A CASE STUDY OF THE DEVELOPMENT OF A.C.E. STUDENTS' CONCEPT IMAGES OF THE DERIVATIVE

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by

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Abstract

This research focuses on the development of the concept images of the derivative concept of students enrolled in the in-service programme 'Advanced Certificate in Education' at University of KwaZulu-Natal, Pietermaritzburg campus. In addition, two qualified teachers not enrolled in the programme were included.

A theoretical framework which describes the derivative as having three layers – the ratio, limit and function layers – that can be represented by a variety of representations – graphical, rate, physical and symbolic – is used to analyse the development of the students' concept images. This framework was adopted from previous research, but expanded to allow for situations where a student's concept image did not fall into any of the layers or representations. In those cases, the concept image was classified into the non-layer section or the instrumental understanding section.

The findings of this research show that of the five ACE students who were interviewed, only one had a profound concept image in all the three layers of the derivative, with multiple representations as well as connections among representations within the layers. This one student also passed the calculus module with a distinction. The other four students had the ratio layer and graphical representation profound in their concept images, while the other layers and representations were pseudo-structural with very few connections. Two of these students passed the calculus module while the other two failed.

All the students showed progression in their concept images, which can only be credited to the ACE calculus module. However, it is clear that even upon completion of this module, many practicing teachers have concept images of the derivative which are not encompassing all the layers and more than one or two representations. With the function layer absent, it can be difficult to make sense of maximization and minimization tasks. With the limit layer absent or pseudo-structural, the concept itself and the essence of calculus escapes the teachers – and therefore also will be out of reach of their learners.

DEDICATION

I would like to dedicate this research project to the following:

My husband Spencer for his patience, support and understanding throughout my study period.

My children Tracey, Terence and Kelly for their understanding and help when I needed help with diagrams and tables on the computer.

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Firstly I want to thank my supervisor Prof Iben Christiansen for her support, input, guidance, without which I could have never done this project. Thank you for all the help of putting this project together.

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My colleagues at school, Bridget Langley and Sarah Alexander for their support.

The ACE students and teachers in this research for their willingness to be interviewed.

DECLARATION

I declare that "The Development of the ACE students' concept image of the Derivative" is my own work and that it has not been submitted previously for degree purposes at any higher education institution.

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CHAPTER ONE : INTRODUCTION

Focus

Calculus is a very important part of mathematics with wide applications in natural and social sciences. According to Tall(1992), " The calculus represent the first time in which the student is faced with the limit concept, involving calculations that are no longer performed by simple arithmetic and algebra, and infinite processes that can only be carried out by indirect arguments." White and Mitchelmore (1996), in their discussion, point out that it is worrying to notice the memorising that is taking place within the large numbers of students who are taking calculus. Tall (1992) agrees with this when, in his discussion on students' difficulties in calculus, he mentions that the difficulties encountered by students include translating real world problems into calculus formulations and that students prefer procedural methods rather than conceptual understanding.

Bowie (1998) mentions that in the USA during the late 1980s there was an emergence of the calculus reform movement because of the high number of dropouts from the calculus course and poor performances by those who continued with the course. Though this reform did affect South Africa as well, I believe the South African situation is worse given the number of unqualified educators[1] in the schools (cf. Parker 2004). It is on this basis, I have chosen to focus on educators' conceptual understanding in calculus.

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[1] In the post-Apartheid curriculum reform, the term 'educator' has replaced the common term 'teacher', and 'learner' is used to refer to pupils. I use these terms interchangeably throughout the thesis.

Motivation

Seldon et al (2002) conducted a study in which they investigated the non-routine problem solving abilities of students who had just completed their first year of calculus. Their results show that although the students appeared to have enough basic knowledge of the concepts in calculus, they had little success in solving problems in calculus. A follow up study on more advanced and experienced calculus students showed that although the students found it easy to deal with complicated algebraic and arithmetic problems, more than half of them still failed to solve the calculus problems.

According to Bezuidenhuit (2001), many Mathematics educators have acknowledged that, while the first year students can find limits, derivatives and integrals, they lack conceptual understanding of basic calculus. His study on the students from three different South African Universities show that a large number of first year students have a weak understanding of basic calculus concepts and that their understanding is mainly procedural rather than conceptual.

In 2003, I was studying with other educators, many of whom were having difficulties understanding calculus. I believe that, if we are aware of the students' concept images, it will be easier to design the activities so that they promote creation of vivid images that will help the students understand better. In turn, when these students are teaching their learners, they will be sensitive to their learners' responses and do the same. When students have a strong conceptual background, they will not struggle, because the conceptions from their previous experiences will strongly influence how they make sense of the concepts that they come across.

Therefore, I research the Advanced Certificate in Education (ACE) students' concept images of one of the key concepts in calculus, the derivative.

An Outline of the Research Project

Chapter 2 describes the theoretical framework which I have adopted from Zandieh (2000) and expanded in order to suit my project. The theoretical framework comes before research questions and the literature review because in the theoretical framework, I explain the concepts mentioned in the research questions and literature review and this makes it easier for the reader to make sense of the concepts used.

In chapter 3, I have described the research questions and how I intend answering them.

Chapter 4 describes the survey of the literature related to my research, and also describes how my research is related to this literature. In this chapter, I also describe how my research will fill some gaps.

The methodology chapter comes after the literature review and in this chapter I describe the method used for data collection, the interviewees as well as the methodological issues.

In chapter 6, I analyse the transcribed interviews from each interviewee and the results are shown by means of the instrument adopted from Zandieh and expanded to suite my data. Finally it is the discussion and conclusion chapters.

CHAPTER TWO:THEORETICAL FRAMEWORK

The Constructivist Notion of 'Concept Image'

'Concept image' refers to the mental pictures and notions that a student has about that concept. These might be in the form of symbols, diagrams, graphs or words. Working within constructivism, Tall and Vinner (1981) and Thompson (1994) define the concept image as the total cognitive structure which is associated with the concept. The student's concept definition is seen as part of the concept image: Tall and Vinner describe this as the statement that the student will give when asked to define a concept. If compartmentalisation occurs within a concept image, the learner fails to link an idea with the other aspects of the concept and this will result in errors and misconceptions.

According to Sfard (1991), abstract concepts can be understood in two different ways, these being as objects and as processes. She states that seeing a mathematical concept as an object means being able to identify the concept immediately and to take it as a whole without going into details. When one sees a concept as a process, they consider it to be a partial rather than a whole, and the concept is said "to come into existence upon request in a sequence of actions" (p4). If the student does not have the internal structure as part of her/his concept image, then a pseudo-structural conception occurs. According to Sfard (1991), a pseudo-structural object is an intuitive understanding that does not involve an understanding of the process underlying the object. Zandieh (2000) points out that the use of pseudo-structural objects allows the student to formulate basic understandings of the derivative concept which differ from each other, and as a result the students' understanding develops from partial to a more complete understanding with only a few aspects of the concept missing (p.124). I will return to this point briefly.

<u>Concept Representation</u>

Gravemerjer et al (2002) mentions that the role of representational systems in Mathematics is made up of two components, one that supports the cognitive processes and the other that mediates communication. Vinner and Dreyfus (1989) points out that some concepts have strong graphical aspects while others have strong symbolical aspects and also that some concepts have both. The derivative is one that is strong in all aspects. They also agree with Zandieh (1997) and Thompson (1994) that a concept is not acquired in one step, but that the learner goes through several stages for a concept to be reified.

Goldin and Shteingold (2000) agree with the above writers but they refer to the mental images that we create as the internal systems of representation and the symbolic and graphical representations as the external systems of representation. In their discussion, they mention that different representational systems can be found as verbal, imagistic, formal notational and affective. They describe the dual nature of the systems of representations. The student will represent internally the external system and they will also represent externally their internal system of representation. They are convinced that interaction between internal and external systems of representation is the key to effective teaching and learning. They also agree with Sfard's process-object theory, outlined below. This is highlighted in their discussion of the internal system of representation that it goes through three stages before it is reified as a complete system. Bezuidenhuit (2001) stresses the point that well built mental representations of the network among concepts of a particular topic are of great importance for a good understanding of that particular topic.

The Process-Object Duality of Mathematical Concepts

The process- object duality of a mathematical concept is seeing a concept both as a process and as an object. For example, an algebraic expression can be seen as a process - a description of a computation, and as an object it can be said to be a relation between two or more variables. As an object, symmetry can be seen as a property of a geometric shape but as a process it can be seen as transformation of the shape.

The notion of 'reified' concepts come from the seminal work of Sfard (1991) where she states that concepts in mathematics are acquired through three stages which include (1) a process operating on objects that are already known by the learner. (2) The second stage would be to put together all the processes into one process and finally (3) the learner being able to see the process as an object in its own right, thus applying new processes to it. For instance, a student gets used to finding the slope of a line using $\frac{f(x+h)-f(x)}{(x+h)-x}$. When lots of these are performed by the student, they get skilled at this process and eventually see it as an object, the slope. The limiting process can be applied to the slope to give the slope at a point.

These three stages which transform a concept (image) from a process to an object, she refers to as interiorisation, condensation and reification. Sfard describes interiorization as getting used to a process which then leads to a new concept. Condensation is squeezing long sequences into a manageable process and being able to think about a given process as a whole. Reification is seeing something in a totally new light and when a process is seen as an object (p.19).

Zandieh's Framework for the Concept Image of Derivative

Zandieh has worked specifically with the concept image of derivative, along three dimensions, namely representations, conceptual layers, and process-object duality. In the following section I am going to summarize Zandieh's theoretical framework showing how it agrees with other research on students' concept images of the derivative.

Zandieh's framework was informed by the whole community of mathematics, which includes researchers, teachers, students and the way the concept of the derivative is described in textbooks. She also points out that it is not enough to ask if a learner understands or does not understand the concept of the derivative, but instead one needs to look for a description of the way in which the learner understands the concept (i.e, his/her concept images).

Zandieh (1997) describes the derivative as a multifaceted concept. She uses three theoretical frameworks in her research and I wish to adopt them in my research. The three frameworks are concept image framework (already mentioned), process-object framework (already mentioned), and the notions of multiple representations, also briefly addressed above. With regard to the latter, Zandieh, referring to Thompson (1994), makes the point that it is our subjective sense of invariance that makes us see one underlying structure across a number of contexts, which we then perceive to be representations of that structure (p. 105).

In describing the process-object framework Zandieh points out that:

> *The underlying structure of any representation of the concept of the derivative can be seen as a function whose value at any point [x] is the limit of the ratio of differences*
> $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ *(p. 106)*

As mentioned, Sfard (1991) describes the dual nature of mathematical concepts, which means that a concept can be seen as a process and as an object. Generally, the process aspect precedes the object aspect. Zandieh takes this idea to the derivative concept (1997, 2000). Zandieh points out that the derivative has three layers and these are ratio, limit and function. Each of these three layers of the derivative can be viewed as a process and as an object. Also each of these layers can be represented graphically, verbally, paradigmatically (physically) and symbolically.

The ratio layer

The ratio as a process is seen as a process of division of the numerator by the denominator, and as an object a ratio is seen as a pair of integers or as the outcome of the division process. The ratio layer can be represented in different ways. As a ratio, the average gradient can be represented in Leibniz notation as $\frac{\Delta y}{\Delta x}$.

In symbolic representation the ratio as the average gradient can be represented as $\frac{f(x_0 + h) - f(x_0)}{h}$. The ratio layer can also be represented graphically as a slope of a secant which is calculated as $\frac{\text{change in } y}{\text{change in } x}$ and in terms of situational representation, average velocity can be represented as a ratio when one is calculating $\frac{\text{change in displacement}}{\text{change in time}}$. All of the representations have both process and object aspects.

The limit layer

The limit as a process is seen as taking the limit and as an object it is seen as the limit value. The limit layer can be represented in several different ways. The limiting process of the gradient can be represented in Leibniz notation as $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ and in symbolic representation as $f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$. Note the use of x_0 here, to indicate that at the limit layer, the derivative is found at a point.

The limit layer can also be represented graphically. For the limiting process, one focuses at a point on a curve and keeps it fixed while choosing other points on the curve increasingly closer to the fixed point. Connecting the fixed point to each of the other points in turn, a number of secants are drawn, which will approach the tangent of the curve, referred to by Thompson (1994) as the sliding secant. So the limiting process will be to approach the tangent to the curve at a given point. When someone understands that the gradient of the tangent is also the gradient of the curve at that point (a matter of existence and uniqueness which I will not touch upon here, but to note that it is an aspect about which Zandieh's framework remains silent), they will have grasped the fundamental assumption of the limit layer in the derivative concept.

The function layer

As a process, a function is described by Vinner and Dreyfus (1989) as a correspondence between two non-empty sets, and as an object they describe it as a set of ordered pairs. In the derivative's function layer, the order pairs are $(x, f'(x))$. The function layer can be represented in many different ways. The derivative can be represented in Leibniz notation as $\frac{dy}{dx}$. In symbolic representation the derivative function, that is the gradient of the curve at every point $(x, f(x))$ is represented as $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$. In situational representation, one looks at the instantaneous velocity at any point in time.

An Overview of Zandieh's Framework

As mentioned above, Zandieh combines three frameworks in her study and illustrates this in a matrix form. I will describe my adaption of her model in the methodology chapter, under 'analysis instrument'. Before I move on to the literature review, however, I find it appropriate to engage the presented frameworks critically.

Firstly, while Zandieh acknowledges that it is in the perceived connection between the various components of the concept image that the concept lies for the individual, in her framework she does not actually identify the extent to which the students make connections. I want to emphasise what becomes hidden in Zandieh's framework, namely that strong concept images exist in connections across layers and representations. For instance, it is desirable that learners and teachers alike will look at a graphical illustration of a derivative and interpret it symbolically or vice versa. It therefore becomes necessary to enhance the framework's ability to capture such connections.

