UNIVERSITY OF KWAZULU-NATAL

APPLICATIONS OF LIE SYMMETRIES TO GRAVITATING FLUIDS

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Applications of Lie symmetries to gravitating fluids

by

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Abstract

This thesis is concerned with the application of Lie's group theoretic method to the Einstein field equations in order to find new exact solutions. We analyse the nonlinear partial differential equation which arises in the study of non-static, non-conformally flat fluid plates of embedding class one. In order to find the group invariant solutions to the partial differential equation in a systematic and comprehensive manner we apply the method of optimal subgroups. We demonstrate that the model admits linear barotropic equations of state in several special cases. Secondly, we study a shear-free spherically symmetric cosmological model with heat flow. We review and extend a method of generating solutions developed by Deng. We use the method of Lie analysis as a systematic approach to generate new solutions to the master equation. Also, general classes of solution are found in which there is an explicit relationship between the gravitational potentials which is not present in earlier models. Using our systematic approach, we can recover known solutions. Thirdly, we study generalised shear-free spherically symmetric models with heat flow in higher dimensions. The method of Lie generates new solutions to the master equation. We obtain an implicit solution or we can reduce the governing equation to a Riccati equation.

To

Boniwe Ellen : my mother

Ayabonga Ntandoyenkosi: my daughter

Nomkhosi Bongiwe Sizakele: my wife,

for their support.

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DECLARATION 1 - PLAGIARISM

I, Alfred Mvunyelwa Msomi, student number: 201509435, declare that

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DECLARATION 2 - PUBLICATIONS

DETAILS OF CONTRIBUTION TO PUBLICATIONS that form part and/or include research presented in this thesis (include publications in preparation, submitted, in press and published and give details of contributions of each author to experimental work and writing of each publication)

Publication 1

Msomi A M, Govinder K S and Maharaj S D, Gravitating fluids with Lie symmetries, *J. Math. Phys.* **43**, 285203 (2010).

(There were regular meetings between myself and my supervisors to discuss research material for publications. The outline of the research papers and discussion of the significance of the results were jointly done. The papers were mainly written by myself with some input from my supervisors.)

Publication 2

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(There were regular meetings between myself and my supervisors to discuss research material for publications. The outline of the research papers and discussion of the significance of the results were jointly done. The papers were mainly written by myself with some input from my supervisors.)

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Chapter 1

Introduction

1.1 Lie Symmetries

Sophus Lie introduced the idea of a continuous transformation group, which is today known as the Lie group. The theory of Lie groups and Lie algebras has evolved into one of the most developed subject in Mathematics and Physics. One of the most important applications of Lie theory of symmetry group for differential equations is the construction of group invariant solutions. In actual fact, given any subgroup of the symmetry group, one can determine the equations for the group invariant solution with respect to this group. As a result, this reduced system is of fewer variables and easier to solve generally (Ahmad *et al* 2008, Bluman and Anco 2004, Edwards and Nucci 2006, Fushchych and Popowych 1994, Ibragimov 1994-1996). Since a Lie group usually contains infinitely many subgroups of the same dimension, it is not usually feasible to list all possible group invariant solutions. For this reason, one needs an effective, systematic means of classifying these solutions, leading to a fundamental set of group invariant solutions from which every other such solution can be derived. The idea of optimal system of symmetry subgroup was introduced, and some examples can be found in (Ibragimov 1994-1996, Olver 1993, Ovsiannikov 1982). Symmetry groups are also important for nonlinear partial differential equations (PDES). When these PDES admit an infinite set of point symmetries ,they can be linearised. The solutions of nonlinear PDES can be obtained from the linearised equations through the symmetry transformation. Anco *et al* (2008) and Bluman and Kumei (1990) have completed some theoretic studies and performed some examples in this interesting field. We observe that the symmetry generator takes on many different terms depending on the symmetry studied. These different terms include generalised/Lie-Bäcklund symmetries (Anderson and Ibragimov 1979), nonlocal symmetries (Govinder 1993) and potential symmetries (Bluman and Kumei 1989).

1.2 Gravitating fluids

We have been able to study shear-free spherically symmetric relativistic models with heat flow, and also generalised shear-free manifolds in higher dimensions using the method of Lie analysis of differential equations. In particular, we generate a five-parameter family of transformations which enables us to map existing solutions to new solutions such that all known solutions of Einstein's equation with heat flow can therefore produce infinite families of new solutions. In this thesis, we mainly deal with Lie point symmetries and illustrate their usefulness in the reduction of partial differential equations to ordinary differential equations so that we can solve the partial differential equations. This thesis is separated into two major parts presented in the form of chapters. In the first part, we systematically study the master equation for gravitating fluid plates and obtain a deeper insight into the nature of solutions permitted using the

Lie analysis of differential equations. The second part deals with spherically symmetric radiating spacetimes with vanishing shear which are important in relativistic astrophysics, radiating stars and cosmology.

1.3 Outline

The thesis is organised as follows:

- Chapter 1: Introduction.

- Chapter 2: We analyse the underlying nonlinear partial differential equation which arises in the study of gravitating flat fluid plates of embedding class one. Our interest in this equation lies in discussing new solutions that can be found by means of Lie point symmetries. The method utilised reduces the partial differential equation to an ordinary differential equation according to the Lie symmetry admitted. We show that a class of solutions found previously can be characterised by a particular Lie generator. Several new families of solutions are found explicitly. In particular we find the relevant ordinary differential equation for all one-dimensional optimal subgroups; in several cases the ordinary differential equation can be solved in general. We are in a position to characterise particular solutions with a linear barotropic equation of state. This chapter forms a substantial part of this study.

- Chapter 3: We study shear-free spherically symmetric relativistic models with heat flow. Our analysis is based on Lie's theory of extended groups applied to the governing field equations. In particular, we generate a five–parameter family of transformations which enables us to map existing solutions to new solutions. All known solutions of Einstein's equations with heat flow can therefore produce infinite families of new solutions. In addition, we provide two new classes of solutions utilising the Lie infinitesimal generators. These solutions generate an infinite class of solutions given any one of the two unknown metric functions.

- Chapter 4: We consider a shear-free spherically symmetric metric in higher dimensions in the presence of heat flux. In generating new solutions of the master equation, we use the method of Lie analysis of differential equation as a systematic approach. Using the five Lie point symmetries of the master equation, we obtain either an implicit solution or we can reduce the governing equation to a Riccati equation. However two cases give us new solutions regardless of the complexity of functions chosen.

- Chapter 5: Conclusion.

Chapter 2

Symmetries in gravitating fluid plates

2.1 Introduction

The local isometric embedding of four-dimensional Riemannian manifolds M_4 in higher dimensional flat pseudo-Euclidean spaces $E_N(N \leq 10)$ is important for several applications in general relativity. For the basic theory and general results pertinent to embeddings the reader is referred to Stephani *et al* (2003). The invariance of the embedding class naturally generates a classification scheme for all solutions of the field equations in terms of their embedding class. The embedding class p is the minimum number of extra dimensions of the Riemannian manifold M_4, ie. $p = N - 4$. Exact solutions have been found by the method of embedding in particular spacetimes for simple cases of low embedding classes. Some of these exact solutions may not be easily found using other methods and techniques. For example, the embedding method has been utilised to find all conformally flat perfect fluid solutions, in embedding class $p = 1$, of Einstein's field equations (Krasinski 1997, Stephani 1967a, Stephani 1967b). We point out that embedding of four-dimensional Riemannian manifolds in higher dimensional spacetimes with arbitrary Ricci tensors has been investigated by several authors. The physical motivation here is to understand the nature of physics in higher dimensions; the modern view is that the Riemannian manifold M_4 is a hypersurface in the higher dimensional bulk in the brane world scenario and other higher dimensional themes (Dahia and Romero 2002a, 2002b, Dahia *et al* 2008).

Gupta and Sharma (1996) have generated a relativistic model in higher dimensions describing gravitating fluid plates. Advantages of this model are that it is easy to interpret the physical features using embedding in higher dimensions and the underlying differential equation governing the gravitational dynamics is tractable. This model is expanding and not conformally flat. A plane symmetric metric in four-dimensional spacetimes M_4 given by

$$ds^2 = -dR^2 - t^2(d\theta^2 + \theta^2 d\phi^2) + (1 + 2\dot{V})dt^2 + 2V' dR dt \tag{2.1}$$

is embedded in the five-dimensional pseudo-Euclidean space E_5 with metric

$$ds^2 = -(dz^1)^2 - (dz^2)^2 - (dz^3)^2 + (dz^4)^2 - (dz^5)^2 \tag{2.2}$$

This embedding is achieved by setting

$$z^1 = t\theta \cos\phi \tag{2.3a}$$

$$z^2 = t\theta \sin\phi \tag{2.3b}$$

$$z^3 = \frac{\theta^2}{2}t + V \tag{2.3c}$$

$$z^4 = t\left(\frac{\theta^2}{2} + 1\right) + V \tag{2.3d}$$

$$z^5 = R \tag{2.3e}$$

where $V = V(R, t)$ is an arbitrary function.

Consequently this model has embedding class $p = 1$ which allows both conformally flat and non-conformally flat fluid distributions. The solutions admitted may be geodesic or accelerating. For a nonzero conformal (Weyl) tensor a partial differential equation has to be satisfied. This master equation governs the evolution of the system and a particular class of solutions was identified by Gupta and Sharma (1996) by inspection. A detailed analysis of the master equation shows that other classes of solution are possible which contain the Gupta and Sharma (1996) models as a special case.

Our intention is to systematically study the master equation and to obtain a deeper insight into the nature of solutions permitted using the Lie analysis of differential equations. In section 2.2 we discuss the fundamental partial differential equation that governs the gravitational behaviour of the model, and present known solutions. An outline of the basic features of the Lie symmetry analysis and associated concepts are given in sections $2.3 - 2.5$. We regain the Gupta and Sharma (1996) models using the relevant Lie generator in section 2.6. In section 2.7 we consider group invariant solutions admitted by the fundamental equation which are invariant under all the symmetries. The partial differential equation is reduced to an ordinary differential equation in general. This provides the basis for studying the integrability of the ordinary differential equation for each element of the optimal system in the next chapter.

2.2 The model

The embedding of the four-dimensional Riemannian metric (2.1) into the five-dimensional flat metric (2.2) leads to a differential equation that is central to the model. Gupta and Sharma (1996) show that the master equation is

$$-\ddot{V}V'' + \dot{V}'^2 + \frac{1}{t}\left[(1+2\dot{V})V'' - \ddot{V} - 2\dot{V}'V'\right] + \frac{1}{t^2}\left[1+2\dot{V}+V'^2\right] = 0 \quad (2.4)$$

where dots and primes denote derivatives with respect to t and r respectively. This is a nonlinear equation in V and difficult to solve. We need to explicitly solve (2.4) to describe the gravitational dynamics. The expressions of pressure and density given by Gupta and Sharma (1996) are

$$8\pi p = \frac{1}{t^2 P} \quad (2.5a)$$

$$8\pi\rho = \frac{1}{t^2 P} + \frac{2}{P^2}(\ddot{V}V'' - \dot{V}'^2) \quad (2.5b)$$

where

$$P = 1 + 2\dot{V} + V'^2 \quad (2.6)$$

To demonstrate a class of solutions to (2.4), Gupta and Sharma (1996) made the following assumption

$$V = C\left(\frac{f(r)}{t} + h(t)\right) + C_1 \quad (2.7)$$

where C and C_1 are arbitrary constants. Then (2.4) reduces to the separable form

$$4f - \frac{4Cf'^2}{Cf''+1} = t^2\left[\frac{1}{C} - t\ddot{h} + 2\dot{h}\right] = \alpha \quad (2.8)$$

where α is the constant of separability. It is possible to solve the equation (2.8) in terms of t explicitly as

$$h = -\frac{\alpha}{4t} - \frac{t}{2C} - \frac{\alpha_1 t^3}{3} + \alpha_2 \quad (2.9)$$

and to provide four solutions to the equation in terms of r, *viz*

$$f = \frac{1}{2m^2C}\sin X + \frac{1}{2m^2C} + \frac{\alpha}{4} \tag{2.10a}$$

$$f = \frac{\alpha}{4} \tag{2.10b}$$

$$f = \frac{1}{2C}(r+\beta)^2 + \frac{\alpha}{4} \tag{2.10c}$$

$$f = \frac{1}{2m^2C}\cosh X - \frac{1}{2m^2C} + \frac{\alpha}{4} \tag{2.10d}$$

where we have set $X = \sqrt{2}mr + m_0$ and $\alpha_1, \alpha_2, \beta, m$ and m_0 are arbitrary constants.

Equations (2.9) and (2.10a–2.10d) are then combined to provide solutions to the original equation (2.4), *viz*

$$V = \frac{1}{2m^2t}\sin X + \frac{1}{2m^2t} - \frac{t}{2} - \frac{kt^3}{3} + k_1 \tag{2.11a}$$

$$V = -\frac{t}{2} - \frac{kt^3}{3} + k_1 \tag{2.11b}$$

$$V = \frac{1}{2t}(r+\beta)^2 - \frac{t}{2} - \frac{kt^3}{3} + k_1 \tag{2.11c}$$

$$V = \frac{1}{2m^2t}\cosh X - \frac{1}{2m^2t} - \frac{t}{2} - \frac{kt^3}{3} + k_1 \tag{2.11d}$$

where $k = C\alpha_1$ and $k_1 = C\alpha_2$. Thus the assumption (2.7) leads to a simple class of solutions (2.11a–2.11d) which are written in terms of elementary functions. As an aside we observe that α does not appear in the solutions (2.11a–2.11d). Thus the constant of separability α can be taken to be zero, and C can be taken to be unity with no loss of generality. Indeed as we shall demonstrate in later sections, our Lie analysis obviates the need for the introduction of α and C.

We will show that, while (2.7) is an *ad hoc* assumption, the reason for its feasibility lies in the group theoretic properties of (2.4). Utilising the full group properties of (2.4) we can provide further solutions to complement (2.11a–2.11d) as shown later.

2.3 Lie analysis

We briefly summarise the basic features of the Lie analysis that is required to produce our results. The Lie symmetry analysis (Olver 1993) can be applied to a $1+1$ partial differential equation of order n:

$$N(t, r, u, u_t, u_r, u_{rr}, \ldots) = 0 \tag{2.12}$$

where $u = u(t, r)$. The analysis requires the determination of the one–parameter (ε) Lie group of transformations

$$\begin{aligned} \bar{t} &= t + \varepsilon\tau(t, r, u) + O(\varepsilon^2) \\ \bar{r} &= r + \varepsilon\xi(t, r, u) + O(\varepsilon^2) \\ \bar{u} &= u + \varepsilon\eta(t, r, u) + O(\varepsilon^2) \end{aligned} \tag{2.13}$$

that leaves the solution set of (2.12) given by

$$S_N \equiv \{u(t, r) : N = 0\} \tag{2.14}$$

invariant. The generator of the infinitesimal transformations (2.13) is a set of vector fields of the form

$$G = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial r} + \eta\frac{\partial}{\partial u} \tag{2.15}$$

We say that (2.12) possesses the symmetry (2.15) provided

$$\mathrm{pr}^{(n)}G(N)\,|_{N=0} = 0 \tag{2.16}$$

where $\mathrm{pr}^{(n)}G$ is the nth prolongation (or extension) of the vector field (2.14) (Olver 1993)

$$\mathrm{pr}^{(n)}G \quad = \quad G + \sum_{\alpha=1}^{q} \sum_{j} \phi_\alpha^j(x, U^{(n)}) \frac{\partial}{\partial U_j^\alpha} \tag{2.17}$$

with the coefficient functions ϕ_α^j of $\mathrm{pr}^{(n)}G$ given by the following

$$\phi_\alpha^j\left(x, U^{(n)}\right) \quad = \quad D_j\left(\phi_x - \sum_{i=1}^{p} \xi^i U_i^\alpha\right) + \sum_{i=1}^{p} \xi^i U_{j,i}^\alpha \tag{2.18}$$

These expressions are defined in terms of τ, η and ξ. We can use (2.14) to define new variables that reduce the partial differential equation (2.12) into an ordinary differential equation. These variables (which then give rise to group invariant solutions of (2.12)) are obtained by solving the Lagrange's system of ordinary differential equations associated with (2.15) (Prince 1981).

$$\frac{\mathrm{d}t}{\tau} = \frac{\mathrm{d}r}{\xi} = \frac{\mathrm{d}u}{\eta} \tag{2.19}$$

Operating on (2.12) with the nth prolongation of G (*viz* (2.16)) is a straight forward, albeit tedious process. Fortunately, a number of computer algebra packages are available to aid the practitioner (Hereman 1994). While some modern packages have been developed (Dimas and Tsoubelis 2005, Cheviakov 2007) we have found the `PROGRAM LIE` (Head 1993) to be the most useful in practice. Indeed it is quite remarkable how accomplished such an old package is - it often outperforms its modern counterparts!

