AN ALGEBRAIC STUDY OF RESIDUATED ORDERED MONOIDS AND LOGICS WITHOUT EXCHANGE AND CONTRACTION

by

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ABSTRACT

Chapter 1. In classical algebra there arise structures of the form $\mathbf{A} = \langle A; \oplus, 0; \leq \rangle$ consisting of a monoid $\langle A; \oplus, 0 \rangle$ and a $\oplus$–compatible partial order $\leq$ with least element 0, that are *residuated* in the sense that there exists a binary operation $\dot{-}$ on A characterized by the rule

$$x \leq y \oplus z \quad \text{iff} \quad x \dot{-} z \leq y.$$

The structure $\mathbf{A}$ satisfies

$$x \leq y \quad \text{iff} \quad x \dot{-} y \approx 0$$

and is therefore first-order definitionally equivalent to an *algebra* $\langle A; \oplus, \dot{-}, 0 \rangle$, which we call a *polrim*. The examples of ordinal addition and ideal multiplication in rings are discussed. A $\langle \dot{-}, 0 \rangle$–subreduct of a polrim is called a *left residuation algebra*. The class of all polrims [resp. left residuation algebras] is a quasivariety which we denote by $\mathcal{LM}$ [resp. $\mathcal{LR}$]. We investigate (and axiomatize, where necessary) these classes. We also characterize the $\langle \dot{-}, 0 \rangle$–*reducts* of polrims.

Chapter 2. The classes $\mathcal{LM}$ and $\mathcal{LR}$ arise naturally in logic. In [OK85], Ono and Komori defined a Gentzen-style logic L_{BK} that is obtained from Gentzen's formulation LJ of intuitionistic propositional logic by dropping the (structural) rules of exchange and contraction and adding a new 'weak' conjunction &. They also define a Hilbert-style logic H_{BK}[1] that is 'logically equivalent' to L_{BK}. We axiomatize H_{BK} in such a way that its proof theory has a 'separation theorem'. Assume $\{\to\} \subseteq C \subseteq \{\&, \to, \vee, \wedge, \perp\}$. We show that C–H_{BK}, the C–fragment of H_{BK}, is algebraizable in the sense of Blok and Pigozzi [BP89]. We denote its equivalent quasivariety semantics by $\mathcal{H}_{C^*}$, where C^* is a set of operation symbols in $\{\oplus, \dot{-}, \sqcap, \sqcup, 1\}$ corresponding to C. Then $\mathcal{H}_{\{\oplus, \dot{-}\}}$ [resp. $\mathcal{H}_{\{\dot{-}\}}$] is definitionally equivalent to $\mathcal{LM}$ [resp. $\mathcal{LR}$]. (The constant 0 is $x \dot{-} x$.) We axiomatize each $\mathcal{H}_{C^*}$ and show that $\mathcal{H}_{\{\dot{-}, \sqcap\}}$ and $\mathcal{H}_{\{\dot{-}, \sqcap, 1\}}$ are *not* finitely axiomatizable. The results answer a question posed in [OK85].

[1] In [OK85], L_{BK} and H_{BK} are denoted L_{BCC} and H_{BCC}, respectively.

Chapter 3. We prove that $\{\rightarrow\}$–H_{BK} has the finite model property with respect to $\mathcal{LR}$ and also with respect to a class of Kripke-type structures. We deduce that the variety generated by $\mathcal{LR}$ is generated by the *finite* left residuation algebras. These results answer a further question in [OK85].

Chapter 4. We present a proof of Idziak's unpublished result that the quasivariety $\mathcal{H}_{C^*}$ (where we assume that $\dot{-} \in C^*$) is a variety if and only if C^* contains $\oplus$ and at least one of $\sqcap$, $\sqcup$. We characterize the subvarieties of $\mathcal{LR}$ syntactically. For other values of C^*, we infer some sufficient conditions for subclasses of $\mathcal{H}_{C^*}$ to generate subvarieties of $\mathcal{H}_{C^*}$. We prove a result implying that locally finite varieties generated by polrims consist of polrims; the corresponding assertion for left residuation algebras is false. When $\mathcal{H}_{C^*}$ is a variety, we investigate its degree of congruence permutability, proving, e.g., that $\mathcal{H}_{\{\oplus, \dot{-}, \sqcap\}}$ is congruence permutable. When $\oplus \in C^*$, we show that a locally finite subvariety of $\mathcal{H}_{C^*}$ is congruence 3-permutable. We show that each of the classes $\mathcal{H}_{C^*}$ is relatively congruence distributive, relatively 0-regular and lacks the relative congruence extension property (RCEP). An *ideal* of an algebra $\mathbf{A}$ in $\mathcal{H}_{C^*}$ is just the 0–class of a relative congruence of $\mathbf{A}$. We describe ideal generation and deduce a syntactic characterization of the relative subvarieties of $\mathcal{H}_{C^*}$ that have the RCEP. Ordinals α less than ω^ω, considered as well ordered monoids with *left* residuation, illustrate a number of our results. In particular, we show that they generate *subvarieties* of $\mathcal{H}_{C^*}$ which fail to have the congruence extension property when $1 \notin C^*$ and $\alpha > \omega + 1$.

Chapter 5. We characterize syntactically the relative subvarieties of $\mathcal{H}_{C^*}$ that have equationally definable principal [relative] congruences (EDP[R]C). We investigate relative subvarieties $\mathcal{H}^n_{C^*}$ ($n \in \omega$) of $\mathcal{H}_{C^*}$ defined by the identity

$$x \dot{-} (x \dot{-} y) \dot{-} ny \approx 0;$$

these arise naturally as the relative subvarieties in which ideals and a weaker notion 'preideals' coincide. The quasivariety of all BCK–algebras is precisely $\mathcal{H}^1_{\{\dot{-}\}}$, while ordinals less than ω^ω with *right* residuation are natural examples of algebras in $\mathcal{H}^2_{C^*}$. When $\oplus, 1 \notin C^*$, we characterize the relative subvarieties of $\mathcal{H}^n_{C^*}$ among those of $\mathcal{H}_{C^*}$ by their possession of the RCEP and a very weak 'finiteness' condition. We deduce that the locally finite relative subvarieties of $\mathcal{H}_{C^*}$ that have the RCEP are *just* those satisfying the above identity, for some n; they also have EDPRC. We characterize the finitely subdirectly irreducible members of $\mathcal{H}^n_{C^*}$ and axiomatize the quasivariety generated by $\mathcal{H}^n_{C^*}$'s linearly ordered members, showing that this is a *relative subvariety* of $\mathcal{H}^n_{C^*}$. When $\sqcap \in C^*$, we show that $\mathcal{H}^n_{C^*}$ has equationally definable principal relative meets (EDPRM). We also show that the subquasivariety of $\mathcal{H}^n_{C^*}$ generated by its linearly ordered algebras has EDPRM (regardless of the availability of $\sqcap$).

Chapter 6. We investigate the lattice $\mathbf{P}^{\mathrm{V}}(\mathcal{LR})$ of subvarieties of $\mathcal{LR}$. This lattice has a unique atom – the variety of Tarski algebras. We show that this atom has exactly three *finitely generated* covers in $\mathbf{P}^{\mathrm{V}}(\mathcal{LR})$. We show that the only variety of left residuation algebras that covers the atom, has EDPC and is *not* semisimple is $\mathrm{V}(\mathbf{H}_3)$, where $\mathbf{H}_3$ is (dually) isomorphic to the implication reduct of the three-element linearly ordered Heyting algebra. We construct a countably infinite sequence $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3, \ldots$ of infinite left residuation algebras that generate distinct varieties of left residuation algebras that have EDPC, are semisimple and cover the atom. We show that the variety generated by all the $\mathbf{A}_i$'s is a variety of left residuation algebras that has $2^{\aleph_0}$ subvarieties, none of which is generated by its finite members, except the variety of Tarski algebras and the trivial variety. We prove a finite axiomatization of the variety generated by $\mathbf{A}_1$ and show that the variety generated by $\mathbf{A}_2$ is also finitely axiomatized. Our results extend and contrast with Kowalski's recent solution of an open problem about covers of the atom in the lattice of varieties of BCK–algebras.

Chapter 7. We show that if $\mathbf{A} \in \mathcal{H}_{\{\dot{-},\sqcap\}}$ is a 'distributive residuation nearlattice' then its canonical distributive lattice extension has a well defined operation $\dot{-}^{\mathbf{A}^\circ}$ (extending $\dot{-}^{\mathbf{A}}$) enriched with which it is an algebra $\mathbf{A}^\circ \in \mathcal{H}_{\{\dot{-},\sqcap,\sqcup\}}$ and the unique extensions from A to A° of significant morphisms preserve $\dot{-}$ also. Indeed, $\mathbf{A} \mapsto \mathbf{A}^\circ$ induces a simple mono-preserving reflection of categories. We show that the lattices of relative congruences of $\mathbf{A}$ and $\mathbf{A}^\circ$ (with respect to $\mathcal{H}_{\{\dot{-},\sqcap\}}$ and $\mathcal{H}_{\{\dot{-},\sqcap,\sqcup\}}$) are isomorphic. Moreover, we show that if $\mathbf{A}$ is an algebra in $\mathcal{H}^n_{\{\dot{-},\sqcap\}}$ that satisfies

$$(x \dot{-} y) \sqcap (y \dot{-} x) \approx 0$$

then $\mathbf{A}$ and the $\langle \dot{-}, \sqcap, 0\rangle$–reduct of $\mathbf{A}^\circ$ belong to the same varieties.

Appendix. Lattices of topologizing filters of unital rings are interesting examples of (lattice ordered) polrims which are *not* right residuated. An expository appendix discussing these algebras has been included.

Much of the above mentioned material has already been written up in the form of research papers. The papers [RvA97] and [vAR1] contain most of the results of Chapters 1, 2, 4 and 5. The results of Chapters 3 and 7 have been accepted for publication in the form of [vAR2] and [vAl], respectively.

PREFACE

The work described in this thesis was carried out under the supervision of Prof James G. Raftery, Department of Mathematics and Applied Mathematics, University of Natal, Durban, and the co-supervision of Dr J. E. van den Berg, University of Natal, Pietermaritzburg, from April 1995 to December 1997.

The thesis represents original work by the author and has not been submitted in any form to another University. Where use was made of the work of others it has been duly acknowledged in the text.

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CONTENTS

INTRODUCTION

This thesis takes an algebraic approach to the study of ordered monoids that are 'residuated' in the sense that they have a binary operation that acts like a subtraction or division operation. The existence of such an operation permits the calculation of residuals $a - b$ or ab^{-1} in the absence of a unary inverse operation. Such structures arise in areas of classical mathematics such as the ideal theory of rings (the notion of residuation can be traced back to Dedekind's work on modules), algebraic logic and the study of ordinals.

An *integral pomonoid* is a structure $\langle A; \oplus, 0; \leq \rangle$ where $\langle A; \oplus, 0 \rangle$ is a monoid and $\leq$ is a $\oplus$-compatible partial order with least element 0. Such an integral pomonoid is called *left residuated* if there exists a binary operation $\dot{-}$ on A such that, for any $a, b \in A$,

$$a \dot{-} b = \min\{c \in A : a \leq c \oplus b\}.$$

We call $\dot{-}$ the *left residuation operation* of $\langle A; \oplus, 0; \leq \rangle$. A left residuated integral pomonoid $\mathbf{A} = \langle A; \oplus, 0; \leq \rangle$ satisfies

$$x \leq y \quad \text{if and only if} \quad x \dot{-} y \approx 0,$$

so that the partial order $\leq$ is obtainable from the operations $\dot{-}$ and 0, hence $\mathbf{A}$ is first-order definitionally equivalent to an *algebra* $\langle A; \oplus, \dot{-}, 0 \rangle$, which we call a *polrim*. Analogously, one may define a *porrim* as an integral pomonoid with a *right* residuation operation. A residuation–subreduct (i.e., a subalgebra of the $\langle \dot{-}, 0 \rangle$–reduct) of a polrim is called a *left residuation algebra*. The class of all polrims [resp. left residuation algebras] is a quasivariety which we denote by $\mathcal{LM}$ [resp. $\mathcal{LR}$].

The lattice of ideals of a unital ring $\mathbf{R}$, with ideal multiplication as its monoid operation and ordered by reversed set inclusion, is a polrim whose left residuation operation is given by $I : J = \{r \in R : rJ \subseteq I\}$ (I, J ideals of $\mathbf{R}$). In algebraic structures arising from logic, there usually exists a monoid operation corresponding to a logical conjunction or 'fusion of premisses' and a partial order that reflects deducibility. Typically, such structures are polrims whose left residuation operation corresponds to the logical implication. Ordinals, considered as well ordered sets either closed under ordinal addition or with an additively absorptive top element are integral pomonoids. These integral pomonoids are both left and right residuated and, as such, may be considered as polrims or porrims (which are not elementarily equivalent in cases greater than $\omega + 1$). A recently developed tool for the study of rings is the 'lattice of topologizing filters' of a ring, considered as a polrim (using a suitable monoid operation and set inclusion). In general, these polrims provide more information about the underlying ring than the polrim on the ideal lattice. Significantly, such structures need not be right residuated.

Abstract studies of some subclasses of $\mathcal{LM}$ and $\mathcal{LR}$ have appeared in the literature. Bosbach [Bos69] considers *left complemented monoids*, namely, polrims $\langle A; \oplus, 0; \leq \rangle$ that are *(left) complemented* in the sense that, for $a, b \in A$,

$$a \leq b \quad \text{implies} \quad \text{there exists } c \in A \text{ such that } c \oplus a = b.$$

A polrim whose monoid operation is commutative is called a *pocrim*. Pocrims have been studied, e.g., in [Hig84] and [BR97]; their residuation–*reducts* were studied earlier under the name 'BCK–algebras with condition (S)' [Isé79]. Pocrims that are complemented are known as *hoops* and have been studied in [BP94b], [BF93] and [Fer92]. A residuation–subreduct of a pocrim is known as a BCK–*algebra*, that is, an algebra $\langle A; \dot{-}, 0 \rangle$ of type $\langle 2, 0 \rangle$ that satisfies the following identities and quasi-identity:

$$((x \dot{-} y) \dot{-} (x \dot{-} z)) \dot{-} (z \dot{-} y) \approx 0,$$
$$x \dot{-} 0 \approx x,$$
$$0 \dot{-} x \approx 0,$$
$$x \dot{-} y \approx 0 \quad \text{and} \quad y \dot{-} x \approx 0 \quad \text{implies} \quad x \approx y.$$

These algebras were introduced by Iséki (based on the logic BCK of Meredith) and have been extensively studied. Survey articles include [IT78], [Cor82] and, to some extent, the more recent paper [BR95]. That every BCK–algebra is a residuation–subreduct of a pocrim was proved independently by Pałasinski [Pał82], Ono and Komori [OK85] and Fleischer [Fle88]. In [Kom83] and [Kom84] Komori considered the class of algebras obtained by replacing the first identity in the above axiomatization of BCK–algebras with

$$((x \dot{-} y) \dot{-} (z \dot{-} y)) \dot{-} (x \dot{-} z) \approx 0.$$

It was proved by Ono and Komori in [OK85] that every such algebra is a residuation–subreduct of a *polrim*, i.e., a left residuation algebra. The residuation–subreducts of left complemented monoids are among Bosbach's *residuation groupoids* [Bos82].

The study of polrims and left residuation algebras may be considered a unification and extension of the commutative and complemented cases. Ordinal polrims are right but, generally, not left complemented monoids. The only ordinal polrims that are pocrims are those less than $\omega + 2$. The polrim on the lattice of ideals of a noncommutative ring is a pocrim under the arguably artificial demand that ideals commute under multiplication.

Another route to $\mathcal{LM}$ and $\mathcal{LR}$ is via Gentzen's formulation LJ of intuitionistic logic [Gen35]. In [OK85], Ono and Komori present a Gentzen system LJ^* that is essentially the propositional fragment of LJ, but differs in that the conjunction connective $\wedge$ of LJ is duplicated by a further connective & (to be thought of as a 'fusion of premisses') which has a weaker set of inference rules. The logic L_{BK} is obtained from LJ^* by removing the 'structural' rules of contraction and exchange, viz.

$$\frac{\Gamma, \alpha, \alpha, \Delta \Rightarrow \gamma}{\Gamma, \alpha, \Delta \Rightarrow \gamma}(\text{contraction}) \qquad \frac{\Gamma, \alpha, \beta, \Delta \Rightarrow \gamma}{\Gamma, \beta, \alpha, \Delta \Rightarrow \gamma}(\text{exchange}).$$

Thus L_{BK} falls into the class of 'substructural logics' (see [SD93]). Because of the choice of inference rules for LJ^*, the connective $\wedge$ takes on the character of a lattice operation, while & is like a monoid operation. Ono and Komori also define a Hilbert system H_{BK}, which they prove to be 'logically equivalent' to L_{BK}. As expected, H_{BK} lacks Hilbert style exchange and contraction rules, i.e.,

$$(p \to (q \to r)) \to (q \to (p \to r)) \quad \text{and} \quad (p \to (p \to q)) \to (p \to q)$$

are not theorems of H_{BK}. Each C–fragment C–H of H_{BK} (where $\{\to\} \subseteq C \subseteq \{\&, \to, \vee, \wedge, \bot\}$) turns out to be algebraizable in the sense of Blok and Pigozzi [BP89]; we denote its equivalent quasivariety semantics by $\mathcal{H}_{C^*}$, where C^* is a set of operation symbols in $\{\oplus, \dot{-}, \sqcap, \sqcup, 1\}$ corresponding to C. The quasivariety $\mathcal{H}_{\{\oplus, \dot{-}\}}$ [resp. $\mathcal{H}_{\{\dot{-}\}}$] is definitionally equivalent to $\mathcal{LM}$ [resp. $\mathcal{LR}$]. Thus, in particular, left residuation algebras are the natural algebraic semantics of the implicational fragment of Intuitionistic Propositional Calculus without exchange and contraction (as formulated by Ono and Komori).

The members of the equivalent quasivariety semantics of the full logic H_{BK}, denoted $\mathcal{H}$ for brevity, also have operations $\sqcap$ and $\sqcup$, which turn out to be lattice operations, and the constant 1, which turns out to be the maximum element of the associated partial order. Natural axioms describing the interaction between $\oplus$, $\dot{-}$ and these new operations are inherited from H_{BK}. $\mathcal{LM}$ and $\mathcal{LR}$ are both subreduct classes of $\mathcal{H}$; the approach taken in this thesis is to consider *all* the subreduct classes of $\mathcal{H}$ (that contain the residuation connective $\dot{-}$).

One may take the view that it is natural to consider noncommutative ordered monoids having both a left and a right residuation operation (as in [Bos69], for example). Every polrim (and porrim) may be embedded into such a structure, so the study of polrims is a part of such a larger study. The paper [OK85] does *not* take such a "two-sided" approach. This paper was, to some extent, a starting point for our investigations and we have adopted its approach. In fact, a number of the problems solved in this thesis are posed in that paper. The lattice of topologizing filters of a ring, considered as a polrim, need *not* be right residuated (an example is given in the appendix). Thus, such structures are natural models of the theory of "one-sided" residuation, undertaken here.

CHAPTER 0

PRELIMINARIES

Ordered Sets. Let $\mathbf{S} = \langle S; \leq \rangle$ be a partially ordered set. For $X \subseteq S$ and $a \in S$, we define

$$(X] = \{b \in S : b \leq a \text{ for some } a \in X\}, \quad (a] = (\{a\}],$$
$$[X) = \{b \in S : a \leq b \text{ for some } a \in X\}, \quad [a) = [\{a\}).$$

A subset T of S is called *downward closed*, or *hereditary*, in $\mathbf{S}$ if $T = (T]$. A subset T of S is called *upward closed* if $T = [T)$. We shall use the symbols $\sqcap$ and $\sqcup$ for lattice meet and join operations, respectively, and $\bigsqcap$, $\bigsqcup$ for infinitary meet and join operations, respectively.

By *ordinals* we mean Von Neumann ordinals, i.e., each ordinal is identified with its set of predecessors. We use lower case Greek letters as variables for ordinals. The least infinite ordinal is denoted ω; its elements are called the *natural numbers*. The elementhood relation $\in$ between ordinals is also denoted $<$. The relation $\leq$ obtained from $<$ in the usual way therefore coincides with set inclusion $\subseteq$ in every ordinal. For ordinals α, β, we denote the *ordinal sum* [resp. *ordinal product*] of α and β by $\alpha + \beta$ [resp. $\alpha\beta$]. We use α^β to denote *ordinal exponentiation*. In the context of cardinals we shall use $\aleph_0$ instead of ω. Recall that each nonzero ordinal less than ω^ω has the form of a unique 'polynomial' in ω, i.e., $\omega^n a_n + \omega^{n-1} a_{n-1} + \cdots + \omega a_1 + a_0$, where $n, a_0, a_1, \ldots, a_n \in \omega$ and $a_n \neq 0$.

If A is a set, we use $\vec{a}$ to denote a finite sequence $a_0, a_1, \ldots, a_n$ of elements of A, where $n \in \omega$ is arbitrary, understood or unimportant. We sometimes write $\vec{a} \in A$ in this case.

Languages. The countably infinite set $\mathbf{X} = \{\mathrm{x}_0, \mathrm{x}_1, \mathrm{x}_2, \ldots\}$ of *variables* will be fixed throughout this thesis. Lower case Roman letters, possibly with integer subscripts, will be used as metavariables ranging over $\mathbf{X}$. In an algebraic context, we usually use the letters x, y, z for metavariables, and in a logical context, we usually use the letters p, q, r.

Where languages are concerned, we confine ourselves to *algebraic* ones. Thus, by a *language* (or *type*) we mean a pair $\boldsymbol{\mathcal{L}} = \langle \mathcal{L}, ar \rangle$ consisting of a set $\mathcal{L}$ of *operation symbols* and an *arity* function ar that assigns a natural number to each element of $\mathcal{L}$. In a logical context, we shall refer to the elements of $\mathcal{L}$ as *connectives* and to the arity function as the *rank* function. The type $\boldsymbol{\mathcal{L}}$ is called *finite* if $|\mathcal{L}|$ is finite.

Given a language $\boldsymbol{\mathcal{L}}$, the set of *$\boldsymbol{\mathcal{L}}$–terms* is constructed in the usual recursive way from the variables in $\mathbf{X}$ and the operation symbols in $\mathcal{L}$ (see, e.g., [BS81, Definition II.10.1, p62]). We generally use the letters s, t, u, v to denote $\boldsymbol{\mathcal{L}}$–terms. In a logical context, we shall refer to $\boldsymbol{\mathcal{L}}$–terms as *$\boldsymbol{\mathcal{L}}$–formulas* and denote them by lower case Greek letters. The set of all $\mathcal{L}$–terms is the universe of the absolutely free algebra of type $\mathcal{L}$ over $\mathbf{X}$, which we call the *term* (or *formula*) *algebra over $\boldsymbol{\mathcal{L}}$*. We often drop the prefix $\boldsymbol{\mathcal{L}}$– from '$\boldsymbol{\mathcal{L}}$–term' or '$\boldsymbol{\mathcal{L}}$–formula' when the $\boldsymbol{\mathcal{L}}$ is understood. Moreover, we shall often simply write $\mathcal{L}$ for $\boldsymbol{\mathcal{L}}$. We shall often write $t(x_0, \ldots, x_n)$ for a term t if the variables occurring in t are among $x_0, \ldots, x_n$.

Hilbert Systems. Let $\mathcal{L}$ be a language. An *$\mathcal{L}$–substitution* (or *substitution*, if $\mathcal{L}$ is understood) is an endomorphism of the formula algebra over $\mathcal{L}$. Note that, by the universal mapping property, a substitution σ may be identified with its restriction to $\mathbf{X}$. By an *inference rule* (*over $\mathcal{L}$*), we mean any pair $\langle \Gamma, \varphi \rangle$, where Γ is a finite set of formulas and φ is a formula. An *axiom* is an inference rule of the form $\langle \emptyset, \varphi \rangle$; we usually identify an axiom $\langle \emptyset, \varphi \rangle$ with the formula φ. Let I be a set of axioms and inference rules over $\mathcal{L}$. Let $\Delta \cup \{\varphi\}$ be a set of formulas. A *derivation of φ from Δ* (*with respect to I*) is a nonempty finite sequence of formulas $\psi_1, \psi_2, \ldots, \psi_n$ such that $\psi_n = \varphi$ and for $i = 1, 2, \ldots, n$, one of the following conditions holds:

(i) $\psi_i \in \Delta$ or $\psi_i = \sigma(\delta)$ for some axiom δ in I and some substitution σ;

(ii) there exists an inference rule $\langle \Gamma, \delta \rangle$ in I and a substitution σ such that $\psi_i = \sigma(\delta)$ and $\sigma(\gamma) \in \{\psi_1, \ldots, \psi_{i-1}\}$ for each $\gamma \in \Gamma$.

A *Hilbert system S* (*over $\mathcal{L}$*) is determined by a set I of axioms and inference rules; it consists of a pair $\langle \mathcal{L}, \vdash_S \rangle$, where $\vdash_S$ is the relation between sets of formulas and single formulas that is defined by the following condition:

$$\Gamma \vdash_S \varphi \quad \text{iff} \quad \varphi \text{ is derivable from } \Gamma \text{ with respect to } I.$$

The relation $\vdash_S$ is called the *consequence relation* of S. The set I is called an *axiomatization* of S and the axioms and inference rules in I are called the *axioms* and *inference rules of S*, respectively. Of course, a Hilbert system may have more than one axiomatization. A Hilbert system for which there exists a finite axiomatization is said to be *finitely axiomatizable*. An inference rule

$\langle \Gamma, \varphi \rangle$ of S is usually denoted $\Gamma \vdash_S \varphi$. We write $\vdash_S \varphi$ for $\emptyset \vdash_S \varphi$. A formula φ for which $\vdash_S \varphi$ is called a *theorem* of S. We shall also use the following abbreviations for sets of formulas Γ, Δ and formulas $\alpha_1, \dots, \alpha_n, \beta$:

$$\begin{array}{rll} \alpha_1, \dots, \alpha_n \vdash_S \beta & \text{abbreviates} & \{\alpha_1, \dots, \alpha_n\} \vdash_S \beta; \\ \Gamma, \alpha \vdash_S \beta & \text{abbreviates} & \Gamma \cup \{\alpha\} \vdash_S \beta; \\ \Gamma \vdash_S \Delta & \text{abbreviates} & \text{`}\Gamma \vdash_S \delta \text{ for all } \delta \in \Delta\text{'}; \\ \Gamma \dashv\vdash_S \Delta & \text{abbreviates} & \text{`both } \Gamma \vdash_S \Delta \text{ and } \Delta \vdash_S \Gamma\text{'}. \end{array}$$

The consequence relation $\vdash_S$ of a Hilbert system S over a language $\mathcal{L}$ is easily seen to satisfy the following three conditions for all sets of formulas Γ, Δ and all formulas φ, ψ:

(i) $\varphi \in \Gamma$ implies $\Gamma \vdash_S \varphi$;

(ii) $\Gamma \vdash_S \varphi$ and $\Gamma \subseteq \Delta$ implies $\Delta \vdash_S \varphi$;

(iii) $\Gamma \vdash_S \varphi$ and $\Delta \vdash_S \psi$ for each $\psi \in \Gamma$ implies $\Delta \vdash_S \varphi$.

In addition, S is *finitary* in the sense that

(iv) $\Gamma \vdash_S \varphi$ implies $\Gamma' \vdash_S \varphi$ for some finite $\Gamma' \subseteq \Gamma$,

and is *structural* in the sense that

(v) $\Gamma \vdash_S \varphi$ implies $\sigma[\Gamma] \vdash_S \sigma(\varphi)$ for every substitution σ.

Conversely, every relation between sets of $\mathcal{L}$–formulas and $\mathcal{L}$–formulas satisfying conditions (i)–(v) is the consequence relation for some Hilbert system S over $\mathcal{L}$ [LS58]. Consequently, a Hilbert system may be defined as a pair $\langle \mathcal{L}, \vdash_S \rangle$, where $\vdash_S$ is a relation between sets of formulas and single formulas that satisfies (i)–(v); defining axioms and inference rules need not be assumed.

Gentzen systems. Let $\mathcal{L}$ be a language. By an *$\mathcal{L}$–sequent* (or *sequent*, if $\mathcal{L}$ is understood) we mean an expression of the form $\alpha_1, \alpha_2, \dots, \alpha_n \Rightarrow \beta$, where $\alpha_1, \alpha_2, \dots, \alpha_n, \beta$ are formulas. We shall use upper case Greek letters for finite (possibly empty) sequences of formulas, separated by commas. We shall use $\emptyset$ to denote the empty sequence. (Note that in the context of Hilbert systems, we use upper case Greek letters to denote *sets* of formulas.) If $\Delta_1 \Rightarrow \beta_1, \dots, \Delta_n \Rightarrow \beta_n, \Gamma \Rightarrow \alpha$ are sequents, then

$$\frac{\Delta_1 \Rightarrow \beta_1 \quad \dots \quad \Delta_n \Rightarrow \beta_n}{\Gamma \Rightarrow \alpha}$$

is an *inference figure* (*over* $\mathcal{L}$). We identify this figure with

$$\frac{\Delta_{\sigma(1)} \Rightarrow \beta_{\sigma(1)} \quad \dots \quad \Delta_{\sigma(n)} \Rightarrow \beta_{\sigma(n)}}{\Gamma \Rightarrow \alpha}$$

whenever σ is a permutation of $\{1,\dots,n\}$. Formally, therefore, an inference figure is any pair $\langle A; \Gamma \Rightarrow \alpha\rangle$ where A is a finite *multiset* (rather than a finite sequence) of $\mathcal{L}$–sequents, and $\Gamma \Rightarrow \alpha$ is an $\mathcal{L}$–sequent. In the above example, the $\Delta_i \Rightarrow \beta_i$'s are called the *upper sequents* and $\Gamma \Rightarrow \alpha$ the *lower sequent* of the figure. A *Gentzen system* G (*over* $\mathcal{L}$) is determined by a set of inference figures, which we call the *rules of inference of* G, and a set of sequents, which we call the *initial sequents of* G.

A *tree* is a partially ordered set $\langle P; \leq\rangle$ with a (unique) least element $p_0 \in P$ (called the *root* of $\langle P; \leq\rangle$) such that for each $p \in P$, the set $(p]$ is well ordered by $\leq$. The tree $\langle P; \leq\rangle$ is called *finite* if P is a finite set.

A triple $\langle P; \leq; f\rangle$ is called an *$\mathcal{L}$–derivation* if

(i) $\langle P; \leq\rangle$ is a finite tree (with root p_0, say); and

(ii) f is a function from P to the set of all $\mathcal{L}$–sequents.

In this case, $f(p_0)$ is called the *endsequent* of $\langle P; \leq; f\rangle$ and, for each maximal element p of $\langle P; \leq\rangle$, $f(p)$ is called an *initial sequent* of $\langle P; \leq; f\rangle$. If $p \in P$ is not maximal and $p_1, \dots, p_n$ are precisely the covers of p in $\langle P; \leq\rangle$ then

$$\frac{f(p_1) \ \dots \ f(p_n)}{f(p)}$$

is called an *inference figure* of $\langle P; \leq; f\rangle$.

An $\mathcal{L}$–sequent $\Sigma \Rightarrow \gamma$ is said to be *derivable in* G if it is the endsequent of some $\mathcal{L}$–derivation $\langle P; \leq; f\rangle$ all of whose initial sequents are initial sequents of G and all of whose inference figures are rules of inference of G. In this case, $\langle P; \leq; f\rangle$ is called a *derivation of* $\Sigma \Rightarrow \gamma$ *in* G.

Algebraic Preliminaries. For general universal algebraic background we refer the reader to [BS81], [MMT87] or [Grä79]. We usually denote algebras by boldface capitals $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, ... and their respective universes by A, B, C, ... An algebra $\mathbf{A}$ is called *trivial* if $|A| = 1$. We use the letters I, H, S, P, $\mathrm{P_S}$ and $\mathrm{P_U}$ to denote the class operators used to obtain *isomorphic images*, *homomorphic images*, *subalgebras*, *direct products*,[2] *subdirect products* and *ultraproducts*, respectively. If $\mathcal{K}$ is a class of similar algebras (i.e., algebras of the same type), we use $\mathrm{V}(\mathcal{K})$ to denote the *variety generated by* $\mathcal{K}$, i.e., $\mathrm{V}(\mathcal{K}) = \mathrm{H\,S\,P}(\mathcal{K})$. We also use $\mathcal{K}_{\mathrm{NT}}$ to denote the class of all nontrivial algebras in $\mathcal{K}$. For an algebra $\mathbf{A}$ and a subset X of A, we use $\mathrm{Sg}^{\mathbf{A}}(X)$ to denote the *subalgebra of* $\mathbf{A}$ *generated by* X.

[2]The trivial algebra that is the direct product of the empty subfamily of a class $\mathcal{K}$ of algebras of the same type is included in $\mathrm{P}(\mathcal{K})$.

The set of congruences of an algebra $\mathbf{A}$ is denoted $\operatorname{Con}\mathbf{A}$ and the congruence lattice of $\mathbf{A}$ is denoted $\mathbf{Con\,A}$. For $X \subseteq A^2$, we use $\Theta^{\mathbf{A}}(X)$ to denote the *congruence of* $\mathbf{A}$ *generated by* X. For $a, b \in A$, we abbreviate $\Theta^{\mathbf{A}}(\{(a,b)\})$ by $\Theta^{\mathbf{A}}(a,b)$. If Y is any set, we use id_Y to denote the set $\{(b,b) : b \in Y\}$. Thus id_A is the least congruence of an algebra $\mathbf{A}$. If id_A is meet irreducible in $\mathbf{Con\,A}$, we call $\mathbf{A}$ a *finitely subdirectly irreducible algebra.*

The class of all subdirectly irreducible [resp. simple] algebras in a class $\mathcal{K}$ of similar algebras is denoted $\mathcal{K}_{\mathrm{SI}}$ [resp. $\mathcal{K}_{\mathrm{S}}$]. These classes exclude trivial algebras, by definition. The class of all finitely subdirectly irreducible algebras in $\mathcal{K}$ is denoted $\mathcal{K}_{\mathrm{FSI}}$. Thus, $\mathcal{K}_{\mathrm{S}} \subseteq \mathcal{K}_{\mathrm{SI}} \subseteq (\mathcal{K}_{\mathrm{FSI}})_{\mathrm{NT}}$ for every class $\mathcal{K}$ of similar algebras.

A variety $\mathcal{V}$ with an equationally definable constant term 0 is said to be 0–*regular* if, whenever $\mathbf{A} \in \mathcal{V}$ and $\theta_1, \theta_2 \in \operatorname{Con}\mathbf{A}$ with $0^{\mathbf{A}}/\theta_1 = 0^{\mathbf{A}}/\theta_2$, we have $\theta_1 = \theta_2$. We call a variety $\mathcal{V}$ *congruence distributive* if $\mathbf{Con\,A}$ is a distributive lattice for every $\mathbf{A} \in \mathcal{V}$.

If $\mathbf{A}_i$, $i \in I$, is a family of algebras of the same type and $\mathcal{U}$ is an ultrafilter over I, we use $\prod_{i\in I}\mathbf{A}_i/\mathcal{U}$ to denote the ultraproduct of $\{\mathbf{A}_i : i \in I\}$ with respect to $\mathcal{U}$. When $\mathbf{A}_i = \mathbf{A}$ for all $i \in I$, this ultrapower of $\mathbf{A}$ is denoted by $\mathbf{A}^I/\mathcal{U}$. We shall use the following results:

Lemma 0.1. [BS81, Lemma IV.6.5] *If $\{\mathbf{A}_i : i \in I\}$ is a finite set of finite algebras, say $\{\mathbf{B}_1, \ldots, \mathbf{B}_k\}$, and $\mathcal{U}$ is an ultrafilter over I then $\prod_{i\in I}\mathbf{A}_i/\mathcal{U}$ is isomorphic to one of the algebras $\mathbf{B}_1$, …, $\mathbf{B}_k$, namely to that $\mathbf{B}_j$ such that*

$$\{i \in I : \mathbf{A}_i = \mathbf{B}_j\} \in \mathcal{U}.$$

Theorem 0.2. (Jónsson's Theorem) [BS81, Theorem IV.6.8] *Let $\mathcal{V} = \mathrm{V}(\mathcal{K})$ be a congruence distributive variety. Then $\mathcal{V}_{\mathrm{SI}} \subseteq \mathcal{V}_{\mathrm{FSI}} \subseteq \mathrm{H\,S\,P_U}(\mathcal{K})$. Thus, if $\mathcal{K}$ is a finite set of finite algebras then $\mathcal{V}_{\mathrm{SI}} \subseteq \mathcal{V}_{\mathrm{FSI}} \subseteq \mathrm{H\,S}(\mathcal{K})$.*

We shall also use the following well known corollary to Jónsson's Theorem:

Corollary 0.3. *Let $\mathcal{V}_1$, $\mathcal{V}_2$ be subvarieties of a congruence distributive variety. Then $(\mathcal{V}_1 \sqcup \mathcal{V}_2)_{\mathrm{SI}} = (\mathcal{V}_1)_{\mathrm{SI}} \cup (\mathcal{V}_2)_{\mathrm{SI}}$, where the join is taken in the lattice of all varieties of the type of $\mathcal{V}_1$ and $\mathcal{V}_2$.*

Corollary 0.4. [Jón67, Corollary 4.2] *The lattice of subvarieties of a variety $\mathcal{V}$ is distributive if $\mathcal{V}$ is congruence distributive.*

Theorem 0.5. [BS81, Theorem V.2.14] *Every algebra can be embedded into an ultraproduct of its finitely generated subalgebras.*

Model Theoretic Preliminaries. Consider the first order language $\mathcal{L}_{\approx}$ with equality, determined by an algebraic language $\mathcal{L}$. Since $\mathcal{L}$ has operation symbols only, the *atomic (first order) formulas of* $\mathcal{L}_{\approx}$ *(over* $\mathbf{X}$*)* are just all $\mathcal{L}$*–equations (over* $\mathbf{X}$*)*, i.e., all expressions of the form $s \approx t$, where s, t are $\mathcal{L}$–terms. The *first order formulas of* $\mathcal{L}_{\approx}$ *(over* $\mathbf{X}$*)* are defined in the standard recursive way: any atomic formula of $\mathcal{L}_{\approx}$ is a first order formula of $\mathcal{L}_{\approx}$; if $x \in \mathbf{X}$ and Φ, Φ_1, Φ_2 are first order formulas of $\mathcal{L}_{\approx}$ then so are

$$
\begin{array}{l}
(\Phi_1) \text{ and } (\Phi_2), \quad (\Phi_1) \text{ or } (\Phi_2), \quad \neg(\Phi), \\
(\Phi_1) \text{ implies } (\Phi_2), \quad (\Phi_1) \text{ iff } (\Phi_2), \\
(\forall x)(\Phi), \quad (\exists x)(\Phi).^3
\end{array}
$$

We adopt standard bracket omission conventions. For first order formulas $\Phi_1, \Phi_2, \ldots, \Phi_n$ of $\mathcal{L}_{\approx}$, we sometimes abbreviate the formula (Φ_1) and (Φ_2) and $\ldots$ and (Φ_n) by $\bigwedge_{i=1}^{n}(\Phi_i)$.

The *bound* and *free variables* of a first order formula Φ of $\mathcal{L}_{\approx}$ are defined in the usual way (see, e.g., [BS81, Chapter V, p194]); Φ is called a *(first order) sentence of* $\mathcal{L}_{\approx}$ if it has no free variables. On the other hand, if the free variables of Φ (in order of their first occurrence, from left to right, in Φ) are $x_1, \ldots, x_n \in \mathbf{X}$ then the sentence $(\forall x_1)(\forall x_2)\ldots(\forall x_n)(\Phi)$, denoted $\overline{\Phi}$, shall be called the *closure* of Φ. The closure of an $\mathcal{L}$–equation shall be called an $\mathcal{L}$*–identity*.

If the free variables of a first order formula Φ of $\mathcal{L}_{\approx}$ are among the distinct variables $x_1, \ldots, x_n \in \mathbf{X}$, we often write $\Phi(x_1, \ldots, x_n)$ for Φ. In this case, if $t_1, \ldots, t_n$ are $\mathcal{L}$–terms, we may denote by $\Phi[t_1, \ldots, t_n]$ the result of simultaneously replacing each free occurrence of x_i by t_i in Φ, for $i = 1, \ldots, n$.

If $\mathbf{A}$ is an algebra of type $\mathcal{L}$, we let $\mathcal{L}^A$ denote the type obtained by adding (distinct, new) operation symbols a' of arity 0 to $\mathcal{L}$ for each $a \in A$. Thus, for each first order formula $\Phi(x_1, \ldots, x_n)$ of $\mathcal{L}$ and any $a_1, \ldots, a_n \in A$, the language $\mathcal{L}^A_{\approx}$ includes the sentence $\Phi[a'_1, \ldots, a'_n]$, which we usually denote by $\Phi[a_1, \ldots, a_n]$.

Let $\mathcal{K} \cup \{\mathbf{A}\}$ be a class of algebras of type $\mathcal{L}$ and $\Sigma \cup \{\Phi\}$ a set of *sentences* of $\mathcal{L}^A_{\approx}$. The notion that $\mathbf{A}$ *satisfies* Φ (or 'Φ is *true* in $\mathbf{A}$'), denoted $\mathbf{A} \models \Phi$, is defined in the usual recursive way (e.g., [BS81, Definition V.1.10, p195]). In particular, if Φ is an atomic sentence of $\mathcal{L}^A_{\approx}$ then Φ is $s[a_1, \ldots, a_n] \approx t[a_1, \ldots, a_n]$ for some $\mathcal{L}$–terms s, t and some $a_1, \ldots, a_n \in A$. In this case, $\mathbf{A} \models \Phi$ if and

[3] We avoid using $\&, \wedge, \rightarrow$, etc. as logical symbols of the first order language $\mathcal{L}_{\approx}$ because these symbols shall correspond extensively in the sequel to (nonlogical) operation symbols of particular (algebraic) languages. We avoid $\Rightarrow$ here also, as it shall continue to denote the derivability relation of a Gentzen system.

only if $s^{\mathbf{A}}(a_1,\ldots,a_n) = t^{\mathbf{A}}(a_1,\ldots,a_n)$ (where $s^{\mathbf{A}}, t^{\mathbf{A}} : A^n \to A$ are the n-ary term functions on A corresponding to s, t, respectively). The notation $\mathbf{A} \models \Sigma$ abbreviates '$\mathbf{A} \models \Phi'$ for all of the sentences $\Phi' \in \Sigma$', and $\mathcal{K} \models \Phi$ [resp. $\mathcal{K} \models \Sigma$] means that $\mathbf{B} \models \Phi$ [resp. $\mathbf{B} \models \Sigma$] for all $\mathbf{B} \in \mathcal{K}$.

Theorem 0.6. (Łos' Theorem) [BS81, Theorem V.2.9] *Given a family of algebras $\mathbf{A}_i$, $i \in I$, of type $\mathcal{L}$, an ultrafilter $\mathcal{U}$ over I and any first order sentence Φ of $\mathcal{L}_{\approx}$, we have*

$$\prod_{i\in I} \mathbf{A}_i/\mathcal{U} \models \Phi \quad \textit{if and only if} \quad \{i \in I : \mathbf{A}_i \models \Phi\} \in \mathcal{U}.$$

Thus, if a first order sentence of $\mathcal{L}_{\approx}$ is satisfied by all members of a class $\mathcal{K}$ of algebras of type $\mathcal{L}$, then it holds in any ultraproduct of members of $\mathcal{K}$.

Let $\Sigma \cup \{\Phi\}$ be a set of *sentences* of $\mathcal{L}_{\approx}$. We write $\Sigma \models \Phi$ if, whenever $\mathbf{A}$ is an algebra of type $\mathcal{L}$ such that $\mathbf{A} \models \Sigma$, we have $\mathbf{A} \models \Phi$. The following consequence of the Compactness Theorem of first order logic will be needed:

Theorem 0.7. [BS81, Corollary V.2.13] *If $\Sigma \cup \{\Phi\}$ is a set of (first order) sentences of $\mathcal{L}_{\approx}$ and $\Sigma \models \Phi$ then, for some finite subset Σ_0 of Σ, we have $\Sigma_0 \models \Phi$.*

A class $\mathcal{K}$ of algebras of type $\mathcal{L}$ is *axiomatized* by a set Σ of first order sentences if, for all algebras $\mathbf{A}$ of type $\mathcal{L}$, we have $\mathbf{A} \in \mathcal{K}$ if and only if $\mathbf{A} \models \Sigma$. Thus, for example, $\mathcal{K}$ is a variety if and only if it is axiomatized by a set of $\mathcal{L}$–identities. (This is Birkhoff's Theorem: see, e.g., [BS81, Theorem II.11.9].) A class $\mathcal{K}$ of algebras is called an *elementary* [resp. *strictly elementary*] *class* if it is axiomatized by a set [resp. a finite set] of first order sentences. A first order sentence of the form $(\forall x_1)(\forall x_2)\ldots(\forall x_n)(\Phi)$, where the variables $x_1,\ldots,x_n$ are distinct and Φ contains no occurrence of $\forall$ or $\exists$, is called a *universal sentence.* A class $\mathcal{K}$ of algebras is called a *universal class* if it is axiomatized by a set of universal sentences.

Theorem 0.8. [BS81, Theorem V.2.20] *A class $\mathcal{K}$ of algebras of type $\mathcal{L}$ is a universal class if and only if $\mathcal{K}$ is closed under* I, S *and* $\mathrm{P_U}$.

An ($\mathcal{L}$–) *quasi-equation* is an ($\mathcal{L}$–) equation or a first order formula of $\mathcal{L}_{\approx}$ of the form

$$((s_1 \approx t_1) \text{ and } \ldots \text{ and } (s_n \approx t_n)) \text{ implies } (s \approx t),$$

where s_i, t_i, s, t are $\mathcal{L}$–terms for $i = 1,\ldots,n$. The closure $\overline{\Phi}$ of an ($\mathcal{L}$–) quasi-equation Φ is called an ($\mathcal{L}$–)*quasi-identity.*

Quasivarieties. A class $\mathcal{K}$ of algebras of type $\mathcal{L}$ is called a *quasivariety* if it is closed under I, S, P and $\mathrm{P_U}$. Equivalently ([BS81, Theorem V.2.25]), $\mathcal{K}$ is a quasivariety if and only if it is axiomatized by a set of quasi-identities. If $\mathcal{W}$ is a class of similar algebras, we use $\mathrm{Q}(\mathcal{W})$ to denote the *quasivariety generated by* $\mathcal{W}$, i.e., $\mathrm{Q}(\mathcal{W}) = \mathrm{I\,S\,P\,P_U}(\mathcal{W})$.

For a first order formula Φ of $\mathcal{L}_{\approx}$ that is *not* a sentence, and an algebra $\mathbf{A}$ of type $\mathcal{L}$, we define $\mathbf{A} \models \Phi$ to mean that $\mathbf{A} \models \overline{\Phi}$, in which case we say that Φ is *valid* in $\mathbf{A}$ (or that $\mathbf{A}$ *satisfies* Φ). Thus, $\mathbf{A}$ satisfies a quasi-equation Φ if and only if it satisfies the quasi-identity $\overline{\Phi}$. For this reason, particularly in the context of axiomatization, we often abuse terminology by confusing a first order formula Φ with the sentence $\overline{\Phi}$ that is its closure, and by referring to (quasi-) equations as (quasi-) identities.

Now let $\mathcal{K} \cup \{\mathbf{A}\}$ be a class of algebras of type $\mathcal{L}$ and $\Sigma \cup \Sigma' \cup \{\Phi\}$ a set of first order formulas of $\mathcal{L}_{\approx}$ (over $\mathbf{X}$) that are not necessarily sentences. For each function $\overline{a} : \omega \rightarrow A$, with $\overline{a}(i) = a_i \in A$ for each $i \in \omega$, and each $\Psi \in \Sigma \cup \{\Phi\}$, let $\Psi[\overline{a}]$ (or $\Psi^{\mathbf{A}}(\overline{a})$) denote the result of simultaneously replacing each (if any) free occurence of $\mathbf{x}_i$ in Ψ by a_i, for each $i \in \omega$. Note that $\Psi[\overline{a}]$ is a sentence of $\mathcal{L}^A_{\approx}$. We define $\Sigma \models_{\mathbf{A}} \Phi$ to mean that for each $\overline{a} : \omega \rightarrow A$, if $\mathbf{A} \models \{\Psi[\overline{a}] : \Psi \in \Sigma\}$ then $\mathbf{A} \models \Phi[\overline{a}]$. We also define $\Sigma \models_{\mathcal{K}} \Phi$ to mean that $\Sigma \models_{\mathbf{B}} \Phi$ for all $\mathbf{B} \in \mathcal{K}$. We use $\Sigma \models_{\mathcal{K}} \Sigma'$ to abbreviate $\Sigma \models_{\mathcal{K}} \Psi$ for all $\Psi \in \Sigma'$. $\Sigma = \!\mid\!\models_{\mathcal{K}} \Sigma'$ abbreviates 'both $\Sigma \models_{\mathcal{K}} \Sigma'$ and $\Sigma' \models_{\mathcal{K}} \Sigma$'. Note that $\emptyset \models_{\mathcal{K}} \Sigma'$ has the same meaning as $\mathcal{K} \models \Sigma'$. On the other hand, if $\overline{\Sigma} = \{\overline{\Psi} : \Psi \in \Sigma\}$ and $\mathcal{K}$ is the class of all algebras of type $\mathcal{L}$, then $\Sigma \models_{\mathcal{K}} \Phi$ does *not* have the same meaning as $\overline{\Sigma} \models \overline{\Phi}$ (unless $\Sigma \cup \{\Phi\}$ consists of sentences).

For a quasivariety $\mathcal{K}$ and an algebra $\mathbf{A}$ of the same type, the $\mathcal{K}$*–congruences* (or *relative congruences*, if $\mathcal{K}$ is understood) of $\mathbf{A}$ are the congruences θ of $\mathbf{A}$ for which $\mathbf{A}/\theta \in \mathcal{K}$. We use $\mathrm{Con}_{\mathcal{K}}\,\mathbf{A}$ to denote the set of all $\mathcal{K}$–congruences of $\mathbf{A}$. When ordered by set inclusion, these form an algebraic lattice $\mathbf{Con}_{\mathcal{K}}\,\mathbf{A}$. We call $\mathbf{A}$ $\mathcal{K}$*–subdirectly irreducible* (or *relatively subdirectly irreducible*) if $\mathbf{A}$ has a smallest nonidentity $\mathcal{K}$–congruence. We call $\mathbf{A}$ $\mathcal{K}$*–simple* (or *relatively simple*) if $|\mathrm{Con}_{\mathcal{K}}\mathbf{A}| = 2$. We call $\mathbf{A}$ *finitely* $\mathcal{K}$*–subdirectly irreducible* (or *relatively finitely subdirectly irreducible*) if the least $\mathcal{K}$–congruence of $\mathbf{A}$ is meet irreducible in $\mathbf{Con}_{\mathcal{K}}\mathbf{A}$. The classes of all relatively subdirectly irreducible, all relatively finitely subdirectly irreducible and all relatively simple algebras in $\mathcal{K}$ are denoted $\mathcal{K}_{\mathrm{RSI}}$, $\mathcal{K}_{\mathrm{RFSI}}$ and $\mathcal{K}_{\mathrm{RS}}$, respectively. Note that $\mathcal{K}_{\mathrm{RSI}}$ and $\mathcal{K}_{\mathrm{RS}}$ exclude trivial algebras. We always have $\mathcal{K}_{\mathrm{S}} \subseteq \mathcal{K}_{\mathrm{RS}} \subseteq \mathcal{K}_{\mathrm{RSI}} \subseteq (\mathcal{K}_{\mathrm{RFSI}})_{\mathrm{NT}}$ and $\mathcal{K}_{\mathrm{SI}} \subseteq \mathcal{K}_{\mathrm{RSI}}$ and $\mathcal{K}_{\mathrm{FSI}} \subseteq \mathcal{K}_{\mathrm{RFSI}}$.

For $X \subseteq A^2$, the $\mathcal{K}$*–congruence of* $\mathbf{A}$ *generated by* X, denoted $\Theta^{\mathbf{A}}_{\mathcal{K}}(X)$, is the smallest $\mathcal{K}$–congruence of $\mathbf{A}$ containing X. For $a, b \in A$, we abbreviate

$\Theta_{\mathcal{K}}^{\mathbf{A}}(\{(a,b)\})$ by $\Theta_{\mathcal{K}}^{\mathbf{A}}(a,b)$. If $\theta \in \mathrm{Con}_{\mathcal{K}}\mathbf{A}$ and $\theta = \Theta_{\mathcal{K}}^{\mathbf{A}}(a,b)$ for some $a, b \in A$, we call θ a *principal* $\mathcal{K}$–congruence (or relative congruence).

A quasivariety $\mathcal{K}$ with an equationally definable constant term 0 is said to be *relatively* 0*–regular* if, whenever $\mathbf{A} \in \mathcal{K}$ and $\theta_1, \theta_2 \in \mathrm{Con}_{\mathcal{K}}\mathbf{A}$ with $0^{\mathbf{A}}/\theta_1 = 0^{\mathbf{A}}/\theta_2$, we have $\theta_1 = \theta_2$. We call a quasivariety $\mathcal{K}$ *relatively congruence distributive* if $\mathbf{Con}_{\mathcal{K}}\mathbf{A}$ is a distributive lattice for every $\mathbf{A} \in \mathcal{K}$.

We shall use the following quasivarietal analogue of Birkhoff's Subdirect Decomposition Theorem (see, e.g., [BS81, Theorem II.8.6]), which follows as a special case of [BP92, Theorem 9.2], but can be proved by generalizing the proof strategy of the standard Birkhoff Subdirect Decomposition Theorem in a straightforward manner.

Theorem 0.9. *Let $\mathcal{K}$ be a quasivariety. Then every member $\mathbf{A}$ of $\mathcal{K}$ is isomorphic to a subdirect product of $\mathcal{K}$-subdirectly irreducible algebras in $\mathcal{K}$ (which are homomorphic images of $\mathbf{A}$). Thus, $\mathcal{K} = \mathrm{I}\,\mathrm{P_S}(\mathcal{K}_{\mathrm{RSI}})$.*

The original Birkhoff Theorem $\mathbf{A} \in \mathrm{I}\,\mathrm{P_S}((\mathrm{H}(\mathbf{A}))_{\mathrm{SI}})$ may be considered to be the special case where $\mathcal{K} = \mathrm{V}(\mathbf{A})$. Its corollary that $\mathcal{V} = \mathrm{I}\,\mathrm{P_S}(\mathcal{V}_{\mathrm{SI}})$ for every variety $\mathcal{V}$ also follows.

By a *finitely generated quasivariety* [resp. a *finitely generated variety*] we mean a class of the form $\mathcal{K} = \mathrm{Q}(\mathcal{K}')$ [resp. $\mathcal{K} = \mathrm{V}(\mathcal{K}')$] for some finite set $\mathcal{K}'$ of *finite* algebras. A quasivariety $\mathcal{K}$ is called *locally finite* if every finitely generated algebra in $\mathcal{K}$ is finite. It follows easily from [BS81, Theorem II.10.16] that every finitely generated quasivariety or variety is locally finite. A variety [resp. quasivariety] $\mathcal{K}$ is said to be *generated as a variety* [resp. *quasivariety*] *by its finite members* if $\mathcal{K} = \mathrm{V}(\mathcal{K}_{\mathrm{fin}})$ [resp. $\mathcal{K} = \mathrm{Q}(\mathcal{K}_{\mathrm{fin}})$], where $\mathcal{K}_{\mathrm{fin}}$ is the class of all finite algebras in $\mathcal{K}$. This is true whenever $\mathcal{K}$ is locally finite.

Finite Axiomatizability. A quasivariety $\mathcal{K}$ of type $\mathcal{L}$ is said to be *finitely based* (or *finitely axiomatizable*) if $\mathcal{K}$ is axiomatized by some *finite* set Σ consisting of *$\mathcal{L}$–quasi-identities.* When $\mathcal{K}$ is actually a *variety*, it follows that $\mathcal{K}$ is finitely based if and only if it is axiomatized by a finite set of *$\mathcal{L}$–identities.* This is a consequence of the Compactness Theorem (see Theorem 0.7). We shall need the following "finite basis theorems".

Theorem 0.10. [Pi88] *Every finitely generated relatively congruence distributive quasivariety of finite type is finitely based.*

The specialization to varieties of the above result was proved earlier by K. Baker [Bak77]. Baker's theorem also admits the following generalization, which was proved by Jónsson:

Theorem 0.11. [Jón79], [BS81, Theorem V.4.17] *If $\mathcal{V}$ is a congruence distributive variety of finite type and $\mathcal{V}_{\mathrm{FSI}}$ is a strictly elementary class then $\mathcal{V}$ is finitely based.*

Algebraizable Hilbert Systems. In [BP89] a Hilbert system S over a language $\mathcal{L}$ is defined to be *algebraizable* if there exist a finite family $\boldsymbol{\Delta} = \{\Delta_1, \ldots, \Delta_m\}$ of binary formulas and finite families $\boldsymbol{\delta} = \{\delta_1, \ldots, \delta_r\}$ and $\boldsymbol{\varepsilon} = \{\varepsilon_1, \ldots, \varepsilon_r\}$ of unary formulas such that for any connective α (of rank n, say) and formulas $\varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_n, \varphi, \psi, \zeta$, the following five conditions hold for $j = 1, \ldots, m$:

(i) $\vdash_S \Delta_j(\varphi, \varphi)$
(ii) $\{\Delta_i(\varphi, \psi) : i = 1, \ldots, m\} \vdash_S \Delta_j(\psi, \varphi)$
(iii) $\{\Delta_i(\varphi, \psi) : i = 1, \ldots, m\} \cup \{\Delta_i(\psi, \zeta) : i = 1, \ldots, m\} \vdash_S \Delta_j(\varphi, \zeta)$
(iv) $\{\Delta_i(\varphi_k, \psi_k) : i = 1, \ldots, m\, ;\ k = 1, \ldots, n\} \vdash_S$
$\Delta_j(\alpha(\varphi_1, \ldots, \varphi_n), \alpha(\psi_1, \ldots, \psi_n))$
(v) $\zeta \dashv\vdash_S \{\Delta_i(\delta_t(\zeta), \varepsilon_t(\zeta)) : i = 1, \ldots, m\, ;\ t = 1, \ldots, r\}$

In this case, there is a unique quasivariety $\mathcal{K}$ of algebras of type $\mathcal{L}$ in which for any set $\Gamma \cup \{\varphi, \psi\}$ of formulas and for $l = 1, \ldots, r$, we have

(1) $\Gamma \vdash_S \varphi$ iff $\{\delta_t(\zeta) \approx \varepsilon_t(\zeta) : \zeta \in \Gamma\, ;\ t = 1, \ldots, r\} \models_{\mathcal{K}} \delta_l(\varphi) \approx \varepsilon_l(\varphi)$,
(2) $\varphi \approx \psi \mathrel{=\!\mid\!\models_{\mathcal{K}}} \{\delta_t(\Delta_i(\varphi, \psi)) \approx \varepsilon_t(\Delta_i(\varphi, \psi)) : i = 1, \ldots, m\, ;\ t = 1, \ldots, r\}$.

Given any axiomatization $\mathrm{Ax} \cup \mathrm{Ir}$ of S, where Ax is a set of axioms and Ir a set of inference rules with nonempty sets of premisses, the aforementioned quasivariety $\mathcal{K}$ is axiomatized by the identities

(vi) $\delta_t(\varphi) \approx \varepsilon_t(\varphi),\ t = 1, \ldots, r\, ;\ \varphi \in \mathrm{Ax}$, and
$\delta_t(\Delta_i(x, x)) \approx \varepsilon_t(\Delta_i(x, x)),\ i = 1, \ldots, m\, ;\ t = 1, \ldots, r$

together with the quasi-identities

(vii) $(\bigwedge_{u=1}^{n} \bigwedge_{t=1}^{r} \delta_t(\zeta_u) \approx \varepsilon_t(\zeta_u))$ implies $\delta_l(\varphi) \approx \varepsilon_l(\varphi)$,
$l = 1, \ldots, r,\ \langle\{\zeta_1, \ldots, \zeta_n\}, \varphi\rangle \in \mathrm{Ir}$, and
$(\bigwedge_{i=1}^{m} \bigwedge_{t=1}^{r} \delta_t(\Delta_i(x, y)) \approx \varepsilon_t(\Delta_i(x, y)))$ implies $x \approx y$

[BP89, Theorems 2.17 and 4.7]. Following [BP89], we call $\mathcal{K}$ the *equivalent quasivariety semantics* of S. The formulas in $\boldsymbol{\Delta}$ and the equations $\delta_t \approx \varepsilon_t, t = 1, \ldots, r$, are called *equivalence formulas* and *defining equations* for S and $\mathcal{K}$. They are unique up to interderivability over S and $\mathcal{K}$, respectively. If, in addition, $p, q \vdash_S \Delta_j(p, q)$ for $j = 1, \ldots, m$, we say that S has the *Gödel Rule.*

The next lemma extends [BP89, Corollary 5.4]; a full proof appears, e.g., in [vAl95, Theorem 3.2.4, p182].

Lemma 0.12. *Let S be an algebraizable Hilbert system with equivalent quasivariety semantics $\mathcal{K}$, equivalence formulas Δ_j, $j = 1, \ldots, m$, and defining equations $\delta_t \approx \varepsilon_t$, $t = 1, \ldots, r$, such that S has the Gödel Rule. Then $\mathcal{K}$ satisfies $\Delta_i(x,x) \approx \Delta_j(y,y)$ for all i,j, hence there is a constant term $\mathrm{T} = \Delta_i(x,x)$ definable over $\mathcal{K}$ and, moreover, $\mathcal{K}$ is relatively* T*–regular.*

If S is an algebraizable Hilbert system whose equivalent quasivariety semantics $\mathcal{K}$ is a variety, we say that S is *strongly algebraizable* with *equivalent variety semantics* $\mathcal{K}$. A quasivariety $\mathcal{K}$ satisfying (1) (but not necessarily (2)) above, for some $\boldsymbol{\delta}$, $\boldsymbol{\varepsilon}$, is called an *algebraic semantics* for S.

Let S be a Hilbert system with axiomatization I'. A Hilbert system S' over the same language as S that is axiomatized by I' together with some additional axioms is called an *axiomatic extension* of S. If S is an algebraizable Hilbert system then every axiomatic extension of S is also algebraizable with the same defining equations and equivalence formulas as S [BP89, Corollary 4.9].

CHAPTER 1

RESIDUATED ORDERED MONOIDS

In this chapter we define the class $\mathcal{LM}$ of 'polrims' and the class $\mathcal{LR}$ of all 'residuation–subreducts' of polrims, which we call 'left residuation algebras'. Some properties of these classes of algebras are given in Proposition 1.1. In Proposition 1.4(i) we present an axiomatization of the class $\mathcal{LR}$. This result is implicit in [OK85] and made precise in [Wro85]; we present a sketch of the proof here for the sake of completeness. In part (ii) of Proposition 1.4 we present an axiomatization of the class $\mathcal{LM}$. Some natural examples of polrims and left residuation algebras are given. Lastly, we investigate and characterize the class of residuation–*reducts* of polrims, or 'left residuation algebras with condition (S′)'; these are unpublished results of J.G. Raftery and the author.

A *pomonoid* is a structure $\langle A; \oplus, 0; \leq\rangle$ where $\langle A; \leq\rangle$ is a partially ordered set, $\langle A; \oplus, 0\rangle$ is a monoid and $\leq$ is *compatible* with the monoid operation in the sense that for all $a, b, c \in A$,

$$a \leq b \text{ implies } a \oplus c \leq b \oplus c \text{ and } c \oplus a \leq c \oplus b.$$

Such a pomonoid is called *integral* if 0 is the least element of $\langle A; \leq\rangle$; it is called *left residuated* if for any $a, b \in A$, there is a least element $c \in A$ such that $a \leq c \oplus b$. We denote this element by $a \dot{-} b$ and call $\dot{-}$ the *left residuation operation* of $\langle A; \oplus, 0; \leq\rangle$. In this case, therefore,

$$a \leq (a \dot{-} b) \oplus b. \tag{3}$$

Given a left residuated pomonoid $\langle A; \oplus, 0; \leq\rangle$ and any $a, b, d \in A$, we have

$$a \dot{-} b \leq d \quad \text{if and only if} \quad a \leq d \oplus b. \tag{4}$$

Thus $d \oplus b$ is the greatest element of $\{e \in A : e \dot{-} b \leq d\}$, and

$$a \dot{-} b \leq 0 \quad \text{if and only if} \quad a \leq b.$$

When $\langle A; \oplus, 0; \leq \rangle$ is also integral, we have

$$(5) \qquad a \dot{-} b = 0 \quad \text{if and only if} \quad a \leq b,$$

so that the partial order $\leq$ is determined by the operations $\dot{-}$ and 0, making such integral structures amenable to purely algebraic investigation. An algebra $\langle A; \oplus^{\mathbf{A}}, \dot{-}^{\mathbf{A}}, 0^{\mathbf{A}} \rangle$ of type $\langle 2, 2, 0 \rangle$ will be called a *polrim*[4] if the binary relation on A defined by $a \leq^{\mathbf{A}} b$ if and only if $a \dot{-}^{\mathbf{A}} b = 0^{\mathbf{A}}$ is such that $\langle A; \oplus^{\mathbf{A}}, 0^{\mathbf{A}}; \leq^{\mathbf{A}} \rangle$ is a left residuated integral pomonoid whose left residuation operation is $\dot{-}^{\mathbf{A}}$. We drop the superscripts when there is no danger of confusion. The class of all polrims will be denoted $\mathcal{LM}$.

Analogously, we call a pomonoid $\langle A; \oplus, 0; \leq \rangle$ *right residuated* if for any $a, b \in A$, there is a least element $c \in A$ such that $a \leq b \oplus c$. We denote this element by $a \div b$ and call $\div$ the *right residuation operation* of $\langle A; \oplus, 0; \leq \rangle$. As above, we will call an algebra $\langle A; \oplus^{\mathbf{A}}, \div^{\mathbf{A}}, 0^{\mathbf{A}} \rangle$ a *porrim* if the binary relation on A defined by $a \leq^{\mathbf{A}} b$ if and only if $a \div^{\mathbf{A}} b = 0^{\mathbf{A}}$ is such that $\langle A; \oplus^{\mathbf{A}}, 0^{\mathbf{A}}; \leq^{\mathbf{A}} \rangle$ is a right residuated integral pomonoid whose right residuation operation is $\div^{\mathbf{A}}$. Again, we drop the superscripts when there is no danger of confusion. Clearly, every porrim $\mathbf{A} = \langle A; \oplus, \div, 0 \rangle$ is termwise equivalent to its 'opposite' polrim $\langle A; +, \dot{-}, 0 \rangle$, in which $a + b = b \oplus a$ for any $a, b \in A$. In this sense, it suffices to investigate polrims only. On the other hand, infinite ordinals provide examples to show that when an integral pomonoid is left and right residuated, the associated polrim and porrim need not be even termwise equivalent: see Examples 1.8 and 1.10 below.

By a *left residuation algebra* we mean a $\langle \dot{-}, 0 \rangle$–subreduct (i.e., a subalgebra of the $\langle \dot{-}, 0 \rangle$–reduct) of a polrim. The class of all left residuation algebras will be denoted $\mathcal{LR}$. Note that a left residuation algebra $\mathbf{A} = \langle A; \dot{-}, 0 \rangle$ is partially ordered by the relation $\leq$ defined by $a \leq b$ if and only if $a \dot{-} b = 0$ $(a, b \in A)$. We similarly define a *right residuation algebra* to be a $\langle \div, 0 \rangle$–subreduct of a porrim.

We adopt the convention for $\dot{-}$ (and for $\div$) that omitted parentheses are associated to the left, e.g., $x \dot{-} y \dot{-} z \dot{-} w$ abbreviates $((x \dot{-} y) \dot{-} z) \dot{-} w$. The following proposition presents some properties of $\mathcal{LM}$ and $\mathcal{LR}$.

Proposition 1.1. *The class* $\mathcal{LM}$ *satisfies* (A1)–(A8) *below. Thus the class* $\mathcal{LR}$ *satisfies* (A1)–(A4).

(A1) $\quad x \dot{-} y \dot{-} (z \dot{-} y) \dot{-} (x \dot{-} z) \approx 0 \quad$ (i.e. $x \dot{-} y \dot{-} (z \dot{-} y) \leq x \dot{-} z$),

(A2) $\quad x \dot{-} 0 \approx x$,

(A3) $\quad 0 \dot{-} x \approx 0 \quad$ (i.e. $0 \leq x$),

[4] acronym for *partially ordered left residuated integral monoid.*

(A4) $\quad x \dot{-} y \approx 0$ and $y \dot{-} x \approx 0$ implies $x \approx y$,

(A5) $\quad x \dot{-} (y \oplus z) \approx x \dot{-} z \dot{-} y$,

(A6) $\quad (x \oplus y) \dot{-} y \leq x$,

(A7) $\quad ((x \oplus y) \dot{-} y) \oplus y \approx x \oplus y$,

(A8) $\quad ((x \dot{-} y) \oplus y) \dot{-} y \approx x \dot{-} y$.

Proof. By (3), $\mathcal{LM}$ satisfies $z \leq (z \dot{-} y) \oplus y$. Thus, by (3) again, the compatibility of $\leq$ with $\oplus$ and the associativity of $\oplus$, $\mathcal{LM}$ satisfies

$$x \leq (x \dot{-} z) \oplus z \leq (x \dot{-} z) \oplus ((z \dot{-} y) \oplus y) \approx ((x \dot{-} z) \oplus (z \dot{-} y)) \oplus y,$$

and hence also $x \dot{-} y \leq (x \dot{-} z) \oplus (z \dot{-} y)$, by (4). By (4) again, $\mathcal{LM}$ satisfies $x \dot{-} y \dot{-} (z \dot{-} y) \leq x \dot{-} z$, proving (A1). (A2) and (A3) follow easily from the definition of $\dot{-}$ and the fact that 0 is the least element of $\leq$, and (A4) follows easily from (5).

By (3), the compatibility of $\leq$ with $\oplus$ and the associativity of $\oplus$, we have that $\mathcal{LM}$ satisfies

$$x \leq (x \dot{-} z) \oplus z \leq ((x \dot{-} z \dot{-} y) \oplus y) \oplus z \approx (x \dot{-} z \dot{-} y) \oplus (y \oplus z),$$

and hence also $x \dot{-} (y \oplus z) \leq x \dot{-} z \dot{-} y$, by (4). We similarly have that $\mathcal{LM}$ satisfies

$$x \leq (x \dot{-} (y \oplus z)) \oplus (y \oplus z) \approx ((x \dot{-} (y \oplus z)) \oplus y) \oplus z.$$

By (4), therefore, $\mathcal{LM}$ satisfies $x \dot{-} z \leq (x \dot{-} (y \oplus z)) \oplus y$ and hence also $x \dot{-} z \dot{-} y \leq x \dot{-} (y \oplus z)$. Thus (A5) follows immediately by (A4).

(A6) follows from the fact noted after (4). (A7) and (A8) follow from (3), (A6) and (A4). □

In Proposition 1.4 we shall present an axiomatization for each of the classes $\mathcal{LM}$ and $\mathcal{LR}$. We shall need the following results there.

Lemma 1.2. *Let* $\mathbf{A} = \langle A; \dot{-}, 0\rangle$ *be an algebra of type* $\langle 2, 0\rangle$ *that satisfies* (A1), (A2) *and* (A3). *Let* $\leq$ *be the binary relation defined on* A *by* $a \leq b$ *if and only if* $a \dot{-} b = 0$ $(a, b \in A)$. *Then* $\mathbf{A}$ *satisfies the following:*

(A9) $\quad x \dot{-} x \approx 0$,

(A10) $\quad x \leq y$ implies $x \dot{-} z \leq y \dot{-} z$,

(A11) $\quad x \leq y$ implies $z \dot{-} y \leq z \dot{-} x$,

(A12) $\quad x \dot{-} y \leq x$.

In particular, every polrim or left residuation algebra satisfies (A9), (A10), (A11) *and* (A12). *If* **A** *also satisfies* (A4), *then* $\langle A; \leq \rangle$ *is a partially ordered set whose least element is* 0.

Proof. Suppose **A** satisfies (A1), (A2) and (A3). (A9) is derivable from (A1) and (A2) in the following way:

$$x \dot{-} x \approx x \dot{-} 0 \dot{-} (x \dot{-} 0) \approx x \dot{-} 0 \dot{-} (0 \dot{-} 0) \dot{-} (x \dot{-} 0) \approx 0.$$

(A10) and (A11) are easily derivable from (A1) and (A2). By (A3), **A** satisfies $0 \leq y$, hence (A12) follows from (A11) and (A9). By Proposition 1.1, every polrim and left residuation algebra satisfies (A9)–(A12).

Now suppose that **A** also satisfies (A4). By (A9), $\leq$ is reflexive. Symmetry of $\leq$ follows from (A4). Let $a, b, c \in A$ such that $a \leq b$ and $b \leq c$. By (A2) and (A1),

$$a \dot{-} c = ((a \dot{-} c) \dot{-} 0) \dot{-} 0 = ((a \dot{-} c) \dot{-} (b \dot{-} c)) \dot{-} (a \dot{-} b) = 0,$$

i.e., $a \leq c$. Thus $\langle A; \leq \rangle$ is a partially ordered set whose least element, in view of (A3), is 0. □

The following construction for producing polrims and left residuation algebras is due to Ono and Komori. Let $\mathbf{M} = \langle M; +, 0; \leq \rangle$ be an integral pomonoid. Let $C(M)$ denote the set of all upward closed subsets of $\langle M; \leq \rangle$. For $X, Y \in C(M)$, define

$$X \oplus Y = [\{a + b : a \in X \text{ and } b \in Y\}), \tag{6}$$

$$X \dot{-} Y = \{c \in M : c + Y \subseteq X\}, \tag{7}$$

where

$$c + Y = \{c + b : b \in Y\}.$$

Lemma 1.3. [OK85] *Let* $\mathbf{M} = \langle M; +, 0; \leq \rangle$ *be an integral pomonoid and let* $\oplus$ *and* $\dot{-}$ *be the binary operations on* $C(M)$ *defined in* (6) *and* (7), *respectively. Then* $\langle C(M); \oplus, \dot{-}, M \rangle$ *is a polrim whose associated partial order is* $\supseteq$. *In particular,* $\langle C(M); \dot{-}, M \rangle$ *is a left residuation algebra.*

Proposition 1.4. (i) [OK85], [Wro85] *An algebra* $\langle A; \dot{-}, 0 \rangle$ *of type* $\langle 2, 0 \rangle$ *is a left residuation algebra if and only if it satisfies* (A1), (A2), (A3) *and* (A4). *Consequently,* $\mathcal{LR}$ *is a quasivariety.*
(ii) *An algebra* $\langle A; \oplus, \dot{-}, 0 \rangle$ *of type* $\langle 2, 2, 0 \rangle$ *is a polrim if and only if it satisfies* (A1), (A2), (A3), (A4) *and* (A5). *Consequently,* $\mathcal{LM}$ *is a quasivariety.*

Proof. (i) By Proposition 1.1, we know that a left residuation algebra satisfies (A1)–(A4). We sketch here a proof of the converse, for the sake of completeness. Let $\mathbf{A} = \langle A; \dot{-}, 0\rangle$ be an algebra of type $\langle 2, 0\rangle$ that satisfies (A1)–(A4). Let $\leq$ be the relation defined on A by $a \leq b$ if and only if $a \dot{-} b = 0$ $(a, b \in A)$. By Lemma 1.2, $\langle A; \leq\rangle$ is a partially ordered set whose least element is 0 and $\mathbf{A}$ satisfies (A9)–(A12).