Zandieh's framework focuses on the conceptual layers of the derivative. However, many learners and educators have procedural (distinct from process aspects of concepts) knowledge of the derivative. For instance, some of the interviewees can find the derivative of a given function without necessarily showing concept images falling within Zandieh's framework. To capture this aspect, I found it necessary to expand the theoretical framework, so I had to include Instrumental Understanding in-order to include the only learning that most of the interviewees of my study exhibited. Lithner (2003) describes instrumental understanding as mastering of a rule or procedure. A student can master a rule or a procedure without any insight into the reasons that make it work, but some students can master a rule or procedure and at the same time have insight into the reasons that make it work. Here, I use it simply to refer to the ability to use as a rule. If a student also understands the background to the rule, I will indicate this in my coding of the data (see 'analysis instrument' in Methodology chapter).

In this chapter, the derivative is described as a multifaceted concept and a description of the outline of the construct of the concept is given which will be used to analyse the students' concept images. This includes the layers which are ratio, limit and function layers and their representations which are graphical, verbal, physical and symbolical. The next chapter presents the research questions which guided my research, after which I discuss the literature related to this research.

CHAPTER THREE : KEY QUESTIONS

1. Which layers of the derivative concept appear to be part of the students' concept images?
    - Which layers are present in the learners' concept definition?
    - What are consistencies and inconsistencies across representations within the layers of the derivative?
2. Do students understand both the process and object nature of each layer?
3. How do the students' concept image change over time?

In this chapter, I will describe how the research questions help meet the purpose of this research.

The purpose of this research is to gain insight into the students' concept image of the derivative concept. As mentioned in chapter two, the derivative has three layers, the ratio, limit and function layers. The order of conceptualization does not matter. Students' preference of layers may vary, as shown by Zandieh (2000)'s results. Each layer can be seen as a process and as an object, as explained in chapter two. If a student does not see the process involved in a layer, but can refer to it as an object in some respect, it is said that the pseudo-structural conception occurs. Some students will work on surface resemblance because they have instrumental understanding of a concept. This helps them follow procedures without worrying about the process underlying the concept.

Each layer can be represented by means of various representations, which are graphical as slope, verbal as rate of change, physical as velocity and symbolical as difference quotient. Connections within representations of a layer will be seen when a student is able to describe the derivative using a representation and at the same time being able to link the representation to another representation. For example, when a student mentions that the derivative is a slope which is given by $\frac{f(x_0+h)-f(x_0)}{h}$, this example shows the connection between the graphical representation and symbolical representation.

Studies that have been conducted here in South Africa on the derivative concept were mainly on one aspect of the derivative using one representation, as will become evident in the review of the literature. This study involves all three layers and various representations of the derivative concept.

CHAPTER FOUR: LITERATURE REVIEW

Various studies have been conducted focusing on the derivative concept, but these studies have been done focusing on a particular aspect of the derivative concept and representations of the aspect. The one exception to most of these studies is the relatively recent work by Zandieh (1997, 2000). She develops a theoretical framework for her study based on a thorough analysis of "what the mathematical community considers to be part of the understanding of the concept of the derivative" (Zandieh 2000: p113), describing the three layers of the derivative concept and their representations (see the section on the theoretical framework). Her theoretical framework provides an excellent frame of reference for considering other studies on the teaching and learning of 'derivative'. One can use it to analyze the development of the students' concept images. Also one can also use it as a guide to teaching the derivative concept.

In this literature review, I will discuss the studies and some teaching experiments that have given insight into the students' concept image of the derivative concept.

Graphical and Symbolical representations

Orton (1983), Ubuz (2001), Amit and Vinner (1990) are among the researchers who conducted studies on students' misconceptions of and errors involving the derivative concept. Orton (1983) did a similar research to Zandieh, but he was mainly looking at students' errors, which he classified as structural, arbitrary and executive errors. He describes structural errors as those errors that are due to a student's failure to grasp some basic concepts. Arbitrary errors are errors whereby a student's answer does not show any logic and they disregard information given to them. Executive errors are errors in manipulation. Orton (1983) conducted a study on 110 students. The students were given tasks which in retrospect can be seen to involve the three layers of Zandieh's framework. He used the tasks to identify the errors and he classified them into different categories. Some of the errors were said to be structural, for example, when students were required to apply the idea of the ratio $\frac{\textit{difference in } y}{\textit{difference in } x}$ to rate of change at a point, and the students could not apply this idea. Orton classified this error as structural, but if one looks at this through Zandieh's framework, one realizes that the students were still operating at the ratio layer and that is why they could not answer the question which required them to take the limit. His results show that most students did not have the limit layer in their concept images because they could not answer the question which involved instantaneous rate of change but Orton called this a structural error which fails to identify the aspect of the concept which had not been grasped. Some errors were classified as executive while others were classified as arbitrary, but still this does not provide insights into the underlying structures of students' concept images.

Thompson (1994) in his research introduced the derivative concept physically to his students. He did this by using average speed and instantaneous speed. When one looks through Zandieh's framework, one notices that this would be the ratio and the limit layers. His research focused on symbolic representation. He states that his students thought of the difference quotient as the derivative and they chose to write it as

$$\frac{f(x+h)-f(x)}{(x+h)-x}.$$

He also mentions that his students expressed the instantaneous rate of change in the same way and this is referred to as the first layer of the derivative concept by Zandieh (1997). His intentions were to orient his students to thinking of instantaneous velocity as a limit of average velocities, but he was surprised when his students seemed to all agree that 'instant' did not really mean instant 'but an amount so small that it was virtually indistinguishable from zero.' This reflects the absence of the limit layer in the students' concept images.

Orton (1983) and Thompson (1994) completed similar studies. Their students had difficulties in finding instantaneous rate of change. While Orton called this a structural error, Thompson described his students as not having schemes for rate and average rate of change but when one uses Zandieh's framework, one notices that the students did not have the limit layer in their concept images. Both Orton and Thompson's results show that the symbolic representation was not very strong in their students' concept images. Thompson mentions that his students used notation blindly and interpreted notation in a way that suited them, while when Orton's students showed misunderstandings that involved symbols, he referred to these errors as fundamentally structural errors.

Orton acknowledges that it would not be easy to use the same classifications of errors in more advanced and older students because it would be difficult to distinguish between arbitrary, structural or executive errors. These results provide us with insights into the learners' concept image on symbolic representation of the derivative at the ratio and limit layers. However, they do not show the consistencies and inconsistencies in the representations because they mainly focus on symbolic representation and they both do not deal with the derivative as a function (Zandieh's third layer).

Amit and Vinner (1990) and Ubuz (2001) also report on studies with similar results. In Ubuz's study, students thought that the derivative at a point gives the function of the derivative and that the tangent equation was the derivative function. Amit and Vinner's research was based on the ratio and limit layers using graphical and symbolic representations. Their results show how a student in their case study ends up confusing the formula of the derivative with the equation of a tangent at a certain point. They point out that this may be due to several possibilities, one of them being that of the typical tangent drawing that is mostly used to introduce the derivative concept. They point out that this visual image makes it easy for a student to confuse the derivative with the tangent. The other possibility mentioned by Amit and Vinner is that of the language used to define the derivative. The derivative is defined by many as the slope of the graph of a function at a certain point. Their argument is that this is confusing to a student. However, these results give us insight in the students' concept image of the derivative concept at the ratio and limit layers using graphical and symbolic representation.

Aspinwall et al (1997) conducted a case study on one student, focusing on the derivative function and this was based on graphical and symbolic representations of the derivative function. The student in this study had instrumental understanding of the derivative because he could differentiate functions and explain what he did and why with ease. When this student was given a graph of a parabola, and asked to draw the graph of its derivative, he did it incorrectly because the picture of a parabola was so strong in his mind and prevented him from drawing the correct graph of the derivative. This study revealed the disadvantages of vivid and uncontrollable mental images that end up being a hindrance in constructing meaning for a concept. Just like the other researchers that I have already mentioned, Aspinwall and co-researchers focused on one layer of the derivative concept and one way of representing the layer.

Just as Aspinwall et al, Tall and Blackett (1986) had students investigate the derivative function by means of graphical representation, they used guided discovery, so that the students could find out the graphs of derivatives of different functions. Their results show that the students ended up being very good at sketching graphs of derivatives of functions. The results also show that the students developed a dynamic concept image of the derivative and were able to sketch some of the graphs in a single procedure. When one looks at this study using Zandieh's framework, one notices that the students had the function layer in the concept image that had strengthened over time. While these results give us insight of the students' concept image at the function layer, using graphical representation, they do not give us insight at the ratio or limit layers.

Bezuidenhout's (1997) research was based on students' understanding of rate of change. His results show that a lot of the students equated the idea of average with finding the arithmetic mean. These students had the idea of average dominant in their minds, and they used it to tackle the problems involving rate of change. This result gives us insight on the students' concept image of the derivative at the ratio layer only within the symbolic representation.

Tall (1986) conducted a research similar to Amit and Vinner. His research was based on graphical representation of the derivative. The students were exposed to numerous of examples and non-examples of finding the gradient and drawing the tangent of a graph at a point. By exposing the students to numerous examples and non-examples, Tall was aiming at developing a rich concept image so that the students would be capable of responding responsibly to new situations and know boundaries of concept. The students were given a test which tested their knowledge of gradient at a point and drawing a tangent at a point. The results show that most students showed insight into problems and managed them with ease. The students in this research were working at the function layer with graphical representation. They also had instrumental understanding in their concept images because they could find the derivative of a function.

In summary, these studies have highlighted common aspects of the concept images of students, and in particular of how these concept images may have deviated from the concept as it is accepted by mathematicians. This proves most useful in relating to students' concept images, gaining a better understanding of their origin, and applying this knowledge in teaching. However, all these studies have only addressed certain aspects of the concept of the derivative. Thus, they have failed to give a comprehensive picture of the full extent of these concept images for students – partially because their focus has been another, partially because their starting point was not a comprehensive look at the concept of the derivative.

The role of Technology

In the following paragraphs, I will discuss some of the very few studies that have been conducted with the use of technology.

Heid's research (1988) was on two groups of students. One group was introduced to the derivative concept by means of graphical and symbol manipulation computer programs. The other group was introduced to the derivative concept by means of traditional methods. Her results show that in both groups the students were not operating at the limit layer because they defined the derivative as the approximation for the slope of a tangent, but not as equal to the slope of the actual tangent. It seems the limit layer causes many difficulties. This makes sense, since the limit layer works shifts from the approach students have been used to from arithmetic and algebra; for instance, equality is now 'redefined': $A = B$ when $\forall\delta > 0, |A - B| < \delta$. (See also the discussion in Chapter 7.)

Most studies conducted on understanding rate of change has focused on students. Castro (2000) focuses on a teacher's conceptual understanding. Despite the fact that the teacher in the study had been teaching for eleven years, he had viewed rate of change mainly as a numeric quantity and had difficulty discussing its use as a physical quantity (velocity). As a result, the learners' opportunities for learning were limited, because the teacher's actions affect the learners' learning.

The teacher in this study went through a number of workshops with activities that included content exploration and the introduction of technological tools such as motion detectors. His conceptual development was monitored by means of interviews, and was also reflected in his classroom practice. He ended up being able to discuss with his students the difference between distance time graphs and velocity and time graphs, also the rate of change being a physical quantity such as velocity. The study showed how teachers can be influenced by their own experiences and that teachers need to experience the concepts as learners before using the activities with the learners. If the teacher has a strong concept image they are more likely to find it enjoyable to design suitable activities for their learners.

Ubuz (2002) and Giraldo et al (2003) used computers in their studies. In Ubuz's study which involved written tasks and interviews, the results show that the students were able to state the difference between the derivative at a point and the derivative of a function. Using Zandieh's framework one realizes that the students were working at the ratio and limit layers. Ubuz acknowledges that although the students in her study reacted positively to the idea of using computers, there is not enough time for both computers and the learning of calculus. Giraldo et al (2003) focused on the pedagogical role of the computer's natural limitations on the development of the learner's concept image of the derivative concept. They argue just like Vinner and Dreyfus (1989) that each representation gives emphasis to certain aspects of the concept. Hence they say "Teaching the concept of the derivative must comprise of different approaches and representations to enable learners build up multiple and flexible connections of the cognitive units and therefore, a rich concept image."(2003: 2).

Their study involved using theoretical-computation conflict to enrich the students' concept image of the derivative. Previous research had shown the negative effects caused by the misuse of computational environments in the learning of Mathematical concepts. Six undergraduate students were chosen for this study.

Their results show that students were affected differently and therefore revealed a wide range of concept images. Students were given a graphical representation of a function on a computer as well as a symbolical representation of the same function, and they were asked to find the derivative of the function. One student could see that the function was differentiable symbolically and therefore visualized the important ideas without looking at the computer, or try to change the graphic window in order to be certain. Another student could see that the function could be differentiated symbolically, but he zoomed into the diagram to confirm this. Thus he realized the conflict and zoomed in to have a broader understanding of the graph. The third student realized that the graph as seen on the screen did not match with the given symbolic expression and then gave up because he did not know what to do. The fourth student carried out a deeper exploration of the graph but ended up choosing the symbolic expression. The last two students showed no interest on the computer. They trusted the symbolic expression completely. The computer seemed to be irrelevant to them and they avoided dealing with it.

The results of this study reveal that theoretical-computational conflict plays noticeably different roles for different students. Some will look first at the picture on the screen; others will focus first on symbolic representation. For some students the conflict may immediately be resolved, for others it may be barely noticed. For some students the conflict may stimulate a deeper exploration leading to correct answers for some and incorrect answers for others. This also shows that encouraging the building of a rich concept image of the derivative happens in various ways with different students.

