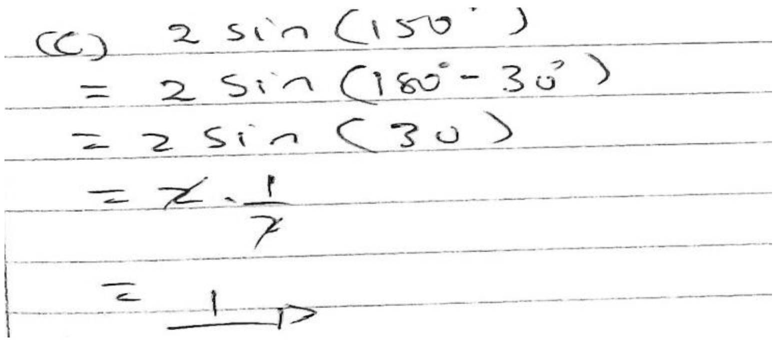


Figure 7. 24: Correct answer provided by P1

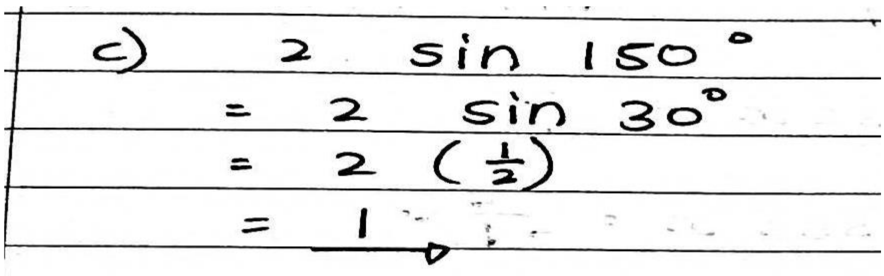


Figure 7. 25: Correct answer provided by P3

P17 had a misconception about the multiplication rule of a variable to a trigonometric function. The participant stated that $2\sin(150°) = \sin(2 \times 150°)$, she seemed to believe that $2\sin(x) = \sin(2x)$. However, $2\sin(150°)$ means first you take the sine of 150˚, then you multiply the result by 2. In the other and $\sin(2 \times 150°)$ means first you multiply 150˚ by 2, then you take the sine of the result. Additionally, $\sin(2x) = \sin(x)\cos(x)$. Where x is a reference angle in a right-angled triangle.

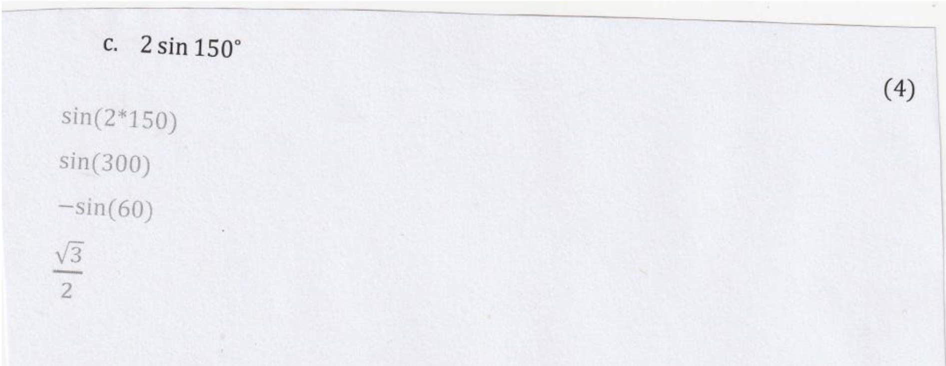


Figure 7. 26: Incorrect answer provided by P17

7.3 Simplification of trigonometric expression

The aim of the task was to examine the PSTs' knowledge of simplifying trigonometric expressions by using basic identities or the factorization method. The task was to simplify the expressions $\tan(x)\csc(x)$ and $\frac{(\cos^2(x))-4}{(\cos(x))-2}$ for $(\cos(x)) - 2 \neq 0$.

7.3.1 Using basic identities to simplify a trigonometric expression

The participants were required to simplify the expression $\tan(x)\csc(x)$. Ninety-one participants correctly initiated the process of defining $\tan x$ in terms of $\sin(x)$ *and* $\cos(x)$, and $\csc(x)$ as $\frac{1}{\sin(x)}$. These simplifications led to the correct answer, as shown in Figure 7.27. However, among these correct answers, 53 participants provided an answer from Figure 7.28. These participants did not arrive at the final simplification. Two participants did not provide any response, and 22 participants provided an incorrect answer.

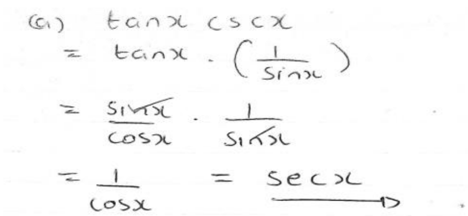


Figure 7. 27: Correct answer provided by P1

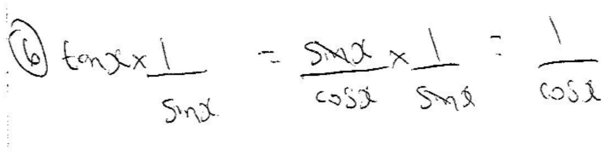


Figure 7. 28: Correct answer provided by P67

Twelve students provided an answer shown in Figures 7.29. The student's response in Figure 7.29 shows a partially correct but incomplete understanding of trigonometric simplification. The student begins with the expression $\tan(x) . \csc(x)$, and correctly substitute $\csc(x) = \frac{1}{\sin(x)}$. However, the student leaves $\tan(x)$ unchanged in the multiplication, resulting in the expression $\frac{\tan(x)}{\sin(x)}$, which they present as the final answer.

This reveals a missed opportunity to simplify the expression further. Had the student recalled and applied the identity $\tan(x) = \frac{\sin(x)}{\cos(x)}$, they would have been able to continue simplifying:

$$\frac{\tan(x)}{\sin(x)} = \frac{\frac{\sin(x)}{\cos(x)}}{\sin(x)} = \frac{1}{\cos(x)} = \sec(x)$$

Instead, they stopped midway, indicating that while they understood the reciprocal identity for cosecant, they did not complete the simplification by substituting the identity for tangent. This incomplete response suggests that the student has some foundational knowledge of trigonometric identities but may lack the fluency or confidence to proceed beyond the initial step. It could also point to a compartmentalized understanding of identities, knowing them in isolation but struggling to apply them in combination to achieve full simplification. To improve, the student should be encouraged to always attempt to express all trigonometric functions in terms of sine and cosine and simplify systematically until reaching the most reduced form.

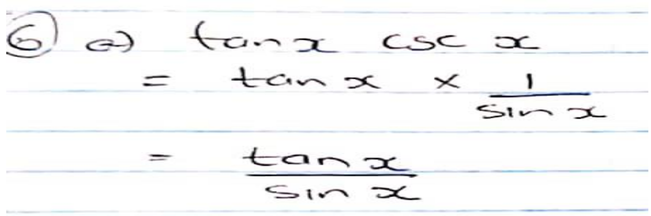


Figure 7. 29: An incorrect answer provided by P51

The answer in Figure 7.30 was provided by ten students. In Figure 7.30, the students' response demonstrates a correct initial approach but ends with a critical simplification error. They correctly rewrite $\tan(x)$ as $\frac{\sin(x)}{\cos(x)}$ and $\csc(x)$ as $\frac{1}{\sin(x)}$, leading to:

$$\tan(x) \times \csc(x) = \frac{\sin(x)}{\cos(x)} \times \frac{1}{\sin(x)} = \frac{\sin(x)}{\cos(x)\sin(x)}$$

So far, the steps are accurate and reflect a solid understanding of fundamental trigonometric identities and multiplication of fractions. However, in the final step, the students incorrectly simplify $\frac{\sin(x)}{\cos(x)\sin(x)}$ to $\cos(x)$, which is mathematically incorrect. In fact, the correct simplification would involve cancelling out $\sin(x)$ from the numerator and denominator: $\frac{\sin(x)}{\cos(x)\sin(x)} = \frac{1}{\cos(x)} = \sec(x)$.

The students seem to have mistakenly inverted the expression or misapplied the cancellation rule, leading to the final incorrect answer of $\cos(x)$ instead of $\frac{1}{\cos(x)} = \sec(x)$. This kind of error suggests a procedural misunderstanding in fraction simplification, particularly in dealing with trigonometric functions.

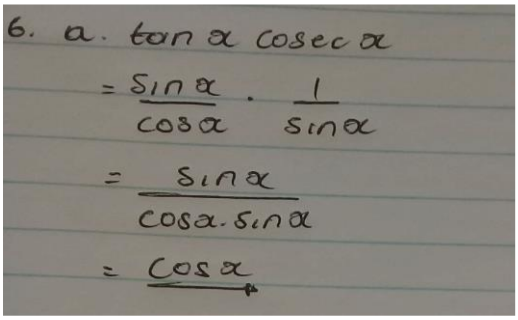


Figure 7. 30: An incorrect answer provided by P55

7.3.2 Using algebraic factorisation to simplify trigonometric expression

To simplify $\frac{(\cos^2(x))-4}{(\cos(x))-2}$ for $(\cos(x)) - 2 \neq 0$, the participants were required to use their knowledge of relationship between a quadratic expression and a trigonometric expression. This coordination between the two registers (algebraic and trigonometric) can complicate a problem complex because one is forced to navigate from a trigonometric expression to an underlying algebraic structure. The close coordination of the two representations is essential to solving this problem. The participants were required to first recognize that the expression $(\cos^2(x)) - 4$ could be represented by an underlying quadratic expression $a^2 - b^2$ and then apply the algebraic factorization to simplify it into the expression $(\cos(x) - 2)(\cos(x) +$

2). Secondly, they would then recognize the common factors across the numerator and denominator and hence simplify the ratio $\frac{(\cos(x)-2)(\cos(x)+2)}{(\cos(x)-2)}$ into the expression $(\cos(x)+2)$. Eighty-seven participants provided a correct answer, as shown in Figure 7.31. Six participants did not answer the question, and 32 provided an incorrect answer.

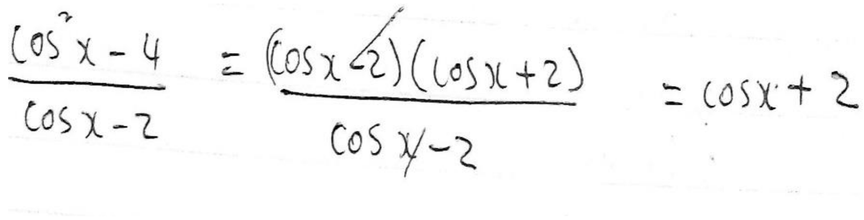


Figure 7. 31: Correct answer provided by P1

Eight participants correctly initiated the process of simplifying the numerator of the given expression. However, they held a misconception related to the notation of a trigonometric function with compound angles, where they interpreted $\cos(x)-2$ *as* $\cos(x-2)$, and similarly interpreted $\cos(x)+2$ *as* $\cos(x+2)$. Figure 7.32 shows an incorrect answer provided by these participants.

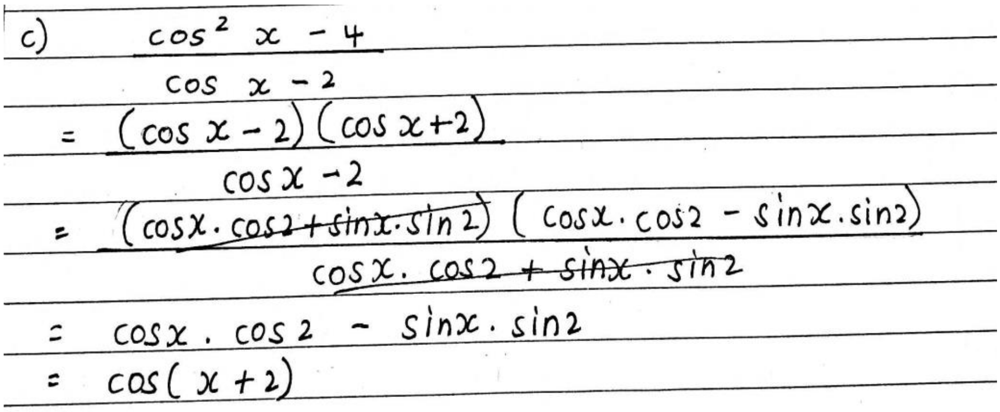


Figure 7. 32: An incorrect answer provided by P2

In Figure 7.33, 13 participants had a misconception about the quadratic expression a^2-b^2 and $(a-b)^2$. They took $a^2-b^2 as$ $(a-b)(a-b)$ instead of $a^2-b^2=(a-b)(a+b)$, This misconception led to the incorrect simplification of the given expression.