Let M be the set of all finite nonempty sequences of elements of A. If $\vec{a} = a_1, \ldots, a_n \in M$ and $c \in A$ then we shall abbreviate $c \dot{-} a_1 \dot{-} \ldots \dot{-} a_n$ by $c \dot{-} \vec{a}$. Define a relation $\preceq'$ on M by

$$\vec{a} \preceq' \vec{b} \quad \text{if and only if, for all } c \in A \text{ and all } \vec{d} \in M,$$
$$c \dot{-} \vec{a} \dot{-} \vec{d} = 0 \quad \text{implies} \quad c \dot{-} \vec{b} \dot{-} \vec{d} = 0.$$

Then $\preceq'$ is a quasiorder on M, so $\preceq' \cap (\preceq')^{-1}$ is an equivalence relation on M. We shall abbreviate $\preceq' \cap (\preceq')^{-1}$ by $\equiv$ and use $[\vec{a}]$ to denote the equivalence class of $\vec{a}$ with respect to $\equiv$. The relation $\preceq$ on $M/\equiv$ defined by

$$[\vec{a}] \preceq [\vec{b}] \quad \text{if and only if, for all } c \in A \text{ and all } \vec{d} \in M,$$
$$c \dot{-} \vec{a} \dot{-} \vec{d} = 0 \quad \text{implies} \quad c \dot{-} \vec{b} \dot{-} \vec{d} = 0$$

is therefore a well-defined partial ordering of $M/\equiv$. Define a binary operation $+$ on $M/\equiv$ by

$$[\vec{a}] + [\vec{b}] = [\vec{b}, \vec{a}].$$

It follows easily from the definitions that $\preceq$ is compatible with $+$ and hence that $\langle M/\equiv; +, [0]; \preceq\rangle$ is an integral pomonoid. By Lemma 1.3, therefore, $\langle C(M/\equiv); \oplus, \dot{-}, M/\equiv\rangle$, where $\oplus$ and $\dot{-}$ are defined as in (6) and (7), is a polrim.

For $a, b \in A$, we have $[a] \preceq [a \dot{-} b] + [b] = [b, a \dot{-} b]$. For, if $c \in A$ and $\vec{d} \in M$ such that $c \dot{-} a \dot{-} \vec{d} = 0$, then by (A1) and repeated applications of (A10),

$$c \dot{-} b \dot{-} (a \dot{-} b) \dot{-} \vec{d} \leq c \dot{-} a \dot{-} \vec{d} = 0,$$

hence $c \dot{-} b \dot{-} (a \dot{-} b) \dot{-} \vec{d} = 0$. Moreover, $[a \dot{-} b]$ is the least $[\vec{e}] \in M/\equiv$ for which $[a] \preceq [\vec{e}] + [b]$. For suppose that

$$[a] \preceq [\vec{e}] + [b] = [b, \vec{e}] \tag{8}$$

and that for some $c \in A$ and $\vec{d} \in M$,

$$c \dot{-} (a \dot{-} b) \dot{-} \vec{d} = 0. \tag{9}$$

From $a \dot{-} a = 0$ and (8), we get $a \dot{-} b \dot{-} \vec{e} = 0$, hence

$$\begin{aligned} c \dot{-} \vec{e} \dot{-} \vec{d} &= c \dot{-} \vec{e} \dot{-} (a \dot{-} b \dot{-} \vec{e}) \dot{-} \vec{d} \\ &\leq c \dot{-} (a \dot{-} b) \dot{-} \vec{d} \\ &\quad \text{(by repeated applications of (A1) and (A10))} \\ &= 0 \quad \text{(by (9))}. \end{aligned}$$

Thus $[a \dot{-} b] \preceq [\vec{e}]$.

Define a map $f : A \to C(M/\equiv)$ by $f(a) = [[a])$. For $a, b \in A$ we have, by (7), that

$$f(a) \dot{-} f(b) = [[a]) \dot{-} [[b]) = [\{[\vec{e}] \in M/\equiv : [\vec{e}] + [[b]) \subseteq [[a])\}).$$

By the above observations, we know that $[a \dot{-} b]$ is the least $[\vec{e}] \in M/\equiv$ that satisfies $[a] \preceq [\vec{e}] + [b]$, i.e. $[\vec{e}] + [[b]) \subseteq [[a])$, hence $f(a) \dot{-} f(b) = [[a \dot{-} b]) = f(a \dot{-} b)$. That f is a one-to-one map follows from the definition of $\preceq$ and (A9) and (A4). We therefore have that f embeds $\mathbf{A}$ into $\langle C(M/\equiv); \dot{-}, M/\equiv\rangle$, which is the $\langle \dot{-}, M/\equiv\rangle$-reduct of the polrim $\langle C(M/\equiv); \oplus, \dot{-}, M/\equiv\rangle$, which completes the proof of (i).

(ii) By Proposition 1.1 we know that a polrim satisfies (A1)–(A5). Conversely, suppose $\mathbf{A} = \langle A; \oplus, \dot{-}, 0\rangle$ is an algebra of type $\langle 2, 2, 0\rangle$ satisfying (A1)–(A5). Let $\leq$ be the relation defined on A by $a \leq b$ if and only if $a \dot{-} b = 0$ ($a, b \in A$). By Lemma 1.2, $\langle A; \leq\rangle$ is a partially ordered set whose least element is 0 and $\mathbf{A}$ satisfies (A9)–(A12). If $d = a \oplus (b \oplus c)$ and $e = (a \oplus b) \oplus c$ then, by (A5),

$$\begin{aligned} d \dot{-} e &= d \dot{-} c \dot{-} (a \oplus b) = d \dot{-} c \dot{-} b \dot{-} a = d \dot{-} (b \oplus c) \dot{-} a = d \dot{-} (a \oplus (b \oplus c)) = 0, \\ e \dot{-} d &= e \dot{-} (b \oplus c) \dot{-} a = e \dot{-} c \dot{-} b \dot{-} a = e \dot{-} c \dot{-} (a \oplus b) = e \dot{-} ((a \oplus b) \oplus c) = 0, \end{aligned}$$

so by (A4), $\oplus$ is associative. Using (A2), (A3) and (A5), we have

$$\begin{aligned} a \dot{-} (a \oplus 0) &= a \dot{-} 0 \dot{-} a = a \dot{-} a = 0, \\ (a \oplus 0) \dot{-} a &= (a \oplus 0) \dot{-} 0 \dot{-} a = (a \oplus 0) \dot{-} (a \oplus 0) = 0, \\ a \dot{-} (0 \oplus a) &= a \dot{-} a \dot{-} 0 = 0, \\ \text{and} \quad (0 \oplus a) \dot{-} a &= (0 \oplus a) \dot{-} a \dot{-} 0 = (0 \oplus a) \dot{-} (0 \oplus a) = 0, \end{aligned}$$

so by (A4), $a = a \oplus 0 = 0 \oplus a$ and $\langle A; \oplus, 0\rangle$ is a monoid. Suppose $a \leq b$. By (A5), (A2) and (A1),

$$\begin{aligned} (a \oplus c) \dot{-} (b \oplus c) &= (a \oplus c) \dot{-} c \dot{-} b \dot{-} 0 \dot{-} 0 \\ &= (a \oplus c) \dot{-} c \dot{-} b \dot{-} (a \dot{-} b) \dot{-} ((a \oplus c) \dot{-} c \dot{-} a) = 0; \end{aligned}$$

$$\begin{aligned}
(c \oplus a) \dot{-} (c \oplus b) &= (c \oplus a) \dot{-} b \dot{-} 0 \dot{-} c \dot{-} 0 \dot{-} 0 \\
&= (c \oplus a) \dot{-} b \dot{-} (a \dot{-} b) \dot{-} c \dot{-} ((c \oplus a) \dot{-} a \dot{-} c) \\
&\quad \dot{-} ((c \oplus a) \dot{-} b \dot{-} (a \dot{-} b) \dot{-} ((c \oplus a) \dot{-} a)) = 0,
\end{aligned}$$

so $\leq$ is compatible with $\oplus$. Finally, from the fact that $a \leq c \oplus b$ if and only if $0 = a \dot{-} (c \oplus b) = a \dot{-} b \dot{-} c$, if and only if $a \dot{-} b \leq c$, it follows that $\dot{-}$ is a left residuation operation for the integral pomonoid $\langle A; \oplus, 0; \leq \rangle$, and hence that $\langle A; \oplus, \dot{-}, 0 \rangle$ is a polrim. $\square$

(This proof also shows that a *finite* algebra in $\mathcal{LR}$ is embeddable into the $\langle \dot{-}, 0 \rangle$-reduct of a *finite* polrim.)

In [Kom83], Komori introduced the term 'BCC-algebra' for an algebra of type $\langle 2, 0 \rangle$ that satisfies (A1)–(A4). It follows that left residuation algebras coincide with 'BCC-algebras'. For reasons concerning deductive systems to be discussed later, we prefer, in this thesis, to avoid the term BCC-algebra. The quasivarieties $\mathcal{LM}$ and $\mathcal{LR}$ are not varieties: see Section 4.1.

We present some examples of polrims and left residuation algebras.

Example 1.5. A polrim $\langle A; \oplus, \dot{-}, 0 \rangle$ whose monoid operation is commutative is called a *pocrim.* Pocrims have been studied in [Hig84], [Fle88] and [BR97]; their residuation– (i.e., $\langle \dot{-}, 0 \rangle$–) reducts were studied earlier under the name 'BCK–algebras with condition (S)' [Isé79]. The class of all pocrims, denoted $\mathcal{M}$, was shown in [Isé80] to be axiomatized by (A2), (A3), (A4), (A5) and

(A13) $\qquad x \dot{-} y \dot{-} (x \dot{-} z) \dot{-} (z \dot{-} y) \approx 0.$

Thus $\mathcal{M}$ is a quasivariety; it is not a variety [Hig84]. An algebra $\langle A; \dot{-}, 0 \rangle$ of type $\langle 2, 0 \rangle$ is called a *BCK–algebra* if it satisfies (A2), (A3), (A4) and (A13). The class of all BCK–algebras, denoted $\mathcal{BCK}$, is precisely the class of all residuation–subreducts of pocrims [Pał82], [OK85], [Fle88], [Wro85]. The class $\mathcal{BCK}$ is also a quasivariety that is not a variety [Wro83]. The following are important examples of identities that hold in $\mathcal{BCK}$ but not in $\mathcal{LR}$:

(A14) $\qquad x \dot{-} y \dot{-} z \approx x \dot{-} z \dot{-} y,$

(A15) $\qquad x \dot{-} (x \dot{-} y) \dot{-} y \approx 0, \quad$ (i.e., $x \dot{-} (x \dot{-} y) \leq y$).

In fact, it is well known that $\mathcal{BCK}$ is axiomatized by (A1)–(A4) and (A14). We also have the following result:

Lemma 1.6. *The quasivariety $\mathcal{BCK}$ is axiomatized by* (A1)–(A4) *and* (A15).

Proof. Let $\mathcal{K}$ be the class of algebras of type $\langle 2, 0 \rangle$ over the language $\langle \dot{-}, 0 \rangle$ axiomatized by (A1)–(A4) and (A15). Then $\mathcal{K}$ is a quasivariety of left residuation algebras. By (A1), $\mathcal{K}$ satisfies $x \dot{-} y \dot{-} (z \dot{-} y) \leq x \dot{-} z$ hence, $\mathcal{K}$ satisfies

$$\begin{aligned} x \dot{-} y \dot{-} (x \dot{-} z) &\leq x \dot{-} y \dot{-} (x \dot{-} y \dot{-} (z \dot{-} y)) \quad \text{(by (A11))} \\ &\leq z \dot{-} y \quad \text{(by (A15))}, \end{aligned}$$

so $\mathcal{K}$ satisfies (A13), which completes the proof. □

A wealth of literature on BCK–algebras exists and includes survey articles [IT78], [Cor82]; the more recent paper [BR95] also serves partially as a survey.

Example 1.7. Let $\mathbf{R} = \langle R; +, \cdot, -, 0, 1\rangle$ be a ring with identity and let $\operatorname{Id}\mathbf{R}$ denote the lattice of (two-sided) ideals of $\mathbf{R}$. For $I, J \in \operatorname{Id}\mathbf{R}$, define $I \cdot J$ to be the ideal of $\mathbf{R}$ generated by $\{i \cdot j : i \in I; j \in J\}$ and $I : J$ to be the ideal $\{r \in R : r \cdot J \subseteq I\}$. Then $\langle \operatorname{Id}\mathbf{R}; \cdot, :, R\rangle$ is a polrim whose partial order is that of *reversed* set inclusion. A recent development in ring theory is the study of the lattices of *topologizing filters of rings*, considered as polrims; these extend the ideal lattices in respect of all interesting operations [Gol87], [vdB1], [vdB2]. The definition of a topologizing filter on a ring $\mathbf{R}$ and a discussion of the lattice of all such filters on $\mathbf{R}$ appear in the Appendix. These lattices provide natural examples of integral pomonoids that are residuated on the left but *not* on the right.

Example 1.8. Let α be a nonzero ordinal. Recall that for $\beta, \gamma \in \alpha$, we write $\beta + \gamma$ for the usual sum of the ordinals β and γ. If α is closed under $+$ then $\langle \alpha; \oplus, 0; \subseteq\rangle$, where $\oplus$ is $+$, is an integral pomonoid residuated on both sides. Thus α gives rise to a polrim $\langle \alpha; \oplus, \dot{-}, 0\rangle$ and a porrim $\langle \alpha; \oplus, \underset{\cdot}{-}, 0\rangle$. If α is a successor ordinal, say $\alpha = \kappa + 1$, then the structure $\langle \alpha; \oplus, 0; \subseteq\rangle$ is also an integral pomonoid residuated on both sides if we define $\beta \oplus \gamma$ to be the minimum of κ and $\beta + \gamma$ $(\beta, \gamma \in \alpha)$. Clearly only ordinals not exceeding $\omega + 1$ give rise, in this sense, to pocrims. For example, in $\omega + 2$ we have

$$1 \oplus \omega = 1 + \omega = \omega \neq \omega + 1 = \omega \oplus 1.$$

Any other ordinals have different left and right residuation operations. For example, in $\omega + 2$ again,

$$(\omega + 1) \dot{-} 1 = \omega, \quad \text{while} \quad (\omega + 1) \underset{\cdot}{-} 1 = \omega + 1.$$

Note that ordinals that are not successor ordinals and are not closed under $+$, such as $\omega + \omega$, do not give rise, in this way, to polrims (or porrims). For each nonzero ordinal α however, there does exist a left [resp. right] residuation algebra, whose universe is α, that is a left [resp. right] residuation–subreduct of some ordinal polrim.

Example 1.9. A polrim $\mathbf{A} = \langle A; \oplus, \dot{-}, 0\rangle$ is called a *left* [resp. *right*] *complemented monoid* if, whenever $a, b \in A$ with $a \leq b$, there exists $c \in A$ such that $c \oplus a = b$ [resp. $a \oplus c = b$]. Left (and right) complemented monoids were investigated by Bosbach in a series of papers (e.g., [Bos69], [Bos70], [Bos82]). The residuation–subreducts of left complemented monoids belong to a class of algebras that Bosbach called *residuation groupoids.* (More precisely, a residuation groupoid satisfying $x \dot{-} x \approx 0$ and $x \dot{-} y \dot{-} x \approx 0$ is such a subreduct.) Commutative complemented monoids (considered as pocrims) are called *hoops*; they were first investigated in an unpublished work of Büchi and Owens (c1975) and their structure is now well understood [BP94a], [BF93], [Fer92]. The identity

$$(x \dot{-} y) \oplus y \approx (y \dot{-} x) \oplus x \tag{10}$$

distinguishes left complemented monoids [resp. hoops] among polrims [resp. pocrims] and may be used to account for the (well known) fact that these subclasses are varieties. The variety of left complemented monoids has a definable join operation: if $\mathbf{A}$ is a left complemented monoid and $a, b \in A$, then $(a \dot{-} b) \oplus b$ $(= (b \dot{-} a) \oplus a)$ is the join $a \sqcup b$ of a and b in $\langle A; \leq\rangle$ [BP94b]. The identity

$$z \dot{-} x \dot{-} (y \dot{-} x) \approx z \dot{-} y \dot{-} (x \dot{-} y)$$

similarly distinguishes residuation groupoids and the residuation–subreducts of hoops among left residuation algebras and BCK–algebras, respectively. Whereas the residuation–subreducts of hoops form a variety [Fer92, Theorem 3.15, p96], the residuation groupoids in $\mathcal{LR}$ do not (see the final example in [Wro85]).

Example 1.10. A polrim is *left cancellative* if it satisfies the quasi-identity

$$z \oplus x \approx z \oplus y \quad \text{implies} \quad x \approx y.$$

Analogously, a polrim is *right cancellative* if it satisfies the quasi-identity

$$x \oplus z \approx y \oplus z \quad \text{implies} \quad x \approx y,$$

which is equivalent (by an easy application of (A7)) to the identity

$$(x \oplus y) \dot{-} y \approx x.$$

Evidently, in pocrims, left and right cancellativity coincide. The quasivariety generated by the pocrim on the ordinal ω coincides with the variety of cancellative hoops [BF93] and is axiomatized, relative to pocrims, by the previous identity together with

$$x \dot{-} (x \dot{-} y) \approx y \dot{-} (y \dot{-} x)$$

((10) is redundant: see [BR97, Proposition 8.7]). For example, the ideals of a Dedekind domain (considered as in Example 1.7) form a cancellative hoop.

The positive cone of a lattice ordered abelian group may be viewed, in a natural way, as a cancellative hoop, and every cancellative hoop arises from a lattice ordered abelian group in this way (see, e.g., [Bos82], [BF93], [Fer92]). The ordinal polrims mentioned in Example 1.8 are right complemented monoids, the unbounded ones being left cancellative. Thus the [unbounded] ordinal porrims are [right cancellative] left complemented monoids. (As polrims, however, ordinals exceeding $\omega + 1$ are neither right cancellative nor left complemented.)

Definition 1.11. Let $\mathbf{A} = \langle A;\, \dot{-}\,, 0\rangle \in \mathcal{LR}$. For each $a, b \in A$, define

$$S_{a,b} = \{c \in A : c \dot{-} b \leq a\}.$$

We say $\mathbf{A}$ has *condition* (S) if $\max S_{a,b}$ exists for all $a, b \in A$. In this case, we denote $\max S_{a,b}$ by $a + b$. We say that $\mathbf{A}$ has *condition* (S$'$) if $\mathbf{A}$ also satisfies the following condition:

$$(\forall a, b, c \in A)\ (\exists d \in A)\ (c \dot{-} b \dot{-} a = c \dot{-} d \ \text{ and } \ d \dot{-} b \leq a).$$

Suppose that $\mathbf{A}$ has condition (S) and set $\mathbf{A}' = \langle A; +, \dot{-}\,, 0\rangle$. By definition, $(y + z) \dot{-} z \leq y$, hence

$$\begin{aligned} x \dot{-} z \dot{-} y &\approx x \dot{-} z \dot{-} y \dot{-} ((y + z) \dot{-} z \dot{-} y) \quad \text{(by (A2))} \\ &\leq x \dot{-} z \dot{-} ((y + z) \dot{-} z) \quad \text{(by (A1))} \\ &\leq x \dot{-} (y + z) \quad \text{(by (A1))}, \end{aligned}$$

i.e.,

$$\mathbf{A}' \models x \dot{-} z \dot{-} y \leq x \dot{-} (y + z). \tag{11}$$

For all $a, b \in A$, set $D_{a,b} = \{c \in A : a \leq c + b\}$. Then $a \dot{-} b = \min D_{a,b}$. For if $c \in D_{a,b}$ then $a \leq c + b$, so by (11), $a \dot{-} b \dot{-} c \leq a \dot{-} (c + b) = 0$, i.e., $a \dot{-} b \leq c$. Also, $(a \dot{-} b) + b$ is $\max S_{a \dot{-} b, b}$ and $a \in S_{a \dot{-} b, b}$, so $a \leq (a \dot{-} b) + b$. Thus $a \dot{-} b \in D_{a,b}$, so $a \dot{-} b = \min D_{a,b}$.

Calculating over $\mathbf{A}'$ and using (11), we have

$$(x + (y + z)) \dot{-} z \dot{-} y \dot{-} x \leq (x + (y + z)) \dot{-} (y + z) \dot{-} x \approx 0,$$

hence $(x + (y + z)) \dot{-} z \dot{-} y \leq x$, hence $(x + (y + z)) \dot{-} z \leq x + y$, so

$$\mathbf{A}' \models x + (y + z) \leq (x + y) + z. \tag{12}$$

The proofs of the following are straightforward:

$$\mathbf{A}' \models x + 0 \approx x \approx 0 + x,$$
$$\mathbf{A}' \models x \leq y \ \textsf{implies} \ (z + x \leq z + y \ \textsf{and} \ x + z \leq y + z).$$

Suppose that $\mathbf{A}$ has condition (S′). Then

$$\text{(13)} \qquad \mathbf{A}' \models x \dot{-} z \dot{-} y \approx x \dot{-} (y + z).$$

For, if $a, b, c \in A$ such that $c \dot{-} b \dot{-} a = c \dot{-} d$ and $d \dot{-} b \leq a$, then $d \in S_{a,b}$ so $d \leq \max S_{a,b} = a + b$. Then $c \dot{-} (a + b) \leq c \dot{-} d = c \dot{-} b \dot{-} a$, so by (11), we conclude (13).

By (13), therefore, $\mathbf{A}'$ is a polrim. Note that the reduct of a polrim satisfies condition (S′) (set $d = a \oplus b$). Thus we have the following:

Proposition 1.12. *The* $\langle \dot{-}, 0\rangle$*-reducts of polrims are just the left residuation algebras with condition* (S′).

A BCK–algebra $\mathbf{A}$ with condition (S) has condition (S′): for $a, b, c, d \in A$ we have, by (A14) and (A15),

$$c \dot{-} (c \dot{-} b \dot{-} a) \dot{-} b = c \dot{-} b \dot{-} (c \dot{-} b \dot{-} a) \leq a.$$

Thus $c \dot{-} (c \dot{-} b \dot{-} a) \in S_{a,b}$, so $c \dot{-} (c \dot{-} b \dot{-} a) \leq \max S_{a,b} = a + b$. By (A11) and (A15),

$$c \dot{-} (a + b) \leq c \dot{-} (c \dot{-} (c \dot{-} b \dot{-} a)) \leq c \dot{-} b \dot{-} a,$$

hence, by (11), $c \dot{-} (a + b) = c \dot{-} b \dot{-} a$, and we may take $d = a + b$ in (S′). We therefore obtain the known result that *the* $\langle \dot{-}, 0\rangle$*-reducts of pocrims are just the BCK–algebras with condition* (S) [Isé79].

Example 1.13. As a final observation, we present an example to show that in $\mathcal{LR}$, condition (S) does not imply condition (S′). Let A be a disjoint union $\{a_i : i \in \omega\} \cup \{0\}$, where $\{a_i : i \in \omega\}$ is a one-to-one sequence. Define a binary operation $\dot{-}$ on A by setting $0 \dot{-} 0 = 0$, $a_i \dot{-} 0 = a_i$ and $0 \dot{-} a_i = 0$ for all $i \in \omega$, and for $i, j \in \omega$,

$$a_i \dot{-} a_j = \begin{cases} 0 & \text{if } j \leq i, \\ a_{i+2} & \text{if } j = i + 1, \\ a_{i+1} & \text{if } j \geq i + 2. \end{cases}$$

One routinely checks that $\mathbf{A} = \langle A; \dot{-}, 0\rangle$ is a left residuation algebra; the associated partial order corresponds to the Hasse diagram in Figure 1.

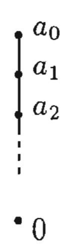


Figure 1.

A satisfies condition (S) where, for $i, j \in \omega$, we have

$$a_i + a_j = \begin{cases} a_0 & \text{if } i = 0 \text{ or } j = 0 \\ a_{i-1} & \text{if } i, j \geq 1 \text{ and } i \leq j \\ a_{i-2} & \text{if } i \geq 2 \text{ and } i = j + 1 \\ a_j & \text{if } i > j + 1. \end{cases}$$

Note, however, that **A** violates condition (S′): $a_0 \dot{-} a_1 \dot{-} a_3 = a_2 \dot{-} a_3 = a_4$ and there does not exist a $d \in A$ such that $a_0 \dot{-} d = a_4$. In general, therefore, when $\mathbf{A} \in \mathcal{LR}$ has condition (S), $\mathbf{A}'$ need not be a polrim.

CHAPTER 2

THE LOGIC IPC WITHOUT EXCHANGE AND CONTRACTION

In [OK85], Ono and Komori present a Gentzen system LJ^* that is a formulation of Intuitionistic Propositional Calculus (**IPC**). In a sense to be made clear below, LJ^* is essentially the 'propositional fragment' of Gentzen's system LJ [Gen35]. Strictly speaking, the two systems differ in that the conjunction connective $\wedge$ of LJ is duplicated by a further connective & in LJ^*. The connectives $\wedge$ and & are interchangeable in any derivable sequent of LJ^* and their identification in all such sequents yields just the derivable (propositional) sequents of LJ. The (explicit) inference rules of LJ^* include only two inference figures dealing with the connective &, however. The system LJ^* has four 'structural rules', i.e., rules of inference in which no connectives occur explicitly; they are 'cut', 'exchange', 'contraction' and 'weakening'. The paper [OK85] is devoted to the study of systems L_{BK} and L_{BCK} that are obtained from LJ^* by removing, respectively, the rules of exchange and contraction, and the rule of contraction alone. Thus L_{BK} and L_{BCK} fall into the class of 'substructural logics' (see [SD93]). Because of the choice of inference rules for LJ^*, the connectives $\wedge$ and & are no longer interchangeable in the derivable (propositional) sequents of L_{BK}, nor in those of L_{BCK}. Roughly speaking, $\wedge$ takes on the character of a lattice operation, while & is like a monoid operation. Ono and Komori also define two Hilbert systems H_{BK}[5] and H_{BCK}, which they prove to be 'logically equivalent' to L_{BK} and L_{BCK}, respectively. In this chapter, we shall consider the logics L_{BK} and H_{BK}, which we shall call L and H, and their fragments.

In Section 1 we summarize some results from [OK85], namely that the 'cut elimination theorem' holds for L and that for a subset C of its connectives that contains $\rightarrow$ and either does not contain $\vee$ or does contain $\wedge$, the C-fragments

[5]In [OK85], L_{BK} and H_{BK} are denoted L_{BCC} and H_{BCC}, respectively.

of L and H are logically equivalent. The Hilbert system H is defined by a finite set I of (axioms and) inference rules for which the 'separation theorem' does not hold. We present a set J of inference rules which we prove to axiomatize H. The remainder of Section 1 is devoted to showing that the C-fragments of L and H are logically equivalent for *all* sets C of connectives that contain $\to$ (we call these the 'superimplicational fragments'), and hence that the separation theorem holds for the axiomatization J of H. This answers a question posed in [OK85].

In Section 2 we show that H (and each of its superimplicational fragments) is 'algebraizable' in the sense of Blok and Pigozzi [BP89]. The 'equivalent quasivariety semantics' of the $\{\to\}$- and $\{\&,\to\}$-fragments of H turn out to be $\mathcal{LR}$ and $\mathcal{LM}$, respectively. We axiomatize the equivalent quasivariety semantics of each of the superimplicational fragments of H. To conclude, we show that the $\{\to,\vee\}$- and $\{\to,\vee,\bot\}$-fragments of H (and their equivalent quasivariety semantics) are *not* finitely axiomatizable.

In Section 3 we show that the '$\langle C,I\rangle$-subsystem' of H, namely the Hilbert system defined by those axioms and inference rules of I that contain only the connectives in C (where $\to\in C$), is algebraizable *only* when C does not contain $\vee$ or does contain $\wedge$.

2.1. The Separation Theorem. The Gentzen system L has a language consisting of the binary connectives $\&$, $\to$, $\vee$ and $\wedge$ and the constant $\bot$. Initial sequents of L are either of the form $\bot\Rightarrow\alpha$ for any formula α, or of the form $p\Rightarrow p$ for any variable p. Rules of inference of L are as follows: for all finite sequences Γ,Δ,Σ of formulas and all formulas α,β,γ,

$$\frac{\Gamma,\Delta\Rightarrow\gamma}{\Gamma,\alpha,\Delta\Rightarrow\gamma}(\text{weakening})\qquad\frac{\Gamma\Rightarrow\alpha\qquad\Delta,\alpha,\Sigma\Rightarrow\gamma}{\Delta,\Gamma,\Sigma\Rightarrow\gamma}(\text{cut})$$

$$\frac{\Gamma,\alpha\Rightarrow\beta}{\Gamma\Rightarrow\alpha\to\beta}(\Rightarrow\to)\qquad\frac{\Gamma\Rightarrow\alpha\qquad\Delta,\beta,\Sigma\Rightarrow\gamma}{\Delta,\alpha\to\beta,\Gamma,\Sigma\Rightarrow\gamma}(\to\Rightarrow)$$

$$\frac{\Gamma\Rightarrow\alpha}{\Gamma\Rightarrow\alpha\vee\beta}(\Rightarrow\vee 1)\qquad\frac{\Gamma\Rightarrow\beta}{\Gamma\Rightarrow\alpha\vee\beta}(\Rightarrow\vee 2)$$

$$\frac{\Gamma,\alpha,\Delta\Rightarrow\gamma\qquad\Gamma,\beta,\Delta\Rightarrow\gamma}{\Gamma,\alpha\vee\beta,\Delta\Rightarrow\gamma}(\vee\Rightarrow)$$

$$\frac{\Gamma,\alpha,\Delta\Rightarrow\gamma}{\Gamma,\alpha\wedge\beta,\Delta\Rightarrow\gamma}(\wedge\Rightarrow 1)\qquad\frac{\Gamma,\beta,\Delta\Rightarrow\gamma}{\Gamma,\alpha\wedge\beta,\Delta\Rightarrow\gamma}(\wedge\Rightarrow 2)$$

$$\frac{\Gamma\Rightarrow\alpha\qquad\Gamma\Rightarrow\beta}{\Gamma\Rightarrow\alpha\wedge\beta}(\Rightarrow\wedge)$$

$$\frac{\Gamma \Rightarrow \alpha \qquad \Delta \Rightarrow \beta}{\Gamma, \Delta \Rightarrow \alpha \,\&\, \beta}(\Rightarrow \&) \qquad \frac{\Gamma, \alpha, \beta, \Delta \Rightarrow \gamma}{\Gamma, \alpha \,\&\, \beta, \Delta \Rightarrow \gamma}(\& \Rightarrow).$$

Note that L is *not* equipped with the structural rules

$$\frac{\Gamma, \alpha, \alpha, \Delta \Rightarrow \gamma}{\Gamma, \alpha, \Delta \Rightarrow \gamma}(\text{contraction}) \qquad \frac{\Gamma, \alpha, \beta, \Delta \Rightarrow \gamma}{\Gamma, \beta, \alpha, \Delta \Rightarrow \gamma}(\text{exchange}).$$

One obtains L_{BCK} by adding the exchange rule to L and one obtains LJ^* (essentially **IPC**) by adding the contraction rule to L_{BCK}. In the derivable sequents of LJ^*, the connectives & and $\wedge$ are interchangable. As the following derivation shows, the exchange rule is 'derivable' in the extension of L by the contraction rule. This fact is noted without proof in [OK85]. For any formula α, the sequent $\alpha \Rightarrow \alpha$ is easily seen, by induction on the complexity of α, to be derivable in L.

$$\dfrac{\dfrac{\beta \Rightarrow \beta \qquad \alpha \Rightarrow \alpha}{\beta, \alpha \Rightarrow \beta \,\&\, \alpha}(\Rightarrow \&) \qquad \dfrac{\dfrac{\dfrac{\Gamma, \alpha, \beta, \Delta \Rightarrow \gamma}{\Gamma, \beta, \alpha, \beta, \alpha, \Delta \Rightarrow \gamma}(\text{weakening})}{\Gamma,\ \beta \,\&\, \alpha,\ \beta \,\&\, \alpha,\ \Delta \Rightarrow \gamma}(\& \Rightarrow)}{\Gamma,\ \beta \,\&\, \alpha,\ \Delta \Rightarrow \gamma}(\text{contraction})}{\Gamma, \beta, \alpha, \Delta \Rightarrow \gamma}(\text{cut})$$

Let G be a Gentzen system whose rules of inference include the cut rule. If, for every sequent $\Gamma \Rightarrow \gamma$ derivable in G, there exists a derivation of $\Gamma \Rightarrow \gamma$ that does not invoke the cut rule, we say that the *cut elimination theorem* holds for G.

Theorem 2.1. [OK85, Theorem 2.3] *The cut elimination theorem holds for L.*

(We remark, however, that the extension of L by the contraction rule does not have the cut elimination theorem [OK85].)

Let C be a subset of $\{\&, \rightarrow, \vee, \wedge, \bot\}$ containing at least $\rightarrow$. A formula α is called a *C–formula* if any connective appearing in α belongs to C. A sequent consisting of only C–formulas is called a *C–sequent.* Let C–L denote the Gentzen system whose language consists of the connectives in C, whose initial sequents are $p \Rightarrow p$ for any propositional variable p and, when $\bot \in C$, also $\bot \Rightarrow \alpha$ for any C–formula α, and whose rules of inference are those of L that use only connectives from C. Theorem 2.1 has the following corollary (see, for example, [Tak75, Theorem 6.3]):

Corollary 2.2. *If a sequent $\Gamma \Rightarrow \alpha$ is derivable in L then it is derivable in C–L, whenever C contains $\rightarrow$ and all the connectives occurring in $\Gamma \cup \{\alpha\}$.*

Proof. In each rule of inference other than the cut rule, the set of connectives occurring in the upper sequents is a subset of the set of connectives occurring in the lower sequent. Thus the result follows by Theorem 2.1. □

We therefore call C–L the *C-fragment* of L.

The Hilbert system $H = H_{\mathrm{BK}}$ has a language consisting of the binary connectives $\&$, $\rightarrow$, $\vee$, $\wedge$ and the constant $\bot$. We reserve the symbol $\vdash$ for the consequence relation of H. We adopt the convention for $\rightarrow$ that omitted parentheses are associated to the right, e.g., $\alpha \rightarrow \beta \rightarrow \gamma$ abbreviates $\alpha \rightarrow (\beta \rightarrow \gamma)$.

The system H is defined by the axioms

(H1) $p \rightarrow q \rightarrow p$
(H2) $\bot \rightarrow p$
(H3) $(p \rightarrow q) \rightarrow (r \rightarrow p) \rightarrow r \rightarrow q$
(H4) $((p \rightarrow r) \wedge (q \rightarrow r)) \rightarrow (p \vee q) \rightarrow r$
(H5) $p \rightarrow (p \vee q)$
(H6) $q \rightarrow (p \vee q)$
(H7) $(p \wedge q) \rightarrow p$
(H8) $(p \wedge q) \rightarrow q$
(H9) $((r \rightarrow p) \wedge (r \rightarrow q)) \rightarrow r \rightarrow (p \wedge q)$
(H10) $p \rightarrow q \rightarrow (p \wedge q)$
(H11) $(p \rightarrow q \rightarrow r) \rightarrow (p \,\&\, q) \rightarrow r$
(H12) $p \rightarrow q \rightarrow (p \,\&\, q)$.

and the following two variants of modus ponens as additional inference rules:

$$\text{(m.p.1)} \quad p,\ p \rightarrow q \ \vdash\ q \quad \text{and} \quad \text{(m.p.2)} \quad q,\ p \rightarrow q \rightarrow r \ \vdash\ p \rightarrow r.$$

Let I be the above axiomatization of H, i.e., the set consisting of (H1)–(H12), (m.p.1) and (m.p.2). The Hilbert system H_{BCK} has the same language as H and is axiomatized by I together with the axiom

(H13) $\qquad (p \rightarrow q \rightarrow r) \rightarrow q \rightarrow p \rightarrow r$

(essentially the exchange rule), which makes the inference rule (m.p.2) redundant. The axioms (H3), (H13) and (H1) are usually called (B), (C) and (K), respectively. For this reason, the Hilbert system with language $\{\rightarrow\}$ that is axiomatized by these three axioms and (m.p.1) is usually called **BCK**.

Let C be a subset of $\{\&, \rightarrow, \vee, \wedge, \bot\}$ that contains $\rightarrow$. The *C-fragment* of H, denoted C–H, is the Hilbert system whose language consists of the connectives of C and whose consequence relation, denoted $\vdash_C$, is defined by:

$$\Gamma \vdash_C \varphi \text{ if and only if } \Gamma \vdash \varphi \text{ and } \Gamma \cup \{\varphi\} \text{ is a set of } C\text{–formulas.}$$

Here our nomenclature differs from that of [OK85]. What Ono and Komori refer to there as the 'C–fragment' of H will be called here the '$\langle C, I\rangle$–subsystem'. This notion, unlike the one just defined, depends not only on H and C but also on the axiomatization I. In particular, for C as above, the *$\langle C, I\rangle$–subsystem* of H, denoted $\langle C, I\rangle$–H, is a Hilbert system whose language consists of the connectives of C; it is axiomatized by the formulas among (H1)–(H12) that use only connectives from C, and (m.p.1) and (m.p.2). We denote its consequence relation by $\vdash_{C,I}$. (Our usage of 'fragment' is consistent, for example, with [BP89]. No corresponding notational distinction need be drawn for L, in view of Corollary 2.2.)

A *superimplicational* fragment shall mean a C–fragment, for some C containing $\rightarrow$.

Let G and S be a Gentzen and a Hilbert system, respectively, with a common language that includes a specified binary connective $\rightarrow$. Suppose that for all formulas $\alpha_1, \ldots, \alpha_n, \gamma$, the sequent $\alpha_1, \ldots, \alpha_n \Rightarrow \gamma$ is derivable in G exactly when the formula $\alpha_1 \rightarrow \alpha_2 \rightarrow \ldots \rightarrow \alpha_n \rightarrow \gamma$ is a theorem of S. Then we say that G and S are *logically equivalent.*

The following lemma appears without proof in [OK85]. We include a sketch of a proof for the sake of completeness.

Lemma 2.3. [OK85, Lemma 2.1] *Let C be a subset of $\{\&, \rightarrow, \vee, \wedge, \bot\}$ that contains $\rightarrow$ and let $\alpha_1, \ldots, \alpha_n, \gamma$ be C-formulas. The following are equivalent:*

(i) *the sequent $\alpha_1, \ldots, \alpha_n \Rightarrow \gamma$ is derivable in C–L,*

(ii) *the sequent $\emptyset \Rightarrow \alpha_1 \rightarrow \alpha_2 \rightarrow \ldots \rightarrow \alpha_n \rightarrow \gamma$ is derivable in C-L.*

Proof. That (i) implies (ii) follows from repeated applications of $(\Rightarrow \rightarrow)$. Conversely, suppose there exists a (cut-free) derivation of $\emptyset \Rightarrow \alpha_1 \rightarrow \alpha_2 \rightarrow \ldots \rightarrow \alpha_n \rightarrow \gamma$ in C–L. The only (non-cut) rule with $\emptyset \Rightarrow \alpha_1 \rightarrow \alpha_2 \rightarrow \ldots \rightarrow \alpha_n \rightarrow \gamma$ as lower sequent is $(\Rightarrow \rightarrow)$. Thus there must exist a derivation of the sequent $\alpha_1 \Rightarrow \alpha_2 \rightarrow \alpha_3 \rightarrow \ldots \rightarrow \alpha_n \rightarrow \gamma$ in C–L. Continuing in this way, we obtain (i). □

Theorem 2.4. [OK85, Corollary 2.8.2] *L and H are logically equivalent. Moreover, for any subset C of $\{\&, \rightarrow, \vee, \wedge, \bot\}$ which contains $\rightarrow$ and either does not contain $\vee$ or does contain $\wedge$, the systems C–L and $\langle C, I\rangle$–H are logically equivalent.*

We say that the *separation theorem* holds for an axiomatization I' of a Hilbert system S over a language $\mathcal{L}$ that includes a binary connective $\rightarrow$ provided that for any theorem α of S, there exists a derivation of α (with respect to I') in which all occurring connectives other than $\rightarrow$ are among the connectives occurring in α. In this case, it is evident that each (superimplicational) C-fragment of S is axiomatized by the axioms and inference rules of I' that contain only the connectives in C. In [OK85, §9, Remark 1], Ono and Komori ask whether there exists an axiomatization of H for which the separation theorem holds. (H does not have the separation theorem for I: see Section 3.) We shall present such an axiomatization here.

Let J be the set obtained by removing from I the axiom (H4) and adding the axioms

(X_0) $(q \rightarrow s) \rightarrow (r \rightarrow s) \rightarrow (q \vee r) \rightarrow s,$

(X_n) $(p_1 \rightarrow \ldots \rightarrow p_n \rightarrow q \rightarrow s) \rightarrow (p_1 \rightarrow \ldots \rightarrow p_n \rightarrow r \rightarrow s) \rightarrow$
$p_1 \rightarrow \ldots \rightarrow p_n \rightarrow (q \vee r) \rightarrow s,$

for each natural number $n \geq 1$. In [OK85, Theorem 2.9] it is proved that H_{BCK} has the separation theorem for the axiomatization obtained from I by excluding (H4) and including (H13) and (X_0). Thus, the $\{\rightarrow\}$-fragment of H_{BCK} is **BCK** and all superimplicational fragments of H_{BCK} are finitely axiomatizable.

Let $H' = H'_{\mathrm{BK}}$ be the Hilbert system with the same language as H that is axiomatized by J and let $\vdash'$ denote its consequence relation. For a subset C of $\{\&, \rightarrow, \vee, \wedge, \bot\}$ which contains $\rightarrow$, we define the C-fragment and $\langle C, J\rangle$-subsystem of H' just as for H and I. We denote them by C–H' and $\langle C, J\rangle$–H', and their consequence relations by $\vdash'_C$ and $\vdash'_{C,J}$, respectively.

Lemma 2.5. *$\langle\{\rightarrow, \vee, \wedge\}, I\rangle$–$H$ and $\langle\{\rightarrow, \vee, \wedge\}, J\rangle$–$H'$ coincide. Thus, H and H' are the same Hilbert system.*

Proof. First we show that each (X_n) is a theorem of $\langle\{\rightarrow, \vee, \wedge\}, I\rangle$–$H$. We begin with the following observation:

$$(14) \qquad \zeta \rightarrow \eta,\ \alpha \rightarrow \beta \rightarrow \zeta \vdash_{\{\rightarrow\},I} \alpha \rightarrow \beta \rightarrow \eta.$$

For, by (H3),

$$\vdash_{\{\rightarrow\},I} (\zeta \rightarrow \eta) \rightarrow (\beta \rightarrow \zeta) \rightarrow \beta \rightarrow \eta,$$

and

$$\vdash_{\{\rightarrow\},I} ((\beta \rightarrow \zeta) \rightarrow (\beta \rightarrow \eta)) \rightarrow (\alpha \rightarrow \beta \rightarrow \zeta) \rightarrow \alpha \rightarrow \beta \rightarrow \eta,$$

so (14) follows by (m.p.1). By (H4),

$$\vdash_{\{\rightarrow,\vee,\wedge\},I} ((q \rightarrow s) \wedge (r \rightarrow s)) \rightarrow (q \vee r) \rightarrow s,$$

and by (H10),

$$\vdash_{\{\to,\vee,\wedge\},I} (q \to s) \to (r \to s) \to ((q \to s) \wedge (r \to s)).$$

Thus (X_0) is a theorem of $\langle\{\to,\vee,\wedge\},I\rangle$-$H$ by (14). Using (H9), (14) and induction, we obtain

$$\vdash_{\{\to,\vee,\wedge\},I} ((p_1 \to \ldots \to p_n \to q \to s) \wedge (p_1 \to \ldots \to p_n \to r \to s)) \to \\ p_1 \to \ldots \to p_n \to ((q \to s) \wedge (r \to s)).$$

Using (H4), (H3) and induction, we have

$$\vdash_{\{\to,\vee,\wedge\},I} (p_1 \to \ldots \to p_n \to ((q \to s) \wedge (r \to s))) \to p_1 \to \ldots \to p_n \to \\ (q \vee r) \to s,$$

hence, by (H3) and (m.p.1),

$$\vdash_{\{\to,\vee,\wedge\},I} ((p_1 \to \ldots \to p_n \to q \to s) \wedge (p_1 \to \ldots \to p_n \to r \to s)) \to \\ p_1 \to \ldots \to p_n \to (q \vee r) \to s.$$

Now, by (H10),

$$\vdash_{\{\to,\vee,\wedge\},I} (p_1 \to \ldots \to p_n \to q \to s) \to (p_1 \to \ldots \to p_n \to r \to s) \to \\ ((p_1 \to \ldots \to p_n \to q \to s) \wedge (p_1 \to \ldots \to p_n \to r \to s)),$$

so, using (14), we obtain (X_n) as a theorem of $\langle\{\to,\vee,\wedge\},I\rangle$-$H$.

Next, we derive (H4) in $\langle\{\to,\vee,\wedge\},J\rangle$-$H'$. By substituting $(q \to s) \wedge (r \to s)$ for p in (X_1), we get

$$\vdash'_{\{\to,\vee,\wedge\},J} (((q \to s) \wedge (r \to s)) \to (q \to s)) \to (((q \to s) \wedge (r \to s)) \to \\ (r \to s)) \to ((q \to s) \wedge (r \to s)) \to (q \vee r) \to s,$$

so, using (H7), (H8) and (m.p.1), we may prove (H4).

This establishes the first assertion of the lemma, from which the second follows immediately since the axioms by which H and H' differ contain only the connectives $\to$, $\vee$ and $\wedge$. □

Henceforth, therefore, we shall drop the $'$ from H', $\vdash'$, etc. Note that in the derivation of (H4) in the above lemma, we used only (X_1), (H7), (H8) and (m.p.1). Thus, H is also axiomatized by the finite set J_1 obtained from J by replacing the $(X_n), n \in \omega$, by the single formula (X_1), while $\langle\{\to,\vee,\wedge\},I\rangle$-$H$ is axiomatized by the corresponding subset of J_1. As we shall see in the next section, however, H does not have the separation theorem for J_1, nor for any axiomatizating subset of J in which only finitely many of the (X_n)'s are present.

Evidently, if C contains $\rightarrow$ and either does not contain $\vee$ or does contain $\wedge$ then $\langle C, I\rangle$–H and $\langle C, J\rangle$–H coincide. In this case Theorem 2.4 implies that C–L and $\langle C, J\rangle$–H are logically equivalent. We show that the same is true in the other cases.

Lemma 2.6. *$\{\rightarrow, \vee\}$–L and $\langle\{\rightarrow, \vee\}, J\rangle$–$H$ are logically equivalent. Moreover, if C is a subset of $\{\&, \rightarrow, \vee, \wedge, \perp\}$ that contains $\rightarrow$ and $\vee$ but not $\wedge$, then C–L and $\langle C, J\rangle$–H are logically equivalent.*

Proof. We first show that the sequent

$$p_1 \rightarrow \ldots \rightarrow p_n \rightarrow q \rightarrow s,\ \ p_1 \rightarrow \ldots \rightarrow p_n \rightarrow r \rightarrow s,\ \ p_1, \ldots, p_n, q \vee r \Rightarrow s,$$

which corresponds to the axiom (X_n), is derivable in $\{\rightarrow, \vee\}$–L. Consider the following derivation in $\{\rightarrow, \vee\}$–L. Each step is an application of the rule $(\rightarrow\, \Rightarrow)$.

$$\dfrac{p_{n-1} \Rightarrow p_{n-1} \qquad \dfrac{p_n \Rightarrow p_n \qquad \dfrac{q \Rightarrow q \qquad s \Rightarrow s}{q \rightarrow s, q \Rightarrow s}}{p_n \rightarrow q \rightarrow s, p_n, q \Rightarrow s}}{\begin{array}{c}\vdots\\ \hline p_1 \rightarrow \ldots \rightarrow p_n \rightarrow q \rightarrow s,\ \ p_1, \ldots, p_n, q \Rightarrow s\end{array}}$$

Similarly we can derive

$$p_1 \rightarrow \ldots \rightarrow p_n \rightarrow r \rightarrow s,\ \ p_1, \ldots, p_n, r \Rightarrow s.$$

Set

$$\Gamma = p_1 \rightarrow \ldots \rightarrow p_n \rightarrow q \rightarrow s,\ \ p_1 \rightarrow \ldots \rightarrow p_n \rightarrow r \rightarrow s,\ \ p_1, \ldots, p_n.$$

By (weakening), we derive $\Gamma, q \Rightarrow s$ and $\Gamma, r \Rightarrow s$, and hence, by $(\vee\, \Rightarrow)$, also $\Gamma, q \vee r \Rightarrow s$.

The sequents corresponding to the other axioms of $\langle\{\rightarrow, \vee\}, J\rangle$–$H$ are easily derivable in $\{\rightarrow, \vee\}$–L. Suppose $\alpha_1 \rightarrow \ldots \rightarrow \alpha_n \rightarrow \gamma$ is a theorem of $\langle\{\rightarrow, \vee\}, J\rangle$–$H$. We use φ to denote $\alpha_2 \rightarrow \ldots \rightarrow \alpha_n \rightarrow \gamma$. Then there exists a derivation of $\alpha_1 \rightarrow \varphi$ from $\emptyset$ in $\langle\{\rightarrow, \vee\}, J\rangle$–$H$. Suppose that the last step of a derivation of $\alpha_1 \rightarrow \varphi$ uses the inference rule (m.p.2). Then there exists a $\{\rightarrow, \vee\}$-formula β such that $\vdash_{\{\rightarrow,\vee\},J} \beta$ and $\vdash_{\{\rightarrow,\vee\},J} \alpha_1 \rightarrow \beta \rightarrow \varphi$. Proceeding inductively, we assume that the sequent $\alpha_1, \beta, \alpha_2, \ldots, \alpha_n \Rightarrow \gamma$ is derivable in $\{\rightarrow, \vee\}$–L and, using Lemma 2.3, that the sequent $\emptyset \Rightarrow \beta$ derivable in $\{\rightarrow, \vee\}$–L. We have the following derivation:

$$\dfrac{\emptyset \Rightarrow \beta \qquad \alpha_1, \beta, \alpha_2, \ldots, \alpha_n \Rightarrow \gamma}{\alpha_1, \alpha_2, \ldots, \alpha_n \Rightarrow \gamma}\text{(cut)}.$$

Thus the sequent $\alpha_1, \ldots, \alpha_n \Rightarrow \gamma$ is derivable in $\{\rightarrow, \vee\}$–L. One deals similarly with the (m.p.1) case, thereby completing the inductive proof.

Conversely, suppose that $\alpha_1, \ldots, \alpha_n \Rightarrow \gamma$ is derivable in $\{\rightarrow, \vee\}$–L. To show that $\alpha_1 \rightarrow \ldots \rightarrow \alpha_n \rightarrow \gamma$ is a theorem of $\langle\{\rightarrow, \vee\}, J\rangle$–$H$, we proceed by induction on the length of the derivation of $\alpha_1, \ldots, \alpha_n \Rightarrow \gamma$. Suppose that there exists a derivation in which $\alpha_1, \ldots, \alpha_n \Rightarrow \gamma$ occurs as the lower sequent of the inference rule ($\vee \Rightarrow$). Then we have

$$\alpha_1, \ldots, \alpha_n = \beta_1, \ldots, \beta_m, \zeta \vee \eta, \delta_1, \ldots, \delta_r,$$

where $m + r + 1 = n$, and

$$\frac{\beta_1, \ldots, \beta_m, \zeta, \delta_1, \ldots, \delta_r \Rightarrow \gamma \qquad \beta_1, \ldots, \beta_m, \eta, \delta_1, \ldots, \delta_r \Rightarrow \gamma}{\beta_1, \ldots, \beta_m, \zeta \vee \eta, \delta_1, \ldots, \delta_r \Rightarrow \gamma}(\vee \Rightarrow)$$

occurs in the derivation. By the induction hypothesis,

$$\vdash_{\{\rightarrow,\vee\},J} \beta_1 \rightarrow \ldots \rightarrow \beta_m \rightarrow \zeta \rightarrow \delta_1 \rightarrow \ldots \rightarrow \delta_r \rightarrow \gamma$$

and

$$\vdash_{\{\rightarrow,\vee\},J} \beta_1 \rightarrow \ldots \rightarrow \beta_m \rightarrow \eta \rightarrow \delta_1 \rightarrow \ldots \rightarrow \delta_r \rightarrow \gamma.$$

Thus, using the axiom (X_m) of $\langle\{\rightarrow, \vee\}, J\rangle$–$H$ and (m.p.1), we obtain

$$\vdash_{\{\rightarrow,\vee\},J} \beta_1 \rightarrow \ldots \rightarrow \beta_m \rightarrow (\zeta \vee \eta) \rightarrow \delta_1 \rightarrow \ldots \rightarrow \delta_r \rightarrow \gamma.$$

If there exists a derivation in which $\alpha_1, \ldots, \alpha_n \Rightarrow \gamma$ occurs as the lower sequent of any of the other inference rules (other than the cut rule), the same result follows easily, so the first assertion of the lemma follows by Theorem 2.1. The rest of the lemma is proved similarly. □

The above lemma, in conjunction with Lemma 2.5, gives us the following:

Corollary 2.7. *For each subset C of $\{\&, \rightarrow, \vee, \wedge, \perp\}$ containing $\rightarrow$, C–L and $\langle C, J\rangle$–H are logically equivalent.*

Corollary 2.8. *The separation theorem holds for the axiomatization J of H.*

Proof. Let C be a subset of $\{\&, \rightarrow, \vee, \wedge, \perp\}$ containing $\rightarrow$ and suppose that a C–formula α is a theorem of H. By Corollary 2.7, the sequent $\emptyset \Rightarrow \alpha$ is derivable in L. In view of Corollary 2.2, this derivation can be assumed to belong to C–L. By Corollary 2.7 again, α is a theorem of $\langle C, J\rangle$–H. □

Note that as a consequence of the separation theorem for J, the C-fragment C–H of H (where $\rightarrow \in C$) is axiomatized by those axioms and inference rules in J that contain only the connectives in C. Thus, the systems C–H and $\langle C, J\rangle$–H coincide for each C containing $\rightarrow$.

Remark. The $\{\to\}$–fragment of H has been given the rather misleading name **BCC** in the literature. By Corollary 2.8, this system is axiomatized by (H1), (H3), (m.p.1), (m.p.2), so we have used the more natural name **BK** in [RvA97], [vAR1] and in subsequent chapters of this thesis.

2.2. Algebraizability and Axiomatization of Fragments of H. In this section we use notions that are defined under the heading 'Algebraizable Hilbert Systems' in Chapter 0. Algebraic semantics for superimplicational fragments of H are given implicitly in [OK85, Theorem 8.1]. The next proposition sharpens this result.

Proposition 2.9. [6] *Let C be a subset of $\{\&, \to, \vee, \wedge, \perp\}$ containing $\to$. Then the Hilbert system C–H is algebraizable with the Gödel Rule, having defining equation $p \approx p \to p$ and equivalence formulas $\mathbf{\Delta}(p,q) = \{p \to q, q \to p\}$.*

Proof. Let ψ be any theorem of C–H. Then $\vdash_C \psi$ and, by (H1), $\vdash_C \varphi \to \psi \to \varphi$, so (m.p.2) yields

$$\vdash_C \varphi \to \varphi. \tag{15}$$

Thus property (i) in the definition of algebraizability holds. Property (ii) is satisfied by symmetry. By (H3) and (m.p.1),

$$q \to r,\ p \to q \vdash_C p \to r$$

and by symmetry of the variables, we obtain (iii). By (H3), we have

$$\vdash_C (p \to r) \to (q \to p) \to q \to r,$$

so by (m.p.2),

$$q \to p \vdash_C (p \to r) \to q \to r.$$

By (H3) and (m.p.1), we also have

$$r \to s \vdash_C (q \to r) \to q \to s \text{ and}$$

$$(q \to r) \to q \to s \vdash_C ((p \to r) \to q \to r) \to (p \to r) \to q \to s,$$

hence, by repeated applications of (m.p.1),

$$q \to p,\ r \to s \vdash_C (p \to r) \to q \to s,$$

and (iv) follows for the connective $\to$ by symmetry of the variables.

Suppose that C contains $\&$. By (H3), we have

$$\vdash_C (s \to (q \,\&\, s)) \to (r \to s) \to r \to (q \,\&\, s),$$

---

[6] Proposition 2.9 combines results from our papers [RvA97] and [vAR1]. After the publication of [RvA97] and the submission of [vAR1], we obtained a copy of the preprint [Agl] in which the algebraizability of the *full* system H is also remarked upon.

so by (m.p.2),

$$r \to s \vdash_C (s \to (q \,\&\, s)) \to r \to (q \,\&\, s).$$

Also by (H3) and (m.p.1),

$$(s \to (q \,\&\, s)) \to r \to (q \,\&\, s) \vdash_C (p \to (s \to (q \,\&\, s))) \to p \to r \to (q \,\&\, s),$$

while (H11) and (m.p.1) entail

$$p \to r \to (q \,\&\, s) \vdash_C (p \,\&\, r) \to (q \,\&\, s).$$

Applying (m.p.1) to the previous three inferences, we obtain

$$\text{(16)} \qquad r \to s,\ p \to s \to (q \,\&\, s) \vdash_C (p \,\&\, r) \to (q \,\&\, s).$$

Furthermore, (H3) yields

$$\vdash_C (q \to s \to (q \,\&\, s)) \to (p \to q) \to p \to s \to (q \,\&\, s),$$

the first premiss of which is an instance of (H12), so by two applications of (m.p.1),

$$p \to q \vdash_C p \to s \to (q \,\&\, s),$$

hence by (16),

$$r \to s,\ p \to q \vdash_C (p \,\&\, r) \to (q \,\&\, s).$$

By symmetry of the variables, we infer property (iv) for the connective &.

Suppose that C contains $\wedge$. By (H3),

$$\vdash_C (p \to q) \to ((p \wedge r) \to p) \to (p \wedge r) \to q,$$

hence, by (H7) and (m.p.1),

$$p \to q \vdash_C (p \wedge r) \to q.$$

Similarly, by (H3), (H8) and (m.p.1),

$$r \to s \vdash_C (p \wedge r) \to s.$$

By (H10) and (m.p.1), $p, q \vdash_C p \wedge q$, so by structurality and the above,

$$p \to q,\ r \to s \vdash_C ((p \wedge r) \to q) \wedge ((p \wedge r) \to s).$$

By (H9),

$$\vdash_C (((p \wedge r) \to q) \wedge ((p \wedge r) \to s)) \to (p \wedge r) \to (q \wedge s),$$

hence, by (m.p.1),

$$p \to q,\ r \to s \vdash_C (p \wedge r) \to (q \wedge s).$$

By symmetry of the variables, we infer property (iv) for the connective $\wedge$.

Suppose that C contains $\vee$. By (H3),

$$\vdash_C (q \to (q \vee s)) \to (p \to q) \to p \to (q \vee s),$$

so by (H5) and (m.p.1),

$$p \to q \vdash_C p \to (q \vee s).$$

Similarly, using (H3), (H6) and (m.p.1), we have

$$r \to s \vdash_C r \to (q \vee s).$$

By (X_0),

$$\vdash_C (p \to (q \vee s)) \to (r \to (q \vee s)) \to (p \vee r) \to (q \vee s),$$

hence, by (m.p.1),

$$p \to q,\ r \to s\ \vdash_C\ (p \vee r) \to (q \vee s),$$

By symmetry of the variables, we infer property (iv) for the connective $\vee$. The case of $\perp$ is an immediate consequence of (i).

For (v), we note that by (H1) and (m.p.1),

$$p \vdash_C (p \to p) \to p \text{ and } p \vdash_C p \to p \to p,$$

while (15) and (m.p.1) yield $(p \to p) \to p \vdash_C p$, so

$$p \dashv\vdash_C (p \to p) \to p,\ p \to p \to p.$$

To see that the Gödel rule holds, observe that from (H1) and (m.p.1) we obtain $p \vdash_C q \to p$ and hence also $q \vdash_C p \to q$, so

$$p,\ q \vdash_C q \to p,\ p \to q,$$

as required. □

When discussing the equivalent quasivariety semantics of H and C–H, we shall use the more natural algebraic symbols $\oplus$, $\dot{-}$, $\sqcap$, $\sqcup$, 1 for the connectives $\&, \to, \vee, \wedge, \perp$, respectively. Let C be a subset of $\{\&, \to, \vee, \wedge, \perp\}$. By C^* we mean the subset of $\{\oplus, \dot{-}, \sqcap, \sqcup, 1\}$ obtained by replacing each connective in C by its corresponding algebraic symbol. For each C–formula α we define a C^*–term α^* inductively as follows: first replace $\perp$ by 1. If α is a variable, let α^* be α. Suppose that β^* and γ^* have been defined. If α is $\beta \,\&\, \gamma$, let α^* be $\beta^* \oplus \gamma^*$; if α is $\beta \to \gamma$, let α^* be $\gamma^* \dot{-} \beta^*$; if α is $\beta \vee \gamma$, let α^* be $\gamma^* \sqcap \beta^*$; if α is $\beta \wedge \gamma$, let α^* be $\gamma^* \sqcup \beta^*$. Note that our convention for $\to$ of omitting parentheses by association to the right is consistent with our convention for $\dot{-}$ of omitting parentheses by association to the left.

Using the above notation, the equivalent quasivariety semantics of H, denoted $\mathcal{H}$, is a class of algebras of type $\langle 2, 2, 2, 2, 0 \rangle$, with fundamental operation symbols $\oplus$, $\dot{-}$, $\sqcap$, $\sqcup$, 1. Similarly, for a subset C of the connectives containing $\rightarrow$, the equivalent quasivariety semantics of C–H, denoted $\mathcal{H}_{C^*}$, is a class of algebras whose set of fundamental operation symbols is C^*. By Proposition 2.9 and Lemma 0.12, a constant is defined over $\mathcal{H}_{C^*}$ by $0 = x \dot{-} x$ and each $\mathcal{H}_{C^*}$ is relatively 0–regular. For convenience, *we shall now assume that* 0 *is in the language of each* $\mathcal{H}_{C^*}$. By (1) (see page 14), we have that for all sets Γ of C–formulas and all C–formulas α,

(17) $\quad \Gamma \vdash_C \alpha \quad$ if and only if $\quad \{\beta^* \approx 0 : \beta \in \Gamma\} \models_{\mathcal{H}_{C^*}} \alpha^* \approx 0.$

In view of Corollary 2.7, $\mathcal{H}$ [resp. $\mathcal{H}_{C^*}$] coincides with the class of *full–BCC–algebras* [resp. '*C^*–BCC–algebras*'] defined in [OK85]. Several embedding theorems for these algebras appear in [OK85], exemplifying a general correspondence between fragments of an algebraizable Hilbert system S and subreduct classes of the equivalent quasivariety semantics of S [BP89, Corollary 2.12]. In particular, when $\dot{-} \in C^*$, the class of all C^*–subreducts of members of $\mathcal{H}$ is precisely $\mathcal{H}_{C^*}$. Thus, any algebra in $\mathcal{H}_{C^*}$ is embeddable into an algebra in $\mathcal{H}$.

Let C contain $\rightarrow$. As observed after Corollary 2.8, C–H is axiomatized by those axioms and inference rules in J that contain only the connectives in C. An explicit axiomatization of each $\mathcal{H}_{C^*}$ therefore follows from the algebraizability of $\mathcal{H}_{C^*}$ (Proposition 2.9) and properties (vi) and (vii) after the definition of algebraizability (see page 14). In particular, $\mathcal{H}_{\{\dot{-}\}}$ is axiomatized by the following identities and quasi-identities:

(B1) $\quad x \dot{-} y \dot{-} x \approx 0,$

(B2) $\quad x \dot{-} y \dot{-} (z \dot{-} y) \dot{-} (x \dot{-} z) \approx 0,$

(B3) $\quad x \dot{-} x \approx 0,$

(B4) $\quad x \approx 0$ and $y \dot{-} x \approx 0$ implies $y \approx 0,$

(B5) $\quad y \approx 0$ and $z \dot{-} y \dot{-} x \approx 0$ implies $z \dot{-} x \approx 0,$

(B6) $\quad x \dot{-} y \approx 0$ and $y \dot{-} x \approx 0$ implies $x \approx y.$

$\mathcal{H}_{\{\oplus, \dot{-}\}}$ is axiomatized by (B1)–(B6) and

(B7) $\quad (x \oplus y) \dot{-} y \dot{-} x \approx 0,$

(B8) $\quad z \dot{-} (x \oplus y) \dot{-} (z \dot{-} y \dot{-} x) \approx 0.$

Proposition 2.10. *The quasivarieties* $\mathcal{H}_{\{\dot{-}\}}$ *[resp.* $\mathcal{H}_{\{\oplus, \dot{-}\}}$*] and* $\mathcal{LR}$ *[resp.* $\mathcal{LM}$*] are the same.*

Proof. We shall first show that $\mathcal{H}_{\{\dot{-}\}} \subseteq \mathcal{LR}$. Since (A1) and (A4) are precisely (B2) and (B6), respectively, we need only show that (A2) and (A3) hold in

$\mathcal{H}_{\{\dot{-}\}}$. By (B1), $\mathcal{H}_{\{\dot{-}\}} \models 0 \dot{-} z \dot{-} 0 \approx 0$, so by (B4), we infer that $\mathcal{H}_{\{\dot{-}\}} \models 0 \dot{-} z \approx 0$. By (B1) again, $\mathcal{H}_{\{\dot{-}\}} \models z \dot{-} 0 \dot{-} z \approx 0$. By (B3), $\mathcal{H}_{\{\dot{-}\}} \models z \dot{-} 0 \dot{-} (z \dot{-} 0) \approx 0$, hence by (B5), $\mathcal{H}_{\{\dot{-}\}} \models z \dot{-} (z \dot{-} 0) \approx 0$. Thus by (B6), $\mathcal{H}_{\{\dot{-}\}} \models z \dot{-} 0 \approx z$. That $\mathcal{LR} \subseteq \mathcal{H}_{\{\dot{-}\}}$ follows from Proposition 1.4 and properties of left residuation algebras that are either obvious or have been noted.

To show that $\mathcal{H}_{\{\oplus, \dot{-}\}} \subseteq \mathcal{LM}$, it now suffices to show that $\mathcal{H}_{\{\oplus, \dot{-}\}}$ satisfies (A5). In view of (B8) and (B6), we need only show that $\mathcal{H}_{\{\oplus, \dot{-}\}} \models z \dot{-} x \dot{-} y \dot{-} (z \dot{-} (y \oplus x)) \approx 0$. Using (A2), (B2) and (B7), we calculate (over $\mathcal{H}_{\{\oplus, \dot{-}\}}$)

$$\begin{aligned}
& z \dot{-} x \dot{-} y \dot{-} (z \dot{-} (y \oplus x)) \\
\approx\ & z \dot{-} x \dot{-} y \dot{-} 0 \dot{-} (z \dot{-} (y \oplus x)) \\
\approx\ & z \dot{-} x \dot{-} y \dot{-} ((y \oplus x) \dot{-} x \dot{-} y) \dot{-} (z \dot{-} (y \oplus x)) \\
\approx\ & z \dot{-} x \dot{-} y \dot{-} ((y \oplus x) \dot{-} x \dot{-} y) \dot{-} (z \dot{-} (y \oplus x)) \dot{-} 0 \dot{-} 0 \\
\approx\ & z \dot{-} x \dot{-} y \dot{-} ((y \oplus x) \dot{-} x \dot{-} y) \dot{-} (z \dot{-} (y \oplus x)) \\
& \dot{-} (z \dot{-} x \dot{-} ((y \oplus x) \dot{-} x) \dot{-} (z \dot{-} (y \oplus x))) \\
& \dot{-} (z \dot{-} x \dot{-} y \dot{-} ((y \oplus x) \dot{-} x \dot{-} y) \dot{-} (z \dot{-} x \dot{-} ((y \oplus x) \dot{-} x))) \\
\approx\ & 0.
\end{aligned}$$

Conversely, by (A5), $\mathcal{LM}$ satisfies (B8) and hence also (B7):

$$\mathcal{LM} \models (x \oplus y) \dot{-} y \dot{-} x \approx (x \oplus y) \dot{-} (x \oplus y) \approx 0.$$

□

This proposition confirms that it would have been consistent to call a left residuation algebra a 'BK–algebra' (rather than a 'BCC–algebra'), but we prefer the more descriptive term that we have adopted. It follows from this proposition that the Hilbert system **BCK** is algebraizable with equivalent quasivariety semantics $\mathcal{BCK}$ (also see [BP89, Section 5.2.3], for example) and the $\{\&, \rightarrow\}$–fragment of H_{BCK} is algebraizable with equivalent quasivariety semantics the class of all pocrims (see Example 1.5).

We noted after Lemma 2.5 that H is axiomatized by the set J_1, namely (H1)–(H3), (H5)–(H12), (X_1), (m.p.1) and (m.p.2). By Proposition 2.10, an axiomatization of $\mathcal{H}$ is therefore given by (A1)–(A5) and the following:

(C1) $\quad x \dot{-} 1 \approx 0$

(C2) $\quad (x \sqcap y) \dot{-} y \approx 0$

(C3) $\quad (x \sqcap y) \dot{-} x \approx 0$

(C4) $\quad x \dot{-} (y \sqcap z) \dot{-} w \dot{-} (x \dot{-} y \dot{-} w) \dot{-} (x \dot{-} z \dot{-} w) \approx 0$

(C5) $\quad x \dot{-} (x \sqcup y) \approx 0$

(C6) $\quad y \dot{-} (x \sqcup y) \approx 0$

(C7) $\quad (x \sqcup y) \dot{-} z \dot{-} ((x \dot{-} z) \sqcup (y \dot{-} z)) \approx 0$

(C8) $\quad (x \sqcup y) \dot{-} y \dot{-} x \approx 0.$

We similarly obtain from the axiomatization of $\{\rightarrow, \vee\}$–H that $\mathcal{H}_{\{\dot{-}, \sqcap\}}$ is axiomatized by (A1)–(A4), (C2), (C3) and the identities

(Y_0) $\quad x \dot{-} (y \sqcap z) \dot{-} (x \dot{-} y) \dot{-} (x \dot{-} z) \approx 0,$

(Y_n) $\quad x \dot{-} (y \sqcap z) \dot{-} w_1 \dot{-} \ldots \dot{-} w_n \dot{-} (x \dot{-} y \dot{-} w_1 \dot{-} \ldots \dot{-} w_n)$
$\quad \dot{-} (x \dot{-} z \dot{-} w_1 \dot{-} \ldots \dot{-} w_n) \approx 0$

(corresponding to (X_n)) for each natural number $n \geq 1$. Note that we can derive (Y_n) from (Y_{n+1}) and (A2) in general by setting $w_{n+1} = 0$. As we shall demonstrate later, however, the converse is false. The class $\mathcal{H}_{\{\dot{-}, \sqcap, 1\}}$ is axiomatized by the axioms of $\mathcal{H}_{\{\dot{-}, \sqcap\}}$ together with (C1).

If C^* contains both $\oplus$ and $\sqcap$, then for each $n \geq 2$, (Y_n) may be derived from (A5) and (Y_1) (i.e. (C4)) in the following way:

$$\begin{aligned} & x \dot{-} (y \sqcap z) \dot{-} w_1 \dot{-} \ldots \dot{-} w_n \dot{-} (x \dot{-} y \dot{-} w_1 \dot{-} \ldots \dot{-} w_n) \\ & \quad \dot{-} (x \dot{-} z \dot{-} w_1 \dot{-} \ldots \dot{-} w_n) \\ \approx\; & x \dot{-} (y \sqcap z) \dot{-} (w_n \oplus \cdots \oplus w_1) \dot{-} (x \dot{-} y \dot{-} (w_n \oplus \cdots \oplus w_1)) \\ & \quad \dot{-} (x \dot{-} z \dot{-} (w_n \oplus \cdots \oplus w_1)) \\ \approx\; & 0. \end{aligned}$$

When C^* contains both $\sqcap$ and $\sqcup$ (i.e., C contains $\vee$ and $\wedge$), the proof of Lemma 2.5 shows that the formulas (X_n), $n \geq 2$, are redundant in our axiomatization of C–H. We may summarize the above results as follows:

Corollary 2.11. *For each $C^* \subseteq \{\oplus, \dot{-}, \sqcap, \sqcup, 1\}$ that contains $\dot{-}$, other than $\{\dot{-}, \sqcap\}$ and $\{\dot{-}, \sqcap, 1\}$, $\mathcal{H}_{C^*}$ is axiomatized by* (A1)–(A4) *and those identities among* (A5), (C1)–(C8) *that use only the operation symbols in C^*. $\mathcal{H}_{\{\dot{-}, \sqcap\}}$ is axiomatized by* (A1)–(A4) *and* (Y_n), *$n \in \omega$, while $\mathcal{H}_{\{\dot{-}, \sqcap, 1\}}$ is axiomatized by* (C1) *and the identities and quasi-identity axiomatizing $\mathcal{H}_{\{\dot{-}, \sqcap\}}$.*

Corollary 2.12. $\{\&, \rightarrow, \vee\}$*–$H$ is axiomatized by* (H1), (H3), (H5), (H6), (H11), (H12), (X_1), (m.p.1) *and* (m.p.2).

For each C^* containing $\dot{-}$, the $\langle \dot{-}, 0\rangle$-reduct of each $\mathbf{A} \in \mathcal{H}_{C^*}$ is a left residuation algebra. Thus each such $\mathbf{A}$ is partially ordered by the relation $\leq$ defined by $a \leq b$ if and only if $a \dot{-} b = 0$ $(a, b \in A)$.

