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An Overview of Hidden Symmetries

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Abstract

Approaches to finding solutions to differential equations are usually *ad hoc*. One of the more successful methods is that of group theory, due to Sophus Lie. In the case of ordinary differential equations, the subsequent symmetries obtained allow one to reduce the order of the equation. In the case of partial differential equations, the symmetries are used to find (particular) group invariant solutions by reducing the number of variables in the original equation. In the latter case, these solutions are particularly popular in applications as they are often the only physically significant ones obtainable. As a result, it is now becoming traditional to apply this symmetry method to find solutions to differential equations in a systematic manner.

Based upon the Lie algebra of symmetries of the equation, we expect a certain number of symmetries after the reductions. However, it has become increasingly observed that, after reduction, more symmetries than expected are often obtained. These are called Hidden Symmetries and they provide new routes for further reduction. The idea of our research is to give an overview of this phenomenon. In particular, we investigate the possible origins of these symmetries. We show that they manifest themselves as nonlocal symmetries (or potential symmetries), contact symmetries or nonlocal contact symmetries of the original equation as well as point symmetries of another equation of same order.

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Ntobeko Isaac Bujela

December 2012

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Contents

Chapter 1

Introduction

Suppose we are given a second order ordinary differential equation of the form

$$y'' = (x - y)y'^3 \tag{1.0.1}$$

or

$$y'' = x^n y^2 \tag{1.0.2}$$

or a partial differential equation of the form

$$u_{xx}u_{yy} - u_{xy}^2 - 1 = 0, \tag{1.0.3}$$

where primes denote the derivatives with respect to x and subscripts denote appropriate derivatives [45]. To obtain the solutions to these differential equations, we can start by checking if they belong to a known solvable class. Alternatively, we could try to transform the given differential equations to a family of linear differential equations. Especially for partial differential equations, many powerful methods to solve them exist. However, these methods are not always successful. We could try to solve them using the separation of variables or reduction of order methods [45]. These two methods might work but only for known differential equations [45].

It appears that in classifying differential equations and to obtain their solutions one has to employ different techniques. Among those methods Lie symmetry analysis is the most successful method, especially for non-linear equations [32]. It is an easy task to find symmetries of any differential equation. We can utilise these symmetries systematically to obtain exact solutions [16]. Most physical problems are solved and simplified using symmetry analysis methods. Even if they do not solve the problem, they clarify the problem and thereafter a precise method can be applied [16]. It is clear that symmetry analysis methods are the key to solve almost every differential equation [32].

It is our firm belief that every mathematician, physicist, chemist, biologist and engineer who works with differential equations should be familiar with symmetries in order to solve most problems in their respective fields [16]. The idea of our work is to develop the techniques of symmetry analysis based on Lie groups and to employ these techniques and obtain symmetry properties of many differential equations and the solutions that they possess [16]. Our work is devoted to study the symmetries of differential equations and their use to obtain solutions of a given differential equation [45]. In particular, we focus on the idea of hidden symmetries as popularised by Abraham-Shrauner and co-workers [1, 2, 3, 5, 22, 6].

1.1 Background of Symmetries

Evariste Galois was a mathematician and a well known revolutionary who died at the age of twenty-one [48]. He introduced the concept of solvability of algebraic equations in terms of radicals. He also introduced the terminology of group theory that is in common use today such as *group*, *subgroup*, *simple* *group*, *order*, *normal* group, *simple* subgroup and *solvable* group [48].

Camille Jordan was impressed and absorbed by Galois' ideas. As a result the papers he published in 1860s included similar but elaborated ideas of Galois [48]. Ultimately he decided to write a large monograph about the branch of mathematics; the book was published in 1870. In that period Jordan had two postgraduate students (the German Felix Christian Klein and the Norwegian Sophus Marius Lie) who came from their respective universities to enlarge their visions [48]. Jordan was an expert in the theory of groups and he believed that the theory of groups would play a vital role in the future of mathematics. He imparted his strong beliefs to Klein and Lie. They were also influenced by the work of Gaston Darboux [48].