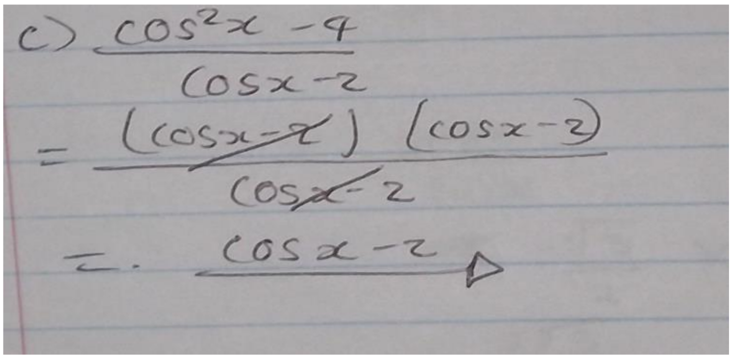


Figure 7. 33: An incorrect answer provided by P14

Five participants provided an answer from Figure 7.34, where they replaced $\cos^2(x)$ with $1 - \sin^2(x)$ which is a correct replacement; however, it did not lead them any further in the simplification.

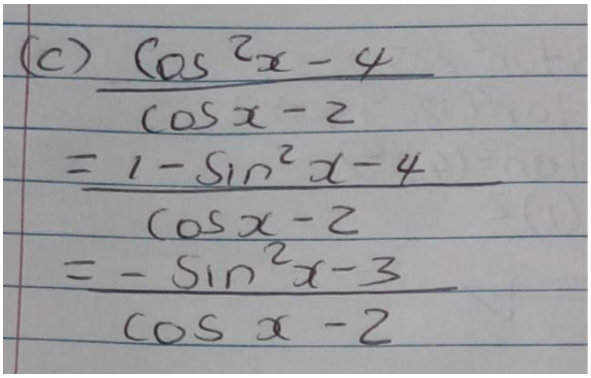


Figure 7. 34: An incorrect answer provided by P48

Three students expressed $\cos^2(x)$ as $\frac{\cos^2(x)}{2} + \frac{1}{2}$, which is not a correct identity, and did not pursue it further as shown in Figure 7.35.

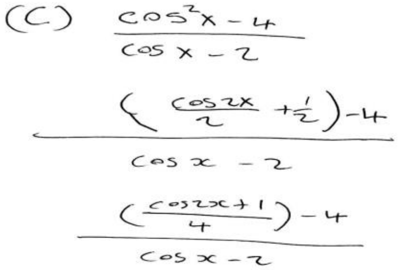


Figure 7. 35: An incorrect answer provided by P31

Three students made an incorrect substitution of $\cos^2(x) - 4$ as $(2\cos^2(x) - 1) - 4$ and then simplified it to $4\cos^2(x) - 1$ as shown in Figure 7.36. They incorrectly attempt to rewrite the numerator $\cos^2(x) - 4$ as a step-by-step transformation: first as $\cos^2(x) - 2$, then inexplicably as $(2\cos^2(x) - 1) - 2$, and finally as $4\cos^2(x) - 2$.

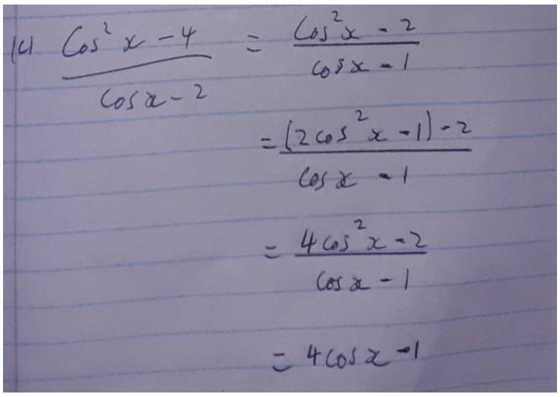


Figure 7. 36: An incorrect answer provided by P87

These manipulations have no algebraic or trigonometric justification and appear to be arbitrarily formed rather than based on identities or structured factoring.

The final expression, $4\cos^2(x) - 1$, is not logically connected to the original expression and is entirely incorrect. The students likely confused different identities (such as the double-angle formula for cosine or the identity $\cos(2x) = 2\cos^2(x - 1)$ but applied them out of context. This indicates a lack of understanding of how to factor quadratic expressions and no clear grasp of the identities being used.

CHAPTER 8
DISCUSSION, CONCLUSION AND RECOMMENDATIONS

The detailed analysis and close examination of the results presented in Chapters 6 and 7 are used in this chapter to address the two research questions:

- *What errors do preservice mathematics teachers commonly make when working on tasks involving trigonometric functions?*
- *How can Duval theory be used to explain these errors?*

In the discussion that follows, the terms "participants," "students," and "PSTs" will be used interchangeably to refer to the individuals involved in the study. The chapter concludes with a summary of key findings, followed by recommendations based on the insights gained.

8.1 Common errors made by PSTs when dealing with trigonometric functions

In trying to answer the first research question related to the PSTs' errors, the discussion of the errors is organized into five main areas: the distinction between a function that is trigonometric and one that is not; the calculation of the period and amplitude of given sine and cosine functions; the interpretation of graphical representations of the trigonometric functions; the formulation of the trigonometric functions when given the parameter values; and the manipulation of trigonometric expressions. These are discussed below:

8.1.1 Errors encountered in distinguishing between a trigonometric and non-trigonometric function

Chapter 6 examined the PSTs ' responses in distinguishing between a graph representing a trigonometric function and a non-trigonometric function. A trigonometric function is a function that connects the angles of a right-angled triangle with the ratios of the lengths of its sides, while a non-trigonometric function does not involve angles or triangles. Examples of non-trigonometric functions include linear ($f(x) = ax + b$, where b is a constant), quadratic ($g(x) = ax^2 + bx + c$, where a, b, and c are constants), and exponential ($h(x) = a^x$, where a is a constant) functions.

Out of the 107 participants, 9% of the responses were left blank, while 10% of the students who responded were unable to explain how the graph of a trigonometric function differs from that of a non-trigonometric function. Some participants chose to sketch only a trigonometric graph, while others sketched graphs of non-trigonometric functions exclusively. This variation in response suggests that several students may have struggled to distinguish between the two types of functions or may not have fully understood the task. A common misconception appeared to be the assumption that all functions share similar graphical properties and behaviours, regardless of their type. This indicates a gap in conceptual understanding that may have originated from earlier stages of learning. Sarwadi and Shahrill (2014) argued that such errors and misconceptions in mathematics often emerge during the foundational phases of learning, and if left unaddressed, they can hinder students' progress and accuracy in more advanced mathematical tasks, such as graphing functions, interpreting inequalities, or solving problems involving trigonometric relationships.

Distinguishing between trigonometric and non-trigonometric functions is critical because each type exhibits unique characteristics and serves distinct purposes in mathematical modelling and real-world applications. Trigonometric functions, such as sine, cosine, and tangent, are defined by their periodic nature, oscillating between specific values and repeating at regular intervals. These functions are central to modelling phenomena that involve cyclical behaviour, such as sound and light waves, tidal movements, and the mechanics of rotating objects (Stein et al., 2019). In disciplines such as physics and engineering, trigonometric functions are used extensively to simplify complex systems, describe harmonic motion, and analyse alternating current circuits, among many other applications (Nguyen & Kulm, 2020). Their periodicity makes them essential tools in any context involving repetitive or angular processes.

In contrast, non-trigonometric functions include linear, quadratic, exponential, and logarithmic functions, among others, which behave differently from their trigonometric counterparts. These functions model a wide range of real-world contexts, such as population growth (using exponential functions), economic forecasting (using polynomial models), or data analysis in statistics (using linear or logistic models) (Kastberg et al., 2022). For example, in biology, exponential and logistic functions are frequently used to describe population dynamics or the spread of diseases, while in economics, polynomial and exponential models are used to represent financial growth and decline trends.

Recognizing the differences between these classes of functions allows students to choose appropriate mathematical tools and strategies when solving problems. This ability enhances their mathematical reasoning and ensures that their approaches to problem-solving are both relevant and accurate. The conceptual distinction between trigonometric and non-trigonometric functions is foundational not only for success in mathematics but also for the application of mathematical thinking across disciplines. As such, addressing misconceptions and reinforcing the unique features of each function type is vital for developing students' overall mathematical competence and confidence (Özgür & Eren, 2023).

8.1.2 Errors in finding the amplitude and period when given equations of the form

$f(x) = A\sin(Bx - C) + D$ (Equation 1), or

$g(x) = A\cos(Bx - C) + D$ (Equation 2).

When working with the graphs of $f(x) = 2\sin\left(\frac{\pi x}{4} + 3\right) + 4$ and $g(x) = -6\cos\left(\frac{x}{3} - \frac{\pi}{4}\right) + 1$ within the interval $[0\,;2\pi]$, the errors related to the calculation of the amplitude and period are discussed in further detail below:

8.1.2.1 Errors related to the amplitude

Thirty participants correctly identified the amplitude values from the given symbolic representations of two trigonometric functions.

During follow-up interviews, students S13 and S37 indicated that they believed the amplitude determines the points at which the graph of a trigonometric function turns. This suggests that they perceive amplitude as something that can only be identified after sketching the graph. Such a misconception highlights a gap in their conceptual understanding of trigonometric functions and should be addressed to strengthen their foundational knowledge.

The amplitude of a trigonometric function is half the vertical distance between its highest and lowest points. Thirty students determined the amplitude of the graph $f(x) = 2\,sin\left(\frac{\pi x}{4} + 3\right) + 4$ accurately and found that the amplitude of the graph $g(x) = -6\,cos\left(\frac{x}{3} - \frac{\pi}{4}\right) + 1$ is -6. Some students mistakenly assumed that the coefficient A within the function (Equations 1 or 2) represents the amplitude itself. However, note that coefficient A does not necessarily equate to the actual amplitude, e.g., if A is a negative number. Students ignored the negative sign in front of the trigonometric function. These students have overlooked or dismissed this sign and thus erroneously concluded that the amplitude is equal to the negative value of A, rather than correctly understanding it as an indication of reflection across the $x - axis$.

Five students calculated the amplitude of both graphs of $f(x)$ and $g(x)$ by adding the value A and D. This error arises from confusion between the vertical shift value and the amplitude value which are both related to changes in the dependent value of the function. The students took D as contributing to the calculation of the amplitude instead of recognizing it as denoting the vertical shift of the graph and used the method of finding the range of a trigonometric function to determine the amplitude. However, the amplitude of a trigonometric function is the maximum distance a function moves from its mean or equilibrium position. For a standard sine or cosine function, the amplitude is calculated from the coefficient of the sine or cosine term. Mathematically, for a function of the type given in Equations 1 or 2, the amplitude is |A|. It measures how "tall" the waveform is from the centreline (mean position) to its peak or trough. The vertical shift (D) represents how much the entire wave is shifted upward or downward along the vertical axis. In the Equations 1 and 2, the parameter D represents the number of units these respective graphs have shifted vertically when compared to the respective graphs $y = A\cos(Bx - C)$ and $y = A\sin(Bx - C)$.

The amplitude of a trigonometric function is independent of the vertical shift; it is solely determined by the coefficient A. The vertical shift D adjusts the baseline around which the function oscillates but does not affect the height from this baseline to the function's peaks. To illustrate, consider the trigonometric function $f(x) = 3\sin(x) + 2$ as shown in Figure 8.1. Here, the amplitude is |3|, which means the function oscillates 3 units above and below its mean position, not above and below D. The vertical shift is 2, shifting the entire sine wave up by 2 units. Therefore, the peaks and troughs of the function are 3 units away from the central position, regardless of how high or low that central position is due to D. If we applied the students' method, we would incorrectly add A and D to get the amplitude (3 + 2 = 5), which does not reflect the actual behaviour of the sine wave. The proper understanding shows that the amplitude remains 3, and the vertical shift simply modifies the central line of oscillation to $y = 2$.