Proposition 2.13. *Let* $\mathbf{A} \in \mathcal{H}_{C^*}$*, where* $\dot{-} \in C^*$*, and let* $\leq$ *be the partial order on* A *defined by* $a \leq b$ *iff* $a \dot{-} b = 0$ $(a, b \in A)$*.*

(i) *If* $\sqcap \in C^*$ *then* $\leq$ *is a meet semilattice order and the meet operation is* $\sqcap$*.*

(ii) *If* $\sqcup \in C^*$ *then* $\leq$ *is a join semilattice order and the join operation is* $\sqcup$*.*

(iii) *If* $\{\sqcap, \sqcup\} \subseteq C^*$ *then* $\leq$ *is a lattice order with* $\sqcap$ *and* $\sqcup$ *as its meet and join operations, respectively.*

(iv) *If* $1 \in C^*$ *then* 1 *is the greatest element with respect to* $\leq$*.*

Proof. (i) Suppose $\mathbf{A} \in \mathcal{H}_{C^*}$ and $a, b \in A$. Then, by (C2) and (C3), $a \sqcap b \leq b$ and $a \sqcap b \leq a$. Moreover, if $c \in A$ such that $c \leq a$ and $c \leq b$ then, by (Y_1),

$$\begin{aligned} c \dot{-} (a \sqcap b) &= c \dot{-} (a \sqcap b) \dot{-} 0 \dot{-} 0 \\ &= c \dot{-} (a \sqcap b) \dot{-} (c \dot{-} a) \dot{-} (c \dot{-} b) \\ &= 0, \end{aligned}$$

hence $c \leq a \sqcap b$. Thus, $a \sqcap b$ is the meet of a and b.

(ii) Suppose $\mathbf{A} \in \mathcal{H}_{C^*}$ and $a, b \in A$. Note first that $\mathbf{A}$ satisfies $(0 \sqcup 0) \dot{-} 0 = (0 \sqcup 0) \dot{-} 0 \dot{-} 0 = 0$ by (C8), hence $0 \sqcup 0 = 0$ by (A3) and (A5). By (C5) and (C6), $a \leq a \sqcup b$ and $b \leq a \sqcup b$. Moreover, if $c \in A$ such that $a \leq c$ and $b \leq c$ then, by (C7),

$$\begin{aligned} (a \sqcup b) \dot{-} c &= (a \sqcup b) \dot{-} c \dot{-} 0 \\ &= (a \sqcup b) \dot{-} c \dot{-} (0 \sqcup 0) \\ &= (a \sqcup b) \dot{-} c \dot{-} ((a \dot{-} c) \sqcup (b \dot{-} c)) \\ &= 0, \end{aligned}$$

hence $a \sqcup b \leq c$. Thus, $a \sqcup b$ is the join of a and b.

That (iii) holds is an immediate consequence of (i) and (ii), and (iv) follows immediately from (C1). $\square$

Lemma 2.14. *When* $\{\dot{-}, \sqcup\} \subseteq C^*$*,* $\mathcal{H}_{C^*}$ *satisfies*

(C9) $\quad (x \sqcup y) \dot{-} z \approx (x \dot{-} z) \sqcup (y \dot{-} z)$,

(C10) $\quad (x \sqcup y) \dot{-} y \approx x \dot{-} y$.

When $\{\dot{-}, \sqcap\} \subseteq C^*$*,* $\mathcal{H}_{C^*}$ *satisfies*

(C11) $\quad x \dot{-} (x \sqcap y) \approx x \dot{-} y$.

When $\{\dot{-}, \sqcap, \sqcup\} \subseteq C^*$*,* $\mathcal{H}_{C^*}$ *satisfies*

(C12) $\quad x \dot{-} (y \sqcap z) \approx (x \dot{-} y) \sqcup (x \dot{-} z)$.

When $\{\oplus, \dot{-}, \sqcap\} \subseteq C^*$, $\mathcal{H}_{C^*}$ *satisfies*

(C13) $\quad x \oplus (y \sqcap z) \approx (x \oplus y) \sqcap (x \oplus z)$ *and*

(C14) $\quad (y \sqcap z) \oplus x \approx (y \oplus x) \sqcap (z \oplus x)$.

Proof. Suppose $\{\dot{-}, \sqcup\} \subseteq C^*$. By (C7), $\mathcal{H}_{C^*}$ satisfies

$$(x \sqcup y) \dot{-} z \leq (x \dot{-} z) \sqcup (y \dot{-} z)$$

and, by (A10) (see page 18) and Proposition 2.13(ii), also $(x \dot{-} z) \sqcup (y \dot{-} z) \leq (x \sqcup y) \dot{-} z$, hence (C9) follows by (A4). By (C9), $\mathcal{H}_{C^*}$ satisfies

$$(x \sqcup y) \dot{-} y \approx (x \dot{-} y) \sqcup (y \dot{-} y) \approx (x \dot{-} y) \sqcup 0.$$

By Proposition 2.13(ii), $\mathcal{H}_{C^*}$ satisfies $(x \dot{-} y) \sqcup 0 \approx x \dot{-} y$, and (C10) follows.

Suppose $\{\dot{-}, \sqcap\} \subseteq C^*$. By (A11), $\mathcal{H}_{C^*}$ satisfies $x \dot{-} y \leq x \dot{-} (x \sqcap y)$ and, by (Y_0), also

$$x \dot{-} (x \sqcap y) \dot{-} (x \dot{-} x) \dot{-} (x \dot{-} y) \approx 0,$$

so (C11) follows by (A4).

Suppose $\{\dot{-}, \sqcap, \sqcup\} \subseteq C^*$. That $\mathcal{H}_{C^*}$ satisfies $x \dot{-} (y \sqcap z) \leq (x \dot{-} y) \sqcup (x \dot{-} z)$ follows immediately if one sets $w = (x \dot{-} y) \sqcup (x \dot{-} z)$ in (C4) and, by (A11) and Proposition 2.13(iii), $\mathcal{H}_{C^*}$ satisfies $(x \dot{-} y) \sqcup (x \dot{-} z) \leq x \dot{-} (y \sqcap z)$, so (C12) follows by (A4).

Suppose $\{\oplus, \dot{-}, \sqcap\} \subseteq C^*$. Let $\mathbf{A} \in \mathcal{H}_{C^*}$ and $a, b, c \in A$. To see that (C13) holds, note that since $b \sqcap c \leq b$ and $b \sqcap c \leq c$, we have $a \oplus (b \sqcap c) \leq a \oplus b$ and $a \oplus (b \sqcap c) \leq a \oplus c$ so that $a \oplus (b \sqcap c) \leq (a \oplus b) \sqcap (a \oplus c)$. If $d \in A$ such that $d \leq a \oplus b$ and $d \leq a \oplus c$ then, by (A5) and (Y_1),

$$\begin{aligned}
d \dot{-} (a \oplus (b \sqcap c)) &= d \dot{-} (a \oplus (b \sqcap c)) \dot{-} 0 \dot{-} 0 \\
&= d \dot{-} (a \oplus (b \sqcap c)) \dot{-} (d \dot{-} (a \oplus b)) \dot{-} (d \dot{-} (a \oplus c)) \\
&= d \dot{-} (b \sqcap c) \dot{-} a \dot{-} (d \dot{-} b \dot{-} a) \dot{-} (d \dot{-} c \dot{-} a) \\
&= 0,
\end{aligned}$$

hence $a \oplus (b \sqcap c) = (a \oplus b) \sqcap (a \oplus c)$. For (C14), note that $(b \sqcap c) \oplus a \leq (b \oplus a) \sqcap (c \oplus a)$. If $d \in A$ such that $d \leq b \oplus a$ and $d \leq c \oplus a$ then, by (A5) and (Y_0),

$$\begin{aligned}
d \dot{-} ((b \sqcap c) \oplus a) &= d \dot{-} ((b \sqcap c) \oplus a) \dot{-} 0 \dot{-} 0 \\
&= d \dot{-} ((b \sqcap c) \oplus a) \dot{-} (d \dot{-} (b \oplus a)) \dot{-} (d \dot{-} (c \oplus a)) \\
&= d \dot{-} a \dot{-} (b \sqcap c) \dot{-} (d \dot{-} a \dot{-} b) \dot{-} (d \dot{-} a \dot{-} c) \\
&= 0,
\end{aligned}$$

hence $(b \sqcap c) \oplus a = (b \oplus a) \sqcap (c \oplus a)$. $\square$

Corollary 2.15. *When* $\{\dot{-}, \sqcup\} \subseteq C^*$, *the identity* (C7) *may be replaced by the identity* (C9) *in the axiomatization of* $\mathcal{H}_{C^*}$ *given in Corollary* 2.11. *Moreover, when* $\{\dot{-}, \sqcap, \sqcup\} \subseteq C^*$, *the identity* (C4) *(i.e.* (Y_1)*) may be replaced by the identity* (C12) *in the axiomatization of* $\mathcal{H}_{C^*}$ *given in Corollary* 2.11.

Proof. That (C7) is derivable from (C9) is obvious. When $\{\dot{-}, \sqcap, \sqcup\} \subseteq C^*$, one may derive (C4) from (C12) and (C9) in the following way:

$$\begin{aligned} & x \dot{-} (y \sqcap z) \dot{-} w \dot{-} (x \dot{-} y \dot{-} w) \dot{-} (x \dot{-} z \dot{-} w) \\ \approx\ & ((x \dot{-} y) \sqcup (x \dot{-} z)) \dot{-} w \dot{-} (x \dot{-} y \dot{-} w) \dot{-} (x \dot{-} z \dot{-} w) \\ \approx\ & ((x \dot{-} y \dot{-} w) \dot{-} (x \dot{-} y \dot{-} w) \dot{-} (x \dot{-} z \dot{-} w)) \sqcup \\ & \quad ((x \dot{-} z \dot{-} w) \dot{-} (x \dot{-} y \dot{-} w) \dot{-} (x \dot{-} z \dot{-} w)) \\ \approx\ & 0 \sqcup 0. \end{aligned}$$

Using Proposition 2.13(ii) (whose proof does not invoke (C4)), one may derive $0 \sqcup 0 \approx 0$ and the result follows. $\square$

We present some examples of algebras in $\mathcal{H}$ and $\mathcal{H}_{C^*}$.

Example 2.16. Let $\mathbf{R} = \langle R; +, \cdot, -, 0, 1\rangle$ be a ring with identity. The polrim $\langle \operatorname{Id} \mathbf{R}; \cdot, :, R\rangle$ defined in Example 1.7 has an underlying lattice order $\supseteq$ whose meet and join operations are $\sqcup$ and $\cap$, respectively, where $I \sqcup J = \{i + j : i \in I;\ j \in J\}$ $(I, J \in \operatorname{Id} \mathbf{R})$. One may easily check that the algebra $\langle \operatorname{Id} \mathbf{R}; \cdot, :, \sqcup, \cap, R, \{0\}\rangle$ satisfies the identities (C1)–(C8), and therefore is a member of $\mathcal{H}$. It can be shown that this remains true for the algebras of topologizing filters on $\mathbf{R}$ mentioned in Example 1.7 and studied in the Appendix.

Example 2.17. Let α be a nonzero ordinal that is closed under ordinal addition. Then the algebra $\langle \alpha; \oplus, \dot{-}, 0\rangle$ [resp. $\langle \alpha; \oplus, \div, 0\rangle$] defined in Example 1.8 is a polrim [resp. porrim]. Let $\sqcap$ and $\sqcup$ be the meet and join (i.e., minimum and maximum) operations associated with the order $\subseteq$ on α. Then $\langle \alpha; \oplus, \dot{-}, \sqcap, \sqcup, 0\rangle$ and $\langle \alpha; \oplus, \div, \sqcap, \sqcup, 0\rangle$ are members of $\mathcal{H}_{\{\oplus, \dot{-}, \sqcap, \sqcup\}}$. Indeed, checking that each of the identities (C2)–(C8) holds is straightforward since $\subseteq$ is a linear order. Thus, for each C^* not containing 1, we may define the algebra $\boldsymbol{\alpha}_{C^*}$ [resp. $\boldsymbol{\alpha}^R_{C^*}$] to be the C^*–reduct of $\langle \alpha; \oplus, \dot{-}, \sqcap, \sqcup, 0\rangle$ [resp. $\langle \alpha; \oplus, \div, \sqcap, \sqcup, 0\rangle$]. Evidently, $\boldsymbol{\alpha}_{C^*}, \boldsymbol{\alpha}^R_{C^*} \in \mathcal{H}_{C^*}$.

When α is a successor ordinal, say $\beta+1$, then α has a largest element β, hence the algebras $\langle \alpha; \oplus, \dot{-}, \sqcap, \sqcup, 0, \beta\rangle$ and $\langle \alpha; \oplus, \div, \sqcap, \sqcup, 0, \beta\rangle$ are members of $\mathcal{H}$ (where $\oplus$ is defined as in Example 1.8). In this case, for each C^*, we may define the algebra $\boldsymbol{\alpha}_{C^*}$ [resp. $\boldsymbol{\alpha}^R_{C^*}$] to be the C^*–reduct of $\langle \alpha; \oplus, \dot{-}, \sqcap, \sqcup, 0, \beta\rangle$ [resp. $\langle \alpha; \oplus, \div, \sqcap, \sqcup, 0, \beta\rangle$]. Evidently, $\boldsymbol{\alpha}_{C^*}, \boldsymbol{\alpha}^R_{C^*} \in \mathcal{H}_{C^*}$.

We have seen that each of the quasivarieties $\mathcal{H}_{C^*}$ (where $\dot{-} \in C^*$) except possibly $\mathcal{H}_{\{\dot{-},\sqcap\}}$ and $\mathcal{H}_{\{\dot{-},\sqcap,1\}}$ is finitely axiomatizable (Corollary 2.11). We show now that this is the best possible result, i.e. $\mathcal{H}_{\{\dot{-},\sqcap\}}$ and $\mathcal{H}_{\{\dot{-},\sqcap,1\}}$ are *not* finitely axiomatizable, from which it follows that $\{\rightarrow,\vee\}$–H and $\{\rightarrow,\vee,\perp\}$–$H$ are not finitely axiomatizable.

Theorem 2.18. *The quasivarieties $\mathcal{H}_{\{\dot{-},\sqcap\}}$ and $\mathcal{H}_{\{\dot{-},\sqcap,1\}}$ are not finitely axiomatizable.*

Proof. Recall that $\mathcal{H}_{\{\dot{-},\sqcap\}}$ is axiomatized by the set Σ consisting of (A1)–(A4), (C2), (C3) and (Y_n), $n \in \omega$. We shall show that no finite subset of Σ axiomatizes $\mathcal{H}_{\{\dot{-},\sqcap\}}$.

First we define an algebra $\mathbf{A} = \langle A; \dot{-}, \sqcap, 0\rangle$ of type $\langle 2,2,0\rangle$ that satisfies (A1)–(A4), (C2), (C3) and (Y_0) but does not satisfy (Y_1). Let $\langle A; \leq\rangle$ be the partially ordered set defined by the Hasse diagram in Figure 2. Let $\sqcap$ be the meet operation on A determined by $\leq$ and let $\dot{-}$ be defined on A as follows: for $x, y \in A$, $x \dot{-} y = 0$ if $x \leq y$, $x \dot{-} 0 = x$; $1 \dot{-} a = d$, $1 \dot{-} b = a \dot{-} b = c \dot{-} b = c \dot{-} d = c$, $1 \dot{-} c = a \dot{-} c = b \dot{-} c = b \dot{-} d = b$, $1 \dot{-} d = 1$ and $a \dot{-} d = a$.

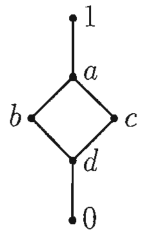


Figure 2.

One checks routinely that $\mathbf{A}$ satisfies (A1)–(A4), (C2), (C3) and (Y_0), but $\mathbf{A}$ does not satisfy (Y_1), since

$$\begin{aligned} 1 \dot{-} (b \sqcap c) \dot{-} a \dot{-} (1 \dot{-} b \dot{-} a) \dot{-} (1 \dot{-} c \dot{-} a) &= 1 \dot{-} a \dot{-} (c \dot{-} a) \dot{-} (b \dot{-} a) \\ &= d \dot{-} 0 \dot{-} 0 \\ &= d \neq 0. \end{aligned}$$

Next, for each $n \geq 1$, we define an algebra $\mathbf{A} = \langle A; \dot{-}, \sqcap, 0\rangle$ that satisfies (A1)–(A4), (C2), (C3) and (Y_n) (and hence (Y_m) for each $m \leq n$) but does not satisfy (Y_{n+1}). First consider the structure $\langle B; \dot{-}', d; \leq'\rangle$ where $\leq'$ is the partial order on the four-element set $B = \{a, b, c, d\}$ defined by the Hasse diagram in Figure 3 and $\dot{-}'$ is a binary operation on B defined in the following way: for $x, y \in B$, $x \dot{-}' y = d$ if $x \leq' y$, $x \dot{-}' d = x$; $a \dot{-}' b = c \dot{-}' b = c$ and $a \dot{-}' c = b \dot{-}' c = b$. One easily checks that $\langle B; \dot{-}', d\rangle \in \mathcal{H}_{\{\dot{-}\}}$.

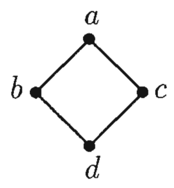


Figure 3.

Next we define the algebra $\mathbf{A} = \langle A; \dot{-}, \sqcap, 0 \rangle$. Let N be the set of positive natural numbers and A the following union of five mutually disjoint sets:

$$A = \{a_i : i \in N\} \cup \{b_i : i \in N\} \cup \{c_i : i \in N\} \cup \{d_i : i \in N\} \cup \{e, f, 0\}.$$

Let $\leq$ be the partial order on A depicted in Figure 4 and $\sqcap$ the meet semilattice operation on A induced by $\leq$. For all $u, v \in A$, let $u \dot{-} 0 = u$ and if $u \leq v$, define $u \dot{-} v = 0$. For all $i, j \in N$ and all $x, y \in \{a, b, c, d\}$, define

$$x_i \dot{-} y_j = \begin{cases} (x \dot{-}' y)_{i(n+1)} & \text{if } i < j \\ (x \dot{-}' y)_{(i+1)(n+1)} & \text{if } i \geq j \text{ and } x \not\leq' y \\ 0 & \text{if } i \geq j \text{ and } x \leq' y, \end{cases}$$

$$a_{2n} \dot{-} f = e \text{ and } x_i \dot{-} f = x_{i+1} \text{ for } x_i \neq a_{2n},$$

$$e \dot{-} a_{2n+1} = d_{2n(n+1)},$$

$$e \dot{-} u = a_{2n+1} \dot{-} u \text{ for } u \neq a_{2n+1}, \text{ and}$$

$$u \dot{-} e = u \dot{-} a_{2n+1} \text{ for } u \neq e.$$

One checks routinely that $\mathbf{A}$ satisfies (A1)–(A4), (C2), (C3) and (Y_n), but $\mathbf{A}$ does not satisfy (Y_{n+1}) because

$$\begin{aligned} &(a_1 \dot{-} (b_2 \sqcap c_2) \dot{-} nf \dot{-} a_{2n+1}) \dot{-} (a_1 \dot{-} b_2 \dot{-} nf \dot{-} a_{2n+1}) \dot{-} (a_1 \dot{-} c_2 \dot{-} nf \dot{-} a_{2n+1}) \\ &= (a_{n+1} \dot{-} nf \dot{-} a_{2n+1}) \dot{-} (c_{n+1} \dot{-} nf \dot{-} a_{2n+1}) \dot{-} (b_{n+1} \dot{-} nf \dot{-} a_{2n+1}) \\ &= (e \dot{-} a_{2n+1}) \dot{-} (c_{2n+1} \dot{-} a_{2n+1}) \dot{-} (b_{2n+1} \dot{-} a_{2n+1}) \\ &= d_{2n(n+1)} \dot{-} 0 \dot{-} 0 \\ &= d_{2n(n+1)} \neq 0. \end{aligned}$$

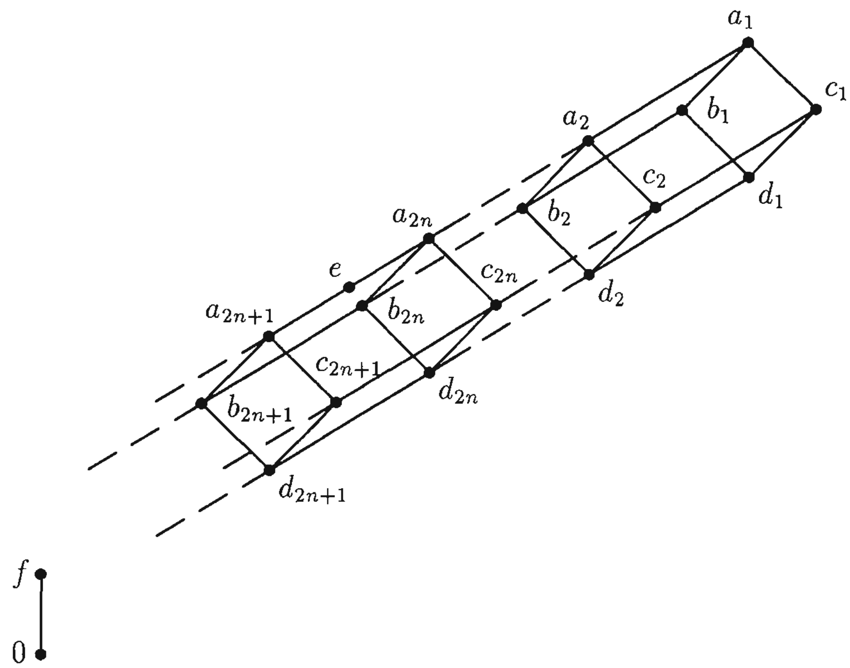


Figure 4.

Suppose that there exists a finite set Σ' of quasi-identities that axiomatizes $\mathcal{H}_{\{\dot{-},\sqcap\}}$. Let $\Phi \in \Sigma'$. Obviously, $\mathcal{H}_{\{\dot{-},\sqcap\}} \models \Phi$, hence $\Sigma \models \Phi$. By Theorem 0.7, therefore, there exists a finite subset Σ_Φ of Σ such that $\Sigma_\Phi \models \Phi$. Let $n(\Phi)$ be the largest $m \in \omega$ such that $(Y_m) \in \Sigma_\Phi$. Since Σ' is finite, the set $\{n(\Phi) : \Phi \in \Sigma'\}$ must have a largest element, say k. So, the set Σ_0 consisting of (A1)–(A4), (C2), (C3) and (Y_n) for each $n \leq k$ satisfies $\Sigma_0 \models \Phi$ for each $\Phi \in \Sigma'$. Thus, Σ_0 axiomatizes $\mathcal{H}_{\{\dot{-},\sqcap\}}$, which contradicts our earlier observations, hence $\mathcal{H}_{\{\dot{-},\sqcap\}}$ is not finitely axiomatizable.

Since each of the algebras constructed in this proof has a top element, we may similarly conclude that $\mathcal{H}_{\{\dot{-},\sqcap,1\}}$ is not finitely axiomatizable. □

By algebraizability (Proposition 2.9), we immediately obtain:

Corollary 2.19. *The* $\{\rightarrow, \vee\}$- *and* $\{\rightarrow, \vee, \bot\}$-*fragments of* H *are not finitely axiomatizable.*

This contrasts with the fact (noted before Lemma 2.5) that *all* superimplicational fragments of H_{BCK} are finitely axiomatizable.

2.3. Some Non-algebraizability Results. For a Hilbert system S over a language $\mathcal{L}$, the condition that S be algebraizable was defined in Chapter 0. and shown to apply to all superimplicational *fragments* of H. In the context of this chapter it is natural to ask whether the $\langle C, I\rangle$-subsystems of H (with respect to Ono and Komori's axiomatization I of H: see page 31) are also algebraizable. If C contains $\to$ and either does not contain $\vee$ or does contain $\wedge$, then $\langle C, I\rangle$–H coincides with $\langle C, J\rangle$–H (as remarked prior to Lemma 2.6) which coincides with C–H (as remarked after Corollary 2.8). Thus we need only consider the cases when C contains $\to$ and $\vee$ but not $\wedge$. We show that $\langle C, I\rangle$–H is *not* algebraizable in each of these cases.

For a Hilbert system S with language $\mathcal{L}$, the *S–filters* of an algebra $\mathbf{A}$ of type $\mathcal{L}$ are defined to be those subsets F of A for which the following is true: for any set $\Gamma \cup \{\varphi\}$ of $\mathcal{L}$–formulas for which $\Gamma \vdash_S \varphi$, and for any function $\overline{a} : \omega \to A$,

$$\text{if } \gamma^{\mathbf{A}}(\overline{a}) \in F \text{ for all } \gamma \in \Gamma, \text{ then } \varphi^{\mathbf{A}}(\overline{a}) \in F.$$

It clearly suffices to verify this closure property for the (axioms and) inference rules $\Gamma \vdash_S \varphi$ in some given axiomatization of S. The set $\mathrm{Fi}^S\,\mathbf{A}$ of all S–filters of $\mathbf{A}$ becomes an algebraic lattice $\mathbf{Fi}^S\,\mathbf{A}$ when ordered by inclusion. A congruence θ of an algebra $\mathbf{A}$ of type $\mathcal{L}$ is said to be *compatible* with $F \subseteq A$ provided that $b \in F$ whenever both $a \in F$ and $(a, b) \in \theta$. Let $\Omega_{\mathbf{A}}F$ denote the largest congruence of $\mathbf{A}$ that is compatible with F. The map $\Omega_{\mathbf{A}} : \mathbf{Fi}^S\,\mathbf{A} \to \mathbf{Con}\,\mathbf{A}$ thus defined is called the *Leibniz operator* of $\mathbf{A}$. If $\Omega_{\mathbf{A}}$ is order preserving from $\mathbf{Fi}^S\,\mathbf{A}$ into $\mathbf{Con}\,\mathbf{A}$ for all algebras $\mathbf{A}$ of type $\mathcal{L}$, we call S *protoalgebraic.*

Theorem 2.20. [BP89, Theorem 5.1] *A Hilbert system S is algebraizable with equivalent quasivariety semantics $\mathcal{K}$ if and only if for each algebra $\mathbf{A}$ of type $\mathcal{L}$, the map $\Omega_{\mathbf{A}} : \mathbf{Fi}^S\,\mathbf{A} \to \mathbf{Con}\,\mathbf{A}$ is a (lattice) isomorphism from $\mathbf{Fi}^S\,\mathbf{A}$ onto $\mathbf{Con}_{\mathcal{K}}\mathbf{A}$. In this case, in particular, each $\Omega_{\mathbf{A}}$ is injective and S is protoalgebraic.*

Proposition 2.21. *Let C be a subset of $\{\&, \to, \vee, \wedge, \bot\}$ that contains $\to$ and $\vee$ but not $\wedge$. Then $\langle C, I\rangle$–H is not algebraizable.*

Proof. Consider $\langle\{\to, \vee\}, I\rangle$–$H$. Define an algebra $\mathbf{A} = \langle A; \dot{-}, \sqcap, 0\rangle$ of type $\langle 2, 2, 0\rangle$ as follows. Let A be a four-element set $\{0, a, b, 1\}$ and $\leq$ the partial order on A defined by the Hasse diagram in Figure 5. For $x, y \in A$, set $x \dot{-} y = 0$ if $x \leq y$ and $x \dot{-} 0 = x$; $1 \dot{-} a = b \dot{-} a = b$ and $1 \dot{-} b = a$. Also set $1 \sqcap 1 = 1$ and $x \sqcap y = 0$ otherwise. One easily checks that $\{0\}$, $\{0, a\}$ and A are $\langle\{\to, \vee\}, I\rangle$–$H$–filters of $\mathbf{A}$.

Note that $(a, 0) = (a \dot{-} 0, a \dot{-} 1) \in \Theta^{\mathbf{A}}(0, 1)$ and, similarly, $(b, 0) \in \Theta^{\mathbf{A}}(0, 1)$, so $\Theta^{\mathbf{A}}(0, 1) = A^2$. Since $(0, 1) = (b \sqcap 1, 1 \sqcap 1) \in \Theta^{\mathbf{A}}(b, 1)$, we have $\Theta^{\mathbf{A}}(b, 1) = A^2$. Since $(b, 1) = (1 \dot{-} a, 1 \dot{-} 0) \in \Theta^{\mathbf{A}}(0, a)$, we have $\Theta^{\mathbf{A}}(0, a) = A^2$. Since $(0, a) = (a \dot{-} b, a \dot{-} 0) \in \Theta^{\mathbf{A}}(b, 0)$, we have $\Theta^{\mathbf{A}}(b, 0) = A^2$. Since $(b, 0) = (b \dot{-} a, b \dot{-} b) \in \Theta^{\mathbf{A}}(a, b)$, we have $\Theta^{\mathbf{A}}(a, b) = A^2$. Lastly, since $(b, 0) =$

$(b \dot{-} a, b \dot{-} 1) \in \Theta^{\mathbf{A}}(a,1)$, we have $\Theta^{\mathbf{A}}(a,1) = A^2$. Thus, there are only two congruences of $\mathbf{A}$, namely id_A and A^2, so the Leibniz operator $\Omega_{\mathbf{A}}$ is not injective. Thus, $\langle\{\rightarrow, \vee\}, I\rangle$–$H$ is not algebraizable. The same example shows that $\langle\{\rightarrow, \vee, \bot\}, I\rangle$–$H$ is not algebraizable.

Figure 5.

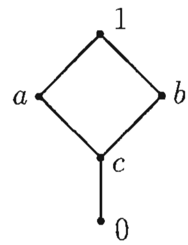


Figure 6.

Consider $\langle\{\&, \rightarrow, \vee\}, I\rangle$–$H$. Define $\mathbf{A} = \langle A; \oplus, \dot{-}, \sqcap, 0\rangle$ as follows. Let A be a five-element set $\{0, a, b, c, 1\}$ and $\leq$ the partial order on A defined by the Hasse diagram in Figure 6. For $x, y \in A$, set $0 \oplus x = x = x \oplus 0$, $c \oplus c = c$, $a \oplus c = a$, $b \oplus c = b$ and $x \oplus y = 1$ otherwise. Also, set $x \dot{-} y = 0$ if $x \leq y$; $x \dot{-} 0 = x$, $1 \dot{-} a = 1 \dot{-} b = a \dot{-} b = b \dot{-} a = c$, $1 \dot{-} c = 1$, $a \dot{-} c = a$ and $b \dot{-} c = b$. Let $\sqcap$ be the meet operation on A determined by $\leq$. One easily checks that $\{0\}$, $\{0, c\}$ and A are $\langle\{\&, \rightarrow, \vee\}, I\rangle$–$H$–filters of $\mathbf{A}$.

Note that $(a, 1) = (0 \oplus a, c \oplus a) \in \Theta^{\mathbf{A}}(0, c)$ and $(b, 1) = (0 \oplus b, c \oplus b) \in \Theta^{\mathbf{A}}(0, c)$, so $(c, 1) = (a \sqcap b, 1 \sqcap 1) \in \Theta^{\mathbf{A}}(0, c)$. Thus $(0, 1) \in \Theta^{\mathbf{A}}(0, c)$ and it follows that $\Theta^{\mathbf{A}}(0, c) = A^2$. Next, since $(c, 0) = (a \dot{-} b, c \dot{-} b) \in \Theta^{\mathbf{A}}(a, c)$ and $(c, 0) = (b \dot{-} a, c \dot{-} a) \in \Theta^{\mathbf{A}}(b, c)$, we have $\Theta^{\mathbf{A}}(a, c) = \Theta^{\mathbf{A}}(b, c) = A^2$. Since $(b, c) = (1 \sqcap b, a \sqcap b) \in \Theta^{\mathbf{A}}(1, a)$ and $(a, c) = (1 \sqcap a, b \sqcap a) \in \Theta^{\mathbf{A}}(1, b)$, we have $\Theta^{\mathbf{A}}(1, a) = \Theta^{\mathbf{A}}(1, b) = A^2$. It follows that the only congruences of $\mathbf{A}$ are id_A and A^2, so $\Omega_{\mathbf{A}}$ is not injective. Thus, $\langle\{\&, \rightarrow, \vee\}, I\rangle$–$H$ is not algebraizable, and neither is $\langle\{\&, \rightarrow, \vee, \bot\}, I\rangle$–$H$. □

This confirms, for example, that the deductive systems $\langle\{\rightarrow, \vee\}, I\rangle$–$H$ and $\{\rightarrow, \vee\}$–H are not the same (the latter being a proper extension of the former) and that $\langle\{\rightarrow, \vee\}, I\rangle$–$H$ is not logically equivalent to $\{\rightarrow, \vee\}$–L. It also confirms that H does not have the separation theorem for the axiomatization I. Let $\mathcal{K}$ be the class of algebras $\mathbf{A} = \langle A; \dot{-}, \sqcap, 0\rangle$ of type $\langle 2, 2, 0\rangle$ such that $\langle A; \dot{-}, 0\rangle \in \mathcal{LR}$, the partial order on A defined by $a \leq b$ if and only if $a \dot{-} b = 0$ $(a, b \in A)$ is a meet semilattice order, and $\mathbf{A}$ satisfies (C2) and (C3). (Thus, $\mathcal{K}$ models the algebraic analogues of all axioms of $\langle\{\rightarrow, \vee\}, I\rangle$–$H$.) The first example of the previous proof shows that even in members of $\mathcal{K}$, the operation $\sqcap$ and the meet operation induced by the definable partial order need not coincide. (This contrasts with Proposition 2.13(i).)

CHAPTER 3

A FINITE MODEL PROPERTY

In [OK85, §9, Question 3], Ono and Komori asked whether various fragments of L ($= L_{\mathrm{BK}}$) and of L_{BCK} have the finite model property (with respect to suitable semantics). In Section 1 of this chapter we show that **BK** (i.e. $\{\rightarrow\}$-H) has the finite model property with respect to the quasivariety of left residuation algebras. It follows that the variety generated by all left residuation algebras is generated by the finite left residuation algebras. In Section 2 we establish that **BK** has the finite model property with respect to a class of structures that constitute a Kripke-style relational semantics for it.

By Theorem 2.4, **BK** and $\{\rightarrow\}$-L are logically equivalent. Establishing a finite model property for **BK** therefore proves the same for $\{\rightarrow\}$-L, so our results settle a part of [OK85, §9, Question 3]; they are to be published in [vAR2].

Recently, Meyer and Ono settled a related question by showing that the $\{\rightarrow\}$-fragment of H_{BCK} (i.e. **BCK**) and the $\{\rightarrow, \wedge\}$-fragment of H_{BCK} have a finite model property. Their paper [MO94] stimulated our interest in the problem solved here. We conclude by noting that our solution may be adapted to give an alternative proof of their result for **BCK**. We have not been able to adapt our strategy (nor that of Meyer and Ono) to proving the finite model property for $\{\rightarrow, \wedge\}$-H, nor for any other fragment.[7]

3.1. The Finite Model Property for BK with respect to $\mathcal{LR}$. We shall prove:

[7] Added in proof: in a personal communication, Professor H. Ono has drawn our attention to the currently unpublished manuscripts [OT1], [OT2], in which Okada and Terui recently proved a finite model property for H (and other logics) with respect to a class of 'intuitionistic phase spaces'. These models are not algebras, and the results do not appear to imply our Corollary 3.6.

Theorem 3.1. *An $\{\rightarrow\}$-formula φ is a theorem of* **BK** *if and only if the identity $\varphi^* \approx 0$ holds in every finite left residuation algebra.*

(The definition of φ^* is on page 39.) The implication from left to right (validity) follows immediately from the fact that $\mathcal{LR}$ is an algebraic semantics for **BK**.

Let φ be an $\{\rightarrow\}$-formula that is not a theorem of **BK**, i.e. $\nvdash_{\{\rightarrow\}} \varphi$. We shall construct a finite left residuation algebra which does not satisfy the identity $\varphi^* \approx 0$, thereby proving Theorem 3.1.

Let $Sub(\varphi)$ be the set of all subformulas of φ. Let M be the set of all finite sequences of elements of $Sub(\varphi)$. Throughout this chapter we shall denote the empty sequence $\langle\rangle$ by $\emptyset$ and use the capital letters S, T, U (possibly with integer subscripts) as variables for elements of M. For $\langle\psi_1, \psi_2, \ldots, \psi_n\rangle$, $\langle\sigma_1, \sigma_2, \ldots, \sigma_m\rangle \in M$, set

$$\langle\psi_1, \psi_2, \ldots, \psi_n\rangle + \langle\sigma_1, \sigma_2, \ldots, \sigma_m\rangle = \langle\psi_1, \psi_2, \ldots, \psi_n, \sigma_1, \sigma_2, \ldots, \sigma_m\rangle.$$

That is, $+$ denotes concatenation of sequences of subformulas of φ. Clearly, $+$ is an associative operation and $\emptyset$ is its identity element. For $S, T \in M$, set

$$S \leq T \text{ if and only if } S \text{ is a subsequence of } T.$$

One easily sees that $\leq$ is a partial order that is compatible with $+$ and has $\emptyset$ as its least element, hence $\langle M; +, \emptyset; \leq\rangle$ is an integral pomonoid. As usual, $S < T$ shall mean $S \leq T$ and $S \neq T$. Note that for each $S \in M$ there are only finitely many $T \in M$ such that $T \leq S$.

If $S = \langle\sigma_1, \ldots, \sigma_n\rangle \in M$ and ζ is any $\{\rightarrow\}$-formula, we shall use $S \rightarrow \zeta$ to abbreviate the formula $\sigma_1 \rightarrow \sigma_2 \rightarrow \ldots \rightarrow \sigma_n \rightarrow \zeta$. We also identify $\emptyset \rightarrow \zeta$ with ζ. Let $\psi \in Sub(\varphi)$. An element S of M is called *ψ-critical* if

(i) $\vdash_{\{\rightarrow\}} S \rightarrow \psi$ and

(ii) for every $T \in M$ such that $T < S$, $\nvdash_{\{\rightarrow\}} T \rightarrow \psi$.

Note that if $U \in M$ and $\vdash_{\{\rightarrow\}} U \rightarrow \psi$, then there exists a ψ-critical element S such that $S \leq U$. Denote by $R(\psi)$ the set of all ψ-critical elements. Note that any two distinct elements S, T of $R(\psi)$ are incomparable, i.e., $S \nleq T$ and $T \nleq S$.

Meyer and Ono's proof [MO94] of the finite model property for **BCK** made crucial use of 'Kripke's Lemma'. The applicability of this lemma in [MO94] depends strongly on the presence of the exchange rule, whose unavailability in our context amounts, roughly speaking, to the noncommutativity of the operation $+$ on M. Instead, we shall use the following result, generalizations of which are proved, e.g., in [Kru60] and [Nas63][8].

Theorem 3.2. (Finite Sequence Theorem) *Let $X \subseteq M$ be such that for distinct $S, T \in X$, $S \not\leq T$ and $T \not\leq S$. Then X is a finite set.*

Set
$$R = \bigcup\{R(\psi) : \psi \in Sub(\varphi)\}.$$

Let R' be the downward closure of R in M, i.e.,
$$R' = \{S \in M : S \leq T \text{ for some } T \in R\}.$$

Let $D(R')$ denote the set of all nonempty downward closed subsets of R'. The Finite Sequence Theorem implies that $R(\psi)$ is finite for each $\psi \in Sub(\varphi)$. Thus R, R' and $D(R')$ are all finite. For $X, Y \in D(R')$, set
$$X \oplus Y = \{S : S \in R' \text{ and } (\exists T \in X)(\exists U \in Y) \text{ such that } S \leq T + U\}.$$

Evidently, $X \oplus Y$ is a downward closed subset of R', hence $\oplus$ is a binary operation on $D(R')$.

Lemma 3.3. *Let $X, Y \in D(R')$. If $S \in X \oplus Y$, then there exist $S_1 \in X$ and $S_2 \in Y$ such that $S = S_1 + S_2$.*

Proof. Let $S = \langle\psi_1, \ldots, \psi_n\rangle \in X \oplus Y$ and let $T = \langle\sigma_1, \ldots, \sigma_m\rangle \in X$ and $U = \langle\zeta_1, \ldots, \zeta_k\rangle \in Y$ such that $S \leq T + U$. So $\langle\psi_1, \ldots, \psi_n\rangle$ is a subsequence of $\langle\sigma_1, \ldots, \sigma_m, \zeta_1, \ldots, \zeta_k\rangle$. Thus there exists $j \leq n$ such that $\langle\psi_1, \ldots, \psi_j\rangle$ is a subsequence of $\langle\sigma_1, \ldots, \sigma_m\rangle$ and $\langle\psi_{j+1}, \ldots, \psi_n\rangle$ is a subsequence of $\langle\zeta_1, \ldots, \zeta_k\rangle$. Let $S_1 = \langle\psi_1, \ldots, \psi_j\rangle$ and $S_2 = \langle\psi_{j+1}, \ldots, \psi_n\rangle$. Then $S_1 \leq T$ and $S_2 \leq U$, so $S_1 \in X$ and $S_2 \in Y$ and $S = S_1 + S_2$. □

Lemma 3.4. $\mathbf{D(R')} = \langle D(R'); \oplus, \{\emptyset\}; \subseteq\rangle$ *is an integral pomonoid.*

Proof. Let $X, Y, Z \in D(R')$. Let $S \in (X \oplus Y) \oplus Z$. By Lemma 3.3, $S = S_1 + S_2$, where $S_1 \in X \oplus Y$ and $S_2 \in Z$. Again by Lemma 3.3, $S_1 = T_1 + T_2$, where $T_1 \in X$ and $T_2 \in Y$. Thus $S = (T_1 + T_2) + S_2 = T_1 + (T_2 + S_2)$, by associativity of $+$, and therefore $S \in X \oplus (Y \oplus Z)$. Thus $(X \oplus Y) \oplus Z \subseteq X \oplus (Y \oplus Z)$. The converse inclusion is similarly proved, hence $\oplus$ is associative. One easily checks that $X \oplus \{\emptyset\} = \{\emptyset\} \oplus X = X$. Moreover, $\subseteq$ is a partial order with least element $\{\emptyset\}$.

Suppose $X \subseteq Y$ and $S \in X \oplus Z$. By Lemma 3.3, $S = S_1 + S_2$, for some $S_1 \in X$ and $S_2 \in Z$. Thus $S_1 \in Y$ as well, so $S \in Y \oplus Z$. Therefore

[8] We thank Professor W.J. Blok for assistance in locating the work [Nas63], and thereby, [Kru60].

$X \oplus Z \subseteq Y \oplus Z$. One similarly shows that $Z \oplus X \subseteq Z \oplus Y$, hence $\subseteq$ is compatible with $\oplus$, which proves the lemma. $\square$

Denote by $C(D(R'))$ the set of all upward closed subsets of $D(R')$. By the above lemma and Lemma 1.3, $\mathbf{C(D(R'))} = \langle C(D(R')); \dot{-}, D(R') \rangle$ is a left residuation algebra, where $\dot{-}$ is defined, for $\mathcal{X}, \mathcal{Y} \in C(D(R'))$, by

$$\mathcal{X} \dot{-} \mathcal{Y} = \{X \in D(R') : X \oplus \mathcal{Y} \subseteq \mathcal{X}\},$$

where

$$X \oplus \mathcal{Y} = \{X \oplus Y : Y \in \mathcal{Y}\}.$$

Moreover, $\mathbf{C(D(R'))}$ is a finite algebra since $D(R')$ is finite. We shall show that the identity $\varphi^* \approx 0$ fails in $\mathbf{C(D(R'))}$.

For each variable p occurring in φ^*, set

$$f(p) = \{X \in D(R') : X \text{ contains at least one } p\text{–critical element}\}.$$

Note that $f(p)$ is upward closed. Thus $f(p) \in C(D(R'))$, so f assigns elements of $C(D(R'))$ to the variables of φ^*. For each subterm ψ^* of φ^*, we use $f(\psi^*)$ to denote the evaluation of ψ^* in $\mathbf{C(D(R'))}$ under this assignment. Suppose that $\sigma^* \dot{-} \psi^*$ is a subterm of φ^* and that $f(\psi^*)$ and $f(\sigma^*)$ have been evaluated. Then

$$\begin{aligned} f(\sigma^* \dot{-} \psi^*) &= f(\sigma^*) \dot{-} f(\psi^*) \\ &= \{X \in D(R') : X \oplus f(\psi^*) \subseteq f(\sigma^*)\}. \end{aligned}$$

Lemma 3.5. *For each subterm ζ^* of φ^*,*

$$f(\zeta^*) = \{X \in D(R') : X \textit{ contains at least one } \zeta\textit{–critical element}\}.$$

Proof. The proof is by induction on the complexity of ζ^*. If ζ^* is a variable, then the result holds by definition of f. Assume, inductively, that ζ^* is $\sigma^* \dot{-} \psi^*$ (i.e., ζ is $\psi \to \sigma$) and that

$$f(\psi^*) = \{X \in D(R') : X \text{ contains at least one } \psi\text{–critical element}\},$$

$$f(\sigma^*) = \{X \in D(R') : X \text{ contains at least one } \sigma\text{–critical element}\}.$$

Let $X \in f(\zeta^*) = f(\sigma^*) \dot{-} f(\psi^*)$, i.e.,

$$X \oplus f(\psi^*) \subseteq f(\sigma^*).$$

Now either $\emptyset$ or $\langle\psi\rangle$ is a ψ–critical element, so either $\{\emptyset\}$ or $\{\langle\psi\rangle, \emptyset\}$ is a downward closed subset of R' hence, by our assumption, $\{\emptyset\} \in f(\psi^*)$ or $\{\langle\psi\rangle, \emptyset\} \in f(\psi^*)$. Thus,

$$X \oplus \{\emptyset\} \in f(\sigma^*) \quad \text{or} \quad X \oplus \{\langle\psi\rangle, \emptyset\} \in f(\sigma^*).$$

By assumption, therefore, $X \oplus \{\langle\psi\rangle, \emptyset\}$ contains a σ–critical element, say S (since $X \oplus \{\emptyset\} \subseteq X \oplus \{\langle\psi\rangle, \emptyset\}$). So $S \in R'$ and, by Lemma 3.3, there exist $T \in X$ and $U \in \{\langle\psi\rangle, \emptyset\}$ such that $S = T + U$. Thus, for some $T \in X$, we have $S = T$ or $S = T + \langle\psi\rangle$. Since S is σ–critical, $\vdash_{\{\rightarrow\}} S \rightarrow \sigma$. We claim $\vdash_{\{\rightarrow\}} T + \langle\psi\rangle \rightarrow \sigma$. This is immediate if $S = T + \langle\psi\rangle$. Suppose $S = T$ and let $S = \langle\eta_1, \ldots, \eta_n\rangle$. By (17) (see page 40), we have that

$$\mathcal{LR} \models \sigma^* \dot{-} \eta_n^* \dot{-} \ldots \dot{-} \eta_1^* \approx 0,$$

hence, by (A12) and (A10) (see page 18),

$$\mathcal{LR} \models \sigma^* \dot{-} \psi^* \dot{-} \eta_n^* \dot{-} \ldots \dot{-} \eta_1^* \approx 0.$$

Thus, by (17), $\vdash_{\{\rightarrow\}} S \rightarrow \psi \rightarrow \sigma$. We now have $\vdash_{\{\rightarrow\}} T + \psi \rightarrow \sigma$, i.e., $\vdash_{\{\rightarrow\}} T \rightarrow \zeta$. Thus there must exist a ζ–critical element, U say, such that $U \leq T$. Since $T \in X$ and X is downward closed, we have $U \in X$ as well, so X contains a ζ–critical element.

Conversely, suppose that $X \in D(R')$ contains a ζ–critical (i.e., a $(\psi \rightarrow \sigma)$–critical) element, say S. Let Y be any element of $f(\psi^*)$. Then Y contains a ψ–critical element, say T. Thus,

$$\vdash_{\{\rightarrow\}} S \rightarrow \psi \rightarrow \sigma \quad \text{and} \quad \vdash_{\{\rightarrow\}} T \rightarrow \psi.$$

We claim that $\vdash_{\{\rightarrow\}} S \rightarrow T \rightarrow \sigma$. Suppose $S = \langle\eta_1, \ldots, \eta_n\rangle$ and $T = \langle\mu_1, \ldots, \mu_m\rangle$. By (17), we have that $\mathcal{LR}$, and hence also $\mathcal{LM}$, satisfies

$$\sigma^* \dot{-} \psi^* \dot{-} \eta_n^* \dot{-} \ldots \dot{-} \eta_1^* \approx 0$$

and

$$\psi^* \dot{-} \mu_m^* \dot{-} \ldots \dot{-} \mu_1^* \approx 0.$$

Thus, by (A5), $\mathcal{LM}$ satisfies

$$\sigma^* \leq \eta_1^* \oplus \cdots \oplus \eta_n^* \oplus \psi^* \quad \text{and} \quad \psi^* \leq \mu_1^* \oplus \cdots \oplus \mu_m^*,$$

hence $\mathcal{LM} \models \sigma^* \leq \eta_1^* \oplus \cdots \oplus \eta_n^* \oplus \mu_1^* \oplus \ldots \mu_m^*$. We therefore have that

$$\mathcal{LR} \models \sigma^* \dot{-} \mu_m^* \dot{-} \ldots \dot{-} \mu_1^* \dot{-} \eta_n^* \dot{-} \ldots \dot{-} \eta_1^* \approx 0,$$

so $\vdash_{\{\rightarrow\}} S \rightarrow T \rightarrow \sigma$, i.e., $\vdash_{\{\rightarrow\}} S + T \rightarrow \sigma$. Thus there must exist a σ–critical element, say U, such that $U \leq S + T$. Since $S \in X$ and $T \in Y$ and $U \in R'$, we have $U \in X \oplus Y$. Since U is σ–critical, $X \oplus Y \in f(\sigma^*)$ by our assumption. It follows that $X \oplus f(\psi^*) \subseteq f(\sigma^*)$, hence $X \in f(\sigma^*) \dot{-} f(\psi^*) = f(\zeta^*)$, as required. □

By the above lemma,

$$f(\varphi^*) = \{X \in D(R') : X \text{ contains at least one } \varphi\text{–critical element }\}.$$

Since $\{\emptyset\} \in D(R')$ and $\nvdash_{\{\rightarrow\}} \varphi$ (i.e., $\nvdash_{\{\rightarrow\}} \emptyset \rightarrow \varphi$), we have that $\{\emptyset\} \notin f(\varphi^*)$. Thus $f(\varphi^*) \neq D(R')$ and therefore the finite algebra $\mathbf{C}(\mathbf{D}(\mathbf{R}'))$ does not satisfy $\varphi^* \approx 0$, proving Theorem 3.1.[9]

Recall that $\mathcal{LR}_{\text{fin}}$ denotes the class of all finite left residuation algebras.

Corollary 3.6. *The variety generated by* $\mathcal{LR}$ *(i.e., the class* $\mathrm{H}(\mathcal{LR})$*) is generated by the class* $\mathcal{LR}_{\text{fin}}$. *Briefly,* $\mathrm{H}(\mathcal{LR}) = \mathrm{HSP}(\mathcal{LR}_{\text{fin}})$.

Proof. Let Σ_1 [resp. Σ_2] be the set of all identities satisfied by $\mathrm{H}(\mathcal{LR})$ [resp. $\mathrm{HSP}(\mathcal{LR}_{\text{fin}})$]. Since $\mathcal{LR}_{\text{fin}}$ is a subclass of $\mathrm{H}(\mathcal{LR})$ we have $\Sigma_1 \subseteq \Sigma_2$. Conversely, suppose the identity $u \approx v$ is not a member of Σ_1, i.e., $\mathrm{H}(\mathcal{LR}) \not\models u \approx v$, hence also $\mathcal{LR} \not\models u \approx v$. Then, by (A4), either $\mathcal{LR} \not\models u \dot{-} v \approx 0$ or $\mathcal{LR} \not\models v \dot{-} u \approx 0$. Assume, without loss of generality, that the former holds. Let φ and ψ be $\{\rightarrow\}$-formulas such that φ^* is u and ψ^* is v. By (17), $\nvdash_{\{\rightarrow\}} \psi \rightarrow \varphi$, hence, by Theorem 3.1, there exists a finite left residuation algebra $\mathbf{A}$ such that $\mathbf{A} \not\models u \dot{-} v \approx 0$. Thus $\mathbf{A} \not\models u \approx v$, implying that $\Sigma_2 \subseteq \Sigma_1$. Thus $\Sigma_1 = \Sigma_2$ and the result follows. □

The variety $\mathrm{H}(\mathcal{LR})$ is finitely axiomatized (by (A1), (A2) and (A3)) [Kom83], as is the logic **BK**. It is well known that a finitely axiomatized Hilbert system over a finite language, having the finite model property with respect to an equivalent quasivariety semantics, is 'decidable' (i.e., has a decidable set of theorems) [Har58]. We therefore have an alternative proof of the following known result:

Corollary 3.7. [Kom84] **BK** *(hence also the equational theory of* $\mathcal{LR}$*, i.e. of* $\mathrm{H}(\mathcal{LR})$*) is decidable.*

3.2. Kripke Semantics for BK. Next, we shall establish the finite model property for **BK** with respect to a class of Kripke-type structures. We refer the reader to [OK85], [Doš88] and [Doš89] for more information on the use of relational semantics of this kind (but the account to follow is self-contained). A **BK**-*structure* is a structure $\langle A; +, 0; \leq \rangle$, where $\langle A; +, 0 \rangle$ is a monoid and $\leq$ is a binary relation on A that is compatible with $+$ and satisfies $0 \leq a$ for all $a \in A$. For all $a, b \in A$, we therefore have $b \leq a + b$ and $b \leq b + a$. From $0 \leq 0$ we obtain $a \leq a$, so $\leq$ is reflexive. ($\leq$ is not necessarily a partial order, however.)

[9] A direct (syntactic) proof of Lemma 3.5 is also possible (see [vAR2]), as is a Gentzen-style argument using Theorem 2.4.

A *valuation* $\models$ on a **BK**–structure $\langle A; +, 0; \leq \rangle$ is a relation between elements of A and variables satisfying (for any $a, b \in A$):

$$a \models p \text{ and } a \leq b \text{ implies } b \models p.$$

(One reads $b \models p$ as "p is true at b".) A valuation $\models$ can be extended to a relation between elements of A and $\{\rightarrow\}$–formulas by setting

$$(18) \qquad a \models \psi \rightarrow \sigma \text{ iff for any } b \in A,\ b \models \psi \text{ implies } a + b \models \sigma.$$

One easily obtains, by induction on the complexity of formulas, that for all formulas φ, and $a, b \in A$,

$$(19) \qquad a \models \varphi \text{ and } a \leq b \text{ implies } b \models \varphi.$$

By a straightforward inductive proof, one can show that for all $\{\rightarrow\}$–formulas $\sigma, \psi_1, \dots, \psi_n$, and all $a \in A$,

$$(20) \qquad \begin{array}{c} a \models \psi_1 \rightarrow \dots \rightarrow \psi_n \rightarrow \sigma \text{ iff for any } b_1, \dots, b_n \in A, \\ b_i \models \psi_i \text{ for } i = 1, \dots, n \text{ implies } a + b_1 + \dots + b_n \models \sigma. \end{array}$$

An $\{\rightarrow\}$–formula φ is *valid* in a **BK**–structure if $0 \models \varphi$ for any valuation $\models$ on the **BK**–structure.

Lemma 3.8. *If* $\vdash_{\{\rightarrow\}} \varphi$ *then* φ *is valid in every* **BK**–*structure.*

Proof. Let $\mathbf{A} = \langle A; +, 0; \leq \rangle$ be a **BK**–structure and let $\models$ be any valuation on $\mathbf{A}$. First consider substitution instances of the axioms (H1) and (H3) of **BK** (see page 31). By (20), we have

$$\begin{array}{c} 0 \models \varphi \rightarrow \psi \rightarrow \varphi \text{ iff for any } b_1, b_2 \in A, \\ b_1 \models \varphi \text{ and } b_2 \models \psi \text{ implies } b_1 + b_2 \models \varphi. \end{array}$$

So, suppose $b_1, b_2 \in A$ such that $b_1 \models \varphi$ and $b_2 \models \psi$. Since $b_1 \leq b_1 + b_2$ we have, by (19), that $b_1 + b_2 \models \varphi$, hence $0 \models \varphi \rightarrow \psi \rightarrow \varphi$.

By (20) again, we have

$$\begin{array}{c} 0 \models (\varphi \rightarrow \psi) \rightarrow (\sigma \rightarrow \varphi) \rightarrow \sigma \rightarrow \psi \text{ iff for any } b_1, b_2, b_3 \in A, \\ b_1 \models \varphi \rightarrow \psi,\ b_2 \models \sigma \rightarrow \varphi, \text{ and } b_3 \models \sigma \text{ implies } b_1 + b_2 + b_3 \models \psi. \end{array}$$

Suppose $b_1, b_2, b_3 \in A$ such that $b_1 \models \varphi \rightarrow \psi$, $b_2 \models \sigma \rightarrow \varphi$ and $b_3 \models \sigma$. By (18), we obtain that $b_2 + b_3 \models \varphi$ and hence that $b_1 + b_2 + b_3 \models \psi$, so $0 \models (\varphi \rightarrow \psi) \rightarrow (\sigma \rightarrow \varphi) \rightarrow \sigma \rightarrow \psi$.

Suppose there exists a derivation of φ in **BK** whose last step is an application of the inference rule (m.p.1). Then there exists an $\{\rightarrow\}$–formula ψ such that

$\vdash_{\{\to\}} \psi$ and $\vdash_{\{\to\}} \psi \to \varphi$. Assume, inductively, that $0 \models \psi$ and $0 \models \psi \to \varphi$. Then, for any $b \in A$,

$$b \models \psi \text{ implies } 0 + b \models \varphi.$$

Since $0 \models \psi$ we obtain that $0 \models \varphi$ as well.

Suppose there exists a derivation of $\varphi \to \psi$ in **BK** whose last step is an application of the inference rule (m.p.2). Then there exists an $\{\to\}$-formula σ such that $\vdash_{\{\to\}} \sigma$ and $\vdash_{\{\to\}} \varphi \to \sigma \to \psi$. Assume, inductively, that $0 \models \sigma$ and $0 \models \varphi \to \sigma \to \psi$. Then, by (20), we have that for any $b_1, b_2 \in A$,

$$b_1 \models \varphi \text{ and } b_2 \models \sigma \text{ implies } 0 + b_1 + b_2 \models \psi.$$

Since $0 \models \sigma$, we have, for any $b_1 \in A$,

$$b_1 \models \varphi \text{ implies } 0 + b_1 \models \psi,$$

i.e., $0 \models \varphi \to \psi$, which completes the proof. □

We claim **BK** is complete with respect to the class of **BK**–structures. In fact, this will be established by showing that **BK** has the *finite* model property with respect to the class of **BK**–structures.

Suppose that $\nvdash_{\{\to\}} \varphi$. Then we can define $\mathbf{D}(\mathbf{R}') = \langle D(R'); \oplus, \{\emptyset\}; \subseteq\rangle$ as in Lemma 3.4. As noted, $\mathbf{D}(\mathbf{R}')$ is a finite integral pomonoid. In particular, this means that $\mathbf{D}(\mathbf{R}')$ is a finite **BK**–structure. Define a valuation $\models$ on $\mathbf{D}(\mathbf{R}')$ as follows: for $X \in D(R')$, set

$$X \models p \text{ if and only if } X \text{ contains at least one } p\text{–critical element.}$$

In the manner of Lemma 3.5 we can show that for all $\psi \in Sub(\varphi)$ and all $X \in D(R')$,

$$X \models \psi \text{ if and only if } X \text{ contains at least one } \psi\text{–critical element.}$$

Thus, $\{\emptyset\} \not\models \varphi$, so φ is not valid in $\mathbf{D}(\mathbf{R}')$, proving the following:

Theorem 3.9. *An $\{\to\}$-formula is a theorem of* **BK** *if and only if it is valid in every finite* **BK***-structure.*

Remark. Factoring the integral pomonoid $\langle M; +, 0; \leq\rangle$ of Section 1 by the equivalence relation which identifies two sequences if they are identical apart from the order of the subformulas produces, in a natural way, a *commutative* integral pomonoid, say $\mathbf{M}'$. If **BCK** and $\mathbf{M}'$ are used in place of **BK** and $\langle M; +, 0; \leq\rangle$ in the arguments of Section 1 [resp. Section 2] we obtain an alternative proof to that in [MO94] of the finite model property for **BCK** with respect to the class of BCK–algebras [resp. the appropriate class of Kripke structures].

CHAPTER 4

ALGEBRAIC PROPERTIES OF THE QUASIVARIETY SEMANTICS

We have seen that each superimplicational fragment C–H of the logic H ($= H_{\mathrm{BK}}$) has an equivalent quasivariety semantics $\mathcal{H}_{C^*}$. In this chapter, we investigate various algebraic properties of the classes $\mathcal{H}_{C^*}$ and, where possible, interpret the results in the context of the logics C–H. *Throughout this chapter, C^* will therefore denote an arbitrary subset of $\{\oplus, \dot{-}, \sqcap, \sqcup, 1\}$ that contains $\dot{-}$.*

In Section 1 we show that the quasivariety $\mathcal{H}_{C^*}$ is a variety if and only if C^* contains $\oplus$ and at least one of $\sqcap$, $\sqcup$. This is an unpublished result of P.M. Idziak; earlier, Komori had proved that $\mathcal{H}_{\{\dot{-}\}}$ is not a variety [Kom84]. We characterize the subvarieties of $\mathcal{LR}$ syntactically. For other values of C^*, we infer some sufficient conditions for subclasses of $\mathcal{H}_{C^*}$ to generate subvarieties of $\mathcal{H}_{C^*}$ and illustrate these results in the case of ordinals less than ω^ω. We prove a result implying that locally finite varieties generated by algebras in $\mathcal{H}_{C^*}$, where $\oplus \in C^*$, are subvarieties of $\mathcal{H}_{C^*}$.

In Section 2 we show that every subvariety of $\mathcal{LR}$ is congruence 3-permutable, while the variety $\mathcal{H}_{\{\oplus, \dot{-}, \sqcap\}}$ [resp. $\mathcal{H}_{\{\oplus, \dot{-}, \sqcup\}}$] is congruence permutable [resp. 4-permutable]. We give a (syntactic) sufficient condition for congruence permutability for subvarieties of $\mathcal{H}_{C^*}$, where $\oplus \in C^*$, and interpret it for left complemented monoids.

In Section 3 we investigate the notion of an 'ideal' of an algebra $\mathbf{A}$ in $\mathcal{H}_{C^*}$. An ideal of $\mathbf{A}$ is just a C–H-filter of $\mathbf{A}$. The ideals of $\mathbf{A}$ therefore form a lattice isomorphic to the lattice of all $\mathcal{H}_{C^*}$–congruences of $\mathbf{A}$ and, as such, play a central role in our study of second order properties of $\mathcal{H}_{C^*}$. We show that in $\mathcal{H}_{C^*}$, our notion of ideal coincides with one defined for more general classes of algebras by Ursini [GU84]. We also consider the weaker notion of a 'preideal' and prove a characterization of ideal generation. Using the theory of ideals, we show that all relatively [finitely] subdirectly irreducible algebras of $\mathcal{H}_{C^*}$ are [finitely] subdirectly irreducible, and hence (using a criterion of Nurakunov) that the lattice $\mathbf{Con}_{\mathcal{H}_{C^*}}\mathbf{A}$ is distributive for each C^* and all $\mathbf{A} \in \mathcal{H}_{C^*}$.

The classes $\mathcal{H}_{C^*}$ turn out not to have the 'relative congruence extension property' (RCEP), which corresponds to the failure of a 'local deduction detachment theorem' (LDDT) for C–H. This is shown in Section 4, where we also provide a characterization of the relative subvarieties of $\mathcal{H}_{C^*}$ that *do* have the RCEP. These correspond to the axiomatic extensions of C–H with the LDDT. Using the results of this section, we show that the relative subvarieties of $\mathcal{H}_{C^*}$ generated by ordinals greater than $\omega + 1$ (with left residuation) do not have the RCEP (provided that $1 \notin C^*$).

4.1. Varieties and Quasivarieties. In [OK85, §9, Remark 7], Ono and Komori asked which of the classes $\mathcal{H}_{C^*}$ are varieties. In other words, which superimplicational fragments of H are *strongly* algebraizable? The next proposition answers this question. Professor Ono has pointed out to us that this proposition is an unpublished result of P.M. Idziak. We include a proof for the sake of completeness and for future reference.

Proposition 4.1. (Idziak) *$\mathcal{H}_{C^*}$ is a variety if and only if C^* contains $\oplus$ and at least one of $\sqcap$, $\sqcup$.*

Proof. Consider the algebra $\mathbf{D} = \langle \omega + 2; \div, \sqcap, \sqcup, 0, \omega + 1\rangle$ that is the $\langle \div, \sqcap, \sqcup, 0, \omega + 1\rangle$–reduct of the algebra $(\boldsymbol{\omega} + \mathbf{2})^R_{\{\oplus, \dot{-}, \sqcap, \sqcup, 0, 1\}}$ defined in Example 2.17. Since $(\omega + 1) \div 1 = \omega + 1$; $(\omega + 1) \div \omega = 1$ and $\omega \div 1 = \omega$, the set $A = \{0, 1, \omega, \omega + 1\}$ is the universe of a subalgebra $\mathbf{A}$ of $\mathbf{D}$. Thus $\mathbf{A} \in \mathcal{H}_{\{\dot{-}, \sqcap, \sqcup, 1\}}$. (We stress that $\dot{-}^{\mathbf{A}}$ is the restriction to A of $\omega + 2$'s *right* residuation operation $\div$ and $1^{\mathbf{A}}$ is $\omega + 1$.) The relation $\theta = \{(0,1), (1,0)\} \cup \mathrm{id}_A$ is a congruence of $\mathbf{A}$, but $\mathbf{A}/\theta$ violates (A4), since

$$(\omega + 1)/\theta \div \omega/\theta = 1/\theta = 0/\theta$$
$$\text{and} \quad \omega/\theta \div (\omega + 1)/\theta = 0/\theta, \quad \text{but } (\omega + 1)/\theta \neq \omega/\theta.$$

Thus $\mathcal{H}_{\{\dot{-}, \sqcap, \sqcup, 1\}}$ is not a variety. Considering reducts of the same algebra, we infer that $\mathcal{H}_{C^*}$ is not a variety whenever $\{\dot{-}\} \subseteq C^* \subseteq \{\dot{-}, \sqcap, \sqcup, 1\}$. (This last result for $\{\dot{-}\}$ was proved by a syntactic argument in [Kom84].)

Recall from Example 1.5 that the quasivariety of pocrims is the class of all members of $\mathcal{H}_{\{\oplus, \dot{-}\}}$ whose monoid operation $\oplus$ is commutative. Higgs exhibited in [Hig84] a pocrim that has a homomorphic image in which (A4) fails, whence $\mathcal{H}_{\{\oplus, \dot{-}\}}$ is not a variety.

Suppose that C^* contains $\oplus$ and $\sqcap$. Since the axiomatization of $\mathcal{H}_{C^*}$ given in Corollary 2.11 consists of identities and the single quasi-identity (A4), it suffices to check that any homomorphic image $\mathbf{B}$ of an algebra in $\mathcal{H}_{C^*}$ satisfies (A4). By (3) (see page 16) and Proposition 2.13(i), $\mathcal{H}_{C^*}$ (and hence also $\mathbf{B}$) satisfies the identity

$$x \sqcap ((x \dot{-} y) \oplus y) \approx x, \tag{21}$$

hence

$$\mathbf{B} \models x \dot{-} y \approx 0 \text{ implies } x \sqcap y \approx x$$

and (exchanging variables)

$$\mathbf{B} \models y \dot{-} x \approx 0 \text{ implies } y \sqcap x \approx y.$$

Since $\sqcap$ is commutative in $\mathbf{B}$, we infer that

$$\mathbf{B} \models x \dot{-} y \approx 0 \text{ and } y \dot{-} x \approx 0 \text{ implies } x \approx y,$$

hence $\mathcal{H}_{C^*}$ is a variety. More precisely, (21) may replace (A4) in the axiomatization of $\mathcal{H}_{C^*}$. If C^* contains $\sqcup$ and $\oplus$, the result follows similarly from a consideration of the identity

$$x \sqcup ((x \dot{-} y) \oplus y) \approx (x \dot{-} y) \oplus y,$$

which may be deduced from (3) and Proposition 2.13(ii). □

The above result contrasts with findings of Idziak [Idz84] (also see [Wro83], [Hig84]) essentially concerning the Hilbert system H_{BCK}: all superimplicational C–fragments of H_{BCK} are algebraizable; the strongly algebraizable ones are just those where C contains at least one of $\vee, \wedge$.

The following lemma summarizes some facts about $\langle \dot{-}, 0\rangle$–terms; we omit the proofs which are by straightforward induction on the complexity of terms. For $n \in \omega$ and C^*–terms $u_1, \ldots, u_n$ in the variables $\vec{x}$, we shall use the abbreviation $x \dot{-} \Sigma_{i=1}^{n} u_i(\vec{x})$ for $x \dot{-} u_1(\vec{x}) \dot{-} \ldots \dot{-} u_n(\vec{x})$.

Lemma 4.2. *Let $\mathcal{K}$ be any class of algebras of type $\langle 2, 0\rangle$ with language $\langle \dot{-}, 0\rangle$ that satisfies* (A1), (A2) *and* (A3). *Let $t(x_0, \ldots, x_n)$ be a $\langle \dot{-}, 0\rangle$-term, $n \in \omega$.*

(i) *There exists a $\langle \dot{-} \rangle$-term $s(x_0, \ldots, x_n)$ such that $\mathcal{K} \models s \approx t$.*
(If t is 0 we may take $s = x \dot{-} x$ for any variable x.)

(ii) *There exists $i \in \{0, \ldots, n\}$ and $\langle \dot{-} \rangle$-terms $u_1, \ldots, u_m$, $m \in \omega$, in the variables $x_0, \ldots, x_n$ such that*

$$\mathcal{K} \models t(x_0, \ldots, x_n) \approx x_i \dot{-} \Sigma_{j=1}^{m} u_j(x_0, \ldots, x_n).$$

(iii) *If $y \in \{x_0, \ldots, x_n\}$ and $y_i \in \{0, y\}$ for $i = 1, \ldots, n$, then*

$$\mathcal{K} \models t(y_0, \ldots, y_n) \approx 0 \ \ or \ \mathcal{K} \models t(y_0, \ldots, y_n) \approx y.$$

Recall from Lemma 1.2 that if $\mathbf{A} = \langle A; \dot{-}, 0\rangle$ is an algebra of type $\langle 2, 0\rangle$ that satisfies (A1), (A2) and (A3), then $\mathbf{A}$ satisfies (A9) and (A12) as well.

Lemma 4.3. *Let* $\mathbf{A} = \langle A; \dot{-}, 0\rangle$ *be an algebra of type* $\langle 2, 0\rangle$ *that satisfies* (A1), (A2) *and* (A3). *If* $u(x, y)$ *is any binary* $\langle \dot{-}, 0\rangle$*-term for which* $\mathcal{LR}$ *satisfies* $u(x, x) \approx 0$, *then*

(22) $$\mathbf{A} \models x \dot{-} y \approx 0 \text{ and } y \dot{-} x \approx 0 \text{ implies } u(x, y) \approx 0.$$

Proof. This is proved by induction on the complexity of the term u. If u contains no occurrence of $\dot{-}$ then u must be the term 0 and (22) holds, trivially. Suppose that $\dot{-}$ occurs in u and that for any binary $\langle \dot{-}, 0\rangle$–term $v(x, y)$ for which $\mathcal{LR}$ satisfies $v(x, x) \approx 0$ and which contains fewer occurrences of $\dot{-}$ than u, (22) holds with v replacing u.

By Lemma 4.2(ii), we may assume that $u(x, y) = z \dot{-} \Sigma_{i=1}^{n} v_i(x, y)$, where $z \in \{x, y\}$ and $n \in \omega$. By assumption,

(23) $$\mathcal{LR} \models u(x, x) \approx x \dot{-} \Sigma_{i=1}^{n} v_i(x, x) \approx 0.$$

For each $i \in \{1, \ldots, n\}$ we have, by Lemma 4.2(iii), that $\mathcal{LR}$ satisfies either $v_i(x, x) \approx 0$ or $v_i(x, x) \approx x$. It follows from (23) that there must exist $k \in \{1, \ldots, n\}$ such that $\mathcal{LR}$ satisfies $v_k(x, x) \approx x$. Using Lemma 4.2(ii) again, we may assume that $v_k(x, y) = w \dot{-} \Sigma_{j=1}^{m} s_j(x, y)$, where $w \in \{x, y\}$ and $m \in \omega$. Since $\mathcal{LR}$ satisfies $v_k(x, x) \approx x \dot{-} \Sigma_{j=1}^{m} s_j(x, x) \approx x$, it follows that $\mathcal{LR}$ satisfies $s_j(x, x) \approx 0$ for each j. By our induction assumption, therefore,

$$\mathbf{A} \models x \dot{-} y \approx 0 \text{ and } y \dot{-} x \approx 0 \text{ implies } s_j(x, y) \approx 0,$$

for each j, hence

(24) $$\mathbf{A} \models x \dot{-} y \approx 0 \text{ and } y \dot{-} x \approx 0 \text{ implies } v_k(x, y) \approx w.$$

Note that $z \dot{-} w \in \{x \dot{-} x, y \dot{-} y, x \dot{-} y, y \dot{-} x\}$, so, by (A9),

(25) $$\mathbf{A} \models x \dot{-} y \approx 0 \text{ and } y \dot{-} x \approx 0 \text{ implies } z \dot{-} w \approx 0.$$

Now, by (A12),

$$\mathbf{A} \models z \dot{-} \Sigma_{i=1}^{k-1} v_i(x, y) \dot{-} z \approx 0,$$

hence, by (A1) and (A2),

$$\mathbf{A} \models z \dot{-} \Sigma_{i=1}^{k-1} v_i(x, y) \dot{-} w \dot{-} (z \dot{-} w) \approx 0,$$

so

$$\mathbf{A} \models z \dot{-} w \approx 0 \text{ implies } z \dot{-} \Sigma_{i=1}^{k-1} v_i(x, y) \dot{-} w \approx 0.$$

By (A3), therefore,

(26) $\mathbf{A} \models z \dot{-} w \approx 0$ implies $z \dot{-} \Sigma_{i=1}^{k-1} v_i(x,y) \dot{-} w \dot{-} \Sigma_{i=k+1}^{n} v_i(x,y) \approx 0.$

The result now follows from (25), (26) and (24). □

Let $\mathbf{C}_2 = \langle\{0,1\};\ \dot{-}\,, 0\rangle$ be the unique two-element left residuation algebra; that is, $\dot{-}$ is defined by $0 \dot{-} 0 = 0 \dot{-} 1 = 1 \dot{-} 1 = 0$ and $1 \dot{-} 0 = 1$. Note that, with meet and join operations $\sqcap$, $\sqcup$ corresponding to the partial order defined by $0 \leq 1$, $\langle\{0,1\};\ \dot{-}\,, \sqcap, \sqcup, 0, 1\rangle$ is a member of $\mathcal{H}_{\{\dot{-}\,,\sqcap\,,\sqcup\,,1\}}$. Note also that $\mathbf{C}_2$ is a BCK–algebra. Every non-trivial left residuation algebra contains $\mathbf{C}_2$ as a subalgebra (since 0 together with any other element form a subalgebra isomorphic to $\mathbf{C}_2$). The variety generated by $\mathbf{C}_2$ is termwise equivalent to the variety of *Tarski* (alias *implication*) *algebras*; it is the smallest nontrivial subquasivariety of $\mathcal{LR}$ and consists just of the $\langle\dot{-}\,, 0\rangle$–subreducts of Boolean algebras, where $x \dot{-} y = x \sqcap (y')$. This variety is axiomatized, relative to BCK–algebras, by the equation $x \dot{-} (y \dot{-} x) \approx x$ (sometimes called Peirce's Law). Also, note that $\mathbf{C}_2$ is the $\mathcal{LR}$–free algebra on one free generator: this is essentially the content of Lemma 4.2(iii).