Utilising `PROGRAM LIE`, we can demonstrate that (2.4) admits the following Lie point symmetries/vector fields:

$$G_1 = \frac{\partial}{\partial V} \tag{2.20a}$$

$$G_2 = \frac{\partial}{\partial r} \tag{2.20b}$$

$$G_3 = t^3\frac{\partial}{\partial V} \tag{2.20c}$$

$$G_4 = t\frac{\partial}{\partial r} + r\frac{\partial}{\partial V} \tag{2.20d}$$

$$G_5 = t\frac{\partial}{\partial t} - (V+t)\frac{\partial}{\partial V} \tag{2.20e}$$

$$G_6 = r\frac{\partial}{\partial r} + (t+2V)\frac{\partial}{\partial V} \tag{2.20f}$$

with the nonzero Lie bracket relationships

$$\begin{array}{ll} [G_1, G_5] = -G_1 & [G_1, G_6] = 2G_1 \\ [G_2, G_4] = G_1 & [G_2, G_6] = G_2 \\ [G_3, G_5] = -4G_3 & [G_3, G_6] = 2G_3 \\ [G_4, G_5] = -G_4 & [G_4, G_6] = G_4 \end{array} \tag{2.21}$$

for the given fields. As a result, the symmetries (2.20a–2.20f) form a six–dimensional indecomposable solvable Lie algebra L (Rand *et al* 1988). While L is not nilpotent, its first derived Lie subalgebra $L^{(1)} =< G_1, G_2, G_3, G_4 >$ is nilpotent and also represents the nilradical of L. Further information about such Lie algebras can be found in (Turkowiski 1990). The commutation relations between these vector fields is given by the Table 2.1, the entry at row i and column j representing $[G_i, G_j]$ such that

$$[G_i, G_j] = \sum_{k=1}^{j-1} c_{ij}^k G_k \tag{2.22}$$

whenever $i < j$. This format is used to develop a commutation Table 2.1 which is important in the determination of Lie algebras and optimal subgroups (see

	G_1	G_2	G_3	G_4	G_5	G_6
G_1	0	0	0	0	$-G_1$	G_1
G_2	0	0	0	G_1	0	G_2
G_3	0	0	0	0	$-4G_3$	$2G_3$
G_4	0	$-G_1$	0	0	$-G_4$	0
G_5	G_1	0	$4G_3$	G_4	0	0
G_6	$-G_1$	$-G_2$	$2G_3$	0	0	0

Table 2.1: Commutation table for vector fields G_1–G_6

later). For simplicity we will replace

$$\begin{aligned} \bar{G}_6 &= G_5 + G_6 \\ &= r\frac{\partial}{\partial r} + t\frac{\partial}{\partial t} + V\frac{\partial}{\partial V} \end{aligned} \tag{2.23}$$

and obtain commutation Table 2.1. The representation of an adjoint of a Lie group on its Lie algebra is usually developed from its infinitesimal generators. We develop the adjoint representation by simply summing up the Lie series:

$$\begin{aligned} \mathrm{Ad}\left(\exp\left(\varepsilon G\right)\right) W_0 &= \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} \left(\mathrm{Ad}(G)\right)^n \left(W_0\right) \qquad (2.24) \\ &= W_0 - \varepsilon\left[G, W_0\right] + \frac{\varepsilon^2}{2}\left[G,\left[G, W_0\right]\right] - \cdots \end{aligned}$$

By following this format for all the given vector fields in the commutation Table 2.1, we develop an adjoint Table 2.2 having a structure similar to that of a commutator table.

2.4 Use of symmetries

A symmetry group of a system of differential equations is a Lie group acting on the space of independent and dependent variables in such a way that solutions

Ad	G_1	G_2	G_3	G_4	G_5	G_6
G_1	G_1	G_2	G_3	G_4	$G_5 + \varepsilon G_1$	$G_6 - \varepsilon G_1$
G_2	G_1	G_2	G_3	$G_4 - \varepsilon G_1$	G_5	$G_6 - \varepsilon G_2$
G_3	G_1	G_2	G_3	G_4	$G_5 + 4\varepsilon G_3$	$G_6 + 2\varepsilon G_3$
G_4	G_1	$G_2 + \varepsilon G_1$	G_3	G_4	$G_5 + \varepsilon G_4$	G_6
G_5	$G_1 e^{-\varepsilon}$	G_2	$G_3 e^{-4\varepsilon}$	$G_4 e^{-\varepsilon}$	G_5	G_6
G_6	$G_1 e^{\varepsilon}$	$G_2 e^{\varepsilon}$	$G_3 e^{-2\varepsilon}$	G_4	G_5	G_6

Table 2.2: Table of adjoint operators for G_1–G_6

are mapped into other solutions. Knowing the symmetry group allows one to determine some special types of solutions that are invariant under a subgroup of the full symmetry group, and in some cases one can solve the equations completely. The symmetry approach to solving differential equations can be found, for example in the excellent books of Olver (1993), Bluman and Cole (1974), Bluman and Kumei (1989), Fushchich and Nikitin (1994) and Ovsiannikov (1982). Einstein's general theory of relativity is based on the most fundamental way of the concepts of symmetry and general covariance for gravity. Lie's theory on the other hand is the most systematic mathematical way to study symmetries of differential equations. So it is obviously interesting to apply the Lie methods to a theory fundamentally related with the idea of symmetry and general covariance.

2.5 Inequivalent subalgebras

In determining the inequivalent subalgebras, whose adjoint representation is determined by Table 2.2, we begin with the following nonzero vector:

$$G = a_1 G_1 + a_2 G_2 + a_3 G_3 + a_4 G_4 + a_5 G_5 + a_6 G_6 \tag{2.25}$$

We try to reduce as many of the coefficients, a_i of G, as possible through judicious applications of adjoint maps to G. We will achieve this by considering the applications of adjoint maps to G. We first choose $a_6 \neq 0$ and set $a_6 = 1$. Thus

$$G = a_1G_1 + a_2G_2 + a_3G_3 + a_4G_4 + a_5G_5 + G_6 \tag{2.26}$$

We act on G with $\text{Ad}(\exp(\varepsilon G_5))$ to obtain

$$\begin{aligned} &\text{Ad}(\exp(\varepsilon G_5))(a_1G_1 + a_2G_2 + a_3G_3 + a_4G_4 + a_5G_5 + G_6) \\ &= a_1G_1e^{\varepsilon} + a_2G_2 + a_3G_3e^{-4\varepsilon} + a_4G_4e^{-\varepsilon} + a_5G_5 + G_6 \end{aligned} \tag{2.27}$$

Again by referring to Table 2.2, if we act on such a G with $\text{Ad}(\exp(\varepsilon G_5))$, we can see that our choice does not work and hence there is no reduction. By referring to Table 2.2, we act on this G by $\text{Ad}(\exp(\varepsilon G_4))$ so that we can attempt to cancel the coefficient of G_4. This lead us to

$$\begin{aligned} &\text{Ad}(\exp(\varepsilon G_5))(a_1G_1 + a_2G_2 + a_3G_3 + a_4G_4 + a_5G_5 + G_6) \\ &= a_1G_1 + a_2G_2 + a_2\varepsilon G_1 + a_3G_3 + a_4G_4 + a_5G_5 + a_5\varepsilon G_4 + G_6 \end{aligned} \tag{2.28}$$

We set $\varepsilon = \frac{-a_4}{a_5}$ and obtain

$$\bar{G} = \bar{a}_1G_1 + \bar{a}_2G_2 + \bar{a}_3G_3 + \bar{a}_5G_5 + G_6 \tag{2.29}$$

with the constants redefined. Acting on this $\bar{G}$ by $\text{Ad}(\exp(\varepsilon G_3))$ we have

$$\begin{aligned} &\text{Ad}(\exp(\varepsilon G_3))(\bar{a}_1G_1 + \bar{a}_2G_2 + \bar{a}_3G_3 + \bar{a}_5G_5 + G_6) \\ &= \bar{a}_1G_1 + \bar{a}_2G_2 + \bar{a}_3G_3 + \bar{a}_5G_5 + 4\bar{a}_5\varepsilon G_3 + G_6 + 2\varepsilon G_3 \end{aligned} \tag{2.30}$$

When we substitute $\varepsilon = \frac{-\bar{a}_3}{4\bar{a}_5+2}$ in equation (2.30) to cancel the coefficient of G_4, we obtain

$$\vec{G} = \vec{a}_1G_1 + \vec{a}_2G_2 + \vec{a}_5G_5 + G_6 \tag{2.31}$$

Again, acting on this $\vec{G}$ by $\mathrm{Ad}(\exp(\varepsilon G_2))$ with the hope of canceling the coefficient of G_2, we obtain

$$\begin{aligned} &\mathrm{Ad}(\exp(\varepsilon G_2))(\vec{a}_1 G_1 + \vec{a}_2 G_2 + \vec{a}_5 G_5 + G_6) \\ &= \vec{a}_1 G_1 + \vec{a}_2 G_2 + \vec{a}_5 G_5 + G_6 - \varepsilon G_2 \end{aligned} \tag{2.32}$$

and if we substitute $\varepsilon = \vec{a}_2$ in equation (2.32), we have

$$\tilde{G} = \tilde{a}_1 G_1 + \tilde{a}_5 G_5 + G_6 \tag{2.33}$$

Finally, if we act on this $\tilde{G}$ by $\mathrm{Ad}(\exp(\varepsilon G_1))$ in order to cancel the coefficient of G_1, we find

$$\mathrm{Ad}(e^{\varepsilon G_1})(\tilde{a}_1 G_1 + \tilde{a}_5 G_5 + G_6) = \tilde{a}_1 G_1 + \tilde{a}_5 G_5 + \tilde{a}_5 \varepsilon G_1 + G_6 - \varepsilon G_1 \tag{2.34}$$

Now when we substitute $\varepsilon = \frac{-\tilde{a}_1}{\tilde{a}_5}$ in equation (2.34), we obtain

$$\overleftarrow{G} = \overleftarrow{a}_5 G_5 + G_6 \tag{2.35}$$

If we further try to reduce the right hand side of the above equation, we discover that this cannot be done under the adjoint representation. In this case, we can conclude that every one-dimensional subalgebra given by a G with coefficient $a_4 \neq 0$ is equivalent to the subalgebra spanned by $\overleftarrow{a}_5 G_5 + G_6$ which we can express in general as

$$G = aG_5 + G_6 \tag{2.36}$$

We now set $a_4 = 0$ and we choose another parameter, a_3 say, and set that equal to one. Continuing in this fashion, we can reduce all the coefficient of G. Recapitulating, we have found an optimal system of one-dimensional subalgebras to be those spanned by the symmetry combinations given in subsection 2.7.2.

2.6 Known solutions

We first demonstrate that the solutions (2.11a–2.11d) are a natural consequence of a subset of these symmetries. If we take the combination

$$\begin{aligned} \tilde{G} &= c_1 G_1 + c_2 G_2 + G_5 \\ &= t\frac{\partial}{\partial t} + (c_1 - t + c_2 t^3 - V)\frac{\partial}{\partial V} \end{aligned} \tag{2.37}$$

we have to solve the system

$$\frac{\mathrm{d}t}{t} = \frac{\mathrm{d}r}{0} = \frac{\mathrm{d}V}{c_1 - t + c_2 t^3 - V} \tag{2.38}$$

to find corresponding variables to reduce (2.4) to an ordinary differential equation. The invariants of (2.38) are

$$y = r \tag{2.39a}$$

$$V = c_1 - \frac{t}{2} + \frac{c_2 t^3}{4} + \frac{U}{t} \tag{2.39b}$$

with constants suitably relabeled. The partial differential equation (2.4) is reduced by the transformation (2.39a–2.39b) to the form of an ordinary differential equation

$$UU'' - U'^2 + U = 0 \tag{2.40}$$

as the essential equation governing the gravitational dynamics.

On comparing (2.8) and (2.40) we can identify the function f with U. We are now in a position to make a number of comments relating to the underlying assumption (2.7) in the Gupta and Sharma (1996) solutions. We observe that the t-dependence in the equation (2.39b) arises naturally because of the choice of the symmetry $\tilde{G}$. Thus the temporal dependence is not arbitrary as suggested by the function $h(t)$ in the choice (2.7). It is not necessary to solve any differential equation to obtain the form of $h(t)$ given by (2.9). Also, for consistency we need to set

$$\alpha = 0, \qquad C = 1 \tag{2.41}$$

which indicates that the introduction of the parameter α is redundant. The solutions of (2.40) are the same as (2.10a–2.10d) with $\alpha = 0$ and $C = 1$. As stated earlier, the final solutions (2.11a–2.11d) do not include these constants. Thus the Lie symmetry $\tilde{G}$ leads directly to the canonical form of the solution to (2.4) without the need to introduce spurious arbitrary functions and parameters.

2.7 Group invariant solutions

A solution of the system of partial differential equations is said to be G-invariant if it is unchanged by all the group transformations in G. In general, to each s-parameter subgroup H of the full symmetry group g of a system of differential equations, there will correspond a family of group invariant solutions. Since there are almost always an infinite number of such subgroups, it is not usually feasible to list all possible group invariant solutions to the system. We need an effective systematic means of classifying these solutions, leading to an optimal system of group invariant solutions from which every other solution can be derived. In this section, we now seek to utilise the Lie point symmetries in a systematic manner to generate new solutions. These new solutions are termed group invariant solutions as they will be invariant under the group generated by the symmetry used to find them. The advantage of using Lie point symmetries is that we are guaranteed that the variable combinations obtained will always result in an equation in the new variables -

no further “consistency” conditions are needed. As we are dealing with a $1+1$ partial differential equation here, we will always be able to find an ordinary differential equation in the new variables defined by the symmetries.

2.7.1 The general case

We first attempt to find solutions of (2.4) which are invariant under all the symmetries $G_1, \ldots, G_6$. We combine the symmetries into the single general nonzero vector

$$G = a_1 G_1 + a_2 G_2 + a_3 G_3 + a_4 G_4 + a_5 G_5 + G_6. \tag{2.42}$$

We determine the invariants of G from the invariant surface condition associated with the symmetry

$$(a_1 + a_3 t^3 + a_4 r - a_5(V+t) + V)\frac{\partial}{\partial V} = (a_2 + a_4 t + r)\frac{\partial}{\partial r} = (a_5+1)t\frac{\partial}{\partial t} \tag{2.43}$$

where

$$b = \frac{a_2}{a_5+1} \tag{2.44}$$

$$c = \frac{a_4}{a_5+1} \tag{2.45}$$

$$d = \frac{1}{a_5+1} \tag{2.46}$$

$$e = \frac{a_1}{a_5+1} \tag{2.47}$$

$$f = \frac{a_3}{a_5+1} \tag{2.48}$$

$$g = \frac{1-a_5}{1+a_5} \tag{2.49}$$

$$h = \frac{a_5}{a_5+1} \tag{2.50}$$

are constants. The system has invariants y and $U(y)$ given via

$$y = \frac{r}{t^d} + \frac{b}{dt^d} - \frac{ct^{1-d}}{1-d} \tag{2.51}$$

$$\begin{aligned} V &= -\frac{e}{g} + \frac{ft^3}{3-g} + \frac{bc}{dg} + \frac{c^2t}{(1-d)(1-g)} + \frac{cr}{d-g} + \frac{cb}{d(d-g)} \\ &\quad - \frac{c^2t}{(d-g)(1-d)} - \frac{ht}{1-g} + t^g U(y) \end{aligned} \tag{2.52}$$

However, since $h = 1 - d$ and $g = -1 + 2d$, the invariant (2.52) becomes

$$\begin{aligned} V &= \frac{bc}{d(1-d)} + \frac{bc}{d(2d-1)} - \frac{e}{2d-1} + \frac{cr}{1-d} - \frac{c^2t}{2(1-d)^2} \\ &\quad - \frac{t}{2} + \frac{ft^3}{2(2-d)} + t^{(-1+2d)}U(y) \end{aligned} \tag{2.53}$$

Using the above transformation in equation (2.4), the partial differential equation is reduced to the form

$$\begin{aligned} -(d-2)^2U_y^2 + d^2y^2U_{yy} + 2(2d-1)(d-2)U(1+U_{yy}) \\ +dyU_y(5-3d-(d-1)U_{yy}) = 0 \end{aligned} \tag{2.54}$$

Unfortunately, this ordinary differential equation has only one symmetry given by `PROGRAM LIE` (Head 1993) as

$$G = \frac{y}{1-2d}\frac{\partial}{\partial y} + \frac{2U}{1-2d}\frac{\partial}{\partial U} \tag{2.55}$$

As a result of having just one symmetry, we can reduce (2.54) to a first order ordinary differential equation, but there is little hope for a solution.

Remark: It is clear, from (2.53) that one must also consider the special values $d = \frac{1}{2}, 1$ and $d = 2$. However, in all these cases, the reduced equation still possesses only one symmetry and there is little hope for a general solution.

As can be seen above, it is difficult to find a solution to (2.4) that is invariant under all the symmetries. However, one can still make progress. In the next subsection, we show how to find the solutions of the original equation invariant under a single symmetry in a systematic manner using the optimal system rather than all linear combinations of symmetries.

2.7.2 The optimal system

Given that equation (2.4) has the six symmetries (2.20a–2.20f), we can find group invariant solutions using each symmetry individually, or any linear combination of symmetries. However, taking all possible combinations into account is overly excessive. It turns out (Olver 1993), that one only need to consider a subspace of this vector space. We use the subalgebraic structure of the symmetries (2.20a–2.20f) of the system (2.4) to construct an optimal system of one-dimensional subgroups. Such an optimal system of subgroups is determined by classifying the orbits of the infinitesimal adjoint representation of a Lie group on its Lie algebra obtained by using its infinitesimal generators. All group invariant solutions can be transformed to those obtained via this optimal system. The process is algorithmic and can be found in (Olver 1993). Here we only summarise the final results.