Klein focused more on discrete transformation groups and published some books in non-Euclidean geometry [48]. On the other hand Lie devoted his entire life to the theory of continuous groups (Lie groups) [48]. Lie introduced the theory of Lie groups and Lie algebras with "rare completeness and thoroughness" [48]. Lie gave the relationship between Lie groups and Lie algebras; defined solvable Lie algebras corresponding to so-called solvable Lie groups and also assigned these continuous groups to differential equations. It appeared that "those and only those equations which correspond to solvable continuous groups have solutions in quadratures" [48]. The Lie algorithm can be applied to almost any system of ordinary differential equations and partial differential equations.

Lie's theory of differential equations was highly appreciated by many researchers and was extremely popular. It was introduced as one of the university courses in many institutions [48]. In the 1940s, the first computers were introduced to solve differential equations and Lie's theory was then considered old-fashioned to young researchers. It was almost completely forgotten [48]. However, in the late 1960s and early 1970s, physicists and mathematicians suddenly remembered that the Lie group of a differential equation does not only classify the solvability or unsolvability of a given problem, but it also determined and described the symmetries of a given equation [16, 48]. The field was again active and the Lie's ideas were again valued.

Lie's ideas were extend tremendously by Ovsiannikov [42] and his students in the Soviet Union and Garret Birkhoff [13] in the United States of America. New ideas and new applications were being developed by many mathematicians and physicists (including Ibragimov [35] in the Soviet Union, Bluman and Cole [14] at Caltech, and Anderson *et al* [9] at the University of the Pacific) [16]. Today we observe a sudden and great increase of interest in the Lie theory of differential equations and many publications [8, 10, 12, 15, 30, 32, 34, 40, 44, 45].

1.2 Thesis outline

In chapter 2 we will introduce the concept of Lie symmetry analysis. We will provide some of the important algorithms to obtain and analyse symmetries of differential equations. We will also demonstrate these algorithms by considering examples. The reduction of order and group invariant of solutions are also discussed and their algorithms are given in chapter 2. In chapter 3 we discuss the concept of hidden symmetries applied to ordinary differential equations. We also explain where they originate. In chapter 4 we present similar results, but for partial differential equations. We summarise our work in chapter 5.

Chapter 2

Lie Symmetry Analysis

2.1 Ordinary Differential Equations

An ordinary differential equation (ODE) is an equation which has only one independent variable and one or more derivatives with respect to that variable [36]. In this section we will introduce Lie's techniques to determine and utilize the symmetries of ODEs.

2.1.1 One–Parameter Group of Transformations

Let us consider the following transformation

$$x_1 = f(x, t, \varepsilon), \qquad y_1 = g(x, y, \varepsilon) \tag{2.1.1}$$

on the (x, y) plane. By transformation we mean the mapping which depends on one or more parameters [32].

Definition 2.1.1. *The transformation (2.1.1) is called a one parameter group of transformations if and only if the properties below hold* [15]*:*

1. *Identity property: When* $\varepsilon = 0$,

$$x = f(x, y, 0), \qquad y = g(x, y, 0). \tag{2.1.2}$$

2. *Inverse property: When* $\varepsilon = -\varepsilon$

$$x = f(x_1, y_1, -\varepsilon), \qquad y = g(x, y, -\varepsilon). \tag{2.1.3}$$

3. *Closure property: If*

$$x_2 = f(x_1, y_1, \delta), \qquad y_2 = g(x_1, y_2, \delta), \tag{2.1.4}$$

the product of the two transformations is also a member of the transformations (2.1.1) and is characterised by $\varepsilon + \delta$,

$$x_2 = f(x, y, \varepsilon + \delta), \qquad y_2 = g(x, y, \varepsilon + \delta). \tag{2.1.5}$$

The functions $f(x, t, \varepsilon)$ and $g(x, y, \varepsilon)$ are collectively called the global form of the group [15].