Part (i) of the following proposition characterizes (equationally) those subclasses of $\mathcal{LR}$ that generate subvarieties of $\mathcal{LR}$. A less precise version of the 'necessity' assertion of (i) is stated without proof in [Kom84] and, for BCK–algebras, in [Idz83]. (For related results for BCK–algebras, see [BR95][10] and [Nag94].)

Proposition 4.4. *(i) Let $\mathcal{K}$ be a variety of algebras of type* $\langle 2,0\rangle$ *with language* $\langle\dot{-}\,, 0\rangle$. *$\mathcal{K}$ is a subvariety of $\mathcal{LR}$ if and only if $\mathcal{K}$ satisfies* (A1), (A2), (A3) *and there exist $n, m \in \omega$ and binary $\langle\dot{-}\rangle$–terms $u_1,\ldots,u_n, v_1,\ldots,v_m$ such that $\mathcal{LR}$ satisfies $u_i(x,x) \approx 0 \approx v_j(x,x)$ for each i, j, and $\mathcal{K}$ satisfies*

(27) $x \dot{-} u_1(x,y) \dot{-} \ldots \dot{-} u_n(x,y) \approx y \dot{-} v_1(x,y) \dot{-} \ldots \dot{-} v_m(x,y).$

(ii) Suppose that C^* contains $\oplus$ but neither of $\sqcap$, $\sqcup$ or that C^* does not contain $\oplus$ (i.e., $\mathcal{H}_{C^*}$ is not a variety). Let $\mathcal{K}$ be a class of algebras with language C^* satisfying the identities *in the axiomatization of $\mathcal{H}_{C^*}$ given in Corollary* 2.11 *and an identity of the form* (27), *where $\mathcal{LR}$ satisfies $u_i(x,x) \approx 0 \approx v_j(x,x)$ for each i, j. Then* HSP($\mathcal{K}$) *is a subvariety of $\mathcal{H}_{C^*}$.*
(iii) For any C^*, let $\mathcal{K}$ be a subvariety of $\mathcal{H}_{C^*}$. The quasivariety of $\langle\dot{-}\,, 0\rangle$–subreducts of members of $\mathcal{K}$ is a subvariety of $\mathcal{LR}$ if and only if $\mathcal{K}$ satisfies an identity of the form (27) *for which $\mathcal{LR}$ satisfies $u_i(x,x) \approx 0 \approx v_j(x,x)$ for each i, j.*

[10]The proof of necessity in (i) given here is modelled on a corresponding argument for BCK–algebras from [BR95] but avoids using a lemma from [BR95] that does not transfer to the generality of $\mathcal{LR}$.

Proof. (i) ($\Rightarrow$) Assume, without loss of generality, that $\mathcal{K}$ is nontrivial (so $\mathbf{C}_2 \in \mathcal{K}$). Let $\mathbf{B} = \langle B; \dot{-}, 0\rangle$ be the algebra of type $\langle 2,0\rangle$ with B the three-element set $\{0,a,b\}$, $a \dot{-} 0 = a$, $b \dot{-} 0 = b$ and $x \dot{-} y = 0$ in all other cases. Note that $\mathbf{B} \notin \mathcal{LR}$ since $\mathbf{B}$ does not satisfy (A4) ($a \dot{-} b = 0$ and $b \dot{-} a = 0$, but $a \neq b$). Also note that $\mathbf{C}_2$ may be considered a subalgebra of $\mathbf{B}$ with universe $C_2 = \{0, a\}$. Let $\varepsilon : \mathbf{B} \to \mathbf{C}_2$ be the homomorphism defined by $\varepsilon(a) = \varepsilon(b) = a$.

Let $\mathbf{T} = \langle T(x,y); \dot{-}, 0\rangle$ be the term algebra of type $\langle 2,0\rangle$ over distinct variables x, y and let $\mathbf{F}$ be the $\mathcal{K}$-free algebra on two free generators $\overline{x}, \overline{y}$. Define the homomorphisms $\mu : \mathbf{T} \to \mathbf{B}$ by $\mu(x) = a$, $\mu(y) = b$ and $\lambda : \mathbf{T} \to \mathbf{F}$ by $\lambda(x) = \overline{x}$, $\lambda(y) = \overline{y}$.

Now $\ker\lambda \nsubseteq \ker\mu$, otherwise $\mathbf{B} \cong \mathbf{T}/\ker\mu \in \mathrm{H}(\mathbf{T}/\ker\lambda) = \mathrm{H}(\mathbf{F}) \subseteq \mathcal{K}$, so $\mathbf{B} \in \mathcal{K} \subseteq \mathcal{LR}$, a contradiction. So there exist $s, t \in T$ such that $\mathcal{K}$ satisfies $s \approx t$ and $\mu(s) \neq \mu(t)$. If $\{\mu(s), \mu(t)\} \in \{\{0,a\},\{0,b\}\}$ then

$$\{s^{\mathbf{C}_2}(a,a), t^{\mathbf{C}_2}(a,a)\} = \{\varepsilon\mu(s), \varepsilon\mu(t)\} = \{0, a\},$$

so $\mathbf{C}_2$ does not satisfy $s \approx t$ and $\mathbf{C}_2 \in \mathcal{K}$, so $\mathcal{K}$ does not satisfy $s \approx t$, a contradiction. Thus $\{\mu(s), \mu(t)\} = \{a, b\}$, so $s^{\mathbf{C}_2}(a,a) = a = t^{\mathbf{C}_2}(a,a)$.

By Lemma 4.2(ii), we may assume that

$$\{s(x,y), t(x,y)\} = \{w \dot{-} \Sigma_{i=1}^{n} u_i(x,y), z \dot{-} \Sigma_{j=1}^{m} v_j(x,y)\},$$

where $z, w \in \{x, y\}$ and $n, m \in \omega$. From

$$a = \varepsilon\mu(s) = \varepsilon\mu(t) = a \dot{-} \Sigma_{i=1}^{n} u_i^{\mathbf{C}_2}(a,a) = a \dot{-} \Sigma_{j=1}^{m} v_i^{\mathbf{C}_2}(a,a),$$

we infer that, for each i, j, $u_i^{\mathbf{C}_2}(a,a) = 0 = v_j^{\mathbf{C}_2}(a,a)$, hence $\mathcal{LR} \models u_i(x,x) \approx 0 \approx v_j(x,x)$ (because $\mathbf{C}_2$ is the $\mathcal{LR}$-free algebra on one free generator). Now, for each i, j, we have $\varepsilon\mu(u_i) = 0^{\mathbf{C}_2} = \varepsilon\mu(v_j)$, hence $\mu(u_i) = 0^{\mathbf{B}} = \mu(v_j)$, so $\{\mu(w), \mu(z)\} = \{a, b\}$, so $\{w, z\} = \{x, y\}$, completing the proof.

($\Leftarrow$) To prove this implication, we need only show that $\mathcal{K}$ satisfies (A4). By Lemma 4.3, $\mathcal{K}$ satisfies

$$x \dot{-} y \approx 0 \text{ and } y \dot{-} x \approx 0 \text{ implies } u_i(x,y) \approx 0 \approx v_j(x,y),$$

for each i, j. By (27), therefore, $\mathcal{K}$ satisfies (A4), as required.

(ii) and (iii) follow easily from (i). $\square$

We illustrate part (ii) of the above proposition in the case of ordinals not exceeding ω^ω.

Proposition 4.5. *Let* $\boldsymbol{\omega}^{\boldsymbol{\omega}}{}_{\{\dot-\}} = \langle \omega^\omega; \dot-, 0\rangle$ *be the left residuation algebra defined in Example* 1.8. *Set*

$$t(x,y) = x \dot- (x \dot- y) \dot- (y \dot- x).$$

Then $\boldsymbol{\omega}^{\boldsymbol{\omega}}{}_{\{\dot-\}}$ *satisfies the identity*

$$t(x,y) \dot- (t(x,y) \dot- t(y,x)) \approx t(y,x) \dot- (t(y,x) \dot- t(x,y)), \tag{28}$$

which is of the form of (27).

Proof. Let $\alpha, \beta \in \omega^\omega$. If $\alpha = 0$ or $\beta = 0$ or $\alpha = \beta$, then (28) holds in $\boldsymbol{\omega}^{\boldsymbol{\omega}}{}_{\{\dot-\}}$ when we interpret x as α and y as β. By the symmetry of (28), we may assume that $\alpha > \beta > 0$. Thus $\beta \dot- \alpha = 0$, so

$$t(\alpha,\beta) = \alpha \dot- (\alpha \dot- \beta) \quad \text{and} \quad t(\beta,\alpha) = \beta \dot- (\alpha \dot- \beta).$$

The ordinals α and β take the following forms:

$$\alpha = \omega^n a_n + \omega^{n-1} a_{n-1} + \cdots + \omega a_1 + a_0,$$
$$\beta = \omega^m b_m + \omega^{m-1} b_{m-1} + \cdots + \omega b_1 + b_0,$$

where $n, m, a_0, \ldots, a_n, b_0, \ldots, b_m \in \omega$, $a_n \neq 0$ and $b_m \neq 0$.

Case (i) Suppose $n > m$. Then

$$\alpha \dot- \beta = \omega^n a_n + ((\omega^{n-1} a_{n-1} + \cdots + a_0) \dot- \beta),$$

hence $\beta \dot- (\alpha \dot- \beta) = 0$, so $t(\beta, \alpha) = 0$ and (28) holds in $\boldsymbol{\omega}^{\boldsymbol{\omega}}{}_{\{\dot-\}}$ when we interpret x as α and y as β.

Case (ii) Suppose $n = m$. Note that if $a_n = b_n$, then we must have $\omega^{n-1} a_{n-1} + \cdots + a_0 > \omega^{n-1} b_{n-1} + \cdots + b_0$ (since $\alpha > \beta$). Now,

$$\alpha \dot- \beta = \begin{cases} \omega^n (a_n - b_n) & \text{if } a_n > b_n \text{ and} \\ & \omega^{n-1} a_{n-1} + \cdots + a_0 \leq \omega^{n-1} b_{n-1} + \cdots + b_0 \\ \omega^n (a_n - b_n + 1) & \text{if } a_n > b_n \text{ and} \\ & \omega^{n-1} a_{n-1} + \cdots + a_0 > \omega^{n-1} b_{n-1} + \cdots + b_0 \\ \omega^n & \text{if } a_n = b_n. \end{cases}$$

Thus $\alpha \dot- \beta$ is a nonzero multiple of ω^n, so $t(\alpha, \beta) = \alpha \dot- (\alpha \dot- \beta) = \omega^n k$ for some $k \in \omega$ and $t(\beta, \alpha) = \beta \dot- (\alpha \dot- \beta) = \omega^n l$ for some $l \in \omega$. Without loss of generality, suppose $k \geq l$; so $k = l + p$ for some $p \in \omega$. Then

$$\begin{aligned} t(\alpha,\beta) \dot{-} (t(\alpha,\beta) \dot{-} t(\beta,\alpha)) &= \omega^n(l+p) \dot{-} (\omega^n(l+p) \dot{-} \omega^n l) \\ &= \omega^n(l+p) \dot{-} (\omega^n p) \\ &= \omega^n l \\ &= \omega^n l \dot{-} (\omega^n l \dot{-} \omega^n(l+p)) \\ &= t(\beta,\alpha) \dot{-} (t(\beta,\alpha) \dot{-} t(\alpha,\beta)), \end{aligned}$$

as required. □

The above proposition and Proposition 4.4(ii) imply the following:

Corollary 4.6. *For each C^* and for each ordinal $\alpha \leq \omega^\omega$ for which $\boldsymbol{\alpha}_{C^*}$ (as defined in Example* 2.17*) exists, the variety generated by $\boldsymbol{\alpha}_{C^*}$ is a subvariety of $\mathcal{H}_{C^*}$.*

It is well known, and easily checked, that for any C^*, any finite algebra in $\mathcal{H}_{C^*}$ that satisfies $x \dot{-} y \dot{-} z \approx x \dot{-} z \dot{-} y$ (i.e., that has a BCK–algebra reduct) generates a subvariety of $\mathcal{H}_{C^*}$. In general, however, as the algebras defined in the first paragraph of the proof of Proposition 4.1 show, when $\{\dot{-}\} \subseteq C^* \subseteq \{\dot{-}, \sqcap, \sqcup, 1\}$, the variety generated by a finite algebra in $\mathcal{H}_{C^*}$ need not be a subvariety of $\mathcal{H}_{C^*}$. The following proposition shows that, when $\oplus \in C^*$, any finite algebra in $\mathcal{H}_{C^*}$ will generate a subvariety of $\mathcal{H}_{C^*}$. Since the $\langle \dot{-} \rangle$–reduct of the algebra $\mathbf{A}$ in the proof of Proposition 4.1 is also the $\langle \dot{-} \rangle$–reduct of a (finite) polrim (with ordinal addition, except that $1 \oplus 1 = 1$ and $\omega + 1$ is absorptive under $\oplus$), it follows from Propositions 4.7 and 4.4(i) that a variety of polrims need not satisfy an identity of the form of (27). Thus the converse of Proposition 4.4(ii) is false (whereas its analogue in the 'commutative' case, with $C^* = \{\oplus, \dot{-}\}$, is an open problem [BR97, §9, Problem 4]).

Proposition 4.7. *Let $\mathcal{K}$ be a variety generated by a class of algebras in $\mathcal{H}_{C^*}$, where $\oplus \in C^*$. Let $\mathbf{F}$ be the $\mathcal{K}$–free algebra on two free generators and suppose that $\langle F; \leq \rangle$ satisfies the ascending chain condition. Then $\mathcal{K}$ is a subvariety of $\mathcal{H}_{C^*}$. Thus, every finite algebra in $\mathcal{H}_{C^*}$ generates a subvariety of $\mathcal{H}_{C^*}$.*

Proof. First observe that $\mathbf{F} \in \mathcal{H}_{C^*}$ since $\mathbf{F} \in \mathrm{I\,S\,P}(Z)$ for some subclass Z of $\mathcal{H}_{C^*}$ with $\mathcal{K} = \mathrm{H\,S\,P}(Z)$, so the reference to $\langle F; \leq \rangle$ makes sense. Define binary terms $t_n(x,y)$, $n \in \omega$, over $\langle \oplus, \dot{-} \rangle$ in the following manner:

$$t_0(x,y) = x, \qquad t_{2k+1}(x,y) = (t_{2k}(x,y) \dot{-} y) \oplus y,$$
$$t_{2k+2}(x,y) = (t_{2k+1}(x,y) \dot{-} x) \oplus x, \quad k \in \omega.$$

By (3) (see page 16), $\mathcal{LM}$ satisfies $t_n(x,y) \leq t_{n+1}(x,y)$ for all $n \in \omega$. If $\overline{x}, \overline{y}$ are the free generators of $\mathbf{F}$ then the ascending chain condition forces the existence of an $n \in \omega$ such that $t_n^{\mathbf{F}}(\overline{x}, \overline{y}) = t_{n+1}^{\mathbf{F}}(\overline{x}, \overline{y})$. Thus,

$$\mathcal{K} \models t_n(x,y) \approx t_{n+1}(x,y). \tag{29}$$

Let $\mathbf{A} \in \mathcal{K}$, $\theta \in \operatorname{Con} \mathbf{A}$ and $a, b \in A$ such that $(a \dot{-} b, 0^{\mathbf{A}}), (b \dot{-} a, 0^{\mathbf{A}}) \in \theta$. It follows easily by induction that $(t_{2k}^{\mathbf{A}}(a,b), a) \in \theta$ and $(t_{2k+1}^{\mathbf{A}}(a,b), b) \in \theta$ for all $k \in \omega$ hence, by (29), $(a,b) \in \theta$. This shows that $\mathcal{K}$ satisfies the quasi-identity (A4), which implies the first assertion of the proposition. The second assertion follows from the first and the fact that finitely generated varieties are locally finite (see, e.g., [BS81, Theorem 10.16]); in particular, their 2-generated free algebras are finite. □

4.2. Congruence Permutability. In this section we investigate congruence n-permutability in subvarieties of $\mathcal{H}_{C^*}$. Let $\mathbf{A}$ be an algebra, $\theta_1, \theta_2 \in \operatorname{Con} \mathbf{A}$ and $n \in \omega$. By $\theta_1 \circ^n \theta_2$ we mean the relational product $\theta_1 \circ \theta_2 \circ \theta_1 \circ \ldots$ of n factors (alternating between θ_1 and θ_2). If, for all $\theta_1, \theta_2 \in \operatorname{Con} \mathbf{A}$,

$$\theta_1 \circ^n \theta_2 = \theta_2 \circ^n \theta_1,$$

then $\mathbf{A}$ is said to be *congruence n-permutable*. A variety is said to be *congruence n-permutable* if every algebra in it is congruence n-permutable. We shall use the term *congruence permutable* for congruence 2-permutable. The standard criterion for congruence n-permutability of a variety is stated in the following theorem.

Theorem 4.8. [HM73] *For any variety $\mathcal{V}$ and $2 \leq n \in \omega$, the following conditions are equivalent:*

(i) *$\mathcal{V}$ is congruence n-permutable;*

(ii) *there exist ternary terms $t_1(x,y,z), \ldots, t_{n-1}(x,y,z)$ such that $\mathcal{V}$ satisfies*

$$\begin{aligned} &t_1(x,y,y) \approx x, \\ &t_{i-1}(x,x,y) \approx t_i(x,y,y) \quad \textit{for } i = 2, \ldots, n-1, \\ &t_{n-1}(x,x,y) \approx y \end{aligned}$$

The last claim of the following proposition is a generalization of Idziak's corresponding result for BCK-algebras [Idz83]. Modulo Proposition 4.4, Idziak's proof works without modification.

Proposition 4.9. *Let $\mathcal{K}$ be a subvariety of $\mathcal{H}_{C^*}$. If the quasivariety of $\langle \dot{-}, 0 \rangle$-subreducts of $\mathcal{K}$ is a subvariety of $\mathcal{LR}$ then $\mathcal{K}$ is congruence 3-permutable. In particular, every subvariety of $\mathcal{LR}$ is congruence 3-permutable.*

Proof. Let $\mathcal{K}$ be a subvariety of $\mathcal{H}_{C^*}$ that satisfies the conditions of this proposition. By Proposition 4.4(iii), there exist $n, m \in \omega$ and binary $\langle \dot{-} \rangle$-terms $u_1, \dots, u_n, v_1, \dots, v_m$ such that $\mathcal{LR}$ satisfies $u_i(x,x) \approx 0 \approx v_j(x,x)$ for each i, j, and $\mathcal{K}$ satisfies

$$x \dot{-} u_1(x,y) \dot{-} \dots \dot{-} u_n(x,y) \approx y \dot{-} v_1(x,y) \dot{-} \dots \dot{-} v_m(x,y).$$

If we define

$$\begin{aligned} t_1(x,y,z) &= x \dot{-} u_1(y,z) \dot{-} \dots \dot{-} u_n(y,z) \\ \text{and} \quad t_2(x,y,z) &= z \dot{-} v_1(x,y) \dot{-} \dots \dot{-} v_m(x,y), \end{aligned}$$

then $\mathcal{K}$ satisfies

$$\begin{aligned} t_1(x,y,y) &\approx x, \\ t_1(x,x,y) &\approx t_2(x,y,y) \\ \text{and} \qquad t_2(x,x,y) &\approx y, \end{aligned}$$

and the result follows by Theorem 4.8. □

Since the two-element BCK–algebra $\mathbf{C}_2$ embeds into every nontrivial left residuation algebra and generates a variety termwise equivalent to the (non-congruence permutable) variety of Tarski algebras, *no nontrivial subvariety of $\mathcal{LR}$ is congruence permutable.* The same argument is valid if we enrich the language of $\mathbf{C}_2$ with its meet operation $\sqcap$, so we may also conclude that *no nontrivial subvariety of $\mathcal{H}_{\{\dot{-}, \sqcap\}}$ is congruence permutable.*

The following proposition provides a sufficient condition for a subvariety of $\mathcal{H}_{C^*}$ (when $\oplus \in C^*$) to be congruence permutable. The corollary that follows shows that this condition is not artificial. (Among natural syntactic conditions on subvarieties of $\mathcal{H}_{\{\oplus, \dot{-}\}}$ implying congruence permutability, it is the most general condition known to us.)

Proposition 4.10. *Let C^* contain $\oplus$ and let $\mathcal{K}$ be a subvariety of $\mathcal{H}_{C^*}$ which satisfies an identity of the form*

(30)
$$(y \dot{-} (x \dot{-} (u_1(x,y) \oplus \dots \oplus u_n(x,y)))) \oplus (x \dot{-} (v_1(x,y) \oplus \dots \oplus v_m(x,y))) \approx y$$

where $n, m \in \omega$ and u_i and v_j are $\langle \dot{-} \rangle$-terms such that $\mathcal{LR}$ satisfies $u_i(x,x) \approx 0 \approx v_j(x,x)$ for each i, j. Then $\mathcal{K}$ is congruence permutable.

Proof. Set

$$\begin{aligned} t(x,y,z) = [z \dot{-} (v_1(y,x) \oplus \dots \oplus v_m(y,x)) \dot{-} (x \dot{-} (u_1(y,z) \oplus \dots \\ \oplus u_n(y,z)))] \oplus (x \dot{-} (v_1(y,z) \oplus \dots \oplus v_m(y,z))). \end{aligned}$$

By (30), $\mathcal{K}$ satisfies $x \dot{-} (v_1(x,y) \oplus \cdots \oplus v_m(x,y)) \leq y$. Thus we calculate, over $\mathcal{K}$, that

$$\begin{aligned} &t(x,y,y) \approx [y \dot{-} (v_1(y,x) \oplus \cdots \oplus v_m(y,x)) \dot{-} (x \dot{-} (0 \oplus \cdots \oplus 0))] \oplus \\ &(x \dot{-} (0 \oplus \cdots \oplus 0)) \approx 0 \oplus x \approx x \text{ and} \\ &t(x,x,y) \approx [y \dot{-} (0 \oplus \cdots \oplus 0) \dot{-} (x \dot{-} (u_1(x,y) \oplus \cdots \oplus u_n(x,y)))] \oplus \\ &(x \dot{-} (v_1(x,y) \oplus \cdots \oplus v_m(x,y))) \approx y \text{ (by (30))}. \end{aligned}$$

This shows that $\mathcal{K}$ is congruence permutable. □

Recall the definition of a left complemented monoid from Example 1.9.

Corollary 4.11. *Let $\mathcal{K}$ be a variety of left complemented monoids. If $\mathcal{K}$ satisfies an identity of the form $x \dot{-} (u_1(x,y) \oplus \cdots \oplus u_n(x,y)) \leq y$ for suitable $\langle \dot{-} \rangle$-terms u_i, where $\mathcal{LR}$ satisfies $u_i(x,x) \approx 0$ for each i, then $\mathcal{K}$ is congruence permutable. In particular, if the quasivariety of residuation subreducts of $\mathcal{K}$ is a variety then $\mathcal{K}$ is congruence permutable.*

Proof. Recall that $\mathcal{K}$ has a definable join operation $\sqcup$ defined by $x \sqcup y = (x \dot{-} y) \oplus y$, hence $\mathcal{K}$ satisfies

$$\begin{aligned} & (y \dot{-} (x \dot{-} (u_1(x,y) \oplus \cdots \oplus u_n(x,y)))) \oplus (x \dot{-} (u_1(x,y) \oplus \cdots \oplus u_n(x,y))) \\ \approx\ & y \sqcup (x \dot{-} (u_1(x,y) \oplus \cdots \oplus u_n(x,y))) \\ \approx\ & y \end{aligned}$$

so the first assertion follows from Proposition 4.10. If the $\langle \dot{-}, 0 \rangle$–subreducts of $\mathcal{K}$ form a variety, it follows from Proposition 4.4(iii) that $\mathcal{K}$ satisfies an identity of the form $x \dot{-} (u_1(x,y) \oplus \cdots \oplus u_n(x,y)) \leq y$ for suitable u_i as described in the statement of the present corollary. The second assertion therefore follows from the first. □

Since the class of hoops (see Example 1.9) is a variety of left complemented monoids satisfying $x \dot{-} (x \dot{-} y) \leq y$, Corollary 4.11 generalizes the fact that hoops are congruence permutable [BP94b, Theorem 1.10]. (In fact, the $\langle \dot{-}, 0 \rangle$–subreducts of hoops also form a variety: see [Fer92, Theorem 3.15, p96].)

Proposition 4.12. *If $\oplus \in C^*$, then a locally finite subvariety of $\mathcal{H}_{C^*}$ is congruence 3-permutable.*

Proof. Let $\mathcal{K}$ be a locally finite subvariety of $\mathcal{H}_{C^*}$, where $\oplus \in C^*$. Then $\mathcal{K}$ is a variety satisfying the conditions of Proposition 4.7, hence $\mathcal{K}$ satisfies $t_n(x,y) \approx t_{n+1}(x,y)$, where $n \in \omega$ and t_n, t_{n+1} are the terms defined in the proof of Proposition 4.7. Note that $\mathcal{H}_{\{\oplus, \dot{-}\}}$ (hence also $\mathcal{K}$) satisfies $t_i(x,x) \approx x$ for each $i \in \omega$.

Case 1. Suppose n is even. If $n = 0$ then $\mathcal{K}$ satisfies $x \approx (x \dot{-} y) \oplus y \geq y$, so $\mathcal{K}$ is trivial. So assume $n > 0$. Then $t_n(x,y) = (t_{n-1}(x,y) \dot{-} x) \oplus x$ and we set $s_1(x,y,z) = (t_{n-1}(y,z) \dot{-} y) \oplus x$. Since $t_{n+1}(x,y) = (t_n(x,y) \dot{-} y) \oplus y$, we may set $s_2(x,y,z) = (t_n(x,y) \dot{-} y) \oplus z$. Then $\mathcal{K}$ satisfies $s_1(x,y,y) \approx x$, $s_2(x,x,y) \approx y$ and

$$\begin{aligned} s_1(x,x,y) &\approx (t_{n-1}(x,y) \dot{-} x) \oplus x \\ &\approx t_n(x,y) \\ &\approx t_{n+1}(x,y) \\ &\approx (t_n(x,y) \dot{-} y) \oplus y \approx s_2(x,y,y). \end{aligned}$$

Case 2. Suppose n is odd. Then $t_n(x,y) = (t_{n-1}(x,y) \dot{-} y) \oplus y$ and $t_{n+1}(x,y) = (t_n(x,y) \dot{-} x) \oplus x$. Set $s_1(x,y,z) = (t_{n-1}(z,y) \dot{-} y) \oplus x$ and $s_2(x,y,z) = (t_n(y,x) \dot{-} y) \oplus z$. Then $\mathcal{K}$ satisfies $s_1(x,y,y) \approx x$, $s_2(x,x,y) \approx y$ and

$$\begin{aligned} s_1(x,x,y) &\approx (t_{n-1}(y,x) \dot{-} x) \oplus x \\ &\approx t_n(y,x) \\ &\approx t_{n+1}(y,x) \\ &\approx (t_n(y,x) \dot{-} y) \oplus y \approx s_2(x,y,y). \end{aligned}$$

In both cases, therefore, s_1, s_2 satisfy condition (ii) of Theorem 4.8. Thus, $\mathcal{K}$ is congruence 3-permutable. □

Proposition 4.13. *If C^* contains $\{\oplus, \dot{-}, \sqcap\}$ then the variety $\mathcal{H}_{C^*}$ is congruence permutable. If C^* contains $\{\oplus, \dot{-}, \sqcup\}$ then the variety $\mathcal{H}_{C^*}$ is congruence 4-permutable.*

Proof. Let C^* contain $\{\oplus, \dot{-}, \sqcap\}$. For the term

$$u(x,y,z) = ((z \dot{-} y) \oplus x) \sqcap ((x \dot{-} y) \oplus z),$$

the variety $\mathcal{H}_{C^*}$ satisfies the identities $u(x,y,y) \approx x \approx u(y,y,x)$ (using (3), page 16), and is therefore congruence permutable. Let C^* contain $\{\oplus, \dot{-}, \sqcup\}$. If we define

$$\begin{aligned} t_1(x,y,z) &= (z \dot{-} y) \oplus x, \\ t_2(x,y,z) &= ((y \dot{-} z) \oplus z) \sqcup ((y \dot{-} x) \oplus x), \\ t_3(x,y,z) &= (x \dot{-} y) \oplus z, \end{aligned}$$

then $\mathcal{H}_{C^*}$ satisfies

$$x \approx t_1(x, y, y),$$
$$t_i(x, x, y) \approx t_{i+1}(x, y, y) \ (i = 1, 2),$$
$$t_3(x, x, y) \approx y,$$

and the result follows by Theorem 4.8. □

If a variety $\mathcal{K}$ satisfies the hypotheses of any one of the Propositions 4.9, 4.10, 4.12, Corollary 4.11 or the first assertion of Proposition 4.13, and $\mathbf{A} \in \mathcal{K}$ and $\theta \in \operatorname{Con} \mathbf{A}$ and $h : \mathbf{A} \mapsto \mathbf{B}$ is a surjective homomorphism, then $h[\theta] := \{(h(a_1), h(a_2)) : (a_1, a_2) \in \theta\} \in \operatorname{Con} \mathbf{B}$, since this is a general consequence of congruence 3-permutability (see [MMT87, Theorem 4.68]).

4.3. Relative Congruences and Ideals. Let S be a Hilbert system with language $\mathcal{L}$ and $\mathbf{A}$ an algebra of type $\mathcal{L}$. Recall the definition of an S–filter of $\mathbf{A}$ from Section 2.3. For $X \subseteq A$, we define the *S–filter of $\mathbf{A}$ generated by X*, denoted $\langle X \rangle_{\mathbf{A}}$, as the intersection of all S–filters of $\mathbf{A}$ containing X. In the case where X consists of a single element, say a, we write $\langle a \rangle_{\mathbf{A}}$ instead of $\langle \{a\} \rangle_{\mathbf{A}}$. An S–filter F of $\mathbf{A}$ for which there exists an $a \in A$ such that $F = \langle a \rangle_{\mathbf{A}}$ is called a *principal S-filter* of $\mathbf{A}$.

Recall from Theorem 2.20 that when S is an algebraizable Hilbert system with equivalent quasivariety semantics $\mathcal{K}$ there is an isomorphism $\Omega_{\mathbf{A}}$ from the lattice $\mathbf{Fi}^S \mathbf{A}$ to the lattice $\mathbf{Con}_{\mathcal{K}} \mathbf{A}$ of relative congruences of $\mathbf{A}$. More explicitly, the bijections are defined (for $F \in \operatorname{Fi}^S \mathbf{A}$ and $\theta \in \operatorname{Con}_{\mathcal{K}} \mathbf{A}$) by

(31) $$F \mapsto \Omega_{\mathbf{A}} F = \{(a, b) \in A^2 : \Delta_j^{\mathbf{A}}(a, b) \in F \text{ for } j = 1, \ldots, m\},$$

(32) $$\theta \mapsto \{a \in A : (\delta_i^{\mathbf{A}}(a), \varepsilon_i^{\mathbf{A}}(a)) \in \theta \text{ for } i = 1, \ldots, n\},$$

where the Δ_j and $\delta_i \approx \varepsilon_i$ are equivalence formulas and defining equations for S and $\mathcal{K}$ [BP89, Theorem 5.1].

Since our notational conventions for $\mathcal{H}_{C^*}$ dualize those of C–H, we shall use term *$\mathcal{H}_{C^*}$–ideal* in place of 'C–H–filter'.

By Corollary 2.8, each C–H (where $\rightarrow \in C$) is axiomatized by a set of axioms and the two rules of inference (m.p.1) and (m.p.2). Let $\mathbf{A} \in \mathcal{H}_{C^*}$. For each axiom $\varphi(\vec{p})$ of C–H and all $\vec{a} \in A$, it follows from (17) (see page 40) that $\varphi^{*\mathbf{A}}(\vec{a}) = 0$. Thus an $\mathcal{H}_{C^*}$–ideal of $\mathbf{A} \in \mathcal{H}_{C^*}$ is just a subset I of A such that $0 \in I$ and conditions (I1) and (I2) (corresponding to (m.p.1) and (m.p.2)) hold:

(I1) for all $a, b \in A$, if $b, a \dot{-} b \in I$ then $a \in I$, (I2) for all $a, b, c \in A$, if $b, a \dot{-} b \dot{-} c \in I$ then $a \dot{-} c \in I$.

Evidently, if one sets c equal to 0 in (I2), then one obtains (I1), so (I1) is redundant. Note that I is an $\mathcal{H}_{C^*}$–ideal of $\mathbf{A} \in \mathcal{H}_{C^*}$ if and only if I is an $\mathcal{H}_{\{\dot{-}\}}$–ideal of the $\langle \dot{-}, 0\rangle$–reduct of $\mathbf{A}$. Thus, when the $\langle \dot{-}, 0\rangle$–reduct of $\mathbf{A}$ is in $\mathcal{H}_{\{\dot{-}\}}$, the definition of an $\mathcal{H}_{C^*}$–ideal of $\mathbf{A}$ does not depend on C^*, so in the absence of any possible confusion, we shall refer to an $\mathcal{H}_{C^*}$–ideal simply as an *ideal.* Consequently, we may use $\mathbf{Id\,A}$ instead of $\mathbf{Fi}^{\mathcal{H}_{C^*}}\mathbf{A}$ to denote the lattice of ideals (i.e., C–H–filters) of $\mathbf{A} \in \mathcal{H}_{C^*}$. The smallest ideal of $\mathbf{A} \in \mathcal{H}_{C^*}$ is $\{0^{\mathbf{A}}\}$, which we refer to as the *trivial* (or *zero*) *ideal.* The largest ideal of $\mathbf{A}$ is A; any ideal of $\mathbf{A}$ different from A is called a *proper ideal.*

In view of Proposition 2.9, (31) and (32), we have the following:

Proposition 4.14. *Let $\mathbf{A} \in \mathcal{H}_{C^*}$. For $I \in \mathrm{Id}\,\mathbf{A}$ and $\theta \in \mathrm{Con}_{\mathcal{H}_{C^*}}\mathbf{A}$, the maps*

$$I \mapsto \Omega_{\mathbf{A}} I = \{(a, b) \in A^2 : a \dot{-} b, b \dot{-} a \in I\},$$

$$\theta \mapsto \{a \in A : (a, 0) \in \theta\} = 0/\theta,$$

are mutually inverse isomorphisms between the ideal and relative congruence lattices of $\mathbf{A}$.

Thus the ideals of an algebra in $\mathcal{H}_{C^*}$ are precisely the 0-classes of its relative congruences. The isomorphisms between $\mathbf{Con}_{\mathcal{K}}\mathbf{A}$ and $\mathbf{Id\,A}$ show that $\mathcal{K}$ is relatively 0–regular. This result was deduced earlier using the fact that each superimplicational C–H has the Gödel rule (see page 40) and Lemma 0.12.

It is clear from the definition of a C–H–filter and (17) (see page 40) that the ideals of an algebra $\mathbf{A}$ in $\mathcal{H}_{C^*}$ are characterized by the property that whenever $\mathcal{H}_{C^*}$ satisfies a quasi-identity of the form

$$\bigwedge_{i=1}^{m} u_i(\vec{y}) \approx 0 \;\; \textsf{implies} \;\; v(\vec{x}, \vec{y}) \approx 0$$

and $\vec{a}, \vec{b} \in A$ with $u_i^{\mathbf{A}}(\vec{b}) \in I$ for each i, then $v^{\mathbf{A}}(\vec{a}, \vec{b}) \in I$. As a special case, ideals I of an algebra $\mathbf{A}$ in $\mathcal{H}_{C^*}$ have the following (algebraically simpler) closure property: whenever $\mathcal{H}_{C^*}$ satisfies a quasi-identity of the form

$$\bigwedge_{i=1}^{m} y_i \approx 0 \;\; \textsf{implies} \;\; t(\vec{x}, \vec{y}) \approx 0$$

(i.e. $t(\vec{x}, \vec{0}) \approx 0$) and $\vec{a} \in A$ and $\vec{b} \in I$, then $t^{\mathbf{A}}(\vec{a}, \vec{b}) \in I$.

Let $\mathcal{K}$ be a class of algebras of type $\mathcal{L}$ that contains a constant term 0 and let $\mathbf{A}$ be an algebra of type $\mathcal{L}$. A nonempty subset I of A will be called an

Ursini-ideal of $\mathbf{A}$ if it satisfies the following closure property: whenever $t(\vec{x}, \vec{y})$ is an $\mathcal{L}$–term for which

$$\mathcal{K} \models t(\vec{x}, \vec{0}) \approx 0 \tag{33}$$

and $\vec{a} \in A$ and $\vec{b} \in I$, then $t^{\mathbf{A}}(\vec{a}, \vec{b}) \in I$. This definition was introduced by Ursini (see, e.g., [GU84]). The Ursini-ideals of an algebra $\mathbf{A}$ depend, in general, on the choice of $\mathcal{K}$ but they are independent of $\mathcal{K}$ in the event that $\mathbf{A}$ belongs to a 'subtractive' variety: a variety is called *subtractive* if, for a suitable binary term s, it satisfies the identities $s(x, x) \approx 0$ and $s(x, 0) \approx x$. For each C^*, the variety $\mathrm{H}(\mathcal{H}_{C^*})$ is evidently subtractive with respect to the term $s(x, y) = x \dot{-} y$. It is easy to see that for any reflexive compatible binary relation τ on $\mathbf{A}$, the class $0^{\mathbf{A}}/\tau$ is an Ursini-ideal of $\mathbf{A}$. The Ursini-ideals of $\mathbf{A}$ form an algebraic lattice under inclusion.

The requirement that the maps $\theta \mapsto 0^{\mathbf{A}}/\theta$ $(\theta \in \operatorname{Con} \mathbf{A})$ be isomorphisms from the congruence lattices onto the Ursini-ideal lattices of each $\mathbf{A}$ in a variety $\mathcal{K}$ (with 0) is equivalent to the requirement that $\mathcal{K}$ be both 0-regular and subtractive [GU84]. This result does *not* extend to relatively 0-regular subtractive *quasivarieties* $\mathcal{K}$. (Consider, e.g., the group $\mathbf{Z}$ of integers as a member of the relatively 0-regular quasivariety of torsion-free abelian groups. The Ursini-ideals of $\mathbf{Z}$ are just its subgroups but $\mathbf{Z}$ has no nontrivial relative congruences.) In general, it is a nontrivial question whether S–filters in the equivalent quasivariety semantics of an algebraizable deductive system S with the Gödel rule coincide with Ursini-ideals (even if subtractivity of the semantics is given), but it is desirable for the sake of algebraic simplicity that they should do so. We show that this is indeed the case in our present context:

Proposition 4.15. *Let $\mathbf{A} \in \mathcal{H}_{C^*}$ and $I \subseteq A$. Then I is an ideal of $\mathbf{A}$ if and only if it is an Ursini-ideal of $\mathbf{A}$.*

Proof. As noted above, the variety $\mathrm{H}(\mathcal{H}_{C^*})$ is subtractive for each C^*, so the proposition's reference to Ursini-ideals needs no varietal qualification. Necessity is immediate, in view of the previous discussion. Conversely, suppose I is an Ursini-ideal of $\mathbf{A}$. By interpreting t (in (33)) as the constant term 0, we obtain $0 \in I$. Suppose that $a, b, c \in A$ and that $b, a \dot{-} b \dot{-} c \in I$. Set $t(x, z, y_1, y_2) = x \dot{-} z \dot{-} (x \dot{-} z \dot{-} y_1 \dot{-} (x \dot{-} (x \dot{-} y_2)))$ and observe that $\mathcal{H}_{C^*}$ satisfies

$$t(x, z, 0, 0) \approx x \dot{-} z \dot{-} (x \dot{-} z \dot{-} (x \dot{-} x)) \approx x \dot{-} z \dot{-} (x \dot{-} z \dot{-} 0) \approx 0$$

and so $t^{\mathbf{A}}(a, c, a \dot{-} b \dot{-} c, b) \in I$ by the definition of Ursini-ideal. But

$$\begin{aligned} t^{\mathbf{A}}(a, c, a \dot{-} b \dot{-} c, b) &= a \dot{-} c \dot{-} (a \dot{-} c \dot{-} (a \dot{-} b \dot{-} c) \dot{-} (a \dot{-} (a \dot{-} b))) \\ &= a \dot{-} c \dot{-} 0 \quad \text{(by (A1))} \\ &= a \dot{-} c, \end{aligned}$$

so I is an ideal of $\mathbf{A}$. □

The theory of 'ideal determined' varieties developed in [GU84] and subsequent papers is therefore applicable in a natural way to the *subvarieties* of $\mathcal{H}_{C^*}$.

Let $\mathbf{A} \in \mathcal{H}_{C^*}$. A subset I of A will be called a *preideal* of $\mathbf{A}$ if $0 \in I$ and I satisfies (I1) (see page 71). Obviously an ideal of $\mathbf{A}$ is a preideal of $\mathbf{A}$. In Section 5.2, we shall see that the converse is false and we shall characterize the 'relative subvarieties' of $\mathcal{H}_{C^*}$ in whose members the sets of ideals and preideals coincide. One easily sees that the set of all preideals of $\mathbf{A}$ is the universe of an algebraic lattice when ordered by set inclusion. For $X \subseteq A$, we may therefore define the *preideal of* $\mathbf{A}$ *generated by* X, denoted $pre_{\mathbf{A}}(X)$, as the intersection of all preideals of $\mathbf{A}$ containing X.

Lemma 4.16. *Let* $\mathbf{A} \in \mathcal{H}_{C^*}$.

(i) *A preideal* I *of* $\mathbf{A}$ *is a hereditary subset of* $\langle A; \leq \rangle$.

(ii) *If* I *is a preideal of* $\mathbf{A}$, $a \in A$ *and* $b_1, \ldots, b_n \in I$ *such that* $a \dot{-} b_1 \dot{-} \ldots \dot{-} b_n = 0$, *then* $a \in I$.

(iii) *When* $\oplus \in C^*$, *the preideals of* $\mathbf{A}$ *are precisely the subuniverses of the monoid* $\langle A; \oplus, 0 \rangle$ *that are hereditary subsets of* $\langle A; \leq \rangle$.

(iv) *Let* $X \subseteq A$. *Then* $pre_{\mathbf{A}}(X) =$ $\{a \in A : (\exists n \in \omega)(\exists c_1, \ldots, c_n \in X)$ *such that* $a \dot{-} c_1 \dot{-} \ldots \dot{-} c_n = 0\}$.

Proof. (i) If $b \in I$ and $a \leq b$, then $b, a \dot{-} b = 0 \in I$, so $a \in I$ by (I1).

(ii) Since $0 \in I$, the result follows by repeated application of (I1).

(iii) By (A5), $\mathcal{H}_{C^*}$ satisfies $(x \oplus y) \dot{-} y \dot{-} x \approx 0$, hence it follows by (ii) that a preideal of $\mathbf{A}$ is closed under $\oplus$. Conversely, suppose I is a subuniverse of $\langle A; \oplus, 0 \rangle$ that is a hereditary subset of $\langle A; \leq \rangle$. Evidently, $0 \in I$. If $a, b \in A$ with $b, a \dot{-} b \in I$, then, by (3),

$$a \leq (a \dot{-} b) \oplus b \in I,$$

hence $a \in I$. Thus I satisfies (I1).

(iv) Set $Y = \{a \in A : (\exists n \in \omega)(\exists c_1, \ldots, c_n \in X)$ such that $a \dot{-} c_1 \dot{-} \ldots \dot{-} c_n = 0\}$. By (ii), $Y \subseteq pre_{\mathbf{A}}(X)$. Also, $X \subseteq Y$, so we need only establish that Y is a preideal of $\mathbf{A}$. Let $a \in A$ and $b, a \dot{-} b \in Y$, say

$$a \dot{-} b \dot{-} c_1 \dot{-} \ldots \dot{-} c_n = 0 = b \dot{-} d_1 \dot{-} \ldots \dot{-} d_m,$$

where the c_i, d_j are in X. Then

$$\begin{aligned} & a \dot{-} d_1 \dot{-} \ldots \dot{-} d_m \dot{-} c_1 \dot{-} \ldots \dot{-} c_n \\ = \; & a \dot{-} d_1 \dot{-} \ldots \dot{-} d_m \dot{-} 0 \dot{-} c_1 \dot{-} \ldots \dot{-} c_n \\ = \; & a \dot{-} d_1 \dot{-} \ldots \dot{-} d_m \dot{-} (b \dot{-} d_1 \dot{-} \ldots \dot{-} d_m) \dot{-} c_1 \dot{-} \ldots \dot{-} c_n \\ \leq \; & a \dot{-} b \dot{-} c_1 \dot{-} \ldots \dot{-} c_n \quad \text{(by (A1) and (A10))} \; = \; 0, \end{aligned}$$

so $a \in Y$, as required. $\square$

The next lemma provides us with a number of characterizations of ideals. As noted earlier, an ideal is a preideal, so parts (i) and (ii) of the above lemma hold for ideals as well. The characterization that will prove most important is (iii) below.

Lemma 4.17. *Let $\mathbf{A} \in \mathcal{H}_{C^*}$ and let $I \subseteq A$. The following conditions are equivalent:*

(i) *I is an ideal of $\mathbf{A}$;*

(ii) *I is a preideal of $\mathbf{A}$ and, for all $a, b \in A$, if $b \in I$ then $a \dot{-} (a \dot{-} b) \in I$;*

(iii) *$0 \in I$ and, for all $a, b, c \in A$, if $a, b \in I$ then $c \dot{-} (c \dot{-} a \dot{-} b) \in I$;*

(iv) *$0 \in I$ and, for all $a, b, c \in A$, if $b, a \dot{-} b \in I$ then $c \dot{-} (c \dot{-} a) \in I$.*

If $\oplus \in C^*$ then these conditions are also equivalent to each of:

(v) *I is a subuniverse of the monoid $\langle A; \oplus, 0\rangle$ and, for all $a, b \in A$, if $b \in I$ then $a \dot{-} (a \dot{-} b) \in I$;*

(vi) *I is a subuniverse of $\langle A; \oplus, 0\rangle$ and a hereditary subset of $\langle A; \leq\rangle$ and, for all $a, b \in A$, if $b \in I$ then $(a \oplus b) \dot{-} a \in I$.*

Moreover, for any ideal I of $\mathbf{A}$ and $n \geq 1$, if $a_1, \ldots, a_n \in I$ and $c \in A$ then $c \dot{-} (c \dot{-} a_1 \dot{-} \ldots \dot{-} a_n) \in I$.

Proof. Let $a, b, c \in A$. (i) $\Rightarrow$ (ii) If $b \in I$, then $a \dot{-} (a \dot{-} b) \in I$ by (I2) and the fact that $a \dot{-} b \dot{-} (a \dot{-} b) = 0 \in I$.

(ii) $\Rightarrow$ (iii) Suppose that $a, b \in I$. Then $c \dot{-} (c \dot{-} a) \in I$ and $c \dot{-} a \dot{-} (c \dot{-} a \dot{-} b) \in I$. By (A1),

$$c \dot{-} (c \dot{-} a \dot{-} b) \dot{-} (c \dot{-} a \dot{-} (c \dot{-} a \dot{-} b)) \dot{-} (c \dot{-} (c \dot{-} a)) = 0.$$

Since $0 \in I$, it follows that $c \dot{-} (c \dot{-} a \dot{-} b) \in I$, as required.

(iii) $\Rightarrow$ (iv) Suppose that $b, a \dot{-} b \in I$. Then $a = a \dot{-} 0 = a \dot{-} (a \dot{-} b \dot{-} (a \dot{-} b)) \in I$. Thus, since $0 \in I$, we have $c \dot{-} (c \dot{-} a) = c \dot{-} (c \dot{-} a \dot{-} 0) \in I$, as required.

(iv) $\Rightarrow$ (i) Suppose that $b, a \dot{-} b \dot{-} c \in I$. Then $0, b \dot{-} 0 \in I$ so $a \dot{-} (a \dot{-} b) \in I$. Also, I is a preideal of $\mathbf{A}$, since if $e, d \dot{-} e \in I$ then $d = d \dot{-} (d \dot{-} d) \in I$. By (A1),

$$a \dot{-} c \dot{-} (a \dot{-} b \dot{-} c) \dot{-} (a \dot{-} (a \dot{-} b)) = 0 \in I,$$

from which we may conclude by (I1) that $a \dot{-} c \in I$, so (I2) holds.

Suppose now that $\oplus \in C^*$.

[(i)–(iv)] $\Rightarrow$ (v) Since I is a preideal of $\mathbf{A}$, I is a subuniverse of $\langle A; \oplus, 0\rangle$ (by Lemma 4.16(iii)), hence (v) holds.

(v) $\Rightarrow$ (vi) Suppose $a \leq b \in I$. Then $a = a \dot{-} 0 = a \dot{-} (a \dot{-} b)$, so $a \in I$ by (v). This shows that I is hereditary in $\langle A; \leq\rangle$. By Lemma 4.16(iii), I is a preideal of $\mathbf{A}$, so $0 \in I$. From the fact that (ii) implies (i), we infer that I is an ideal of $\mathbf{A}$. If $b \in I$ then from $(a \oplus b) \dot{-} b \dot{-} a = 0 \in I$ we conclude, by (I2), that $(a \oplus b) \dot{-} a \in I$.

(vi) $\Rightarrow$ (ii) By Lemma 4.16(iii), I is a preideal of $\mathbf{A}$. If $b \in I$ then, by (3) and (A10), $a \dot{-} (a \dot{-} b) \leq ((a \dot{-} b) \oplus b) \dot{-} (a \dot{-} b) \in I$, hence $a \dot{-} (a \dot{-} b) \in I$.

The remaining statement of the lemma is true for $n \in \{1, 2\}$, by (ii) and (iii). Assume, inductively, that it holds for some $n \geq 2$ and that $a_1, \ldots, a_{n+1} \in I$. By assumption, $c \dot{-} (c \dot{-} a_1 \dot{-} \ldots \dot{-} a_n) \in I$; by (ii),

$$c \dot{-} a_1 \dot{-} \ldots \dot{-} a_n \dot{-} (c \dot{-} a_1 \dot{-} \ldots \dot{-} a_n \dot{-} a_{n+1}) \in I,$$

and, by (A1),

$$\begin{aligned} c \dot{-} (c \dot{-} a_1 \dot{-} \ldots \dot{-} a_{n+1}) \dot{-} (c \dot{-} a_1 \dot{-} \ldots \dot{-} a_n \dot{-} (c \dot{-} a_1 \dot{-} \ldots \dot{-} a_n \dot{-} a_{n+1})) \\ \dot{-} (c \dot{-} (c \dot{-} a_1 \dot{-} \ldots \dot{-} a_n)) = 0, \end{aligned}$$

hence $c \dot{-} (c \dot{-} a_1 \dot{-} \ldots \dot{-} a_{n+1}) \in I$ by (I1), which completes the inductive proof. $\square$

The following corollary is an immediate consequence of part (iii) of the above lemma.

Corollary 4.18. *Let* $\mathbf{A} \in \mathcal{H}_{C^*}$ *and let* X *be a nonempty subset of* A. *Then the ideal of* $\mathbf{A}$ *generated by* X *is* $\langle X\rangle_{\mathbf{A}} = \{a \in A :$ *there exist* $n \in \omega$ *and* $a_0, \ldots, a_n \in A$ *such that* $a_n = a$ *and for each* $i \in \{0, \ldots, n\}$, $a_i \in X \cup \{0\}$ *or* $a_i = c \dot{-} (c \dot{-} a_j \dot{-} a_k)$ *for some* $c \in A$ *and some nonnegative integers* $j, k < i\}$.

Let $\{x_0, x_1, x_2, \ldots\}$ be a fixed, countably infinite set of variables. We define the sets of terms T_i inductively as follows: set $T_0 = \{0, x_0\}$ and for each $n \in \omega$, set

$$T_{n+1} = \{x_i \dot{-} (x_i \dot{-} u \dot{-} v) : i \leq n+1 \text{ and } u, v \in T_n\}.$$

Now set $T = \bigcup\{T_n : n \in \omega\}$.

Lemma 4.19. *For* $\mathbf{A} \in \mathcal{H}_{C^\bullet}$ *and* $a \in A$*, the ideal of* $\mathbf{A}$ *generated by* a *is*

$$\langle a\rangle_{\mathbf{A}} = \{t^{\mathbf{A}}(a, a_1, \ldots, a_n) : t(x_0, x_1, \ldots, x_n) \in T \textit{ and } a_1, \ldots a_n \in A\}.$$

Proof. Set $B = \{t^{\mathbf{A}}(a, a_1, \ldots, a_n) : t(x_0, x_1, \ldots, x_n) \in T \text{ and } a_1, \ldots a_n \in A\}$. By induction on n and condition (iii) of Lemma 4.17 (or Corollary 4.18), it follows that $t^{\mathbf{A}}(a, a_1, \ldots, a_n) \in \langle a\rangle_{\mathbf{A}}$ for all $n \in \omega$, all $t(x_0, x_1, \ldots, x_n) \in T_n$ and all $a_1, \ldots, a_n \in A$, hence $B \subseteq \langle a\rangle_{\mathbf{A}}$. For the converse inclusion, it is evident that B satisfies condition (iii) of Lemma 4.17, hence B is an ideal of $\mathbf{A}$. Since $a \in B$, we have that $\langle a\rangle_{\mathbf{A}} \subseteq B$. □

By considering the ideal of an algebra $\mathbf{A} \in \mathcal{H}_{C^\bullet}$ generated by $0^{\mathbf{A}}$ in the above lemma, one sees that

$$\mathcal{H}_{C^\bullet} \models t(0, x_1, \ldots, x_n) \approx 0 \tag{34}$$

for each $t(x_0, x_1, \ldots, x_n) \in T$.

Recall that a quasivariety $\mathcal{K}$ is called relatively congruence distributive if the lattice $\mathbf{Con}_{\mathcal{K}}\mathbf{A}$ is distributive for every $\mathbf{A} \in \mathcal{K}$. Also recall that $\mathcal{K}_{[\mathrm{R}]\mathrm{FSI}}$ denoted the class of all algebras in a quasivariety $\mathcal{K}$ that are [relatively] finitely subdirectly irreducible.

Lemma 4.20. *If* $\mathbf{A} \in \mathcal{H}_{C^\bullet}$ *and* τ *is a reflexive compatible binary relation on the universe of* $\mathbf{A}$ *then the relative congruence* θ *of* $\mathbf{A}$ *generated by* τ *is* $\{(a, b) \in A^2 : a \dot{-} b, b \dot{-} a \in 0^{\mathbf{A}}/\tau\}$. *Consequently,* $0^{\mathbf{A}}/\theta = 0^{\mathbf{A}}/\tau$.

Proof. Let $I = 0^{\mathbf{A}}/\tau$ and $\eta = \{(a, b) \in A^2 : a \dot{-} b, b \dot{-} a \in 0^{\mathbf{A}}/\tau\}$. By Proposition 4.15, I is an ideal of $\mathbf{A}$ (since it is an Ursini-ideal). Thus, by Proposition 4.14, $\eta \in \mathrm{Con}_{\mathcal{H}_{C^\bullet}} \mathbf{A}$ and $0^{\mathbf{A}}/\eta = I$. That $\tau \subseteq \eta$ follows from the identity (A9). That any relative congruence of $\mathbf{A}$ containing τ must contain η is an immediate consequence of the quasi-identity (A4). □

Proposition 4.21. $(\mathcal{H}_{C^\bullet})_{\mathrm{RSI}} = (\mathcal{H}_{C^\bullet})_{\mathrm{SI}}$ *and* $(\mathcal{H}_{C^\bullet})_{\mathrm{RFSI}} = (\mathcal{H}_{C^\bullet})_{\mathrm{FSI}}$. *Thus,* $\mathbf{A} \in \mathcal{H}_{C^\bullet}$ *is finitely subdirectly irreducible [resp. subdirectly irreducible] if and only if* $\{0^{\mathbf{A}}\}$ *is meet irreducible [resp. completely meet irreducible] in* $\mathbf{Id}\,\mathbf{A}$.

Proof. Let $\mathbf{A} \in (\mathcal{H}_{C^\bullet})_{\mathrm{RSI}}$ and, for $\mathrm{id}_A \neq \theta \in \mathrm{Con}\,\mathbf{A}$, let $I(\theta) = 0^{\mathbf{A}}/\theta$ and let $\eta(\theta)$ be the relative congruence of $\mathbf{A}$ generated by θ. We have $\bigcap\{\eta(\theta) : \mathrm{id}_A \neq \theta \in \mathrm{Con}\,\mathbf{A}\} \neq \mathrm{id}_A$, by relative subdirect irreducibility. By Lemma 4.20, each $I(\theta)$ is $0^{\mathbf{A}}/\eta(\theta)$, so $\bigcap\{I(\theta) : \mathrm{id}_A \neq \theta \in \mathrm{Con}\,\mathbf{A}\} \neq \{0^{\mathbf{A}}\}$, by relative 0-regularity. But $\bigcap\{I(\theta) : \mathrm{id}_A \neq \theta \in \mathrm{Con}\,\mathbf{A}\} = 0^{\mathbf{A}}/(\bigcap\{\theta : \mathrm{id}_A \neq \theta \in \mathrm{Con}\,\mathbf{A}\})$, so $\bigcap\{\theta : \mathrm{id}_A \neq \theta \in \mathrm{Con}\,\mathbf{A}\} \neq \mathrm{id}_A$. This shows that $\mathbf{A} \in (\mathcal{H}_{C^*})_{\mathrm{SI}}$. The reverse inclusion is trivial.

Let $\mathbf{A} \in (\mathcal{H}_{C^*})_{\mathrm{RFSI}}$ and $\mathrm{id}_A \neq \theta_1, \theta_2 \in \mathrm{Con}\,\mathbf{A}$. Let $I_j = 0^{\mathbf{A}}/\theta_j$ and let η_j be the relative congruence of $\mathbf{A}$ generated by θ_j for $j = 1, 2$. Just as in the above argument, we infer from $\eta_1 \cap \eta_2 \neq \mathrm{id}_A$ that $I_1 \cap I_2 \neq \{0^{\mathbf{A}}\}$ and then that $\theta_1 \cap \theta_2 \neq \mathrm{id}_A$, so $\mathbf{A} \in (\mathcal{H}_{C^*})_{\mathrm{FSI}}$. The reverse inclusion is trivial.

The remaining statement of the lemma is implicit in the above proofs but also follows from the fact that, for each $\mathbf{A} \in \mathcal{H}_{C^*}$, the lattices $\mathbf{Con}_{\mathcal{K}}\mathbf{A}$ and $\mathbf{Id}\,\mathbf{A}$ are isomorphic (see Proposition 4.14). $\square$

By the same argument, an algebra $\mathbf{A} \in \mathcal{H}_{C^*}$ is *relatively simple* (meaning $|\,\mathrm{Con}_{\mathcal{H}_{C^*}}\mathbf{A}\,| = 2$) if and only if $\mathbf{A}$ is *simple* (i.e., $|\,\mathrm{Con}\,\mathbf{A}\,| = 2$), if and only if $|\,\mathrm{Id}\,\mathbf{A}\,| = 2$.

Proposition 4.22. *Each of the quasivarieties $\mathcal{H}_{C^*}$ is relatively congruence distributive. Thus, every subvariety of $\mathcal{H}_{C^*}$ is congruence distributive. In particular, when C^* contains $\oplus$ and at least one of $\sqcap$ and $\sqcup$, then the variety $\mathcal{H}_{C^*}$ is congruence distributive.*

Proof. We use a criterion of Nurakunov [Nur90a][11]: a quasivariety $\mathcal{K}$ is relatively congruence distributive if $\mathcal{K}_{\mathrm{RFSI}} \subseteq \mathcal{K}_{\mathrm{FSI}}$ and there exist nonempty finite sets Σ_1 and Σ_2 of pairs $\langle u, v\rangle$ of ternary terms $u(x,y,z), v(x,y,z)$ such that $\mathcal{K}$ satisfies the identities

$$\begin{aligned} u(x,y,x) &\approx v(x,y,x), \quad \langle u,v\rangle \in \Sigma_1 \cup \Sigma_2,\\ u(x,x,y) &\approx v(x,x,y), \quad \langle u,v\rangle \in \Sigma_1,\\ u(x,y,y) &\approx v(x,y,y), \quad \langle u,v\rangle \in \Sigma_2, \end{aligned}$$

and the quasi-identity

$$\bigwedge\{u(x,y,z) \approx v(x,y,z) : \langle u,v\rangle \in \Sigma_1 \cup \Sigma_2\} \text{ implies } x \approx z.$$

The requirement $(\mathcal{H}_{C^*})_{\mathrm{RFSI}} \subseteq (\mathcal{H}_{C^*})_{\mathrm{FSI}}$ was proved in Proposition 4.21. Define $t(x,y,z) = 0$,

$$\begin{aligned} u_1(x,y,z) &= x \dot{-} z \dot{-} (x \dot{-} z \dot{-} (y \dot{-} z)), \quad v_1(x,y,z) = x \dot{-} z,\\ u_2(x,y,z) &= z \dot{-} x \dot{-} (z \dot{-} x \dot{-} (z \dot{-} y)), \quad v_2(x,y,z) = z \dot{-} x \end{aligned}$$

and set $\Sigma_1 = \{\langle u_1, v_1\rangle, \langle u_2, v_2\rangle\}$ and $\Sigma_2 = \{\langle u_1, t\rangle, \langle u_2, t\rangle\}$. It is a straightforward consequence of (A2), (A9) and (A4) that $\mathcal{H}_{C^*}$ satisfies Nurakunov's

[11]The proof of one implication of Nurakunov's result requires a correction that appears in [BR96, Lemma 4.2]; only the converse implication is used here, however, and it is correct. See [KM92] for an alternative characterization of relative distributivity.

identities and quasi-identity, with respect to this choice of Σ_1 and Σ_2, so the proof is complete. $\square$

For each $\mathbf{A} \in \mathcal{H}_{C^*}$, $\mathbf{Id}\,\mathbf{A}$ and $\mathbf{Con}_{\mathcal{H}_{C^*}}\mathbf{A}$ are isomorphic, hence the lattice $\mathbf{Id}\,\mathbf{A}$ is distributive; we say that $\mathbf{A}$ is *ideal distributive.* Noting that for each $\mathbf{A} \in \mathcal{H}_{C^*}$, $\mathbf{Id}\,\mathbf{A}$ and the lattice of ideals of the $\langle \dot{-}, 0\rangle$–reduct of $\mathbf{A}$ are isomorphic, we see that $\mathbf{A}$ is [finitely] subdirectly irreducible if and only if the $\langle \dot{-}, 0\rangle$–reduct of $\mathbf{A}$ is [finitely] subdirectly irreducible.

Pigozzi's finite basis theorem (Theorem 0.10, which says that any finitely generated relatively congruence distributive quasivariety of finite type is finitely axiomatizable) clearly applies to the finitely generated subquasivarieties of the $\mathcal{H}_{C^*}$'s. A subtler consequence of Proposition 4.22 is:

Corollary 4.23. *If $\oplus \in C^*$, then the variety generated by a finite algebra in $\mathcal{H}_{C^*}$ is finitely axiomatized.*

Proof. By Propositions 4.7 and 4.22, such a variety is congruence distributive and the result follows from Baker's Finite Basis Theorem (see the note following Theorem 0.10). $\square$

A variety that is both congruence distributive and congruence permutable is called *arithmetical*; such varieties are characterized by a variant of the Chinese Remainder Theorem (see [Grä79, Chapter 5, §35, Exercise 68, p.221]). Proposition 4.22 and Proposition 4.13 therefore imply:

Corollary 4.24. *For each C^* which contains $\{\oplus, \dot{-}, \sqcap\}$, the variety $\mathcal{H}_{C^*}$ is arithmetical.*

4.4. Relative Congruence Extensibility and Local Deduction Theorems. A quasivariety $\mathcal{K}$ has the *relative congruence extension property* (RCEP) if for every $\mathbf{A} \in \mathcal{K}$, every subalgebra $\mathbf{B}$ of $\mathbf{A}$ and every relative congruence θ of $\mathbf{B}$, there is a relative congruence θ' of $\mathbf{A}$ such that

$$\theta' \cap (B \times B) = \theta.$$

(We drop the qualification 'relative' and speak of the 'CEP' if $\mathcal{K}$ is a variety.) Suppose $\mathcal{K}$ is the equivalent quasivariety semantics of an algebraizable Hilbert system S. In the second paragraph of Section 3, we remarked that for any $\mathbf{A} \in \mathcal{K}$, the lattices $\mathbf{Con}_{\mathcal{K}}\,\mathbf{A}$ and $\mathbf{Fi}^S\,\mathbf{A}$ are isomorphic with respect to the mutually inverse isomorphisms given in (31) and (32). It follows immediately that the RCEP for $\mathcal{K}$ is equivalent to the property that for every $\mathbf{A} \in \mathcal{K}$, every subalgebra $\mathbf{B}$ of $\mathbf{A}$ and every S–filter F of $\mathbf{B}$, there is an S–filter F' of $\mathbf{A}$ such that $F' \cap B = F$. This property is known as the *filter extension property* (FEP). By [BP88, Theorem 3.2], $\mathcal{K}$ has the FEP if and only if it has the *principal filter extension property* (PFEP), by which we mean that for every $\mathbf{A} \in \mathcal{K}$, every subalgebra $\mathbf{B}$ of $\mathbf{A}$ and every $a \in A$,

$$\langle a\rangle_{\mathbf{A}} \cap B = \langle a\rangle_{\mathbf{B}}.$$

A Hilbert system S is said to possess a *local deduction detachment theorem* (LDDT) if there is a family $\mathcal{E} = \{\Sigma_i(p,q) : i \in Y\}$ of finite sets $\Sigma_i(p,q) = \{\zeta^j(p,q) : j = 1,\ldots,n_i\}$ of binary formulas ζ^j of S such that for any set $\Gamma \cup \{\alpha,\beta\}$ of formulas of S,

$$\Gamma,\alpha \vdash_S \beta \quad \text{if and only if} \quad \Gamma \vdash_S \Sigma_i(\alpha,\beta) \text{ for some } i \in Y.$$

In this case, $\mathcal{E}$ is called a *local deduction detachment system* for S.

Theorem 4.25. [BP88, Corollary 5.3] *Let S be an algebraizable Hilbert system and $\mathcal{K}$ its equivalent quasivariety semantics. Then S has a LDDT if and only if $\mathcal{K}$ has the RCEP.*

A subquasivariety $\mathcal{K}'$ of a quasivariety $\mathcal{K}$ is called a *relative subvariety* of $\mathcal{K}$ if $\mathcal{K}'$ is axiomatized by a set $\Sigma_1 \cup \Sigma_2$ of first order formulas, where Σ_1 consists of identities, Σ_2 consists of quasi-identities and $\mathcal{K} \models \Sigma_2$. (Equivalently, a relative subvariety of $\mathcal{K}$ is a class of the form $\mathcal{K} \cap \mathrm{HSP}(\mathcal{K}_1)$ for some $\mathcal{K}_1 \subseteq \mathcal{K}$.) For example, suppose S is an algebraizable Hilbert system with equivalent quasivariety semantics $\mathcal{K}$ and let S' be an axiomatic extension of S. As noted after the definition of algebraizability (see page 14), S' is algebraizable as well, with equivalent quasivariety semantics $\mathcal{K}'$, say. By (vi) and (vii) in the discussion of algebraizability, it follows that $\mathcal{K}'$ is a relative subvariety of $\mathcal{K}$. Moreover, it follows from (2), (vi) and (vii) that if $\mathcal{K}''$ is any relative subvariety of $\mathcal{K}$, then there exists an axiomatic extension S'' of S whose equivalent quasivariety semantics is $\mathcal{K}''$.

Clearly, for a relative subvariety $\mathcal{K}$ of $\mathcal{H}_{C^*}$, we may read 'ideal' for 'S–filter', and refer to the [P]FEP as the *[principal] ideal extension property* ([P]IEP).

Proposition 4.26. *Let $\mathbf{B}$ be any algebra in $\mathcal{H}_{C^*}$ that has a nontrivial proper ideal I (e.g. $\mathbf{B} = \mathbf{D}\times\mathbf{D}$ for nontrivial $\mathbf{D} \in \mathcal{H}_{C^*}$). Then there is an algebra $\mathbf{A} \in \mathcal{H}_{C^*}$ such that $\mathbf{B}$ is a subalgebra of $\mathbf{A}$ and for every ideal J of $\mathbf{A}$ with $I \subseteq J$, we have $I \neq J \cap B$. Thus, $\mathcal{H}_{C^*}$ lacks the IEP (hence also the RCEP).*

Proof. Suppose first that $1 \notin C^*$. Define $A = B \cup \{c,d,e\}$, where c,d,e are three distinct non-elements of B, and define $\leq$ to be the partial order on A extending the given partial order of $\mathbf{B}$, such that $b < c < d < e$ for all $b \in B$. When $\sqcap \in C^*$, we may define an operation $\sqcap^{\mathbf{A}}$ as the meet-semilattice operation on A corresponding to $\leq$. Evidently, $\sqcap^{\mathbf{A}}$ extends the operation $\sqcap^{\mathbf{B}}$. When $\sqcup \in C^*$ we may similarly define an operation $\sqcup^{\mathbf{A}}$ on A extending $\sqcup^{\mathbf{B}}$.

When $\oplus \in C^*$, define an operation $\oplus^{\mathbf{A}}$ on A by setting $\oplus^{\mathbf{A}}|_{(B\times B)} = \oplus^{\mathbf{B}}$,

$$b \oplus^{\mathbf{A}} c = c \oplus^{\mathbf{A}} b = c, \quad d \oplus^{\mathbf{A}} b = e, \quad b \oplus^{\mathbf{A}} d = d \text{ for all } b \in B \text{ and}$$
$$x \oplus^{\mathbf{A}} y = e \text{ for all } x, y \in A \text{ such that } c \leq x, y.$$

Then $\oplus^{\mathbf{A}}$ is an associative operation with which $\leq$ is compatible. The integral pomonoid $\langle A; \oplus^{\mathbf{A}}, 0; \leq \rangle$ is left residuated and its left residuation operation $\dot{-}^{\mathbf{A}}$ extends that of $\mathbf{B}$ as follows:

$$d \dot{-}^{\mathbf{A}} c = c \dot{-}^{\mathbf{A}} b = e \dot{-}^{\mathbf{A}} d = e \dot{-}^{\mathbf{A}} c = c \text{ for all } b \in B,$$
$$e \dot{-}^{\mathbf{A}} b = d \dot{-}^{\mathbf{A}} b = d \text{ for all } b \in B \setminus \{0\},$$
$$x \dot{-}^{\mathbf{A}} 0 = x \text{ and } x \dot{-}^{\mathbf{A}} y = 0 \text{ for all } x, y \in A \text{ with } x \leq y.$$

When $\oplus \notin C^*$, we may define the operation $\dot{-}^{\mathbf{A}}$ as above.

In all cases, one easily checks that $\mathbf{A}$ satisfies each of the axiomatizing identities and quasi-identity of $\mathcal{H}_{C^*}$, (see Corollary 2.11) hence $\mathbf{A} \in \mathcal{H}_{C^*}$. Clearly, $\mathbf{B}$ is a subalgebra of $\mathbf{A}$. Choose $b' \in I \setminus \{0\}$ and $a \in B \setminus I$. If J is an ideal of $\mathbf{A}$ with $I \subseteq J$ then $c = e \dot{-} d = e \dot{-} (e \dot{-} b') \in J$. But $a \leq c$, so this forces $a \in J$. Consequently, $I \neq J \cap B$. This clearly disproves the IEP for $\mathcal{H}_{C^*}$ when $1 \notin C^*$.

Suppose next that C^* contains 1 and let $\mathbf{D}$ denote the $(C^* \setminus \{1\})$-reduct of some $\mathbf{E} \in \mathcal{H}_{C^*}$. There is, up to isomorphism, a unique extension of $\mathbf{D}$ by a new top element f satisfying $f \dot{-} x = f$ whenever $x \in D$. Enriching this extension by the designated constant f, we have an algebra $\mathbf{D}_1 \in \mathcal{H}_{C^*}$ whose proper ideals coincide with the ideals of $\mathbf{D}$ (or of $\mathbf{E}$). Applying the above construction to $\mathbf{B}$, $\mathbf{A}$ and I of the previous paragraph, which witnessed failure of the IEP for $\mathcal{H}_{C^* \setminus \{1\}}$, we infer that $\mathbf{B}_1, I$ and $\mathbf{A}_1$ witness failure of the IEP in $\mathcal{H}_{C^*}$. □

In view of Theorem 4.25, we may conclude the following:

Corollary 4.27. *No superimplicational fragment of H possesses a local deduction detachment theorem.*

In contrast with this corollary, every superimplicational fragment of H_{BCK} (see page 31) does have a LDDT. (This follows easily from [BP88, Example 2.1] and will be generalized in Corollary 5.9.) By Theorem 4.25, an axiomatic extension of C–H that has a LDDT has as its equivalent quasivariety semantics a relative subvariety of $\mathcal{H}_{C^*}$ with the IEP (equivalently, the RCEP). The following theorem characterizes such relative subvarieties syntactically. We shall make reference to the set T defined on page 77.

Theorem 4.28. *Let $\mathcal{K}$ be a relative subvariety of $\mathcal{H}_{C^*}$. The following are equivalent.*

(i) $\mathcal{K}$ *has the IEP.*

(ii) *For every* $t(\vec{x}) = t(x_0, \ldots, x_n) \in T$ *there exists a term* $u^t(x,y) = v^t(x, s_1^t(x,y), \ldots, s_{m(t)}^t(x,y))$, *where* $v^t(x_0, \ldots, x_{m(t)}) \in T$ *and* $s_1^t(x,y), \ldots, s_{m(t)}^t(x,y)$ *are* C^**-terms, such that* $\mathcal{K} \models t(\vec{x}) \approx u^t(x_0, t(\vec{x}))$.

(iii) *For every* $t(\vec{x}) = t(x_0, \ldots, x_n) \in T$ *there exist a positive integer* k^t *and, for* $i = 1, \ldots, k^t$, *terms* $v_i^t(x_0, \ldots, x_{m_i(t)}) \in T$ *and binary* C^**-terms* $s_{i,1}^t, \ldots, s_{i,m_i(t)}^t$ *such that if* $u_i^t(x,y) = v_i^t(x, s_{i,1}^t(x,y), \ldots, s_{i,m_i(t)}^t(x,y))$ *for* $i = 1, \ldots, k^t$, *then* $\mathcal{K} \models t(\vec{x}) \dot{-} u_1^t(x_0, t(\vec{x})) \dot{-} \ldots \dot{-} u_{k^t}^t(x_0, t(\vec{x})) \approx 0$.