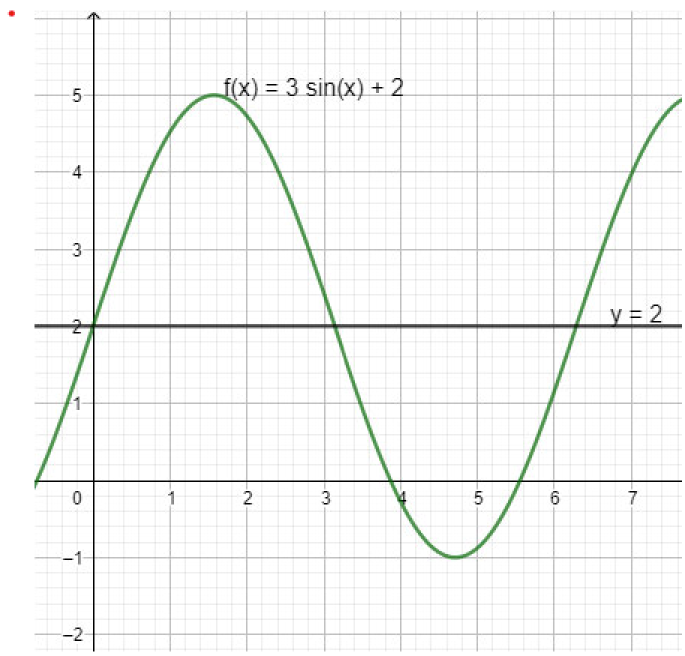


Figure 8. 1: A graph of f(x) = 3 sin(x) + 2 and y = 2

Three students stated that the amplitudes of both functions were 6 which is the amplitude of $g(x) = -6\,cos\left(\frac{x}{3} - \frac{\pi}{4}\right) + 1$ only. Two students concluded that the amplitude of both functions can be represented by the interval [-2, 2] for $f(x) = 2\,sin\left(\frac{\pi x}{4} + 3\right) + 4$ and [-4, 4] for $g(x) = -6\,cos\left(\frac{x}{3} - \frac{\pi}{4}\right) + 1$. It seems that students picked out numbers from the functions and expressed them as intervals. The interval mentioned represents the range of values the function takes and includes the midline central value plus and minus the amplitude. The range describes the set of all possible output values (y-values) a function can take. For the basic sine and cosine functions, the range is always between (-1) and 1. This range can be modified by vertical shifts and stretching or compressing the graph. For example, if the function is $f(x) = 3\sin(x) + 2$, the amplitude is 3, but the range of the function is from $3 \times (-1) + 2$ to $1 \times 3 + 2$, which simplifies to [-1, 7]. Therefore, the range of this function is [-1, 7]. The interval made up of y-values cannot be related to the value of the amplitude, which is a distance expressed as a number.

8.1.2.2 Errors related to the period of a trigonometric function

The period of a trigonometric function refers to the interval over which the function completes one full cycle before repeating itself. In other words, it is the length of one complete oscillation or wave of the function in radians or degrees.

There were 52 students who incorrectly identified the periods. Among these, seven individuals made errors related to working with radians in the formula $Period = \frac{2\pi}{|B|}$. The transition from degrees to radians presents a challenge, particularly when exposure to radians is limited until advanced stages of learning. In South Africa, for instance, the DoBE curriculum (2011) traditionally emphasizes the use of degrees when teaching trigonometric functions at the school level. Consequently, students often first encounter radians at the university level, which can lead to potential confusion. The abrupt introduction of radians at the university level can feel jarring, as students must unlearn their reliance on degrees and adapt to a new, albeit more efficient, system. This transition may be smoother if the curriculum incorporates radians earlier in the educational process. Introducing radians alongside degrees from the outset can help students develop a dual understanding of angle measurement, thereby facilitating a smoother transition to more advanced mathematical concepts in later learning stages.

Understanding that 2π radians represent a full circle is fundamental, but applying this in various mathematical contexts requires a deeper comprehension. For example, students struggled to recognize how the value of B in the formula affects the period of a trigonometric function. If B is greater than 1, the period becomes shorter, and if B is between 0 and 1, the period lengthens. These implications of B's value must be internalized, which can be difficult without a solid grounding in the unit circle and radian measures.

Furthermore, students also had difficulties with algebraic manipulation involving π. Students found it cumbersome to work with π in calculations, confusing it for a parameter instead of a number, and the manipulations were more complicated when B was a fraction or a negative number.

Thirty-three students believed that the period for trigonometric functions such as sine and cosine always equals 360 degrees. This misunderstanding was also evident during interviews, where Student S13 confidently expressed that the period of these functions typically represents a full 360-degree rotation. Student S37 provided a more nuanced yet still incorrect response, stating that the period signifies the repetition of a distinct pattern when visualized on a graph. From these responses, these students demonstrate that they possess a prototype image of a sine and cosine function. The periods for $y = \sin(x)\ or\ y = \cos(x)$ functions are 2π radians, which corresponds to 360 degrees, but this can change when the parameters involved in the functions take different values. The key is recognizing that these periods can be modified by certain factors within the function, such as frequency adjustments, which the students failed to consider. For instance, the period of the function $f(x) = 2\sin\left(\frac{\pi x}{4} + 3\right) + 4$ is equal to 8 and not 360 degrees. The period of the function $g(x) = -6\cos\left(\frac{x}{3} - \frac{\pi}{4}\right) + 1$ is equal to 6π and not 2π.

Additionally, misunderstanding was also present regarding the parameter "B" within the symbolic representation of trigonometric functions. Two students took the value of this parameter "B" as the period itself instead of using the formula $\frac{2\pi}{|B|}$ to calculate it.

Lastly, five students could not provide the correct value of the period of both functions, suggesting that they did not know how to identify it.

Overall, in our sample size of 107 students, a considerable portion struggled with various aspects related to determining the periods for both functions. Their errors stemmed from difficulties involving radian manipulation, misconceptions surrounding universal period values, misinterpretation of key parameters within symbolic representations, and even a complete lack of awareness regarding which components directly influenced periodicity.

8.1.3 Errors related to the graphical representation of a trigonometric function

Understanding and teaching transformations of functions have been central to mathematics education, although research has often neglected horizontal transformations in favour of other types. Typically, transformations are introduced by examining the graph of a function $f(x)$ on a Cartesian plane. Dilation, reflection, vertical translation, and horizontal translation are represented by $f(ax), f(-x), f(x) + k, and\ f(x + k)$, respectively. Traditional instruction starts with parabolas and extends to other quadratic relations and functions (Zazkis et.al., 2003).

In the given task, participants were asked to sketch the graphs of $q(x) = \cos x$ and $h(x) = 2\cos\left(\frac{x}{2} - 30°\right) + 1$ on the same set of axes within the interval $[0°; 360°]$. Only 43 students correctly represented both functions accurately on the same set of axes, while 54 students provided incorrect representations of the functions. These incorrect responses contained errors ranging from the periodicity of the functions to their amplitudes, their horizontal shifts, and vertical shifts.

8.1.3.1 Incorrect representation of the vertical shift of each graph

The standard cosine function, $q(x) = \cos x$, fluctuates between (-1) and 1, but the coefficient of 2 in $h(x) = 2\cos\left(\frac{x}{2} - 30°\right) + 1$ alters this range, causing the function $h(x)$ to fluctuate between (-1) and 3. Twenty-eight students correctly represented the fluctuation of the graph of the function q(x) but incorrectly represented the fluctuation of the graph of $h(x)$ which they depicted as being between (-2) and 2. Two students represented the wrong fluctuation for both function $q(x)$ and $h(x)$. They represented the graph of h to fluctuate between (-2) and 2, and the graph of the function f to fluctuate between (-1) and. This indicates a misunderstanding of the changes introduced by the coefficient of 2 and the alterations within the function. Two students made errors in both functions, misrepresenting the fluctuation of $h(x)$ as being between (-2) and 2 and misrepresenting $q(x)$ as fluctuating between (-1) and $\sqrt{3}$. This indicates an overall confusion about the vertical shift.

These two groups of students held a misconception that the amplitude in the function $h(x)$ is the only parameter that affects the oscillation of its graph. The students did not first identify the midline of the oscillation which is at $y = 1$ this is determined by the constant term D in the function $A\cos(Bx - C) + D$ which shifts the entire graph vertically by one unit. This means the students do not understand how the amplitude and vertical shift affect the range of the function.

As for the two students who represented the oscillation of the graph of $q(x)$ as being between (-1) and $\sqrt{3}$, they seem to be confused about the cosine function, which has a well-defined range that oscillates between (-1) and 1. Similarly, they made mistakes by conflating the concepts of amplitude and vertical shift.

These discrepancies suggest a need for a deeper comprehension of trigonometric transformations, specifically how vertical shifts, along with amplitude modifications, affect the graph's fluctuations. These errors highlight a common difficulty in applying theoretical concepts of function transformation to practical graphing tasks.

Based on the research conducted by Clement (1985), it has been identified that there are two distinct categories of misconceptions when it comes to graphing functions. The first type involves individuals treating the graph solely as a picture, neglecting to understand its underlying mathematical properties and relationships. This misconception hampers their ability to grasp the true essence of the function being represented. The second category, known as slope-height confusion, pertains to individuals mistakenly equating the steepness or incline of a graph with its overall height. In a similar, manner in this context, some students mistakenly equate the vertical shift with the amplitude values and are unable to distinguish between the effects of these two parameters on the graph. Such misunderstanding can lead to erroneous interpretations and misrepresentations of the data portrayed in the graph.

8.1.3.2 Incorrect representation of the horizontal shift of each graph

In mathematics, particularly in the study of trigonometric functions, a "phase shift" refers to a horizontal displacement of the graph of a function. Hence, when we consider the mathematical use of the following sinusoidal formulas: $y = A\sin\big(B(x - C)\big) + D$, the horizontal shift or phase shift equals C and for $y = A\sin(Bx - C) + D$, the horizontal shift or phase shift equals $\frac{C}{B}$.

The basic function $y = \cos(x)$ is a standard cosine wave with a period of 360°, an amplitude of 1, and no vertical or horizontal shift as represented in Figure 8.2. Within the interval from 0° to 360°, it completes exactly one cycle, starting from $(x = 0°; y = \cos(0°) = 1)$, to $(x = 90°; y = \cos(90°) = 0)$, $(x = 180°; y = \cos(180°) = -1)$, and back to $(x = 270°; y = \cos(270°) = 0)$, $(x = 360°; y = \cos(360°) = 1)$. These are the four critical points students were requested to represent to plot the correct graph of the function q.

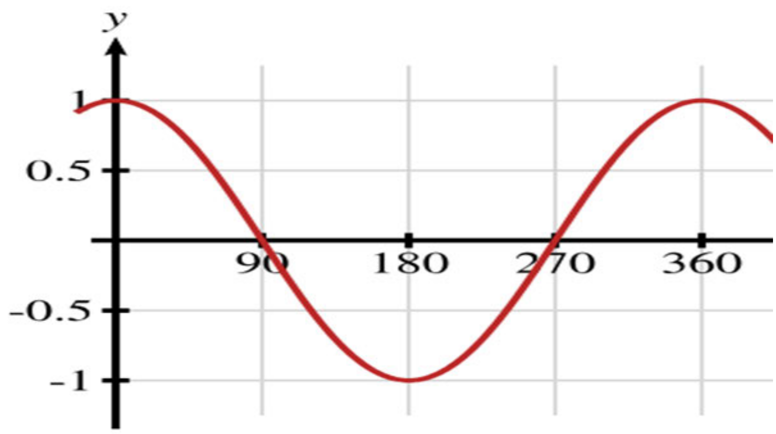


Figure 8. 2: A graph views of y = cos(x) on an interval [0°; 360°]

In contrast, the function $h(x) = 2\cos\left(\frac{x}{2} - 30°\right) + 1$ involves several more complex transformations: a change in amplitude, a horizontal shift, a compression, and a vertical shift.

Given that factor $B = \frac{1}{2}$, the period of $h(x)$ is 360° times 2, which is equal to 720°. However, within the interval [0°; 360°], we can only see half of a complete cycle of the cosine wave. This cycle is completed from ($x = 0°; y = \sqrt{3} + 1$), to the maximum point ($x = 60°; y = 3$), then decreasing to the intersection point ($x = 300°; y = 0$), and finally reaching the endpoint ($x = 360°; y = 1 - \sqrt{3}$). Therefore, the function $h(x)$ presents a horizontal shift of 60° to the right. Within the interval of 0° to 360° interval, this shift is consistently by 60° to the right, altering the coordinate values of key points but remaining within one cycle of the original period due to the constraint of the interval.