In order to obtain group invariant solutions of (2.4) explicitly, we only need to consider the following symmetry combinations (determined by following the procedure in section 2.5):

$$G_2 = \frac{\partial}{\partial r} \tag{2.56a}$$

$$G_3 = t^3\frac{\partial}{\partial V} \tag{2.56b}$$

$$G_4 = t\frac{\partial}{\partial r} + r\frac{\partial}{\partial V} \tag{2.56c}$$

$$G_5 = t\frac{\partial}{\partial t} - (V+t)\frac{\partial}{\partial V} \tag{2.56d}$$

$$G_2 + G_3 = \frac{\partial}{\partial r} + t^3\frac{\partial}{\partial V} \tag{2.56e}$$

$$G_2 + G_4 = (1+t)\frac{\partial}{\partial r} + r\frac{\partial}{\partial V} \tag{2.56f}$$

$$G_2 + G_5 = \frac{\partial}{\partial r} + t\frac{\partial}{\partial t} - (V+t)\frac{\partial}{\partial V} \tag{2.56g}$$

$$G_3 + G_4 = (t^3 + r)\frac{\partial}{\partial V} + t\frac{\partial}{\partial r} \tag{2.56h}$$

$$aG_5 + G_6 = r\frac{\partial}{\partial r} + (a+1)t\frac{\partial}{\partial t} + [(1-a)V - at]\frac{\partial}{\partial V} \tag{2.56i}$$

$$G_2 - G_3 + G_4 = (t+1)\frac{\partial}{\partial r} + (r - t^3)\frac{\partial}{\partial V} \tag{2.56j}$$

$$G_2 + G_3 + G_4 = (t+1)\frac{\partial}{\partial r} + (t^3 + r)\frac{\partial}{\partial V} \tag{2.56k}$$

All solutions of (2.4) which are obtained via other combinations of point symmetries can be transformed into the solutions obtained from the combinations above. Here, we have also taken into account the fact that (2.4) is invariant under the following involutions: $t \to -t, r \to -r$ and $V \to -V$ and so were able to restrict the optimal system further.

It is clear that the optimal system consists of single elements of the Lie algebra, combinations of two elements and combinations of three elements only. We divide our discussion of the solutions based on this separation in the next section. We do not consider the generators G_1 and G_6 as they do not appear in the optimal system (2.56a–2.56k). The procedure of generating the invariants, the resulting ordinary differential equation and finally the exact solution to the partial differential equation is standard. We only provide the relevant details of the calculations.

2.8 Solutions invariant under one generator

In this section we generate solutions to the master equation (2.4) when it admits the individual Lie symmetries G_2, G_3, G_4 and G_5. We do not consider the generators G_1 and G_6 as they do not appear in the optimal system (2.56a–2.56k). The procedure of generating the invariants, the resultant ordinary differential equation and finally the solution to the partial differential equation is shown.

2.8.1 Invariance under G_2

Using the generator G_2 we determine the invariants from the invariant surface condition

$$\frac{dr}{1} = \frac{dt}{0} = \frac{dV}{0} \tag{2.57}$$

The system has the invariants given by

$$t = y \tag{2.58a}$$

$$V = U(t) \tag{2.58b}$$

From the above transformation, the partial differential equation (2.4) is reduced to the linear ordinary differential equation

$$yU_{yy} - 2U_y - 1 = 0 \tag{2.59}$$

with the solution to ordinary differential equation

$$U = A + \frac{B}{3}y^3 - \frac{y}{2} \tag{2.60}$$

The solution to the resulting ordinary differential equation in the explicit form for the function V is given by

$$V = A + \frac{B}{3}t^3 - \frac{t}{2} \tag{2.61}$$

This is a new solution to equation (2.4) which has not been obtained previously.

Substituting the solution (2.61) into (2.5a) and (2.5b), we evaluate the pressure and energy density to give equation (2.5a) reduced to

$$p = \frac{1}{16\pi t^4} \tag{2.62}$$

while equation (2.5b) is reduced to

$$\rho = \frac{1}{16\pi t^4} \tag{2.63}$$

Therefore

$$\rho = p \tag{2.64}$$

We observe from the above equations that $\rho > 0$ and $p > 0$. This is a desirable feature because we expect that barotropic matter in cosmological models should have positive pressures and positive energy densities. The solution obtained displays the very interesting feature of a linear barotropic equation of state. At later times $t \to \infty$ the model approaches a vacuum state.

2.8.2 Invariance under G_3

Using G_3, we determine the invariants from the invariant surface condition

$$\frac{dr}{0} = \frac{dt}{0} = \frac{dV}{t^3} \tag{2.65}$$

which gives the invariants

$$t = y \tag{2.66a}$$

$$r = U(t) \tag{2.66b}$$

From the transformation (2.66b), there is only a single characteristic invariant so that it is not possible to generate an ordinary differential equation. Thus there is no solution possible with G_3.

2.8.3 Invariance under G_4

Using G_4, we determine the invariants from the invariant surface condition

$$\frac{dr}{t} = \frac{dt}{0} = \frac{dV}{r} \tag{2.67}$$

The system has the invariants given by

$$t = y \tag{2.68a}$$

$$V = \frac{r^2}{2t} + U(t) \tag{2.68b}$$

Using the above transformation in equation (2.4), the master equation is reduced to the second order linear differential equation

$$yU_{yy} - 2U_y - 1 = 0 \tag{2.69}$$

with the solution

$$U = A + B\frac{1}{3}y^3 - \frac{y}{2} \tag{2.70}$$

The explicit form for the function V is given by

$$V = \frac{r^2}{2t} + A + B\frac{1}{3}t^3 - \frac{t}{2} \tag{2.71}$$

The solution in V is again, a new solution to equation (2.4) and it has not been obtained previously.

Using the equation (2.71) together with equation (2.5a) and (2.5b), the pressure (2.5a) reduces to

$$p = \frac{1}{16B\pi t^4} \tag{2.72}$$

and the energy density (2.5b) is

$$\rho = \frac{1+2\pi}{32B\pi^2 t^4} \tag{2.73}$$

and therefore, the equation of state is given by

$$\rho = p\left[\frac{1}{2\pi}+1\right] \tag{2.74}$$

which is linear.

2.8.4 Invariance under G_5

Using G_5, we determine the invariants from the invariant surface condition

$$\frac{dt}{t} = -\frac{dV}{V+t} = \frac{dr}{0} \tag{2.75}$$

which gives the invariants

$$r = y \tag{2.76a}$$

$$V = -\frac{t}{2}+\frac{1}{t}U(r) \tag{2.76b}$$

The partial differential equation (2.4) is reduced by the transformation to the form

$$UU_{yy} - U_y^2 + U = 0 \tag{2.77}$$

This form of an equation, is an ordinary differential equation with two symmetries given by `PROGRAM LIE` (Head 1993) as follows:

$$G_1 = \frac{\partial}{\partial y} \tag{2.78a}$$

$$G_2 = 2U\frac{\partial}{\partial U} + y\frac{\partial}{\partial y} \tag{2.78b}$$

Gupta and Sharma (1996) solved the equation similar to equation (2.77) to give

$$U = \frac{1}{2m^2}\sin X + \frac{1}{2m^2} \tag{2.79a}$$

$$U = \frac{1}{2}(r+\beta)^2 \tag{2.79b}$$

$$U = \frac{1}{2m^2}\cosh X - \frac{1}{2m^2} \tag{2.79c}$$

where $X = \sqrt{2}mr + m_0$ and β, m and m_0 are arbitrary constants. Consequently, we obtain the following solutions of V. However it is important to observe that the characteristics here are different from (2.10a–2.10d). Consequently the solutions V generated by the Lie symmetry G_5, comprise a new class of exact solutions to equation (2.4):

$$V = -\frac{t}{2} + \frac{1}{t}\left[\frac{1}{2m^2}(\sin(\sqrt{2}mr + m_0) + 1)\right] \tag{2.80a}$$

$$V = -\frac{t}{2} \tag{2.80b}$$

$$V = -\frac{t}{2} + \frac{1}{t}\left[\frac{1}{2}(r+\beta)^2\right] \tag{2.80c}$$

$$V = -\frac{t}{2} + \frac{1}{t}\left[\frac{1}{2m^2}(\cosh(\sqrt{2}mr + m_0) - 1)\right] \tag{2.80d}$$

which have simple forms.

We now consider the expressions of pressure and energy density for the solutions (2.80a) and (2.80d) of the partial differential equation. This distinguishes when there is no barotropic equation of state and the linear equation of state. Using the equation (2.80a), the pressure (2.5a) becomes

$$p = \frac{m^2}{4\pi\left(\cos^2(\sqrt{2}mr + m_0)\right) - 2\left(1 + \sin(\sqrt{2}mr + m_0)\right)}$$

and the density (2.5b) is given by

$$\rho = \frac{m^2(1 + 2\pi)}{4\pi^2\left(-3 + \cos(2\sqrt{2}mr + m_0)\right) - 4\left(\sin(\sqrt{2}mr + m_0)\right)}$$

There is no simple relationship between pressure and energy density. Using the equation (2.80d), the pressure becomes

$$p = \frac{m^2\cosh^4(1/2(\sqrt{2}mr + m_0))}{16\pi}$$

and the density is given by

$$\rho = \frac{m^2(1 + 2\pi)\cosh^4(1/2(\sqrt{2}mr + m_0))}{32\pi^2}$$

Therefore

$$\rho = p\left(\frac{1}{2\pi} + 1\right) \tag{2.81}$$

which describes the linear relationship between the pressure and energy density.

2.9 Solutions invariant under two generators

We now consider the combinations of two generators which arise in the optimal system (2.56a–2.56k).

2.9.1 Invariance under $G_2 + G_3$

Using the generator

$$G_2 + G_3 = \frac{\partial}{\partial r} + t^3 \frac{\partial}{\partial V} \tag{2.82}$$

we determine the invariants from the invariant surface condition

$$\frac{dr}{1} = \frac{dV}{t^3} = \frac{dt}{0} \tag{2.83}$$

The invariants are

$$t = y \tag{2.84a}$$

$$V = rt^3 + U(t) \tag{2.84b}$$

The partial differential equation (2.4) is reduced by the transformation to the linear ordinary differential equation

$$yU_{yy} - 2U_y - 4y^6 - 1 = 0 \tag{2.85}$$

Equation (2.85) has solution

$$U = A + B\frac{1}{3}y^3 + \frac{y^7}{7} - \frac{y}{2} \tag{2.86}$$

so that the solution of equation (2.4) is

$$V = rt^3 + A + B\frac{1}{3}t^3 + \frac{t^7}{7} - \frac{t}{2} \tag{2.87}$$

This is a new solution to equation (2.4) with pressure (2.5a) and energy density (2.5b) given by

$$p = \frac{1}{16B\pi t^4 + 48\pi t^4 + 24\pi t^8}$$

$$\rho = \frac{4\pi(B + 3r) + 3(-3 + 2\pi)t^4}{16\pi^2 t^4(2B + 6r + 3t^4)^2}$$

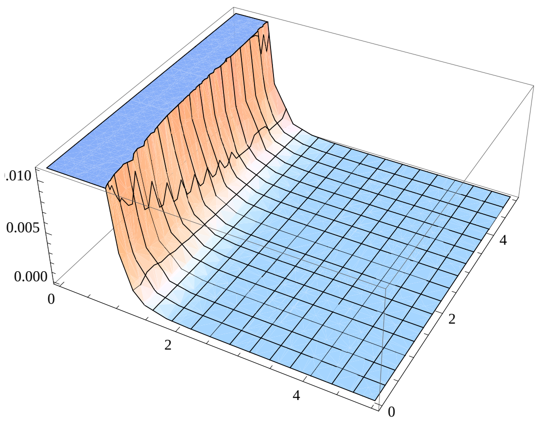

Figure 2.1: Graph of pressure for $G_2 + G_3$.

Clearly there is no barotropic equation of state connecting the energy density and the pressure. However it is possible to describe the thermodynamical behaviour graphically. We have generated the plots in Figures 2.1 and 2.2 with the help of `Mathematica` (Wolfram 1996) and it is clear that there exist regions of spacetime in which ρ and p are well behaved, remaining finite, continuous and bounded. It is then viable to study the behaviour of the thermodynamical quantities such as temperature over this region.

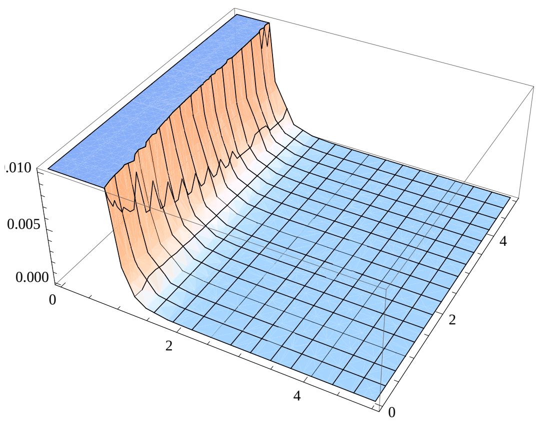

Figure 2.2: Graph of energy density for $G_2 + G_3$.

2.9.2 Invariance under $G_2 + G_4$

Using the generator

$$G_2 + G_4 = (1+t)\frac{\partial}{\partial r} + r\frac{\partial}{\partial V} \tag{2.88}$$

we determine the invariants from the invariant surface condition

$$\frac{dr}{1+t} = \frac{dV}{r} = \frac{dt}{0} \tag{2.89}$$

to be

$$t = y \tag{2.90a}$$

$$V = \frac{r^2}{2(1+t)} + U(t) \tag{2.90b}$$

This leads to the linear ordinary differential equation

$$(1+2y)(-1-2U_y+yU_{yy}) = 0 \tag{2.91}$$

with solution

$$U = A + B\frac{1}{3}y^3 - \frac{y}{2} \tag{2.92}$$

Therefore the solution to equation (2.4) is

$$V = A\frac{r^2}{2(1+t)} + B\frac{1}{3}t^3 - \frac{t}{2} \tag{2.93}$$

This form for V is another new solution to equation (2.4) with pressure (2.5a) and energy density (2.5b):

$$p = \frac{(1+t)^2}{8\pi t^2\left((-1+A)Ar^2 + 2Bt^2(1+t)^2\right)}$$

$$\begin{aligned}\rho \;=\; & \frac{(1+t)^2\left((-1+A)A\pi r^2 + 2B\pi t^2 + B(A+4\pi)t^3\right)}{8\pi^2 t^4\left((-1+A)Ar^2 + 2Bt^2(1+t)^2\right)^2} \\ & + \frac{(1+t)^2\left(B(A+2\pi)t^4\right)}{8\pi^2 t^4\left((-1+A)Ar^2 + 2Bt^2(1+t)^2\right)^2}\end{aligned}$$

Clearly there is no barotropic equation of state in this case. The behaviour of energy density and pressure has been plotted in Figures 2.3 and 2.4.

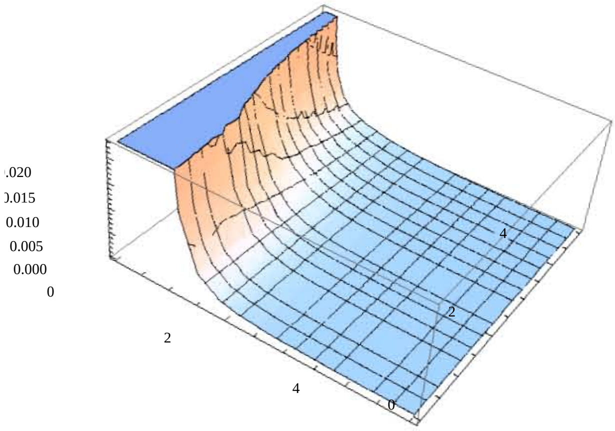

Figure 2.3: Graph of pressure for $G_2 + G_4$.

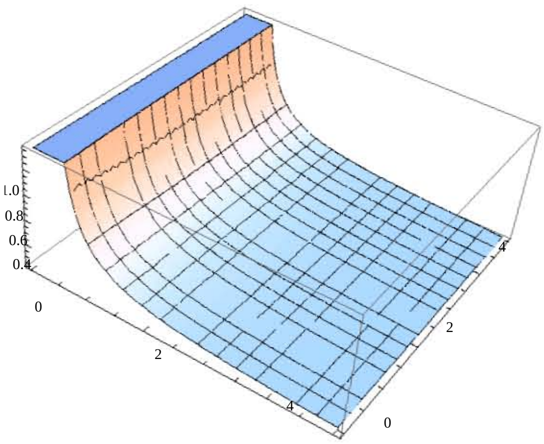

Figure 2.4: Graph of energy density for $G_2 + G_4$.

It is a also clear that there exist regions of spacetime in which ρ and p are well behaved.

2.9.3 Invariance under $G_2 + G_5$

Using the generator

$$G_2 + G_5 = \frac{\partial}{\partial r} + t\frac{\partial}{\partial t} - (V + t)\frac{\partial}{\partial V} \tag{2.94}$$

we determine the invariants from the invariant surface condition

$$\frac{dr}{1} = \frac{dt}{t} = -\frac{dV}{(V+t)} \tag{2.95}$$

and the system gives the invariants

$$r = \ln t + y \tag{2.96a}$$

$$V = -\frac{t}{2} + \frac{1}{t}U(r - \ln t) \tag{2.96b}$$

Equation (2.4) is then reduced to the nonlinear form

$$U_{yy} + 4U + 4UU_{yy} + 5U_y - 4U_y^2 + U_yU_{yy} = 0 \tag{2.97}$$

This nonlinear equation is difficult to solve. We observe that equation (2.97), using `PROGRAM LIE` (Head 1993), admits the symmetry

$$\bar{G} = \frac{\partial}{\partial y} \tag{2.98}$$

This indicates that we can reduce the order of equation (2.97) if we let

$$p = U \tag{2.99a}$$

$$q = U_y \tag{2.99b}$$

Then equation (2.97) becomes

$$\frac{dq}{dp} = \frac{4q^2 - 5q - 4p}{q + 4pq - q^2} \tag{2.100}$$

which is a first order equation. Equation (2.100) cannot be easily integrated. Therefore we could not get an analytical solution to the field equation and this tells us that not every optimal symmetry leads to a solution. However using `Mathematica` (Wolfram 1996) it is possible to produce plots indicating the graphical behaviour of the solution. In Figure 2.5 we generate the plot for U over the interval $0 \leq y \leq 1$ which indicates that the solution is well behaved in this range.