Proof. (i) $\Rightarrow$ (ii) Let $t(\vec{x}) = t(x_0, \ldots, x_n) \in T$. Let $\mathbf{F}$ denote the $\mathcal{K}$–free algebra on the $n+1$ free generators $\overline{x}_0, \ldots, \overline{x}_n$. We use $\overline{t}$ to denote $t(\overline{x}_0, \ldots, \overline{x}_n)$. Let $\mathbf{A}$ be the subalgebra of $\mathbf{F}$ generated by $\{\overline{x}_0, \overline{t}\}$. By Lemma 4.19, $\overline{t} \in \langle \overline{x}_0 \rangle_{\mathbf{F}}$. Since $\mathcal{K}$ has the IEP,

$$\langle \overline{x}_0 \rangle_{\mathbf{F}} \cap A = \langle \overline{x}_0 \rangle_{\mathbf{A}},$$

hence $\overline{t} \in \langle \overline{x}_0 \rangle_{\mathbf{A}}$. Thus, by Lemma 4.19, there exists a term $v^t(x_0, \ldots, x_{m(t)}) \in T$ and $s_1^t(\overline{x}_0, \overline{t}), \ldots, s_{m(t)}^t(\overline{x}_0, \overline{t}) \in A$ such that

$$\overline{t} = v^t(\overline{x}_0, s_1^t(\overline{x}_0, \overline{t}), \ldots, s_{m(t)}^t(\overline{x}_0, \overline{t})).$$

Set $u^t(x,y) = v^t(x, s_1^t(x,y), \ldots, s_{m(t)}^t(x,y))$. By properties of free algebras it follows that $\mathcal{K} \models t(\vec{x}) \approx u^t(x_0, t(\vec{x}))$, as required.

(ii) $\Rightarrow$ (iii) It follows trivially from the assumptions in (ii) that

$$\mathcal{K} \models t(\vec{x}) \dot{-} u^t(x_0, t(\vec{x})) \approx 0.$$

(iii) $\Rightarrow$ (i) To show that $\mathcal{K}$ has the IEP it suffices to show that for $\mathbf{A} \in \mathcal{K}$, $\mathbf{B}$ a subalgebra of $\mathbf{A}$ and $a \in B$, $\langle a \rangle_{\mathbf{A}} \cap B = \langle a \rangle_{\mathbf{B}}$ (see page 80). Trivially, $\langle a \rangle_{\mathbf{B}} \subseteq \langle a \rangle_{\mathbf{A}} \cap B$. Let $b \in \langle a \rangle_{\mathbf{A}} \cap B$. Since $b \in \langle a \rangle_{\mathbf{A}}$ there exists a term $t(x_0, \ldots, x_n) \in T$ and $a_1, \ldots, a_n \in A$ such that $b = t^{\mathbf{A}}(a, a_1, \ldots, a_n)$ (by Lemma 4.19). By assumption there exist $k^t \in \omega$ and $u_i^t(x,y) = v_i^t(x, s_{i,1}^t(x,y), \ldots, s_{i,m_i(t)}^t(x,y))$ for $i = 1, \ldots, k^t$, such that

$$b \dot{-} u_1^{t\mathbf{A}}(a,b) \dot{-} \ldots \dot{-} u_{k^t}^{t\mathbf{A}}(a,b) = 0.$$

Now $a, b \in B$, so $s_{i,j}^{t\mathbf{A}}(a,b) \in B$ for each i,j, hence $u_i^{t\mathbf{A}}(a,b) \in \langle a \rangle_{\mathbf{B}}$ for $i = 1, \ldots, k^t$ and so $b \in \langle a \rangle_{\mathbf{B}}$. Thus $\langle a \rangle_{\mathbf{A}} \cap B = \langle a \rangle_{\mathbf{B}}$, so $\mathcal{K}$ has the IEP. $\square$

Corollary 4.29. *Let $\mathcal{K}$ be a relative subvariety of $\mathcal{H}_{C^*}$ that has the IEP and for each $t(\vec{x}) = t(x_0, \ldots, x_n) \in T$, let $u^t(x, y)$ be as in Theorem 4.28(ii). Then, for all $\mathbf{A} \in \mathcal{K}$ and $a, b \in A$,*

$$\begin{aligned} b \in \langle a \rangle_{\mathbf{A}} \quad & \textit{if and only if} \quad b = u^{t\mathbf{A}}(a, b) \textit{ for some } t \in T \\ & \textit{if and only if} \quad b \dot{-} u^{t\mathbf{A}}(a, b) = 0 \textit{ for some } t \in T. \end{aligned}$$

Proof. If $b \in \langle a \rangle_{\mathbf{A}}$ then, by Lemma 4.19, there exist $t(x_0, \ldots, x_n) \in T$ and $a_1, \ldots, a_n \in A$ with $b = t^{\mathbf{A}}(a, a_1, \ldots, a_n)$. Thus, by Theorem 4.28(ii),

$$b = t^{\mathbf{A}}(a, a_1, \ldots, a_n) = u^{t\mathbf{A}}(a, t^{\mathbf{A}}(a, a_1, \ldots, a_n)) = u^{t\mathbf{A}}(a, b),$$

whence $b \dot{-} u^{t\mathbf{A}}(a, b) = 0$. Conversely, if $b \dot{-} u^{t\mathbf{A}}(a, b) = 0$ for some $t \in T$ then $b \in \langle a \rangle_{\mathbf{A}}$ because, by Lemma 4.19, $u^{t\mathbf{A}}(a, b) \in \langle a \rangle_{\mathbf{A}}$. □

Let S be a Hilbert system over language $\mathcal{L}$. An *S–matrix* is an ordered pair $\langle \mathbf{A}, F \rangle$, where $\mathbf{A}$ is an algebra of type $\mathcal{L}$ and F is an S–filter of $\mathbf{A}$. The class of all S–matrices is denoted $\operatorname{Mat} S$. Recall the definition of $\Omega_{\mathbf{A}}$ from page 49. An S–matrix $\langle \mathbf{A}, F \rangle$ is *reduced* if $\Omega_{\mathbf{A}} F = \mathrm{id}_A$. The class of all reduced S–matrices is denoted $\operatorname{Mat}^* S$. If S is an algebraizable Hilbert system with equivalent quasivariety semantics $\mathcal{K}$, then the class of all algebra reducts of reduced S–matrices is precisely $\mathcal{K}$ [BP89, Corollary 5.3], i.e.,

$$\mathcal{K} = \{\mathbf{A} : \langle \mathbf{A}, F \rangle \in \operatorname{Mat}^* S \text{ for some } S\text{–filter } F \text{ of } \mathbf{A}\}. \tag{35}$$

Let S be an axiomatic extension of C–H with equivalent quasivariety semantics $\mathcal{K}$, a relative subvariety of $\mathcal{H}_{C^*}$. If $\mathbf{A} \in \mathcal{K}$ and I is any ideal of $\mathbf{A}$, then $\langle \mathbf{A}, I \rangle$ is an S–matrix. By (35), a reduced S-matrix $\langle \mathbf{A}, I \rangle$ consists of an algebra $\mathbf{A} \in \mathcal{K}$ and an ideal I of $\mathbf{A}$. By Proposition 4.14, $\Omega_{\mathbf{A}}$ is an isomorphism from $\mathbf{Id}\,\mathbf{A}$ (i.e. $\mathbf{Fi}^S\mathbf{A}$) to $\mathbf{Con}_{\mathcal{K}}\mathbf{A}$, hence an S–matrix $\langle \mathbf{A}, I \rangle$ is reduced if and only if $\mathbf{A} \in \mathcal{K}$ and $I = \{0\}$.

Let S be a Hilbert system over language $\mathcal{L}$ and let M be a class of S–matrices. Let J be any index set and for each $j \in J$, let $\Sigma_j = \Sigma_j(x, y)$ be a finite set of binary $\mathcal{L}$–formulas. Following [BP88], we say that M has *locally formula definable principal S–filters* (LFDPF) with *defining system*

$$\mathcal{E} = \{\Sigma_j : j \in J\}$$

if for all $\langle \mathbf{A}, F \rangle \in M$ and $a, b \in A$,

$$b \in \langle F \cup \{a\} \rangle_{\mathbf{A}} \quad \text{iff} \quad \{\zeta^{\mathbf{A}}(a, b) : \zeta \in \Sigma_j\} \subseteq F \text{ for some } j \in J.$$

Let S be an axiomatic extension of C–H with equivalent algebraic semantics $\mathcal{K}$ and suppose that $\operatorname{Mat}^* S$ has LFDPF with defining system $\mathcal{E} = \{\Sigma_j : j \in J\}$.

Then, for all $\mathbf{A} \in \mathcal{K}$ and $a, b \in A$,

$$b \in \langle a \rangle_{\mathbf{A}} \quad \text{if and only if} \quad \{\zeta^{\mathbf{A}}(a,b) : \zeta \in \Sigma_j\} = \{0\} \text{ for some } j \in J.$$

Corollary 4.30. *Let $\mathcal{K}$ be a relative subvariety of $\mathcal{H}_{C^*}$ and let S be the axiomatic extension of C–H with equivalent quasivariety semantics $\mathcal{K}$. Then*

(i) *S has a LDDT if and only if $\mathcal{K}$ has the IEP.*

Suppose $\mathcal{K}$ has the IEP and for each $t(\vec{x}) = t(x_0, \ldots, x_n) \in T$, let $u^t(x,y)$ be as in Theorem 4.28(ii). For each $t \in T$, let $\zeta^t(x,y)$ be a C-formula such that $\zeta^t(x,y)^* = u^t(x,y)$. Set $\mathcal{E} = \{\{\zeta^t(x,y) \to y\} : t \in T\}$ and set $\mathcal{E}^* = \{\{y \dot{-} u^t(x,y)\} : t \in T\}$. Then

(ii) Mat$^*S$ *has LFDPF with defining system $\mathcal{E}^*$;*

(iii) Mat S *has LFDPF with defining system $\mathcal{E}^*$;*

(iv) *$\mathcal{E}$ is a local deduction detachment system for S.*

Proof. (i) follows from Theorem 4.25; (ii) reformulates Corollary 4.29 and (iii) and (iv) follow directly from (ii) and [BP88, Theorem 2.4]. □

An alternative local deduction detachment system for S can be given in terms of the terms u_i^t whose existence is asserted by Theorem 4.28 (iii).

We give an application of Theorem 4.28 in a very natural setting. Let α be a nonzero ordinal and let $\boldsymbol{\alpha}_{C^*}$ be the algebra defined in Example 2.17 (when it exists). We shall consider the relative subvariety $\mathrm{HSP}(\boldsymbol{\alpha}_{C^*}) \cap \mathcal{H}_{C^*}$ of $\mathcal{H}_{C^*}$ generated by $\boldsymbol{\alpha}_{C^*}$. Recall from Corollary 4.6 that when $\alpha \le \omega^\omega$, $\boldsymbol{\alpha}_{C^*}$ generates a subvariety of $\mathcal{H}_{C^*}$, i.e., $\mathrm{HSP}(\boldsymbol{\alpha}_{C^*}) \cap \mathcal{H}_{C^*} = \mathrm{HSP}(\boldsymbol{\alpha}_{C^*})$. If $\alpha \le \omega + 1$ then $\mathrm{HSP}(\boldsymbol{\alpha}_{C^*})$ is a variety of (possibly enriched) BCK–algebras and, as such, has the IEP (i.e., the CEP). From a direct consideration of algebras, it is difficult to see whether the relative subvariety of $\mathcal{H}_{C^*}$ generated by $\boldsymbol{\alpha}_{C^*}$ has the IEP when $\alpha \ge \omega + 2$, but the next proposition settles this question (when $1 \notin C^*$).

Proposition 4.31. *Suppose C^* does not contain 1 and let α be an ordinal greater than or equal to $\omega+2$ for which $\boldsymbol{\alpha}_{C^*}$ exists. Then the relative subvariety $\mathrm{HSP}(\boldsymbol{\alpha}_{C^*}) \cap \mathcal{H}_{C^*}$ of $\mathcal{H}_{C^*}$ generated by $\boldsymbol{\alpha}_{C^*}$ does not have the IEP (i.e., the RCEP). In particular, when $\alpha \le \omega^\omega$, the variety $\mathrm{HSP}(\boldsymbol{\alpha}_{C^*})$ does not have the CEP.*

Proof. We show that there exists $t(\vec{x}) = t(x_0, x_1, \ldots, x_n) \in T$ such that for all $v(x_0, x_1, \ldots, x_m) \in T$ and all binary C^*-terms $s_1(x,y), \ldots, s_m(x,y)$,

$$\boldsymbol{\alpha}_{C^*} \not\models t(\vec{x}) \approx v(x_0, s_1(x_0, t(\vec{x})), \ldots, s_m(x_0, t(\vec{x}))).$$

Set $t(x_0, x_1, x_2) = x_2 \dot{-} (x_2 \dot{-} (x_1 \dot{-} (x_1 \dot{-} x_0)))$. Then for any $a \in \omega$,

(36) $t^{\boldsymbol{\alpha}_{C^*}}(1, \omega + 1, a) = a \dot{-} (a \dot{-} ((\omega + 1) \dot{-} ((\omega + 1) \dot{-} 1))) = a \dot{-} (a \dot{-} \omega) = a.$

Claim: For every $v(x_0, x_1, \ldots, x_m) \in T$ there exists $k^v \in \omega$ such that for all $a_1, a_2, \ldots, a_m \in \omega$, $v^{\boldsymbol{\alpha}_{C^*}}(1, a_1, a_2, \ldots, a_m) \leq k^v$.

This is proved by induction on the complexity of v: If $v(x_0)$ is x_0 or 0, then $v^{\boldsymbol{\alpha}_{C^*}}(1) \leq 1$. Suppose that $v(x_0, \ldots, x_m) = x_l \dot{-} (x_l \dot{-} v_1 \dot{-} v_2)$, where $l \leq m$ and, for $i = 1, 2$, $v_i(x_0, x_1, \ldots, x_m) \in T$ and $v_i^{\boldsymbol{\alpha}_{C^*}}(1, a_1, \ldots, a_m) \leq k^{v_i} \in \omega$ for all $a_1, \ldots, a_m \in \omega$. Then

$$\begin{aligned} v^{\boldsymbol{\alpha}_{C^*}}(1, a_1, \ldots, a_m) &= a_l \dot{-} (a_l \dot{-} v_1^{\boldsymbol{\alpha}_{C^*}}(1, a_1, \ldots, a_m) \dot{-} v_2^{\boldsymbol{\alpha}_{C^*}}(1, a_1, \ldots, a_m)) \\ &\leq a_l \dot{-} (a_l \dot{-} k^{v_1} \dot{-} k^{v_2}) \\ &\leq k^{v_1} + k^{v_2} \quad (\text{since } a_l, k^{v_1}, k^{v_2} \in \omega). \end{aligned}$$

Thus the Claim holds.

Let $v(x_0, \ldots, x_m) \in T$ and let $s_1(x, y), \ldots, s_m(x, y)$ be binary C^*-terms. Set $a = k^v + 1$. By (36), $s_i^{\boldsymbol{\alpha}_{C^*}}(1, t^{\boldsymbol{\alpha}_{C^*}}(1, \omega + 1, a)) = s_i^{\boldsymbol{\alpha}_{C^*}}(1, a)$ for $i = 1, \ldots, m$. Moreover, $1, a \in \omega$ and ω is a subuniverse of $\boldsymbol{\alpha}_{C^*}$ (since $1 \notin C^*$), so $s_i^{\boldsymbol{\alpha}_{C^*}}(1, a) \in \omega$. Thus, by the Claim and (36),

$$\begin{aligned} &v^{\boldsymbol{\alpha}_{C^*}}(1, s_1^{\boldsymbol{\alpha}_{C^*}}(1, t^{\boldsymbol{\alpha}_{C^*}}(1, \omega + 1, a)), \ldots, s_m^{\boldsymbol{\alpha}_{C^*}}(1, t^{\boldsymbol{\alpha}_{C^*}}(1, \omega + 1, a))) \\ &= v^{\boldsymbol{\alpha}_{C^*}}(1, s_1^{\boldsymbol{\alpha}_{C^*}}(1, a), \ldots, s_m^{\boldsymbol{\alpha}_{C^*}}(1, a)) \\ &\leq k^v \\ &< k^v + 1 = a = t^{\boldsymbol{\alpha}_{C^*}}(1, \omega + 1, a). \end{aligned}$$

Thus

$$\boldsymbol{\alpha}_{C^*} \not\models t(\vec{x}) \approx v(x_0, s_1(x_0, t(\vec{x})), \ldots, s_m(x_0, t(\vec{x}))),$$

so Theorem 4.28 (ii) implies that $\mathcal{H}_{C^*} \cap \mathrm{HSP}(\boldsymbol{\alpha}_{C^*})$ does not have the IEP. □

CHAPTER 5

EQUATIONAL DEFINABILITY OF PRINCIPAL RELATIVE CONGRUENCES AND THEIR INTERSECTIONS

As in Chapter 4, *throughout this chapter, C^* will denote an arbitrary subset of* $\{\oplus, \dot{-}, \sqcap, \sqcup, 1\}$ *that contains* $\dot{-}$.

In Section 1 of this chapter, we discuss a property of quasivarieties that is generally stronger than the RCEP, namely the property of having 'equationally definable principal relative congruences' (EDPRC). We characterize the relative subvarieties of $\mathcal{H}_{C^*}$ that have EDPRC; this condition corresponds to the existence of a 'deduction detachment theorem' for the corresponding extension of C–H.

In Section 2, we investigate a natural sequence $\mathcal{H}^n_{C^*}$, $n \in \omega$, of relative subvarieties of $\mathcal{H}_{C^*}$ defined by the identity

$$x \dot{-} (x \dot{-} y) \dot{-} ny \approx 0.$$

In these classes, ideals coincide with preideals. These quasivarieties turn out to have the RCEP. In fact, we characterize the relative subvarieties of the $\mathcal{H}^n_{C^*}$'s (when $\oplus, 1 \notin C^*$) as precisely the relative subvarieties of $\mathcal{H}_{C^*}$ that have the RCEP and satisfy a certain very weak 'finiteness condition'. It follows that each locally finite relative subvariety of $\mathcal{H}_{C^*}$ with the RCEP lies in some $\mathcal{H}^n_{C^*}$. We characterize the finitely subdirectly irreducible algebras in $\mathcal{H}^n_{C^*}$, and we provide a (relative) equational base for the subquasivariety of $\mathcal{H}^n_{C^*}$ generated by its linearly ordered members, i.e., the quasivariety of 'representable' algebras in $\mathcal{H}^n_{C^*}$.

In Section 3 we investigate subquasivarieties of $\mathcal{H}_{C^*}$ with 'equationally definable principal relative meets' (EDPRM). In particular, we show that the quasivariety of representable members of $\mathcal{H}^n_{C^*}$ and each of the classes $\mathcal{H}^n_{C^*}$, where $\sqcap \in C^*$, have EDPRM, and we exhibit a system of 'principal intersection terms' for each of these classes.

5.1. EDPRC and Deduction Theorems. A quasivariety $\mathcal{K}$ has *equationally definable principal relative congruences* (EDPRC) if there exists a finite set $\{u_i(x_0, x_1, x_2, x_3) \approx v_i(x_0, x_1, x_2, x_3) : i = 1, \ldots, k\}$ of equations in four variables such that for every $\mathbf{A} \in \mathcal{K}$ and all $a, b, c, d \in A$,

$$(c, d) \in \Theta_{\mathcal{K}}^{\mathbf{A}}(a, b) \text{ if and only if } u_i^{\mathbf{A}}(a, b, c, d) = v_i^{\mathbf{A}}(a, b, c, d) \text{ for } i = 1, \ldots, k.$$

If in addition, $\mathcal{K}$ is a variety, we drop the qualification 'relative' and speak of 'EDPC'.

The following lemma collects some facts about quasivarieties with EDPRC.

Lemma 5.1. *For any quasivariety $\mathcal{K}$, the following are equivalent* [BP, Theorem IV.4.1]*:*

(i) *$\mathcal{K}$ has EDPRC;*

(ii) *for each $\mathbf{A} \in \mathcal{K}$, the finitely generated relative congruences of $\mathbf{A}$ form a Brouwerian semilattice.*

Moreover, if $\mathcal{K}$ has EDPRC, then

(iii) *$\mathcal{K}$ is relatively congruence distributive* [BP, Theorem IV.4.2]*;*

(iv) *the classes $\mathcal{K}_{\mathrm{SI}}$ and $\mathcal{K}_{\mathrm{S}}$ are both closed under ultraproducts* [BP, Theorem IV.2.1]*;*

(v) *$\mathcal{K}$ has the RCEP* [BP, Theorem IV.3.1].

A Hilbert system S is said to possess a *deduction detachment theorem* (DDT) if there exists a finite set $\Sigma = \Sigma(x, y) = \{\zeta_i(x, y) : i = 1, \ldots, k\}$ of formulas of S such that for any set $\Gamma \cup \{\alpha, \beta\}$ of formulas of S,

$$\Gamma, \alpha \vdash_S \beta \quad \text{if and only if} \quad \Gamma \vdash_S \Sigma(\alpha, \beta). \tag{37}$$

In this case, Σ is called a *deduction detachment set* for S.

Theorem 5.2. [BP, Theorem VI.1.3] *Let S be an algebraizable Hilbert system and $\mathcal{K}$ its equivalent quasivariety semantics. Then S has a DDT if and only if $\mathcal{K}$ has EDPRC.*

Let S be a Hilbert system and let M be a class of S–matrices (see page 83). For a finite set $\Sigma = \Sigma(x, y) = \{\zeta_i(x, y) : i = 1, \ldots, k\}$ of binary formulas, we say that M has *formula definable principal S-filters* (FDPF) with *defining set* Σ if, for all $\langle \mathbf{A}, F\rangle \in M$ and $a, b \in A$,

$$b \in \langle F \cup \{a\}\rangle_{\mathbf{A}} \quad \text{if and only if} \quad \zeta_i^{\mathbf{A}}(a,b) \in F \text{ for } i = 1, \dots, k.$$

Consider a relative subvariety $\mathcal{K}$ of $\mathcal{H}_{C^*}$. Let S be the axiomatic extension of C–H whose equivalent algebraic semantics is $\mathcal{K}$. As observed in the previous section, the reduced S-matrices are the matrices $\langle \mathbf{A}, \{0\}\rangle$, where $\mathbf{A} \in \mathcal{K}$. Thus, Mat$^*S$ has FDPF with defining set $\Sigma = \{u_i(x,y) : i = 1, \dots, k\}$ if and only if the following condition holds: for all $\mathbf{A} \in \mathcal{K}$ and $a, b \in A$,

$$b \in \langle a\rangle_{\mathbf{A}} \text{ if and only if } u_i^{\mathbf{A}}(a,b) = 0 \text{ for } i = 1, \dots, k.$$

We find it more natural here to say that $\mathcal{K}$ has *term definable principal ideals* (TDPI) with *defining set* Σ if the above condition holds. By [BP88, Theorem 4.6], we know that S has a DDT if and only if Mat$^*S$ has FDPF, hence Theorem 5.2 implies the following:

Corollary 5.3. *Let $\mathcal{K}$ be a relative subvariety of $\mathcal{H}_{C^*}$. Then $\mathcal{K}$ has EDPRC if and only if $\mathcal{K}$ has TDPI.*

A more concrete syntactic characterization of relative subvarieties of $\mathcal{H}_{C^*}$ that have TDPI (equivalently, EDPRC) follows. (T is the set of terms defined on page 77.)

Theorem 5.4. *Let $\mathcal{K}$ be a relative subvariety of $\mathcal{H}_{C^*}$. The following conditions are equivalent:*

(i) $\mathcal{K}$ *has TDPI (equivalently, EDPRC).*

(ii) *There exists a term* $u(x,y) = v(x, s_1(x,y), \dots, s_m(x,y))$, *where* $v(x_0, x_1, \dots, x_m) \in T$ *and* $s_1(x,y), \dots, s_m(x,y)$ *are* C^**-terms, such that for every* $t(\vec{x}) = t(x_0, \dots, x_n) \in T$,
$\mathcal{K} \models t(\vec{x}) \approx u(x_0, t(\vec{x}))$.

(iii) *There exist* $k \in \omega$ *and terms*
$u_i(x,y) = v_i(x, s_{i,1}(x,y), \dots, s_{i,m_i}(x,y))$, $i = 1, \dots, k$, *where each* $v_i(x_0, x_1, \dots, x_{m_i}) \in T$ *and each* $s_{i,j}(x,y)$ *is a* C^**-term, such that for every* $t(\vec{x}) = t(x_0, \dots, x_n) \in T$,
$\mathcal{K} \models t(\vec{x}) \dot{-} u_1(x_0, t(\vec{x})) \dot{-} \dots \dot{-} u_k(x_0, t(\vec{x})) \approx 0$.

Proof. (i) $\Rightarrow$ (ii) (and (iii)) Since $\mathcal{K}$ has TDPI, it also has the IEP (by Lemma 5.1(v) and Corollary 5.3), so Corollary 4.29 implies that for $\mathbf{A} \in \mathcal{K}$

and $a, b \in A$,

(38) $\qquad b \in \langle a \rangle_{\mathbf{A}}$ if and only if $b = u^t(a,b)$ for some $t \in T$,

where $u^t(x,y)$ is as in Theorem 4.28(ii).

Let $\mathcal{P}(T)$ denote the set of all subsets of T and let $\mathcal{P}_\omega(T)$ be the set of all finite subsets of T.

Claim: There exists $J \in \mathcal{P}_\omega(T)$ such that for all $\mathbf{A} \in \mathcal{K}$ and $a, b \in A$,

(39) $\qquad b \in \langle a \rangle_{\mathbf{A}}$ if and only if $b = u^t(a,b)$ for some $t \in J$.

Suppose the Claim is false. Then for each $J \in \mathcal{P}_\omega(T)$, there exist $\mathbf{A}_J \in \mathcal{K}$ and $a_J, b_J \in A_J$ such that

(40) $$b_J \in \langle a_J \rangle_{\mathbf{A}_J}$$

but

(41) $$b_J \neq u^t(a_J, b_J) \text{ for all } t \in J.$$

For each $J \in \mathcal{P}_\omega(T)$, set

$$Q_J = \{K \in \mathcal{P}_\omega(T) : J \subseteq K\} \text{ and}$$
$$\mathcal{F} = \{X \subseteq \mathcal{P}_\omega(T) : \exists J \in \mathcal{P}_\omega(T) \text{ such that } Q_J \subseteq X\}.$$

Noting that $Q_J \cap Q_K = Q_{J \cup K} \in \mathcal{F}$ for all $J, K \in \mathcal{P}_\omega(T)$, we see that $\mathcal{F}$ is a filter of the Boolean algebra on the set of all subsets of $\mathcal{P}_\omega(T)$, i.e., a filter over $\mathcal{P}_\omega(T)$. Let $\mathcal{U}$ be an ultrafilter (necessarily nonprincipal) over $\mathcal{P}_\omega(T)$ with $\mathcal{F} \subseteq \mathcal{U}$, and let $\mathbf{A} = \prod_{J \in \mathcal{P}_\omega(T)} \mathbf{A}_J$ and $\mathbf{D} = \mathbf{A}/\mathcal{U}$. Since $\mathcal{K}$ is a quasivariety, $\mathbf{D} \in \mathcal{K}$.

Define $\overline{a}, \overline{b} \in A$ by $\overline{a}(J) = a_J$, $\overline{b}(J) = b_J$ $(J \in \mathcal{P}_\omega(T))$. By TDPI, there is a finite set $\{v_i(x,y) : i = 1, \ldots, k\}$ of binary C^*–terms such that for any $\mathbf{B} \in \mathcal{K}$ and $c, d \in B$, we have

(42) $$d \in \langle c \rangle_{\mathbf{B}} \text{ if and only if } \mathbf{B} \models \bigwedge_{i=1}^{k} v_i[c,d] \approx 0.$$[12]

By (40) and (42), $\bigwedge_{i=1}^{k} v_i[a_J, b_J] \approx 0$ is true in $\mathbf{A}_J$ for each $J \in \mathcal{P}_\omega(T)$, hence $\bigwedge_{i=1}^{k} v_i[\overline{a}/\mathcal{U}, \overline{b}/\mathcal{U}] \approx 0$ is true in $\mathbf{D}$ by Los' Theorem. By (42), $\overline{b}/\mathcal{U} \in \langle \overline{a}/\mathcal{U} \rangle_{\mathbf{D}}$,

[12] This notation may be interpreted either as $v_i^{\mathbf{B}}(c,d) = 0$ for $i = 1, \ldots, k$, or, equivalently, as an assertion of the *truth* of the (first order) sentence $\bigwedge_{i=1}^{k} v_i[c', d'] \approx 0$ in the structure $\langle \mathbf{B}; \langle b : b \in B \rangle \rangle$ for the first order language with equality determined by the expansion $C^* \cup \{b' : b \in B\}$ of C^* by (distinct) constant symbols b' corresponding to the elements of B.

so by (38), there exists $t' \in T$ such that $\overline{b}/\mathcal{U} = u^{t'}(\overline{a}/\mathcal{U}, \overline{b}/\mathcal{U})$, i.e.,

$$U = \{J \in \mathcal{P}_\omega(T) : b_J = u^{t'}(a_J, b_J)\} \in \mathcal{U}. \tag{43}$$

Let $J_1 = \{t'\}$ and note that $Q_{J_1} = \{K \in \mathcal{P}_\omega(T) : t' \in K\} \in \mathcal{U}$, so $\emptyset \neq U \cap Q_{J_1} \in \mathcal{U}$. Let $K \in U \cap Q_{J_1}$. By (43), since $K \in U$,

$$b_K = u^{t'}(a_K, b_K),$$

but, since $K \in Q_{J_1}$ we have $t' \in K$ hence, by (41),

$$b_K \neq u^{t'}(a_K, b_K),$$

a contradiction. So the Claim is true.

Let the set J whose existence is asserted by the Claim be $\{t_1, \ldots, t_k\} \subseteq T$. It follows from the Claim that if $\mathbf{A} \in \mathcal{K}$ and $a, b \in A$ then

$$b \in \langle a \rangle_{\mathbf{A}} \text{ if and only if } b \dot{-} u^{t_1}(a, b) \dot{-} \ldots \dot{-} u^{t_k}(a, b) = 0. \tag{44}$$

The implication from right to left follows from Lemma 4.16(ii) and the fact that $u^{t_i}(a, b) \in \langle a \rangle_{\mathbf{A}}$ for $i = 1, \ldots, k$ (Lemma 4.19). Let $t(\vec{x}) = t(x_0, \ldots, x_n) \in T$, let $\mathbf{F}$ be the $\mathcal{K}$–free algebra freely generated by $\overline{x}_0, \ldots, \overline{x}_n$ and let $\overline{t} = t^{\mathbf{F}}(\overline{x}_0, \ldots, \overline{x}_n)$. Since $\overline{t} \in \langle \overline{x}_0 \rangle_{\mathbf{F}}$ (Lemma 4.19), (44) implies that

$$\overline{t} \dot{-} u^{t_1}(\overline{x}_0, \overline{t}) \dot{-} \ldots \dot{-} u^{t_k}(\overline{x}_0, \overline{t}) = 0^{\mathbf{F}},$$

whence we deduce

$$\mathcal{K} \models t(\vec{x}) \dot{-} u^{t_1}(x_0, t(\vec{x})) \dot{-} \ldots \dot{-} u^{t_k}(x_0, t(\vec{x})) \approx 0 \tag{45}$$

(proving (iii)).

Define

$$u(x, y) = y \dot{-} (y \dot{-} u^{t_1}(x, y) \dot{-} \ldots \dot{-} u^{t_k}(x, y)).$$

In $\mathbf{F}(\overline{x}, \overline{y})$, the $\mathcal{K}$–free algebra generated by $\overline{x}, \overline{y}$, we have $u^{\mathbf{F}(\overline{x},\overline{y})}(\overline{x}, \overline{y}) \in \langle \overline{x} \rangle_{\mathbf{F}(\overline{x},\overline{y})}$ (by the last assertion of Lemma 4.17) so, by Lemma 4.19,

$$u^{\mathbf{F}(\overline{x},\overline{y})}(\overline{x}, \overline{y}) = v^{\mathbf{F}(\overline{x},\overline{y})}(\overline{x}, s_1(\overline{x}, \overline{y}), \ldots, s_m(\overline{x}, \overline{y}))$$

for some $v(x_0, \ldots, x_m) \in T$ and C^*–terms $s_1(x, y), \ldots, s_m(x, y)$, hence

$$\mathcal{K} \models u(x, y) \approx v(x, s_1(x, y), \ldots, s_m(x, y)).$$

By (45),

$$\mathcal{K} \models v(x_0, s_1(x_0, t(\vec{x})), \ldots, s_m(x_0, t(\vec{x}))) \approx u(x_0, t(\vec{x})) \approx t(\vec{x}) \dot{-} 0 \approx t(\vec{x}),$$

which proves (ii).

That (ii) implies (iii) is trivial.

(iii) $\Rightarrow$ (i) We shall show that $\mathcal{K}$ has TDPI with respect to the set $\{y \dot{-} u_1(x,y) \dot{-} \ldots \dot{-} u_k(x,y)\}$. Let $\mathbf{A} \in \mathcal{K}$ and $a, b \in A$. If $b \dot{-} u_1(a,b) \dot{-} \ldots \dot{-} u_k(a,b) = 0$ then, since each $u_i(a,b) \in \langle a \rangle_{\mathbf{A}}$ (Lemma 4.19), it follows that $b \in \langle a \rangle_{\mathbf{A}}$. Conversely, if $b \in \langle a \rangle_{\mathbf{A}}$, then there exists a term $t(x_0, \ldots, x_n) \in T$ and $a_1, \ldots, a_n \in A$ such that $b = t(a, a_1, \ldots, a_n)$. It follows immediately from our assumptions that $b \dot{-} u_1(a,b) \dot{-} \ldots \dot{-} u_k(a,b) = 0$. □

Corollary 5.5. *Let $\mathcal{K}$ be a relative subvariety of $\mathcal{H}_{C^*}$ that has TDPI and let $u(x,y)$ be as in Theorem 5.4(ii). Then, for all $\mathbf{A} \in \mathcal{K}$ and $a, b \in A$,*

$$\begin{aligned} b \in \langle a \rangle_{\mathbf{A}} \quad &\textit{if and only if} \quad b = u^{\mathbf{A}}(a,b) \\ &\textit{if and only if} \quad b \dot{-} u^{\mathbf{A}}(a,b) = 0. \end{aligned}$$

The proof is analogous to that of Corollary 4.29.

Corollary 5.6. *Let $\mathcal{K}$ be a relative subvariety of $\mathcal{H}_{C^*}$, and let S be the axiomatic extension of C–H whose equivalent quasivariety semantics is $\mathcal{K}$. Then*

(i) $\mathcal{K}$ has TDPI if and only if S has a DDT.

Suppose $\mathcal{K}$ has TDPI and let $u(x,y)$ be as in Theorem 5.4(ii). Let $\zeta(x,y)$ be a C-formula such that $\zeta(x,y)^* = u(x,y)$, set $\Sigma = \Sigma(x,y) = \{\zeta(x,y) \to y\}$ and set $\Sigma^* = \{y \dot{-} u(x,y)\}$. Then

(ii) Mat^*S has FDPF with defining set Σ^*;

(iii) $\mathrm{Mat}\, S$ has FDPF with defining set Σ^*;

(iv) $\mathcal{K}$ has TDPI with defining set Σ^*;

(v) Σ is a deduction detachment set for S.

Proof. (i) follows from Theorem 5.2 and Corollary 5.3; (ii) is proved in Corollary 5.5 and (iii), (iv) and (v) follow from (ii) and [BP88, Theorem 4.6]. □

Another deduction detachment set can be provided in the style of Theorem 5.4(iii), just as in the case of the LDDT.

5.2. The Quasivarieties $\mathcal{H}^n_{C^*}$. For each $n \in \omega$, we shall write $x \dot{-} ny$ for $x \dot{-} y \dot{-} \ldots \dot{-} y$ when there are n y's in the latter expression. Note that if $\oplus \in C^*$, this abbreviation is an identity of $\mathcal{H}_{C^*}$, but its use shall *not* imply that $\oplus \in C^*$.

Proposition 5.7. *Let $\mathcal{K}$ be a relative subvariety of $\mathcal{H}_{C^*}$. The following conditions are equivalent:*

(i) *for every* $\mathbf{A} \in \mathcal{K}$, *every preideal of* $\mathbf{A}$ *is an ideal of* $\mathbf{A}$;

(ii) *for some* $n \in \omega$, $\mathcal{K} \models x \dot{-} (x \dot{-} y) \dot{-} ny \approx 0$.

In this case, $\mathcal{K}$ *has the IEP (and hence also the RCEP) and for* $\mathbf{A} \in \mathcal{K}$ *and* $a \in A$,

$$\langle a \rangle_{\mathbf{A}} = \mathrm{pre}_{\mathbf{A}}(\{a\}) = \{b \in A : \exists m \in \omega \text{ such that } b \dot{-} ma = 0\}.$$

Proof. (i) $\Rightarrow$ (ii) Let $\mathbf{F}$ be the $\mathcal{K}$–free algebra on two free generators $\overline{x}, \overline{y}$. The ideal of $\mathbf{F}$ generated by $\{\overline{y}\}$ coincides with the preideal of $\mathbf{F}$ generated by $\{\overline{y}\}$, and contains $\overline{x} \dot{-} (\overline{x} \dot{-} \overline{y})$, by Lemma 4.17(ii). By Lemma 4.16(iv), there exists $n \in \omega$ such that $\overline{x} \dot{-} (\overline{x} \dot{-} \overline{y}) \dot{-} n\overline{y} = 0^{\mathbf{F}}$, hence $\mathcal{K}$ satisfies $x \dot{-} (x \dot{-} y) \dot{-} ny \approx 0$.

(ii) $\Rightarrow$ (i) Let I be a preideal of $\mathbf{A}$, $a \in A$ and $b \in I$. By (ii), we have $a \dot{-} (a \dot{-} b) \dot{-} nb = 0 \in I$, so $a \dot{-} (a \dot{-} b) \in I$. By Lemma 4.17(ii), I is an ideal of $\mathbf{A}$.

Suppose these equivalent conditions hold, and that $\mathbf{A} \in \mathcal{K}$, $\mathbf{B}$ is a subalgebra of $\mathbf{A}$ and $I \in \mathrm{Id}\,\mathbf{B}$. Let $J = \mathrm{pre}_{\mathbf{A}}(I) = \langle I \rangle_{\mathbf{A}}$. Then $I \subseteq J \cap B$. Conversely, suppose $a \in J \cap B$. By Lemma 4.16(iv), there exist $n \in \omega$ and $c_1, \ldots, c_n \in I$ such that $a \dot{-} c_1 \dot{-} \ldots \dot{-} c_n = 0^{\mathbf{A}} = 0^{\mathbf{B}}$. Since $a \in B$, we infer from Lemma 4.16 that $a \in I$. Thus $J \cap B = I$ and $\mathcal{K}$ has the IEP. The last assertion of this proposition is a consequence of Lemma 4.16(iv). □

Proposition 5.7 provides us with some interesting relative subvarieties of $\mathcal{H}_{C^*}$. For each $n \in \omega$, we define $\mathcal{H}^n_{C^*}$ as the class of all algebras in $\mathcal{H}_{C^*}$ that satisfy the identity

$$(Z_n) \qquad x \dot{-} (x \dot{-} y) \dot{-} ny \approx 0.$$

The classes $\mathcal{H}^n_{C^*}$ are also of natural interest. In particular, in view of Lemma 1.6 and the fact that (Z_1) coincides with (A15), $\mathcal{H}^1_{\{\dot{-}\}}$ is the class $\mathcal{BCK}$ of all BCK–algebras. Thus, Proposition 5.7 generalizes the fact that $\mathcal{BCK}$ has the RCEP. We also have that $\mathcal{H}^1_{C^*}$ is the class of all algebras in $\mathcal{H}_{C^*}$ with BCK–algebra reducts, e.g. $\mathcal{H}^1_{\{\oplus, \dot{-}\}}$ is just the class of all pocrims. The following proposition shows that for each ordinal $\alpha \leq \omega^\omega$ and each C^*, the algebra $\boldsymbol{\alpha}^R_{C^*}$ (see Example 2.17), when it exists, is a member of $\mathcal{H}^2_{C^*}$ (and not of $\mathcal{H}^1_{C^*}$ unless $\alpha \leq \omega + 1$).

Proposition 5.8. *The left residuation algebra* $(\boldsymbol{\omega}^{\boldsymbol{\omega}})^R_{\{\dot{-}\}} = \langle \omega^\omega; \div, 0 \rangle$ *defined in Example* 2.17 *satisfies*

$$(46) \qquad x \div (x \div y) \div y \div y \approx 0$$

and does not satisfy

$$x \dot{-} (x \dot{-} y) \dot{-} y \approx 0. \tag{47}$$

Proof. We first show that $(\omega^\omega)^R_{\{\dot{-}\}}$ fails to satisfy (47):

$$\begin{aligned}(\omega+1)\dot{-}((\omega+1)\dot{-}\omega)\dot{-}\omega &= (\omega+1)\dot{-}1\dot{-}\omega \\ &= (\omega+1)\dot{-}\omega = 1 \neq 0.\end{aligned}$$

Let $\alpha, \beta \in \omega^\omega$. If $\alpha \leq \beta$ or $\alpha = 0$ or $\beta = 0$ then (46) holds if we interpret x as α and y as β. Suppose $\alpha > \beta > 0$ and that

$$\alpha = \omega^n a_n + \omega^{n-1} a_{n-1} + \cdots + \omega a_1 + a_0,$$

$$\beta = \omega^m b_m + \omega^{m-1} b_{m-1} + \cdots + \omega b_1 + b_0,$$

where $n, m, a_0, \ldots, a_n, b_0, \ldots, b_m \in \omega$, $a_n \neq 0$ and $b_m \neq 0$.

Case (i) Suppose $n > m$. Then $\alpha \dot{-} \beta = \alpha$ hence $\alpha \dot{-} (\alpha \dot{-} \beta) \dot{-} \beta \dot{-} \beta = 0$.

Case (ii) Suppose $n = m$ and $a_n > b_n$. Then

$$\alpha \dot{-} \beta = \omega^n (a_n - b_n) + \omega^{n-1} a_{n-1} + \cdots + \omega a_1 + a_0,$$

$$\alpha \dot{-} (\alpha \dot{-} \beta) = \omega^n b_n + \omega^{n-1} a_{n-1} + \cdots + \omega a_1 + a_0,$$

$$\alpha \dot{-} (\alpha \dot{-} \beta) \dot{-} \beta < \omega^n,$$

$$\alpha \dot{-} (\alpha \dot{-} \beta) \dot{-} \beta \dot{-} \beta = 0.$$

Case (iii) Suppose $n = m$ and $a_n = b_n$. Since $\alpha > \beta$, there exists $k < n$ such that $a_{n-1} = b_{n-1}, \ldots, a_{k+1} = b_{k+1}$ and $a_k > b_k$. Then

$$\alpha \dot{-} \beta = \omega^k (a_k - b_k) + \omega^{k-1} a_{k-1} + \cdots + \omega a_1 + a_0,$$

$$\alpha \dot{-} (\alpha \dot{-} \beta) = \alpha,$$

$$\alpha \dot{-} (\alpha \dot{-} \beta) \dot{-} \beta = \alpha \dot{-} \beta,$$

$$\alpha \dot{-} (\alpha \dot{-} \beta) \dot{-} \beta \dot{-} \beta = 0.$$

$\square$

It follows that the relative subvariety of $\mathcal{H}^2_{C^*}$ (or of $\mathcal{H}_{C^*}$) generated by $(\omega^\omega)^R_{C^*}$ has the RCEP. When $\oplus \in C^*$, this relative subvariety is a *variety* (with the CEP), since the $\langle \oplus, \dot{-}, 0 \rangle$–reduct of $(\omega^\omega)^R_{C^*}$ is a right complemented monoid: see Example 1.10.

We have given the most economical proof of Proposition 5.7 that we know, but we also wish to show how its condition (ii) illustrates our characterization of the IEP from Theorem 4.28. The next two claims connect the IEP for $\mathcal{H}^n_{C^*}$

to the general characterization. Since $\mathcal{H}^0_{C^*}$ is the trivial variety, we assume $n > 0$.

Claim: For all $n, m \in \omega$ with $n > 0$, $\mathcal{H}^n_{C^*}$ satisfies

$$x \dot{-} (x \dot{-} my) \dot{-} (mn)y \approx 0.$$

(We remark that if $\oplus \in C^*$ this Claim is clearly true. If $C^* = \{\dot{-}\}$ then any $\mathbf{A} \in \mathcal{H}^n_{C^*}$ is a subreduct of a polrim $\mathbf{B} \in \mathcal{H}_{\{\oplus, \dot{-}\}}$ but for $n > 1$, it is not obvious that we can choose $\mathbf{B} \in \mathcal{H}^n_{\{\oplus, \dot{-}\}}$. So a proof is necessary.)

We prove this Claim by induction on m. The cases $m = 0$ and $m = 1$ are trivial. Suppose that the Claim holds for some $m \geq 1$. Then $\mathcal{H}^n_{C^*}$ satisfies $x \dot{-} y \dot{-} (x \dot{-} y \dot{-} my) \dot{-} (mn)y \approx 0$, hence $\mathcal{H}^n_{C^*}$ satisfies

$$\begin{aligned}
& x \dot{-} (x \dot{-} (m+1)y) \dot{-} (mn)y \\
\approx\ & x \dot{-} (x \dot{-} (m+1)y) \dot{-} (mn)y \dot{-} (x \dot{-} y \dot{-} (x \dot{-} (m+1)y) \dot{-} (mn)y) \\
\leq\ & \dots \\
\leq\ & x \dot{-} (x \dot{-} (m+1)y) \dot{-} (x \dot{-} y \dot{-} (x \dot{-} (m+1)y)) \\
& \text{(by repeated application of (A1))} \\
\leq\ & x \dot{-} (x \dot{-} y) \quad \text{(by (A1))}.
\end{aligned}$$

By repeated application of (A10), and by (Z_n), $\mathcal{H}^n_{C^*}$ satisfies

$$x \dot{-} (x \dot{-} (m+1)y) \dot{-} (mn)y \dot{-} ny \leq x \dot{-} (x \dot{-} y) \dot{-} ny \approx 0,$$

which completes the inductive proof.

Claim: For every $t(\vec{x}) \in T$ there exists a $k^t \in \omega$ such that $\mathcal{H}^n_{C^*}$ satisfies $t(\vec{x}) \dot{-} k^t x_0 \approx 0$. (Here T is as defined on page 77.)

To see this, first observe that $\mathcal{H}_{\{\oplus, \dot{-}\}}$ satisfies

$$\begin{aligned}
& x \dot{-} y_1 \dot{-} \dots \dot{-} y_n \dot{-} (z \dot{-} y_1 \dot{-} \dots \dot{-} y_n) \dot{-} (x \dot{-} z) \\
\approx\ & x \dot{-} (y_n \oplus \dots \oplus y_1) \dot{-} (z \dot{-} (y_n \oplus \dots \oplus y_1)) \dot{-} (x \dot{-} z) \approx 0 \quad \text{(by (A1))},
\end{aligned}$$

hence $\mathcal{H}_{\{\dot{-}\}}$ satisfies

$$x \dot{-} y_1 \dot{-} \dots \dot{-} y_n \dot{-} (z \dot{-} y_1 \dot{-} \dots \dot{-} y_n) \dot{-} (x \dot{-} z) \approx 0.$$

Thus $\mathcal{H}_{\{\dot{-}\}}$ (and hence $\mathcal{H}_{C^*}$) satisfies

$$(48) \qquad z \dot{-} y_1 \dot{-} \dots \dot{-} y_n \approx 0 \text{ implies } x \dot{-} y_1 \dot{-} \dots \dot{-} y_n \leq x \dot{-} z.$$

If $t(x_0)$ is x_0 or 0, we may take k^t to be 1. Suppose $t(\vec{x}) = t(x_0, \dots, x_n) = x_l \dot{-} (x_l \dot{-} t_1(\vec{x}) \dot{-} t_2(\vec{x}))$, where $l \leq m$ and, for $i = 1, 2$, $t_i(\vec{x}) \in T$ and there exists $k^{t_i} \in \omega$ such that $\mathcal{H}^n_{C^*}$ satisfies $t_i(\vec{x}) \dot{-} k^{t_i}x_0 \approx 0$. By two applications of (48), we obtain

$$\mathcal{H}^n_{C^*} \models x_l \dot{-} (k^{t_1} + k^{t_2})x_0 \approx x_l \dot{-} k^{t_1}x_0 \dot{-} k^{t_2}x_0 \leq x_l \dot{-} t_1(\vec{x}) \dot{-} t_2(\vec{x}),$$

hence

$$\mathcal{H}^n_{C^*} \models t(\vec{x}) \approx x_l \dot{-} (x_l \dot{-} t_1(\vec{x}) \dot{-} t_2(\vec{x})) \leq x_l \dot{-} (x_l \dot{-} (k^{t_1} + k^{t_2})x_0).$$

Using the first Claim, we obtain that $\mathcal{H}^n_{C^*}$ satisfies

$$\begin{aligned} & t(\vec{x}) \dot{-} ((k^{t_1} + k^{t_2})n)x_0 \\ \leq \ & x_l \dot{-} (x_l \dot{-} (k^{t_1} + k^{t_2})x_0) \dot{-} ((k^{t_1} + k^{t_2})n)x_0 \approx 0. \end{aligned}$$

Thus, setting $k^t = (k^{t_1} + k^{t_2})n$ completes the inductive proof of the Claim.

For relative subvarieties of $\mathcal{H}^n_{C^*}$, therefore, condition (iii) of Theorem 4.28 holds, with $u^t_i(x, y) = x$ for $i = 1, \ldots, k^t$.

For $m \in \omega$ and formulas φ, ψ, we shall use $\varphi \xrightarrow{m} \psi$ to denote the formula $\varphi \to \varphi \to \ldots \to \varphi \to \psi$ with φ and $\to$ each occurring m times. By Proposition 5.7 and the remark following the statement of Corollary 4.30, we obtain the following:

Corollary 5.9. *Let S be an axiomatic extension of C-H such that for some $n \in \omega$, $\vdash_S p \xrightarrow{n} ((p \to q) \to q)$. Then S has a LDDT with local deduction detachment system $\{\{p \xrightarrow{m} q\} : m \in \omega\}$.*

The above corollary captures as a special case (with $n = 1$) the result that every superimplicational fragment of H_{BCK} has a LDDT (with the local deduction detachment system given in the corollary).

The relative subvarieties of $\mathcal{H}^n_{C^*}$ are not the only relative subvarieties of $\mathcal{H}_{C^*}$ that have the IEP. In Example 5.12, we shall exhibit a subvariety of $\mathcal{H}_{\{\dot{-}, \sqcap, \sqcup\}}$ which has TDPI (hence also the IEP) but does not satisfy the identity (Z_n) for any $n \in \omega$. Nevertheless, if $\oplus, 1 \notin C^*$ then, among the relative subvarieties of $\mathcal{H}_{C^*}$, those of $\mathcal{H}^n_{C^*}$ are characterized by the IEP together with a kind of weak finiteness condition. More precisely:

Theorem 5.10. *Let $\mathcal{K}$ be a relative subvariety of $\mathcal{H}_{C^*}$, where $\oplus, 1 \notin C^*$, and let $n \in \omega$. Then $\mathcal{K} \subseteq \mathcal{H}^n_{C^*}$ if and only if $\mathcal{K}$ has the IEP and*

$$\mathcal{K} \models x \dot{-} (x \dot{-} y) \dot{-} ny \approx x \dot{-} (x \dot{-} y) \dot{-} (n+1)y.$$

Proof. We have already observed that if $\mathcal{K} \subseteq \mathcal{H}^n_{C^*}$ then $\mathcal{K}$ has the IEP and, over $\mathcal{K}$, the required identity follows easily from (Z_n). For the converse, we shall consider in detail only the case $C^* = \{\dot{-}, \sqcap, \sqcup\}$. Set $u(x, y) = x \dot{-} (x \dot{-} y) \dot{-} ny$. Let $\mathbf{F}$ be the $\mathcal{K}$–free algebra generated by the free generators

$\overline{x}, \overline{y}$ and let $\mathbf{A}$ be the subalgebra of $\mathbf{F}$ generated by $\{\overline{u}, \overline{y}\}$, where $\overline{u} = u^{\mathbf{F}}(\overline{x}, \overline{y})$. Note that $\overline{u} \dot{-} \overline{y} = \overline{u}$ by our assumptions.

For each binary C^*–term $v = v(x_0, x_1)$, let $f(v)$ be the number of occurrences of connectives in v, and let $\tilde{v} = v^{\mathbf{A}}(\overline{y}, \overline{u}) \in A$. One can show easily by induction on $f(v)$ that

(49) $\tilde{v} \leq \overline{u} \sqcup \overline{y}$ for every binary C^*–term v (hence for every $\tilde{v} \in A$).

Claim: For every binary C^*–term v (hence for every $\tilde{v} \in A$),

$$\tilde{v} \leq \overline{y} \quad \text{or} \quad \tilde{v} \geq \overline{u}. \tag{50}$$

Certainly this is true if $f(v) = 0$. Now let $0 < m \in \omega$ and suppose that (50) holds whenever $f(v) < m$. Consider a binary C^*–term v with $f(v) = m$. Then $\tilde{v} = \tilde{v}_1 \sqcap \tilde{v}_2$ or $\tilde{v} = \tilde{v}_1 \sqcup \tilde{v}_2$ or $\tilde{v} = \tilde{v}_1 \dot{-} \tilde{v}_2$ for binary C^*–terms v_1, v_2 with $f(v_1), f(v_2) < m$. In the first two cases one can easily show that (50) holds. Suppose $\tilde{v} = \tilde{v}_1 \dot{-} \tilde{v}_2$. If $\tilde{v}_1 \leq \overline{y}$ then $\tilde{v} \leq \overline{y}$ as well. If $\tilde{v}_1 \geq \overline{u}$ and $\tilde{v}_2 \leq \overline{y}$ then

$$\tilde{v} = \tilde{v}_1 \dot{-} \tilde{v}_2 \geq \overline{u} \dot{-} \tilde{v}_2 \geq \overline{u} \dot{-} \overline{y} = \overline{u}.$$

If $\tilde{v}_1 \geq \overline{u}$ and $\tilde{v}_2 \geq \overline{u}$ then by (49) and (C9) (see page 43),

$$\tilde{v} = \tilde{v}_1 \dot{-} \tilde{v}_2 \leq (\overline{u} \sqcup \overline{y}) \dot{-} \tilde{v}_2 = (\overline{u} \dot{-} \tilde{v}_2) \sqcup (\overline{y} \dot{-} \tilde{v}_2) = 0^{\mathbf{F}} \sqcup (\overline{y} \dot{-} \tilde{v}_2) \leq \overline{y}.$$

Thus our Claim holds.

Next, we show, using Lemma 4.19, that $\langle \overline{y} \rangle_{\mathbf{A}} = \{\tilde{v} \in A : \tilde{v} \leq \overline{y}\}$. Trivially, $0^{\mathbf{F}}, \overline{y} \leq \overline{y}$. Suppose that $\tilde{v}_1, \tilde{v}_2 \in \langle \overline{y} \rangle_{\mathbf{A}}$ such that $\tilde{v}_1, \tilde{v}_2 \leq \overline{y}$ and let $\tilde{v} \in A$. Then $\tilde{v} \dot{-} (\tilde{v} \dot{-} \tilde{v}_1 \dot{-} \tilde{v}_2) \in \langle \overline{y} \rangle_{\mathbf{A}}$. By our Claim, we know that either $\tilde{v} \leq \overline{y}$ or $\tilde{v} \geq \overline{u}$. In the first case, we have $\tilde{v} \dot{-} (\tilde{v} \dot{-} \tilde{v}_1 \dot{-} \tilde{v}_2) \leq \overline{y}$, and in the second case, we have

$$\begin{aligned}
\tilde{v} \dot{-} (\tilde{v} \dot{-} \tilde{v}_1 \dot{-} \tilde{v}_2) &\leq \tilde{v} \dot{-} (\tilde{v} \dot{-} \overline{y} \dot{-} \overline{y}) \leq \tilde{v} \dot{-} (\overline{u} \dot{-} \overline{y} \dot{-} \overline{y}) \\
&= \tilde{v} \dot{-} \overline{u} \leq (\overline{u} \sqcup \overline{y}) \dot{-} \overline{u} \quad \text{(by (49))} \\
&\leq \overline{y} \quad \text{(by (C8), page 42), as required.}
\end{aligned}$$

Now it is evident that $\overline{u} \in \langle \overline{y} \rangle_{\mathbf{F}}$, hence the IEP implies that $\overline{u} \in \langle \overline{y} \rangle_{\mathbf{A}}$, so $\overline{u} \leq \overline{y}$. Thus $\overline{u} \dot{-} \overline{y} = 0^{\mathbf{F}}$. Since $\overline{u} \dot{-} \overline{y} = \overline{u}$ it follows that $\overline{u} = 0^{\mathbf{F}}$. By properties of free algebras, we have that $\mathcal{K} \models u(x, y) \approx 0$.

For the case $C^* = \{\dot{-}\}$, the above argument may be modified, the essential fact being that $\mathcal{LR}$ satisfies either $v(x_0, x_1) \leq x_0$ or $v(x_0, x_1) \leq x_1$ for any binary C^*–term v. The result for intermediate values of C^* follows from these two cases. $\square$

A quasivariety $\mathcal{K}$ has *definable principal relative congruences* if there exists a formula $\phi(x_0, x_1, y_0, y_1)$ in the first order language with equality $\mathcal{L}_{\approx}$ (whose only non-logical symbols are the connectives of $\mathcal{L}$ interpreted as function symbols), such that for all $\mathbf{A} \in \mathcal{K}$, and all $a, b, c, d \in A$,

$$(c, d) \in \Theta_{\mathcal{K}}^{\mathbf{A}}(a, b) \text{ if and only if } \mathbf{A} \models \phi[a, b, c, d].$$

It is known that a locally finite quasivariety $\mathcal{K}$ with the RCEP has definable principal relative congruences: see [Nur90b] (and [BB75] for varieties). When $\mathcal{K}$ is also relatively congruence distributive, $\mathcal{K}$ has EDPRC by [BP88, Corollary 4.7]. Thus, in particular, every locally finite relative subvariety of $\mathcal{H}_{C^*}$ with the IEP (e.g., each $\mathcal{H}_{C^*}^n$) must have TDPI (see Proposition 4.22). The following result clarifies and strengthens this observation.

A quasivariety $\mathcal{K}$ is called *2–finite* if, for any $\mathbf{A} \in \mathcal{K}$ and $X \subseteq A$ with $|X| \leq 2$, the algebra $\mathrm{Sg}^{\mathbf{A}}(X)$ is finite. In particular, every locally finite quasivariety is 2–finite.

Corollary 5.11. *Let $\mathcal{K}$ be a 2–finite relative subvariety of $\mathcal{H}_{C^*}$, where $\oplus, 1 \notin C^*$, that has the IEP. Then there exist $n, m \in \omega$ such that $\mathcal{K} \subseteq \mathcal{H}_{C^*}^n$ and $\mathcal{K}$ has TDPI with respect to the set $\{x \dot{-} my\}$, i.e., for $\mathbf{A} \in \mathcal{K}$ and $a, b \in A$,*

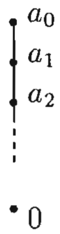
$$b \in \langle a \rangle_{\mathbf{A}} \text{ if and only if } b \dot{-} ma = 0.$$

Proof. Since $\mathcal{K}$ is 2–finite, the $\mathcal{K}$–free algebra $\mathbf{F}$ on two free generators $\overline{x}, \overline{y}$ is a finite algebra in which $\{\overline{x} \dot{-} (\overline{x} \dot{-} \overline{y}) \dot{-} n\overline{y} : n \in \omega\}$ is a descending chain. Thus $\mathcal{K} \models x \dot{-} (x \dot{-} y) \dot{-} ny \approx x \dot{-} (x \dot{-} y) \dot{-} (n+1)y$ for some $n \in \omega$. Now the first part of the corollary follows from the previous theorem.

By a similar argument, since $\mathcal{K}$ is 2–finite, it satisfies an identity $x \dot{-} my \approx x \dot{-} (m+1)y$ for some $m \in \omega$. In view of the last assertion of Proposition 5.7, for $\mathbf{A} \in \mathcal{K}$ and $a \in A$, we have $\langle a \rangle_{\mathbf{A}} = \{b \in A : b \dot{-} ma = 0\}$. □

Example 5.12. We present an example of an algebra $\mathbf{A}$ such that $\mathrm{HSP}(\mathbf{A})$ is a subvariety of $\mathcal{H}_{\{\dot{-}, \sqcap, \sqcup\}}$ that has the IEP but does not satisfy (Z_n) for any $n \in \omega$.[13] Let $\mathbf{A}$ be the algebra defined in Example 1.13. Recall that the underlying partial order $\leq$ on A is given by the following diagram:

[13] We could add 1 to the type of $\mathbf{A}$ without affecting the truth of any claims here, but in the context of Theorem 5.10 and Corollary 5.11, the example is of more interest if we omit 1.



Since $\langle A; \leq \rangle$ is *linearly* ordered, if we define lattice operations $\sqcap$, $\sqcup$ on A corresponding to the meet and join operations on $\langle A; \leq \rangle$, respectively, then the algebra $\mathbf{A}' = \langle A; \dot{-}, \sqcap, \sqcup, 0 \rangle$ lies in $\mathcal{H}_{\{\dot{-}, \sqcap, \sqcup\}}$. Note that $\mathbf{A}' \notin \mathcal{H}^n_{\{\dot{-}, \sqcap, \sqcup\}}$ for any $n \in \omega$, since

$$a_0 \dot{-} (a_0 \dot{-} a_{n+3}) \dot{-} n a_{n+3} = a_2 \dot{-} n a_{n+3} = a_{n+2} \neq 0.$$

More strongly, $\mathbf{A}'$ does not satisfy $x \dot{-} (x \dot{-} y) \dot{-} ny \approx x \dot{-} (x \dot{-} y) \dot{-} (n+1)y$ for any $n \in \omega$. For example, $a_0 \dot{-} (a_0 \dot{-} a_{n+3}) \dot{-} (n+1)a_{n+3} = a_{n+2} \dot{-} a_{n+3} = a_{n+4}$.

One can show that $\mathbf{A}'$ satisfies the identity

$$x \dot{-} 2(x \dot{-} y) \dot{-} (y \dot{-} x) \approx y \dot{-} 2(y \dot{-} x) \dot{-} (x \dot{-} y),$$

which is of the form of (27) hence, by Proposition 4.4(ii), $\mathbf{A}'$ generates a subvariety of $\mathcal{H}_{\{\dot{-}, \sqcap, \sqcup\}}$, say $\mathcal{V}$. We claim that condition (ii) of Theorem 5.4 holds for $\mathcal{V}$ and, hence, that $\mathcal{V}$ has TDPI. Set

$$u(x, y) = y \dot{-} (y \dot{-} x \dot{-} (y \dot{-} (y \dot{-} x \dot{-} (y \dot{-} (y \dot{-} x \dot{-} x))))).$$

Then u is a term of the syntactic form described in Theorem 5.4(ii). By (34) (see page 77), for each $t(x_0, \ldots, x_n) \in T$, $\mathcal{H}_{\{\dot{-}, \sqcap, \sqcup\}}$ satisfies $t(0, x_1, \ldots, x_n) \approx 0$, hence also $u(0, t(0, x_1, \ldots, x_n)) \approx u(0, 0) \approx 0 \approx t(0, x_1, \ldots, x_n)$. Moreover, one easily checks that for all $i, j \in \omega$,

$$(51) \qquad a_i \dot{-} (a_i \dot{-} a_j \dot{-} (a_i \dot{-} (a_i \dot{-} a_j \dot{-} (a_i \dot{-} (a_i \dot{-} a_j \dot{-} a_j))))) = a_i.$$

It follows that for all $t(x_0, \ldots, x_n) \in T$ and $b_1, \ldots, b_n \in A$,

$$u(a_j, t(a_j, b_1, \ldots, b_n)) = t(a_j, b_1, \ldots, b_n)$$

regardless of the value of $t(a_j, b_1, \ldots, b_n)$. Thus our claim holds and so $\mathcal{V}$ has TDPI and therefore also the IEP.

This example therefore shows that we cannot drop from Theorem 5.10 the condition that $\mathcal{K}$ satisfy $x \dot{-} (x \dot{-} y) \dot{-} ny \approx x \dot{-} (x \dot{-} y) \dot{-} (n+1)y$. Of course, this example does not contradict Corollary 5.11, as $\mathbf{A}'$ is 2-generated (by $\{a_0, a_1\}$) but infinite.

Example 5.13. We now present an example which shows that Corollary 5.11 (and Theorem 5.10) would fail if we allowed $\oplus \in C^*$ or $1 \in C^*$. Let B be a four-element set $\{0, a, b, 1\}$, and define a binary operation $\oplus$ on B by $0 \oplus x = x \oplus 0 = x$ for all $x \in B$; $a \oplus a = a$; $a \oplus b = b$ and $x \oplus y = 1$ otherwise. Define a linear order $\leq$ on B by $0 \leq a \leq b \leq 1$ (see Figure 7).

Figure 7.

Then $\langle B; \oplus, 0; \leq \rangle$ is an integral pomonoid, residuated on the left as follows: $x \dot{-} y = 0$ for $x, y \in B$ with $x \leq y$; $b \dot{-} a = 1 \dot{-} a = 1 \dot{-} b = b$. Let $\sqcap$ and $\sqcup$ be the meet and join operations of the poset $\langle B, \leq \rangle$, respectively. Since $\leq$ is a linear order, the algebra $\mathbf{B} = \langle B; \oplus, \dot{-}, \sqcap, \sqcup, 0, 1 \rangle$ is in $\mathcal{H}$. For each C^*, let $\mathbf{B}_{C^*}$ denote the C^*–reduct of $\mathbf{B}$.

The $\langle \dot{-}, 0 \rangle$–reduct of $\mathbf{B}$ is isomorphic to the subalgebra of $\boldsymbol{\omega} + \mathbf{2}_{\{\dot{-}\}}$ (as defined in Example 1.8) with universe $\{0, 1, \omega, \omega + 1\}$. By Corollary 4.6, each $\mathbf{B}_{C^*}$ generates a subvariety of $\mathcal{H}_{C^*}$, say $\mathcal{B}_{C^*}$. Note that $1 \dot{-} (1 \dot{-} a) \dot{-} na = b \neq 0$ for each $n \in \omega$. Suppose C^* contains $\oplus$ and consider the C^*–term

$$u(x, y) = y \dot{-} (y \dot{-} x \dot{-} ((y \oplus x) \dot{-} ((y \oplus x) \dot{-} x))).$$

Evidently u is of the syntactic form described in Theorem 5.4(ii) and for every $t(\vec{x}) \in T$, it may be shown that $\mathbf{B}_{C^*}$ satisfies $t(\vec{x}) \approx u(x_0, t(\vec{x}))$, hence $\mathcal{B}_{C^*}$ has TDPI. Moreover, if $1 \in C^*$, then it can be shown similarly, using the C^*–term

$$u'(x, y) = y \dot{-} (y \dot{-} x \dot{-} (1 \dot{-} (1 \dot{-} x))),$$

that $\mathcal{B}_{C^*}$ has TDPI.

5.3. Finitely Subdirectly Irreducible Algebras in $\mathcal{H}^n_{C^*}$. Recall that for a quasivariety $\mathcal{K}$, the class of all [relatively] finitely subdirectly irreducible algebras in $\mathcal{K}$ is denoted $\mathcal{K}_{\text{[R]FSI}}$ and that, by Proposition 4.21, $(\mathcal{H}_{C^*})_{\text{RFSI}} = (\mathcal{H}_{C^*})_{\text{FSI}}$.

Proposition 5.14. *For $n \in \omega$, an algebra $\mathbf{A} \in \mathcal{H}^n_{C^*}$ belongs to $(\mathcal{H}^n_{C^*})_{\text{FSI}}$ if and only if 0 is meet irreducible in $\langle A, \leq \rangle$.*

Proof. ($\Rightarrow$) As observed after Proposition 4.22, algebras in $\mathcal{H}_{C^*}$ are finitely subdirectly irreducible if and only if their $\langle \dot{-}, 0\rangle$–reducts are. We may therefore assume, without loss of generality, that $C^* = \{\dot{-}\}$.

Let $n \in \omega$ and $\mathbf{A} \in (\mathcal{H}^n_{C^*})_{\mathrm{FSI}}$. Suppose $0 \neq a, b \in A$ and that 0 is the only common lower bound of a and b in $\langle A; \leq\rangle$. Let $\mathbf{B}$ be the subalgebra of $\mathbf{A}$ whose universe is $(a] \cup (b]$. Since $(a] \cap (b] = \{0\}$, $(a]$ and $(b]$ are preideals (hence ideals, by Proposition 5.7) of $\mathbf{B}$.

Now $\mathcal{H}^n_{C^*}$ has the IEP, so $\langle a\rangle_{\mathbf{A}} \cap B = \langle a\rangle_{\mathbf{B}} = (a]$ and $\langle b\rangle_{\mathbf{A}} \cap B = \langle b\rangle_{\mathbf{B}} = (b]$, so

$$\langle a\rangle_{\mathbf{A}} \cap \langle b\rangle_{\mathbf{A}} \cap B = (a] \cap (b] = \{0\}.$$

Let $0 \neq c \in \langle a\rangle_{\mathbf{A}} \cap \langle b\rangle_{\mathbf{A}}$. By the last assertion of Proposition 5.7, there exist $k, l \in \omega$ with

$$c \dot{-} ka = 0 = c \dot{-} lb. \tag{52}$$

Choose k, l minimal such that (52) is true and note that $k, l > 0$. Then,

$$0 \neq c \dot{-} (k-1)a \leq a, \tag{53}$$

$$0 \neq c \dot{-} (l-1)b \leq b, \tag{54}$$

hence

$$c \dot{-} (k-1)a \dot{-} (l-1)b \leq a, b. \tag{55}$$

By assumption, therefore,

$$c \dot{-} (k-1)a \dot{-} (l-1)b = 0. \tag{56}$$

Thus $c \dot{-} (k-1)a \in B \cap \langle b\rangle_{\mathbf{A}}$ and, by (52), $c \dot{-} (k-1)a \in \langle a\rangle_{\mathbf{A}}$, so $c \dot{-} (k-1)a \in \langle a\rangle_{\mathbf{A}} \cap \langle b\rangle_{\mathbf{A}} \cap B$, i.e., $c \dot{-} (k-1)a = 0$, contradicting (53). So $\langle a\rangle_{\mathbf{A}} \cap \langle b\rangle_{\mathbf{A}} = \{0\}$. By Proposition 4.21, therefore, $\langle a\rangle_{\mathbf{A}} = \{0\}$ or $\langle b\rangle_{\mathbf{A}} = \{0\}$, hence $a = 0$ or $b = 0$, as required.

($\Leftarrow$) Let 0 be meet irreducible in $\langle A; \leq\rangle$ and let I and J be ideals of $\mathbf{A}$ with $I \cap J = \{0\}$. Suppose $I \neq \{0\}$ and choose $a \in I$ with $a \neq 0$. Let $b \in J$. Any common lower bound of a and b is in $I \cap J$, so the greatest lower bound of a and b in $\langle A; \leq\rangle$ is 0. Thus $b = 0$, by meet irreducibility of 0 in $\langle A; \leq\rangle$, so $J = \{0\}$ and $\mathbf{A}$ is finitely subdirectly irreducible by Proposition 4.21. □

Consider the following quasi-identity:

$$z \dot{-} (x \dot{-} y) \approx 0 \text{ and } z \dot{-} (y \dot{-} x) \approx 0 \text{ implies } z \approx 0. \tag{57}$$

If an algebra $\mathbf{A} \in \mathcal{H}_{C^*}$ satisfies (57), then for all $a, b \in A$, 0 is the only common lower bound of $a \dot{-} b$ and $b \dot{-} a$ in $\langle A; \leq \rangle$. Evidently, when $\sqcap \in C^*$, (57) is equivalent to the identity

$$(58) \qquad (x \dot{-} y) \sqcap (y \dot{-} x) \approx 0.$$

Moreover, when $\sqcap \in C^*$, the $\langle \dot{-}, \sqcap, 0 \rangle$–identity (58) is equivalent to the $\langle \dot{-}, 0 \rangle$–identity

$$(59) \qquad z \dot{-} (z \dot{-} (x \dot{-} y)) \dot{-} (z \dot{-} (y \dot{-} x)) \approx 0.$$

For, by (Y_0) (see page 42), $\mathcal{H}_{\{\dot{-}, \sqcap\}}$ satisfies

$$z \dot{-} ((x \dot{-} y) \sqcap (y \dot{-} x)) \dot{-} (z \dot{-} (x \dot{-} y)) \dot{-} (z \dot{-} (y \dot{-} x)) \approx 0.$$

Thus, over $\mathcal{H}_{\{\dot{-}, \sqcap\}}$, (58) implies (59). Conversely, over $\mathcal{H}_{\{\dot{-}, \sqcap\}}$, we may derive (58) from (59) if we set $z = (x \dot{-} y) \sqcap (y \dot{-} x)$.

Corollary 5.15. *For $\mathbf{A} \in \mathcal{H}_{C^*}$, consider the following conditions:*

(i) $\mathbf{A}$ *is linearly ordered;*

(ii) $\mathbf{A}$ *satisfies* (59);

(iii) $\mathbf{A}$ *satisfies* (57).

In general, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). If $\mathbf{A} \in (\mathcal{H}^n_{C^*})_{\mathrm{FSI}}$, for some $n \in \omega$, the three conditions are equivalent. If, in addition, $\sqcap \in C^*$, (i)–(iii) are each equivalent to

(iv) $\mathbf{A}$ *satisfies* (58).

Proof. (i) $\Rightarrow$ (ii) Under any interpretation of x, y, z in A, one of $x \dot{-} y$ and $y \dot{-} x$ takes the value 0, hence (59) holds in $\mathbf{A}$.

(ii) $\Rightarrow$ (iii) This implication is clear.

Suppose now that $\mathbf{A} \in (\mathcal{H}^n_{C^*})_{\mathrm{FSI}}$, where $n \in \omega$.

(iii) $\Rightarrow$ (i) By Proposition 5.14, 0 is meet irreducible in $\langle A; \leq \rangle$, hence, for $a, b \in A$, (57) implies that $a \dot{-} b = 0$ or $b \dot{-} a = 0$, so $a \leq b$ or $b \leq a$.

The remaining statement follows from the discussion preceding this corollary. $\square$

The special case of Corollary 5.15 in which $C^* = \{\dot{-}\}$, $\mathbf{A}$ is subdirectly irreducible and $n = 1$ (hence $\mathbf{A}$ is a BCK–algebra) was discovered by M. Pałasinski [Pał80]. Thus, the next corollary also generalizes some known properties of BCK–algebras.

