AN APOS ANALYSIS OF THE UNDERSTANDING OF VECTOR SPACE CONCEPTS BY ZIMBABWEAN IN-SERVICE MATHEMATICS TEACHERS

BY

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THESIS

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PREFACE

The work done in this thesis was carried out in the School of Education, University of KwaZulu - Natal, from August 2015 to December 2018 under the supervision of Professor S. Bansilal.

This study represents original work done by me and has not been submitted in any form for any degree to any tertiary institution. The work used from others is properly acknowledged.

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DECLARATION 1: PLAGIARISM

I, Lillias Hamufari Natsai Mutambara, declare that:

(i) The research report in this dissertation, except where otherwise indicated or acknowledged, is my original work.

(ii) This dissertation has not been submitted in full or in part for my degree or examination to any other university.

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Signed: Mutambara

Date:

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DEDICATION

This work is dedicated to my late husband Tendai Mutambara, my children (Vanessa, Clement, Precious), my parents, brothers (Dunmore and his wife Florence, Washington and his wife Rumbidzai, Tinaye), sisters (Violet, Jessina, Penina and their husbands) and my auntie Rhodhina, Josephine Mukwende. You have been there for me as my constant source of motivation, care and support.

ABSTRACT

University mathematics students often find the content of linear algebra difficult because of the abstract and highly theoretical nature of the subject as well as the formal logic required to carry out proofs. This study explored some specific difficulties experienced by students when negotiating the various vector space concepts. The participants were 73 Zimbabwean mathematics teachers who were enrolled in an in-service programme and who were studying for a Bachelor of Science Education Honours Degree in Mathematics. The Zimbabwean mathematics in-service teachers who were studying these concepts were also teaching some of the concepts at high school. The study was qualitative in nature and it was strengthened by the interpretivist paradigm. Data were generated from the teachers' written responses to three tasks based on the various vector space concepts. The items in the activity sheets probed the participants on the concepts on vector space, subspace, linear combination, linear independence, basis and dimension. Follow-up interviews on the written work were conducted to identify the participants' ways of understanding. Thirteen students volunteered to be interviewed and were probed further about the vector space concepts so as to elicit more information on the way they understood the various vector space concepts, and the connection they seemed to make between these concepts. An APOS (action–process–object–schema) theory was used to unpack the structure of the concepts. The main aim of the study was to identify the mental constructions that the students made when learning the various vector space concepts and the extent to which they concurred with a preliminary genetic decomposition..

The study also employed another theoretical framework, Sfard theory, which was used to describe the in-service teachers cognitive difficulties in the learning of linear algebra which were identified as errors and misconceptions with particular reference to the study of vector space concepts. The errors were categorised in terms of conceptual (deeply seated misunderstandings) procedural (related to using related procedures) and technical (calculation or interpretation) errors.

In terms of APOS theory, the responses revealed that most in-service teachers were operating at the action and process levels, with a few students using some aspects of object level reasoning for some of the questions. Findings revealed that the teachers struggled with the vector space and subspace concepts, mainly because of prior non-encapsulation of prerequisite concepts of sets and binary operations, and difficulties with understanding the role of counter-examples in showing that

a set is not a vector subspace. Most of the students operated at the action level of understanding. The findings revealed that across the items on the concepts on linear combinations, linear independence, basis and dimension, students were comfortable in answering problems that required the use of algorithms, for example carrying out the Gaussian elimination method. However a major hurdle that hindered them from interiorising the actions into a process for the items on linear combination, linear independence and basis was their failure to interpret the solutions to the systems of equations and providing insufficient argumentation in relation to the posed questions. Fifty students struggled with concepts on linear combination and did not provide any evidence in their written responses of moving past an action conception.The results on understanding linear independence revealed that 17 (23%) students were able to make arguments based on the use of theorems that given vectors are linearly dependent without showing the step by step procedures and giving precise descriptions of the procedures used to determine linear independence. There were 46 students who represented their understanding in a manner described as the action conception as they were engaged in a step by step manner in an attempt to show that given vectors are linearly independent. The major drawback that hindered the students to develop their understanding of the concept of linear independence was a failure to distinguish the two terms linear independence/dependence, application of inappropriate theorems and inappropriate methods when solving the problems. Furthermore, the results on understanding of basis and dimension also revealed that the in-service teachers were able to cope with the procedures of row reduction, but struggled to justify whether given vectors formed a basis or not; they also struggled to find the basis of the solution space. Only 9 (12%) of the students were able to develop their mental construction at the process conception of basis of a vector space as they were able to coordinate the two processes of establishing that a given set span the particular vector space and that the set is linearly independent.

On cognitive challenges, the study revealed the distribution pattern of the conceptual errors, technical errors and procedural errors varied across the items. The most errors manifested were the conceptual and technical. It is hoped that the identification of such errors and misconceptions will assist other educators in modifying their planning so that long term learning will take place.

PUBLICATIONS ARISING FROM THIS THESIS

Mutambara, L.H.N; and Bansilal, S. (2018). Abstraction of vector space concept. In S. Stewart, C., Andrews-Larson, A. Berman, & M. Zandieh, *Challenges and Strategies in Teaching Linear Algebra*. Netherlands: Springer.

An exploratory study on the understanding of the vector subspace concept. The paper has been accepted for publication in African Journal of Research in Mathematics, Science and Technology Education subject to some refinement and corrections, see Appendix J.

The publication form part of chapter 5 of this thesis. In some places the published work has undergone minor revisions in order to better accommodate the style, layout and overall objectives of this thesis.

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CHAPTER ONE

INTRODUCTION

1.1 Introduction

This chapter provides the overview of the study. I discuss the background of the study with the motivation for doing this research. The rationale for carrying out the study is discussed together with the problem statement. Detailed research objectives and research questions of the study are provided. The significance of this study is discussed. A discussion of the context of the study is also provided. Lastly, the definitions of key terms used in the study and summaries of successive chapters are provided.

1.2 Background of the study

The value accorded to mathematics the world over cannot be over- emphasized. There is broad consensus among policy makers, curriculum planners, school administrators, business and industry leaders and politicians that mathematics is an important subject and is critical for economic development (Brumbaugh & Rock, 2001).

According to Kolman and Hill (2008) linear algebra is important for two fundamental reasons. The first reason is that linear algebra has a wide range of applications in fields of mathematics, for example in multivariate calculus, differential equations and probability, and also in the fields of physics, biology, chemistry, computer science and engineering. Linear algebra courses are basic for a wide variety of disciplines at tertiary level such as mathematics and physics. The second fundamental reason is that it provides an opportunity for students to learn how to make mathematical abstractions. Even though its importance is clear, educators and learners perceive that the teaching and learning of linear algebra is a difficult and challenging experience (Celic, 2015). Various studies have explored students' conceptual understanding of linear algebra (Bogomolny, 2007; Stewart & Thomas, 2009). Dorier (2003) also argued that the teaching and learning of algebra is a frustrating experience for both teachers and students.

From 2008, I have been teaching mathematics to first year applied mathematics students and the Bachelor of Science Honours Degree in mathematics at a university in Zimbabwe. I have noticed that students excel in the first Linear Algebra module as compared to the second one which looks at vector space concepts. Informal discussions held with other lecturers revealed similar experiences. Stewart and Thomas (2007) argued that students start well and cope with the procedural aspect of first year courses, which is solving systems of linear equations and manipulating matrices, but struggle to understand the concepts on the vector spaces such as subspaces, span and linear independence.

French researchers Dorier, Robert, Robinet, and Rogalski (2000) asserted that in learning concepts on vector spaces students feel that they have landed on another planet. The teachers feel frustrated that their students fail to grasp the concepts. The French researchers referred to the learning of these concepts as an obstacle of formalism whereby students are overwhelmed by learning new definitions, symbols, words and theorems. In addition other researchers (Britton & Henderson, 2009) bewailed the lack of particular mathematical knowledge and skills, for example, showing links between the different representation, are other reasons that students experience obstacles in learning linear algebra. This therefore motivated me to explore why and to what extent undergraduate students struggle to understand the vector space concepts by considering the following aspects: vector space, subspace, linear combination, spanning, linear independence, basis and dimension. As a result this caused the researcher to recall what Hiebert (2013) asserted: that one of the most widely accepted ideas in mathematics education is that students should understand mathematics.

Dubinsky and McDonald (2001) are of the view that the main concern in mathematics should be with the student's construction of schemas for understanding concepts, and thus they concentrate on how a theory of learning mathematics can help researchers understand the learning process. Hence the need arises for research that offers insight into how students construct mathematical concepts related to vector space and this research has not been carried out previously in Zimbabwe.

1.3 Purpose of the study

The main aim of this study was to explore the undergraduate mathematics in-service teachers' understanding of vector space concepts using APOS (Action-Process-Object--Schema) theory, which attempts to describe the mental structures that deals with the nature of mathematical concept and possible ways that students construct certain concepts in mathematics. These students were enrolled in a Bachelor of Science Education Honours Degree in Mathematics and some recommendations were reached which might lead to an improvement in the learning of linear algebra at undergraduate mathematics curricular.

1.4 Statement of the problem

Despite the large body of research being undertaken in the teaching and learning of linear algebra by undergraduate mathematics students, research literature in the area of vector space concepts is limited, especially in the context of Zimbabwe. Globally, Hillel (2000) and Dogan-Dunlop (2010) mentioned that very few studies were conducted on linear algebra concepts, and it does not tally with research in calculus with Possani, Trigueros, Preciado and Lozan (2010) who outline that the learning of vector space concepts has received little attention from researchers. Much work was done on the cognitive difficulties encountered in the teaching of concepts like limit, function and continuity. My personal observation as a mathematics lecturer in Zimbabwe is that students struggle to distinguish these various vector space concepts, and find it difficult to apply the concepts to other areas. This study aims at looking at the cognitive difficulties and the mental constructions and mechanisms involved in the learning of vector spaces, subspace, linear combination, linear independence, basis and dimension. I intend to use APOS (Action-Process-Object-Schema) theory to explore the different levels of mental constructions undergraduate mathematics students operate at when they learn vector space concepts.

1.5 Objectives of the study

The following objectives were explored with the aim to achieve the focus of the study:

1. To identify the APOS constructions students have developed with respect to the various vector space concepts.

2. To investigate the cognitive difficulties that students encounter when constructing the vector space concepts.
3. To find out how the preliminary genetic decomposition could be revised to take into account the students' learning experiences

1.6 Research questions

The study sought to answer the following questions:

- What APOS mental constructions can be inferred from the students' written and verbal responses to items based on vector space concepts?
- What are some cognitive difficulties encountered by the students when trying to construct the necessary vector space concepts?
- How can the preliminary genetic decomposition be revised to take into account the students' learning experiences?

1.7 Rationale of the study

The rationale for this study hinges on my personal interest in identifying impediments to the teaching and learning of vector space concepts as an important component of undergraduate mathematics curriculum. The study is significant in that it has the potential to contribute to more knowledge and literature on the mental constructions that university students can make in the learning of linear algebra. This study is also important as it focuses on students cognitive difficulties in the form of errors and misconceptions that students encounter when learning vector space concepts. This might assist mathematics lecturers to be more effective in their teaching as they will be teaching mathematics using the students' errors and misconception and might help the students to resolve and overcome these errors. Hence it is hoped that the results of the study will constitute a basis for thinking about possible intervention strategies designed to improve students understanding of linear algebra, in particular the vector space concepts. It is also hoped that the study will provide necessary information to all educators concerned with the teaching of linear algebra at undergraduate mathematics level. Those teachers working in the field of mathematics education will benefit in building knowledge on vector space concepts since the key aspect of effective teaching is seen as knowing what and how students are thinking (Dunham & Osborne, 1991). The framework used is the APOS theory which proved to be a useful tool in analysing students work with the help of the preliminary genetic decomposition. The genetic decomposition formulated was used for explaining the students' responses and articulating the level they will be operating at in terms of the APOS theory, as well as ascertaining the mental constructions that the students made with respect to the vector space concepts.

1.8 Context of the study

The study was conducted at a particular university in Zimbabwe. The university was chosen for convenience as the researcher is a full time lecturer at that university. In order to improve the education system and the quality of teachers in the country, the university introduced a Bachelor of Science Education Honours degree in mathematics and the university transformed itself and seeks to produce innovative and highly acclaimed graduates equipped with research and technical skills for the benefit of the nation. The enrollment went down to unsustainable levels especially in 2008. In light of the above a number of STEM initiatives have been embraced. Block release programmes were launched. These programmes were meant to address the problems of the practicing teachers who had no access to university education because of high university fees and could not get time off to attend conventional classes. The teachers needed to upgrade their teaching qualifications thus enabling them to teach mathematics at advanced level in the schools. Due to the stated problems, the government took some initiatives together with an overseas funding organisation (UNICEF) which sponsored upgrading the in-service mathematics teachers. The ministry of secondary education selected these students to study for a Bachelor of Science Education Honours degree in mathematics (HBScEDH) offered for the in-service teachers. Satisfactory research activity was carried out without fear of possible unexpected disruptions, since the mathematics department at the university is fully functional and adequately resourced with effective teaching and learning taking place. The rationale for the courses is that in-service teachers need the same amount of content courses as pre-service teachers on the programme for them to be able to effectively teach 'A' Level content. The in-service teachers' pedagogical knowledge obtained during initial teacher education and teaching experience after qualifying is recognised.

1.9 Contributions made by the study

The study has made a contribution in terms of furthering African scholarship in the area of learning of linear algebra. The findings that have been presented about students understanding of the various vector space concepts including vector space, subspace, linear combination, linear independence, basis and dimension. All these area have not been touched in previous publications in Africa. Furthermore as part of the study detailed genetic decompositions based solely on APOS theory for vector space, subspace, linear combination, linear independence, basis and dimension were arrived at. These genetic decompositions can now be interrogated, refined and applied by other scholars who work with APOS theory. Hence the study has made a contribution to the extension of APOS of the above mentioned areas. It has also provided many concrete instructional implications for each concept which have been detailed at the end of each chapter. Hence these recommendations will guide the revision of these courses when they are taught again at the institution in which the study was conducted. The pedagogical implications can also provide guidance to instructors from other contexts and countries who can take these recommendations into account.

1.10 Limitations of the study

Several limitations were encountered in this study. A case study was used. The first year undergraduate mathematics students who do a Bachelor of Science Education in mathematics from one university in Zimbabwe were used. The sample used was purposively selected hence the results cannot be generalized to other contexts. It is, however, hoped that essential information was found as intended such that the findings are informative enough to give useful information on what can be expected in the learning and teaching of vector space concepts. I administered the interview questionnaire and it was possible that some respondents might not have felt comfortable out of their usual environment. However I tried by all means possible to make sure that the environment was friendly and I gave them opportunities to ask questions and reminded them that they were not examinable.

1.11 Definition of key terms

The following operational terms are defined for the current study in this section. These terms are discussed in more details as the study develops.

APOS – the acronym APOS stands for Action, Process, Object, Schema (Brijlall & Ndlovu, 2013). This theory describes the mental structures that deal with the nature of a mathematical concept and its development in the mind of an individual. It gives a description of the possible process by which a concept can be learnt so as to construct the necessary knowledge. The theory further uses the model of what might be going on in the mind of an individual and uses these models to evaluate students' successes and failures in dealing with mathematical problem situations (Dubinsky & Wilson, 2013) (for further explanation see chapter 3).

Genetic decomposition – it is a structured set of mental constructs which might describe how a particular concept can develop in the mind of an individual (Jojo, 2013). It is regarded as a diagnostic tool which provides the investigator with insight into how a learner can develop a concept throughout the different levels of Action, Process, Object and Schema. At the end of the exercise the investigator must be in a position to come up with suitable activities that will enable the students to develop a better understanding of the concepts being taught.

Conceptual understanding – it is seen as that knowledge that is rich in relationships (more discussion in chapter 3).

Procedural understanding – it is seen as knowledge of the steps, sequence of steps or algorithms required to attain various goals (see further discussion in chapter 3).

Vector space - is a nonempty set V of objects, called vectors, on which are defined two operations, called addition and multiplication by scalars (real numbers), satisfying ten axioms The axioms must hold for all $u, v, w \in V$ and for all scalars $\alpha,\beta \in K$ (see chapter 2, section 2.3 for further discussion) .

Subspace – if V is a vector space over a field K and taking W to be a subset of V then W is a subspace of V if W is itself a vector space over K with respect to the operations of vector addition and scalar multiplication in V.

Student – an individual who attends classes in a school or any institution of higher learning.

Lecturer – an academic expect teaching at an institution of higher learning, university or college.

In-service teacher – any learning opportunity for practicing teachers already teaching in the classroom or something that happens while someone is a full-time employee. For this study the term students and in-service teachers will be used interchangeably.

1.12 Conclusion

In this chapter background of the research was discussed, to try and put the research problem in a well thought out context. In the same chapter the statement of the problem has been outlined followed by the research questions and assumptions of the research study. The significance of the study, the limitations and delimitations of the study was highlighted. In addition to this an attempt to provide definition of terms used in the study was done. The chapter for the thesis are organized below as follows.

1.13 Organisation of the dissertation

The thesis comprises of ten chapters.

Chapter 1 introduces the background to the study, context of the study, statement of the problem, and motivation for doing this research. Furthermore, it introduces the research questions, explains the rationale and purpose of the study, discusses research questions and gives the definition of the key terms.

In **chapter 2** I provide a review of literature related to the learning of linear algebra. The review discusses the concept of understanding in mathematics, what is linear algebra, difficulties encountered in the learning of linear algebra, misconceptions encountered in the learning of linear algebra, some studies done in linear algebra using APOS theory, and the nature of errors in learning mathematics, particularly errors in the learning of linear algebra.

Chapter 3 presents the theoretical frameworks which were used to analyse the data collected. The following theoretical framework was looked at and discussed: APOS theory and Sfard theorem. The initial genetic decomposition for the concepts of subspace, vector space, linear combination, linear independence, basis and dimension are represented.

Chapter 4 I present research methods that were used in the study. These include the research design, research paradigm, the participants, methods of data collection, data generation methods, the issues of reliability and validity as well as trustworthiness. I also discuss the issues of research ethics.

Chapter 5 I present the analysis of students' responses from activity sheets 1 and interviews. I explored the students' mental constructions when learning vector space and subspace concepts. The genetic decomposition was used as an analytical tool. I also identified the cognitive challenges experienced by students as they tried to solve problems involving vector space and subspace. The findings and implications of the research were also explored.

Chapter 6 I present the analysis of students' responses from activity sheet 2 and interviews. A report on the study on the exploration of the conceptions of linear combination using Action Process Object Schema (APOS) theory is outlined. The questions involve an understanding of the concept of linear combination and spanning. A modified genetic decomposition of the discussed concepts is represented. The finding and implications of the research are also explored.

Chapter 7 I present the analysis of students' responses from activity sheet 2 and interviews. A report on the study which explored the students' mental construction of the concept of linear independence/ dependence is outlined using APOS theory. A modified genetic decomposition of the concepts discussed is represented. The finding and implications of the research are also explored.

Chapter 8 I present the analysis of students' responses from activity sheet 3 and interviews and report on the study which explored the students' mental construction of the concept of basis and dimension conceptual using APOS theory. A modified genetic decomposition of the concepts discussed is represented. The finding and implications of the research are also explored.

Chapter 9 presents the results based on the types of error that the students revealed when learning the concepts of vector space, subspace linear combination, linear independence, basis and dimension using Sfard theory (1992). The finding and implications of the research are also explored.

Chapter 10 presents the general summary and conclusions of the study based on the results obtained during content analysis, interview interpretation using APOS as the theoretical framework, and Sfard theory (1992). The limitations of the study and recommendations for future research are also proposed in this chapter.

CHAPTER TWO

REVIEW OF RELATED LITERATURE

2.1 Introduction

The purpose of this study is to explore the mental constructions that undergraduate mathematics students make and the difficulties they experience when learning linear algebra concepts. This review first discusses what understanding in mathematics is in general terms. Learners' difficulties in learning and understanding concepts related to linear algebra and the approaches that are commonly used to teach these concepts are discussed. The present study combines the ideas of the research literature and produces a recommendation that may possibly improve learners' understanding of vector space concepts. Lastly, the chapter concludes with a summary of the sections.

2.2 Nature of linear algebra

Konyahoglu, Sabri and Ahmet (2003) see linear algebra as a crucial course because of its importance and appropriateness for studying the science subjects. They further argued that linear algebra is a branch of modern algebra which deals with the abstract system called vector spaces which is derived from the solution of systems of equations. Consistent with such observation is Tucker (1993) who posits that linear algebra is a powerful mathematical theory that is taught at the university level and it is the first undergraduate mathematics course. He further argued that linear algebra has a well-structured and comprehensive curriculum, and he advocated that any person who ventures into the area of sciences must have grasped the concepts of the vector space and linear transformation which are the first mathematical concepts taught in the first year at the university.

Dorier (2000) points out that the concept of linear algebra is grounded on the notion of a vector space. Similarly Konyahoglu, Cihan, Sabri and Ahmet (2003) also commented that the vector space is the core content for the basis of linear algebra. According to Dorier (2000) the development of linear algebra dates back to the 18th century with Swiss mathematician, Leornard Euler's work on the ideas of solving of simultaneous systems of equation of the form $3x - 2y = 5$ and $4y = 6x - 10$. Euler examined whether any given system of n equations with n uknowns has a unique solution. According to Dorier (2000) Euler noticed that it is not the case that since the second equation is double the first then that does not indicate that the system of equations are undetermined, but it is a result of the use of the elimination method. He further argued that Euler gave more examples with larger systems of equations, and posited that such equations which are "comprised in' or 'contained in' results in equations that are linearly independent Dorier (2002, p.7). Around 1850, the mathematicians of that time came up with the theory of determinants which was unavoidable since they wanted to solve systems of linear equations and the Crammers rule was established. However, according to Dorier (2000) mathematicians during the 1850s neglected the problems that gave rise to inconsistent systems after solving the equation. The concept of rank was developed from 1840 to 1879 since it was embedded in the theory of determinants. However, the birth of the vector space theory can be dated back to the late nineteenth century (Dorier, 2000). This meant that the methods for solving systems of linear equations were reached and were being referred to as the axiomatization of linear algebra. Dorier and Sierpinska (2001) also argued that the axiomatic approach acts as a method that was widely used in other subjects such as geometry, functional analysis and so on, while Dorier, Robert, Robinet and Rogalski (2000) also added that linear algebra is a unifying and generalizing theory which is also a formal theory. The authors further outlined that formalization is an inevitable component of unifying and generalizing concepts meaning that learning of the concepts requires implicit reasoning mainly in the process of proving of axioms. When students are learning the vector space concept, they really struggle to prove the ten axioms especially the multiplication axioms, and this really requires implicit reasoning. Stewart (2007) also commented that the course is very tense for the students who have no prior understanding of the course. Furthermore, Dorier (1995) and Harel (1999) asserted that linear algebra provides an opportunity for students to engage with mathematical abstractions.

Many researchers (Hazzan, 1999; Tall, 1999; Lerron & Dubinsky, 1995; Findell, 2006) have different views about what abstract algebra is. Findell (2006) argued that abstract algebra is seen as a generalization of school algebra, with variables that can represent various mathematical objects, such as vectors, matrices and so on. Furthermore, Findell (2006), consistent with Arnawa, Kartasasmita and Baskoro (2012) further argued that abstract algebra consists of axiomatic theories that provide chances to consider many different mathematical systems as being special cases of the same abstract structure. These structures are defined using axioms and according to Arnawa, et.al. (2012) the group theory is an example of axiomatic theory. For example, in order to show that a given set is a vectors space, this is heavily dependent on proving the group axioms, and the operation of addition and multiplication by scalar must be satisfied. The operation of addition or multiplication combines two elements; it is called a *binary operation* and to generalize the binary for a group the operation is usually denoted by using the symbol *. To be specific now in order to show that a given set is a vector space, the following binary operation $'+'$ and $'\times'$ must be satisfied. The following are the group axioms for any given real number.

1. *Axiom closure*. Given any two elements, x, y in the set $x * y \in S$, that is the resulting element must also lie in the set.

2. *Commutative*. For any two elements, x, y in a given set $x * y = y * \text{x}$.

3. *Associativity*. For any three elements, x, y, and z, in the set $(x*y) *z = x* (y*z)$.

4. *Identity*. There is an element, e, in the set, such that for any x in the set, e*x = x = x*e. (For addition of integers, the identity is 0 and for multiplication the identity is 1)

5. *Inverse.* For each element x in the set, there is an element (x)in the set such that $x * x^{-1} = e$ that is it gives us the identity above.

Any given set, be it a matrix or any n-tuples with its operation, that satisfies these axioms is said to be a *group.* All the above axioms resemble the group axioms with the binary operation "*" for any given set. However, for it to be a vector space it must satisfy the addition axioms as well as the multiplication axioms as shown below. In order to show that a set is a space all the 5 group axioms must be satisfied with a binary operation '+' for $x, y \in V$ and a further 5 more axioms need to be satisfied. Anton (2013) outlined that by scalar multiplication we mean a rule for associating with each scalar *k and l* and any given object say $\boldsymbol{u}, \boldsymbol{v}$ in *V* where k and l are scalars then the following axioms must be satisfied, 1. $k\boldsymbol{v} \in \boldsymbol{V}$ 2. $k(u + \boldsymbol{v}) = k\boldsymbol{u} + k\boldsymbol{v}$ 3. $kl(\boldsymbol{u}) = k\,(l\boldsymbol{v})$ 4. $(k + l)\boldsymbol{v} = k\boldsymbol{v} + l\boldsymbol{v}$ 5. $1.\boldsymbol{v} = \boldsymbol{v}$**.** If all the above ten axioms are satisfied then we say it is a vector space.

Findell (2006) goes on to say the concepts of group, subgroup, and isomorphism constitute the fundamental concepts of group theory together with the other mathematical structures that include *rings* and *fields* and these are the major concepts in abstract algebra with the ring theory having two operations called multiplication and addition. Furthermore, Findell (2006) said that abstract algebra is one of the courses where learners can obtain common characteristics from many mathematical systems that they have encountered or used in previous mathematics courses, such as calculus and algebra and use it to understand new knowledge, and it gives them the chance to develop deeper understandings of different notions such as identity, inverse, equivalence, and function when carrying out logical proofs. However, these researchers believe that the teaching of abstract algebra is a terrible experience and that students have difficulty in comprehending the concept. For example, Leron and Dubinsky (1995, p.19) have this to say, “The teaching of abstract algebra is a disaster, and this remains true almost independently of the quality of the lectures.”

The key problem identified with learning about abstract algebra is the abstract nature of the concepts. Working with abstract algebra requires students to engage with abstraction and generalisation, two skills that Findell (2006) sees as central to the study of mathematics which faces much criticism in advanced mathematics, whilst Dienes (1961) defined abstraction as an inductive process where-by an individual must draw from a number of situations something that is common to all concepts learnt. Hazzan (1999) describes abstraction as a complex process which has many facets, and further said that this is real in the context of mathematics as well as mathematics education. However, Dorier (2000) commented that there are very few studies conducted on abstract algebra as compared to calculus.

2.3 Understanding of mathematics

Before going on to discuss studies focused on the understanding of linear algebra, it is important to look at the concept of understanding in mathematics. Understanding of mathematical concepts is vital in the teaching and learning process of mathematics. One of the main learning principles put forward by the NCTM (2000) is that students must learn mathematics with understanding. The major goal of mathematics teaching is that students must understand mathematical concepts encountered in the mathematics classroom so that they discover for themselves. The main focus of the research study is to improve students’ understanding of the vector space concepts. The interpretation of the definition of understanding is meant to serve as a guiding tool in describing the studies that have been conducted on the teaching of linear algebra. Hence it is necessary to begin by looking at the phrase “understanding of mathematics”.

Romberg (1992) outlines that “there isn’t a common definition of understanding” (Kieran, 1992, p. 590). Sierpinska (1994) and Perkins and Blythe (1994) define understanding as being able to explain and justify, finding evidence and examples, generalizing, applying, analogizing and representing the topic in a new way. On the same note Usiski (2012, p.19) said that a learner has:

“full understanding of a mathematical concept if he/she can deal effectively with the skills and algorithms associated with the concept, with properties and mathematical justifications (proofs) involving the concepts, with uses and applications of the concept, and with representations for the concept.” Usiski (2012) here postulated that understanding is about being able to connect ideas together rather than simply knowing them as isolated facts. This is in line with Barmby (2007) who also argued that if one is able to make connections between mental representations with a mathematical concept then the individual is content that learning has taken place.

Furthermore, in her studies on linear algebra, Sierpinska (1994) argued that there are some indicators of mathematical understanding. These are identification, discrimination, generalization, and synthesis. A student who understands a mathematical concept must be able to recognize it and know what it is. Secondly, one who understands a mathematical concept must be able to discriminate between two concepts, meaning that the student must be able to outline the differences between any two given concepts. Also, to show that the discrimination process has taken place successfully is to consider the degree of abstraction which is indicative of deep understanding. Thirdly, an individual who understands a mathematical concept must have the ability to generalise, that is to identify one mathematical concept as a particular case of another more general one. Lastly, the fourth indicator of mathematical understanding is synthesis. This means that this individual must be able to find ‘similarities between or among several generalizations’ (Sierpinska, 1994). Mathematics teachers always hope that students can develop the kind of understanding described by Sierpinska (1994).

Hierbert and Carpenter (1992) also suggested possible methods that can be used to assess mathematical understanding and these are summarized below:

- Students’ errors.
- Connections made between symbols and symbolic procedures and corresponding referents.
- Connections between symbolic procedures and informal problem solving situations.
- Connections made between different symbol systems

Barmby, et al. (2007) used these suggestions as a starting point to express the possibilities of assessing mathematical understanding and also added two more strategies that of using concept maps and mind maps so as to get the external manifestation for the links that the students can come up with. This shows that it is important in the teaching process for teachers to determine how learners understand the mathematical concepts that they teach.

Durbinsky (1991, p.119) also argued that mathematics teachers should be concerned "with the learners' construction of schemas or networks for understanding mathematical concepts." This means that the mental constructions made by an individual will expose the knowledge that they have constructed about certain concepts and the appropriate activities will be put in place. Furthermore, to support Dubinsky's contention in showing knowledge acquisition Asiala, Brown, DeVries, Dubinsky, Matthews, and Thomas (1996, p. 7) have this to say:

"An individual's mathematical knowledge is her or his tendency to respond to perceived mathematical problem situations by reflecting on problems and their solutions in a social context and by constructing or reconstructing mathematical actions, processes and objects and organizing these in schemas to use in dealing with the situations."

From the above discussions, it seems that understanding in mathematics is difficult to achieve, since it does not require a simple recall of facts. This is because when one solves a mathematical problem, he/she must be able to explain and justify giving evidence as to why certain steps have been done, as well as being in a position to identify the relationships between the new learnt material and the older one. In the mathematics classroom, most students find it difficult to discover new mathematics on their own.

Skemp (1976) also distinguished between two different types of understanding in mathematics which he termed instrumental understanding and relational understanding. Other researchers used the term procedural understanding to refer to instrumental understanding and conceptual understanding to mean relational understanding (Hiebert & Lefevre, 1986; Van de Walle, 2007). Instrumental understanding, according to Skemp (1976, p.20) can be described as knowing "rules without reasons". Hiebert (2013) concurs with Skemp and adds that procedural understanding is the ability to solve problems in a step-by-step manner, logically and with deterministic instructions for how to solve a problem, whereas relational understanding was defined by Skemp (1976, p.20) as "knowing both what to do and why". Hiebert (2013) defines relational understanding as conceptual understanding and sees it as knowledge that is rich in relationships. Let us now look more closely at the main difference between procedural and conceptual understanding.

Other researchers (Rittle-Johnson, Siegler, & Alibali, 2001; Hiebert & Lefevre, 1986; Hierbert, 2013; Van de Walle, 2001) described procedural understanding as the ability to execute predetermined sequences of actions to solve problems that include knowledge of the rules and procedures that one uses in carrying out a mathematical task. A key feature of procedures is that they are executed in a logically linear sequence. Students with procedural understanding of mathematics have the ability to apply an appropriate remembered rule to the solution of a problem, without knowing why they are using that rule, in a series of steps or series of actions. Contrary to the above Hiebert (2013) outlines that a person who possesses conceptual knowledge can link pieces of information in a network of ideas and is capable of understanding and linking relationships.

Donevska-Todorova (2016) used her study to illustrate these ideas of procedural and conceptual understanding according to the content area of linear algebra. Donevska -Todorova (2016) further articulated that if an individual was able to follow specific rules and apply Gauss elimination method, then the student had developed procedural understanding. If the individual goes further to calculate the inverse of the matrices or solve say the systems of equation, and links them to the idea of basis then this is regarded as conceptual understanding of the concept basis. Donevska-Todorova (2015) also said the rules that are used to calculate the determinant of matrices, for example Sarrus rule or method of Laplace expansion, is procedural understanding, and conceptual understanding is knowing how to link it to the idea of the inverse. Furthermore, he explained that conceptual understanding shares a networking of concepts that is established between pieces of existing information and the new knowledge. However, he also outlined that students enjoyed carrying out the step by step procedures, for example finding the determinant of a given matrix, but they could not engage with concepts involving deep knowledge concepts which do not require computational skill such geometrical interpretations. I also agree with Todorovas contention that students know the rules, but they find it difficult to explain the reasons why they are executing certain procedures, as well as fail to fuse the new knowledge and the old one. Van de Walle (2007) explained that conceptual knowledge contains logical knowledge which is made internally and is present in one’s mind as part of a network of ideas of knowledge that Piaget called logico mathematical knowledge. This requires some critical judgments which appears to be a cognitively difficult process because of the abstract axiomatic nature which calls for an ability to explain and give concrete justification with deep interpretations.

Van de Walle (2001) gives some benefits of conceptual understanding. He argues that it is intrinsically rewarding. Students enjoy learning when new information, new concepts and principles connect with ideas that they already know. The new knowledge fits into the learners' schema and it feels good. Li (2004) on the other hand sees procedural knowledge as rote learning and comments that learning is a passive process, which inhibits a learner to interpret information learnt, but the learner only stores the information for future use. Knowledge acquisition is also through memorization of facts. Long (2005) goes on to say that rote learning does not create a skill that can be linked with another skill or knowledge. Students who learn new concepts by rote memorization of facts and rules without understanding must be motivated by external means so that they will understand the concepts. For example, they might need to please their teacher or they fear failure. However, when the reward is not there, then the individual does not work hard. Conceptual understanding has the potential to motivate the students and is more powerful in terms of data acquisition, thereby satisfying the student as well as representing true mathematical sense (Foster, 2014). From the above definitions, the common view of the authors is that a person who possesses conceptual knowledge has the ability to understand the connections between pieces of information and make a coherent link about them. It may be that learning mathematics through discovery methods such as inductive approaches may mean help learners develop a conceptual understanding over time and will be self-motivating. A person who has only developed procedural knowledge will have difficulties in linking concepts because these were learnt as isolated facts and the learning involved a memorisation of the facts and rules.

To conclude, bringing together the above definitions it can be seen that mathematical understanding goes beyond the ability to perform algorithms but an individual must also be able to make connections between different concepts and see the relationships that exist and make a logical judgement. Understanding is not something that is suddenly attained. It is also important to note that the teacher should be at the centre stage of the learning, guiding the learners so that they can make necessary mental construction since understanding is promoted internally, and at the end the teachers should make an assessment to see whether the students have understood the concepts and for planning purposes. I summarise the definition of understanding by considering the viewpoint of a mathematician according to Michener (1978, p. 361) as:

‘’When a mathematician says he understands a mathematical theory, he possesses much more knowledge which concerns the deductive aspect of theorems and proofs. He knows about examples and heuristics and how they are related. He has a sense of what to use and when to use it, and what is worth remembering. He has an intuitive feeling for the subject, how it hangs together, and how it relates to other theories. He knows how not to be swamped by details, but also to reference them when he needs them.’

2.4 Students’ difficulties in linear algebra

Klapsinou and Gray (1999) noted that linear algebra is the first course of advanced mathematics that is offered at university level. However, Stewart and Thomas (2008) stated that, over the last three decades many researchers were concerned with the difficulties encountered in the teaching and learning of linear algebra course at university level. In recent years, the following researchers Bogomolny (2007), Britton and Henderson (2009), Parraguez and Oktaç (2010), Stewart and Thomas (2007), Stewart & Thomas (2009), Stewart (2008), and Wawro, Sweeney and Rabin (2011) were disheartened by the performance of their students when learning linear algebra. In a study by Carlson (1997) he claimed that his students did not encounter problems when manipulating matrices, for example solving linear systems of equations or carrying out the multiplication of matrices. However, the students experienced difficulties when it came to studying the vector space concepts when manipulating subspaces, spanning and linear independence. He outlined that they even become confused without knowing how to go about the problems. As a result many researchers adopted various theories and strategies to explain why students have difficulties in understanding linear algebra and suggestions were made concerning teaching methods for linear algebra issues (Britton & Henderson, 2009).

Almost two decades ago, Dorier, Robert, Robinet and Rogalski (2000) noted that the teaching of vector spaces had completely disappeared in the secondary schools and that teaching had become less formal with no studies on algebraic structures. Some criticisms voiced by students about linear algebra concern the use of formalism and the lack of connection with what they already know, since this is not done at secondary level. Dorier, et al. (2000) elaborated that formalism is experienced when students need to learn new definitions, symbols, words and theorems. They further lamented that the students have a feeling that they have landed on another planet. This is also supported by Dogan-Dunlop (2010) who added that the high level of formalisation plays a wider part in the difficulties that are experienced by the students who feel that what they are taught has no connection with what they already know. He further argued that the axiomatic approach as well as the multiple representational approach used are not essential for the majors.

Hillel (2000) states that linear algebra is the first mathematics course that is taken at university that relies on definitions, justifications and formal proofs. Hillel (2000) noted that there are some sources of conceptual difficulties in carrying out proofs that are typical to linear algebra, which make it difficult for the students and the instructor to comprehend. These difficulties include the existence of different modes of representation in linear algebra. He distinguishes three basic languages that are used in linear algebra, that is, the abstract (concepts of generalized formalized theorems), algebraic (concepts in $\mathbf{R}^n$) and geometric (concepts of 2- and 3- space) languages. He argued that the existence of these several languages and the problem of representation and applicability of theories are the major sources of conceptual difficulties. This concurs with Tall's (2004) theorem whereby abstract language represents the formal world, the algebraic world represents the symbolic world and the geometric languages represent the embodied world. The abstract mode utilizes language of generalized theories, for example when looking at span, linear combination, dimension and subspace. In an attempt to define the term linear combination, the following definition can be given, that is $w = c_1v_1 + c_2v_2 + ... + c_kv_k$, showing the abstract language. The algebraic mode utilizes the language of matrices, rank and systems of linear equations, for example writing $4x = 2y + 4b$, application of axioms when proving the vector space concepts, whereas the geometric mode concentrates on points, lines, planes and geometric transformation, for example representing linear combination of vectors geometrically by showing the resultant as described above. It is noted that students have difficulty in describing vectors as well as moving between the algebraic and the abstract modes. The other difficulty reported was the problem of representation in terms of basis. Students encounter difficulties in switching from the abstract to the algebraic mode when the underlying vector space is $\mathbf{R}^n$. He added that students had problems in identifying a string of numbers representing a vector relative to a given basis, causing the notion of vectors to disintegrate.

Stewart and Thomas (2010) also noted that many students in the first years cope well with the procedural aspects of solving systems of linear equations but struggle to understand the crucial concepts underpinning the material involving the study of vector space concepts such as subspace, linear independence and spanning. Stewart (2007) argued that only those students who are well versed and familiar with the concepts of linear algebra and have a strong background knowledge of the ideas can find the ideas of the axiomatization of the concepts interesting. The rest of the students will struggle with learning the new concepts because the instructors will hurriedly introduce the ideas with a chain of definitions and those students who lack sufficient background of such concepts will fail to connect the new with the old mathematics. Moreso Klapsinou and Gray (1999) noted the presence of a lot of algorithms that are experienced in the learning of linear algebra and they argued that it is very difficult for an individual to choose the appropriate algorithm to use in relation to a specific problem.

Teachers often complain that students have limited skills in elementary cartesian geometry, and display an inconsistent use of the basic tools of logic or set theory (Dorier, et al., 2000). The authors further argued that the lack of prior knowledge in logic and elementary set theory contributed much to the creation of errors in linear algebra. Harel and Brown (2008) also noted that most university students had proof related difficulties that hindered them from understanding linear algebra concepts. Harel and Brown (2008), Harel (1997) and Hillel and Sierpinska (1994) claimed that the students make quick generalisations without enough evidence when attempting to make proofs. On another dimension Dorier and Sierpinska (2001) were of the opinion that “the nature of linear algebra (conceptual difficulties) and the kind of thinking required for the understanding of linear algebra (cognitive difficulties)” were the root cause of some of the sources of difficulties with the learning of the linear algebra course. Stewart (2007) commented that one can adopt new ways to battle with conceptual understanding and research might help to combat cognitive difficulties because it calls for the understanding of concepts. I also concur with Harel and Brown (2008) that students struggle at the university to carry out proofs mainly because these students are not exposed to the ideas of proofs at elementary level.

Dubinsky (1997) gave an interpretation of the difficulties that under- graduate students encounter when studying linear concepts like subspaces, bases, linear independence and the matrices in general that was a slightly different from Carlson's thinking. Dubinsky put much of the blame on the pedagogical approach to learning the course. He believed that the lecturers simply tell the students the mathematics behind solving the problems. The students are not given opportunities to experiment with different types of problems, but they only stick to what they have done in the classroom. Thus students play a passive role and this is due to the traditional way of teaching. He also blamed the mathematics text books for not including worked examples in their written work; this was also heavily criticised by many reviewers of the books.

Secondly, Dubinsky (1997) blamed the students themselves for lack of prior knowledge of the concepts that are crucial to the learning of linear algebra from other subjects such as calculus. The function concept and universal quantification are crucial concepts for the learning of linear algebra. Here the curriculum planners can be blamed for not imparting the prerequisite concepts before embarking on some of the linear algebra concepts. The other problem was placed on the pedagogical weaknesses of the instructor for lack of interaction with the students thus failing to give them the chance to construct their own knowledge. Active participation is advocated for by many researchers as it encourages a dialogue between the students and the lecturers resulting in criticisms and disagreements thereby leading to the construction of rich data.

Britton and Henderson (2009) conducted a study to assess student's conceptual understanding of a subspace. The researchers argued that the abstract "obstacle of formalism" and the theoretical nature of linear algebra are the root cause of the difficulties experienced. They believed that lecturers teach students for procedural rather than conceptual understanding and students have poor backgrounds of the concepts on proofs, logic and set theory. Their sample consisted of 500 students who had completed a first year course in linear algebra and two calculus courses. One of the questions posed was as follows:

Let V = $\{t(1,2,3) \mid t \in R\}$. Show that V is a subspace of R^3.

Two groups of students were assessed separately in two consecutive years. One group of students answered the questions as an assignment and the other group as a test. Poor performance was seen in the group that wrote the questions under exam conditions. Results revealed that most of the students could show that the set is non empty but failed to prove the aspect on the closure property.

Students chose particular vectors instead of arbitrary vectors. Solutions of the form below were popular: Let $u = (a,b,c)$ and $v = (d,e,f)$ then $u + v = (a+d, b+e, c+f) \in X$ were common and some failed to show that u, v ∈ V.

Some had some misconceptions on the definition of a subspace with solution of the form $t(1,0,0) + t(0,2,0) + t(0,0,3)$. V spans $\mathbf{R}^3$ and dim $\mathbf{R}^3 = 3$. This showed that the students were mixing up concepts and showing rote learning of the concepts on vector space. The researchers also claimed that students had problems with logic and set theory, moving from abstract to algebraic mode and failing to write a convincing proof. In the other question the students had difficulties in treating functions as element of a vector space. The researchers agreed that in order to improve teaching there is a need to use more than one representation and to establish links between them.

Klapsinou and Gray (1999) argued that the linear algebra concept has some traits that hamper the students understanding of some of the linear algebra concepts. The peculiar problems included the three types of generalisation in advanced mathematics that were distinguished by Harel and Tall (1991) which included the expansive, reconstructive and disjunctive generalisation. They further commented that expansive generalisation refers to the successive generalisation of vector sum from scalar multiples of R^2 to R^3 to R^n which involved application of the same technique to each coordinate in successive broader system. Furthermore, the geometric aspect required a massive cognitive reconstruction of a vector space over a field F, for example:

"The learner is presented with a name for the concept ("the vector space V") and some of its properties (the axioms) and usually guided by an expert must follow a subtle and difficult process of construction of the meaning of *V* and its properties by deduction from the axioms. This is further complicated in the learner's mind by the fact that the properties *to be deduced in V are known* to hold $\mathbf{R}^n$, causing the problem for the student that, although these properties are 'obvious' in the (only) examples (s)he understands, judgement must be suspended on their truth in *V* until they are shown to follow by deduction from the axioms" (Harel & Tall, 1991, p. 39)

Lastly, Tall (1991) claimed that the challenges faced by students when learning linear algebra are not content based but are a result of the transition from elementary to advanced mathematics. Tall (1991, p.20) has this to say:

"The move from elementary to advanced mathematical thinking involves a significant transition: that from *describing* to *defining*, from *convincing* to *proving* in a logical manner based on those definitions. This transition requires a cognitive reconstruction which is seen during the university students' initial struggle with formal abstractions as they tackle the first year of university. It is the transition from the *coherence* of elementary mathematics to the *consequence* of advanced mathematics, based on abstract entities which the individual must construct through deductions from formal definitions".

Wawro (2014) viewed reasoning as a valuable skill and as part of practice in mathematics. He further defined it as a sense of making connections across ideas as making arguments through the way with justifications. It has extensively been seen that in many of the universities, locally and internationally, the first undergraduate course is on matrix algebra. The course is taught prior to an introduction course on proofs, for example the course on mathematical discourse and structures where proofs on set theorems are concentrated.

2.5 Studies done in linear algebra using APOS theory

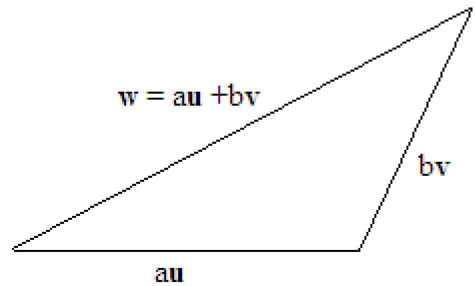
In recent years, various studies have been conducted on the learning of linear algebra concepts. The researchers adopted various theories and strategies to explain why students have difficulties in understanding linear algebra and suggestions were made concerning teaching methods for linear algebra issues (Britton & Henderson, 2009). Some of the researchers used APOS theory in explaining the construction of several concepts in undergraduate mathematics curriculum and it is argued that its use in linear algebra is recent (Stewart & Thomas 2010; Parraguez & Oktaç, 2009; Stewart & Thomas, 2009; Bogomolny, 2007; Maharaj, 2013; Ndlovu & Brijlall, 2015; Jojo, Maharaj & Brijlall, 2013; Maharaj, 2010, Brijlall & Ndlovu, 2013). There have been a few studies conducted in Africa in linear algebra (Maharaj 2015; Ndlovu & Brijlall, 2015; Ndlovu, 2013; Kazunga & Bansilal, 2017, 2018) most of which have focused on students' understanding of matrix algebra concepts. Researchers argued that this theory has been successfully used to explain the mental constructions of concepts in undergraduate mathematics studies. In one of these studies Bogomolny (2007) made use of the APOS theory as a pedagogical strategy to examine how example generating tasks can influence undergraduate mathematics students' understanding of linear dependence and independence of vectors, in particular in $\mathbf{R}^3$. The instruments that were used for data collection were mainly students' written responses and clinical interviews. One hundred and thirteen students participated in the study. Of these students, six of them volunteered to participate in the interviews. Students were asked to generate examples of linear algebra concepts such as linear independence/ dependence. The general findings of the study revealed that most of the students failed to provide the geometric interpretation of spanning. The researcher asserted that the geometric and algebraic representations seem to be completely detached, incomplete and fragmented, yet geometric representation helps in visualizing the concepts of linear algebra. The learners also understood the concept of linear independence as a process rather than as an encapsulated object since most of the students did not have a geometric interpretation of spanning as well as failing to recognize the possible ways to alter a set so as to obtain a linearly independent set.

An APOS study was conducted by Ndlovu and Brijlall (2015) based on pre-service teachers' mental constructions of concepts when learning matrix algebra. The study found that most of the pre-service teachers were operating at the action and process level, with a few operating at the object level. The authors argue that the lack of background knowledge of basic algebra schema hampered the teachers from developing adequate schemas at the object level. Many pre-service teachers could not manipulate numbers correctly when multiplying matrices and some of them failed to use notation correctly. The goal of mathematics teaching is that students understand mathematical concepts that are introduced to them or information that they discover for themselves. Hiebert and Carpenter (1992) asserted that one of the most widely accepted ideas in mathematics education is that students should understand mathematics.

Parraguez and Oktaç (2010) used APOS theory and proposed a genetic decomposition of a vector space concept which included the coordination between two binary operation and relationship of the vector space schema to other concepts. The researchers were of the opinion that in order to construct the concept, students start by activating the construction they already know about the sets and binary operations. The researchers were also interested in observing the extent to which the students coordinated the processes of the two operations through the distributive laws. The major finding from the study was that it was very difficult for learners who lack prerequisite constructions to develop the schema of the vector space concept. The pedagogical suggestion was that more practice questions should be given so that the students experiment with different kinds of sets and binary operations. They also suggested that more activities must be designed to facilitate the coordination of the following axioms $\delta(v+u) = \delta v + \delta u$ and $v(\delta + \lambda) = v\delta + v\lambda$. They also argued that the genetic decomposition needs to be revised, so as to include some of the pre-requisite concepts for effective learning to take place. It is therefore important for any study to take into consideration prior knowledge that supports the development of a concept, and therefore much emphasis should be put on the construction of the binary operation schema through experimenting with different kinds of sets and binary operations.

Stewart and Thomas (2007) argued that many university students in the first years cope well with the procedural aspects of solving systems of linear equations but struggle to understand the crucial concepts involving the study of vector space concepts such as subspace, linear independence and spanning. Stewart and Thomas (2010) combined APOS theory with Tall's (2004) Three Worlds of Mathematics to generalize students' understanding of linear independence, span, and basis according to the authors' genetic decomposition of the concepts. Tall (2004) discusses the three worlds of mathematics, which are: the embodied (physical world or thing that surrounds us), symbolic (world of symbols, algebra and algorithms) and formal worlds (defined objects). The group comprised of ten students of which one was a doctoral student for comparison purposes. The finding revealed that the students could not define the terms span, basis, subspace nor interpret linear independence geometrically. The findings also revealed that students represented their understanding in an embodied and symbolic world. Students were able to do the relatively easy procedures. For example, the students were asked to define the term linear combination. The doctoral student seemed to be operating in the formal world whilst the other two were operating in the embodied symbolic world. One of them defined linear combination as forming a plane or space, showing that he was operating in the embodied world.

Stewart and Thomas (2007) agreed that the embodied view gave a deeper understanding of the concepts taught. They further gave an example of a linear combination of two vectors which may be thought of as a triangle of vector lines symbolized as $au + bv$, $a\,(u_1, u_2, u_2) + b(v_1, v_2, v_3)$. The resultant can be illustrated in the triangle below.



This showed that the resultant of the two scalar multiples is a new vector which we can call **w**. This may be illustrative of an object conception of linear combination in the embodied world (Stewart and Thomas, 2007).

Consistent with such reflections Maharaj (2015) applied APOS framework in the context of instrumental and procedural understanding and carried out a study with two students and discovered that the students developed their mental construction at the action level and their knowledge attainment was mainly procedural. These university students failed to interpret the equality of matrices when solving the problem $\begin{bmatrix} 1 & x \\ 2y & -3 \end{bmatrix} - 4\begin{bmatrix} 2 & -2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 3z & 10 \\ 4 & -4 \end{bmatrix}$. The student was able to do the correct scalar multiplication, bringing the matrix on the RHS and do correct manipulation. Instead of finding the values of unknowns, the solution was left in terms x, y and z since the matrix was equated to the zero vector instead of the zero matrix. The student, therefore, could not equate corresponding elements and thus failed to interpret the aspect of equality of matrices. The same student could not figure out that matrix multiplication is not commutative and failed to expand given square matrices in algebraic form $(A + B)^2$. Instead, the student expanded it as $(A + B)(A - B)$. The author further pointed out that the student's knowledge of mathematical symbolism was fragmented as the student failed to develop his mental construction across different situations encountered in solving mathematical problems. In their research Possani, Trigueros, Preciado and Lozan (2010) used two theories of mathematics education namely, the Models and Model theory and APOS theory, to investigate how students understand the concepts of systems of equations. To develop the conceptual tools the use of 'real life' decision making was used to design the teaching sequence. A genetic decomposition of solving systems of equations was presented where the mental structures and mechanisms that an individual might develop in order to construct mathematical knowledge in different contexts were described, together with an analysis involving a problem related to traffic flow. The results indicated that the use mathematical education theories was very successful and helped students in the understanding of their systems of equations. There is little research on how the concepts of linear combination and spanning can be learned and the types of difficulties that students experience during the learning process as well as how the students reason about the relationships between linear combination and spanning.

Kazunga and Bansilal (2017) used the APOS theory to analyse the undergraduate mental construction of the matrices operation concepts. Data was collected through written responses and interviews. Students were asked to carry out multiplication of matrices. In their findings, they indicated that most of the participants were still operating at the action conception level according to APOS theory. It is noted that most students were able to multiply matrices of the same order but struggled to multiply the matrices of different orders. Some of them even struggled to state the order of the matrix; for example if the matrix is a 3×2 they will use a wrong notation and say that it is a 2×3. They outlined that only 48% of the class were able to multiply the matrices of the order $n \times 1$ and $1 \times n$. An example is that of a student who attempted to multiply the following two matrices $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & -1 \end{pmatrix}$ and obtained $(2 \times 1 + 0 \times 2 + 1 \times (-1)) = 1$. The student had an idea of the procedures of saying row by column but these were incorrectly manipulated showing that the idea of how to find the resulting matrix for an $n \times 1$ and $1 \times n$ matrices was not mastered.

On a similar note, Figueroa, Possani and Trigueros (2017) also used the APOS theory to explore undergraduate students' understanding of the concept of matrix multiplication. The researcher used a modelling problem and didactical sequence which was considered and verified using the proposed genetic decomposition with the aid of students' written responses. The researchers used the modelling problem so that they could construct meaning to the concepts being taught and develop abstraction strategies. The findings of the study revealed that most of these students developed their understanding at the object level of matrix product and the authors regarded the didactical sequence as an effective modelling technique tool for students' understanding of the concepts on matrix multiplication and matrix transformations.

Salgado and Trigueros, (2015) used models and the APOS theory to investigate the way in which 30 undergraduate mathematics students who were enrolled in a mandatory linear algebra course students understood the concepts of eigenvalues and eigenvectors. The researchers argued that they suggested incorporating the APOS theory and the instructional design using models. The modelling cycles were further combined with the activity ACE (classroom discussion and exercises) which is an active instructional methodology consistent with the constructivist methodology that promotes higher level thought processes. They also believe that modelling motivates people and increases students' interest to learn new concepts. The results show that 3 students were able to develop their mental construction at object level of understanding with 26 developing at least a process conception between the geometric and the algebraic representation of eigenvalues and eigenvectors. These findings suggested that the use of APOS theory and the models assisted the undergraduate students as they were able to develop flexibility in thinking about the concept on eigenvalues and eigenvectors since they were able to develop their mental construction at the process or object level of understanding.

DeVries and Arnon (2004) conducted a study to explore students' understanding of the concepts in linear algebra on solutions of a system of equations. The study comprised 15 students at a teachers college who were interviewed after completing a course in linear algebra. 12 of the teachers volunteered to be interviewed. The interviews were conducted individually and each interview lasted for at least 45 minutes. The APOS theory was used as the instructional tool to identify the mental constructions required by the students to learn the concept on solution of equations. The researchers noted that the questionnaire that they used had some weaknesses in the sense that it gave inadequate information so that they could not come up with convincing results based on their research question.

Some authors (Kú, Oktac¸ & Trigueros, 2011; Dorier et al., 2000; Rogalski, 2000) discuss problems students have with the vector space concepts. Kú, Trigueros and Oktaç (2011) reported that students had challenges in distinguishing a spanning set from a basis. They also reported that students had problems with the analysis of a spanning set provided the given vector space is not R^n. Stewart and Thomas (2010) used (APOS) theory, together with the Three Worlds of Mathematics to examine students' difficulties in the learning of linear combination and independence. They noted that most of the students were working in the symbolic mode of thinking and suggested that the embodied world is more worthwhile for concept acquisition. The students encountered various difficulties in an attempt to define the terms linear combination and linear independence, because of the abstract and formal presentation of the concepts. Students were more comfortable with manipulating algorithms when solving systems of linear equations.

2.6 Misconception in the learning of linear algebra

In recent years, various studies have been conducted on errors and misconceptions in mathematics as a way of enhancing students' understanding. These errors and misconceptions have been discussed and are well documented in mathematics education research (Swan, 2001; Molefe & Brodie, 2010; Luneta & Makonye, 2010; Barmby, 2009; Aygor & Ozdag, 2012; Siyepu, 2013, Hansen, 2006, Lannin, Baker, & Townsend, (2007)). It is also noted that some difficulties in linear algebra are related to misconceptions and errors that students can make. Cangelosi, Madrid, Cooper, Olson and Hartter (2013) reported that students memorise algebraic rules with no conceptual understanding attached to the concepts. The students then have difficulty in keeping track and applying the rules appropriately. They also found that language and notation can also hinder or enhance students' mathematical development, such that at the end they manifest several errors. When errors are diagnosed, it is important to identify the root cause and find ways to rectify them (Makhubele & Luneta, 2015).

2.7 Teaching strategies in linear algebra

Many research studies (Harel, 1997; Hillel & Sierpinska, 1994; Hillel, 2000; Carlson, 1993, 1997; Sierpinska, 2000) have indicated that some students experience difficulties when learning certain linear algebra topics and that teaching these had been challenging to many. To address these difficulties, and to address an increasing demand for student understanding of linear algebra courses, Linear Algebra Curriculum Study Group (LACSG) was formed in 1990 to provide recommendations for the first course in linear algebra (Carlson, et al, 1993, 1997). Their work was centered on the pedagogical strategies that can be used in the teaching and learning of linear algebra at university level. The following recommendations were put forward by Carlson et al. (1993):

- They proposed the need for a challenging course to be introduced in the first course in linear algebra, for example the course on mathematical proof with definition and theories so as to enhance critical thinking.

- They advocated for more time to be allocated to the study of linear algebra by introducing basic linear algebra concepts at high school and having a matrix oriented course for the first year undergraduate students. This will then ensure that there is a continuity between high school mathematics and college mathematics; for example, high school students will be introduced to the concepts on matrix algebra and calculating determinants which are prerequisites to the learning of linear independence, basis, dimension, rank and so on. Thus students will be cognitively equipped for the abstraction of the vector space concepts.
- The instructors were encouraged to utilize technology in the first linear algebra course, for example incorporating various software packages in the teaching of matrices such as Matlab, Maple so as to enrich students' lectures and effective teaching.

Lastly, a core syllabus was recommended dealing with the following topics in linear algebra: matrix addition and multiplication, system of linear equations, determinants, linear combination, linear dependence and independence, bases of R^n, subspaces of R^n, inner product space, eigenvalues and eigenvectors. However, on the issue of the central curriculum Aydin (2009) recommended that every mathematics department in the university must have its particular programme and should decide the curriculum model that suits their programme including topics they want their students to learn. He goes on to say that there is no best way to teach linear algebra, but every instructor needs to understand how students learn, and to identify the proper methods that will be useful in imparting the concepts in linear algebra. Ulusoy (2013) outlined that the recommendations motivated many researchers to carry out research on the teaching of linear algebra which has this led to an extensive literature in recent years. Sierpinska (2000) noted that after these recommendations were put in place in various institutions of learning students still had difficulties with linear algebra topics. Researchers such as Hillel (2000), Stewart and Thomas (2008), Dubinsky (1997) and Dorrier (2000) continued to look at the root cause of these difficulties that persisted, but the students still have difficulties with the learning of linear algebra, so some researches were done in the area of linear algebra.

2.7.1 Pedagogical thinking for learning and teaching mathematics

In order to make learning beneficial for the students, Harel (2000) formulated three pedagogical 'principles' for the learning and teaching of mathematics. He was inspired by Piaget's psychological theory of concept development. The pedagogical principles involve the Concreteness Principle, the Necessity Principle and the Generalisability Principle. He observed that the approaches that were used to teach the various concepts in linear algebra did not suit the students' pedagogical needs. He felt that it was unfair for the students in the first years of university who were presented with abstract structures and required to apply the principle of the abstract background to solve problems without wide preparation. He therefore designed the Concreteness Principle. The principle states that, "For students to abstract a mathematical structure from a given model of that structure, the elements of that model must be conceptual entities in the student's eyes, that is to say, the student has mental procedures that can take these objects as inputs". This is based on the Piagetian assumption of the idea of conceptual entity formation. The principal further posits that the first course that is done by students at university is on matrix algebra and vectors and this is taught to students who have not yet constructed the elements of these structures into conceptual entities. This becomes difficult for the students since they need to build their understanding of a concept in a context that is familiar and concrete to them. He further gave an example of a problem on linear independence. He outlined that once a student is able to come up with a homogenous equation involving a polynomial equated to zero on the right hand side of a given polynomial function is not a zero vector, then the student would have formed the concept of linear independence as a mathematical object. This will then be used with ease for other operations.

Harel (2000) carried out a study to determine the effectiveness of using either the geometric or the algebraic systems when teaching the vector space concepts. He observed that a constant emphasis on the embodiment of a geometric system of the abstract linear algebra concepts proved to be effective and produced a strong base for students understanding. Harel further suggested that linear algebra course should start with geometry and build the algebraic concepts through some kind of generalization from geometry. The geometric concept should only act as a pathway to the more abstract algebraic concepts (Harel, 2000).

The Necessity Principle states, "Students must see a need to learn for what they are intended to be taught. By 'need' it is meant an intellectual need as opposed to a social or economic need". The main idea behind this principle is that knowledge develops as a solution to a problem. Problem solving requires learners to identify, define and solve problems using logic as well as lateral creative thinking. In the process, learners arrive at a deep understanding of the topic area and construct new knowledge and understanding on which they are to make decisions. The teacher becomes a facilitator whose role can be a subject matter expert, resource guide and a task consultant. Students feel that they are the owners of the knowledge. If the teacher solves the problems for the students, the students have a tendency of simply reproducing the teacher's solution, without understanding the procedures; for example students have difficulties with proving the vector space axioms. They fail to grasp the meaning of the statement say of (-1)A = -A, and have difficulties in coming to terms with arguments made. This principle is violated whenever one derives the definition of a vector space from a presentation of the properties of $\mathbf{R}^n$.

The last principle that Harel postulates is the Generalisability Principle. It states that "When instruction is concerned with a 'concrete' model that is a model that satisfies the Concreteness Principle, the instructional activities within this model should allow and encourage students to the generalizability of concepts" (Harel, 2000, p.187). The principle is mainly concerned about the teaching material whereby it aims at enabling students to abstract the concepts that they learn in a specific model. Dubinsky (1997) also advocated for learner centered pedagogical strategies when teaching linear algebra. He is of the idea that the instructor should first analyse the particular mental constructions that are required to engage with the understanding of certain concept. Syarifuddin (2013) commented that teaching mathematics in higher institutions is vital as it helps to bridge the gap with elementary mathematics and in the long run it connects the student to the world of work. Syarifuddin (2013) argued that students should be given the chance to increase their communication abilities through interaction with peers, be problem solvers, and be able to justify ideas as they communicate with peers in the classroom so as to deepen their ability to understand mathematics.

Wawro (2014) views reasoning as a valuable skill and part of the practice of mathematics. Wawro, Sweeney and Rabin (2011) noted that in order to attain such reasoning skills, individuals should engage in mathematical activities of defining mathematical concepts, problem solving, proving and making arguments with justifications, as well as example generation. Wawro's et al. study (2011) focused on students' understanding of the concept of a subspace using Tall and Vinner's (1981) theory on concept image and concept definition. The authors found that students had varied definitions of the term 'subspace', and identified common imagery for a subspace as a geometric object, part of a whole, and as an algebraic object. They also noted that students had incorrect conceptions that $\mathbf{R^k}$ is a subspace of $\mathbf{R^n}$ for $k < n$. They concluded that students struggled to understand mathematical ideas especially definitions because of the cognitive conflicts between the concept 'image' and concept 'definition'.

Britton and Henderson (2009) found that in order to improve teaching, it is important to develop representational versatility by making links between different vector spaces. The representational 'versatility' includes the ability to move from one representation to another; for example, a vector may be presented geometrically as an arrow, algebraically as row vectors and abstractly as elements of a vector space. Similarly, Stewart (2018) argues that part of the instructors' role is to help students to move between the different worlds, for example by identifying suitable tasks that can link their intuitive ideas to the formal definitions or to draw upon dynamic software to help provide visualizations of aspects such as linearly independent vectors.
Furthermore, in order to enrich teaching in linear algebra, lecturers use various teaching strategies in their lectures so that the students can understand the concepts. The various methods that lecturers use include use of projects, group activities, inductive and deductive methods, demonstration method and technology. These strategies help students understand the concepts. In particular NCTM (2000) came up with activities that enhance the understanding of mathematics, such as problem solving activities, reasoning and proof, communication, connection and representation.

2.7.2 Role of technology in the teaching of linear algebra

Considering the role of technology in the teaching linear algebra, Dikovic (2007) disagreed with the use of the traditional method of imparting the necessary knowledge to students as he made an assumption that it involves "only talking". He goes further to say it is not a suitable approach to teaching because it does not consider the learners' preferences by overlooking the cognitive development of the learner. He advocated for the active involvement of the learner when acquiring new knowledge so as to improve the levels of mental constructions and that they must acquire knowledge by themselves. Ulusoy (2013) concurred with the above sentiments and pointed out that traditional methods are no longer acceptable in the teaching and learning of linear algebra since they cannot support the learning experiences and meet the demands of higher education.

Many researchers (Ferrara, Pratt & Robutti, 2006; Aydin, 2009; Dikovic, 2007; Ulusoy, 2013) have carried out studies concerning the way in which technology can be used to enrich the teaching and learning of linear algebra. The world over many researchers are now advocating for the inclusion of the use of technology in the teaching and learning of mathematics so as to reduce computational processes. Buteau, Marshall, Jarvis and Lavicza (2010) points out that the use of computer technology has developed rapidly all over the world such that many universities provide network systems for students, and has stimulated the discovery of mathematical ideas in abstract algebra using programmes like maple. This is also supported by Ulusoy (2013) who argued that the use of Information and Communication Technologies (ICT) in mathematics at university level is already widespread and several researchers have adopted its use as they claim that it impacts positively on the students' way of thinking, improving their mathematical skills and thus leading to improved way of understanding and academic achievements. He also added that many teachers acknowledge the use of software programmes like MatLab, Octave and Maple as well as computer demonstrations so as to enrich and improve teaching and learning. Weller, Montgomery, Clark, Trigueros, Arnon and Dubinsky (2002) also noted that some members of RUMEC (Research in Undergraduate Mathematics Education Society) published through Internet some teaching materials based on APOS theory and the materials incorporated some activities and exercises on linear algebra concepts using the computer programming language ISETL. The activities were guided by genetic decomposition of concepts on algebra.

Aydin (2009) emphasised that many of the computer algebra systems permit the students to effortlessly manipulate matrices, matrix inverse, calculations of determinants, carry out elementary row operations and so on. Buteau et al. (2009) further commented that the use of technology was advantageous to the students since it encourages them to focus on the outcome that is the meaning of the computations, develop critical thinking skills in an attempt to justify the end result of the calculated values, thus motivating students to learn the axiomatic theory and to strengthen their conceptual understanding. Thus the linear algebra algorithms can be manipulated hastily, satisfactorily and accurately by computers, and mathematical software encourages the learner to understand the theory. In line with Adrian's contention, Dikovic (2007) further showed that there are several different roles that technology can play in the teaching and learning of linear algebra. This involves computational drudgery in application to providing environments for active exploration of the properties of mathematical structures and objects. He further argued that students use computer programmes to make some tedious calculations in large dimension of matrices thus encouraging active exploration of the mathematical structures since it liberates the students from concentrating on calculation of figures. Ferrarr, Pratt and Robutti (2006) commented that when the machine does the work, the individuals can now do the argumentations, make conjectures and engage in proofs. On the same note Nicaise and Barnes (1999) point out that the use of technology helps to create an information rich classroom since it encourages teacher and students to consolidate and share information, hence it aids deep understanding of concepts with opportunities for discussions thus promoting higher level thinking. In his studies on evaluating the effective utilization of advanced calculators (graphing and computer algebra system) as an assistance tool in furthering the conceptual understanding of diagolisation of matrices.

However, in a study about a framework for monitoring progress and planning teaching towards the effective use of CAS, Pierce and Stacey (2004) discovered that the use of CAS requires knowledge and technical skills pertaining to the use of the machine. They suggested three technical problems that can be encountered. Firstly, the problem in entering the syntax correctly which required mathematical knowledge to analyse the structure of the expression to be entered, for example how to enter a matrix when doing elementary row operations. Students had problems with the use of parenthesis as well as specifying the variables to be manipulated. Secondly, the ability to systematically change representation, for example by moving smoothly from algebraic to graphical form. Lastly, the students must be able to interpret the CAS output because at times the output can be different from the conventional pen-and paper representations. Ulusoy (2013) agreed with the above sentiments and said that the use of technology does not make the learning simpler but it calls for a thorough designing of the teaching environment so that students can understand the processes; this means both the teacher and the student need to be technologically literate. He further said that though the use of technology is a vital tool to learning and teaching some teachers feel that it does not encourage conceptual understanding but will simply turn the students into "unthinking button pushers'

2.8 Analysing the gap from the literature review

This study will use APOS theory which is a constructivist theory of understanding mathematical concepts. Asiala, Brown, De Vries, Dubinsky, Mathews and Thomas (1996) argued that APOS

(Action-Process-Object-Schema) is directly related to the studies of linear algebra and has been widely used in research studies concentrating on the understanding of undergraduate mathematics. Dubinsky and McDonald (2001) claimed that the proposed approach to learning is expansively different from the idea of stating theorems, defining terms and then carrying out the proofs that characterize most of the topics taught at several universities. This study will attempt to use the APOS theory in explaining the construction and mental mechanism involved by the undergraduate students when learning the vector space concepts. Furthermore there are few studies about in-service teachers understanding of linear algebra.

An analysis of the above literature reveals that most of the studies looked at the vector space concepts as separate entities. Research studies conducted were focused on vector space on its own, linear independence on its own, linear combination, spanning and basis on its own and subspace on its own. This study intends to look at the mental construction and the nature of the difficulties encountered when teaching concepts on vector space, subspace, linear combination, linear independence/dependence, basis and dimension. This is because these concepts are well connected and related and there is a need to see the dynamics on how students construct knowledge about the vector space concepts and the connections that the students seem to make among the concepts. I am also interested with this research so as to offer a viable path that students may follow in order to construct the vector space concepts as well as explain the nature of related difficulties while learning it. Informed by theoretical data I also focus on making pedagogical suggestions.

2.9 Conclusion

This chapter provided an overview of the literature concerned with understanding of mathematics in general and the nature of linear algebra. The chapter also provided some background of what other researchers identified as some of the causes of difficulties which are encountered by students in the learning of linear algebra. The literature review also pointed out some of the strategies that other researchers adopted in the teaching and learning of linear algebra. The chapter concluded with by a reflection on the implication of the literature reviewed for this study. In the next chapter the theoretical frameworks informing the study are outlined.

CHAPTER THREE

THEORETICAL CONSIDERATIONS

3.1 Introduction

This study explores the in-service teachers' mental constructions when learning vector space concepts. In this chapter the theoretical framework within which this study is situated will be discussed. The chapter starts with a discussion of the concept of constructivism and its components. I provide a discussion of the specific framework for this research and its components that is considering its theoretical analysis, design, implementation of instruction and data collection and analysis. The framework was developed in an attempt to understand the philosophies of Piaget regarding reflective abstraction. The concept of reflective abstraction is presented together with a discussion of the framework used in this study, i.e. the APOS theory. The description of the terms action, process, object and schema are given in detail. A detailed account of the concept genetic decomposition is given, together with the various genetic decompositions of the vector space concepts. Kiat (2005) construct was another theoretical framework that this study employed to analyse students' errors in learning about vector space concepts, and it is also discussed in this chapter.