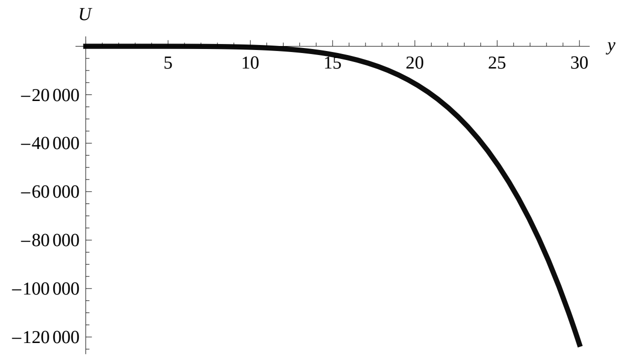

Figure 2.5: Graphical solution for $G_2 + G_5$.

2.9.4 Invariance under $G_3 + G_4$

Using the generator

$$G_3 + G_4 = (t^3 + r)\frac{\partial}{\partial V} + t\frac{\partial}{\partial r} \tag{2.101}$$

we determine the invariants from the invariant surface condition

$$\frac{dr}{t} = \frac{dV}{r + t^3} = \frac{dt}{0} \tag{2.102}$$

The invariants from the system are

$$t = y \tag{2.103a}$$

$$V = rt^2 + \frac{r^2}{2t} + U(t) \tag{2.103b}$$

Therefore in this case the partial differential equation (2.4) is reduced to the linear form

$$2 + y^4 + 4U_y - 2yU_{yy} = 0 \tag{2.104}$$

Equation (2.104) has solution

$$U = A + B\frac{1}{3}y^3 + \frac{y^5}{20} - \frac{y}{2} \tag{2.105}$$

so that we must have

$$V = rt^2 + \frac{r^2}{2t} + A + B\frac{1}{3}t^3 + \frac{t^5}{20} - \frac{t}{2} \tag{2.106}$$

Again we have generated another new solution to equation (2.4) with pressure (2.5a) and energy density (2.5b)

$$p = \frac{1}{4\pi t^2(4Bt^2 + t^4 + 4r^2(3 + 2t^2))}$$

$$\begin{aligned}\rho &= \frac{6(1 + 2\pi)r^2 + 2(B + 2B\pi + 2(-1 + 2\pi)r^2)t^2}{4\pi^2 t^2(4Bt^2 + t^4 + 4r^2(3 + 2t^2))^2} \\ &+ \frac{(1 + 4B + \pi)t^4 + 2t^6}{4\pi^2 t^2(4Bt^2 + t^4 + 4r^2(3 + 2t^2))^2}\end{aligned}$$

For the generator $G_3 + G_4$, there is no simple barotropic equation of state

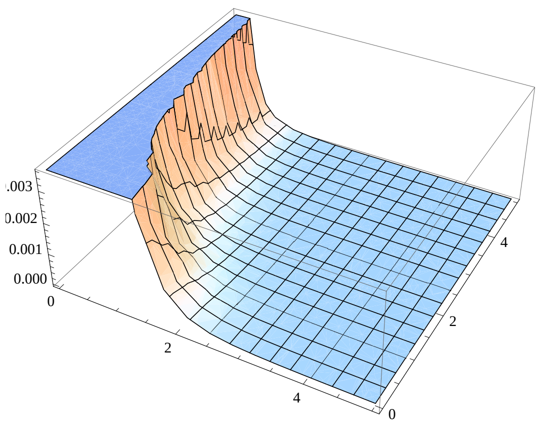

Figure 2.6: Graph of pressure for $G_3 + G_4$.

connecting the energy density and the pressure. The best we can do, will be to describe the matter variables and thermodynamical behaviour graphically. The plots provided in Figures 2.6 and 2.7 indicate that ρ and p are continuous and well behaved.

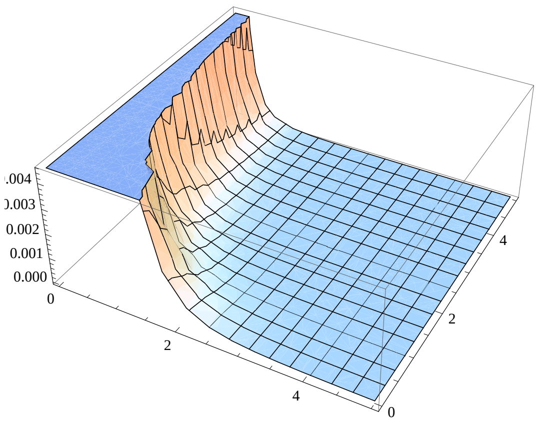

Figure 2.7: Graph of energy density for $G_3 + G_4$.

2.9.5 Invariance under $aG_5 + G_6$.

For the generator

$$aG_5 + G_6 = r\frac{\partial}{\partial r} + (a+1)t\frac{\partial}{\partial t} + [(1-a)V - at]\frac{\partial}{\partial V} \tag{2.107}$$

we find that invariants are given by

$$r = yt^{1/(a+1)} \tag{2.108}$$

$$V = -\frac{t}{2} + t^{(1-a)/(1+a)}U(rt^{-1/(1+a)}). \tag{2.109}$$

Then the partial differential equation (2.4) is reduced to

$$-(1+2a)^2U_y^2 + y^2U_{yy} + 2(-1+a)(1+2a)U(1+U_{yy}) + yU_y(2+5a+aU_{yy}) = 0 \tag{2.110}$$

where $a \neq \pm 1$. This nonlinear equation, in general, admits the only symmetry:

$$\hat{G} = \frac{2U}{(a-1)}\frac{\partial}{\partial U} - \frac{y}{(1-a)}\frac{\partial}{\partial y} \tag{2.111}$$

and so we can only hope to reduce the order of (2.110) once. Unlike the previous case, we can make some progress for the special values $a = -\frac{1}{2}$ and

$a = 1$ (The case $a = -1$ again only yields an equation with a single symmetry.). For $a = -\frac{1}{2}$, equation (2.110) is simplified to

$$y^2 U_{yy} - \frac{1}{2} y U_y (1 + U_{yy}) \quad = \quad 0 \tag{2.112}$$

with solution

$$U = \frac{1}{2}\left(\frac{1}{6} e^{A/2} (e^A - 4y)^{3/2} + e^A y\right) + B \tag{2.113}$$

Then equation (2.4) has the solution

$$V = -\frac{t}{2} + \left(B + \frac{1}{2}\left(-\frac{1}{6} e^{A/2}\left(e^A - \frac{4r}{t^2}\right)^{3/2} + \frac{e^A r}{t^2}\right)\right) t^3 \tag{2.114}$$

The pressure (2.5a) and energy density (2.5b) are

$$p = \frac{1}{48 B \pi t^4 + 4 e^{2A} \pi t^4}$$

$$\rho = \frac{-e^{A/2} + 2\pi \sqrt{e^A - \frac{4r}{t^2}}}{8(12B + e^{2A}) \pi^2 \sqrt{e^A - \frac{4r}{t^2}} t^4}$$

There is no simple relationship between the pressure and energy density. The behaviour of energy density and pressure is graphically presented in Figures 2.8 and 2.9. There exist regions of spacetime in which ρ and p are continuous, well behaved and physically reasonable.

For $a = 1$, we recalculate the invariants to be

$$r \quad = \quad y\sqrt{t} \tag{2.115}$$

$$V \quad = \quad -\frac{t}{2} - \frac{U(y)}{2}. \tag{2.116}$$

Here, (2.4) reduces to

$$-14 y U_y - 9 U_y^2 - 2 y^2 U_{yy} + y U_y U_{yy} = 0 \tag{2.117}$$

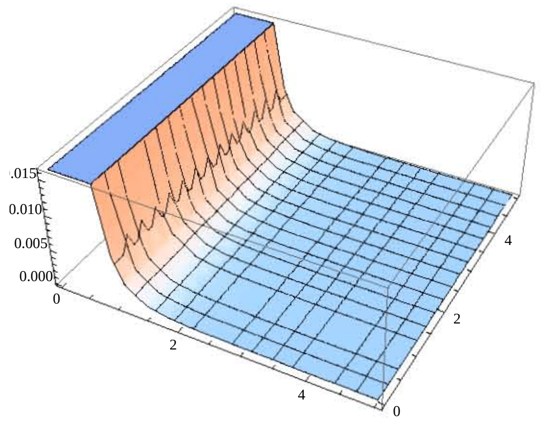

Figure 2.8: Graph of pressure for $aG_5 + G_6$.

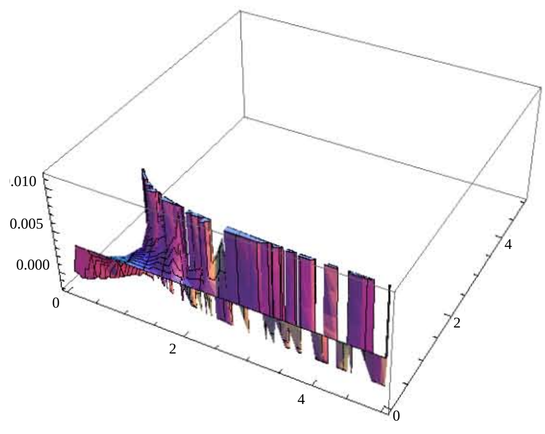

Figure 2.9: Graph of energy density for $aG_5 + G_6$.

with solution

$$U = -y^2 + \frac{C^2y^6}{6} + \frac{2\sqrt{C^2y^4 - 4}}{3C} - \frac{Cy^4\sqrt{C^2y^4-4}}{6} + D. \qquad (2.118)$$

Then equation (2.4) has the solution

$$V = -\frac{t}{2} - \frac{r^2}{2t} + \frac{C^2r^6}{12t^3} + \frac{\sqrt{C^2r^4 - 4t^2}}{3Ct} - \frac{Cr^4\sqrt{C^2r^4-4t^2}}{12t^3} + D. \qquad (2.119)$$

Both (2.114) and (2.119) are new solutions to (2.4).

Thus we have generated new analytic solutions to (2.4) via combinations of two symmetries of (2.4) except for one case when a numerical solution was provided.

2.10 Solutions invariant under three generators

It now remains to consider the two combinations of three generators which appear in the optimal system (2.56a–2.56k).

2.10.1 Invariance under $G_2 + G_3 + G_4$

Using the generator

$$G_2 + G_3 + G_4 = (t+1)\frac{\partial}{\partial r} + (t^3 + r)\frac{\partial}{\partial V} \qquad (2.120)$$

we determine the invariants from the invariant surface condition

$$\frac{dr}{1+t} = \frac{dV}{r+t^3} = \frac{dt}{0} \qquad (2.121)$$

such that the system has the invariants given by

$$t = y \qquad (2.122a)$$

$$V = \frac{r^2}{2(1+t)} + \frac{rt^3}{(1+t)} + U(t) \qquad (2.122b)$$

The partial differential equation (2.4) is reduced to the linear ordinary differential equation

$$\begin{aligned} -1 - y(5 + y(9 + y(7 + y(2 + y^2(2 + y)^2)))) & \\ + (1 + y)^3(1 + 2y)(-2U_y + yU_{yy}) &= 0 \end{aligned} \qquad (2.123)$$

Equation (2.123) can be solved with the assistance of `Mathematica` (Wolfram 1996) to give

$$\begin{aligned} U &= A + \frac{1}{8}\left(\frac{3y}{4} - \frac{19y^2}{4} + \frac{y^4}{2} + \frac{2y^5}{5} + \frac{4}{1+y} + y^3\left(1 + B\frac{8}{3}\right)\right) \\ &\quad -\frac{3}{64}\left((1 + 8y^3)\log[1 + 2y]\right) \end{aligned} \qquad (2.124)$$

The solution to equation (2.4) is then

$$\begin{aligned} V &= \frac{r^2}{2(1+t)} + \frac{rt^3}{(1+t)} + A + \frac{1}{8}\left(\frac{3t}{4} - \frac{19t^2}{4} + \frac{t^4}{2} + \frac{2t^5}{5} + \frac{4}{1+t}\right) \\ &\quad +\frac{1}{8}t^3\left((1 + B\frac{8}{3}) - \frac{3}{8}(1 + 8t^3)\log[1 + 2t]\right) \end{aligned} \qquad (2.125)$$

This is another new solution to equation (2.4) with pressure (2.5a) given by

$$p = \frac{1+t}{M}$$

where

$$\begin{aligned} M &= 2\pi t^4\left(8B(1+t) + 6(-2 - 4r + t + t^3)\right) \\ &\quad -2\pi t^4\left(9(1+t)\log[1+2t]\right) \end{aligned}$$

and energy density (2.5b) given by

$$\rho = \frac{-36(1+t)}{Q} + \frac{(t + 2\pi(1+t))}{4\pi^2 t^4}$$

where

$$Q = 4\pi^2(1+2t)\left(8B(1+t)+6(-2-4r+t+t^3)\right) -4\pi^2(1+2t)\,(9(1+t)\log[1+2t])^2$$

There is no simple barotropic equation of state connecting the energy density and the pressure. The thermodynamical behaviour describing this case graphically, is given in Figures 2.10 and 2.11.

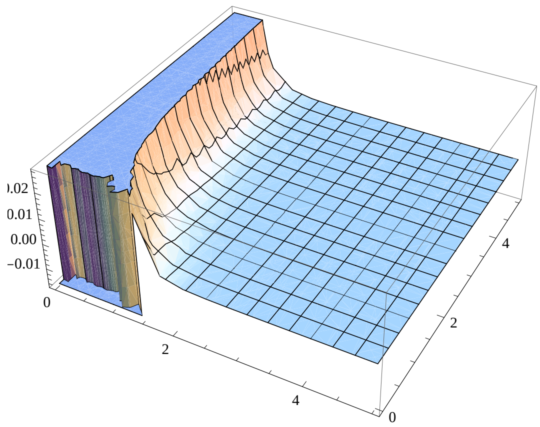

Figure 2.10: Graph of pressure for $G_2 - G_3 + G_4$.

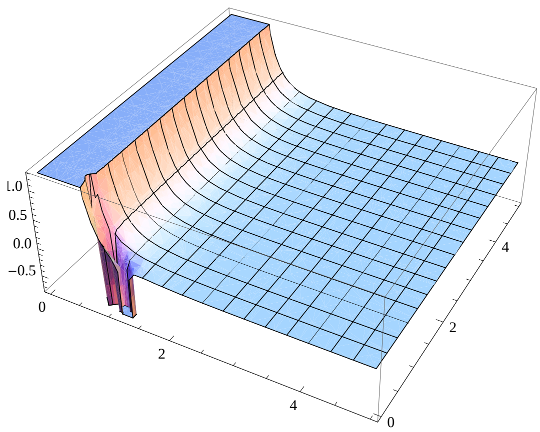

Figure 2.11: Graph of energy density for $G_2 - G_3 + G_4$.

Hence ρ and p are continuous functions and well behaved.

2.10.2 Invariance under $G_2 - G_3 + G_4$

For the generator

$$G_2 - G_3 + G_4 = (t+1)\frac{\partial}{\partial r} + (r - t^3)\frac{\partial}{\partial V} \tag{2.126}$$

we determine the invariants from the invariant surface condition

$$\frac{dr}{1+t} = \frac{dV}{r - t^3} = \frac{dt}{0} \tag{2.127}$$

we obtain the invariants

$$t = y \tag{2.128a}$$

$$V = \frac{r^2}{2(1+t)} - \frac{rt^3}{(1+t)} + U(t) \tag{2.128b}$$

This helps to produce the linear ordinary differential equation

$$\begin{aligned} -1 - y(5 + y(9 + y(7 + y(2 + y^2(2+y)^2)))) \\ + (1+y)^3(1+2y)(-2U_y + yU_{yy}) &= 0 \end{aligned} \tag{2.129}$$

It is possible to solve (2.129) with the help of `Mathematica` (Wolfram 1996) to get

$$\begin{aligned} U &= A + \frac{1}{8}\left(\frac{3y}{4} - \frac{19y^2}{4} + \frac{y^4}{2} + \frac{2y^5}{5} + \frac{4}{1+y} + y^3\left(1 + B\frac{8}{3}\right)\right) \\ &\quad -\frac{3}{64}\left((1+8y^3)\log[1+2y]\right) \end{aligned} \tag{2.130}$$

Hence the solution of equation (2.4) is

$$\begin{aligned} V &= \frac{r^2}{2(1+t)} + \frac{rt^3}{(1+t)} + A + \frac{1}{8}\left(\frac{3t}{4} - \frac{19t^2}{4} + \frac{t^4}{2} + \frac{2t^5}{5} + \frac{4}{1+t}\right) \\ &\quad + \frac{1}{8}t^3\left((1 + B\frac{8}{3}) - \frac{3}{8}(1+8t^3)\log[1+2t]\right) \end{aligned} \tag{2.131}$$

Thus we have found another new solution to equation (2.4) corresponding to the generator $G_2 - G_3 + G_4$ with pressure (2.5a) given by

$$p = \frac{1+t}{M}$$

where

$$\begin{aligned} M &= 2\pi t^4 \left(8B(1+t) + 6(-2-4r+t+t^3)\right) \\ &\quad -2\pi t^4 \left(9(1+t)\log[1+2t]\right) \end{aligned}$$

and energy density (2.5b) given by

$$\rho = \frac{-36(1+t)}{Q} + \frac{(t+2\pi(1+t))}{4\pi^2 t^4}$$

where

$$\begin{aligned} Q &= 4\pi^2(1+2t)\left(8B(1+t) + 6(-2-4r+t+t^3)\right) \\ &\quad -4\pi^2(1+2t)\left(9(1+t)\log[1+2t]\right)^2 \end{aligned}$$

Clearly there is no barotropic equation of state in this case. We have shown

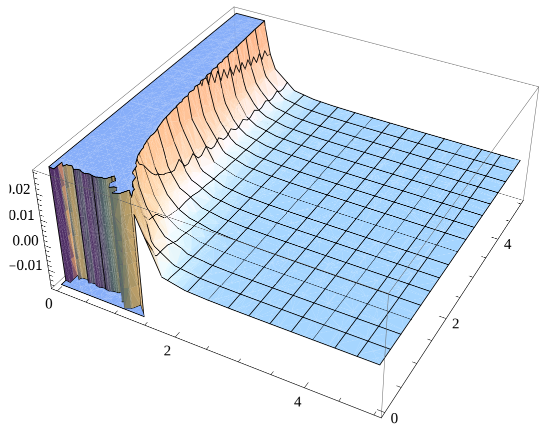

Figure 2.12: Graph of pressure for $G_2 - G_3 + G_4$.

the behaviour of energy density and pressure using `Mathematica` (Wolfram 1996) in Figures 2.12 and 2.13. It is clear that there exist regions of spacetime in which ρ and p are well behaved, remaining finite, continuous and bounded.