2.1.2 Infinitesimal Generators

The infinitesimal transformations of function x_1 and y_1 can be estimated by Taylor serious expansions for small ε (about $\varepsilon = 0$) of (2.1.1), as

$$x_1 = x + \varepsilon \left(\frac{dx_1}{d\varepsilon} \right)_{\varepsilon=0} + \mathrm{O}(\varepsilon), \tag{2.1.6}$$

$$y_1 = y + \varepsilon \left(\frac{dy_1}{\varepsilon} \right)_{\varepsilon=0} + \mathrm{O}(\varepsilon). \tag{2.1.7}$$

Now, if we introduce new functions $\eta\,(x, y)$ and $\xi\,(x, y)$ as

$$\left(\frac{dx_1}{d\varepsilon} \right)_{\varepsilon=0} = \xi\,(x, y)\,, \tag{2.1.8}$$

$$\left(\frac{dy_1}{d\varepsilon}\right)_{\varepsilon=0} = \eta\left(x, y\right), \tag{2.1.9}$$

then we obtain

$$\begin{aligned} x_1 &= x + \varepsilon\xi\left(x, y\right), & (2.1.10) \\ y_1 &= y + \varepsilon\eta\left(x, y\right). & (2.1.11) \end{aligned}$$

In general (2.1.10)–(2.1.11) is an approximation since we neglected higher order terms. The above equations are called the infinitesimal form of the group. Note that when we know the infinitesimal form of the group, we can integrate it to obtain the global form. In other words, we solve the system (2.1.10)–(2.1.11) subject to the initial conditions

$$x_1 = x, \qquad y_1 = y \tag{2.1.12}$$

when $\varepsilon = 0$.

If we define the operator G as

$$G = \xi\left(x, y\right)\partial_x + \eta\left(x, y\right)\partial_y, \tag{2.1.13}$$

where $\partial_x = \frac{\partial}{\partial x}$ we can then re-write (2.1.10)–(2.1.11) as [32]

$$\begin{aligned} x_1 &= (1 + \varepsilon G)x, & (2.1.14) \\ y_1 &= (1 + \varepsilon G)y. & (2.1.15) \end{aligned}$$

2.1.3 The Extended Generator

So far we have been dealing with transformations of the variables only. Since we are dealing with differential equations, one needs to consider the transformations of the derivatives too. To accommodate derivatives, we need to consider the prolongation or extension of G. The first and second extensions of G are given by (from now on $\eta(x,y)$ and $\xi(x,y)$ will be denoted by η and ξ) [?]

$$G^{[1]} = G + (\eta' - y'\xi')\,\partial_{y'}, \tag{2.1.16}$$

$$G^{[2]} = G^{[1]} + (\eta'' - 2y''\xi' - y'\xi'')\,\partial_{y''}. \tag{2.1.17}$$

In general the nth extension is given by [?]

$$G^{[n]} = G^{[n-1]} + \eta_n \partial y^{(n)} \tag{2.1.18}$$

$$= G + \sum_{i=1}^{n} \left\{ \eta^{(i)} - \sum_{j=1}^{i-1} \begin{pmatrix} i \\ j \end{pmatrix} y^{j+1}\xi^{i-j} \right\} \partial y^{(i)}. \tag{2.1.19}$$

Since ξ and η are functions of x and y we note that in the case of the first derivative we have

$$\xi' = \xi_x + y'\partial\xi_y \tag{2.1.20}$$

and

$$\xi'' = \partial\xi_{xx} + 2y'\partial\xi_{xy} + y'^2\partial\xi_{xy} + y''\partial\xi_y \tag{2.1.21}$$

in the case of second derivative, and so on. We have similar expressions for the derivatives of η.