3.2 The constructivist paradigm

Constructivism forms the basis of how learners learn, and the APOS theory is basically a constructivist theory (Dubinsky & Macdonald, 2001; Brijlall, 2013). According to Jaddah (2000) the general principal of constructivism and how it is translated into the classroom is largely based on the work of Piaget (1964) and Vygotsky (1978) among others. Constructivism is a view of learning based on the belief that knowledge is constructed by learners through an active process when looking for meaning during the mental process of development (Jonassen, Howland, Moore, & Marra, 2003). Hamlet, Carr and Steinruck (2015) argued that constructivism is centered on the learners' active participation in problem solving activities and critical thinking, in an endeavor to construct their own understanding and will become an integral part of their cognitive network. Rovai (2004) commented that constructivism makes an attempt to create an environment that is conducive for learning, for example by making use of open ended questions during class discussion so that there is dialogue among the learners which encourages them to construct their own knowledge. The teacher is the person who creates the environment suitable for the students to learn and supports learning by providing specific guidelines for learning. Furthermore, Faulkenbury and Faulkenbury (2010) argued that constructivists believe that students build their understanding of concepts through self-modification of cognitive structures. However, they stated that those who are able to master the concepts have the opportunity and means of reflecting the existing cognitive structures to a higher plane of thought so as to accommodate new pieces of information. Faulkenbury and Faulkenbury (2010) referred to cognitive structures as reflective abstraction. This notion of reflective abstraction is Piaget's original idea and it forms a central theoretical concept in APOS theory which will be explained further in section 3.6. Constructivism also focusses on mental processes that construct meaning, for example Van de Walle (2007) advocated for the general principal used by Piaget (1972) of the process of accommodation and assimilation. Assimilation refers to an individual's use of his existing schemas so as to give meaning to experiences whilst accommodation refers to a process of altering schemas that do not fit into the existing schemas. This means that learners come into the classroom with some prior knowledge and not as empty vessels (Von Glaserfeld, 1987).

I was involved in the teaching of concepts of a vector space, and I used some of the principles so as to provide a framework for a conducive learning environment, such as acting as a facilitator of knowledge, giving the students the opportunity to work in groups solving given tasks, involving them in class discussions and giving them chance to solve problems on the chalk board. This led to a productive learning process. This was done so that the students could create mathematical knowledge in a relaxed environment.

3.3 The research framework

This study is based on the specific framework for research and curriculum development in mathematics education according to Asiala, Brown, DeVries, Dubinsky, Mathews and Thomas (2004). APOS theory is based on this framework for research and curriculum development, which emphasizes the cognitive growth of an individual trying to construct the necessary knowledge when learning mathematical concepts. The framework consists of three components: theoretical analysis, instructional treatment, and observations and assessment of student learning. According to Dubinsky (2004) APOS theory based research should be led according to the paradigm illustrated in Figure 1, with the three components which are cyclic and influence one another.

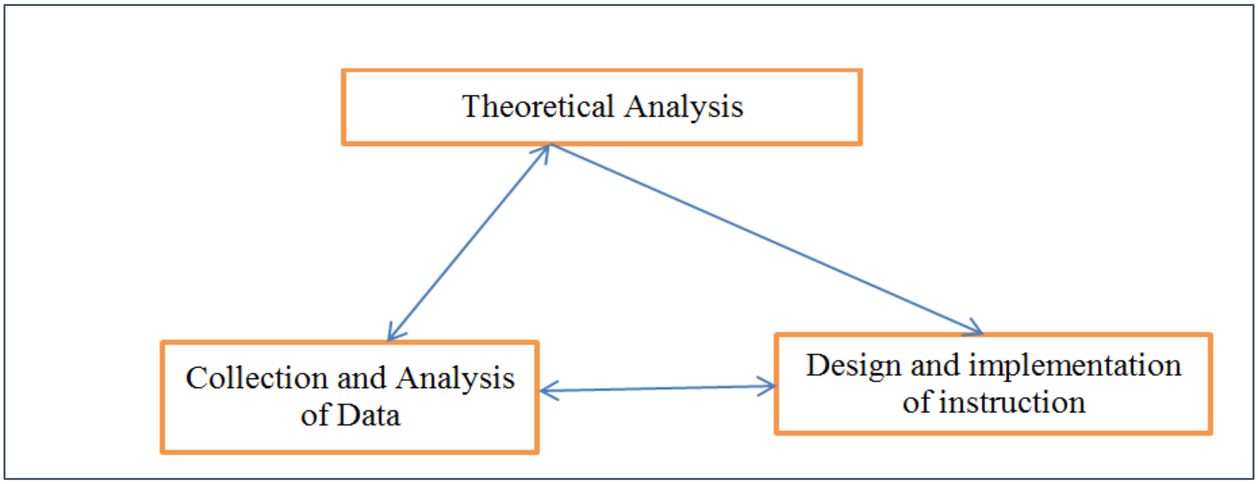


Figure 3.1: The Research Framework (adapted from Asiala et al., 1996)

The understanding of a mathematical concept is constructed by continuous modifications of the material as the individual repeatedly progresses through the three component activities. In order to develop an understanding of the vector space concepts, an initial theoretical analysis of the concept in relation to the specific mental construction that an individual makes will take place. The individuals' understanding of a concept is based on the researchers' knowledge of the concept, i.e. prior experiences, as this give rise to a preliminary genetic decomposition of the concepts (Arnon, Cottrill, Dubinsky, Oktac, Fuentes, Trigueros & Weller, 2014). The theoretical analysis will in turn will inform the design and implementation of instruction as shown in Figure 3.1 and this is achieved by designing appropriate activities and exercises that encourage the individual to make the relevant mental construction in developing the vector space concepts. Arnon et al. (2014) further upheld that some pedagogical strategies are required to assist the individuals in making the necessary mental constructions such as cooperative learning, group discussion as well as lecturing. The last step involves implementation of instruction which leads to the collection and analysis of data. The analysis stage is crucial so as to provide the answer as to whether the students were able to make the necessary mental constructions called for by the theoretical analysis (Arnon, et al., 2014). If the students fail to make the necessary mental constructions then it necessary to revise the instruction.

3.4 APOS theory as an extension of reflective abstraction

The concept of reflective abstraction was introduced by Piaget to describe the construction of logico-mathematical structures by an individual during the process of cognitive development. According to Dubinsky (1991a) Piaget felt that reflective abstraction is a powerful tool which is necessary for the development of more advanced concepts in mathematics since it contributes to an individual's understanding of what this thing is and helps students develop the ability to engage in it.

Piaget made two important observations while examining the way in which reflective abstraction leads to the construction of logico-mathematical structures. According to Beth and Piaget (1966) the first feature of this was that reflective abstraction does not have an absolute beginning but is present at the earliest ages in the coordination of sensory-motor structures. The implication here was that an individual cannot determine the time at which a child starts to develop logical thinking. The second feature of this observation was that reflective abstraction continues on up through higher mathematics to the extent that the entire history of the development of mathematics from antiquity to the present day may be considered as an example. Dubinsky (1991) commented that as a mathematics teacher, one must understand the ideas of reconstruction and this will help the individual to relate it when designing instructional methodology.

According to Dubinsky (1991a), Piaget distinguished three types of abstractions which are not independent of each other and he talks of Empirical abstraction, Pseudo-empirical abstraction, and Reflective abstraction. According to Dubinsky (1991) the focus of empirical abstraction is on the general characteristics of objects, which has to do with experiences that appear to the subject to be external. Piaget and Gracia (1983) posit that empirical abstraction also derives common characteristics from a class of objects by combining abstraction and simple generalization. Dubinsky (1991) also confirmed that the knowledge of these properties is internal although the experiences are external.

The pseudo-empirical abstraction is an intermediate one that lies between empirical and the reflective abstraction. (Piaget, 1985) outlines that the main focus of pseudo-empirical abstraction is that it attempts to outline the properties that the actions of the subject have introduced into objects. Lastly, reflective abstraction is drawn from what Piaget (1980) called general co-ordination of actions by the subject, internally. This means the main focus is on the actions or operations done by the subject on (mental) objects. Beth and Piaget further outline that this kind of abstraction leads to a very different sort of generalisation, which is constructive, and which results in a ‘new synthesis in the midst of which particular laws acquire new meaning’ (Piaget & Gracia, 1983). This is also in line with Beth and Piaget (1966, p.188 -189) who state that reflective abstraction consists in deriving from a system of actions or operations at a lower level, certain characteristics whose reflection (in the quasi-physical sense of the term) upon actions or operations of a higher level it guarantees, for it is only possible to be conscious of the process of an earlier construction through a reconstruction on a new plane. Reflective abstraction proceeds by reconstructions which transcend whilst integrating previous constructions.

The acquisition of mathematical knowledge was considered to be associated with reflective abstraction. To illustrate an instance of reflective abstraction, for example an individual is asked to show that the vector say w = (2,3,4) can be written as a linear combination of say the vectors $v_1 = (2, 1,4), v_2 = (5,1,0)$ and $v_3 = (2, 2.4)$. The individual can perform numerous individual actions in his mind so as to come up with the vector equation, equate corresponding elements, come up with a system of three equations in three unknowns equations and then formulate the augmented matrix (that is, drawing knowledge from given objects). The individual will then interiorise and coordinate the action so as to form a total organization of carrying out the Gauss elimination method and relate the solution to the given vectors. Dubinsky (1991) further outlined that these ultimately formed objects may no longer be physical but rather mathematical such as a function or group. According to Beth and Piaget (1996) these new mathematical constructions only progress through reflective abstraction but Piaget (1978) noted that this role is not restricted to the intellectual development of children. Piaget (1972) noted that it is reflective abstraction in its most advanced form that leads to the kind of mathematical thinking by which form or process is separated from content and that processes themselves are converted, in the mind of the mathematician, to objects of content. Berth and Piaget (1966) also outlined that the first part of reflective abstraction consists of drawing properties from mental or physical properties at a particular level of thought. Thus empirical abstractions deals with actions as opposed to objects and pseudo-structural is not concerned so much with the action but with interrelationships among actions (Dubinsky, 1991).

3.5 APOS theory

APOS theory involves four primary stages, namely an action, process, object, and schema stage. These four primary stages are called the mental structures for the construction of mathematical knowledge. The main mental mechanism for building the mental structures are interiorisation and encapsulation. The theory advocates that in order to form a mathematical concept there is a need to transform an entity to obtain another entity. Thus a transformation begins with an action. The action is interiorised to form a process and in turn the process is encapsulated to form objects (Dubinsky et al., 2005). The mental structure and the mental mechanisms are described in Figure 3.2 below according to an extract from Arnon, et al. (2014, p.10).

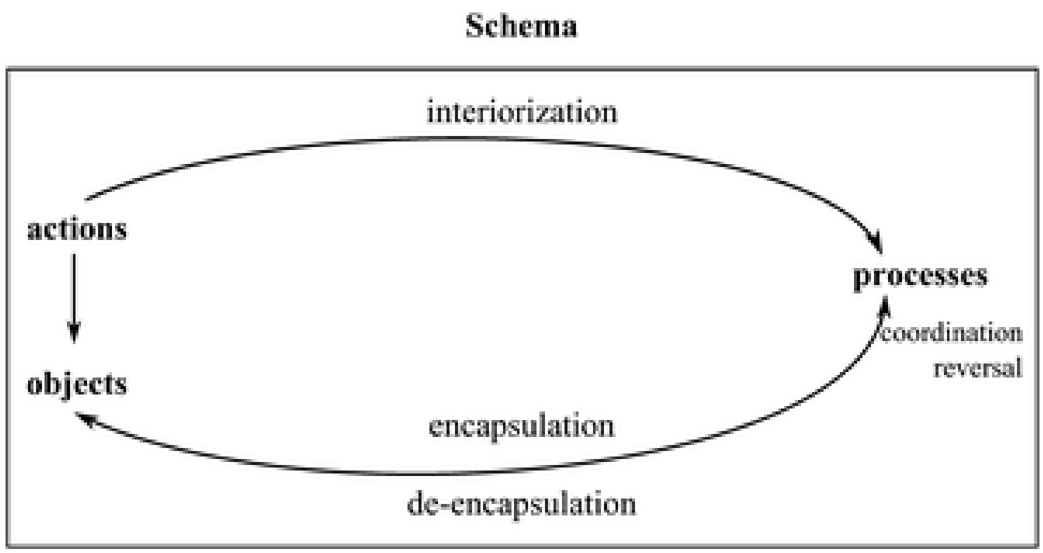


Figure 3.2: Mental structures and mechanisms for the construction of mathematical concept

3.5.1 Description of the mental structures

The mental structures for the construction of mathematical concept is described below:

- **Action**: According to the APOS theory, the development of every concept begins in the learner's mind with an action (Dubinsky and McDonald, 2001). The transformation of objects is thought of as a reaction to stimuli which an individual perceives as external. A mathematical concept suffices as one applies a transformation of an object to obtain another object, and therefore it requires specific instructions or rules as well as the ability to perform each step of the transformation explicitly. Thus a student can think about the problem in a step by step manner and look at one step at a time, for example when an individual reduces the following matrix $\begin{bmatrix} 1 & 2 & -3 & 1 \\ 2 & 4 & 3 & 1 \\ 4 & 2 & 1 & 0 \end{bmatrix}$ to echelon form. The individual will use a_{11}=1 as the pivot element to obtain zeros below a_{11} that is add $-2r_1$ to r_2, and show the working for all the elements in row 2. The individual will continue to carry out the row operations until (s)he obtains zeros below all the obtained pivot entries. Another simple example is that of substitution into a formula. However, Dubinsky et al. (2005) noted that when working with patterns, students in the action stage can find variations in a pattern but may fail to come up with a rule that produces the pattern.
- **Process**: Dubinsky and McDonald (2001) state that a process conception involves a form of understanding that performs the same operation as the action but takes place wholly in the mind of the individual to produce a completely new object without external stimuli. The repeated action interiorises into a mental process (Dubinsky, 2001). The individual takes an object and transforms it and produces a new object without having to execute each step explicitly. At this level, the student has the potential to reverse and compare the processes or use it with other processes. A student is able to describe the action verbally, verify results using a different method from the one used and predict an outcome (Stewart, 2008). When solving a system of linear equations, the row reduction procedure is a process when an individual only show the row that (s)he will be working on, but all the additions and subtractions are done mentally and are carried out wholly in the mind of the individual without indicating any step by step procedures. A student with a process conception of say linear independence, will think of the resulting row reduction of given vectors and make arguments based on given theorems, and this will take place in the mind of the individual.

- **Object**: Dubinsky and McDonald (2001) talk of a process being encapsulated into an object when a student becomes aware of a process as a totality and also realizes that transformation can act on that totality explicitly. The process is perceived as an entity upon which actions and processes can be made. The student at this stage can perform a new mathematical operation on the process. Students operating at this stage can represent the solution using a different representation. They can also explain why a chosen method cannot be used to solve a given problem (Stewart, 2008). The following examples illustrate the students' conception of the concept linear independence at the object stage. The individual can think of a set of linearly independent vectors as an entity upon which other operations can be carried out such as checking if the set forms a basis and establishing the dimension of a vector space. Arnon et al. (2014) asserted that it is necessary to de-encapsulate an object back to the process that led to it only when there is a need to do so. Thus, in mathematics, it is important for an individual to be able to move back and forth between an object and a process. The objects can be de-encapsulated to obtain the process from where they come from, for example, when solving problems on the basis of a vector space, which is a result of it being linearly independent and spans given vectors. Weyer (2010) defines encapsulation as the mental construction of a process into a cognitive object which can be viewed as a total entity. If the learner appreciates this and can actually build the changes, then the learner has encapsulated the process into a cognitive object (Dubinsky 2001).
- Schema: Many actions, processes and objects are interconnected in the individual's mind and these will be organized to form a coherent framework called a schema. This is supported by Possani, Trigueros, Preciado and Lozan (2010) who argued that the presence of new relations between the new and the previous actions, process, object, and other schemata leads to the formation of schemas. This connection provides an individual with decisions when presented with a mathematical problem (Dubinsky, 2001). At this stage, one has a clear understanding of the concept and can apply it in real life situations. Maharaj (2013) notes that explanations offered by an APOS analysis are limited to descriptions of the thinking which an individual might be capable of. Maharaj (2013) further observes that it does not necessarily mean that if an individual possesses a certain mental structure, then he or she will apply it in a given situation as this depends on other factors. The main

objective of an APOS analysis is to point to possible pedagogical strategies for helping students learn it.

3.5.2 Description of the mental mechanisms

Dubinsky (1991) describes five kinds of mental mechanisms namely: interiorisation, encapsulation, coordination, reversal and generalization and these are summarized below**:**

- Interiorisation: This is when an individual does not depend on external cues when solving mathematical problems but can construct internal processes without carrying out the step by step procedures as a way of making sense of the perceived phenomena.
- Encapsulation: the ability to apply an action to a process and consider the process as a totality.
- Coordination: two or more processes are coordinated to form a new process.
- Reversal: the ability to reverse thought processes of previous interiorised processes.
- Generalisation: the ability to apply existing schema to a wider range of contexts.

3.6 The genetic decomposition

Theoretical analyis is concerned with the manner in which learners cognitively construct the necessary knowledge in mathematics. Dubinsky (2001) said that the purpose of theoretical analysis is to plan the specific mental construction that are needed for a student to make the necessary construction and this gives rise to the construction of a genetic decomposition of the concept. Furthermore, Asiala, Brown, DeVries, Dubinsky, Matthews and Thomas (1997) argued that in order to apply the APOS theory to describe particular constructions by students, researchers are required to develop a genetic decomposition. DeVries and Arnon (2004) described a genetic decomposition as a diagnostic tool, a detailed description of the types of APOS mental structures and the associated mechanisms, which gives insight to researchers into how learners develop various concepts, with Brijlall and Ndlovu (2013) arguing that it is a structured set of mental constructs, which might describe how a concept can develop in the mind of an individual. This is also supported by Possani et al. *(*2010) who claimed that a genetic decomposition is a description of specific mental constructions (action, process, and object) that an individual may make in the process of understanding mathematical concepts and their relationship. It also models how students learn so as to design teaching strategies and analyse the knowledge that students display when solving specific activities in an endeavor to improve instruction. This preliminary genetic decomposition is designed based on the researchers' experiences of a particular concept. It is important to note that a genetic decomposition should be given in terms of cognitive constructions, but should not be given in terms of mathematical results (Parraguez & Oktaç, 2010). Ndlovu (2013) also outlined that a genetic decomposition is usually presented in a linear form but learning is not linear. What is essential is how individuals cognitively construct appropriate knowledge of a given concept. The genetic decomposition for the various vector space concepts is presented below, and this is based on the research framework under theoretical analysis. Implementation of the instruction gives the researcher the chance to gather data, usually in the form of written instruments and can be followed by an in-depth analysis of the instruments through the use of interviews. Possani et al. (2010) further argued that the genetic decomposition guides the development of the instructional treatment such that at the end, if students fail to make the necessary construction called for by the genetic decomposition, this will lead to the revision and modification of the genetic decomposition or the pedagogical strategies. A summary of the preliminary genetic decomposition of the various vector space concepts is given below:

3.6.1 Hypothesized or preliminary genetic decomposition for vector space/subspace

The genetic decomposition of a vector space and vector subspace is proposed based on the author's own experiences and a review of literature (Arnon et al., 2014; Parraguez, & Oktaç, 2010).

Set schema: At an action level, an individual conceives of a set when given a specific listing of a particular condition of set membership. The action of gathering and putting objects together in a collection according to some condition is interiorised into a process. This is encapsulated into an object when an individual can apply actions or processes to the process such as compare two sets, consider a set to be an element of another and analyse properties of the set (Arnon et al., 2014).

Binary operation schema: A binary operation is a function of two variables defined on a single set or on a Cartesian product of two sets. At an action level, given a binary operation, an individual can take two specific elements of the sets and apply the formula. The individual interiorises the action into a process that takes two objects (elements) and acts on these to produce a new object (element) that is the result of the binary operation. At the object level, an individual can distinguish between two binary operations, check whether a binary operation satisfies an axiom and compare objects arising from two different binary operations (Arnon et al., 2014).

Parraguez and Oktac (2010) describe how these two schema can be drawn together to form the notion of vector space: The objects that are sets with two kinds of operations (addition and multiplication by a scalar) can be coordinated through the related processes and the vector space axioms that involve both operations, to give rise to a new object that can be called a vector space (Parraguez & Oktac, 2010, p. 2116).

3.6.2 The hypothesized or preliminary genetic decomposition for linear combinations

The genetic decomposition for linear combinations that guided this study was derived from the help of the work of Dubinsky (1999).

Action: An action conception of linear combinations is evident if an individual, when asked to show that a given vector say **w** can be written as a linear combination of the vectors, say $\boldsymbol{u}$ and $\boldsymbol{v}$, carries out a series of steps where the one step acts as a prompt for the next one. The term linear combination acts as external stimulus of what needs to be done. The first step is to form vector equations of the form

$$\mathbf{v} = k_1\boldsymbol{u} + k_2\boldsymbol{v}.$$

where k_1 and k_2 are scalars that need to be calculated. The next step is to express the given vectors in coordinate system and then come up with a system of linear equations in two unknowns (scalars). Solutions to the system of equations are then calculated or it is concluded that the system has no solution.

Process: The action is interiorised into a process when the individual is able to think through or describe the steps without having to perform each step explicitly. For example, an individual might think of expressing a given vector in terms of two given vectors as something like: write the vector as a linear combination of the two other vectors with unknown coefficients and use the coordinate form to express this as a system of equations and then solve the system for the coefficients. This means that the individual can think of an action without specific vectors or even without specifying the number of coordinates.

Object: At this stage the individual can see the object as a totality, that is, s/he must be able to see and relate the structures that make up a linear combination (specifying the role of vectors and scalars). The individual can carry out further actions on linear combinations such as working out sums of linear combinations or determining properties of linear combination.

3.6.3 The hypothesized or preliminary genetic decomposition for linear independence

Action: At the action level, if an individual is asked to show whether a given set of vectors say $v_1, v_2 \dots v_n$ in R^n are linearly independent or not, the transformation involves a number of multiple steps that need to be followed. The term linear independence acts as an external stimulus of what needs to be done. The individual formulates a vector equation of the form

$$k_1 v_1 + k_2 v_2 + \cdots k_n v_n = 0$$

where $k_1 \;\; k_2$, $\dots k_n$ are scalars that need to be calculated. The given vectors are expressed in coordinate system and then the individual comes up with a homogenous system of linear equations in n unknowns. Suitable methods are selected to determine whether the system is linearly independent or not.

Process: An action is interiorised into a process when the individual can think of the procedures described above without specific vectors or even without specifying the number of coordinates.

The individual can make arguments based on the use of a theorem whether given vectors are linearly independent or not without performing the steps.

The individual can relate linear independence/dependence to row reduced echelon form of a relevant matrix or to the calculated determinant.

Object level: At the object level the individual can reflect on the operations applied above and becomes aware of the process as a totality.

The individual can distinguish the difference between the two methods that can be used to test for linear independence.

The individual can think of a set of linear independent vectors v_i as an entity and can use for other operations such for basis and dimension.

The individual can interpret linear independence/dependence geometrical.

3.6.4 The hypothesized or preliminary genetic decomposition for basis of a vector space

Action: At the action level, if an individual is asked to show whether a given set of vectors say R^n forms a basis, the transformation involves multiple steps. The term basis acts as external stimulus of what needs to be done. The first step is to form vector equations of the form

$$k_1 v_1 + k_2 v_2 + \cdots k_n v_n = 0 \text{ and}$$

$$k_1 v_1 + k_2 v_2 + \cdots k_n v_n = b$$

where k_1 k_2 , ...k_n are scalars that need to be calculated. The next step is to express the given vectors in coordinate system and represent $Ax = 0$ and $Ax = b$ in matrix solution, then translate to a matrix which consists of matrix A. Suitable methods are selected to determine whether these vectors are linearly independent and span R^n.

Process: An action is interiorised into a process when an individual can describe and generalize the method for finding a basis for vector space. This means that the individual can think of the procedure without specific vectors or even without specifying the number of coordinates.

The individual can reflect on how to find basis and dimension of the solution space without specific vectors.

The individual can make arguments based on the use of a theorem whether given vectors forms a basis of a vector space without performing the steps.

The individual can relate linear independence/dependence to row reduced echelon form of a relevant matrix or to the calculated determinant.

Object: At the object level the individual can reflect on the operations applied above and becomes aware of the process as a totality

The processes are encapsulated into an object if the individual can see that a set of vectors $\{v_1, v_2, \ldots, v_n\}$ form a basis for all of R^n if they are linearly independent and span R^n, adapted from Stewart (2007).

The individual will be able to carry out further transformation by extending $\{\boldsymbol{u_1},\ \boldsymbol{u_2}\ \dots \boldsymbol{u_n}\}$ to a basis of R^n.

The individual must be able to apply process or further transformation on the basis of the solution space. The individual must be able to distinguish the two methods that can be used to test for basis.

This part of this study also aims to use this data to present a revised genetic decomposition of all the concepts described above.

3.7 The theoretical framework for analysing students errors and misconceptions

The analysis of the learner's errors and misconceptions in the study is guided by the work of Kiat (2005), so as to identify the three types of errors that occur when solving problems based on vector space concepts.

- ***Conceptual errors*** are errors made because of not grasping the concepts involved in the problem or errors that arise from not understanding the relationships involved in the problem. For example consider an item where the student is asked to show that a given vector $\boldsymbol{v}$ can be written as a linear combination of a set of vectors $\boldsymbol{u_1}, \boldsymbol{u_2}$ and $\boldsymbol{u_3}$; An incorrect response was classified as a conceptual error if students were not able to identify the specific vector and/or scalar quantities forming the linear combination relationship or to correctly set up a vector equation that represents the linear combination relationship (e.g. $\boldsymbol{v} = k_1\boldsymbol{u}_1 + k_2\boldsymbol{u_2} + k_3\boldsymbol{u_3}$**).** With respect to a question which asked whether a given set of vectors, $\boldsymbol{u_1}, \boldsymbol{u_2}$ and $\boldsymbol{u_3}$ are linearly independent or not, an incorrect response was classified as a conceptual error if students were not able to consider the equation $k_1\boldsymbol{u}_1 + k_2\boldsymbol{u_2} + k_3\boldsymbol{u_3} = \boldsymbol{0}$. For an item where they were asked to show that a set was a vector subspace, responses which did not express the subspace conditions correctly were considered as conceptual errors or incorrect.
- ***Procedural errors*** are errors which arise while trying to carry out a procedure or implement a particular approach to solving a problem, despite having understood the main concepts behind the problem. In this study procedural errors refer to those responses where participants passed the first hurdle of recognising the relationship between the vectors, but where further progress was hampered, for example, in the linear combination problem

(above), a procedural error was evident in those response where students set up the original vector equation showing the linear combination expression but were not able to represent the vector equation either as a system of equations where the scalars were explicitly recognised as the unknowns. Responses where teachers were not able to get to the stage of representing the system of equations in terms of the matrix equation, that is by forming an appropriate augmented matrix, were also considered as procedural errors. For the questions requiring them to show that a set was a subspace, failure to prove all the axioms was regarded as a procedural error.

- ***Technical errors*** are not misconceptions but are due to carelessness while carrying out an appropriate or relevant procedure or because of a misapplication of a previously learnt procedure or concept that is being applied in the new concept. Godden, Mbekwa and Julie (2013) further reports that these errors are a result of slips or silly mistakes that learners make. In this study some examples of technical errors were those that arose in the process of solving the systems of equations or in reducing the augmented matrix. Some errors were due to slips or carelessness or the incorrect procedure of solving systems of equations while others were related to the misinterpretation of solutions to the system of equations and so on.

It is often the case that errors may be a result of persistent misconceptions which may be related to concepts encountered beforehand or concepts that they may have learnt later. De Lima and Tall (2008 p.6) define a 'met before' as "a mental construct that an individual uses at a given time based on experiences they have met before" which forms part of an individual's concept image. 'Met befores' may impact existing learning either positively or negatively and are seen as impediments to learning if they are used outside their domain of validity. This leads to the development of what Sfard (1992) refers to as the pseudo-structural conceptions. The authors added that new knowledge can also disturb the way in which old knowledge is perceived because experiences at a later stage can also disturb the memories of prior knowledge. Like any other newly introduced concept, the misapplication of 'met befores' can interfere with the learning of new vector space concepts and can often be identified in the errors and misconceptions displayed by students.

3.8 Conclusion

This chapter begun with a discussion of what a theory of learning is in mathematics. This was followed by the discussion of the research framework for this study. The APOS theory is the theoretical framework that underpins this study and it was discussed at length. The instructional design, the genetic decomposition of the various vector space matrix algebra concepts, was discussed, together with the various mental constructions expected. Kiat (2005) construct was engaged to illuminate the nature of students' errors. The chapter presented the elementary preliminary genetic decomposition which is what the study started with. As the study progressed, each of the genetic decomposition changed dramatically. The revised genetic decompositions that emerged as a result of the insights gained from the analysis. The differences between the preliminary and the revised genetic are presented in chapter ten. The next chapter gives an outline of the research methodology.

CHAPTER FOUR

RESEARCH DESIGN AND METHODOLOGY

4.1 Introduction

In this chapter, the critical research questions are introduced. The chapter presents the study methodology used to gather data to answers the research questions. The methodology is comprised of the discussion of the concept qualitative research, research design, research paradigm, study context, sampling procedures, data generation instruments, data analysis procedures, as well as trustworthiness and ethical consideration which are discussed. Thereafter I highlight the limitations of the study before concluding the chapter.

4.2 Critical research questions

The study proposed to use the APOS (Action, Process, Object, Schema) approach in explaining the undergraduate students' mental constructions in the learning of the vector space concepts. The study also sought to unpack the difficulties that students experience when learning vector space concepts. A genetic decomposition was suggested, and the study attempted to use the genetic decomposition in detecting whether the students were able construct the relevant mental constructions when learning vector space concepts. The following questions will be answered by this research study:

- What APOS mental constructions can be inferred from the students written and verbal responses to questions based on vector space concepts?
- What are some cognitive difficulties encountered by the students when trying to construct the necessary vector space concepts?
- How can the preliminary genetic decomposition be revised to take into account the students' learning experiences?

4.3 Research design

The study used the interpretive research paradigm in order to interpret data on how students construct knowledge when learning vector space concepts. This interpretive paradigm is suited for the study since the main viewpoint of the study is to describe the specific mental constructions that the undergraduate in-service teachers make so as to develop their understanding when learning the vector space concepts. According to Antwi and Hamza (2015) the interpretivist paradigm is concerned with an individual's understanding of the world around them,. This is in line with Cohen, Manion, and Morris (2011) who asserted that the interpretive paradigm is characterized by modeling the works from the human perspective. It aims at understanding the experiences of participants. .

The methodological framework adopted for this study is a qualitative approach. Denzin and Lincoln (2011) indicated that qualitative research is a method of data collection which attempts to build a holistic and narrative description so as to report the researcher's understanding of social or cultural phenomena. Creswell (2014) points out that qualitative research is an approach that is used to address difficulties that involves a person carrying out the study to explore and understand the meaning of a phenomenon by relying on the views of the participants. This study aimed to explore the students' understanding of vector space when learning vector space concepts

In this study the researcher used both inductive and deductive analyses. The researcher identified codes and these were grouped into categories. The researcher becomes aware of themes and patterns that were emerging from the data. These themes were analysed using the genetic decomposition in section 3.8. The genetic decomposition is a model that consists of a mental construction that a student can make in order to develop conceptual understanding of vector space concepts.

The writer has utilized a case study research design to address the research questions. The case study design allows the use of multiple methods of data collection for triangulation purposes to obtain corroborating evidence (Johansson, 2007; Lincolin & Guba, 1985; Miles & Huberman, 1944; & Yin, 2003). The variety of evidence is obtained from different sources which include artefacts, documentary evidence, interviews and observations (Rowley, 2002). In this study the methods that were used were document analysis and interviews.

4.4 Context of the study

The study was conducted at one of the universities in Zimbabwe. In 2015, the Zimbabwe government embarked on an intervention in partnership with the United Nations Child Care and Food Security (UNICEF), which is a large global funding organization, to upgrade the qualifications of practicing science and mathematics teachers. Teachers were selected by the ministry of education and thereafter registered with the particular university to study a Bachelor of Science Education Honours degree, or Bachelor of Education relevant to their area(s) of expertise, which is offered over a period of three years. The mode of learning delivery was open and distance education. These in-service mathematics teachers were holders of a diploma in Education from the various teachers' colleges in Zimbabwe, who did not have degree qualifications. Thus these teachers were enrolled in a part-time in-service course that was designed to upgrade their qualifications so that they could attain degrees. The design of the programme was such that the teachers would complete the equivalent of an undergraduate three-year degree programme except that the lectures were offered in two intensive block sessions for each semester.

These teachers were qualified to teach mathematics up to grade 7 for those who attended the primary teachers training. Those who were trained to teach at secondary level were required to teach mathematics up to Ordinary level. If they attain the degrees, then they are now qualified to teach mathematics up to Advanced level. At the time of the study, the teachers had already completed a first course in linear algebra and calculus and were engaged in a second course in linear algebra that included the concepts of vector spaces, subspace, linear combinations, linear independence, basis, dimension, linear transformation and diagonalisation, eigenvalues and eigenvectors as well as solving systems of linear differential equations. However, it was noted that this module was taught concurrently with a module on mathematical discourse and structures. This module on mathematics discourse and structures introduces students to the concepts on sets and relations, operations and structures, logic, mathematical proofs, and numbers.

The Faculty of Science in this university is fully functional and adequately resourced where effective teaching and learning take place. Hence satisfactory research activity was carried out without fear of possible unexpected disruptions.

4.5 Participants

Sampling is the process of thoroughly selecting that which will be examined during the course of the study. 73 participants were selected for the study. The study used purposive sampling. Purposive sampling is a feature of qualitative research whereby researchers select cases that are accessible and have in-depth knowledge about a particular issue (Cohen, Manion & Morris, 2007). In this study the participants were the in-service teachers studying for a Bachelor of Science Education Honours degree in Mathematics at a chosen university in Zimbabwe. The researcher works at that institution such that she has easy access to the participants. These in-service teachers were studying the second module where the vector space concepts are embedded and have already looked at the prerequisite concepts on matrix algebra done in the first course. This contention suits Kombo and Tromp (2006) who advise that the choice of purposive sampling should lie in choosing information-rich cases. This means that these first year students were chosen for relevance to the topic under investigation.

It is important to note that the participants include male and female and are holders of different academic qualifications at elementary level. Primary level in Zimbabwe is from ECD and is followed by grade one to grade 7; that means primary level education is completed in 8 years. After completion of primary education, learners proceed to secondary school where they spend four years doing ordinary level. An Ordinary level holder must pass at least five subjects including English. If they wish, they can proceed to Advanced level and have areas of speciality, i.e. one can choose to do sciences, commercial subjects or arts subjects, and this is done over 2 years. The choice of subjects is dependent on one’s performance at ordinary level. Students do a minimum of three subjects at Advanced level. To enter the primary teacher’s college for training, one must have a minimum of 5 O levels including English and Mathematics. To enter the secondary teacher’s colleges, some colleges have embarked on post O levels and some post A level criterion. For those with ordinary level mode of study the course duration is 3 years. The candidate must have passed English and Mathematics at Ordinary level, so that he or she can major in mathematics. For those who embark on the Advanced level mode of study, the course duration is 2 years, and for one to be able do mathematics, a pass in A level mathematics is a must as well as a pass in English at Ordinary level. For one to pass at A Level one must have at least grade E or better. Table 4.1 shows the distribution of the 73 participants by gender, academic qualification, and level taught.

Table 4.1 Distribution of the 73 participants

		Female	Male
Academic Qualification	Advanced Level	25	27
	Ordinary Level	14	7
Level Taught	Primary	0	1
	Secondary School	39	33

4.6 Data Generation methods

Students are then given three structured worksheets to work individually, followed by semi structured interviews of the in-service mathematical teachers so as to capture insights into their experiences and understandings gained through the activity sheets, and make mental structures suggested in the genetic decomposition. For the sake of anonymity, pseudonyms were used. The participants were coded using tags 'T1', 'T2' and so forth, where the order did not have any significance. This was done so that the responses of the in service teachers could not be linked in publications to the original participants, while enabling an organization of the data. The following instruments were used for data collection:

- Structured individual activity sheets for in-service mathematics teachers.
- Interviews (individual) of in-service mathematics teachers.

4.6.1 Structured activity sheet

The study employed the use of a structured activity sheet to generate the necessary data which helped the researcher to understand the mental constructions that the undergraduate mathematics students make when learning vector space concepts. The activity sheets for the main study were divided into 3 sections and the students solved these problems individually. The problems set were similar to the problems covered in class. Note that the words items and questions are used interchangeably in this thesis where both denote the same thing.

4.6.1.1 Items based on vector space and subspace

The questions based on the vector space and subspace are taken from activity sheet 1 which consisted of nine questions. One of the questions distinguished the difference between a subspace and a vector space. Furthermore, two questions tested the aspects of a vector space and four questions tested the aspects on subspace. Two of the questions tested process conception and seven of the questions tested the object conception according to the APOS theory. The questions of activity sheet 1 appear in Appendix A. Questions 4 and 7 are covered in chapter 5 and questions 4 and 7 are also covered in chapter 9 as well as question 2.

4.6.1.2 Items based on linear combination

The questions based on linear combination are taken from activity sheet 2 which consisted of 8 questions. One of the question required students to give the difference between linear combination and spanning, 4 questions tested about linear combination of vectors and 3 questions tested about span of vectors. The question on the difference between terms linear combination and spanning (question 1) tested the object understanding of the concept. Questions 2 , 3 5 and 6 tested the process level understanding of linear combination whilst questions 7 and 8 tested the object conception of understanding. The questions on activity sheet 2 appear in Appendix B. Questions that is 1, 2, 7, and 8 are covered in chapter 6 and questions 1, 2, 7 and 8 also appear in chapter 9 as well as questions 3, 5 and 6.

4.6.1.3 Items based on linear independence, basis and dimension

The questions based on linear independence, basis and dimension are taken from activity sheet 3 and consisted of 12 questions. Six of the questions tested the concepts on linear independence of vectors and 6 of the question tested on basis and dimension of a vector space. On question 1, the analysis considered the definition of linear independence, which tested the object conception according to APOS theory. Questions 2 and 6 tested the process understanding of linear independence whilst question 3 and 8 tested the process understanding of basis. Five questions tested the object understanding according to APOS theory that is question 1, 5 and 7 tested the object understanding of linear independence and question 9 and 12 tested the object understanding of basis. The other three questions that is 4, 10 and 11 were not discussed here. The questions on activity sheet 2 appear in Appendix B. Questions 1, 2, 5, 6 and 7 are covered in chapter 7 and Chapter 9, whilst questions 3, 8, 9 and 12 are covered in chapter 8 and chapter 9.

While administering these activity sheets, I made sure that I explicitly gave relevant instructions to the students, especially on making sure that they justified their solutions. I also made sure that the students would not copy each other's work or give guidance to their colleagues. This was done to ensure that data generated was a reflection of students' own efforts at making the necessary mental constructions. A pseudonym was used to identify all the written work collected as data on each activity sheet. The first two activity sheets were written in one hour 30 minutes each to make sure that the students had enough time to demonstrate their conceptual understanding of vector space, subspace and linear combination. Tutorial work sheet 3 was written in 2 hours since it was a little longer than the other two. These were administered on three separate days so as to give them time to revise the concepts. This was in a bid to see whether the mental constructions that the in-service teachers made were linked with the genetic decomposition.

4.7 Interviews

This study employed the use of semi-structured interviews. Kvale and Brinkmann (2009) noted that in this kind of interview, the researcher prepares beforehand a set of questions to know more about specific issues, and sometimes identifies new issues that were not originally part of the interview. The interviewer explains the purpose of the interview and stresses the issues of confidentiality and anonymity. Semi-structured interviews are more flexible and generate more useful data. It also gives the respondent flexibility and freedom in deciding what needs to be described or argued, and how much explanation to offer (Pathak & Intratat, 2016; Cresswell, 2012). Creswell (2014) argues that one to one interviews are useful because they give interviewees the chance to ask questions or even go beyond the proposed questions.

Descombe (2014) points out that the most common feature of semi-structured or unstructured interviews is the use of the face to face interview where two people, that is the interviewer and the interviewee, are involved. He further outlined four advantages of using this method of data collection. He advocated that it is relatively easy to arrange because only two people should meet. He further believes that the opinions and the views only come from one source, that is the interview, and this is easy to control since the interviewer will be dealing with one person meaning that s/he only needs to grasp and interrogate his/her ideas through the process. Finally, the author believed that it is easy to transcribe the data of only person at a time.

The task-based interviews were implemented in conjunction with written tasks. An in-depth analysis of the activity sheet was employed before conducting the interviews. Task-based interviews were used to explore the undergraduate mathematics students understanding of vector space to gain more insight about the ways in which students construct the appropriate knowledge when solving the vector space concepts. Above all, the interviews were directed towards finding out more about the skills which the students employ when solving mathematical problems on vector space concepts. The main aim for conducting the interview was to explore the mental construction made by the students when learning the concepts on vector space, subspace, linear combination, linear independence, basis and dimension.

Another reason for using task- based interviews is that these enable researchers to find out the methods used by students, difficulties that they encountered and to see misconceptions that they manifested. 18 students volunteered to participate in the interviews. This large number was necessary to cater for those who might withdraw since they were aware that any person was given the room to withdraw from the study at any time they wished since participation was voluntary. However, 5 of the students did not turn up for the interviews for various reasons. The major reason they outlined was mainly the time factor. Their lessons were heavily packed due the nature of their programme as indicated. I noted that within the group of volunteers there were those who performed very well, average performers and below average performers.

The interviews were conducted in four weeks that is from mid-August to the first week of September, and also in the last week of November when the students came for revision lessons in preparation for the November/ December examinations. In-order to obtain rich descriptions of the work covered, I made sure that I first addressed the easier questions at the beginning of the interview session. I also made an effort to set the scene appropriately by first explaining to the respondent the purpose of the research, and the likely duration of the interview session, and that they were free to ask questions. Cohen, Manion and Morris (2011) and Denzin and Lincoln (2011) highlighted that it is the responsibility of the interviewer to establish rapport with the participants so as to obtain authentic data. I tactfully probed the students responses, followed up students' ideas and asked open ended questions flexibly so as to maintain the flow and at the same time getting a better understanding of how the undergraduate mathematics students construct their knowledge while learning the vector space concepts. Probes are an essential feature of semi structured interviews in the sense that that the respondents are given a chance to clarify their response and add more to the solution thus addressing the issues of comprehensiveness of the response (Cohen, Manion & Morris, 2011; Creswell, 2013). This can be in the sense that a researcher can ask for a clarification of a concept that is not expressed well, or repeating a question. Other questions posed elicit the kind data that I sought is useful in understanding the level of APOS conception they are in for example I asked the students explain how they can express a given vector as a linear combination of the other vectors. Here I wanted to check whether the students had developed the process level understanding of linear combination. This question was more of an open question since the student could come up with the various methods that can be used to go about the question. Patton (1990) advocated for the use of the "how" and the "why" questions when conducting structured interviewing and emphasized that the best thing is to start with the 'what' questions as he believes that the 'how' and 'why' questions are more difficult to comprehend.

The main objective of interviewing the in-service mathematics teachers was to probe the thinking and reasoning behind their responses to the three activity sheets. Each interviewee was interviewed after checking the performance of the written activity sheet and asked to explain some of their responses to selected questions. I noted the areas of difficulty and reasons for failing to make the necessary mental constructions suggested by the genetic decomposition during the interview session. Whenever I posed a question, I would give the participants more time to develop their ideas and elaborate their responses. I also used some prompts to clarify the questions that some of the participants seemed not to have understood.

During the interview sessions, the data was video as well tape recorded with permission from the participants. This was done after obtaining appropriate consent from the students. For the sake of anonymity, the participants were coded using tags 'T1', 'T2' and so forth, where the order did not have any significance. This was done so that the responses could not be linked in publications to the original participants, while enabling an organization of the data. Since participation was voluntary, five opted not to participate, leaving 13 (T4, T7, T13, T21, T23, T25, T27, T33, T44, T57, T62, T63, T69) who responded to the interview invitation.

4.8 Data analysis method

Descombe (2014) noted that when carrying out data analysis, the conclusions should be firmly rooted in the collected data so that the meanings are drawn from the raw data. The author further outlined that data analysis should involve an iterative process whereby, in order to develop a concept or a theory, the researcher need to compare empirical data and categories formed. Descombe (2014) also commented that data analysis involves inductive logic, where-by the logic of discovering things involves moving from the data to the theory and to a more generalized conclusion. Thus in order to develop a concept the researcher should move back and forth comparing the empirical data with categories.

Thus data analysis was mainly based on identification of themes, patterns, similarities and differences, and these were used for the organisation and presentation of the results. The other data from the written exercises was coded and this was done according to the various levels of mental construction that the students made in the learning of vector space concepts. The coding system of Asiala, Cottrill, Dubinsky and Schingendorf (1997) was imitated to evaluate some of students' responses and is as follows: categories for no response, category for responses that show some progress towards solution but far from the correct solution, category for almost correct responses with minor flaws in the solution, and category for totally correct responses.

To add to the categories formed, an in-depth content analysis was carried out. The participants' areas of difficulties were noted and errors and misconceptions were identified in order to answer research questions. Kiat (2005) constructs was used to analyse the data. Data analysis was also accompanied by images of the students' written work so as to generate rich data. Some of the transcripts of the interviews were also analysed so that the findings could verify the written work. The analysis was supported by the preliminary genetic decomposition which was part of the theoretical framework. In addition, coding of responses from three activity sheets was done according to different levels of mental construction making and conceptual development.

4.9 Validity and reliability

The issues of validity and reliability are considered since the study is qualitative in nature. Patton (1990) pointed out that in order to judge the quality of the study the issues of validity and reliability must be addressed, so that the research findings are worth reading. According to Cohen, et al. (2011) reliability and validity assess the credibility of a research. The major goal of reliability is to minimise errors and biases in study. To assure validity and reliability of the instruments in this study, a pilot study was conducted. Leon, Davis, and Kraemer (2011) outline that a pilot study is conducted before the intended study so as to provide the researcher with ideas and hints that he/she may not have seen before the main study and to try to examine the viability of the study. The pilot study was conducted for validating the activity sheets for the purpose of the main study.

4.10 Trustworthiness

To ensure reliability and validity in qualitative research, Guba and Lincoln (1994) asserted that consideration of trustworthiness of a study is essential in evaluating its worth. Trustworthiness is a term used in qualitative research and it refers to the extent the research is credible, dependable, confirmable and transferable (Bertram & Christiansen, 2014; Lincoln & Guba; 1985, LeCompte, 2000). It describes the extent to which data analysis are believable. Bertram and Christiansen (2014) commented that triangulation can be used to increase trustworthiness. This study was strengthened by data triangulation so that research findings are considered worthwhile. This is in line with Patton (2002) who asserted that triangulation toughens a study by using different methods of data collection such as interviews, observations and recordings so that it is valid and reliable. Thus I enhanced trustworthiness by employing the following data collection methods, interviews, and document reviews in the form of students' written work.

The naturalist believed that to ensure reliability trustworthiness is important, and to ensure validity, rigour, quality and trustworthiness is called for (Stenback, 2001). The rigour in this study was achieved by using the criteria suggested by Guba and Lincoln (1985) to address Guba's criteria for trustworthiness and is summarised in Table 4.2 below.

Table 4.2: Criteria to enable trustworthiness in the study

Quality Criterion	Criteria	Possible provision made by researcher
Credibility	Prolonged field experience Triangulation Interview Technique	Triangulation- I used different methods of data collection such as document analysis of written responses from three activity sheets and interviews (individual and focus groups). This showed an adoption of appropriate well organised research methods. The semi-structured interviews were audio and video recordings. Throughout the research pseudonyms were used. This was to ensure that data analysis was believable and trustworthy.
Transferability	Dense description	Detailed description of phenomena involves quotes from interviews and participants' written responses from three activity sheets on vectors space, subspace, linear combination, linear independence, basis and dimension. These were scanned and detailed descriptions were given.
Dependability	Dependability audit Triangulation	In-depth methodological descriptions were given, that is, transcripts of interviews, document analysis of students' written responses and detailed explanation of coding system used were given. This was done at each stage of data collection and analysis.
Conformability	Confirmability audit Triangulation	To reduce investigator bias, at any stage we can go back and confirm results from the semi-structured interviews and written responses. Even though we use pseudonyms we can go back and confirm our results. At any time, I can trace back our original data.

4.11 Ethical issues and limitation of the study

Marshall and Rossman (2011) asserted that the privacy of individuals and protection in directing a research report of their identities becomes vital. In carrying out this study the issues of ethical considerations were considered. These conform to the ethical requirements of the University of Kwazulu-Natal's ethical committee. To address the ethical issues properly, permission was granted by the Dean of the Faculty of Science of the institution where the research was conducted. A letter of confirmation was granted by the human resources; see the Gatekeepers letters, Appendix E. Furthermore, an ethical clearance from the university research committee of UKZN was granted before the commencement of the study (see Appendix F).

The major key ethical issues that I considered in the study were informed consent and confidentiality issues. The researcher approached the students who were enrolled for the Linear Algebra 2 class and explained the proposed research project. I also explained that participation was voluntary. Participants had a choice to participate, not participate or stop participating in the research. They were then asked if they wanted to participate and informed consent letters were given to them. They could then choose if they wanted to participate or not. Each participant was provided with his/her signed consent form to sign. A copy of this letter is shown in the appendix section. Furthermore, the students were assured that the data collected would only be used for the purpose of the study. To guarantee their anonymity the researcher used pseudonyms. The participants' responses were conserved with strict secrecy, and this is supported by Denzin and Lincoln (2011) who commented that one of the moral measures that must be taken in any study is to respect the respondents' rights and secrecy. Thus they were free to withdraw at any time from the study without any drawbacks, and would not be penalized. Participants were also assured that data collected would be stored in the university and would only be used for the purpose of the study.

As I carried out the study, one limitation I had was that of time. This was a result of the nature of the block release programme that was only done during the holiday period. The in-service teachers did not have enough time since some groups did not manage to finish tutorial activities, and time to do homework was limited because their time table was heavily packed, learning from 8 am to 6 pm on week days as well as on Saturdays. Another limit was that the study was a case study carried out at just one university in Zimbabwe with 73 participants. This means that it is not possible to generalize the findings of the study to all the universities in Zimbabwe. However, an attempt was made to ensure that the findings are credible and trustworthy, and the data gathered will be informative enough to the students studying the vector space concepts, lectures and mathematics community

4.12 Conclusion

The chapter focused on the methodology and instruments used in the study and this was covered in detail. The chapter started with an overview of the critical research questions and a discussion of the research design and the research paradigm used in study were detailed and justified. I discussed the sampling techniques, suitability of the participants and the context of the study. The data collection instruments and the procedures of data collection were discussed at length in the chapter. Furthermore, I considered issues pertaining to validity and reliability of the findings, trustworthiness and ethical considerations in line with qualitative research approaches. In the next chapter I will present the findings and the analysis of the vector space concepts.

CHAPTER 5

ANALYSIS OF STUDENTS' RESPONSES TO ITEMS BASED ON VECTOR SPACE AND SUBSPACE

5.1 Introduction

This chapter reports on the analysis of students' written responses to activity sheet 1. The data was collected from the teachers' written responses to an activity sheet consisting of nine items which were intended to probe their understanding of vector spaces and vector sub-spaces. The analysis is based on the mental constructions indicated in the genetic decomposition in chapter 4. Some students volunteered to be interviewed and the interview transcriptions are presented. The purpose of the interviews was to develop a deeper understanding of the ways in which the teachers responded to the tasks, and this was done in an effort to offer more understanding on how they constructed the various mental structures. The chapter is based on the published paper by Mutambara and Bansilal (2018). To ensure that the discussion of the students' responses makes sense, I present some of the definitions and theorems that commonly appear in this discussion. The definitions for vector space and subspace appears in section 1.10.

In order to construct the concepts of a subspace the items required the application of the following theorem: Theorem 5.1: W is a subspace of V if and only if:

(i) W is non empty (or: $0 \in W$)

(ii) For any $u, w \in W$, W is closed under vector addition meaning that $v + w \in W$.

(iii) For every $k \in K$, and $v \in W$, W is closed unders scalar multiplication meaning that $kv \in W$.

Thus the responses of the students are analysed and presented and this is based on the genetic decomposition presented in section 3.8.1.

5.2 Analysis and discussion of data

In this chapter we focus on the responses to two tasks that were based on the vector space consisting of 2×2 matrices over the real field $\mathbb{R}$, and which were considered as having different levels of demand: one task required the teachers to confirm that a subset is a vector space and the second required them to show that a given set was not a subspace. The two tasks based on the vector space of 2×2 matrices appear in Table 5.1, together with comments.

Table 5.1: Research tasks

Question	**Comments**
4. Let **V** be the vector space over of all 2×2 matrices over the real field ℝ. Show that **W** is not a subspace of **V**, where **W** consists of all 2×2 matrices which have a zero determinant.	For this, teachers were expected to find a counter-example to show that the set W is not closed under vector addition.
7. Show that the set of all $M_{2\times2}$matrices of the form $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ is a vector space.	For, this teachers could argue that since $M_{2\times2}$ is already a vector space, then it was only required to show that the given subset formed a vector subspace of $M_{2\times2}$. Alternatively, teachers could show that the ten axioms for a vector space were satisfied.

5.3 Results for question 4

Question 4 required the teachers to generate a counter-example to show that the set of 2x2 matrices with a zero determinant is not a subspace. Question 4 was intended to provide insight as to whether teachers were on the path of developing strong schema of vector spaces. However, most teachers' responses to this question showed that they had considerable difficulties with the formal reasoning required to present an argument why the set W did not fulfil the condition of being a subspace. To do that they needed to understand how counter-examples function in the process of rejecting conjectures (Zaslavsky & Ron, 1998; Bansilal, 2015).

The overall analysis for question 4 showed that 16 (22%) of the teachers did not attempt the question, while 15 (21%) had completely incorrect responses. These teachers were quite lost in the task; 27 out of 73 (37%) of the teachers attempted to add two matrices with zero determinants and showed that the sum had a zero determinant. Most of them proceeded to show the closure property of multiplication, and made various conclusions, many of which were incorrect. We now present more detail about some of the issues that emerged from the analysis of the written responses and some interviews.

5.3a) Comfortable engagement with the higher abstract layers of reasoning

Some teachers' responses suggested that they were comfortable with the reasoning required at the higher abstract layers. Seven students were able to present an example that did not satisfy the closure condition for vector addition. An example of such a response was that given by T6, who chose the two matrices: $\mathbf{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, and wrote (ii) $\mathbf{x} + \boldsymbol{y} \in V$: $\mathbf{x} + \mathbf{y} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \notin V$.

She concluded that the determinant of the sum $\boldsymbol{x} + \boldsymbol{y}$ was not zero, and hence did not satisfy the condition, hence the set **V** was not a subspace. She demonstrated understanding of the concept as she displayed a clear understanding of the determinant of a matrix. The student T6 should have made a comment that the choice of the matrices denoted by x and y belongs to the set W before carrying out the verification of the closure property. The teachers' approaches and coherent arguments showed that they were comfortable with coordinating the binary operation and set processes to present arguments about why the subspace criteria were not fulfilled. These seven teachers' responses are aligned to that required by object-level reasoning about vector spaces. The students were able to coordinate the binary operation process and the set process and seem to have encapsulated the process into an object when they presented the argument about why the subspace criteria were not fulfilled. This supports Dubinsky's (1991) contention that an individual operating at the object level is able to take the process as a whole and create clear linkages between the concepts. However, most other teachers were unable to demonstrate such ease with the abstractions of the vector subspace concept and displayed different degrees of uncertainty.

5.3b) Tried to show the set was a subspace, contrary to the instruction

The analysis identified some students who used examples to show that the set was a subspace, despite the instruction to the contrary. For example, student T4 did not attempt question 4. When probed in the interview, she explained her reasoning about why she concluded that the set was a subspace. R stands for the researcher and T4 represents Student 4. Note that the dialogue is captured verbatim and language errors have been left unchanged.

R*: I understand you did not attempt this question. What exactly does this question require us to do?*

T4*: To find a matrix that gives determinant zero.*

R: Oh O K, can you give an example of such a matrix?

T4*: (Writing down)* $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ *det = 0.*

R*: So we have shown that it is not a subspace of the vector space.*

T4*: No, no, no, we find another matrix; we can use a multiple* $\begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix}$*. So the closure property u+v =* $\begin{bmatrix} 3 & 0 \\ 3 & 0 \end{bmatrix}$ *det = 0, satisfied therefore subspace.*

R*: What else? Are we done? The question said, "Show that it is not a subspace." Is the set not a subspace of the given set?*

T4: *No, No, No take positive scalar k = 2 to give* $2\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix}$ *det = 0 therefore subspace.*

The excerpt above shows that student T4 chose two vector elements whose sum belonged to the set, and took the single example as evidence that the set was a subspace. When probed further, she considered an example of a scalar multiple of a vector which also satisfied the condition on the subset. She was aware of the conditions that are needed to be satisfied by a subspace, that is showing the closure property for addition and scalar multiplication. However, she was confused about the role of examples in proving or disproving a statement. Taking an example which satisfies a condition is not evidence that the condition is always true (Bansilal, 2015). At first she may have thought that she was showing that it is a subspace, which is why she then wanted to go on to the scalar multiplication condition. The student was also not clear about the determinant of a matrix, and did not view it as a function whose input is a matrix (Donevska-Todorova, 2014) but seemed to take it as a detached calculation.

5.3c) Used an illustrative example to show that the result of the binary operations belonged to the set, but concluded it was not a subspace

T13 considered four matrices, **v**, **u, s and t,** each with zero determinants as shown below in Figure 5.1.

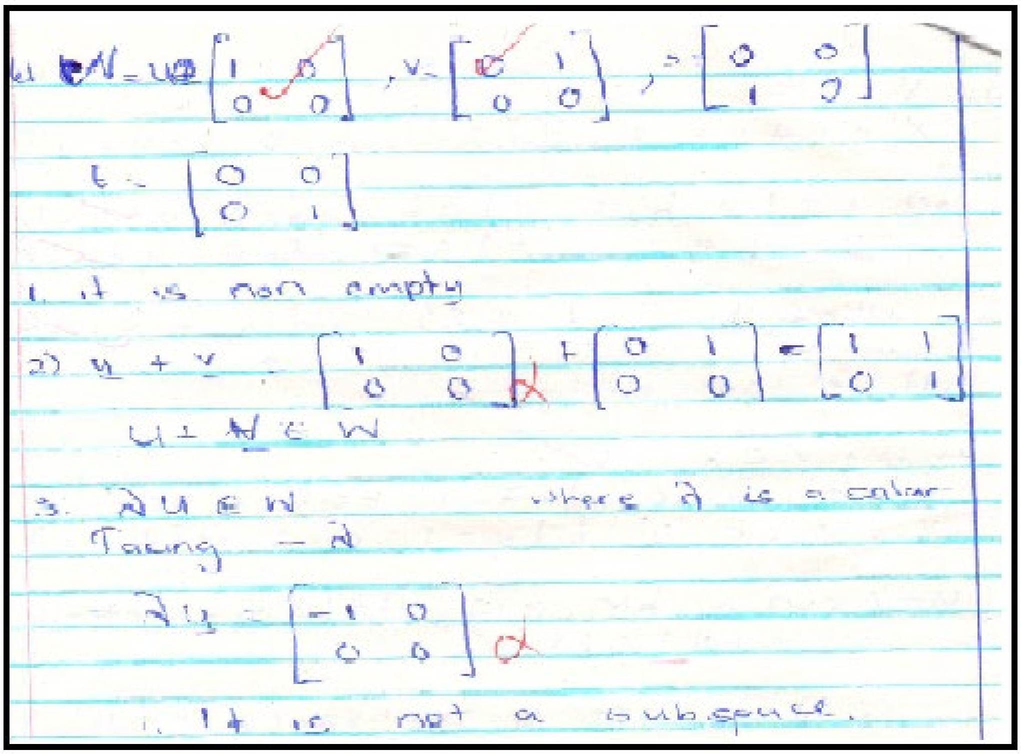


Figure 5.1: Written response of student T13 for question 4

T13 attempted to use two vectors and tried to show that the closure property of addition and scalar multiplication was not satisfied. However, for the vector addition, the determinant of the sum that she obtained was not zero, but she concluded that the sum belonged to W. The student made a computational error when trying to add the two matrices. The scalar multiple had zero determinant and these results were interpreted to mean that the set was not a subspace. T13's written response reveals her difficulty in showing that the set is not a subspace. She was just following procedures without understanding, which indicates action-level reasoning. Teaching students to engage in mathematics by applying a set of memorised algorithms is seen as hindering their mathematical procedures (Foster, 2014). In an attempt to better understand why she was struggling, she was interviewed. The interview with student T13 showed that she was still struggling to understand what the question really asked for:

R: *You are talking of two vectors,* ***v*** *and* ***u****, so why did you choose three vectors for* ***v*** *and one vector for* ***s*** *in your solution?*

T13: *If the determinant is zero it is no longer a subspace. Maybe I confused myself because I see now that I must get a non-zero determinant. So I will choose matrix* $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ *and matrix* $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$. *Determinant is zero*

R: *So what can we do?*

T13: *I have another matrix A =* $\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}$, *det = 8-1 =7. Choose another one; I think so.*

Then I will choose a negative scalar, will change for example $-2\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix}$ *and the determinant is zero. It doesn't, no it doesn't work – it is a scalar multiple.*

The interview responses showed her struggle to show that the set is not a subspace. She stated that if the determinant [of the sum] is zero then it means that the set is not a subspace. Two matrices were then chosen, one of which did not belong to the set W since it did not have a zero determinant. She added the matrices and said the determinant was zero, which was not true. She then rushed to attempt to show the closure property of scalar multiplication, saying that a negative scalar should provide the counter-example. In her written response she also used (-λ) as her scalar. Her ability to continue to use rules without reasoning is an indicator that she was still operating at the action level of understanding.

5.3d) Confused about the role of the counter-example

Some students knew they needed to produce a counter-example, but seemed not to know what the counter-example should show as shown in the written work of student T39 appearing in Figure 5.2.

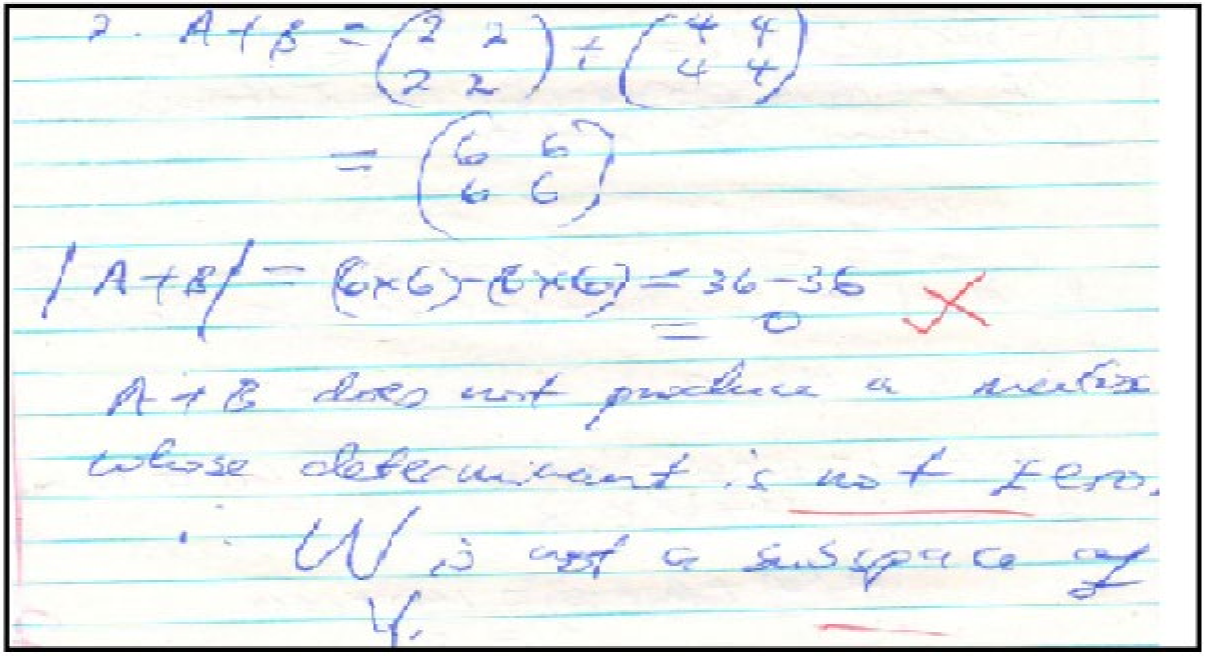


Figure 5.2: Written response of student T39 showing confused argument

He proceeded in a similar manner as explained in the interview by student T4, except at the end T39 tried to twist the result to imply that closure property of addition was not satisfied on W. After adding the two matrices, he proceeded to find the determinant which was equal to zero. He then concluded that it showed that the resultant determinant was not equal to zero and concluded that it meant it could not be a subspace. However his logic was misguided. He chose two elements belonging to the subset, added them and found that the determinant was 0, which does not indicate anything significant in this case. This suggests that he knew he was looking for a counter-example, but was not sure what the counter-example should show.

5.3e) Not able to produce an argument around the appropriate counter-example.

There were some students who presented an appropriate counter-example but struggled to produce the accompanying argument about the counter-example as shown in Figure 5. 3.

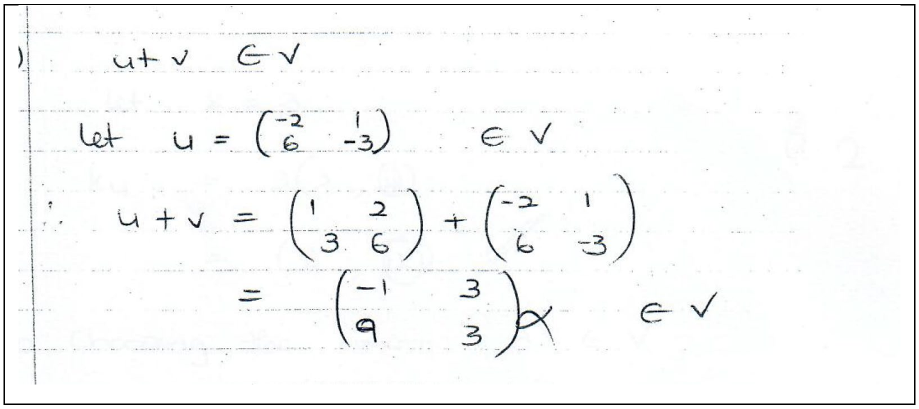


Figure 5.3: Written response of student T46 with a counter-example that was not recognised

The response of student T46 shows a counter-example produced because the determinant of the sum was not equal to zero but the student goes on to incorrectly concluded that the sum belonged to the set V.

5.3f) Chose an inappropriate counter-example

Unlike the case of students such as T46 who were able to identify suitable counter-examples, some students were unable to find an appropriate counter-example. For example, one student, T47, whose response appears in Figure 5.4, elected two elements of W and assumed incorrectly that the determinant of the sum was not equal to 0. Hence she seemed to know what was required but could not identify the appropriate counter-example to fulfil her purpose. Note too that she did not mention explicitly the determinant of the sum, suggesting that she had a limited understanding of the determinant of a matrix and did not see it as a function which acts on a matrix.

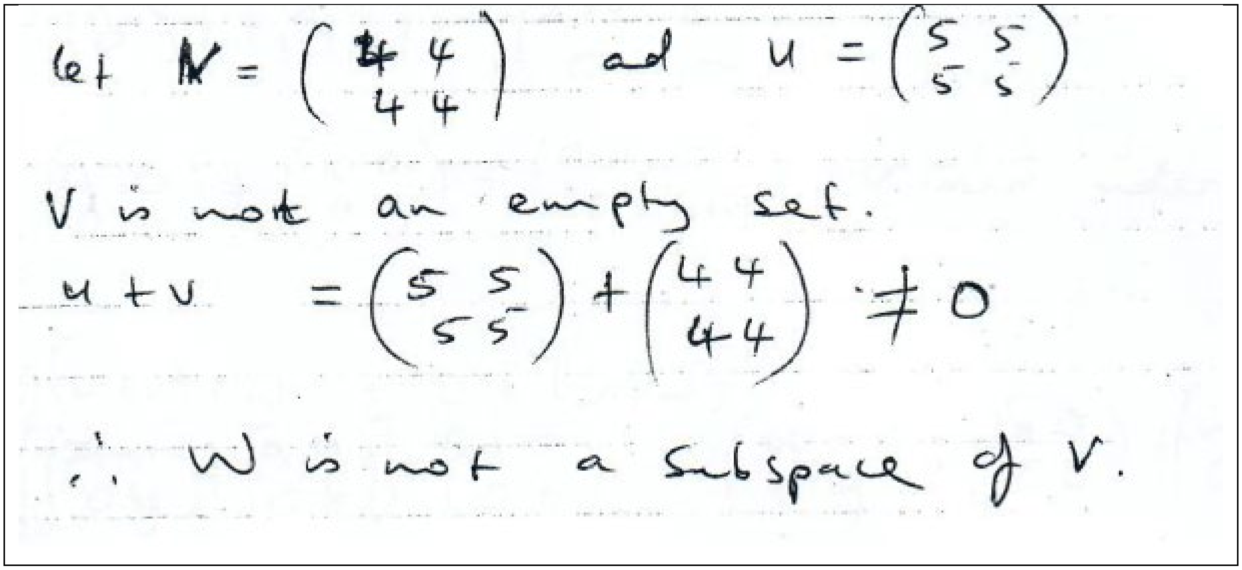


Figure 5.4: Written response of student T47 of matrices with zero determinants

The response of student T47 above shows that she added two matrices belonging to W that were made up of identical entries. The determinant of the sum was zero but student T47 assumed that the determinant was not zero, allowing her to conclude that the set was not a subspace. This shows that student T47 knew what she wanted from her counter-example but was not able to find the appropriate counter-example with the required property. The student T47 did not use the determinant notation. However other students were not even clear about what they wanted to accomplish, as shown by student T7 in Figure 5.5.

5.3g) Uncertainty about what the counterexample should do.

Many students were not clear about the role of a counter-example. The response from student T7 shows a matrix with entries 1, 2, 3 and 4 and the student shows that the determinant of the matrix is not equal to zero. That is, he produces a 2×2 matrix, **u** which does not belong to the given set, and then shows that the determinant of **u** $\neq 0$.

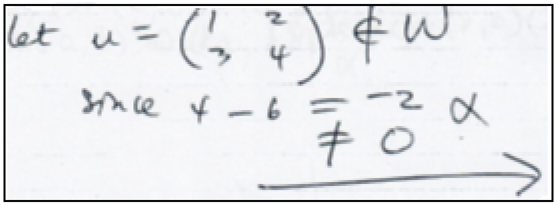


Figure 5.5: Written response of student T7 showing uncertainty

5.3h) General confusion and lack of common content knowledge

Some of the students, such as T69 and T12, produced responses which were unrelated to the questions, as shown in Figures 5.6 and 5.7.

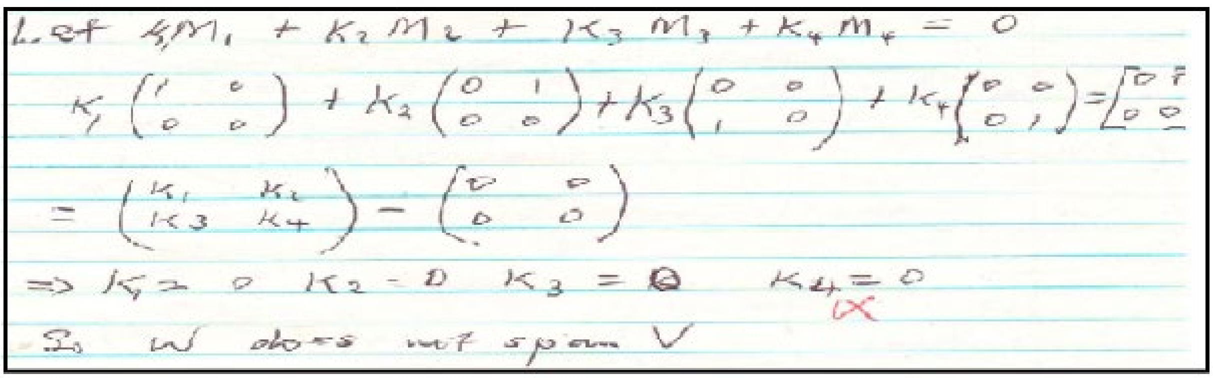


Figure 5.6: Written response of student T69 showing linear independence

T69 incorrectly interpreted the problem and applied wrong procedures to solve the problem. He used the aspect of finding linear independence and concluded that it does not span the space. The answer given indicated that he saw some connection between subspaces, linear independence and spanning, but was not clear about how they are connected.