Corollary 5.16. *For $n \in \omega$ and $\mathbf{A} \in \mathcal{H}^n_{C^*}$, the following are equivalent:*

(i) $\mathbf{A}$ *satisfies* (59);

(ii) $\mathbf{A}$ *is a subdirect product of linearly ordered subdirectly irreducible algebras in* $\mathcal{H}^n_{C^*}$;

(iii) $\mathbf{A}$ *is a subdirect product of linearly ordered algebras in* $\mathcal{H}^n_{C^*}$.

In this case, $\mathbf{A}$ *satisfies* (57). *When* $\sqcap \in C^*$, *we may replace* (59) *by* (58) *in* (i).

Proof. (i) $\Rightarrow$ (ii) By the quasivarietal analogue of Birkhoff's subdirect decomposition theorem (Theorem 0.9), $\mathbf{A}$ is a subdirect product of relatively ($\mathcal{H}_{C^*}$–, or equivalently, $\mathcal{H}^n_{C^*}$–) subdirectly irreducible algebras $\mathbf{A}_i \in \mathrm{H}(\mathbf{A}) \cap \mathcal{H}^n_{C^*}$, $i \in I$. Each $\mathbf{A}_i$ satisfies (59) because $\mathbf{A}$ does. Now, $\mathcal{H}_{C^*}$–subdirectly irreducible algebras in $\mathcal{H}_{C^*}$ are subdirectly irreducible (by Proposition 4.21), hence finitely subdirectly irreducible. By Corollary 5.15, therefore, each $\mathbf{A}_i$ is linearly ordered, and the result follows.

The implication (ii) $\Rightarrow$ (iii) is trivial and (iii) $\Rightarrow$ (i) follows by Corollary 5.15 since subdirect products preserve identities. The remaining statements of the corollary follow from the discussion preceding Corollary 5.15. □

For each $n \in \omega$, let $L^n_{C^*}$ denote the class of all linearly ordered members of $\mathcal{H}^n_{C^*}$. Let $\mathcal{K}^n_{C^*}$ denote the relative subvariety of $\mathcal{H}^n_{C^*}$ generated by $L^n_{C^*}$, i.e., $\mathcal{K}^n_{C^*} = \mathcal{H}^n_{C^*} \cap \mathrm{H\,S\,P}(L^n_{C^*})$. By Corollary 5.16, $\mathcal{K}^n_{C^*}$ satisfies (59).

Corollary 5.17. *The subquasivariety* $\mathrm{Q}(L^n_{C^*})$ *of* $\mathcal{H}^n_{C^*}$, *where* $n \in \omega$, *generated by* $L^n_{C^*}$ *is* $\mathcal{K}^n_{C^*}$. *The class* $\mathcal{K}^n_{C^*}$ *is axiomatized, relative to* $\mathcal{H}^n_{C^*}$, *by* (59) *(equivalently, by* (58), *when* $\sqcap \in C^*$*).*

Proof. Since $\mathcal{K}^n_{C^*}$ is a quasivariety containing $L^n_{C^*}$, $\mathrm{Q}(L^n_{C^*}) \subseteq \mathcal{K}^n_{C^*}$. If $\mathbf{A} \in \mathcal{K}^n_{C^*}$ is $\mathcal{K}^n_{C^*}$–subdirectly irreducible, then it is $\mathcal{H}^n_{C^*}$–subdirectly irreducible, hence finitely subdirectly irreducible (in the absolute sense) in $\mathcal{H}^n_{C^*}$ (by Proposition 4.21), and satisfies (59). By Corollary 5.15, $\mathbf{A}$ is linearly ordered, so $\mathbf{A} \in L^n_{C^*}$. Now,

$$\mathcal{K}^n_{C^*} = \mathrm{I\,P_S}((\mathcal{K}^n_{C^*})_{\mathrm{RSI}}) \subseteq \mathrm{I\,P_S}(L^n_{C^*}) \subseteq \mathrm{Q}(L^n_{C^*}),$$

where RSI refers to $\mathcal{K}^n_{C^*}$–subdirectly irreducible. Thus $\mathcal{K}^n_{C^*} = \mathrm{Q}(L^n_{C^*})$.

Obviously, $\mathcal{K}^n_{C^*}$ is a subclass of $\mathcal{H}^n_{C^*}$ that satisfies (59). Conversely, suppose $\mathbf{A}$ is a member of $\mathcal{H}^n_{C^*}$ that satisfies (59). By Corollary 5.16, $\mathbf{A}$ is a subdirect product of linearly ordered algebras in $\mathcal{H}^n_{C^*}$, hence $\mathbf{A} \in \mathrm{Q}(L^n_{C^*}) = \mathcal{K}^n_{C^*}$. Thus $\mathcal{K}^n_{C^*}$ is axiomatized, relative to $\mathcal{H}^n_{C^*}$, by (59). □

The members of $\mathcal{K}^n_{C^*}$ $(= \mathrm{Q}(L^n_{C^*}))$ shall be called the *representable algebras in* $\mathcal{H}^n_{C^*}$.

5.4. Equationally Definable Principal Relative Meets. A quasivariety $\mathcal{K}$ has *equationally definable principal relative meets* (EDPRM) if there exists a finite system $\Sigma = \{\langle u_i(x,y,z,w), v_i(x,y,z,w)\rangle : i = 1,\ldots,n\}$, of pairs of 4-ary terms such that, for all $\mathbf{A} \in \mathcal{K}$ and $a,b,c,d \in A$,

$$\Theta^{\mathbf{A}}_{\mathcal{K}}(a,b) \cap \Theta^{\mathbf{A}}_{\mathcal{K}}(c,d) = \bigsqcup\{\Theta^{\mathbf{A}}_{\mathcal{K}}(u_i(a,b,c,d), v_i(a,b,c,d)) : i = 1,\ldots,n\}$$

where the join is taken in the lattice $\mathbf{Con}_{\mathcal{K}}\mathbf{A}$. In this case Σ is called a *system of principal intersection terms* for $\mathcal{K}$. (We drop the qualification 'relative' and speak of 'EDPM' if $\mathcal{K}$ is a variety.) For varieties, this notion originates in [BP86]. It has been studied mainly in the context of axiomatization problems: see [BP86], [CD90], [Dzi89]. For example, constructive proofs of some of the finite basis theorems mentioned in Chapter 0 for finitely generated [relatively] congruence distributive [quasi]varieties $\mathcal{K}$ may be simplified considerably under the assumption that $\mathcal{K}$ has EDP[R]M.

Theorem 5.18. [CD90, Theorem 2.3, Corollary 2.4] *For a quasivariety $\mathcal{K}$ of algebras the following are equivalent:*

(i) *$\mathcal{K}$ has EDPRM;*

(ii) *$\mathcal{K}$ is relatively congruence distributive and $\mathcal{K}_{\mathrm{RFSI}}$ forms a universal class;*

(iii) *there exists a finite system $\Sigma = \{\langle u_i(x,y,z,w), v_i(x,y,z,w)\rangle :$ $i = 1,\ldots,n\}$, of pairs of 4-ary terms such that $\mathcal{K}_{\mathrm{RFSI}}$ satisfies*

$$(\forall x)(\forall y)(\forall z)(\forall w)\,[(\bigwedge_{i=1}^{n} u_i(x,y,z,w) \approx v_i(x,y,z,w)) \text{ iff } (x \approx y \text{ or } z \approx w)].$$

In this case Σ is a system of principal intersection terms for $\mathcal{K}$.

Recall that for a class $\mathcal{K}$ of similar algebras, $\mathcal{K}_{\mathrm{NT}}$ denotes the class of all nontrivial algebras in $\mathcal{K}$, and $\mathcal{K}_{\mathrm{S}}$ the class of all simple algebras in $\mathcal{K}$. A variety $\mathcal{V}$ is called *semisimple* if every subdirectly irreducible member of $\mathcal{V}$ is simple, i.e., if $\mathcal{V}_{\mathrm{SI}} = \mathcal{V}_{\mathrm{S}}$. The following result is essentially well known. A proof is included for the reader's convenience.

Proposition 5.19. *Let $\mathcal{V}$ be a variety with EDPC.*
(i) $\mathcal{V}$ is semisimple if and only if $\mathcal{V}$ is generated (as a variety) by a class of simple algebras.
(ii) If $\mathcal{V}$ is semisimple, say $\mathcal{V} = \mathrm{V}(\mathcal{K})$, where $\mathcal{K} \subseteq \mathcal{V}_{\mathrm{S}}$, then

$$(\mathcal{V}_{\mathrm{FSI}})_{\mathrm{NT}} = \mathcal{V}_{\mathrm{SI}} = \mathcal{V}_{\mathrm{S}} = (\mathrm{I}\,\mathrm{S}\,\mathrm{P}_{\mathrm{U}}(\mathcal{K}))_{\mathrm{NT}}$$

and $\mathcal{V}$ has EDPM.

Proof. (i) By Lemma 5.1(iii), (v), $\mathcal{V}$ is congruence distributive with the CEP.

($\Rightarrow$) $\mathcal{V} = \mathrm{V}(\mathcal{V}_{\mathrm{SI}}) = \mathrm{V}(\mathcal{V}_{\mathrm{S}})$.

($\Leftarrow$) Let $\mathcal{K} \subseteq \mathcal{V}_{\mathrm{S}}$ with $\mathcal{V} = \mathrm{V}(\mathcal{K})$. By Lemma 5.1(iv), $\mathrm{P_U}(\mathcal{K}) \subseteq \mathcal{V}_{\mathrm{S}}$ and, by the CEP, $(\mathrm{S}(\mathcal{V}_{\mathrm{S}}))_{\mathrm{NT}} \subseteq \mathcal{V}_{\mathrm{S}}$, hence $(\mathrm{H\,S}(\mathcal{V}_{\mathrm{S}}))_{\mathrm{NT}} \subseteq \mathcal{V}_{\mathrm{S}}$. Now by Jónsson's Theorem (Theorem 0.2), $\mathcal{V}_{\mathrm{SI}} \subseteq (\mathcal{V}_{\mathrm{FSI}})_{\mathrm{NT}} \subseteq (\mathrm{H\,S\,P_U}(\mathcal{K}))_{\mathrm{NT}} \subseteq \mathcal{V}_{\mathrm{S}}$, so $\mathcal{V}$ is semisimple.

(ii) Since $\mathcal{V}_{\mathrm{S}} \subseteq \mathcal{V}_{\mathrm{SI}}$ (for *any* variety), the above proof also establishes the equations in the first assertion of (ii). Thus, when $\mathcal{V}$ is semisimple, $(\mathcal{V}_{\mathrm{FSI}})_{\mathrm{NT}} = \mathcal{V}_{\mathrm{S}} = \mathrm{P_U}(\mathcal{V}_{\mathrm{S}}) = (\mathrm{S}(\mathcal{V}_{\mathrm{S}}))_{\mathrm{NT}}$, so $\mathcal{V}_{\mathrm{FSI}}$ is closed under subalgebras and ultraproducts, i.e., $\mathcal{V}_{\mathrm{FSI}}$ is a universal class (see Theorem 0.8). By Theorem 5.18, $\mathcal{V}$ has EDPM. □

Semisimple varieties with EDPC have also been studied under the name *filtral* varieties: see [BP82] for further information and references.

In Corollary 5.25 we show, for each $n \in \omega$, that $\mathcal{K}^n_{C^*}$ has EDPRM, by presenting a system of principal intersection terms for $\mathcal{K}^n_{C^*}$. We include the following proof, however, as it is an interesting illustration of Theorem 5.18(ii).

Corollary 5.20. *For $n \in \omega$, the quasivariety $\mathcal{K}^n_{C^*}$ of representable algebras in $\mathcal{H}^n_{C^*}$ has EDPRM.*

Proof. Since $\mathcal{K}^n_{C^*}$ is a relative subvariety of $\mathcal{H}_{C^*}$, it is relatively congruence distributive. The class $L^n_{C^*}$ is axiomatized by the axioms of $\mathcal{H}^n_{C^*}$ together with the following universal sentence:

$$(\forall x)(\forall y)\,(x \dot{-} y \approx 0 \text{ or } y \dot{-} x \approx 0).$$

Thus $L^n_{C^*}$ is a universal class, so it is closed under I, S and $\mathrm{P_U}$. Lemma 1.5 of [CD90] states that, for a class $\mathcal{M}$ of similar algebras, every nontrivial member of $\mathrm{Q}(\mathcal{M})_{\mathrm{RFSI}}$ belongs to $\mathrm{I\,S\,P_U}(\mathcal{M})$, hence $\mathrm{Q}((L^n_{C^*})_{\mathrm{RFSI}}) \subseteq \mathrm{I\,S\,P_U}(L^n_{C^*})$. Now, by Corollary 5.17,

$$(\mathcal{K}^n_{C^*})_{\mathrm{RFSI}} = (\mathrm{Q}(L^n_{C^*}))_{\mathrm{RFSI}} \subseteq \mathrm{I\,S\,P_U}(L^n_{C^*}) \subseteq L^n_{C^*}.$$

Conversely, if $\mathbf{A} \in L^n_{C^*}$, then $\mathbf{A}$ is linearly ordered, hence $0^{\mathbf{A}}$ is meet irreducible, so $\mathbf{A} \in (\mathcal{K}^n_{C^*})_{\mathrm{FSI}}$, by Proposition 5.14. By Proposition 4.21, $(\mathcal{K}^n_{C^*})_{\mathrm{FSI}} = (\mathcal{K}^n_{C^*})_{\mathrm{RFSI}}$, so $(\mathcal{K}^n_{C^*})_{\mathrm{RFSI}} = L^n_{C^*}$. Thus $(\mathcal{K}^n_{C^*})_{\mathrm{RFSI}}$ is a universal class. By Theorem 5.18, $\mathcal{K}^n_{C^*}$ has EDPRM. □

Proposition 5.21. *Let $\mathcal{K}$ be a relative subvariety of $\mathcal{H}_{C^*}$. Suppose that there exist terms $t_i(x,y)$, $i = 1,\ldots,n$, such that for all $\mathbf{A} \in \mathcal{K}$ and $a, b \in A$,*

$$\langle a\rangle_{\mathbf{A}} \cap \langle b\rangle_{\mathbf{A}} = \langle\{t_i^{\mathbf{A}}(a,b) : i = 1,\ldots,n\}\rangle_{\mathbf{A}}.$$

Then $\mathcal{K}$ has EDPRM with a system of principal intersection terms

$$\Sigma = \{\langle u_i^j(x,y,z,w), 0\rangle : i = 1,\dots,n;\ j = 1,\dots,4\},$$

where, for $i = 1,\dots,n$,

$$u_i^1(x,y,z,w) = t_i(x \dot- y, z \dot- w), \quad u_i^2(x,y,z,w) = t_i(x \dot- y, w \dot- z),$$
$$u_i^3(x,y,z,w) = t_i(y \dot- x, z \dot- w), \quad u_i^4(x,y,z,w) = t_i(y \dot- x, w \dot- z).$$

Moreover, if $\mathbf{A} \in \mathcal{K}$, $a, b \in A$ and I is an ideal of $\mathbf{A}$ for which $t_i^{\mathbf{A}}(a,b) \in I$ for $i = 1,\dots,n$, then

$$I = \langle I \cup \{a\}\rangle_{\mathbf{A}} \cap \langle I \cup \{b\}\rangle_{\mathbf{A}}.$$

Proof. By Theorem 5.18 and Proposition 4.21, we need only show that

$$\mathcal{K}_{\mathrm{FSI}} \models (\forall x)(\forall y)(\forall z)(\forall w)\,[(\bigwedge_{j=1}^{4}\bigwedge_{i=1}^{n} u_i^j(x,y,z,w) \approx 0) \text{ iff } (x \approx y \text{ or } z \approx w)].$$

Note that, for $\mathbf{A} \in \mathcal{K}$ and $a \in A$,

$$\{0\} = \langle a\rangle_{\mathbf{A}} \cap \langle 0\rangle_{\mathbf{A}} = \langle\{t_i^{\mathbf{A}}(a,0) : i = 1,\dots,n\}\rangle_{\mathbf{A}},$$

so $t_i^{\mathbf{A}}(a,0) = 0$ for $i = 1,\dots,n$. Similarly $t_i^{\mathbf{A}}(0,a) = 0$ and $t_i^{\mathbf{A}}(b,0) = 0 = t_i^{\mathbf{A}}(0,b)$ for $i = 1,\dots,n$. The implication from right to left now follows easily.

Conversely, let $\mathbf{A} \in \mathcal{K}_{\mathrm{FSI}}$ and let $a,b,c,d \in A$ such that $u_i^{j\,\mathbf{A}}(a,b,c,d) = 0$ for each i,j. By assumption,

$$\langle a \dot- b\rangle_{\mathbf{A}} \cap \langle c \dot- d\rangle_{\mathbf{A}} = \langle\{t_i^{\mathbf{A}}(a \dot- b, c \dot- d) : i = 1,\dots n\}\rangle_{\mathbf{A}} = \langle\{0\}\rangle_{\mathbf{A}} = \{0\},$$

hence, by Proposition 4.21 and the fact that $\mathbf{A}$ is finitely subdirectly irreducible, either $\langle a \dot- b\rangle_{\mathbf{A}} = \{0\}$ or $\langle c \dot- d\rangle_{\mathbf{A}} = \{0\}$. Thus, either $a \dot- b = 0$ or $c \dot- d = 0$. By similar considerations, we obtain that: $a \dot- b = 0$ or $d \dot- c = 0$; $b \dot- a = 0$ or $c \dot- d = 0$; and $b \dot- a = 0$ or $d \dot- c = 0$. It follows from (A4) that either $a = b$ or $c = d$, as required.

The last observation follows by the distributivity of ideals (see Proposition 4.22): we have

$$\begin{aligned}\langle I \cup \{a\}\rangle_{\mathbf{A}} \cap \langle I \cup \{b\}\rangle_{\mathbf{A}} &= I \sqcup^{\mathbf{Id\,A}} (\langle a\rangle_{\mathbf{A}} \cap \langle b\rangle_{\mathbf{A}}) \\ &= I \sqcup^{\mathbf{Id\,A}} \langle\{t_i^{\mathbf{A}}(a,b) : i = 1,\dots,n\}\rangle_{\mathbf{A}} = I.\end{aligned}$$

□

Lemma 5.22. *For $n, m \in \omega$, $\mathcal{H}_{C^*}^n$ satisfies*

$$\begin{aligned} z \dot- (x \dot- (x \dot- y)) \dot- n(x \dot- y) \dot- (x \dot- (x \dot- y)) \dot- n(x \dot- y) \dot- \dots \\ \dot- (x \dot- (x \dot- y)) \dot- n(x \dot- y) \le z \dot- mx, \end{aligned}$$

where there are m occurrences of each of $x \dot{-} (x \dot{-} y)$ and $n(x \dot{-} y)$ in the above expression.

Proof. The case $m = 0$ is trivial. For $m = 1$, we have that $\mathcal{H}^n_{C*}$ satisfies

$$\begin{aligned} & z \dot{-} (x \dot{-} (x \dot{-} y)) \dot{-} n(x \dot{-} y) \\ \approx\ & z \dot{-} (x \dot{-} (x \dot{-} y)) \dot{-} n(x \dot{-} y) \dot{-} (x \dot{-} (x \dot{-} (x \dot{-} y)) \dot{-} n(x \dot{-} y)) \quad \text{(by } (Z_n)) \\ \leq\ & z \dot{-} x \quad \text{(by repeated application of (A1)).} \end{aligned}$$

For values of m greater than 1 the result follows from the case $m = 1$. □

Lemma 5.23. *Let $t(\vec{z}, w)$ be any $\langle \dot{-}, 0\rangle$-term. Then, for $n \in \omega$, every representable algebra in $\mathcal{H}^n_{C*}$ satisfies*

$$t(\vec{z}, x \dot{-} y) \approx 0 \text{ and } t(\vec{z}, y \dot{-} x) \approx 0 \text{ implies } t(\vec{z}, 0) \approx 0. \tag{60}$$

Proof. Let $\mathbf{A} \in L^n_{C*}$, let $a, b, \vec{c} \in A$ and suppose that $t^{\mathbf{A}}(\vec{c}, a \dot{-} b) = 0 = t^{\mathbf{A}}(\vec{c}, b \dot{-} a)$. Since $\mathbf{A}$ is linearly ordered, $a \dot{-} b = 0$ or $b \dot{-} a = 0$, so $t^{\mathbf{A}}(\vec{c}, 0) = 0$. Thus L^n_{C*} satisfies the quasi-identity (60). By Corollary 5.17, $\mathcal{K}^n_{C*} = \mathrm{Q}(L^n_{C*})$, hence $\mathcal{K}^n_{C*}$ satisfies (60) as well. □

Lemma 5.24. *Let $n \in \omega$, $\mathbf{A} \in \mathcal{K}^n_{C*}$ and $a, b \in A$. Then*

$$\langle a\rangle_{\mathbf{A}} \cap \langle b\rangle_{\mathbf{A}} = \langle\{a \dot{-} (a \dot{-} b), b \dot{-} (b \dot{-} a)\}\rangle_{\mathbf{A}}.$$

Moreover, if I is an ideal of $\mathbf{A}$ and $a \dot{-} (a \dot{-} b), b \dot{-} (b \dot{-} a) \in I$, then

$$I = \langle I \cup \{a\}\rangle_{\mathbf{A}} \cap \langle I \cup \{b\}\rangle_{\mathbf{A}}.$$

Proof. Since $a \dot{-} (a \dot{-} b) \leq a$, we have $a \dot{-} (a \dot{-} b) \in \langle a\rangle_{\mathbf{A}}$; since $\mathbf{A} \in \mathcal{H}^n_{C*}$, we have $a \dot{-} (a \dot{-} b) \dot{-} nb = 0$ hence $a \dot{-} (a \dot{-} b) \in \langle b\rangle_{\mathbf{A}}$. Thus $a \dot{-} (a \dot{-} b) \in \langle a\rangle_{\mathbf{A}} \cap \langle b\rangle_{\mathbf{A}}$ and, similarly, $b \dot{-} (b \dot{-} a) \in \langle a\rangle_{\mathbf{A}} \cap \langle b\rangle_{\mathbf{A}}$, so $\langle\{a \dot{-} (a \dot{-} b), b \dot{-} (b \dot{-} a)\}\rangle_{\mathbf{A}} \subseteq \langle a\rangle_{\mathbf{A}} \cap \langle b\rangle_{\mathbf{A}}$.

Conversely, let $c \in \langle a\rangle_{\mathbf{A}} \cap \langle b\rangle_{\mathbf{A}}$. By the last assertion of Proposition 5.7, there exists $m \in \omega$ such that $c \dot{-} ma = 0 = c \dot{-} mb$. Set

$$\begin{aligned} t(x, y, z, w) = z \dot{-} (x \dot{-} (x \dot{-} y)) \dot{-} (y \dot{-} (y \dot{-} x)) \dot{-} nw \dot{-} \ldots \dot{-} (x \dot{-} (x \dot{-} y)) \\ \dot{-} (y \dot{-} (y \dot{-} x)) \dot{-} nw, \end{aligned}$$

where there are m occurrences of each of $x \dot{-} (x \dot{-} y)$, $y \dot{-} (y \dot{-} x)$ and nw. Now

$$\begin{aligned} & t^{\mathbf{A}}(a, b, c, a \dot{-} b) \\ \leq\ & c \dot{-} (a \dot{-} (a \dot{-} b)) \dot{-} n(a \dot{-} b) \dot{-} \ldots \dot{-} (a \dot{-} (a \dot{-} b)) \dot{-} n(a \dot{-} b) \\ \leq\ & c \dot{-} ma \quad \text{(by Lemma 5.22)} \\ =\ & 0, \end{aligned}$$

so $t^{\mathbf{A}}(a, b, c, a \dot{-} b) = 0$ and, similarly, $t^{\mathbf{A}}(a, b, c, b \dot{-} a) = 0$. By Lemma 5.23, therefore, $t^{\mathbf{A}}(a, b, c, 0) = 0$, i.e.,

$$c \dot{-} (a \dot{-} (a \dot{-} b)) \dot{-} (b \dot{-} (b \dot{-} a)) \dot{-} \ldots \dot{-} (a \dot{-} (a \dot{-} b)) \dot{-} (b \dot{-} (b \dot{-} a)) = 0,$$

so $c \in \langle\{a \dot{-} (a \dot{-} b), b \dot{-} (b \dot{-} a)\}\rangle_{\mathbf{A}}$. The remaining statement follows from Proposition 5.21. □

The above lemma and Proposition 5.21 give us the following:

Corollary 5.25. *For $n \in \omega$, a system of principal intersection terms for $\mathcal{K}^n_{C^*}$ is given by*

$$\begin{aligned}
u_1(x,y,z,w) &= x \dot{-} y \dot{-} (x \dot{-} y \dot{-} (z \dot{-} w)), & v_1(x,y,z,w) &= 0,\\
u_2(x,y,z,w) &= x \dot{-} y \dot{-} (x \dot{-} y \dot{-} (w \dot{-} z)), & v_2(x,y,z,w) &= 0,\\
u_3(x,y,z,w) &= y \dot{-} x \dot{-} (y \dot{-} x \dot{-} (z \dot{-} w)), & v_3(x,y,z,w) &= 0,\\
u_4(x,y,z,w) &= y \dot{-} x \dot{-} (y \dot{-} x \dot{-} (w \dot{-} z)), & v_4(x,y,z,w) &= 0,\\
u_5(x,y,z,w) &= z \dot{-} w \dot{-} (z \dot{-} w \dot{-} (x \dot{-} y)), & v_5(x,y,z,w) &= 0,\\
u_6(x,y,z,w) &= z \dot{-} w \dot{-} (z \dot{-} w \dot{-} (y \dot{-} x)), & v_6(x,y,z,w) &= 0,\\
u_7(x,y,z,w) &= w \dot{-} z \dot{-} (w \dot{-} z \dot{-} (x \dot{-} y)), & v_7(x,y,z,w) &= 0,\\
u_8(x,y,z,w) &= w \dot{-} z \dot{-} (w \dot{-} z \dot{-} (y \dot{-} x)), & v_8(x,y,z,w) &= 0.
\end{aligned}$$

Lemma 5.26. *Let $n \in \omega$, $\sqcap \in C^*$, $\mathbf{A} \in \mathcal{H}^n_{C^*}$ and $a, b \in A$. Then $\langle a\rangle_{\mathbf{A}} \cap \langle b\rangle_{\mathbf{A}} = \langle a \sqcap b\rangle_{\mathbf{A}}$. Moreover, if I is an ideal of $\mathbf{A}$ and $a \sqcap b \in I$, then $I = \langle I \cup \{a\}\rangle_{\mathbf{A}} \cap \langle I \cup \{b\}\rangle_{\mathbf{A}}$.*

Proof. Let $c \in \langle a\rangle_{\mathbf{A}} \cap \langle b\rangle_{\mathbf{A}}$. By the last assertion of Proposition 5.7, there exists $m \in \omega$ such that $c \dot{-} ma = 0 = c \dot{-} mb$. Note that

$$c \dot{-} d_1 \dot{-} \ldots \dot{-} d_{2m} = 0 \tag{61}$$

whenever $d_i \in \{a, b\}$ for each $i \in \{1, \ldots, 2m\}$. To see this, we need only note that amongst the d_i's there are at least m a's or at least m b's. Now, using two applications of (61) (namely with $d_{2m} = a$ and $d_{2m} = b$) and appealing to (Y_0) (see page 42) and (A2), we infer that

$$c \dot{-} d_1 \dot{-} \ldots \dot{-} d_{2m-1} \dot{-} (a \sqcap b) = 0 \tag{62}$$

whenever $d_i \in \{a, b\}$ for each $i \in \{1, \ldots, 2m-1\}$. Now, using two applications of (62) (with $d_{2m-1} = a$ and $d_{2m-1} = b$) and appealing to (Y_1) and (A2), we find that

$$c \dot{-} d_1 \dot{-} \ldots \dot{-} d_{2m-2} \dot{-} (a \sqcap b) \dot{-} (a \sqcap b) = 0.$$

Continuing in this way, we obtain

$$c \dot{-} (2m)(a \sqcap b) = 0,$$

hence $c \in \langle a \sqcap b\rangle_{\mathbf{A}}$. Thus $\langle a\rangle_{\mathbf{A}} \cap \langle b\rangle_{\mathbf{A}} \subseteq \langle a \sqcap b\rangle_{\mathbf{A}}$. The reverse inclusion is trivial. The remaining statement follows from Proposition 5.21. □

By the above lemma and Proposition 5.21, we therefore obtain the following:

Corollary 5.27. *Suppose $\sqcap \in C^*$. Then, for each $n \in \omega$, a system of principal intersection terms for $\mathcal{H}^n_{C^*}$ is given by*

$$\begin{aligned} u_1(x,y,z,w) &= (x \dot{-} y) \sqcap (z \dot{-} w), & v_1(x,y,z,w) &= 0,\\ u_2(x,y,z,w) &= (x \dot{-} y) \sqcap (w \dot{-} z), & v_2(x,y,z,w) &= 0,\\ u_3(x,y,z,w) &= (y \dot{-} x) \sqcap (z \dot{-} w), & v_3(x,y,z,w) &= 0,\\ u_4(x,y,z,w) &= (y \dot{-} x) \sqcap (w \dot{-} z), & v_4(x,y,z,w) &= 0. \end{aligned}$$

The above results illustrate for $\mathcal{H}^n_{C^*}$ ($\sqcap \in C^*$) and $\mathcal{K}^n_{C^*}$ a general property of quasivarieties $\mathcal{K}$ with EDPRM: for each $\mathbf{A} \in \mathcal{K}$, the compact relative congruences of $\mathbf{A}$ form a sublattice of $\mathbf{Con}_{\mathcal{K}}\mathbf{A}$ [CD90, Theorem 2.3]. Since $\mathcal{H}^1_{\{\dot{-},\sqcap\}}$ is the variety of 'lower BCK-semilattices', the above results (in the case $\sqcap \in C^*$) generalize some theorems of [RS87].

We remark that the variety $\mathcal{V}$ of Example 5.12 and the varieties $\mathcal{B}_{C^*}$ of Example 5.13, when $\sqcap \in C^*$ (which are not in $\mathcal{H}^n_{C^*}$ for any $n \in \omega$), have EDPM with same system of principal intersection terms as in the above corollary. In the case of the first example, it follows from the linear order on $\mathbf{A}$ that $\mathbf{A}$ (hence also $\mathcal{V}$) satisfies

$$u(x,z) \sqcap u(y,z) \approx u(x \sqcap y, z).$$

Now, for $\mathbf{B} \in \mathcal{V}$ and $a, b, c \in B$, we have $c \in \langle a\rangle_{\mathbf{B}} \sqcap \langle b\rangle_{\mathbf{B}}$ if and only if $c = u^{\mathbf{B}}(a,c)$ and $c = u^{\mathbf{B}}(b,c)$. In this case,

$$c = c \sqcap c = u^{\mathbf{B}}(a,c) \sqcap u^{\mathbf{B}}(b,c) = u^{\mathbf{B}}(a \sqcap b, c),$$

hence $c \in \langle a \sqcap b\rangle_{\mathbf{B}}$. Thus $\langle a\rangle_{\mathbf{B}} \cap \langle b\rangle_{\mathbf{B}} = \langle a \sqcap b\rangle_{\mathbf{B}}$. The fact that $\mathcal{V}$ has EDPM now follows from Proposition 5.21.[14]

When C^* contains $\sqcap$ it can be shown in a similar manner that the varieties $\mathcal{B}_{C^*}$ of Example 5.13 also have EDPM. Since $\mathcal{B}_{C^*}$ is finitely generated, Corollary 5.27 and the methods of [BP86] could be employed to construct a finite equational basis for $\mathcal{B}_{C^*}$.

[14]In the next chapter, we shall observe that the algebra $\mathbf{A}$ of Example 5.12 is simple. Since $\mathrm{V}(\mathbf{A})$ has EDPC, the fact that it has EDPM could also have been derived from Proposition 5.19. This does not apply to any of the other varieties with EDPM identified in the present chapter.

CHAPTER 6

THE LATTICE OF VARIETIES OF LEFT RESIDUATION ALGEBRAS

By an $\mathcal{LR}$-*variety* we shall mean a variety consisting of left residuation algebras. We denote by $\mathbf{P}^{\mathrm{V}}(\mathcal{LR})$ the partially ordered "set" of all $\mathcal{LR}$-varieties (ordered by inclusion). In Section 1 we shall show that $\mathbf{P}^{\mathrm{V}}(\mathcal{LR})$ is, in fact, a (distributive) lattice. This lattice has a unique atom, namely, the variety generated by the two-element left residuation algebra $\mathbf{C}_2$. In this chapter we investigate the lattice of $\mathcal{LR}$-varieties and, in particular, the covers of the atom $\mathrm{V}(\mathbf{C}_2)$.

Two finitely generated covers of the atom are the varieties $\mathrm{V}(\mathbf{L}_3)$ and $\mathrm{V}(\mathbf{H}_3)$, where $\mathbf{L}_3$ and $\mathbf{H}_3$ are (dually) isomorphic to the implication reducts of the three-element Lukasiewicz algebra and the three-element linearly ordered Heyting algebra, respectively. These are, in fact, varieties of BCK–algebras and it was recently proved [Kow95] that they are the only covers of the atom $\mathrm{V}(\mathbf{C}_2)$ in the lattice of varieties of BCK–algebras. In Section 2 we describe a third finitely generated cover of the atom in the lattice of $\mathcal{LR}$-varieties. This variety is generated by a five-element algebra $\mathbf{P}_5$; we prove that it, together with $\mathrm{V}(\mathbf{L}_3)$ and $\mathrm{V}(\mathbf{H}_3)$, are the only *finitely generated* covers of the atom in $\mathbf{P}^{\mathrm{V}}(\mathcal{LR})$.

In Section 3 we show that if $\mathcal{V}$ is an $\mathcal{LR}$-variety that covers the atom and contains a nonsimple subdirectly irreducible algebra whose smallest nontrivial ideal consists of two elements, then $\mathcal{V}$ is $\mathrm{V}(\mathbf{H}_3)$ or $\mathrm{V}(\mathbf{P}_5)$. Thus, we conclude that the only $\mathcal{LR}$-variety that covers the atom, has EDPC and is not semisimple is $\mathrm{V}(\mathbf{H}_3)$.

In Section 4 we investigate covers of the atom that are semisimple and have EDPC. We show that among $\mathcal{LR}$-varieties, those that are semisimple with EDPC are precisely the fixed point discriminator $\mathcal{LR}$-varieties. We present a countably infinite sequence $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots$ of infinite left residuation algebras that generate distinct $\mathcal{LR}$-varieties that are semisimple, have EDPC and cover the atom. Also, we show that the variety generated by all the $\mathbf{A}_i$'s is an $\mathcal{LR}$-variety that has $2^{\aleph_0}$ subvarieties, none of which is generated by its finite members, except $\mathrm{V}(\mathbf{C}_2)$ and the trivial variety. The algebra $\mathbf{A}_1$ coincides with the algebra defined in Example 1.13; we establish a finite axiomatization of the variety $\mathrm{V}(\mathbf{A}_1)$ in Section 5.

6.1. The Lattice of $\mathcal{LR}$-Varieties. Recall from Proposition 4.22 that every $\mathcal{LR}$-variety is congruence distributive. Thus, Jónsson's Theorem (Theorem 0.2) applies to every $\mathcal{LR}$-variety, that is, if $\mathcal{K}$ is a class of left residuation algebras and $\mathcal{V} = \mathrm{H\,S\,P}(\mathcal{K})$ is an $\mathcal{LR}$-variety, then $\mathcal{V}_{\mathrm{SI}} \subseteq \mathcal{V}_{\mathrm{FSI}} \subseteq \mathrm{H\,S\,P_U}(\mathcal{K})$; in particular, if $\mathcal{K}$ is a finite set of finite left residuation algebras and $\mathrm{H\,S\,P}(\mathcal{K})$ is an $\mathcal{LR}$-variety, then $\mathcal{V}_{\mathrm{SI}} \subseteq \mathcal{V}_{\mathrm{FSI}} \subseteq \mathrm{H\,S}(\mathcal{K})$.

In general the varietal join of two subvarieties of a quasivariety $\mathcal{K}$ need not be contained in $\mathcal{K}$. (An example can be extracted from [KM92, Example 6.6].) The proof of the following theorem is identical to the proof of [BR95, Theorem 11], which asserts the same result for the poset of all varieties of BCK–algebras. We include the proof for the sake of completeness.

Theorem 6.1. *Let $\mathcal{V}_1$, $\mathcal{V}_2$ be $\mathcal{LR}$-varieties. Then their varietal join $\mathrm{H\,S\,P}(\mathcal{V}_1 \cup \mathcal{V}_2)$ is also an $\mathcal{LR}$-variety. Consequently, the poset $\mathbf{P}^{\mathrm{V}}(\mathcal{LR})$ of all $\mathcal{LR}$-varieties is a lattice. Moreover,*

$$\mathcal{V}_1 \sqcup^{\mathbf{P}^{\mathrm{V}}(\mathcal{LR})} \mathcal{V}_2 = \mathrm{H\,S\,P}(\mathcal{V}_1 \cup \mathcal{V}_2) = \mathrm{I\,S\,P}((\mathcal{V}_1)_{\mathrm{SI}} \cup (\mathcal{V}_2)_{\mathrm{SI}}).$$

Proof. By Proposition 4.4, there are identities

$$x \dot{-} \Sigma_{i=1}^{n} t_i(x,y) \approx y \dot{-} \Sigma_{i=1}^{n} s_i(x,y), \tag{63}$$

$$x \dot{-} \Sigma_{i=1}^{n} u_i(x,y) \approx y \dot{-} \Sigma_{i=1}^{n} v_i(x,y), \tag{64}$$

$$t_i(x,x) \approx s_i(x,x) \approx u_i(x,x) \approx v_i(x,x) \approx 0, \quad \text{for } i = 1, \dots, n, \tag{65}$$

such that $\mathcal{V}_1$ satisfies (63), $\mathcal{V}_2$ satisfies (64) and $\mathcal{LR}$ satisfies (65). (For notational simplicity we use a uniform n; this loses no generality since $\mathcal{LR}$ satisfies $x \approx x \dot{-} (y \dot{-} y)$.) Now, let $\mathcal{V}$ be the variety of type $\langle 2, 0\rangle$ over the language $\langle\, \dot{-}\,, 0\rangle$ satisfying (65) and

$$\begin{aligned} & x \dot{-} \Sigma_{i=1}^{n} t_i(x,y) \dot{-} \Sigma_{j=1}^{n} u_j(x \dot{-} \Sigma_{i=1}^{n} t_i(x,y), y \dot{-} \Sigma_{i=1}^{n} s_i(x,y)) \\ \approx\ & y \dot{-} \Sigma_{i=1}^{n} s_i(x,y) \dot{-} \Sigma_{j=1}^{n} v_j(x \dot{-} \Sigma_{i=1}^{n} t_i(x,y), y \dot{-} \Sigma_{i=1}^{n} s_i(x,y)). \end{aligned} \tag{66}$$

Observe that $\mathcal{LR}$ satisfies

$$\begin{aligned} & u_j(x \dot{-} \Sigma_{i=1}^{n} t_i(x,x), x \dot{-} \Sigma_{i=1}^{n} s_i(x,x)) \approx u_j(x,x) \approx 0 \\ & \approx v_j(x,x) \approx v_j(x \dot{-} \Sigma_{i=1}^{n} t_i(x,x), x \dot{-} \Sigma s_i(x,x)) \end{aligned}$$

for $j = 1, \ldots, n$. This, together with (65) and Proposition 4.4(i), implies that $\mathcal{V}$ is an $\mathcal{LR}$-variety. Now, $\mathcal{V}_2$ satisfies (66), since (66) is a substitution instance of (64), while in $\mathcal{V}_1$, the left-hand side of (66) reduces, by (63) and (65), to

$$\begin{aligned} & x \dot{-} \Sigma_{i=1}^n t_i(x,y) \dot{-} \Sigma_{i=1}^n u_j(x \dot{-} \Sigma_{i=1}^n t_i(x,y), x \dot{-} \Sigma_{i=1}^n t_i(x,y)) \\ & \approx x \dot{-} \Sigma_{i=1}^n t_i(x,y) \dot{-} \Sigma_{i=1}^n 0 \approx x \dot{-} \Sigma_{i=1}^n t_i(x,y). \end{aligned}$$

Similarly, the right-hand side of (66) becomes $y \dot{-} \Sigma_{i=1}^n s_i(x,y)$ in $\mathcal{V}_1$, so that $\mathcal{V}_1$ satisfies (66), as a consequence of (63). Thus $\mathcal{V}_1 \cup \mathcal{V}_2 \subseteq \mathcal{V}$. Write $\mathcal{V}_1 \sqcup \mathcal{V}_2$ for the join of $\mathcal{V}_1$ and $\mathcal{V}_2$ in the lattice of all varieties of type $\langle 2,0 \rangle$. Clearly, $\mathcal{V}_1 \sqcup \mathcal{V}_2$ is an $\mathcal{LR}$-variety. Thus $\mathcal{V}_1 \sqcup \mathcal{V}_2$ is a congruence distributive variety. By Corollary 0.3, $(\mathcal{V}_1 \sqcup \mathcal{V}_2)_{\mathrm{SI}} = (\mathcal{V}_1)_{\mathrm{SI}} \cup (\mathcal{V}_2)_{\mathrm{SI}}$, hence

$$\mathcal{V}_1 \sqcup \mathcal{V}_2 \subseteq \mathrm{I\,S\,P}((\mathcal{V}_1 \sqcup \mathcal{V}_2)_{\mathrm{SI}}) = \mathrm{I\,S\,P}((\mathcal{V}_1)_{\mathrm{SI}} \cup (\mathcal{V}_2)_{\mathrm{SI}}) \subseteq \mathcal{V}_1 \sqcup \mathcal{V}_2.$$

□

Corollary 6.2. $\mathbf{P}^{\mathrm{V}}(\mathcal{LR})$ *is a distributive lattice.*

Proof. Let $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3$ be $\mathcal{LR}$-varieties. By the previous theorem, $\mathcal{V} = \mathrm{H\,S\,P}(\mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}_3)$ is an $\mathcal{LR}$-variety, so $\mathcal{V}$ is a congruence distributive variety and the poset $\mathbf{P}^{\mathrm{V}}(\mathcal{V})$ of subvarieties of $\mathcal{V}$ is clearly a sublattice of $\mathbf{P}^{\mathrm{V}}(\mathcal{LR})$. Since $\mathcal{V}$ is congruence distributive, $\mathbf{P}^{\mathrm{V}}(\mathcal{V})$ is a distributive lattice, by Corollary 0.4. Thus,

$$\mathcal{V}_1 \cap (\mathcal{V}_2 \sqcup \mathcal{V}_3) = (\mathcal{V}_1 \cap \mathcal{V}_2) \sqcup (\mathcal{V}_1 \cap \mathcal{V}_3),$$

where $\sqcup$ is the join operation of $\mathcal{V}$, hence also of $\mathbf{P}^{\mathrm{V}}(\mathcal{LR})$, as required. □

It is well known that there are $2^{\aleph_0}$ varieties of BCK–algebras. Since $\mathcal{LR}$ has finite type, it follows that $|\mathbf{P}^{\mathrm{V}}(\mathcal{LR})| = 2^{\aleph_0}$ also. In [WK84], Wroński and Kabziński proved that there is no largest variety of BCK–algebras, by constructing a denumerable sequence $\mathbf{B}_0$, $\mathbf{B}_1$, $\mathbf{B}_2$, ... of finite BCK–algebras, a subalgebra $\mathbf{B}$ of an ultraproduct of $\{\mathbf{B}_i : i \in \omega\}$ and a homomorphic image $\mathbf{D}$ of $\mathbf{B}$ that violates the quasi-identity (A4) and is therefore not in $\mathcal{LR}$. We have mentioned that the variety generated by any finite BCK–algebra consists of BCK–algebras; in particular $\mathrm{V}(\mathbf{B}_i)$ is an $\mathcal{LR}$-variety for each $i \in \omega$. This argument therefore shows also that the lattice $\mathbf{P}^{\mathrm{V}}(\mathcal{LR})$ *has no greatest element.* Nevertheless, just as in the case of BCK–algebras, we obtain the following result, whose proof reproduces that of [BR95, Corollary 12]:

Corollary 6.3. *The lattice* $\mathbf{P}^{\mathrm{V}}(\mathcal{LR})$ *contains cofinal chains of order type* ω.

Proof. The "set" of all $\mathcal{LR}$-varieties that are axiomatized, relative to $\mathcal{LR}$, by an identity of the form of (63), where $\mathcal{LR}$ satisfies $t_i(x,x) \approx 0 \approx s_i(x,x)$, is cofinal in $\mathbf{P}^{\mathrm{V}}(\mathcal{LR})$, by Proposition 4.4(i), and may be enumerated, say as $\mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2, \ldots$. Define $\mathcal{W}_0 = \mathcal{V}_0$ and for each $n \in \omega$, choose $\mathcal{W}_{n+1}$ to be the variety constructed as in the proof of Theorem 6.1, containing $\mathcal{W}_n$ and $\mathcal{V}_{n+1}$. (Note that $\mathcal{W}_{n+1}$ is one of the $\mathcal{V}_m$.) The sequence $\mathcal{W}_0, \mathcal{W}_1, \mathcal{W}_2, \ldots$ is a cofinal chain in $\mathbf{P}^{\mathrm{V}}(\mathcal{LR})$. $\square$

6.2. Finitely Generated Covers of the Atom. Recall from Section 4.1 that the unique two-element left residuation algebra $\mathbf{C}_2$ is embeddable in every nontrivial left residuation algebra, hence the variety $\mathrm{V}(\mathbf{C}_2)$ (which is a variety of BCK–algebras) is the smallest nontrivial $\mathcal{LR}$-variety, i.e., it is the unique atom of the lattice $\mathbf{P}^{\mathrm{V}}(\mathcal{LR})$. Up to isomorphism, there are exactly two subdirectly irreducible three-element left residuation algebras, namely $\mathbf{L}_3 = \langle\{0,1,2\};\ \dot{-}\,,0\rangle$, where $0 \leq 1 \leq 2$ and $2 \dot{-} 1 = 1$ (thus, $\mathbf{L}_3$ is just the ordinal algebra $\mathbf{3}_{\{\dot{-}\}}$ of Example 2.17); and $\mathbf{H}_3 = \langle\{0,1,2\};\ \dot{-}\,,0\rangle$, where $0 \leq 1 \leq 2$ and $2 \dot{-} 1 = 2$. The algebras $\mathbf{L}_3$ and $\mathbf{H}_3$ are (dually) isomorphic to the implication reducts of the three-element Łukasiewicz algebra and the three-element linearly ordered Heyting algebra, respectively, and are therefore BCK–algebras. It follows from Jónsson's Theorem that $\mathrm{V}(\mathbf{L}_3)$ and $\mathrm{V}(\mathbf{H}_3)$ are covers of the atom $\mathrm{V}(\mathbf{C}_2)$ in the lattice of varieties of BCK–algebras. Of course, $\mathrm{V}(\mathbf{L}_3)$ and $\mathrm{V}(\mathbf{H}_3)$ are therefore also (finitely generated) covers of the atom $\mathrm{V}(\mathbf{C}_2)$ in the lattice $\mathbf{P}^{\mathrm{V}}(\mathcal{LR})$. We shall now describe a third finitely generated cover.

Let $\langle A; \leq\rangle$ be the poset depicted in Figure 8 and define a binary operation $\oplus$ on A by the rules (where $x, y \in A$): $0 \oplus x = x = x \oplus 0$; $x \oplus y = e$ if $x, y \in \{e, d\}$; $c \oplus c = c$; $c \oplus d = c \oplus b = b$; and $x \oplus y = a$ in all the remaining cases. $\langle A; \oplus, 0; \leq\rangle$ is an integral pomonoid, residuated on the left as follows (where $x, y \in A$): $x \dot{-} 0 = x$; $x \dot{-} y = 0$ if $x \leq y$; $a \dot{-} b = a \dot{-} c = b \dot{-} c = d \dot{-} c = e \dot{-} c = e \dot{-} d = e \dot{-} b = d$; $a \dot{-} e = b \dot{-} e = c \dot{-} e = b \dot{-} d = c \dot{-} d = c$ and $a \dot{-} d = b$.

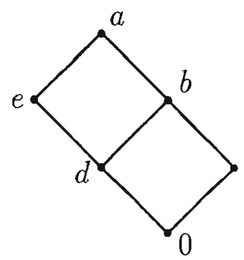


Figure 8.

Thus, $\langle A; \oplus, \dot{-}\,, 0\rangle$ is a polrim with associated order $\leq$. $\mathbf{A}$ has a subalgebra with universe $\{0, d, e\}$ that is isomorphic to the ordinal algebra $\mathbf{3}_{\{\oplus, \dot{-}\}}$. Its only other nontrivial proper subuniverses are $\{0, a\}$, $\{0, c\}$ and $\{0, e\}$. The $\langle \dot{-}\,, 0\rangle$–reduct $\mathbf{A}_{\{\dot{-}\}}$ of $\mathbf{A}$ has as a further subuniverse the set $P_5 =$

$\{0,a,b,c,d\}$. The algebra $\mathbf{P}_5 = \langle P_5; \dot{-}, 0\rangle$ is depicted in Figure 9; its subalgebra on $\{0,b,c,d\}$ is an isomorphic copy of $\mathbf{C}_2 \times \mathbf{C}_2$. An algebra isomorphic to $\mathbf{P}_5$ shall be called a *pendent square.* Notice that every proper subalgebra of $\mathbf{P}_5$ is embeddable in $\mathbf{C}_2 \times \mathbf{C}_2$.

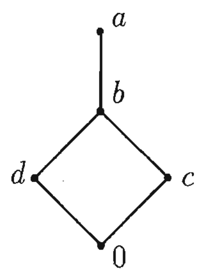


Figure 9.

One may check that $\mathbf{P}_5$ satisfies

$$x \dot{-} 2(x \dot{-} y \dot{-} y) \dot{-} 2(x \dot{-} y) \dot{-} 2(y \dot{-} x \dot{-} x) \dot{-} 2(y \dot{-} x) \approx$$
$$y \dot{-} 2(y \dot{-} x \dot{-} x) \dot{-} 2(y \dot{-} x) \dot{-} 2(x \dot{-} y \dot{-} y) \dot{-} 2(x \dot{-} y),$$

which is an identity of the form of (27), hence $\mathrm{V}(\mathbf{P}_5)$ is an $\mathcal{LR}$-variety, by Proposition 4.4(i). By Jónsson's Theorem, $\mathrm{V}(\mathbf{P}_5)_{\mathrm{SI}} \subseteq \mathrm{H\,S}(\mathbf{P}_5)$. Each proper subalgebra of $\mathbf{P}_5$ is in $\mathrm{V}(\mathbf{C}_2)$ and the only nontrivial ideals of $\mathbf{P}_5$ are P_5 and $\{0,d\}$. Since $|P_5/\Theta^{\mathbf{P}_5}(0,d)| = 2$, the (nonisomorphic) subdirectly irreducible algebras in $\mathrm{H\,S}(\mathbf{P}_5)$ are just $\mathbf{P}_5$ and $\mathbf{C}_2$. Thus $\mathrm{V}(\mathbf{P}_5)_{\mathrm{SI}} = \mathrm{I}\{\mathbf{P}_5, \mathbf{C}_2\}$, so $\mathrm{V}(\mathbf{P}_5)$ is a cover of the atom $\mathrm{V}(\mathbf{C}_2)$ distinct from $\mathrm{V}(\mathbf{H}_3)$ and $\mathrm{V}(\mathbf{L}_3)$. We remark that $\mathrm{V}(\mathbf{P}_5)$ does not have the IEP (nor the CEP, therefore): the subalgebra $\langle\{0,b,c,d\}; \dot{-}, 0\rangle$ of $\mathbf{P}_5$ has an ideal $\{0,c\}$ and no ideal of $\mathbf{P}_5$ meets $\{0,b,c,d\}$ in $\{0,c\}$.

We shall prove the following:

Theorem 6.4. *The only finitely generated $\mathcal{LR}$-varieties that cover the atom* $\mathrm{V}(\mathbf{C}_2)$ *are* $\mathrm{V}(\mathbf{H}_3)$, $\mathrm{V}(\mathbf{L}_3)$ *and* $\mathrm{V}(\mathbf{P}_5)$.

We recall here some facts concerning the variety $\mathrm{V}(\mathbf{C}_2)$ of *Tarski algebras.* Most of these derive from the fact that Tarski algebras are exactly the $\langle \dot{-}, 0\rangle$-subreducts of Boolean algebras $\langle A; \sqcap, \sqcup, ', 0, 1\rangle$ (where $x \dot{-} y = x \sqcap (y')$).

(i) The variety $\mathrm{V}(\mathbf{C}_2)$ consists of all *BCK-algebras* satisfying

(67) $$x \dot{-} (y \dot{-} x) \approx x.$$

It follows by Lemma 1.6 that $\mathrm{V}(\mathbf{C}_2)$ is axiomatized by (67) and

(A1) $x \dot{-} y \dot{-} (z \dot{-} y) \dot{-} (x \dot{-} z) \approx 0$

(A2) $x \dot{-} 0 \approx x$

(A3) $\quad 0 \dot{-} x \approx 0$

(A4) $\quad x \dot{-} y \approx 0$ and $y \dot{-} x \approx 0$ implies $x \approx y$

(A15) $\quad x \dot{-} (x \dot{-} y) \dot{-} y \approx 0.$

The quasi-identity (A4) can be replaced in this axiomatization by

$$x \dot{-} (x \dot{-} y) \approx y \dot{-} (y \dot{-} x), \tag{68}$$

which is of the form of (27) and accounts for the fact that $\mathrm{V}(\mathbf{C}_2)$ is an $\mathcal{LR}$-variety. Moreover, in the presence of the other axioms, (67) and (68) render (A2) and (A15) redundant, so $\mathrm{V}(\mathbf{C}_2)$ is axiomatized by (A1), (A3), (67) and (68).

(ii) The partial order defined on an algebra in $\mathrm{V}(\mathbf{C}_2)$ by $x \leq y$ iff $x \dot{-} y \approx 0$ is a meet-semilattice order; the meet operation is definable by $x \sqcap y = x \dot{-} (x \dot{-} y)$ $(\approx y \dot{-} (y \dot{-} x))$. Also, $\mathrm{V}(\mathbf{C}_2)$ satisfies $x \dot{-} (x \sqcap y) \approx x \dot{-} y$.

Let $\mathbf{A} \in \mathrm{V}(\mathbf{C}_2)$.

(iii) If $a, b \in A$ have an upper bound in $\langle A; \leq \rangle$, then they have a least upper bound $a \sqcup b$, in particular,

$$a \sqcup b = \sqcap \{c \in A : c \text{ is an upper bound of } a \text{ and } b\}.$$

(iv) If $a, b_1, \ldots, b_n \in A$ and $b_1 \sqcup \ldots \sqcup b_n$ exists in $\langle A; \leq \rangle$, then

$$a \dot{-} (b_1 \sqcup \ldots \sqcup b_n) = (a \dot{-} b_1) \sqcap \ldots \sqcap (a \dot{-} b_n).$$

(v) For each $a \in A$, $(\boldsymbol{a}] = \langle (a]; \sqcap, \sqcup, ', 0, a \rangle$ is a Boolean algebra, where $\sqcap$ and $\sqcup$ are the meet and join operations corresponding to the partial order $\leq$ on A, and $b' = a \dot{-} b$ for all $b \in (a]$. In particular, when $\mathbf{A}$ is a finite algebra, $(\boldsymbol{a}]$ is a *finite* Boolean algebra for each $a \in A$.

(vi) Let $\mathbf{A}$ be a finite algebra in $\mathrm{V}(\mathbf{C}_2)$. Then each $a \in A$ is the join of all the atoms e of the (finite) Boolean algebra $(\boldsymbol{a}]$.

Lemma 6.5. *If $\mathbf{A}$ is a left residuation algebra that satisfies* (67), *then $\mathbf{A}$ also satisfies $x \dot{-} y \dot{-} y \approx x \dot{-} y$.*

Proof. By two applications of (67), $\mathbf{A}$ satisfies

$$x \dot{-} y \approx x \dot{-} y \dot{-} (y \dot{-} (x \dot{-} y)) \approx x \dot{-} y \dot{-} y.$$

□

We shall make use the fact that, owing to (A12) (see page 18), if $\mathbf{A}$ is a left residuation algebra and B is a hereditary subset of A (with respect to the associated partial order), then B is a subuniverse of $\mathbf{A}$.

Let $\mathcal{V}$ be a finitely generated $\mathcal{LR}$-variety that is a cover of the atom $\mathrm{V}(\mathbf{C}_2)$. Then $\mathcal{V}$ is generated by a single finite left residuation algebra $\mathbf{A}$. Moreover, $\mathbf{A}$ can be chosen in such a way that no proper subalgebra of $\mathbf{A}$ generates $\mathcal{V}$. Thus, every proper subalgebra of $\mathbf{A}$ is in $\mathrm{V}(\mathbf{C}_2)$ and, therefore, satisfies (A15), (67) and (A14) (see page 22). Let a be a maximal element of $\mathbf{A}$. Then $A\backslash\{a\}$ is a hereditary subset of A, hence it is the universe of a proper subalgebra of $\mathbf{A}$, which must lie in $\mathrm{V}(\mathbf{C}_2)$.

Assume that $\mathcal{V}$ is neither $\mathrm{V}(\mathbf{L}_3)$ nor $\mathrm{V}(\mathbf{H}_3)$. Thus, $\mathbf{A}$ contains no linearly ordered three-element subalgebra. The following four lemmas prove that $\mathbf{A}$ must contain a pendent square, which proves Theorem 6.4.

Lemma 6.6. $\mathbf{A}$ *has a largest element.*

Proof. Suppose that $\mathbf{A}$ has more than one maximal element. We shall infer that $\mathbf{A}$ satisfies (67) and (A15), i.e., that $\mathbf{A} \in \mathrm{V}(\mathbf{C}_2)$ (by (i), page 113), a contradiction.

Claim 1. $\mathbf{A}$ satisfies (67): $x \dot{-} (y \dot{-} x) \approx x$.
Let a be any maximal element. First, we show that for any other maximal element b, we have $a \dot{-} (b \dot{-} a) = a$.

Case 1: $(a] = \{0, a\}$.
Since a and b are maximal, $a \not\leq b$. Thus, $a \not\leq b \dot{-} a$ (since $b \dot{-} a \leq b$), hence $a \dot{-} (b \dot{-} a) \neq 0$. But $a \dot{-} (b \dot{-} a) \in (a]$, hence $a \dot{-} (b \dot{-} a) = a$.

Case 2: $(a]$ contains more than two elements.
Since $(\boldsymbol{a}]$ is a proper subalgebra of $\mathbf{A}$, $(\boldsymbol{a}] \in \mathrm{V}(\mathbf{C}_2)$. By (v) (see page 114), $(\boldsymbol{a}]$ is a finite Boolean algebra which, by assumption, has more than one atom. Let $e_1, \ldots, e_n$ be the atoms of $(\boldsymbol{a}]$; so $a = e_1 \sqcup \ldots \sqcup e_n$. Since $b, e_i \in A\backslash\{a\}$, $e_i \dot{-} (b \dot{-} e_i) = e_i$ for each i, by (67). Now, for each i, $b \dot{-} a \leq b \dot{-} e_i$, by (A11) (see page 18), so, by (A11) and (A10),

$$e_i \dot{-} (b \dot{-} e_i) \leq e_i \dot{-} (b \dot{-} a) \leq a \dot{-} (b \dot{-} a),$$

so $e_i \leq a \dot{-} (b \dot{-} a)$. Thus

$$a = e_1 \sqcup \ldots \sqcup e_n \leq a \dot{-} (b \dot{-} a),$$

so $a = a \dot{-} (b \dot{-} a)$.

Since a, b were arbitrary, it follows that $a \dot{-} (b \dot{-} a) = a$ for all maximal elements a, b. Suppose $c, d \in A$ are not both maximal elements. Let a be a maximal element different from c, d. Then $c, d \in A\backslash\{a\}$, hence $c \dot{-} (d \dot{-} c) = c$. We therefore conclude that $\mathbf{A}$ satisfies (67).

By Lemma 6.5, $\mathbf{A}$ satisfies $x \dot{-} y \dot{-} y \approx x \dot{-} y$.

Claim 2. **A** satisfies (A15): $x \dot{-} (x \dot{-} y) \dot{-} y \approx 0$.
If $a, b \in A$ are such that at most one of a, b is maximal, then there exists another maximal element $c \in A$ such that $a, b \in A \backslash \{c\}$, hence $a \dot{-} (a \dot{-} b) \dot{-} b = 0$, by (A15). So, let a, b be distinct maximal elements. If $a \dot{-} b = a$ then $a \dot{-} (a \dot{-} b) \dot{-} b = 0 \dot{-} b = 0$, as required. Suppose that $a \dot{-} b < a$. Note that $a \dot{-} (a \dot{-} b) \neq a$ otherwise $\{0, a \dot{-} b, a\}$ is a linearly ordered three-element subuniverse of **A**, contradicting our assumptions about V(**A**), so $a \dot{-} (a \dot{-} b) < a$. Thus,

$$\begin{aligned} & a \dot{-} (a \dot{-} b) \dot{-} b \\ = \;& (a \dot{-} (a \dot{-} b)) \dot{-} (a \dot{-} b) \dot{-} b \quad \text{(by Lemma 6.5)} \\ = \;& (a \dot{-} (a \dot{-} b)) \dot{-} b \dot{-} (a \dot{-} b) \quad \text{(by (A14), since } a \dot{-} (a \dot{-} b), a \dot{-} b, b \in A\backslash\{a\}) \\ \leq \;& a \dot{-} b \dot{-} (a \dot{-} b) = 0. \end{aligned}$$

Since a, b were arbitrary, it follows that $a \dot{-} (a \dot{-} b) \dot{-} b = 0$ for all maximal elements of A, hence **A** satisfies (A15). Thus, $\mathbf{A} \in \mathrm{V}(\mathbf{C}_2)$, a contradiction.
□

Let a be the maximum element of **A**. Let $e_1, \ldots, e_n$ be the atoms of $A \backslash \{a\}$ and note that $n > 1$. Since **A** satisfies $x \dot{-} y \leq x$, we have $e_i \dot{-} b = e_i$ for each atom e_i and each $b \in A \backslash \{a\}$ for which $e_i \not\leq b$. Note that for each $b \in A \backslash \{0, a\}$, $a \dot{-} b \neq a$ and $a \dot{-} b \neq b$, otherwise $\{0, b, a\}$ is a linearly ordered three-element subuniverse of **A**, contradicting our assumptions about V(**A**).

Lemma 6.7. *Let I be a proper subset of $\{1, \ldots, n\}$. Then the set $\{e_i : i \in I\}$ has a least upper bound in $A \backslash \{a\}$.*

Proof. We consider only the case $I = \{1, \ldots, n-1\}$. For each $i \in I$, $a \dot{-} e_n \geq e_i \dot{-} e_n = e_i$, hence $a \dot{-} e_n$ is an upper bound for $\{e_1, \ldots, e_{n-1}\}$. Since $a \dot{-} e_n, e_1, \ldots, e_{n-1} \in A \backslash \{a\}$, $\{e_1, \ldots, e_{n-1}\}$ has a least upper bound in $A \backslash \{a\}$, by (iii) (see page 114). □

Lemma 6.8. *The set $\{e_1, \ldots, e_n\}$ has a least upper bound in $A \backslash \{a\}$.*

Proof. Suppose, on the contrary, that $\{e_1, \ldots, e_n\}$ does not have an upper bound in $A \backslash \{a\}$. Then a is the only upper bound of $\{e_1, \ldots, e_n\}$ in A. We show that this implies that $\mathbf{A} \in \mathrm{V}(\mathbf{C}_2)$.

Claim 1. **A** satisfies (67): $x \dot{-} (y \dot{-} x) \approx x$.
If $b, c \in A \backslash \{a\}$, then $b \dot{-} (c \dot{-} b) = b$. For an atom e_i of **A**, we have $a \dot{-} (e_i \dot{-} a) = a \dot{-} 0 = a$. Also, for $j \neq i$, $a \dot{-} e_i \geq e_j \dot{-} e_i = e_j$, so

$$a \dot{-} e_i \geq \bigsqcup \{e_j : j \neq i\} \in A \backslash \{a\} \quad \text{(by Lemma 6.7).}$$

Since $a \dot{-} e_i \neq a$, $a \dot{-} e_i \in A \backslash \{a\}$. Also, $a \dot{-} e_i \not\geq e_i$, otherwise $a \dot{-} e_i$ is an upper bound for $\{e_1, \ldots, e_n\}$ in $A \backslash \{a\}$, contradicting our assumption. Thus, by (vi) (see page 114), $a \dot{-} e_i = \bigsqcup \{e_j : j \neq i\}$, so

$$\begin{aligned} e_i \dot{-} (a \dot{-} e_i) &= e_i \dot{-} (\bigsqcup \{e_j : j \neq i\}) \\ &= \bigsqcap \{e_i \dot{-} e_j : j \neq i\} \text{ (by (iv), page 114)} \\ &= e_i. \end{aligned}$$

Now let $b \in A \backslash \{a\}$ and, without loss of generality, assume b is the join of distinct atoms $e_1, \ldots, e_m$, where $m > 1$. Note that $a \dot{-} (b \dot{-} a) = a \dot{-} 0 = a$. For $i = 1, \ldots, m$, $e_i \leq b$, so $a \dot{-} b \leq a \dot{-} e_i$, hence

$$b \dot{-} (a \dot{-} b) \geq b \dot{-} (a \dot{-} e_i) \geq e_i \dot{-} (a \dot{-} e_i) = e_i.$$

Thus $b \dot{-} (a \dot{-} b) \geq e_1 \sqcup \ldots \sqcup e_m = b$, so $b \dot{-} (a \dot{-} b) = b$, which completes the proof of the claim.

Claim 2. $\mathbf{A}$ satisfies (A15): $x \dot{-} (x \dot{-} y) \dot{-} y \approx 0$.
The only nontrivial case to verify is $a \dot{-} (a \dot{-} b) \dot{-} b = 0$, where $b \in A \backslash \{a, 0\}$. Now

$$\begin{aligned} & a \dot{-} (a \dot{-} b) \dot{-} b \\ = \ & (a \dot{-} (a \dot{-} b)) \dot{-} (a \dot{-} b) \dot{-} b \text{ (by Lemma 6.5)} \\ = \ & (a \dot{-} (a \dot{-} b)) \dot{-} b \dot{-} (a \dot{-} b) \text{ (by (A14), since } a \dot{-} (a \dot{-} b), a \dot{-} b, b \in A \backslash \{a\}) \\ \leq \ & a \dot{-} b \dot{-} (a \dot{-} b) = 0, \end{aligned}$$

as required. Thus $\mathbf{A} \in \mathrm{V}(\mathbf{C}_2)$, which completes the proof of the claim and also the lemma. □

By the above lemma, there exists an element $b \in A \backslash \{a\}$ that is the join of all the atoms of $A \backslash \{a\}$. Since every $c \in A \backslash \{a\}$ is the join of the atoms less than or equal to c (by (vi), page 114), b is the maximum element of $A \backslash \{a\}$.

Lemma 6.9. *The pendent square* $\mathbf{P}_5$ *is embeddable into* $\mathbf{A}$.

Proof. Set $c = a \dot{-} b$, so $c \in A \backslash \{0, a, b\}$. We shall show that $\{0, a, b, c, c'\}$ is the universe of a subalgebra of $\mathbf{A}$ isomorphic to $\mathbf{P}_5$, where $c' = b \dot{-} c$ is the Boolean complement of c in $(\boldsymbol{b}]$. By (v) (see page 114), $b \dot{-} c' = c \dot{-} c' = c$ and $b \dot{-} c = c' \dot{-} c = c'$.

By (A1), $a \dot{-} c' \dot{-} (b \dot{-} c') \dot{-} (a \dot{-} b) = 0$, hence $a \dot{-} c' \dot{-} c \dot{-} c = 0$. Since $a \dot{-} c', c \in A \backslash \{a\}$, $0 = a \dot{-} c' \dot{-} c \dot{-} c = a \dot{-} c' \dot{-} c$, hence $a \dot{-} c' \leq c$. But $a \dot{-} c' \geq b \dot{-} c' = c$, so $a \dot{-} c' = c$.

Now, $a \dot{-} c \in A \backslash \{a\}$, so $a \dot{-} c \leq b$. Also, $a \dot{-} c \geq b \dot{-} c = c'$ and $a \dot{-} c \geq a \dot{-} b = c$, hence $a \dot{-} c \geq c \sqcup c' = b$. Thus $a \dot{-} c = b$. □

This completes the proof of Theorem 6.4. In fact, it follows that among the subvarieties of $\mathcal{LR}$ that are generated by their finite members, only $\mathrm{V}(\mathbf{H}_3)$, $\mathrm{V}(\mathbf{L}_3)$ and $\mathrm{V}(\mathbf{P}_5)$ cover $\mathrm{V}(\mathbf{C}_2)$.

Corollary 6.10. *Let $\mathcal{V}$ be an $\mathcal{LR}$-variety distinct from $\mathrm{V}(\mathbf{H}_3)$, $\mathrm{V}(\mathbf{L}_3)$ and $\mathrm{V}(\mathbf{P}_5)$ such that $\mathcal{V}$ covers $\mathrm{V}(\mathbf{C}_2)$. Then every finite algebra in $\mathcal{V}$ is a Tarski algebra.*

Proof. By the above, if $\mathcal{V}_{\text{fin}}$ is the class of finite algebras in $\mathcal{V}$ then $\mathbf{C}_2 \in \mathrm{H\,S\,P}(\mathcal{V}_{\text{fin}})$ which is a proper subclass of $\mathcal{V}$, so $\mathrm{H\,S\,P}(\mathcal{V}_{\text{fin}}) = \mathrm{V}(\mathbf{C}_2)$. $\square$

6.3. Other Covers of the Atom. Let $\mathbf{A}$ be an algebra in $\mathcal{LR}_{\mathrm{RSI}}$ (i.e., in $\mathcal{LR}_{\mathrm{SI}}$, by Proposition 4.22). By Proposition 4.14, there exists a unique nonzero ideal of $\mathbf{A}$ that is the intersection of all nonzero ideals of $\mathbf{A}$; we call this ideal the *monolith* of $\mathbf{A}$. Recall that a variety $\mathcal{V}$ is *semisimple* if every subdirectly irreducible algebra in $\mathcal{V}$ is simple, i.e., $\mathcal{V}_{\mathrm{SI}} = \mathcal{V}_{\mathrm{S}}$.

Proposition 6.11. *Let $\mathcal{V}$ be an $\mathcal{LR}$-variety that is a cover of the atom $\mathrm{V}(\mathbf{C}_2)$. If $\mathcal{V}$ contains a subdirectly irreducible and nonsimple algebra whose monolith consists of two elements, then $\mathcal{V}$ is $\mathrm{V}(\mathbf{H}_3)$ or $\mathrm{V}(\mathbf{P}_5)$.*

Proof. Let $\mathbf{B}$ be a subdirectly irreducible and nonsimple algebra in $\mathcal{V}$ whose monolith is $I = \{0, e\}$, say. So e covers 0 in the partial order of B and $e \dot{-} b \in I$ for all $b \in B$.

Observe that for $a \in B \backslash I$ and $m, n \in \omega$,

$$a \dot{-} ne \dot{-} (a \dot{-} me) \in I, \tag{69}$$

since $a \dot{-} ne \dot{-} (a \dot{-} me) \leq a \dot{-} (a \dot{-} me) \in I$, by Lemma 4.17.

For each $a \in B \backslash I$, set $B_a = I \cup \{a \dot{-} ne : n \in \omega\}$. It follows from (69) that B_a is a subuniverse of $\mathbf{B}$.

Suppose there exists $a \in B \backslash I$ such that B_a is infinite. Then, for all $n \in \omega$, $a \dot{-} (n+1)e < a \dot{-} ne$. Moreover, $a \dot{-} ne \dot{-} (a \dot{-} (n+1)e) \in \langle e \rangle_{\mathbf{B}} = I$, so $a \dot{-} ne \dot{-} (a \dot{-} (n+1)e) = e$. For each $n \in \omega$, let a_n denote $a \dot{-} ne$.

Let $u(x, y)$ be any $\langle \dot{-} \rangle$-term such that $\mathcal{LR}$ satisfies $u(x, x) \approx 0$. We claim that $u(a_n, a_m) \in I$ for all $n, m \in \omega$. By (A4), $\mathcal{LR}$ satisfies the quasi-identity: $x \dot{-} y \approx 0$ **and** $y \dot{-} x \approx 0$ **implies** $u(x, y) \approx 0$. Since $I \in \mathrm{Id}\,\mathbf{B}$ and $a_n \dot{-} a_m, a_m \dot{-} a_n \in I$ by (69), it follows that $u(a_n, a_m) \in I$ (see page 72).

Let $x \dot{-} \Sigma_{i=1}^{k} u_i(x, y) \approx y \dot{-} \Sigma_{j=1}^{l} v_j(x, y)$ be any identity for which $\mathcal{LR}$ satisfies $u_i(x, x) \approx 0 \approx v_j(x, x)$ for each i, j. By the above claim, for each i and all

$m, n \in \omega$, $u_i(a_n, a_m) \leq e$, so

$$a_0 \dot{-} \Sigma_{i=1}^{k} u_i(a_0, a_{k+1}) \geq a_0 \dot{-} ke = a_k > a_{k+1} \geq a_{k+1} \dot{-} \Sigma_{j=1}^{l} v_j(a_0, a_{k+1}),$$

so $\mathbf{B}$ fails to satisfy any identity of the form (27). This contradicts Proposition 4.4(i) since $\mathcal{V}$ is an $\mathcal{LR}$-variety. Thus, for each $a \in B \backslash I$, B_a is a finite subuniverse of $\mathbf{B}$.

For each $a \in B \backslash I$, let $\mathbf{B}_a$ denote the algebra $\langle B_a; \dot{-}, 0\rangle$. Since $\mathcal{V}$ is a cover of $\mathrm{V}(\mathbf{C}_2)$ and $\mathbf{B}_a \in \mathcal{V}$, $\mathrm{V}(\mathbf{B}_a)$ is either $\mathcal{V}$ or $\mathrm{V}(\mathbf{C}_2)$. If, for some $a \in B \backslash I$, $\mathrm{V}(\mathbf{B}_a) = \mathcal{V}$, then $\mathcal{V}$ is finitely generated, hence $\mathcal{V}$ is $\mathrm{V}(\mathbf{H}_3)$, $\mathrm{V}(\mathbf{L}_3)$ or $\mathrm{V}(\mathbf{P}_5)$. We can exclude $\mathrm{V}(\mathbf{L}_3)$, however, since it is semisimple.

Suppose that for each $a \in B \backslash I$, $\mathrm{V}(\mathbf{B}_a) = \mathrm{V}(\mathbf{C}_2)$. Set $J = \{b \in B : e \not\leq b\}$. Note the following:

(i) If $b \in J$ (i.e., $e \not\leq b$), $c \in B$ and $c \leq b$, then $c \in J$ as well.