This compares well to Zandieh's study. Her results (2000) show that the nine students in her study had a wide variety of representational preferences. These students had different concept images of the derivative concept and these included the basic representations that focuses on rate and slope as a way of describing the derivative. The results also show that there is no hierarchy in the layers; this means that the students may learn the layers in any order. The results show that the top learners had a complete understanding of the formal definition of the derivative including all layers across representations of the formal definition, while the other learners had memorized it and could not explain it when asked to do so. There were two learners who did not see the relevance of the definition and hence refused to learn it (p.120).The results also showed the presents of pseudo-objects in the students' concept images. Her results also show that most of the students did not see the connections among the layers except for one who managed to link two layers in her explanation of the limiting process (p.123).

Using Zandieh's framework, my research will focus on all three layers of the derivative concept and the multiple representations. Zandieh brings out an important point about our personal feeling leading us to think that there must be something similar within the representations, which makes them integrate in what seems to be one concept. Her research instrument did not include looking for connections across representations and this seems to be a shortcoming reflected in her research results. This is why my research will focus on connections between representations.

Summary

Researcher/article	Main findings	Relate to Zandieh	Fall short
Orton (1983)	Errors classified as structural, executive and arbitrary	Ratio and limit layers using graphical and symbolic representations	No insight in function layer and not all representations were dealt with. Could not be used in more advanced students.
Thompson (1994)	Students in his research had no schemes for rate of change	Ratio and limit layers using symbolic and physical representation	No insight in function layer and not all representations were dealt with
Aspinwall et al (1997)	Strong vivid images hinder concept formation	Function layer using graphical and symbolic representations	
Tall and Blacket (1986)	Students developed dynamic concept images at function layer using graphical representation	Function layer using graphical representation	No insight in the ratio and limit layers and other representations

Bezouidnohout – correct spelling! (1997)	Students had the idea of average dominant in their concept images and this prevented them from tackling problems involving rate of change correctly	Ratio layer using verbal and symbolic representations	No insight in the other representations and the limit and function layers
Amit and Vinner (1990)	Students confuse the formula of the derivative with the equation of a tangent at a certain point	Ratio and limit layers using graphical and symbolic representations	No insight in function layer and other representations
Giraldo et al (2003)	Students who were exposed to computational environment showed a wide range of concept images because they were affected differently by the computer and by the intended theoretical-computational conflict	Function layer using graphical and symbolic representations	No insight in ratio and limit layers
Tall (1986)	Students can develop a strong and rich concept image by exposing them to a variety of examples and non-examples	Ratio and limit layers using graphical and symbolic representations	No insight in function layer
Heid (1988)	Students were describing the derivative as an	Limit layer using graphical representation	No insight in ratio and function layers

	approximation of the slope		and other representations

Table 1

In this survey, I have looked at the studies that have dealt with different aspects of the derivative concept using different representations. It is noticeable that very few studies have dealt with the derivative at all the three layers using different representations. My study will deal with the derivative at all the three layers using various representations to gain insight of the students' concept image of the derivative in the South African context. My intention is to show connections between the representations within the layers of the derivative. Of all these studies, only the Bezuidenheut one was done with South African students. Thus, there is a clear need to gain insight into the concept images of South African teachers in particular, as this will inform our understanding of and actions on the teaching and learning of calculus in both schools and teacher education programmes.

CHAPTER FIVE: METHODOLOGY

Paradigm

I will be working within an interpretivist paradigm because I want to understand how students understand the derivative concept, while I acknowledge that my understanding will always be tentative, open to interpretation and dependant on my methodology.

Working within the interpretivist paradigm, one's aim is to understand and describe people's actions, since people create meaning and are constantly making sense of their environment. Within this paradigm, one wants to describe how people construct meaning and how they sustain it. The interviewee (in this case the ACE students), have concept images that they have constructed, experienced and shared and which will be elicited as they interact with the researcher (compare to the discussion in Guba and Lincoln (1989)). Therefore, the qualitative method of analysing the data will be used so as to gain insight into the student's concept images. This will be achieved by interviewing the students because each and every person understands and makes meaning of experiences differently from others. Through interacting with their lecturer and classmates, the students have created meaning and made sense of the derivative concept.

Therefore, I assume that students have concept images which do not change substantially because they are being asked about them. Also that what the students say during the interviews, can be assumed to reflect their concept images reasonably – though I do acknowledge that it is possible that only parts of their concept images will be elicited during the interview.

I will use a case study approach and my unit of analysis will be the ACE students at the University of Kwazulu-Natal at the Pietermaritzburg campus. This was chosen as being the most convenient. However, the students attending the ACE in Pietermaritzburg come from a wide range of backgrounds, though mostly historically disadvantaged due to Apartheid's unequal educational system. In that sense, there is not reason to believe that this group of students will be markedly different in their concept knowledge from students at other higher education institutions across South Africa.

All ethical guidelines were followed. The study is in no way harmful to the participants. I obtained permission from these students' lecturer to interview them and the students signed a consent form agreeing to be interviewed (see appendix A). The students were also given a letter explaining the research because they have the right to know why I wanted to interview them. A pseudonym was used for each of the interviewee.

Interviewees

The interviewees were 5 educators in the in-service ACE programme[2] at University of KwaZulu-Natal (referred to as 'the ACE students', and 2 educators not in the programme).

[2] The ACE programme is a retraining and upgrading in-service programme for teachers. It is offered in a range of specializations, one of which is mathematics for FET level (grade 10-12). At University of KwaZulu-Natal, the programme is a two year part time programme, with a total of eight modules. Two modules are general education modules, four are content modules, and two are specialized education modules- in this case mathematics education. The students were in their third semester of the programme.

The ACE students in this research had all done matric[3] to obtain their school leaving certificates. Two of them had been taught introductory calculus at high school and the other three had not, so there might be differences in the development of their concept images. All five students completed a three year Teaching Diploma at different colleges of education, where they learnt differential calculus and they all specialised in the teaching of mathematics. Two of the students have never taught calculus since they completed their diploma. During the ACE programme, they were all reintroduced to differential calculus, especially the derivative concept.

I randomly selected students with the only criteria that there would be at least one high performing, one medium performing, and one low performing student. More than five students agreed to participate, but failed to make themselves available for interviewing, despite repeated prompting, and I ended with a sample of 2 high performing, one medium performing, and 2 low performing students.

I conducted three open ended interviews with each student. These students were interviewed at the Pietermaritzburg campus of University of KwaZulu-Natal. The interviews were recorded and the tapes were transcribed.

[3] For a student to have completed matric, they would have done five years of high school with a minimum of six subjects which includes English and one other of the eleven official languages of South Africa. It is worth mentioning that the teachers in the ACE had all completed their matric, because there are many underqualified teachers practicing in South African schools who do not have this school leaving certificate. However, the admission criteria for the ACE are such that it would be highly unlikely for a student not to have completed matric as well as some higher education.

The two educators outside of the ACE programme who took part in this research are practicing educators who are qualified. They both have a BSc in Mathematics and an HDE from the University of Natal. The two educators are teaching at a historically advantaged school. Megan has been teaching for more than ten years, while Deanne has been teaching for two years.

Table 2 below gives an overview of the interviewees' background and current situation.[4]

Pseudonym of Interviewee	Did Calculus in Matric?	Type of school where matriculated	Did calculus at college?	Number of years teaching	Teach matric (grade 12)?	Number of years teaching calculus
Matthew	Yes	Ex Model C	Yes	Five	Yes	Three
Themba	No	Village School	Yes	Ten	Yes	Ten
Nompilo	Yes	Township School	Yes	Five	No	None
Ayanda	No	Mission School	Yes	Six	No	None
Mandisa	No	Township School	Yes	Ten	Yes	None
Megan	Yes	Ex Model C	Yes	More than 10	Yes	More than 10
Deanne	Yes	Ex Model C	Yes	Two	Yes	One

Table 2

[4] An 'Ex-Model C' school is a school that during Apartheid catered for White students, and thus was more advantaged in terms of both financial and human resources. These schools are often still better off than many of their counterparts in the townships, with their better building and equipment and more financially affluent parent body which can then contribute financially to the running of the school. Thus, these schools are often more capable of attracting the best qualified teachers. Many parents from historically disadvantaged background choose to send their children to Ex-Model C schools in the hope that the advantage will benefit them. Thus, the learner body has become racially diversified, while in many cases the educator body remains dominantly White.

Data Collection Methods

Three interviews were conducted with each one of the students. Matthew's first interview was done one week before his exams, the second interview was done just after the exams and the last interview was done four months after the second interview. For the other four students, the first interview was done halfway into the course and the second interview at the end of the course just before the exams. The third interview was done four months after the second interview. The interviews were recorded and transcribed.

Name of Student	Interview One	Interview Two	Interview Three
Matthew	30 June 2005	8 July 2005	12 November 2005
Themba	30 April 2005	2 July 2005	12 November 2005
Nompilo	30 April 2005	2 July 2005	26 November 2005
Ayanda	30 April 2005	2 July 2005	8 December 2005
Mandisa	30 April 2005	2 July 2005	8 December 2005

Table 3

The two educators were interviewed once, each, as there was no reason to assume that their concept images would change drastically over time.

During the interviews, the students and educators were asked open-ended questions to allow them to provide information about their understanding of the derivative concept. Choosing suitable questions to ask the students was a big challenge. The question, "What is a derivative?" seemed to be threatening to the students because it may be perceived as an assessment question. Thus, it could have limited the possible explanations the students could have given, which is why it was only asked at the end of the interview. Hence the question "What comes to your mind when I say derivative?" was used, as it also directly searches for concept image rather than concept definition. During their answer, where necessary, I followed up with prompts. Examples were asked for, where I deemed it necessary or useful in obtaining more insight into the concept images. Questions like, "What else comes to your mind?", "Tell me more about that"," How are all the parts of the formula connected?" were asked as deliberately vague follow-up prompts, so as to limit the possibility to which a desired type of answer would be read into the question by the interviewees.

Methodological Issues

The fear of asking leading questions in some cases prevented me from probing more, and hence had limited the information I got from the students. Questions like, "Is a derivative of a function also a function?" was a leading question because students could answer in the affirmative, yet not have the function layer as part of their concept image. "Would you like to use a diagram?" was also a leading question because I would be putting ideas in their head. Instead, "Would you like to give an example?" was asked as a follow up question to an explanation. "What is instantaneous rate of change?" was too much of a leading question because rate of change is one of the representations of the derivative. So this question was only asked if the student mentioned instantaneous rate of change. The danger of probing too much made me hold back, at times resulting in not getting more information from the students, and as a result finding it difficult to code the data accurately. One example of this is during Nompilo's second interview:

Botshiwe:	What else comes to your mind?
Nompilo:	The formula of the derivative comes to my mind.
Botshiwe:	Tell me more about the formula.
Nompilo:	It is (she writes $\frac{dy}{dx}$) we use it to find the derivative.
Botshiwe:	What else comes to your mind when I say derivative?
Nompilo:	The chain rule and implicit differentiation

I should have probed more and asked how the formula was used to find the derivative and also asked her to explain more about the chain rule and implicit differentiation. As it is, I cannot tell whether Nompilo is just reciting these words or she has instrumental or even conceptual understanding. However, there were many occasions where following up on a statement made by the interviewee allowed me to gain more information without being leading.

There was also a danger of students feeling threatened by the interview, since they did not know exactly what questions I was going to ask them. After the first interview, Mandisa said that she was so nervous it felt like she was going to write an exam. This could have negatively impacted on the results of the interview in that she was not able to answer the questions freely and hence not bringing out her concept image.

In terms of wanting to compare the students' concept images, the time of the interviews presents a problem. Four students had their first interview on the same day at different times. Two months later, Matthew managed to meet his appointment for the first interview, and this was at the end of the module so his results of the first interview cannot be compared to the other four students because their first interview was midway during the module.

Each interview was by appointment so there is a possibility that the students could have prepared for the interview and conveyed the information learned by heart rather than their own concept image. It is also possible that the students conferred with each other, affecting the results of the interviews.

Analysis Instrument

I will use the qualitative method of analyzing the data. Zandieh's instrument (see below) will be used, with three additions: (a) instrumental understanding has been added as a separate category, (b) arrows will be added if there connections among the representations are revealed in the interviews, (c) a non-layer has been added as a separate layer if the students' responses cannot be classified in any of the three layers, the ratio layer, limit layer and function layer. Below, I discuss how my first engagement with the interviews lead me to change the instrument in these respects.

REPRESENTATION						Instrumental understanding
Process Object Layer	**Graphical**	**Verba l**	**Paradig-matic/ Physical**	**Symbolic**	**Other**	
	Slope	Rate	Velocity			
Ratio **Limit** **Function** **Non-layer**						

Figure 1

Following Zandieh, there will be shaded circles that will denote a complete understanding of the representation and the process involved. There will also be un-shaded circles that will denote pseudo-structural understanding. Instrumental understanding is characterized by the interviewee not showing an understanding of reasons behind a rule, and thus it will always be illustrated by an un-shaded circle. The arrows from one representation to the other will show connections, if there are any.

The need to expand the instrument

While attempting to analyze the data, it was evident that some of the students' responses could not be analyzed using Zandieh's theoretical framework and instrument. The following are examples, which illustrates why I had to expand the theoretical framework and the instrument.

Botshiwe:	What comes to your mind when I say derivative?
Nompilo:	The skill and technique of how to find the derivative
Botshiwe:	What else comes to your mind?
Nompilo:	It is challenging to me, but I know that if I have (she writes $y = x^2$) I must say $\frac{dy}{dx} = 2x$

This is an example that illustrates instrumental understanding of the derivative concept, which is not in Zandieh's framework and instrument. Nompilo knows how to find the derivative but she cannot describe the derivative, and she does not give reasons for the rule. I felt it was unduly representing the student as having learned nothing from her experiences with the derivative to simply leave out this aspect.