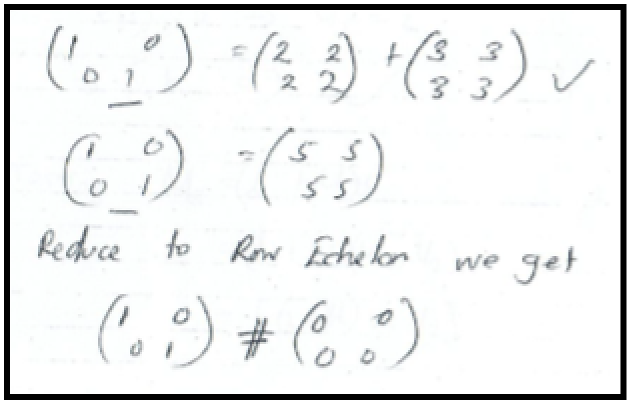


Figure 5.7: Written response of student T12 trying to use row echelon form

The response of student T12 shows that he had considered a particular type of 2×2 matrices that have identical entries. These matrices belong to the given set, because their determinants are zero. The sum of the matrices does satisfy the condition of having a zero determinant, but the student was confused about what he was trying to do. He seemed to be trying to show that the sum should be equal to the identity matrix. He was also confused about equal matrices and brought in the aspect of reducing to row echelon form. These misconceptions had accumulated in a number of areas and emerged when the students were asked a question that required object-level reasoning about a vector space.

5.3.1 APOS insights from the responses to question 4

Some students, such as studentT46, showed some progress towards interiorisation of the process of checking the subspace axioms, but struggled with articulating the arguments about why the set was not a subspace. However, most of the students had fundamental problems relating to the prerequisite schema for binary operations and set. The students did not seem to have developed object-level conceptions of those prerequisites and hence could not cope with the demands of the task which required them to identify that the closure property of the binary operation on the set of 2×2 matrices was not fulfilled. Showing that a condition is not fulfilled requires sophisticated reasoning and arguments and this is not available to those who have not moved past a process conception of all the axioms. Many of these students were limited to carrying out procedures in a step-by-step manner. Dubinsky (1991) asserts that the ability to carry out procedures is at the action level of the APOS theory. The lack of the prerequisite construction of the set schema and binary operation schema hampered the students in developing a sufficiently strong schema for a subspace.

5.4 Results for question 7

There were six students who went through each of the ten axioms and showed that they were satisfied by the elements of the given set. Three (3%) students presented totally incorrect responses, indicating that they had no idea of what was expected in this task. It seemed that these students had not reached the action level using the genetic decomposition as they were still operating at the pre-action level. A further 12 (16%) of the students were able to identify and tried to prove some of the axioms, indicating action-level engagement with the set of 2×2 matrices. Fifty-three (73%) students attempted to show that the ten axioms for a vector space were satisfied; however, most of them had problems identifying exactly what they wanted to show for different axioms. There are some pertinent issues that emerged in the analysis related to how the students solved the problem. We identified five issues that emerged in the analysis of the responses to question 7, and these issues are discussed below.

5.4a) Difficulties in recognizing what needed to be shown for particular axioms

Question 7 was intended to provide insight as to whether the students had developed a coherent vector space schema. However, some of them had problems identifying exactly what they wanted to show, for example the response by student T11 indicates the student's attempt at showing that the set is closed under vector addition (Figure 5.8).

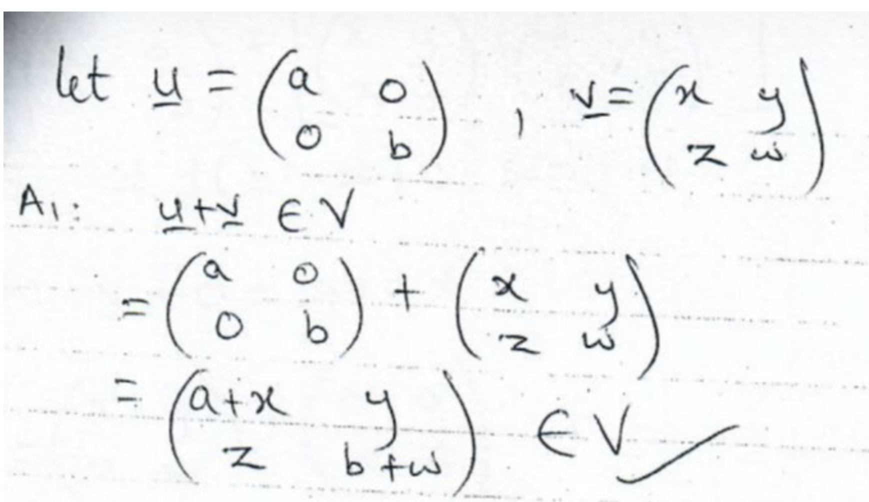


Figure 5.8: Written response of student T11 using one element from the set and another general 2×2 matrix

The response of student T11 shows that the student found the sum of an element of V and another arbitrary 2×2 matrix and then concluded that the sum belongs to V, without considering whether it satisfied the condition for elements to be in the set, similar to the finding reported by Britton and Henderson (2009) but which was for a different vector space. On the choice of the set V, this student seemed to be reproducing an example done in class. In terms of APOS, her actions of carrying out the binary operation for addition had not been interiorised into a process.

5.4b) Using specific elements to illustrate axioms

There were many students who considered specific elements from the set, and showed that they satisfied the conditions of the axioms. The response from student T8 in Figure 5.9 shows such a response.

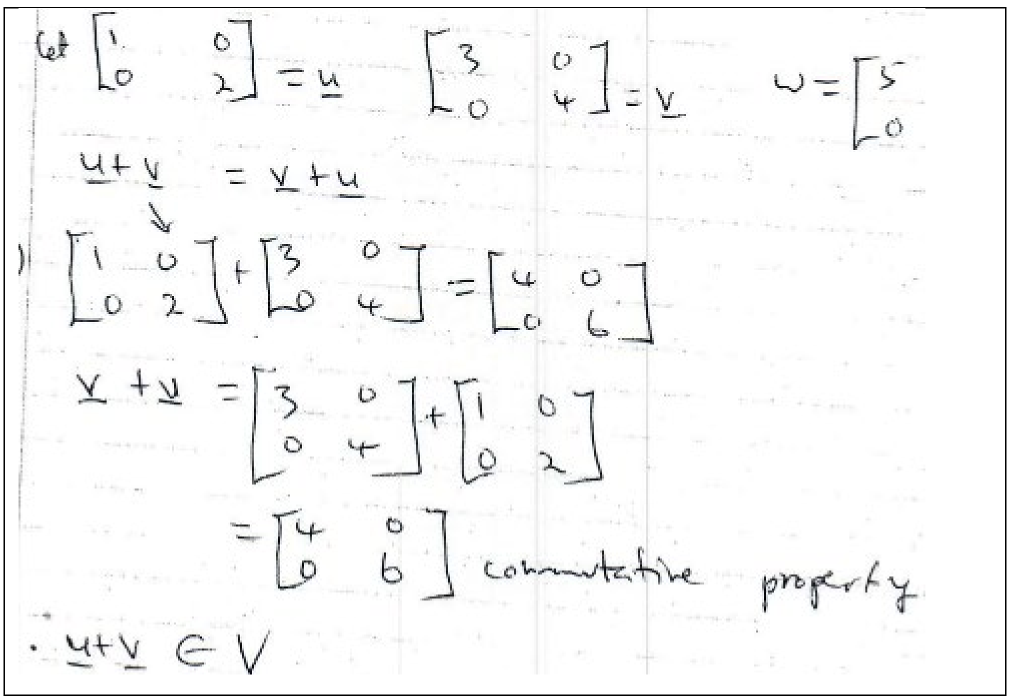


Figure 5.9: Written response of student T8 considering only specific elements

Figure 5.9 shows that the student had tested the axioms for specific elements of V, instead of considering generalised examples, an approach which was also identified in the study by Britton and Henderson (2009), using a different vector space. Furthermore, student T11 was confused about the relationship between the property of commutativity of the binary operation and that of closure because he concluded that because $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$, it is true that $\boldsymbol{u} + \boldsymbol{v} \in \boldsymbol{V}$.

5.4c) Confusion about the inverse element for vector addition

Some students were evidently confused about what the identities for the different operations were. With respect to the additive inverse for vector addition, one student tried to show existence of the inverse of the 2×2 matrix as shown in Figure 5.10.

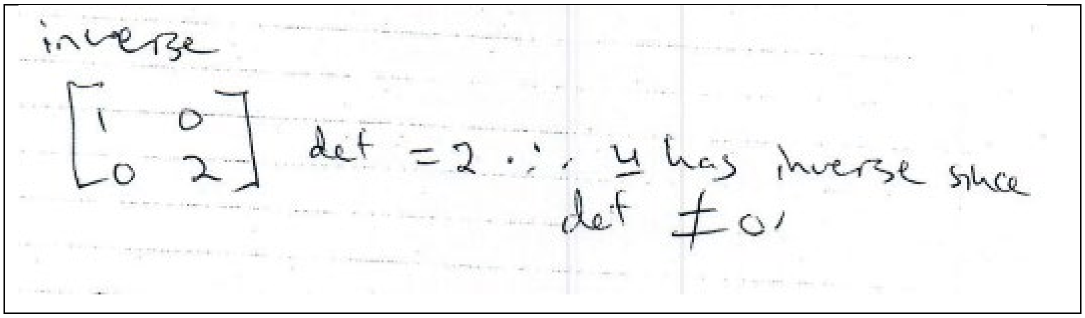


Figure 5.10: Written response of student T8 showing confusion between identity for vector addition and existence criterion for matrix inverse

Student T8, whose response appears in Figure 10, showed that the determinant of the matrix is not zero. If the determinant of the matrix is zero, then it implies that the inverse of the matrix does not exist. However, the required element was the inverse element for vector addition.

5.4d) Confusion about the identity for scalar multiplication

The students' responses to question 7 further showed that many of them were able to state all the ten axioms but were unable to prove some of them. The responses showed that most of the students were unable to prove axiom ten which states that $\forall\, \boldsymbol{v} \in V,\ 1.\boldsymbol{v} = \boldsymbol{v}$ – only six students managed to prove that the axiom held. The students were not clear about what '1' in the axiom referred to, in the scalar multiplication $1.\boldsymbol{u}$ when the vector elements were matrices. The most common misconception was taking the scalar 1 as the identity matrix that is $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ which was shown by 44 students. One student, T20, took the identity matrix instead of the scalar value 1, and then carried out matrix multiplication, as shown in Figure 5.11, hence he did not apply the scalar identity property.

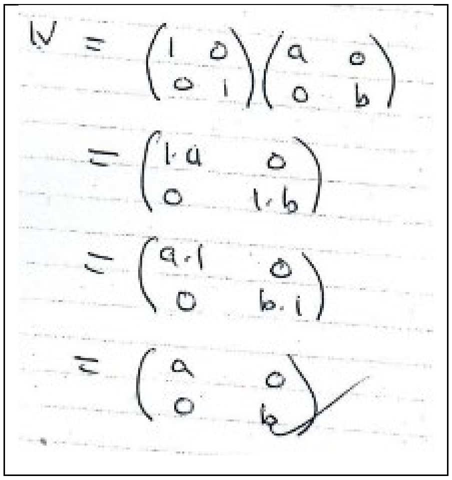


Figure 5.11: Written response of student T20 taking the identity matrix instead of the scalar identity

In the interview, student T7 revealed his confusion between the identity matrix and that of the identity for scalar multiplication:

R: *[Referring to a question in the activity sheet]. How do you show axiom 10 that 1.**v** = **v**?*

T7: *The* 1 *is represented by the identity matrix which is* $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

R: *What if I just multiply by the scalar* 1*?*

T7: *Aah no, it is not still correct but this* 1 *is an identity we are multiplying with an identity, so this one must take the form of **v**.*

R: *So we cannot use the scalar* 1*?*

T7: *Yes* 1 *is a scalar. Here we are trying to show that eh … if we multiply a matrix with its identity.*

R: *But it is possible to multiply by* 1*?*

T7: *It is very possible if **v** is not a matrix.*

The above excerpt shows that student T7 had a misconception about scalar multiplication – he did not accept that it is possible to multiply a 2×2 matrix with a scalar and he therefore replaced the identity for scalar multiplication with the identity matrix I_2 so that multiplication of the matrix **v** by I_2 leaves **v** unchanged. This shows confusion between scalar multiplication and matrix multiplication – showing that the binary operation of scalar multiplication had not yet been interiorised.

Some students tried to use the identity matrix, but did not even identify it correctly. Two of them took $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ as an identity matrix, as shown in the response of T13 shown in Figure 5.12.

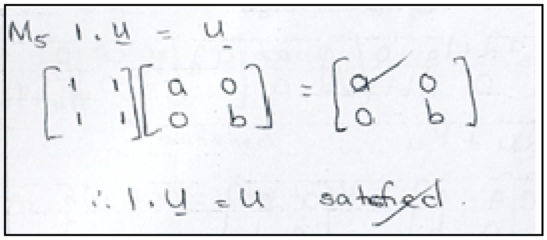


Figure 5.12: Written response of student T13 taking matrix with 1's as the scalar identity

One student presented the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ as the identity, while another wrote $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \cdot \begin{bmatrix} \frac{-1}{ab} \end{bmatrix}$, i.e. v $\times\, v^{-1} = 1$. There were other students who took the identity element for scalar multiplication as the zero matrix. The widespread confusion and misconceptions related to the identity for scalar multiplication arose mainly from their weak background in set theory and binary operations.

5.4e) Confusion about the binary operations

Confusion about the operation of vector addition was identified in the response by student T14, who took vector addition as pairwise multiplication of corresponding elements.

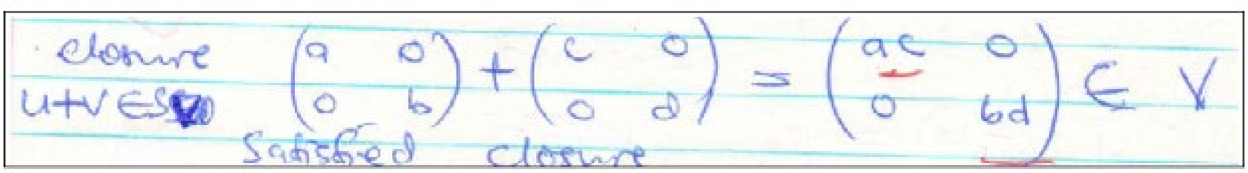


Figure 5.13: Written response of student T14 showing confusion between addition of matrices and pairwise multiplication of corresponding elements

The response in Figure 5.13 above shows that the student was aware that the property of addition needs to be satisfied, but instead of adding the two matrices, corresponding elements were multiplied. The student displayed the same misconception when he tried to prove the commutative as well as the associative property of addition, confirming that he had not developed even an action conception of the binary operation for addition and this did not allow him to develop the necessary construction for a strong vector space schema. Another student took the additive identity as the identity matrix and wrote:

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}.$$

Some of the students wrote $u.0 = 1$ for the additive identity.

5.4.1 APOS insights for question 7

Only 6 (9%) of the students were able to move back and forth and managed to show correctly that the given set was a vector space. This required the coordination of the concepts of set, and the two binary operations, to show that the axioms were satisfied. In terms of the genetic decomposition, it is only these students who had made progress towards the development of the vector space schema. Most other students were waylaid at many different places, indicating uneven interiorisation of some but not all concepts. Many students were able to state the axioms for the distributive and associative property of scalar multiplication, and had challenges in proving them, while many could not even state them. Students displayed a number of misconceptions peculiar to the different axioms. The widespread confusion between the identities and operations indicates that many of the students had not developed even process conceptions of the binary operations, because they were unable to carry out a binary operation if the two elements were not presented to them. It can be seen here that the students were just following procedure without understanding how some of the axioms are proved. In his research Harel (2000) noted that students struggle to understand the vector space axioms. The students whom he interviewed could not prove that for any A in a vector space V $(-1)A = -A.$ The students could not articulate the argument indicated in the axiom, but Harel (2000) indicated that this proof was even done in class.

5.5 General observations

This chapter attempted to uncover the conceptual difficulties that in-service students experience when learning the vector space concepts. We also attempted to understand some common misconceptions and errors that the students made. From the responses of the students, it was evident that many of them were experiencing problems with the abstraction level of the vector space concepts with its axiomatic approach as well as formalism required for communicating the arguments.

Thomas (2011) posits that conceptual understanding, which is strictly related to the goal of becoming mathematically competent, is evident when one is able to solve problems in an unfamiliar situation. This is in line with Stewart and Thomas's (2010) argument that many students experience difficulties when learning linear algebra because of its abstract and epistemological nature. This meant that the students in our study could not coordinate the set schema or the binary operation schema and so they could not carry out the process of checking all the axioms, and could not construct a robust proof. Some students were not clear about what the elements of the subset in question 4 were. Students who were unable to identify elements of the set showed that they had not developed a conception of the set of matrices beyond an action level. Furthermore, some also displayed a weak algebraic background, as shown by student T13 in Figure 5.1. This confirms some of the work reported in the literature that students find the abstract nature of concepts of a vector subspace challenging. The existence of different modes of representation contributes to their difficulties. The question on the subspace is represented in algebraic mode, whilst the definition of a subspace is phrased in abstract mode (Britton & Henderson, 2009). This is in agreement with Hillel (2000) who conducted a study on five experienced lecturers teaching concepts in linear algebra. He argued that the lecturers themselves confused the students because they persistently moved within the modes without explanations. The in-service students were unable to connect the algebraic and abstract mode of representation. Similarly, Hazzan and Zaski (2005) argue that abstractness of mathematics is complex because abstract concepts have many facets, with some concepts being more abstract than others.

We found that the students had some misconceptions emanating from previous concepts that they had encountered. Some students confused matrix addition with pairwise multiplication of the matrix elements. Others confused the identity elements for the binary operations with the identity matrix. It appeared as if many students did not understand that determinant is actually a function. This problem regarding determinants has led students to improper usage of the symbolic mathematical language as seen, for example, by student T13 writing $det = 8 - 7 = 1$. The question is: determinant of what? It seems as if they do not understand that determinant is actually a function. The definition is substantiated by Donevska-Todorova (2014) who explains the determinant as a function. It may also be that the student (T8's) confusion about the existence of an inverse of a matrix (a non-zero determinant) in Figure 5.10 arose from the student's misconception of the determinant.

In terms of APOS theory, the responses produced by many students showed that they did not even have an action conception of binary operations. For example, the response of student T14 in Figure 5.13 showed that the student struggled to carry out a binary operation. It may be that for such students, the instructor may have moved too quickly to the more abstract treatments of binary operations which required process or object conceptions. However, APOS theory emphasises that a conception begins with an external action. The action level is a very important building block upon which other conceptions develop, and instructors must take care that enough attention is paid to practising the binary operation using various vector spaces before moving on to more complicated questions. Ndlovu and Brijlall (2015) argue similarly that when abstract algebra is introduced via definitions and axioms only, it can become a source of conceptual difficulty.

The analysis revealed that many students were confused about the identity elements for the binary operations. Students who struggled with identifying the identity elements but were able to carry out the binary operations also illustrated action conceptions, because they could only carry out operations on elements that were presented to them. Action-level conceptions were seen to be limiting because students could only carry out an operation in a step–by-step manner with the elements in front of them. In order to identify possible identity elements, it is necessary for the student to have moved past looking at the binary operation as a step–by-step procedure (action) to one that has been interiorised (process) which allows the student to imagine the result of the operation. Those who resorted to the use of specific examples in question 7 demonstrated action-level reasoning or to what Hazzan (2001) calls 'canonical procedures'. T8's use of specific elements, as shown in her response when attempting to show that the ten vector space axioms are satisfied, indicates that she has not moved past an action conception because she needs the comfort of the concrete matrices to carry out the vector addition operations.

For question 7, some students showed that certain actions had been interiorised into processes as some of them were able to prove some of the ten axioms. However, it was clear that not even a process-level engagement with binary operations was sufficient to show that the axioms were satisfied. The response of student T8 shows how the student confused the commutativity property and the closure property. Commutativity of operations and the distributive law requires that students are able to compare the results arising from different binary operations, which requires object-level reasoning.

Although some students demonstrated process-level engagement with the binary operations, this was not sufficient in providing a justification that the set W was not a vector subspace of V. The understanding of the role of examples in proving or disproving a statement was crucial in this task. If one wants to prove that a proposal does not hold, it is sufficient to produce one counter-example. However, if one produces an illustrative example of a proposal it is not sufficient to prove that the statement holds true. The responses of student T39 (Figure 5.2) and student T4 (interview in 5.1b) show that they produced illustrative examples which satisfied the condition (added two vector elements from a set W and showed that the sum belonged to W). The statements are that given any two elements of the non-empty W, the sum belongs to W and the scalar multiple of an element of W also belongs to W. To disprove this, it is sufficient to produce a counter-example for any of the statements. However, to prove the statements, one would need to show that for any general elements each of the statements is true. The issue of the determinant being zero seemed to have caused some confusion in the minds of students such as student T39 (in Figure 5.2) who got entangled in the argument. Zaslavsky and Ron (1998) as well as Bansilal (2015) highlight the confusion experienced by students in distinguishing between an example that satisfies the condition of a statement and a counter-example that provides evidence that a statement does not hold. In Zaslavsky and Ron's study, the content was high school algebra and geometry and students struggled with using examples and counter-examples appropriately. In our study, the content was the highly abstract vector space concepts and providing counter-examples required a sound understanding of these concepts which explains why so many of the students struggled with the first task.

Some participants, such as student T4 in this study, argued that W was a subspace of V contrary to what the question asked for. This was a similar response as those in the study by Zaslavsky and Ron (1998) where they found that almost a third of the students were not able to determine that certain statements were false. Similarly, Bansilal (2015) reported that more than half the participants took the statement that 'every real number is rational' as true.

5.6 Implication for teaching

In this study we presented responses from 73 students who were enrolled in a linear algebra course at a Zimbabwean university. The chapter attempted to unpack some of the cognitive difficulties experienced by the students in negotiating the meaning of the various vector space concepts. The study showed that many problems were related to the students' understanding of the underlying concepts of binary operations and sets. I found that many students were confused about the identity elements for the binary operations and matrix operations. Further research is needed to help us understand why and how students become confused about these different operations, and to understand the extent of the difficulty in other vector spaces. Furthermore, most students struggled with explaining why a given set did not form a vector subspace because of the increased demand of using counter-examples appropriately. Further research may help us understand how to set out the teaching of proofs relating to when a subset of a vector space does not form a vector space. If the teaching of proof relating to this and other properties and relationships could be successfully scaffolded for students, they would be better prepared to deal with the abstractness of these concepts.

5.7 Modification of the genetic decomposition

I noted that although the genetic decomposition was useful as a diagnostic tool, there are some concepts that are important for the conceptual development of subspace and vector space that were not included, as well as some of the participants' responses that were not captured in the genetic decomposition. The preliminary genetic decomposition looked at the coordination of objects of sets and the vector space axioms which was based on the work of Parraguez and Oktac (2010) whilst in the revised genetic decomposition I also included the role of the axiom schema as well as the vector subspace schema. Hence I advocated for the modification of the genetic decomposition which is illustrated below. I now present a revised genetic decomposition based on some of the issues that emerged in this study.

The modified genetic decomposition is represented in the form of a table shown in Table 5.2.

Table 5.2. Preliminary and modified genetic decomposition for vector space/subspace

Set schema. At an action level, an individual conceives of a set when given a specific listing of a particular condition of set membership. The action of gathering and putting objects together in a collection according to some condition is interiorised into a process. This is	**Set Schema**. With an action conception, an individual conceives of a set when given a specific listing if a particular condition of set membership. The action of gathering and putting objects together in a collection according to some condition is interiorised into a process. This is encapsulated into an object when an individual can apply actions or processes to the process such as compare two

encapsulated into an object when an individual can apply actions or processes to the process such as compare two sets, consider a set to be an element of another and analyse properties of the set. (Arnon et al., 2014) **Binary operation schema.** A binary operation is a function of two variables defined on a single set or on a Cartesian product of two sets. At an action level, given a binary operation, an individual can take two specific elements of the sets and apply the formula. The individual interiorises the action into a process that takes two objects (elements) and acts on these to produce a new object (element) that is the result of the binary operation. At the object level, an individual can distinguish between two binary operations, check whether a binary operation satisfies an axiom and compare objects arising from two different binary operations. (Arnon et al., 2014) Parraguez and Oktac (2010) describe how these two schema can be drawn together to form the notion of vector space: the objects that are sets with two kinds of operations (addition and multiplication by a scalar) can be coordinated through the related processes and the vector space axioms that involve both operations, to give rise to a new object that can	sets, consider a set to be an element of another and analyse properties of the set. (Arnon et al., 2014) **Binary operation Schema.** A binary operation is a function of two variables defined on a single set or on a Cartesian product of two sets. With an action conception, given a binary operation, an individual can take two specific elements of the sets and apply the operation. The individual interiorises the action into a process that can consider any two objects (elements) without the need to work on specific vectors and without having to make specific calculations, and can consider the result of the binary operation. With an object conception, an individual can distinguish between two binary operations, check whether a binary operation satisfies an axiom and compare objects arising from two different binary operations. (Arnon et al., 2014) **Axiom Schema**. For vector space, an axiom can be considered as a Boolean-valued function, which accepts a set (or Cartesian product of sets), and a binary operation defined on the set (or sets) and checks whether the axiom property is satisfied. In order to apply the axiom to a set and binary operation, the set and binary operation objects must be de-encapsulated and coordinated with the process of checking the axiom property in question. **Vector Space Schema**. The concept of vector space concept is constructed by "coordinating the three schemas" of set, binary operations (vector addition; scalar multiplication) and

be called a vector space. (Parraguez & Oktac, 2010, p. 2116)	axioms (Parraguez, , & Oktaç, 2010 p, 2114). The ten instances of the operation of checking if each axiom is satisfied are coordinated into a single process that can establish if the system is a vector space, which is then encapsulated into an object vector space, that is a set with binary operations that satisfies axioms (Arnon et al., 2014; Parraguez & Oktaç, 2010). The vector subspace concept emerges from this schema – students will not be able to see the connections between a vector space and vector subspace if they have not developed the vector space schema. **Vector Subspace Schema** The instances of checking if each of the two axioms (closure condition of binary operations) is satisfied, is coordinated into a single process that can establish if a non-empty subset is a vector subspace. The process of verifying that a set satisfies the sub-space conditions can then be encapsulated into a vector subspace object, making it possible to determine properties of vector subspaces and to see relationships between vector subspaces. At this stage an individual can judge the equivalence between different definitions of subspaces. Counter-examples can also be constructed to show that a set does not satisfy the properties of a vector subspace.

CHAPTER 6

ANALYSIS OF STUDENTS' RESPONSES TO ITEMS BASED ON LINEAR COMBINATION

6.1 Introduction

The analysis in this chapter is based on students' responses to activity sheet two and transcripts from the interviews. In this chapter, it was important to explore the students understanding of the notion linear combination. Students were interviewed with the intention of understanding more on how they construct various mental structures when solving problems involving linear combination of vectors. The transcriptions of the students' interviews are based on students' written responses and some of the students' interview transcriptions are discussed in this chapter.

To ensure that the discussion of the students' responses makes sense, I present the definitions and theorems that commonly appear in this discussion.

Definition 6.1. Linear combination of a set of vectors: If $\boldsymbol{w}$ is a vector in a vector space V, then **w** is said to be a linear combination of the vectors $v_1, v_2, \ldots v_r$ in V if $\boldsymbol{w}$ can be expressed in the form

$w = k_1v_1 + k_2v_2 + \cdots k_nv_r,$ where $k_{1,}k_2 \ldots k_r$ are scalars, … (equation 6.1)

These scalars are called the coefficients of the linear combination.

Definition 6.2. Spanning: The subspace of a vector space V that is formed from all possible linear combinations of the vectors in a nonempty set S is called the span of S, and we say that the vectors in S span that subspace. If $S = w_1, w_2 \ldots w_r$, then we denote the span of S by span $w_1, w_2 \ldots w_r$.

Theorem 6.1: If A is an $n \times n$ matrix, then the following statements hold: $Ax = b$ is consistent for every$_{nx1}$ matrix b, that is the system is consistent if and only if the coefficient matrix A has a non zero determinant, ($detA \neq 0$).

Thus the responses of the students are analysed and presented and this is based on the genetic decomposition presented in section 3.8.2.

6.2 Analysis and discussion of data

The activity sheet consisted of eight questions of which four are discussed below. Data analysis is based on the sequencing of the concepts categorized in the following three groups: students understanding of the definition of linear combination and spanning, linear combination of vectors, and spanning. In order to make a comprehensive discussion, we made use of students' supported sample pictures and interview excerpts so as to add evidence of the APOS level at which the in-service teachers were operating in terms of their understanding of the concepts on linear combination and spanning. Questions are presented below, and the scores are categorized and summarized in Tables below.

6.3 Question 1

Students' understanding of the definitions of linear combination and spanning

1. Distinguish between the term linear combination and spanning.

Students were asked to explain their understanding of linear combinations and how the process of spanning of vectors was linked to that of forming linear combinations. We first look at the students explanations of linear combinations and thereafter consider their understanding of spanning.

6.3.1 Descriptions of linear combinations

There were seven students who did not attempt to define the term linear combination with some simply transcribing the question. Nine students provided disjointed explanations as shown by the response of student T15 in Figure 6.1.

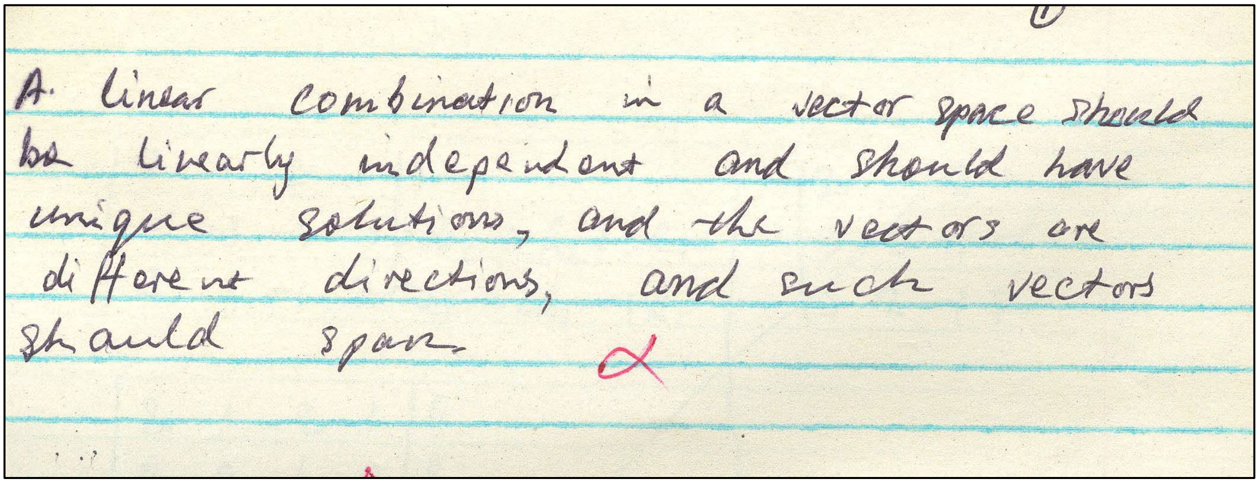
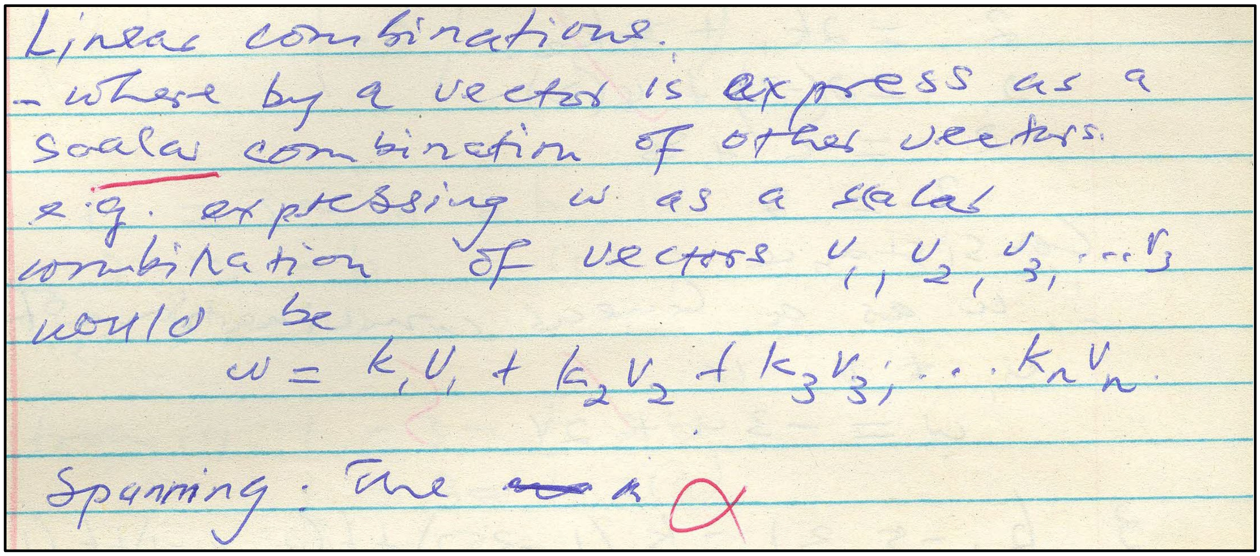

A linear combination in a vector space should be linearly independent and should have unique solutions, and the vectors are different directions, and such vectors should span

Figure 6.1: Written response of student T15

In Figure 6.1, the term linear combination was defined incorrectly in terms of linear independence. The students also brought in the idea of unique solutions which is related to systems of linear equations which can be part of the procedure of trying to express a vector as a linear combination of a set of vectors. One student, T35, said that linear combination is whereby a set of vectors can be expressed as one being added together. Another student, T45, said that linear combination is when vectors lie on the same line, also showing conceptual difficulties. This idea may have cropped up from the concepts taught on geometric interpretation of linear independence, whereby two vectors in R^2 or R^3 are linearly independent if and only if they lie on the same line and when they have their initial points at the origin.

Forty five students provided partially correct definitions although they had not made the necessary constructions suggested for by the genetic decomposition. We noted that 12 of these 45 students attempted to define the term linear combination using symbolic expressions, that is by just writing that $w = k_1v_1 + k_2v_2 + \cdots k_nv_n,$ without further explanations. There were 33 students who expressed their definition using symbols with further explanation but exhibited varying difficulties in trying to support the definition, for example student T37 whose response appears in Figure 6.2 below.

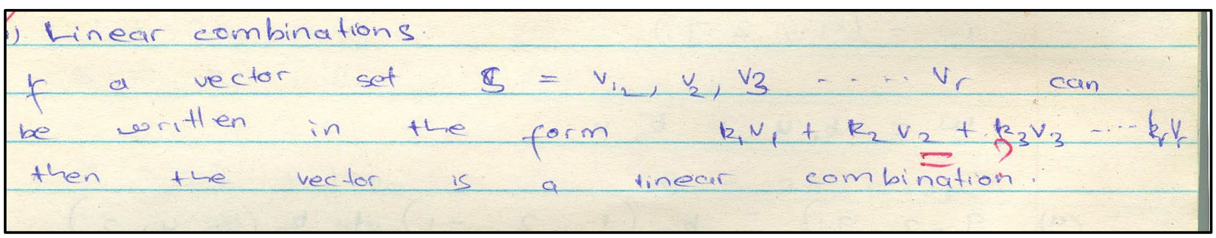


Linear combinations.
- where by a vector is express as a scalar combination of other vectors.
e.g. expressing w as a scalar combination of vectors $v_1, v_2, v_3, \ldots v_3$ would be
$w = k_1v_1 + k_2v_2 + k_3v_3; \ldots k_nv_n$.

Spanning: The

Figure 6.2: Written response of student T37

Student T37 provided the symbolic equation $v = k_1v_1 + k_2v_2 + \cdots + k_nv_n$, but used the phrase scalar combination of another vector instead of taking it as a sum of scalars of multiples of the vectors. The students also did not specify where the vector belongs and the role of the vectors and scalars. Student T54 provided a disconnected explanation as shown in Figure 6.3.



Linear combinations.
If a vector set $S = v_1, v_2, v_3 \ldots v_r$ can be written in the form $k_1v_1 + k_2v_2 + k_3v_3 \ldots k_rv_r$ then the vector is a linear combination.

Figure 6.3: Written response of student T54

The student uses the term 'if a vector set' without showing that the elements of the set S belongs to the vector space V. The expression $k_1v_1 + k_2v_2 \ldots k_rv_r$ is presented as a vector set, and without defining the variables $k_1, k_2 \ldots, k_n$. Another example is that of student T53 shown in Figure 6.4 who was able to give the definition in symbolic form as indicated, specifying where the vector **v** belongs.

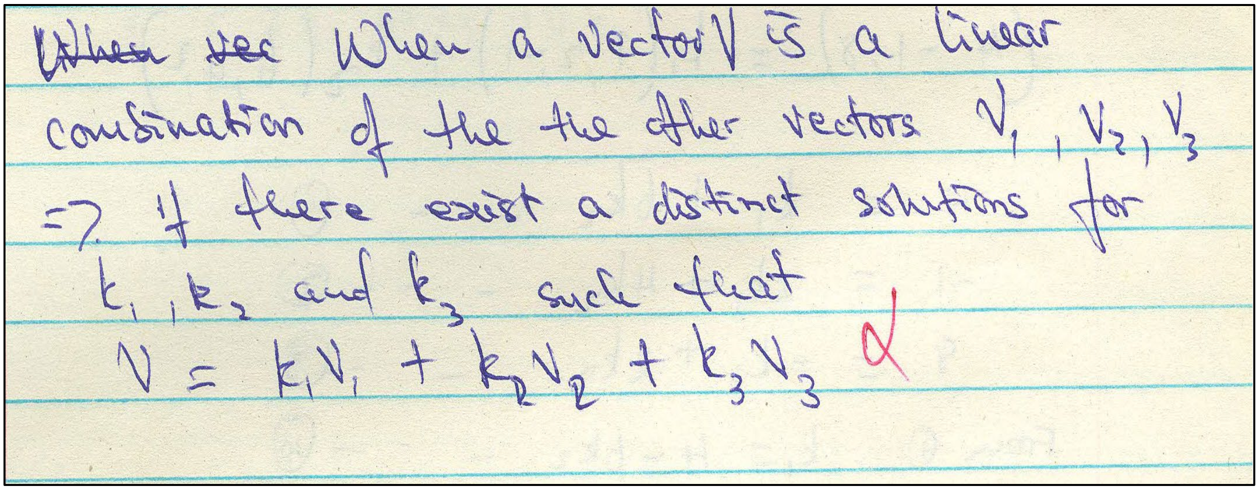

~~When vec~~ When a vector V is a linear combination of ~~the~~ the other vectors V_1, V_2, V_3 => if there exist a distinct solutions for k_1, k_2 and k_3 such that $V = k_1V_1 + k_2V_2 + k_3V_3$

Figure 6.4: Written response of student T53

However, the student did not explain what the variables $k_1, k_2 \ldots, k_n$ were. Student T53 also did not attempt to write the definition of spanning, hence showing that he is not able to make any link between the two concepts. However, 12 out of the 73 students provided clear definitions of the term linear combination which explained the role of scalars and where the vectors belong, showing progression towards the object level understanding according to APOS theory. This is also supported by Stewart's (2008) argument that learners operating at the object level of understanding of linear combination should also give a strong argument for $v_i \in V$, $c_i \in F$. This shows that these students may have encapsulated some of the processes into an object conception and have developed some of the mental structures in place with regards to the meaning of linear combination.

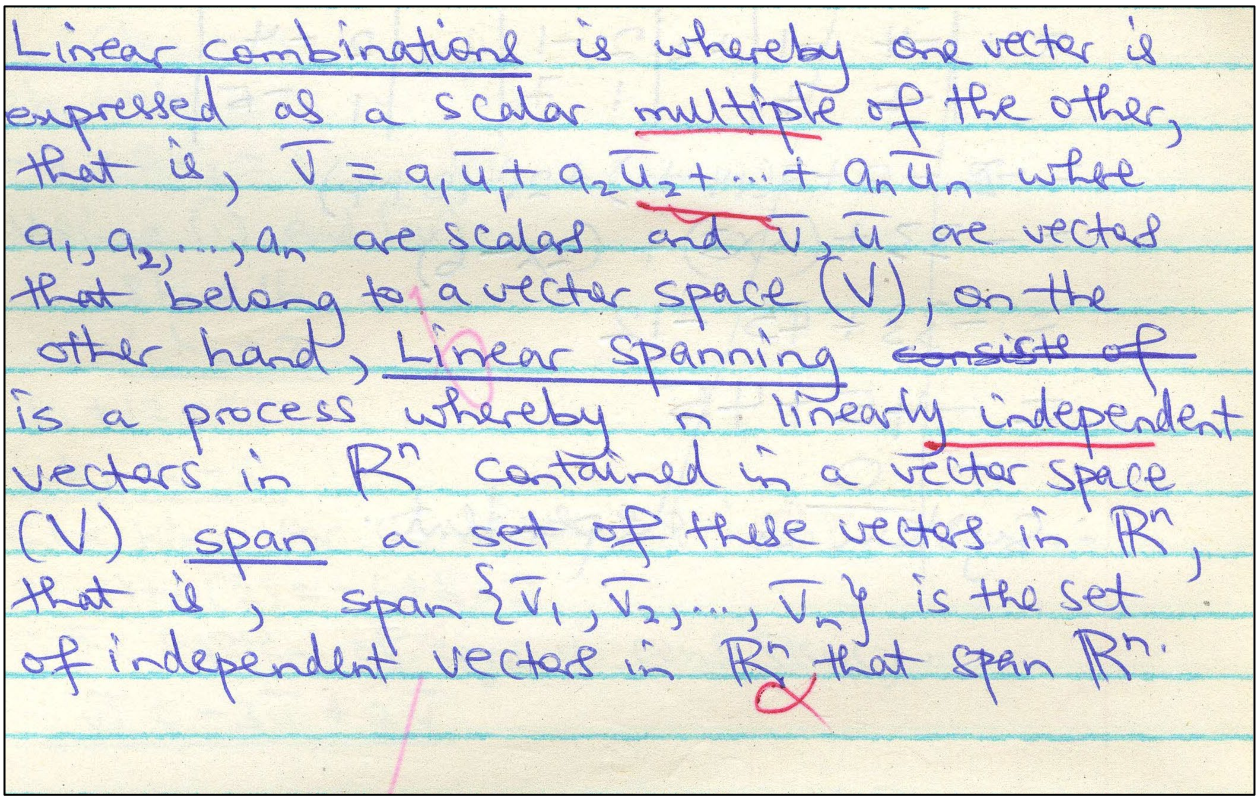
The students' responses for the explanation of linear combination can be summarized in table 6.1.

Table 6.1: Students' responses on the explanation of linear combination

Categories		**Frequency**
No response		7
Incorrect response		9
Partially Correct	Using symbols without further explanation	12
	Using symbols with further explanations, which were not complete	33
Correct responses		12

6.3.2 Explanation of spanning

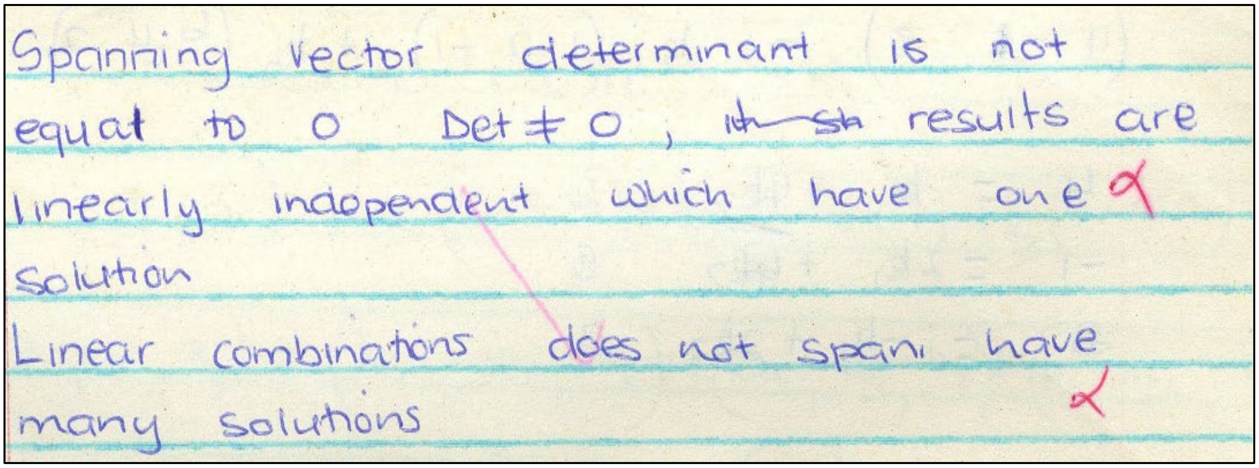
The students' responses showed that they encountered considerable difficulties with explaining the term spanning with 16 students not even trying to explain the term spanning. There were nine students who provided irrelevant explanations, as shown in the response by student T65, whose written response appears in Figure 6.5. Student T65 defined the term spanning in terms of linear independence and also coined a new phrase 'linear spanning'.



Linear combinations is whereby one vector is expressed as a scalar multiple of the other, that is, $\bar{V} = a_1\bar{u}_1 + a_2\bar{u}_2 + \dots + a_n\bar{u}_n$ where $a_1, a_2, \dots, a_n$ are scalars and $\bar{V}, \bar{u}$ are vectors that belong to a vector space (V), on the other hand, Linear Spanning ~~consists of~~ is a process whereby n linearly independent vectors in R^n contained in a vector space (V) span a set of these vectors in R^n, that is, span$\{\bar{V}_1, \bar{V}_2, \dots, \bar{V}_n\}$ is the set of independent vectors in R^n that span R^n.

Figure 6.5: Written response of student T65

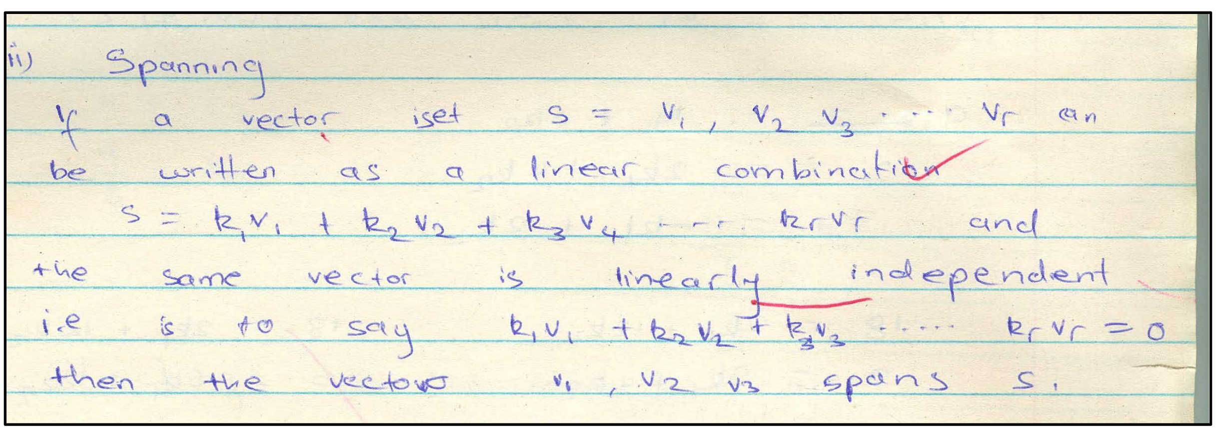
This and other similar responses showed that these students had not developed an object understanding of the concept on spanning. Another student, T5, (whose response is not shown here) said that spanning is when k_1, k_2, k_3 has at least one solution that is linearly independent and $k_1, k_2, k_3 = 0$ and she concluded that these are unique solutions. This response showed that the students did not have a meaningful understanding of the term spanning. Although spanning is a concept that can be related to linear independence, the relationship has to be explained. Another incorrect response is shown by student T31 in Figure 6.6.



Spanning vector determinant is not
equat to 0 Det ≠ 0 , ~~ith sh~~ results are
linearly indopendent which have one
solution
Linear combinations does not span have
many solutions

Figure 6.6: Written response of student T31

The response by student T31 above shows that she tried to define the term spanning by mentioning disconnected rules such as specifying that the determinant is non-zero but it not clear what determinant she was referring to. The same student had said that linear combinations have many solutions. There were 43 students who provided an algebraic relationship of the spanning vectors and a general vector. However, 27 of these responses did not provide any further explanation. The other 16 out of 43 students provided an unexpected error, whereby they further defined the term spanning in terms of linear independence. For example, the response shown by student T54 is shown in Figure 6.7.



ii) Spanning
If a vector iset S = V_1, V_2 V_3 . . . V_r can
be written as a linear combination
S = $k_1V_1 + k_2V_2 + k_3V_4$ . . . . k_rV_r and
the same vector is linearly independent
i.e is to say $k_1V_1 + k_2V_2 + k_3V_3$ . . . . $k_rV_r = 0$
then the vectors V_1, V_2 V_3 spans S.

Figure 6.7: Written response of student T54

This indicates that the student was able to set up the vector equation but the variables were not explicitly defined. The set S is not explicitly defined. There is a mix up between the different terms, for example set and equation. The student also equates a set of vectors with a linear combination of vectors (which is a vector). She mixed up different concepts, for example scalars, vectors and solutions of systems of equations.

The students' responses for the explanation of spanning can be summarized in the Table 6.2.

Table 6.2: Students' responses for the explanation of spanning

Categories		Frequency
No response		16
Incorrect response		9
Partially correct	Correct use of symbols but made errors by further explaining it in terms linear independence	16
	Use of symbols without further explanations	27
Completely correct response		5

6.3.3 Interview responses to question 1

In trying to understand more about the possible APOS levels of understanding at which these students were operating, the students were interviewed. They also revealed low levels of understanding in distinguishing the terms spanning and linear independence, for example T57 and T21. Student T57 was interviewed and her written response for the linear combination showed that she tried to express the linear combination relationship equation using symbols which were not completely correct, while the response for spanning was very vague. Her response was "Linear combination of vector is when a set of vectors $\mathrm{w} = \mathrm{v}_1, \mathrm{v}_2, \ldots \mathrm{v}_\mathrm{r}$ can be expressed in the form of $\mathrm{w} = \mathrm{k}_1\mathrm{v}_1 + \mathrm{k}_2\mathrm{v}_2 + \cdots \mathrm{k}_\mathrm{n}\mathrm{v}_\mathrm{n}$ where $\mathrm{k}_1 \ldots \mathrm{k}_\mathrm{n}$ are scalars, and spanning is where the vectors is linearly independent".

The following question was posed:

R: *What do you understand by the term linear combination and spanning in a vector space?*

T57*:* *Linear combination is whereby we have a set of vectors which can be expressed in terms of the vectors* v_1, v_2, v_3 *and we have scalars* $k_1,\ k_2\ \ldots\ k_n,$ *so when we say the vectors can be expressed as a linear combination, is when we say a set of vectors, we can say* $w = k_1v_1 +$

$k_2v_2 + \cdots k_nv_n$ *that is a linear combination we are multiplying the scalars and multiplying the vectors and can be added together to come up with the set of vectors.*

It is interesting to note that her interview response no longer includes the incorrect expression "$w = v_1, v_2, \dots v_r$", which appeared in her written response, but her formulation is still not precise since she speaks about a linear combination being "added together to come up with a set of vectors". In her written response about spanning, she had mentioned linearly independent vectors but in her interview response this changed to "arbitrary vectors". Her description of spanning appears below:

T57: *Spanning now I thought eh when we talked about spanning there was this issue of arbitrary vectors, can be expressed in the form an arbitrary vectors* b_1, b_2 *....*b_n *(frowning and looking at the interviewer for help)*
R: *Just try, tell me what you think.*

T57*:* *I now have a confusion but the issue of an arbitrary vector I know it is there, but are we saying we are supposed to express our vector in terms of* $b_1, b_2 \dots b_n$*, I now have a confusion.*

The interview provided more insight into the student's confusion about the term "arbitrary vectors". She identified that the word arbitrary was associated with "spanning" but was not sure what the role was. This shows that student T57 is stuck at the action level since she only has an action conception of linear combination. She can only think of actions with specific vectors.

Another student, T21, whose written response was very brief, when interviewed tried to link the term to determinants. The written response was: "Linear combination is a vector which can be expressed in the form $v = k_1v_1 + k_2v_2 + \cdots k_rv_r$ where k_r are scalars". For spanning, she simply wrote the word "spanning" but did not write anything further. The interview with student T21 went as follows:

R: *What do you understand by the term linear combination and spanning in a vector space?*

T 21: *I thought they are similar. Eh I thought to find linear combination we find the determinant and if the answer is not zero is means the vector is a linear combination hmm (laughing for a moment) no linear combination is where all the vectors give the same answer and spanning is where the determinant is not zero.*

***R**: The determinant of what?*

***T 21**: The determinant of the vectors. I come up with a matrix of vectors that you are given, then you find the determinant. If the determinant is zero then it spans.*

***R**: What about a linear combination?*

***T**: For the linear combination is when we come up with equations and the solution must be the same.*

***R:** The solution must be the same with what?*

***T 21**: The solution must be same with hmm (quiet for a moment). I can't explain.*

Student T21 was quite confused and made reference to finding a determinant. When probed about what determinant she was referring to, the student spoke about the determinant of vectors, which does not exist. She further reduced the concept to the solution of the system of equations which often forms part of the procedure of identifying the scalar in the original linear combination vector equation. She opted to talk about the methods that are used to determine whether a set of vectors in a vector space, say V, can be a written as a linear combination, or to determine if a set of vectors spanned V. According to the genetic decomposition being able to describe the general procedures for determining linear combination without actually carrying out each step has been placed at the process level according to APOS theory. Hence this student has clearly not interiorised the actions of determining a linear combination expression into a process.

Another student, T33, gave the following written response in the activity sheet [Linear combinations: Let **w**, a vector of vector space V such that $\boldsymbol{w} = w_1, w_2 \dots w_n,$ then $\boldsymbol{w} = k_1 w_1 + k_2 w_2 + \cdots k_n w_n$ is called the linear combination of **w** and let v, a vector of s, a subspace **v** = $v_1, v_2, \dots, v_n$. If $\boldsymbol{v}$ can be written as a linear combination, then v spans s, written, span $(v_1, v_2, \dots, v_n)$. In the interview student T33 defined the terms as indicated below:

T33: *Linear combination is expressing a vector as a sum of product of scalar and constitute scalar for example* $w = u_1 v_1 + v_2 w_2$ *(writing it down) and spanning it has something like element of a given set and trying to express it as linear combination and now taking it as linear dependent and independent.*

The student has attempted to define the term linear combination in terms of the original vector equation making up the linear combination relationship. However, the student seemed to experience difficulty with the understanding of the term which may be because of the tendency of cramming these definitions without understanding them. In terms of distinguishing the two terms I can make an argument that the student might be operating at the pre-action level according to APOS theory.

6.3.4 Summary for question 1

In question 1, the students were asked to explain what was meant by linear combination and to explain the notion of spanning in a vector space. An understanding of spanning according to our genetic decomposition entails an object understanding of linear combinations. The students could have explained what spanning of vectors meant or they could have described more precisely the span of a set S (Span S) as a set of linear combinations of the vectors in the set S which forms a vector subspace. They could also have considered spanning in terms of the set of vectors forming the spanning set (S), that is, those vectors which generate the linear combinations The idea of a spanning set is an important concept because it gives rise to the notion of a basis which is a spanning set consisting of linearly independent vectors. However, only five of the students were able to provide a description of the two terms while also explaining the differences between the procedures used to determine if a given vector could be expressed as a linear combination of a given set of vectors and that to determine if a given set of vectors spanned another set of vectors. According to the genetic decomposition, these students might be working at the object level of understanding of linear combinations. However, it will be shown later that not all of them were able to handle the object- level demands of the concept.

Most of the students' explanations were based on the procedure used to solve for the scalars in the vector equation $\boldsymbol{v} = k_1\boldsymbol{v_1} + k_2\boldsymbol{v_2} + \cdots + k_n\boldsymbol{v_n}$ expressing the linear combination relationship. However this equation is used for many purposes. It can be used to show that a given specific vector **v** is (or is not) a linear combination of the vectors on the RHS of the equation. It can be used with $\boldsymbol{v} = 0$, to show that the set of vectors on the RHS are linearly independent. It can also be used to show that the set $v_1, v_2 \dots v_n$ is a set of linearly dependent vectors. It can also be used to show that any arbitrary vector $\boldsymbol{v}$ is a linear combination of the vectors on the RHS (which would imply that the vectors on the RHS of the equation span the vector space $\boldsymbol{V}$). However, none of the students was able to express their explanation in terms of the different purposes of the equation. Many students were side tracked with issues about the augmented matrix arising from the systems of equations to solve for the scalars. Depending on the purpose, the augmented matrix could be different, but the students spoke generally about det A, referring to a general matrix A.

6.4 Question 2

Question 2 was intended to provide insight about students' possible progress towards developing a process conception of the concept of linear combination. Question 2 is presented below, and the students' responses are categorized and summarized in Table below:

Consider the vectors $\boldsymbol{u} = (1, 2, -1)$ and $\boldsymbol{v} = (6, 4, 2)$. Show that $\boldsymbol{w} = (9, 2, 7)$ is a linear combination of $\boldsymbol{u}$ and $\boldsymbol{v}$ and show that $w = (4, -1, 8)$ is not a linear combination of $\boldsymbol{u}$, $\boldsymbol{v}$.

To solve this problem the students could work through the following steps:

- Set up a vector equation $\boldsymbol{w} = k_1 u + k_2 \boldsymbol{v}$ where k_1, k_2 and k_3 are arbitrary scalars.
- Set up a system of three equations with two unknowns (k_1, k_2).
- Solve for the scalars, using an augmented matrix approach or by manipulation of the equations.
- Check for consistency across the three equations , since there are 2 equations in three unknowns. It is necessary to check whether the solution satisfies all the three.

6.4.1 Results for question 2, part 1

Two students did not attempt the question. 12 of the students attempted to answer the question but were caught up in different levels of difficulty leading to incorrect responses. Two students did not write the usual vector equations for the linear combination expression, for example student T47's response shown below in Figure 6.8.

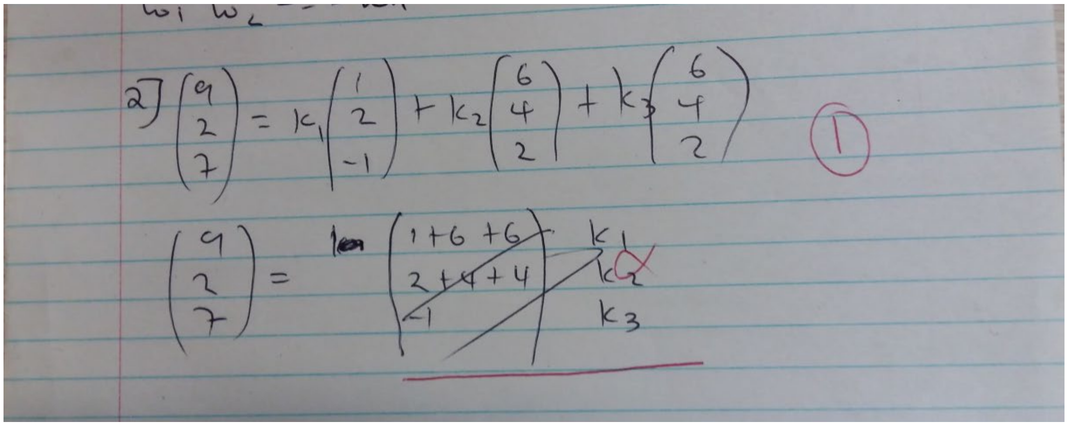


Figure 6.8: Written response of student T47

It is interesting to note that the student firstly had a conception of coming up with a vector equation, and forced in three variables by repeating one of the vectors in making up the equation. He tried to do scalar multiplication, and tried to come up with an augmented matrix, but he was stuck. This response showed that the student's response is externally directed as he tried to show the step by step procedures but could not execute it correctly. This indicated that the actions of forming the correct vector equation, scalar multiplication and formulation of systems of linear equations had not yet developed.

The other 10 students were able to set up the vector equations, formulate three equations [non homogenous] using scalar coefficients in two unknowns. However, the students had difficulties in trying to find the scalars. These students used the method of solving the equations simultaneously. Some students made careless errors. Since the students did not reflect by checking the values for consistency across the three equations, they were unable to identify their errors. This shows that these students are acting on a step by step procedure, and did not check whether it is correct. The response by these students showed that the action stage had not fully developed because they were only able to carry out repeatable mental manipulation and did not reflect whether their solution satisfied the equations.

The remaining 59 students were able to set up the vector equation, express the given vectors in coordinate system and then come up with a system of linear equations in two unknowns. The students solved two of the equations simultaneously and obtained the correct scalars say k_1 and k_2. It is important to check for consistency using the values of k_1 and k_2 in the third equations before making some conclusions. However, three of the students did not check the third equation for consistency. Here it indicates that the action level according to APOS theory has not fully developed because these students were able to carry out the steps but they could not see beyond the current steps that they were working in. They were supposed to reflect whether the answer was correct by checking for consistency. They did not see the need to reflect on the process on the answer and go beyond the steps of the procedure. These students might be working within the action level of understanding. The remaining 56 students checked for consistency in the third equation, using the two scalars. This showed that each step is acting as a prompt for the next procedure. However, 16 of the students simply showed consistency and did not make a concluding statement to the effect that w can be written as a linear combination as well as writing a concluding statement that is $w = -3\boldsymbol{u} + 2\boldsymbol{v}$, making a conclusive deduction that the vector w can be expressed as a linear combination of the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, where -3 and 2 were the correct scalars. The remaining 40 students made the following deductions: the system is consistent since the scalars k_1 and k_2, exist and that **w** is a linear combination of the vectors **u** and **v** and were able to write a concluding statement that $w = -3\boldsymbol{u} + 2\boldsymbol{v}$, and they thus provided a complete argument. This suggests that these students were working at the process level according to APOS theory. The above results are summarized in the Table 6.3 below:

The student responses for question 2 part 1 are summarized in the Table below:

Table 6.3: Student's responses for question 2 part 1 on linear combination

<table>
<tr><th colspan="3">Category</th><th>Frequency</th></tr>
<tr><td colspan="3">No response</td><td>2</td></tr>
<tr><td colspan="3">No correct vector equation</td><td>2</td></tr>
<tr><td rowspan="4">Set up vector equation</td><td colspan="2">Obtained incorrect scalars</td><td>10</td></tr>
<tr><td rowspan="3">Correct Scalars</td><td>Did not verify scalars</td><td>3</td></tr>
<tr><td>Verified scalars, no concluding statement</td><td>16</td></tr>
<tr><td>Verified scalars, concluding statement</td><td>40</td></tr>
</table>

6.4.2 Results for question 2, part 2

16 of the students did not attempt the question at all while only two did not attempt the question in the first part. The phrase 'is not a linear combination' seems to have confused the students or they struggled to interpret it. According to APOS theory, these students have not shown evidence of developing the necessary mental construction with regards to a vector failing to be expressed as

a linear combination of the other set of vectors. 19 of the students in the next category were able to set the vector equations, express the given vectors in coordinate system and then come up with a system of linear equations in two unknowns. However, these students then had difficulties. I8 students used the simultaneous equation method to solve for the scalars. Seven out of the 18 students confused these methods; an example is that of student T28 who used the elimination method, and she calculated the value of k_1 using 2 equations and then concluded that it is not a linear combination. She did not try to calculate the other unknown, showing lack of understanding of concepts on solving of simultaneous equations that were done at elementary level. In Zimbabwe, the concepts on simultaneous equations are first encountered at ZJC, that is form two level, where the aspect on elimination or substitution method is introduced. 11 out of the 18 students could not manipulate figures due to difficulties with calculation of integers, and the errors that were mainly encountered were silly mistakes which Siyepu (2013) referred to as slips. One of the 19 students, that is student T5, used the Gaussian elimination method but encountered a number of procedural errors as shown in Figure 6.9.

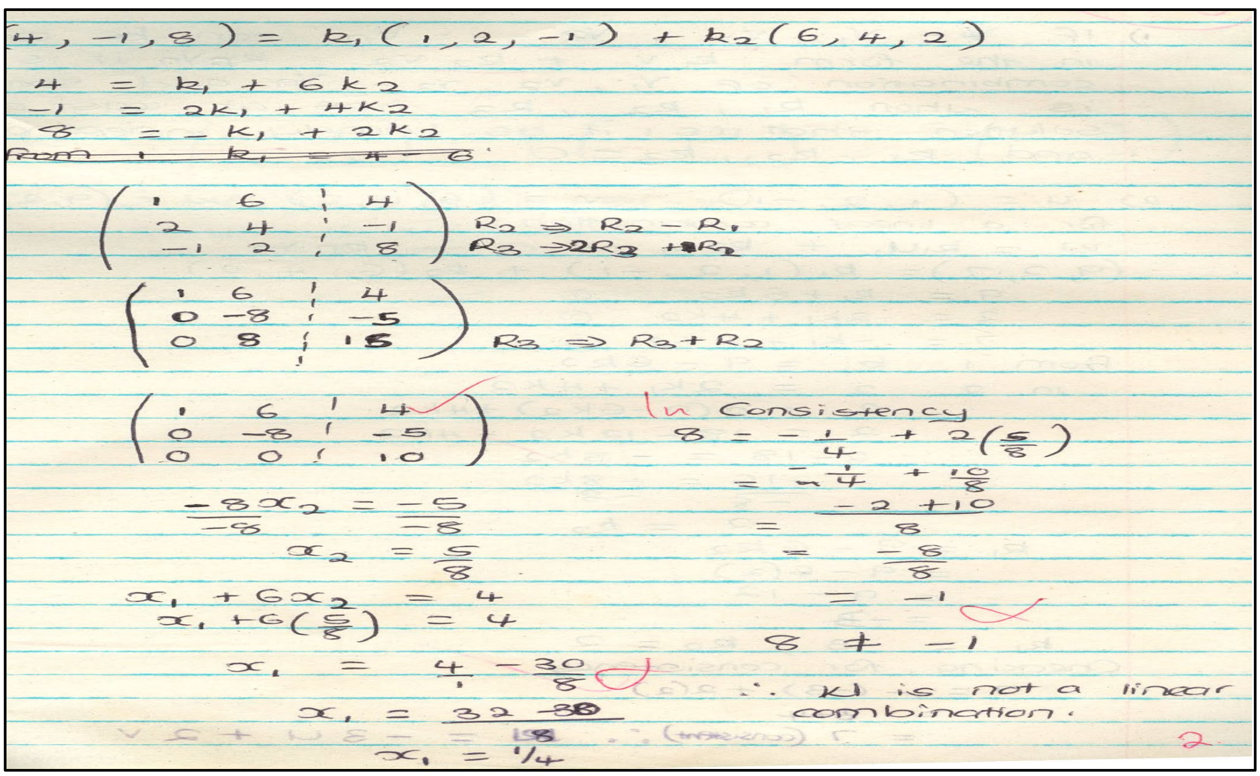


Figure 6.9: Written response of student T5

Firstly, the student wanted to introduce a zero below the element 1. The student executed the correct elementary row operation that is $r_2 \to r_2 - 2r_1$, but wrote an incorrect row operation: $r_2 \to r_2 - r_1$. The student here demonstrated a careless error. The student work continued to show some knowledge gaps and irregularities on the work on elementary row operations. She struggled to carry out a correct row operation, that is she had: $r_3 \to 2r_3 + r_2$. The student did not realise that she had formed a new row 2, but proceeded to use the original elements of row 2 that is: $[2 \quad 4 \quad \vdots -1]$ instead of using row 1. After completing the row operation, instead of making a conclusion, the student had problems with procedural fluency of row reduction. This shows that the action stage had not yet fully developed according to APOS theory.

The next category involved 10 students who were able to solve the systems of equations by either solving the equations simultaneously or using the Gaussian elimination method. Those who used the elimination method solved the two equations, but did not test for consistency in the third equations hence they made some incorrect conclusions. Another example is that of student T66 who used the Gaussian elimination method and obtained the following matrix $\begin{bmatrix} 1 & 6 & \vdots\, 4 \\ 0 & -8 & \vdots -9 \\ 0 & 0 & \vdots -2 \end{bmatrix}$.

She wrote $\to \therefore \boldsymbol{w} = (4, -1, 8)$ is not a linear combination of $\boldsymbol{u}$ and $\boldsymbol{v}$. The conclusion is correct but the student did not provide reasons for her deductions. It was important here to give reasons why the vector $\boldsymbol{w}$ cannot be written as a linear combination of the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$. It is important to outline that the system is inconsistent, since no such scalars, say k_1 and k_2 exist, hence the vector $\boldsymbol{w}$ is not a linear combination of the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$.

The remaining 28 students provided correct, complete responses and were able to interpret the solution to the systems of equations and make correct judgements about the solution. This revealed that the students were able to interiorise the action into a process as they were able to interpret the solution to the system of linear equations in terms of linear combinations. The results are summarised in the Table below.

The students' responses for question 2 part 2 are summarised in the Table 6.4.

Table 6.4: Students responses for question 2 part 2 on not a linear combination

Category			Frequency
No response			16
Set up vector equation	Obtain incorrect Scalars	Confuses the method	7
		Failure to solve the equations correctly	12
	Obtain correct scalars	In correct deductions/ insufficient conclusions	10
		Completely correct deductions and conclusion	28

6.4.3 Interview responses to question 2

A student, T63, who obtained the correct solutions for question 2 part 2 was interviewed. He had used the elimination method to calculate the scalars in order to find scalars in the written response. Student T63 was presented with a hypothetical augumented matrix, which represented the situation where the students were asked to write one vector **v** as a linear combination of three given vectors. The matrix that was presented was reduced to echelon form so that the last row was zero as shown: $\begin{bmatrix} 1 & 2 & 3 & \vdots & 3 \\ 0 & 1 & 0 & \vdots & 2 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$. Student T63 was prompted about what could be deduced from such a matrix which has 1 row of zeros. The student responded as follows:

***T63**: This has many solutions meaning that it cannot be expressed as a linear combination.*

Student T63 was able to correctly infer that the systems of linear equations have many solutions. This shows that he has an object understanding of systems of linear equations. However, the student demonstrated a serious misconception of the implication of this in terms of linear combinations thinking that if it has an infinite number of solutions then this implies that it cannot be expressed as a linear combination. In the hypothetical example the systems of equations is consistent, meaning that such scalars exist.

Student T62 had this to say:

***R**: Can you briefly explain how you can determine whether a given vector can be written as a linear combination of the other given vectors, in R^n?*

T62: *We are saying we have this vector when you express it you must get constants if it is possible that is* $Ax = 0$. *We then find constants by using the method of row reduction.*

R: *Suppose we have equation of the form Ax = b, then we form an augmented matrix. The matrix is then reduced to row echelon form. Let's say we obtain the matrix of the form* $\begin{bmatrix} 1 & 6 & \vdots 4 \\ 0 & -2 & \vdots -5 \\ 0 & 0 & \vdots 1 \end{bmatrix}$ *after row reduction. What is your conclusion in terms of linear combinations? Does the given vector form a linear combination of the given vectors?*

T62: *Then what is it oh, we should put a constant, that is* $x_3 = 1$.

R: *How did you get the value,* $x_3 = 1$?

T62: *Probably that one cannot be expressed as a linear combination because we have failed to get the value for the last part.*

From this interview, it was noted that T62 had a vague idea about the concept of linear combination, which he introduced by using the homogenous system $Ax = 0$, instead of a non-homogenous system one. This shows that the student is operating at the pre action according to the APOS theory. From the hypothetical example, instead of outlining that $0 \neq 1$ the student goes on to say $x_3 = 1$. This student has not been able to interpret the solutions to the system of equations given the reduced row form of the augmented matrix. It is clear that student T62 has not encapsulated solutions to systems of equations as an object. This is evident from his assumption that $x_3 = 1$, instead of recognizing that the last row represents an inconsistent equation. His response is in contrast to that of Student T25 who obtained the correct solution and whose explanation appears below:

R: *As you were testing for linear combination, suppose you obtain the following last row, [showing her the answer script], you obtain the following result* $[0\ 0\ 0\ \vdots\ 1]$. *What would be your conclusion in terms of linear combination?*

T25: *The system of equations is inconsistent because* $0 \neq 1$ *therefore the vectors cannot be written as a linear combination because for a linear combination, the system of equations must be consistent.*

R: *Which other method apart from the method of row reduction can you use to check whether a given vector is a linear combination of given vectors.*

T25: *Hmm the determinant method.*

R: *Can you briefly explain how you go about it?*

T25: *We make use of the matrix on the left hand side [the student here was referring to the coefficient matrix]. If the determinant is not equal to det* $\neq 0$ *then the vector can be written as a linear combination of given vectors.*

R: *What happens if the det = 0?*

T25: *Hmmm, I am not sure now, can we use the determinant method [quiet for a moment], no no I am confused.*

From the discussion above, it is clearly shown that the student has an object understanding of the systems of linear equations since she is able to interpret the solution to the system of equations with respect to the linear combination. The student also has a good understanding of the idea of using the determinant method to check for linear combination. However, she is not able to explain that this method cannot be used to test if there is no linear combination, because if the coefficient matrix of the systems is not invertible, the system then could have many solutions or no solution. Hence this method cannot be solely used to test for not a linear combination.