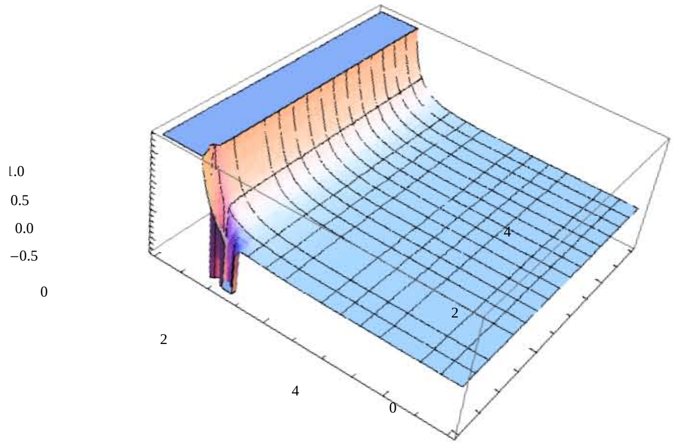

Figure 2.13: Graph of energy density for $G_2 - G_3 + G_4$.

2.11 Discussion

We summarise the results obtained in this chapter for convenience. In Table 2.3 we present the ODES that are generated when a single Lie symmetry generator is present. In Table 2.4 we give solutions to the partial differential equation (2.4) when it is invariant under a single generator from the optimal system (2.56a–2.56k). For the generator G_3 we do not obtain an invariant involving V and so it is not possible to generate an ordinary differential equation. Thus there is no solution possible invariant under G_3 alone. For the generators G_2 and G_4 it is possible to solve the resulting ordinary differential equations and obtain explicit forms for the function V given in Table 2.4. These are new solutions to equation (2.4) which have not been obtained previously. For the generator G_5 we have obtained the ordinary differential equation

$$UU_{yy} - (U_y)^2 + U = 0 \qquad (2.132)$$

which is of the same form as (2.40) in section 2.6. However it is important to observe that the characteristics here are different from those in section 2.6. Consequently the solutions V generated by the Lie symmetry G_5, and listed in

Generator	Invariants	ODE
G_2	$y = t$	$yU_{yy} - 2U_y - 1 = 0$
	$V = U(t)$	
G_3	$y = t$	No ODE exists
	$U(t) = r$	
G_4	$y = t$	$yU_{yy} - 2U_y - 1 = 0$
	$V = \frac{r^2}{2t} + U(t)$	
G_5	$y = r$	$UU_{yy} - (U_y)^2 + U = 0$
	$V = -\frac{t}{2} + \frac{1}{t}U(r)$	

Table 2.3: One symmetry: generators and ODES.

Generator	Solution to PDE
G_2	$V = A + \frac{1}{3}Bt^3 - \frac{t}{2}$
G_3	No solution to PDE
G_4	$V = \frac{r^2}{2t} + A + \frac{1}{3}Bt^3 - \frac{t}{2}$
G_5	$V = -\frac{t}{2} + \frac{1}{t}\left[\frac{1}{2m^2}(\sin(\sqrt{2}mr + m_0) + 1)\right]$
	$V = -\frac{t}{2}$
	$V = -\frac{t}{2} + \frac{1}{t}\left[\frac{1}{2}(r + \beta)^2\right]$
	$V = -\frac{t}{2} + \frac{1}{t}\left[\frac{1}{2m^2}(\cosh(\sqrt{2}mr + m_0) - 1)\right]$

Table 2.4: One symmetry: PDE solutions.

Table 2.4, comprise a new class of exact solutions to equation (2.4). Secondly, we consider the combinations of two generators which arise in the optimal system (2.56a–2.56k). Table 2.5, with invariants and reduced ODE, and Table 2.6, with analytic solution of the PDE, contain all cases that we were able to obtain explicit general solutions. For the generator $G_2 + G_5$, we have not been able to find a solution to the resultant ODE which is highly nonlinear. For the generator $aG_5 + G_6$, we have obtained explicit solutions only for the special parameter values $a = -\frac{1}{2}$ and $a = 1$. It is unlikely that the ODE will yield closed form solutions for other values of a.

It finally remains to consider the two combinations of three generators which

Generator	Invariants	ODE
$G_2 + G_3$	$t = y$	$yU_{yy} - 2U_y - 4y^6 - 1 = 0$
	$V = rt^3 + U(t)$	
$G_2 + G_4$	$t = y$	$(1 + 2y)(-1 - 2U_y + yU_{yy}) = 0$
	$V = \frac{r^2}{2(1+t)} + U(t)$	
$G_3 + G_4$	$t = y$	$2 + y^4 + 4U_y - 2yU_{yy} = 0$
	$V = rt^2 + \frac{r^2}{2t} + U(t)$	
$G_2 + G_5$	$r = \ln t + y$	$U_{yy} + 4U + 4UU_{yy} + 5U_y - 4U_y^2 + U_yU_{yy} = 0$
	$V = -\frac{t}{2} + \frac{1}{t}U(r - \ln t)$	
$aG_5 + G_6$	$r = yt^{1/(a+1)}$	$-(1 + 2a)^2U_y^2 + y^2U_{yy} + 2(-1 + a)(1 + 2a)U(1 + U_{yy}) + yU_y(2 + 5a + aU_{yy}) = 0$
	$V = -\frac{t}{2} + t^{(1-a)/(1+a)}U(rt^{-1/(1+a)})$	

Table 2.5: Two symmetries: generators and ODES.

Generator	Solution to PDE
G_2+G_3	$V=rt^3+A+\frac{1}{3}Bt^3+\frac{t^7}{7}-\frac{t}{2}$
G_2+G_4	$V=A\frac{r^2}{2(1+t)}+\frac{1}{3}Bt^3-\frac{t}{2}$
G_3+G_4	$V=rt^2+\frac{r^2}{2t}+A+\frac{1}{3}Bt^3+\frac{t^5}{20}-\frac{t}{2}$
G_2+G_5	No solution to PDE
aG_5+G_6	$V=-\frac{t}{2}+\left(B+\frac{1}{2}\left(-\frac{1}{6}e^{A/2}\left(e^A-\frac{4r}{t^2}\right)^{3/2}+\frac{e^Ar}{t^2}\right)\right)t^3, a=-\frac{1}{2}$
	$V=-\frac{t}{2}-\frac{r^2}{2t}+\frac{C^2r^6}{12t^3}+\frac{\sqrt{C^2r^4-4t^2}}{3Ct}-\frac{Cr^4\sqrt{C^2r^4-4t^2}}{12t^3}+D, a=1$

Table 2.6: Two symmetries: PDE solutions.

appear in the optimal system (2.56a)-(2.56k). In both cases it is possible to generate the invariants and reduced ODE which are presented in Table 2.7. The explicit solutions of the PDE are given in Table 2.8. Both functions in Table 2.8 are new solutions to equation (2.4).

For many applications in cosmology it is necessary that there exist barotropic equations of state in the form $p=p(\rho)$ (Stephani *et al* 2003). We find that for the case of a single generator of the optimal system (2.56a–2.56k) considered, there exists a linear equation of state. The relevant equations of state are presented in Table 2.9.

The equations of state $p=\rho$ and $p=\frac{1}{3}\rho$ were identified by Gupta and Sharma (1996) for their class of solutions. We have demonstrated that their result follows because of the existence of the symmetry G_5. The Lie symmetry G_2 produces a new solution with equation of state $p=\rho$. The generator G_4 gives another new solution with the linear equation of state $p=\frac{2\pi}{1+2\pi}\rho$. Such linear equations of state are of importance in relativistic stellar structures and arise in models of quark stars (Komathiraj and Maharaj 2007, Mak and Harko 2004, Sharma and Maharaj 2007, Witten 1984). Also, in the modeling of anisotropic relativistic matter in the presence of the electromagnetic field for strange stars and matter distributions, we need a linear barotropic equation of state (Lobo

Generator	Invariants	ODE
$G_2 + G_3 + G_4$	$t = y$	$-1 - y(5 + y(9 + y(7 + y(2 + y^2(2 + y)^2)))) + (1 + y)^3(1 + 2y)(-2U_y + yU_{yy}) = 0$
	$V = \frac{r^2}{2(1+t)} + \frac{rt^3}{(1+t)} + U(t)$	
$G_2 - G_3 + G_4$	$t = y$	$-1 - y(5 + y(9 + y(7 + y(2 + y^2(2 + y)^2)))) + (1 + y)^3(1 + 2y)(-2U_y + yU_{yy}) = 0$
	$V = \frac{r^2}{2(1+t)} - \frac{rt^3}{(1+t)} + U(t)$	

Table 2.7: Three symmetries: generators and ODES.

Generator	Solution to PDE
$G_2 + G_3 + G_4$	$V = \frac{r^2}{2(1+t)} + \frac{rt^3}{(1+t)} + A + \frac{1}{8}\left(\frac{3t}{4} - \frac{19t^2}{4} + \frac{t^4}{2} + \frac{2t^5}{5} + \frac{4}{1+t}\right) + \frac{1}{8}t^3\left((1 + B\frac{8}{3}) - \frac{3}{8}(1 + 8t^3)\log(1 + 2t)\right)$
$G_2 - G_3 + G_4$	$V = \frac{r^2}{2(1+t)} + \frac{rt^3}{(1+t)} + A + \frac{1}{8}\left(\frac{3t}{4} - \frac{19t^2}{4} + \frac{t^4}{2} + \frac{2t^5}{5} + \frac{4}{1+t}\right) + \frac{1}{8}t^3\left((1 + B\frac{8}{3}) - \frac{3}{8}(1 + 8t^3)\log(1 + 2t)\right)$

Table 2.8: Three symmetries: PDE solutions.

Generator	Solution to PDE	Equation of state
G_2	$V = A + \frac{1}{3}Bt^3 - \frac{t}{2}$	$p = \rho$
G_3	No solution to PDE	No equation of state
G_4	$V = \frac{r^2}{2t} + A + \frac{1}{3}Bt^3 - \frac{t}{2}$	$p = \frac{2\pi}{1+2\pi}\rho$
G_5	$V = -\frac{t}{2}$	$p = \rho$
	$V = -\frac{t}{2} + \frac{1}{t}\left[\frac{1}{2}(r+\beta)^2\right]$	$p = \frac{1}{3}\rho$

Table 2.9: Equation of state.

2006, Thirukkanesh and Maharaj 2009).

For the generators considered in section 2.9 and section 2.10 there are no simple barotropic equations of state connecting the energy density and the pressure. However it is possible to describe the thermodynamical behaviour graphically. As an example we consider the solution corresponding to the generator G_2+G_3. The energy density is

$$\rho = \frac{4\pi(B+3r) + 3(-3+2\pi)t^4}{16\pi^2 t^4(2B+6r+3t^4)^2}$$

and pressure is

$$p = \frac{1}{8\pi t^4(2B+6r+3t^4)}.$$

Clearly there is no barotropic equation of state in this case. The behaviour of pressure has been plotted in Figure 2.1 and the energy density is represented in Figure 2.2. We have generated these plots with the help of `Mathematica` (Wolfram 1996). It is clear that there exist regions of spacetime in which ρ and p are well behaved, remaining finite, continuous and bounded. It is then viable to study the behaviour of the thermodynamical quantities such as the temperature over this region. We point out that plots for ρ and p for the other combinations of generators in the optimal system have similar behaviour.

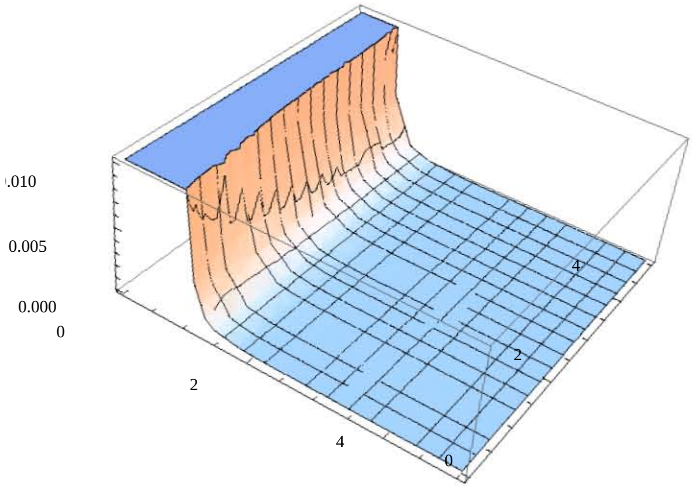

Figure 2.14: Graph of energy density for $G_2 + G_3$.

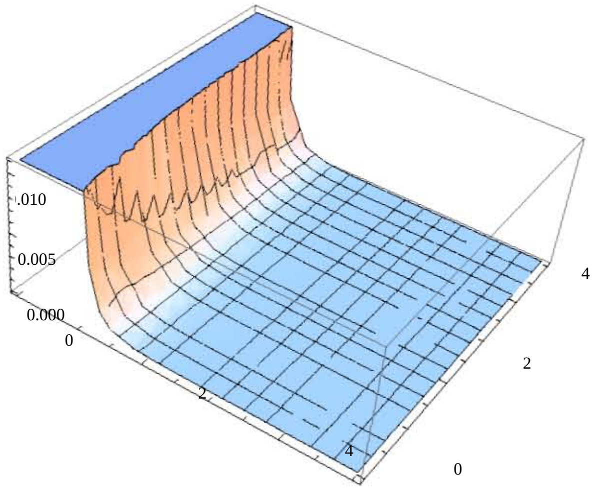

Figure 2.15: Graph of pressure for $G_2 + G_3$.

Chapter 3

New shear-free relativistic models with heat flow

3.1 Introduction

In this chapter, we consider spherically symmetric radiating spacetimes with vanishing shear which are important in relativistic astrophysics, radiating stars and cosmology. The assumption that the shear vanishes in a spherically symmetric spacetime, in the presence of nonvanishing heat flux, is often made to describe the dynamics of cosmological models. The importance of relativistic heat conducting fluids in modelling inhomogeneous processes, such as galaxy formation and evolution of perturbations, has been pointed out by Krasinski (1997). Some of the early exact solutions in the presence of heat flow were given by Bergmann (1981), Maiti (1982) and Modak (1984). Deng (1989) provided a general method of generating solutions to the Einstein field equations which contains most previously known exact solutions. Heat conducting exact solutions are necessary to generate temperature profiles in dissipative processes by integrating the heat transport equation as shown by Triginer and Pavon (1995). Bulk viscosity with heat flow affects the dynamics of inhomogeneous cosmological models as shown by Deng and Mannheim (1990). Recently Banerjee and Chatterjee (2005) and Banerjee *et al* (2003) have investigated heat conducting fluids in higher dimensional cosmological models when considering spherical collapse, the appearance of singularities and the formation of horizons. The role of heat flow in gravitational dynamics and perturbations in the framework of brane world cosmological models has been highlighted by Davidson and Gurwich (2008) and Maartens and Koyama (2010).

The presence of heat flux is necessary for a proper and complete description of radiating relativistic stars. The result of Santos *et al* (1985), in one of the first complete relativistic radiating models, indicates that the interior spacetime should contain a nonzero heat flux so that the matching at the boundary to the exterior Vaidya spacetime is possible. Models containing heat flow in astrophysics have been applied to problems in gravitational collapse, black hole physics, formation of singularities and particle production at the stellar surface in four and higher dimensions. The study of Chang *et al* (2008) showed that the process of gravitational collapse of a spherical star with heat flow may serve as a possible energy mechanism for gamma-ray bursts.

Herrera *et al* (2004), Maharaj and Govender (2004), and Misthry *et al* (2008) have shown that relativistic radiating stars are useful in the investigation of the cosmic censorship hypothesis and in describing collapse with vanishing tidal forces. Wagh *et al* (2001) presented solutions to the Einstein field equations for a shear-free spherically symmetric spacetime, with radial heat flux by choosing a barotropic equation of state. For particles in geodesic motion a general analytic treatment is possible and solutions are obtainable in terms of elementary and special functions as demonstrated by Thirukkanesh and Maharaj

(2009). Herrera *et al* (2006) found analytical solutions to the field equations for radiating collapsing spheres in the diffusion approximation. These authors demonstrated that the thermal evolution of the collapsing sphere which can be modeled in causal thermodynamics requires heat flow. Note that stellar models with shear are difficult to analyse but particular exact solutions have been found by Naidu *et al* (2006) and Rajah and Maharaj (2008).