The following example shows why the non-layer had to be added.

Botshiwe:	I want to know what comes to your mind when I say derivative.
Ayanda:	Velocity. Because I know that in Physics, when we talk about acceleration, if you want to find the derivative of that you get velocity.

Ayanda knows that the derivative has something to do with velocity. It is not clear if Ayanda refers to the velocity at a given time (limit layer, pseudo-structural) or at any time (function layer, pseudo-structural).

Thus, this response cannot be classified in any of the layers of Zandieh's framework, which is why the non-layer had to be added. It is a way of recognizing that a relevant representation was evoked, while it was not possible to identify the relevant layer(s).

Botshiwe:	What comes to your mind when I say derivative?
Nompilo:	Slope or steepness of a graph.
Botshiwe:	Tell me more about the slope.
Nompilo:	The slope is $\frac{y_2 - y_1}{x_2 - x_1}$ and this is the gradient. But the gradient at a point is $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.
Botshiwe:	In the formula for gradient at a point, how the are different parts connected to each other?
Nompilo:	$f(x+h) - f(x)$ is change in y and h is change in x. The limit helps us to find the gradient at a point.

This shows the connections between the graphical and symbolic representations in the ratio layer as well as the limit layer. She knows that the derivative is the slope at a point which is represented symbolically as $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$. Generally, we would consider a concept image stronger if it is has such connections between representations (or to other concepts). Therefore, I have fore-grounded these connections.

CHAPTER SIX: ANALYSIS - CONCEPT IMAGES OF INDIVIDUAL INTERVIEWEES

In this chapter I am going to analyze the data using the qualitative tool outlined in the previous chapter. I will be looking at the concept images of each student because I want see what layers and representations of the derivative are exhibited by each student in their concept images.

Nompilo's concept images in the three interviews

Nompilo attended a township high school. She had no Mathematics teacher in grade 12. She used to go to the University for Mathematics lessons on Saturdays. She did her teacher training course at Adams College of Education (historically black and thus disadvantaged[5]). While at college, she did Calculus. She has been teaching the junior classes for the past five years. She failed her calculus module in the ACE programme.

Nompilo Interview 1: 30 April 2005

Botshiwe:	What comes to your mind when I say derivative?
Nompilo:	The skill and technique of how to find the derivative
Botshiwe:	What else comes to your mind?
Nompilo:	It is challenging to me, but I know that if I have (she writes $y = x^2$) I must say $\frac{dy}{dx} = 2x$
Botshiwe:	Is there a situation where one could apply this?
Nompilo:	I don't know but in the Calculus book I have seen that they use it in Geography and Economics.
Botshiwe:	I want to know what comes to your mind when I say derivative.
Nompilo:	Finding the derivative of $\log x$ and $\ln x$ and many more.

[5] Not only were the historically black higher education institution financially disadvantaged, the curricula were also different from that at the White counter-parts.

REPRESENTATION					
Process Object Layer	**Graphi-cal**	**Verbal**	**Paradig-matic/ Physical**	**Symbolic**	**Instrume ntal understa nding**
	Slope	Rate	Velocity		
Ratio					
Limit					
Function					
Non-Layer					

Figure 2

Figure 2 illustrates that Nompilo exhibits instrumental understanding of the derivative concept only. What comes to her mind is what to do in order to find the derivative of a function. At this stage her concept image of the derivative does not appear to contain any of the representations or layers.

Interview two 2 July 2005

Botshiwe: When I say derivative what comes to your mind?
Nompilo: Slope comes to my mind.
Bosthiwe: Is there a situation where we can use this?
Nompilo: We use the derivative to find the slope of a graph.
Botshiwe: What else comes to your mind?
Nompilo: The formula of the derivative comes to my mind.
Botshiwe: Tell me more about the formula
Nompilo: It is (she writes $\frac{dy}{dx}$), we use it to find the derivative.
Botshiwe: What else comes to your mind when I say derivative?
Nompilo: The chain rule and implicit differentiation.
Botshiwe: Anything else?
Nompilo: Nothing.

REPRESENTATION					
Process Object Layer	**Graphical**	**Verbal**	**Paradigmatic/ Physical**	**Symbolic**	Instrumental understanding
	Slope	Rate	Velocity	Difference Quotient	
Ratio					
Limit					
Function					
Non-layer					

Figure 3

By placing an un-shaded circle in the slope column and the non-layer row, the diagram reflects that Nompilo describes the derivative as slope. If she had described the slope as a ratio, the circle would have been in the ratio row. An circle with a question mark is in the instrumental understanding column because Nompilo mentions that the formula comes to her mind, but it is not clear if Nompilo is able to use the formula.

<u>Interview Three 26 November 2005</u>

Botshiwe: What comes to your mind when I say derivative?

Nompilo: Slope or steepness of a graph.

Botshiwe: Tell me more about the slope.

Nompilo: The slope is $\frac{y_2 - y_1}{x_2 - x_1}$ and this is the gradient. But the gradient at a point is $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

Botshiwe: In the formula for gradient at a point, how are the different parts connected to each other?

Nompilo: $f(x+h) - f(x)$ is change in y and h is change in x. The limit helps us to find the gradient at a point.

Botshiwe:	Is there any situation where we can use the slope in real life?
Nompilo:	There is but I don't remember.
Botshiwe:	Is there anything else that comes to your mind about the derivative?
Nompilo:	I know we find the derivatives in Trigonometry.
Botshiwe:	How do you do that?
Nompilo:	I don't remember.
Botshiwe:	Sum up and tell me what is a derivative?
Nompilo:	A derivative tells us about the slope of a function.

REPRESENTATION					
Process Object Layer	**Graphi-cal**	**Verbal**	**Paradig-matic/ Physical**	**Symbolic**	Instrumen tal understan ding
	Slope	Rate	Velocity	Diffe-rence Quotient	
Ratio	●			●	
Limit **Function**	○			○	
Non-layer					

Figure 4

The shaded circle in the ratio row and slope column indicates that Nompilo describes the derivative as a slope which is change in y divided by change in x which means that she has the ratio layer which is represented graphically in her concept image. The un-shaded circle in the slope column and limit row indicates that this layer of her concept image is pseudo-structural; she knows it involves the limit but does not explain therefore she has a pseudo-structural representation of the limit layer in her concept image. The un-shaded circle in the limit row and the difference quotient column indicates that Nompilo has symbolic representation of slope in her concept image but this is pseudo-structural.

She knows the limit is involved but does not explain how. Nompilo explains the calculations that are involved in the symbolic expression of the derivative at the ratio layer, that is why the circle in the ratio row and difference quotient is shaded.

The arrows indicate that she makes connections between the graphical representation and the symbolic representation, because she describes the derivative as a slope and then gives the symbolic representation of the slope at the ratio layer. The arrows at the limit layer show that there is a connection graphical representation and the symbolic representation, because she describes the derivative as the gradient at a point and then gives the symbolic representation of the gradient at a point.

Development in Nompilo's concept image

While looking at the results of Nompilo's three interviews one notices that generally the slope is the main representation in her concept image because during the second and third interviews she mentions the slope first. The ratio layer seems to be dominant in her concept image. Nompilo's concept image did not seem to change during the module but seems to have been strengthened after the module. The results show her responses during the interview. While it is possible that she could have the function layer and other representations of the derivative as part of her concept image and not exhibited them during the interview, I find this unlikely given Nompilo's stated difficulties with recalling anything else of relevance.

Mandisa's three interviews

Mandisa Interview 1: 30 April 2005

Mandisa attended a township high school. She did not do Calculus at school. She trained as a teacher at Indumiso College of Education (historically black) and calculus was part of the course. She has been teaching for the past nine years. She has been teaching grade 12 for ten years and she finds it difficult to teach calculus so her colleagues teach calculus to her classes. She passed the ACE module on Calculus.

Botshiwe:	When I say derivative what comes to your mind?
Mandisa:	The limit comes to my mind. Derivative is finding the limit of a function.
Botshiwe:	Tell me more about it.
Mandisa:	What I know is that when you have the value of x approaching something, then the function also approaches it.
Botshiwe:	Is there any situation where we can apply this?
Mandisa:	I don't know.
Botshiwe:	What else comes to your mind when I say derivative?
Mandisa:	Nothing.

	REPRESENTATION				
Process Object Layer	**Graphi-cal**	**Verbal**	**Paradig-matic/ Physical**	**Symbolic**	Instrumen tal understan ding
	Slope	Rate	Velocity	Diffe-rence Quotient	
Ratio					
Limit					
Function					
Non-layer					

Figure 5

The results of Mandisa's first interview cannot be classified in diagram because they do not show any concept image of the derivative. She seems to conflate the notion of the derivative with limits of functions as in $\lim_{x \to 5} f(x)$.

Mandisa Interview 2: 2 July 2005

Botshiwe: When I say derivative what comes to your mind?

Mandisa: Slope comes to my mind.

Botshiwe: Is there a situation where we can apply this?

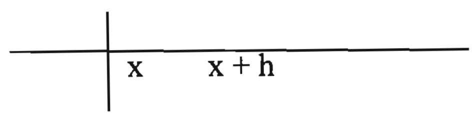
Mandisa: I can't think of any real life situation.

Botshiwe: What else comes to your mind when I say derivative?

Mandisa: The limit comes to my mind.

Botshiwe: So the limit comes to your mind?

Mandisa: Yes. The limit of a function is when the value of x approaches a certain number. For example (she draws a diagram and says she cannot remember the rest of the diagram) to find the gradient we say



$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$ and we use the formula $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Botshiwe: How is the formula connected to your diagram?

Mandisa: This is the formula for finding the derivative, which is the slope at a point.

Botshiwe: I want to know what comes to your mind when I say derivative.

Mandisa: Nothing else comes to my mind.

Mandisa does mention the gradient, but uses symbolic notation which is standard for finding the gradient of a given straight line, and does not link that to the derivative. Therefore the un-shaded circle in the ratio row and symbolic column (see diagram next page). The circle would have been shaded had she explained the process involved. The un-shaded circle in the non-layer row and the slope column indicates that Mandisa has an idea that the derivative has something to do with the slope. Mandisa mentions the slope at a point, but does not explain it, hence the un-shaded circle in the limit row and the slope column. She also gives the formula for derivative but does not explain the calculations involve that is why there is an un-shaded circle in the limit row and difference quotient column.

REPRESENTATION					
Process Object Layer	**Graphi-cal**	**Verbal**	**Paradig-matic/ Physical**	**Symbolic**	Instrumen tal understan ding
	Slope	Rate	Velocity	Diffe-rence Quotient	
Ratio				○	
Limit				○	
Function					
Non-layer	○				

Figure 6

At the time of the second interview, Mandisa's concept image mainly contained pseudo-structural element, but she showed connections between the graphical and symbolic representations. This is evident when she states a formula and mentions that it is the formula for finding the slope at a point. However, she still seems to confuse the limit of a function and the derivative, as the diagram she draws was used in the teaching of the former.

Interview three 9 December 2005

Botshiwe: What comes to your mind when I say derivative?
Mandisa: Limits and slope.
Botshiwe: Tell me more about the slope.

Mandisa:	The slope is the change in y over the change in x. So the derivative is the change in y over the change in x at a certain point.
Botshiwe:	Where in real life can we use the slope?
Mandisa:	In building bridges but I don't know how.
Botshiwe:	You also mentioned limits. Tell me more.
Mandisa:	I just know that limits have something to do with the derivative but I don't know more.
Botshiwe:	Is there anything else that comes to your mind about the derivative?
Mandisa:	If I am asked to find the derivative of $2x^2$, I know it is 4x because we use the rules of differentiation.
Botshiwe:	Tell me about these rules.
Mandisa:	I know that when you are finding the derivative, the exponent multiplies the coefficient of the variable then you subtract one from the exponent to get the new exponent.
Botshiwe:	What else comes to your mind?
Mandisa:	Nothing.

	REPRESENTATION				
Process Object Layer	**Graphi-cal**	**Verbal**	**Paradig-matic/ Physical**	**Symbolic**	Instrumental understanding
	Slope	Rate	Velocity	Diffe-rence Quotient	
Ratio	●				
Limit	○				○
Function					
Non-layer					

Figure 7

The shaded circle in the ratio row and slope column illustrates that Mandisa describes the derivative as the slope and goes on to mention that it is change in y divided by change in x. The un-shaded circle in the limit row and slope column show that she has pseudo-structural conception of the derivative, at the limit layer because she only mentions that limits have something to do with the derivative but does not explain it further. Mandisa also has instrumental understanding because she mentions that she knows how to find the derivative of a (polynomial) function but clearly treats it as a rule unrelated to the other aspects of the derivative she mentions.

The development in Mandisa's concept image

While looking at Mandisa's results, I notice that the slope is her main representation of the derivative concept which gets strengthened towards the end of the module. Although she is a high performing student, her understanding is mainly pseudo-structural. The results of the third interview show that she also has instrumental understanding in her concept image. From these results I can only comment on what Mandisa exhibits during the interview, she could have had more representations and more layers of the derivative concept in her concept image which she did not reveal during the interviews. Still, her concept image appears rather weak, which does highlight a potential short-coming of the ACE module; it appears possible to pass the course with reasonably good results without having a comprehensive concept image of the derivative.

Ayanda's concept image in the three interviews

Ayanda went to a mission school, she did not learn calculus at high school. She trained as a teacher at Esikhawini College of Education (historically black) where she did calculus as part of the course. She has been teaching Physics for the past five years, and has never taught Mathematics. Ayanda is one of the low performing students. At the time of the interview, she was repeating the calculus module and she failed the module for the second time.

Ayanda Interview 1: 30 April 2005

Botshiwe: When I say derivative what comes to your mind?