6.5 Question 7

The item attempt to identify the students' understanding of $span(\boldsymbol{u})$, which is seen as a scalar multiple of $\boldsymbol{u}$, and has its origin at the point 0, and whether they could link the visualization of the *span* $(\boldsymbol{u},\boldsymbol{v})$ to a linear combination of the vectors $\boldsymbol{u}$ and the vector $\boldsymbol{v}$. Thus we intended to provide insight into whether the students had developed the object conception of the concept of linear combination as indicated in the genetic decomposition, section 3.8.2. In part b, the students should be able to explain properties of linear combinations, that, is, looking at whether they have constructed this aspect of linear combination as an object, where *span*$(\boldsymbol{u},\boldsymbol{v})$ is the set of all possible linear combinations of the two vectors.

Describe geometrically
(a) span$(\boldsymbol{u})$ where $\boldsymbol{u}$ is a non zero vector in $\mathbf{R}^3$**.**
(b) span$(\boldsymbol{u},\boldsymbol{v})$ where $\boldsymbol{u}$ and $\boldsymbol{v}$ are non zero vectors in $\mathbf{R}^3$ which are not multiples of each other.

6.5.1 Results for question 7a

We noted that 27 of the students did not attempt to answer the two questions or simply transcribed the question. This is in line with Stewart's (2008) study on geometrical interpretation of linear independence, where she outlined that students struggled to interpret it geometrically. It is important to note that the students were more comfortable with answering the questions where they needed to apply rules and algorithms, for example questions 3 and 8. 34 students partially attempted to answer the first part of the question, but their responses indicated that they struggled to represent the span of $\boldsymbol{u}$ geometrically. The students were simply drawing diagrams showing the xyz plane, thus showing a geometric representation in R^3 but with vectors drawn haphazardly, starting from the origin, for example, the one as shown by student T46 below. Two students demonstrated such an error.

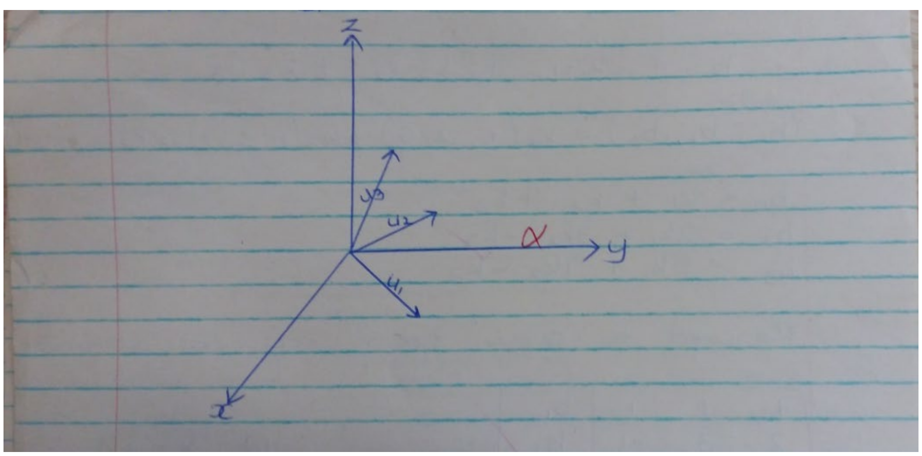


Figure 6.10: Written response of student T46

Two out of the 34 students attempted to draw the span($\boldsymbol{u}$) in a plane, however this shows a violation of the definition of spanning. This showed that the students did not encapsulate the process of linear combinations into an object level of understanding according to the APOS theory. Most of the students simply drew lines passing through the origin and they were not labelled. Eight students revealed such an error. 22 students could draw the vector $\boldsymbol{v}$ passing through the origin but could not explain the idea of being scalar multiples of the vector $\boldsymbol{v}$. It is surprisingly to note, when making reference to definition of spanning, that most of the students viewed linear combination as a scalar multiple, but could not even make a link here with the definition of $\text{span}(\boldsymbol{u})$. This indicates that the students simply learnt these concepts by heart. The diagrams drawn indicated that the students were struggling to picture the span of **v**. 12 of the students were able to draw the correct diagram, passing through the origin, with the idea of scalar multiple e.g. $(c\boldsymbol{u})$ in mind and the vector was in R^3. It is important to note that the students struggled to add an explanation of what the c in the vector was. However, it is important to spell out that span$(\boldsymbol{u})$ consists of all scalar multiples of $\boldsymbol{u}$. However, this prevented these students to develop their mental construction at the object level of understanding of linear combination. The students' responses are summarised in the Table 6.5.

Table 6.5: Students' responses for question 7a

Categories		Frequency
No response		27
Incorrect representations	Diagrams haphazardly drawn starting from the origin	2
	Vector $\boldsymbol{u}$ passing through the origin and drawn in a plane	2
	Vector $\boldsymbol{u}$ not labelled	8
Partially correct representation	Vector $\boldsymbol{u}$ of fixed length starting from origin	22
	Diagrams passing through the origin with scalar multiple representation e.g k$\boldsymbol{u}$ without explanations	12
Completely correct		0

6.5.2 Results for question 7b

The results indicated that 20 of the in-service students did not even attempt to answer the question. The students' responses revealed that the students struggled to understand the notion of $span\ (\boldsymbol{u}, \boldsymbol{v})$, with many students not being able to link it to a linear combination of two vectors resulting in incorrect diagrams. 42 of the students in category 2 attempted the question but had completely incorrect responses. 8 of the students drew 2 free vectors passing through the origin, and 2 of them drew 3 vectors originating from the origin. A large number of students seem to have been misled by the three dimensional nature of R^3 and forced in three vectors instead of the two that were given. These students drew three vectors say $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ starting from the origin instead of the two vectors that were given. This showed a misconception with R^3 expecting that they needed three vectors for R^3. 32 students had such diagrams with one of them drawing one of the vectors found outside the plane. One simply drew the plane passing through the origin.

11 students had partially correct representations. It is interesting to note that the students were able to draw a plane passing through the origin, with the two vectors pointing in different directions. A parallelogram of vectors were also seen but with vectors of fixed length originating from the origin. No explanations were made. An example is that of student T39 who attempted to make reference to the idea of linear combinations, but it was wrongly written. This is shown in Figure 6.11.

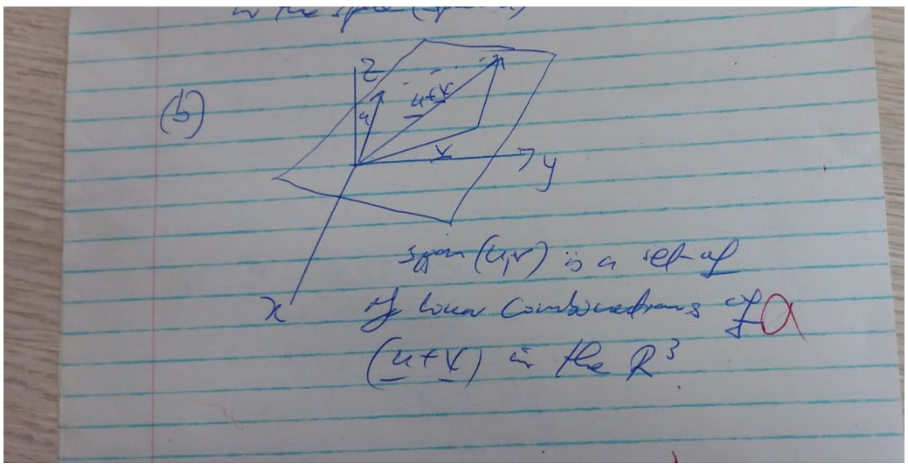


Figure 6.11: Written response of student T39

The student did not recognize that span $\boldsymbol{v}$ is a scalar multiple of **v**, and that span u is a scalar multiple of u which also means that span {**u, v**} consists of all linear combination of $\boldsymbol{u}$ and $\boldsymbol{v}$, which is a plane through the origin determined by the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$. The student then recognised span $(\boldsymbol{u}, \boldsymbol{v})$ as a set of linear combinations of $(\boldsymbol{u} + \boldsymbol{v})$ in the $\mathbf{R}^3$, as shown in Figure 6.11 above. It is important to talk about span$(\boldsymbol{u}, \boldsymbol{v})$ as consisting of all the vectors of the form $a\boldsymbol{u} + b\boldsymbol{v}$ where $a, b \in \boldsymbol{R}$. The students showed some understanding of the connection between the phrase of span$(\boldsymbol{u}, \boldsymbol{v})$ and a set of linear combinations $\boldsymbol{u}$ and $\boldsymbol{v}$ but this was not explicitly outlined. This, therefore, hampered the students from encapsulating the processes into an object level of understanding according to APOS theory. The results are summarized in Table 6.6.

Table 6.6: Students' responses for question 7b

Categories			Frequency
No response			20
Incorrect response	No plane drawn	2 free vectors passing through the origin	8
		3 free vectors passing through the origin	2
	Plane passing through the origin	3 free vectors lying inside plane	30
		2 free vectors lying inside plane	1
		Plane passing through origin	1
Partially correct	Plane passing through origin with 2 vectors without further explanations or incorrect explanations		11
Correct response			0

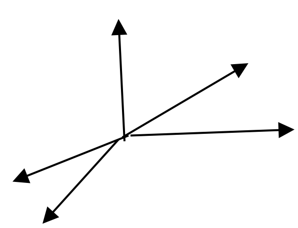
6.5.3 Interview responses to question 7

The following students were probed further so as to understand the APOS level understanding of spanning. In the written response, student T69 did not attempt the question.

R: *Describe geometrically span (**u**), where u is a non zero vector in R^3?* ***T69****: In this case the row in R^3, alright, this one will be a plane (drawing the x and y plane) and hmm and this will mean that from the origin you draw a plane and you show a vector hmm u_1 and u_2 and all those vectors, they should eh be within that plane.*

R: *Can you outline the difference between span(**u**) and span (**u**,**v**), where u and v are none zero vectors equations in R^3, which are not multiples of each other?*

T69: *It means that we have a vector **u** and hmm another vector **v** and all have their initial points at the same point but vector **v** will be having its own direction but will be lying on the same plane like this (showing it on the plane).*



***R**: Can you relate linear combination of vectors to spanning?*

***T69:** I am not sure here.*

In the first instance, instead of talking about $span\ (\boldsymbol{u})$, the student was already referring to the two vectors, that is $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$ meaning that he was attempting to describe $span(\boldsymbol{u_1}, \boldsymbol{u_2})$ unknowingly. I probed further so that he could clarify, but he repeated what he had said initially. Furthermore, the student clearly could not explain the link between spanning and linear combination of the vectors of the two vectors hence demonstrating up a mix up of ideas.

6.6 Question 8

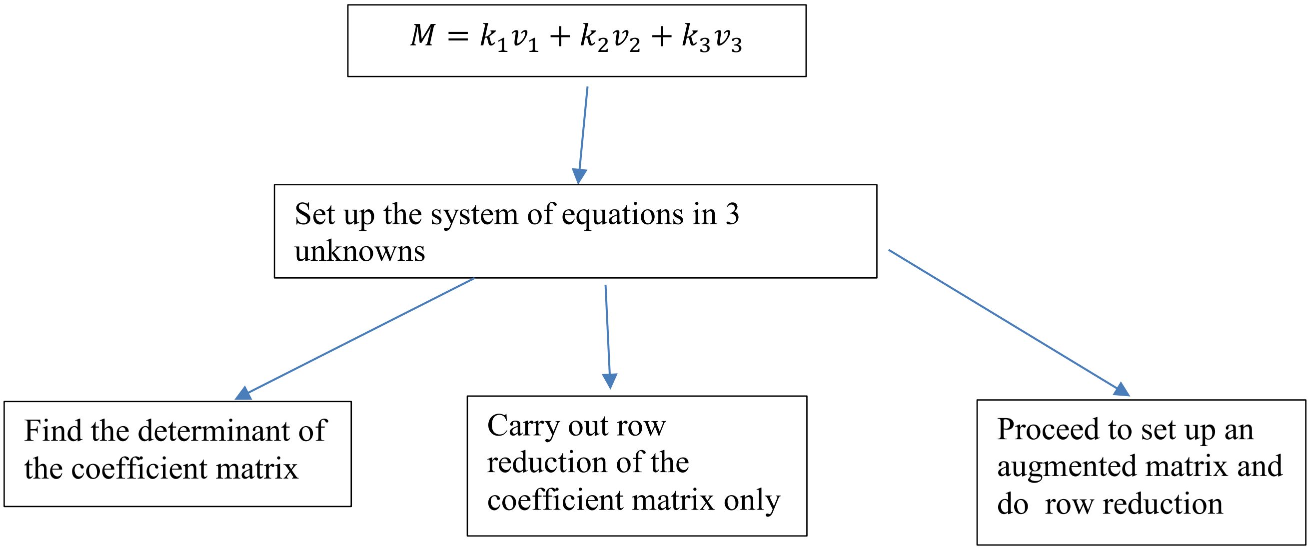
Question 8 addressed the last part of the preliminary genetic decomposition. It was intended to provide insight and aimed at identifying whether students have constructed the object conception of linear combination. This category focuses on exploring students' mental constructions in identifying their understanding of the term linear combination. The item involves identifying whether the students are able to set the vector equation, and determine whether an arbitrary vector of the form $b = (b_1, b_2, b_3)$ in R^3 can be expressed as a linear combination of the vectors $\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}$. The problem then reduces to check and ascertain whether the system is consistent for all values of b_1, b_2 and b_3. The task and the different approaches that can be used to solve the problem are illustrated in Table 6.7.

Table 6.7: Question 8 with the possible ways for solving the question

Task	Steps to follow
8. Show that $\boldsymbol{u_1} = (1, 2, 5)$, $\boldsymbol{u_2} = (1, 3, 7)$ and $\boldsymbol{u_3} = (1, -1, -1)$ do not span $\mathbf{R^3}$.	Students could opt for three methods here: **Method 1.** • Set up a vector equation $\boldsymbol{M} = k_1 u_1 + k_2 u_2 + k_3 u_3$ **where M is arbitrary.** • Set up a system of three equations with three unknowns (k_1, k_2 and k_3). • Represent the system as an augmented matrix.

	• Reduce the matrix to row echelon form and then deduce that the three vectors do not span the space $\mathbf{R}^3$**,** giving a reason. **Method 2** Immediately from the system of equation, calculate the determinant of the coefficient matrix and make a conclusion. **Method 3** Immediately from the system of equation, reduce the matrix to row echelon form, without the arbitrary vector and make a conclusion. **Method 4** Simply consider the vectors as row vectors or column vectors and come up with a 3×3 square matrix A. Find the determinant of A to be zero. Deduce that the system of equations is inconsistent and therefore the three vectors do not span.

As explained in the Table above, the students could have opted for three different methods depending on when they would have recognised the solution which is also expressed as the flow diagram below.



A flow chart of the different approaches that can be used to determine the span of R^n.

6.6.1 Results for question 8

The results indicated that 14 of the students did either not the answer the question, or had an incorrect response. Those with incorrect responses indicated that they had not made the necessary mental construction of what linear is according to genetic decomposition. However, a number of students struggled to come up with the correct vector equation. Unexpectedly 11 of the students formulated a homogenous system of equations instead of non-homogenous, given that the vector (0, 0, 0) was not given. The student below, T70, came up with an equation of the form $Ax = 0$ as indicated in Figure 6.12.

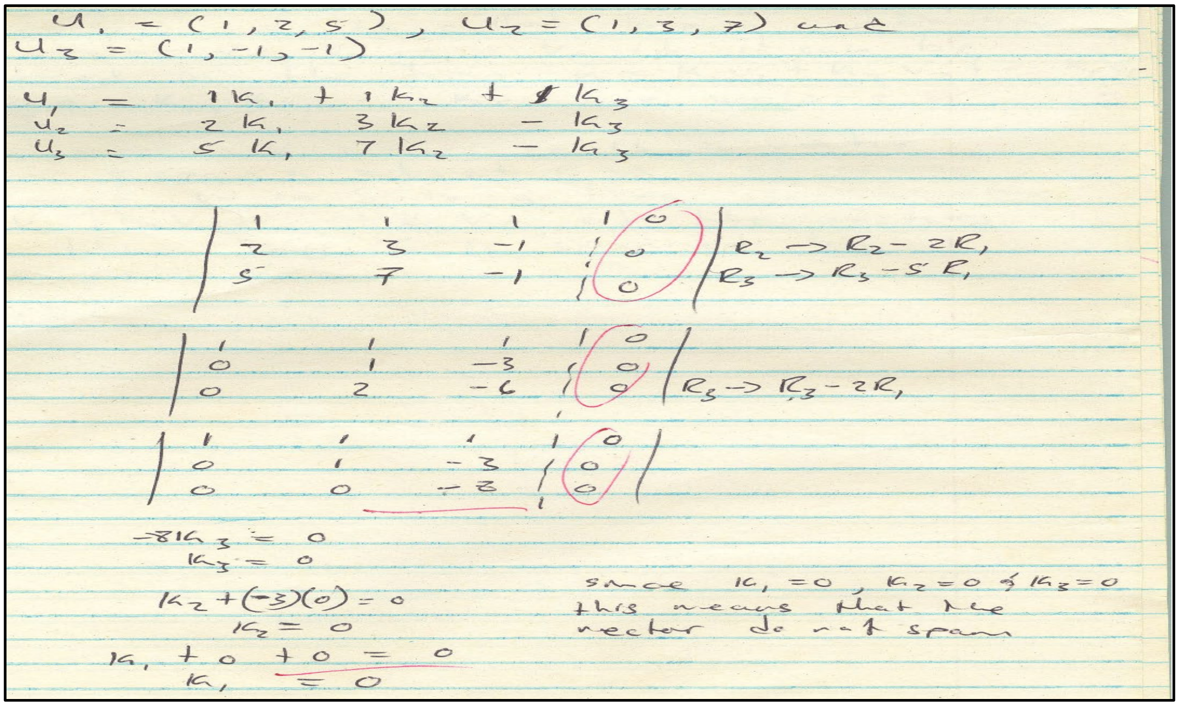


Figure 6.12: Written response of student T70

The above written response showed that student T70 had a poor conceptualization of what needed to be done. The student followed inappropriate procedures in an endeavor to show that the given three vectors spans R^3. Firstly, the student seems to have a mixed up idea as he tried to set up the vector equations. The response indicated the following flaws: the first equation was equated to the vector $\boldsymbol{u_1}$. The same error also appeared in the second and third equations, and also the plus sign is missing in between $2k_1$ and $3k_2$ as well as $5k_1$ and $7k_2$. The student is aware that he must come up with an augmented matrix, which was wrongly written, with the zero vector. This further shows that the student did not construct the meaning of spanning. The procedures for row reduction were done, showing that the student was acting on a step by step manner without recognizing the error he had made. The student here struggled to carry out a correct calculation when attempting to apply the elementary row operation for row three where he wrote: $r_3 \rightarrow r_3 - 2r_1$. The student obtained the value -8, instead of getting 0 and proceeded to say the vectors do not span, without giving a reason. It showed that the student had simply memorized the algorithms with little understanding of the concepts behind spanning. It is clear that the action of representing vectors in coordinate form, formulating the augmented matrix and carrying out elementary row operation has not fully developed.

Another student, T08, applied the procedure for showing that a set of vectors is linearly dependent/independent, instead of trying to show that the set of vectors span R^3. The student formulated the vector equation and equated it to the zero vector. He further expressed the equations of the form $Ax = 0$ and obtained the following reduced matrix after elementary row operations $\begin{bmatrix} 1 & 2 & 5 \vdots 0 \\ 0 & 1 & 2 \vdots 0 \\ 0 & 0 & 0 \vdots 0 \end{bmatrix}$. This student proceeded to do back substitution in a bid to find values of the scalars. When doing back substitution, he obtained the following results: $k_3 = t, k_2 = -2t$. Now to calculate k_1 he obtained the following expression $k_1+2k_2 + 5k_3 = 0$, and wrote $k_1 = -2k_2 + 5k_3$ and obtained k_1= 9t instead of $k_1 = -t$. It is shown that the student solved the vector equation $k_1v_1 + k_2v_2 + k_3v_3 = 0$, and he then deduced (correctly) that the vectors were linearly dependent. However, that was not the question that was posed to him.

I noted that 28 students simply treated the vectors as row vectors and came up immediately with a 3×3 square matrix as shown by student T6 in Figure 6.13.

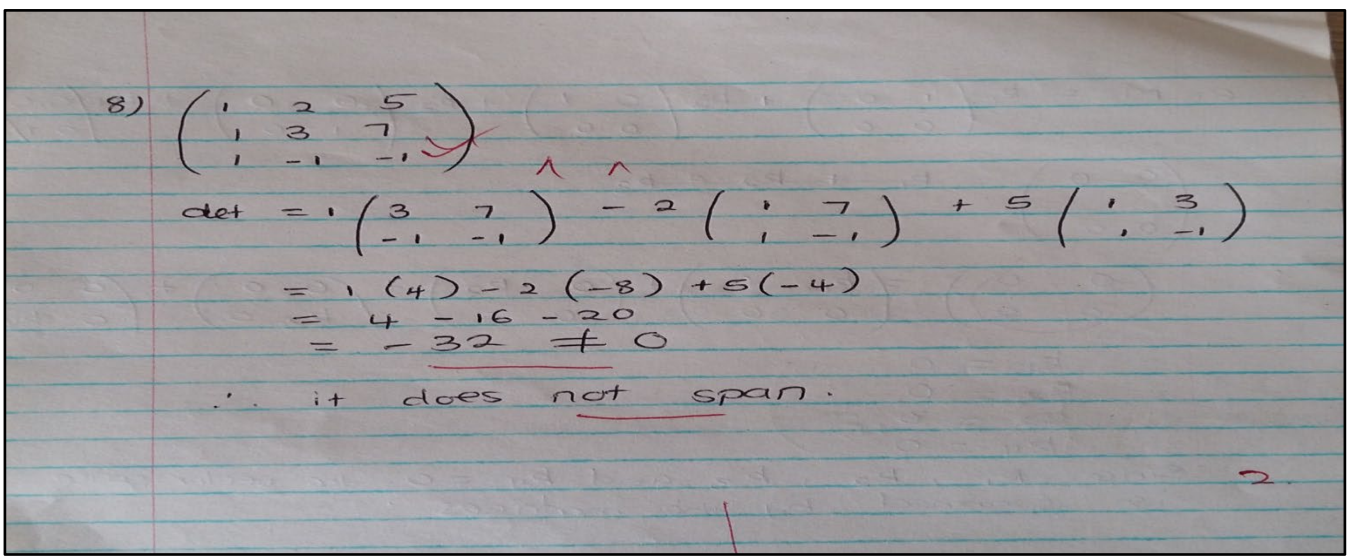


Figure 6.13: Written response of student T6

The student proceeded to find the determinant. But the question is, they are finding the determinant of what? This was indicative of a misconception with the concept of determinant which is a function which needs an input of a matrix such as det (A). The system $Ax = b$, say, is consistent if and only if the coefficient matrix A has a non zero determinant. Correct procedures for determining the determinant were seen, using the method of Laplace transformation. The student tried to apply the theorem but made a careless error with the calculation of the determinant. Even though she obtained the determinant -32, she further made a wrong deduction saying that the vectors do not span R^3. This showed that the student struggled to write the correct arguments as to whether the given vectors span R^3, and this indicated that the object conception of the concept spanning is not fully developed. Four of the students did not treat the determinant as a function and obtained wrong determinants as well as wrong deductions.

20 of the students also used the method of calculating the determinant. Thereafter they obtained the correct determinant, which was zero. Different interpretations of the result were seen. Since the determinant is zero, it means that the system is inconsistent for all values of say b_1, b_2 and b_3, hence it does not span the vectors $\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}$. However, many students made the following deduction: for example student T4 wrote since the determinant is zero, it shows it is linearly dependent and hence it does not span in R^3. Some simply wrote that $det = 0$ hence $\boldsymbol{u_1}, \boldsymbol{u_2}$ and $\boldsymbol{u_3}$ do not span R^3. Four out of the 28 students used the Gaussian elimination method on the coefficient matrix. Two of the students simply manipulated the coefficient matrix incorrectly and obtained the following matrix after row reduction, $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$. The student said 'it does not span R^3', without giving any further explanations. There was the need to provide proper arguments so as to justify their solutions. These difficulties show that the students have problems with the pre-requisite concept of solutions to systems of equations; they need to first deduce from the reduced matrix what the nature of the solution is and only then can they reflect on the implications that it has on the issue of whether the vectors span the vector space R^3. It is clear that they have not constructed the solution to systems of equations as an object; this therefore hampered the in-service students to develop their understanding at the object level according to APOS theory.

The other 19 expressed the vector equation in the form $k_1v_1 + k_2v_2 + \cdots k_nv_n = b$, where b is the arbitrary vector (b_1, b_2, b_3). These students were acting on a step by step procedure. However nine of the students proceeded to find the determinant of the coefficient matrix but without specifying the determinant of what. Only one student, T55, was able to write the following statement: finding the determinant of the coefficient matrix. All these nine students and student T55 were able to find the correct value of the determinant that was equal to zero. The following conclusions were seen in the students' solutions: therefore $det = 0$ therefore it does not span R^3. This resembles a partially correct response because more elaborations were required in-order to come up with a convincing and coherent conclusion. Others wrote that it is not invertible and linearly independent and therefore it does not span. These students seem not to have formed cognitive structures on how to apply the theorems on spanning, especially its meaning. This shows that they had learnt the rules by rote learning and showed surface learning. They concluded by referring to the aspect of linear independence but they did not even have the equation of the form $k_1 v_1 + k_2 v_2 + \cdots k_n v_n = 0$, thus showing a mix up of ideas. Based on the written responses, it seems as if the students were unable to see the determinant as a function, and failure to write correct deduction, prevented the processes to be encapsulated into an object understanding.

The other four students proceeded to consider the coefficient matrix and applied the Gauss elimination method to obtain correct matrix $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$. Student T48 made the following conclusion: that is $b_3 \neq 0$ $\therefore$ do not span. This response shows that the students are able to carry out the steps and carry out some correct algebraic manipulation; however they were not able to use the results from matrix reduction to deduce results related to the issue of spanning. Student T28 said that since R^3 have zeros it is independent $\therefore$ it does not span. The other two students, T62 and T22, simply concluded that it does not span, referring to the answer already given without further elaborations. This perhaps shows a lack of knowing the definition of spanning which was introduced formally, and lack of skill in making corrective deductions. This confirms what the literature alluded to about carrying out deductions in mathematics; for example Star and Stylianide (2013) asserted that it involves some arguments and logical sequence of steps for or against a mathematical assertion.

Five of the nineteen in-service teachers made a successful link between the definition of spanning and Gauss elimination. They converted the system of equations to an augmented matrix in terms of b_1, b_2, b_3 and carried out the elementary row operations up to reduced row echelon. Three of them faltered on the way and two of them obtained the correct reduced row operation as shown $\begin{bmatrix} 1 & 1 & 1 \vdots & b_1 \\ 0 & 1 & -3\vdots & b_2 - 2b_1 \\ 0 & 0 & 0 \vdots & b_3 - b_1 - 2b_2 \end{bmatrix}$. Each step was acting as a prompt for the next procedure. Student T43 concluded that it has many solutions, it does not span and therefore it is linearly independent. The student's misuse of terms and incomplete deduction prevented student T43 to have the relevant structures of linear combination at the object level according to APOS theory.

One student, T25, had a completely correct response. She was able to get the correct reduced row echelon as shown above (i.e. the same result with the two students in category 3) and concluded that it is inconsistent therefore does not span R^3. This indicated that T25 has provided a completely correct result as well as the correct argument as to why the given three vectors do not span R^3. This indicated that the student was able to encapsulate the process into an object. She has developed a conceptual understanding of the concepts on a spanning set. The results are summarized in Table 6.8.

Table 6.8: Students' responses for question 8 on spanning

<table>
<tr><th colspan="5">Categories</th><th>Frequency</th></tr>
<tr><td>No response/Incorrect</td><td colspan="4"></td><td>14</td></tr>
<tr><td rowspan="4">No vector equation</td><td rowspan="2">Use of determinant method</td><td colspan="3">Obtain wrong determinant</td><td>4</td></tr>
<tr><td colspan="3">Obtain correct determinant/insufficient deduction</td><td>20</td></tr>
<tr><td rowspan="2">Using Gaussian elimination method</td><td colspan="3">Incorrect manipulations</td><td>2</td></tr>
<tr><td colspan="3">Correct manipulation but wrong deductions</td><td>2</td></tr>
<tr><td rowspan="6">Set up vector Equation</td><td>Inappropriate vector equation</td><td colspan="3"></td><td>11</td></tr>
<tr><td rowspan="5">Appropriate vector equation</td><td>No augmented matrix</td><td colspan="2">Correct procedures Gauss elimination but wrong/insufficient deductions</td><td>4</td></tr>
<tr><td rowspan="4">Augmented Matrix</td><td>Use of determinant method</td><td>Correct determinant/incomplete or wrong deductions</td><td>10</td></tr>
<tr><td rowspan="3">Using Gaussian elimination method</td><td>Incorrect manipulation</td><td>3</td></tr>
<tr><td>Correct manipulations but insufficient deductions</td><td>2</td></tr>
<tr><td>Correct manipulations and complete deductions</td><td>1</td></tr>
</table>

6.6.2 Interview responses to question 8

In an interview student T25 was asked the following question,

R: *Can you briefly outline the procedures to be taken in order to determine spanning, using another method apart from the one that you have used?*

T25: *Hmm we formulate the vector equations and can use the determinant method on the coefficient matrix but I don't know why. Spanning isn't it the one that hmm no it is for basis and linearly independent.*

In the interview question student T25 does not answer concisely because she seems to be reflecting while talking, for example she clarifies for herself the relations between spanning, linear independence and basis confirming her object understanding of linear independence.

Students T4 and T44 were asked to describe the procedures to be followed in order to determine spanning, given a set of vectors in R^n. Student T4 used the determinant method and obtained the correct determinant, but the conclusion was not sufficient. The following responses were given:

T4*: You choose an arbitrary vector, come up with a system of linear equations and reduce it to row echelon form. If you get unique solutions it means it spans* R^3.

Student T44 had this to say:

T44: *Alright if you can find, do we call it the inverse. If we have to find the inverse of that vectors, then it can be spanned, if it we cannot find the inverse then it cannot be spanned.*
R*: (Giving hints), do we find the inverse or the determinant of the coefficient matrix?*
T44*: The determinant sorry we need to find the determinant first for us to get the inverse, so it's after finding the determinant, if it has the determinant, then it can span the vector space.*
R*: Which method can we us to determine spanning?*
T44: *(Repeating the same method). We can only use the method of finding the determinant.*

From the discussion, student T4 demonstrated a robust understanding of spanning in his brief discussion *"You choose an arbitrary vector, come up with a system of linear equations and reduce it to row echelon form. If you get unique solutions it means it spans R^3.''*

On the other hand student T44's interview revealed his confusion with the various concepts.
The following question was also posed to student T33:

T33 was presented with a hypothetical augumented matrix of the form: $\begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$, that has been reduced so that the last row was zero. The augumented matrix represented the situation where the students were asked to determine whether three given vectors spans R^3. Student T33 was prompted about what could be deduced from such a matrix which has 1 row of zeros. The student responded as follows:

T33: *Taking the reduced matrix, right hmmm* $x_3 = t$ *, the other one* $x = -3t,$ *and the other matrix will be eeeh hey ...eeeh, it does not span.*

R: *Why? Give reasons for your answer.*

T33: *Because hmmm our* $x_3 \neq x_2 \neq x_1$.

Student T33 revealed a misconception of the definition of spanning. The deduction was correct but did not have a correct justification. This indicated that student T33 had not constructed the suitable mental construction for the development of the understanding of the concept of linear independence.

6.7 General observations

The analysis in this chapter attempted to unpack the cognitive difficulties that the undergraduate in-service students possessed when learning the concepts on linear combination. The study revealed that the majority of the students struggled to understand the basic concepts on linear combination and spanning. We were able to analyse students' written work, scrutinize it, conduct interviews and figured out the areas that needed urgent attention.

We noted that many of the students struggled to remember the key concepts when differentiating the two terms linear combination and spanning. The term linear independence and dependence featured very often as students attempted to define the term spanning. Most of the students could not make a clear connection between the two concepts and showed an inability to link any representations. Instead of making a connection between spanning and linear combination, they found themselves immersed into the concepts of homogenous linear systems of equations of the form $k_1v_1 + k_2v_2 + ... + k_nv_n = 0$ instead of coming up with non- homogenous systems of equations. Here it is seen that some experiences that have been leant on linear independence, have greatly impacted on prior learning. de lima and Tall (2008) referred to these infringing factors as the 'met after' which can affect learning in a positive way only if they are used within their domain. The topic on linear independence was taught after teaching the topic linear combination and spanning, but the students did not realise that the concept of linear combination plays an important role when understanding the idea on spanning.

In order to determine spanning most of the students formulated homogenous system of equations instead of non-homogenous system of equations. Some of the students were able to carry the row operations, but they struggled to write the correct justifications for the solutions due to a failure to correctly interpret solutions to the system of equations. When calculating the determinant or carrying out row reduction, many of students demonstrated some calculation errors.

6.8 The APOS analysis emerging from the chapter

The APOS analysis was guided by the preliminary genetic decomposition which appears in Chapter 3 section 3.8.2. From the genetic decomposition, the student who is at action level is able to come up with the vectors equations, express the vectors in coordinated systems, come up with systems of linear equations, and calculate the scalars. The student who is at the process level is able to think through the steps without having to perform each of the steps explicitly and is able to tell whether the given vector forms a linear combination of a set of vectors. The individual must also be able to interpret solutions to the systems of equations. At the object level the individual should be able to carry out further actions on linear combination, for example being able to determine whether given vectors spans R^n. At the object conception students should also be able to represent the span(**u**) and span(**u**,**v**) geometrically.

For the concept on the definition of linear independence the responses from the documentary analysis and interviews suggest that most of the participants had developed at least an action conception. The vector equations of the form $w = k_1u_1 + k_2u_2 + \cdots k_nu_n$ were seen in some students' written work without further explanations. The students struggled to relate the structures that make up linear combination (i.e. indicating the description in terms of the role of vectors and the scalars) which would indicate possible object levels of understanding linear combinations, as purported by Stewart (2008). The students did not make any references to the aspects of the field where the scalars belong or the vector space. These omissions show that the students' understanding of the concept is externally directed, since they did not engage with the meaning of the expression because no further explanations were made, indicating that they simply learnt it by rote. According to the genetic decomposition, this showed that many of the students had mental constructions at the action level according to APOS theory. The results revealed that an action conception was not enough to make a link between the two terms linear combination and spanning. Many students were not able to visualize that the term spanning has no meaning if it is not described in terms of linear combination and a subspace. A set S is said to span a vector subspace V if every element of V can be expressed as a linear combination of elements of S. Only 5 of the students' written responses showed the connections between the definition of the two terms linear combination and spanning suggesting that they had interiorised some of the actions according to the genetic decomposition, as they were able to see the link between the two concepts in terms of definition. Edwards and Ward (2010) commented that if students are able to recall and use the appropriate definition then they are likely to accomplish any given task successfully.

We also noted that some students were comfortable in answering the questions that required the use of algorithms without having constructed the meaning of the concepts and without really understanding them. This was reflected when the students were answering questions 2 and 8. Many of the students were able to set up vector equations i.e. equation 6.1 and carry out step by step procedures, in an attempt to find the values of the unknowns. However, in question 2 some of the students did not check for consistency in the third equation. The other obstacle that hampered the development of the mental constructions fully at the process level was that many of the students struggled to interpret solutions of the systems of linear equations. The work on solving of systems of linear equations was prerequisite concepts to the learning of vector space concepts. However, the presence of these solutions of system of equations on the work on linear combination and spanning greatly affected the performance of students to interiorise the actions into a process. Mhlolo and Schafer (2014) argued that educators must consider the dubious 'met befores' that inhibit learning so as to encourage effective learning. If this is not controlled then knowledge structures will become disjointed and the learners will fail to build connections between prior knowledge and the new, and knowledge gaps will lack consistency.

From the written responses, some of the students' proved that their thinking went beyond the action conception as they were able to interpret the solution to the systems of linear equations and they obtained the correct solution to question 2. From the interviews conducted, most of these students could not interiorise actions into a process as they struggled to describe the steps that could be followed in order to express a given vector as a linear combination of the others without specific vectors or thinking of linear combination without having to perform any operations. The interviews revealed that these students were simply using rules to answer the questions without fully engaging with the concepts.

Evidence from item 8 revealed that many of the participants' were not able to engage with the object understanding of the concept linear combination. Many of the students were able to come up with 3×3 matrices, and carry out row reduction or calculated the determinant in an effort to show that three vectors do not span R^3. Correct methods were seen for showing that the given vectors do not span R^3. 48 (66%) either used the Gaussian elimination method or method of finding the determinant. However, some of these students did not express themselves fully in an attempt to show that the vectors do not span R^3. For the students to make a sensible conclusion, they needed to at least have developed a schema of interpretation of solutions of systems of linear equations and correct manipulations of figures (effective schema of basic algebra). The students lacked the schemas stated such that only one student, T25 seemed to have developed the necessary mental constructions enabling her to carry out further actions on linear combinations, and was able to explicitly outline why the given vectors do not span R^3. The students interviewed could not carry out further actions on linear combinations such as working out sums of linear combinations or determining properties of linear combination. From item 7, none of the students could encapsulate the process into an object as they struggled to represent the span$(\boldsymbol{u})$ and span$(\boldsymbol{u}, \boldsymbol{v})$ geometrically. The students' responses indicated that they did not develop their mental construction at the object level terms of the presented genetic decomposition as they were only able to react from external stimuli of being able to draw lines/plane through the origin showing span($\boldsymbol{u}$)/span($\boldsymbol{u}, \boldsymbol{v}$) respectively. Maharaj (2010) purported that in order for students to develop appropriate mental structures, the aim of teaching is to come up with approaches that help them in understanding the concepts.

However, from an analysis of all the 6 questions, we noted that across the questions, four students showed that they had no idea of the concept of linear combinations and hence are still operating at the pre-action level. These four students attempted some of the questions but had incorrect responses with some of the questions left blank. Therefore overall we concluded that these students were operating within the pre-action stage since they did not respond to an external stimulus of what needed to be done.

50 of the students did not provide any evidence in their written responses of moving past an action conception because they did not see the need to make some explanations at the end or provide arguments to prove for spanning or linear combination. Some of them were able to perform Gaussian elimination, calculating the determinant as well as using the method of solving systems of equation, but were caught out as they were unable to explain whether the given vector is a linear combination or not, nor whether any given vectors span R^n or not, suggesting action conception.

From the written responses, it is likely that 18 of the students were able to construct a process conception as they were able to reflect upon the actions in such a way that they did not require an external stimulus, showing that actions were interiorised into a process. These students were able to justify and give strong arguments as to why a given vector can be written as a linear combination of given vectors by being able to interpret the solutions to the systems of equations as well giving the correct reasoning as to whether the system of equations is consistent or inconsistent. They were able to relate the determinant or Gauss elimination procedures to linear combination and spanning, and had more control over these transformations.

I also noted that one student, T25 might have encapsulated the processes into an object as she was able to recognize the interrelationships and differences between linear combination and spanning. The student had an object understanding of linear combination as a totality as she demonstrated a deeper understanding of the connections between the two concepts linear combination and spanning in terms of the definition. Teacher T25 also have the relevant structures of linear combinations at the object level because she was able relate the concept of linear combination to the notion of span. As evidenced from the interviews, when asked to state the methods used to determine whether a given vector can be written as a linear combination of given vectors, though at first she also included the determinant method, after some probing she was able to explain that the determinant method could not be used to determine concept of linear combination.

6.9 Implications for teaching

In this chapter I suggested a genetic decomposition which shows how students may construct the concepts on linear combinations. The APOS theory was used as a lens to understand how the undergraduate students construct their mathematical knowledge. Mathematical constructs were categorized into their components of actions, process and object. We contend that APOS theory provides an insight into how students understand the given concepts on vector spaces. Maharaj (2010) outlined that the APOS theory calls for the detection of the relevant mental structures that an individual must possess, so that one can design pedagogical strategies to substantiate the constructions of the mental structures. The findings of this study confirms that students find it difficult to understand the concepts of linear combination and spanning that have been introduced using the formal definition. Many of the students did not have appropriate mental structures at the process and object conception. Maharaj (2010) noted that more time is needed or should be devoted to help students develop the mental structures at the process and object level and the teaching should focus on unpacking the structures given in symbolic form. Tall (2014) on the other hand noted that it is important to take into consideration the students' prior knowledge so as to enhance the learning of new knowledge. He also added that instructors need to look closely at the met befores so that they can influence learning in a positive manner.

Bogomonly (2009) comments that the pedagogical weakness in linear algebra is that students are exposed to the use of algorithms that work but the students cannot verbalise the meaning behind what they will be calculating, and do not understand its meaning. She also added that the concepts on the vector spaces are connected, so students need to use them with understanding to enable them to move from one point to the other. The teaching strategies advocated for is to attend to some of the basic concepts that they encountered at elementary age or at high school, before being introduced to these abstract concepts. Ulusoy (2013) advocated for the use of ICT in mathematics at university and further argued that it impacted positively on the students' mathematical skills. He emphasized the use of computer software such as Matlab, Octave and Maple. Consistent with that Aydin (2007) further said such computer algebra systems permit the calculation of elementary row operations, thereby calculating the determinant without making errors. Students can concentrate more on ideas rather than doing the tedious calculations (Dikovic, 2007). Ferrer (2006) further commented that students can only do argumentations rather than concentrating on manipulation skills. Brijall and Maharaj (2010) said that poor performance is a result of the students struggling to adequately have a good grip with the concepts that are expressed in symbolic form, since they represent abstract entities. We then suggested that we need to have a variety of questions that push students so that they develop their understanding at the object level.

We next present a revised genetic decomposition based on some of the issues that emerged in this study.

6.10 Modification of the genetic decomposition

It is noted that there is the need to revise the genetic decomposition so as to capture items that came out of the data analysis or to respond to the demands of that analysis. The revised genetic decomposition will serve as a way of making instruction more meaningful so that it can improve students' understanding. It is evident that students could not construct the structures of linear combination and spanning, mainly because they lacked the prerequisite concepts necessary for the construction of these concepts. The students have the following weak schemas which need to be developed first: working with the binary operations in a vector space and solving systems of equations. It became clear from the written responses and the interviews that many students struggled to interpret the different types of solutions to system of equations in terms of linear combinations and spanning. Hence I have revised the GD to reflect that students need object conceptions of binary operations as well as that of solutions to systems of equations. A further change is in the more explicit descriptions of the actual actions that become interiorised and then encapsulated into an object. From the study it became clear that if the actual actions are not specified it is difficult to use the GD in the analysis.

The modified genetic decomposition is represented in the form of a Table shown below.

Table 6.9: Preliminary and modified genetic decomposition for linear combination

Preliminary Genetic Decomposition	Modified Genetic Decomposition
	Prerequisite concepts The prerequisite concepts to start the construction of linear combinations are object conceptions of the concepts of binary operations of scalar multiplication and vector addition as well as solutions to systems of equations. Since a linear combination is a vector sum of scalar multiples of vectors, an object conception of these binary operations is necessary to conceive the notion of a linear combination. In terms of solutions to systems of equations, the individual needs to be able to see the resulting solutions to systems of equations as a totality irrespective of whether the vectors are in the form of matrices or vectors in R^n . The result from the augmented matrix reduction can be used to deduce results related to linear combination as well as spanning of vectors.
Linear combination **Action** An action conception of linear combinations is evident if an individual, when asked to show that a given vector say $\mathbf{w}$ can be written as a linear combination of the vectors say $\boldsymbol{u}$ and $\boldsymbol{v}$, carries out a series of steps where the one step	**Linear combination** **Action** An action conception of linear combination is evident when the term linear combination acts as an external stimulus of what needs to be done. The first step is to identify the vector that need to be expressed as a linear combination,

acts as a prompt for the next one. The term linear combination acts as external stimulus of what needs to be done. The first step is to form vector equations of the form $$\mathrm{v} = k_1\boldsymbol{u} + k_2\boldsymbol{v}.$$ where k_1 and k_2 are scalars that need to be calculated. The next step is to express the given vectors in coordinate system and then come up with a system of linear equations in two unknowns (scalars). Solutions to the system of equations are then calculated or it is concluded that the system has no solution.	the existence of scalars say k_1 and k_2, k_3 etc The individual uses scalar multiplication and addition to generate a non-homogenous system of equations from the form w = $\boldsymbol{u}k_1 + \boldsymbol{v}k_2$ where or w = $\boldsymbol{u}k_1 + \boldsymbol{v}k_2 + \boldsymbol{z}\boldsymbol{k_3}$ where k_1, k_2 and k_3 are scalars that need to be calculated. A suitable method is selected to determine the values of the scalars and the resulting solutions are checked for consistency if necessary. If the individual uses the augmented matrix method, the matrix is reduced until the solution is identified or until an inconsistency is evident.
Process. The action is interiorised into a process when the individual is able to think through or describe the steps without having to perform each step explicitly; for example an individual might think of expressing a given vector in terms of two given vectors as something like: write the vector as a linear combination of the two other vectors with unknown coefficients and use the coordinate form to express this as a system of equations and then solve the system for the coefficients. This means that the individual can think of an action without specific vectors or even without specifying the number of coordinates.	**Process** The action is interiorised into a process of verifying if a given vector is a linear combination of a given set of vectors, when the individual can think of the actions without specific vectors or even without specifying the number of coordinates. At this stage the individual may not need to perform each step explicitly, and may, for example, make a deduction that the vector can or cannot be expressed as a linear combination of the given set of vectors by predicting the nature of the solution of the augmented matrix. At this stage the individual can also consider a situation of whether an arbitrary vector can be expressed as a linear combination of a set of given vectors (spanning).

	The individual will interpret the results in terms of the scalars appearing in the original vector equation, and use that to decide whether the linear combination is valid. They can see that for a set S to span a vector space V, any vector **v** from V will be able to be written as a linear combination of the vectors in S.
Object: At this stage the individual can see the object as a totality that is, s/he must be able to see and relate the structures that make up a linear combination, (specifying the role of vectors and scalars). The individual can carry out further actions on linear combinations such as working out sums of linear combinations or determining properties of linear combination. Can see the resulting plane as an object in its own right.	**Object** The process of verifying that a vector is a linear combination of other vectors is encapsulated into an object called linear combination, when other actions or processes can be carried out on it, such as working with sums of linear combinations. At this stage the individual is able to explain the properties of linear combination such as showing that a set of vectors is linearly dependent or not. The individual will be able to show that Span S, for a given set S forms a subspace; and to use properties of linear combinations to show that a set is the basis of a vector space. An individual can use a method to find a solution, and verify using another method. The individual must be able to represent the span ($\boldsymbol{u}$) and span ($\boldsymbol{u}, \boldsymbol{v}$) geometrically, linking it to linear combinations. .

CHAPTER 7

ANALYSIS OF STUDENTS' RESPONSES TO ITEMS BASED ON LINEAR INDEPENDENCE/DEPENDENCE

7.1 Introduction

The analysis in this chapter is based on students' responses to activity sheet three and transcripts from the interviews. In this chapter, some of the transcriptions of the students' interviews are based on students' written responses and these are discussed. The interviews were incorporated so that I could understand how the in-service students constructed their mental structures. To ensure that the discussion of the students' responses makes sense, I present some of the definitions and theorems that commonly appear in this discussion.

Definition 7.1. Linearly independent set of vectors

Let V be a vector space over a field k. If S = $\{v_1, v_2, \dots v_r\}$ is a nonempty set of vectors in a vector space V, then the vector equation $k_1v_1 + \ k_2v_2 + \cdots k_nv_r = 0$ ... [Equation 7.1] has at least one solution namely $k_1 = 0,\ k_2 = 0, \dots k_r = 0$. These solutions are called the trivial solution. If this is the only solution, then *S* is said to be a linearly independent set. If there are other solutions in addition to the trivial solution, then *S* is said to be a linearly dependent.

In order to construct the concepts of linear independence/dependence, some of the items required the application of the following theorems:

Theorem 7.1: Let S = $\{v_1, v_2, \dots v_r\}$ in R^n, where $v_1, v_2, \dots v_r$ is a set of vectors in set S. If $r > n$, then the set S is linearly dependent.

Theorem 7.2: A set with exactly two vectors is linearly independent if and only if neither vector is a scalar multiple of the other.

Theorem 7.3: A set S with two or more vectors is linearly dependent if and only if at least one of the vectors in S is expressible as a linear combination of the other vectors in S.

Theorem 7.4: If A is an $n \times n$ matrix, then the following statement is equivalent. $Ax = 0$ has only the trivial solution if and only if the $det\ (A) \neq 0$, meaning that the given set of vectors will be linearly independent. Thus the responses of the students are analysed and presented and this is based on the genetic decomposition presented in section 3.8.3.

7.2 Analysis and discussion of data

The activity sheet consisted of 11 questions. For the purpose of this chapter 5 questions which tested the aspects on linear independence/dependence are discussed here. The results of the study are categorized and reported in terms of question 1 definition of linear independence, question 2 testing linear independence/dependence of which students were required to use inspection, without showing any working, question 5 geometrical representation of linear independence and dependence, question 6 and question 7 testing for linear independence using multiple step procedures and students were required to show their understanding of linear independence or dependence. Question 6 dealt with a matrix representation of vectors and question 7 dealt with the vectors in R^3.

7.3 Question 1

The question was intended to provide insight into whether students had developed the conceptual understanding of the definition of linear independence. The question further addressed the object level of understanding according to the genetic decomposition in chapter 4 and students' difficulties are also analysed. Question 1 is presented below and the scores are summarized in Table 1.

1. Define the term linear independence

7.3.1 Results for question 1

The written responses of students showed that 2 (3%) of the students did not attempt to answer the question, and 10 (14%) of them attempted to answer the question but had incorrect responses. These students showed that they did not make any mental constructions of the term linear independence according to the genetic decomposition. Some students' incorrect responses ranged from incorrectly outlining the algorithms that can be used to determine linear independence to making incorrect links to the notion linear combination and linear independence. Incorrect deductions about the nature of the solution system of equations in relationship to linear independence was evident. Five of the ten students grabbed any words that they thought of in an attempt to define the term linear independence. An example is that of T71 who said that linear independent vectors are vectors where the determinant is not zero. This is similar to the definition given by student T2 shown below. In her explanation she attempted to describe the procedures that are used to determine linear independence, which are not explicitly detailed.

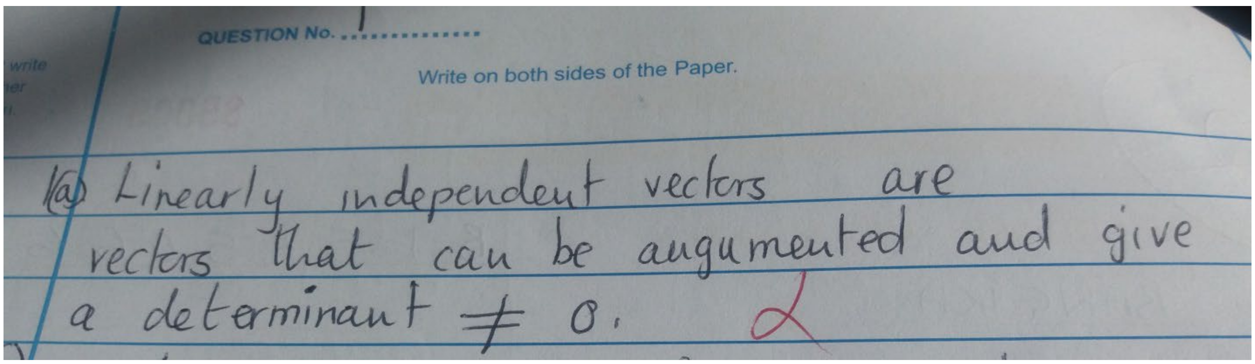
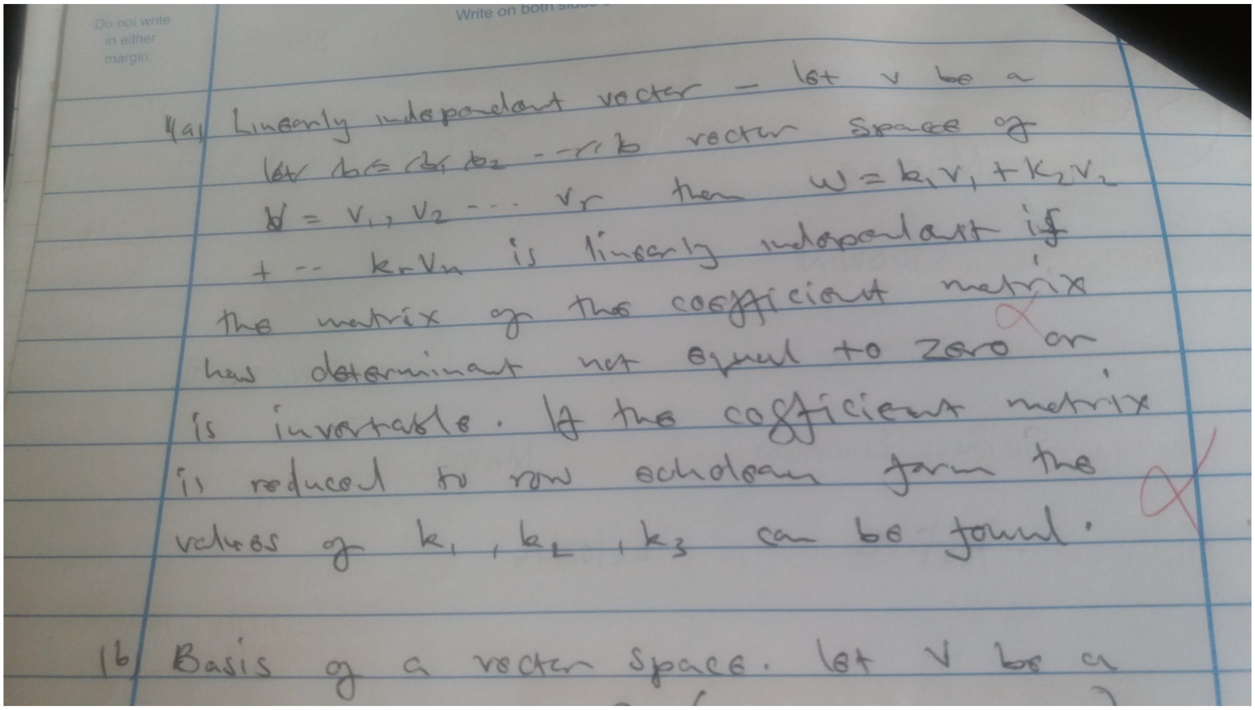


Figure 7.1: Written response of student T2

Here the student attempted to outline part of the procedures that need to be followed in order to determine whether given vectors are linearly independent. This explanation is vague and the improper use of incorrect terminology 'determinant of augmented matrix' hampered the student to develop her understanding at the action level. Another student, T30, said that linear independent vectors are vectors which have a unique solution, which is also similar to student T17 who outlined that linear independent vectors are a set of vectors with only one unique solution if and only if S = $\{v_1, v_2 \dots v_{rn}\}$ and V = $\{k_1 v_1 k_2 v_2 \dots \dots k_{rn}\}$. From the two responses above, it is clear that the students did not have a clear picture of the term linear independence. They defined it in terms of unique solution instead of saying you should obtain only trivial solutions to the vector equation 7.1. and there was the need to be more specific on the exact solution. The two given sets S and V are not described at all by student T17.

A further analysis of students' responses showed that 5 out of the ten students had expressions of the form $w = k_1 v_1 + k_2 v_2 + \cdots k_n v_n,$ within their explanations. This shows that these students could not distinguish between the terms linear combination and linear independence, see student T36's written response in Figure 2.


1(a) Linearly independent vector – let V be a
vector space of
V = v_1, v_2 --- v_r then $w = k_1v_1 + k_2v_2$
+ -- k_rv_n is linearly independent if
the matrix of the coefficient matrix
has determinant not equal to zero or
is invertable. If the coefficient matrix
is reduced to row echolean form the
values of k_1, k_2, k_3 can be found.

1b Basis of a vector space. let V be a

Figure 7.2: Written response of student T36

In the above extract, student T36 exhibited a number of misconceptions. Firstly he wrote that V is obtained from the vector space V = $v_1, v_2 \ldots v_r$. Secondly the vector **w** is not explicitly defined. From his definition above, he has simply expressed the vector $\boldsymbol{w}$ as a linear combination of the vectors $v_1, v_2 \ldots v_r$. This shows that the student could not distinguish between linear independence and linear combination of vectors. There was a need to explain that the vector equation was supposed to be equated to the zero vector. The student then further expressed her thoughts in terms of the procedure that can be used to determine linear independence instead of defining what linear independence is. Basing on the procedures outlined, there are quite a number of mixed steps. Firstly, it is not clear where the coefficient matrix is coming from yet there is the set of the sum of scalar multiples of given vectors. If the elementary row echelon procedure is done, it is clear that many of the essential actions have not been interiorised into a process of what linear independence is.

Based on the students' responses in this category, it is clear that the action conception of the definition of linearly independent/dependent had not developed. The terms that they met before, i.e. on linear combination and the step by step procedures that are used to determine linear independence, seemed to be a barrier in understanding the concept of linear independence and this hindered the students' development of the necessary mental constructions at the object level. This is an indication that the students have made no mental constructions as deliberated in the APOS theory and can be placed on the pre-action stage.

22 (30%) of the students showed some idea about the definition of linear independence by presenting the equation similar to 7.1. However students revealed some misconceptions and some important information supporting the definition was missing as evidenced by student T12's response below.

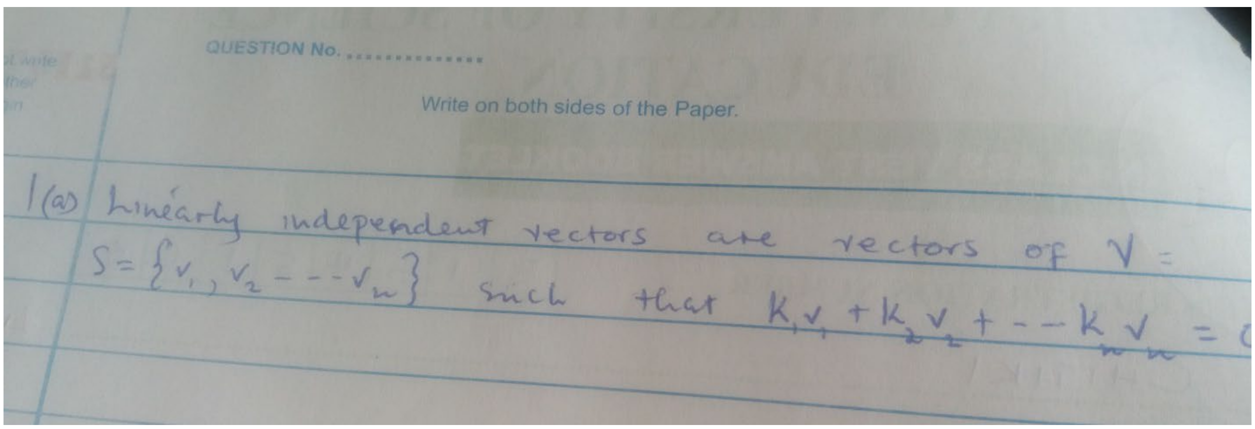


Figure 7.3: Written response of student T12

An analysis of student T12's response revealed that some important information is missing. Student T21 outlined that vectors are linearly independent if $k_1v_1 + k_2v_2 + \cdots + k_nv_n = 0$ such that k_1; k_2 and k_3 have unique value. Both definitions above show that the two students missed a number of important details such as where the vectors $v_1, v_2 \ldots v_n$ belong, as well as the types of solutions that can be obtained. Also student T21 needed to explain further the idea of unique values for the scalars k_1, k_2 and k_3, which was incorrectly stated. The students met the term unique value when dealing with the concepts on solutions of systems of equation but the term is misplaced in this context. The students could not fully provide an explanation of the structure of linear independence of a set of vectors.

30 of the students provided a correct representation of vector equation 7.1. This is illustrated by student T73's written response shown in Figure 7.4, but the accompanying explanations were incomplete.

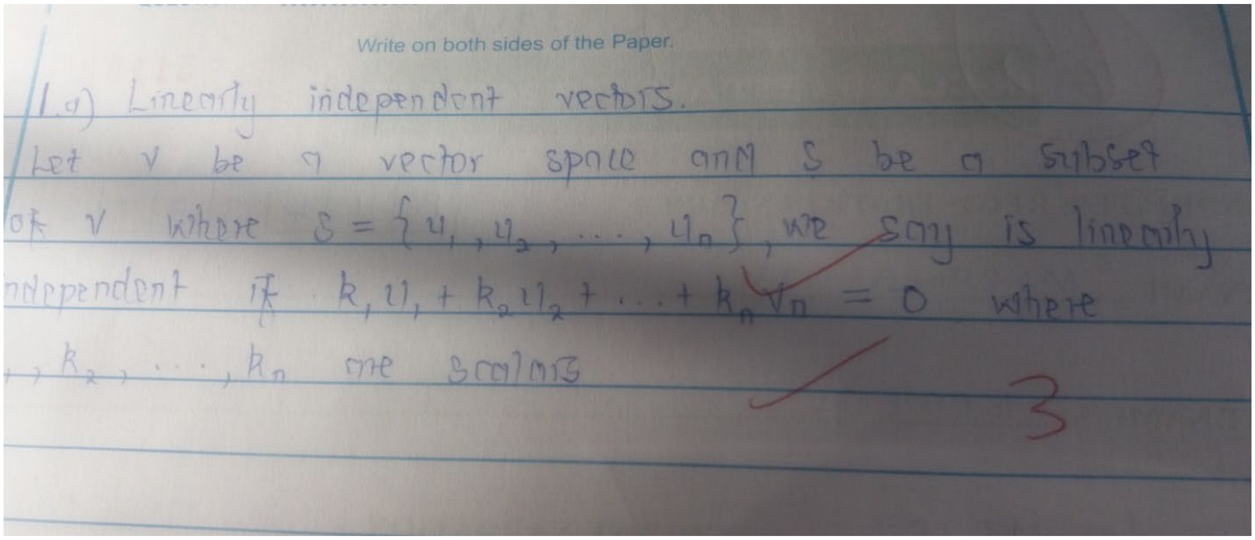


Figure 7.4: Written response of student T73

From the above written response by student T73, it can be clearly seen that the student's definition is incomplete. The crucial condition that the equation was only satisfied when the scalars $k_1 = k_2 = k_n = 0$ was not provided. Ndlovu (2013) outlined that failure to build a meaningful and coherent understanding of some notation used in mathematical concepts means that the concept is not conceptually understood and this could be an obstacle to meaningful learning. The response indicated that the meaning of the concept linear independence was not fully encapsulated into object.

Nine of the students provided complete and correct definitions suggesting that these students had made some of the necessary constructions called for by the preliminary genetic decomposition involving the definition of linear independence. The students expressed the definition in symbolic form and were able to demonstrate its link to the linear combination of vectors. The students seemed to encapsulated the processes into an object when they constructed and presented a coherent definition of the term linear independence, giving a strong argument and incorporating the following aspects: they were able to explicitly outline the weights that must be obtained as purported by Stewart (2007), that is showing that the scalars $c_1, c_2, \dots c_n = 0$, outlining were the vectors belongs to as well as scalars. The students were able to use the language appropriately when constructing the definition. This indicated a high level of mathematical abstraction in the formal world as outlined by Dorier (2000). The results of question 1 are summarized in Table 7.1.

Table 7.1: Allocation of scores for question 1

Categories		**Frequency**
No response		2
Incorrect response	Incorrect explanation in terms of step by step procedures for determining linear independence/dependence wrongly written	5
	Expressions of the form $w = k_1v_1 + k_2v_2 + \cdots k_nv_n$	5
Some correct ideas	Vector equations of the form $0 = k_1v_1 + k_2v_2 + \cdots k_nv_n$ but with wrong statements supporting it.	22
Correct ideas but incomplete	Vector equations of the form $k_1v_1 + k_2v_2 + \cdots k_nv_n = 0$ but without supporting statements.	30
Completely correct response		9

7.3.2 Interviews responses to question 1

Student T69 in the written response wrote: linearly independent vectors are such that, if S = $\{v_1, v_2, v_3 \ldots, v_n$ a set of vector in a vector space V, then $w = (a, b, c \; \ldots . n) = k_1v_1 + k_2v_2 + k_3v_3 \ldots + k_nv_n$ where $k_1, k_2, k_3 \; \ldots k_n$ are scalars, only if $k_1 = k_2, = \; \ldots = k_n = 0$. The student here revealed a misconception by equating the vector equation to an arbitrary vector. In the interview student T69 has this to say:

T69: "I think linear independence is when the solution vector gives one solution, or when we use the determinant method and if the determinant is not zero, the vector is linearly independent."

Student T69 attempted to combine the two methods that can be used to determine linear independence. The first method involves using Gaussian elimination of the matrix say A, representing a homogenous system of equations arising from equation 7.1. He also referred to the determinant method which entails showing that det $A \neq 0$. However, he got confused in his expression and the ideas were not explicitly outlined. This showed that the student is still operating at the action level according to APOS theory. The following intercept reveals the interview done with T57. In the written response, student T57 defined the term linearly independent as follows.

Given Was a set of vectors $v_1, v_2 \ldots, v_n$, W can be expressed as in $k_1v_1 + k_2v_2 + \cdots k_nv_n = 0$. If $k_1, k_2, \ldots k_n = 0$, where $k_1, k_2, \ldots k_n$ are constants this indicates a trivial solution which indicates that the vectors are linearly independent. Her definition was correctly written. The following interview excerpt took place:

R: *In your own words can you define the term linear independence*

T57: *Definition in terms of wording?*

R: *Can use any form that you are comfortable with.*

T57: *Linear independent vectors [quiet for a moment] thus when we have ma trivial solution (using mother language), whereby we get k_1, k_2 and k_3.*

R: *How do we get the trivial solutions?*

T57: *Thus when we have the unique solution.*

R: *How do we get the unique solutions?*

T57: *We have the vectors then after that we test for linear independence e,g. by doing row reduction, then after doing row reduction, we get the unique solutions for the scalars k_1, k_2 and k_3.*

R: *What else?*

T57: *Kept quiet*

From the conversation with T57 the student has the idea that we must obtain the trivial solution but is confusing the terms trivial and unique solution. She also attempted to use the procedural way in an attempt to define the term linear independence when she included the ideas of doing Gaussian elimination method. Her response shows that even though she provided the correct definition she did not understand all the connections in the structure of the notion linear independence. This shows that T57 could not see the aspect of linear independence as a totality, and hence did not encapsulate the processes into an object understanding of the definition linear independence. The student is operating at process conception which is not fully developed.

Another student who also obtained the correct definition in the written response, that is student T25, was also interviewed. When asked to define the term, she first outlined that given a set say S $= \{v_1, v_2 \ldots v_n\}$ then the vector $k_1v_1 + k_2v_2 + \cdots k_nv_n = 0$, and it must give the same solution for the scalars that is zero, zero, zero, from the augmented matrix $Ax = 0$. She also outlined that either the determinant method or the method of elementary row operations can be used to determine linear independence/dependence. However she was further asked to attach the meaning if the coefficient matrix gives say $det(A) = 0$. She outlined that it means that the set of vectors are linearly dependent but she could not provide a convincing argument of the meaning in terms of the systems of equations being consistent or not. However, from the discussion with student T25, it is evident that she showed a correct concept image of the definition of linear independence. So based on the explanations given, the student was moving towards an object understanding of the concept linear independence/dependence. In their study about students' understanding of matrix, Kazunga and Bansilal (2018) noted that the deductions about APOS levels should not be based on students' written responses only. The processes were, however, not completely encapsulated into thinkable objects due to a failure to give a strong argument on reflection of why the system of equations is linearly dependent if $det(A) = 0$.

7.4 Question 2

Question 2 and the results based on finding whether given vectors are linearly independent or not using inspection are shown below.

Table 7.2: Allocation of scores for question 2

2. Explain whether the following are linearly independent or not justifying your result. (Solve the problem by inspection).

(a) $\boldsymbol{u_1} = (-1,2,4)$ and $\boldsymbol{u_2} = (5,-10,-20)$ in R^3.
(b) $\boldsymbol{u_1} = (3,-1), \boldsymbol{u_2} = (4,5)$ and $\boldsymbol{u_3} = (-4,7)$ in R^2.
(c) $p_1 = 3 - 2x + x^2,\ p_2 = (6 - 4x + 2x^2)$ in p_2.
(d) $A = \begin{bmatrix} -3 & 4 \\ 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -4 \\ -2 & 0 \end{bmatrix}$.
(e) $(2,-3,7)$, (0,0,0) , (3, -1, -4)

	Percentage with correct result and interpretation
2a	42 (57%)
2b	13 (17%)
2c	54 (74%)
2d	35 (48%)
2e	28 (39%)

The question addressed the process level of understanding according to the genetic decomposition**.** According to the question, the individuals were supposed to be able to explain without showing any working whether the given vectors were linearly independent or not giving correct reasoning, and choosing the most appropriate theorem. Thus the item aimed at providing insight and exploring the students' mental constructions to see if they could make arguments based on the applications of the theorems whether the given vectors are linearly independent or not without performing the steps.

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The following questions items 2a, 2c and 2d required the use of theorem 7.2. For task 2a, 42 (57 %) (Figure 2) students were able to state that the set is linearly dependent and made a correct deduction based on the aspect of one vector being a scalar multiple of the other. It is also noted that 54 (73%) and 35 (48%) students for question 2c and 2d respectively were able to make correct deductions with correct reasoning without needing to carry out any algebraic manipulations . This showed that these students have possibly interiorised the actions that define linear independence by constructing a mental process that is perceived as being internal to the individual rather than responding to external cues. The students thus had a process conception of linear independence.

However, the following shortcomings were evident in the students' responses to the three items: (a) individuals trying to come up with vector equations and augmented matrix; (b) individuals trying to calculate the determinant inappropriately; c) wrong theorems were applied. More details of these difficulties are now presented. From the analysis above, it is important to note that the students were more comfortable with engaging with vectors in R^n (part 2a), the polynomial functions (part 2c) as compared to the vectors in matrix form (part 2d).

7.4.1 Results for question 2a

There were nine students who did not use inspection but formulated a vector equation of the form $k_1v_1 + k_2v_2 = 0$, in a step by step manner. The given equation was expressed in coordinate system form and a system of 3 equations in two unknowns was formulated. T72, T29, T65, T61, T48, T38 and T21 tried to solve them simultaneously but were stuck and some made computational errors and they wrote linearly independent since $k_1 = k_2 = k_3 = 0$. Another student, T7, proceeded to formulate the augmented matrix but did not attempt to reduce it but wrote: It is linearly independent since cofactor matrix has rows that are less than columns. The word cofactor matrix used here is inappropriately used. These students could be operating at the action stage since they are engaged in a step by step procedure which, however, is not fully developed. Another student, T62, simply said by inspection there are many solutions implying that linearly dependent.

Some students viewed the given vectors as row vectors and the students came up with a 2× 3 matrix, as shown by the response of student T27 in Figure 7.5.

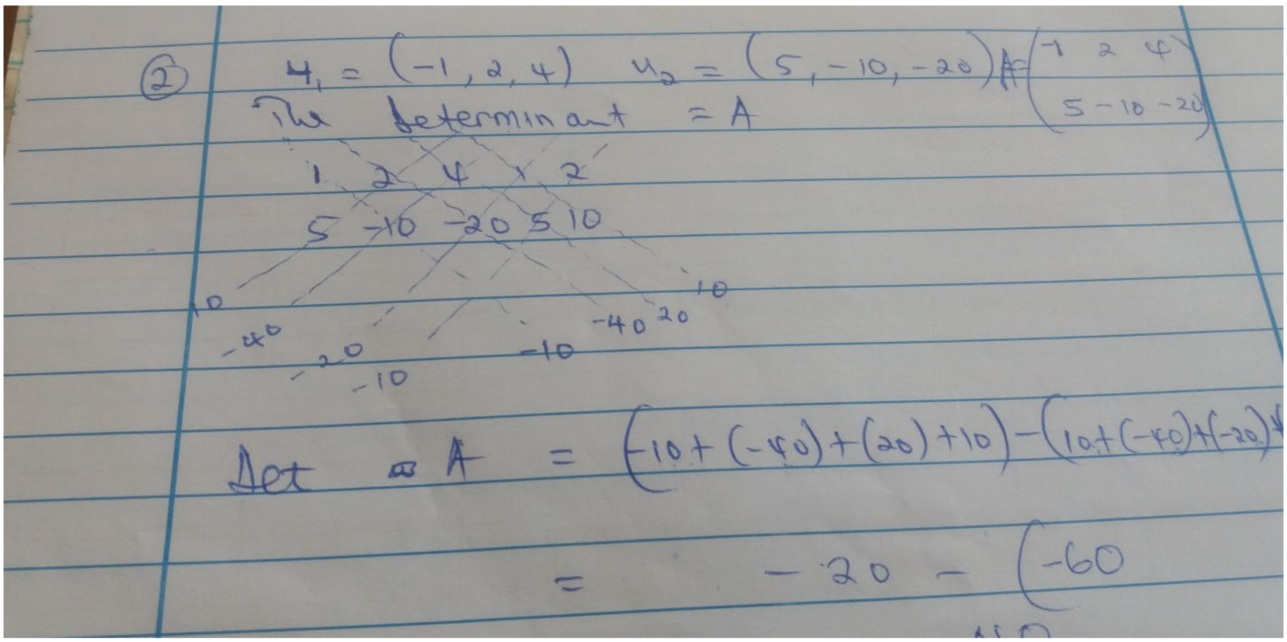


Figure 7.5: Written response of student T27

The student above treated the given vectors as row vectors and came up with a matrix which he named matrix A, and tried to calculate the determinant using Sarrus rule. He did not realise that matrix A is not a square matrix. Sarrus rule is only applicable to a 3×3 matrices. He did not see that the determinant method was not applicable and it acted as a barrier to the learning of linear independence which was used out of its domain. The wrong conceptions are emanating from prior knowledge on the notion on determinants which was not well captured. Thus students' prior experiences can greatly affect new learning either positively or negatively (Tomito, 2008). This has a great negative impact when constructing appropriate knowledge of the concept on linear independence. However, some of the students were able to identify that it is not a square matrix but revealed more errors by adding another row of zeros so as to force it into an $n \times n$ matrix form as shown by student T70 in Figure 7.6.