Shear-free fluids are also essential in modeling inhomogeneous cosmological processes. Krasinski (1997) points out the need for radiating models in the formation of structure, evolution of voids, the study of singularities and cosmic censorship. Heat conducting fluids are important in cosmological models in higher dimensions and permits collapse without the appearance of an event horizon; this aspect has been studied by Banerjee and Chatterjee (2005). In brane world models the presence of heat flow allows for more general behaviour than in standard general relativity; the analogue of the Oppenheimer-Snyder model of a collapsing dust on the brane radiates which was proved by Govender and Dadhich (2002).

In this chapter we intend to analyse the pivotal equation previously studied by Deng (1989). He developed a general (though *ad hoc*) method to generate solutions and obtained a new class of solutions which included various known special cases (see section 3.2.). We adopt a systematic approach (using Lie theory) to generalise known solutions and generate new solutions of the same equation. The basic features of Lie symmetry analysis are outlined in section 3.3. In section 3.4, we extend the Deng (1989) known solutions to find new solutions to the fundamental equation utilising Lie theory. In section 3.5, we systematically study other group invariant solutions admitted by the fundamental equation. For most of the symmetries, we obtain either an implicit solution or we can reduce the governing equations to a Riccati equation which is difficult to solve in general (though particular solutions can always be found). There are two cases in which we find new exact solutions regardless of the complexity of the generating function chosen.

3.2 Radiating Model

Due to the requirements of spherical symmetry and the shear-free condition, the line element can be written as

$$ds^2 = -D^2 dt^2 + \frac{1}{V^2}\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{3.1}$$

where D and V are functions of t and r. In the study of solutions of the Einstein equations with heat flux, Deng (1989) studied a shear-free spherically symmetric cosmological model where he considered the energy-momentum tensor as

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu} + q_\mu U_\nu + q_\nu U_\mu \tag{3.2}$$

where U_μ, ρ, p are the four-velocity of the fluid, energy density and pressure, respectively, and q_μ is the heat flux vector. The Einstein field equations are given by

$$\rho = \frac{3V_t^2}{D^2V^2} + V^2\left[\frac{2V_{rr}}{V} - \frac{3V_r^2}{V^2} + \frac{4V_r}{rV}\right] \tag{3.3a}$$

$$p = \frac{1}{D^2}\left[\frac{2V_{tt}}{V} - \frac{5V_t^2}{V^2} - \frac{2D_tV_t}{DV}\right] + V^2\left[\frac{V_r^2}{V^2} - \frac{2D_rV_r}{DV} + \frac{2D_r}{rD} - \frac{2V_r}{rV}\right] \tag{3.3b}$$

$$p = \frac{2V_{tt}}{D^2V} - \frac{5V_t^2}{D^2V^2} - \frac{2D_tV_t}{D^3V} + \frac{D_rV^2}{rD} - \frac{VV_r}{r} + \frac{D_{rr}V^2}{D} + V_r^2 - VV_{rr} \tag{3.3c}$$

$$q = -2V^2\left[\frac{V_{tr}}{DV} - \frac{V_tV_r}{DV^2} - \frac{D_rV_t}{D^2V}\right] \tag{3.3d}$$

Equations (3.3*b*) and (3.3*c*) together imply

$$VD_{uu} + 2D_uV_u - DV_{uu} = 0 \tag{3.4}$$

which is the condition of pressure isotropy with $u = r^2$. Glass (1990) and Bergmann (1981), also discovered that in the comoving system, Einstein field equations generate the pressure isotropy condition given by the equation (3.4), which is the master equation for the system (3.3a–3.3d).

A number of authors have obtained various solutions to (3.4), among which is the conformally flat class

$$D = \frac{c(t)u + d(t)}{a(t)u + b(t)} \tag{3.5a}$$

$$V = a(t)u + b(t) \tag{3.5b}$$

where a, b, c, and d are arbitrary functions of t. Sanyal and Ray (1984) and Modak (1984) obtained this class along with other solutions, while Bergmann (1981) and Maiti (1982) obtained special cases of the class.

A method of generating more general solutions to the master equation (3.4) has been developed by Deng (1989) who found solutions when simple forms of V or D are chosen. In finding solutions, Deng (1989) considered the master equation as an ordinary differential equation with respect to u. He treated (3.4) as a linear equation of V if D is a known function and vice versa. His approach was as follows:

- Choose a simple function $D = D_1$ and find the most general solution $V = V_1$ of (3.4).
- Take $V = V_1$ and find the most general solution $D = D_2$ obeying (3.4).
- Take $D = D_2$ and find the most general solution $V = V_2$.

This procedure can be continued indefinitely. By alternating between D and V, this process can go on forever generating infinite series of solutions expressed in terms of integrals. This is a powerful method as all known solutions can be generated using this algorithm

In this chapter, we show the Deng (1989) general method of generating solutions, that gives a general class of solutions which include (3.5a–3.5b) as special cases, may be extended by a simple invariant transformations. In addition, we reduce the order of (3.4) via Lie analysis to obtain new solutions not obtainable via the Deng approach.

3.3 Lie analysis

The basic feature of Lie analysis for a system of ordinary differential equations in n dependent variables requires the determination of the (infinitesimal form of the) one–parameter (ε) Lie group of transformations

$$\begin{aligned} \bar{u} &= u + \varepsilon\xi(u, V, D) + O(\varepsilon^2) \\ \bar{V} &= V + \varepsilon\eta(u, V, D) + O(\varepsilon^2) \\ \bar{D} &= D + \varepsilon\zeta(u, V, D) + O(\varepsilon^2) \end{aligned} \tag{3.6}$$

that leaves the solution set of the system invariant. The full details can be found in a number of excellent texts (Bluman and Kumei 1989, Olver 1993). In order to obtain (3.6) we need to determine their "generator"

$$Y = \xi\frac{\partial}{\partial u} + \eta\frac{\partial}{\partial V} + \zeta\frac{\partial}{\partial D} \tag{3.7}$$

(also called a symmetry of the differential equation) which is a set of vector fields. Having found the symmetries, the finite (global) form of the transformation that leaves the system invariant is obtained by solving

$$\begin{aligned} \tfrac{\mathrm{d}\bar{u}}{\mathrm{d}\varepsilon} &= \xi(\bar{u}, \bar{V}, \bar{D}) \\ \tfrac{\mathrm{d}\bar{V}}{\mathrm{d}\varepsilon} &= \eta(\bar{u}, \bar{V}, \bar{D}) \\ \tfrac{\mathrm{d}\bar{D}}{\mathrm{d}\varepsilon} &= \zeta(\bar{u}, \bar{V}, \bar{D}) \end{aligned} \tag{3.8}$$

subject to

$$\bar{u}\,\Big|_{\varepsilon=0} = u, \qquad \bar{V}\big|_{\varepsilon=0} = V, \qquad \bar{D}\Big|_{\varepsilon=0} = D \tag{3.9}$$

The determination of the generators is a straight forward process, greatly aided by computer algebra packages (Dimas and Tsoubelis 2005, Cheviakov 2007). We have found the package `PROGRAM LIE` (Head 1993) to be the most useful in practice. Indeed it is quite remarkable how accomplished such an old package is – it often outperforms its modern counterparts!

Utilising `PROGRAM LIE`, we can demonstrate that (3.4) admits the following Lie

	Y_1	Y_2	Y_3	Y_4	Y_5
Y_1	0	Y_1	0	0	$Y_4 + 2Y_2$
Y_2	$-Y_1$	0	0	0	Y_5
Y_3	0	0	0	0	0
Y_4	0	0	0	0	0
Y_5	$-Y_4 - 2Y_2$	$-Y_5$	0		0

Table 3.1: Commutation table for vector fields Y_1–Y_5

point symmetries/vector fields:

$$Y_1 = \frac{\partial}{\partial u} \tag{3.10a}$$

$$Y_2 = u\frac{\partial}{\partial u} \tag{3.10b}$$

$$Y_3 = D\frac{\partial}{\partial D} \tag{3.10c}$$

$$Y_4 = V\frac{\partial}{\partial V} \tag{3.10d}$$

$$Y_5 = u^2\frac{\partial}{\partial u} + uV\frac{\partial}{\partial V} \tag{3.10e}$$

with the non-zero Lie Bracket relationships:

$$\begin{array}{ll} [Y_1, Y_2] = Y_1 & [Y_2, Y_5] = Y_5 \\ [Y_1, Y_5] = Y_4 + 2Y_2 & [Y_5, Y_1] = -Y_4 - 2Y_2 \\ [Y_2, Y_1] = -Y_1 & [Y_5, Y_2] = -Y_5 \end{array} \tag{3.11}$$

for the given fields. The commutation relations between these vector fields is given by Table 3.1. At this stage, it is usual to use these symmetries to reduce the order of the equation in the hope of finding solutions. Before we proceed with this approach, we show how the finite form of the transformations generated by these symmetries can generate new solutions from known solutions. According to Lie's theory, the construction of a one-parameter group Y is equivalent to the determination of the corresponding infinitesimal transformations:

$$\begin{aligned} \bar{u}^i &= u^i + a\xi^i(u, D, V) \\ \bar{D}^\beta &= D^\beta + a\eta^\beta(u, D, V) \\ \bar{V}^\delta &= V^\delta + a\zeta^\delta(u, D, V) \end{aligned} \tag{3.12}$$

These are obtained by the Taylor series expression in a of

$$\begin{aligned} Ta : \bar{u}^i &= f^i(u, D, V, a), \qquad i = 1, ..., n \\ \bar{D}^\beta &= \psi^\beta(u, D, V, a), \qquad \beta = 1, ..., n \\ \bar{V}^\delta &= \phi^\delta(u, D, V, a), \qquad \delta = 1, ..., n \end{aligned} \tag{3.13}$$

about $a = 0$, taking into account the initial conditions:

$$f^i|_{a=0} = u^i, \qquad \psi^\beta|_{a=0} = D^\beta, \qquad \phi^\delta|_{a=0} = V^\delta \tag{3.14}$$

Therefore

$$\begin{aligned} \xi^i(u, D, V, a) &= \tfrac{\partial f^i(u,D,V,a)}{\partial a}|_{a=0} \\ \eta^\beta(u, D, V, a) &= \tfrac{\partial \psi^i(u,D,V,a)}{\partial a}|_{a=0} \\ \zeta^\delta(u, D, V, a) &= \tfrac{\partial \phi^i(u,D,V,a)}{\partial a}|_{a=0} \end{aligned} \tag{3.15}$$

It is possible to introduce the symbol X of the infinitesimal transformations by rewriting (3.12) as

$$\bar{u}^i = (1 + aX)u^i, \qquad \bar{D}^\beta = (1 + aX)D^\beta, \qquad \bar{V}^\delta = (1 + aX)V^\delta \tag{3.16}$$

where

$$X = \xi^i(u, D, V)\frac{\partial}{\partial u^i} + \eta^\beta(u, D, V)\frac{\partial}{\partial D^\beta} + \zeta^\delta(u, D, V)\frac{\partial}{\partial V^\delta} \tag{3.17}$$

which is also known as the infinitesimal operator or generator of the group X.

3.4 Extending known solutions

Deng's approach was *ad hoc* as a result, we use Lie analysis as a systematic approach to generate a solution to the equation

$$VD_{uu} + 2D_uV_u - DV_{uu} = 0 \tag{3.18}$$

Using the Lie equations, we get the following symmetry transformations for the individual symmetry. From the generator:

$$Y_1 = \frac{\partial}{\partial u} \tag{3.19}$$

we have

$$\xi^i(u, D, V) = 1 \tag{3.20a}$$

$$\eta^\beta(u, D, V) = 0 \tag{3.20b}$$

$$\zeta^\delta(u, D, V) = 0 \tag{3.20c}$$

Using (3.20a), we find

$$\frac{d\bar{u}}{da}|_{a=0} = 1 \tag{3.21a}$$

$$\bar{u} = a + k_1 \tag{3.21b}$$

such that, at $a = 0$, implies that $\bar{u} = u$, hence $k_1 = u$, then we have the transformation

$$\bar{u} = u + a \tag{3.22}$$

Again, using (3.20b), we find

$$\frac{d\bar{D}}{da}|_{a=0} = 0 \tag{3.23a}$$

$$\bar{D} = k_2 \tag{3.23b}$$

so that at $a = 0$, implies $\bar{D} = D$ and $k_2 = D$, then we have the transformation

$$\bar{D} = D \tag{3.24}$$

Lastly, from (3.20c), we find

$$\frac{d\bar{V}}{da}|_{a=0} = 0 \tag{3.25a}$$

$$\bar{V} = k_3 \tag{3.25b}$$

where, at $a = 0$, implies that $\bar{V} = V$, so that $k_3 = V$, then we have the transformation

$$\bar{V} = V \tag{3.26}$$

Having found the symmetries of (3.18) we know that they generate transformations of the form (3.8) that leaves (3.18) invariant.

We illustrate the approach using the infinitesimal generator Y_1. First we observe that

$$\xi = 1, \qquad \eta = 0, \qquad \zeta = 0 \tag{3.27}$$

We solve equations (3.8), subject to (3.9), to obtain

$$\bar{u} = u + a_1 \tag{3.28a}$$

$$\bar{V} = V \tag{3.28b}$$

$$\bar{D} = D \tag{3.28c}$$

This means that (3.28a–3.28c) maps equation (3.18) to the form

$$\bar{V}\bar{D}_{\bar{u}\bar{u}} + 2\bar{D}_{\bar{u}}\bar{V}_{\bar{u}} - \bar{D}\bar{V}_{\bar{u}\bar{u}} = 0. \tag{3.29}$$

As a result, any existing solution to equation (3.18) can be transformed to a solution of (3.29) (and so a solution of (3.18) itself) by (3.28a–3.28c). Note that, usually, a_1 is an arbitrary constant. However, since V and D depend on u and t we take a_1 to be an arbitrary function of time, $a_1 = a_1(t)$.

If we now take each of the remaining symmetries successively, we obtain the general transformation as follows:

$$Y_1 : \bar{u} = a_1 + u, \qquad \bar{D} = D, \qquad \bar{V} = V \tag{3.30a}$$

$$Y_2 : \bar{u} = e^{a_2}(a_1 + u), \qquad \bar{D} = D, \qquad \bar{V} = V \tag{3.30b}$$

$$Y_3 : \bar{u} = e^{a_2}(a_1 + u), \qquad \bar{D} = e^{a_3}D, \qquad \bar{V} = V \tag{3.30c}$$

$$Y_4 : \bar{u} = e^{a_2}(a_1 + u), \qquad \bar{D} = e^{a_3}D, \qquad \bar{V} = e^{a_4}V \tag{3.30d}$$

$$Y_5 : \bar{u} = \frac{e^{a_2}(a_1 + u)}{1 - a_5 e^{a_2}(a_1 + u)}, \bar{D} = e^{a_3}D, \qquad \bar{V} = \frac{e^{a_4}V}{1 - a_5 e^{a_2}(a_1 + u)} \tag{3.30e}$$

The combination of all the transformation of symmetries, therefore leads to the general relationship:

$$\bar{u} = \frac{e^{a_2}(a_1 + u)}{1 - a_5 e^{a_2}(a_1 + u)} \tag{3.31a}$$

$$\bar{D} = e^{a_3}D \tag{3.31b}$$

$$\bar{V} = \frac{e^{a_4}V}{1 - a_5 e^{a_2}(a_1 + u)} \tag{3.31c}$$

where the a_i are all arbitrary function of time.

Thus any known solution of equation (3.18) can be transformed to a new solution of equation (3.18) via (3.31a–3.31c). For example, the particular Deng (1989) solution

$$D = 1, \qquad V = \alpha(t)u + \beta(t) \tag{3.32}$$

is transformed to the new solution

$$\bar{u} = \frac{e^{a_2}(a_1 + u)}{1 - a_5 e^{a_2}(a_1 + u)} \tag{3.33a}$$

$$\bar{V} = \frac{e^{a_4}(\alpha(t)u + \beta(t))}{1 - a_5 e^{a_2}(a_1 + u)} \tag{3.33b}$$

$$\bar{D} = e^{a_3} \tag{3.33c}$$

All the solutions in the Deng (1989) class, the conformally flat models (3.5a–3.5b), the result listed in Krasinski (1997) and Stephani *et al* (2003) can be extended by (3.31a–3.31c) to produce new solutions of (3.18). Also note that all the new results that we derive in the next section can be similarly extended via (3.31a–3.31c).

3.5 New solutions via Lie symmetries

The usual use of symmetries of ordinary differential equations is to reduce the order of a differential equation. For symmetries (3.10a–3.10e) we obtain either an implicit solution of (3.18) or we can reduce the governing equations to a Riccati equation. Both these results are no improvement to that of Deng (1989), *i.e.* we need to choose simple forms for one of the functions (either D or V) in order to solve for the other. However there are two cases in which we can find new solutions regardless of the complexity of the function chosen.