(ii) If $b \notin J$ then $b \dot{-} e \in J$: since $e, b \in B_b \in \mathrm{V}(\mathbf{C}_2)$, (67) implies that $e \dot{-} (b \dot{-} e) = e \neq 0$, so $e \not\leq b \dot{-} e$.

(iii) If $b \in J$ then $e \dot{-} b = e$: $e \dot{-} b \in \{0, e\}$ and $e \dot{-} b \neq 0$ since $e \not\leq b$.

Claim. J is a nonzero ideal of $\mathbf{B}$.
To see this, first note that $0 \in J$. Let $b, c \in J$ and $a \in B$. If $a \in J$ then $a \dot{-} (a \dot{-} b \dot{-} c) \leq a \in J$, hence $a \dot{-} (a \dot{-} b \dot{-} c) \in J$, by (i).

If $a \notin J$, then $e \leq a$, so

$$\begin{aligned} a \dot{-} (a \dot{-} b \dot{-} c) \leq a \dot{-} (e \dot{-} b \dot{-} c) &= a \dot{-} (e \dot{-} c) \quad \text{(by (iii))} \\ &= a \dot{-} e \quad \text{(by (iii))} \\ &\in J \quad \text{(by (ii))}, \end{aligned}$$

hence $a \dot{-} (a \dot{-} b \dot{-} c) \in J$, by (i). By Lemma 4.17(iii), J is an ideal of $\mathbf{B}$. Also, J is not $\{0\}$ since for any $b \in B \backslash I$, either $b \in J$ or $0 \neq b \dot{-} e \in J$, by (ii).

Now $e \notin J$, so $I \not\subseteq J$, contradicting the assumption that I is the monolith of $\mathbf{B}$. □

Corollary 6.12. *The only $\mathcal{LR}$-variety that covers* $\mathrm{V}(\mathbf{C}_2)$*, has EDPC and is not semisimple is* $\mathrm{V}(\mathbf{H}_3)$.

Proof. Let $\mathcal{V}$ be an $\mathcal{LR}$-variety that covers $\mathrm{V}(\mathbf{C}_2)$, has EDPC and is not semisimple. Note that $\mathcal{V}$ is not generated by a simple algebra, otherwise, by Proposition 5.19(i), $\mathcal{V}$ would be semisimple. Thus, since $\mathcal{V}$ covers $\mathrm{V}(\mathbf{C}_2)$, each simple algebra in $\mathcal{V}$ must lie in $\mathrm{V}(\mathbf{C}_2)$, so, up to isomorphism, the only simple algebra in $\mathcal{V}$ is $\mathbf{C}_2$.

Let $\mathbf{A} \in \mathcal{V}_{\mathrm{SI}} \backslash \mathcal{V}_{\mathrm{S}}$ and let I be the monolith of $\mathbf{A}$. We claim that the subalgebra $\mathbf{I} = \langle I; \dot{-}, 0\rangle$ of $\mathbf{A}$ is simple. Let J be an ideal of $\mathbf{I}$. Since $\mathcal{V}$ has EDPC, it follows from Proposition 5.1(v) that it also has the CEP. It follows from Proposition 4.14 that there exists an ideal K of $\mathbf{A}$ such that $J = K \cap I$. But $K \cap I$ is I or $\{0\}$ since I is the monolith of $\mathbf{A}$, so J is I or $\{0\}$. By Proposition 4.14 again, I is simple. Thus $\mathbf{I}$ is isomorphic to $\mathbf{C}_2$. By Proposition 6.11, $\mathcal{V}$ is $\mathrm{V}(\mathbf{H}_3)$ or $\mathrm{V}(\mathbf{P}_5)$, but we can exclude $\mathrm{V}(\mathbf{P}_5)$ as it does not have the CEP. □

6.4. Covers that are Semisimple with EDPC. Corollary 6.12 prompts us to investigate $\mathcal{LR}$-varieties that cover $\mathrm{V}(\mathbf{C}_2)$, are semisimple and have EDPC. (We have seen that $\mathrm{V}(\mathbf{L}_3)$ is the only such variety that is generated by its finite members.) We start with a more general discussion.

A *discriminator variety* is a variety $\mathcal{V}$ generated by a class $\mathcal{K}$ of algebras, having a ternary term p such that $\mathcal{V}$ satisfies $p(x, x, y) \approx y$, and $p^{\mathbf{A}}(a, b, c) = a$ whenever $\mathbf{A} \in \mathcal{K}$, $a, b, c \in A$ and $a \neq b$. Discriminator varieties have strong and desirable properties and behave, in several respects, like the variety of Boolean algebras. In particular, every such variety is semisimple and arithmetical [BS81, Theorem IV.9.4]. Since a nontrivial $\mathcal{LR}$-variety is never congruence permutable, it cannot be a discriminator variety. Parts of the theory of discriminator varieties were extended to a hierarchy of wider classes of varieties in [BP94b], from which the following definitions have been taken:

A ternary term $p = p(x, y, z)$ is a *ternary deductive (TD) term* for a class $\mathcal{K}$ of similar algebras if

(i) $\quad \mathcal{K} \models p(x, x, z) \approx z$,

and for all $\mathbf{A} \in \mathcal{K}$ and $a, b, c, d \in A$,

(ii) $\quad p^{\mathbf{A}}(a, b, c) = p^{\mathbf{A}}(a, b, d)$ if $(c, d) \in \Theta^{\mathbf{A}}(a, b)$.

(Note that the converse of (ii) follows from (i).) A TD term p on $\mathcal{K}$ is *commutative* if $\mathcal{K}$ satisfies the identity

$$p(x, y, p(x', y', z)) \approx p(x', y', p(x, y, z)). \tag{70}$$

A ternary operation p on a set A is called a *fixed point discriminator* if there exists $d \in A$ such that

$$p(a, b, c) = \begin{cases} c & \text{if } a = b; \\ d & \text{otherwise.} \end{cases}$$

A variety $\mathcal{V}$ is a *fixed point discriminator variety* if there is a ternary term function p of $\mathcal{V}$ and a subclass $\mathcal{K}$ of $\mathcal{V}$ such that $\mathcal{V} = \mathrm{V}(\mathcal{K})$ and p is a fixed point discriminator on each $\mathbf{A} \in \mathcal{K}$; p is called a *fixed point discriminator term* for $\mathcal{V}$, in this case.

A variety with a TD term must have EDPC [BP94b, Corollary 2.5]. By [BP94b, Theorem 3.4], a variety generated by simple algebras which has a commutative TD term is a fixed point discriminator variety for which the TD term is a fixed point discriminator term. Conversely, every fixed point discriminator variety $\mathcal{V}$ is a semisimple variety for which the fixed point discriminator term is a commutative TD term (so $\mathcal{V}$ has EDPC).

It is not known whether every semisimple, congruence 3-permutable variety with EDPC has a TD term. Even if congruence permutability is assumed, a semisimple variety with EDPC need not have a *commutative* TD term (i.e., it need not be a fixed point discriminator variety) [BP94b]. We now show, however, that for $\mathcal{LR}$-varieties, there is no such distinction.

Proposition 6.13. *Let $\mathcal{V}$ be an $\mathcal{LR}$-variety that is semisimple and has EDPC. Let $u(x,y)$ be a term as described in Theorem 5.4(ii) that witnesses TDPI for $\mathcal{V}$. Then*

(i) $p(x,y,z) = z \dot{-} u(x \dot{-} y, z) \dot{-} u(y \dot{-} x, z)$ is a commutative TD term for $\mathcal{V}$,

(ii) $\mathcal{V}$ is a fixed point discriminator variety with fixed point discriminator term p.

Proof. It suffices, by the above discussion, to show that $p^{\mathbf{A}}$ is a fixed point discriminator on every simple algebra $\mathbf{A} \in \mathcal{V}$, with $0^{\mathbf{A}}$ as the 'fixed point' d.

Recall that for any $\mathbf{A} \in \mathcal{V}$ and $a, b \in A$, we have $b \in \langle a \rangle_{\mathbf{A}}$ if and only if $b = u^{\mathbf{A}}(a,b)$ (Corollary 5.5). It follows that $\mathcal{V}$ satisfies $u(x,0) \approx 0$, while $\mathcal{LR}$ satisfies $u(0,x) \approx 0$ (by (34), page 77); therefore $\mathcal{V}$ satisfies $p(x,x,z) \approx z$.

Now let $\mathbf{B}$ be a simple algebra in $\mathcal{V}$ and $a, b, c \in B$ with $a \neq b$. Then $a \dot{-} b \neq 0$ or $b \dot{-} a \neq 0$, so $\langle a \dot{-} b \rangle_{\mathbf{A}} = B$ or $\langle b \dot{-} a \rangle_{\mathbf{A}} = B$, by the simplicity of $\mathbf{B}$. Thus, $c \in \langle a \dot{-} b \rangle_{\mathbf{A}}$ or $c \in \langle b \dot{-} a \rangle_{\mathbf{A}}$, so $c = u(a \dot{-} b, c)$ or $c = u(b \dot{-} a, c)$, whence $p(a,b,c) = c \dot{-} c = 0$, as required. □

In view of the remarks preceding the above proposition, we infer the following characterization of semisimple $\mathcal{LR}$-varieties with EDPC.

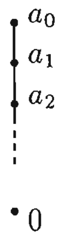
Corollary 6.14. *For an $\mathcal{LR}$-variety $\mathcal{V}$, the following are equivalent:*

(i) $\mathcal{V}$ is semisimple with EDPC,

(ii) $\mathcal{V}$ is semisimple with a commutative TD term,

(iii) $\mathcal{V}$ is a fixed point discriminator variety.

We shall exhibit a countably infinite sequence of $\mathcal{LR}$-varieties that are semisimple with EDPC and covers of $\mathrm{V}(\mathbf{C}_2)$. Set $A = \{0\} \cup \{a_i : i \in \omega\}$, where $\langle a_i : i \in \omega \rangle$ is a one-to-one sequence whose range excludes 0. Let $\leq$ be the partial order on A defined by the following Hasse diagram:



We shall define a countably infinite sequence $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, ... of algebras each with universe A. Let $n \geq 1$. Define a binary operation $\dot{-}_n$ on A as follows: for $x, y \in A$, set $x \dot{-}_n 0 = x$, $x \dot{-}_n y = 0$ iff $x \leq y$ (thus $a_i \dot{-}_n a_j = 0$ iff $i \geq j$); for $i, j \in \omega$ with $j \geq 1$, set

$$\begin{aligned} a_i \dot{-}_n a_{i+1} &= a_{i+n+1} \\ a_i \dot{-}_n a_{i+2} &= a_{i+n+1} \\ &\vdots \\ a_i \dot{-}_n a_{i+n} &= a_{i+n+1} \\ a_i \dot{-}_n a_{i+n+j} &= a_{i+n}. \end{aligned}$$

Let $\mathbf{A}_n$ denote the algebra $\langle A; \dot{-}_n, 0 \rangle$. (Observe that $\mathbf{A}_1$ is the algebra defined at the end of Chapter 1 and discussed in Example 5.12, page 97). Then each $\mathbf{A}_n$ is a left residuation algebra that satisfies

$$(71) \qquad x \dot{-}_n 3(x \dot{-}_n y) \dot{-}_n 3(y \dot{-}_n x) \approx y \dot{-}_n 3(y \dot{-}_n x) \dot{-}_n 3(x \dot{-}_n y),$$

which is of the form of (27). (In fact, as noted in Example 5.12, $\mathbf{A}_1$ satisfies the simpler identity of the same form:

$$x \dot{-}_1 2(x \dot{-}_1 y) \dot{-}_1 (y \dot{-}_1 x) \approx y \dot{-}_1 2(y \dot{-}_1 x) \dot{-}_1 (x \dot{-}_1 y).)$$

Thus each $\mathbf{A}_n$ generates an $\mathcal{LR}$-variety $\mathrm{V}(\mathbf{A}_n)$, which we denote by $\mathcal{V}_n$. In particular, $\mathcal{V}_n$ is a congruence distributive variety for each n.

For the next three lemmas, we shall assume that $n \geq 1$ is fixed. We drop the subscript from $\dot{-}_n$.

Lemma 6.15. $\mathbf{A}_n$ *is simple.*

Proof. Evidently $\langle a_0\rangle_{\mathbf{A}_n} = A$. Let $i \geq 1$. Then

$$a_{i-1} \dot{-} (a_{i-1} \dot{-} a_i \dot{-} a_i) = a_{i-1} \dot{-} (a_{i-1+n+1} \dot{-} a_i) = a_{i-1} \dot{-} 0 = a_{i-1},$$

so $a_{i-1} \in \langle a_i\rangle_{\mathbf{A}_n}$, by Lemma 4.17(iii). It follows that $\langle a_i\rangle_{\mathbf{A}_n} = A$. Thus the only ideals of $\mathbf{A}_n$ are $\{0\}$ and A, so $\mathbf{A}_n$ is simple, by Proposition 4.14. □

Lemma 6.16. *Condition (ii) of Theorem* 5.4 *holds for $\mathcal{V}_n$ with respect to the term*

$$u(x,y) = y \dot{-} (y \dot{-} x \dot{-} (y \dot{-} (y \dot{-} x \dot{-} (y \dot{-} (y \dot{-} x \dot{-} x))))).$$

Thus $\mathcal{V}_n$ has EDPC and EDPM and is semisimple. A system of principal intersection terms for $\mathcal{V}_n$ is given by

$$\begin{aligned}
s_1(x,y,z,w) &= (x \dot{-} y) \dot{-} (z \dot{-} w), & t_1(x,y,z,w) &= x \dot{-} y,\\
s_2(x,y,z,w) &= (x \dot{-} y) \dot{-} (w \dot{-} z), & t_2(x,y,z,w) &= x \dot{-} y,\\
s_3(x,y,z,w) &= (y \dot{-} x) \dot{-} (z \dot{-} w), & t_3(x,y,z,w) &= y \dot{-} x,\\
s_4(x,y,z,w) &= (y \dot{-} x) \dot{-} (w \dot{-} z), & t_4(x,y,z,w) &= y \dot{-} x.
\end{aligned}$$

Proof. The situation for $\mathbf{A}_1$ (see Example 5.12, page 97) generalizes fully: for any $t(x_0,\ldots,x_m) \in T$, $\mathcal{LR}$ satisfies $u(0,t(0,x_1,\ldots,x_m)) \approx t(0,x_1,\ldots,x_m)$. Moreover, for all $i,j \in \omega$, we still have

$$a_i \dot{-} (a_i \dot{-} a_j \dot{-} (a_i \dot{-} (a_i \dot{-} a_j \dot{-} (a_i \dot{-} (a_i \dot{-} a_j \dot{-} a_j))))) = a_i.$$

Thus, for all $t(x_0,\ldots,x_m) \in T$ and $b_1,\ldots,b_m \in A$, $u(a_j,t(a_j,b_1,\ldots,b_m)) = t(a_j,b_1,\ldots,b_m)$ regardless of the value of $t(a_j,b_1,\ldots,b_m)$. By Theorem 5.4, $\mathcal{V}_n$ has EDPC. By Lemma 6.15, $\mathcal{V}_n$ is also semisimple so, by Proposition 5.19, $\mathcal{V}_n$ has EDPM. To see that $\{\langle s_i(x,y,z,w), t_i(x,y,z,w)\rangle : i = 1,\ldots,n\}$ is a system of principal intersection terms for $\mathcal{V}_n$, we need (in view of Theorem 5.18) to show that $(\mathcal{V}_n)_{\mathrm{FSI}}$ satisfies the following universal sentence:

(72)

$$(\forall x)(\forall y)(\forall z)(\forall w)\,[(\bigwedge_{i=1}^{4} s_i(x,y,z,w) \approx t_i(x,y,z,w)) \text{ iff } (x \approx y \text{ or } z \approx w)].$$

By Proposition 5.19, $(\mathcal{V}_n)_{\mathrm{FSI}} = \mathrm{I\,S\,P_U}(\mathbf{A}_n)$; in particular, $(\mathcal{V}_n)_{\mathrm{FSI}}$ is a universal class. Thus, we need only show that $\mathbf{A}_n$ satisfies (72). The implication from right to left in (72) is trivial. For the other implication, let $a,b,c,d \in A_n$ such that $s_i^{\mathbf{A}_n}(a,b,c,d) = t_i^{\mathbf{A}_n}(a,b,c,d)$ for $i = 1,\ldots,4$, and note that

$$\mathbf{A}_n \models (\forall x)(\forall y)\,[(x \dot{-} y \approx x) \text{ iff } (x \approx 0 \text{ or } y \approx 0)].$$

Thus, $a \dot{-} b = 0$ or $c \dot{-} d = 0$, and $a \dot{-} b = 0$ or $d \dot{-} c = 0$, and $b \dot{-} a = 0$ or $c \dot{-} d = 0$, and $b \dot{-} a = 0$ or $c \dot{-} d = 0$. It follows by (A4) that $a = b$ or $c = d$, as required. □

In the first order sentence (73) below, $y \leq x$ abbreviates $y \dot{-} x \approx 0$, $y < x$ abbreviates $((y \leq x)$ and $(\neg(x \approx y)))$ and

$$\mathrm{Pre}^1(y,x) \quad \text{abbreviates} \quad (y < x) \text{ and } (\forall z)((z < x) \text{ implies } (z \leq y)).$$

In the proof of Lemma 6.17 we also use the following inductively defined abbreviations:

$$\text{for } n \geq 1, \quad \mathrm{Pre}^{n+1}(y,x) \text{ abbreviates } (\exists z)((\mathrm{Pre}^n(z,x)) \text{ and } (\mathrm{Pre}^1(y,z))).$$

It follows from the definition of $\dot{-}$ that $\mathbf{A}_n$ satisfies the first order sentence

$$\begin{aligned}(73) \qquad (\forall x)\ [(\neg(x \approx 0)) \text{ implies } &(\exists y_1)(\exists y_2)\ldots(\exists y_{n+1})((\mathrm{Pre}^1(y_1,x)) \\ &\text{and } (\mathrm{Pre}^1(y_2,y_1)) \text{ and } \ldots \text{ and } (\mathrm{Pre}^1(y_{n+1},y_n)) \\ &\text{and } (x \dot{-} y_1 \approx y_{n+1}) \text{ and } (x \dot{-} y_2 \approx y_{n+1}) \text{ and } \ldots \\ &\text{and } (x \dot{-} y_n \approx y_{n+1}) \text{ and } (\forall y)((y < y_n) \text{ implies } (x \dot{-} y \approx y_n)))]\end{aligned}$$

By Łos's Theorem, $\mathrm{P_U}(\mathbf{A}_n)$ satisfies the above sentence as well.

That $\mathbf{A}_n$ is linearly ordered is expressible by the universal sentence

$$(\forall x)(\forall y)(x \leq y \text{ or } y \leq x),$$

hence every algebra in $\mathrm{S\,P_U}(\mathbf{A}_n)$ is linearly ordered as well.

Lemma 6.17. *Every finitely subdirectly irreducible algebra in $\mathcal{V}_n$ is isomorphic to $\mathbf{C}_2$ or has a subalgebra isomorphic to $\mathbf{A}_n$.*

Proof. By Proposition 5.19, Lemmas 6.15 and 6.16, $((\mathcal{V}_n)_{\mathrm{FSI}})_{\mathrm{NT}} = (\mathcal{V}_n)_{\mathrm{S}} = (\mathcal{V}_n)_{\mathrm{SI}} = (\mathrm{S\,P_U}(\mathbf{A}_n))_{\mathrm{NT}}$. Let $\mathbf{B} \in \mathrm{P_U}(\mathbf{A}_n)$. The result will follow if we can show that every subalgebra of $\mathbf{B}$ on at least 3 elements has a subalgebra isomorphic to $\mathbf{A}_n$. For each nonzero $b \in B$, the truth of (73) in $\mathbf{B}$ says that there are unique elements $c_1, \ldots, c_n \in B$ such that $\mathrm{Pre}^i(c_i, b)$ is true in $\mathbf{B}$ for $i = 1, \ldots, n$. Let us write $c_i = \mathrm{Pre}^i(b)$ is this case.

Claim. Let $0 \neq b \in B$. Then $\mathrm{Sg}^{\mathbf{B}}(\{b, \mathrm{Pre}^1(b), \ldots, \mathrm{Pre}^n(b)\})$ is isomorphic to $\mathbf{A}_n$.

By (73),

$$\begin{aligned} b \dot{-} \mathrm{Pre}^1(b) &= \mathrm{Pre}^{n+1}(b), \\ \mathrm{Pre}^1(b) \dot{-} \mathrm{Pre}^2(b) &= \mathrm{Pre}^{n+2}(b), \\ &\vdots \\ \mathrm{Pre}^{n-1}(b) \dot{-} \mathrm{Pre}^n(b) &= \mathrm{Pre}^{n+n}(b), \end{aligned}$$

hence $\{b, \mathrm{Pre}^1(b), \ldots, \mathrm{Pre}^{2n}(b)\} \subseteq \mathrm{Sg}^{\mathbf{B}}(\{b, \mathrm{Pre}^1(b), \ldots, \mathrm{Pre}^n(b)\})$. By induction and (73), $\{0, b\} \cup \{\mathrm{Pre}^m(b) : m \geq 1\}$ is a subuniverse of $\mathrm{Sg}^{\mathbf{B}}(\{b, \mathrm{Pre}^1(b), \ldots, \mathrm{Pre}^n(b)\})$ isomorphic to $\mathbf{A}_n$, hence the Claim holds.

Next, we show that every subalgebra of $\mathbf{B}$ on at least 3 elements contains a subset of the form $\{b, \mathrm{Pre}^1(b), \ldots, \mathrm{Pre}^n(b)\}$ for some $b \in B$. A subset of B of the form $\{b, \mathrm{Pre}^1(b), \ldots, \mathrm{Pre}^{k-1}(b)\}$, where $b \in B$ and $k \geq 1$, will be called a *k-element chain in* $\mathbf{B}$. Let $\mathbf{C}$ be a nontrivial subalgebra of $\mathbf{B}$ not isomorphic to $\mathbf{C}_2$ and let $0 \neq b, c \in C$. Without loss of generality, suppose $b < c$. By (73), either $c \dot{-} b = \mathrm{Pre}^{n+1}(c)$, in which case $c \dot{-} (c \dot{-} b) = \mathrm{Pre}^n(c)$, or $c \dot{-} b = \mathrm{Pre}^n(c)$, in which case $c \dot{-} (c \dot{-} b) = \mathrm{Pre}^{n+1}(c)$. In either case, $\{c \dot{-} b, c \dot{-} (c \dot{-} b)\}$ forms a 2-element chain in $\mathbf{C}$. Suppose that $\{d, \mathrm{Pre}^1(d), \ldots, \mathrm{Pre}^{k-1}(d)\}$ is a k-element chain in $\mathbf{C}$, where $2 \leq k < n$. By (73),

$$
\begin{aligned}
d \dot{-} \mathrm{Pre}^1(d) &= \mathrm{Pre}^{n+1}(d), \\
\mathrm{Pre}^1(d) \dot{-} \mathrm{Pre}^2(d) &= \mathrm{Pre}^{n+2}(d), \\
&\;\;\vdots \\
\mathrm{Pre}^{k-2}(d) \dot{-} \mathrm{Pre}^{k-1}(d) &= \mathrm{Pre}^{n+k-1}(d), \\
d \dot{-} \mathrm{Pre}^{n+1}(d) &= \mathrm{Pre}^n(d), \\
\mathrm{Pre}^{k-1}(d) \dot{-} \mathrm{Pre}^n(d) &= \mathrm{Pre}^{n+k}(d).
\end{aligned}
$$

Thus $\{\mathrm{Pre}^n(d), \ldots, \mathrm{Pre}^{n+k}(d)\}$ is a $(k+1)$-element chain in $\mathbf{C}$. By induction, $\mathbf{C}$ contains an n-element chain. By the Claim, $\mathbf{C}$ contains a subalgebra isomorphic to $\mathbf{A}_n$. □

Proposition 6.18. *For each $n \geq 1$, $\mathcal{V}_n$ is a cover of the atom* $\mathrm{V}(\mathbf{C}_2)$.

Proof. Suppose $\mathcal{W}$ is a nontrivial $\mathcal{LR}$-variety such that $\mathcal{W} \subseteq \mathcal{V}_n$. Let $\mathbf{B} \in \mathcal{W}_{\mathrm{SI}}$, so $\mathbf{B} \in (\mathcal{V}_n)_{\mathrm{SI}}$ as well. Then, by Lemma 6.17, $\mathbf{B}$ is isomorphic to $\mathbf{C}_2$, or $\mathbf{B}$ contains a subalgebra isomorphic to $\mathbf{A}_n$. Thus, $\mathcal{W}$ is $\mathrm{V}(\mathbf{C}_2)$ or $\mathcal{V}_n$. □

Set $\mathcal{X} = \mathrm{V}(\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots\})$. Each $\mathbf{A}_i$ satisfies (71), hence $\mathcal{X}$ does as well, so $\mathcal{X}$ is an $\mathcal{LR}$-variety. Moreover, $p(x, y, z) = z \dot{-} u(x \dot{-} y, z) \dot{-} u(y \dot{-} x, z)$ is a fixed point discriminator on each $\mathbf{A}_i$, hence $\mathcal{X}$ is a fixed point discriminator variety. Thus $\mathcal{X}$ is semisimple with EDPC and so, by Proposition 5.19, $\mathcal{X}_{\mathrm{SI}} = \mathcal{X}_{\mathrm{S}} = (\mathrm{S}\,\mathrm{P}_{\mathrm{U}}(\{\mathbf{A}_1, \mathbf{A}_2, \ldots\}))_{\mathrm{NT}}$.

Proposition 6.19. *There are precisely $2^{\aleph_0}$ subvarieties of $\mathcal{X}$. Moreover, $\mathcal{X}$ contains no finite algebra other than members of* $\mathrm{V}(\mathbf{C}_2)$.

Proof. Set $N = \{1, 2, 3, \ldots\}$. Let $I_1, I_2 \subseteq N$, $I_1 \neq I_2$. Without loss of generality, suppose $n \in I_1 \backslash I_2$. We claim that $\mathbf{A}_n \notin \mathrm{V}(\{\mathbf{A}_i : i \in I_2\}) = \mathcal{V}$. $\mathcal{V}$ is also semisimple with EDPC, hence $\mathcal{V}_{\mathrm{SI}} = \mathcal{V}_{\mathrm{S}} = (\mathrm{S}\,\mathrm{P_U}(\{\mathbf{A}_i : i \in I_2\}))_{\mathrm{NT}}$. Consider the following first order sentence ϕ, which is equivalent to a universal sentence:

$$\neg[(\exists x_0)\ldots(\exists x_n)[(x_0 > \cdots > x_n) \text{ and } (x_0 \dot{-} x_1 \approx x_{n+1}) \text{ and } \ldots \\ \text{and } (x_0 \dot{-} x_n \approx x_{n+1}) \text{ and } (x_0 \dot{-} x_{n+1} \approx x_n)]].$$

Evidently, each $\mathbf{A}_m$, where $m \neq n$, satisfies ϕ. Thus ϕ is satisfied by $\mathcal{V}_{\mathrm{S}}$. But $\mathbf{A}_n$ does not satisfy ϕ, so $\mathbf{A}_n \notin \mathcal{V}_{\mathrm{S}}$. Since $\mathbf{A}_n$ is a simple algebra, we must have $\mathbf{A}_n \notin \mathcal{V}$.

It follows from the claim that $\mathrm{V}(\mathbf{A}_n) \not\subseteq \mathrm{V}(\{\mathbf{A}_i : i \in I_2\})$, hence that $\mathrm{V}(\{\mathbf{A}_i : i \in I_1\}) \neq \mathrm{V}(\{\mathbf{A}_i : i \in I_2\})$. Since I_1 and I_2 we arbitrary, it follows that there must exist as many subvarieties of $\mathcal{X}$ as there are subsets of N, i.e., $2^{\aleph_0}$ subvarieties.

Let $\mathbf{B}$ be any finite subdirectly irreducible left residuation algebra with $|B| > 2$. Let φ be a first order sentence such that for any algebra $\mathbf{A}$ of type $\langle 2, 0\rangle$ we have: $\mathbf{A}$ satisfies φ if and only if $\mathbf{A}$ has a subalgebra isomorphic to $\mathbf{B}$. Evidently, for each $n \geq 1$, $\mathbf{A}_n$ does not satisfy φ, hence $\mathbf{A}_n$ satisfies $\neg\varphi$. The sentence $\neg\varphi$ is equivalent to a universal sentence, hence $\mathrm{S}\,\mathrm{P_U}(\{\mathbf{A}_1, \mathbf{A}_2, \ldots\})$ satisfies $\neg\varphi$. Thus $\mathbf{B} \notin \mathrm{S}\,\mathrm{P_U}(\{\mathbf{A}_1, \mathbf{A}_2, \ldots\})$, hence $\mathbf{B} \notin \mathcal{X}$. It follows that $\mathcal{X}$ contains no finite algebras other than members of $\mathrm{V}(\mathbf{C}_2)$. $\square$

6.5. Finite Axiomatization: $\mathbf{V(A_1)}$ and $\mathbf{V(A_2)}$. We present here a finite axiomatization of $\mathrm{V}(\mathbf{A}_1)$. We also deduce from theoretical considerations that $\mathrm{V}(\mathbf{A}_2)$ is a finitely based variety.

We shall write $\dot{-}$ instead of $\dot{-}_1$. As in Lemma 6.16, we set

$$u(x, y) = y \dot{-} (y \dot{-} x \dot{-} (y \dot{-} (y \dot{-} x \dot{-} (y \dot{-} (y \dot{-} x \dot{-} x))))).$$

Consider the following identities:

(D1) $\quad x \dot{-} 2(x \dot{-} y) \dot{-} (y \dot{-} x) \approx y \dot{-} 2(y \dot{-} x) \dot{-} (x \dot{-} y)$,

(D2) $\quad u(x, y \dot{-} (y \dot{-} z \dot{-} w)) \approx y \dot{-} (y \dot{-} u(x, z) \dot{-} u(x, w))$,

(D3) $\quad x \dot{-} (y \dot{-} (y \dot{-} x)) \approx x \dot{-} y$,

(D4) $\quad x \dot{-} y \dot{-} (y \dot{-} x) \approx x \dot{-} y$,

(D5) $\quad x \dot{-} (x \dot{-} (y \dot{-} u(x, y))) \approx 0$,

(D6) $\quad x \dot{-} y \dot{-} 2(x \dot{-} 2y) \dot{-} 2(y \dot{-} (x \dot{-} y)) \approx 0$,

(D7) $\quad x \dot{-} y \dot{-} (x \dot{-} y \dot{-} (x \dot{-} z \dot{-} (x \dot{-} z \dot{-} (z \dot{-} (x \dot{-} y) \dot{-} (z \dot{-} (x \dot{-} y) \dot{-} (z \dot{-} (x \dot{-} (x \dot{-} y)))))))) \approx 0$, (D8) $\quad x \dot{-} (x \dot{-} (x \dot{-} y)) \approx x \dot{-} y,$

(D9) $\quad x \dot{-} y \dot{-} (x \dot{-} y \dot{-} (x \dot{-} 3(x \dot{-} y))) \approx 0.$

We shall prove the following:

Theorem 6.20. $V(\mathbf{A}_1)$ *is axiomatized by* (A1)-(A3) *and* (D1)-(D9).

We know that $\mathbf{A}_1$ satisfies (A1)-(A3) and (D1). It is routine to check that $\mathbf{A}_1$ satisfies (D3)-(D9). By the proof of Lemma 6.16, $\mathbf{A}_1$ satisfies $x \approx 0$ or $u(x,y) \approx y$, and $\mathcal{LR}$ satisfies $u(0,y) \approx 0$. It follows that $\mathbf{A}_1$ satisfies (D2).

Conversely, let $\mathcal{W}$ be the variety axiomatized by (A1)-(A3) and (D1)-(D9). Since $\mathcal{W}$ satisfies (A1)-(A3) and (D1), it follows from Proposition 4.4(i) that $\mathcal{W}$ is an $\mathcal{LR}$-variety.

Lemma 6.21. *For each* $\mathbf{C} \in \mathcal{W}$ *and all* $a, b \in C$, $b \in \langle a \rangle_{\mathbf{C}}$ *if and only if* $u(a,b) = b$. *Thus,* $\mathcal{W}$ *has EDPC.*

Proof. ($\Leftarrow$) If $u(a,b) = b$ then, since $u(a,b) \in \langle a \rangle_{\mathbf{C}}$, we have $b \in \langle a \rangle_{\mathbf{C}}$.

($\Rightarrow$) Suppose $b \in \langle a \rangle_{\mathbf{C}}$. By Corollary 4.18, there exist $n \in \omega$ and $b_0, \ldots, b_n \in C$ such that $b_n = b$ and for each $i \in \{0, \ldots, n\}$, $b_i = a$ or $b_i = c \dot{-} (c \dot{-} b_j \dot{-} b_k)$ for some $c \in C$ and $j, k < i$. If $n = 0$ then $b = b_0 = a$ and $u(a,b) = b$. Suppose that $n > 0$. Either $b_n = a$ (so $u(a,b) = b$), or $b_n = c \dot{-} (c \dot{-} b_j \dot{-} b_k)$, where $j, k < n$ and $c \in C$. In the second case, by an induction hypothesis, $u(a, b_j) = b_j$ and $u(a, b_k) = b_k$, and

$$\begin{aligned} u(a,b) &= u(a, c \dot{-} (c \dot{-} b_j \dot{-} b_k)) \\ &= c \dot{-} (c \dot{-} u(a,b_j) \dot{-} u(a,b_k)) \quad \text{(by (D2))} \\ &= c \dot{-} (c \dot{-} b_j \dot{-} b_k) = b, \end{aligned}$$

as required. By Theorem 5.4, $\mathcal{W}$ has EDPC. □

In the Lemmas 6.22 to 6.30, $\mathbf{E}$ denotes an arbitrary subdirectly irreducible algebra in $\mathcal{W}$ with monolith I.

Lemma 6.22. *For all* $a \in E$ *and* $b \in I$, $a \dot{-} b = a$ *if and only if* $b = 0$ *or* $a = 0$.

Proof. Sufficiency follows from (A2) and (A3), respectively. Conversely, suppose $a \dot{-} b = a$ and $a \neq 0$. By (D3), $b \dot{-} a = b \dot{-} (a \dot{-} (a \dot{-} b)) = b \dot{-} 0 = b$. Now, $I \subseteq \langle a \rangle_{\mathbf{C}}$ and $b \in I$, so $b \in \langle a \rangle_{\mathbf{C}}$. By Lemma 6.21, $u(a,b) = b$, i.e.,

$$b \dot{-} (b \dot{-} a \dot{-} (b \dot{-} (b \dot{-} a \dot{-} (b \dot{-} (b \dot{-} a \dot{-} a))))) = b,$$

from which it follows that $0 = b$. □

Lemma 6.23. *For all* $a \in E \backslash I$ *and* $b \in I$, *we have* $b < a$.

Proof. By (D4), $a \dot{-} b \dot{-} (b \dot{-} a) = a \dot{-} b$. Since $b \in I$, we have $b \dot{-} a \in I$ as well. By Lemma 6.22, either $a \dot{-} b = 0$ or $b \dot{-} a = 0$. But $a \dot{-} b = 0$ implies $a \in I$, a contradiction, so $b \dot{-} a = 0$, i.e., $b \leq a$ (hence $b < a$). □

Lemma 6.24. $\mathbf{E}$ *is simple.*

Proof. Let $0 \neq b \in I$ and let $a \in E$. By (D5), $b \dot{-} (b \dot{-} (a \dot{-} u(b,a))) = 0$, so $b = b \dot{-} (a \dot{-} u(b,a))$. Since $b \neq 0$, we have $b \dot{-} (a \dot{-} u(b,a)) \neq 0$, so $b \not\leq a \dot{-} u(b,a)$. By Lemma 6.23, since $b \in I$ we have $b \leq c$ for each $c \in E \backslash I$. Thus $a \dot{-} u(b,a) \notin E \backslash I$, so $a \dot{-} u(b,a) \in I$. Since $u(b,a) \in \langle b \rangle_{\mathbf{E}} = I$, we have $a \in I$, so $E = I$ and $\mathbf{E}$ is simple. □

Lemma 6.25. $\mathbf{E}$ *is linearly ordered.*

Proof. Let $0 \neq a, b \in E$ such that $a \neq b$. We cannot have both $a \dot{-} b = 0$ and $b \dot{-} a = 0$ hence, since $\mathbf{E}$ is simple, either $a \dot{-} b \in \langle b \dot{-} a \rangle_{\mathbf{E}}$ or $b \dot{-} a \in \langle a \dot{-} b \rangle_{\mathbf{E}}$. Suppose the former holds. By Lemma 6.21, $u(b \dot{-} a, a \dot{-} b) = a \dot{-} b$, i.e.,

$$\begin{aligned} a \dot{-} b &= a \dot{-} b \dot{-} (a \dot{-} b \dot{-} (b \dot{-} a) \dot{-} (a \dot{-} b \dot{-} (a \dot{-} b \dot{-} (b \dot{-} a) \\ &\qquad \dot{-} (a \dot{-} b \dot{-} (a \dot{-} b \dot{-} (b \dot{-} a) \dot{-} (b \dot{-} a)))))) \\ &= a \dot{-} b \dot{-} (a \dot{-} b \dot{-} (a \dot{-} b \dot{-} (a \dot{-} b \dot{-} (a \dot{-} b \dot{-} (a \dot{-} b))))) \quad \text{(by (D4))} \\ &= 0, \end{aligned}$$

so $a \leq b$. If $b \dot{-} a \in \langle a \dot{-} b \rangle_{\mathbf{E}}$ then we similarly obtain that $b \leq a$. □

The next lemma follows from Lemmas 6.22 and 6.24.

Lemma 6.26. *For all* $a, b \in E$, $a \dot{-} (a \dot{-} b) = 0$ *if and only if* $a = 0$ *or* $b = 0$.

Lemma 6.27. *For all* $a, b \in E$, $a \dot{-} b = b$ *if and only if* $a = 0$ *and* $b = 0$.

Proof. Suppose $a \dot{-} b = b$. Then $a \dot{-} b \dot{-} b = 0$ and $b \dot{-} (a \dot{-} b) = 0$, so (D6) implies that $a \dot{-} b = 0$. Thus, $b = a \dot{-} b = 0$ and $a = a \dot{-} 0 = a \dot{-} b = 0$. The other implication is trivial. □

Lemma 6.28. *Let* $a, b \in E$ *such that* $0 < b < a$. *Then* a *has a unique predecessor that is either* $a \dot{-} b$ *or* $a \dot{-} (a \dot{-} b)$.

Proof. Evidently, both $a \dot{-} b \leq a$ and $a \dot{-} (a \dot{-} b) \leq a$. By Lemma 6.26, we cannot have $a \dot{-} b = a$ or $a \dot{-} (a \dot{-} b) = a$, so $a \dot{-} b < a$ and $a \dot{-} (a \dot{-} b) < a$. By Lemma 6.27, $a \dot{-} b \neq a \dot{-} (a \dot{-} b)$, so the linear order on E implies that $a \dot{-} b < a \dot{-} (a \dot{-} b)$ or $a \dot{-} (a \dot{-} b) < a \dot{-} b$. Let $c \in E$ such that $c < a$. By (D7),

$$\begin{aligned} a \dot{-} b \dot{-} (a \dot{-} b \dot{-} (a \dot{-} c \dot{-} (a \dot{-} c \dot{-} (c \dot{-} (a \dot{-} b) \dot{-} (c \dot{-} (a \dot{-} b) \\ \dot{-} (c \dot{-} (a \dot{-} (a \dot{-} b))))))) = 0. \end{aligned}$$

Thus, by Lemma 6.26, either $a \dot{-} b = 0$ or

$$a \dot{-} c \dot{-} (a \dot{-} c \dot{-} (c \dot{-} (a \dot{-} b) \dot{-} (c \dot{-} (a \dot{-} b) \dot{-} (c \dot{-} (a \dot{-} (a \dot{-} b)))))) = 0.$$

Since $a \dot{-} b \neq 0$, the latter holds. By Lemma 6.26 again, either $a \dot{-} c = 0$ or $c \dot{-} (a \dot{-} b) \dot{-} (c \dot{-} (a \dot{-} b) \dot{-} (c \dot{-} (a \dot{-} (a \dot{-} b)))) = 0$. Since $a \dot{-} c \neq 0$, the latter holds so, by Lemma 6.26 once more, either $c \dot{-} (a \dot{-} b) = 0$ or $c \dot{-} (a \dot{-} (a \dot{-} b)) = 0$. Thus $c < a$ implies $c \leq a \dot{-} b$ or $c \leq a \dot{-} (a \dot{-} b)$. Now, if $a \dot{-} b < a \dot{-} (a \dot{-} b)$ then $c < a$ implies $c \leq a \dot{-} (a \dot{-} b)$, i.e., $a \dot{-} (a \dot{-} b)$ is the predecessor of a; if $a \dot{-} (a \dot{-} b) < a \dot{-} b$ then $c < a$ implies $c \leq a \dot{-} b$, i.e., $a \dot{-} b$ is the predecessor of a. □

Note that Lemma 6.28 implies that every nonzero element of E has a unique predecessor, which we shall denote by $\mathrm{Pre}^{\mathbf{E}}(a)$.

Lemma 6.29. *Let $a, b \in E$ such that $0 < b < a$ and suppose that $\mathrm{Pre}^{\mathbf{E}}(a) = a \dot{-} (a \dot{-} b)$. Then $\mathrm{Pre}^{\mathbf{E}}(\mathrm{Pre}^{\mathbf{E}}(a)) = a \dot{-} b$.*

Proof. Clearly, $a \dot{-} b < a$, so by Lemma 6.27, $a \dot{-} b < a \dot{-} (a \dot{-} b)$. By Lemma 6.28, either

$$\mathrm{Pre}^{\mathbf{E}}(a \dot{-} (a \dot{-} b)) = a \dot{-} (a \dot{-} b) \dot{-} (a \dot{-} b) \quad \text{or} \tag{74}$$

$$\mathrm{Pre}^{\mathbf{E}}(a \dot{-} (a \dot{-} b)) = a \dot{-} (a \dot{-} b) \dot{-} (a \dot{-} (a \dot{-} b) \dot{-} (a \dot{-} b)). \tag{75}$$

By (A1) and (D8),

$$a \dot{-} (a \dot{-} b) \dot{-} (a \dot{-} (a \dot{-} b) \dot{-} (a \dot{-} b)) \leq a \dot{-} (a \dot{-} (a \dot{-} b)) = a \dot{-} b. \tag{76}$$

By (D9), $a \dot{-} b \dot{-} (a \dot{-} b \dot{-} (a \dot{-} 3(a \dot{-} b))) = 0$, so, by Lemma 6.26, either $a \dot{-} b = 0$ or $a \dot{-} 3(a \dot{-} b) = 0$. Since $a \dot{-} b \neq 0$, the latter holds, so

$$a \dot{-} 2(a \dot{-} b) \leq a \dot{-} b. \tag{77}$$

By (74), (75), (76) and (77), we have that $\mathrm{Pre}^{\mathbf{E}}(a \dot{-} (a \dot{-} b)) \leq a \dot{-} b$. Since $a \dot{-} b < a \dot{-} (a \dot{-} b)$, we have $\mathrm{Pre}^{\mathbf{E}}(a \dot{-} (a \dot{-} b)) = a \dot{-} b$, i.e., $\mathrm{Pre}^{\mathbf{E}}(\mathrm{Pre}^{\mathbf{E}}(a)) = a \dot{-} b$. □

Lemma 6.30. *Let $a, b \in E$ such that $0 < b < a$. If $b = \mathrm{Pre}^{\mathbf{E}}(a)$ then $a \dot{-} b = \mathrm{Pre}^{\mathbf{E}}(\mathrm{Pre}^{\mathbf{E}}(a))$; if $b < \mathrm{Pre}^{\mathbf{E}}(a)$ then $a \dot{-} b = \mathrm{Pre}^{\mathbf{E}}(a)$.*

Proof. By Lemma 6.28, either $a \dot{-} b = \mathrm{Pre}^{\mathbf{E}}(a)$ or $a \dot{-} (a \dot{-} b) = \mathrm{Pre}^{\mathbf{E}}(a)$. If $b = \mathrm{Pre}^{\mathbf{E}}(a)$ then $a \dot{-} b \neq \mathrm{Pre}^{\mathbf{E}}(a)$ otherwise $b = a \dot{-} b$, contradicting Lemma 6.27, so $a \dot{-} (a \dot{-} b) = \mathrm{Pre}^{\mathbf{E}}(a)$. By Lemma 6.29, therefore, $\mathrm{Pre}^{\mathbf{E}}(\mathrm{Pre}^{\mathbf{E}}(a)) = a \dot{-} b$.

Suppose $b < \mathrm{Pre}^{\mathbf{E}}(a)$. If $a \dot{-} (a \dot{-} b) = \mathrm{Pre}^{\mathbf{E}}(a)$ then $b < a \dot{-} (a \dot{-} b)$ so $b \leq \mathrm{Pre}^{\mathbf{E}}(\mathrm{Pre}^{\mathbf{E}}(a)) = a \dot{-} b$ by Lemma 6.29. But $b \leq a \dot{-} b$ implies $\mathrm{Pre}^{\mathbf{E}}(a) =$

$a \dot{-} (a \dot{-} b) \leq a \dot{-} b = \mathrm{Pre}^{\mathbf{E}}(\mathrm{Pre}^{\mathbf{E}}(a))$, a contradiction. Thus $a \dot{-} b = \mathrm{Pre}^{\mathbf{E}}(a)$, as required. □

As in Section 6.4, let $\mathcal{V}_1 = \mathrm{V}(\mathbf{A}_1)$. Let $\mathcal{U}$ be a nonprincipal ultrafilter over ω, let $\mathbf{C} = \mathbf{A}_1^{\omega}$ and $\mathbf{D} = \mathbf{A}_1^{\omega}/\mathcal{U}$. Łos' Theorem (Theorem 0.6) implies the following properties of $\mathbf{D}$, because they are first order definable properties of $\mathbf{A}_1$: $\langle D; \leq \rangle$ is linearly ordered with a top element, T say, and every nonzero $d \in D$ has a *nonzero* (unique) predecessor $\mathrm{Pre}^{\mathbf{D}}(d)$ in $\langle D; \leq \rangle$. If $0^{\mathbf{D}} \neq d, d' \in D$, then

$$d \dot{-} \mathrm{Pre}^{\mathbf{D}}(d) = \mathrm{Pre}^{\mathbf{D}}(\mathrm{Pre}^{\mathbf{D}}(d)) \quad (\text{denoted } \mathrm{Pre}^{2\,\mathbf{D}}(d))$$

$$\text{and} \qquad \text{if } d' \leq \mathrm{Pre}^{2\,\mathbf{D}}(d) \text{ then } d \dot{-} d' = \mathrm{Pre}^{\mathbf{D}}(d).$$

Lemma 6.31. *If* $\mathbf{E} \in \mathcal{W}_{\mathrm{SI}}$ *and* $\mathbf{B}$ *is a finitely generated subalgebra of* $\mathbf{E}$ *then there exists a one-to-one homomorphism* $\varphi : \mathbf{B} \to \mathbf{D}$.

Proof. Let $\mathbf{E} \in \mathcal{W}_{\mathrm{SI}}$. By Lemmas 6.24 and 6.25, $\mathbf{E}$ is simple and linearly ordered. By Lemma 6.21, $\mathcal{W}$ has EDPC, hence also the CEP, so every nontrivial subalgebra $\mathbf{B}$ of $\mathbf{E}$ is also simple and linearly ordered and Lemmas 6.28 and 6.30 apply to $\mathbf{B}$.

Now, assume that $\mathbf{B} \in (\mathrm{S}(\mathbf{E}))_{\mathrm{NT}}$ is finitely generated. Let $\{b_0, b_1, \ldots, b_n\}$ be a minimal subset of E such that $\mathbf{B} = \mathrm{Sg}^{\mathbf{E}}(\{b_0, b_1, \ldots, b_n\})$ and $b_0 > b_1 > \cdots > b_n > 0^{\mathbf{E}}$. If $n = 0$, we may choose $\varphi = \{(b_0, \mathrm{T}), (0^{\mathbf{E}}, 0^{\mathbf{D}})\}$ so assume $n > 0$. If $n > 1$, by repeated use of Lemmas 6.28 and 6.30, $\{\mathrm{Pre}^{i\,\mathbf{E}}(b_0) : i \in \omega\} \subseteq \mathrm{Sg}^{\mathbf{E}}(\{b_0, b_2\})$ (where we define $\mathrm{Pre}^{0\,\mathbf{E}}(b_0) = b_0$ and $\mathrm{Pre}^{k+1\,\mathbf{E}}(b_0) = \mathrm{Pre}^{\mathbf{E}}(\mathrm{Pre}^{k\,\mathbf{E}}(b_0))$ for $k \in \omega$). In this case if $b_1 = \mathrm{Pre}^{i\,\mathbf{E}}(b_0)$ for some $i \in \omega$ then $i \neq 0$ and $b_1 = (\mathrm{Pre}^{i-1\,\mathbf{E}}(b_0)) \dot{-} b_2$, so $b_1 \in \mathrm{Sg}^{\mathbf{E}}(\{b_0, b_2, \ldots, b_n\})$, contradicting the minimality of $\{b_0, b_1, \ldots, b_n\}$. Thus, $b_1 < \mathrm{Pre}^{i\,\mathbf{E}}(b_0)$ for all $i \in \omega$, or $n = 1$.

Repeating this argument, we conclude that $B = \{0^{\mathbf{E}}\} \cup (\bigcup_{r=0}^{n} B_r)$ where $B_r = \{\mathrm{Pre}^{i\,\mathbf{E}}(b_r) : i \in \omega\}$ for $r < n$ and either (Case (i)) $B_n = \{\mathrm{Pre}^{i\,\mathbf{E}}(b_n) : i \in \omega\}$ or (Case (ii)) $B_n = \{b_n\}$.

For each $r, s \in \omega$, define $c_r^s \in C$ by $c_r^s(j) = a_{rj+s}$ $(j \in \omega)$ and note that $c_r^s/\mathcal{U} = \mathrm{Pre}^{s\,\mathbf{D}}(c_r^0/\mathcal{U}) > c_{r+1}^0/\mathcal{U}$ (because $\{j \in \omega : c_r^s(j) > c_{r+1}^0(j)\} = \{j \in \omega : j > s\} \in \mathcal{U}$ for any $r, s \in \omega$).

Define $\varphi = \{(0^{\mathbf{E}}, 0^{\mathbf{D}})\} \cup \{(\mathrm{Pre}^{i\,\mathbf{E}}(b_r), c_r^i/\mathcal{U}) : i \in \omega, r < n\} \cup \psi$, where, in Case (i), $\psi = \{(\mathrm{Pre}^{i\,\mathbf{E}}(b_n), c_n^i/\mathcal{U}) : i \in \omega\}$ while, in Case (ii), $\psi = \{(b_n, c_n^0/\mathcal{U})\}$. Then φ is an embedding of $\mathbf{B}$ into $\mathbf{D}$, as required. □

Corollary 6.32. $\mathcal{W}_{\mathrm{SI}} \subseteq \mathrm{I\,S\,P_U}(\mathbf{A}_1)$.

Proof. This follows because every $\mathbf{E} \in \mathcal{W}_{\mathrm{SI}}$ is embeddable in an ultraproduct of its finitely generated subalgebras (Theorem 0.5) and the class operator $\mathrm{I\,S\,P_U}$

is idempotent. $\square$

By Birkhoff's Subdirect Decomposition Theorem and the above corollary, we have $\mathcal{W} \subseteq \mathcal{V}_1$, hence $\mathcal{W} = \mathcal{V}_1$, completing the proof of Theorem 6.20.

We turn our attention to the variety $\mathcal{V}_2$ $(= \mathrm{V}(\mathbf{A}_2))$. We have been unable to find a finite set of axiomatizing identities for $\mathcal{V}_2$, but we shall use Theorem 0.11 to prove the following:

Theorem 6.33. *$\mathcal{V}_2$ is a finitely based variety.*

We shall use $\dot{-}$ instead of $\dot{-}_2$ and we shall use the abbreviations $y \leq x$, $y < x$ and $\mathrm{Pre}^1(y, x)$ that were defined in the previous section. We show that $(\mathcal{V}_2)_{\mathrm{FSI}}$ is a strictly elementary class by showing that $(\mathcal{V}_2)_{\mathrm{FSI}}$ is axiomatized by the first order sentences which are the closures of (A1)–(A4),

(E1) $(\forall x)(\forall y)((x \leq y)$ or $(y \leq x))$

(E2) $(\forall x)[(\neg(x \approx 0))$ implies $(\exists y)(\mathrm{Pre}^1(y, x))]$

(E3) $(\forall x)[((\neg(x \approx 0))$ and $(\neg\mathrm{Pre}^1(0, x))$ and $(\forall w)(\neg\mathrm{Pre}^1(x, w)))$ implies $(\phi_1(x)$ or $\phi_2(x))]$

(E4) $(\forall x)[((\neg(x \approx 0))$ and $(\exists w)(\mathrm{Pre}^1(x, w)))$ implies $\phi_2(x)]$

where $\phi_1(x)$ is the first order formula

$$(\exists x_1)(\exists x_2)[(\neg(x_1 \approx 0)) \text{ and } (\neg(x_2 \approx 0)) \text{ and } \mathrm{Pre}^1(x_1, x) \text{ and } \mathrm{Pre}^1(x_2, x_1) \text{ and } (x \dot{-} x_1 \approx x_2) \text{ and } (\forall y)((y \leq x_2) \text{ implies } (x \dot{-} y \approx x_1))],$$

and $\phi_2(x)$ is the first order formula

$$(\exists x_1)(\exists x_2)(\exists x_3)[(\neg(x_1 \approx 0)) \text{ and } (\neg(x_2 \approx 0)) \text{ and } (\neg(x_3 \approx 0)) \text{ and } \mathrm{Pre}^1(x_1, x) \text{ and } \mathrm{Pre}^1(x_2, x_1) \text{ and } \mathrm{Pre}^1(x_3, x_2) \text{ and } (x \dot{-} x_1 \approx x_3) \text{ and } (x \dot{-} x_2 \approx x_3) \text{ and } (\forall y)((y \leq x_3) \text{ implies } (x \dot{-} y \approx x_2))].$$

Let $\mathcal{Z}$ be the class of all algebras of type $\langle 2, 0\rangle$ over the language $\langle \dot{-}, 0\rangle$ satisfying (A1)–(A4) and (E1)–(E4). By (A1)–(A4), $\mathcal{Z} \subseteq \mathcal{LR}$.

By Proposition 5.19 and Lemma 6.16, $(\mathcal{V}_2)_{\mathrm{FSI}} = \mathrm{I}\,\mathrm{S}\,\mathrm{P_U}(\mathbf{A}_2)$. Evidently, $\mathbf{A}_2$ (hence $(\mathcal{V}_2)_{\mathrm{FSI}}$) satisfies the universal sentences (A1)–(A4) and (E1). By Łos' Theorem (Theorem 0.6), any algebra $\mathbf{B} \in \mathrm{P_U}(\mathbf{A}_2)$ shares the following (first order) properties of $\mathbf{A}_2$: $\langle B; \leq\rangle$ is linearly ordered with a top element and every nonzero $b \in B$ has a nonzero (unique) predecessor $\mathrm{Pre}^{\mathbf{B}}(b)$ in $\langle B; \leq\rangle$. If $0^{\mathbf{B}} \neq b, b' \in B$, then

$$b \dot{-} \mathrm{Pre}^{\mathbf{B}}(b) = b \dot{-} \mathrm{Pre}^{2\,\mathbf{B}}(b) = \mathrm{Pre}^{3\,\mathbf{B}}(b)$$
$$\text{and} \quad \text{if } b' \leq \mathrm{Pre}^{3\,\mathbf{B}}(b) \text{ then } b \dot{-} b' = \mathrm{Pre}^{2\,\mathbf{B}}(b)$$

(where $\mathrm{Pre}^{2\,\mathbf{B}}(b) = \mathrm{Pre}^{\mathbf{B}}(\mathrm{Pre}^{\mathbf{B}}(b))$ and $\mathrm{Pre}^{3\,\mathbf{B}}(b) = \mathrm{Pre}^{\mathbf{B}}(\mathrm{Pre}^{2\,\mathbf{B}}(b)))$. Thus, $\mathbf{A}_2$ and $\mathrm{P_U}(\mathbf{A}_2)$ satisfy $(\forall x)((\neg(x \approx 0))$ **implies** $\phi_2(x))$. $\mathbf{A}_2$'s subuniverses on three or more elements are just its hereditary subsets, together with the sets obtained from these by deleting the second-to-top element. By considering which elements may be omitted in taking subalgebras of ultrapowers we find that $\mathrm{S\,P_U}(\mathbf{A}_2)$ satisfies (E2), (E3) and (E4). Thus, $(\mathcal{V}_2)_{\mathrm{FSI}} \subseteq \mathcal{Z}$.

Let $\mathcal{U}$ be a nonprincipal ultrafilter over ω, let $\mathbf{C} = \mathbf{A}_2^{\omega}$ and $\mathbf{D} = \mathbf{A}_2^{\omega}/\mathcal{U}$.

Lemma 6.34. *If* $\mathbf{E} \in \mathcal{Z}$ *and* $\mathbf{B}$ *is a finitely generated subalgebra of* $\mathbf{E}$ *then there exists a one-to-one homomorphism* $\varphi : \mathbf{B} \to \mathbf{D}$.

Proof. Let $\mathbf{B} = \mathrm{Sg}^{\mathbf{E}}(X)$, where X is a finite subset of E and $0^{\mathbf{E}} \notin X$. If $X = \emptyset$ then $\mathbf{B}$ is the trivial algebra; if $|X| = 1$ then $\mathbf{B}$ is isomorphic to $\mathbf{C}_2$, which embeds into $\mathbf{D}$. For each nonzero $e \in E$, we use $\mathrm{Pre}^{\mathbf{E}}(e)$ to denote the predecessor of e, and we set $\mathrm{Pre}^{0\,\mathbf{E}}(e) = e$ and $\mathrm{Pre}^{i+1\,\mathbf{E}}(e) = \mathrm{Pre}^{\mathbf{E}}(\mathrm{Pre}^{i\,\mathbf{E}}(e))$, for each $i \in \omega$. Suppose $|X| > 1$ and let b_0 be the greatest element of X. If $\mathbf{E} \models \phi_1[b_0]$ then $\{\mathrm{Pre}^{i\,\mathbf{E}}(b_0) : i \in \omega\} \subseteq B$. If $\mathbf{E} \models \phi_2[b_0]$ then $\{b_0\} \cup \{\mathrm{Pre}^{i\,\mathbf{E}}(b_0) : i \geq 2\} \subseteq B$, and $\mathrm{Pre}^{1\,\mathbf{E}}(b_0) \in B$ if and only if $\mathrm{Pre}^{1\,\mathbf{E}}(b_0) \in X$. Set $X_1 = X \backslash \{\mathrm{Pre}^{i\,\mathbf{E}}(b_0) : i \in \omega\}$. If $|X_1| > 1$, let b_1 be the greatest element of X_1 and proceed as above. We repeat the above process until, for some $n \in \omega$, we have $|X_n| \leq 1$. If $|X_n| = 1$, we denote the unique element of X_n by b_n.

The above process will account for all nonzero elements of B. Thus, B is the disjoint union $\{0^{\mathbf{E}}\} \cup (\bigcup_{r=0}^{n} B_r)$ where, for each $r \in \{0, \ldots, n-1\}$,

$$B_r = \{b_r\} \cup \{\mathrm{Pre}^{i\,\mathbf{E}}(b_r) : i \geq 2\} \tag{78}$$

or

$$B_r = \{\mathrm{Pre}^{i\,\mathbf{E}}(b_r) : i \in \omega\}, \tag{79}$$

and $B_n = \emptyset$ or $B_n = \{b_n\}$.

As in Lemma 6.31, for each $r, s \in \omega$, we define $c_r^s \in C$ by $c_r^s(j) = a_{rj+s}$ ($j \in \omega$). Then $c_r^s/\mathcal{U} = \mathrm{Pre}^{s\,\mathbf{D}}(c_r^0/\mathcal{U}) > c_{r+1}^0/\mathcal{U}$. Define $\varphi = \{(0^{\mathbf{E}}, 0^{\mathbf{D}})\} \cup (\bigcup_{r=0}^{n} \psi_r)$, where, for each $r \in \{0, \ldots, n-1\}$, ψ_r is defined in one of the following ways. If $\mathbf{E} \models \phi_1[b_r]$ then (79) holds and we set $\psi_r = \{(b_r, c_r^0/\mathcal{U})\} \cup \{(\mathrm{Pre}^{i\,\mathbf{E}}(b_r), c_r^{i+1}/\mathcal{U}) : i \geq 1\}$. If $\mathbf{E} \models \phi_2[b_r]$ then (78) or (79) holds. In the first case, set $\psi_r = \{(b_r, c_r^0/\mathcal{U})\} \cup \{(\mathrm{Pre}^{i\,\mathbf{E}}(b_r), c_r^i/\mathcal{U}) : i \geq 2\}$. In the second case, set $\psi_r = \{(\mathrm{Pre}^{i\,\mathbf{E}}(b_r), c_r^i/\mathcal{U}) : i \in \omega\}$. Set $\psi_n = \{(b_n, c_n^0/\mathcal{U})\}$ if $B_n = \{b_n\}$ and set $\psi_n = \emptyset$ otherwise. Then φ is an embedding of $\mathbf{B}$ into $\mathbf{D}$, as required. $\square$

The proof of the following corollary is analogous to that of Corollary 6.32.

Corollary 6.35. $\mathcal{Z} \subseteq \mathrm{I\,S\,P_U}(\mathbf{A}_2) = (\mathcal{V}_2)_{\mathrm{FSI}}$.

Thus $(\mathcal{V}_2)_{\mathrm{FSI}} = \mathcal{Z}$, so $(\mathcal{V}_2)_{\mathrm{FSI}}$ is a strictly elementary class. Since $\mathcal{V}_2$ is congruence distributive, we may deduce Theorem 6.33 from Theorem 0.11.

It appears that this proof of Theorem 6.33 may be extended to the varieties $\mathcal{V}_n$ ($= \mathrm{V}(\mathbf{A}_n)$), for all $n \geq 3$, but that the sentence playing the role of (E3) becomes very complicated as n increases. (Its conclusion has five disjuncts even when $n = 3$.) We have therefore not been able to formulate an elegant uniform generalization of the proof.

CHAPTER 7

RESIDUATION NEARLATTICES AND AN EMBEDDING THEOREM

The problem of how 'best' to embed a semilattice into a distributive lattice was probably first addressed by MacNeille [Mac37]. Note that a hereditary subset of a distributive lattice $\mathbf{L} = \langle L; \sqcap, \sqcup \rangle$ is necessarily a meet-subsemilattice of $\mathbf{L}$ in which all nonempty (upper) bounded subsets have joins (the 'upper bound' property); also the operation $\sqcap$ distributes over all such joins. A meet semilattice with these (abstracted) properties is called a distributive 'nearlattice'. Implicit in [Mac37] is an important converse: every distributive nearlattice $\mathbf{A}$ embeds as an initial segment (in fact as a hereditary partial sublattice) in a distributive lattice $\mathbf{B}$. The sublattice $\mathbf{A}^\circ$ of $\mathbf{B}$ generated by $\mathbf{A}$ may be considered a canonical distributive lattice extension of $\mathbf{A}$. (Also see [Fle76], [FS79] and [Stu92] for much additional information.)

In this chapter, we consider 'distributive residuation lattices', namely, those algebras in $\mathcal{H}_{\{\dot{-}, \sqcap, \sqcup\}}$ whose underlying lattice order is distributive. A *hereditary* subalgebra of the $\langle \dot{-}, \sqcap, 0 \rangle$–reduct of a distributive residuation lattice is an algebra in $\mathcal{H}_{\{\dot{-}, \sqcap\}}$ whose underlying meet semilattice is a distributive nearlattice and that satisfies

$$(x \sqcup y) \dot{-} z \approx (x \dot{-} z) \sqcup (y \dot{-} z),$$

whenever the left hand join exists. Such an algebra we shall call a 'distributive residuation nearlattice'. The main result of this chapter will be the converse of the above observation, namely that every distributive residuation nearlattice is embeddable, as a hereditary partial subalgebra, into a distributive residuation lattice.

Section 1 contains some necessary definitions. In Section 2 we show that the residuation operation of a distributive residuation nearlattice $\mathbf{A}$ can be extended to the canonical lattice extension $\mathbf{A}^\circ$ in such a way that $\mathbf{A}^\circ$ becomes a distributive residuation lattice. Since the natural morphisms between residuation nearlattices are not purely algebraic homomorphisms, we phrase in category-theoretic terms a result to the effect that the association $\mathbf{A} \mapsto \mathbf{A}^\circ$ behaves well with respect to the extension of suitable morphisms. In Section 3 we show that the ideal lattices of $\mathbf{A}$ and $\mathbf{A}^\circ$ are isomorphic. Consequently, second order algebraic properties such as simplicity and subdirect irreducibility are preserved by the canonical extension. We also show that when $n \in \omega$ and $\mathbf{A}$ is a distributive residuation nearlattice that lies in $\mathcal{H}^n_{\{\dot{-},\sqcap\}}$ (i.e., that satisfies $x \dot{-} (x \dot{-} y) \dot{-} ny \approx 0$) and satisfies $(x \dot{-} y) \sqcap (y \dot{-} x) \approx 0$, then $\mathbf{A}$ and the $\langle \dot{-}, \sqcap, 0\rangle$-reduct of $\mathbf{A}^\circ$ belong to the same varieties.

The results here generalize several results from [RS88] (and hence also [CST84] and [Stu82]) which concern BCK-algebras. An example is given at the end of Section 4 to show that these generalizations are essential.

In Section 4 we show that the $\langle \dot{-}, 0\rangle$-subreducts of distributive residuation lattices form a proper subquasivariety of $\mathcal{H}_{\{\dot{-},\sqcap\}}$.

7.1. Preliminaries. Recall that the quasivariety $\mathcal{H}_{\{\dot{-},\sqcap,\sqcup\}}$ was defined and axiomatized in Section 2.2 (see Corollaries 2.11 and 2.15). We denote by $\mathcal{D}$ the class of all algebras in $\mathcal{H}_{\{\dot{-},\sqcap,\sqcup\}}$ whose $\langle \sqcap, \sqcup, 0\rangle$-reducts are distributive lattices, i.e., that satisfy

(D) $\quad (x \sqcup y) \sqcap z \approx (x \sqcap z) \sqcup (y \sqcap z)$.

An algebra in $\mathcal{D}$ is called a *distributive residuation lattice.* $\mathcal{D}$ is a proper relative subvariety of $\mathcal{H}_{\{\dot{-},\sqcap,\sqcup\}}$ [OK85] (see also Example 7.14); it is not a variety (as witnessed by the algebra in the proof of Theorem 4.1). We remark that no explicit axiomatization of the class of $\langle \dot{-}, \sqcap, 0\rangle$-subreducts of elements of $\mathcal{D}$ is known. This is an open problem posed in [OK85].

Let S be a set. We define

$$S_\omega = \{X \subseteq S : 0 < |X| < \aleph_0\}.$$

Now let $\mathbf{S} = \langle S; \leq\rangle$ be a partially ordered set. If every element of S has the form $\bigsqcup^{\mathbf{S}} X$ for some $X \in T_\omega$ then T is said to be a *finitely-join-dense subset* of S. $\mathbf{S}$ is said to have the *upper bound property* if for each $X \in S_\omega$, $\bigsqcup^{\mathbf{S}} X$ exists whenever the elements of X have a common upper bound in S (equivalently, for every $a \in S$, $(a]$ is a join semilattice).

If $\mathbf{S} = \langle S; \sqcap\rangle$ is a meet semilattice then for $X, Y \subseteq S$ and $c \in S$, we define

$$X \sqcap Y = \{a \sqcap b : a \in X; b \in Y\} \;\; ; \;\; c \sqcap Y = \{c\} \sqcap Y.$$

A meet semilattice satisfying the upper bound property is called a *nearlattice.* If $\mathbf{S}$, $\mathbf{U}$ are nearlattices then a semilattice homomorphism $f : \mathbf{S} \to \mathbf{U}$ is called a *nearlattice homomorphism* if

$$f(\bigsqcup^{\mathbf{S}} X) = \bigsqcup^{\mathbf{U}} f(X) \tag{80}$$

for all $X \in S_\omega$ for which $\bigsqcup^{\mathbf{S}} X$ exists. We shall often write $\bigsqcup$ (without superscript) when the underlying nearlattice is understood.

A nearlattice $\mathbf{S}$ which satisfies $(a \sqcup b) \sqcap c = (a \sqcap c) \sqcup (b \sqcap c)$ whenever $a, b, c \in S$ and $a \sqcup b$ exists is called a *distributive nearlattice.* It follows easily that a distributive nearlattice $\mathbf{S}$ satisfies

$$(\bigsqcup X) \sqcap b = \bigsqcup (X \sqcap b) \tag{81}$$

for all $X \in S_\omega$ for which $\bigsqcup X$ exists and all $b \in S$. (Equivalently, a meet semilattice $\mathbf{S}$ is a distributive nearlattice if and only if $(a]$ is a distributive lattice for each $a \in S$.)

An algebra $\mathbf{A} = \langle A; \dot{-}, \sqcap, 0\rangle \in \mathcal{H}_{\{\dot{-}, \sqcap\}}$ for which $\langle A; \sqcap, 0\rangle$ is a [distributive] nearlattice and which satisfies

$$(a \sqcup b) \dot{-} c = (a \dot{-} c) \sqcup (b \dot{-} c)$$

whenever $a, b, c \in A$ and $a \sqcup b$ exists is called a [*distributive*] *residuation nearlattice.* Note that every hereditary subalgebra of the $\langle \dot{-}, \sqcap, 0\rangle$–reduct of a [distributive] residuation lattice is a [distributive] residuation nearlattice[15].

Let $\mathbf{A}$ be a residuation nearlattice. If we define

$$X \dot{-} b = \{a \dot{-} b : a \in X\}$$

for $X \subseteq A$ and $b \in A$, then it follows easily that $\mathbf{A}$ satisfies

$$(\bigsqcup X) \dot{-} b = \bigsqcup (X \dot{-} b) \tag{82}$$

for every $X \in A_\omega$ for which $\bigsqcup X$ exists and all $b \in A$. If $a, b, c \in A$ then the upper bound property implies that $(a \dot{-} b) \sqcup (a \dot{-} c)$ exists (since a is a common upper bound of $a \dot{-} b$ and $a \dot{-} c$). It follows from (A11) (see page 18) that $(a \dot{-} b) \sqcup (a \dot{-} c) \leq a \dot{-} (b \sqcap c)$. Moreover,

$$\begin{aligned}
& a \dot{-} (b \sqcap c) \dot{-} ((a \dot{-} b) \sqcup (a \dot{-} c)) \\
= \; & a \dot{-} (b \sqcap c) \dot{-} ((a \dot{-} b) \sqcup (a \dot{-} c)) \dot{-} (a \dot{-} b \dot{-} ((a \dot{-} b) \sqcup (a \dot{-} c))) \dot{-} \\
& \quad (a \dot{-} c \dot{-} ((a \dot{-} b) \sqcup (a \dot{-} c))) \quad \text{(by (C5) and (C6), page 42)} \\
= \; & 0 \quad \text{(by } (Y_1)\text{, page 42).}
\end{aligned}$$

Thus $a \dot{-} (b \sqcap c) \leq (a \dot{-} b) \sqcup (a \dot{-} c)$, hence

[15] A residuation lattice must not be confused with the much less general notion of a *residuated* lattice appearing, e.g., in [WD39].

$$(83) \qquad a \dot{-} (b \sqcap c) = (a \dot{-} b) \sqcup (a \dot{-} c).^{16}$$

If $\mathbf{A}$ and $\mathbf{B}$ are residuation nearlattices [resp. lattices] and $f : \mathbf{A} \to \mathbf{B}$ is a nearlattice [resp. lattice] homomorphism, then f is called a *residuation nearlattice* [resp. *lattice*] *homomorphism* if $f(a \dot{-}^{\mathbf{A}} b) = f(a) \dot{-}^{\mathbf{B}} f(b)$ for all $a, b \in A$.

Another example: if $\langle A; \oplus, \dot{-}, \sqcap, 0\rangle \in \mathcal{H}_{\{\oplus, \dot{-}, \sqcap\}}$ and $\langle A; \sqcap, 0\rangle$ is a [distributive] nearlattice then $\langle A; \dot{-}, \sqcap, 0\rangle$ is a [distributive] residuation nearlattice. To see this, note first that, for all $x, y \in A$, $x \oplus y$ (and $y \oplus x$) is an upper bound of x, y, hence $x \sqcup y$ exists in A by the upper bound property. Let $a, b, c \in A$, so $a \sqcup b$ and $(a \dot{-} c) \sqcup (b \dot{-} c)$ exist. Suppose $d \in A$ such that $a \dot{-} c \leq d$ and $b \dot{-} c \leq d$, so $a \leq d \oplus c$ and $b \leq d \oplus c$. Then

$$(a \sqcup b) \dot{-} c \dot{-} d \leq (d \oplus c) \dot{-} c \dot{-} d = 0$$

by (A6) (see page 18), so $(a \sqcup b) \dot{-} c \leq d$. Thus, $(a \sqcup b) \dot{-} c = (a \dot{-} c) \sqcup (b \dot{-} c)$, as required.

7.2. The Construction. We outline here a construction which appears in detail in [FS79] (also see [Fle76]). Earlier work on this construction can be found in [Mac37].

Let $\mathbf{D} = \langle D; \leq\rangle$ be a distributive nearlattice. The relation $\leq'$ on D_ω defined by

$$X \leq' Y \text{ if and only if } a = \bigsqcup a \sqcap Y \text{ for each } a \in X,$$

where $X, Y \in D_\omega$, is a quasiorder, hence $(\leq') \cap (\leq')^{-1}$ is an equivalence relation on D compatible with $\leq$. Let $D^\circ = D_\omega / (\leq' \cap (\leq')^{-1})$. We denote by $\varepsilon = \varepsilon_{\mathbf{D}} : D_\omega \to D^\circ$ the canonical surjection, i.e.,

$$\varepsilon(X) = \xi \text{ if and only if } X \in \xi \in D^\circ.$$

Also, we define a mapping $e = e_{\mathbf{D}} : D \to D^\circ$ by

$$e(a) = \varepsilon(\{a\}) \qquad (a \in D).$$

Finally, the relation on D° (also denoted $\leq$) defined by

$$\xi \leq \eta \text{ if and only if } (\forall X \in \xi)(\forall Y \in \eta)\ X \leq' Y$$

[16] By results from Chapter 2, a residuation nearlattice $\mathbf{A}$ is also embeddable into an algebra $\mathbf{B} \in \mathcal{H}_{\{\dot{-}, \sqcap, \sqcup\}}$ which satisfies (83) universally (see Corollary 2.15). The temptation to infer the above result about $\mathbf{A}$ directly from this identity of $\mathbf{B}$ must be resisted, however, because the partial join operation of $\langle A; \leq\rangle$ need not coincide with the restriction to A of the operation $\sqcup^{\mathbf{B}}$. (This phenomenon manifests itself repeatedly in the present chapter.)