Ayanda: You mean generally?

Botshiwe: Yes when I say derivative what comes to your mind?

Ayanda: Maybe if you can make an example, maybe tell me to give a derivative of this, then I can answer.

Botshiwe: If I say derivative, I just want to know what comes to your mind.

Ayanda: I think of the formula, or the thing that I need to do to find the derivative of a sum. Otherwise, its very abstract to me. I can't link it with anything.

Botshiwe: Tell me more about the formula you just mentioned.

Ayanda: If the sum is (she writes $3x^4$) and if I have to find the derivative, I know that this is what I have to do (she writes $12x^3$).

Botshiwe: Is there anything else that comes to your mind about the derivative?

Ayanda: I remember when my lecturer was explaining and trying to link the derivative to everyday life, but now I can't relate to anything.

	REPRESENTATION				
Process Object Layer	**Graphical**	**Verbal**	**Paradigmatic/ Physical**	**Symbolic**	Instrumental understanding
	Slope	Rate	Velocity	Difference Quotient	
Ratio					
Limit					
Function					
Non-layer					

Figure 8

The results of Ayanda's first interview show that she has instrumental understanding of the derivative concept only. At this stage her concept image could not be classified in any of the layers.

Ayanda Interview two: 2 July 2005

Botshiwe: What comes to your mind when I say derivative?

Ayanda: The first thing that comes to my mind is that I picture a certain problem and I know what I need to do, in order to find the derivative of that particular problem.

Botshiwe: What kind of problem do you picture?

Ayanda: A problem like (she writes $y = 2x + 2$). I know that the derivative is 2.

Botshiwe: Where can one apply this kind of example?

Ayanda: Like if we have a graph, I understand that it has something to do with the steepness of the graph. It is the slope of the graph.

Botshiwe: So the derivative is the slope of a graph?

Ayanda: Yes.

Botshiwe: Is there a situation where we can use the derivative to find the slope?

Ayanda: I am not sure, but I am just thinking of something in motion. Like a car going on a steep hill.

Botshiwe: So how do we use the derivative in this situation?

Ayanda: To calculate.

Botshiwe: To calculate what?

Ayanda: The motion of this car, but I don't know how.

Botshiwe: I want to know what comes to your mind when I say derivative.

Ayanda: Velocity. Because I know that in Physics, when we talk about acceleration, if you want to find the derivative of that you get velocity.

Botshiwe: Is there anything else that comes to mind about the derivative?

Ayanda: Nothing.

REPRESENTATION					
Process Object Layer	**Graphical**	**Verbal**	**Paradigmatic/ Physical**	**Symbolic**	Instrumental understanding
	Slope	Rate	Velocity	Difference Quotient	
Ratio					○
Limit					
Function					
Non-layer	○		○		

Figure 9

The circle in the instrumental understanding column show that Ayanda knows how to find the derivative of a particular (type of) function but she does not explain the process involved or link it to other aspects of the concept. The un-shaded circle in the non-layer row and slope column show that she knows that the derivative has something to do with slope but does not indicate that ratio or limits are involved. The un-shaded circle in the non-layer row and velocity column show that she knows that the derivative has something to do with velocity though her linking of acceleration and velocity is incorrect. At the time of the second interview her concept image of the derivative was mainly pseudo-structural and very vague. Ayanda might have had the ratio, limit and function layers as well as more representations of the derivative in her concept image which she did not exhibit during this interview, but given her self-expressed problems with grasping the concept, I find this unlikely.

Ayanda Interview three: 9 December 2005

Botshiwe:	When I say derivative what comes to your mind?
Ayanda:	The slope comes to my mind.
Botshiwe:	Tell me more about the slope.
Ayanda:	The slope of a graph is change in y over change in x.
Botshiwe:	Is there a situation where we can use the slope?
Ayanda:	When we are calculating velocity we say distance divided by time.
Botshiwe:	Tell me more about velocity.
Ayanda:	Velocity is distance divided by time.
Botshiwe:	What else comes to your mind about the derivative?
Ayanda:	Nothing.

The shaded circle in the ratio row and slope column in the diagram on the next page illustrates that Ayanda describes the derivative as slope and goes on to explain that it is change in y divided by change in x. Also the shaded circle in the ratio row and velocity column indicates that she describes the derivative as velocity which she explains as distance divided by time. In that respect, Ayanda really only exhibits the layer of the derivative concept which it has in common with average rate of change; she does not appear to have made the shift to recognizing the fundamental characteristics of the derivative as a limiting concept.

			REPRESENTATION		
Process Object Layer	**Graphi-cal**	**Verbal**	**Paradig-matic/ Physical**	**Symbolic**	Instrumen tal understan ding
	Slope	Rate	Velocity	Diffe-rence Quotient	
Ratio	●		●		
Limit					
Function					
Non-layer					

Figure 10

Development in Ayanda's concept image

From Ayanda's three interviews, it seems that the slope is her main representation of the derivative concept because she mentions it first. Velocity is mentioned after being prompted. Her two representations are strengthened towards the end of the module, but only insofar as she manages to clarify the ratio layer – so in general she seems to be lacking the notion of the derivative as anything different from average rate of change. It is noticeable that instrumental understanding seems to be less prevalent in her concept image after completion of the course.

Themba's concept image in the three interviews

Themba went to a village school. He did not do calculus at high school. He went to (the historically black and thus disadvantaged) Indumiso Technicon where he trained as a teacher. He did calculus as part of the course. He has been teaching for the past nine years. He has taught grade 12 for six years. He says he struggles to teach calculus. Themba passed the ACE calculus module.

Themba Interview one: 30 April 2005

Botshiwe:	When I say derivative, what comes to your mind?
Themba:	According to my understanding, the derivative gives us the gradient at a certain point.
Botshiwe:	Tell me of a situation where one can use the derivative to find the gradient.
Themba:	I think when there are constructing roads, and also in the manufacturing of cars, it's also about slopes.
Botshiwe:	How do they use the derivative?
Themba:	When they are working out the aerodynamics of a car, they have to work out the exact slopes so that no friction occurs.
Botshiwe:	I want to know what comes to your mind when I say derivative.
Themba:	The derivative starts with average gradient between two points and we end up having to find the gradient at a point. Because it's normally used to find the gradient of a straight line but the derivative helps us to find the gradient of a curve.
Botshiwe:	How is the average gradient connected to the gradient at a point?
Themba:	It is connected when we use the limits.
Botshiwe:	Tell me more about this.
Themba:	The limit is when we approach a certain point and try to find the gradient at that point.
Botshiwe:	To sum up, tell me what is meant by derivative?
Themba:	The derivative is the limit of a function as we approach a certain point.

REPRESENTATION					
Process Object Layer	**Graphi-cal**	**Verbal**	**Paradig-matic/ Physical**	**Symbolic**	Instrumen tal understan ding
	Slope	Rate	Velocity	Diffe-rence Quotient	
Ratio	○				
Limit	●				
Function					
Non-layer					

Figure 11

In figure 11, the un-shaded circle indicates Themba's description of the derivative as the average gradient between two points. The shaded circle shows that he describes the derivative as the gradient at a certain point explaining the process involved. He exhibits the ratio layer pseudo-structurally and the limit layer. Both layers are represented graphically. These are the results of what he says during the interview but he might have more representations in his concept image which he does not exhibit during the interview.

<u>Themba Interview two: 2 July 2005</u>

Botshiwe: When I say derivative, what comes to your mind?

Themba: Rate of change, since differentiation is about finding the gradient at a certain point.

Botshiwe: Is there a situation where we can use rate of change?

Themba: Normally with curve graphs we can find the gradient at a certain point. For example, if we have the graph of (he writes $y = x^2$) we can differentiate that and when given the coordinates of a point, we can find the gradient at that point. With rate of change we are able to calculate acceleration and velocity.

Botshiwe: What else comes to your mind when I say derivative?

Themba:	It is about what you get when you differentiate.
Botshiwe:	Tell me more about that.
Themba:	It's either you differentiate to get gradient, or you can differentiate to calculate the cost of a number of items or you can differentiate to get acceleration. I can't remember the other one.
Botshiwe:	Is there anything else that comes to your mind when I say derivative?
Themba:	The rules of differentiation. The product rule, the sum and difference rule, and the product rule.

REPRESENTATION					
Process Object Layer	**Graphi-cal**	**Verbal**	**Paradig-matic/ Physical**	**Symbolic**	Instrumental understanding
	Slope	Rate	Velocity	Diffe-rence Quotient	
Ratio					
Limit					
Function					
Non-layer					

Figure 12

Themba mentions that rate of change comes to his mind, but does not explain further, that is why there is the un-shaded circle in the non layer row and rate column. The un-shaded circle in the non layer row and velocity column is a result of him mentioning that with rate of change we are able to calculate velocity and acceleration without explaining further. He also mentions that differentiation is about finding the gradient at a certain point, but does not explain how, hence the un-shaded circle in the limit row and slope column. Themba exhibits instrumental understanding of the derivative, this is evident when he says that the derivative is what you get when you differentiate. All the un-shaded circles in this diagram indicate that the concept image demonstrated in the second interview was pseudo-structural.

Although he exhibits a pseudo-structural understanding, Themba demonstrates connections among the representations. The connection between the slope and rate representations as well as instrumental understanding, is shown when he says, "Rate of change, since differentiation is about finding the gradient at a certain point." He also exhibits a connection between the rate and velocity representations as he mentions them as related.

Themba Interview 3: 12 November 2005

This interview was done four months after the second interview, and thus quite a long time after the exam.

Botshiwe:	What comes to your mind when I say derivative?
Themba:	Derivative is about rate of change. When we speak about graphs, the derivative is the gradient at a certain point, rather than getting average gradient.
Botshiwe:	Tell me more about rate of change.
Themba:	Suppose we have a graph of a parabola, normally we can get the gradient between two points by drawing a straight line and finding the gradient of this line which gives us the average gradient of the parabola between these two points. But the derivative helps us get the gradient of the parabola at any point.
Botshiwe:	Is there any real life situation where we can apply the gradient of a parabola?
Themba:	In real life we normally speak about maximum and minimum.
Botshiwe:	Tell me more about maximum and minimum.
Themba:	Suppose we want to construct a cube, when given limited material and we want to use the material to its maximum, we then use some calculations, using the derivative to find the dimensions of the cube that we want to construct.
Botshiwe:	Is there anything else you would like to tell me about the derivative?
Themba:	In teaching before we go to the derivative we start with limits.

Botshiwe: How are limits connected to the derivative?

Themba: The limit of a function, tells us what is happening to a function when we are approaching a certain point either from the left hand side or from the right hand side. We are interested to what is happening when we are getting closer and closer to that point.

Botshiwe: How is all this related to the derivative?

Themba: Let's say we have $\lim\limits_{x\to 2}\dfrac{x^2-4}{x-2}$ we factorise the numerator and get $\lim\limits_{x\to 2}\dfrac{(x-2)(x+2)}{x-2}$ and this will give us –4. So I would like to believe that when we differentiate, $\lim\limits_{x\to 2}\dfrac{x^2-4}{x-2}$, we get –4.

Bothiwe: Tell me, what is a derivative?

Themba: The derivative is what we get when we differentiate a function or an equation.

Botshiwe: How do we differentiate?

Themba: We use the rules of differentiating, the quotient rule, sum and difference rule and the chain rule.

Botshiwe: Sum up what you just told me about the derivative.

Themba: To get the derivative you need to know the rules of differentiation and the derivative is used in real life situations to get the maximum and minimum.

		REPRESENTATION			
Process Object Layer	**Graphical**	**Verbal**	**Paradigmatic/ Physical**	**Symbolic**	Instrumental understanding
	Slope	Rate	Velocity	Difference Quotient	
Ratio		○			
Limit	○				○
Function					
Non-layer					

Figure 13

In figure 13, the un-shaded circle in the ratio row and rate column show that Themba knows that the derivative has something to do with rate. The un-shaded circle in the limit row and slope column illustrates his description of the derivative as the gradient at a certain point. Themba exhibits instrumental understanding since he mentions that the derivative is what you get when you differentiate but without describing how this relates to the previously mentioned aspects of the derivative. Themba mentions rate and slope together, which indicates a connection between the two representations.

Developments in Themba's concept image

While looking at the results of analyzing Themba's interview statements, one notices that his concept image is mainly pseudo-structural and the main representation is the slope, since he mentions in all the three interviews. During the first interview he has two layers in his concept images, a very strong limit layer and a pseudo-structural ratio layer both in graphical representation. During the second interview his limit layer appears to have weakened but he exhibits more representations of the limit and ratio layers. His third interview shows a weak concept image of the limit and ratio layers with pseudo-structural graphical and verbal representations as well as instrumental understanding. This is what he exhibited during the interviews, he might have had more layers and representations in his concept image but did not reveal them during the interviews.

Comparing Themba's interview with those of Nompilo, Mandisa and Ayanda, he appears more confident, which is likely to make one read more understanding into his explanations. However, careful analysis using the given framework helps to clarify his concept image, which, though slightly more inclusive than those of the previous students, does not include many representations and layers and certainly does not engage the processes involved in these aspects.

Matthew's concept image in the three interviews

Matthew Interview one: 30 June 2005

Matthew went to a Model C (historically White, thus advantaged) school and he did calculus in grade 12. He trained as a teacher at (the historically White) Edgewood college of Education and calculus was part of the course. He majored in Mathematics and Physics. He has been teaching for three years, has taught calculus for two years and feels comfortable teaching calculus.

Botshiwe: When I say derivative what comes to your mind?

Matthew: Lots of things, like rate of change. The derivative is the gradient of a tangent. It is a ratio as well, because it is the rate of change of one thing over another.

Botshiwe: Tell me more about the derivative being a ratio.