3.5.1 Generator Y_1

We consider the generator

$$Y_1 = \frac{\partial}{\partial u} \tag{3.34}$$

Using the first extension of G_1:

$$\begin{aligned} Y^{[1]} &= \xi\frac{\partial f}{\partial u} + \eta_1\frac{\partial f}{\partial V} + \eta_2\frac{\partial f}{\partial D} + (\eta_1' - V'\xi')\frac{\partial f}{\partial V'} \\ &\quad + (\eta_2' - D'\xi')\frac{\partial f}{\partial D'} \end{aligned} \tag{3.35}$$

we obtain

$$\frac{\partial f}{\partial u} + 0\frac{\partial f}{\partial V} + 0\frac{\partial f}{\partial D} + 0\frac{\partial f}{\partial V'} + 0\frac{\partial f}{\partial D'} = 0$$

with the associated Lagrange's system given by:

$$\frac{du}{1} = \frac{dV}{0} = \frac{dD}{0} = \frac{dV'}{0} = \frac{dD'}{0} \tag{3.36}$$

Therefore $p = V, r = D, q = V_u$ and $s = D_u$ are the four characteristics. The generator Y_1 (3.34) admits the invariants

$$\begin{aligned} p &= V \\ q(p) &= V_u \\ r(p) &= D \end{aligned} \tag{3.37}$$

In the transformation of the master equation to a new ordinary differential equation we have excluded the fourth invariant $s = D_u$ so that we can concentrate on the dependent variable V. The transformation leads to the equation

$$q'(p)(r(p) - pr'(p)) = q(p)(pr''(p) + 2r'(p)) \tag{3.38}$$

We treat equation (4.14) as an equation in q which is first order and linear. This can be integrated to give

$$q = q_0 e^{\int \frac{2r' + pr''}{r - r'p}} dp \tag{3.39}$$

since $q = V_u$, we can integrate one more time to give the solution

$$\int \left[e^{-\int \frac{2D_V + VD_{VV}}{D - VD_V} dV} \right] dV = q_0 u + u_0 \tag{3.40}$$

where q_0 and u_0 are arbitrary functions of time. This means that, given any function V depending on D, we can work out V explicitly from (3.40). Such a relationship between V and D has not been found previously. This is a new solution to equation (3.18).

Using Deng (1989) solution $D = 1$, we evaluate (3.40) to obtain

$$V = q_0(t)u + u_0 \tag{3.41}$$

as obtained by Deng (1989). If we take $D = V^2$, equation (3.40) is reduced to

$$\int V^6 dV = q_0 u + u_0 \tag{3.42}$$

and hence

$$\begin{aligned} V_1 &= (7q_0 u + 7u_0)^{1/7} \\ &= (\bar{q}_0 u + \bar{u}_0)^{1/7} \end{aligned} \tag{3.43}$$

with

$$D = (\bar{q}_0 u + \bar{u}_0)^{2/7} \tag{3.44}$$

Note that, having found V from (3.40) we can generate a new solution via the reduction of order. The second independent solution is given by the form

$$V_2 = y\,(\bar{q}_0 u + \bar{u}_0)^{1/7} \tag{3.45}$$

with y being a function to be found. The transformation reduces (3.18) to the form

$$y'' - \frac{2q_0}{7\,(q_0 u + u_0))} y' = 0 \tag{3.46}$$

The solution of (3.46) is

$$y = \frac{7C_1}{9q_0} (q_0 u + u_0)^{9/7} + C_2 \tag{3.47}$$

Hence

$$V_2 = \frac{7C_1}{9q_0} (q_0 u + u_0)^{10/7} + C_2\,(q_0 u + u_0)^{1/7} \tag{3.48}$$

Therefore

$$V = \frac{C_1}{q_0} (q_0 u + u_0)^{10/7} + C_2\,(q_0 u + u_0)^{1/7} \tag{3.49}$$

where C_1 and C_2 are arbitrary functions of time, is the general solution to (3.18) when $D = (\bar{q}_0 u + \bar{u}_0)^{2/7}$.

3.5.2 Generator Y_2

The partial set of invariants of

$$Y_2 = u\frac{\partial}{\partial u} \tag{3.50}$$

are given by

$$\begin{aligned} p &= V \\ q(p) &= uV_u \\ r(p) &= D \end{aligned} \tag{3.51}$$

Then the master equation (3.18) is reduced to

$$q'(p) + \left(\frac{pr''(p) + 2r'(p)}{pr'(p) - r(p)}\right) q(p) = 1 \tag{3.52}$$

We treat (3.52) as an equation in q which is first order and linear. This can be integrated to give

$$qe^{\int \left(\frac{pr''+2r'}{pr'-r}\right) dp} = \int e^{\int \left(\frac{pr''+2r'}{pr'-r}\right) dp} + A \tag{3.53}$$

In terms of the original variables, the solution to equation (3.18) is

$$\int e^{-\int \frac{2D_V + VD_{VV}}{D - VD_V} dV} dV = \bar{B}u + A \tag{3.54}$$

which is the same as (3.40).

3.5.3 Generator Y_3

A partial set of invariants for

$$Y_3 = D\frac{\partial}{\partial D} \tag{3.55}$$

is

$$\begin{aligned} p &= u \\ q(p) &= V \\ r(p) &= \tfrac{D_u}{D} \end{aligned} \tag{3.56}$$

These invariants reduce equation (3.18) to the Riccati equation

$$r'(p) = \frac{q''(p)}{q(p)} - \frac{2q'(p)}{q(p)} r(p) - r^2(p) \tag{3.57}$$

If we use the full set of invariants which includes $s(p) = V_u$, we obtain the equation

$$s_p - 2rs = q(r_p + r^2) \tag{3.58}$$

which can be integrated to obtain

$$s = e^{2\int r dp} \int \left[q(r_p + r^2) e^{-2\int r dp} dp + s_0 \right] \tag{3.59}$$

since $s(p) = V_u$ we can integrate further to obtain the implicit solution

$$\begin{aligned} V &= \int \left[e^{2\int (D_u/D) du} \int V \left(\frac{d(D_u/D)}{du} + \left(\frac{D_u}{D} \right)^2 \right) e^{-2\int (D_u/D) du} du + A \right] du + B \\ &= \int \left[e^{2\int (D_u/D) du} \int V \frac{D_{uu}}{D} e^{-2\int (D_u/D) du} du + A \right] du + B \end{aligned} \tag{3.60}$$

3.5.4 Generator Y_4

The generator

$$Y_4 = V \frac{\partial}{\partial V} \tag{3.61}$$

has the invariants

$$\begin{aligned} p &= u \\ q(p) &= \tfrac{V_u}{V} \\ r(p) &= D \end{aligned} \tag{3.62}$$

Using these invariants, equation (3.18) is transformed to the Riccati equation

$$q'(p) = \frac{r''(p)}{r(p)} + 2\frac{r'(p)}{r(p)}q(p) - q^2(p) \tag{3.63}$$

Then if we use the full set of invariants which includes $s(p) = D_u$, we obtain the equation

$$s_p + 2sq = r(q_p + q^2) \tag{3.64}$$

which can be integrated to obtain

$$s = e^{-2\int q dp} \int \left[r(q_p + q^2)e^{2\int q dp} dp + s_0 \right] \tag{3.65}$$

since $s(p) = D_u$ we can integrate further to obtain the implicit solution

$$\begin{aligned} D &= \int \left[e^{-2\int (V_u/V) du} \int D \left(\frac{d(V_u/V)}{du} + \left(\frac{V_u}{V} \right)^2 \right) e^{2\int (V_u/V) du} du + A \right] du + B \\ &= \int \left[e^{-2\int (V_u/V) du} \int D \frac{V_{uu}}{V} e^{2\int (V_u/V) du} du + A \right] du + B \end{aligned} \tag{3.66}$$

3.5.5 Generator Y_5

Using the generator

$$Y_5 = u^2 \frac{\partial}{\partial u} + uV \frac{\partial}{\partial V} \tag{3.67}$$

we obtain the invariants

$$\begin{aligned} p &= \tfrac{V}{u} \\ q(p) &= u^2 D_u \\ r(p) &= D \end{aligned} \tag{3.68}$$

The differential equation (3.18) is reduced to

$$q'(p) + \left[\frac{2r'^2(p) + r(p)r''(p)}{pr'^2(p) - r(p)r'(p)} \right] q(p) = 0 \tag{3.69}$$

which can be integrated to obtain

$$q = e^{\int\left(\frac{2r'^2+rr''}{rr'-pr'^2}\right)dp+q_0} \tag{3.70}$$

since $q = u^2 D_u$ we can integrate further to obtain the solution

$$D = \int -\frac{1}{u}\left[e^{\int\left(\frac{2D_u^2+DD_{uu}}{DD_u-(V/u)D_u^2}\right)\frac{uV_u-V}{u^2}du+A}\right]du + B \tag{3.71}$$

3.5.6 Generator $Y_3 + Y_4$

For the generator

$$Y_3 + Y_4 = D\frac{\partial}{\partial D} + V\frac{\partial}{\partial V} \tag{3.72}$$

the invariants are

$$\begin{aligned} p &= u \\ q(p) &= \tfrac{V}{D} \\ r(p) &= \tfrac{D_u}{V} \end{aligned} \tag{3.73}$$

The differential equation (3.18) is reduced to

$$q'' = 2qr^2 \tag{3.74}$$

after eliminating $s(p) = \frac{V_u}{D_u}$. If we consider the ratio

$$r^2 = \frac{q''}{2q} \tag{3.75}$$

which can also be expressed as

$$\left(\frac{D_u}{V}\right)^2 = \frac{\frac{V_{uu}}{D} - \frac{2V_uD_u}{D^2} - \frac{VD_{uu}}{D^2} + \frac{2VD_u^2}{D^3}}{\frac{2V}{D}} \tag{3.76}$$

Then the solution to equation (3.18) is

$$D = \pm\int\left(V\sqrt{\frac{V_{uu}}{2V} - \frac{V_uD_u}{DV} - \frac{D_{uu}}{2D} + \frac{D_u^2}{D^2}}\right)du + A \tag{3.77}$$

3.5.7 The choice $W = \frac{V}{D}$

We also consider the ratio of V to D (and later D to V) to generate a new solution. Incidentally both ratios arise as a result of a combination of the generators Y_3 and Y_4 given by

$$Y_3 + Y_4 = D\frac{\partial}{\partial D} + V\frac{\partial}{\partial V} \tag{3.78}$$

This symmetry combination gives rise to the invariant

$$W = \frac{V}{D} \tag{3.79}$$

Then equation (3.18) is transformed by (3.79) to the form

$$-2WD_u^2 + D^2 W_{uu} = 0 \tag{3.80}$$

with solution

$$D = C_1(t)\exp\left(\int \pm\frac{\sqrt{W_{uu}}}{\sqrt{2W}}du\right) \tag{3.81}$$

Given any function $W = W(u)$ we can integrate the right hand side and find a form for D. If we take $W = a(t)u + b(t)$, then (3.81) gives

$$D = \bar{C}_1(t) \tag{3.82}$$

and

$$V = \bar{C}_1(t)(a(t)u + b(t)) \tag{3.83}$$

which corresponds to a Deng (1989) solution.

Alternatively, we could substitute the inverse of (4.25), *i.e.*

$$\widehat{W} = \frac{D}{V} \tag{3.84}$$

into (3.18) and obtain

$$2D'^2\widehat{W}^2 - 2D^2\widehat{W}'^2 + D^2\widehat{W}\widehat{W}'' = 0 \tag{3.85}$$

with solution

$$D = C_2(t)\exp\left(\pm\int\frac{\sqrt{\widehat{W}'^2 - \frac{1}{2}\widehat{W}\widehat{W}''}}{\widehat{W}}\mathrm{d}u\right) \tag{3.86}$$

Again, given any function $W = W(u)$ we can integrate the right hand side and find a form for D. If we take $W = a(t)u + b(t)$ as before, in (3.86), we find that

$$D_1 = C_2(t)(a(t)u + b(t)) \tag{3.87}$$

and

$$V_1 = C_2(t) \tag{3.88}$$

which again is essentially a solution of Deng (1989). However, we also have

$$D_2 = \frac{\bar{C}_2(t)}{a(t)u + b(t)} \tag{3.89}$$

and

$$V_2 = \frac{\bar{C}_2(t)}{(a(t)u + b(t))^2} \tag{3.90}$$

satisfies (3.4), thus obtaining two different solutions from the same seed function.

Observe that (3.81) and (3.86) will contain all solutions of Deng (1989) for appropriately chosen seed functions W or $\widehat{W}$ which are ratios of the metric functions. While Deng's approach requires simply chosen forms of either D or V in order to integrate (3.18), we have no such requirement. We are always able to reduce (3.18) to the quadratures (3.81) or (3.86) regardless of the complexity of the seed functions, a result not obtainable via Deng's approach.

3.6 Discussion

We have been able to extend Deng's solutions (Deng 1989) of the Einstein field equations governing shear-free heat conducting fluids in general relativity. This was accomplished by using simple transformations based on the invariance properties of the equation under study *viz.* (3.4). In addition, motivated by the invariants of the symmetries admitted by (3.4) we were able to reduce (3.4) to quadratures for *any* given seed function. This leads to three new classes of solutions for infinite families of functional forms involving $D(V)$, W and $\widehat{W}$. This is an improvement on the approach of Deng (1989) who required 'simple' functional forms for D or V to be chosen before (3.4) could be solved. This again promotes the use of Lie's theory of extended groups to analyse the Einstein field equations arising in different applications/models in general relativity.

Chapter 4

Applications of Lie symmetries to higher dimensional manifolds

4.1 Introduction

In this chapter, we consider higher dimensional spherically symmetric radiating spacetimes with vanishing shear which are important in relativistic astrophysics, radiating stars and cosmology. Shear-free spacetimes are mostly used in the modeling of relativistic stars which get rid of null radiation in the form of a radial heat flow. In the literature, there exists a large number of studies of various models involving gravitational collapse with radiative processes. The presence of heat flux is necessary for a proper and complete description of radiating relativistic stars. The result of Santos *et al* (1985) indicates that the interior spacetime should contain a nonzero heat flux so that the matching at the boundary to the exterior Vaidya spacetime is possible. Models containing heat flow in astrophysics have been applied to problems in the gravitational collapse, black hole physics, formation of singularities and particle production at the stellar surface in four and higher dimensions.

Herrera *et al* (2004), Maharaj and Govender (2004), Misthry *et al* (2008) in the literature show that relativistic radiating stars are also useful in the investigation of the cosmic censorship hypothesis and radiating collapse with vanishing tidal forces. Wagh *et al* (2001) presented solutions to the Einstein field equations for a shear-free spherically symmetric spacetime, with radial heat flux by choosing a barotropic equation of state. Herrera *et al* (2006) in their study, found analytical solutions to the field equations for radiating collapsing spheres in the diffusion approximation. These authors demonstrated that the thermal evolution of the collapsing sphere can be modeled in causal thermodynamics which requires heat flow.

Shear-free fluids are also essential in modeling inhomogeneous cosmological processes. Krasinski (1997) points out the need for radiating models in the formation of structure, evolution of voids, the study of singularities and cosmic censorship. Heat conducting fluids are important in cosmological models in higher dimensions and permits collapse without the appearance of an event horizon as demonstrated by Banerjee and Chatterjee (2005). In brane world models the presence of heat flow allows for more general behaviour than in standard general relativity, the analogue of the Oppenheimer - Snyder model of a collapsing dust permits a radiating brane which was established by Govender and Dadhich (2002).

Lie analysis methods are useful to determine the solutions to partial and ordinary differential equations in an algorithmic manner (Bluman and Kumei 1989, Olver 1993). In the case of partial differential equations, we can reduce them to ordinary differential equations. We can also reduce the order of ordinary differential via this technique. Ultimately the method maps every solution of a differential equation into another solution.

In this chapter we intend to analyse the master equation arising from a radiating $(n+2)$-dimensional shear-free metric. We use the Lie theory of extended group as a systematic approach to generalise known solutions and generate new solutions of the same equation. The master equation is given in section 4.2. A brief outline of the Lie theory is given in section 4.3. In section 4.4, we extend known solutions to new solutions of the fundamental equation utilising Lie theory. In section 4.5, we systematically study other group invariant solutions admitted by the fundamental equation. For most of the symmetries, we obtain either an implicit solution or we can reduce the governing equations to a Riccati equation which is difficult to solve in general (though particular solutions can always be found). There are two cases in which we find new exact solutions regardless of the complexity of the generating function chosen. Some brief concluding remarks are made in section 4.6.

4.2 Radiating Model

We consider the shear-free, spherically symmetric line element in an $(n+2)$-dimensional manifold given by

$$ds^2 = -A^2dt^2 + \frac{1}{F^2}\left[dr^2 + r^2dX_n^2\right] \tag{4.1}$$

where $A = A(t, r)$ and $F = F(t, r)$ and

$$X_n^2 = d\theta_1^2 + \sin^2\theta_1 d\theta_2^2 + ... + \sin^2\theta_1 \sin^2\theta_2 \sin^2\theta_{n-1} d\theta_n^2 \tag{4.2}$$

The energy-momentum tensor for a non-viscous heat conducting fluid is given by

$$T_{ij} = (\rho + p)v_iv_j + pg_{ij} + q_iv_j + q_jv_i \tag{4.3}$$

where ρ is the energy density of the fluid, p the isotropic fluid pressure, v_i is the $(n+2)$-velocity and q_i is the heat flow vector. Using equation (4.1) and (4.3) the non-trivial Einstein field equations in comoving co-ordinates are

$$\rho = \frac{n(n+1)F_t^2}{2A^2F^2} - \frac{n(n+1)F_r^2}{2} + nFF_{rr} + \frac{n^2FF_r}{r} \tag{4.4a}$$

$$\begin{aligned} p = & -\frac{nA_rFF_r}{A} + \frac{nA_rF^2}{rA} + \frac{n(n-1)F_r^2}{2} - \frac{n(n-1)FF_r}{r} \\ & + \frac{nF_{tt}}{A^2F} - \frac{n(n+3)F_t^2}{2A^2F^2} - \frac{nA_tF_t}{A^3F} \end{aligned} \tag{4.4b}$$

$$\begin{aligned} p = & \frac{F^2A_{rr}}{A} - (n-1)FF_{rr} + \frac{n(n-1)F_r^2}{2} + \frac{(n-1)F^2A_r}{rA} \\ & - \frac{(n-1)^2FF_r}{r} - \frac{(n-2)FF_rA_r}{A} + \frac{nF_{tt}}{A^2F} \\ & - \frac{n(n+3)F_t^2}{2A^2F^2} - \frac{nA_tF_t}{A^3F} \end{aligned} \tag{4.4c}$$

$$q = -\frac{nFF_{tr}}{A} + \frac{nF_tF_r}{A} + \frac{nFF_tA_r}{A^2} \tag{4.4d}$$

The isotropy of pressure is given by equations (4.4b) and (4.4c) in the form

$$FA_{xx} + 2A_xF_x - (n-1)AF_{xx} = 0 \tag{4.5}$$

with $x = r^2$. Equation (4.5) is the master equation for the system of Einstein field equations in the case $n \geq 2$. Observe that when $n = 2$ then equation (4.5) reduces to (3.4) which is the fundamental equation in four dimensions with heat flow. In this chapter, we reduce the order of (4.5) via Lie analysis in the hope of finding general solutions in the higher dimensions.