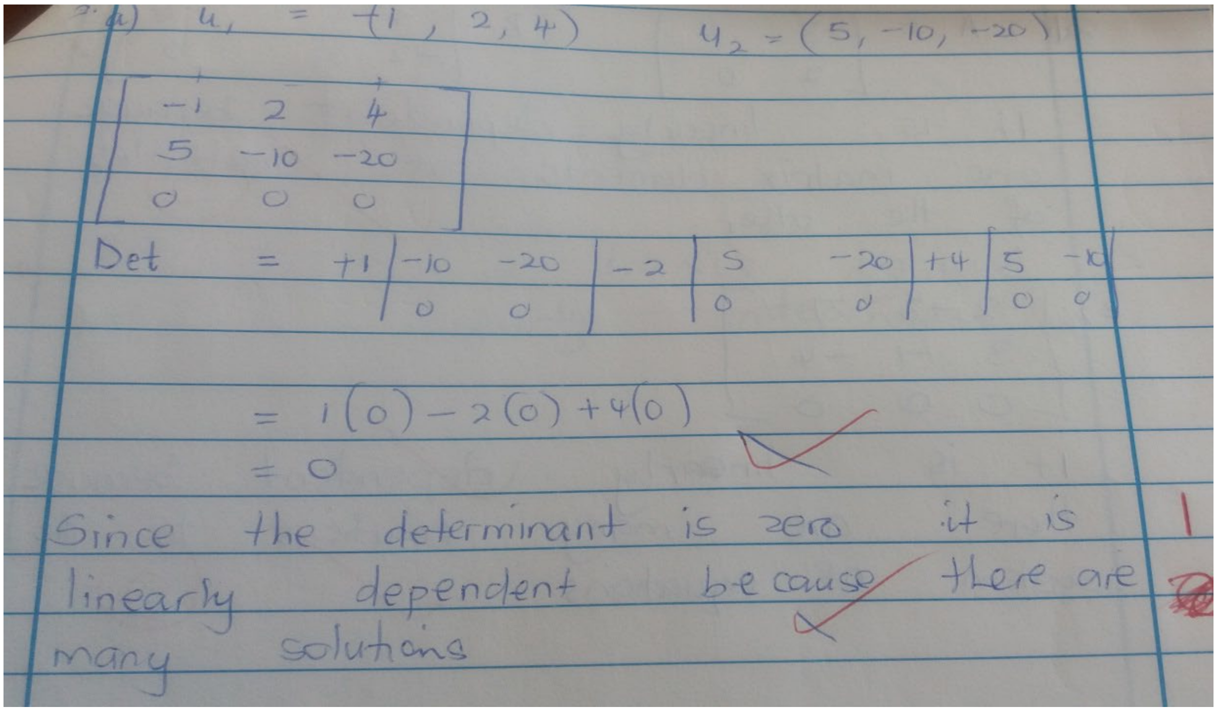


Figure 7.6: Written response of student T70

This suggests that the students can only perform the actions linked to the determinant method (finding the determinant of the coefficient matrix so as to make deductions about linear independence). Hence the use of the determinant method hindered the students to interiorising the actions into a process understanding of linear independence/dependence. Thus six of the students attempted to find the determinant, and these only developed an action conception linked to the determinant method.

6 of the students attempted to apply theorem 7.1 but it was incorrectly interpreted, for example T39 wrote it is linearly independent because the number of components in vectors are less than the no of vectors since $r > n$, with r columns and n rows. 10 (13%) of the students were grappling and just wrote any words that came to mind for example T13 wrote linearly independent there are no multiples in the vectors, T5 wrote since $r < n$ and scalar product therefore linearly independent. The above students could not build their mental construction at the process level of understanding of the concept linear independence/dependence.

7.4.2 Results for question 2c

Most students got this question correct. For this task the problem reduces to ascertain that the polynomial P2 is a scalar multiple of P1 and therefore the system is linearly dependent. However, students provided different interpretations of the given vectors. Eight of the students used inspection but made an incorrect deduction. They recognized that one vector was a scalar multiple of the other but made wrong deductions saying that the vectors are linearly independent. Perhaps these students had memorized the algorithms for determining linear independence/dependence without understanding the difference between the two terms. The students seemed to have constructed their mathematical knowledge as isolated facts because they could not see the difference between these two concepts. The remaining 11 (15%) of the students simply said it is linearly dependent giving wrong or no reasoning. Among those giving wrong deductions examples was T48 who said that it is linearly dependent since $r < n$, vector is less that the number of equations, T49 said it is linearly dependent because of x^2 (quadratic aspect). From Figure 7.1 it is evident that 54 (74%) of the students were not bound to a sequence of steps and were able to give correct deductions with correct reasoning suggesting that they may be operating at the process level of understanding according to APOS theory.

7.4.3 Results for question 2d

In order to determine that the two given matrices are linearly independent or not, quite a number of flaws were evidenced. Ten of the students used guess work by simply saying linearly dependent/ linearly independent and grabbing any terms inorder to justify the answer, with some simply transcribing the problem. The other students seemed to confuse the methods used to determine linear independence. Nine students preferred to use step by step procedures and applied the determinant method wrongly. They calculated the determinant of matrix A and the determinant of matrix B separately and then concluded that the two matrices were linearly independent because the determinants were both equal to -8. They confused this with the idea of finding the determinant of the coefficient matrix. The response of T72 illustrates this error.

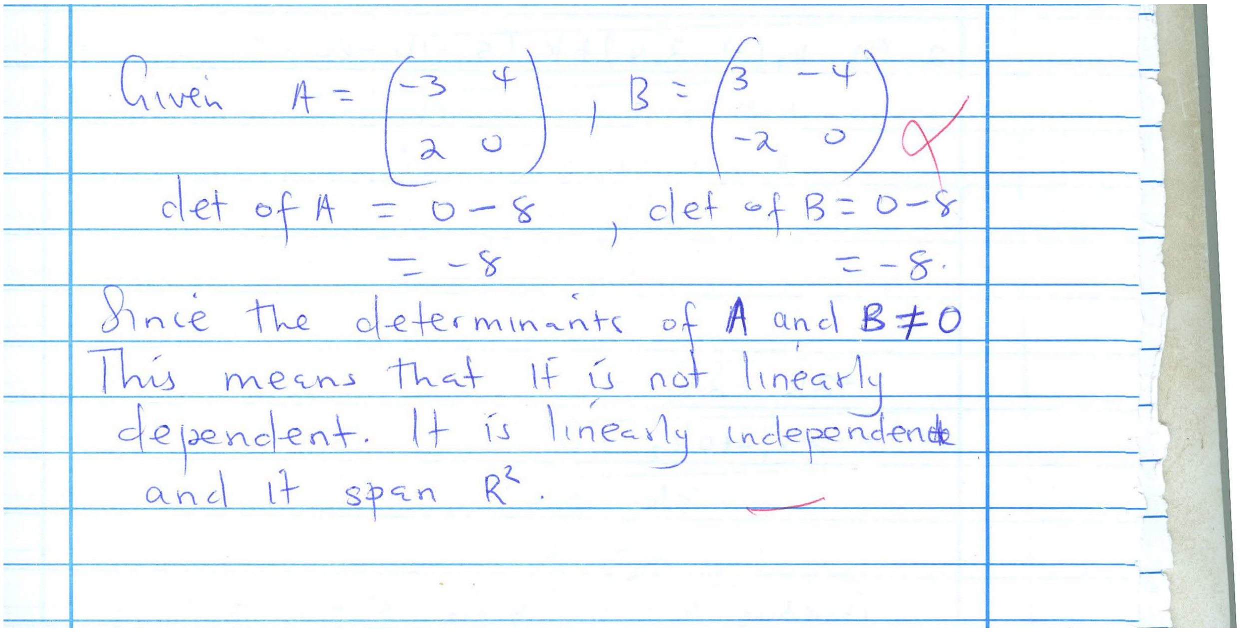


Figure 7.7: Written response of student T72

Here the student was only concerned with the idea that if $det \neq 0$ then it implies that the matrices are linearly independent. The student proceeded to deduce even further that the vectors spanned R^2 again giving the result in terms of spanning. The student intended to present the work in a step by step manner, but wrong methods were evident. These responses are completely incorrect. This analysis showed that the student did not interiorise the operation of linear independence to a process level engagement.

Five of the students came up with a vector equation of the form $k_1v_1 + k_2v_2 = 0$, and then two of them were able to do scalar multiplication and they attempted to equate corresponding elements. However solutions of the form $k_1 = 0, k_2 = 0$ were seen without providing any algebraic manipulations and justification. The other three did not do scalar multiplication but gave solutions of the form $k_1 = 0, k_2 = 0$. The students concluded that was linearly dependent. Two students, T12 and T14, wrote expressions of the form $k_1v_1 + k_2v_2$ without equating them to zero. They simply wrote 'it is linearly dependent'. By formulating the vector equations, this shows that the students are operating at the action level which is not fully developed. 12 of the students wrote that $A = -B$ and concluded that they were linearly dependent, without further explanations. This shows that these students had not made the necessary mental constructions in order to determine the notion of linear independence/dependence at the process level of understanding according to the genetic decomposition. The results of question 2a.2c and 2d are summarised in Table 7.3.

Table 7.3: Allocation of scores for question 2a, 2c and 2d.

Categories		Frequency		
		Part 2a	Part 2c	Part 2d
Incorrect responses	Used inspection wrongly / grabbed any terms leading to illogical deductions	16	19	10
	Used the determinant method	6		9
	Used method of elementary row operation/attempt to formulate vector equations	9		7
	Expression of the form $A = -B$			12
Correct responses	Use of correct theory and justification	42	54	35

7.4.4 Interviews responses to question 2a, 2b, 2c

During the interviews, the students were again given the vectors and they were asked to describe in their own words how they could express the given vectors for item 2a, 2c, and 2d. It is interesting to note that some of students interviewed were able to foresee that there is no need to do any calculation since one of the vectors is a scalar multiple of the other to the three given items. The following students T57 and T13 obtained incorrect solutions were interviewed and have this to say,

T57: *Item 2a was linearly dependent because the number of vectors is less than the vector in a vector space.*

Her imprecise use of language hindered her to develop her understanding at the process level.

T13: In item 2a this one is a multiple of this one, pointing to the vectors.

Student T13's response is not explicitly defined. Another student T23 when asked to tell whether the matrices are linearly independent or not said,

T23: Hmm they are the same or one is a multiple of the other hence linearly dependent because we multiplied by a negative.

Student T23 obtained the correct solutions in the written response. By using the word or, this shows that the student is not confident as to what the solution is. Another student, T21, who obtained incorrect solutions for the whole of question number 2 actually said that she didn't have an idea. It is evident that T27 continued to cling to the misconceptions that he had in the written response when asked to explain whether the following were linearly independent or not see Figure 7.5. The following interview took place with student T27.

R: *Pointing to question 2a, can you explain whether the following vectors are linearly independent/ linearly dependent?*

T27: *I need to write the vector equation hmmm and come up with systems of equations and then calculate the determinant of the matrix obtained.*

R: *[Probing further] What is the order of the matrix, and how would we calculate the determinant. [trying to draw attention to the fact that W is not a square matrix]*

T27: *[writing down the matrices, and attempting again to use Sarrus rule], The order of the matrix is* 2×3.

R: *Is it possible to calculate the determinant? [probing again about the size of the of the matrix]*

T27: *Yes can find the determinant of any matrix, isn't it by using Sarrus rule and the method of cofactors.*

From the discussion above it can be seen that student T27 still struggled to make an argument without showing the step by step procedures in an endeavor to show whether the given vectors were linear independent or not. However student T27 is still operating beyond the action conception since he argued that he can find the determinant of an $m \times n$ matrix in order to determine whether given vectors are linearly independent or not. The other four students provided incorrect explanations. They have an incomplete conception of whether the three vectors are linearly independent/ dependent. This indicates that the process level is not developed because of their inability to recognise that we can only find the determinant of a square matrix. This interview excerpt alerted to the important role played by the determinant in constructing the understanding of linear independence and hence it was added as a prerequisite concept in the genetic decomposition.

7.4.5 Summary of question 2a, 2c and 2d

These three questions were based on the use of theorem 7.2. The students did very well in question 2c, which was in the form of a polynomial. The reason might be that they easily identified the scalar multiple 2. For parts 2a and 2c the scalar multiple was not as obvious as compared to that of part a. The other reason was that it was a question about the matrix representation in part d that made it difficult to realise that $A = -B.$ Also in part a and part c the students used the determinant method inappropriately. The students also used the step by step procedures such as formulating the vector equations and attempting to do row reduction, showing that they could not interiorise the actions into a process level understanding according to the genetic decomposition.

7.4.6 Results for question 2b

The problem required the students to judge the relationship between r and n given a set of vectors: say S = $\{v_1, v_2, \dots v_r\}$ in R^n. If $r > n$, then S is linearly dependent. Only 13 (18%) of the students were able to outline that the set is linearly dependent with correct reasoning. The question was more difficult than the others most probably because the students could not see the connections between the symbols that were used that is the r and the n. Students could not link it to a system of equations with r unknowns and n equations. The following shortcomings were popular: (a) two attempted to find the determinant of the 3 vectors; and (b) wrong theorem being applied/ deduction. The rest of the students i.e. 58, had wrong deductions of varying levels and applied wrong theorems. Some students simply wrote that it is linearly dependent since $r > n$ without specifying what the r stands for and what the n stands for, or specifying how many equations and how many unknowns. Student T62 wrote it in a different way. He said that it is linearly dependent because $r < n$ in R^n. This a misunderstanding of the concept because the number of unknowns r is greater that the number of equations, that is n. This showed that the students simply memorized the rule without understanding the concept put forward. Another student, T2, said that it is linearly independent since it is not connected by any scalar, with student T53 writing that they are linearly dependent because they are on the same plane. The two deductions showed that these students had some ideas which were expressed vaguely. Student T53 attempted to define the term in terms of geometrical interpretation. Another student, T45, put it as follows: it is linearly independent for there are no parallel vectors, since the no of rows is greater than the number of columns in the matrix $\begin{bmatrix} 3 & -1 \\ 4 & 5 \\ -4 & 7 \end{bmatrix}$. It was important for the students to build some internal representation in the mind, formulate a homogenous system of equations in say r unknowns and n equations, then if r > n, it means that the system has nontrivial solutions meaning that the vectors are linearly independent, without any step by step procedure. Thus the use of theorem 7.1 has hindered many of the in-service teachers from developing their understanding at the process level. This supports the contention by Ndlovu (2013) that the instant abstract algebra is embedded in definitions and axioms, students will struggle to understand the language and the concepts put forward. The results of question 2b are summarised in Table 7.4 .

Table 7.4: Allocation of scores for question 2b

Categories		**Frequency**
Incorrect responses	Used inspection wrongly/ grab any terms leading to illogical deductions	58
	Use of the determinant method	2
Correct Response	Correct theorem and correct deduction	13

7.4.7 Interviews responses to question 2b

Some of the students who were interviewed struggled to obtain a completely correct solution. During the interview sessions T25 and T4 were able to give a convincing argument why the set is linearly independent, showing that indeed they are able to interiorise the actions into a process.

The following excerpts were arrived at with different students for example:

T25: *We consider the number of vectors that we are given and also the number of elements in R^n in a given vector. Let r be the no of vectors and n be elements in a given vector. If r > n it means the set is linearly dependent that is if say $S = \{v_1, v_2, \dots v_r\}$ in R^n. If $r > n$, then S is linearly dependent, or we have n equations in r uknowns from the formulated homogenous system of equations.*

T23: *We say that it is not linearly dependent hmm dependent because we have three vectors here and we are using* R^2.

T13: *Quiet for a moment …. right we can use scalars to express one of the vectors in terms of the other or we can use the general vector to express this one, then we check for consistency at the end.*

T44: *These are linearly dependent, because there is no vector which is a multiple of each other and there is a certain theorem that we use that is when r is greater than* n, *thus all.*

T57: *Linearly independent because the number of vectors is greater than the number of vectors in a vector space*

R: *In your response on the activity sheet you simply wrote since r > n, therefore they are linearly independent. What were you referring to?*

T57: *There is this theorem where-by* r *stand for number of elements in a given vector and* n *stands for the geometric part where you say* R^2 *and* R^3, *where say* R^2 *it's a 2 dimension* x *and* y, *where* R^3 *is three dimension.*

R: *Oh ok can you further illustrate this using the vectors* u_1, u_2 *and* u_3.

T57: *Our vectors are in two dimension so, hmm, because we have the vectors in* 2 *dimension and we have number of rows and number of vectors they are three, (quiet for a while). I am failing to state the theory.*

From the interview excerpts the students had an idea about the use of theorem 7.1 which was introduced to them using the formal world according to Tall (2004). However the students' responses showed that they struggled to relate the theorem in terms of the link of the number of equations, and the number of unknowns say r, (in a given system of equations) meaning that if $r > n$ the system will result in non-trivial solution and then the given set say $\{S = v_1, v_2 \dots v_r\}$ will be linearly dependent hampered them to develop their understanding at the process level, for example see interview excerpt with students T57, T44, and T23. Student T44 used theorem 7.2 inappropriately, since this is applicable to a set with exactly two vectors. Student T57 struggled to outline theorem 7.1 appropriately as well as student T23. Student T23 did not explicitly explain the relationship between the three vectors and R^2. The ability by student T13 to explain the step by step procedures indicates that she is still operating at the action conception of linear independence.

7.4.8 Results for question 2e

In this question 28 (38%) students had the correct result and correct interpretation. The students were able to identify the zero vectors so that they were able to tell that the given set is linearly dependent. Some of the students were also able to tell that the determinant of the relevant matrix would be zero, since there is a row of zeros. In the written response student T54 wrote that the presence of zeros means it is inconsistent therefore linearly dependent. A large number of students, instead of using inspection, proceeded to calculate the determinant, contrary to the given instruction. The presence of the zeros was sufficient to tell the students that the determinant of the coefficient matrix is zero. Eight of the students treated the vectors as row vectors and proceeded to come up with a 3×3 matrix. The students calculated the determinant of the matrix as zero and then concluded that the vectors are linearly dependent. The students' inability to use inspection in order to explain whether the given vectors are linearly independent or linearly dependent indicates that they are still operating at the action level of determinants. These students required the comfort of the step by step procedures that was not necessary. 19 of the students really struggled to decide whether it was linearly dependent or linearly independent with correct reasoning. Out of the 19 students 4 of them gave the argument that it is linearly independent because it has a row of zero [instead of saying it is linearly dependent] whilst student T8 said that it is linearly independent because there is no inverse so determinant is zero. The deduction that it is linearly independent is wrong but the idea of the zero vector is appropriate. The other 15 out of the 19 students students simply used terms that came into their minds for example some gave solutions like 'it is linearly independent because it is not connected by any scalar' and also 'it is linearly independent because there is no corresponding value'. The other 18 students said the vectors are linearly dependent but used wrong arguments for example, student T62 said vectors are linearly dependent since there is a row vector, and T17 said they are linearly dependent because they are not scalar multiples of the other. This shows that the teachers had not developed the process conception of linear independence or dependence. The results of question 2e are summarised in Table 7.5.

Table 7.5: Allocation of scores for question 2e

Categories		Frequency
Incorrect response	Use of the determinant method	8
	Use of wrong theorem and deduction	19
	Correct result (Linearly dependent) with wrong justification/illogical deduction	18
Correct response	Use correct theory and deduction	28

7.4.9 Interviews responses to question 2e

In the written responses student T69 simply took the vectors as row vectors and came up with a 3×3 square matrix and attempted to carry out elementary row operations. He obtained the following matrix after row reduction $\begin{bmatrix} 1 & -1,5 & 3,5 \\ 0 & -3,5 & 6,5 \\ 0 & 0 & 0 \end{bmatrix}$ and he wrote the vectors are a basis of R^3 and the basis are $(1 \;\; -1\frac{1}{2} \; 3\frac{1}{2})$ and $\left(0 \;\; -3\frac{1}{2} \;\; 6\frac{1}{2}\right)$. This shows that the student was now confusing the terms basis of vector space and linearly dependent or independent. The following students were interviewed:

R*: May you explain whether the given vectors in part e are linearly independent or not?*

T69: *Hmmm part e, [quiet for a moment]. Let me see there is a zero zero zero so hmmm it is not linearly independent.*

R: Can you explain further why it is not linearly independent?

T69: *It is linearly independent because if you do back substitution, you obtain many solutions, k_3 will be a parameter.[referring to the solution of the form $Ax = 0$.]*

R: *Oh ok if we get many solutions, do we talk about linearly independent or linearly dependent?*

T69: *Linearly independent hmmm [laughing]. I am now confusing myself. I am confusing the terms. If we have many solutions, it is linearly dependent because we are saying it depends on the parameter now.*

Another student, T63, whose written response was as follows: since $k_1(2,-3,7)+k_3(3,-1,4)=0$ has many solutions for k_1 and k_3. When asked to explain whether the set of vectors were linearly independent/linearly dependent, student T63 changed his perspective and has this to say:

T63: *I think it is linear independent, because one of the vectors are zeros, then it is linear independent. [note that in his original response he left out the zero vector]*

R: *What types of solutions are you going to get and what will your conclusion be in terms linear independent.*

T63: *I am not sure. I think you get many solutions. If I get many solutions then it will be linearly dependent.*

From the task above we noted that the students exchanged the terms 'it is linearly independent' or 'it is linearly dependent' although they were able to see the idea of the zero vector in the matrix, or having vector equations equated to the zero showing correct reasoning but the reasoning which did not match with the justification. These students struggled to state the correct reasoning indicated that they had not yet fully developed the mental constructions of linear independence at the process level according to the genetic decomposition. This interview shows that although student T63 answered the question incorrectly in the written responses, he still has not constructed the process conception of linear dependence.

7.5 Question 5

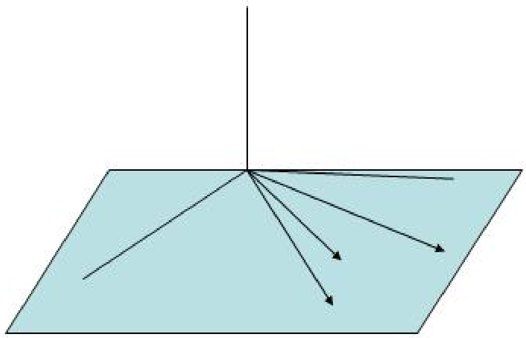
In the activity sheet, question 5 comprises two sub-questions. These two sub-questions addressed the object level understanding of linear independence and dependence in the genetic decomposition in section 3.8.3. The item was intended to provide insight as to whether the students had developed the object conception of the concept linear independence or linear dependence. This question addressed the object conception as was observed by Donevska-Todorova (2015) who argued that the geometric problems solutions does not require any calculating abilities, but it relies on visualisation and interpretation of data, and such problems are viewed as cognitively difficult. The question is presented in table 7.6 below and the possible answers.

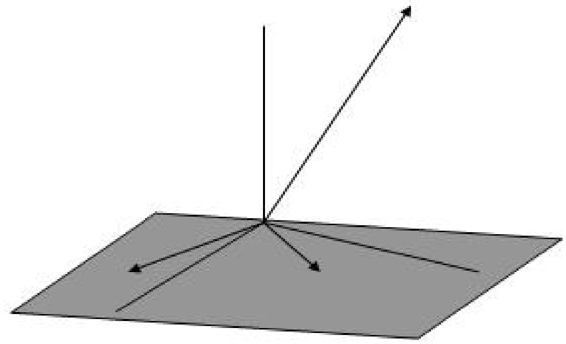
Table 7.6 Question 5 and the possible solution

Task	Possible Steps
1. Given three vectors in $\boldsymbol{R^3}$ interpret and (describe) geometrically with the aid of a diagram (a) Linearly independent vectors. (b) Linearly dependent vectors.	(a) The students need to explain that any three vectors say u, v and w in $\mathbf{R^3}$ are linearly independent if and only if they do not lie in the same plane when they have their initial points at the origin, with the aid of a diagram. (b) The students need to explain that any three vectors say u, v and w in $\mathbf{R^3}$ are linearly dependent if and only if they do not lie in the same plane when they have their initial points at the origin and conclude that at least one of the vectors would be a linear combination of the other, with the aid of a diagram.

7.5.1 Results for question 5.

Nine students did not attempt the two questions. 43 and 40 of the students attempted to answer questions 5a and 5b but they had more difficulties in trying to give a geometric representation or geometric interpretation of three vectors in R^3 indicating that they had not yet developed their understanding at the object level. Stewart (2007) carried out a study whereby students were asked to match the correct image for the concept of linear independence/independence from the two diagrams shown.





Stewart (2007) noted that the students in one study struggled to link the correct image with the description. She noted that two of the students did not even respond to the question while four out of the eleven students had incorrect responses. In the case of this study, the students were required to draw the diagrams and to support their thinking and give a description so as to probe whether the students had developed an object understanding of linear independence/linear dependence. I observed the following: (1) students representing two vectors in a plane instead of three; (2) representing the three vectors u, v and w on the same line passing through the origin; (3) diagrams showing a linearly independent set of vectors lying in a plane; (4) diagrams showing a linearly dependent set of vectors not lying on the same plane; and (5) pictures with all the three vectors lying outside the plane. These shortcomings hampered the students' ability to develop the mental constructions necessary to understand the concept of linear independence and linear dependence at the object level of understanding according to APOS theory.

More details of these difficulties are now presented. Instead of giving a description with the help of a diagram student T50 wrote: three vectors in R^3 are linearly independent if $k_1v_1 + k_2v_2 + k_3v_3 = 0$ yields a common solution for k_1, k_2 and k_3 that is if $k_1 = k_2 = k_3 = 0$ and if the determinant of the resultant matrix is not a zero. For part (b) she wrote: Three vectors u_1, u_2 and u_3 in R^3 are linearly dependent vectors if $k_1u_1 + k_2u_2 + k_3u_3 = 0$ yields different values for k_1, k_2 and k_3 or when the determinant of the matrix is formed. From the above, it is evident that student T50 simply defined the terms and attempted to outline the methods that are used to determine linear independence and dependence. Consequently it is evident that student T50 was unable to use the geometric representation of what linear independence and dependence mean and her concept image in terms of geometric representation was very limited. Most of the students attempted to simply draw pictures with two or three vectors passing through the origin with wrong or no explanations as shown below by student T36's written response.

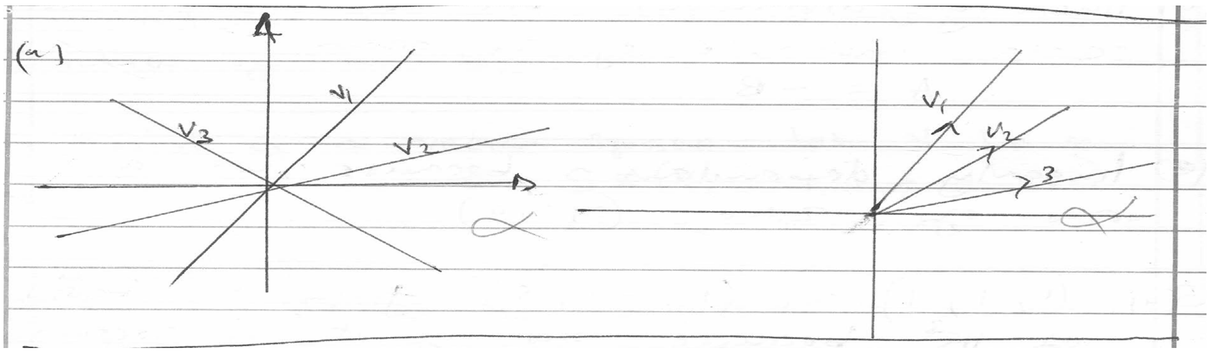


Figure 7.8: Written response of student T36

The above student did not develop his understanding at the object level as expected by the preliminary genetic decomposition, thus displaying a weak geometric representation of linear dependence and independence. Student T46 attempted the question but encountered a number of difficulties as shown in Figure 7.9 below.

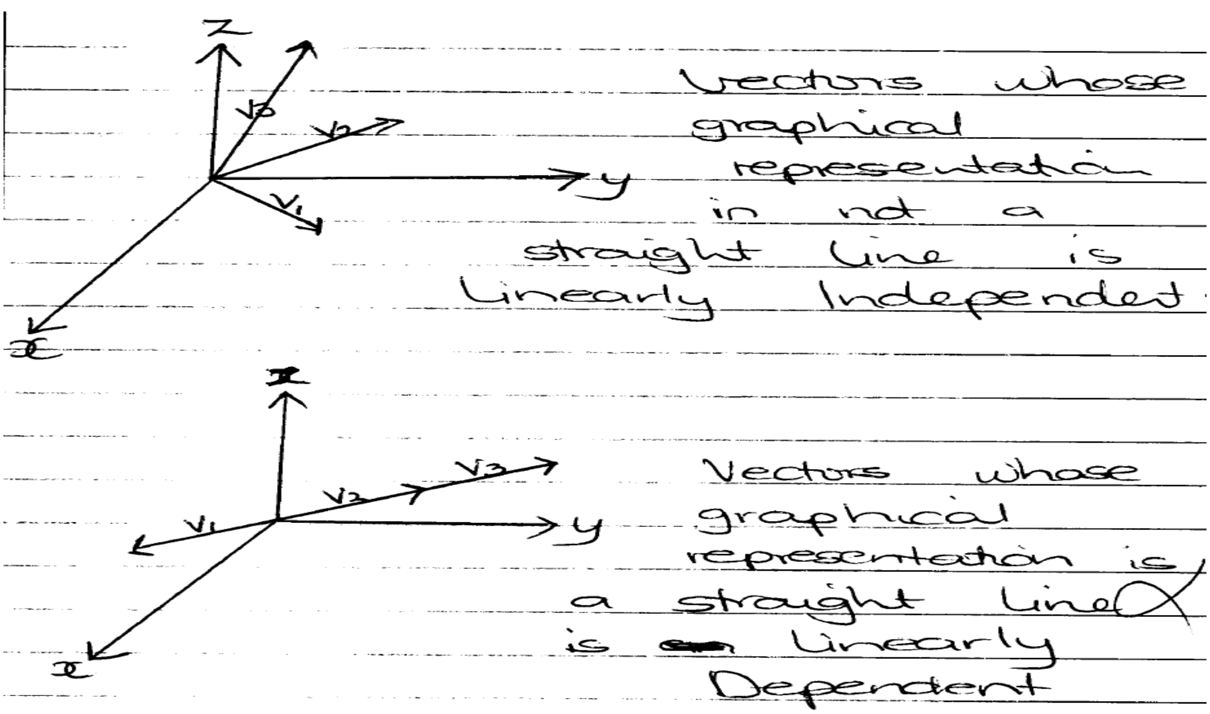


Figure 7.9: Written response of student T46

The word vectors whose graphical representation is not a straight line or is a straight line was used inappropriately. From the diagrams, the student was able to represent the vectors originating from the origin. However, errors were made in representing the vectors u_1, u_2 and u_3 and the pictures drawn as well as the interpretation were incorrect. The misconception manifested here is that the student did not notice that it is impossible to have three linearly dependent or three linearly independent vectors drawn in a two dimensional vector space. This means that there was a need for a three dimensional plane. It can be seen that the student attempted to link her interpretation to linear independence or linear dependence of two vectors in R^2 or R^3. Her explanations lacked some important information so that we might make an assumption that the student was making references about the vectors lying on the same line (scalar multiple of the other) or vectors lying on different lines with their initial points at the origin. This showed that the in-service students struggled to express the notions of linear independence or dependence geometrically as shown in Figure 7.10.

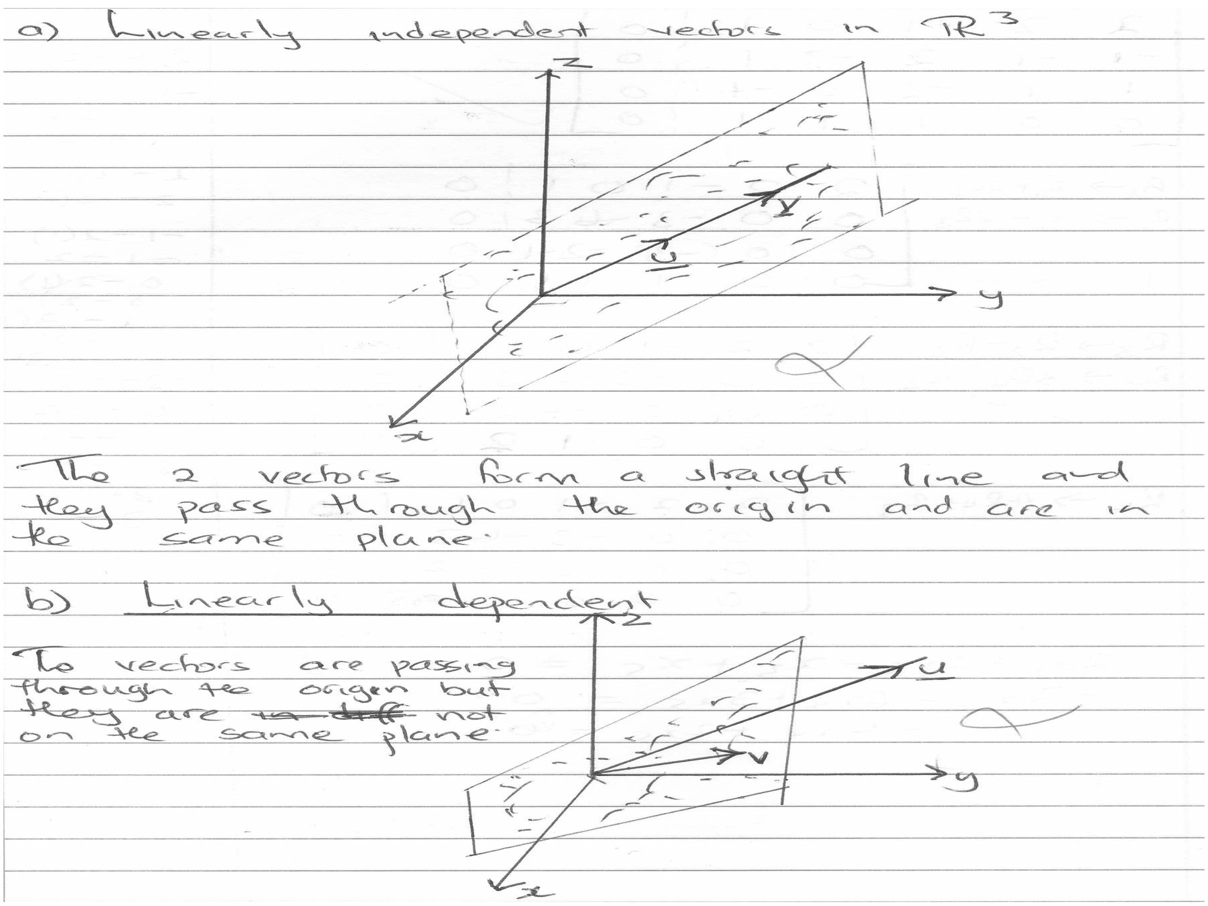


Figure 7.10: Written response of student T29

The written response by student T29 showed that he was aware that the vectors must be lying in a plane and passing through the origin. In part (a) the scalar multiple of u and v was associated with linearly independent vectors instead of linearly dependent vectors. However the following shortcomings are evident: struggling to represent the three vectors geometrically, instead two vectors are represented and the written explanation given are not correct.

I also noted that in questions 5a, 13 students were able to draw the correct diagrams but did not give a correct written description with some not even attempting to give a written explanation, whilst in question 5b, 15 students managed to draw the correct diagram, see table number 7.7. These students might have simply memorised how to represent the vectors with the use of diagrams. We noted that T48 was able to give the correct explanations for both questions but the pictures drawn were incorrect with T39 having the correct diagrams and no explanations. T43 could draw the correct diagram for question 5a and wrongly interpreted the constructions, but had a completely correct solution for 5b. T63 wrote that linearly independent vectors are found in a plane and linearly dependent vectors are found in the same plane but did not show the plane in the picture but all three vectors were emanating from the origin.   These students could not develop their understanding at the object conception according to APOS theory.

I noted that only eight and nine students gave a complete response to Item 5a and item 5b respectively. Their responses indicated that they had developed a concept image of the ideas on geometric interpretation of linear independence/dependence showing clear understanding of the differences between linear independence and linear dependence. Their responses showed encapsulations of process into the object linear independence/linear dependence. This confirms Aydin (2014) who pointed out that the two concepts linear dependence and independence are directly linked concepts in linear algebra implying that a conceptual understanding of one of the concept means having an understanding of the other. Here these students were able to make a comprehensive link of linear independence and dependence to a geometric perspective by drawing the pictures and by pointing out that for linear independence both vectors lie in the same plane through the origin; i.e. for linear dependent vectors and for linear independence they were able to draw the correct diagrams and pointed out that one of the vectors does not lie in the plane. The results of question 5 are summarised in Table 7.8.

Table 7.7: Allocation of scores for question 5

Category		Frequency	
		5(a)	5(b)
No responses		9	9
Incorrect Representation		43	40
Partially correct	Correct verbal description, incorrect or no diagram	4	5
	Correct diagram, incorrect or no verbal description	9	10
Completely correct	Correct diagrams and description	8	9

7.5.2 Interviews responses to question 5

Interviews were conducted in order to investigate the level of thinking of the students and the following conversations took place with various participants.

R: *Given any two vectors in R^2, can you interpret geometrically linear independent or dependent vectors?*

T69: *Linear independent vectors , those vectors in R^2, if they are in the x , y plane they will have hmmm [repeating again] if they are linearly independent they will be facing different directions but hmmm their initial point, will be sharing point to their initial point.*

R: *What if they are linearly dependent?*

T69: *It means that the other one will be a multiple of each other.*

T69: *All the two because they are facing different directions.*

R: *What if you are given three vectors in R^3, interpret geometrically linearly independent vectors?*

R69: *Hmm these vectors will be on the same plane.*

R: What about for linearly dependent vectors?

R69: *For linearly dependent vectors will not be on the same plane.*

R: *So which of the following is linearly independent, [showing him the two pictures] in figure 7.11 below?*

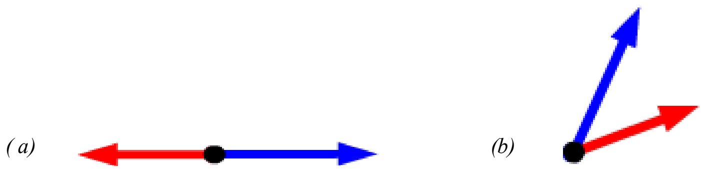


Figure 7.11: Geometrical representation of linear independence/ dependence

T69: *Both diagrams because they are facing different directions. Isn't it?*

From the discussion, student T69 showed much improvement especially in an attempt to describe the geometrical interpretation of three vectors in R^3 though he confused the properties of linear independence and linear dependence. In his written response he did not mention any aspect about the vectors lying in a plane, he simply drew the diagrams without any explanations. We explored further and probed on the ideas of the different directions to see if he was able to engage constructively with them. The two diagrams were drawn by the researcher. The response showed that T69 was still holding on to a conceptual error. His statement "multiples of each other" seemed to be referring to the diagrams whereby the arrows will be pointing to one direction in Figure 7.11. This was also evident from his written response in the activity sheet, whereby when he represented geometrically three vectors in R^3 he drew the diagram similar to those drawn by student T29 but they were not in a plane. No description was given. The conception of failing to recognise that the vectors in Figure 7.11 diagram (a) above are parallel hence are multiples of the other, and confusion about the geometric interpretation of linear independence or dependence hampered him to make the necessary mental construction to make sense of the notion on geometric interpretation of linear independence or dependence, hence did not encapsulate the processes into an object.

Another student, T62, with an incorrect solution in his written response was also interviewed and we have the following interview excerpt:

R: *[reading the question from the activity sheet], how can you interpret geometrically three vectors in R^3?*

T62: *[repeating geometrically] hmm… you can draw.*

R: *Oh ok you can draw or can give an explanation.*

T62: *Yes, the vectors in R^3 [quiet for a moment] this is now a vector hmm space so, they will be lying in the same plane for linear dependence and this one (pointing to activity sheet) will be in different planes, indicating the one for linear independence.*

R: *What else?*

T62: *[laughing] that is all mem.*

R: *If the vectors are linearly dependent, what conclusions can you make in terms of spanning or linear combination?*

T62: *Hmmm that one I have no idea but linearly dependent vectors are vectors whose determinant is zero.*

From the discussion it is evident that the undergraduate in-service teachers were struggling to see the relationship between the two concepts, showing rote memorization of facts. Initially student T62 was of the idea that to represent something geometrically is simply to draw a diagram. Student T62 was probed further so as to elicit more information on whether she was able see the object linear independence as a totality. More could have been said for instance the student did not talk about the issue on three dimensional space, where the vectors originated from, argue whether the three vectors are coplanar and outline the relationship between linear dependence and linear combination. The student was not very confident in the discussion. The errors made showed that these students had not yet advanced their mental structures to work at the object level of understanding according to APOS theory with regards to geometric interpretation of linear independency or dependency. Ertekin, Solak and Yazici (2010) carried out a similar study whereby they investigated the relationship between the definition of the concepts of linear dependency/independency and the algebraic and geometric interpretations of these concepts. The results revealed that the students were generally less successful in geometric interpretation than in algebraic interpretation. The researchers concluded that the students could not appropriately link the formal definition of linear dependency/independency to a geometric interpretation.

We also noted that students T4 and T25, when interviewed were able to represent the two aspects diagrammatically as well as giving a description of each of the concepts geometrically. However, in the written responses student T4 was able to explain in words and presented a correct diagram when illustrating linearly independent vectors and linearly dependent vectors, whilst student T25 obtained incorrect representation. The two students might be working at an object level of the APOS theory. Hence student T25 seemed to have now developed the appropriate metal structures.

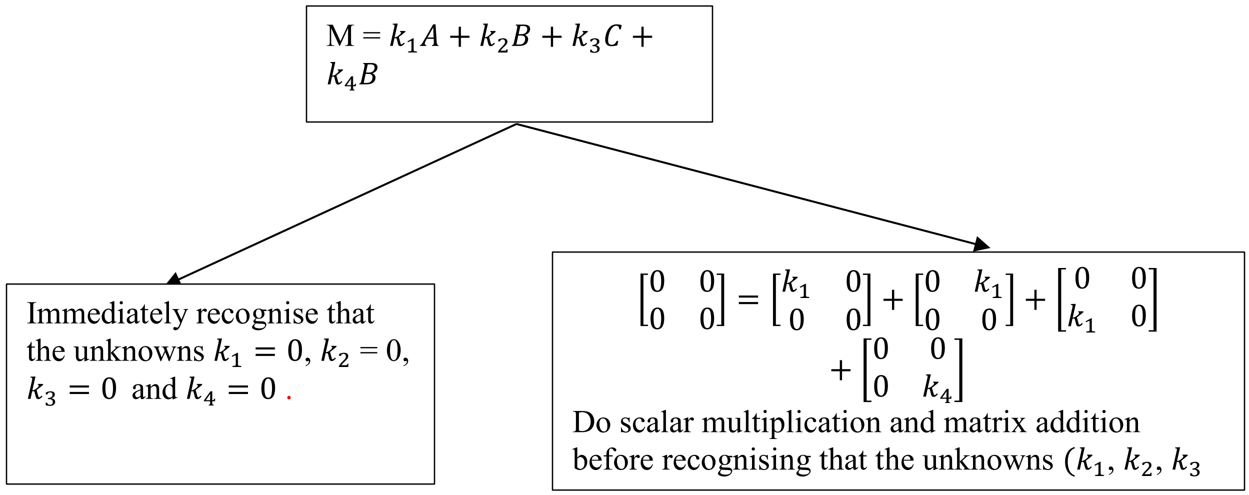
7.6 Question 6

Question 6 addressed the action/process view of the APOS theory according to the genetic decomposition. It was intended to provide insight into whether students had developed an action/process conception of determining whether given matrices are linearly independent. Here we expected the students to give a sequence of steps together with some form of logic, giving high level arguments in spelling out why the matrices are linearly independent. Question 6 and the possible ways that can be used to solve the problem are presented below in table 7. 8.

Table 7.8 Question 6 and possible ways for solving the question

6. Prove that the following matrices $\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$. are linearly independent. Explain the result.	This question required the students to work through the following steps: As a first step consider 2×2 zero matrix M = $\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}$ suppose the given matrices are **A, B, C** and **D**. Set up a vector equation $\boldsymbol{M} = k_1\,\boldsymbol{A} + k_2\boldsymbol{B} + k_3\boldsymbol{C} + k_4\,\boldsymbol{D}$. **Method 1.** Immediately recognise that the unknowns $k_1 = 0$, $k_2 = 0$, $k_3 = 0$ and $k_4 = 0$. **Method 2**: They may proceed from step 1 to set up a system of four equations with four unknowns (k_1, k_2, k_3 and k_4) and thereafter do scalar multiplication and matrix addition before recognising that the unknowns (k_1, k_2, k_3 and k_4) equal 0, 0,0 and 0 respectively. **The following explanation must be given**: Linearly independent because of the existence of one solution only, or we obtain only the zero solution, or zero vector, or trivial solution that is $k_1 = k_2 = k_3 = k_4 = 0$.

As explained in the table above, the students could have opted for two different methods depending on when they would have recognised the solution which is expressed as the flow diagram below.



7.6.1 Results for question 6

From the students' written response, 3 of the students did not attempt the question. 9 of them did not have correct ideas. For example, I noted that student T32 attempted to find separate determinants of the four matrices. All the matrices gave a determinant that is equal to zero then he concluded that they are linearly dependent. This shows a wrong conception of what linear independence is. It seems like the word "determinant " and "determinant not equal to zero'' which in turn was associated with theorem on solving systems of linear equations which outlines that if given an $n \times n$ matrix then it is equivalent that $\det A \neq 0$ led the student to think that he was proving for linear independence. The student confused this with the idea of calculating the determinant of the coefficient matrix not the determinant of the separate matrices. This means that representational form of the vectors (matrix form) confused the student. Two students constructed a wrong matrix. They came up with an augmented matrix where the coefficient matrix was an identity matrix of order 4×4, and the solution was the vector (a, b, c, d) and six other students also came up with an augmented matrix where the coefficient matrix was an identity matrix of order 4×4 equated to the vector (0,0,0,0) as shown by student T57 in Figure 7.12.

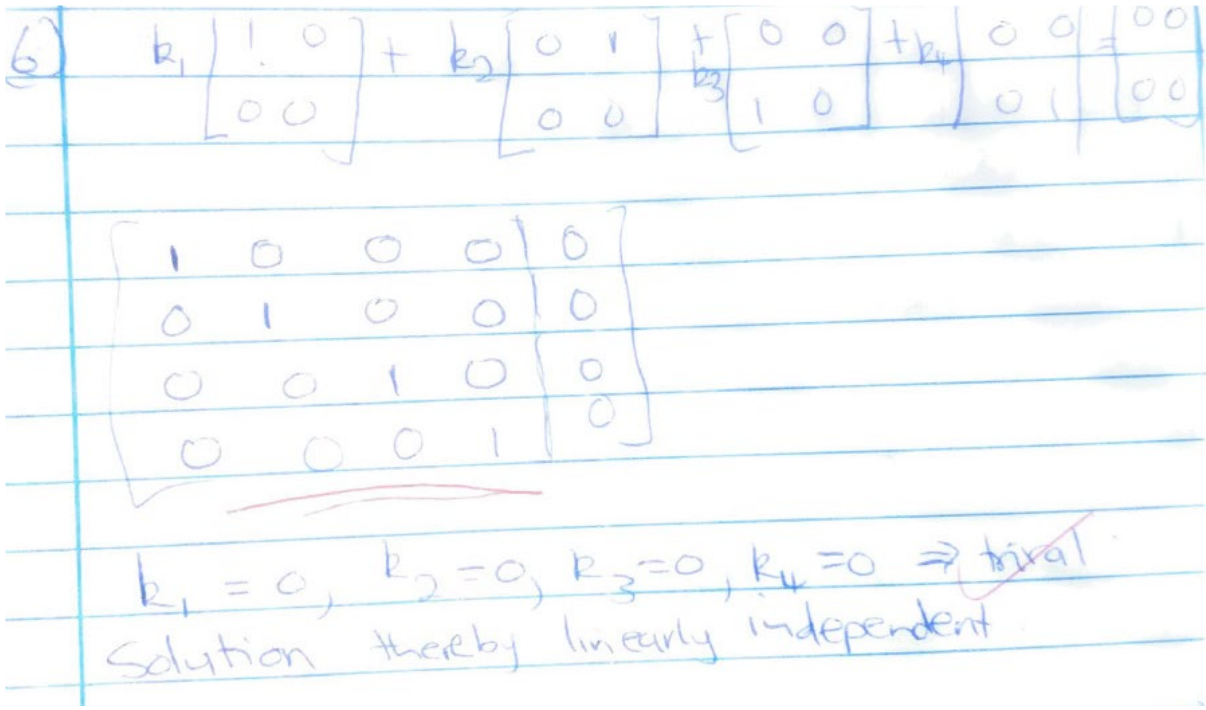


Figure 7.12: Written response of student T57

The student T57 did not explain how she came up with the 4×4 identity matrix despite the fact that she had the equation $\begin{bmatrix} k_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & k_1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ k_1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & k_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Student T57 formulated an augmented matrix and then solved it to get the answer $k_1 = k_2 = k_3 = k_4 = 0$. The scalars k_1, k_2, k_3 and k_4 just appeared in the solution. It seems this teacher took the sum of the 2×2 square matrix wrongly and converted it to an augmented matrix of the form $I_4 x = 0$. Those with an augmented matrix where the coefficient matrix was an identity matrix of order 4, and the solution was the vector (a, b, c, d) also obtained the result $k_1 = a, k_2 = b, k_3 = c$ and k_4= d, and they concluded that it was linearly independent. These other students did not formulate the vector equations but straight away simply came up with augmented matrix as shown by student T54 in Figure 7.13.

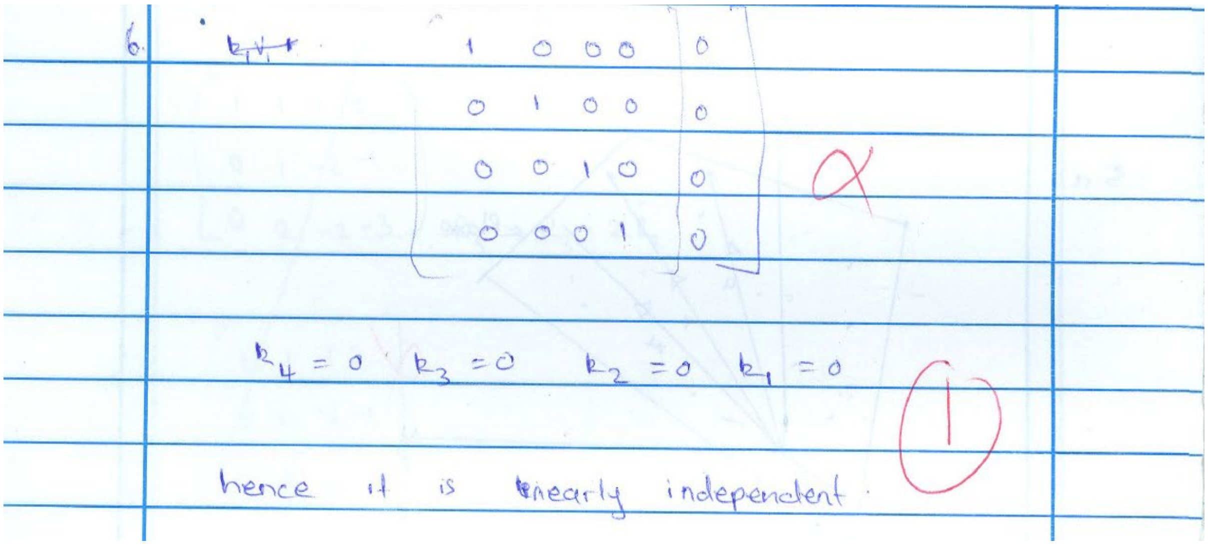


Figure 7.13: Written response of student T54

These students provided incorrect procedures hence they struggled to build coherent cognitive structures around the concept linear independence.

Another group of 14 students had some correct ideas but these were inappropriately presented. We noted that 12 of the students equated the vector equation to an arbitrary vector instead of the zero vector as shown by student T44 in Figure 7.14.

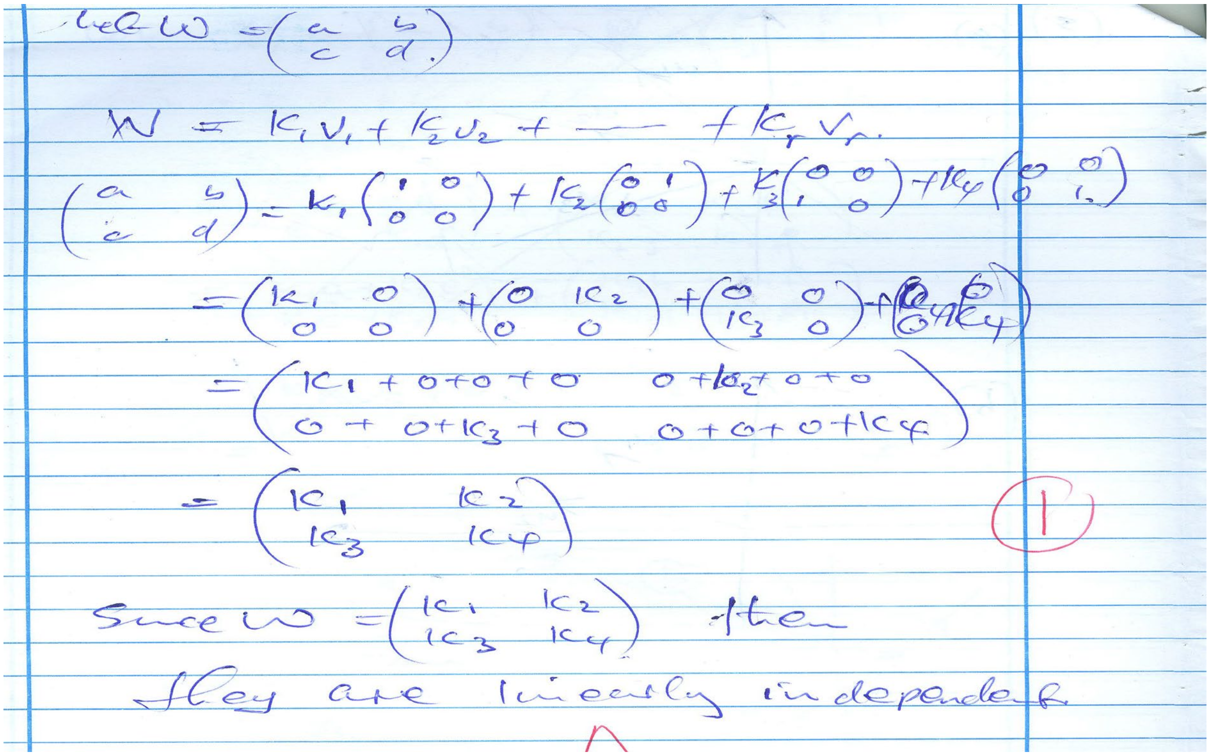


Figure 7.14: Written response of student T44

The response by the student showed that he was able to do scalar multiplication and come up with a linear combination of the vectors but equated to the wrong vector. Step by step procedures are evident, though resulting from a wrong statement. This response indicates that the student's conceptual difficulties might have originated from the confusion with the ideas of showing linear independence and spanning, because he did not have a sufficient basis of what linear independence is. Two of the students had some correct ideas. Student T30 and student T40 were able to come up with the linear combination expression in terms of matrices equated to the zero-matrix. The word linear independence acted as an explicit instruction on what to do. Students T30 was able to carry out the usual operations of scalar multiplication and matrix addition for only three matrices and omitted the other matrix so that at the end she had three scalars. She should have noticed that all the corresponding elements were not used up. This indicates that student T30 is still operating at the action level according to the genetic decomposition. Student T40 was able to come up with the correct vector equations, carry out scalar multiplication and the operations of matrix addition and left the result in the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ without any concluding statement. This indicated that student T40 is also limited to an action conception of what linear independence is but which is more developed than that of T30 because he was able to show the step by step procedures that were triggered by what he had come across before. He, however, did not make a logical deduction and some arguments why the vectors form a linearly independent set. This question required more than just carrying out a procedure as shown, but there was the need of a logical deduction. 23 of the students had correct ideas but did to provide proper arguments to justify their answers. They simply wrote $k_1 = 0, k_2 = 0, k_3 = 0$ and $k_4 = 0$ without a concluding statement. This was a prove question which needed justifications as to why the given matrices are linearly independent.

24 of the students carried out the correct procedures and provided complete correct explanations in showing that the matrices are linearly independent. This showed that these students were able to build cognitive structures around the concept of linear independence. These individuals have developed a process level understanding according to APOS theory. The results of question 6 are summarised in Table 7.9.

Table 7.9: Allocation of scores for question 6

Categories		**Frequency**
No response		3
Incorrect response	Calculated the determinant of separate matrices	1
	Came up with an augmented matrix which was an identity matrix of order 4×4 , and the solution was the arbitrary vector or the zero vector.	8
	Equated the vector equation to an arbitrary vector	12
Got some idea	Equated the vector equation to the zero vector but did not simplify solution to determine the scalars	2
Correct ideas but incomplete	Did not justify the final result.	23
Completely correct result	Solutions justified	24

7.6.2 Interviews responses to question 6

An interview was conducted with student T57 whose response is shown in Figure 8.

R*: In order to show that it is linearly independent, you were able to write the equation* $k_1 v_1 + k_2 v_2 + \cdots k_n v_n = 0$. *How then did you proceed from there?*

T57: *I took the components in the first row first column. This one was matching it with this one. [attempt to come up with the* 4×4 *matrices augmented to the zero matrix]. I think thus the same confusion with the other ones we have discussed earlier.*

R: *So what did you do next?*

T57: *Like what I said, I was taking the first element in the first row first column, then I write zeros, I take again the next element in the next row, this one is the second row number first row second column, was this one, and I continued. Then I used back substitution to obtain the values of* k_1, k_2, k_3 *and* k_4. *Since the values of* k_1, k_2, k_3 *and* k_4 *were the same then it is linearly independent.*

The discussion with T57 formulations of the 4×4 square matrix is an indication of a possible misconception and she continued to hold on to that error. The student applied inappropriate rules in an attempt to simplify the vector equations and demonstrated confusion in an attempt to show that the given matrices are linearly independent. However the student had some idea that at the end she should have trivial solutions so that she could determine linearly independent. However because the student's procedural understanding of linearly independent is inappropriate this hindered student T57 to interiorise the actions into a process level understanding of linear independence.

Another discussion was done with T44 who still struggled to outline whether the vector equation was supposed to be equated to an arbitrary vector or to the zero vector. However, with much probing, he was able to state that he was supposed to equate the vector equation to the zero vector. The interview helped him to identify his misconception.

7.7 Question 7

The question addressed process and object view of the APOS theory according to the genetic decomposition. It was intended to provide insight into whether students had developed a process conception of determining whether given vectors are linearly independent or not, and the object view of the link between linearly dependent vectors and linear combination. Question 7 is presented below and the scores are summarized in Table 7.10 below.

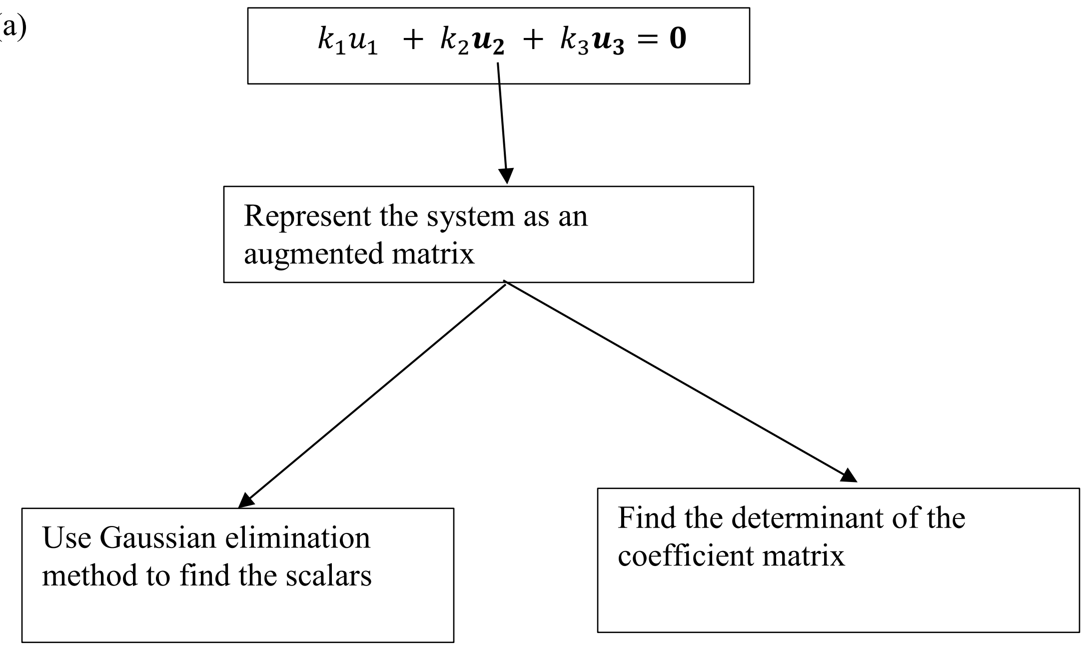
> 7. Determine whether or not the vectors $u = (1, 1, 2)$, $v = (2, 3, 1)$ and $w = (4, 5, 5)$ in R^3 are linearly independent. Explain further the result obtained in terms of linear combination**.**

To solve this problem the students could work through the following steps:

- Set up a vector equation $\mathbf{0} = k_1 u_1 + k_2 \boldsymbol{u_2} + k_3 \boldsymbol{u_3}$**,** where k_1, k_2 and k_3 are scalars
- Set up a system of three homogenous equations with three unknowns (k_1, k_2 and k_3).
- Represent the system as an augmented matrix,
- Carry out row reductions on the matrix and interpret the reduced matrix as indicating that that the third column has no pivot.
- Can also find the determinant of the coefficient matrix.

As explained above, the students could have opted for two different methods which are expressed in the flow diagram below.

(a)



7.7.1 Results for question 7

I noted that two students did not attempt the question whilst only seven had incorrect responses. They applied inappropriate rules and provided incorrect responses indicating an inability to apply correct procedures; there is a possibility that the learners' procedural understanding is flawed, as their responses seemed to struggle to link to the demands of the questions. This indicates that the students' mental construction has not yet developed.

It is evident that 29 of the participants used the method of elementary row operations. When using the method of row reduction, only three elementary row operations were required to reach the correct conclusions as shown below $\begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 5 \\ 2 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & -3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and the following interpretation was necessary: the third column does not have a pivot. Hence the third vector $\boldsymbol{w}$ is a linear combination of the two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$. Thus the vectors are linearly dependent. However, the analysis showed that students struggled to carry out the correct elementary row operations. 15 of those who used the method of elementary row operation encountered problems of failing to carry out correct row operations. An example is that of student T8 who was able to come up with the correct vector equation, express the given vector in coordinate system and then come up with a system of linear equations in 3 unknowns, which is a homogenous system of equations. He proceeded to come up with an augmented matrix and carry out elementary row operations. However, he made an error during elementary row operation and obtained the following result $\begin{bmatrix} 1 & 2 & 4 & & 0 \\ 0 & 1 & 1 & : & 0 \\ 0 & 0 & -3 & & 0 \end{bmatrix}$ which is incorrect. He further made an incorrect interpretation based on the wrong solution; he wrote that $-3 \neq 0$, there is inconsistency hence u, v, w cannot be expressed as linear combination. This incorrect interpretation based on his solution further shows that the student has not encapsulated the prerequisite concept of solutions to systems of equations. Some of the students only carried out row reduction, obtained incorrect solutions and did not write any conclusions basing on their solutions. 14 of the students were able to carry out the correct row operations but did not write down the correct deduction. Some of the participants, like T29, T38, T40, T52, and T55, made the following conclusions respectively: the system is inconsistent therefore it is linearly independent; the vectors cannot be expressed as a linear combination because it has many solutions. This showed a mix up of ideas. I was able to detect that there is a serious weakness in the students' understanding of the concepts. Based on that we safely concluded from their explanations that these students were still operating at the action level according to APOS theory. They could not abstractly denote the conditions necessary to reach the correct arguments for linearly independent and linearly dependent. They only used the terms interchangeably. This indicated that the students had developed at least an action conception of the notion and the possible answers are due to the existence of the step by step procedures.

Three students used the determinant method. Two of the students treated the row as the columns in coming up with the augmented matrix whilst one of them just treated it as the row vectors. An example is that of student T7. The student was able to set up the correct vector equation and straight away came up with augmented matrix as shown $\begin{bmatrix} 1 & 2 & 4 \vdots 0 \\ 1 & 3 & 5 \vdots 0 \\ 2 & 1 & 5 \vdots 0 \end{bmatrix}$. The student used the Laplase transform method to calculate the determinant without specifying the determinant of what or without using the correct notation (brackets) to show that he was finding the determinant. All three students did not view the determinant as a function and just proceeded to calculate the determinant without stating what it was the determinant of. Student T7 obtained the wrong determinant due to a failure to use the lapse transform method correctly. Student T40 used the determinant method correctly but he transcribed the vectors incorrectly, instead of writing the vector (1, 1, 2) he wrote (1, 1, 1), and he obtained the wrong determinant 2 instead of zero as well as did student T60 who obtained a determinant of 5. They struggled to recognize the difference in meaning between the following brackets ( ) and | |. These results coincide with the findings in the literature by Kazunga and Bansilal's (2017) study. The students are still operating within the action level since step by step procedures are evident, but the process stage has not developed.

The other 15 students used the determinant method and obtained the determinant zero. However these students encountered a number of misconception/slips on the way. 6 of the participants simply came up with a 3×3 matrix by simply treating the vectors as row vectors; they calculated the determinant but did not treat the determinant as a function. Correcting deductions were made basing on the calculated determinant that the set of vectors are linearly dependent. Failure to treat the determinant as a function indicated that these participants were not yet reasoning comfortably at a process level of showing that given vectors are linearly dependent. However nine of the students were able to treat the determinant as a function and calculated the determinant using Laplase transform or Sarrus rule and obtained zero determinant. They were able to give the correct argument that the vectors are linearly dependent and this shows that their concept image and reasoning indicated that the action conception has fully developed and they are now operating within the process levels. 17 of the students also used the method of elementary row operations correctly and came up with a correct argument indicating that the vectors are linearly dependent. This again showed that the students are operating at the process level of understanding. However, I also noted that none of the students was able to make the correct interpretation in terms of linear combination indicating that the object conception has not developed at all. This suggests that these students are engaged in a process level reasoning as they were able interpret correctly the results of row reduced matrix in terms of linear dependence. The concept on linear dependence is not seen as an object upon which transformations can be carried out and they are unable to distinguish between the object arising from the process of row reduction and calculation of the determinants and the object linear combination arising from the process. The results of question 7 are summarised in Table 7.10.

Table 7.10: Allocation of scores for question 7

Category			**Frequency**
No response			9
Using the method row operations	Some correct ideas but made errors/ incorrect deductions	Failure to carry out correct row operation	15
		Correct row operations but wrong deductions made	14
	Correct techniques	Correct row reduction and correct justification that the vectors are linearly dependent	17
Using the determinant method	Some correct ideas but made errors or incorrect deductions	Getting wrong determinant and failed to view the determinant as a function	3
	Correct techniques	Obtained correct determinant and deduction but failed to view the determinant as a function	6
		Obtained correct determinant and deduction and viewed the determinant as a function	9
Completely correct	Being able to relate the result to the linear combination of the vectors		0

7.7.2 Interview responses to question 7

The following exchanges took place with student T23.

R: *How do we show that the following vectors are linearly independent? (Showing the student question 7)*

T23: *[quiet for a while] …. hmm this one pointing to the vectors, [the student nodded so that he can proceed], I will come up with a matrix.*

R: *What do you do with the matrix?*

T23: *I will reduce it to row echelon form.*

R: *Can you explain the end result so that you can tell that the set of vectors is linearly independent.*

T23: *I must get a row of zero at the end.*

R: *So what is your conclusion?*

T23: *Hmm I should get what do we call these, you must get hmm the solutions.*

R: *What types of solutions?*

T23: *Trivial solutions*

R: *What are trivial solutions?*

T23: *[laughing for a moment]. Hmmm the questions are too many now. I don't know how to explain this now but what I am saying now is that we should not be found to be using parameters at the end of the solutions.*

Student T69's response above indicated that he was not able to give a precise description of the procedures that are used to determine whether given vectors are linearly independent or not. Though the student attempted to react from a series of instructions, his explanations demonstrated an incomplete conception of the concept indicating that he is still operating at the action stage. The student struggled to accommodate the new learnt material.

The other students had this to say:

T63: *This is a* 3×3 *matrix therefore I can find the determinant.*

R: *The determinant of what?*

T63: *The determinant of the vectors. If the determinant is zero, then the vectors are linearly dependent.*

R: *Which other method can we use to check for linearly independence?*

T63*: The determinant method only.*

R: *Let's say you are given a hypothetical matrix reduced to row echelon form as shown below what would be your conclusion in terms of linear independence.*

$$\begin{bmatrix} 1 & 1 & 2 & : & 0 \\ 0 & 1 & 2 & : & 0 \\ 0 & 0 & 4 & : & 0 \end{bmatrix}$$

T63: *Will be having solutions that are different. This will mean the matrices are linearly dependent.*

T25: *[writing it down and started to do back substitution] the solution are unique solutions, we get* x_3 *= 0…all of them are zeros meaning that the set of vectors is consistent hence the vectors are linearly independent.*

R: *Which other method can you use to determine linearly independent?*

T25. *The determinant method, that if the* $\det = 0$ *the vectors are linearly dependent and if the determinant is not equal to zero, then it is linearly dependent.*

From the dialogue with T63 we observed that he attempted to describe in words the procedures to be followed when determining linear independence. However T63 was not very fluent in the discussions and missed some points showing that he did not have control over the transformations that he was carrying. Furthermore we observed that T63 could not construct the concept of the solution of systems of equations to a process level because he could not bring together the existence of trivial solutions and relate it to the concept of linearly independent. Hence the student did not interiorise the actions into a process as well as come up with another method that could be used to test for linear independence.

From the hypothetical question, student T25's explanations were convincing, precise and gave an argument that is mathematically convincing. Her written response indicated that she used the determinant method correctly and was also able to view the determinant as function. She obtained the correct determinant with correct interpretation. However, she did not manage to express the result in terms of linear combination. Her ability to describe why the vectors are linearly independent as well as being able to distinguish the method that can be used to determine linearly independent is an indication that student T25 has constructed the actions into a process. However student T25's did not recognise that one of the vectors that is w can be expressed as a linear combination of the others hindered her to encapsulate the processes into an object linearly independent. The notion of linear combination is prerequisite for linear independence.