Matthew: Well, the derivative when you look at it specifically in Physics, it is the rate of change. In Mathematics, we can say $\frac{dy}{dx}$; it doesn't mean very much to anybody but they can see it's a ratio. In chemistry, concentration would be the number of change in moles over the number of change in liquid and stuff like that. So it's a comparison between two rates of change.

Botshiwe: Is there anything else that comes to mind when I say derivative?

Matthew: It is applied very often in engineering. We can also use it to work out how fast you have taken, or how long you have taken to travel a distance. You can work out your velocity and acceleration by taking the derivative and the double derivative. You start with the distance graph. The gradient of the graph gives you the velocity, and the gradient of velocity gives you acceleration.

Botshiwe: What else comes to your mind?

Matthew: I don't know, I can't remember. Oh, we use the derivatives in trigonometry.

Botshiwe: Sum up and tell me what is a derivative?

Matthew: The derivative is a ratio, it is a rate of change, it is velocity and acceleration and it is a gradient of a tangent.

REPRESENTATION					
Process Object Layer	**Graphi-cal**	**Verbal**	**Paradig-matic/ Physical**	**Symbolic**	Instrumen tal understan ding
	Slope	Rate	Velocity	Diffe-rence Quotient	
Ratio		●			
Limit	○		○		
Function					
Non-layer		○	○		

Figure 14

From these results it is evident that Matthew's concept image of the derivative, as it is reflected in the interview, is mostly pseudo-structural. That is why the circles are not filled in. The limit layer in his concept image is not a process but an object because he says "the derivative is the gradient of a tangent" and does not explain any further. The circle in the limit row and slope column would have been closed if he had explained the gradient as the limit of a series of average gradients. Matthew has the non- layer with velocity representation in his concept image and this is illustrated when he mentions that "you can use the derivative to work out how fast you have travelled a distance", that is why the circle in the non-layer row and the velocity column is un-shaded because he does not explain further. The circle would have been shaded had he explained how velocity is calculated. He also has a strong ratio layer with verbal representation in his concept image, which is why the circle in the ratio row and rate column is shaded.

It is possible that Matthew might have had other representations as well as the function layer in his concept image, but chose not to explain them. So I can only comment on what he exhibited during the interview.

Matthew: Interview two: 8 July 2005

At the time of this interview, Mathew had just written his exams.

Botshiwe: What comes to your mind when I say derivative?

Matthew: Rate of change. The derivative is the change in y over the change in x. It is the gradient of a tangent. Derivative is velocity. When we look at the distance thing, velocity is the gradient of the distance graph. It is instantaneous rate of change.

Botshiwe: Tell me more about instantaneous rate of change.

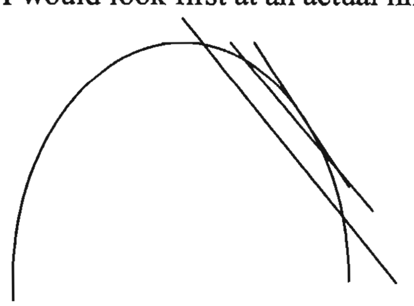
Matthew: We look at rate of change over a period of time. When we look at instantaneous rate of change, we want to know how that graph is changing at that particular moment. … We want to know what is the velocity at that moment.

Botshiwe: You said something about change in y over change in x.

Matthew: Yes. Change in y over change in x gives us the gradient of a graph. When we look at the derivative, we are looking at the change in y over change in x at a specific point.

Botshiwe: How would you explain derivative to a learner who doesn't know anything about it?

Matthew: I would look first at an actual hill (he draws)



We can see that the steepness is not the same throughout the hill. Then we look at how we can measure the steepness at this one point. And then we come up with the idea that if we get a tangent, we can calculate exactly how steep it is. So if we have a line and bring it closer and closer to the tangent until we finally get a tangent we can calculate the steepness of a hill at that particular point.

Botshiwe: Sum up and tell me what is a derivative?

Matthew: A derivative is a function that tells us how another function changes. It explores the gradient of that function at all points. Sometimes it is a constant function. The gradient is rate of change at a particular moment.

REPRESENTATION					
Process Object Layer	**Graphical**	**Verbal**	**Paradigmatic/ Physical**	**Symbolic**	Instrumental understanding
	Slope	Rate	Velocity	Difference Quotient	
Ratio	●				
Limit	●	○	○		
Function	●	○			
Non-layer					

Figure 15

The shaded circle in the ratio row and the slope column indicates that at the time, he had the ratio layer with graphical representation in his concept image and this is illustrated when he said, "Change in y over change in x gives us the gradient of a graph." He also describes the derivative as instantaneous rate of change at a specific moment. The circle in the limit row and rate of change column would have been shaded had he explained the underlying process. The un-shaded circle in the limit row and velocity column indicates that Matthew describes the derivative as instantaneous velocity. Matthew explains how the derivative is the gradient of the tangent using an example of a steep hill, and this shows clearly that the limit layer in his concept image is a process not a pseudo-object, which is why the circle in the limit row and the slope column is shaded. He describes the derivative as a function which means he has the function layer in his concept image. The circle is shaded because he explains fully that the derivative gives the gradient of a function at all different points.

The strength of Matthew's concept image at the time of the second interview lies not only in that he evoked all layers and three out of four representations, but in the clear links he makes between the representations.

<u>*Matthew Interview 3:12 November 2005*</u>

This interview was done four months after the second interview.

Botshiwe:	When I say derivative, what comes to your mind?
Matthew:	Derivative is the rate of change, it is the speed at which things change, usually when we talk about it we are looking at velocity or something like that, at how quickly the velocity increases or decreases so we are looking at acceleration. For instance, if you have a distance-time graph you can work out the velocity and acceleration by taking the derivative twice. So a derivative is the slope of a graph.
Botshiwe:	You have mentioned a lot of things here, one of them is velocity. Tell me more about it.
Matthew:	Velocity is rate of change in itself. You normally start with the distance time graph, and that tells you how far you go in a certain amount of time, then your velocity time graph tells you how fast you are going. And then you have your acceleration which tells you how quickly your speed is changing. The velocity tells you how quickly the distance will change and acceleration tells you how quickly velocity will change.
Botshiwe:	You also mentioned the slope of a graph. Tell me about it.
Matthew:	For the slope of graph, if we look at the gradient equation $\frac{y_2 - y_1}{x_2 - x_1}$, its actually the change in y divided by the change in x. So the slope of a graph is actually looking at the change, when we look at the derivative, we are looking at instantaneous change, the change at a specific moment. It's not so much the change from half way up the graph. It is the change at one specific point.
Botshiwe:	You mentioned rate of change, tell me more about it.

Matthew: Instantaneous rate of change, if you are looking at a distance time graph, the gradient of a distance time graph is velocity. Well, the derivative of it is velocity which tells you that if you look at one particular instant, it tells you how quickly the distance is changing at that point. If you look at the velocity graph at that exact point, it tells you how quickly the velocity is changing at that point which is the acceleration.

Botshiwe: O.K. Is there anything else that comes to your mind abut the derivative?

Matthew: You can use the derivative when calculating population growth. You get the derivative from something it tells you about another situation.

Botshiwe: I want to know what comes to your mind when I say derivative.

Matthew: Nothing. But I can tell you how to get the derivative.

Botshiwe: Please tell me.

Matthew: O.K to get the derivative, well, the method of getting the derivative which is not quite the same as how the derivative is derived.

Botshiwe: What is the difference between the two?

Matthew: How the derivative is derived relates to the gradient equation again. Because you have limits, you are approaching something and you want to see how the graph is behaving as you get closer to a point and how you do that is you look at $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ so basically you are looking at the difference between the two divided by h. This is the formula for finding the derivative using the first principles, and it comes from the gradient equation initially.

Botshiwe: How are all these parts of the formula connected to the derivative?

Matthew: Well, $f(x+h) - f(x)$ is the change in y and h is the change in x.

Botshiwe: How is the limit part connected to the derivative?

Matthew: What the limit is, is if you look at this equation $f(x+h) - f(x)$, if you let h = 0, you will have something minus itself and it won't give you an answer. So what we do is we see how it behaves as it gets closer to that point, where we can't actually work out, because it cancels itself out and that's when we take the limit.

Botshiwe: You said there is also a method of finding the derivative.

Matthew: Yes, the short method of finding the derivative which was used by doing lots of these $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ you see that there is a pattern, is you take the exponent of the variable and you multiply it by the coefficient of the variable and then you subtract one from the exponent, for example, the derivative of $2x^3$ is $6x^2$.

Botshiwe: Lets go back to $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$. How is it related to velocity and acceleration?

Matthew: Yes, because if you take this formula, ignore the limit for now $\frac{f(x+h)-f(x)}{h}$ this will be the change in distance divided by the change in time and that will give you velocity. If you have change in velocity divided by change in time you get acceleration.

Botshiwe: How does the limit fit in?

Matthew: The limit allows us to calculate the instantaneous rate of change. If we just use change in distance divided by change in time, we get the average velocity. But with the limit we can work out the velocity at a specific moment.

Botshiwe: To sum up tell me what is a derivative?

Matthew: The derivative is rate of change. It allows us to calculate the slope of a graph at a certain point, and that graph can come from a number of things or models from real life.

The shaded circle in the ratio row and slope column illustrates that Matthew describes the derivative as the slope and explains that it is change in y divided by change in x.(see next page). He also describes the derivative as velocity, which is given by change in distance divided by change in time. That is why there is a shaded circle in the ratio row and the velocity column. The shaded circles in the ratio row and the difference quotient column as well as the limit row and difference quotient column indicate that Matthew gives the formula of the derivative and explains the processes that are involved. He also describes the derivative as the slope at a specific point and explains how the limit is involved, hence the shaded circle in the limit row and the slope column. Matthew mentions that the derivative is instantaneous rate of change; the circle in the limit row and rate of change column would have been shaded had he explained more about the process of finding the limit. The shaded circle in the limit row and the velocity column illustrates that Matthew describes how the limit is involved in finding instantaneous velocity. The circle in the instrumental understanding column indicates that he has instrumental understanding in his concept image. He can actually explain the processes underlying the standard rule for derivatives of polynomial functions, therefore the shaded circle in this column.

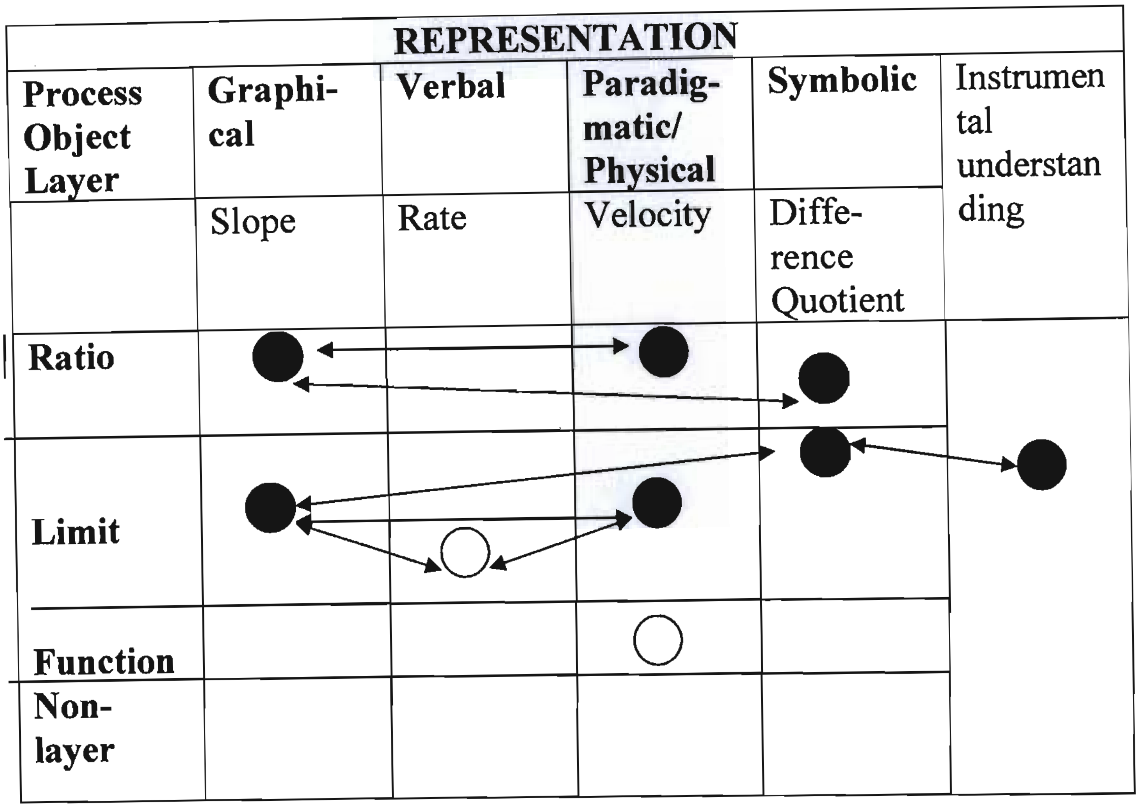


Figure 16

Matthew's concept image is spread across all the representations and he has connections between representations. Matthew describes the derivative as the slope and gives the formula for the derivative which shows that he has a connection between the graphical and symbolic representations at the ratio layer. He also describes the derivative as slope at a specific point and explains using the formula which is an indication of a connection between the graphical and symbolic representations. The connection between the physical and symbolic representations is illustrated when he mentions that velocity is change in distance divided by change in time. The connection between verbal and physical representation is evident when he mentions that velocity is rate of change. He also mentions that the short method of finding the derivative is a result of a pattern that comes from performing lots of manipulations using $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$. This shows that in this case he made a connection between the symbolic representation and instrumental understanding.