4.3 Lie analysis

The symmetry analysis for a system of ordinary differential equations in two dependent variables requires the determination of the one–parameter (ε) Lie group of transformations

$$\begin{aligned}\bar{x} &= f(x, F, A, \varepsilon) \\ \bar{F} &= g(x, F, A, \varepsilon) \\ \bar{A} &= h(x, F, A, \varepsilon)\end{aligned} \tag{4.6}$$

that leaves the solution set of the system invariant. It is difficult to calculate these transformations directly, and as such, we must resort to approximations via

$$\begin{aligned}\bar{x} &= x + \varepsilon\xi(x, F, A) + O(\varepsilon^2) \\ \bar{F} &= F + \varepsilon\eta(x, F, A) + O(\varepsilon^2) \\ \bar{A} &= A + \varepsilon\zeta(x, F, A) + O(\varepsilon^2).\end{aligned} \tag{4.7}$$

The transformations (4.7) can be obtained once we find the (symmetry) operator

$$Z = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial F} + \zeta\frac{\partial}{\partial A} \tag{4.8}$$

which is a set of vector fields. Once these symmetries are determined, it is possible to regain the finite (global) form of the transformation, (4.6), by solving Lie equations

$$\begin{aligned}\tfrac{\mathrm{d}\bar{x}}{\mathrm{d}\varepsilon} &= \xi(\bar{x}, \bar{F}, \bar{A}) \\ \tfrac{\mathrm{d}\bar{F}}{\mathrm{d}\varepsilon} &= \eta(\bar{x}, \bar{F}, \bar{A}) \\ \tfrac{\mathrm{d}\bar{A}}{\mathrm{d}\varepsilon} &= \zeta(\bar{x}, \bar{F}, \bar{A})\end{aligned} \tag{4.9}$$

subject to initial conditions

$$\bar{x}\,\big|_{\varepsilon=0} = x, \qquad \bar{F}|_{\varepsilon=0} = F, \qquad \bar{A}\big|_{\varepsilon=0} = A. \tag{4.10}$$

The full details on the symmetry approach to solving differential equations can be found in a number of excellent texts (Bluman and Kumei 1989, Olver 1993).

The determination of the generators is a straight forward process and has been automated by computer algebra packages (Dimas and Tsoubelis 2005, Cheviakov 2007). In practice, we have found the package `PROGRAM LIE` (Head 1993) to be the most useful. It is quite accomplished given its age – it often yields results when its modern counterparts fail.

Utilising `PROGRAM LIE`, we show that (4.5) admits the following Lie point symmetries/vector fields:

$$Z_1 = \frac{\partial}{\partial x} \tag{4.11a}$$

$$Z_2 = x\frac{\partial}{\partial x} \tag{4.11b}$$

$$Z_3 = A\frac{\partial}{\partial A} \tag{4.11c}$$

$$Z_4 = \frac{F}{n-1}\frac{\partial}{\partial F} \tag{4.11d}$$

$$Z_5 = x^2\frac{\partial}{\partial x} + \frac{xF}{n-1}\frac{\partial}{\partial F} \tag{4.11e}$$

where $n \geq 2$. It is a normal practice to use the symmetries (4.11a–4.11e) to reduce the order of the equation in the hope of finding solutions of the master equation. We need to proceed with some caution due to the overdetermined nature of (4.5). Thereafter we indicate how known solutions can be extended using these symmetries.

4.4 New solutions via Lie symmetries

One of the main purpose for calculating symmetry is to use them for obtaining symmetry reductions and hopefully group invariant solutions. The goal of this section is to apply the symmetries calculated in section 4.3 to obtain symmetry reductions and exact solutions where possible. The application of symmetries (4.11a–4.11e) to the master equation result in either an implicit solution of (4.5) or we can reduce the governing equations to complicated Riccati equations that are difficult to solve. However there are two cases in which we can find new solutions regardless of the complexity of the function chosen.

4.4.1 The choice $A = A(F)$

An obvious case to consider in this subsection, is when one dependent variable in (4.5) is a function of the other. Usually such an approach results in a more complicated equation to solve. In spite of this, we can make significant progress if we use the Lie symmetry Z_1 (which gives the same result as Z_2). For our purposes we use the partial set of invariants of

$$Z_1 = \frac{\partial}{\partial x} \tag{4.12}$$

given by

$$\begin{aligned} p &= F \\ q(p) &= F_x \\ r(p) &= A \end{aligned} \tag{4.13}$$

This transformation reduces equation (4.5) to

$$q'(p)\left[(n-1)r(p) - pr'(p)\right] = q(p)\left[pr''(p) + 2r'(p)\right] \tag{4.14}$$

which can be integrated to give

$$q = q_0 e^{\int \frac{2r' + pr''}{(n-1)r - r'p}} dp \tag{4.15}$$

Substituting for the metric functions via (4.13), we can integrate one more time to give the solution

$$\int \left[e^{-\int \frac{2A_F + FA_{FF}}{(n-1)A - FA_F} dF}\right] dF = q_0 x + x_0 \tag{4.16}$$

where q_0 and x_0 are arbitrary functions of time. Equation (4.16) suggests that, given any function F depending on A, we can work out F explicitly from (4.16). Such an explicit relationship between F and A has not been found previously. Note that since (4.5) is linear, once we obtain F via (4.16) we can use it to obtain the general solution of (4.5) using standard techniques for solving linear equations.

We illustrate this method with simple examples. Using $A = 1$, we evaluate (4.16) to obtain

$$F = q_0(t)x + x_0(t) \tag{4.17}$$

We can easily generate the general solution to (4.5) as

$$A = \frac{-C_1}{q(x_0 + qx)} + C_2 \tag{4.18}$$

$$F = \frac{-1+n}{qC_1(1+n)}(x_0+qx)^{\frac{2}{1-n}} \left[\frac{1+n}{-1+n} q(x_0+qx)^{\frac{1+n}{-1+n}} C_1 + (-C_1 + q(x_0+qx)C_2)^{\frac{1+n}{-1+n}} C_2\right] \tag{4.19}$$

If we take $A = F^2$, then equation (4.16) is reduced to

$$\int F^{\left(\frac{6}{3-n}\right)} dF = q_0 x + x_0 \tag{4.20}$$

and hence

$$\begin{aligned} F &= \left[\tfrac{9-n}{3-n}(\bar{q}_0 x + \bar{x}_0)\right]^{\frac{3-n}{9-n}} \\ A &= \left[\tfrac{9-n}{3-n}(\bar{q}_0 x + \bar{x}_0)\right]^{\frac{6-2n}{9-n}} \end{aligned} \tag{4.21}$$

The functional form of F can be easily extended to obtain

$$F = C_1 \frac{\left(\frac{9-n}{3-n}\right)^{\frac{3-n}{9-n}} (9 - 10n + n^2)(q_0 x + x_0)^{\frac{3-n}{9-n} + \frac{-9+2n-n^2}{9-10n+n^2}}}{q_0(-9+2n-n^2)} + C_2 \left[\frac{9-n}{3-n}(q_0 x + x_0)\right]^{\frac{3-n}{9-n}} \tag{4.22}$$

where C_1 and C_2 are arbitrary functions of time, which is the general solution to (4.5) when

$$A = \left[\frac{9-n}{3-n}(\bar{q}_0 x + \bar{x}_0)\right]^{\frac{6-2n}{9-n}} \tag{4.23}$$

4.4.2 The choice $W = \frac{F}{A^{1/(n-1)}}$

The combination of the symmetries given by

$$Z_3 + Z_4 = A\frac{\partial}{\partial A} + \frac{F}{n-1}\frac{\partial}{\partial F} \tag{4.24}$$

gives rise to the invariant

$$W = \frac{F}{A^{1/(n-1)}} \tag{4.25}$$

Then equation (4.5) is transformed by (4.25) to the form

$$-nWA_x^2 + (n-1)A^2 W_{xx} = 0 \tag{4.26}$$

with solution

$$A = C_1(t)\exp\left(\int \pm\frac{\sqrt{(n-1)^2 W_{xx}}}{\sqrt{nW}}dx\right) \tag{4.27}$$

which comes as a result of treating equation (4.26) as a nonlinear first order ordinary differential equation in A.

Given any function W we can integrate the right hand side of (4.27) and find a form for A. If we take $W = a(t)x + b(t)$, then (4.27) gives

$$A = \bar{C}_1(t) \tag{4.28}$$

and

$$F = \bar{C}_1(t)^{1/n-1}(a(t)x + b(t)) \tag{4.29}$$

which are new solutions of (4.5) in the case $n > 2$.

Alternatively, we could substitute the inverse of (4.25), *i.e.*

$$\widehat{W} = \frac{A^{1/(n-1)}}{F} \tag{4.30}$$

into (4.5) and obtain

$$nA_x{}^2\widehat{W}^2 - 2(n-1)^2A^2\widehat{W}_x{}^2 + (n-1)^2A^2\widehat{W}\widehat{W}_{xx} = 0 \tag{4.31}$$

with solution

$$A = C_2(t)\exp\left(\pm\int\frac{\sqrt{2(n-1)^2\widehat{W}_x{}^2 - (n-1)^2\widehat{W}\widehat{W}_{xx}}}{\sqrt{n}\widehat{W}}\mathrm{d}x\right) \tag{4.32}$$

Again, given any function W we can integrate the right hand side of (4.32) and find a form for A. If we take $W = a(t)x + b(t)$ as before we find that

$$A_1 = C_2(t)\left[a(t)x + b(t)\right]^{\frac{\sqrt{2}(n-1)}{\sqrt{n}}} \tag{4.33}$$

and

$$F_1 = \frac{\left[C_2(t)(a(t)x + b(t))^{\frac{\sqrt{2}(n-1)}{\sqrt{n}}}\right]^{\frac{1}{n-1}}}{a(t)x + b(t)} \tag{4.34}$$

which is essentially a new solution of the master equation in higher dimensional space. However, we can also have

$$A_2 = \frac{\bar{C}_2(t)}{(a(t)x + b(t))^{\frac{\sqrt{2}(n-1)}{\sqrt{n}}}} \tag{4.35}$$

and

$$F_2 = \frac{\left[\bar{C}_2(t)(a(t)x + b(t))^{\frac{-\sqrt{2}(n-1)}{\sqrt{n}}}\right]^{1/n-1}}{a(t)x + b(t)} \tag{4.36}$$

thus obtaining two different solutions from the same seed function.

Observe that (4.27) and (4.32) will contain all solutions of the master equation (4.5) for appropriately chosen seed functions W or $\widehat{W}$ which are ratios of the metric functions. We are always able to reduce (4.5) to the quadratures (4.27) or (4.32) regardless of the complexity of the seed functions.

4.5 Extending known solutions

Another use of Lie point symmetries is the extension of known solutions of differential equations. This is possible due to the fact that the symmetries generate transformations that leave the equations invariant. As a result, applying those transformations to known solutions will (usually) result in new solutions.

We illustrate the approach by using the simple infinitesimal generator Z_1, where we observe that

$$\xi = 1, \qquad \eta = 0, \qquad \zeta = 0 \tag{4.37}$$

We solve the Lie equations (4.9), subject to initial conditions (4.10), to obtain

$$\begin{aligned} \bar{x} &= x + a_1 \\ \bar{F} &= F \\ \bar{A} &= A \end{aligned} \tag{4.38}$$

This means that using (4.38) we can map the equation (4.5) to the form

$$\bar{F}\bar{A}_{\bar{x}\bar{x}} + 2\bar{A}_{\bar{x}}\bar{F}_{\bar{x}} - (n-1)\bar{A}\bar{F}_{\bar{x}\bar{x}} = 0. \tag{4.39}$$

As a result of this mapping, any existing solution to equation (4.5) can be transformed to a solution of (4.39) by (4.38). Usually, a_1 is an arbitrary constant. However, since F and A depend on x and t we take a_1 to be an arbitrary function of time, $a_1 = a_1(t)$.

If we now take each of the remaining symmetries successively, we obtain the general transformation

$$\begin{array}{rcl}\bar{x} & = & \frac{e^{a_2}(a_1+x)}{1-a_5e^{a_2}(a_1+x)} \\ \bar{F} & = & \frac{e^{\frac{a_4}{n-1}}F}{1-a_5e^{a_2}(a_1+x)} \\ labelmh2\bar{A} & = & e^{a_3}A\end{array} \qquad (4.40)$$

where the a_i are all arbitrary functions of time and $n \geq 2$.

Thus any known solution of equation (4.5) can be transformed to a new solution of equation (4.5) via (4.40). For example, if we start with the solution

$$A = 1, \qquad F = \alpha(t)x + \beta(t) \qquad (4.41)$$

the transformation (4.40) yields the new solution

$$\begin{array}{rcl}\bar{x} & = & \frac{e^{a_2}(a_1+x)}{1-a_5e^{a_2}(a_1+x)} \\ \bar{F} & = & \frac{e^{\frac{a_4}{n-1}}(\alpha(t)x+\beta(t))}{1-a_5e^{a_2}(a_1+x)} \\ \bar{A} & = & e^{a_3}\end{array} \qquad (4.42)$$

All the new results that we derived in the previous section can be similarly extended via (4.40).

4.6 Discussion

We have provided symmetry reductions and exact solutions of the Einstein field equations governing shear-free heat conducting fluids in higher dimensions. Explicit relationships were provided between the gravitational potentials, obviating a need to start with “simple” forms for one to calculate the other. We were also able to provide a general transformation to extend our (and any other) known solution into new solutions.

We remark that equation (4.5) is a higher–dimensional generalisation of that analysed by Deng (1989) (see also Msomi *et al* (2010)):

$$F A_{xx} + 2A_x F_x - A F_{xx} = 0 \tag{4.43}$$

As a result, all solutions obtained by Deng (1989) and Msomi *et al* (2010) for which $F_{xx} = 0$ will apply in the case of (4.5).

Chapter 5

Conclusion

Many partial differential equations of physical importance are nonlinear partial differential equations. While there is no existing general theory for solving such equations the methods of point transformations are a powerful tool. Knowing the symmetry group allows one to determine some special types of solutions that are invariant under a subgroup of the full symmetry group, and in some cases one can solve the equations completely. The primary purpose of this dissertation has been to provide exact solutions to the Einstein field equations presented in this work, by using the Lie analysis.

We now provide an overview of the main results obtained during the course of our investigations:

- In Chapter 2, we presented the master equation developed by Gupta and Sharma (1996) and discussed their *ad hoc* approach of finding solutions to the master equation. We provided the fundamental theory of Lie analysis as the systematic Lie approach to be followed in the next chapter to find exact solutions to this master equation. Our major achievement in this chapter was to demonstrate that the master equation admits six Lie point symmetries. On using these symmetries we developed the optimal system that reduces the partial differential equation to an ordinary differential equation in the hope of finding solutions to the master equation. In addition a variety of new exact solutions of the governing equation (2.4), using the Lie method of infinitesimal generators, have been obtained. Previously known solutions were shown to be characterized by a particular Lie generator and are contained, as special cases, in our new family of solutions. We considered each element in the optimal system of one-dimensional subgroups and reduced the master equation to an ordinary differential equation. We were in a position to solve the resulting equations and obtain several new solutions for the gravitating model. A pleasing feature of our analysis is that several models generated admit a linear barotropic equation of state.

The master equation has six Lie point symmetries with eight nonzero Lie bracket relationships which generates the optimal system. It is this geometric structure which has enabled us to show that equation (2.4) has a rich structure.

We have demonstrated that equations of state $p = \rho$ and $p = \frac{1}{3}\rho$ identified by Gupta and Sharma (1996) follows because of the existence of the symmetry G_5. Using generators G_2 and G_4, we found the linear equations of state $p = \rho$ and $p = \frac{2\pi}{1+2\pi}\rho$ respectively which are important in relativistic stellar structures that arise in models of quark stars (Komathiraj and Maharaj 2007, Mak and Harko 2004, Sharma and Maharaj 2007, Witten 1984). Graphical representations of the energy density ρ and the pressure p indicate that the model is well behaved in the spacetime manifolds even in cases where there is no simple barotropic equation of state connecting the energy density and pressure.

- In Chapter 3, we have analysed the gravitational behaviour of shear-free spacetimes with heat flow. We discussed the Deng (1989) general method of generating solutions. Then we reduced the order of the master equation via Lie analysis to obtain new solutions not obtainable via the Deng (1989) *ad hoc* approach. We found solutions of the master equation using the systematic Lie approach.

- In Chapter 4, we consider the generalised situation of shear-free spacetimes with heat flow in higher dimensions. We analysed the master equation for $(n+2)$-dimensional spherically symmetric metric, using Lie theory of extended groups as a systematic Lie approach to generate new solutions of this equation.

We believe that the exact solutions, using the Lie method, found in this thesis will be helpful in modelling astrophysical and cosmological processes.

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