7.8 General observations

The chapter attempted to uncover the difficulties that the in-service students encountered when learning the concepts of linear independence/dependence. The analysis of the students' responses both in written work and interviews revealed that some students struggled to understand the concepts on linear independence. I noted that most of students could not give an explicit definition of linear independence. Some of the students did not define the term but they just wrote down the procedures that are used to determine linear independence/dependence. Equations of the form $w = k_1v_1 + k_2v_2 \dots k_nv_n$ were also seen, indicating confusion between the terms linear independence and linear combination. The definition of linear combination acts as a met before that has been used out of its domain. De lima and Tall (2008) argued that such problematic met befores always cause problems because they have a tendency of impeding generalisations at the end and cause confusion. When attempting to solve problems on linear independence, students usually confused the concept of linear independence/ dependence. They struggled in making the justification whether the given systems are linearly independent/linearly dependent. A justification for linear independence might be given, for example saying from the given vectors, one vector might be a scalar multiple of the other, then a conclusion given might be it is linearly dependent. This shows that the student confuses the two terms. Another drawback was that the students also confused the different theorems and struggled to explain them after identifying that it was the correct theorem to use, for example theorem 7.1. Students could not distinguish the variables r and n and others had a tendency of reversing the theorem, that is saying if $r < n$ then it means it linearly independent. Instead of using inspection a large of students immersed themselves in the step by step procedures such that they ended up calculating the determinants of $m \times n$ matrices or using the Gaussian elimination method. The other difficulties that hindered the students to develop their conceptual understanding of the concept of linear independence/dependence was that they applied inappropriate theorems, did not to view the determinant as a function, added a row of zeros so that they created an $n \times n$ matrix, failed to recognize the meaning of the brackets ( ) and | | for general matrices or to indicate that they were finding the determinant, that is viewing the determinant as function.

On question 6 some of the students struggled to come up with the correct vector equations. They applied an incorrect procedure, as they could not figure out how they should deal with the question involving the set of $M_{2\times 2}$ matrices. Other students equated the vector equation to an arbitrary vector instead of the zero vector, with some of the students calculating the determinant of the separate matrices. Lastly for question 7, many of the students were able to figure out the correct approaches when showing whether the vectors are linearly independent or not. Across the questions, students struggled to perform multi-step computations as well as explain and justify their solutions whether the set of vectors are linearly independent or not.

7.9 APOS insight emerging from the chapter

The APOS analysis was guided by the preliminary genetic decomposition which appears in Chapter 3 section 3.8.3. This means that at the action level, the student must be able to apply the correct methods for determining linear independence for example formulating vector equations and expressing the vectors in coordinated systems form and then come up with systems of equations. The individual will come up with the coefficient matrix and could calculate the determinant of the coefficient matrix or reducing the coefficient matrix to echelon form. At the process level the student will think through the steps without having to perform the steps, for example by applying the correct theorems. At the object level the individual can interpret linear independence geometrically and becomes aware of the processes as a totality and be able to relate linear dependence and linear combination.

The findings of this study suggested that some actions conception of the concept linear independence and dependence were introduced but not all. Some of the actions had not developed for some of the students.

An analysis of question 2 revealed that some of the students resorted to the use of the Gaussian elimination method or calculating the determinant instead of using arguments based on the use of a theorem whether given vectors are linearly independent or not without performing multiple steps computation. Some of the students who attempted to use theorems encountered problems of confusing the theorems such that many of these students did not have a process view of what linear independence is. They struggled to understand theorem 7.1 as compared to theorems theorem 7.2 and 7.3. Some of the students could not communicate their thinking such that they could not explicitly outline the theorems. Another major contributing factor that hampered the students to develop their understanding at the process level was failure to relate the theorem in terms of it being linearly independent or linearly dependent. These terms were used interchangeably, leading to inappropriate conclusions.

Considering questions 6 and 7, most of the students were operating at an action level as they were able to set vector equations, equate the corresponding elements and come up with the systems of equations. The students were comfortable in carrying out tasks where algorithms are required which reinforces procedural understanding, for example carrying out the Gauss elimination algorithm and finding the determinant in a step by step manner, but they encountered a number of flaws that hampered them to develop their mental conception fully at the action level, and they could not construct the meaning of linear independence. Furthermore the students' ability to use rules and struggling to justify their answers with inadequate argumentation showed that they are still operating at the action level according to APOS theory. In addition this shows that the students had developed procedural rather than conceptual understanding of the concepts. The analysis also shows that the process stage was still developing for the majority of students, with some of the students operating at the process level. The process of failing to verify whether the given vectors are linearly independent or linearly dependent hampered the actions to be interiorised into a process thus indicating inadequate conceptual understanding of two concepts. This is in line with studies by Dorier, et al. (2000) who noted that students struggle with the concepts on the vector space because of lack of prior knowledge and the basic tools of logic. This is also in line with contentions by Donevsk-Todorova (2016) who echoed that some of the sources of conceptual difficulties are a result of abstract axiomatic nature of the subject which makes decision making and justifications difficult processes.

The study further revealed that only 24 (33%) of the students were able to interiorise the action of showing that given vectors of the form $M_{2\times2}$ [question 6] as they were able to give a correct description as to why the given vectors are linearly independent. Also in question 7 only 26 (35%) were able to carry out correct row reduction of the coefficient matrix or calculating the determinant, and providing appropriate descriptions as to why the given vectors are not linearly independent. We noted that none of the students appeared to have encapsulated the process stage into an object, as they struggled to apply procedures to problems in unfamiliar context [were not able to relate the result of linear dependency to the linear combination of vectors] and make meaning of their solution.   Panasuk and Beyranevand (2011) commented that fluent procedural skills are not reinforced by conceptual understanding but are demonstrated and arrived at according to fixed rules. This is further supported by the contention of Donevska- Todorova (2010) who outlined that hasty calculations or fluent procedural abilities do not require conceptual understanding.

For the concepts on the definitions of linear independence and geometrical interpretation of linear independence/dependence a large number of the students did not develop their understanding at the object level according to APOS theory. Evidence showed that 5 (7%) of the students had a tendency of just outlining the procedures that are used to determine linear independence instead of describing what linear independence is. 5 (6%) of the students had equations of the form $w = k_1v_1 + k_2v_2 \dots k_nv_n$ [or arbitrary vector] and they did not outline whether the vector **w** was a zero vector. Although some of the students had the correct vector equations in symbolic form, they revealed some misconceptions or used incorrect terminology in an attempt to give more explanations about the issue on the types of solution that one must get, or in explaining the role of vectors and scalars. According to Stewart (2007, p. 94) an object understanding of the concept linear independence involves an understanding of the equation $k_1v_1 + k_2v_2 + \cdots k_nv_n = 0$, which gives the trivial solution only that $k_1 = k_2 = \cdots k_n = 0$ and $v_i \in V$ and $k_i \in F$. I noted that most of the students were able to write down the vector equation but did not make an explicit explanation of the types of solution that one must get, as well as failing to describe where the vectors and the scalars belong. This then hindered the in-service teachers to encapsulate the notion of linear independence into an object. Only nine of the students might have made the necessary mental construction as they were able to explicitly give the definition of linear independence and further give an explanation on the types of solution that one gets as well giving a description of where the vectors and the scalars belong. The other difficult part that the students encountered was to interpret linear independence/dependence geometrically given three vectors in R^3. Some of the students simply drew the diagrams but could not explicitly describe them in words so as to understand what they had drawn. This shows that these students have learnt the concepts on linear independence/dependence by rote memorization of facts. On question 7, students did not make the necessary mental constructions by failing to link the concepts of linear dependence to a linear combination whereby the concept on linear combination is a prerequisite to the learning of linear dependence. According to Bogomolny (2009) encapsulation of linear dependence/independence at an object level requires a movement beyond the procedures of row reduction toward a conceptual understanding of linear dependence/independence relations of vectors.

I noted that across the questions, 46 (63%) of the students' responses indicated that they were bound by the external stimulus of the steps involved in determining if a given set is linearly independent. These students resorted to the use of Gaussian elimination method and did not provide correct reasoning and justification as to why given vectors are linearly independent or not thus indicating that they are still operating at the action level. Their poor computational skills meant that they obtained wrong solutions, hence did not interiorise the actions into a process. In question 2, some of the students relied on the multiple step sequence in trying to identify whether the sets were linear independent, instead of applying the related theorems illustrating that they required the external stimulus of the step by step by procedures associated with an action level conception. I also noted that 10 of the students could not make any mental constructions of what linear independence and some of the questions were left blank.

I further noted that across the questionss only 17 (23%) of the students were able to explain whether the given vectors are linearly independent or not by applying the correct theorems or making the correct deductions without showing all the steps. Many of the students did very well and developed their understanding at the process level in question 2a, 2c and 2d, as compared to question 2b and 2d. These students were able to interiorise the processes of linear independence into a process as they were able to make the correct and convincing logical deductions as to whether given vectors are linearly independent or not. Also in question 6, these students were also able to justify why the given matrices are linearly independent. Some of the students when interviewed, for example student T23 could not give a precise description of the procedures used to determine linear independence, whilst student T63 developed a better understanding of the procedures to determine linear independence as compared to the written response. This is revealed when student T63 now included the idea about the zero vector and outlining the types of solutions that one gets after row reduction e.g. if one gets many solutions, it means the set of vectors are linearly dependent. Also student T25 was interviewed using a hypothetical matrix that was reduced to row echelon form and she was able to explain why the set of vectors are linearly independent, showing that she has developed the appropriate mental structures at the process level.

However, none of the students showed evidence that they had encapsulated the process of linear independence/dependence into an object. This was mainly because of their struggles with Question 7. The students were unable to make links between the concepts related to linear independence learnt. According to Bogomolny (2009) the object conception of linear independence relation take account of mastery of all possible characterization of linear dependence set of vectors, which students were not able to evidence here. The students did not notice that after row reduction, one of the vectors, that is **w,** could be written as a linear combination of the vectors **u** and **v**. This, therefore, means that the students were not able to reflect on the operations applied to the processes of linear independence/dependence, so as to encapsulate them into an object.

7.10 Implications for teaching

In this research we explored the undergraduate students' mental constructions related to the concepts of linear independence using APOS theory. A preliminary genetic decomposition was constructed which served as a diagnostic tool and it proved to be useful in supporting the researchers in identifying the mental constructions the undergraduate students were operating at. The research consisted of 73 students' responses to 6 questions set on the fundamentals of linear independence, and thereafter unstructured interviewers were carried out based on the students' responses. The study identified various misconceptions such as failure to interpret correctly solutions to systems equations, distinguish between the notation used for a matrix and that for determinants. Brijllal and Ndlovu (2013) outlined that the students need to be aware of learners' learning conflicts so as to reinforce the new concepts they encounter. Aygor and Ozdag (2012) noted that in order to decrease the percentage of students having misconceptions during linear algebra courses, lecturers should take up most of the time explaining the noted misconceptions so as to prevent "long-term trouble" as well as imparting the concrete concepts before the abstract ones. We also concluded that the students could not appropriately link the formal definition of linear dependency/independency to a geometric interpretation. Our instructional suggestion is that the students should be given an opportunity to experiment with diverse examples of diagrams representing linear dependency/independency of two or three vectors in R^2 and then move onto geometric representation of two or three vectors in R^3. We also concur with Ertekin, Solak and Yazici (2010) that the concept on linearly independent and dependent can be introduced using the geometric means before moving onto the formal definition so that the students will develop a adequately strong schema of linear dependency and dependence.

7.11 Modification of the genetic decomposition

However, we noted that as researchers that although the genetic decomposition was useful as our diagnostic tool, there are some concepts essential for the conceptual development of linear independence or dependence that were not included as well as some of the participants' responses that were not captured in the genetic decomposition. Hence we suggested that there is the need to modify the genetic decomposition. Hence the researchers advocated for the modification of the genetic decomposition. The modified genetic decomposition is presented below in the form of a Table. Some aspects considered useful for conceptual development of the linearly independent or linearly dependent are not included. These include the schema for solving systems of equations with particular reference to interpretation of the solutions of systems of equations, basic algebra schema, basic notation when calculating determinants and also the important role of linear combination. These aspects are useful for the conceptual understanding of linear independence/linear dependence.

We next present a revised genetic decomposition based on some of the issues that emerged in this study.

Table 7.11: Preliminary and modified genetic decomposition for linear independence

Preliminary Genetic Decomposition	**Modified Genetic Decomposition**
	Prerequisite concepts The prerequisite concepts to start the construction of linear independence are object conceptions of the concepts of binary operations of scalar multiplication and vector addition as well as solutions to systems of equations. The process conception of determinants is also necessary to comprehend the notion of a linear independence, as well as the object understanding of linear combination. In terms of solutions to systems of equations, the individual needs to be able to see the resulting solutions to systems of equations as a totality irrespective of whether the vectors

<table>
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<td>are in the form of matrices or vectors in R^n. The result from the augmented matrix reduction can be used to deduce results related to linear independence/dependence.</td>
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Linear independence
Action

At the action level if an individual is asked to show whether a given set of vectors say $v_1, v_2 \dots v_n$ in R^n are linearly independent or not, the transformation involves a number of multiple steps that need to be followed. The term linear independence acts as an external stimulus of what needs to be done. The individual formulate a vector equation of the form

$$k_1v_1 + k_2v_2 + \cdots k_nv_n = 0$$

where $k_1 \;\; k_2$, $\dots k_n$ are scalars that need to be calculated. The given vectors are expressed in coordinate system and then come up with a homogenous system of linear equations in n unknowns. Suitable methods are selected to determine whether the system is linearly independent or not.

Process

An action is interiorised into a process when the individual can think of the procedures described above without specific vectors or even without specifying the number of coordinates.

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Action

When asked to show that a set of vectors S=$\{v_1, v_2, v_3\}$ is linearly independent, the term linear independence acts as an external stimulus for the series of steps (actions) that need to be taken. The first step is to set up a vector equation similar to that of Equation 7.1: $k_1v_1 + k_2v_2 + k_3v_3 = 0$. This equation is then represented as a system of homogeneous equations in three unknowns, k_1, k_2 and k_3. Thereafter the system is represented as a matrix which can then be reduced to determine if the system of equations has only the trivial solution or whether it has non-trivial solutions. The student then concludes that the set of vectors is independnt or not independent.

Process

The action is interiorised into a process when the individual can think of the actions without specific vectors or even without specifying the number of coordinates.

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Can make arguments based on the use of a theorem whether given vectors are linearly independent or not without performing the steps. Can relate linear independence/dependence to row reduced echelon form of a relevant matrix or to the calculated determinant.	The action of reducing the augmented matrix is interiorised into a process if the individual is able to predict the solution sets of the homogenous systems of equations, without necessarily going through each step, or by invoking Theorem 7.1, that is the individual can make a deduction that the vectors are linearly independent/dependent by predicting the nature of the solution of the augmented matrix without carrying step by step procedures. Alternatively, the individual can apply theorem 7.2 to predict whether the vectors are linearly independent or not without performing the steps.
Object level At the object level the individual can reflect on the operations applied above and becomes aware of the process as a totality. Can distinguish the difference between the two methods that can be used to test for linear independence. Can think of a set of linear independent vectors v_i as an entity and can use for other operations such for basis and dimension. Individual can interpret linear independence/ dependence geometrically.	**Object level** The process of verifying that a vector is linearly independent/dependent is encapsulated into an object when other actions or processes can be carried out on a set of linearly independent vectors. At this stage the individual is able to explain the properties of linear independent or dependence. They can see that for a set S that is linearly independent, one vector can be written as a linear combination of the other. An individual can use one method to find a solution, and verify using another method. Can reason about properties of linearly independant vectors such as considering any

	two vectors defining a plane, and if the third vector does not lie on the same plane, then being able to reason that the three vectors are independent, otherwise it is linearly dependent.

CHAPTER 8

ANALYSIS OF STUDENTS' RESPONSES TO ITEMS BASED ON BASIS AND DIMENSION

8.1 Introduction

In this chapter students' written responses to activity sheet 3, on the concepts on basis and dimension, are analysed and discussed. Interview transcriptions of the students are also presented. The APOS theory was used to describe the level at which students were operating with the help of the preliminary genetic decomposition discussed in section 3.8.4 so as to examine the students' mental constructions. This was done so as to gain an insight into the students' understanding of the concepts on basis and dimension. Moreover, during the interviews the students were asked several questions aimed at discovering the mental constructions that students make in understanding of concepts on basis and dimension of a vector space. To ensure that the discussion of the students' responses makes sense, I present some of the definitions and theorems that commonly appear in this discussion.

Definition 8.1: A set S = $\{ v_1,\ v_2, \dots, v_n \}$ of vectors is a basis of V if the following two conditions hold:

(i) $v_1, v_2, \dots, v_n$ are linearly independent

(ii) $v_1, v_2, \dots, v_n$ span V.

Definition 8.2: The usual basis of R^n has n vectors meaning that dim $R^n = n$.

In order to construct the concept of basis, some of the items required the application of Theorem 7.1.

8.2 Analysis and discussion of data

The analysis in this section was established using students' responses to an activity sheet and transcriptions from the interviews. For the purpose of this study, we attempted to explore the students' understanding of the concepts on basis of a vector space and dimension. The activity sheet consisted of nine items of which five are discussed in this section. The results of the study are presented as follows: question 3 which is intended to explore whether the in-service teachers could explain whether given vectors forms a basis in R^3 giving a strong argument for the solution; question 8 testing for basis using multiple step procedures; question 9 testing the concept of basis using a learner generated example; and question 12 testing the concept of basis and dimension of the solution space.

8.3 Question 3

The item was intended to provide insight into whether or not the students had developed the process conception of the notion of a basis of a vector space. To be able to find the basis of a vector space, the student must understand first that a basis of a vector space must be linearly independent and secondly the vector must span R^n, hence there is the need to bring together the two concepts and be able to explain their thinking without the application of rules. Furthermore, the students must understand the concept that a basis of $\boldsymbol{R^n}$ must contain exactly n elements, since dim $\boldsymbol{R^n} = n$ [definition 8.2]. Theorem 7.1 can be used here to determine whether the vectors are linearly independent or not. The question is represented below.

3. Explain whether or not each of the following forms a basis of $\mathbf{R^3}$. [Solve by using inspection] $\{(1, 2, 3), (1, 3, 5), (1, 0, 1), (2, 3, 0)\}$.

8.3.1 Results for question 3

The results indicated that 9 (12%) of the students did not attempt the question or even bother to make guess work by simply outlining whether it is a basis or not. I also noted that 13 (18%) of the students formulated vector equations of the form $k_1v_1 + k_2v_2 + k_3v_3 + k_4v_4 = 0$ with some of them having again equations of the form $k_1v_1 + k_2v_2 + k_3v_3 + k_4v_4 = b$ that is equated to an arbitrary vector, yet the question required them to use inspection. The students then expressed the given vectors in coordinate system and then came up with a system of 3 linear equations in four unknowns. They further formulated an augmented matrix and carried out elementary row operations. Two students did not carry out row reductions and did not make any conclusions based on the result obtained. The other 11 students out of 13 attempted to carry out row reduction, but they faltered on the way, for example T66 obtained the following matrix: $\begin{bmatrix} 1 & 1 & 1 & 7 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and he concluded that cannot form a basis because there are rows of zeros. I also noted that most of the students simply made some conclusions based on the term basis rather than talking about whether the vectors are linearly independent or spans the vectors in R^3.

25 of the students simply treated the given vectors as row vectors or the given vectors as column vectors and came up with a 4×3 matrix or 3×4 matrix. The following shortcomings were evident: (1) simply drawing conclusions / or no conclusions from the formulated matrix; (2) carrying out elementary row operations, encountering calculation errors by failing to manipulate figures correctly; and (3) from the incorrect row reduced matrix, taking the non-zero rows as a basis of a vector space. Two students simply made some conclusions based on the formulated matrix, for example student T14 who came up with a 3×4 matrix and said that the matrix does not form a basis. Student T28 said since we are not able to reduce, it does not span hence it is not a basis. Three of the students did not make any conclusion after carrying out Gaussian elimination. The other 20 students carried out the Gaussian elimination and they experienced some difficulties. An example is that of student T43 who obtained the following result $\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 2 & -4 \end{bmatrix}$ and he simply made the following conclusion based on his solution: do not form a basis. No reason was given to show that it does not form a basis, with student T27 who obtained the following matrix after row reduction, $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 6 \\ 0 & 0 & 4 \\ 0 & 0 & -10 \end{bmatrix}$ and wrote that the vectors forms a basis (1,0,1, 0,0,6 , 0,0,4, 0,0,0,-10). Student T27 listed the non zeros as if is now finding basis of subspace W of R^4 spanned by the given vectors. 8 of the students listed the non-zero rows and referred to it as the basis, showing a misconception. Student T57 had an idea that there is the need to test for linear independence. However the student did not make a proper justification of the result that she obtained, see figure 8.1.

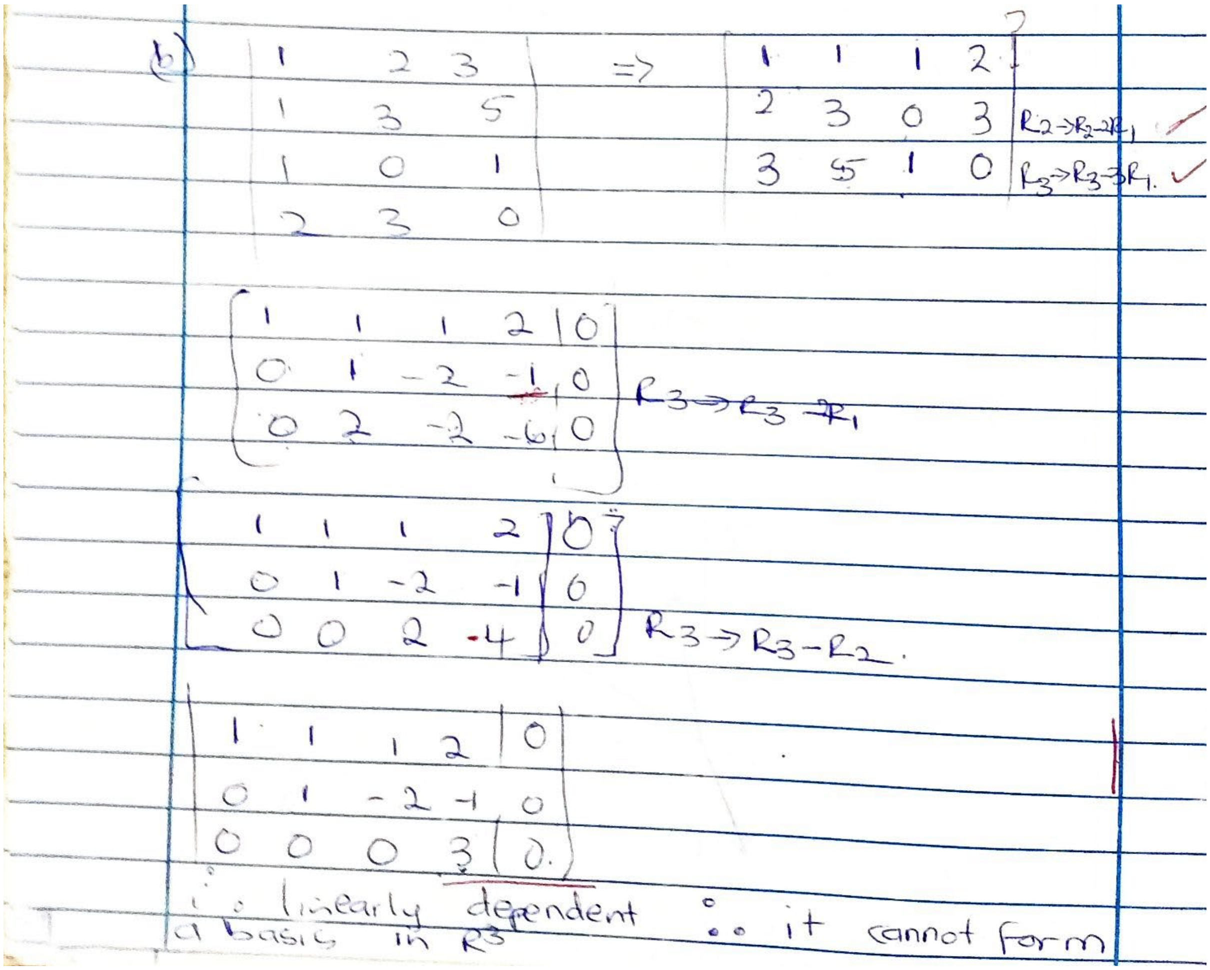


Figure 8.1: Written response of student T57

Examining her response, student T57 treated the vectors as row vectors and came up with a 3×4 matrix. Step by step procedures are evident as she then transposed the matrix and came up with a 4×3 matrix. Wrong parentheses are seen, [see first, second and last steps]. Correct row operations are evident [see first, second and third steps]. Furthermore, a wrong manipulation is seen on the third step showing challenges in working with directed numbers, hence showing absence of a basic algebra schema. The fourth row operation [$r_3 \rightarrow r_3 - r_2$] is inappropriate since it will create a -1 in the third row second column, and this again showed that teacher T57 lacked the skills of doing elementary row operations correctly. The student's knowledge construction seems to be inconsistent and computational errors seemed to be embedded in her cognitive structures. The student then made a conclusion about linear dependence based on the result obtained without giving a reason why it does not form a basis. Another student, T66, with a similar result to T57 said that it cannot form a basis because there is no row of zeros, with T65 taking the non-zero row after row reduction as the basis of the vector space. All these students' responses indicated that the students were struggling to explain whether the given vector forms a basis of R^4. The students' responses showed that they did not interiorise the actions into a process level according to APOS theory as they did not effectively make use of rules repeatedly. I observed that the students built the concept of basis as an action since they were skillful in transforming previously known concepts as external and applying a step by step sequence of calculations. However, they were unable to make meaningful deductions on the results they obtained after carrying out these steps. This showed that the undergraduate in-service teachers were able to do the transformation of the problem through row reduction but did not interiorise the actions into a process level of what basis of a vector space is. Many of these students who used the Gaussian elimination method did not specify at the end whether they were checking for linear independence or spanning except for a few, for example student T57 and student T28.

The other 10 (14%) attempted to use inspection but they provided incorrect responses. The students showed no meaningful understanding of the basic facts covered when learning the basic concepts of basis. Four students used theorem 8.1 wrongly and had an expression of the form $r < n$ and were accompanied by the statement: it does not form a basis since they are linearly independent [some wrote since they were linearly dependent]. The letters r and n were not described. Although the students seemed to have an idea that they need to incorporate theorem 8.1 in the discussion, they showed that they have difficulties in articulating their thought processes clearly. The other 6 students simply grabbed any terms that they had met before during the study, for example student T67 said that it is a basis since the vectors must lie in the same line, and student T39 said it is a basis because it is linearly independent and they can span R^3. Student T50 came up with 4×3 matrix and wrote that it forms a basis since the entries are less than the number of vectors. The terms spanning, linear independence and linear dependency were used interchangeably with no meaningful understanding and all the written statements were incorrect. This showed that the in-service teachers struggled to grasp the concept of basis that has been introduced through a formal definition. It became clear that these students did not develop their mental construction at the process level.

7 (10%) of the students were able to use inspection and noticed that the vectors do not form a basis of R^3 but they were not able to explicitly give a convincing deductive argument. Most of the students simply said it does not form a basis since r > n, meaning it is linearly dependent without starting what the r and n stand for. An example is that of student T22 who wrote that it is not a basis for R^3 because it is not linearly independent because the number of elements is less than the number of vectors.

The above statements showed that the students experienced more difficulties in attempting to give the reasons why it is not a basis. There was the need to explain what the r stands for and what the n stands for rather than simply saying $r > n$ and therefore linearly dependent. The students were silent about the idea of spanning. The extract in Figure 8.2 below illustrates the written response of student T34 who almost got the solution correct.

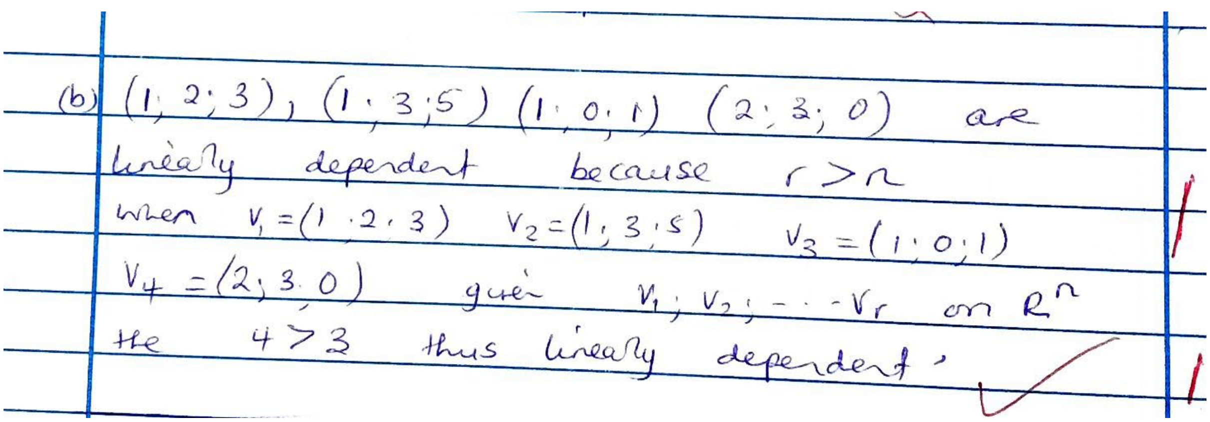

(b) (1; 2; 3), (1; 3; 5) (1; 0; 1) (2; 3; 0) are linearly dependent because r > n
when $V_1 = (1; 2; 3)$ $V_2 = (1; 3; 5)$ $V_3 = (1; 0; 1)$
$V_4 = (2; 3; 0)$ given $V_1; V_2; \ldots V_r$ on R^n
the 4 > 3 thus linearly dependent.

Figure 8.2: Written response of student T34

The student attempted to define the terms r and n explicitly but she did not at the end tell us whether the vectors forms a basis or not, hence the response was incomplete because it is not clear now whether the vectors forms a basis or not, which was the main issue required in the question. 9 (12%) of the students obtained the correct solution showing that they have developed their mental construction at the process level of understanding according to the genetic decomposition. In Table 8.1 the allocation of scores for question 3 is displayed.

Table 8.1: Allocation of scores for question 3

Categories			Frequency
No response			9
Formulated vector equations	Did not carry row reduction		2
	Used Gauss elimination	Wrong deductions made in terms of basis/ no deduction made	11
Treated vectors as row vectors or columns vectors	Did not carry row reduction		2
	Use Gaussian elimination	No conclusions made	3
		Listed the non-zero rows as basis	8
		Incorrect deductions	12
Used inspection	Inspection used wrongly		10
	Almost correct response but failed to explicitly give a convincing deduction as to why it form a basis		7
Completely correct			9

8.3.2 Interviews responses to question 3

The following interview excerpt took place with student T27

R: *Explain whether or not the following forms a basis or not [showing the student the vectors for question 3]*

T27*: It is not a basis*

R: *Why are you saying it is not a basis?*

T27: *Let me see now because for a basis we have to test for linearly dependent or for spanning. So it means we need to test whether they are linearly independent and we test whether they span, so we do row reduction.*

R: *Is it possible to carry out these tests without carrying row reduction?*

T27: *[Quiet for a moment] Because of the number of vectors that we have there and the space that we are given there*

R: *What about the space?*

T27: *We are given R^3 and we have counted 1, 2, 3 vectors.*

R*: So what?*

T27: *So which means the other vectors won't be needed.*

From the discussion it is evident that T27 had again constructed a number of rules, in order to deal with this problem on basis. In his cognitive structures he has built the knowledge that when proving for a basis, an individual ought to do row reduction. However, after probing him it seems he had the idea of how to go about it, but was struggling in trying to clearly explain the procedures. From his explanations it seems in his mind he had the idea that dim $R^3 = 3$ which explains why he was saying I now have extra vectors, referring to the fact that the question has 4 vectors. From the interview, it seems student T27 was now moving towards the process understanding of basis of a vector space. The incorrect use of language has hindered the student from constructing the necessary mental construction required for a basis of a vector space at the process level.

8.4 Question 8

Question 8 was aimed at exploring students' conceptual understanding of what a basis of a vector space is. In order to show that the vectors form a basis of R^4, students must show that these vectors are linearly independent and span R^4. To prove linear independence students must show that the vector equation $k_1v_1 + k_2v_2 + k_3v_3 + k_4v_4 = 0$ have only the trivial solution. To prove that the vectors span R^4 they must show that every vector b = (b_1, b_2, b_3, b_4) in R^4 can be expressed as $k_1v_1 + k_2v_2 + k_3v_3 + k_4v_4 = b$. By equating corresponding components on the two sides, these two equations can be expressed as the linear systems. The problem reduces to show that the homogeneous system has only the trivial solution and that the nonhomogeneous system is consistent for all values of b_1, b_2, b_3, b_4 or the problem reduces to show that you form the matrix whose rows are the given vectors, and row reduce to echelon form. The question addressed the process level of the concept of a basis as expected by the preliminary genetic decomposition. The question is shown below.

8. Determine whether {(1,1,1,1), (1,2,3,2) , (2,5,6,4), (2, 6, 8,5)} form a basis of $\boldsymbol{R^4}$.

8.4.1 Results for question 8

This question revealed that 6 (8%) of the students did not attempt the question with some of them simply transcribing the question. 20 (27%) of the students were able to come with an augmented matrix with some treating the given vectors as row vectors and coming up with a 4×4 square matrix. Step by step procedures were seen as the students carried out elementary row operation. However, the students committed a lot of computational errors as they attempted to carry out elementary row operations. Some of the students lacked the appropriate procedures to be executed when carrying out elementary row operations. The students' failure to reduce the given matrices to elementary row operations showed that they did not have the appropriate schema of solving systems of equations thereby hampering the successful accomplishment of what a basis of a vector space is. This is illustrated by student T44 in Figure 8.3.

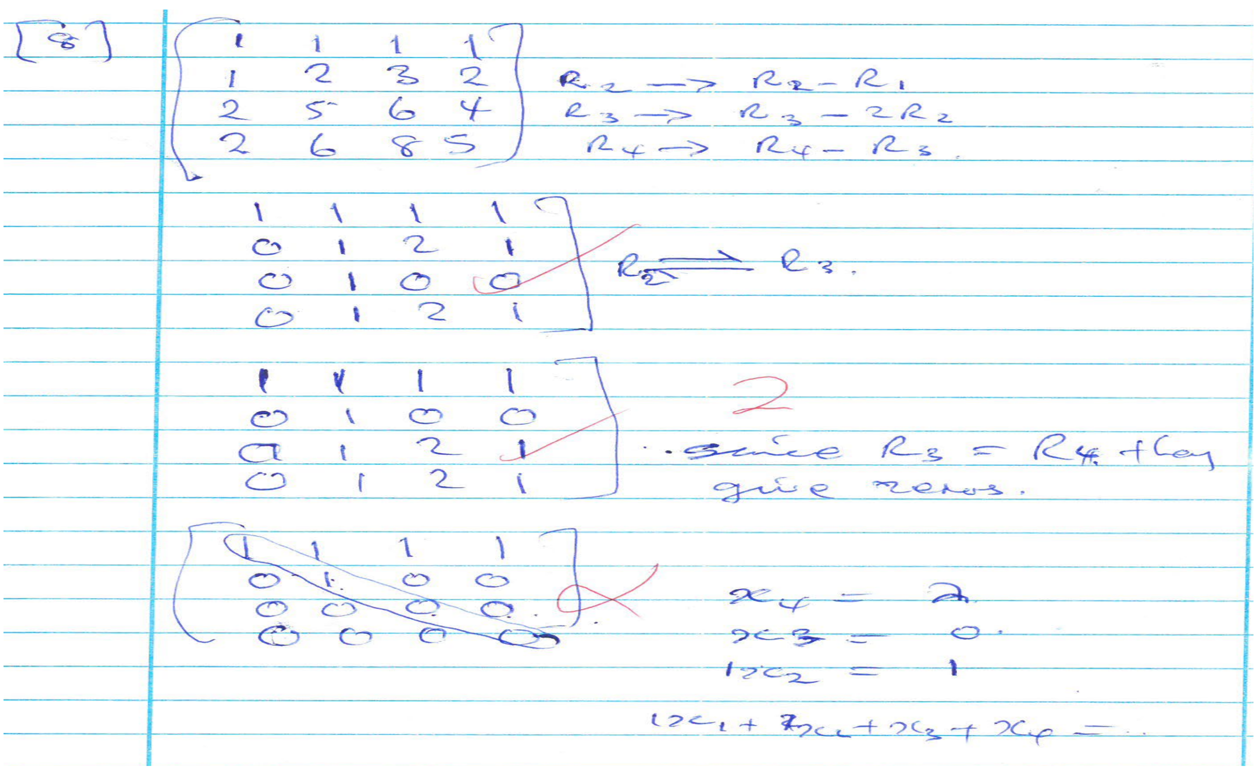


Figure 8.3: Written response of student T44

The teacher treated the given vectors as row vectors and proceeded to apply row reduction. Examining the row operations, the first row operation was correctly executed that is $r_2 \rightarrow r_2 - r_1$. However, the second row operation and the third row operation were inappropriate since he now had a new row 2 and row 3 respectively but was still using the same row 2 and row 3. Most of the students struggled to carry out the correct elementary row operations. Following up the next stage, the stage on row interchange that is $r_2 \leftrightarrow r_3$ is inappropriate, and only the last row was supposed to be zero. The last statement written is: since $r_3 = r_4$ it means we obtain zeros is incorrect and only the last rows will be zeros. Instead of stating whether the reduced matrices forms a basis or not, the student proceeded to find solutions to the system of homogenous system of equations which is inappropriately executed. This showed that the student had learnt the rules wrongly. By examining his final solution, the student had many errors. The variables x_3 and x_4 do not have leading elements but only the result of x_4 is equated to a parameter. The reason behind writing $x_2 = 1$ is not justifiable. This showed that the student is not all that competent with the ideas of solving systems of linear equations. The student did not draw any conclusions whether it forms a basis or not. The students seem not to have mastered the techniques of showing the concept of basis of a vector space. This indicated that these students were still operating at the action level with regards to the understanding of basis of a vector space.

28 of the students in category 2 relied on external stimuli of what a basis of a vector space is. Step by step procedures were seen as 10 of these students were able to come up with vector equations equated to the zero vector to an arbitrary vector or both. The students were able to equate corresponding elements and came up with four equations in four unknowns. The students formulated an augmented matrix and carried out elementary row operations. The other 18 students treated the vectors as row vectors or column vectors and came up with 4×4 matrix and proceeded to carry out elementary row operations. All 28 students were able to carry out the correct elementary row operations. However, a number of shortcomings were evident as the students struggled to make the corrective conclusions. An example of a shortcoming is illustrated by T13 below. The last part of the reduced matrix is shown below in Figure 8.4.

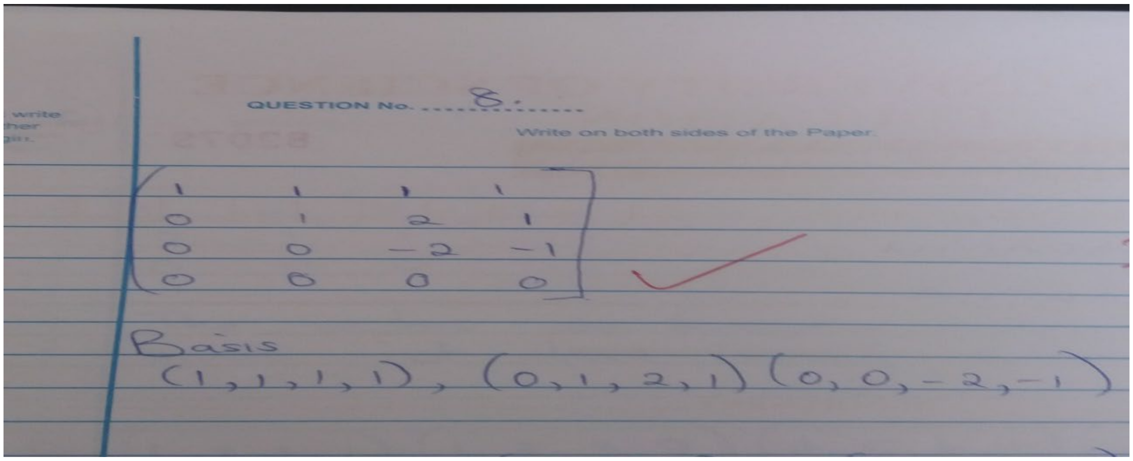


Figure 8.4: Written response of student T13

Instead of making a comprehensive conclusion based on the reduced matrix, the student did not bother to explain whether the vectors form a basis in R^4 or not. Instead the student went on to write basis and listed down the non-zero rows, showing the manifestation of a serious misconception. Here the student seemed to confuse the methods for finding basis of a vector space and basis of a subspace/ row space. 10 of the students manifested such a misconception. Some of the students simply said it is a basis without stating any reasons why it was a basis. From the remaining 18 students, some of the students simply reduced the matrix to echelon form without outlining whether it forms a basis or not. Some simply said it is linearly independent therefore is not a basis, for example T34 said that since it is linearly independent and spans, they form a basis for R^4. This showed that these students confused the terms linear dependent and linearly independent, hence they could not come up with a strong argument as to whether the vectors forms a basis or not. Hence failure to link the aspects of linear independence and spanning hampered the successful accomplishment of examining whether the given vectors forms a basis of R^4. This further confirms that these students did not have the mental structures in place concerning basis of a vector space. This also shows that these students had not progressed past an action level conception of what linear independence and spanning are and are still operating at the action stage according to APOS theory. Another example is that of T4 who had the correct elementary row operation as shown above but had wrong brackets. The following brackets were seen | |. The student presented the correct row reductions and conclusion. She said that: It is not linearly independent as it has zero vectors hence it does not span therefore does not form a basis for R^4. However, the wrong usage of brackets hampered T4 to develop her mental constructions at the process level of understanding.

19 (26%) of the students in this item provided a complete response. The students' responses indicated they recognised the relationship between basis of a vector space in terms of it being linearly independent and spans R^4. The students also demonstrated an understanding of being able to link the structure of a matrix in reduced row echelon form to a basis of a vector space as well as making connections to linear independence and spanning. This showed that these students had developed the process conception of what a basis of a vector space is. Table 8.2 shows the allocation of scores for Question 8.

Table 8.2: Allocation of scores for question 8

Category			**Frequency**
No response			6
Augmented matrix or simply treating given vectors as row vectors and come up with a matrix	Encountered computational errors		20
	Correct row reductions	Listed the non zero rows as basis of vector space	10
		Wrong deductions in terms of linear independence	18
Complete correct response			19

8.4.2 Interview responses to question 8

T4 and T13 were further interviewed so as to explore further their understanding of a basis of a vector space and on the issue of the usage of the brackets.

R: *How do you determine whether given vectors forms a basis of a vector space?*

T4: *The vectors must be linearly independent and must span say R^3 or R^4. We can use the determinant method or the method of elementary row operation to determine whether vectors are linearly independent or spans and then we can make a conclusion from the results.*

R: *Oh ok. You reduced this matrix to row echelon. What then can you conclude basing on this result?*

T4: *Since there is a row of zeros, hence the vectors are not linearly independent and thus they do not form a basis.*

The following discussion took place with T13

R: *Can you outline how we can determine whether given vectors form a basis of say R^4?*

T13: *I will first come up with a matrix. Then I will carry out elementary row operations. Then I will take the non-zero rows and this will form a basis for R^4.*

R: *How then do we find a basis of the subspace say W of R^4 spanned by a given set of vectors?*

T13: *We simply do the same isn't it?*

From the discussion student T13 still struggled to explicitly outline the procedures to be followed in order to determine whether given vectors forms a basis of say R^n. During the interview session, it seems teacher T4 was moving towards the process level engagement of basis of a vector space. She was able to describe the two processes and to distinguish between them and reflect on how they are concluded. However, student T13 was still operating within the action level of understanding according to APOS theory.

8.5 Question 9

Question 9 was intended to explore students' conceptual understanding of what a basis of a vector space is by using a learner generated example. The item is more of an application question that addressed the object conception of the concept as expected by the preliminary genetic decomposition in chapter 3. The question required the students to find other vectors say u_3 and u_4 so as to obtain four vectors that are linearly independent, since a basis of R^4 must have four linearly independent vectors in such a way that they form a matrix that is in echelon form. Question 9 is represented below.

9. Extend $\{\boldsymbol{u_1},\ \boldsymbol{u_2}\}$ to a basis of R^4, where: $\boldsymbol{u_1} = (1,1,1,1)$ and $\boldsymbol{u_2} = (2,2,3,4)$.

8.5.1 Results for question 9

From the analysis, it was evident that only a few students had constructed the object understanding of the concepts. This question also required the students to have a strong understanding of the relationship between the ideas of the concept of basis, for example that dim $R^4 = 4$ and they should be able to link it to the idea of linearly independent and span R^4 and also have a strong foundation of the idea of elementary row operations. It is surprising to note that 28 (38%) of the students did not attempt the question at all. Some of them simply transcribed the question and left it blank. It seems the question had created such difficulties for them that they were not able to see how they could go about it. However, 19 of the students attempted to answer the question but the analysis of the question showed that the students did not have the correct ideas and had completely incorrect responses. Equations of the form $(b_1, b_2, b_3, b_4) = k_1(1,1,1,1) + k_2(2,2,3,4)$ were seen. Some students proceeded to come up with 4 equations in two unknowns and formed an augmented matrix, for example student T23. Some students attempted to do row reduction but were stuck. Some of the equations were equated to zero and students wrote that we need to check for consistency. An example is the response of student T67 who wrote that that $k_1 = 0$ and $k_2 = 0$. He further wrote the conclusion that it is independent and it forms a basis. Some students simply treated the given vectors as row vectors and came up with a 2×4 matrices. Some moved a step further to come up with a basis of subspace. The in correct responses revealed problems of a mix up of ideas. Students here seemed not to understand the demands of the question. The question required critical thinking so that the students were able to figure out what the question was really asking for. I noted that out of the 19 students, 2 did not carry out row operations, 7 tried to carry out some row operations but were stuck, and the other 10 simply treated the formed row vectors as basis of subspace.

20 (27%) of the students in category 3 had some idea that they needed to generate a 4×4 square matrix. However the question further required some high level cognitive thinking in an attempt to come up with the 4×4 matrix. The major challenge encountered by the students was struggling to reduce the 4×4 matrix to row echelon form. An example is shown by the response of student T4 in Figure 8.5.

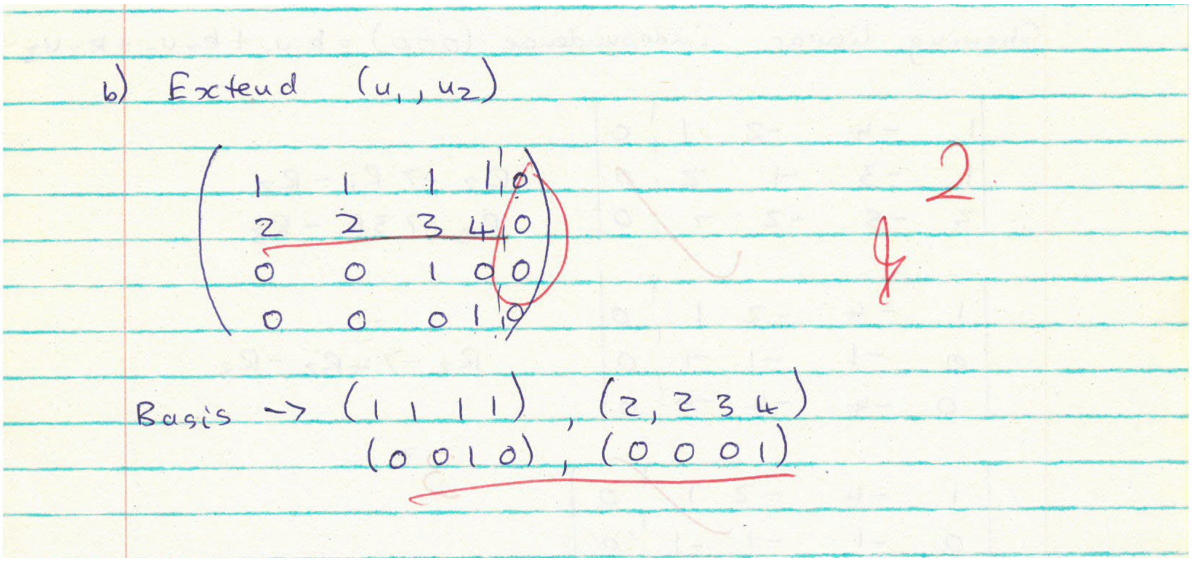


Figure 8.5: Written response of student T4.

Here I observed that student T4 had some idea that in order to form a basis of R^4, then dim $R^4 = 4$ but it seems she could not figure out that she needed to carry row reduction. This learner also has some incorrect mathematical ideas since she further formulated an augmented matrix. Failure to reduce the matrix to reduced row echelon form and come up with an upper triangle matrix hindered the student to develop an understanding of the concept basis. This shows that these students are still operating within the action stage of what a basis of vector space is.

However, 6 (8%) of the students provided the correct answer to the question, demonstrating that they had made the necessary mental constructions according to the genetic decomposition. Their responses show that they had constructed the knowledge of the relationship between basis of vector space and the method of elementary row operations. This is indicated by the attempt to first come up with a matrix say of the form $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ as shown by T62. Row reduction and row interchange took place until the students obtained the matrix of the form $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

The 6 students started with 4×4 matrices but had different elements and the correct procedure was seen. All the six students correctly identified the correct matrix at the end. This showed that these students had encapsulated the process of what a basis of a vector space is to an object level understanding according to the genetic decomposition. A correct schema for basis of R^4 was evident. Table 8.3 below shows the allocation of scores for question 9.

Table 8.3: Allocation of scores for question 9

Category			**Frequency**
No response			28
Having an augmented matrix from a given vector equation or treating the vectors as row vectors	Simply left it blank		2
	Carried out elementary row operation	Stuck or wrong deduction	7
		Coming up with basis of subspace	10
Generated a 4×4 matrix	Failed to carry out row reduction		20
	Completely correct response		6

8.5.2 Interview responses to question 9

The following interview excerpt was done with student T7 who did not write anything in the written responses.

R: *Extend this to a basis of R^4, [showing the student the vectors]*

***T7**: I didn't have an idea*

***R:** It means add more vectors in such a way that you form a basis of R^4.*

***T7**: How many rows do you need?*

***R**: Oh ok. Lets say you are given a set of vectors, how do you determine whether it forms a basis of R^n? Then I am sure from there you will be able to tell how many rows that you need.*

***T7**: The basis hmmm … I will reduce the matrix. Basis are those vectors [laughing] the vectors that are non-zero after reducing and does not contain zeros.*

The above interview excerpt showed that the student did not have correct ideas about the concept of basis of a vector space. It was evident that even when I tried to probe the student, he did not have an idea of how to go about the question and could not make a link with the problem. He, however, knew the procedure for determining a basis of vector space but this was not explicitly explained. The terms liner independence or spanning did not feature in the explanations. Based on the interview this indicated that the concept had not been developed. This meant that student T7 in terms of APOS theory, he has not constructed the meaning of the term basis of a vector space as I observed that he struggled to explain explicitly the procedures necessary to determine a basis.

The following interview was done with T62 who obtained the correct solution in the written responses in the activity sheet.

R: *May you define the term basis of a vector space?*

T62: *From the discussion in class, S is a basis if and only if it is linearly independent and if it spans V.*

R: *Can you extend the following to a basis of R^4? [Showing the student the question]*

T62: *Hmmm we treat the given vectors as row vectors, add 2 more rows so that you come up with a 4×4 matrix. [kept quiet]*

R: *So does that mean to say that those vectors now forms a basis of R^4?*

T62*: If the vectors are not in echelon form we need to take a step further. This means we carry some elementary row operations and reduce the matrix to row echelon form.*

R: *If the matrix is now in echelon form what does that tell you about basis of a vector space?*

T62: *If there is a row of zeros then it means that the vectors are linearly independent.*

R: *Does a row of zeros indicate linear independence or dependent?*

T62: *[Thinking aloud] Oooh zero determinant means linear dependent. Therefore a matrix with a zero row implies that the vectors are linearly dependent. So it means if there are non-zero rows, then it means that the vectors are linearly independent.*

Student T62's written response and his explanations in the interview excerpt revealed that he was able to explain the procedures to be used to determine whether given vectors form a basis of R^4. He was also able to describe the relationship between concepts and outline in brief another method that can be used to show that given vectors are linearly independent, that of finding the determinant. This helped him to realise that if a matrix has been reduced to echelon matrix, and if there is a row of zeros, then it means that the vectors are linearly dependent, otherwise it will be linearly independent.

8.6 Question 12

The question is an application question on basis and dimension to a linear system of equations. The question addressed the action, process and object levels of the preliminary genetic decomposition in section 3.8.4. This question involves a multiple step procedures. The action stages involve forming an augmented matrix and carrying out Gaussian elimination method. The processes are interiorised into action when the individual is able to explain how to find the solution space and express the solution in vector form, and then a deduction should be made to show which part of the solution is the basis and which one is the dimension. The processes are encapsulated into an object when the individual is able to link the aspect on the vector being linearly independent in terms of one vector not a scalar multiple of each other and hence thus forming a basis. To be able to evaluate the problems on basis and dimension of the solution space, students need to have an object understanding of the procedures of solving systems of equations, and a strong understanding of the concepts linear independence, linear dependence and spanning.

Question 12 is represented below

12. Determine a basis for and the dimension of the solution space of the homogenous system of equations.

$$
\begin{aligned}
2x_1 + 2x_2 + -x_3 + x_5 &= 0 \\
-x_1 - x_2 + 2x_3 - 3x_4 + x_5 &= 0 \\
x_1 + x_2 - 2x_3 - x_5 &= 0 \\
\vdots \quad x_3 + x_4 + x_5 &= 0
\end{aligned}
$$

Justify your result why it forms a basis of the solution space

8.6.1 Results for question 12

An analysis of students' responses revealed that two students did not attempt to answer the question and the other two had completely totally incorrect responses. An example is that of student T17. She attempted to come up with the coefficient matrix which was wrongly transcribed. She proceeded to write that: Basis is the number of non-zero vectors. Basis = (-2, 2, -1,1), (-1,-1,2,1), (1,1,-2,-1) and (0,0,1,1). She also wrote that dimension is the number of the vectors in a basis of the row space, Dimension = 4. This is also supported by the following extract by student T11 in Figure 8.6.

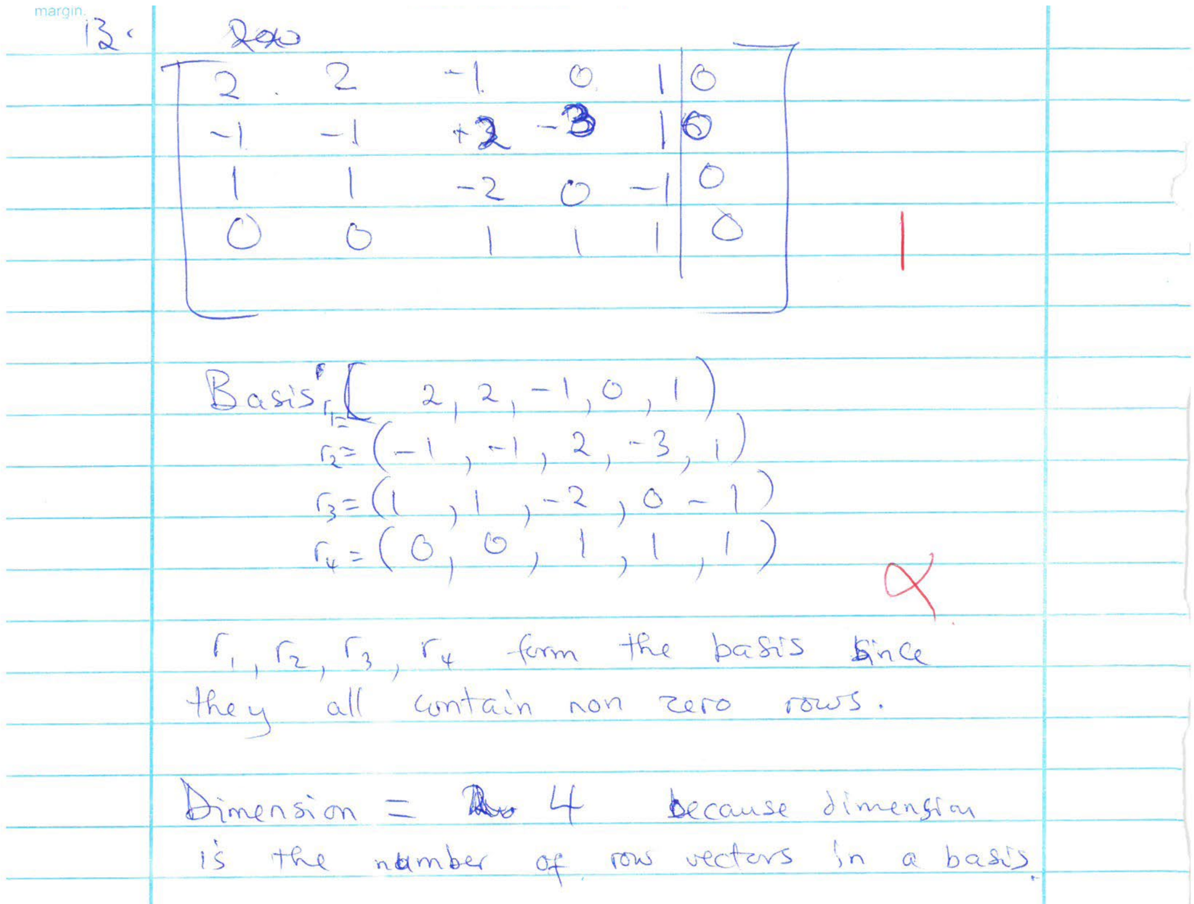


Figure 8.6: Written response of student T11

It is evident that student T11 was able to come up with the correct augmented matrix. Instead of proceeding to carry out elementary row operations, the in-service teacher inappropriately said that the row vectors forms the basis, since they contain non zero rows. This shows a serious misconception about the concept of basis of a solution space. The student could not appropriately acknowledge the defining and distinguishing characteristics of a basis, that of linearly independent and spanning. He also failed to grasp the definition of dimension of the solution space as evidenced by the solution given that is dimension is 4. The student's response established that he had not yet advanced to the necessary reasoning required for answering the questions involving finding basis and dimension of the solution space. The student could not figure out that there is the need to carry out the Gaussian elimination method. The teacher's reasoning seems to be still beyond the action level of understanding of basis and dimension of a solution space.

The solutions of 23 students had the following shortcomings: (1) procedural errors, where some students could not carry out the correct elementary row operations, and (2) finding basis of subspace instead of basis and dimension of solution space. We noted that these 23 students were aware that they must carry out step by step procedures in an attempt to find the basis and the dimension for the solution space. The 23 teachers were able to come up with the augmented matrix. However, some of the students encountered some calculations errors during the row reduction procedure. However, at the end of the row reduction process we discovered that the teachers did not assimilate the correct schema and appropriate knowledge of what of a basis of solution space is. This is supported by the following part extracts by T13 and T50 respectively.

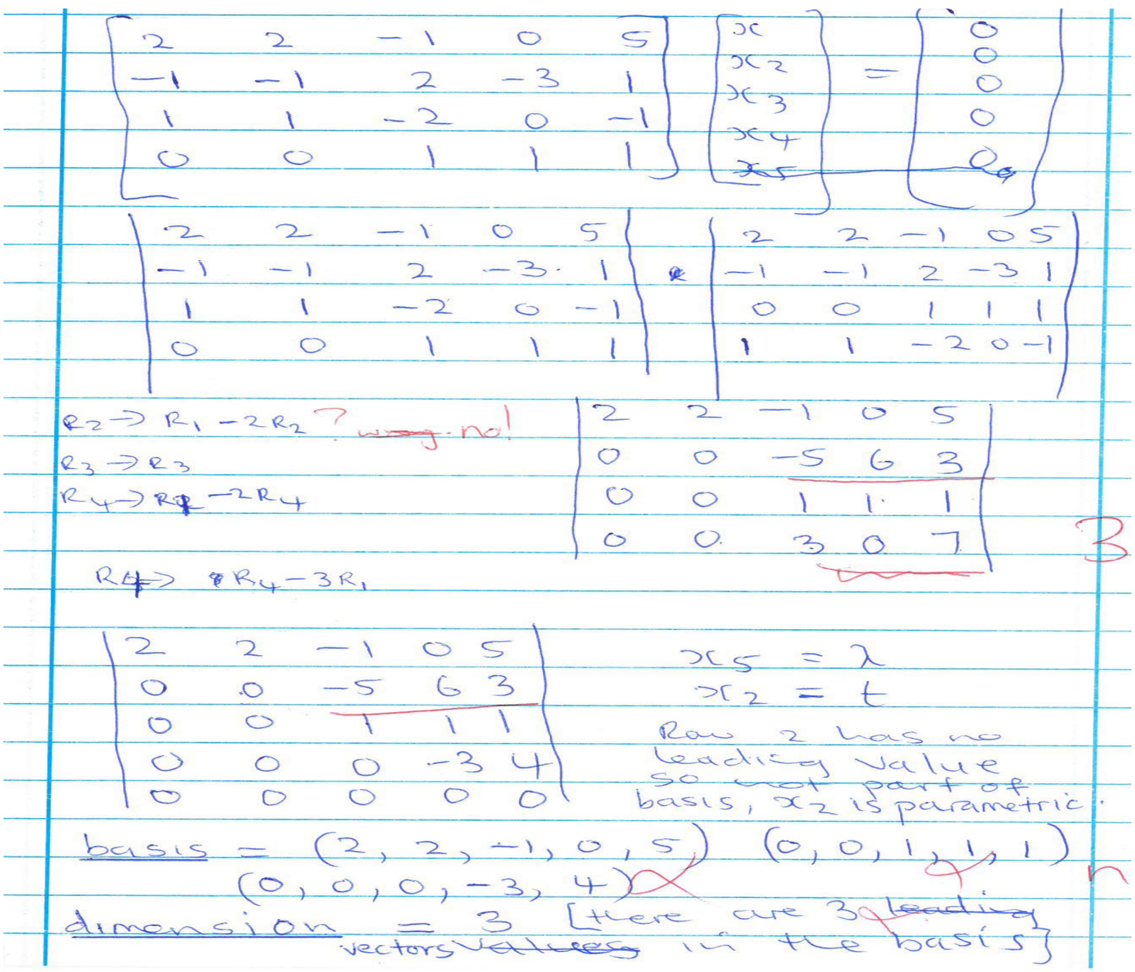


Figure 8.7: Written response of student T13

From the extract, see Figure 8.7, it is clearly evident that student T13 was able to come up with the correct coefficient matrix and used the correct brackets. After that, the student used the following type of bracket, $|\ |$ in place of continuing with the following types of brackets $[\ ]$. It seems he wanted to find the determinant of the matrix but she continued with elementary row operations. The two teachers were able to demonstrate the step by step procedures in an attempt to do row reduction. It is evident that row reduction process presented a challenge to the students. It is evident that student T13, during row reduction, firstly did row interchange, that is interchanged row 3 and row 4 that is [$r_3 \leftrightarrow r_4$]. This procedure was not necessary at all since the rows with the most number of zeros must appear at the bottom row in the matrix. However, her next row operation was $r_2 \rightarrow r_1 - 2r_2$. This row operation was not correct because it did not give her the zero in the second row first column that is $2 - 2(-1) = 4 \neq 0$. Her last row operation that is $r_4 \rightarrow r_4 - 3r_1$ did not give her the zeros again in row 4 first column that she must obtain $0 - 3(2) = -6 \neq 0$. This shows that student T13 struggled to manipulate figures and at the same time confuses the method of elementary row operations. She lacked the mathematical skills and appropriate knowledge in dealing with directed numbers, despite the fact that she is teaching the concepts at secondary school. A lot of errors were committed, showing poor conceptualization of the concepts on row reduction. We also noted that the student had fundamental misconceptions in the use of parenthesis. The aspects on usage of brackets as well as procedures for elementary row operation were covered in depth in the first module. The other student, T50, was able to do the correct row operation but also could not manipulate figures for example the following was a correct row operation $r_3 \rightarrow 2r_3 - r_1$. However, considering column 3 we should obtain $2(-2) - (-1) = -3 \neq -5$. The two students did not manage to reduce the matrices to reduced row echelon form as shown again by T50.

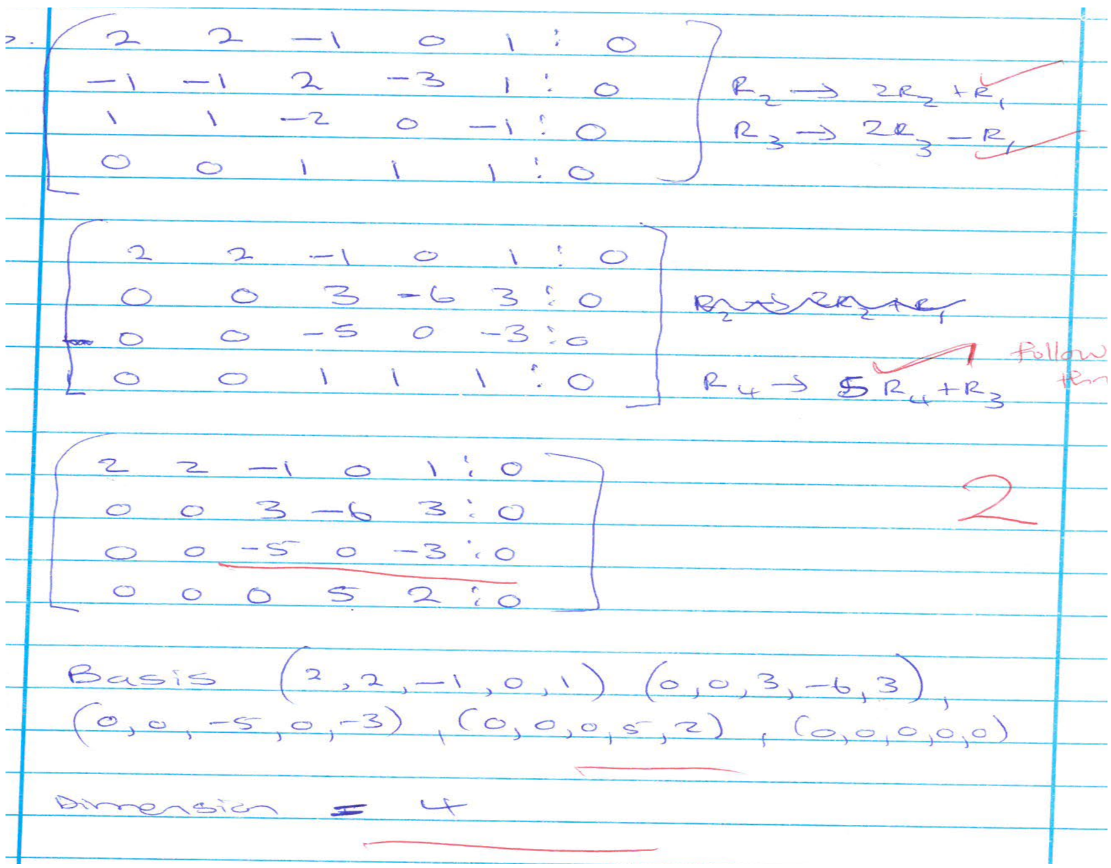


Figure 8.8: Written response of student T50

Student T50 proceeded to take the row vectors as the basis of the solution space as shown in Figure 8.8, adding some more confusion by adding the zero vector, and this whole procedure was incorrect. To find the dimension of the solution space, it seems the student simply counted the number of non-zero row vectors whilst student T13 seemed at first calculated the values of the scalars by saying that $x_5 = \gamma$ and $x_2 = t.$ The student later on listed the non zeros rows with the exception of row 2. This showed some confusion on the learnt concepts. The teacher proceeded to count the non- zero row and referred to it as the dimension. In their cognitive structures it seems the students have constructed the schema of the basis of a subspace. This shows that the teachers were just memorizing the procedures with lack of understanding and not taking cognisance of the significance to the question. This show a serious indication of a misunderstanding of the concepts learnt. It indicated that the concept of basis was not resolutely established. This hindered the students to encapsulate the processes into an object understanding according to the genetic decomposition.

16 of the students were able to come up with the augmented matrix and carried out elementary row operation but encountered calculation errors on the way. All 16 students did not bother to find the solution space after elementary row operations. The students only engaged with row reduction and made various explanations and drew conclusions based on the results obtained and made incorrect deductions. In order to gain more understanding of students' thinking some more responses were further scrutinized. I examined further student T2's response who gave the following result as the final solution, $\begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & -2 & -2 \end{bmatrix}$. The student wrote the following conclusion: It is linearly independent, it spans therefore it forms a basis. The explanations made lacked detail and showed confusion between the concept of linear independence, basis and dimension. This showed that the students were experiencing some challenges and confused these terms. There is no relationship between linear independence and the reduced matrix formed. In their studies Goris & Dyrenfurth (2010) discovered that students' misconceptions emanate from prior learning. Student T11, with a similar solution, wrote: It is not consistent, and has many solutions. No further explanations were given. The terms linear independent, linear dependent and spanning were used interchangeably and incorrect conclusions were arrived in an attempt to show that it is a basis of the solution space. Another teacher, T41, simply wrote since it is linearly independent it does not form a basis of R^4.

The other 14 students attempted to find the scalars but they encountered some calculations errors so that some of them ended up having three parameters instead of two. The students saw these transformations as external and hence it hindered them to move past an action level conception Five of the students were able to apply the correct algorithm for finding the basis of the solution space. They were able were able to carry out correct manipulations and find the correct solution space that is $X = t\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} + s\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and listing the vectors say $v_1 = (1, 0, -1, 0, 1)$ and $v_2 =$

$(-1, 1, 0, 0, 0)$ and they referred to it as the basis of the solution space without giving the reason why it is basis. However, these students encountered challenges in stating the dimension. An example is student T14 who after finding the basis of the solution space, proceeded to write that basis is 2, linear independence and spans instead of articulating that the dimension of the space spanned by a linearly independent set of vectors is equal to the number of vectors in that set. This indicates that he did not state the dimension, showing more confusion of the terms. The other four students were able to state the basis of the solution space but it seems they misused the defining aspect of dimension. They wrote that the dimension is three meaning that they simply added the total number of rows with leading elements and said it is the dimension. This indicates that these students were not reasoning favorably at the object conception. No proper justification was given as to why it was a basis.

Only 11 (73%) of the students gave a correct response on what the basis of solution space is and what the dimension of the solution space is but without justifications as to why it was a basis of the solution space. The students' responses showed a mathematical understanding of the procedures to be followed when calculating the basis of the solution space as well as the dimension without having sound knowledge as to why it formed a basis. I can argue that the students had built the correct concept image of the procedures for finding the basis of the solution space but the responses also exposed that they did not have a vibrant understanding of the relationship between the basis and the dimension of the solution space. This shows that the process was not encapsulated into an object of the basis of a solution space.

It is important to note that none of the students gave a complete response to item 12. The students responses indicated that they had not yet constructed the necessary mental constructions, as anticipated in the preliminary genetic decomposition. It is important to note that after row reduction, the following results were supposed to be obtained for the scalars x_1 up to x_5 together with the stated deductions:

$$X = t\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} + s\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

This indicates that the vectors $\boldsymbol{v_1} = (1, 0, -1, 0,1)$ and $\boldsymbol{v_2}$= (-1, 1, 0,0, 0) span the solution space. It is clearly seen that neither of the vectors is a scalar multiple of the other illustrating that the vectors are linearly independent and hence form a basis for the solution space. The dimension of the solution space is 2, since the number of free variables is 2. However, none of the students was able to explicitly explain why it formed a basis for the solution space. The students simply wrote the solution space and stated the dimension. Some were silent on justifying the result obtained and others simply used different terms interchangeably. This showed that the students struggled to apply the learnt material and come up with the convincing deduction. This is in line with Stewart and Thomas's (2008) studies where they discovered that students need to build on a number of previous concepts so as to construct the concept of basis. The students' imagination did not integrate the ideas of spanning and linear independence. It is important to note that the set of the vectors $\{ v_1, v_2\}$ forms a basis of R^2 because it is linearly independent and spans R^2. Table 8.4 shows the allocation of scores for question 9.