While looking at the analysis of Matthew's statements, one notices that he has a strong concept image that is spread across all representations. The graphical, verbal and physical are his main representations because he mentions them in all three interviews. The symbolic representation and instrumental understanding are mentioned in the last interview. It is evident that he has a connection between the graphical and verbal representations at the limit layer when he mentions that the slope is rate of change at a particular moment. He also describes velocity as rate of change and this shows that he has a connection between the verbal and physical representations.

Developments in Matthew's concept image

While looking at Matthew's results one notices that during the first interview, his concept image was mainly pseudo-structural with one connection between the slope representation and the velocity representation. He also had two representations in the non-layer. The second interview shows that his concept image had become more profound at all three layers within the slope representation while the other representations remained pseudo-structural. The second interview also shows more connections among representations. The third interview shows that his concept image had become even stronger. His concept image showed a clear consolidation which is highlighted by the way in which he addresses more underlying processes and in the way he makes links across more representations. Matthew also adds symbolic representation but the function layer seems to be less active in the third interview.

Megan's concept image

Megan is a qualified, experienced educator; she has been teaching mathematics for more than ten years. She did her high schooling at an ex model C school (historically white) and completed her BSc at the University of Rhodesia and a Higher Diploma in Education (HDE) at the University of Natal. She is currently teaching at an ex model C school.

Botshiwe:	What comes to your mind when I say derivative?
Megan:	The derivative is represented by $\frac{dy}{dx}$ which means the difference in y over the difference in x at a specific point, which is the slope at a specific point. The derivative is a gradient function and you can apply it to anything, a straight line, parabola or cubic function. We use it to find the gradient at any specific point. For a straight line, the derivative is a constant because the gradient is the same anywhere along the straight line. But in a curve the derivative has a variable in it and that is why we substitute the specific variable to find the gradient at that point.
Botshiwe:	Where can one use the slope?
Megan:	Well, the most useful place is when you are finding the maximum or minimum area or volume.
Botshiwe:	Is there anything else that comes to your mind about the derivative?
Megan:	Velocity.
Botshiwe:	Tell me more about velocity.
Megan:	Velocity is change in distance divided by change in time, this gives you average velocity. But instantaneous velocity is change in distance divided by change in time at a specific point.

The shaded circle in the ratio row and velocity column (see next page) illustrates that Megan describes the derivative as velocity which is change in distance divided by change in time. She also describes the derivative as difference in y over the difference in x, hence the shaded circle in the ratio row and the difference quotient column. This indicates that her concept image of the ratio layer is strongly represented by the velocity and symbolic representations.

REPRESENTATION					
Process Object Layer	**Graphi-cal**	**Verbal**	**Paradig-matic/ Physical**	**Symbolic**	Instrumental understanding
	Slope	Rate	Velocity	Diffe-rence Quotient	
Ratio			●	●	
Limit	○		○	○	
Function	○				
Non-layer					

Figure 17

The circle in the limit row and the slope column indicates that Megan describes the derivative as the gradient at any specific point, while the un-shaded circle in the limit row and the velocity column illustrates that Megan describes velocity as instantaneous velocity. She also describes the derivative as a gradient function that can be applied to anything. That is why there is a shaded circle in the function row and the slope column.

Though it appears from the diagram constructed on the basis of the interview with Megan that her concept image includes several representations, linked to some extent, and all three layers in at least one representation, it disguises the fundamental misunderstanding of seeing the instantaneous rate of change as a ratio and not as a limit. This is only implied by the un-shaded circles in the limit layer.

Deanne's Concept Image

Deanne is a qualified educator who has been teaching for two years. She has a BSc in mathematics and an HDE, both from the University of Natal. She is currently teaching at an ex model C school.

Botshiwe: When I say derivative what comes to your mind?

Deanne: Slope and rate of change.

Botshiwe: Tell me more about the slope.

Deanne: We use the derivative to find more about the slope. If you know the formula for the slope then you can find out the rate of change whether it's time, it depends on what you are looking at.

Botshiwe: Tell me more about the formula for the slope.

Deanne: The formula I was talking about is something like $f(x)=3x^3-2x^2+x-5$ which is the formula of a graph. Do you want to know the formula of finding the slope of this graph?

Botshiwe: Yes.

Deanne: To find the derivative of $f(x)=3x^3-2x^2+x-5$, initially one would do it from first principles which would be finding the average gradient between two points on the graph. $f(x)=3x^3-2x^2+x-5$ would be at x on the graph, then you would find another point on the graph say f(x+h) . To do that you would substitute $x+h$ into $3x^3-2x^2+x-5$ which means $f(x+h)=3(x+h)^3-2(x+h)^2+(x+h)-5$. To find the derivative you would then find the difference between these two and divide by h. And then we take the limit of that as h tends to zero. This is written as $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$.

Botshiwe: How are the different parts of the formula connected?

Deanne: $f(x+h)-f(x)$ is the difference in y and h is the difference in x.

Botshiwe: How does the limit fit in?

Deanne: The limit fits in because we are looking at average slope between two points and we want to find the slope at a specific point.

Botshiwe: How would one use the slope in real life?

Deanne: If you were trying to model population growth you might have a formula for that population growth and you might want to find what the rate of change of population is over time. So then you would find

	the derivative of that formula and that would let you see at various points in time what the rate of change is when the population was increasing at that stage or decreasing.
Botshiwe:	You mentioned rate of change. Tell me more about it.
Deanne:	OK the rate of change. What we are looking at, is the rate of change between two points and the derivative is the instantaneous rate of change. Which means we are looking at how the function is changing at a specific point.
Botshiwe:	Is there anything else that comes to your mind about the derivative?
Deanne:	Looking at maximum and minimum values in a graph is one of the things that one would use the derivative for. And one does that by looking at local maxima and local minima. When the derivative is zero you will find the local maximum or local minimum in a graph because your instantaneous rate of change becomes zero as the graph reaches a maximum or minimum.
Botshiwe:	What is a derivative?
Deanne:	It is the instantaneous rate of change.

REPRESENTATION					
Process Object Layer	**Graphi-cal**	**Verbal**	**Paradig-matic/ Physical**	**Symbolic**	Instrumental understanding
	Slope	Rate	Velocity	Diffe-rence Quotient	
Ratio	●			●	
Limit	●	●		●	
Function		●			
Non-layer					

Figure 18

Deanne describes the derivative as the slope and gives the formula of finding the slope. That is why there is a shaded circle in the ratio row and slope column. She is able to explain the underlying idea behind the common rule of finding the derivative of a function. This shows that she has a strong ratio layer which is represented graphically in her concept image. She also describes the derivative as rate of change and explains that "it is a formula that would let you see at various points in time what the rate of change is", so she has the function layer in her concept image. Deanne gives the definition of the derivative function and explains the calculations that are involved. This is shown by the shaded circle in the ratio row and the difference quotient column and also the limit row and difference quotient column which shows that she has a strong symbolic representation of the ratio layer in her concept image. The shaded circle in the limit row and slope column illustrates that Deanne explains how the limit is involved in finding the derivative. She also explains instantaneous rate of change, that is why there is a shaded circle in the limit row and rate of change column.

Her concept image of the derivative is spread across all but one representation. Deanne describes the derivative as slope and gives the formula of the slope which shows that she has connections between graphical and symbolic representations at the ratio layer. She also describes the derivative as the slope at a point and explains the process using the formula. This indicates that she has connections between the graphical and symbolic representations at the limit layer. Deanne has connections between the graphical and verbal representations because she mentions that if one knows the formula of the slope they can find rate of change.

From these results, one can conclude that Deanne has a strong concept image of the derivative which is spread across all representations except velocity which does not occur. She might have had velocity in her concept image but chosen not to mention it during the interview. The function layer is least strong in her concept image because it has one representation. These results also show that she has several connections among representations in her concept image.

Overview of the Interview Results

The table shows the layers and representations of the derivative concept, preferred by the interviewees when they were asked, "What comes to your mind when I say derivative?" This offers an overview of the first level of analysis. In the next chapter, I will discuss patterns across the interviews in more detail.

Interviewee	Interview One	Interview Two	Interview Three
Matthew			
Representation	R+S	R+S+V	R+V+S
Dominant Layer	Ratio	ratio/limit	ratio/limit
Themba			
Representation	S	R+S	R+S
Dominant Layer	Limit	Ratio	limit
Mandisa			
Representation	None	S	S
Dominant Layer			
Ayanda			
Representation	I	I	S
Dominant Layer	None	non-layer	ratio
Nompilo			
Representation	I	S	S
Dominant Layer	None	non-layer	ratio
Megan			
Representation	S		
Dominant Layer	ratio/limit		
Deanne			
Representation	S+R		
Dominant Layer	ratio/limit		

Table 4

R= Rate of change S= Slope V= Velocity I= Instrumental understanding

CHAPTER SEVEN: DISCUSSION

Patterns Across Interviewees

Nompilo and Ayanda are low performing students, according to their results in the module. There is a pattern in the progression of their concept images. Their first interviews only showed instrumental understanding of the derivative concept. They show that they have mastered the rule of differentiating polynomial functions without insight into the reasons why this rule works. According to Lithner (2003) a rule can be mastered by students and used in solving problems by mainly identifying surface resemblance with some previous experiences, and this appears to be what these students are doing.

By the second interview, both students still had instrumental understanding but they had added the non-layer in the graphical representation (Nompilo) or for velocity (Ayanda). By the third interview, their instrumental understanding was replaced by the ratio layer which was expressed in the graphical and symbolical representations. Nompilo had added the limit layer where she had pseudo-structural understanding of the graphical and symbolic representations.

All five students at some stage showed that they had instrumental understanding of the derivative concept. It is also evident from the interviews that all five students have the slope as (one of) their main representation(s) of the derivative, while the limit layer is pseudo-structural for most of the interviewees and the function layer rarely exhibited. The analysis of Matthew, Deanne and Megan's results show that they display all the three layers. Unlike Zandieh's students, it appears that these students are more likely to develop the ratio layer before the limit layer, with the function layer as the last aspect to be added.

The students' responses do not illustrate the function layer with the exception of Matthew's second interview which shows that he has a pseudo-structural understanding, although Matthew's function layer is not evident in the third interview.

The results of Mandisa, Nompilo, Ayanda and Themba show consistencies across representations within the layers. This means that in the ratio layer for example, if their understanding is pseudo-structural in one representation, then it will be pseudo-structural in the other representations of the same layer. These four students have one or two representations of a layer. Their representations do not exceed two, while Matthew's interviews show that he has more than two representations of a layer.

In all the three interviews, Ayanda and Themba do not show that they have symbolic representations in their concept images, although Ayanda is repeating the module and Themba is a medium performing student. Mandisa, Ayanda and Nompilo do not have verbal representation in their concept images while Themba and Matthew mainly have pseudo-structural verbal representation.

Matthew shows a steady development of the velocity representation which starts as pseudo-structural in the ratio layer by the first interview.

During the second interview, velocity has become more profound in the ratio layer and pseudo-structural in the limit layer. The third interview shows a strong understanding of the derivative in the ratio and limit layers with velocity as a representation. Themba and Ayanda have a pseudo-structural understanding of the velocity representation in the non-layer during their second interviews, while Mandisa and Nompilo do not have velocity as their representation. These students have been through the same course and the slope has been a strong part of their concept image which has been evoked by the course. Even though this is the case, the students display obvious differences in their concept images. What is needed, however, is more insight into how students' prior knowledge and experiences influence how they interpret their experiences in a calculus class.

Misconceptions and Confusions at the Limit Layer

As mentioned before, a lot of research has indicated that students have difficulties with the limit concept Orton (1983) and Thompson (1994). Zandieh's results show that some of the students in her research found it difficult to explain the limiting process. This is consistent with Mandisa's concept image as she confuses the limit of a function with the limit of the derivative function. This is evident when she says "the limit of a function comes to my mind" in response to the question "What comes to your mind when I say derivative?" And she goes on to say "What I know is that when you have the value of x approaching something, the function also approaches it".

Ayanda's second interview shows that she confuses velocity and acceleration because she mentions that the derivative of acceleration is velocity. This fits with that she does not evoke the verbal representation in her interview; she may not have related the notion of derivate to rate of change all together.

Themba's third interview shows a confusion of the limit of a function and the derivative. This is evident when he says " Lets say we have $\lim_{x \to -2} \frac{x^2 - 4}{x - 2}$, we factorise the numerator and get $\lim_{x \to -2} \frac{(x-2)(x+2)}{x-2}$ and this will give us -4. So I would like to believe that when we differentiate $\lim_{x \to -2} \frac{x^2 - 4}{x - 2}$, we get –4."

Such confusions stand in the way of developing a strong concept image in the limit layer.

But even without such confusions, the limit layer is difficult to come to grips with. When Megan says, "The derivative is represented by $\frac{dy}{dx}$ which means the difference in y over the difference in x at a specific point, which is the slope at a specific point." it appears that she has not comprehended that $\frac{dy}{dx}$ is not a ratio at a specific point – where it would be impossible to have a difference in y and a difference in x – but the result of a limiting process applied to a ratio. Mandisa has the same problem: "So the derivative is the change in y over the change in x at a certain point." As does Matthew in his first interview: "In Mathematics, we can say $\frac{dy}{dx}$; it doesn't mean very much to anybody but they can see it's a ratio."

This is in line with the reflections by Artigue on students' difficulties in calculus. She claims that students need to engage in a reconstruction of their understanding of real numbers to exclude successor relations (for instance, many students see 0.999... as the predecessor of 1, thus not having made the shift to an actual infinity) (Artigue, 2001: 312); and a reconstruction of 'equal' from the successive equivalence of algebra to the ε-proximity – which again affect the modes of reasoning (Artigue, 2001: 215). It is unclear if these aspects are at play in these interviewees' relations to the limit layer, as Zandieh's framework does not capture these underlying aspects, but it is worth further investigation.