There were 50 participants, who made various errors in representing the horizontal shifts. Two students represented the complete cycle of $q(x)$ from ($x = 30°; y = \frac{\sqrt{3}}{2}$), to ($x = 90°; y = \cos(90°) = 0$), ($x = 180°; y = \cos(180°) = -1$), and back to ($x = 290°; y = \cos(290°) = 0$), concluding at ($x = 360°; y = \cos(360°) = \sqrt{3}$). The errors stem from a misunderstanding of unit circle properties, angle measurement, or a more fundamental grasp of trigonometric functions. For example, incorrectly stating the starting point of the graph as ($x = 30°; y = \frac{\sqrt{3}}{2}$) indicates a misunderstanding that the domain values have shifted, whereas it is the function values that shift. Placing the maximum point at ($x = 30°; y = \frac{\sqrt{3}}{2}$) demonstrates a misconception about where and what the maximum value of the cosine function is. The given intersection points are (90°; 0) and (290°; 0). While 90° is correct as an intersection point at $y = 0$, the next correct intersection should be at 270° (not 290°), and having an endpoint at ($x = 360°; y = \sqrt{3}$) suggests that students might be confused about the periodic nature and symmetrical properties or the values of cosine at specific standard angles. This indicates a fundamental error in understanding the periodic property of the cosine function and a deeper misunderstanding of the domain of the cosine function. It seems that the students have difficulty correlating the (a; $\cos(a)$) original coordinates with the transformed function values (a; $2\cos\left(\frac{a}{2} - 30°\right) + 1$). The x-coordinate remains the same, but students appear to want to shift the x-coordinate as well.

Thirty students represented an incorrect graph of $h(x)$. There were 28 students who plotted the graph from the point (0°; $\sqrt{3}$ + 1) to the maximum point (60°; 2), then decreased to (240°; 0) and ended at (360°; $1 - \sqrt{3}$). The other two students plotted the graph from the point (60°;

2) decreasing to (240°; 0) and ending at (360°; $-\sqrt{3}$). These students plotted the maximum point incorrectly; for instance, $h(60°) = 3$. They missed the horizontal shift of 30°, which led some to start their function improperly, directly influencing the entire graph. Both Eisenberg and Dreyfus (1994) and Baker et al. (2000) address the challenges associated with horizontal transformation, but their explanations for the source of this difficulty diverge. Eisenberg and Dreyfus focus on the visual processing aspect, suggesting that the difficulty lies in how our visual system interprets and manages spatial information. They argue that horizontal transformations require a particular kind of visual coordination that can be cognitively demanding. On the other hand, Baker et al. (2000) approach the issue from a cognitive construction standpoint. They believe that the complexity arises from the mental effort needed to comprehend and perform a horizontal transformation. According to their perspective, the challenge is not solely visual but also deeply cognitive, involving intricate mental constructs that must be navigated and manipulated accurately. In this case, we see that students confused the starting point with the value of the horizontal shift. The parameters A, B, and D are also added to their confusion. However, Zazkis et al. (2003) argue that a potential source of difficulty in learning function transformations lies within the traditional instructional sequence used in teaching these concepts. Typically, the curriculum focuses heavily on algebraic representation, dictated by a conventional approach to teaching transformations. This method embeds discussions on transformations of specific functions, such as parabolas, quadratic relations, or polynomial functions, within the broad context of functions in general. Consequently, this can inadvertently reinforce the challenges students face rather than mitigate them. The researchers believe that these difficulties can be minimized by adopting a pedagogical approach that situates function transformations within the context of transformations themselves. By doing so, the focus shifts from merely understanding algebraic representations to comprehending the broader and more intuitive concept of transformations. This approach encourages a deeper understanding and potentially makes the learning process more intuitive for students (Zazkis et al., 2003).

8.1.3.3 Incorrect starting point

The students' misconceptions about the starting point of the graph of the function $h(x) = 2\cos\left(\frac{x}{2} - 30°\right) + 1$ stem from a misunderstanding that the domain of the function is not changed by the transformation shifts. To understand the starting point of $h(x)$, it's important to analyse the transformations applied to the basic cosine function $\cos(x)$. The term

$\left(\frac{x}{2} - 30°\right)$ in $h(x)$ indicates a phase shift. The expression $\frac{x}{2}$ indicates a stretch of the argument by a factor of $\frac{1}{2}$. The 30° translates the graph horizontally to the right by 30°. The coefficient 2 in front of the cosine function stretches the graph by a factor of 2. The +1 at the end translates the entire graph upward by 1 unit. The term "starting point", as indicated by the domain stipulated in the instruction, commonly refers to a notable point on the graph, such as where x begins mapping out the periodic nature of the trigonometric function. One key point to check is when the cosine function equals 1, since $\cos(0) = 1$:

$$\frac{x}{2} - 30° = 0$$

$$\frac{x}{2} = 30°$$

$$x = 60°$$

Therefore, at $x = 60°$, the point (60°, 3) identifies the phase shift of the function $h(x)$ rather than the starting point of the graph over the interval [0°, 360°]. In this context, the point (60°,3) indicates the translation of the point (0, 1). Hence, the function has been delayed or shifted by 60° on the horizontal axis while maintaining its vertical displacement at this point. This phase shift helps in understanding where the cycles of the wave-like function begin relative to the standard position. Therefore, despite being in the designated interval, the actual visualization of $h(x)$ still depends on the amplitude, period, and vertical shifts specified in its general equation.

Twenty students represented the graph of $h(x)$ with a starting point at $x = 30°$, while two students represented the graph of $h(x)$ starting at $x = 60°$ which is the phase shift of the function. For the function $h(x)$, the phase shift point is indeed 60°, and students seem to believe that the graph starts shifting at this point. However, this does not equate to the function's starting point when considering the graph over an interval, which, in this case, is [0°, 360°]. When questioned about this error, student S85 reasoned that since the cosine function yields positive values and has undergone a rightward shift of 30°, it logically follows that the graph of the function should start at $x = 30°$. Similarly, for the two students who placed the starting point at $x = 60°$, it is evident that they understood the concept of phase shift.

However, their plotting of the function over the designated interval was misguided. This mistake highlights a common issue where students grasp the individual components of a mathematical concept but struggle to integrate them correctly within a broader context. The key error lies in not acknowledging that the starting point is at (0°; 1+ $\sqrt{3}$) which requires that the graph of $h(x)$ to be considered from $x = 0°$ onwards, regardless of the phase shift. This is a misconception in understanding how trigonometric functions interact with their intervals and phase shifts. Accurate interpretation involves knowing that while phase shifts indicate horizontal displacements, they do not reset the plotting interval (domain) of the graph.

8.1.3.4 Incorrect representation of the domain of the graphs

Generally, the concept of a domain pertains to the set of all possible inputs that are defined for a given function or relation. Essentially, the range comprises the resultant values that the function can produce based on its defined inputs. (Rockswold, 2012).

In the given exercise, the domain specified for the trigonometric functions was [0°; 360°]. Among the 107 students, 28 students erroneously represented the endpoints of the graphs with an arrow, implying that the function continued beyond 360°, which contradicts the stated closed interval. This misunderstanding highlights a common misconception about the nature of trigonometric graphs. While these functions are indeed periodic and repeat indefinitely, they are bound by defined intervals when specified.

Placing an arrow at the end of a graph of a function carries significant implications and serves a few important purposes. Firstly, the arrow denotes that the function continues indefinitely in that direction. This is especially useful for functions such as linear functions, exponential functions, or any other type whose domain extends towards positive or negative infinity. The arrow effectively communicates to the reader that the function does not abruptly end at the plotted portion, but rather, it extends beyond the visible range. Moreover, the use of arrows helps convey asymptotic behaviour. For example, in cases of rational functions where the function approaches a certain value but never quite reaches it, the arrow can show the trend toward the asymptote.

Moreover, four students completely disregarded the specified interval and chose to draw their graphs over the range [-360°; 360°]. One student, S54, attempted to justify her answer by explaining that she used a calculator to plot the functions and determined their intervals based on their periods. Aziz and Kurniasih (2019) argue that students widely misunderstand Cartesian coordinates and corresponding limit sets, attributing this to a lack of clarity about the arrows at the ends of curved lines. The researchers found that students particularly had trouble identifying the domain and range of sine functions, whether symbolically or graphically. Beyond understanding sine functions, students also faced challenges in translating information from graphs to different formats, such as two-set arrow diagrams. This issue is exacerbated by their inaccurate grasp of functions within Cartesian coordinates, hindering their ability to extract and translate necessary information correctly.

Abdullah (2010) also observed similar challenges with students using Cartesian graphs. He noted that an overemphasis on integer numbers might lead students to focus solely on integer coordinates, further contributing to their misconceptions. This fixation on integers overshadows the broader understanding required for effectively using Cartesian coordinates and interpreting various functions.

8.1.4 Challenges in formulating trigonometric equation from given parameter values

In one task, participants were asked to derive a symbolic expression for two trigonometric functions, namely sine and cosine. These expressions were expected to be derived based on the values provided for certain parameters associated with said functions, such as their respective amplitudes, periods, and any potential shifts. There were 107 individual responses, with 10 respondents who did not provide any answers.

Out of all the students who answered the question, 81 students incorrectly wrote the equation for the cosine function from the given problem. Similarly, 95 students were unable to derive the correct equation for the sine function in the same problem.

There were four notable errors in finding the values of A, B, C, and D in Equations 1 or 2, based on the provided parameters, such as an amplitude of 3, a period of π, a phase shift of $\frac{\pi}{4}$ to the left, and a vertical translation of 1 unit upward for the cosine function. The same exercise was given to determine the equation for the sine function when it has an amplitude of 3, a period of 5π, a phase shift of $\frac{3\pi}{4}$ to the right, and a vertical translation of 6 units downward.

8.1.4.1 Errors in determining the value of the horizontal shift

Usually, the horizontal phase shift for the function $y = A\cos(Bx - C) + D$ with respect to the function $y = A\cos(x) + D$, denoted by the symbol θ, is equal to the ratio of C divided by $B\left(\frac{C}{B}\right)$. Many students incorrectly assume that the phase shift in the above functions was C. Understanding horizontal transformations can be challenging for students because these transformations often behave counterintuitively. Note that the numerical value for θ will be positive if the function in question is shifted to the right on a coordinate plane, while conversely, it will be negative if said function undergoes a shift to the left instead. Research on errors in horizontal transformations of trigonometric functions is limited, with most studies focusing on learning methods and strategies for understanding these transformations. Educators sometimes emphasize recognizing patterns and memorizing them instead of relying solely on conceptual understanding, as this approach can be more effective in grasping the complexities of such transformations (Axler, 2013; Barton, 2003). Strategies such as visual aids, interactive tools, and exploratory activities have been recommended to aid students in comprehending these transformations more deeply. Such approaches aim to foster critical thinking, allowing students to move beyond rote memorization and truly understand the underlying principles that govern horizontal transformations in graphical representations (Borba and Confrey 1996; Hall and Giacin 2013). Kunene and Bansilal (2015) study found that learners experienced horizontal shifts to be more difficult than vertical shifts, when sketching graphs and producing equations for the new graphs. Despite being taught various methods, many prefer the table method for graphing, which limits their ability to grasp the broader properties of functions. Learners struggle to connect transformed functions to their original counterparts, treating them as unrelated. Studies show that horizontal transformations are difficult, with many learners misapplying the rules for horizontal shifts and confusing symbolic manipulations with physical shifts. Horizontal shifts are less understood than vertical ones, with only a small percentage of learners correctly answering related questions. This difficulty is attributed to a lack of understanding of how equations change when graphs are transformed (Eisenberg & Dreyfus 1994, Kunene & Bansilal, 2015; Zazkis et al., 2003). Zazkis et al. (2003) suggest that including teaching transformations by focusing on individual points before generalizing to new equations, will resolve the epistemological obstacles faced by students.