(equivalently, $\xi \leq \eta$ if and only if $(\exists X \in \xi)(\exists Y \in \eta)\ X \leq' Y$)
where $\xi, \eta \in D^\circ$, is a partial order. The following results are proved in [FS79]:

a. $\mathbf{D}^\circ = \langle D^\circ; \leq \rangle$ *is a distributive lattice.*

If for each $\xi \in D^\circ$ *we choose a representative* $\nu(\xi) \in \xi$ *then for all* $\eta, \zeta \in D^\circ$,

$$\eta = \varepsilon(\nu(\eta)) = \bigsqcup^{\mathbf{D}^\circ} e[\nu(\eta)],$$

(hence $e(D)$ *is a finitely-join-dense subset of* D°*) and*

$$\eta \sqcup \zeta = \varepsilon(\nu(\eta) \cup \nu(\zeta)), \tag{84}$$

$$\eta \sqcap \zeta = \varepsilon(\{a \sqcap b : a \in \nu(\eta); b \in \nu(\zeta)\}). \tag{85}$$

b. $e : D \to D^\circ$ *is a one-to-one nearlattice homomorphism from* $\mathbf{D}$ *into* $\mathbf{D}^\circ$. *If* $Y \subseteq D$ *and* $\sqcap Y^{\mathbf{D}}$ *exists, then* $e(\sqcap^{\mathbf{D}} Y) = \sqcap^{\mathbf{D}^\circ} e[Y]$. *If* D *has a least element* 0 *then* $e(0)$ *is the least element of* $\mathbf{D}^\circ$. *Moreover,* $e[D]$ *is a hereditary subset of* $\mathbf{D}^\circ$.

c. *If* $\mathbf{L}$ *is a distributive lattice and* $f : \mathbf{D} \to \mathbf{L}$ *is a nearlattice homomorphism, then there exists a unique lattice homomorphism* $g : \mathbf{D}^\circ \to \mathbf{L}$ *such that* $ge = f$. *The mapping* g *is one-to-one if* f *is.*

Since e is one-to-one and $e[D]$ is clearly a generating set for the lattice $\langle D^\circ; \sqcap, \sqcup \rangle$, we call $\mathbf{D}^\circ$ the *canonical (lattice) extension* of $\mathbf{D}$.

Next, we show that the above construction can be used to embed distributive *residuation* nearlattices into distributive *residuation* lattices; i.e., we show that the canonical extension $\mathbf{A}^\circ$ of a distributive residuation nearlattice $\mathbf{A}$ may be enriched with a residuation operation (extending that of $\mathbf{A}$) which makes $\mathbf{A}^\circ$ a distributive residuation lattice.[17]

For the rest of this section, $\mathbf{A} = \langle A; \dot{-}, \sqcap, 0 \rangle$ *will be a distributive residuation nearlattice.* Consider $\langle A^\circ; \sqcap, \sqcup, e(0) \rangle$, where $\langle A^\circ; \sqcap, \sqcup \rangle$ is the canonical extension of $\langle A; \leq \rangle$. We can define a binary operation $\dot{-}^{\mathbf{A}^\circ}$ on A° in the following way: for $\xi, \eta \in A^\circ$, $X \in \xi$ and $Y \in \eta$, define

$$\xi \dot{-}^{\mathbf{A}^\circ} \eta = \varepsilon(\{a \dot{-} \bigsqcup a \sqcap Y : a \in X\}). \tag{86}$$

This definition originates in [Stu82]. That $\dot{-}^{\mathbf{A}^\circ}$ is well-defined (i.e., does not depend on the choice of X and Y) can be proved using the arguments of [RS88, Lemmas 2.2 and 2.3], which were stated in the framework of BCK–algebras, but generalize effortlessly. We shall show that the algebra $\mathbf{A}^\circ = \langle A^\circ; \dot{-}^{\mathbf{A}^\circ}, \sqcap, \sqcup, e(0) \rangle$ satisfies (A1)–(A4), (C2), (C3), (C5), (C6), (C8), (C9),

[17]Once again, the embeddings of Chapter 2 cannot be usefully invoked here, as they may fail to preserve the partial join operation of $\langle A; \leq \rangle$.

(C10) (see pages 41–43) and (D), i.e., that $\mathbf{A}^\circ \in \mathcal{D}$ (by Corollary 2.15). Henceforth we shall drop the superscript from $\dot{-}^{\mathbf{A}^\circ}$. It will always be clear from the context to which operation we are referring.

We shall need the following fact: for $\xi, \eta \in A^\circ$,

$$\xi \dot{-} \eta = e(0) \quad \text{if and only if} \quad \xi \leq \eta. \tag{87}$$

This fact is derived as follows:

$$\begin{aligned} \xi \leq \eta \quad &\text{iff} \quad (\exists X \in \xi)(\exists Y \in \eta)\, X \leq' Y \\ &\text{iff} \quad (\exists X \in \xi)(\exists Y \in \eta)\, a = \bigsqcup a \sqcap Y \text{ for each } a \in X \\ &\text{iff} \quad (\exists X \in \xi)(\exists Y \in \eta)\, a \dot{-} \bigsqcup a \sqcap Y = 0 \text{ for each } a \in X. \end{aligned}$$

The last observation follows from the fact that $\bigsqcup a \sqcap Y \leq a$ for each $a \in X$. Now

$$\begin{aligned} \xi \dot{-} \eta = e(0) = \varepsilon(\{0\}) \quad &\text{iff} \quad \{a \dot{-} \bigsqcup a \sqcap Y : a \in X\} = \{0\} \\ &\text{iff} \quad a \dot{-} \bigsqcup a \sqcap Y = 0 \text{ for each } a \in X, \end{aligned}$$

and the result follows.

That $\mathbf{A}^\circ$ satisfies the identities (A2), (A3), (A4), (C2), (C3), (C5), (C6) and (C9) is a straightforward consequence of the definitions of $\dot{-}$, $\sqcap$ and $\sqcup$ on A° (see [RS88, Section 2] for some of the details). That (D) holds is evident from **a** on page 138. We give proofs that the remaining identities also hold.

Lemma 7.1. *For $a, b, c, d \in A$ and $Y, Z \in A_\omega$ the following hold:*

(i) $a \dot{-} (b \sqcap d) \dot{-} (c \dot{-} b) \dot{-} (a \dot{-} c) = a \dot{-} d \dot{-} (c \dot{-} b) \dot{-} (a \dot{-} c)$,

(ii) $(a \sqcap c) \dot{-} ((\bigsqcup a \sqcap Y) \sqcap (\bigsqcup c \sqcap Y)) = (a \sqcap c) \dot{-} \bigsqcup c \sqcap Y$,

(iii) $(a \sqcap c) \dot{-} \bigsqcup a \sqcap Y \leq c \dot{-} \bigsqcup c \sqcap Y$,

(iv) $a \dot{-} (\bigsqcup a \sqcap Y) \dot{-} (\bigsqcup (a \dot{-} \bigsqcup a \sqcap Y) \sqcap \{c \dot{-} \bigsqcup c \sqcap Y : c \in Z\}) \leq a \dot{-} \bigsqcup a \sqcap Z$.

Proof. (i)

$$\begin{aligned} & a \dot{-} (b \sqcap d) \dot{-} (c \dot{-} b) \dot{-} (a \dot{-} c) \\ = \; & ((a \dot{-} b) \sqcup (a \dot{-} d)) \dot{-} (c \dot{-} b) \dot{-} (a \dot{-} c) \quad \text{(by (83))} \\ = \; & ((a \dot{-} b \dot{-} (c \dot{-} b)) \sqcup (a \dot{-} d \dot{-} (c \dot{-} b))) \dot{-} (a \dot{-} c) \quad \text{(by (82))} \\ = \; & (a \dot{-} b \dot{-} (c \dot{-} b) \dot{-} (a \dot{-} c)) \sqcup (a \dot{-} d \dot{-} (c \dot{-} b) \dot{-} (a \dot{-} c)) \quad \text{(by (82))} \\ = \; & 0 \sqcup (a \dot{-} d \dot{-} (c \dot{-} b) \dot{-} (a \dot{-} c)) \quad \text{(by (A1))} \\ = \; & a \dot{-} d \dot{-} (c \dot{-} b) \dot{-} (a \dot{-} c). \end{aligned}$$

(ii) We have $\bigsqcup a \sqcap c \sqcap Y \leq (\bigsqcup a \sqcap Y) \sqcap (\bigsqcup c \sqcap Y)$, hence, by (A11) (see page 18)

$$\begin{aligned}
& (a \sqcap c) \dot{-} ((\bigsqcup a \sqcap Y) \sqcap (\bigsqcup c \sqcap Y)) \\
\leq\ & (a \sqcap c) \dot{-} \bigsqcup a \sqcap c \sqcap Y \\
=\ & (a \sqcap c) \dot{-} \bigsqcup a \sqcap c \sqcap c \sqcap Y \\
=\ & (a \sqcap c) \dot{-} ((a \sqcap c) \sqcap \bigsqcup c \sqcap Y) \quad \text{(by (81))} \\
=\ & (a \sqcap c) \dot{-} \bigsqcup c \sqcap Y \quad \text{(by (C11), page 43)} \\
\leq\ & (a \sqcap c) \dot{-} ((\bigsqcup a \sqcap Y) \sqcap (\bigsqcup c \sqcap Y)) \quad \text{(by (A11))},
\end{aligned}$$

so (ii) holds.

(iii) It follows from (i) and the fact that $(a \sqcap c) \dot{-} c = 0$ that

$$\begin{aligned}
& (a \sqcap c) \dot{-} (\bigsqcup a \sqcap Y) \dot{-} (c \dot{-} \bigsqcup c \sqcap Y) \\
=\ & (a \sqcap c) \dot{-} ((\bigsqcup a \sqcap Y) \sqcap (\bigsqcup c \sqcap Y)) \dot{-} (c \dot{-} \bigsqcup c \sqcap Y) \\
=\ & (a \sqcap c) \dot{-} (\bigsqcup c \sqcap Y) \dot{-} (c \dot{-} \bigsqcup c \sqcap Y) \quad \text{(by (ii))} \\
\leq\ & (a \sqcap c) \dot{-} c = 0 \quad \text{(by (A1))},
\end{aligned}$$

hence (iii) follows.

(iv)

$$\begin{aligned}
& (\bigsqcup (a \dot{-} \bigsqcup a \sqcap Y) \sqcap \{c \dot{-} \bigsqcup c \sqcap Y : c \in Z\}) \sqcap ((\bigsqcup a \sqcap Z) \dot{-} \bigsqcup a \sqcap Y) \\
=\ & \bigsqcup \{(a \dot{-} \bigsqcup a \sqcap Y) \sqcap (c \dot{-} \bigsqcup c \sqcap Y) \sqcap (\bigsqcup (a \sqcap Z) \dot{-} \bigsqcup a \sqcap Y) : c \in Z\} \\
& \text{(by (81))} \\
=\ & \bigsqcup \{(a \dot{-} \bigsqcup a \sqcap Y) \sqcap (c \dot{-} \bigsqcup c \sqcap Y) \sqcap (\bigsqcup \{(a \sqcap c') \dot{-} \bigsqcup a \sqcap Y : c' \in Z\}) : c \in Z\} \\
& \text{(by (82))} \\
=\ & \bigsqcup \{\bigsqcup \{(a \dot{-} \bigsqcup a \sqcap Y) \sqcap (c \dot{-} \bigsqcup c \sqcap Y) \sqcap ((a \sqcap c') \dot{-} \bigsqcup a \sqcap Y) : c' \in Z\} : c \in Z\} \\
& \text{(by (81))} \\
=\ & \bigsqcup \{\bigsqcup \{(c \dot{-} \bigsqcup c \sqcap Y) \sqcap ((a \sqcap c') \dot{-} \bigsqcup a \sqcap Y) : c' \in Z\} : c \in Z\} \\
=\ & \bigsqcup \{\bigsqcup \{(c \dot{-} \bigsqcup c \sqcap Y) \sqcap ((a \sqcap c') \dot{-} \bigsqcup a \sqcap Y) : c \in Z\} : c' \in Z\} \\
=\ & \bigsqcup \{(((a \sqcap c') \dot{-} \bigsqcup a \sqcap Y) \sqcap (c' \dot{-} \bigsqcup c' \sqcap Y)) \sqcup (((a \sqcap c') \dot{-} \bigsqcup a \sqcap Y) \sqcap \\
& \quad \bigsqcup \{c \dot{-} \bigsqcup c \sqcap Y : c \in Z \text{ and } c \neq c'\}) : c' \in Z\} \quad \text{(by (81))} \\
=\ & \bigsqcup \{((a \sqcap c') \dot{-} \bigsqcup a \sqcap Y) \sqcup (((a \sqcap c') \dot{-} \bigsqcup a \sqcap Y) \sqcap \\
& \quad \bigsqcup \{c \dot{-} \bigsqcup c \sqcap Y : c \in Z \text{ and } c \neq c'\}) : c' \in Z\} \qquad \text{(by (iii))} \\
=\ & \bigsqcup \{(a \sqcap c') \dot{-} \bigsqcup a \sqcap Y : c' \in Z\} \\
=\ & (\bigsqcup \{a \sqcap c' : c' \in Z\}) \dot{-} \bigsqcup a \sqcap Y \quad \text{(by (82))} \\
=\ & (\bigsqcup a \sqcap Z) \dot{-} \bigsqcup a \sqcap Y.
\end{aligned}$$

Thus

$$\bigsqcup a \sqcap Z \dot{-} \bigsqcup a \sqcap Y \leq \bigsqcup (a \dot{-} \bigsqcup a \sqcap Y) \sqcap \{c \dot{-} \bigsqcup c \sqcap Y : c \in Z\},$$

hence, by (A11),

$$\begin{aligned} & a \dot{-} (\bigsqcup a \sqcap Y) \dot{-} (\bigsqcup (a \dot{-} \bigsqcup a \sqcap Y) \sqcap \{c \dot{-} \bigsqcup c \sqcap Y : c \in Z\})) \\ \leq\ & a \dot{-} (\bigsqcup a \sqcap Y) \dot{-} ((\bigsqcup a \sqcap Z) \dot{-} \bigsqcup a \sqcap Y) \\ \leq\ & a \dot{-} \bigsqcup a \sqcap Z \quad \text{(by (A1))}. \end{aligned}$$

□

Lemma 7.2. *Let $\xi, \eta, \zeta \in A^\circ$ and let $X \in \xi$, $Y \in \eta$, $Z \in \zeta$. Then*

$$\xi \dot{-} \eta \dot{-} (\zeta \dot{-} \eta) \dot{-} (\xi \dot{-} \zeta) = e(0).$$

Proof. We shall show that $\xi \dot{-} \eta \dot{-} (\zeta \dot{-} \eta) \leq \xi \dot{-} \zeta$ and appeal to (87). By definition (86),

$$\xi \dot{-} \zeta = \varepsilon(\{a \dot{-} \bigsqcup a \sqcap Z : a \in X\}).$$

Also by (86) we have $\{a \dot{-} \bigsqcup a \sqcap Y : a \in X\} \in \xi \dot{-} \eta$ and $\{c \dot{-} \bigsqcup c \sqcap Y : c \in Z\} \in \zeta \dot{-} \eta$ hence, by (86),

$$\begin{aligned} & \xi \dot{-} \eta \dot{-} (\zeta \dot{-} \eta) = \\ & \varepsilon(\{a \dot{-} (\bigsqcup a \sqcap Y) \dot{-} \bigsqcup \{(a \dot{-} \bigsqcup a \sqcap Y) \sqcap \{c \dot{-} \bigsqcup c \sqcap Y : c \in Z\}\} : a \in X\}). \end{aligned}$$

To show that $\xi \dot{-} \eta \dot{-} (\zeta \dot{-} \eta) \leq \xi \dot{-} \zeta$ it is only necessary to show that $\{\beta(a) : a \in X\} \leq' \{\alpha(a) : a \in X\}$, where for each $a \in X$,

$$\begin{aligned} & \alpha(a) = a \dot{-} \bigsqcup a \sqcap Z, \\ & \beta(a) = a \dot{-} (\bigsqcup a \sqcap Y) \dot{-} \bigsqcup (a \dot{-} \bigsqcup a \sqcap Y) \sqcap \{c \dot{-} \bigsqcup c \sqcap Y : c \in Z\}. \end{aligned}$$

Let $a \in X$. Then $\beta(a) \leq \alpha(a)$ by Lemma 7.1 (iv), so

$$\begin{aligned} & \beta(a) \sqcap \bigsqcup \beta(a) \sqcap \{\alpha(a') : a' \in X\} \\ =\ & \beta(a) \sqcap ((\beta(a) \sqcap \alpha(a)) \sqcup \bigsqcup \{\beta(a) \sqcap \alpha(a') : a' \in X \text{ and } a' \neq a\}) \\ =\ & \beta(a) \sqcap (\beta(a) \sqcup \bigsqcup \{\beta(a) \sqcap \alpha(a') : a' \in X \text{ and } a' \neq a\}) \\ =\ & \beta(a). \end{aligned}$$

□

Lemma 7.3. *Let $\xi, \eta, \zeta \in A^\circ$ and let $X \in \xi$, $Y \in \eta$, $Z \in \zeta$. Then*

$$\xi \dot{-} (\eta \sqcap \zeta) = (\xi \dot{-} \eta) \sqcup (\xi \dot{-} \zeta)$$

Proof. For each $a \in X$,

$$\begin{aligned}
\bigsqcup a \sqcap Y \sqcap Z &= \bigsqcup \{a \sqcap b \sqcap c : b \in Y \text{ and } c \in Z\} \\
&= \bigsqcup \{a \sqcap b \sqcap a \sqcap c : b \in Y \text{ and } c \in Z\} \\
&= \bigsqcup \{\bigsqcup (\{a \sqcap b : b \in Y\} \sqcap (a \sqcap c)) : c \in Z\} \\
&= \bigsqcup \{(a \sqcap c) \sqcap \bigsqcup \{a \sqcap b : b \in Y\} : c \in Z\} \quad \text{(by (81))} \\
&= (\bigsqcup \{a \sqcap b : b \in Y\}) \sqcap (\bigsqcup \{a \sqcap c : c \in Z\}) \quad \text{(by (81))} \\
&= (\bigsqcup a \sqcap Y) \sqcap (\bigsqcup a \sqcap Z).
\end{aligned}$$

Thus

$$\begin{aligned}
a \dot{-} (\bigsqcup a \sqcap Y \sqcap Z) &= a \dot{-} ((\bigsqcup a \sqcap Y) \sqcap (\bigsqcup a \sqcap Z)) \\
&= (a \dot{-} \bigsqcup a \sqcap Y) \sqcup (a \dot{-} \bigsqcup a \sqcap Z) \quad \text{(by (83))}.
\end{aligned}$$

Now, by (85) and (84), respectively,

$$\begin{aligned}
\xi \dot{-} (\eta \sqcap \zeta) &= \varepsilon(\{a \dot{-} (\bigsqcup a \sqcap Y \sqcap Z) : a \in X\}) \\
&= \varepsilon(\{(a \dot{-} \bigsqcup a \sqcap Y) \sqcup (a \dot{-} \bigsqcup a \sqcap Z) : a \in X\}),
\end{aligned}$$

$$(\xi \dot{-} \eta) \sqcup (\xi \dot{-} \zeta) = \varepsilon(\{a \dot{-} \bigsqcup a \sqcap Y : a \in X\} \cup \{a \dot{-} \bigsqcup a \sqcap Z : a \in X\}).$$

Set $\alpha(a) = a \dot{-} \bigsqcup a \sqcap Y$, $\beta(a) = a \dot{-} \bigsqcup a \sqcap Z$. We need to show that

$$\{\alpha(a) \sqcup \beta(a) : a \in X\} \leq' \{\alpha(a) : a \in X\} \cup \{\beta(a) : a \in X\}.$$

Now, for any $a \in X$,

$$\begin{aligned}
& (\alpha(a) \sqcup \beta(a)) \sqcap \bigsqcup (\alpha(a) \sqcup \beta(a)) \sqcap (\{\alpha(a') : a' \in X\} \cup \{\beta(a') : a' \in X\}) \\
= \; & (\alpha(a) \sqcup \beta(a)) \sqcap [(((\alpha(a) \sqcup \beta(a)) \sqcap \alpha(a)) \sqcup ((\alpha(a) \sqcup \beta(a)) \sqcap \beta(a))) \sqcup \\
& \quad \bigsqcup (\alpha(a) \sqcup \beta(a)) \sqcap (\{\alpha(a') : a' \in X; a' \neq a\} \cup \{\beta(a') : a' \in X; a' \neq a\})] \\
= \; & (\alpha(a) \sqcup \beta(a)) \sqcap [(\alpha(a) \sqcup \beta(a)) \sqcup \\
& \quad \bigsqcup (\alpha(a) \sqcup \beta(a)) \sqcap (\{\alpha(a') : a' \in X; a' \neq a\} \cup \{\beta(a') : a' \in X; a' \neq a\})] \\
= \; & \alpha(a) \sqcup \beta(a).
\end{aligned}$$

Thus $\xi \dot{-} (\eta \sqcap \zeta) \leq (\xi \dot{-} \eta) \sqcup (\xi \dot{-} \zeta)$.

Since $\mathbf{A}^\circ$ satisfies (A1), (A2) and (87), it follows that it also satisfies (A11), from which we can deduce that $(\xi \dot{-} \eta) \sqcup (\xi \dot{-} \zeta) \leq \xi \dot{-} (\eta \sqcap \zeta)$, hence the result holds. □

Lemma 7.4. *Let $\xi, \eta \in A^\circ$ and let $X \in \xi$, $Y \in \eta$. Then*

$$(\xi \sqcup \eta) \dot{-} \eta \dot{-} \xi = e(0).$$

Proof.

$$
\begin{aligned}
& (\xi \sqcup \eta) \dot{-} \eta \dot{-} \xi \\
= \;& \varepsilon(X \cup Y) \dot{-} \eta \dot{-} \xi \quad \text{(by (84))} \\
= \;& \varepsilon(\{a \dot{-} \bigsqcup a \sqcap Y : a \in X \cup Y\}) \dot{-} \xi \\
= \;& \varepsilon(\{a \dot{-} \bigsqcup a \sqcap Y : a \in X\} \cup \{a \dot{-} \bigsqcup a \sqcap Y : a \in Y\}) \dot{-} \xi \\
= \;& \varepsilon(\{a \dot{-} \bigsqcup a \sqcap Y : a \in X\} \cup \{0\}) \dot{-} \xi \\
= \;& \varepsilon(\{b \dot{-} \bigsqcup b \sqcap X : b \in \{a \dot{-} \bigsqcup a \sqcap Y : a \in X\} \cup \{0\}\}) \\
= \;& \varepsilon(\{0 \dot{-} \bigsqcup 0 \sqcap X\} \cup \{a \dot{-} (\bigsqcup a \sqcap Y) \dot{-} \bigsqcup (a \dot{-} \bigsqcup a \sqcap Y) \sqcap X : a \in X\}) \\
= \;& \varepsilon(\{0\} \cup \{a \dot{-} (\bigsqcup a \sqcap Y) \dot{-} [((a \dot{-} \bigsqcup a \sqcap Y) \sqcap a) \sqcup \\
& \quad \bigsqcup (a \dot{-} \bigsqcup a \sqcap Y) \sqcap (X \backslash \{a\})] : a \in X\}) \\
= \;& \varepsilon(\{0\} \cup \{a \dot{-} (\bigsqcup a \sqcap Y) \dot{-} [(a \dot{-} \bigsqcup a \sqcap Y) \sqcup \\
& \quad \bigsqcup (a \dot{-} \bigsqcup a \sqcap Y) \sqcap (X \backslash \{a\})] : a \in X\}) \\
= \;& \varepsilon(\{0\} \cup \{0\}) = \varepsilon(\{0\}) = e(0).
\end{aligned}
$$

□

Theorem 7.5. *The algebra* $\mathbf{A}^\circ$ *is a distributive residuation lattice; the mapping* $e : A \longrightarrow A^\circ$ *is a one-to-one residuation nearlattice homomorphism and* $e[A]$ *is a hereditary, finitely-join-dense subset of* $\mathbf{A}^\circ$.

Remark. As noted after the definition of a residuation nearlattice, every hereditary subalgebra of the $\langle \dot{-}, \sqcap, 0\rangle$-reduct of $\mathbf{A}^\circ$ is a distributive residuation nearlattice (since $\mathbf{A}^\circ$ is a distributive residuation lattice). Thus the assumption that $\mathbf{A}$ be a distributive residuation nearlattice cannot be weakened if the above theorem is to hold. Consequently, the hereditary $\langle \dot{-}, \sqcap, 0\rangle$-subreducts of distributive residuation lattices are *exactly* the distributive residuation nearlattices.

Proof. By Lemmas 7.2, 7.3 and 7.4 and the remarks preceeding Lemma 7.1, we have that $\mathbf{A}^\circ$ is a distributive residuation lattice. Let $a, b \in A$. We have that $e(a) \dot{-} e(b) = \varepsilon(\{a\}) \dot{-} \varepsilon(\{b\}) = \varepsilon(\{a \dot{-} (a \sqcap b)\}) = \varepsilon(\{a \dot{-} b\})$ (by (C11)) $= e(a \dot{-} b)$. Thus, by **b** on page 138, e is a one-to-one residuation nearlattice homomorphism. By **a** and **b** on page 138, $e[A]$ is a hereditary, finitely-join-dense subset of $\mathbf{A}^\circ$. □

We may now refer to $\mathbf{A}^\circ = \langle A^\circ; \dot{-}, \sqcap, \sqcup, e(0)\rangle$ as the *canonical residuation lattice extension* of $\mathbf{A}$.

If $\mathbf{C}$ is a distributive residuation nearlattice and $f : \mathbf{A} \longrightarrow \mathbf{C}$ is a residuation nearlattice homomorphism, then $e_{\mathbf{C}} f : \mathbf{A} \longrightarrow \mathbf{C}^\circ$ is also a residuation nearlattice homomorphism. By **c** on page 138, $e_{\mathbf{C}} f$ may be extended to a unique lattice homomorphism from $\mathbf{A}^\circ$ to $\mathbf{C}^\circ$, which we shall denote by f°. (The rule $f^\circ(\bigsqcup^{\mathbf{A}^\circ} W) = \bigsqcup^{\mathbf{C}^\circ} f^\circ[W]$, $W \in A^\circ_\omega$, defines f° unambiguously.) In [RS88, Lemma 2.7], where $\langle A; \dot{-}, 0\rangle$ is assumed to be a BCK–algebra, it is shown that f° preserves $\dot{-}$, i.e., for all $\xi, \zeta \in A^\circ$, $f^\circ(\xi) \dot{-} f^\circ(\zeta) = f^\circ(\xi \dot{-}^{\mathbf{C}^\circ} \zeta)$. The proof used there holds for distributive residuation nearlattices as well, hence we have the following:

Proposition 7.6. *If* $\mathbf{C}$ *is a distributive residuation nearlattice and* $f : \mathbf{A} \to \mathbf{C}$ *is a residuation nearlattice homomorphism, then* $f^\circ : \mathbf{A}^\circ \to \mathbf{C}^\circ$ *is a residuation lattice homomorphism.*

The next corollary follows from the above proposition and **c** on page 138.

Corollary 7.7. *If* $\mathbf{B}$ *is a distributive residuation lattice and* $f : \mathbf{A} \to \mathbf{B}$ *is a residuation nearlattice homomorphism, then there exists a unique residuation lattice homomorphism* $f^\circ : \mathbf{A}^\circ \to \mathbf{B}$ *such that* $f^\circ e = f$. *Moreover, the mapping* f° *is one-to-one if* f *is.*

Since nearlattice morphisms are not describable as homomorphisms of a purely algebraic type, the relationship between classes of structures established here is a category theoretic one.

Indeed, the class of all distributive residuation nearlattices [resp. lattices] together with all residuation nearlattice [resp. lattice] homomorphisms forms a category, which we denote DRN [resp. DRL]. Inasmuch as residuation nearlattice homomorphisms between objects of DRL are residuation lattice homomorphisms, DRL is a full subcategory of DRN and the association $\mathbf{A} \mapsto \mathbf{A}^\circ$, $f \mapsto f^\circ$, defines a functor $^\circ$ from DRN to DRL. The fact that morphisms f from objects of DRN to those of DRL have unique extensions f° in the morphism class of DRL with $f^\circ e = f$ makes $^\circ$ a *reflection* of DRN in DRL. It is called a *simple reflection* because e is one-to-one. We say that the functor $^\circ$ is *mono-preserving* because the last assertion of Corollary 7.7 is true. Summarizing Corollary 7.7 in category theoretic terms, we have

Corollary 7.8. *The functor* $^\circ$ *is a simple mono-preserving reflector, hence* DRL *is a full simple reflective subcategory of* DRN.

7.3. Algebraic Properties of the Canonical Extension. The following identity is satisfied by algebras in $\mathcal{H}_{\{\dot{-}, \sqcap, \sqcup\}}$ (by (C9), page 43) and also by residuation nearlattices whenever the required join exists (by (82), page 136):

$$(\bigsqcup_{i=1}^n x_i) \dot{-} x_1 \dot{-} \ldots \dot{-} x_n \approx 0. \tag{88}$$

It follows by Lemma 4.16(ii) that ideals of such algebras are closed under the formation of existent finite joins.

Let $\mathbf{A}$ be a residuation nearlattice and let $a, b, c \in A$ such that $c \sqcup b$ exists. Then

$$\begin{aligned} a \dot{-} b \dot{-} c &= a \dot{-} b \dot{-} c \dot{-} ((c \sqcup b) \dot{-} b \dot{-} c) \quad \text{(by (88))} \\ &\leq a \dot{-} b \dot{-} ((c \sqcup b) \dot{-} b) \quad \text{(by (A1))} \\ &\leq a \dot{-} (c \sqcup b) \quad \text{(by (A1))}. \end{aligned}$$

It follows by (A10) and (A11) that the following identity holds in all algebras in $\mathcal{H}_{\{\dot{-},\sqcap,\sqcup\}}$ and all residuation nearlattices whenever the required joins exist:

$$(89) \quad x \dot{-} (x \dot{-} (\textstyle\bigsqcup_{i=1}^{n} y_i) \dot{-} \bigsqcup_{j=1}^{m} z_j) \leq x \dot{-} (x \dot{-} y_1 \dot{-} \dots \dot{-} y_n \dot{-} z_1 \dot{-} \dots \dot{-} z_m).$$

For the rest of this section we will assume that $\mathbf{A}$ *is a fixed distributive residuation nearlattice and* $\mathbf{A}^\circ$ *is its canonical distributive residuation lattice extension.*

By the proof of Proposition 4.26, the class of distributive residuation nearlattices that are subreducts of a member of $\mathcal{D}$ need not have the ideal extension property. Nevertheless, we have:

Lemma 7.9. *Let I be an ideal of $\mathbf{A}$. Then $\langle I \rangle_{\mathbf{A}^\circ}$ consists of all elements of A° of the form $\bigsqcup_{i=1}^{n} a_i$, where $n \in \omega$ and $a_1, \dots, a_n \in I$. Thus $\langle I \rangle_{\mathbf{A}^\circ} \cap A = I$.*

Proof. Set $K = \{\bigsqcup_{i=1}^{n} a_i : n \in \omega \text{ and } a_1, \dots, a_n \in I\}$. It follows by (88) that $K \subseteq \langle I \rangle_{\mathbf{A}^\circ}$. Conversely, let $a \in \langle I \rangle_{\mathbf{A}^\circ}$. By Corollary 4.18, there exists a sequence $a_0, a_1, \dots, a_n$ of elements of A° such that $a_n = a$ and for each $i \leq n$, either $a_i \in I$, or $a_i = b \dot{-} (b \dot{-} a_j \dot{-} a_k)$, where $j, k < i$ and $b \in A^\circ$.

We prove by induction on n that $a \in K$. If $n = 0$ then $a_n = a_0 \in I \subseteq K$. Suppose that $n > 0$ and that for all $j < n$, a_j is a finite join of elements of I. Either $a_n \in I \subseteq K$, or $a_n = b \dot{-} (b \dot{-} a_j \dot{-} a_k)$, where $j, k < n$ and $b \in A^\circ$. Moreover, we can assume that

$$\begin{aligned} b &= \textstyle\bigsqcup_{i=1}^{m} b_i && \text{where each } b_i \in A, \\ a_j &= \textstyle\bigsqcup_{s=1}^{p} c_s && \text{where each } c_s \in I, \\ a_k &= \textstyle\bigsqcup_{t=1}^{q} d_t && \text{where each } d_t \in I. \end{aligned}$$

Now,

$$\begin{aligned}
a_n &= b \dot{-} (b \dot{-} a_j \dot{-} a_k) \\
&= (\textstyle\bigsqcup_{i=1}^m b_i) \dot{-} ((\bigsqcup_{r=1}^m b_r) \dot{-} a_j \dot{-} a_k) \\
&= (\textstyle\bigsqcup_{i=1}^m b_i) \dot{-} \bigsqcup_{r=1}^m (b_r \dot{-} a_j \dot{-} a_k) \quad \text{(by (82), page 136)} \\
&= \textstyle\bigsqcup_{i=1}^m (b_i \dot{-} \bigsqcup_{r=1}^m (b_r \dot{-} a_j \dot{-} a_k)) \quad \text{(by (82))} \\
&\leq \textstyle\bigsqcup_{i=1}^m (b_i \dot{-} (b_i \dot{-} a_j \dot{-} a_k)) \quad \text{(by (A11))} \\
&= \textstyle\bigsqcup_{i=1}^m (b_i \dot{-} (b_i \dot{-} (\bigsqcup_{s=1}^p c_s) \dot{-} \bigsqcup_{t=1}^q d_t)) \\
&\leq \textstyle\bigsqcup_{i=1}^m (b_i \dot{-} (b_i \dot{-} c_1 \dot{-} \ldots \dot{-} c_p \dot{-} d_1 \dot{-} \ldots \dot{-} d_q)) \quad \text{(by (89))}.
\end{aligned}$$

Set $e_i = (b_i \dot{-} (b_i \dot{-} c_1 \dot{-} \ldots \dot{-} c_p \dot{-} d_1 \dot{-} \ldots \dot{-} d_q))$ for $i = 1, \ldots, m$. Since $b_i \in A$ and $c_1, \ldots, c_p, d_1, \ldots, d_q \in I$, we have that $e_i \in I$ for each i by Lemma 4.17. Now $a_n \leq \bigsqcup_{i=1}^m e_i$ so, by (81),

$$a_n = (\textstyle\bigsqcup_{i=1}^m e_i) \sqcap a_n = \bigsqcup_{i=1}^m (e_i \sqcap a_n).$$

For $i = 1, \ldots, m$ we have $e_i \sqcap a_n \leq e_i \in I \subseteq A$ hence, by the heredity of A in A° and the heredity of I in A, we infer that $e_i \sqcap a_n \in I$. Thus a_n is a finite join of elements of I, so $a_n \in K$. Thus, $K = \langle I \rangle_{\mathbf{A}^\circ}$. This proves the first statement of the lemma; the second statement follows easily from the first. $\square$

Theorem 7.10. *The lattices* $\mathbf{Id\,A}$ *and* $\mathbf{Id\,A}^\circ$ *are isomorphic. Thus the lattices of* $\mathcal{H}_{\{\dot{-},\sqcap\}}$*-congruences of* $\mathbf{A}$ *and* $\mathcal{H}_{\{\dot{-},\sqcap,\sqcup\}}$*-congruences of* $\mathbf{A}^\circ$ *are isomorphic.* $\mathbf{A}$ *is subdirectly irreducible [resp. simple] if and only if* $\mathbf{A}^\circ$ *is subdirectly irreducible [resp. simple].*

Proof. Define a map $\varphi : \mathrm{Id}\,\mathbf{A} \to \mathrm{Id}\,\mathbf{A}^\circ$ by $\varphi(I) = \langle I \rangle_{\mathbf{A}^\circ}$. For $I, J \in \mathrm{Id}\,\mathbf{A}$ we have that $\varphi(I) \sqcup^{\mathbf{Id\,A}^\circ} \varphi(J) \subseteq \varphi(I \sqcup^{\mathbf{Id\,A}} J)$. Conversely,

$$I \sqcup^{\mathbf{Id\,A}} J = \langle I \cup J \rangle_{\mathbf{A}} \subseteq \langle \varphi(I) \cup \varphi(J) \rangle_{\mathbf{A}^\circ} = \varphi(I) \sqcup^{\mathbf{Id\,A}^\circ} \varphi(J),$$

therefore $\varphi(I \sqcup^{\mathbf{Id\,A}} J) \subseteq \varphi(I) \sqcup^{\mathbf{Id\,A}^\circ} \varphi(J)$, so φ preserves joins.

We have that $\varphi(I \cap J) \subseteq \varphi(I) \cap \varphi(J)$. Conversely, let $a \in \varphi(I) \cap \varphi(J) = \langle I \rangle_{\mathbf{A}^\circ} \cap \langle J \rangle_{\mathbf{A}^\circ}$. By Lemma 7.9, $a = \bigsqcup_{i=1}^n a_i = \bigsqcup_{j=1}^m b_j$, where $n, m \in \omega$ and $a_1, \ldots, a_n \in I$, $b_1, \ldots, b_m \in J$. Thus

$$\begin{aligned}
a = a \sqcap a &= (\textstyle\bigsqcup_{i=1}^n a_i) \sqcap (\bigsqcup_{j=1}^m b_j) \\
&= \textstyle\bigsqcup_{i=1}^n (a_i \sqcap (\bigsqcup_{j=1}^m b_j)) \quad \text{(by (81))} \\
&= \textstyle\bigsqcup_{i=1}^n \bigsqcup_{j=1}^m (a_i \sqcap b_j) \quad \text{(by (81))}.
\end{aligned}$$

Since I and J are hereditary subsets of A, $a_i \sqcap b_j \in I \cap J$ for all i and j. Thus, $a \in \langle I \cap J \rangle_{\mathbf{A}^\circ} = \varphi(I \cap J)$. So $\varphi(I) \cap \varphi(J) \subseteq \varphi(I \cap J)$ and so φ preserves intersections.

Suppose $\varphi(I) = \varphi(J)$, i.e., $\langle I\rangle_{\mathbf{A}^\circ} = \langle J\rangle_{\mathbf{A}^\circ}$. Then, by Lemma 7.9, $I = \langle I\rangle_{\mathbf{A}^\circ} \cap A = \langle J\rangle_{\mathbf{A}^\circ} \cap A = J$, so φ is one-to-one.

Let $K \in \operatorname{Id}\mathbf{A}^\circ$. Then $K \cap A \in \operatorname{Id}\mathbf{A}$ and $\varphi(K \cap A) \subseteq K$. Suppose $a \in K$. Since $\mathbf{A}$ is finitely join dense in $\mathbf{A}^\circ$ and K is hereditary in $\langle A^\circ; \leq\rangle$, $a = \bigsqcup_{i=1}^{n} a_i$, where each $a_i \in K \cap A$. Thus $a \dot{-} a_1 \dot{-} \ldots \dot{-} a_n = (\bigsqcup_{i=1}^{n} a_i) \dot{-} a_1 \dot{-} \ldots \dot{-} a_n = 0$ by (88), hence $a \in \varphi(K \cap A)$. So $K = \varphi(K \cap A)$ and φ is onto, which completes the proof of the first statement of the theorem.

That the lattices of $\mathcal{H}_{\{\dot{-}, \sqcap\}}$-congruences of $\mathbf{A}$ and $\mathcal{H}_{\{\dot{-}, \sqcap, \sqcup\}}$-congruences of $\mathbf{A}^\circ$ are isomorphic follows from Lemma 4.14. The remaining statements of the theorem follow from Proposition 4.21. □

Let $\mathbf{C}$ be a residuation nearlattice that satisfies

$$(Z_n) \qquad x \dot{-} (x \dot{-} y) \dot{-} ny \approx 0.$$

for some $n \in \omega$ and also

$$(59) \qquad z \dot{-} (z \dot{-} (x \dot{-} y)) \dot{-} (z \dot{-} (y \dot{-} x)) \approx 0$$

(equivalently (58): $(x \dot{-} y) \sqcap (y \dot{-} x) \approx 0$). Suppose that $s \approx t$ is an identity over the language $\langle \dot{-}, \sqcap, 0\rangle$ that is satisfied by $\mathbf{C}$. By Birkhoff's Subdirect Decomposition Theorem, $\mathbf{C}$ is a subdirect product of subdirectly irreducible algebras, say $\{\mathbf{C}_i : i \in I\}$, which are homomorphic images of $\mathbf{C}$. By Corollary 5.15, each $\mathbf{C}_i$ is linearly ordered, hence each $\mathbf{C}_i$ (enriched with the natural $\sqcup$) is in $\mathcal{D}$. Moreover, each $\mathbf{C}_i$ satisfies $s \approx t$. Thus $\prod_{i\in I} \mathbf{C}_i \in \mathcal{D}$ and satisfies $s \approx t$. Let f be the subdirect embedding of $\mathbf{C}$ into $\prod_{i\in I} \mathbf{C}_i$. By Corollary 7.7, there exists a one-to-one residuation lattice homomorphism $f^\circ : \mathbf{C}^\circ \to \prod_{i\in I} \mathbf{C}_i$ such that $f^\circ e_{\mathbf{C}} = f$, hence $\mathbf{C}^\circ$ satisfies $s \approx t$ as well. Thus we have the following:

Corollary 7.11. *Let $\mathbf{C}$ be a residuation nearlattice that satisfies (Z_n) for some $n \in \omega$ and also (59), and let $s \approx t$ be an identity over the language $\langle \dot{-}, \sqcap, 0\rangle$. Then $\mathbf{C}$ satisfies $s \approx t$ if and only if $\mathbf{C}^\circ$ does. Thus, $\mathbf{C}$ and the $\langle \dot{-}, \sqcap, 0\rangle$-reduct of $\mathbf{C}^\circ$ belong to the same varieties.*

The following example presents a distributive residuation nearlattice whose $\langle \dot{-}, 0\rangle$-reduct is not a BCK–algebra, thereby showing that the results contained in this chapter are essential generalizations of the results in [RS88] (and therefore, also of [CST84]).

Example 7.12. Let $\mathbf{A} = \langle A; \dot{-}, \sqcap, 0\rangle$, where $A = \{0, a, b, c, d\}$ is a five-element set, $\sqcap$ is the meet operation on A determined by the Hasse diagram in Figure 10; for $x, y \in A$, $x \dot{-} y = 0$ if $x \leq y$; $x \dot{-} 0 = x$, $a \dot{-} b = d$, $a \dot{-} c = a \dot{-} d = a$, $b \dot{-} c = b \dot{-} d = b$, and $c \dot{-} a = c \dot{-} b = c \dot{-} d = c$. One checks routinely that $\mathbf{A} \in \mathcal{H}_{\{\dot{-},\sqcap\}}$ and $\langle A; \sqcap, 0\rangle$ is a distributive nearlattice, so $\mathbf{A}$ is a distributive residuation nearlattice. Moreover, $a \dot{-} b \dot{-} c = 0 \neq d = a \dot{-} c \dot{-} b$, so $\langle A; \dot{-}, 0\rangle$ is not a BCK–algebra.

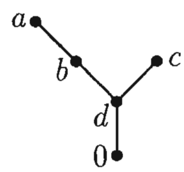


Figure 10.

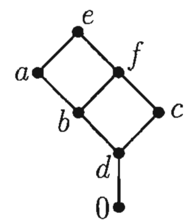


Figure 11.

The algebra $\mathbf{A}^\circ = \langle A^\circ; \dot{-}^{\mathbf{A}^\circ}, \sqcap, \sqcup, 0\rangle$ obtained from $\mathbf{A}$ using the construction in Section 2 has the lattice order described by Figure 11. The residuation operation $\dot{-}^{\mathbf{A}^\circ}$ extends $\dot{-}^{\mathbf{A}}$ as follows: $a \dot{-} f = d$, $e \dot{-} a = e \dot{-} b = c$, $e \dot{-} c = a$, $e \dot{-} d = e$, $e \dot{-} f = d$, $f \dot{-} a = f \dot{-} b = c$, $f \dot{-} c = b$ and $f \dot{-} d = f$.

7.4. Subreducts of Distributive Residuation Lattices. For the remainder of this chapter we consider the $\langle \dot{-}, \sqcap, 0\rangle$– and $\langle \dot{-}, \sqcup, 0\rangle$–subreducts of distributive residuation lattices (i.e., algebras in $\mathcal{D}$). Ono and Komori have shown in [OK85, Theorem 5.15] that the class of all $\langle \dot{-}, \sqcup, 0\rangle$–subreducts of algebras in $\mathcal{D}$ coincides with the quasivariety $\mathcal{H}_{\{\dot{-},\sqcup\}}$; that is, every algebra in $\mathcal{H}_{\{\dot{-},\sqcup\}}$ is embeddable into an algebra in $\mathcal{D}$. Finding an explicit axiomatization of the $\langle \dot{-}, \sqcap, 0\rangle$–subreducts of algebras in $\mathcal{D}$ is an open problem that is implicit in [OK85, §9, Question 6], which effectively asks for an axiomatization *with the separation theorem* of the Hilbert system H $(= H_{\mathrm{BK}})$ extended by the distributive law:

$$(p \wedge (q \vee r)) \rightarrow ((p \wedge q) \vee (p \wedge r)).$$

A first step toward solving this problem is given by the following proposition and subsequent example, which present a quasi-identity not satisfied by $\mathcal{H}_{\{\dot{-},\sqcap\}}$ but satisfied by the $\langle \dot{-}, \sqcap, 0\rangle$–subreducts of algebras in $\mathcal{D}$.

Proposition 7.13. *The* $\langle \dot{-}, \sqcap, 0\rangle$*–quasi-identity*

(90) $\quad x \sqcap (w \dot{-} z) \leq w \dot{-} y$ and $x \leq w \dot{-} (y \sqcap z)$ implies $x \leq w \dot{-} y$

holds in every distributive residuation lattice and hence in every $\langle \dot{-}, \sqcap, 0\rangle$*–subreduct of a distributive residuation lattice.*

Proof. Suppose $\mathbf{A} \in \mathcal{D}$ and $a, b, c, d \in A$ such that

(i) $\quad a \sqcap (d \dot{-} c) \leq d \dot{-} b$,

(ii) $\quad a \leq d \dot{-} (b \sqcap c)$.

By (i) and (D) (see page 135),

$$d \dot{-} b = (a \sqcap (d \dot{-} c)) \sqcup (d \dot{-} b) = (a \sqcup (d \dot{-} b)) \sqcap ((d \dot{-} c) \sqcup (d \dot{-} b)).$$

Thus, since $d \dot{-} c \leq (d \dot{-} c) \sqcup (d \dot{-} b)$,

$$d \dot{-} b \geq (a \sqcup (d \dot{-} b)) \sqcap (d \dot{-} c).$$

Since $d \dot{-} c \geq (a \sqcup (d \dot{-} b)) \sqcap (d \dot{-} c)$,

$$(d \dot{-} b) \sqcap (d \dot{-} c) \geq (a \sqcup (d \dot{-} b)) \sqcap (d \dot{-} c).$$

From the fact that

$$(d \dot{-} b) \sqcap (d \dot{-} c) \leq (a \sqcup (d \dot{-} b)) \sqcap (d \dot{-} c),$$

it follows that

$$(d \dot{-} b) \sqcap (d \dot{-} c) = (a \sqcup (d \dot{-} b)) \sqcap (d \dot{-} c).$$

Now

$$\begin{aligned} (a \sqcup (d \dot{-} b)) \sqcup (d \dot{-} c) &= a \sqcup ((d \dot{-} b) \sqcup (d \dot{-} c)) \\ &= a \sqcup (d \dot{-} (b \sqcap c)) \quad \text{(by (C12), page 44)} \\ &= d \dot{-} (b \sqcap c) \quad \text{(by (ii))} \\ &= (d \dot{-} b) \sqcup (d \dot{-} c) \quad \text{(by (C12))}. \end{aligned}$$

Since $a \sqcup (d \dot{-} b)$ and $d \dot{-} b$ form equal meets and equal joins with $d \dot{-} c$, and $\langle A; \leq \rangle$ is distributive, it follows that $a \sqcup (d \dot{-} b) = d \dot{-} b$, i.e., $a \leq d \dot{-} b$. $\square$

Example 7.14. Let A be a six-element set $\{0, a, b, c, d, e\}$ and let $\leq$ be the partial order on A defined by the Hasse diagram in Figure 12.

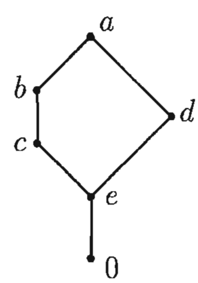


Figure 12.

Let $\mathbf{A} = \langle A; \dot{-}, \sqcap, \sqcup, 0 \rangle$, where $\sqcap$ and $\sqcup$ are the meet and join operations determined by $\leq$; for all $x, y \in A$, $x \dot{-} y = 0$ if $x \leq y$; $x \dot{-} 0 = x$, $a \dot{-} b = a \dot{-} c = d \dot{-} b = d \dot{-} c = d$, $a \dot{-} d = b \dot{-} d = c \dot{-} d = b \dot{-} e = c$, $b \dot{-} c = e$, $a \dot{-} e = a$, $c \dot{-} e = c$ and $d \dot{-} e = d$. It is straightforward to check that $\mathbf{A} \in \mathcal{H}_{\{\dot{-}, \sqcap, \sqcup\}}$ and it is evident from Figure 12 that $\mathbf{A} \notin \mathcal{D}$. We now consider whether the $\langle \dot{-}, \sqcap, 0 \rangle$–reduct of $\mathbf{A}$, $\mathbf{A}^{\sqcap}$ say, is embeddable in an element of

$\mathcal{D}$. By the previous proposition, this is not possible, since $\mathbf{A}^{\sqcap}$ does not satisfy (90):

$$b \sqcap (a \dot{-} c) = b \sqcap d = e \leq c = a \dot{-} d,$$

and

$$b \leq a = a \dot{-} (d \sqcap c),$$

but

$$b \not\leq c = a \dot{-} d.$$

This shows that *the* $\langle \dot{-}, \sqcap, 0 \rangle$*-subreducts of members of* $\mathcal{D}$ *form a proper subquasivariety of* $\mathcal{H}_{\{\dot{-}, \sqcap\}}$.

APPENDIX

TOPOLOGIZING FILTERS ON RINGS

Topologizing filters on rings with identity (mentioned briefly in Example 1.7) are a relatively recent focus of research in ring theory. This expository appendix on them is included because they provide natural and useful examples of (lattice ordered) polrims that are not generally residuated on the right.

8.1. Lattice Ordered Monoids. Let $\langle M; \oplus, 0; \leq \rangle$ be a pomonoid. For $B \subseteq M$ and $a \in M$, we define

$$B \oplus a = \{b \oplus a : b \in B\}, \quad a \oplus B = \{a \oplus b : b \in B\}$$

and we denote the infimum of B in $\langle M; \leq \rangle$ by $\sqcap B$, if this exists. For $a, c \in M$, let

$${}^{c}_{a}B = \{b \in M : c \leq b \oplus a\} \text{ and } B^{c}_{a} = \{b \in M : c \leq a \oplus b\}.$$

Clearly, $\langle M; \oplus, 0; \leq \rangle$ is left [resp. right] residuated if and only if, for each $a, c \in M$, $\sqcap\ {}^{c}_{a}B$ [resp. $\sqcap B^{c}_{a}$] exists and

$$(\sqcap {}^{c}_{a}B) \oplus a = \sqcap (({}^{c}_{a}B) \oplus a)$$
$$[\text{resp.} \quad a \oplus (\sqcap B^{c}_{a}) = \sqcap (a \oplus (B^{c}_{a}))].$$

We may deduce:

Proposition 8.1. *Let* $\mathbf{M} = \langle M; \oplus, 0; \leq \rangle$ *be a pomonoid such that* $\langle M; \leq \rangle$ *is a complete lattice. Then* $\mathbf{M}$ *is left [resp. right] residuated if and only if, for any* $B \cup \{a\} \subseteq M$,

$$(\sqcap B) \oplus a = \sqcap (B \oplus a)$$
$$[\textit{resp.} \quad a \oplus (\sqcap B) = \sqcap (a \oplus B)].$$

Following [Bir73] and [Gol87], we call the pomonoid $\langle M; \oplus, 0; \leq \rangle$ a *lattice ordered monoid* if $\langle M; \leq \rangle$ is a lattice (with binary meet and join operations $\sqcap$, $\sqcup$, say) and $\langle M; \oplus, \sqcap \rangle$ satisfies both of

(C13) $\qquad x \oplus (y \sqcap z) \approx (x \oplus y) \sqcap (x \oplus z);$

(C14) $\qquad (y \sqcap z) \oplus x \approx (y \oplus x) \sqcap (z \oplus x).$

Thus, a pomonoid $\langle M; \oplus, 0; \leq \rangle$ is a lattice ordered monoid whenever it is residuated on both sides and $\langle M; \leq \rangle$ is a complete lattice. Also, by Proposition 2.15, if $\mathbf{A} = \langle A; \oplus, \dot{-}, \sqcap, \sqcup, 0, 1 \rangle \in \mathcal{H}$ has associated order $\leq$ then $\langle A; \oplus, 0; \leq \rangle$ is an integral lattice ordered monoid.

8.2. Ideals of Rings. By a *ring* we mean an associative ring with identity, throughout. Let $\mathbf{R} = \langle R; +, \cdot, -, 0, 1 \rangle$ be a ring. Let $\mathrm{Id}\,\mathbf{R_R}$, $\mathrm{Id}_{\mathbf{R}}\mathbf{R}$ and $\mathrm{Id}\,\mathbf{R}$ denote the sets of right, left and two-sided ideals of $\mathbf{R}$, respectively. (Henceforth, 'ideal' shall always mean two-sided ideal.) For $I \in \mathrm{Id}\,\mathbf{R_R}$, $J \in \mathrm{Id}_{\mathbf{R}}\mathbf{R}$, $B \subseteq R$ and $a \in R$, we define

$$\begin{gathered} Ba = \{ba : b \in B\}, \quad aB = \{ab : b \in B\}, \\ (I :^r B) = \{r \in R : Br \subseteq I\} \in \mathrm{Id}\,\mathbf{R_R}, \\ (J :^l B) = \{r \in R : rB \subseteq J\} \in \mathrm{Id}_{\mathbf{R}}\mathbf{R}, \\ (I :^r a) = (I :^r \{a\}) \text{ and } (J :^l a) = (J :^l \{a\}). \end{gathered}$$

If $I \in \mathrm{Id}\,\mathbf{R}$ then $I \subseteq (I :^r B) \cap (I :^l B)$ for any $B \subseteq R$. If in addition, $B \in \mathrm{Id}\,\mathbf{R_R}$ [resp. $B \in \mathrm{Id}_{\mathbf{R}}\mathbf{R}$] then $(I :^r B) \in \mathrm{Id}\,\mathbf{R}$ [resp. $(I :^l B) \in \mathrm{Id}\,\mathbf{R}$].

Recall (Example 1.7) that $\langle \mathrm{Id}\,\mathbf{R}; \cdot, R; \supseteq \rangle$ (where $\cdot$ is ideal multiplication) is an integral pomonoid and $\langle \mathrm{Id}\,\mathbf{R}; \supseteq \rangle$ is a complete (in fact, a modular, dually algebraic and atomic) lattice. This pomonoid is both left and right residuated, the left and right residuation operations being $:^l$ and $:^r$, respectively. In fact, $\langle \mathrm{Id}\,\mathbf{R}; \cdot, :^l, \sqcup, \cap, R, \{0\} \rangle \in \mathcal{H}$, where $I \sqcup J = \bigcap \{K : I \cup J \subseteq K \in \mathrm{Id}\,\mathbf{R}\} = \{i + j : i \in I; j \in J\}$ for any $I, J \in \mathrm{Id}\,\mathbf{R}$. In particular, $\mathbf{Id\,R} := \langle \mathrm{Id}\,\mathbf{R}; \cdot, :^l, R \rangle$ is a polrim. The distributive identities of Section 1 manifest themselves here as

$$(\bigsqcup \mathcal{B})J = \bigsqcup_{B \in \mathcal{B}} BJ \quad \text{and} \quad J(\bigsqcup \mathcal{B}) = \bigsqcup_{B \in \mathcal{B}} JB$$

for $\mathcal{B} \cup \{J\} \subseteq \mathrm{Id}\,\mathbf{R}$. In particular, $\langle \mathrm{Id}\,\mathbf{R}; \cdot, R; \supseteq \rangle$ is a lattice ordered monoid.

For each $I \in \mathrm{Id}\,\mathbf{R}$, let $\eta(I)$ be the principal filter of the lattice $\mathbf{Id\,R_R}$ generated by I, viz., $\eta(I) = \{K \in \mathrm{Id}\,\mathbf{R_R} : I \subseteq K\}$.

8.3. Topologizing Filters. Again, let $\mathbf{R} = \langle R; +, \cdot, -, 0, 1 \rangle$ be a ring. Consider the (algebraic, modular) lattice $\mathbf{Id\,R_R} = \langle \mathrm{Id}\,\mathbf{R_R}; \subseteq \rangle$ of right ideals of $\mathbf{R}$.

A (nonempty) filter (in the lattice theoretic sense) $\mathcal{F}$ of $\mathbf{Id\,R_R}$ is called a *(right) topologizing filter*[18] on $\mathbf{R}$ if $(I :^r a) \in \mathcal{F}$ whenever $I \in \mathcal{F}$ and $a \in R$. The set of all topologizing filters on $\mathbf{R}$ is denoted by Fil–$\mathbf{R}$. Clearly, $\eta(I) \in$ Fil–$\mathbf{R}$ for every ideal I of $\mathbf{R}$.

The poset $\langle \text{Fil–}\mathbf{R}; \subseteq\rangle$ is an algebraic lattice which is also modular [vdB95, Proposition II.1.6, p68] and atomic [Gol87, Corollary 2.24, p24]. Its greatest and least elements are $\operatorname{Id}\mathbf{R_R}$ and $\{R\}$, respectively. For $\mathcal{F}, \mathcal{G} \in$ Fil–$\mathbf{R}$, define

$$\begin{aligned}\mathcal{F} \oplus \mathcal{G} = \{K \in \operatorname{Id}\mathbf{R_R} :\ & \text{there exists } H \in \mathcal{F} \text{ such that} \\ & (K :^r a) \in \mathcal{G} \text{ for all } a \in H\}.\end{aligned}$$

Then $\{IJ : I \in \mathcal{F}, J \in \mathcal{G}\} \subseteq \mathcal{F} \oplus \mathcal{G} \in$ Fil–$\mathbf{R}$ (hence $\mathcal{F}, \mathcal{G} \subseteq \mathcal{F} \oplus \mathcal{G}$). Now, $\langle \text{Fil–}\mathbf{R}; \oplus, \{R\}; \subseteq\rangle$ is an integral pomonoid. Whenever $\Sigma \cup \{\mathcal{F}\} \subseteq$ Fil–$\mathbf{R}$, we have

$$(\textstyle\bigcap\Sigma) \oplus \mathcal{F} = \bigcap(\Sigma \oplus \mathcal{F});$$

if, in addition, Σ is finite then, also,

$$\mathcal{F} \oplus (\textstyle\bigcap\Sigma) = \bigcap(\mathcal{F} \oplus \Sigma).$$

Thus, $\langle \text{Fil–}\mathbf{R}; \oplus, \{R\}; \subseteq\rangle$ is a lattice ordered monoid and, since $\langle \text{Fil–}\mathbf{R}; \subseteq\rangle$ is a complete lattice, we deduce from Proposition 8.1 that this pomonoid is *left residuated.* We denote its left residuation operation by $\dot{-}$. Thus, $\mathcal{F} \dot{-} \mathcal{G} = \bigcap\{\mathcal{H} \in \text{Fil–}\mathbf{R} :$ for each $F \in \mathcal{F}$, there exists $H \in \mathcal{H}$ such that $(F :^r a) \in \mathcal{G}$ for all $a \in H\}$ for any $\mathcal{F}, \mathcal{G} \in$ Fil–$\mathbf{R}$, and **Fil–R** $:= \langle \text{Fil–}\mathbf{R}; \oplus, \dot{-}, \{R\}\rangle$ is a (lattice ordered) polrim. Moreover, if $\sqcup$ is the join operation of $\langle \text{Fil–}\mathbf{R}; \subseteq\rangle$ then $\langle \text{Fil–}\mathbf{R}; \oplus, \dot{-}, \cap, \sqcup, \{R\}, \operatorname{Id}\mathbf{R_R}\rangle \in \mathcal{H}$.

Proposition 8.2. [Gol87, Propositions 2.7, 3.4, pp17, 31] *For any ring* $\mathbf{R}$, *the map* $\eta : \operatorname{Id}\mathbf{R} \mapsto$ Fil–$\mathbf{R}$ *defined by* $I \mapsto \eta(I) = \{K \in \operatorname{Id}\mathbf{R_R} : I \subseteq K\}$ *(*$I \in \operatorname{Id}\mathbf{R}$*) is a one-to-one homomorphism from* $\langle \operatorname{Id}\mathbf{R}; \cdot, :^l, \sqcup, \cap, R, \{0\}\rangle$ *into* $\langle \text{Fil–}\mathbf{R}; \oplus, \dot{-}, \cap, \sqcup, \{R\}, \operatorname{Id}\mathbf{R_R}\rangle$ *and* $\eta(\bigsqcup\mathcal{B}) = \bigcap \eta[\mathcal{B}]$ *for any* $\mathcal{B} \subseteq \operatorname{Id}\mathbf{R}$.

Thus, $\langle \operatorname{Id}\mathbf{R}; \subseteq\rangle$ is dually isomorphic to a sublattice (also a meet-complete subsemilattice) of $\langle \text{Fil–}\mathbf{R}; \subseteq\rangle$, while **Id R** is isomorphic to a subalgebra of the polrim **Fil–R**. It follows that **Fil–R** carries at least as much information about $\mathbf{R}$ as does $\langle \operatorname{Id}\mathbf{R}; \subseteq\rangle$. Of course, $\langle \operatorname{Id}\mathbf{R}; \subseteq\rangle \cong$ **Con R**, and congruence lattices are the universal algebraist's standard tool for analysing algebras. The next two results illustrate the fact that **Fil–R** is a strictly sharper tool for the

[18] The name derives from the fact that the topologizing filters of a ring $\mathbf{R}$ are just the subsets of $\operatorname{Id}\mathbf{R_R}$ that form neighbourhood bases at 0 for the so-called *linear topologies* on $\mathbf{R}$. Here, a topology τ on R is called *(right) linear* (on $\mathbf{R}$) if the binary operation $+$, the unary operation $-$ and, for each $r \in R$, the operation $a \mapsto ra$ ($a \in R$) are continuous in τ and there exists a neighbourhood base at 0 for τ that consists of right ideals of $\mathbf{R}$. For further details, see [Gol87].

analysis of $\mathbf{R}$ than is $\langle \mathrm{Id}\,\mathbf{R}; \subseteq\rangle$ (and hence $\mathbf{Con}\,\mathbf{R}$). The first of these contrasts with the fact that the condition $|\mathrm{Id}\,\mathbf{R}| = 2$ (i.e., $\mathbf{R}$ is simple) gives very little information about $\mathbf{R}$.

Proposition 8.3. *Let* $\mathbf{R}$ *be a ring.*
(i) (e.g., [Kat83, Theorem 3.2]*)* $|\text{Fil–}\mathbf{R}| = 2$ *if and only if* $\mathbf{R}$ *is a simple artinian ring (i.e.,* $\mathbf{R}$ *is isomorphic to the ring of* $n \times n$ *matrices over some division ring, for some positive integer* n*).*
(ii) $\langle \text{Fil–}\mathbf{R}; \cap, \oplus, ', \{R\}, \mathrm{Id}\,\mathbf{R_R}\rangle$ *is a Boolean algebra (where* $\mathcal{F}' = (\mathrm{Id}\,\mathbf{R_R}) \dot{-} \mathcal{F}$ *for* $\mathcal{F} \in \text{Fil–}\mathbf{R}$*) if and only if* $\mathbf{R}$ *is a semisimple ring (i.e., a finite direct product of simple artinian rings).*

For any $\mathcal{F} \in \text{Fil–}\mathbf{R}$, we have $\bigcap \mathcal{F} \in \mathrm{Id}\,\mathbf{R}$, so $\mathcal{F} \subseteq \eta(\bigcap \mathcal{F})$. If, in addition, $\bigcap \mathcal{F} \in \mathcal{F}$ (so that $\mathcal{F} = \eta(\bigcap \mathcal{F})$), we call $\mathcal{F}$ a *Jansian* topologizing filter.

Proposition 8.4. [BB78, Corollary 3.3] *The following conditions on a ring* $\mathbf{R}$ *(with identity) are equivalent.*
(i) Every (right) topologizing filter on $\mathbf{R}$ *is Jansian;*
(ii) $\eta : \mathbf{Id}\,\mathbf{R} \cong \mathbf{Fil–R}$*;*
(iii) $\mathbf{R}$ *is a right artinian ring, i.e.,* $\langle \mathrm{Id}\,\mathbf{R_R}; \subseteq\rangle$ *satisfies the descending chain condition.*

The next example shows that the polrim $\mathbf{Fil–R}$ need not be right residuated, even if $\mathbf{R}$ is a commutative ring. In particular, the commutativity of $\mathbf{R}$ does not guarantee the commutativity of $\oplus$ in $\mathbf{Fil–R}$. If $\mathbf{R}$ is both commutative and noetherian (i.e., $\langle \mathrm{Id}\,\mathbf{R}; \subseteq\rangle$ satisfies the ascending chain condition) then $\mathbf{Fil–R}$ is a pocrim. This is a consequence of more general results in [vdB1].

Example 8.5. [19] Let $\mathbf{R} = \mathbf{F}[x_0, x_1, x_2, \ldots]$ be the ring of all polynomials over a field $\mathbf{F}$ in denumerably many (commuting) indeterminates $x_0, x_1, x_2, \ldots$. Thus, $\mathbf{R}$ is a commutative ring. For $S \subseteq R$, we write $\langle S\rangle$ for the ideal of $\mathbf{R}$ generated by S, and $\langle s_1, s_2, \ldots\rangle$ for $\langle\{s_1, s_2, \ldots\}\rangle$.

Let $I_n = \langle x_0, x_1, \ldots, x_n\rangle$ for $n \in \omega$, and let $I_\omega = \langle x_0, x_1, \ldots\rangle$. Then $I_0 \subset I_1 \subset \cdots \subset I_\omega$ (where $\subset$ denotes *proper* subset). Let $\mathcal{F}_n = \eta(I_n)$ for $n \in \omega + 1$. Observe that $\bigcap_{n\in\omega} \mathcal{F}_n = \eta(\bigcup_{n\in\omega} \mathcal{F}_n) = \mathcal{F}_\omega$.

Define $\mathcal{G} = \{K \in \mathrm{Id}\,\mathbf{R} : K$ contains some cofinite subset of $\{x_n : n \in \omega\}\}$. We show that

$$\mathcal{G} \oplus (\bigcap_{n\in\omega} \mathcal{F}_n) \neq \bigcap_{n\in\omega} (\mathcal{G} \oplus \mathcal{F}_n),$$

[19]The author thanks Dr J.E. van den Berg for drawing attention to this example.

which shows, by Proposition 8.1, that the pomonoid $\langle \text{Fil–}\mathbf{R}; \oplus, \{R\}; \subseteq \rangle$ is not right residuated. Note that

$$\begin{aligned} \mathcal{G} \oplus (\bigcap_{n\in\omega} \mathcal{F}_n) &= \mathcal{G} \oplus \mathcal{F}_\omega \\ &= \{K \in \operatorname{Id}\mathbf{R} : K \supseteq HI_\omega \text{ for some } H \in \mathcal{G}\} \end{aligned}$$

and

$$\begin{aligned} \mathcal{G} \oplus \mathcal{F}_n &= \mathcal{G} \oplus \eta(I_n) \\ &= \{K \in \operatorname{Id}\mathbf{R} : K \supseteq HI_n \text{ for some } H \in \mathcal{G}\}. \end{aligned}$$

Take $K = \langle \{x_i x_j : i \neq j \text{ and } i, j \in \omega\} \rangle$ and $H = \langle x_{n+1}, x_{n+2}, \ldots \rangle$. Then $K \supseteq HI_n$ and $H \in \mathcal{G}$ as $K \in \mathcal{G} \oplus \mathcal{F}_n$. Thus, $K \in \bigcap_{n\in\omega}(\mathcal{G} \oplus \mathcal{F}_n)$. But $K \not\supseteq HI_\omega$ for any $H \in \mathcal{G}$ because K contains no element of the form x_i^2, whereas HI_ω clearly does. Thus, $K \notin \mathcal{G} \oplus (\bigcap_{n\in\omega} \mathcal{F}_n)$, so $\mathcal{G} \oplus (\bigcap_{n\in\omega} \mathcal{F}_n) \neq \bigcap_{n\in\omega}(\mathcal{G} \oplus \mathcal{F}_n)$, as required.

As with any polrim, **Fil–R** can be embedded in an integral pomonoid that is residuated on both sides, but such a pomonoid would bear little relation to **R**, in general.[20]

For a ring **R**, a (unital) right **R**–module **M** and $\mathcal{F} \in$ Fil–**R**, define

$$\sigma_{\mathcal{F}}(\mathbf{M}) = \{m \in M : mA = \{0^{\mathbf{A}}\} \text{ for some } a \in \mathcal{F}\};$$

$\sigma_{\mathcal{F}}(\mathbf{M})$ is a submodule of **M** (and $\sigma_{\mathcal{F}}(\mathbf{R})$ an ideal of **R**). The functor on the category of right **R**–modules induced by the maps $\mathbf{M} \mapsto \sigma_{\mathcal{F}}(\mathbf{M})$ is called a *torsion preradical* (alias *kernel functor*); it determines $\mathcal{F}$. Torsion preradicals can also be defined abstractly and are often taken as the starting point in a study of topologizing filters. The following conditions on $\mathcal{F} \in$ Fil–**R** are equivalent:

(i) $\mathcal{F} \oplus \mathcal{F} = \mathcal{F}$

(ii) $\sigma_{\mathcal{F}}(\mathbf{M}/\sigma_{\mathcal{F}}(\mathbf{M})) = \{0^{\mathbf{M}/\sigma_{\mathcal{F}}(\mathbf{M})}\}$ for all right **R**–modules **M**.

We call $\mathcal{F}$ a *Gabriel filter* (and $\sigma_{\mathbf{M}}$ a *torsion radical*) if these conditions hold. Such filters facilitate generalizations of the abelian group theoretic notions 'torsion' and 'torsion free', and the ring theoretic notion 'localization', i.e., the formation of various kinds of 'rings of fractions'. Historically, this was the subject's original motivation: see [Gab62], [Mar64], [Gol69], [Lam71]. There is a wealth of literature on Gabriel filters. The condition that every member of Fil–**R** be a Gabriel filter (i.e., that **Fil–R** be a Brouwerian semilattice) has no

[20]Thus, an advantage of the perspective of this thesis (which takes polrims, rather than structures with two residuations, as its starting point) is that the natural models of the theory include the topologizing filter lattices of *all* rings with identity.

known purely ring theoretic characterization but the commutative rings with this property are just the finite direct products of fields [Vio75]. For further results connecting properties of **R** with properties of **Fil**–**R**, see [vdB1], [vdB2], [Gol87] and their bibliographies.

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INDEX

LIST OF SYMBOLS

INDEX OF IDENTITIES

(A1)	$x \dot{-} y \dot{-} (z \dot{-} y) \dot{-} (x \dot{-} z) \approx 0$ (i.e. $x \dot{-} y \dot{-} (z \dot{-} y) \leq x \dot{-} z$)
(A2)	$x \dot{-} 0 \approx x$
(A3)	$0 \dot{-} x \approx 0$ (i.e. $0 \leq x$)
(A4)	$x \dot{-} y \approx 0$ and $y \dot{-} x \approx 0$ implies $x \approx y$
(A5)	$x \dot{-} (y \oplus z) \approx x \dot{-} z \dot{-} y$
(A6)	$(x \oplus y) \dot{-} y \leq x$
(A7)	$((x \oplus y) \dot{-} y) \oplus y \approx x \oplus y$
(A8)	$((x \dot{-} y) \oplus y) \dot{-} y \approx x \dot{-} y$
(A9)	$x \dot{-} x \approx 0$
(A10)	$x \leq y$ implies $x \dot{-} z \leq y \dot{-} z$
(A11)	$x \leq y$ implies $z \dot{-} y \leq z \dot{-} x$
(A12)	$x \dot{-} y \leq x$
(A13)	$x \dot{-} y \dot{-} (x \dot{-} z) \dot{-} (z \dot{-} y) \approx 0$
(A14)	$x \dot{-} y \dot{-} z \approx x \dot{-} z \dot{-} y$
(A15)	$x \dot{-} (x \dot{-} y) \dot{-} y \approx 0$ (i.e. $x \dot{-} (x \dot{-} y) \leq y$)
(C1)	$x \dot{-} 1 \approx 0$
(C2)	$(x \sqcap y) \dot{-} y \approx 0$
(C3)	$(x \sqcap y) \dot{-} x \approx 0$
(C4)	$x \dot{-} (y \sqcap z) \dot{-} w \dot{-} (x \dot{-} y \dot{-} w) \dot{-} (x \dot{-} z \dot{-} w) \approx 0$
(C5)	$x \dot{-} (x \sqcup y) \approx 0$
(C6)	$y \dot{-} (x \sqcup y) \approx 0$
(C7)	$(x \sqcup y) \dot{-} z \dot{-} ((x \dot{-} z) \sqcup (y \dot{-} z)) \approx 0$
(C8)	$(x \sqcup y) \dot{-} y \dot{-} x \approx 0$
(C9)	$(x \sqcup y) \dot{-} z \approx (x \dot{-} z) \sqcup (y \dot{-} z)$
(C10)	$(x \sqcup y) \dot{-} y \approx x \dot{-} y$
(C11)	$x \dot{-} (x \sqcap y) \approx x \dot{-} y$
(C12)	$x \dot{-} (y \sqcap z) \approx (x \dot{-} y) \sqcup (x \dot{-} z)$
(C13)	$x \oplus (y \sqcap z) \approx (x \oplus y) \sqcap (x \oplus z)$
(C14)	$(y \sqcap z) \oplus x \approx (y \oplus x) \sqcap (z \oplus x)$
(Y_0)	$x \dot{-} (y \sqcap z) \dot{-} (x \dot{-} y) \dot{-} (x \dot{-} z) \approx 0$
(Y_n)	$x \dot{-} (y \sqcap z) \dot{-} w_1 \dot{-} \ldots \dot{-} w_n \dot{-} (x \dot{-} y \dot{-} w_1 \dot{-} \ldots \dot{-} w_n)$ $\dot{-} (x \dot{-} z \dot{-} w_1 \dot{-} \ldots \dot{-} w_n) \approx 0$