Table 8.4: Allocation of scores for question 12

Category				**Frequency**
No response				2
Formulated augmented matrix	Taking the non-zero rows as basis of solution space and dimension			2
	Carrying out row reduction	Incorrect row reduction and taking non zero rows		23
		Failure to calculate scalars, simply made deductions from the row reduced echelon matrix		16
		Calculate scalars but obtained wrong scalars		14
		Correct Scalars	Correct basis of solution space but wrong dimension	5
			Correct basis and dimension but wrong justification	11

8.6.2 Interview responses for question 12

An interview excerpt with T13 included the following exchange:

***R**: In your own words, can you briefly outline how you can find the basis and dimension of the solution space?*

***T13**: I will form an augmented matrix, and carry out the Gaussian elimination method and reduce the matrix.*

***R**: After reducing the matrix what do you do?*

***T13**: I will then take none zero rows. These will form the basis of the solution space.*

***R:** How will you find the dimension of the solution space?.*

***T13**: I will now count the total number of non-zero rows and this will become the dimension.*

Another student, T44, who simply reduced the matrix and obtain the matrix was also interviewed.

$$\begin{bmatrix} 1 & 1 & -2 & 0 & -1 & : & 0 \\ 0 & 0 & 0 & -3 & 0 & : & 0 \\ 0 & 0 & 0 & -9 & -4 & : & 0 \\ 0 & 0 & 0 & 0 & 0 & : & 0 \end{bmatrix}.$$

R: *In an attempt to find the basis and dimension of the solution space, you obtained the following matrix, but you did not tell us what the basis of solution space and the dimension is.*

***T44**: [Remained quiet for some time]. I don't still remember. These terms confuses me.*

***R**: Which terms confuses you?*

T44: *The terms dimension of a vector space and rank of a matrix.*

***R**: Can you try to define the term dimension of a vector space?*

***T44**: The dimension of a vector space I think it is the number of basis, the number of elements in the basis gives the dimension. There is a thin line between these two terms. This usually confuses me but what I have notice is that sometimes the dimension of a matrix is equal to its rank.*

The above extracts show that the students showed some challenges and some gaps in the construction of the knowledge on basis and dimension of a vector space. Student T13 knew some of the procedures that needed to be executed in order to determine basis and dimension of a vector space. However, we observed that he confused the last part of the procedure. Instead of finding the solution to the systems of equations, she simply listed the non-zero rows. This shows that the concept of basis of the solution space had not fully developed at the object level understanding according to the APOS theory. Student T44 simply diverted the question and brought in a new term, rank of a matrix. The student could not define the terms. The extract with student T44 shows that the students were able to carry out the step by step procedures without really understanding the needs of the questions. The students have only developed a procedural understanding of the concepts, hence this has greatly affected the development of their mental constructions at the object level.

8.7 General observation

This chapter reveals some of the challenges that the students encounter when learning the concepts on basis and dimension of a vector space. We noted that students struggled to explain the reason whether given vectors forms basis of R^3. The students struggled to apply the theorem that says if V is an n dimensional vector space, and let S be a set in V with exactly n vectors, then S is a basis for V if and only if S spans V or S is linearly independent. Most of the students resorted to using step by step procedures, for example attempting to use Gaussian elimination method. I also noted that in order to show that given vectors form a basis of a vector space, most of the students were aware of the procedures to follow, however a number of the students were stuck and could not tell whether it formed a basis or not. A large number of students simply listed the non-zero rows after doing row reduction. The students also struggled to find the basis and dimension of the solution space. They knew the method of row reduction, but some ended up simply making a conclusion from the row reduced matrix without calculating the scalars, which was incorrect. Again a large number of students simply took the non-zero row after row reduction and concluded that it was the basis of the solution space. A few students were able to find the basis and dimension of a vector space, but could not justify the result why it formed a basis of a solution space.

8.8 APOS analysis emerging from the chapter

For the concept of basis the genetic decomposition was revised substantially. While reflecting upon the preliminary genetic decomposition it became clear in the process of analysing the data, that the concept of basis arose as a result of the coordination of the two process of establishing that a given set spans the particular vector space and that of establishing that the set is linearly independent. In the preliminary genetic decomposition it was proposed that the process of establishing whether a set formed a basis arose as a result of the interiorisation of actions. Hence in the revised genetic decomposition there are no actions proposed as part of the genetic decomposition. Hence we will use the revised genetic decomposition to direct the discussion in this subsection about the APOS insights arising.

Question 3 required the students to imagine internally and use inspection when showing whether given vectors forms a basis, in the process of checking whether the set of vectors is linearly independent and spans R^4. This meant that there was no need to go through the step by step procedures explicitly, thus showing a process level engagement. However, I noted that 19 (26%) of the students did not develop their mental construction at the process level of what a basis. Some of them left the question blank with others having completely incorrect responses. This was a result of rote memorization of concepts as the students struggled to apply Theorem 7.1. It is evident that solutions of the form $r < n$, the vectors are linearly dependent were popular, where the r and the n were not defined at all. 38 (52%) of the students also did not develop their understanding at the process stage. These students used the step by step procedures, showing that they did not develop the necessary mental construction at the process level of understanding according to APOS theory. However, 7 (8%) also showed that they had not fully developed a process level understanding of basis even though they internally visualized that the four given vectors do not form a basis for R^4 since it is not linearly independent. The process conception was not fully developed since these students did not describe the reasons explicitly. All the solutions had slips as illustrated by student T34's response who used the theorem effectively but failed to outline whether the vectors forms a basis or not. Many of the students struggled to explicitly explain the notations used for Theorem 7.1 showing that the process level has not fully developed, but they could see that the vectors are linearly dependent. This hindered the in-service teachers to fully develop their thinking at the process level of understanding. Only 9 (12%) of the students developed the necessary mental construction at the process level of understanding as they were able to give a correct explanation as to why the vectors do not form a basis without showing any step by step procedures. Theorem 7.1 was used explicitly.

Considering the task of the learner generated examples, it was apparent that most of the students struggled to come up with 4×4 matrix that illustrates a basis of R^4. Brijlall and Ndlovu (2013) outlined that students are less comfortable and cannot solve application problems where no rules are applicable. This forms a major cognitive obstacle to the learning of the concept basis, and this question showed a high level of abstraction of learnt concepts. Only 6(8%) of the in-service represented their understanding in a manner described as an object understanding of a basis of a vector space. The students were able to figure out that a basis of R^4 must have four linearly independent vectors in such a way that they form a matrix that is in echelon form.

In terms of operating with the application question on basis and dimension of the solution space, the question involved a system of linear equations. I noted that 4 (5%) of the students were not able to engage with the concept hence they did not develop the necessary mental construction at the object level understanding of the concept of basis and dimension of the solution space. However 53 (73%) of the students realized that there was a need to show step by step procedures of row reduction, in the process of checking if the set of vectors are linearly independent and spans the solution set. The item, however, provided some challenges as 30 (41%) of the students failed to carry out the correct manipulations as they struggled with the basic algebra calculations and 23(32%) of the students obtained incorrect row operations and further made reference to a basis of a subspace thus manifesting a serious misconception. These challenges hampered the students to develop their mental constructions at the object level conception of basis of the solution space. 5 (7%) of the students obtained the correct basis of the solution space with wrong dimension. Furthermore, I found out that only 11 (15%) were able to carry out the correct row operations and were able to deduce the appropriate basis of the solution space and dimension of the solution space. However, none of the students was able to encapsulate the process into an object level as they did not make further transformation on the vector obtained or use the ideas of it being linearly independent and being able to span the solution set so that they developed their understanding at the object level. This study further confirms some of Dorier and Sierpinsk's (2001) argument that students struggle to understand concepts in the first linear algebra courses because of the high level abstraction which is a result of struggling to connect what they already know and linking it to the new knowledge. Furthermore, they commented that the students fail to connect school level mathematics with the new knowledge.

From an analysis across the questions, four students showed that they had no idea of the concept of basis of a vector space showing that they did not develop the necessary mental constructions at the process level of basis of a vector space. These students provided completely incorrect responses or no responses. I noted that 9 (12%) of the students were able to develop their mental constructions at the process level understanding across the questions. These students were able to interiorise the processes of checking if a set of vectors is linearly independent /dependent and they were able to make correct and convincing logical deductions as to whether whether the given vectors formed a basis or not of a vector space. Also when interviewed, some of the students were also able to outline the procedures that can be used to determine a basis of a vector space, that is outlining the process of checking whether the given vectors are linearly independent and spans a given space, without specific vectors, for example student T4, and she was also able to state the methods that can be used to determine the concept of basis of a vector space. These teachers could also interpret correctly whether given vectors form a basis from a row reduced echelon matrix, indicating that they have developed the necessary mental constructions at the process level according to APOS theory.

I also noted that across the questions, two questions tested the object level understanding of the concepts of basis of a vector space. None of the students was able to develop his/her understanding at the object level understanding of basis in question 12. However, in the question on the learner generated example, only 6 (8%) of the students were able to develop the necessary mental constructions at the object level as they were able to recognize the possible ways to come up with a 4×4 square matrix and modify it and make the necessary adjustments so as to obtain a basis for R^4. However for question 12 none of the students was able to give the correct justification why the two vectors obtained formed a basis of the solution space. This indicated a failure to master all the characterization of the concept of basis. The students in questions did to think of the processes as an object due to a failure to link the concepts learnt on linear independence and spanning and the obtained vectors.

8.9 Implications for teaching

The study shows that some aspects described in the genetic decomposition theory were not fully functional because the in-service teachers were not successful with the questions that required flexibility in thinking. The schema part of identifying what a basis is and basis of a solution space was missing in most of the students' cognitive structures. The study managed to categorize the possible obstacles to learning of basis. The subsequent difficulties that emerged from an analysis of students' written responses and interviews conducted were:

1. Incorrect use of parenthesis,

2. Calculations errors, for example incorrect row operations and trapped by failure to manipulate the directed numbers,
3. Finding basis of subspace instead of basis of a vector space,
4. Simply taking the row vectors as basis of the solution space,
5. Taking the total number of non-zero rows in a row echelon form as the dimension of the solution space.

In order to construct the necessary mental construction for a basis, the individual needs to come up with the coefficient matrix and carry out row reduction and at the end make some corrective judgements whether it is a basis or not. The study revealed that the students were successful and confident in coping with the procedures of row reduction but they encountered difficulties when answering the last part of the question that required logical reasoning when making conclusive judgments. Most of the students functioned at the action level according to APOS theory. Kuzle (2013) argued that the concepts imparted in mathematics need not dwell on the calculation abilities only, but that students also need an additional understanding of prior knowledge relative to the task at hand and flexibility in thinking. Similarly, Noyer (2007) argued that learners should be taught to think mathematically, rather than being taught to do mathematical calculations. In order to develop meaningful understanding of concepts students must be able to link the new mathematical knowledge and make connections with the old knowledge in order to solve new problems (Stylianides & Stylianides, 2007). Hence it is important that the students should examine the relationship between the given concepts and then choose the appropriate angles that he or she can use to go about the problem. Dorier and Sierpinska (2001) made a similar observation and they said that students must adhere to a "cognitive flexibility" so that they have a deeper understanding of the linear algebra concepts. From the study, most the students were operating within the action level engagement. Therefore the researcher recommends that lecturers should be aware of the students' errors, misconceptions and learning struggles so that they are able to take the students past an action level engagement of APOS theory.

8.10 Modification of the genetic decomposition

It is also noted that some of the students could not construct the correct structures of a basis of a vector space and basis of the solution space. This is mainly because these students do not possess some of the prerequisite concepts that are required for the construction of the concept of concepts.

The weak schemas of solutions to systems of equations need to be developed first. The specific actions for all the levels were specified so as to capture items that came from the data analysis. I also noted the concept of basis arises as the coordination of the process of checking if the vectors are linearly independent and the process of checking if the vectors span the vector space. Therefore I noted that there is no action for basis, see table 8.5 below. Table 8.5 below shows the modified genetic decomposition.

8.5: Preliminary and modified genetic decomposition for basis and dimension

Preliminary Genetic Decomposition	Modified Genetic Decomposition
	Prerequisite concepts The prerequisite concepts to start the construction of basis are object conceptions of solutions to systems of equations. In terms of solutions to systems of equations, the individual needs to be able to see the resulting solutions to systems of equations or calculated determinant as a totality for given vectors in R^n. The result from the coefficient matrix can be used to deduce results related to linear independence/dependence and spanning.
Action At the action level if an individual is asked to show whether a given set of vectors say R^n forms a basis, the transformation involves multiple steps. The term basis acts as external stimulus of what needs to be done. The first step is to form vector equations of the form $k_1v_1 + k_2v_2 + \cdots k_nv_n = 0$ and $k_1v_1 + k_2v_2 + \cdots k_nv_n = b$	**Action** No action because the process of checking if a set of vectors forms a basis of a vector space arises as the coordination of the Process of checking if the vectors are linearly independent and the Process of checking if the vectors span the vector space.

where k_1 k_2 , …k_n are scalars that need to be calculated. The next step is to express the given vectors in coordinate system and represent $Ax = 0$ and $Ax = b$ in matrix solution, then translate to a matrix which consists of matrix A. Suitable methods are selected to determine whether these vectors are linearly independent and span R^n.	
Process An action is interiorised into a process when an individual can describe and generalize the method for finding a basis for vector space. This means that the individual can think of the procedure without specific vectors or even without specifying the number of coordinates. The individual can reflect on how to find basis and dimension of the solution space without specific vectors. Can make arguments based on the use of a theorem whether given vectors forms a basis of a vector space without performing the steps. Can relate linear independence/dependence to row reduced echelon form of a relevant matrix or to the calculated determinant.	**Process** The process of checking if a set of vectors is linearly independent and the process of checking if a set of vectors span the given space is coordinated into a single process that can establish if the set of vectors forms a basis for a given space. The individual can apply Theorem 7.1 or definition 8.2 to predict whether the given vectors form a basis of R^n without performing the steps.
Object At the object level the individual can reflect on the operations applied above and becomes aware of the process as a totality The processes are encapsulated into an object if the individual can see that a set of vectors $\{v_1, v_2, . . . , v_n\}$ form a basis for all of R^n if	**Object** The process of verifying that a set of vectors forms a basis is encapsulated into an object basis, making it possible to determine properties of basis and see relationships. The individual will be able to solve abstract systems for example can apply further

they are linearly independent and span R^n, adapted from Stewart (2007). The individual will be able to carry out further transformation by extending $\{\boldsymbol{u_1},\ \boldsymbol{u_2}\ \ldots \boldsymbol{u_n}\}$ to a basis of R^n. The individual must be able to apply process or further transformation on basis of the solution space. The individual must be able to distinguish the two methods that can be used to test for basis.	transformation by extending $\{\boldsymbol{u_1},\ \boldsymbol{u_2}\}$, that is to say a basis of R^n has n linearly independent vectors. The individual should be able to encapsulate the obtained basis of solution space and be able to link the result of the basis of the solution space to spanning and linear independence. Describe the relationship between basis for the space spanned by given vectors in R^n and basis for the solution space. Describe the relationship between basis for the subspace spanned by given vectors in R^n and basis for a vector space. The individual should be able distinguish and compare the methods for finding basis of a vector space.

CHAPTER 9

ANALYSIS OF STUDENTS' ERRORS AND MISCONCEPTION

9.1 Introduction

The main aim of this chapter is to examine the in-service teacher's errors and misconceptions in the learning of linear algebra. The analysis focused on the teacher's responses and interviews based on the results already been discussed in chapters 5, 7, and 8, as well at some more written responses on the work written in chapter 6. Errors that were displayed by students in the four chapters were classified as; conceptual; procedural; and technical errors. In this study I adopted the work of Kiat (2005) to identify the three types of errors that occur when solving problems based on vector space concepts.

9.2 Discussion of the errors displayed by students on the question on vector space and subspace

The discussion is based on questions from activity sheet 1. Question 2 is presented below and the results for questions 4 and 7 were discussed in chapter 5.

9.2.1 Results for question 2

The results for each of the three questions are presented in terms of the three types of errors.

2. Let $V = R^3$. Show that W is a subspace of R^3, where: W $=\{(a,b,c)\colon\ a = b = c\}$, that is, W consists of all vectors having three equal components.

Out of the 73 students, four students gave a completely correct solution with correct reasoning. All four students were able to show the three part procedures for a subspace with sufficient explanations of why it forms a subspace of R^3. From the analysis there were 63 responses whose errors were classified as being conceptual, six were classified as procedural errors and there were zero technical errors. We now discuss the nature of those errors identified in this question in more detail.

9.2.1.1 Conceptual errors for question 2 on vector space and subspace

There were 14 responses where the students were confused about what they really wanted to show. These students simply identified the conditions that needed to be satisfied when showing that a given set is a subspace. They simply listed the three conditions and some went further by transcribing the question. This shows that these students revealed conceptual errors. The other type of conceptual error identified in students' written responses was a failure to come up with a vector that satisfied all the three equal elements. 48 responses revealed such an error. An example of the conceptual error is illustrated by the written response of student T46 in Figure 9.1.

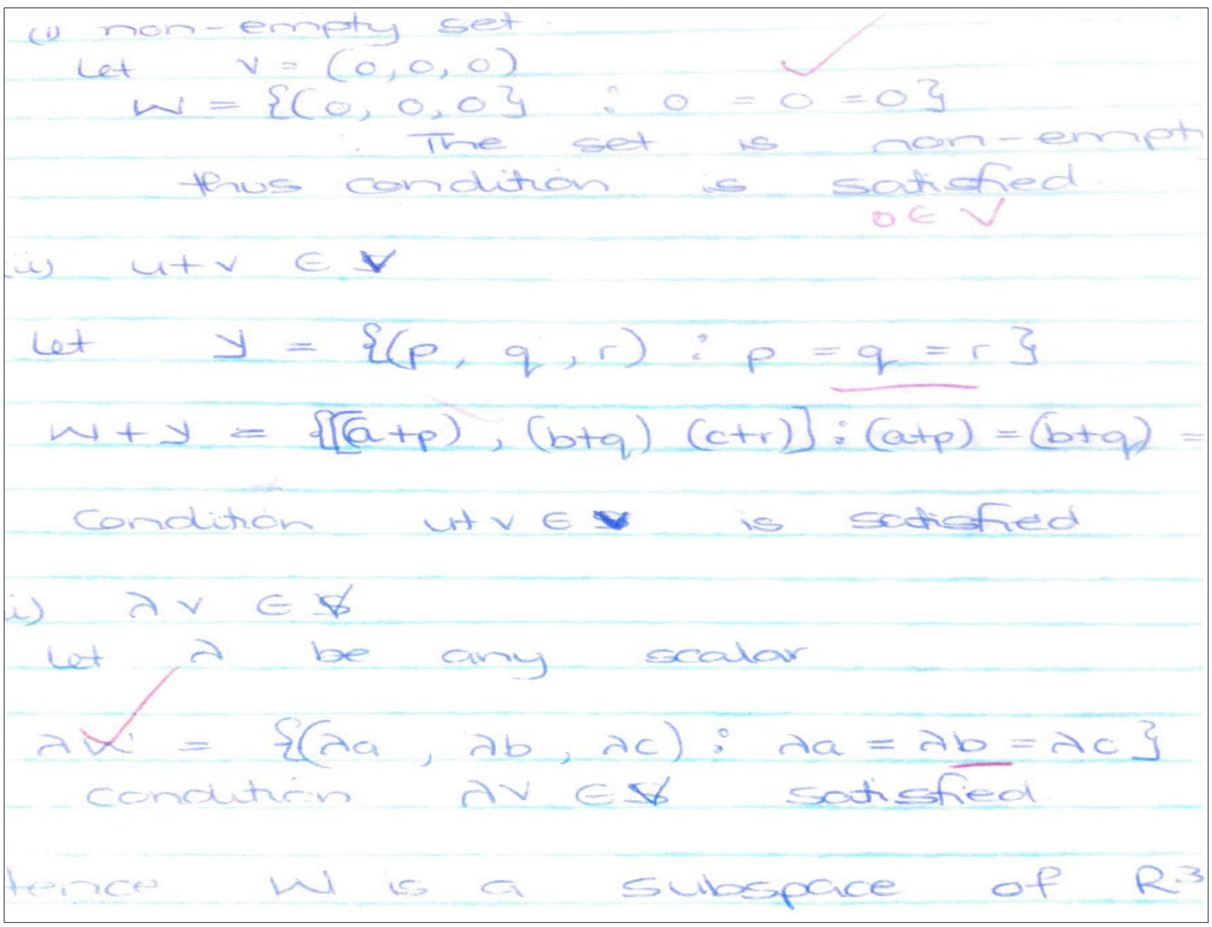


Figure 9.1 Written response of student T46

From the students' written responses, it was clear that the student were able to prove the first axiom. They were able to follow the three part procedure, but failed to come up with the correct sets with all three equal components. Thus these students are revealing a conceptual error, since correct procedures are evident. The challenges experience here by the students was the issue of failing to understand the ideas of what it means to say equal components and then write $w = \{(a, b, c): a = b = c\}$ means equal components. Failure to list the correct sets illustrates that these students are experiencing conceptual errors. One student, T35, wrote that if it is a subspace it satisfies (i) linearly independent (ii) $k_1v_1 + k_2v_2 + k_3v_3 \dots k_nv_n = 0$. This shows that the met afters that is the concepts on linear independence, have affected the conceptualization of the concept subspace.

9.2.1.2 Procedural errors for question 2 on vector space and subspace

Six students manifested procedural errors. The common error manifested by these students was failure to carry out the three part procedure, and they also failed to come up with all three equal components. For example students T14 and T30 tested the addition axiom only and illogically concluded that it was a subspace of R^3 without showing the existence of the zero vector, and scalar multiplication. This shows that the students did not really know the conditions for the existence of a subspace or had just memorised the conditions without understanding how to prove them. Another type of procedural error was manifested by students T23 and T53. These students were able to state the three conditions necessary for a set to be a subset. However, when testing these conditions, instead of testing the stated conditions, they instead tested the commutative property of addition and the closure property of addition, see the written response by student T53 in Figure 9.2.

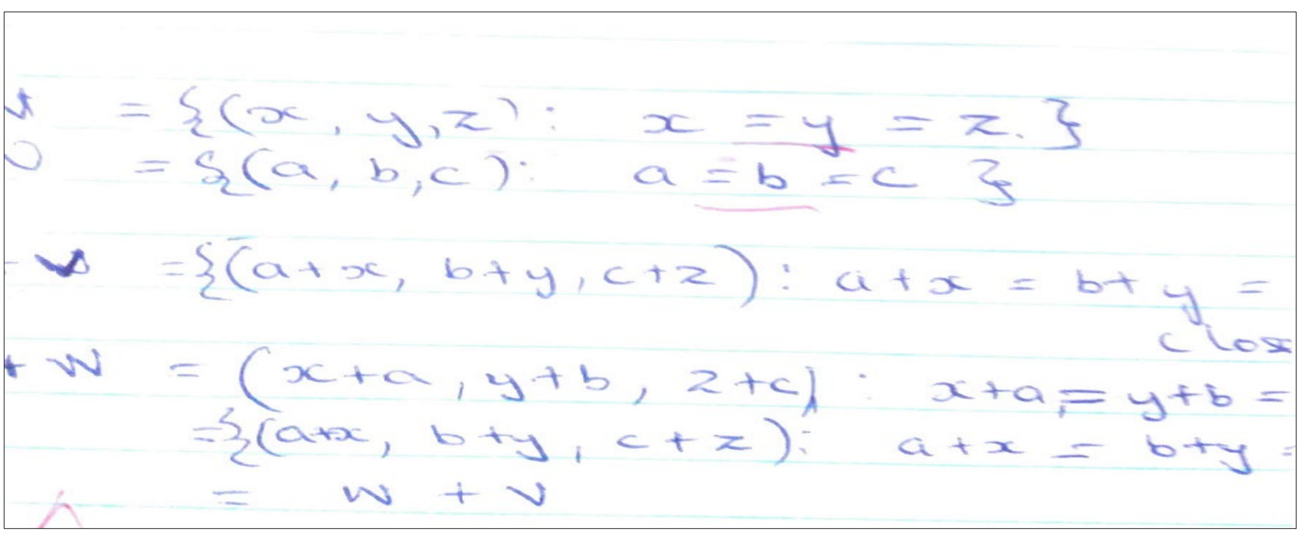


Figure 9.2 Written response of student T53

Student T23 then concluded that the set is a subspace of R^3 but student T53 did not make any conclusions. The other students, T20 and T44, were able to state the conditions for a subspace but copied the rules for testing the addition axioms for a vector space as well as the multiplication axioms and also concluded that it was a subspace of R^3. This showed confusion on the concepts done on subspace and a vector space.

A follow up interview was done with student T4 who used the same elements as T46. The following exchange took place:

R: *May you state the conditions that must be satisfied by a subspace?*
T4: *[writing down] . 0*$\in W$, *or non empty*

2. $u + v \in S$, *closed under vector addition*

3. $\gamma V \in S$, *closed under scalar multiplication*

R: *Question 2 required you to show that* W *consists of all vectors having three equal components.*
T4: *Yes I came up with another set V and I wrote that { (a,b,c):* $a = b = c\}$ *meaning that the three are the same, but is not marked. Maybe is it that I was supposed to say {*$1 = 1 = 1$*} or {*$2 = 2 = 2\}$ *or {*$x = x = x$*}. What mem? I am now confused.*
The response by the student showed that though she was able to make an effort to have the new set of element, it seems the equality sigh was good enough for her that these element are equal.

9.2.2 Results for question 4 on vector space and subspace

The item required the teachers to generate a counter-example to show that the set of 2×2 matrices with a zero determinant was not a subspace. There were 49 responses whose errors were classified as being conceptual and one was considered as a procedural error. We now discuss the nature of these errors identified in this question in more detail.

9.2.2.1 Conceptual errors for question 4 on vector space and subspace

The students had difficulties in trying to come up with a set W of $M_{2\times2}$ matrices which did not fulfil the condition of being a subspace. Many of the students could not come up with two matrices that fulfilled the condition of being a subspace, thus establishing conceptual errors. 27 students manifested this error. They choose inappropriate counter examples, which gave determinant equals to zero, but after carrying vector addition, the condition of not being a subspace was not fulfilled. An example is that of student T13 Figure 5.1. These students were aware of the conditions necessary when showing that a given set is a subset as they could list them, but I noted that some of the students struggled to prove them. However, student T13's response showed that she was able to carry out the correct procedures but got confused since she said the set was not a subspace yet it is a subspace. This shows that the students have a tendency to apply rote learning and they memorise rules and procedures for showing that any given set is a subset without trying to make sense of what is really asked for. When student T13 was probed to think about her response, she showed that she was able to identify the mistakes that she had displayed in the activity sheet. She was now aware of what she was required to prove when she said that "Maybe I confused myself because I see now that I must get a non-zero determinant". The student was given a chance to try and come up with two more matrices but still could not make it. She was able to carry out the step by step procedures in an attempt to show that the given matrices was still a subspace instead of not being a subspace.

15 students were uncertain about what they were supposed to show, thus revealing conceptual errors, as well as 7 students who simply listed the conditions for a subspace and could not find there way. An examples of such students is T5 who simply came up with a 2×2 square matrix which gave a determinant which was not zero, and then concluded that it was not an element of V without even attempting to show the techniques for a subspace. This demonstrates that the student is not aware of the conditions for the existence of a subspace, see also written responses by student T12 in Figure 5.7 and student T69 in Figure 5.6.

9.2.2.2 Procedural errors for question 4 on vector space and subspace

One student, T46, revealed a procedural error. This student was able to come up with appropriate vectors and was able to do vector addition. However, the student failed to produce an argument around the appropriate counter-example. The student did not make a conclusion to show that the set of vectors chosen were not subspace. This caused an obstacle to construct meaningful understanding of what a subspace is not.

9.2.3 Question 7

For this item the students were required to show that the ten axioms for a vector space were satisfied. Six students managed to go over all the ten axioms and showed that it was a subspace. The main errors exhibited were mainly conceptual with 56 students, 11 were procedural errors and there was no technical error.

9.2.3.1 Conceptual errors for question 7 on vector space and subspace

Many of the teachers were confused about how to go about proving some of the axioms and what exactly they wanted to show. 56 students exhibited conceptual errors where for example some students tested the axioms for specific elements of V, instead of considering generalised examples, see Figure 5.9, written response by student T8. Some teachers were evidently confused about what the identities for the different operations were, for example in axiom 5 which says that for every number $a \in S$ there exist a number $-a \in S$ such that $a + (-a) = (-a) + a = 0$. Thus $-a$ is the inverse for addition. Student T8 came up with a specific matrix which gives a determinant equals to zero. This showed confusion on the work done on vectors space. The other conceptual error was on showing that $\forall\, \mathbf{v} \in V,\ 1.\mathbf{v} = \mathbf{v}$**.** The students confused the 1 in the scalar multiplication $1.\boldsymbol{v}$ since **v** was a matrix. They thought that 1 must also take the form of a matrix. This was evidenced by the interview intercept with student T27. The following matrices was used as the identity matrices $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ by 44 students and the other two used the matrix $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ as the identity whilst two others used $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ as well as another student who expressed it as $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} . \left[\frac{-1}{ab}\right]$, i.e. $\text{v} \times v^{-1} = 1$. This suggests confusion with notion on scalar identity. This shows that the students were only relying on rules and could not understand how to prove it. Another kind of conceptual error was identified in student T14's response. The student was aware that the closure property must be satisfied. The students failed to add the following matrices $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}$ and obtained $\begin{bmatrix} ac & 0 \\ 0 & bd \end{bmatrix}$, see Figure 5.13. The student was making an attempt to prove the closure property of addition. The student here multiplied the corresponding elements instead of adding corresponding elements thereby revealing some technical errors. He should also have stated where the vector $\boldsymbol{u}, \boldsymbol{v}$ belonged to. There is a misapplication of a wrong learnt procedure of scalar addition which had confused the student. The student progressed with this error as he tried to prove for commutativity as well as the associative property.

9.2.3.2 Procedural errors for question 7 on vector space and subspace

11 of the students were able to identify the axioms of addition and were able to prove them. However, these students could not prove the axioms of multiplication thus revealing procedural errors.

9.2.4 Discussion

An overview of the different kinds of responses for the three questions appears in Table 9.1.

Table 9.1: Summary of the number of different response types for the three questions on vector space and subspace concepts.

	Not attempted	Conceptual errors	Procedural errors	Technical errors	Almost complete	Complete
Question 2	0	63	6	0	0	4
Question 4	16	49	1	0	0	7
Question 7	0	56	11	0	0	6

9.2.4.1 Summary of commonly identified conceptual errors for vector space and subspace

The most common error displayed in this section was the conceptual error. This was mainly due to the students failing to interpret the nature of the question, especially question 4, and failing to come up with a vector equation that satisfied three equal components. The students did not really understand what was needed on question 4. This was more of an application question done on the work on subspace of a vector space. The students struggled to find appropriate counter examples, with some of the students being uncertain about what the counter example must do see Figure 5.5, written response by student T7. Some students were confused about what really needed to be done, such that one student, T69, came up with a vector equations in matrix form equated to the zero vector, and there was no relationship of this equation with the concept of subspace. This shows that the student manifested conceptual errors. Another student who had confusion of what needed to be done was student T12, see Figure 5.7. He had an expression of the form $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}2 & 2\\2 & 2\end{bmatrix} + \begin{bmatrix}3 & 3\\3 & 3\end{bmatrix}$. He added the expression on the right hand side of the equation, and said after row reduction we obtain $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \neq \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$. This showed a general confusion on the work done on a subspace, thus exhibiting conceptual errors. This shows that these students failed to connect the different mathematical ideas, which is linking the notion of zero determinant and not a subspace of a vector space. The interviews with students T4 and T13 also revealed that the students, even if they were now aware of what must be shown after probing could not come up with the two matrices. The matrices that they still chose satisfy the given conditions that it is a subspace. Sfard (1991) alleged that if the student fails to connect such ideas, it leads to the formation of a misconception, which in this case is a technical error. In question 2, 63 responses revealed conceptual errors mainly because the students failed to come up with the appropriate set and the other students simply listed the axioms mainly because they could not come up with the appropriate set. In question 7, a large number of the students were aware that they must show that the ten axioms must be satisfied. However, the major conceptual error manifested in item 2 was a failure to prove some of the axioms. For example, a large number of students failed to prove axiom ten on showing that $\forall\, \mathbf{v} \in V,\ 1.\mathbf{v} = \mathbf{v}$**.** Instead of multiplying the vector $\boldsymbol{v}$ by a scalar, the 1 was taken as a matrix since $\boldsymbol{v}$ was also a matrix as revealed by student T7 in the interview. This shows that this choice of matrices that was used by the students that is $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$ and $\begin{bmatrix}a & 0\\0 & b\end{bmatrix}$ as explained above are hereby seen as obstacles to the construction of new mathematical knowledge, thereby preventing the in-service teachers to develop their understanding at the structural level according to Sfard (1991). The other conceptual error that was manifested was mainly because there were no tedious calculations and simplifications involved in these two questions. The student who demonstrated the technical error in item two failed to prove the closure property of addition, the commutative and associative property of addition. Instead of doing vector addition, the student carried out multiplication of corresponding elements, see written response by student T14 in Figure 5.13.

9.2.4.2 Summary of commonly identified procedural errors for vector space and subspace

In question 2, 6 students demonstrated some procedural errors because they confused the axioms of a subspace and those of a vector space. In question 4 one student demonstrated procedural error. This showed that those students who attempted to answer the question were aware of the procedures that needed to be followed and most of them were already trapped in conceptual errors, with 11 manifesting procedural errors in question 7.

9.3 Discussion of the errors displayed by students on questions on linear combination

9.3.1 Results for question 3

3. Can you express $\boldsymbol{v} = (2,-5,3)$ in R^3 as a linear combination of the vectors $\boldsymbol{u_1} = (1,-3,2)$, $\boldsymbol{u_2} = (2,-4,-1)$, $\boldsymbol{u_3} = (1,-5,7)$.

To solve this problem the teachers could work through the following steps:

- Set up a vector equation $\boldsymbol{v} = k_1 u_1 + k_2 \boldsymbol{u_2} + k_3 \boldsymbol{u_3}$, where k_1, k_2 and k_3 are arbitrary scalars.
- Set up a system of three equations with three unknowns (k_1, k_2 and k_3).
- Represent the system as an augmented matrix, or solve equations simultaneously using the elimination method.
- Carry out row reductions on the matrix and interpret the reduced matrix as indicating that the system has no solution.

Out of the 73 students, none of them gave a completely correct solution while nine teachers did not provide any response. Three students responded correctly that it could not be expressed as a linear combination, with two of them not giving any reason and one giving an incorrect reason that it had infinitely many solutions. These were considered as almost complete. There were 48 responses whose errors were classified as being technical errors, nine were considered as procedural errors and four errors were conceptual in nature. We now discuss the nature of these errors identified in this question in more detail.

9.3.1.1 Conceptual errors for question 3 on linear combination

There were four responses which showed conceptual errors where the teachers were confused about exactly which quantities were part of the vector equation. Therefore they were not able to get to the stage of representing the vector equation as a system of equations, where it was clear that the scalars were the unknowns. An example of such a conceptual error is illustrated by the response of teacher T35 which appears in Figure 9.3.

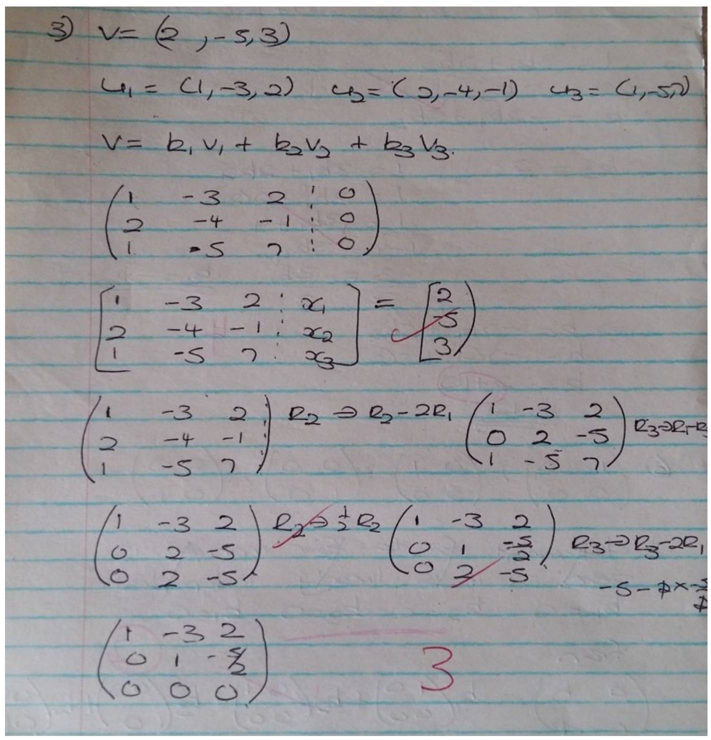


Figure 9.3: Written response of student T35

From Figure 9.3, it can be seen that teacher T35 attempted a solution without using any connecting phrases or statements. Because of the vagueness in the setup of the vector equation, the teacher struggled to translate the vector equation into a system of equations where the unknowns were the scalars, which then resulted in a meaningless augmented matrix This suggests rote learning of the procedures without attaching any meaning to the various steps. As seen in Figure 9.3, the teacher moved from the vector equation $\boldsymbol{v} = k_1\boldsymbol{v}_1 + k_2\boldsymbol{v_2} + k_3\boldsymbol{v_3}$ to an augmented matrix represention of a different vector equation $\mathbf{0} = k_1u_1 + k_2\boldsymbol{u_2} + k_3\boldsymbol{u_3}$**.** In the next step the zeros were replaced by x_1, x_2 and x_3 and the augmented matrix was equated to the vector **u**. This limited concept image understandably did not allow the teacher to make further progress. Initially the teacher seemed to have an idea that she should come up with an augmented matrix, but it was written incorrectly. The teacher continued to reduce the coefficient matrix, but was unable to even provide any interpretation of the resulting matrix. There were four teachers who displayed similar conceptual errors.

9.3.1.2 Procedural errors for question 3 on linear combination

Some teachers coped with the algorithmic procedure of setting up the vector equation expressing the linear combination relationship using the specific quantities but did not proceed further. There were nine such responses which we classified as procedural errors. These teachers proceeded to set up the system of three equations in three unknowns, but could not represent the system correctly in terms of the augmented matrix. This shows that they did not understand the connections between the system of equations and the augmented matrix. An example of such a response by T65 is shown in Figure 9.4.

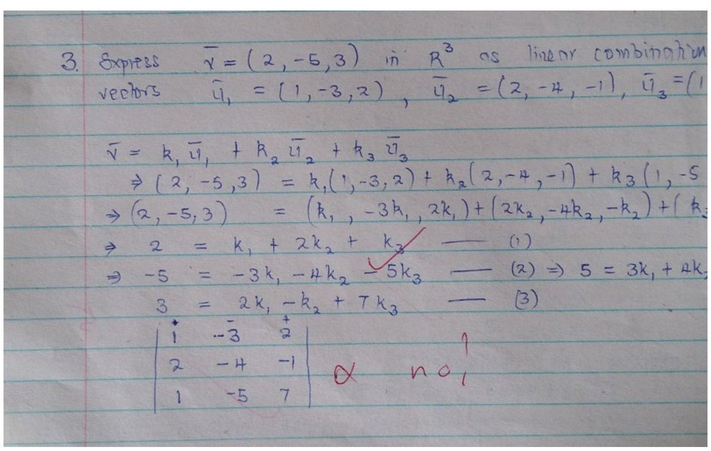


Figure 9.4: Written response of student T65

As seen in the Figure above, teacher 65 was able to come up with the correct vector equation, and also used correct procedures to come up with the system of equation in three unknowns. However, the student T65 came up with the coefficient matrix and attempted to find the determinant of the 3×3 matrix, instead of considering the augmented matrix corresponding to the system of equations.

9.3.1.3 Technical errors for question 3 on linear combination

The most common error type in this question was technical, with 48 responses revealing various technical errors. Two teachers' technical errors were identified when they tried to solve the three equations simultaneously using the elimination method and did not carry out any correct calculation, resulting in obtaining wrong solutions which led to incorrect deductions.

Most of the teachers went on to carry out row reduction on the augmented matrix, attempting to reduce the matrix to row echelon form, but made technical errors. An analysis of the augmented matrix shows that only three elementary row operations were required to get the matrix to reduced row echelon form, yet the teachers struggled with the process. There were 20 teachers who made calculation errors or applied inappropriate row operations in working with the correct augmented matrix. Eight teachers made technical errors that involved applying incorrect or inappropriate row operations, while 12 teachers made careless errors in manipulating the figures. Surprisingly, these 20 teachers obtained unique solutions, and substituted these back into the vector equations, without checking consistency or whether it made sense, incorrectly deducing that the original vector **v** could be expressed as a linear combination of the three vectors $\boldsymbol{u_1}$, $\boldsymbol{u_2}$ and $\boldsymbol{u_3}$.

However, 26 teachers managed to carry out the correct row operation and manipulations but encountered cognitive challenges in interpreting the third row (0 0 0:3) of the augmented matrix. Of these 26 teachers, 12 of them ended up with the reduced rows but did not make any conclusive deduction because they could not identify the inconsistency that $0 \neq 3$, and make valid conclusions.

Some teachers proceeded to use back substitution and wrongly calculated the values of the unknowns as shown by student T6 in figure 9.5.

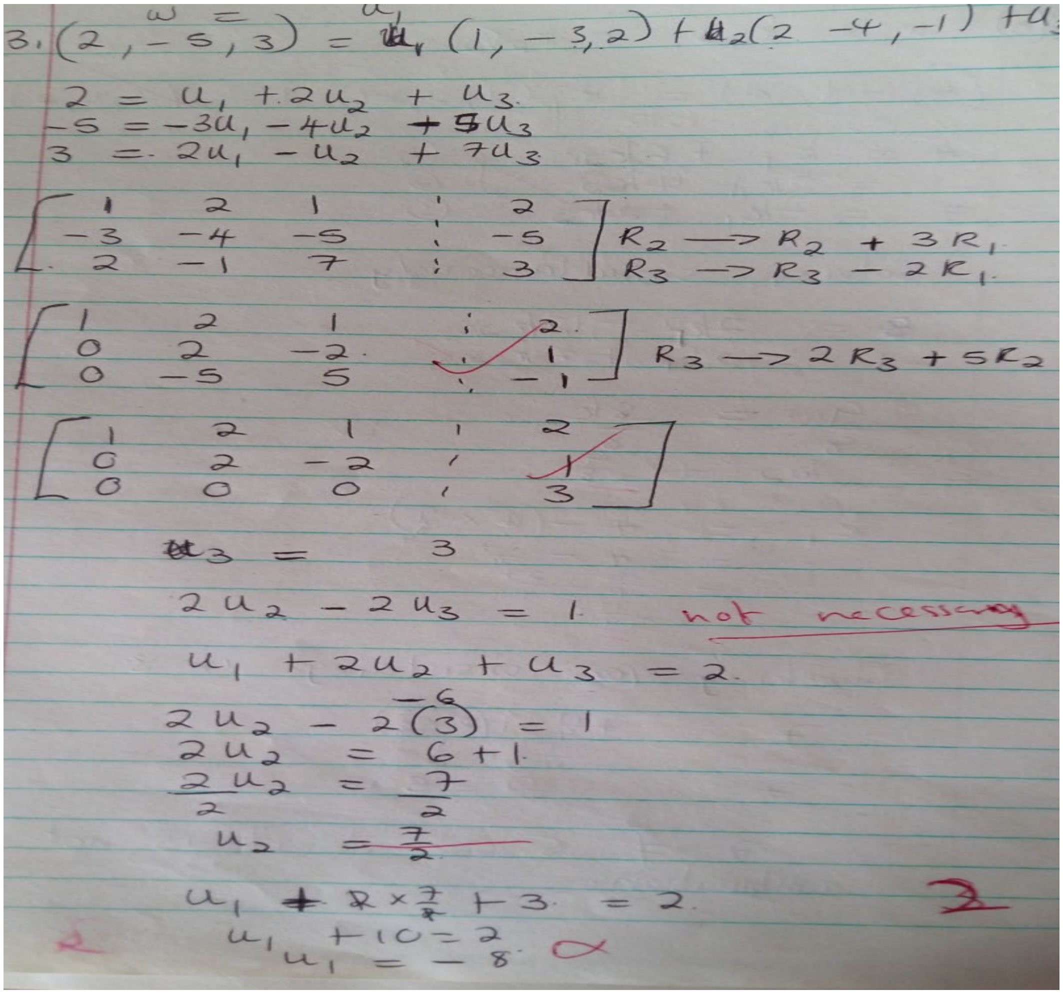


Figure 9.5: Written response of student T6

From Figure 9.5 above, it can be shown that row operations were explicitly carried out by T6. The teacher obtained the correct result after doing the elementary row operations. The next step would have been to conclude based on the results obtained that the system is inconsistent and so has no solution and accordingly, $\boldsymbol{v}$ cannot be written as a linear combination of the vectors $\boldsymbol{u_1}, \boldsymbol{u_2},$ and $\boldsymbol{u_3}$. However, the teacher revealed her confusion in moving forward in an attempt to find solutions to the inconsistent system of equations. The teacher attempted to use back substitution, incorrectly using $u_3 = 3$, instead of noting the inconsistent equation 0 = 3. This teacher' technical errors were because of the problems related to solving systems of linear equations. 14 of the 26 teachers did back substitution.

A follow up interview was done with T57 who presented the same written response as T6 except that she just ended at the reduced row form and did not go any further.

R: *Looking at the question ... (showing the student her solution). You were able to reduce the matrix to row echelon form. Why didn't you proceed neither did you make a conclusion about whether the given vector* **v** *can be written as a linear combination or not?*

T 57*: Since we have a row of zeros it means we now have many solutions, we cannot express* k_1 *because we don't have unique solutions, so that these vectors cannot be expressed as a linear combination.*

R*: The last row is not all zeros we have (0 0 0:3), (showing again the student her solution)*

T57*: Yes on the other side there is a 3 but on this side there are zeros. So three zeros we cannot say ahh it was a confusion we cannot say* k_3. *Can I say* $k_3 = 3$?.

R*: What do you think?*

T57*: Ahh, yes because normally when we have a zero we usually say* $k_3 = t$, *but now can we say* k_3= *3, then we say aah it does not make sense.*

R: *It doesn't make sense?*

T57*: Yes.*

R*: So it means it doesn't have many solutions?*

T57*: Yaa it doesn't have many solutions.*

R*: So what type of solutions are there?[silence]*

T 57*: Because if I am to say* $k_3 = 3$ *then* $k_2 + k_3 = 3$ *, so it means every* $k_1 + k_2 + k_3 = 3$.

Student T57 struggled to answer the question and was unable to interpret the last row (0 0 0:3) as representing an inconsistent equation, despite being prodded by the interviewer. She was not able to give a coherent reason why it is impossible to express the vector as a linear combination. We attribute this to a superficial understanding of the concepts on solving systems of equations that was covered in the first linear algebra module. She continued making links to irrelevant procedures in an endeavour to show the existence of many solutions. In fact none of the teachers that were interviewed was able to recognize that the system is inconsistent, as they thought that it gave many solutions, except for T4, whose response is given below. There was also a misconception that if the system of equations has an infinite number of solutions then the original vector cannot be expressed as a linear combination of the given set of vectors. However, this is not necessarily true since if the vectors in the given set are not linearly independent then there can be an infinite number of ways of expressing the given vector as a linear combination of the set of vectors.

***R**: As you are testing for linear combination you obtained the following last row* $[0 \quad 0 \quad 0 \ \vdots \ 3]$. *[Showing the teacher the matrix], what would be your conclusion?*

***T4**: The system of equations is inconsistent therefore those vectors does not give a linear combination because [for a] linear combination [we] should obtain unique solutions to the vector equation.*

The teacher did not provide a comprehensive response by spelling out the implications of the augmented matrix becoming reduced to an inconsistent system, that is, an inconsistent system means that no scalars exist that can be solutions to the original vector equation. However, the teacher showed that she had conceptualized the ideas on the types of solutions of a system of equations and had been able to use that to determine whether the given set of vectors constituted a linear combination of the given vector. It is interesting to note that the teacher's original written response was left blank after carrying out the correct elementary row operations for this item showing that her understanding of the procedures and about the connections between the procedures had deepened because of the time she spent working through the different problems. However, the remaining 12 teachers who were interviewed assumed that the last row was equivalent to a row of zeroes, and proceeded to try to find solutions to the remaining two equations by introducing scalars as they assumed that the system had many solutions. Their misconception was that if the first three entries of the last row in the augmented matrix row was zero, then it meant that the associated system of three equations in three unknowns had many solutions

9.3.2 Results for question 5

Express M = $\begin{bmatrix} 4 & 7 \\ 7 & 9 \end{bmatrix}$ **as a linear combination of the matrices A=** $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$**, B =** $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$**, C =** $\begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix}$**.**

To solve this problem the teachers could work through the following steps:

- Set up a vector equation $\boldsymbol{M} = k_1 \boldsymbol{A} + k_2\boldsymbol{B} + k_3\boldsymbol{C}$.
- Set up a system of four equations with three unknowns (k_1, k_2 and k_3).
- Represent the system as an augmented matrix.
- Reduce the matrix and solve for the unknowns k_1, k_2 and k_3.

There were 15 teachers who did not attempt to answer the question. Many of the teachers were able to set up the initial vector equation and expressed a relationship between M and the three given matrices but displayed many problems thereafter. Twelve of the teachers provided complete solutions to the problem. Two teachers were able to reduce the matrix to row echelon form and find correct scalars but did not make any conclusions, nor did they use the scalars to show the relationships between the vectors. This indicated that they obtained an almost complete solution to the problem.

9.3.2.1 Conceptual errors for question 5 on linear combination

There were two responses which showed conceptual errors. T4 set up the vector equation, $M = k_1A + k_2B + K_3C$ but he was not able to convert this to a system of linear equations. He went on to write a conclusion "theorem when r > n means they are linearly dependent and cannot be expressed as a linear combination." This teacher seems to have taken r as the number of components in the vector and n as the number of vectors in the set, and made a conclusion that the vectors A, B and C are linearly dependent, which is true. However, this does not preclude M from being expressed as a linear combination of these three vectors. Her conclusion was made too soon without examining the set of equations, since one of the rows could be reduced to zero, resulting in the system having a unique solution. Teacher T52 went further to set up the system of linear equations but made the same conclusion as T4. These two teachers demonstrated a conceptual error which arose from a misunderstanding of the conditions related to linear independence.

9.3.2.2 Procedural errors for question 5 on linear combination

Three teachers exhibited procedural errors where they were able to set up the vector equation, but failed to equate corresponding elements as shown by the response of T8.

$$\begin{bmatrix} 4 & 7 \\ 7 & 9 \end{bmatrix} = \begin{bmatrix} a+b+c & a+2b+c \\ a+3b+3c & a+3b+5c \end{bmatrix}$$

This teacher was able to form an expressions using the scalars and form the vector equation $M = aA + bB + cC$ using the matrix entries. However she was not able to convert these matrix expressions into a set of four linear equations. This demonstrates a procedural error of failing to equate corresponding elements, a concept which is learned at secondary school. Eleven of the teachers were able to set up a system of four linear equations in three unknowns but could not come up with the augmented matrix, and hence were stuck.

9.3.2.3 Technical errors for question 5 on linear combination

The most common error types manifested in this question were the technical ones of which there were 28. 11 of the teachers carried out other incorrect row operations, or made careless errors in manipulating figures, showing that they lacked fluency in carrying out the row operations. An example is that of student T2 whose error is shown in Figure 9.6.

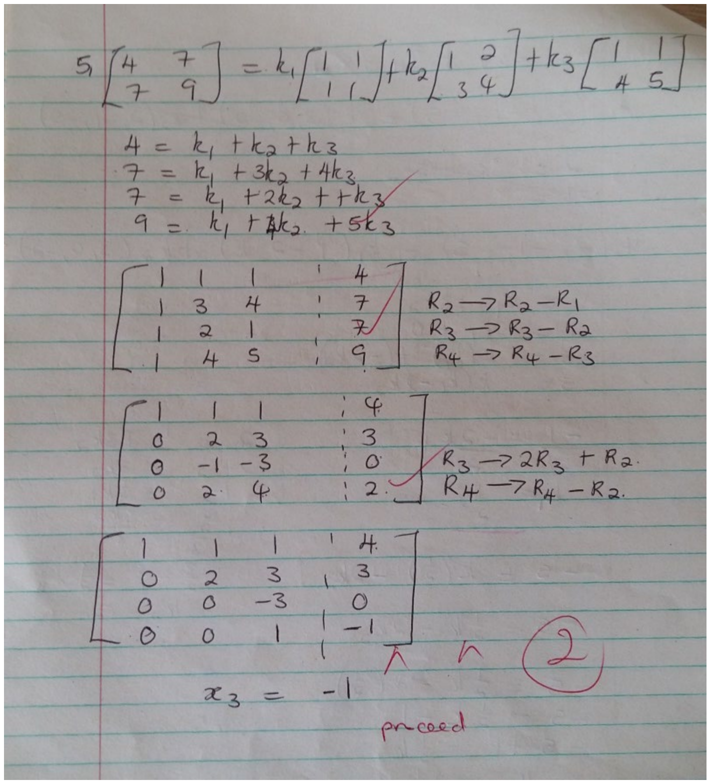


Figure 9.6: Written response of student T2

From Figure 9.6 above, student T2 interpreted the question correctly, and correctly set up the vector equation, formulating the systems of four linear equations in three unknowns, representing these equations in an augmented matrix. However, in attempting to reduce the matrix to row echelon form, the teacher made some errors with row operations. For example, the second and third row operations are incorrect because they ignore the preceding row operations which resulted in a new row 2 which he had already worked on as well as row three. Usually to ensure fluency in row reduction operations, it is row 1 (if it has a leading 1) that is used as a basis for row operations, that is to say, $r_2 \rightarrow r_2 - r_1$, and $r_3 \rightarrow r_3 - r_1$, before moving to other manipulations. This teacher seemed to be stuck, instead of carrying out further row operation, he tried to do back substitution starting with $x_3 = -1$. However her errors led to an inconsistent system and she did not seem to know how to proceed since she was also going to get another $x_3 = 0$ as well. Like student T2, none of the other 10 teachers was able to make conclusions based on the results they got. It can be argued that these teachers did not develop fluency in the row reduction procedures, which formed a focus in the first linear algebra module, based on matrix operations and systems of equations.

Twelve teachers revealed another technical error. Although these teachers carried out row operations without making mistakes, like student T2 they did not complete the execution of the elementary row operations, which would have left them with a row of zeros in the last row. They attempted to do back substitution but were unable to make progress. One teacher (T3) found the correct scalars but presented his final solution as $M = -2A + 3B - C$, instead of $M = 2A + 3B - C$.

Another technical error was revealed in four responses where the teachers were able to set up the vector equations. In an attempt to set up the system of equations, they omitted a pair of corresponding elements or transcribed the problem incorrectly resulting in the formation of a system of three equations in three unknowns, hence indicating a technical error due to carelessness. An example is that of student T69 who was able to come up with the vector equation in matrix form and after equating them he obtained 3 equations in 3 unknowns instead of 4 equations in three unknowns. He proceeded to retrieve an augmented matrix which was reduced to row echelon form and obtained the following results, $k_1 = 1,\ k_2 = 6, k_3 = -3$ but did not provide a conclusion. Teacher T69 was interviewed:

R*: The question requires you to express the given vector M as a linear combination of the given vectors A, B, and C. How can you show this is correct using the values that you have obtained?*

T69*: [Quiet for a while] er wanted to find er... I wanted to find the solution space.*

R*: Ok, let's say we take the values of say* k_1, k_2 *and* k_3 *that you have obtained. What do you do after finding these values so as to determine that the vector M can be written as a linear combination of the vectors A, B and C?*

T69*: Now if* $k_1 = k_2 = k_3$ *then it will mean that er… that the vectors is a linear combination*

R*: Are you saying* k_1, k_2, k_3 *must have equal value*

T69*: Yes for linear combination.*

R*: What if the values are not the same, like the ones you have?*

T69*:Then it means the solutions is non trivial it means that er they are linearly independent.*

R*: But we are talking about a linear combination.*

T69*: Eer it means that it is not, it is not a linear combination.*

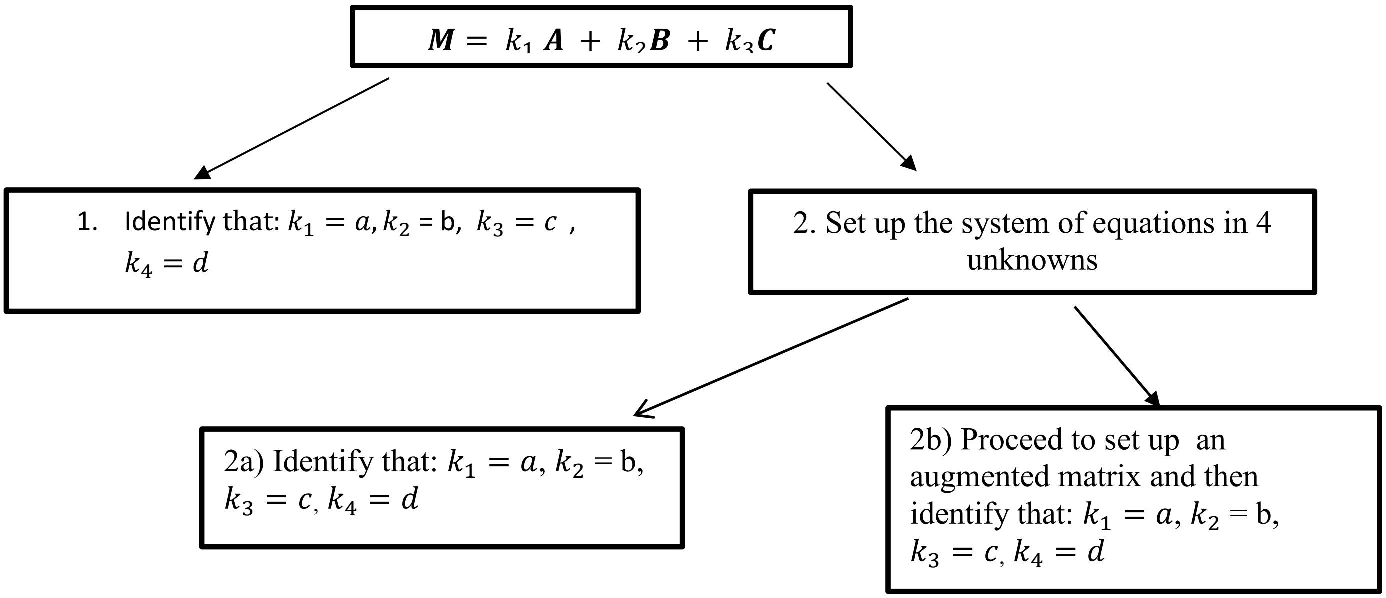
The teacher above demonstrated a poor understanding of the concept of linear combination and revealed many misconceptions in the short extract. Firstly, T69 felt that for M to be written as a linear combination of the three vectors, the scalars in the vector equation should be equal, suggesting confusion between the term unique solution and identical values. Furthermore, he believed that in the case of unequal scalars, it meant that the vectors were linearly independent. He was also very confused about what a non-trivial solution meant in this context. It seems that he had trouble distinguishing between a system of equations that is set up when trying to show a set of vectors are linearly independent and a system of equations that is set up when trying to show that a vector is a linear combination of given vectors (as in question 5 here).

9.3.3 Results for Question 6

Table 9.2 Question 6 and the possible ways for solving the question

3.**Show that the vector space M_{2x2} of all 2×2 matrices is spanned by the matrices** $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$	This question required the teachers to work through the following steps: As a first step consider an arbitrary 2×2 matrix M = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, suppose the given matrices are **A, B, C** and **D**. Set up a vector equation $\boldsymbol{M} = k_1 \boldsymbol{A} + k_2 \boldsymbol{B} + k_3 \boldsymbol{C} + k_4 \boldsymbol{D}$. **Method 1.** Immediately recognise that the unknowns $k_1 = a$, $k_2 =$ b, $k_3 = c$ and $k_4 = d$ **Method 2.** They may proceed from Step 1 to set up a system of four equations with four unknowns (k_1, k_2, k_3 and k_4) and recognise $k_1 = a$, $k_2 =$ b, $k_3 = c$ and $k_4 = d$ **Method 3**: They may proceed from Step 1 to set up a system of four equations with four unknowns (k_1, k_2, k_3 and k_4) and thereafter reduce the matrix before recognising that the unknowns (k_1, k_2, k_3 and k_4) equal a, b, c and d respectively.

As explained in the Table 9.3 above, the teachers could have opted for three different methods depending on when they would have recognised the solution which is expressed as the flow diagram.



There were 23 teachers who did not respond to this question, seemingly not knowing where to start. Most teachers opted for the second method described in the flow diagram above. There were 27 responses that were considered as showing conceptual errors, 14 were taken as procedural errors, there were zero technical errors and these are described in detail below.

9.3.3.1 Conceptual errors for question 6 on linear combination

Conceptual errors were exhibited in 27 responses. The teachers in this category did not seem to understand the principles in checking whether a given set of vectors span a vector space. Instead of equating the linear combination of the matrices in the equation to an arbitrary vector of $M_{2\times2}$, 20 of the teachers set the vector equation to zero as illustrated by the response of T6 in Figure 9.7.

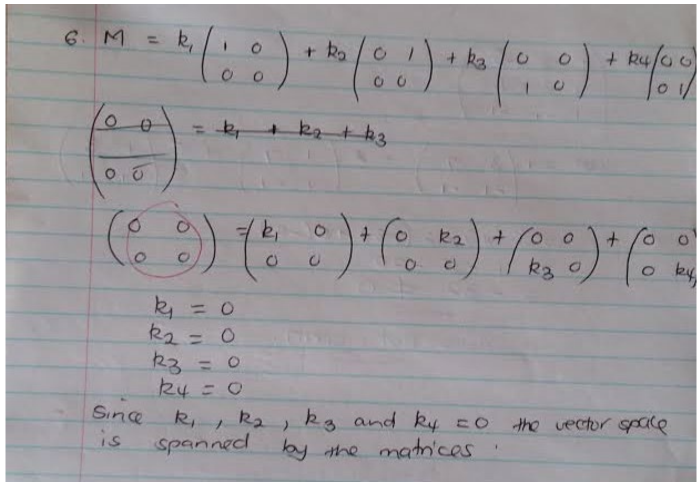


Figure 9.7: Written response of student T6

It seems that the teacher confused himself with regards to properties of a set being linearly independent. Note that if $S = v_1, v_2, \ldots v_r$ is a nonempty set of vectors in a vector space *V*, and the vector equation $k_1v_1 + k_2v_2 \ \ldots k_rv_{r.} = 0$ has only the trivial solution, namely k_1,= 0, $k_2 = 0, \ldots k_r = 0$, then the vectors in *S* are said to be *a* linearly independent**.** Twenty teachers made a similar error, where they set up a vector equation taking the zero vector and tried to express it as linear combinations of the four given vectors and then showed that it had a trivial solution only. They then concluded that the vector space $M_{2\times 2}$ was spanned by the set of four matrices.

One of the teachers who made such an error (T7) was interviewed:

R*: May you explain to me why you have equated the vector equation to the matrix* $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$*, yet this matrix was not given.*

T7*: Umm I think it was not supposed to be equated to zero since it is equal to err the general matrix written on the vectors matrix.*

R*: So were you supposed to equate it to the matrix* $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ *or to the general matrix say* $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$*. [Trying to give a hint]*

T7*: [Nodding his head] I am not sure now.*

***R**: Which condition would we be testing if we equate the set of linear equations to the matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ as you did [showing the student his response]?*

***T**7: Oh ok maybe you can try to ask me the other way?*

***R:** Oh ok, [giving the student a more specific hint] may you define the term linear independence?*

***T**7: Eer… linear independence eer… if maybe the vectors are linearly independent it means the formulated matrix eer… cannot maybe expressed or can be reduced into echelon form.*

***R**: Oh ok say after solving the system of equations [pointing to students' work] what must be the values of your scalars be if say we have scalars k_1, k_2 and k_3, when testing for linear independence?*

***T**7: Scalars must produce eer… maybe a single solution for each.*

***R**: Ok if I get $k_1 = 1, k_2 = 1, k_3 = 1$?*

***T**7: [Also repeating the writing the values of the scalars]. I should think so because at one time these scalars should have a single value if all are equal to zero, or if all equal to some other value. I am not really sure.*

This extract showed that the teacher did not grasp the concepts on spanning. The interviewer tried to draw his attention to the fact that in his equations, he had equated the linear combination to zero. In explicitly mentioning the term linear independence, the interviewer hoped that the teacher would get a clue that he had selected the wrong matrix, but these hints did not work. The teacher was not clear about the different ways in which the systems of equations were set up in regard to the concepts of linear combinations, spanning as well as linear independence.

Five teachers attempted to set up an incorrect augmented matrix of the form as shown in Figure 9.8.

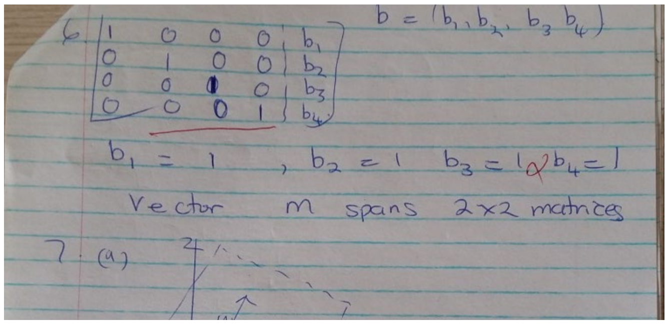


Figure 9.8: Written response of student T13

The response by teacher T13 in Figure 9.8 above reveals that the teacher had some idea about choosing an arbitrary vector but it seemed to have been a vector in R^4 and not a matrix in $M_{2\times2}$, It is also not known how she came up with the $M_{4\times4}$ identity matrix. It is evident that the teacher had a problem with identifying what needed to be done which is because the initial setup of the vector equation was not done.

Two other kinds of conceptual errors were identified in the responses of T24 and T66. Teacher T24 equated the linear combination of the four given matrices to the matrix $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, perhaps taking it as the identity matrix. Teacher T66 used an arbitrary $M_{2\times2}$ element in the setup of the initial vector equation but equated the arbitrary element to the sum of the three matrices only, and left out the scalars. This reduced the vector equation to a statement so there was nothing for him to solve.

9.3.3.2 Procedural errors for question 6 on linear combination

The researcher identified 14 procedural errors. The teachers were able to come up with a suitable vector equation but they faltered on the way, as seen in the work of T41 for example shown in Figure 9.9.

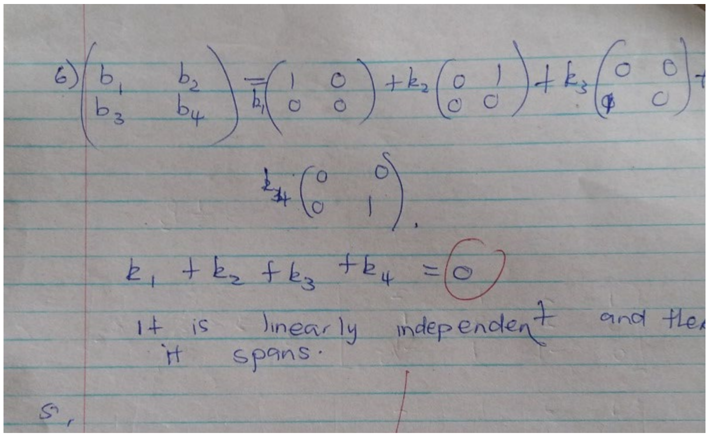


Figure 9.9: Written response of student T12

The teacher was able to express an arbitrary vector as a linear combination of the given matrices, but thereafter was unable to translate this vector equation into a set of four linear equations. That is, he was unable to move from a matrix representation of an equation to the corresponding system of linear equations. The teacher further incorrectly provided one equation, by equating the sum of the scalars to zero, and then further interpreting the incorrect result in terms of linear independence. The conclusion is completely out of place, showing that the teacher had confused the terms spanning and linear independence.

No technical errors that were identified in this question.

9.3.4 Discussion

An overview of the different kinds of responses for the three questions appears below.

Table 9.3: Summary of the number of different response types for the three questions on linear combination

	Not attempted	Conceptual errors	Procedural errors	Technical errors	Almost complete	complete
Question 3	9	4	9	48	3	0
Question 5	15	2	14	28	2	12
Question 6	23	27	14	0	0	9

Taking the numbers of people who did not make an attempt or who immediately chose a wrong approach (conceptual error) from Table 9.2 above, it can be seen that fewer teachers had fundamental misunderstandings in Question 3 as compared to the other two questions. Noting that Question 3 was based in the space R^3 while the other two questions were set in $M_{2\times2}$, it is clear that more teachers struggled with the questions based in $M_{2\times2}$ compared to the space R^3. Perhaps this suggests that the representation of the vectors influenced the teachers' attitude and approach to the questions.

9.3.4.1 Summary of commonly identified technical errors for linear combination

There were no technical errors in question 6, because the question did not require any simplification of an augmented matrix. If the teachers used an augmented matrix, the solution was immediately apparent. Most of the technical errors occurred while working out question 3 where none of the students were able to explicitly express the vector (-2, 5, 3) as a linear combination of the given three vectors. Most of the responses to question 3 did not provide a reasonable interpretation about whether there was a solution or not for the scalars making up the linear combination expression. This was crucial to making a correct deduction as to whether the given vector **v** could be written as a linear combination of the vectors $\boldsymbol{u_1}, \boldsymbol{u_2}$ and $\boldsymbol{u_3}$. The teachers interviewed also revealed the same error. There were fewer technical errors in question 5 than in question 3, probably because teachers were more comfortable with those expressions that gave unique solutions. 12 of teachers were able to carry out correct elementary row operations on question 5, but were stuck on the very last two rows. Their technical errors were responsible for them not reducing it successfully to row echelon form. However, it is important to note that 12 of the teachers were able to obtain the correct solution to question 5 compared to none for question 3. It seems that it was easier for the teachers to compute the solution to a system of equations when it had a unique solution as in the case of question 5, rather than no solution. Picking out a solution from a consistent system requires one to carry out calculations in a step by step manner to arrive at a final answer which can be associated with the unknowns. However, when the system is not consistent, one is required to engage in some reflection when the inconsistency is encountered in order to proceed further. The teachers who obtained a last row with (0, 0, 0 :3) in question 3 often could not tell whether the system was consistent or inconsistent. Some teachers arrived at unique solutions by using back substitution incorrectly and did not check whether their solution made sense. Cangelosi (2013) claimed that if an error persists, then the associated concept developed will be stalled at the operational stage, hence an individual will not be able to move to the structural level of understanding, which is evidently the situation here. These teachers stalled when they encountered the inconsistent system and could not move further in interpreting the inconsistency in terms of the linear combination relationship. This is in line with de Lima and Tall (2008) who also argued that it is very difficult for students to move from the process level to the object level of understanding. This also concurs with Sfard (1991) who maintained that it is crucial for students to have a deep understanding of mathematical concepts, but this only occurs when the student sees a concept as both a process and an object. From the responses, it was evident that many students did not have an adequate foundational knowledge of solving systems of linear equations. The interview with T57 illustrated the teachers' confusion about the meaning of a row of zeros and then a subsequent interpretation thereof. Many teachers, like T57, were not sure how they could interpret the equation represented by the row [000:3] in the last row of an augmented matrix. Teachers such as T69, T7 and T57 showed deep seated confusion about the meaning of a unique solution which they conflated with identical values of the unknowns. Such errors in the interpretation of systems of equations meant that they failed to conclude whether the given vectors could be expressed as a linear combination of the given vectors or whether the set of vectors spanned the given set.

9.3.4.2 Summary of commonly identified procedural errors for linear combination

There were nine procedural errors identified in question 3 where the teachers could not arrive at an appropriate augmented matrix or translate the vector equation into the appropriate systems of equations. For question 5, three teachers were able to write the vector equations but could not equate corresponding elements. Another procedural error appeared in question 6, also based in the vector space $M_{2\times2}$ where 14 of the teachers were able to equate the linear combination to an arbitrary matrix from $M_{2\times2}$, but were thereafter unable to translate this vector equation into a system of four linear equations based on the corresponding entries of each matrix. They were also not able to pick out the values of the scalars that would make the equation true. A substantial number of teachers could not carry out scalar multiplication, add the matrices and equate corresponding elements, resulting in them obtaining incorrect solutions. Sfard (1991) argued that at interiorisation stage, a student must make a proper connection with those processes that lead to a higher degree of structural thinking, which in turn lead to the construction of an actual entity. These teachers' errors showed that they were not able to reach the structural level of understanding; which Sfard (1991) refers to as a fully-fledged mathematical object.

9.3.4.3 Summary of commonly identified conceptual errors for linear combination

It can be noted that most conceptual errors occurred in question 6, where teachers had to consider an arbitrary matrix, and show that this could be expressed as a linear combination of the given matrices. Most teachers could not set up the initial vector equation. Instead of equating the linear combination to an arbitrary matrix of $M_{2\times2}$, 20 teachers equated it to the zero vector. A similar situation prevailed in Question 3, where most conceptual errors occurred because they could not proceed further from the vector equation expressing the linear combination relationship. Some teachers were confused and transformed the vector equation into an incorrect matrix equation of the form $Ax = 0$, where A was the coefficient matrix arising from the system of equations instead of $Ax = b$, where b was the given vector. One of those who expressed the augmented matrix in the form $Ax = 0$ went further to write an expression indicating an attempt to find the determinant. These teachers seemed to have confused the different processes used to show that: a) a given vector could be expressed as a linear combination; b) a given set was linearly independent; c) a given set was linearly dependent; or, d) a given set spanned another set independence. It is important to note that to show linear independence for a set of vectors, the initial vector equation is set to 0 and the resulting augmented matrix is set up to represent the equation $Ax = 0$. Then the system of equations has a trivial solution if and only if det (A)$\neq 0$, where A is the coefficient matrix arising from the system of equations. For checking if a vector b is a linear combination the equation considered is of the form $Ax = b$ and the intention is to find a solution to the system. For checking if a set of vectors span a set B, then the equation considered is also of the form $= b$, where b is any vector in the set B.