8.1.4.2 Errors in determining the value of the amplitude

There were 13 incorrect answers for the value of A in Equations 1 and 2. Many students believed that A is equal to B in Equations 1 and 2. However, another perspective was that A should be exactly twice the size of B. One common misconception that students have when finding the value of A in the standard form of a cosine or sine function is that A represents the period of the function. However, the period is determined by other factors, such as B (the coefficient of x), and can be calculated using $\frac{2\pi}{B}$. Studies show that students are generally able to determine the amplitude of trigonometric functions when the amplitude, represented by the value A, is a whole number (Masomeh, 2017). Tools like TrigReps have been found to aid students in understanding transformations of trigonometric functions, as they provide multiple representations, including algebraic, graphical, unit circle, and audio forms. This multifaceted approach helps students grasp the concept of amplitude more effectively and make insightful observations about function transformations (Bornstein,2020). However, difficulties arise when students and even in-service teachers encounter trigonometric functions where the amplitude is a fraction. For instance, determining the amplitude of a function like $p(x) = \frac{-2\sin x}{4}$ can be challenging. In the study by Mkhwanazi et al. (2023), 77 out of 204 in-service teachers scored zero on a question involving finding the amplitude of such a function, suggesting this is a common difficulty.

8.1.4.3 Errors in determining the value of B

There were 19 incorrect answers for the value of B. From the general form of the cosine or sine function, which is represented as $y = A\cos(Bx + C) + D$ or $y = A\,sin(Bx + C) + D$, students could not differentiate between the value of B in the general form and the period of the function. Some used the formula $B = \frac{2\pi}{Period}$. Moreover, there were certain individuals among these students who displayed a lack of understanding regarding the distinction between the commutative property and the distributive property when dealing with multiplication, particularly in relation to its application within a trigonometric function.

Masomeh (2017)'s study highlights common errors among students interpreting trigonometric graphs. Students were presented with graphs of cosine and sine functions and tasked with determining their equations. All six undergraduate participants successfully identified the periodicity of the function $f(x) = \sin(2x)$. However, four out of five struggled to understand the inverse relationship between the period and the value of B in the Equations 1 or 2. The study found that students could easily determine the coefficient B when it was a whole number but faced challenges when B was a fraction. This indicates that students have a limited understanding of the relationship between B and the period of the function, pointing to an educational need to better explain this relationship, especially in cases involving fractional values.

8.1.4.4 Errors in determining the value of the vertical shift

There were 23 incorrect answers regarding the vertical shift. Specifically, when considering the sine function, it was observed that 20 students mistakenly concluded that the value of D equated to a negative one. On the other hand, concerning the cosine function, only four students provided an erroneous response by stating D to be zero. The value of D determines the vertical shift of the graph. The common misconception here is that students assumed D should always be negative, disregarding the possibility of positive values that would cause an upward shift on the graph. Masomeh (2017) identified this prevalent misconception among students who believed that the vertical shift represented by the parameter D in the general equations $y = A\cos(Bx + C) + D$ or $y = A\cos(Bx + C) + D$ automatically resulted in an upward shift, regardless of whether D was positive or negative. Similarly, Bornstein (2020) explored students' abilities to identify vertical shifts in cosine and sine functions across various tasks. While students could recognize these shifts, they struggled to articulate the distinction between vertical shifts and amplitudes clearly. These findings underscore the need for teaching strategies that not only address procedural competencies but also deepen students' conceptual understanding of how each parameter in trigonometric functions influences their graphs. Instruction could benefit from emphasizing the role of D in shifting the graph vertically, both upward and downward, based on its sign, and differentiating this from the role of A, which affects the amplitude of the function.

8.1.5 Errors related to the manipulation of trigonometric expressions

In Chapter 7, we examined the responses of 125 students who were asked to find an equation that represents trigonometric functions concerning a variable. Additionally, they were asked to calculate the values of the trigonometric functions given in both radians and degrees and to tackle the task of simplifying complex trigonometric expressions.

8.1.5.1 Errors in expressing a trigonometric function in terms of a variable and simplifying a trigonometric expression

8.1.5.1.1 Expressing the trigonometric function with respect to a variable

For the expression of a trigonometric function in terms of a variable, the participants were instructed to determine the numerical values of $\tan(40°)$ and $\sec^2(130°)$ when $\cos(40°) = a$. All students who correctly found the value of $\tan(40°)$ did not use the Pythagorean identity. Instead, they employed the unit circle method, constructing a right-angled triangle where the hypotenuse was equal to 1 (the radius of the circle), and using the $y - value$ of the endpoint on the unit circle to represent the sine of an angle t, while using the $x - value$ to represent the cosine of an angle t.

Out of the 12 students who answered incorrectly, all of them utilized the unit circle approach. However, one student made an error by employing the Pythagorean theorem when attempting to determine the value of $\sin(40°)$ given that $\cos(40°) = a$. Instead of correctly calculating y as $\sqrt{1 - a^2}$, the student found y to be $\sqrt{a^2 - 1}$, which is undefined since $a^2 - 1 < 0$. This shows a misconception that the difference between two numbers is commutative when applying the Pythagorean theorem. On the other hand, six students believed that $\tan 40° = \sqrt{1 - a^2}$. When using the unit circle approach, they saw that $\frac{y}{\tan(x)}$, or equivalently $\tan(x)$, represents the relationship between the vertical $(y - axis)$ and horizontal $(x - axis)$ distances on the unit circle. This understanding led them to believe that y = tan(x). Two students did not understand the notation used for trigonometric functions. Due to their misunderstandings or misinterpretations, they mistakenly came to the wrong conclusion about how $\tan(x), \cos(x)$ and $\sin(x)$ are related. As a result, they incorrectly believe that $\tan 40° = \frac{\cos(40°)}{\sin(40°)}$. Three students mistakenly think that since they are given the value of cos (40°), they can directly substitute it into the formula to obtain tan (40°). However, this approach is incorrect because substituting only one value does not provide us enough information to determine the other trigonometric function accurately. Fifteen students did not answer, and 51 students provided an incorrect response for finding the value of $\sec^2(130°)$. Out of these wrong answers, 39 students made errors in applying algebraic methods, while some incorrectly manipulated cofunction identities. Seven students used incorrect identities, and five students utilized the wrong notation for trigonometric functions. In the interview, Student P79 was confused about the concept of reciprocal functions and how they relate to the unit circle, whereas Student P57 made mistakes with the minus sign.

All these mistakes can be attributed to the challenges students face with the definition of trigonometric functions in a right triangle. This trigonometric concept is difficult as it involves abstract ratios and requires students to work with rotated right triangles without any specific pattern (Blackett and Tall, 1991).

Nurmeidina and Rafidiyah (2019) found that students often misinterpret given problem information, leading to numerous errors in applying trigonometric concepts. The lack of familiarity with trigonometric questions contributes to this confusion, often resulting in incomplete solutions.

Kamber and Takaci (2018) found that students struggle with the periodicity of trigonometric functions and the fact that these functions are not one-to-one functions. This difficulty is compounded by a reliance on the unit circle as a primary tool for solving problems, without a deep understanding of its concepts.

Kamber and Takaci (2018) emphasize the importance of understanding why certain mathematical relationships hold in trigonometry, which is often overlooked when algorithms are memorized without consideration of their foundational principles (for example, given that $3^x > 3^y$ we get $x > y$, however, when $\cos x > \cos y$, this does not imply $x > y$).

Both sets of researchers stress the need for a stronger conceptual foundation in trigonometry education.

8.1.5.1.2 Simplifying a trigonometric function

The main goal of this task was for PSTs to simplify trigonometric expressions, using fundamental identities or the factorization method. The importance of fully grasping basic trigonometric identities, effectively distributing operations, and recognizing common factors when dealing with complex trigonometric expressions was emphasized in this task. The task was to simplify the expressions $\tan(x)\csc(x)$ and $\frac{(\cos^2(x)-4)}{(\cos(x)-2)}$ for $(\cos(x)-2) \neq 0$. Thirty-three students successfully simplified $\tan(x)\csc(x)$, while 53 students failed because they did not recognize that $\frac{1}{\cos(x)}$ simplifies to $\sec(x)$. Two students did not answer, and 22 provided other incorrect responses. Among them, 12 overlooked the identity $\tan(x) = \frac{\sin(x)}{\cos(x)}$, which would aid in simplification. Additionally, 10 participants mistakenly applied the rule "$\frac{1}{a} = a$," leading to incorrect simplifications. Concerning another expression, $\frac{cos^2(x)-4}{(\cos(x)-2)}$, 32 gave incorrect answers. Eight participants misunderstood trigonometric notation, mistaking $\cos(x) - 2$ for $\cos(x - 2)$. Thirteen confused $a^2 - b^2$ and $(a - b)^2$, resulting in miscalculations. Five correctly substituted $\cos^2(x)$ with $1 - \sin^2(x)$ but did not progress, while three made an incorrect identity substitution, further complicating their work.

Simplifying quotient quadratic trigonometric expressions can be challenging for students due to a lack of skills in algebraic manipulation and misunderstanding trigonometric functions. Common mistakes include not factoring out common terms in the numerator and denominator, which complicates expressions further. Another error is misidentifying like terms, leading to incorrect additions or subtractions. Moreover, students sometimes fail to correctly apply algebraic techniques like factoring or expanding expressions, missing chances for simplification. To improve, students should develop a solid foundation in both algebra and trigonometry and practice these skills regularly. Educators must ensure that students understand the connection between algebra and trigonometry, enhancing problem-solving skills. By providing examples of algebraic operations that frequently apply to trigonometry, educators can demystify trigonometric functions. This approach allows students to recognize patterns and make connections, reducing mistakes.

Mkhwanazi et al. (2023) investigated high school math teachers' conceptual difficulties in trigonometry and coordinate geometry. The study reveals that teachers struggle with converting between different mathematical representations, affecting their ability to solve problems accurately. Many teachers found it difficult to translate graphical information into algebraic equations, particularly when dealing with parallel lines, intersections, and trigonometric concepts. This inability to switch between visual and algebraic interpretations leads to misconceptions and errors. Similarly, teachers faced challenges in trigonometry when they failed to recognize algebraic structures within problems, impacting their problem-solving efficacy. Furthermore, the research highlights a lack of representational fluency, which inhibits the teachers' ability to help students make connections between different representations of mathematical concepts. This deficiency poses difficulties in fostering a deeper understanding of mathematics among learners, as teachers themselves struggle to bridge different conceptual views within the subject.

8.1.5.2 Errors in evaluating a trigonometric function expressed in radians or degrees

The participants were given the task of calculating $\sin\left(\frac{13\pi}{6}\right)$ and $2\sin(150°)$ without using a calculator. When evaluating the expression in degrees, most students completed the task. Only six students did not answer, and one student provided an incorrect answer due to an algebraic mistake involving multiplication. On the other hand, when evaluating a trigonometric function expressed in radians, several errors occurred. It is important to note that out of the 107 students who found the correct value of $\sin\left(\frac{13\pi}{6}\right)$, only 12 did so without converting radians to degrees. During interviews with the students, Student P108 explained that she converted to degrees because it is easier to work with without a calculator. Initially, Student P55 suggested using a calculator but was reminded by the researcher that the question specifically asked for a solution without one. Student P55 then admitted to not knowing any other method and relied solely on using degrees.

Out of the total of eighteen students, nine did not answer, while the other nine provided incorrect answers when asked to calculate $\sin\left(\frac{13\pi}{6}\right)$. These wrong answers can be categorized into three groups: three students incorrectly converted radians to degrees, two students made mistakes in breaking down the function as a sum, and four students used incorrect cofunction identities. During the interview, Student P79 encountered difficulties in simplifying fractions and confessed to using an erroneous method found online. Moreover, he was unable to recall the correct approach for solving a similar problem, suggesting a misuse of information obtained from the internet

Tuna (2013) found that 60% of the PSTs in his study could correctly define degrees, while 40% had misconceptions, suggesting only partial comprehension. In contrast, just 8.6% accurately defined radians, often misinterpreting them as expressions involving π or confusing them with degrees. Although capable of performing calculations with radians, the participants struggled with conceptual understanding. Interviews revealed their curiosity about radians but also confusion due to unclear definitions. Many incorrectly used radian values as degrees in computations. Tuna (2013) recommended emphasizing definitions and exercises to illustrate the relationship, such as π radians equalling 180 degrees, to enhance comprehension. Similarly, Topçu et al. (2006) examined how preservice and in-service mathematics teachers understand radians and identified the sources affecting this comprehension. The study indicated that many participants had a limited grasp of radians, often rooted in their familiarity with degrees. The researchers argued that this understanding was further complicated by the misconception that radians are not real numbers, despite the exposure to trigonometric functions in real numbers. The interviews with the participants revealed that the equation $\frac{D}{180^\circ} = \frac{R}{\pi}$ was a common reference point, yet few related it to radians being a length ratio. Teachers with a solid understanding connected the unit circle with other trigonometric concepts, while those focused-on degrees emphasized the right triangle. Additionally, there appears to be a conceptual difference in how π is viewed in trigonometry compared to its role as a real number.