We say that G is a symmetry of the differential equation

$$E(x, y, y', \ldots, y^{(n)}) = 0, \tag{2.1.22}$$

if and only if

$$G^{[n]}E\big|_{E=0} = 0. \tag{2.1.23}$$

This means that the action of the nth prolongation of G on E is zero when the original equation is satisfied.

Let us consider [16]

$$y'' = \frac{1}{y^3}. \tag{2.1.24}$$

We require

$$G^{[2]}E = 0,$$

where $E = y'' - \frac{1}{y^3}$ and $G^{[2]}$ is given by (2.1.17). Then

$$\left(G + (\eta' - y'\xi')\,\partial_{y'} + (\eta'' - 2y''\xi' - y'\xi'')\,\partial_{y''}\right)\left(y'' - \frac{1}{y^3}\right)\bigg|_{E=0} = 0, \quad (2.1.25)$$

produces

$$\left[\frac{3\eta}{y^4} + (\eta'' - 2y''\xi' - y'\xi'')\right]\bigg|_{y''=\frac{1}{y^3}} = 0 \quad (2.1.26)$$

from which

$$\begin{aligned}\eta_{xx} + 2y'\eta_{xy} + y'^2\eta_{yy} + \frac{1}{y^3}\eta_y - \frac{2}{y^3}(\xi_x + y'\xi_y)& \\ - y'\left(\xi_{xx} + 2y'\xi_{xy} + y'^2\xi_{yy} + \frac{1}{y^3}\xi_y\right) &= -\frac{3\eta}{y^4}. \quad (2.1.27)\end{aligned}$$

Note that ξ and η do not depend on y'. As a result we can equate all the coefficients of functions of y' to zero and obtain linear PDEs of the form

$$\begin{aligned}\xi_{yy} &= 0 \\ \eta_{yy} - 2\xi_{xy} &= 0 \\ 2\eta_{xy} - \frac{3}{y^3}\xi_y - \xi_{xx} &= 0 \\ \eta_{xx} + \frac{1}{y^3}\eta_y - \frac{2}{y^3}\xi_x = \frac{3\eta}{y^4} &= 0.\end{aligned}$$

Solving these PDEs we obtain

$$\xi = a_0 + a_1 x + a_2 x^2, \quad (2.1.28)$$

$$\eta = \left(\tfrac{1}{2}a_1 + a_2 x\right) y. \quad (2.1.29)$$

By setting $a_0 = 1$ and $a_1 = a_2 = 0$, we obtain

$$G_1 = \partial_x, \quad (2.1.30)$$

setting $a_1 = 1$ and $a_0 = a_2 = 0$, we obtain

$$G_2 = x\partial_x + \tfrac{1}{2}y\partial_y \tag{2.1.31}$$

and finally, we set $a_2 = 1$ and $a_0 = a_1 = 0$, to obtain

$$G_3 = x^2\partial_x + xy\partial_y. \tag{2.1.32}$$

G_1, G_2 and G_3 are the Lie point symmetries of (2.1.24).

2.1.4 Lie Algebra

A Lie algebra is one of the most important concepts to consider in the study of symmetry analysis of differential equations.

Definition 2.1.2. *A Lie algebra $\mathcal{L}$ is a vector space over some field $\mathcal{F}$ with additional law of combination of elements $\mathcal{L}$, satisfying the following axioms* [40]*:*

1. *Bilinearity*

$$[c_1G_1 + c_2G_2, G_3] = c_1[G_1, G_3] + c_2[G_2, G_3],$$

$$[G_1, c_1G_2 + c_2G_3] = c_1[G_1, G_2] + c_2[G_1, G_3],$$

where c_1 and c_2 are constant.

2. *Skew-symmetry (Anti-commutative)*

$$[G_1, G_2] = -[G_2, G_1]. \tag{2.1.33}$$

3. *Jacobi Identity*

$$[G_1, [G_2, G_3]] + [G_3, [G_1, G_2]] + [G_2, [G_3, G_1]] = 0 \tag{2.1.34}$$

for all G_1, G_2, G_3 in $\mathcal{L}$.