In Question 5, student T4 and T52 were able to set up the vector equation but did not attempt to come up with a system of linear equations, but simply went on to write that: theorem when $r > n$ means they are linearly dependent and cannot be expressed as a linear combination. Across these questions, it is evident that the manifestation of met afters, that is the concepts taught on linear independence, are hindering the development of structural conceptions. They have been used out of their domain and form obstacles in the development of advanced mathematics (Tall &Vinner,1991, Vinner 1991).

The teachers' concept image seems to have conflicting ideas formed in their minds which inhibited the construction of new mathematical knowledge. For question 6, one teacher equated the linear combination in the vector equation to the matrix $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ assuming that it was an identity matrix. Another teacher did not multiply the matrices by a scalar, presenting a sum of vectors instated of a linear combination. These conflicting ideas acted as obstacles which interfered with the new learning thereby stopping learners from developing an adequate concept image of linear combination. Owusu (2015) notes similarly that a learner's existing schemas of a given mathematical concept significantly interferes with the construction of new knowledge.

9.4 Discussion of the errors displayed by students on questions on linear independence

The discussion here is based on the results discussed in chapter 8 for questions 2, 6 and 9.

9.4.1 Results for question 2a

As stated in chapter 7 this item incorporated theorem 7.2. 42 of the students gave a completely correct solution to task 2a. The types of errors revealed across the item are mainly conceptual errors. 22 responses revealed conceptual errors, and 8 revealed some technical errors and one revealed a procedural error.

9.4.1.1 Conceptual errors for question 2a on linear independence

6 of the students attempted to use inspection by applying theorem 7.1. However, these students did not explicitly use this theorem correctly thus displaying conceptual errors. The theorem was not mastered well, see chapter 7. The remaining 10 students also attempted to use inspection but simply grabbed the different theorems or terms that they had learnt which were wrongly interpreted and used, for example the written response by student T13 who wrote that its linearly independent because there are no multiples in the vectors, or student T5 who wrote that since $r < n$ and scalar product therefore linearly independent. 6 students formulated matrices of order of 2×3 and attempted to calculate the determinant using Sarrus rule which is only applicable for a 3×3 square matrix, see the written response by student T27 in Figure 7.5. However, some of the students were able to identify that the matrix was not a square matrix, and they proceeded to reveal more

conceptual errors by adding another row of zeros in an attempt to come up with a square matrix and then proceeded to calculate the determinant using method of lapse transformation, the written response by student T70 in Figure 7.6. Out of the 6 teachers, only 2 of them viewed the determinant as a function. The other 4 simply calculated the determinant without saying the determinant of what, thus showing more conceptual errors.

9.4.1.2 Technical errors for question 2a on linear independence

8 of the students did not use inspection but formulated vector equation of the form $k_1 v_1 + k_2 v_2 = 0$ expressed it in coordinate system form and a system of 3 equations in two unknowns was formulated. The students failed to solve the equation due to computational errors and they simply wrote that it is linearly independent since $k_1 = k_2 = k_3 = 0$.

9.4.1.3 Procedural errors on question 2a on linear independence

One student, T7, formulated the augmented, did not attempt to carry row reduction and simply wrote that it linearly independent.

9.4.2 Results for question 2c

This question reduces to establish that polynomial P2 is a scalar multiple of the polynomial P1 meaning that the system is linearly dependent [application of theorem 7.2]. Out of the 73 students, 54 gave a completely correct response, and 19 students revealed conceptual errors.

9.4.2.1 Conceptual errors for question 2c on linear independence

8 of the students attempted to use inspection but these students made incorrect deductions. Though they had discovered that one of the vectors was a scalar multiple of the other they made wrong deductions saying that the vectors were linearly independent. This shows that the students had learnt the concepts by rote memorization of facts. 11 of the students simply used guess work and said the vectors are linearly dependent giving wrong or no reasoning. This showed that these students demonstrated conceptual errors.

9.4.3 Results for question 2d

This task reduces to establish that matrix B is a scalar multiple of matrix A. I noted that out of the 73 students, only 35 managed to give a completely correct solution, whilst 21 made some errors which are classified as conceptual errors and 17 revealed some errors which are classified as procedural errors.

9.4.3.1 Conceptual errors for question 2d on linear independence

Ten of the students displayed some conceptual errors as they simply said the vectors are linearly dependent or linearly independent without any justification as to why it was linearly independent and some made wrong arguments. This shows that the students did not quite grasp the concepts. Two out of the 10 students wrote expressions of the form $k_1v_1 + k_2v_2$ without equating it to the zero vector, also revealing conceptual errors. Nine students also manifested conceptual errors as they attempted to use the determinant method wrongly. It was surpring to note that the students calculated the separate determinants, that is the determinant of A and the determinant of B. For example see Figure 7.7, that is the written response by student T72. The students were aware that the determinant method can be used to determine linear independence, but it was not applicable here. Thus this method was wrongly used, showing the manifestation of conceptual errors

9.4.3.2 Procedural errors for question 2d on linear independence

Five of the students were able to come up with a vector equation of the form $k_1v_1 + k_2v_2 = 0$ However, three of these students encountered procedural errors as they struggled to do scalar multiplication correctly and could not solve the equations correctly. The other two students were able to do scalar multiplication but did not solve the equations. The students simply wrote the following solutions $k_1 = 0, k_2 = 0$, and wrote that they are linearly dependent. However, 12 of the students wrote the statement that $A = -B$ and concluded that it was linearly dependent, without further explanations. This showed that the students had an idea of what needed to be done but could not explicitly make the correct conclusions, thus exhibiting procedural errors. They had an idea of what needed to be done, but did not relate it to one being a scalar multiple of the other.

9.4.4 Results for question 2b

As stated in section 7.4.6 the problem required the teachers to judge the relationship between r and n given a set of vectors V that is S = $\{v_1, v_2, \ldots v_r\}$ in R^n. If $r > n$, then S is linearly dependent.

However, out of the 73 students, only 13 gave the correct solution. I noted that 56 students demonstrated conceptual errors and only 4 demonstrated technical errors.

9.4.4.1 Conceptual errors for question 2b on linear independence

56 students attempted to use inspection and applied wrong theorems, thus could not establish the reasons why the three vectors are linearly dependent, whilst others did not see the difference between linearly independent vectors and dependent vectors and the conditions that need to be satisfied for linear independence, for example student T66 said that it is not linearly independent because it is not square matrix. Some of the students were aware that they should use theorem 7.1 but struggled to put the ideas forward explicitly, hence failing to argue why it was linearly dependent. See section 7.4.6 for the various struggles experienced by the students in answering the question.

9.4.4.2 Technical errors for question 2b on linear independence

Four students revealed some technical errors. Two students formulated vector equations correctly and came up with three equations in two unknowns. However, the students could not solve the two equations simultaneously, thus failing to deduce whether the set of vectors was linearly dependent or not. The other two students attempted to find the determinant of a 3×2 matrix using Sarrus rule thus revealing technical errors since the wrong method is used.

9.4.5 Results for question 2e

Out of the 73 students, 28 were able to correctly follow the given instructions and gave a completely correct solution. There were 42 responses whose errors were classified as being conceptual and three were considered as technical errors.

9.4.5.1 Conceptual errors for question 2e on linear independence

8 of the students failed to follow the given instruction and preferred to engage with the step by step procedures. They treated the vectors as row vectors and came up with a 3×3 square matrix and they proceeded to calculate the determinant of the matrix instead of using inspection. Though they were able to get the correct determinant and the correct argument as to why the vectors were linearly independent, the students showed a conceptual error by failing to state why it was linearly dependent without carrying out the step by step procedures. 34 students used inspection and out

of these 34 students, four said that there was a row of zeros meaning that the vectors were linearly independent. This shows that the students confused the terms linear dependence/dependence. The students revealed conceptual difficulties, see section 7.4.8 for some of written responses given by the students. The other 15 students were able to tell that the vectors were linearly dependent but they also demonstrated conceptual errors when they failed to give the correct argument as to why the vectors were linearly dependent, for example student T64 who wrote that it is linearly dependent because $k_1 + k_2 + k_3 \neq 0$. This indicates that the in-service teachers experienced conceptual difficulties in an attempt to show that the given vectors were linearly dependent. The remaining 15 students exhibited conceptual errors when they completely failed to outline any relationship between the zero row and linearly dependent. They just grabbed any term that came their way.

9.4.5.2 Technical errors for question 2e on linear independence

There were three students whose errors were classified as technical errors. An example is the one illustrated in section 7.4.9, that is the written response by student T69. The student treated the vectors as row vectors and they carried out row reduction and obtained the following matrix after row reduction $\begin{bmatrix} 1 & -1{,}5 & 3{,}5 \\ 0 & -3{,}5 & 6{,}5 \\ 0 & 0 & 0 \end{bmatrix}$. However, the teacher could not figure out again the presence of zeros, hence proceeded to list the non-zero rows as the basis of R^3. This indicated that the student was now mixing up the concepts that had been learnt after the concepts on row space which Tall (2008) referred to it as the met after which negatively impacted on the understanding of the concepts on linear independence. This revealed a technical error since the student could not interpret the solutions to systems of equations and make a relationship with the concept linear dependence.

9.4.6 Results for question 6

3 of the students did not attempt the question showing a failure to grasp the concept on linear independence and 24 were able to obtain the correct result. There were 20 responses whose errors were classified as being conceptual, 26 were considered as procedural errors and there were no technical errors. I now discuss the nature of these errors identified in this question in more detail.

9.4.6.1 Conceptual errors for question 6 on linear independence

21 of the students revealed conceptual errors. The students were supposed to come up with a vector equation of the form $k_1v_1 + k_2v_2 + k_3v_3 + k_4v_4 = 0$. The first conceptual error depicted in section 7.6.1 was student T32 who failed to come up with the vector equation but resorted to finding the determinant of the separate matrices, hence showing a failure to appreciate the relationship involved in the problem. Another conceptual error was shown when two students failed to construct the vector equations and simply came up with an augmented matrix with an identity matrix of order 4×4, whose solution was an arbitrary vector (a, b, c, d) and six other students also came up with an augmented matrix with an identity matrix of order 4×4, whose solution was the zero vector. The other type of conceptual error manifested by the students was that they were able to come up with the vector equation but the equation was equated to the arbitrary vector instead of the zero vector, that is $k_1v_1 + k_2v_2 + k_3v_3 + k_4v_4 = b$. This is evidenced by student T44 in Figure 7.13. 12 of the students revealed such a conceptual error.

9.4.6.2 Procedural errors for question 6 on linear independence

I noted that two of the teachers demonstrated procedural errors. One student, T30, was aware that he should come up with a vector equation equated to the zero vector. However, the equation had three scalars instead of 4, but he did not even notice that at the end the solution should have four scalars, this showed a manifestation of a procedural error. Student T40 was able to come up with the correct vector equation, and step by step procedures were seen but he could not complete carrying out the algebraic manipulations as he failed to see that he should solve the equations and he left the result in form $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ without any concluding statement and without stating the values of the unknowns $a, b, c,$ and d. This indicated that student T40 displayed a procedural error and he could not come up with a logical deduction and some arguments as to why the vectors formed a linearly independent set.

Furthermore, 23 of the teachers had correct ideas and obtained the following as the final answer $k_1 = 0, k_2 = 0, k_3 = 0$ and $k_4 = 0$ without a concluding statement. However, the question required them to argue as to why the vectors were linearly independent, since it was a prove question which needed justifications. This again demonstrated that these students showed a procedural error because of insufficient argumentation of why the vectors are linearly independent.

The interview carried out in section 7.6.2 with student T57 revealed that she continued to manifest a procedural error. Student T57 was able to construct the vector equations but came up with an augmented matrix with an identity matrix of order 4, whose solution was the zero vector. A wrong procedure was used to come up with such an augmented matrix, see chapter 7 for the explanation on how she obtained the augmented matrix. She failed to carry out the correct scalar multiplication and vector addition. This inadequate conception showed that the students struggled to show that given vectors in matrix form are linearly independent, and such errors have not been identified in any studies about linear independence.

9.4.7 Results for question 7

Out of the 73 students, 2 did not answer the question, 26 of the students got a partially correct result, and none of the students was able to give a completely correct result due to a failure to make a link of the concept of linear dependence and linear combinations. I further noted that 7 revealed conceptual errors and 38 revealed technical errors.

9.4.7.1 Conceptual errors for question 7 on linear independence

Two of the students did not attempt the question and seven of them applied inappropriate rules so provided incorrect responses indicating some conceptual errors. Examples of some of the students with inappropriate techniques are shown by T44 and 46. The two students formulated vector equations of the form $w = k_1v + k_2u$. They made the necessary substitution and came up with three equations in two unknowns. They solved the first two equations and obtained the values of $k_1 = 2$ and $k_2 = 2$. These were then substituted into the third equation and obtained the statement $5 \neq 4$. Student T44 then wrote no linear independence with student T46 writing that it is linearly independent since it is inconsistent. This reveals that the two students applied the wrong method thereby demonstrating a conceptual error. The method executed is that of trying to show that vector **w** can be expressed as a linear combination of the vectors $\boldsymbol{u}$ and vectors $\boldsymbol{v}$.

9.4.7.2 Technical errors for question 7 on linear independence

This was the most common error manifested. 15 students used the Gaussian elimination method in order to determine whether the given vectors were linearly independent or not. However, these fifteen students encountered a number of technical errors such as problems with basic manipulations of figures. A lot of computational errors were made due to a failure to negotiate with directed numbers, hence making slips and calculations errors. Another technical error that was common was failing to carry out the correct row operations so that at the end, the students failed to end with the matrix of the form $\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. An example in section 7.7.1 is that of teacher T8 who obtained the following matrix after row reduction, $\begin{bmatrix} 1 & 2 & 4 & & 0 \\ 0 & 1 & 1 & : & 0 \\ 0 & 0 & -3 & & 0 \end{bmatrix}$ as a result of manifesting a lot of computational errors. After getting the reduced matrix, she further revealed another technical error by writing that $-3 \neq 0$ therefore there is inconsistency as $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ cannot be expressed as a linear combination which means there is linear independence. She failed to see that she could use back substitution so as to obtain trivial solution. The argument made shows some confusion as a result of failing to interpret solutions to system of equations.

Amongst the students who used the determinant method, student T40 failed to transcribe the vectors correctly and at the end failed to get the correct determinant zero. This demonstrated an error which Siyepu (2010) referred to as a slip. Another student, T60, failed to carry out the basic algebra manipulations and obtained the determinant 5. This student also revealed a technical error. Another example of a technical error is revealed by students who failed to use the determinant method correctly, for example student T7 who failed to use the Laplase transform correctly and obtained the wrong determinant, again revealing a technical error. There were also 6 instances where the participants calculated the determinant without treating the determinant as a function thus showing a manifestation of a technical error. This was a result of the students failing to grasp the concepts taught on determinants and this concept was covered in the first module.

Another category of 14 students also revealed some technical errors. These students were able to carry out some correct row operations whilst executing the Gauss elimination method and obtained the following the result $\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. However, some of these students did not write any conclusions with some of them failing to write the correct deduction, that is indicating whether the vectors were linearly independent or not. This was a result of their failure to interpret the solutions to the systems of equations, and relating it to the aspect of linearly independent/ dependent, see section 7.7.1.

9.4.8 Discussion

An overview of the different kinds of responses for the three questions appears below.

Table 9.4: Summary of the number of different response types for the three questions on linear independence/dependence

	Not attempted	Conceptual errors	Procedural errors	Technical errors	Partially correct	Completely correct
Question **2a**	**0**	**22**	**1**	**8**	**0**	**42**
Question **2b**	**0**	**56**	**0**	**4**	**0**	**13**
Question **2c**	**0**	**19**	**0**	**0**	**0**	**54**
Question **2d**	**0**	**21**	**17**	**0**	**0**	**35**
Question **2e**	**0**	**42**	**0**	**3**	**0**	**28**
Question **6**	**3**	**21**	**26**	**0**	**0**	**24**
Question **7**	**2**	**7**	**0**	**38**	**26**	**0**

9.4.8.1 Summary of commonly identified technical errors for linear independence

From the Table above, there is a general structure in the type of misconceptions inherited by the students in question 2. It is clearly seen that very few students revealed some technical errors in that question since there was no need to carry out any algebraic manipulations. 8 students in question 2 part (a) who attempted to use row reduction failed to reveal technical errors. Also in question 2, part b, two students failed to solve equations simultaneously, and the other 2 attempted to find the determinant of a 3×2 matrix. In question 2 part e three students attempted to carry out row reduction but could not come up with proper arguments to show that it is linearly dependent.

The number of algebraic manipulations in question 6 was very limited such that none of the students revealed technical errors. Some students could quickly figure out the solution after carrying out a few steps for question 6. However, in question 7 it is evident that 38 of the teachers revealed some technical errors where they attempted to show the three vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ were linearly independent or not. This question involved the use of Gauss elimination method or using the method of calculating the determinant which involves a series of step by step procedures. An analysis of the students' responses showed that they made a lot of computational errors as a result of failing to carry out the correct algebraic manipulations. On top of that, a further analysis of the incorrect result was done and it was evident that the students had some more challenges as they failed to interpret the solution to the system of equations so that they would make a link with the concept on linearly dependent/independent. Considering the final answer obtained by student T8, as shown, $\begin{bmatrix} 1 & 2 & 4 & & 0 \\ 0 & 1 & 1 & : & 0 \\ 0 & 0 & -3 & & 0 \end{bmatrix}$ the student was supposed to use back substitution and obtained the values of $k_1 = k_2 = k_3 = 0$. This shows that the system is consistent hence the set of vector s are linearly independent. Instead, student T8 wrote that $-3 \neq 0$ therefore there is inconsistency hence the vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ cannot be expressed as linear combination which means there is linear independence. 15 students revealed calculations errors, which is part of technical errors. According to Sfard and Linchevski (1994) the prior knowledge on the solution to systems of equations has acted as a barrier to the development of the abstract concept of what a basis of a vector space is.

Another group of 14 students also demonstrated technical errors due to a failure to link a matrix with a zero vector obtained with the concept of linear dependence. This is a result of a failure to interpret the solution of systems of equations and their relationship to linear independence/dependence. The other drawback that led to the development of technical errors was failure to calculate the determinant correctly using Sarrus rule or the method of Laplace transformation. This again brings to mind that the students' concept images might have some contradictory ideas which cause them not to apply the appropriate techniques thus leading to the development of technical errors outlined above.

9.4.8.2 Summary of commonly identified conceptual errors for linear independence

I noted that most of the conceptual errors occurred in questions 2b and 2e and question 6. Considering question 2, the students manifested a number of conceptual errors. I noted that few conceptual errors were manifested in items 2a, 2c and 2d which required the students to apply the theorem which says that if one vector is a scalar multiple of another, then the vectors are linearly dependent. Most of the students were able to obtain the correct responses on such items. 22, 19 and 21 students respectively manifested conceptual errors. The main cognitive difficulty encountered was an attempt to make use of the step by step procedures and they ended up calculating the determinants of an $m \times n$ matrices, calculating separately the determinants of matrix A and the determinant of matrix B instead of coming up with a vector equations, see question 2d written response by student T72 Figure 7.7 as stated in section 7.4.1. A large number of students simply made some guess work and mixed up the different types of theorems. However, a substantial number of conceptual errors was revealed in question 2b which required the students to determine whether the given three vectors in R^2 were linearly independent/linearly dependent, by applying the theorem 7.1. 56 students showed some conceptual errors of various forms. Many of the students used the theorem incorrectly with most of the students failing to give a convincing result. The r and the s appear in most of the students' responses but no explanations surfaced. Some of the theorems were twisted, for example looking at student T2's solution where he outlined that it is linearly independent since it is not connected by any scalar, with T45 talking of no parallel vectors and some trying to explain it in terms of geometrical interpretation. All the above obstacles resulted in the students' concept images having some contradicting ideas, thus leading to the manifestation of conceptual errors. In question 2e, 42 conceptual errors were manifested, where eight students calculated the determinant but were supposed to use inspection, four said is linearly independent because we have a row of zeros, 15 attempted to use inspection but grabbed any terms that came in their way and finally the other 15 students said it is linearly dependent but with wrong arguments. This revealed conceptual errors.

In question 6 the students could formulate the vector equations but equated it to the wrong vector, that is they equated it to an arbitrary vector instead of the zero vector of an $M_{2\times2}$. 12 students manifested such an error which is conceptual in nature. Some of the students were confused and they constructed a 4×4 square matrix, and augmented it to either a $M_{4\times1}$ zero matrix or an arbitrary one, see Figure 7.13 written response by student T54. Here we see that the in-service teachers support the contention by De Lima (2008) whereby they were building on the experiences that they had learnt before on the aspect on spanning, where the vector equation is equated to an arbitrary vector, instead of equating to the zero vector. The specific concept image has been used out of its domain of validity, and according to Makonye (2012) the existence of such errors in concept leads to some obstacles that hinder the students to construct new mathematical knowledge. However, after formulating the incorrect vector equations, some students were able to carry out correct procedures as they were now able to carry out scalar multiplication and vector action, but at the end obtained the results $k_1 = a,\ k_2 = b, k_3 = c$ and $k_4 = d.$ Makgakga (2016) argued that students should have both procedural and conceptual knowledge so that they are able to solve more complex problems and obtain correct solutions. Here we can see that the students were able to execute the correct procedures, but because they had manifested a conceptual error, at the end the result obtained is incorrect.

9.4.8.3 Summary of commonly identified procedural errors for linear independence

From the Table, it can be seen that the procedural errors were not very widespread. Most of the procedural errors are revealed in question 2d and question 6 with only one student in question 2a. In question 2a the student formulated an augmented matrix but did not carry out row reduction. In question 2d some of the students preferred to use the step by step procedures and formulated vector equations of the form $k_1v_1 + k_2v_2 = 0$, were v_1 and v_2 are 2×2 square matrices, see question 2 above. The students failed to carry out the correct procedures of scalar multiplication and addition and therefore failed to solve the equations. The larger proportion of students, 12, wrote the statement that $A = -B$ and concluded that linearly dependent, without further explanations, indicated that the learners' concept image of what linear independent is, is not complete. The procedure was incomplete because there is the need to give proper justification why one says it is linearly dependent. 26 students in question 6 demonstrated some procedural error. Another procedural error appeared in question 7 where six students failed to view the determinant as a function as well as failing to use the correct notation or brackets indicating that they are finding the determinant. In question 6, 26 students revealed some procedural error, for example one student failed to simplify the statement $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ in order to show that the system of equations was linearly independent. The other 23 students were able to solve and showed that $a =$

$0, b = 0, c = 0$ and $d = 0$ but did not complete the procedure since this was a prove question. There was the need to justify why the vectors are linearly independent. Sfard (1991) noted that even though some students are able to perform some processes, the secondary processes must be seen to be totally arbitrary, and failing to do that, the students' understanding will remain instrumental.

9.5 Discussion of the errors displayed by students on question on basis and dimension

The discussion in this section is based on the results obtained in chapter 8.

9.5.1 Question 3

Students' responses showed that nine students did not answer the question, and 16 almost got the correct response. There were 10 responses whose errors were classified as being conceptual, 4 were procedural errors and 34 were technical errors.

9.5.1.1 Conceptual errors for question 3 on basis of a vector space

10 (14%) of these students showed some conceptual errors as they demonstrated some confusion with regards to the wrong usage of theorems, especially theorem 7.1. Students failed to define the terms r and n explicitly and some confused the inequality sign having $r < n$ and concluded that it did not form a basis. Terms met during the study on vector space concepts were used haphazardly in an attempt to go about the question.

9.5.1.2 Technical errors for question 3 on basis of a vector space

11 students demonstrated technical errors in the sense that instead of showing whether the given vectors form a basis using inspection, step by step procedures were seen as the students attempted to come up with a vector equation equated to the zero vector and/or arbitrary vectors. However, these students revealed technical errors as they demonstrated some computational errors with some of them failing to interpret solution to systems of equations in relationship to whether the vectors are linearly independent or not. 23 students treated the vectors as row vectors and attempted to use the method of Gaussian elimination. These students encountered some technical errors also as a result of demonstrating calculation errors, with some failing to relate the concept of linear independence to the row reduced matrix, due to a failure in interpreting solutions to the systems

of equations. 16 attempted to use inspection but failed to explicitly give a convincing deduction as to why it formed a basis but their solutions were almost correct. The technical error revealed here was just a slip.

9.5.1.3 Procedural Errors for question 3 on basis of a vector space

Two teachers formulated an augmented matrix and did not carry out row reduction, and simply made wrong deductions revealed procedural errors. Two more students who treated vectors as row vectors, came up with a matrix but simply made wrong deductions, for example student T28 who said since we are not able to reduce, it does not span hence it is not a basis and thereby also revealed procedural errors.

9.5.2 Question 8

From an analysis of question 8, 6 of the students did not attempt the question and 19 of the students were able to follow the correct procedures, and obtained the correct matrices reduced to row echelon matrices, and made the correct justifications and conclusion. However, I noted that 48 students revealed technical errors.

9.5.2.1 Technical errors for question 8 on basis of a vector space

48 (66 %) of the students demonstrated correct procedures for finding basis of a vector space. All 48 students used the Gaussian elimination method and obtained reduced row echelon matrices. However, the students struggled to deduce whether the vectors formed a basis of R^4 or not. 18 (25%) of the students could not interpret the results of the reduced matrix obtained due to a failure to link the solution of the system of equations to a basis, and some of them could not even give a reason for whether it formed a basis or not. 10 (14%) of the students went further to list the non-zero vectors in the reduced matrix and concluded that it was the basis, see Figure 8.4 written response by student T13. This again shows that they cannot make a correct link between the reduced matrix and the concept of basis. We noted that when interviewed, see section 8.4.2, the response by student T13 showed that she possessed procedural understanding of what the basis of a vector space is. This is also supported by Siyepu (2013) who argued that the student's responses

showed that the students entered the university with procedural understanding as the main factor for the understanding of mathematical concepts. 20 (27%) of the students also revealed technical errors, see the written response by student T44 in Figure 8.3. The major error manifested was that of failing to manipulate figures or failure to carry out the correct row operation. This is a result of failing to work with the directed numbers, hence manifesting technical errors.

9.5.3 Results for Question 9

The question was a learner generated example which required students to come up with a 4×4 matrix since a basis of R^4 must have four linearly independent vectors in such a way that they form a matrix that is in echelon form. 28 (38%) of the students did not attempt the question or they simply transcribed the question and left it blank. Six students obtained the correct solution. There were 19 responses whose errors were classified as being conceptual, and 20 were considered as procedural errors.

9.5.3.1 Conceptual errors for question 9 on basis of a vector space

We noted that 19 of the students exposed conceptual errors. These students formed vector equations of the form $(b_1, b_2, b_3, b_4) = k_1(1,1,1,1) + k_2(2,2,3,4)$ and showed that they were completely off track with the demands of the question. Some went to the extent of doing row reduction for a 2×4 matrix or 4×2 matrices but did not yield any useful data, thus revealing conceptual errors.

9.5.3.2 Procedural errors for question 9 on basis of a vector space

20 of the students displayed some procedural errors. They were aware that they should come up with 4×4 matrices, but did not have the correct procedures of coming up with four linearly independent vectors in R^4. For an example see written response by student T4, Figure 8.5 who failed to carry elementary row operation so that at the end she could do some row interchange, and then have the four linearly independent vectors in R^4. This then hindered them to develop a conceptual understanding of what a basis of a vector is. The interview done with student T7 in section 8.5.2 further shows that the students struggled and did not know how to go about the question and could not distinguish between the procedures for finding the basis of a vector space and that for finding the row space of a matrix.

9.5.4 Results for question 12

The question was an application question on the work done on the basis of a vector space. 2 students did not answer the question. There were 69 responses whose errors were classified as being technical, and two errors were conceptual in nature. I noted that none of the students could give a justification of the result obtained in terms of spanning and linearly independent.

9.5.4.1 Conceptual errors for question 12 on basis and dimension

Two students did not attempt the question. They did not know how to go about the question and this showed some conceptual errors, as shown in Figure 8.6 section written response of student T11. The student did not know how to go about the problem. The two students formulated the augmented matrix and simply listed the non-zero rows from the augmented matrix and said it is the basis. This demonstrated some conceptual errors.

9.5.4.2 Technical errors for question 12 on basis and dimension

The most common error manifested in this question was the technical error. 69 of the students were aware of the procedures that are necessary to determine basis of a solution space. Step by step procedures were seen as they formulated an augmented matrix and carried out row reduction of the matrix. However, during row reduction, 23 of the students revealed some misapplication of the rules when carrying out row reduction. These students displayed some technical errors, see written response by T13 in Figure 8.6 section 8.5.2 who carried out some incorrect row operations and committed a lot of calculation errors. The written response by T50 also revealed similar errors whereby he failed to manipulate figures. Instead of calculating the scalars, all 23 students listed the non-zero rows. Here the students demonstrated a misapplication of the learnt procedures that are applied to a new concept. 16 of the students displayed some calculation errors, and then made some conclusion based on the row reduced matrix instead of calculating the scalars. This again showed that the students applied the techniques on solution of systems of equation wrongly and could not get the correct solution. The other 14 students displayed calculation errors such that at the end they obtained three scalars instead of two, thus also displaying technical errors. Five of the students also made technical errors as they failed to state the dimension of the solution space despite the fact that they were able to obtain the correct basis of the solution space. 11 of the students were able to give a correct response on what the basis of solution space is and dimension of the solution space. However, these students experienced difficulties in order to justify why it was a basis of the solution space. Failure to give the correct justification showed that these students experienced some technical errors, as they failed to apply what they had learnt on the theorems on linear independence.

9.5.5 Discussion

An overview of the different kinds of responses for the four questions appears in Figure 9.6.

Table 9.5: Summary of the number of different response types for the four questions on basis and dimension

	Not attempted	Conceptual errors	Procedural errors	Technical errors	Almost complete	Complete
Question 3	9	10	4	34	16	0
Question 8	6	0	0	48	0	19
Question 9	28	19	20	0	0	6
Question 12	2	2	0	69	0	0

From the Table, we noted that many of the errors that were revealed are technical errors as compared to procedural and conceptual errors.

9.5.5.1 Summary of commonly identified technical errors for basis and dimension

There were no technical errors in questions 3 and 9 since most of the students did not know the procedures to be followed in order to go about the question. It is also important to note that a large number of students (28) did not attempt to answer the question. Most of the technical errors revealed in questions 8 and 11 were mainly due to calculation errors, failure to carry out the proper procedures on elementary row operations as well as failing to interpret the solutions to the system obtained after carrying out row reduction. This is most evident in questions 8 and 11. 18 of the students in question 8 wrongly interpreted the obtained row reduced echelon matrix. They could not correctly link it to the concept of what a basis is and 14 of the students went to the extent of listing the non-zero rows of the echelon matrix instead of making conclusions whether it is a basis or not. In question 11 more technical errors were revealed as the students, instead of calculating the scalars, simply drew conclusions based on the obtained row reduced matrix, or again took the non-zero row of the reduced echelon matrix. A total of 39 students revealed those errors. A substantial number of students obtained the correct basis and dimension but failed to apply the concepts learnt on linear independence and spanning so that they could to come up with the correct justification as to why it was basis, and this hindered them to reach the structural level.

9.5.5.2 Summary of commonly identified conceptual errors for basis and dimension

Most of the conceptual errors were displayed in questions 3 and 9. In question 3 students struggled to figure out the correct method that they could use to solve the problem. Most of the students intended to use the step by step procedures, which was incorrect according to the given instructions. Those who attempted to use inspection struggled to explain why it did not form a basis. Most of the students grabbed the terms that they met during the study and used them interchangeably in an attempt to find their way out. This shows that the students did not have the correct concept image of the concept of linear independence and spanning, which gives rise to the concept of a basis. In question 9, a considerable number of students revealed conceptual errors. These students formulated vector equations, with some treating the given vectors as row vectors or column vectors, and then attempted to do row reduction. These students manifested conceptual errors as a result of poor conceptualization of the concepts learnt on basis. It seems students are only interested in carrying out elementary row operations without understanding the concepts and fail to come up with the proper techniques of solving systems problem on basis of a vector space. According to Sfard (1994) these students manifested such an error as they were trying to generate knowledge which was not applicable to the situation. Makonye and Nhlanhla (2014) referred to this type of "cognitive chunk of ideas'' as this is what comes into the students' minds at that specific time. Sfard (1992) comments that this has led to the manifestation of conceptual errors. The students did not show any conceptual errors in question 8, and most of these students could have a correct concept image of what a basis is. There were only two students who manifested conceptual errors in question 11, for example the students simply came up with augmented matrix and simply made a deduction from it, showing poor conceptualization of learnt concepts.

9.5.5.3 Summary of commonly identified procedural errors for basis and dimension

Few students displayed procedural errors. For instance, in this section, students demonstrated procedural errors in question 9. The students did not realize that the problem required the application of the Gaussian elimination method, and they simply came up with a 4×4 that was not row reduced to row echelon form for example see student T4. 20 students demonstrated this error. The error originated from the students poor background concepts on linear independence. This supports the contention by Sfard and Linchevski (1994) who commented that if the students assumed knowledge is not vibrant, students would fail to link the new knowledge and the old knowledge, thus the students will not reach the structural level understanding of the given concepts.

9.6 Conclusion

This study has highlighted and recognized some error patterns in terms of conceptual, procedural and technical errors displayed by a group of undergraduate mathematics teachers in their responses to items based on subspace, linear combinations, linear independence, basis and dimension. The written responses provided an insight into the nature of the conceptual, procedural and technical errors. The interviews also added an in depth understanding into what the students think, and helped identify whether their misconceptions are persistent.

However, it is evident that some of these misconceptions were more serious than others. Seng (2010) and Brodie (2010) outline that error analysis helps in interpreting the nature of mental processes that individuals encounter during their mathematical thinking and practices. Teachers were seen making multiple errors on a single question. Most of the errors across the items were conceptual in nature, meaning that the students did not have a broad understanding of the concept and did not know where to start. From my count I found 413 conceptual errors, 297 technical errors and 123 procedural errors. Many of the technical errors were because of inappropriate interpretations of possible solutions to systems of linear equations. One common area where students displayed persistent misconceptions was that of interpreting the results after carrying out Gaussian elimination (the method of solving systems of equations). It is clear that teachers need to develop high proficiency in the foundational concepts that form the basis of higher mathematical abstraction in the concepts of linear combination, linear independence and basis.

Ricommi (2005) asserts that it is the responsibility of the lecturers to provide necessary and appropriate instruction so as to rectify students' misconceptions and errors. Based on this recommendation, it is important that learners must be given opportunities that allow them to experiment with different types of system of equations, sets, addition and subtraction of matrices before engaging with the concepts of the vector space concept. It must be noted that the class taught was also very large, which may have inhibited them from engaging constructively with the concepts. It is advisable to have smaller groups of the students during tutorials so that they get focused attention as they grapple with these concepts. This can be seen as a major goal that will help students to reach the structural level of understanding. Makgakga et al (2011) as well as Brodie (2010) agree that it is essential for teachers to understand students' errors, and then plan their instruction so that it takes these anticipated errors into account.

It is important to note that as pointed out in the Literature Review chapter, there are very few studies that have been conducted on the types of errors that students encounter when learning vector space concepts. This scarcity of studies in the area is possibly because in the conceptual structures of theories such as APOS these words do not need to be used. In such theories, " errors" are cognitively or institutionally explained, and elements to help students construct their knowledge are provided. In APOS theory, this would entail developing activities so that students can perform and reflect on new actions, or de-encapsulate objects in terms of the process that gave rise to them to work on new processes or coordinate new processes into a new object which is in accordance to the expected mathematical concept. Unfortunately, this aspect was beyond the scope of the study, but it will be pursued in further studies that can be focused on addressing these "errors". The next chapter presents the conclusion of the study and recommendations for further studies.

CHAPTER 10

CONCLUSIONS AND RECOMMENDATIONS

10.1 Introduction

This chapter focuses on the conclusions and recommendations of the research study. The chapter also suggests themes for further research.

The main aim of this study was to explore the in-service teachers' understanding of vector space concepts and pay particular attention on how they made the necessary mental constructions and mental mechanisms as they dealt with the concepts on vector space. This course is a compulsory course as well as a core course that must be studied by all students that who intend to teach mathematics at advanced level. This is because these teachers need to have the relevant content knowledge and the knowhow for teaching so that they will develop conceptual understanding of mathematical concepts. The goal of the course is to consolidate and extend knowledge and understanding of linear algebra, where linear algebra is a subject that grew out of the business of solving systems of linear equations. This course will enable these teachers to acquire the necessary knowledge that will assist them in executing their duties and teach the advanced learners with ease. It is hoped that the knowledge gained will play a vital role in the learning process, as well as in the teaching of advanced level mathematics and the world of work. This is in line with Brijllal's (2015 p. 23) contention that:

‘‘Mathematics student performance has been for decades recognized as a problem in society. This is the case not only in schools but also at university, especially at the undergraduate mathematics. It is thought that if one understands how students think when engaging in mathematics activities then one might be able to improve on the ways of making the learning of mathematics more meaningful. Hence exploring student mathematical thinking is important not only to mathematics education research, but the country and the global society as a whole.”

10.2 Conclusion

This study used an APOS theory which, according to Dubinsky (2000), is a theory of learning. The study gives a description of the possible processes by which the concept of vector spaces can be learnt. APOS theory was further used to provide cognitive explanations of the appropriate mental constructions that the in-service teachers made in order to understand the vector space concepts by making use of the mental mechanisms of interiorisation and encapsulation (Dubinsky, 2013). Furthermore, the main aim of applying APOS theory is to divulge the nature of students' mental construction that they make in order understand certain concepts rather than to consider a statistical comparison of their performances (Weller, Dubinsky, McDonald & Merkovsky, 2003). The APOS theory was used in the study in conjunction with a diagnostic tool called the genetic decomposition so that the specific mental structures could be detected. This diagnostic tool helped to evaluate students' successes and failures when learning the vector space concepts. The results were based on students' written responses from three activity sheets and interviews. Data was video and audio recorded. 13 students out of a group of 73 participants volunteered to be interviewed. The interviews conducted helped to shed more light on the written responses given by the students. The study aimed to answer the following research questions:

- What APOS mental constructions can be inferred from the students written and verbal responses to items based on vector space concepts?
- What are some cognitive difficulties encountered by the students when trying to construct the necessary vector space concepts?
- How can the preliminary genetic decomposition be revised to take into account the students' learning experiences?

In chapters 5, 6, 7, 8 and 9 the results of students' thinking processes when learning the vector space concept were presented and examined. I presented the main findings of the study and attempted to address the three research questions stated above. The following concepts were looked at as I tried to explore how the students make the necessary mental constructions when constructing the vector space concepts that is the vector space, vector subspace, linear combination, linear independence, basis and dimension. In learning the vector space concepts, students' knowledge of describing the various vector space concepts was tested such that making the distinctions between linear combination, linear independence and so on was covered. The in-service teachers were further tested on the knowledge of understanding of the ten vector space axioms, the notion subspace theory, showing whether given a given vector is a linear combination of the other given vectors as well as showing whether given vectors are linearly independent/ dependent or form a basis of the given vectors. Aspects on dimension of solution space were also tested. An attempt was made to see how the in-service teachers attempted to make a connection of the above concepts, since they are closely connected. This was done in an endeavor to improve mathematics learning.

Thus the aim of this chapter is to highlight some of the main findings of this study by addressing the research questions, and to suggest what contribution has been made to the body of knowledge. I now present in more detail about some of the issues that emerged from the analysis.

10.2.1 Students' mental constructions of vector space concepts

One of the major aims of this study was to answer the following research question:

- What APOS constructions have the students developed with respect to the various vector space concepts?

The main aim of the study was to reveal the mental construction that the students make when learning the vector space concepts, guided by the theoretical framework APOS theory discussed in chapter 4. The findings from the various chapters are discussed below.

10.2.1.1 Findings on the concept of vector space and subspace

Evidence from chapter 5 questions 1 and 4 revealed that many of the students did not attempt the questions with some of them having completely incorrect responses. These students had difficulties in moving from the abstract mode that is from which the question is phrased and they struggled represent it in algebraic mode by failing to identify the set and failing to understand the notion of sets with respect to showing that the given set is not a subspace. They were unable to identify elements of the set. This, therefore, hindered the students' efforts to develop their understanding even at the action level conception according to APOS theory. Many of them did not even have an action conception of binary operations; an example is the written response of student T14 in Figure 5.13 who struggled to carry out a binary operation. I noted that 27 (37%) of the students who attempted to take two objects whose determinant were zeros, and attempted to produce a new object that resulted into in a zero determinant were limited to action level which was not fully developed. The action level is bound by multiple step algorithms which are a result of external stimuli motivated. 8 (11%) of the students were able to interiorise the action by taking two objects (elements) and finding the sum to produce a new object whose determinant is not zero. However, these students did not fully conceive of the binary operations as a process but showed some progress towards interiorisation as they were able to check the conditions that needed to be satisfied by axioms. However, these individuals struggled to make the justification on why the set was not a subspace. However, 7 (10%) did cope with the demands of the question and were able to identify that the closure property of the binary operation on the set of 2×2 matrices was not fulfilled, hence they were able to encapsulate the processes into an object understanding of what a subspace is.

The second task required an object level engagement when showing that the given $M_{2\times2}$ is a vector space. Ten axioms needed to be satisfied. 11 (15%) could carry out the step by step procedures in an attempt to show the set of 2×2 matrix was a vector space, but these students simply developed some action conception because they could not identify some of the axioms, according to the genetic decomposition. Most of the students were able to state the ten axioms but showed interiorisation of some of the axioms into a process level engagement of what a vector space is. This is because these students struggled to prove some of the axioms. Many of them struggled to prove the distributive and the associative property of scalar multiplication. Also a number of students could not prove the scalar identity property, see Figure 5.11, written response by T20. It can be seen here that the teachers were just following procedure without understanding how some of the axioms are proved. The problem of struggling to carry out the correct binary operations of scalar multiplication hampered most of the students' efforts to develop fluent proof about a vector space concept. Furthermore, the data from interviews with student T7 who said that, $1.v = 1$ meaning, 'i*t is very possible if* $\boldsymbol{v}$ *is not a matrix'* showed that he continued to cling to his misconception from the written response, thus did not encapsulate the processes into what a vector space is. Only 6 (9%) of the students were seen to have made the appropriate mental structures according to the genetic decomposition as these students were able to coordinate the concepts of sets and binary operations in showing that the ten axioms were satisfied.

10.2.1.2 Findings on the concept of linear combination of vectors

The findings in chapter 6 for question 1 indicated that only 5 (7%) of the students might have made the necessary mental construction as they were to explain the difference between the definition of linear independence and spanning as well as seeing the connections between them, but later on some of them could not continue to use the concept in answering successive questions as was shown in chapter 7. Most of the students were operating at the action level according to APOS theory, as they displayed an inadequate understanding of the definitions. The students had a tendency of describing the procedures for determining linear combination or spanning instead of giving the definition of the terms, for example see Figure 6.6, written response by T31 who wrote that spanning vector, determinant is not equal to zero, $Det \neq 0$.

Question 2 required the teachers to carry out Gaussian elimination method or use the method of solving the systems of equations by the elimination method. These problems involved multiple step procedures in order to determine whether given vectors could be written as a linear combination of the other vectors. The students were able to carry out row reduction, but could not understand why they were executing such procedures. The major drawback in question 2 was that the students simply had a collection of rules, that of carrying the step by step procedures, but at the end they could not go beyond the procedures of checking for consistency so as make sufficient judgement whether the given vector was a linear combination of the given vectors. The other problem that hindered the students' development of the mental constructions at the process level was that they did not apply the knowledge acquired from concepts on solving of systems of linear equations as well as struggling to give a verbal description of how to determine that a given vector could be written as a linear combination of given vectors.

40 (55%) of the students' responses showed that they were able to carry out the correct row operations, found the correct scalars and were able to obtain the correct deductions after checking for consistency in the third equations. However, the number of students who obtained the correct deductions dropped down to 28 (38%) for question 2b. The students either could not generalise their arguments appropriately about the reduced rows to echelon in relationship to a given vector not being a linear combination of the given vectors, or they did not check for consistency in the third equation with some struggling to have the starting point since 16 (22%) did not attempt to answer the question. Results for question 7 which required students to describe and represent

geometrically span (u) and span (u, v) showed that the students could not construct the necessary mental constructions called for by the genetic decomposition. Many of the students had an action conception which could have been cognitively triggered by external stimuli, seeing the term geometrical. Lines and planes passing through the origin were seen, and the students did not recognise that span $\boldsymbol{v}$ is a scalar multiple of **v**, and that span u is a scalar multiple of **u** which also means that span $\{\mathbf{u}, \mathbf{v}\}$ consists of all linear combination of $\boldsymbol{u}$ and $\boldsymbol{v}$**.**

Although many of the students were able to carry out the procedures in question 8, that is using Gaussian elimination method or method of calculating the determinant, the item required internalization of these procedures. The results showed that one student, T25, had made the necessary mental structure called for by the genetic decomposition as she was able to carry out further transformation on linear combinations. It was noted that 48 (66%) of the students could not unpack cognitively the structure of linear combination by failing to generalise the correct arguments about solutions to system of equations or the obtained determinant. The interviews also confirmed this. For example, T33 who was given a hypothetical augmented matrix $\begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ had this to say, the original vectors do not span R^3 since $x_3 \neq x_2 \neq x_1$. The justification was not complete since the problems reduced to ascertain whether the system was consistent for say b_1, b_2 and b_3 that is the arbitrary values.

10.2.1.3 Findings on the concept of linear independence

Evidence in question 2 revealed that some of the students could generalise arguments about whether given vectors in R^n, in the form of a polynomial or matrix form were linearly independent or not without external stimulus. It was noted that students performed well in the items which required the use of theorem 7.2 as compared to the use of theorem 7.1. The inappropriate use of notations r and n in theorem 7.1 hindered the in-service teachers to develop their understanding at the process level according to the genetic decomposition. Also, while applying theorem 7.2 the representational nature of the vectors played a greater role in influencing the in-service teachers' mental constructions of linear independence/dependence. Students struggled to see the scalar multiple for part 2a which was a negative in the form R^n and part 2d which was in the form of a matrix as compared to part 2c which had a positive scalar.

The results of the 73 students who responded to question 6 (showing that a set of $M_{2\times2}$ matrices were linearly independent) revealed that 25 (34%) were able to perform the actions of setting up vector equations, equate the corresponding elements and attempt to find the scalars. However, most of the students could not identify the scalars, and some of them did not make any conclusion whether the set of vectors were linearly independent or not. However, only 24 (33%) of the students represented their understanding in a manner described as a process level conception of verifying that given vectors are linearly independent. It was also noted that in question 7 (which required the students to show whether given vectors in R^3 were linearly independent or not) students could perform the procedures of reducing the matrices to row echelon form or calculating the determinants but some of the students were not skilled at carrying out the algebraic manipulations. 15 of the students could not manipulate figures correctly as they struggled to carry out correct row operations and three used the determinant method and got wrong figures. Thus 18 (25%) of the students were considered to be at an action level which is not fully developed according to the genetic decomposition, as they struggled to carry out the correct manipulations. Evidence also revealed that 14 (19%) of the students had correct row operations but made wrong deductions and this showed that the students had simply memorised the rules for determining linear independence and could not interiorise the actions of showing that given vectors were linearly independent or not into a process. In order to construct the necessary mental construction, an individual must be able to relate the result obtained from the calculated determinant to the notion of linear independence and make a corrective deduction and interpret correctly the reduced matrix in terms of linearly independent/linear dependent. 17 (23%) of the students were able to carry out the correct row operations and 15 (20%) obtained the correct determinant. These students were able to carry a correct deduction in terms of the vectors being linearly dependent, but none of the students was able to tell that since the vectors are linearly dependent, then one of the vectors can be written as a linear combination of the other vectors. This therefore hindered the in-service teachers to develop their understanding at object level according to the genetic decomposition.

10.2.1.4 Findings on the concept of basis and dimension

The results for the concept on basis and dimension exposed that most of the undergraduate students worked at an action level conception of what a basis of a vector space is. For example, question 3 required students to use inspection, but surprisingly, a total of 38 (52%) preferred to use the step by step procedures. Only 9 (12%) of the students attempted to use inspection and obtained the correct solution, showing interiorisation of actions into a process. The rest of the students struggled in coordinating the process of checking if given vectors are linearly independent and the process of checking if the vectors the given vectors. Their knowledge acquisition was limited to an action conception as the students could only manage to carry out row operations. Questions 9 and 12 tested the object understanding of basis of a vector space and only 6 (8%) of the students in question 9 were able to carry out further transformation, for example being able to extend $\{u_1, u_2 \dots u_n\}$ to a basis of R^n , showing some objects conception of the concept of basis. However I not that for question 12 none of the students was able to carry out the fully flegged further transformation on the basis of solution space, hence they failed to encapsulate the concept basis

10.2.2 Students' difficulties in the learning of vector space concepts

The second goal of the study was to answer the following research question:

- What are some cognitive difficulties encountered by the students when trying to construct the necessary mental on vector space concepts?

It is evident from the discussion in chapters 5 – 8 that the students were experiencing difficulties in making the necessary mental constructions when learning the vector space concepts. Many of these difficulties were manifested as errors. The errors were grouped into conceptual, procedural and technical categories. However, from the discussions it was seen that most of errors revealed were mainly conceptual and technical, showing that the students were very comfortable with engaging with the algorithms such as carrying out row reduction and calculating the determinants, but without making sense of why they were engaging with the calculations.

10.2.2.1 Conceptual errors displayed by students when learning vector space concepts

Students demonstrated conceptual errors that originated from poor understanding of some of the concepts on vector space concepts. This suggests that these students did not understand some of the concepts and applied incorrect mathematical procedures. Students revealed conceptual errors when they failed to prove some of the axioms for a vector space, for example the multiplication axiom $1.v = v$, where the students took 1 as an identity matrix. On the concept of subspace, the students failed to come up with appropriate sets, for example, a set which consists of all vectors having three equal components, or a set of two matrices that gives a determinant equal to zero, but which is not a subset. This is hindered by a failure to understand the demands of the question. These students revealed conceptual errors as they become confused about what they really wanted to show for given axioms. The other conceptual error manifested was that students came up with a vector equation and equated it to the zero when solving problems on linear combination, i.e. $k_1v_1 + k_2v_2 \dots k_nv_n = 0$, despite the fact that the vector was given. On the same note a similar error was also revealed as the students equated the vector equation to an arbitrary vector, i.e. $k_1v_1 + k_2v_2 \dots k_nv_n = b$, when solving problems on linear independence instead of equating to the zero vector. This was a result of the met afters and met befores which had hindered the development of concepts at the structural level of understanding according to Sfard (1992). Another serious conceptual error was manifested in question 2d, see chapter 7, whereby student T72 calculated the determinant of the separate matrices that is matrix A and matrix B in an attempt to show that given vectors are linearly dependent. In the same section i.e. question 6, in order to show that the four matrices are linearly independent, a student formulated augmented matrix, showing a failured to grasp the concepts on linear independence, as well as finding the determinant of separate matrices. Students also revealed conceptual errors when they fail to extend the vectors $\{u_1, u_2\}$ to a basis of R^4 as they attempted to formulate vector equations.

10.2.2.2 Procedural errors displayed by students when learning vector space concepts

The main procedural errors revealed were failing to prove some of the axioms of multiplication and confusing the axioms of a vector space and those of a subspace. When solving matrices students failed to equate corresponding elements, see section 9.3.2.2. Some students failed to translate given vector equations into appropriate systems of equations and augmented matrix. When extending the vectors $\{u_1, u_2\}$ to a basis of R^4 some students were able to come up with a 4×4 square matrix but failed to carry out the correct procedure of carrying out row reduction, see written response by student T4 Figure 8.5. They did not know the appropriate procedures to be applied to get a basis of R^4.

10.2.2.3Technical errors displayed by students when learning vector space concepts

The most common technical errors manifested were computational errors. Most of the students failed to carry out correct row operations and some had problems with carrying out algebraic manipulations despite the fact that they were teaching these concepts. The other misconception that hindered the conceptual understanding of the concepts on linear combination, linear independence and basis was that the students did not have an adequate basis of reflecting on the solution that they obtained after carrying out row reduction or solving a system of equation in terms of linear combination, linear independence/linear dependence and basis. Some of the students went to the extent of listing the non-zero rows from reduced row echelon matrices and said it was the basis of solution space or basis of a vector space, thus demonstrating technical errors. In order to find the dimension of the solution space, students were seen adding the total number of the non-zero rows. Such obstacles interfered with appropriate learning and hindered the proper construction of correct knowledge. So many errors were manifested due to a failure to interpret solutions to systems of equation. The other technical error manifested was that students applied some wrong techniques, for example attempting to find the determinant of $m \times n$ matrix as well failure to use Sarrus rule and the method of Laplace transformation appropriately.

10.2.3 Modification of the genetic decomposition

The third goal of the study was to answer the following research question

- How can the preliminary genetic decomposition be revised to take into account the students' learning experiences?

To answer this question, I noted that the genetic decomposition did not capture some of the items that came out of the data analysis, so there was a need to revise the genetic decomposition.

10.2.3.1 Modification of the genetic decomposition of the vector space concept

This is represented in section 5.6. The modified genetic decompositions attempted to specify the actual actions since the students had problems in showing the axioms in an attempt to show that the given sets were subspace or the vector space. The axiom schema and vector space schema whereby the ten instances for checking each axiom as well as the two instances for showing that the two axioms are satisfied was included in the modified genetic decomposition. In the preliminary genetic decomposition, we relied on the work of Paraguez and Oktac (2010) which looked at the co ordination on objects of sets and the vector space axioms. In the revised genetic decomposition, I made a full grounded on the role of axiom and schema and the vector subspace schema.

10.2.3.2 Modification of the genetic decomposition of linear combination

This is represented in section 6.10. This was modified so as to include the students' weak schemas that needed to be developed first, such as how to solve and interpret solutions to systems of equations as well as working with binary operations. The actual actions were also specified for example at the object level the individual must be able to link the span(**u**) and span(**u,v**) to linear combinations, section 6.10.

10.2.3.3 Modification of the genetic decomposition of linear independence

This is represented in section 7.11. The schema for solving systems of equations and basic algebra notations were incorporated in the modified genetic decomposition. The actual actions were specified, see section 7.11 such as at the process level the individual must be able to apply relevant theorems for example theorem 7.1, 7.2, 7.3 etc.

10.2.3.4 Modification of the genetic decomposition of basis and dimension

This is represented in section 8.5. The schema for solving systems of equations and calculating the determinant were incorporated in the modified genetic decomposition. Since the concept of basis arises as a coordination of the process of establishing that a given set of vectors is linear independent and that it spans the particular set of vectors, I noted that there is no action for basis. see section 8.5.

10.3 Recommendations

By interpreting students' difficulties in terms of the genetic decomposition and Kiats (2005)'s framework specific areas that need improvement were identified and some implications for pedagogy are further discussed here.

I observed that the in-service teachers had difficulties with conceptualizing the concepts of vector space and vector subspace whereby they lacked the necessary logic in carrying out proofs. Students had problems of identifying the sets and its elements, as well as moving from the abstract mode in which the question is phrased to the algebraic mode. Hence it is essential that students are given opportunities to interact with different sets and different types of binary operations. This will help students to move swiftly between the different languages and modes of representation. It is also important to note that the main aim of the course mathematical discourse and structures is to help students come to grips with abstract notations at an early stage in their training so as to gain a realistic view of what mathematics as a discipline really is and it introduces the students to the concepts on set theory, logic and various proofs. This course was done concurrently with the second course on linear algebra. This suggests that this course should be taught before introducing the students to the concepts on the vector space.

The results indicate that the in-service teachers were more comfortable in carrying out algorithms (very few procedural errors were manifested), for example applying the Gaussian elimination, but at the end they failed to make corrective justifications and arguments on the results obtained. This suggests that lecturers should stimulate the students to coopt these procedures in a critical and interrogative manner so as to make meaningful deductions.

The results show that some 'met befores' and the 'met afters' are seen as problematic in the learning of the various vector space concepts as they cause confusion among the concepts that are heavily connected and they hinder some appropriate generalization of the concepts learnt. It is vital for educators to consider the awkward 'met befores 'which impede successful learning, and these must be addressed (Mhlolo, 2014).

As pointed out in the literature review, there are very few studies that have been conducted on the types of errors that students encounter when learning the vector space concepts. This study dwells much on the cognitive difficulties and the types of errors that students make in the learning of vector space concepts. Many of the misconceptions identified have not been previously reported in many studies. This will help the lecturers to put in place corrective mechanisms and some instructional strategies so as to enhance the students understanding of the various vector space concepts and this will help to understand the students' thinking.

I also noted that the in-service mathematics teachers did not have enough time to engage meaningfully with the concepts on the vector space due to the mode of delivery of their programme. The students also did not have enough time to finish tasks such as group work assignments or homework since their time table was fully packed without any free periods in between. In addition it was also difficult for the lecturers to diagnose individual differences during class discussions due to large numbers in the classes. The university must consider this so that during tutorials students are split into different groups so that have opportunities to interact with the lecturer.

Future courses can also be taught using computer programmes such as Maple since according to Dubinsky (2002) these are useful in advanced mathematics in promoting reflective abstractions. The use of CAS will also free students from dwelling on tiresome calculations, and much more time will then be profitably spent on conceptual understanding of the concepts and on the argumentations and justification required to bring to light whether given vectors are linearly independent, or form a basis, or that one vector can be written as a linear combination of the other vectors. This should also be incorporated as the use of CAS also allows learners to tackle more complex mathematical objects (Thomas and Hong, 2006).

10.4 Suggestions for further studies

A sample of 73 volunteering students doing Bachelor of Science Education Honours degree in mathematics on the block release mode of lesson delivery were participants of the study. Only 13 students volunteered to be interviewed. It is recommended that a wider research should also include the first year students doing the conventional mode of delivery studying the Bachelor of Science Education Degree in Mathematics (preservice teachers) and those doing Bachelor of Science in Applied Mathematics/Physics as part of the study. Further studies should be planned that can test the revised genetic decomposition for each of the vector space concepts covered. A study taking an alternate approach to the study of errors and misconceptions, by using APOS language and descriptions could be a useful contribution. Finally a study which takes all the Genetic Decompositions presented in this study as a starting point to describe the Vector Space Schema can be carried out.

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APPENDIX A: ACTIVITY SHEET 1

Vector space and subspace

Questions

1. Define the following terms (a) vector space (b) subspace
2. Let $V = R^3$. Show that W is a subspace of R^3, where: W =$\{(a, b, c)$: $a = b = c\}$, that is, W consists of all vectors having three equal components.
3. Let $V = R^4$. Determine whether $X = \{(a, b, c, d) \mid a = 1,\ b = 0,\ a + d = 1\}$ is a subspace of R^4.
4. Let V be the vector space over of all 2x2 matrices over the real field R. Show that W is not a subspace of V, where W consists of all matrices with zero determinant.
5. Let V = R^2 and define addition and scalar multiplication operation as follows: if u = (u_1, u_2) and v = (v_1, v_2) then define u + v = $(u_1 + v_1, u_2 + v_2)$ and if k is an real number, then define($(ku_1, 0)$. Show that one of the axioms is not satisfied.
6. Determine whether the set equipped with the given operations is a vector space. If it is not a vector spaces, identify the vector space axioms that fail.
   (a) The set of all pairs of real numbers of the form (x,y) and $x \geq 0$ and $y \geq 0$ with the standard operations on R
7. Given the set of all $M_{2\times2}$ matrices of the form $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$. Show if it is a vector space.
8. Let V be a vector space over a field k. Prove that for any scalar k$\in K$ and 0$\in V$, $k0 = 0$.
9. Illustrate geometrically that a line through the origin of R^3 is a subspace of R^3.

APPENDIX B: ACTIVITY SHEET 2

Linear combination of vectors

Questions

1. Distinguish the terms linear combination and spanning of vectors.
2. Consider the vectors $\boldsymbol{u} = (1, 2, -1)$ and $\boldsymbol{v} = (6, 4, 2)$. Show that $\boldsymbol{w} = (9, 2, 7)$ is a linear combination of $\boldsymbol{u}$ and $\boldsymbol{v}$ and that $w = (4, -1, 8)$ is not a linear combination of $\boldsymbol{u}$ and $\boldsymbol{v}$.
3. Express v = (2, -5, 3) in R^3 as a linear combination of the vectors $u_1 = (1, -3, 2)$, $u_2 = (2, -4, -1)$ $u_3 = (1, -5, 7)$.
4. For which values of k will the vector $\boldsymbol{u} = (1, -2, k)$ in R^3 be a linear combination of $\boldsymbol{v} = (3, 0, -2)$ and $\boldsymbol{w} = (2, -1, -5)$?
5. Express M = $\begin{bmatrix} 4 & 7 \\ 7 & 9 \end{bmatrix}$ as a linear combination of the matrices

   A = $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, B = $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, C = $\begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix}$.
6. Show that the vector space M_{2x2} of all 2×2 matrices is spanned by the matrices
   $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$
7. Describe geometrically
   (a) span$(\boldsymbol{u})$ where $\boldsymbol{u}$ is a non zero vector in $\mathbf{R^3}$
   (b) span$(\boldsymbol{u}, \boldsymbol{v})$ where u and v are non zero vectors in $\mathbf{R^3}$ which are not multiples of each other.
8. Show whether $u_1 = (1, 2, 5)$, $u_2 = (1, 3, 7)$ and $u_3 = (1, -1, -1)$ do span $\mathbf{R^3}$**.**

APPENDIX C: ACTIVITY SHEET 3

Linear Independence basis and dimension

Questions

1. Describe the following terms :Linearly independent, basis and dimension of a vector space
2. Determine whether the following are linearly independent or not justifying your result.
   (f) $\boldsymbol{u_1} = (-1,2,4)$ and $\boldsymbol{u_2} = (5,-10,-20)$ in R^3.
   (g) $\boldsymbol{u_1} = (3,-1), \boldsymbol{u_2} = (4,5)$ and $\boldsymbol{u_3} = (-4,7)$ in R^2.
   (h) $p_1 = 3 - 2x + x^2,\ p_2 = (6 - 4x + 2x^2)$ in p_2.
   (i) $A = \begin{bmatrix} -3 & 4 \\ 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -4 \\ -2 & 0 \end{bmatrix}$.
3. Explain whether or not each of the following forms a basis of $\mathbf{R^3}$
   (a) $(1,1,1),\ (1,-,1,5)$.
   (b) (1, 2, 3), (1, 3, 5), (1, 0, 1), (2, 3, 0).
4. Given two vectors in $\boldsymbol{R^2}$ or $\boldsymbol{R^3}$ interpret geometrically
   (a) Linearly independent vectors.
   (b) Linearly dependent vectors.
5. Given three vectors in $\boldsymbol{R^3}$ interpret geometrically
   (a) Linearly independent vectors.
   (b) Linearly dependent vectors.
6. Prove that the following matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. are linearly independent.
7. Determine whether or not the vectors $\boldsymbol{u} = (1,1,2), (2,3,1)$ and $\boldsymbol{w} = (4,5,5)$ in $\mathbf{R^3}$ are linearly independent. Explain the result in terms of a linear combination.
8. Determine whether (1,1,1,1), (1,2,3,2) , (2,5,6,4), (2, 6, 8,5) form a basis of $\boldsymbol{R^4}$**.**
9. Extend $\{\boldsymbol{u_1},\ \boldsymbol{u_2}\}$ to a basis of R^4, where: $\boldsymbol{u_1} = (1,1,1,1)$ and $\boldsymbol{u_2} = (2,2,3,4)$.
10. Find a basis and the dimension of the subspace W of $\mathbf{R^4}$ spanned by

$$\boldsymbol{u_1} = (1,-4,-2,1), \qquad \boldsymbol{u_2} = (1,-3,-1,2), \quad \boldsymbol{u_3} = (3,-8,-2,7).$$

11. Find the rank and the nullity of the matrix $\begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$ and verify that the results obtained satisfy the dimension theory.

.

12. Determine a basis for and the dimension of the solution space of the homogenous system of equations.

$$2x_1 + 2x_2 + -x_3 + x_5 = 0$$
$$-x_1 - x_2 + 2x_3 - 3x_4 + x_5 = 0$$
$$x_1 + x_2 - 2x_3 - x_5 = 0$$
$$\vdots \quad x_3 + x_4 + x_5 = 0$$

Justifying you result.

APPENDIX D: INTERVIEW GUIDE

Questions:

1. In your own words define and distinguish the following terms:
   (a) vector space and subspace
   (b) linear combination and spanning,
   (c) linear independence/dependence and basis of a vector space
2. Describe how you can determine whether a given vector can be written as a linear combination of given vectors.
3. Describe how you can determine whether given vectors are linearly independent or not.
4. Describe how you can determine whether a given vector can form a basis of R^n.

APPENDIX E: CONSENT LETTERS

University of Kwazulu Natal

Edgewood Campus

Private Bag X03

Ashwood 3605, South Africa

Dear Participant

INFORMED CONSENT LETTER

My name is Lillias Hamufari Natsai Mutambara. I am a Mathematics Education PhD candidate studying at the University of KwaZulu-Natal, Edgewood campus, South Africa. I am looking at the APOS analysis of the understanding of vector space concepts: A case study of Zimbabwean undergraduate mathematics students.

To gather the information, I am interested in asking you some questions.

Please note that:

1. The researcher is going to use activity sheets and individual interviews.
2. The participants are expected to answer the questions to the best of their ability.
3. The participants will not speak over others so that everyone can be heard on tape.
4. All the interviews will be video and tape recorded with prior consultation and permission, and only researchers will have access to the video and tape.
5. The identity of the participants will not be revealed under any circumstance.
6. All responses will be treated with strict confidentiality.
7. The data will not be used for any purposes, except for this study.
8. Participation is voluntary.
9. The participants are free to withdraw from the research at any time without any negative or undesirable consequences to them.
10. There will be no financial benefits that participants may receive as part of their participation in this study.

11. Data will be stored in the university locked cupboards and will be destroyed after five years.
12. Your involvement is purely for academic purposes only.
13. If you are willing to be interviewed, please indicate (by ticking as applicable) whether or not you are willing to allow the interview to be recorded by the following equipment:

	willing	Not willing
Audio equipment		
Photographic equipment		
Video equipment		

I can be contacted at:

Email: tendaimutambara@gmail.com

Cell:+27 749797260 +263 773 239 164

My supervisor is Professor Sarah Bansilal who is located at the School of Education, Edgewood campus of the University of KwaZulu-Natal.

Contact details**: E-mail**: BansilalS@ukzn.ac.za

C**ell no:**+27832795916

You may also contact the Research Office through:

P. Mohun

HSSREC Research Office,

Tel: 031 260 4557 E-mail: mohunp@ukzn.ac.za

Thank you for your academic support, co-operation and valuable time and contribution to this research.

DECLARATION

I………………………………………………………………………… (full names of participant) hereby confirm that I understand the contents of this document and the nature of the research project, and I consent to participating in the research project.

I understand that I am at liberty to withdraw from the project at any time, should I so desire.

SIGNATURE OF PARTICIPANT DATE

………………………………………… …………………………………

APPENDIX F: GATE KEEPERS LETTER

REGISTRY DEPARTMENT

P Bag 1020
BINDURA, Zimbabwe

Tel: 0271 – 7531-6, 7621-4
Fax: 263 – 271 – 7534

BINDURA UNIVERSITY OF SCIENCE EDUCATION

COMMITMENT DISCIPLINE EXCELLENCE

HUMAN RESOURCES

14 December 2015

Mrs Lilias Mutamabara
Bindura University of Science Education
P Bag 1020
BINDURA

Dear Mrs Mutambara

RE: APPLICATION FOR PERMISSION TO CARRY OUT EDUCATIONAL RESEARCH IN THE UNIVERSITY

Permission to carry out Research on:

AN APOS ANALYSIS OF THE UNDERSTANDING OF VECTOR SPACE CONCEPTS. A CASE STUDY OF ZIMBABWEAN UNDERGRADUATE MATHEMATICS STUDENTS

Bindura University of Science Education has granted you the permission on the following conditions.

a) That in carrying out this research you do not disturb the programmes of the University.
b) That you avail to the University a copy of your research findings.
c) That the permission can be withdrawn at any time by the Registrar or by any higher officer.

I wish you success in your research work and in your University College studies.

Yours faithfully

……………………
TF Rumhuma (Mrs)
REGISTRAR

APPENDIX G: ETHICAL CEARANCE CERTIFICATE

UNIVERSITY OF KWAZULU-NATAL
INYUVESI YAKWAZULU-NATALI

28 January 2016

Mrs LHN Mutambara 215081400
School of Education
Edgewood Campus

Dear Mrs Mutambara

Protocol reference number: HSS/1849/0156D
Project title: An APOS analysis of the understanding of vector space concepts: A case study of Zimbabwean undergraduate mathematics students

Full Approval – Expedited Application

In response to your application received 15 December 2015, the Humanities & Social Sciences Research Ethics Committee has considered the abovementioned application and the protocol has been granted **FULL APPROVAL**.

Any alteration/s to the approved research protocol i.e. Questionnaire/Interview Schedule, Informed Consent Form, Title of the Project, Location of the Study, Research Approach and Methods must be reviewed and approved through the amendment /modification prior to its implementation. In case you have further queries, please quote the above reference number.

PLEASE NOTE: Research data should be securely stored in the discipline/department for a period of 5 years.

The ethical clearance certificate is only valid for a period of 3 years from the date of issue. Thereafter Recertification must be applied for on an annual basis.

I take this opportunity of wishing you everything of the best with your study.

Yours faithfully

....

Dr Shenuka Singh (Chair)
Humanities & Social Sciences Research Ethics Committee

/pm

Cc Supervisor: Prof Sarah Bansilal
Cc Academic Leader Research: Professor P Morojele
Cc School Administrator: Ms T Khumalo

Humanities & Social Sciences Research Ethics Committee
Dr Shenuka Singh (Chair)
Westville Campus, Govan Mbeki Building
Postal Address: Private Bag X54001, Durban 4000
Telephone: +27 (0) 31 260 [illegible] / **Facsimile:** +27 (0) 31 260 [illegible] **Email:** [illegible] / [illegible] / [illegible]
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APPENDIX H: TURN IT IN REPORT

Turnitin Originality Report

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AN APOS ANALYSIS OF THE UNDERSTANDING OF VECTOR SPACE CONCEPTS BY IN-SERVICE MATHEMATICS TEACHERS By Lillias Mutambara

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APPENDIX I: EDITORS CERTIFICATE

The Revd Mabel Jean Dalby (B.Th.)
77 Carey Road
Pelham
Pietermaritzburg
3201

mjmccd@gmail.com
082 487 2627

29 November 2018

TO WHOM IT MAY CONCERN

TITLE OF PAPER: AN APOS ANALYSIS OF THE UNDERSTANDING OF VECTOR SPACE CONCEPTS BY IN-SERVICE MATHEMATICS TEACHERS

STUDENT/AUTHOR: LILLIAS HAMUFARI NATSAI MUTAMBARA (215081400)

The above-mentioned thesis submitted for the Degree of Doctor of Philosophy in Mathematics Education was proofread by me for English language, grammar, spelling, punctuation and formatting errors. I endeavoured throughout the process to retain the writing style of the student/author and to remain true to her research content and intentions.

Please note that Tables, Figures, Charts, Graphs, Calculations and References (Bibliography) were not checked for accuracy although obvious errors, for example in headings, were corrected.

My suggested changes may be accepted or not at the student's and your discretion.

M J Dalby

APPENDIX J: MANUSCRIPT FORWARDED TO AJRMSTE

CC: michael.askew@wits.ac.za, bansilals@ukzn.ac.za

Nov 26, 2018

Ref.: Ms. No. RMSE-2018-0101R1
An exploratory study on the understanding of vector subspace concepts
African Journal of Research in Mathematics, Science and Technology Education

Dear Mrs Mutambara,

Dear Lillias (and Sarah)

Thank you for the revisions to your paper 'An exploratory study on the understanding of vector subspace concepts'. Please accept our apologies for the delay in getting back to you.

Two editors have read your paper carefully and want to thank you for the thoughtful responses made not only to the original reviewers' comments but also the editorial comments. Both editors are satisfied that you have largely addressed the comments made and we would like to provisionally accept the paper for publication, subject to some refinements and corrections.
If you are prepared to undertake the work required, we would be pleased to publish your paper in the African Journal of Research in Mathematics, Science and Technology Education (AJRMSTE).

If you decide to revise the work, please use the file attached, accept all our track changes and highlight your new text in colour (this method worked well for us, thank you). Also please note the maximum length of a revised AJRMSTE manuscript of 6300 words, including the title, author detail, abstract, key words, text, figures, tables, appendices, and reference list.
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Yours sincerely

Mike Askew and Fred Lubben
Editorial Team members
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