The findings of both studies are not surprising because degrees are typically introduced as the primary way to measure angles in early education, while radians are introduced at the university level and are seen as a relatively new and abstract form of measurement. In high school, radians are briefly explained in one sentence, highlighting their connection to angles. Additionally, most tasks involving radians focus on converting radian measures to degree measures and vice versa.

8.2 Using Duval's framework to gain a deeper insight into the errors committed by the PSTs

The other aspect explored in this research is to tackle the second question: *What is the explanation for the error commonly made by PSTs when dealing with problems related to trigonometric functions, as per Duval's theory?*

According to Duval (1998), trigonometric representations can be divided into three categories: iconic, enactive, and symbolic. Iconic representation involves visual depictions that resemble the object or concept being represented. Enactive representation relies on physical actions that imitate the characteristics of the object. Symbolic representation utilizes abstract symbols conventionally associated with the object. In the context of trigonometry, enactive representation may involve using physical objects or models, while iconic representation includes diagrams or drawings. Symbolic representation employs mathematical language and symbols such as $\sin(\theta)$ and $\cos(\theta)$.

Duval (1993, 1995) defines a register of representation as a mechanism enabling three cognitive processes. The first process uses a representation to access specific characteristics of an object. The second process modifies the representation by incorporating additional information or data. The third process converts the original representation into another system to extract multiple interpretations from it. Essentially, cognitive processing activity refers to working within the same register, while conversion involves switching registers. Duval's concept of processing can be categorized into three types: operative treatment, process-object treatment, and object-treatment relationship. These approaches allow students to engage with mathematical concepts by interacting with objects mentally or physically. Using treatment and conversion methods can be advantageous for adapting trigonometric functions to specific requirements or problem-solving purposes.

8.2.1 Errors related to treatment

Students made various mistakes while working with treatment. These errors appeared in three different approaches to treatment: operative treatment, process-object treatment, and object-treatment relationships.

8.2.1.1 Errors made in the process of operative treatment

The process of determining the value of $\sec^2(130°)$ when $\cos(a) = a$ and finding the value of $\sin\left(\frac{13\pi}{6}\right)$ can be seen as an operative treatment. This means that it involves strictly following the symbolic trigonometric rules and using only properties and relationships of trigonometric functions. In Figure 7.10, we can see that four students made errors in their answers. They correctly simplified $\sec^2(130°)$ to $\sec^2(50°)$, but then mistakenly simplified $\sec^2(50°)$ as $\sin^2(50°)$ instead of $\frac{2}{\cos^2(50°)}$. Two other students, P79 and P76, incorrectly broke down $\frac{13\pi}{6}$ in the calculation of $\sin\left(\frac{13\pi}{6}\right)$.

These mistakes can be attributed to cognitive difficulties faced by students when trying to understand these problems. This supports Duval's (2006) findings, which suggest that students often rely on superficial features or visual cues rather than truly understanding the underlying concepts. They may memorize formulas without grasping their meanings or mechanically apply algorithms without comprehending the principles behind them. This shallow understanding hinders their ability to solve more complex problems or apply knowledge to new situations.

8.2.1.2 Errors made in the process-object treatment

An example of this treatment is simplifying the expression $\frac{(\cos(x)-2)(\cos(x)+2)}{(\cos(x)-2)}$ into the expression $(\cos(x)+2)$. This involves applying various algebraic and trigonometric techniques to manipulate and simplify the given expression. Participants may have recognized that the expression $(\cos^2 x) - 4$ can be represented as a quadratic expression $a^2 - b^2$ and used algebraic factorization to simplify it into $(\cos(x)-2)(\cos(x)+2)$ (process). They would then substitute this simplified expression back into the original one (object), identify common factors in the numerator and denominator, and simplify the ratio to just $(\cos(x)+2)$. Out of the 32 participants who provided incorrect answers, eight successfully followed the process treatment but made mistakes in the object treatment by using another process treatment instead (see Figure 7.32). These participants applied addition and substitution formulas of trigonometric functions incorrectly. The remaining participants who gave wrong answers did not perform the correct process-object treatment (see Figures 7.33, 34, 35, 36). According to Duval (1995), cognitive activities involve creating a mental connection from a physical element, constructing a representation, processing it using another representation within the same context, and converting it into another context without losing its relation to the physical element. All these errors occurred because students failed to apply proper coordination between algebraic and trigonometric registers and were unable to convert information from one register to another while retaining its original characteristics or meaning.

8.2.1.3 Errors related to object-treatment relationship

Understanding mathematical concepts involves two parts: the actual mathematics content and the cognitive processes used to solve problems. How students perceive, manipulate, and apply mathematical objects using appropriate methods is known as the object-treatment relationship. Mistakes or misunderstandings occur when there is a disconnect between what the object represents and how it is treated (Duval, 2006).

In trigonometry, the idea of treating and transforming trigonometric functions is important when studying their periodic behaviour. This involves actions such as scaling, shifting, or reflecting the functions. By comparing the graphs of the original function $f(x) = \cos(x)$ and the transformed function $h(x) = 2\cos\left(\frac{x}{2} - 30°\right) + 1$, students can understand this concept as defined by Duval. However, in the given problem, 54 out of 107 participants failed to answer correctly, and ten did not provide any answers. Among these incorrect responses, participants mistakenly altered the horizontal stretching or compressing of the graph instead of its vertical aspect. They also applied the phase shift to the wrong axis and neglected to adjust the y-values accordingly. Additionally, they either didn't include or incorrectly applied the scaling factor of 2 when sketching the function and extended the graph beyond its valid intervals (refer to Figures 6.1.24-27).

One mistake commonly made by students is misunderstanding angular units. For example, some incorrectly believe that 30° should be measured from the $x - axis$ instead of being viewed as an adjustment within the function itself. This causes them to plot points at angles like 60° instead of 30° because they do not understand angular measurements correctly. Instead of shifting the graph leftward by 30 degrees, students mistakenly shift it rightward or do not make any shift at all. Students also had difficulty with scaling and amplitude when graphing the trigonometric functions. The amplitude of a cosine function represents the maximum vertical distance from the x-axis to the curve. For instance, $f(x) = \cos(x)$ has an amplitude of 1, while $h(x) = 2\cos\left(\frac{x}{2} - 30°\right) + 1$ has an amplitude of 2. Students wrongly scaled their $y - axis$ or interpreted the amplitude value as a positional shift rather than a scaling factor. As a result, they drew graphs with exaggerated or compressed amplitudes, leading to inaccurate representations. Another challenge was applying trigonometric properties correctly when graphing the $h(x)$ which involves shifts and stretches. In $h(x)$, the presence of $\left(\frac{x}{2}\right)$ indicates a horizontal stretch by a factor of two. However, students overlooked this property and neglected the necessary adjustments during graphing. Instead of stretching the graph horizontally by a factor of two for $h(x)$, they drew similar curves for both $f(x)$ and $h(x)$, disregarding the required transformation. Furthermore, students ignored the domain restrictions when sketching both functions over the intervals [0°;360°]. Due to oversight or lack of understanding, they extended their graphs beyond the given interval or excluded important components, resulting in incorrect representations. They erroneously connected points outside the given interval or ignored certain sections that are not part of the specified range, distorting their graphs.

These errors indicate a lack of understanding regarding how each transformation affects the original trigonometric function. Students may struggle to visualize and interpret these transformations accurately, leading to mistakes when graphing trigonometric functions.

8.2.2 *Error related to conversion*

Duval's (2006) theory maintains that converting information from one form to another is more difficult than simply manipulating information within the same form. The main reason for this increased complexity is that conversion transformations require individuals to switch to a completely different way of representing information. In other words, when converting, people must navigate and manipulate information in two separate ways and transition between them. In contrast, treatments only involve working with information within a single representation. In this study, we can categorize three forms of conversion and examine the mistakes made by students during their attempts to finalize the conversion process.

8.2.2.1 Mistake occurring when converting trigonometric notation into visual representation

Duval's (1993, 1995, and 1996) research indicates that when learning mathematics, individuals frequently need to convert between different types of representations. These can include numbers, symbols, and even words. However, this requirement for conversion can make it difficult to understand mathematical concepts. Surprisingly, even though these representations belong to separate categories such as tables, graphs, and equations, they are interconnected and not independently developed.

In Chapter 7, the analysis of the answers given to a question about determining the value of $\tan(40°)$ when $\cos(40°) = a$ revealed that all students used a conversion transformation approach. This method involved considering the trigonometric representation of $\cos(40°) = a$ and converting it to an iconic representation on the Cartesian coordinate system. This was accomplished by utilizing a triangle within a unit circle. Once this conversion was completed, students were able to use the Pythagorean Theorem to find the values of all sides of the triangle. This process required coordinating between the symbolic trigonometric representation and the iconic representation. Subsquently, a backward conversion was performed using relevant trigonometric ratio identities to arrive at the correct answer to the question (refer to Figure 7.2).

Of the twelve incorrect responses to the question, all the students were able to correctly complete the initial step of conversion. However, they struggled to finish the second step of the conversion. Specifically, they were unsuccessful in accurately determining the value of

$\sin(40°)$ and substituting correct values or variables into the equation $\tan(40°) = \frac{\sin(40°)}{\cos(40°)}$ (see to Figures 6.3-6).

In Chapter 6, there was a question asking to plot the graphs of $f(x) = \cos(x)$ and $h(x) = 2\cos\left(\frac{x}{2} - 30°\right) + 1$ on the same set of axes using the interval [0°;360°]. To solve this problem, a conversion approach was needed. The students had to change the mathematical expressions into geometric shapes. However, out of 107 students, 54 answered incorrectly. They were able to transform the expressions correctly into shapes but made mistakes in the treatment process within the transformed register. These mistakes included altering the horizontal stretching or compressing of the graphs instead of their vertical aspect, applying the phase shift to the wrong axis and not adjusting the y-values accordingly. Additionally, they either did not include or applied correctly the scaling factor of 2 when sketching the $h(x)$ function and extended the graphs beyond their valid intervals.

The group that performed the activity from Chapter 7 had a lower failure rate (10% incorrect answers) in the conversion process compared to those who completed the Chapter 6 activity (50% incorrect responses). This disparity can be attributed to Chapter 7's requirement of performing double conversions between registers, whereas only one conversion was needed for Chapter 6.

8.2.2.2 Errors made when converting trigonometric notation to algebraic expression

To demonstrate the process of converting trigonometric notation into an algebraic expression, we examine the responses given by the participants to determine the value of $\sin\left(\frac{13\pi}{6}\right)$ without relying on a calculator. The aim of this task was to assess the ability of the PSTs to work with radians. Therefore, a conversion step had to be utilized to determine the accurate solution.

Although, 86% of the participants provided the correct answer, only 11% of the participants who gave the correct answer used a one-way conversion to reach that answer (see Figure 7.16). Other students used a two-way conversion, converting one register (radian) to another register (degree) before converting it the algebraic expression (see Figure 7.16-17).

Out of all the participants, nine gave an incorrect answer to the question. They made these errors because they attempted to solve it using a one-way conversion method. The mistakes occurred because the students had to simplify $\sin\left(\frac{13\pi}{6}\right)$ to $\sin\left(\frac{\pi}{6}\right)$ before converting it to a new value in algebraic form $\left(\frac{1}{2}\right)$. This can be seen in Figures 6.18-23.

This demonstrates why most participants were unable to use a one-way conversion and supports Tuna's (2013) findings that students find it challenging to comprehend and use radians compared to degrees. The challenges of working with radians as a measurement stem from their unfamiliarity, abstract nature, limited relevance in real-life situations, complicated conversions, and increased complexity in mathematical calculations.

8.2.2.3 Errors made when converting a trigonometric graphic representation to algebraic expression

Duval's (1995, 1996b, 1998, and 2006) notion of conversion provides a framework for understanding the process of translating graphical representations into algebraic expressions. It suggests that there are three main types of conversions involved in translating between graphical and algebraic representations: iconic, discursive, and symbolic conversions. Iconic conversion refers to the interpretation of visual elements such as shapes, points, or lines. Discursive conversion involves verbalizing mathematical concepts associated with these visual elements. Symbolic conversion focuses on representing these concepts using symbols and equations.