Historically Advantaged versus Historically Disadvantaged Students

Mandisa, Themba, Ayanda and Nompilo are historically disadvantaged students while Matthew is a historically advantaged student. Megan and Deanne are the two historically advantaged educators. From their results one notices that Mandisa, Themba, Ayanda and Nompilo have concept images that are mainly in the ratio layer with not more than two representations. Themba's concept image is mainly pseudo-structural throughout the three interviews, while Ayanda, Mandisa and Nompilo's concept images become more profound at the ratio layer by the third interview.

Matthew is a historically advantaged student and is the top student in his class. He has a very strong concept image which is in the ratio and limit layers and – in the third interview – with all representations present. He is able to describe the derivative as a process as well as an object. Megan and Deanne are historically advantaged educators who have concept images with three or more representations in each layer but Megan's limit layer is quite weak as discussed previously.

The question arising from these observations is "How can the historically disadvantaged students improve their concept images?" It is a problem that the course does not manage to level out the differences more or even to bring the historically disadvantaged students' concept image up to the same level as the historically advantaged students. On a positive note, some learning appears to have taken place for all students in the course, but the development of the historically advantaged student's concept image is so marked in comparison to the development of the historically disadvantaged students' concept images that it does raise some doubts about the extent to which the course is successful in assisting these students in making sense of the material and relating it to their previous experiences.

The analysis of Themba and Nompilo's results show that their concept images do not differ significantly from Ayanda and Mandisa's concept images, but their exam results show that they scored passing marks while Ayanda and Mandisa's exam results show that they were failing. These analyses then raise the following questions. (1) Did the exam which was written by these students capture conceptual aspects of calculus or mostly instrumental competency? (2) Was the problem with my interviewing? (3) Is it the problem with the theoretical framework?

I have not engaged in an analysis of the exam papers for the module, so the first question remains open for now.

My interviewing had shortcomings in that I might have missed out on some points raised by the interviewee and hence failed to ask follow-up questions that might have revealed more about their concept image. It can therefore not be said with certainty that it is possible to pass the module with (too) limited conceptual knowledge of the derivative. Themba and Nompilo's concept images showed more representations and more connections between representations evoked, and perhaps this is the crucial difference.

Critical Reflections on the Theoretical Framework

Following from the last point, the theoretical framework may fall short in that it does not give a clear indication of what is sufficient knowledge. Although it was expanded, the theoretical framework has its limitations in that it only captures what the students can do, but not what they cannot do. Many learners and educators have procedural conception of the derivative and, as a result, the theoretical framework falls short in that it will only capture the concept image that falls into the layers.

A problem with the framework becomes evident for example, when one looks at Megan's interview results. It is not clear how to distinguish the misconception of Megan, that the instantaneous rate of change is also a ratio, from the ratio layer of the derivative when she says "The derivative is represented by $\frac{dy}{dx}$ which means the difference in y over the difference in x at a specific point, which is the slope at a specific point.

CHAPTER EIGHT: CONCLUSION

The aim of the research is to gain insight into the ACE students' concept image, and to do so I will answer the following questions.

1. What layers of the derivative concept do teachers exhibit when asked?
    - Which layers are present in the students' concept definition?
    - What are consistencies and inconsistencies across representations within the layers of the derivative?
2. Do students understand both the process and object nature of each layer?

<u>What layers of the derivative concept do teachers exhibit when asked?</u>

When one looks at Nompilo's results, one notices that by the first interview there was no layer in her concept image, she only had instrumental understanding, which is why I had to expand the theoretical framework. During the second interview she had a pseudo-structural understanding of the non-layer. Finally by the third interview her concept image was more profound at the ratio layer because she was able to describe the derivative as a process. Her concept image at the limit layer was pseudo-structural. This shows that Nompilo has the ratio layer in her concept image.

Mandisa's concept image at the first interview could not be classified in any of the layers or representations. By the second interview her concept image was pseudo-structural at the ratio and limit layers. By the third interview her concept image had grown stronger because she could describe the derivative as a process at the ratio layer, but remained pseudo-structural at the limit layer.

The analysis of Ayanda's interviews shows that on the first interview, Ayanda's concept image could not be classified under any of the layers but she had instrumental understanding; at this point she knew how to find a derivative of a function using the short method. This means that she is working on surface resemblance only. All she knows is the procedure that she needs to follow in order to find the derivative. This is evident when she says "I think of a formula, or the thing that I need to do to find the derivative. Otherwise its very abstract to me". By the second interview, she had pseudo-structural concept images at the ratio and non-layers. Her concept image became more profound at the ratio layer by the third interview. She could describe the derivative as a process.

In all the three interviews, Themba's concept image was pseudo-structural on both the ratio and limit layers with more representations on the ratio layer.

It is clear the four students, Themba, Ayanda, Mandisa and Nompilo have difficulties with the derivative concept at the limit layer. This is similar to Zandieh's results which show that some of the students in her research had problems with the limit layer especially explaining the limiting process. A lot of research has indicated that students have difficulties with limits. Orton's (1983) research show that students had difficulties with the derivative at the limit layer, which he refers to as structural errors. Bezuidenhout (2001) mentions that "Students' failure to express meaningful ideas on the limit concept may to a large extent be due to inappropriate and weak mental links between knowledge of limit and knowledge of other calculus concepts" (p 487). This statements is consistent with the analysis of the above mentioned students. Their limit layer is pseudo-structural. Bezuidenhout's study shows that a large number of students lack conceptual understanding of the limit concept.

During the first interview, Matthew's concept image was pseudo-structural on the ratio and limit layers. By the second interview, it was more profound at the ratio layer and still mostly pseudo-structural at the limit layer. The analysis of Matthew's third interview show that at the time, his concept image had become very strong because he could describe the derivative as a process at the ratio and limit layers using all the representations, which is consistent with Zandieh's results because, just like Zandieh's students, Matthew was able to build a comprehensive concept image of the derivative, which included basic representations like rate and slope.

Megan's results show that she has a strong ratio layer which is represented physically (velocity) as well as symbolically. Although her limit layer is pseudo-structural, it has more representations which are graphical, physical and symbolical representations. She also has the function layer which is represented graphically.

All three layers in Deanne's concept image are very profound. The ratio layer is represented graphically and symbolically, while the limit layer has the graphical, verbal and symbolic representations. The function layer is represented graphically.

From these results one notices that Matthew was the only student who worked at the ratio, limit and function layers. Also the two teachers Deanne and Megan worked at all three layers of the derivative concept. The other four students in this research Themba, Nompilo, Mandisa and Ayanda had concept images that were more profound at the ratio layer while their concept image at the limit layer is mostly pseudo-structural.

Generally, it appears likely that most ACE students will come with a very limited or no conceptual understanding of the derivative and will require substantial support and facilitation to develop both ratio and limit layers.

Which layers are present in their concept definition?

Nompilo, Mandisa and Ayanda describe the derivative as the slope which is given by change in y divided by change in x. This shows that the ratio layer is present in their concept definition. Themba describes the derivative as the "Limit of a function as we approach a certain point." This shows that he has no layer present in his concept definition. In line with Lithner (2003), it appears that Themba is working on surface resemblance only, and the fact that the limit occurs both places has captured his attention. This ends up standing in the way of him practicing constructive reasoning. Matthew on the other hand has the ratio and limit layers in his concept definition because he describes the derivative as "Rate of change and slope of a graph at a certain point."

Megan has the ratio layer in her concept definition because she describes the derivative as change in y over change in x, while Deanne has the limit layer because she describes the derivative as instantaneous rate of change. These results except for Themba, are consistent with Zandieh's results, whose students had the slope as a representation of the ratio layer in their concept definition.

What are the consistencies and inconsistencies across representations within the layers?

When a student exhibits consistencies across representations within the layers, it means that representations in a layer have to be all pseudo-structural or they all have to be profound. If a representation is profound and the others are pseudo-structural in the same layer then there are inconsistencies in that layer.

Nompilo's third interview shows a consistency between the graphical and symbolic representations within the ratio layer. She describes the derivative as a process in both representations in this layer. She also has two representations within the limit layer where she has a pseudo-structural understanding of the derivative concept of the graphical and symbolic representations which means there are consistencies across representations within the layers of her concept image.

Mandisa's second interview shows that there are consistencies across representations within the ratio and limit layers. Her concept image of the derivative concept in the ratio layer is pseudo-structural of both the graphical and symbolic representations. It is also pseudo-structural of both the graphical and symbolic representations at the limit layer. Her third interview also shows that there is pseudo-structural understanding of the graphical and symbolic representations in the limit layer. So one can conclude that there are consistencies in Mandisa's understanding of the derivative across representations within the layers of the derivative concept which are pseudo-structural.

Ayanda's second interview shows that she has a pseudo-structural understanding of the derivative at the ratio layer which is represented graphically as well as symbolically. Her third interview shows that her concept image is still in the same layer and representations, the only difference is that she describes the derivative as a process in both. This shows that there are consistencies across representations within the ratio layer, because one representation is profound while the other is pseudo-structural.

Themba's concept image of the derivative is pseudo-structural. His second interview shows that there is a consistency in the ratio layer of the verbal and physical representations.

Matthew's first interview does not show consistencies because in the ratio layer he describes the derivative as a process using verbal representation which means his verbal representation is profound. But his graphical and physical representations of the same layer are pseudo-structural. His second interview shows that there is a consistency within the ratio layer across graphical and verbal representations because he describes the derivative as a process using both representations. But in the limit layer, there are inconsistencies because he understands the derivative as a process using graphical representation, while his concept image of the verbal and physical representations are pseudo-structural. The third interview shows that there is a consistency within the ratio layer because he understands the derivative as a process using three representations which are graphical, physical and symbolic. There are inconsistencies in the limit layer because he understands the derivative as a process when using graphical, physical and symbolic representations while pseudo-structural when using verbal representation in the same layer.

Megan's results show that there are consistencies within the ratio layer because her physical and symbolic representations are both profound as she describes the derivative as a process using both representations. There are also consistencies within the limit layer because her graphical, physical and symbolic representations are pseudo-structural.

Deanne' results show that there are consistencies in the ratio layer across the graphical and symbolic representations. The two representations are profound as she describes the derivative as a process using both representations. She has consistencies in her limit layer because the derivative is described as a process using the graphical, verbal and symbolic representations.

As one looks at these results, one realizes that it is possible that students only explain the process underlying a concept in one representation, while still being aware of the process in the other representations. However, it appears that, in general, concept images are consistent within layers. The exception is where a student shows a strong preference for one or more representations, explaining processes in these representations and making links to other representations, as is the case with Matthew.

Do students have both the process and object nature of each layer in their concept image?

From the results of Mandisa and Themba one notices that these students' concept image does not exhibit both the process and object nature of each layer. Their concept image is mainly pseudo-structural in all layers, and even when prompted to explain further, they fail to do so. But Nompilo and Ayanda's third interviews show that their concept images exhibit both the process and object nature of the ratio layer, because they are able to explain the process involved in this layer. Matthew's interviews show that his concept image exhibits both the process and object nature of each layer because he is able to explain the process involved in the ratio layer just as the students in Zandieh's research were able to explain the process involved in all the layers except for a few students who had problems with the limit layer.

From the results of the five students it is noticeable that their concept images improved substantially from the first interview to the third interview with the main representation being the slope. The only exception is Matthew whose concept image improved by the second interview, and became even stronger by the third interview with slope, rate of change and velocity as his main representations.

In summary, it is clear that even upon completion of the ACE module on calculus, many practicing teachers have concept images of the derivative which are not encompassing all the layers and more than one or two representations. With the function layer absent, it can be difficult to make sense of maximization and minimization tasks. With the limit layer absent or pseudo-structural, the concept itself and the essence of calculus escapes the teachers – and therefore also will be out of reach of our learners.

APPENDIX A

Merrivale
3291

Dear University of KwaZulu-Natal PMB Campus A.C.E. Students 2005

Re: Consent for participation in the Masters research project

This letter is to inform you about my Masters research project that involves a case study of how students understand the derivative concept. The aim of this project is to find out how students learn the derivative concept, so that the lecturers and teachers can be able to design their activities so that the students can understand better.

You have been selected as possible research participants because I live in PMB and you are the only A.C.E students group nearest to me. This letter formally invites you as a specialist Mathematics student teacher to participate in the project.

Your participation will involve:

1. A first meeting with me at which an overview of the research will be explained.
2. At this meeting, you will be provided with a consent form that you will be asked to complete.
3. You will be interviewed on three occasions, in April, June and after the exams. The interviews will be taped. The questions will be based on what

you will be learning about the derivative. If however you will feel uncomfortable during the interview, the interview will be stopped immediately. The interview will last at least 30 minutes per person.

The interview will be strictly confidential. Your decision to participate or not to participate will not affect your marks in anyway. If you participate, your lecturer will not have access to the recorded interview. You will not be paid for participating in the project. Your real names will not be used.

A more formal consent form is attached to this letter. If you do agree to participate, please complete the form and return it to me.

Yours sincerely

Botshiwe Likwambe
Masters student University of KwaZulu-Natal PMB
Phone 033 3307358

CONSENT FORM

I...(*please print your full name)* as an A.C.E student specialising in Mathematics, I am aware of the data collection process in the research project as listed in the information letter above.

I give consent to being interviewed at various agreed upon times from April 2005 to after the June exams and having these interviews taped and transcribed.

YES / NO (*please circle the correct one*)

I am aware that the data collected will be used in a research project focused the understanding of the derivative concept.

I know that all the information provided and used in the research report will not be connected to me personally and my name will not be used. Full confidentiality will be adhered to and a suitable pseudonym, selected in consultation with me will be used to identify my contribution to the report.

Signed..

Date

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