In Chapter 6, the participants are instructed to create a cosine function that has an amplitude of 3, a period of π, and is shifted π/4 units to the left, and is translated 1 unit up. Additionally, they need to formulate a sine function with an amplitude of 3, a period of 5π, a phase shift of $\frac{3\pi}{4}$ units to the right, and a vertical translation of 6 units downward. The problem presented involves a cosine function and a sine function. To solve this problem in Duval's theory, we use a one-way conversion combining the iconic, discursive, and symbolic conversions. Of the given responses by the participants, 76% provide a wrong answer for the cosine function's equation and 89% provide a wrong answer for the sine function's equation. They successfully move from one register (the description of the geometric presentation of both functions) to another register (algebraic representation $y = A\cos(Bx + C) + D$, and $y = Asin(Bx + C) + D$) but they fail to correctly apply the treatment process within the three main conversions. Based on Duval's notion of conversion, we identified common errors made during each conversion stage: iconic, discursive, and symbolic conversions. These errors include confusing amplitude with the angular frequency B of the general form of a trigonometric function, $y = A\cos(Bx + C) + D$, and $y = Asin(Bx + C) + D$ (see Figures 6.3.7-9), misunderstanding phase shift directions (Figures 6.3.1-6), and overlooking or misattributing vertical translations (see Figure 6.3.17). These mistakes highlight how misconceptions about visual interpretations and mathematical concepts can result in inaccurate algebraic representations.

8.2.2.4 The influence of the conversion's movement

The notion of the direction in which conversions occur is posited by Duval as a key factor influencing the complexity of mathematical activity. According to Duval (2008), a conversion performed in one direction may not be cognitively linked to its reverse operation. This insight underscores that the orientation or path of a conversion process significantly affects students' comprehension and cognitive load. Duval's empirical findings offer compelling evidence for this claim. In a study conducted within the domain of linear algebra, he found that over 80% of students successfully converted a two-dimensional tabular representation of a vector into its graphical form. However, only 34% could perform the reverse conversion, moving from the graphical representation back to a table. This discrepancy reveals the asymmetry of cognitive demand depending on the direction of conversion, suggesting that some representational shifts are inherently more challenging than others due to the absence of symmetrical mental connections.

Our own investigation supports Duval's framework by revealing how the direction and number of conversions directly affect student performance. Specifically, our analysis of subsections 8.2.2.1, 8.2.2.2, and 8.2.2.3 shows that tasks requiring two-directional conversion movements result in lower failure rates than those requiring only a single, isolated conversion. This reinforces Duval's view that understanding and mastering the directionality of conversions is critical for mathematical success. Furthermore, Bansilal's (2012) study of 290 students engaged with a three-part task involving the normal distribution curve using Duval's semiotic framework supports these conclusions. Bansilal reported that students struggled more with conversions than with treatments, procedures that occur within the same representation. Her findings further demonstrate that Question 2, which required more conversions and treatments than Question 1, posed a greater challenge. Question 3, which involved an inverse problem, was the most difficult, as it necessitated even more complex conversions across representational systems. These results emphasize that the level of difficulty in mathematical problem-solving is not only a matter of content but also of representational movement and the cognitive demands of switching between forms.

Recent studies further substantiate these findings. Güçler (2022) examined how preservice teachers navigate between graphical and algebraic representations of functions and found that reverse conversions, especially from graphs back to algebraic forms, posed significant difficulty for students due to weak conceptual linkages. Similarly, Salgado and Stacey (2020) highlighted that students often experience greater success when converting from symbolic to graphical representations than vice versa, reinforcing the claim that the cognitive paths in conversions are not bidirectional in difficulty. These studies align with Duval's assertion that conversion direction is not merely a procedural choice but a cognitive hurdle that can influence students' mathematical performance and conceptual understanding.

Altogether, these insights emphasize the critical role that conversion direction plays in mathematical reasoning and achievement. Educators must therefore be deliberate in designing instructional strategies that expose students to both forward and reverse conversions, ensuring they develop flexible, robust conceptual links between representations. Recognizing the asymmetry in conversion difficulty can help teachers anticipate student struggles and implement targeted interventions to support deeper mathematical understanding and success.

8.3 Conclusion

This study explores the common mistakes made by PSTs in trigonometric tasks, drawing on Duval's theoretical framework. The investigation highlights the cognitive processes involved in mathematical understanding, particularly emphasizing semiotic representation and the necessity to transition between different forms of these representations.

8.3.1 Research findings and comments

Understanding the common errors made by PSTs in trigonometric tasks offers critical insights into the nature of mathematical learning and the cognitive demands of trigonometry. When examined through the lens of Duval's (2006, 2008) theory of semiotic representation, these mistakes reveal the underlying challenges PSTs face in connecting different representations of mathematical concepts. Duval emphasizes that meaningful mathematical understanding is deeply rooted in the ability to move flexibly between various semiotic registers, symbolic, graphical, algebraic, verbal, and numerical. He posits that comprehension is not simply the result of executing procedures accurately but rather depends on one's capacity to convert representations across these registers. In this context, a student who can solve an equation symbolically but fails to interpret its graphical equivalent lacks true conceptual understanding.

This study's analysis of PSTs' work on trigonometric tasks supports Duval's theoretical claims and provides concrete examples of how difficulties in representational conversion can impede learning. A common pattern among PSTs was their struggle to transition between graphical, algebraic, and numerical representations of trigonometric functions. While many students demonstrated procedural proficiency, such as applying a known identity or executing a routine calculation, they often lacked the deeper conceptual grasp needed to justify or interpret their work across representations.

For instance, a recurrent error involved misinterpretation or incomplete understanding of the unit circle. PSTs frequently failed to relate angle measures with their corresponding sine or cosine values. Rather than visualizing or internalizing the unit circle as a cohesive representation of angle-function relationships, many students resorted to rote memorization of values at specific reference angles. As a result, they were prone to incorrectly identifying trigonometric values for angles such as 120°, 225°, or 315°, particularly when signs were concerned. This observation aligns with Duval's assertion that mathematical errors often stem from a breakdown in conversion, where students are unable to link symbolic information (like $\sin(120°)$) to its geometric or visual representation on the unit circle (Duval, 2006).

Similarly, the transition between algebraic and graphical representations posed substantial challenges. In tasks requiring students to sketch the graph of a trigonometric function or interpret features such as amplitude, period, and phase shift from a given equation, errors were frequent. For example, many PSTs assumed that the period of a sine or cosine function is always 360°, failing to recognize how the coefficient "B" in equations such as $y = Asin(Bx + C) + D$ affects the function's period. Others incorrectly believed that vertical shifts affect amplitude, conflating the vertical displacement (parameter D) with the magnitude of oscillation (parameter A). These misconceptions suggest a lack of understanding of how each parameter contributes to the shape and position of the function's graph, as well as a failure to integrate multiple forms of representation into a coherent understanding.

Another critical area of difficulty was observed in tasks requiring the formulation of trigonometric equations based on given parameters. Many PSTs showed an inability to structure or interpret the general form of trigonometric functions, particularly in identifying the roles of amplitude (A), frequency (B), phase shift (C), and vertical displacement (D). While some could recall the correct equation format from memory, their responses revealed a weak conceptual foundation. For example, they might know the structure of $y = A\cos(Bx + C) + D$, but were unable to use it appropriately when provided with a graph or verbal description. This again illustrates Duval's key idea: mathematical proficiency is not just about internalizing formulas but about using them meaningfully across different representations.

One particularly insightful finding in this study relates to the direction of conversion, a dimension of representation that Duval (2006) emphasized in his empirical research. Duval observed that students are often more successful when converting from symbolic to graphical forms than in the reverse direction. This asymmetry was mirrored in our data: while most PSTs could draw a graph given an equation, far fewer could determine an equation from a graph. For example, when given the graph of a sine curve, many struggled to identify its amplitude, period, or phase shift and translate these features into an algebraic form. This aligns with Duval's observation in linear algebra, where over 80% of students could convert tabular representations into graphical ones, but only 34% succeeded in the reverse process.

This asymmetry has serious implications for instruction. It highlights the necessity of designing curricula and pedagogical strategies that foster bidirectional fluency in representation. Rather than focusing solely on forward processes (e.g., from equations to graphs), educators must emphasize reverse processes as well. By encouraging students to interpret, generate, and critique mathematical representations in both directions, teachers can foster deeper, more connected understanding.

Another notable trend revealed by the study was the extent of students' reliance on procedural memorization rather than conceptual reasoning. PSTs often recalled identities such as $\sin^2(x) + \cos^2(x) = 1$, but struggled to apply them in non-routine contexts. In simplifying trigonometric expressions or solving equations, students frequently failed to recognize opportunities to use identities or made inappropriate substitutions. For example, some attempted to simplify $\tan(x)\csc(x)$ without rewriting $\tan(x) = \frac{\sin(x)}{\cos(x)}$ and $\csc(x) = \frac{1}{\sin(x)}$, missing the opportunity to arrive at a simpler expression such as $\sec(x)$. This suggests that while PSTs may memorize standard identities, they often lack the conceptual flexibility to apply them in unfamiliar or complex tasks.

This finding is consistent with research by Bansilal (2012), who used Duval's framework to analyse the difficulties students experienced with normal distribution tasks. Bansilal found that students struggled more with conversions, changing from one representation to another, than with treatments, which are operations carried out within the same representation. This supports our conclusion that representational transitions, particularly those involving directionality and multiple steps, significantly influence success rates in problem-solving. Bansilal also noted that inverse problems, where students must work backward from a result, were especially difficult, reinforcing Duval's view that reverse conversions are cognitively demanding.

These insights point to a need for more deliberate use of multiple representations in teaching. Instruction should not only expose students to different forms (symbolic, graphical, verbal, numerical) but also actively train them to convert between these forms. One promising pedagogical approach involves the use of dynamic mathematics software such as GeoGebra, which allows students to manipulate a graph and observe real-time changes in its equation. This can strengthen their understanding of how parameters influence a graph and vice versa (Nguyen & Kulm, 2020). Additionally, the incorporation of metacognitive strategies, where students reflect on their thinking and conversion processes, can improve representational fluency and foster deeper learning (Salgado & Stacey, 2020).

Another valuable strategy is the use of contrastive examples, where students examine pairs of problems that differ in one representational element. For example, comparing $y = 2\sin(x)$ and $y = \sin(2x)$ can help highlight the distinct roles of amplitude and frequency. This encourages students to pay closer attention to the impact of each parameter and strengthens their understanding of function behaviour. Research by Özgür and Eren (2023) supports this approach, showing that students who engaged with contrastive representations developed more accurate and lasting understandings of trigonometric transformations.

Additionally, the study revealed a subset of PSTs who failed to recognize the difference between trigonometric and non-trigonometric functions. Approximately 10% of participants were unable to distinguish between the two, and 9% provided no graphical response at all. This highlights a foundational gap in functional understanding. Trigonometric functions are periodic and exhibit oscillatory behaviour, whereas non-trigonometric functions (such as linear, quadratic, or exponential functions) follow different growth patterns. The failure to recognize these distinctions suggests that some students do not fully grasp the defining features of trigonometric functions. Addressing this requires an instructional shift towards explicitly teaching function types, their graphical characteristics, and their applications across different disciplines, including physics, engineering, and economics (Kastberg et al., 2022).

Furthermore, errors related to cofunction identities and notation were also observed. Several students confused identities such as $\sin(90° - x) = \cos(x)$, or misapplied them by substituting values incorrectly. Misinterpretation of these relationships points to gaps in understanding symmetry and reference angles, both of which are crucial for mastering the unit circle and evaluating trigonometric expressions accurately. Instruction that emphasizes geometric interpretations, such as reflecting points on the unit circle or using reference triangles, can help correct these misconceptions.

In conclusion, this study's findings reinforce Duval's assertion that the ability to convert between different representations is not merely a supplementary skill in mathematics, but a foundational one. The frequent errors made by PSTs in trigonometric tasks, whether in graphing, simplification, or problem formulation, can largely be traced back to breakdowns in representational fluency. These errors are not just the result of memory lapses or computational slip-ups but rather reflect deeper issues in conceptual understanding and cognitive processing. By using Duval’s framework to analyse these issues, educators can better identify and address the root causes of students’ difficulties.

To improve PSTs’ understanding of trigonometry and prepare them for effective teaching, instruction must deliberately cultivate representational fluency. This includes emphasizing both directions of conversion, reinforcing foundational identities, and creating opportunities for students to engage actively with multiple forms of mathematical representation. Only by doing so can we hope to develop mathematically competent teachers who can support their future learners in mastering one of the most challenging yet essential areas of secondary mathematics.

8.3.2 Implication of the study

The findings of this study carry important implications for mathematics education, particularly in the teaching and learning of trigonometry. They underscore the need for instructional approaches that promote cognitive flexibility and deepen conceptual understanding among learners. By identifying common errors and tracing their underlying causes, the study highlights the importance of equipping PSTs with the skills to transition effectively between different representations of trigonometric concepts. To address these challenges, educators are encouraged to design teaching strategies and materials that explicitly incorporate multiple representations, symbolic, graphical, numerical, and verbal, and that actively engage students in converting between them. Such an approach fosters the ability to interpret, analyse, and solve problems from diverse perspectives. Strengthening trigonometric instruction through these methods can reduce common misconceptions and errors while cultivating a more integrated and meaningful understanding of mathematics. This, in turn, better prepares prospective teachers to support their future learners with clarity, accuracy, and confidence.

8.3.3 Limitation of the study

While this study offers valuable insights into the common errors made by PSTs in trigonometric tasks, several limitations should be acknowledged. Firstly, the scope of the investigation was confined to specific trigonometric functions and identities, excluding other relevant areas such as word problems involving trigonometry or broader mathematical connections across topics. This narrow focus may limit the applicability of the findings to more complex or integrative problem-solving scenarios. Secondly, the data collection process was conducted online without supervision, which introduces the possibility that participants may have collaborated, consulted online resources, or otherwise received external assistance during task completion. Similarly, the interviews were conducted remotely rather than in person, which may have affected the depth of interaction and observation of participants' reasoning processes.

Additionally, the study concentrated on a specific cohort of PSTs within a single educational context, which may restrict the generalizability of the results to broader or more diverse populations. The heavy reliance on written tasks as the primary mode of assessment may not have fully captured the nuanced nature of PSTs' misconceptions or the complexity of their cognitive processes. This suggests that future research should incorporate more varied and interactive methods, such as real-time problem-solving sessions, think-aloud protocols, or in-depth qualitative interviews, to gain richer insights.

Lastly, although Duval's theory of semiotic representation provided a valuable framework for analysing students' errors, relying solely on this perspective may have limited the scope of theoretical interpretation. Future studies could benefit from integrating complementary cognitive or educational theories, such as constructivism, APOS theory, or metacognitive frameworks, to provide a more holistic understanding of how PSTs learn and internalize trigonometric concepts. Addressing these limitations in subsequent research could contribute to more robust and comprehensive insights into the teaching and learning of trigonometry.

8.3.4 Pedagogical implications and suggestions

This study highlights the critical need for diverse and targeted instructional strategies to effectively address and correct the recurring errors preservice teachers (PSTs) make in trigonometry. Educators are encouraged to place greater emphasis on the dynamic interplay between trigonometric concepts and their various representations, symbolic, graphical, numerical, and verbal. This can be facilitated through interactive, visual, and context-rich teaching approaches that help students accurately interpret and compute key properties such as amplitude, period, phase shift, and vertical displacement.

The integration of technology, particularly dynamic geometry or graphing software offers valuable opportunities for students to engage with trigonometric transformations in a visual and intuitive manner. Such tools enable real-time manipulation of function parameters; deepening students' conceptual understanding of how algebraic changes influence graphical behaviour. Additionally, structured problem-solving activities, scaffolded to encourage both procedural fluency and conceptual reasoning, can enhance PSTs' ability to construct and simplify trigonometric equations and expressions effectively.

Instruction should prioritize the development of conceptual understanding over rote memorization, especially with respect to fundamental trigonometric identities and their interrelationships. Reinforcing these identities through varied contexts and multiple representations can foster deeper comprehension and more accurate application. Regular exposure to carefully designed practice tasks, collaborative discussions, and reflective exercises will further support PSTs in internalizing and applying trigonometric principles meaningfully.

By addressing these pedagogical priorities, educators can significantly strengthen PSTs' understanding of trigonometric functions, better preparing them to teach these concepts competently and confidently in their future classrooms.

8.3.5 Reflection about this study

Reflecting on this study reveals the critical importance of designing instruction that supports PSTs in addressing and overcoming common errors in trigonometry. The findings emphasize that bridging conceptual understanding with practical application is essential for developing competent and confident mathematics educators. A key insight from the study is that misconceptions in trigonometry are often deeply rooted and require deliberate, structured interventions that go beyond procedural teaching. Addressing these challenges demands a comprehensive pedagogical approach that integrates multiple representations, encourages meaningful engagement, and promotes cognitive flexibility.

While the study identifies persistent difficulties PSTs face, such as interpreting trigonometric graphs, applying identities, and transitioning between representations, it also illuminates important opportunities for instructional enhancement. By creating learning environments that prioritize active participation, reflective thinking, and conceptual exploration, educators can foster deeper understanding and better prepare PSTs for their future roles in the classroom.

This preparation is not only relevant to their current academic success but is also vital for their professional development as future teachers. Empowering PSTs to understand and teach trigonometry effectively contributes to improved mathematics education more broadly, as these educators will be responsible for shaping the mathematical understanding of future generations. Ultimately, this study underscores the ongoing need for reflective practice in mathematics education. It calls for a continuous evaluation of teaching methods to ensure they align with the evolving needs of learners and the demands of a complex educational landscape.

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APPENDICES

APPENDIX A: ETHICS CLEARANCE LETTER

22 September 2020

Mr Francis Youbi (SN 220108433) School of Education

College of Humanities Edgewood Campus UKZN

Email: 220108433@stu.ukzn.ac.za

Dear Mr Youbi

RE: PERMISSION TO CONDUCT RESEARCH

Gatekeeper's permission is hereby granted for you to conduct research at the University of KwaZulu-Natal (UKZN) towards your postgraduate studies, provided Ethical clearance has been obtained. We note the title of your research project is:

"Exploring challenges of learning trigonometric functions by senior phase mathematics preservice teachers at a KwaZulu-Natal university."

It is noted that you will be constituting your sample as follows:

- Analyzing results of previous assessments
- By conducting interviews and/or focus group discussions with students on the Edgewood campus (Taking in account the regulations imposed during the lockdown ie restrictions on gatherings, travel, social distancing etc. ZOOM, Skype or telephone interviews recommended

Please ensure that the following appears on your questionnaire/attached to your notice:

- Ethical clearance approval letter.
- Research title and details of the research, the researcher and the supervisor.
- Consent form is attached to the notice/questionnaire and to be signed by user before he/she fills in questionnaire.
- gatekeepers' approval by the Registrar.

You are not authorized to contact staff and students using 'Microsoft Outlook' address book. Identity numbers and email addresses of individuals are not a matter of public record and are protected according to Section 14 of the South African Constitution, as well as the PAIA and POPI Act. For the release of such information over to yourself for research purposes, the University of KwaZulu-Natal will need express consent from the relevant data subjects. Data collected must be treated with due confidentiality and anonymity.

Yours sincerely

[redacted]

___

- **DR KE CLELAND: REGISTRAR (ACTING)**

___

APPENDIX B: DECLARATION OF CONSENT

DECLARATION OF CONSENT

I………………………………………………………………………………………………… … (Full names of participant) hereby confirm that I have been informed about the study entitled "**Exploring the challenges of learning transformations of trigonometric functions by senior phase preservice teachers at a university in KwaZulu-Natal"** by ……………………………………………………………………………………..(provide name of researcher). **I understand the contents of this document and the nature of the research project, and I consent to participating in the research project.**

I understand the purpose and procedures of the study (add these again if appropriate).

I have been given an opportunity to answer questions about the study and have had answers to my satisfaction.

I declare that my participation in this study is entirely voluntary and that I may withdraw at any time without negative consequences.

I voluntarily give permission for the interviews to be audio-recorded.

My identity will not be disclosed, and pseudonyms will be used to protect my identity

If I have any further questions/concerns or queries related to the study I understand that I may contact the researcher at ……………………………………………………………(provide details).

If I have any questions or concerns about my rights as a study participant, or if I am concerned about an aspect of the study or the researcher, then I may contact:

HUMANITIES & SOCIAL SCIENCES RESEARCH ETHICS ADMINISTRATION

Research Office, Westville Campus

Govan Mbeki Building

Private Bag X 54001

Durban

4000

KwaZulu-Natal, SOUTH AFRICA

Email: HSSREC@ukzn.ac.za

Additional consent, where applicable

- I am willing to be part of the paper and pen test and semi-structured interviews. I am also willing to allow recording by the following equipment, and the use of other data:

Digital audio recording of interviews	Willing	Not willing
Semi-structured interview		

……………………………………...

Name of Participant

………………………………………

…………………………………

Signature of Participant **Date**

APPENDIX C: STUDENT INFORMED CONSENT

Mathematics, Science and Technology cluster

School of Education,

College of Humanities,

University of KwaZulu-Natal,

Edgewood Campus, KwaZulu Natal

Date: XX August 2020

Dear Student

INFORMED CONSENT LETTER

My name is **Francis Youbi**; I am a postgraduate student who is currently studying towards a Doctoral of Philosophy in Mathematics Education at the University of KwaZulu- Natal, School of Education, College of Humanities. I am conducting the research titled '**Exploring the challenges of learning transformations of trigonometric functions by senior phase preservice teachers at a university in KwaZulu-Natal**'.

Many studies conducted on the mathematics preservice teachers in the country show that most of the preservice teachers do not have sufficiently strong background in basic mathematics. In the other hand, reports from the National Diagnostic Report on Learners Performance show that learners have a deficiency in the understanding of basic concepts across the topic involving graphs in trigonometry.

In view of the foregoing, I intend to explore preservice teachers' understanding of transformation of trigonometric functions to see how ready they are in terms of trigonometry content knowledge to teach this concept when beginning full time teaching career. The objectives of the research are as follows:

1. To identify and analyse the common mistakes made by the preservice teachers when they answer questions on trigonometric functions
2. To determine the extent to which to which these mistakes impact on preservice teachers' basic knowledge in trigonometric functions; and
3. To explore the challenges of learning transformations of sine and cosine functions when semiotic representation is involved.

To gather the information, I am interested in requesting you participate in this project by reflecting critically on your understanding of transformation of trigonometric functions. You are invited to please participate in this research because you are a student who is studying mathematics, which deals with trigonometric functions and their transformations. In addition, I will also ask you to take part in a semi-structured interview which may last about 15-20 minutes.

This study has been ethically reviewed and approved by the UKZN Humanities and Social Sciences Research Ethics Committee (approval number_____).
Please note that:

- Your participation is voluntary. If you do not participate you will not be penalized in any way.
- You have a choice to participate, not participate or stop participating in the research. Hence you are

free to withdraw from the study at stage for any reason. You will not be penalized for taking such an action.

- Your confidentiality is guaranteed as your input will not be attributed to you in person but reported

only as a population member opinion.

- Any information given by you cannot be used against you, and the collected data will be used for

purposes of this research only.

- The individual semi-structured interviews schedule will last for about 15-20 minutes.
- Data collected will be stored in secure storage and destroyed by shredding after 5 years. Digitally

recorded data will be deleted after five years.

- Your involvement is purely for academic purposes only, and there are no financial benefits

involved. However, it is expected that you will gain insight into challenges regarding teaching of trigonometry functions in schools.

Thank you
Yours faithfully
Francis Youbi
My contact details are as follows:
Email: youfra2@yahoo.fr

My supervisor is Prof. Sarah Bansilal. She is a cluster leader of Mathematics discipline, School of Education, College of Humanities, Edgewood Campus, University of KwaZulu-Natal
My supervisor's contact detail is:
Email: BansilalS@ukzn.ac.za

You may also contact the Research Office at:
University of KwaZulu-Natal
Humanities and Social Sciences Research Ethics
Govan Mbeki Centre
Tel +27312604557
Email: HSSREC@ukzn.ac.za
Thank you for reading this document about this research.

APPENDIX D: TURNITIN REPORT SUMMARY

Turnitin Originality Report

Thesis by Francis Youbi

From thesis (thesis submission)

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