Let us define the product associated with the Lie algebra as that of commutation:

$$[G_\alpha, G_\beta] = G_\alpha G_\beta - G_\beta G_\alpha, \tag{2.1.35}$$

where α, $\beta = 1, 2, \ldots$ [15]. If a differential equation admit the operators G_α and G_β, it also admits their commutator $[G_\alpha, G_\beta]$ [15]. Lie's main result is the proof that it is always possible to assign to a continuous Lie group a corresponding Lie algebra and vice versa. Thus, for the real special linear group $SL(n, R)$ the corresponding Lie algebra is $sl(n, R)$ and for $SO(n, R)$ it is $so(n, R)$ [32]. It follows that

$$[G_\alpha, G_\beta] = C^\gamma_{\alpha\beta} G_\gamma, \tag{2.1.36}$$

where $\alpha, \beta, \gamma = 1, 2 \ldots$ and the coefficients $C^\gamma_{\alpha\beta}$ are constants called structure constants [32].

Definition 2.1.3. *A subspace $\mathcal{J}$ is called a subalgebra of the Lie algebra $\mathcal{L}$ if, for any $G_\alpha, G_\beta \in \mathcal{J}$, $[G_\alpha, G_\beta] \in \mathcal{J}$* [32].

Definition 2.1.4. *The Lie algebra $\mathcal{L}$ is called an Abelian algebra if, for any G_α,$G_\beta \in \mathcal{L}$* [32],

$$[G_\alpha, G_\beta] = 0. \tag{2.1.37}$$

Definition 2.1.5. *An r-dimensional Lie algebra is called a solvable algebra if there is a chain of subalgebras*

$$\{0\} = \mathcal{L}_0 \subset \mathcal{L}_1 \subset \cdots \subset \mathcal{L}_R = \mathcal{L}, \tag{2.1.38}$$

where $dim(\mathcal{L}_k) = k$, such that $\mathcal{L}_{k-1}$ is an ideal of $(\mathcal{L}_k)$ for each k [32].

Note that any Abelian algebra is solvable.

Let us demonstrate the idea of Lie brackets by considering the Lie point symmetries of (2.1.24).

$$
\begin{aligned}
\big[G_1, G_2\big] &= (\partial_x)\Big(x\partial_x + \tfrac{1}{2}y\partial_y\Big) - \Big(x\partial_x + \tfrac{1}{2}y\partial_y\Big)(\partial_x) \\
&= \partial_x - 0 = G_1, \\
\big[G_1, G_3\big] &= (\partial_x)\Big(x^2\partial_x + xy\partial_y\Big) - \Big(x^2\partial_x + xy\partial_y\Big)(\partial_x) \\
&= 2x\partial_x + y\partial_y - 0 = 2G_2, \\
\big[G_2, G_3\big] &= \Big(x\partial_x + \tfrac{1}{2}y\partial_y\Big)\Big(x^2\partial_x + xy\partial_y\Big) - \Big(x^2\partial_x + xy\partial_y\Big)\Big(x\partial_x + \tfrac{1}{2}y\partial_y\Big) \\
&= 2x^2\partial_x + xy\partial_y + \tfrac{1}{2}\partial_y - x^2\partial_x - \tfrac{1}{2}\partial_y \\
&= x^2\partial_x + xy\partial_y = G_3.
\end{aligned}
$$

We use the Lie bracket relation of these symmetries in a systematic manner to choose the one symmetry that will be used for the reduction.

2.1.5 Reduction of Order

Reduction of order is another important concept that we need to consider in our analysis of Lie symmetry. In general one can reduce the nth order ODE with $r \leq n$ Lie point symmetries to an ODE of order $n - r$ or an algebraic equation (if $r = n$). Let us consider the nth order ODE that can be written in a solved form as

$$
E(x, y, y', \ldots, y^{(n)}) = y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}), n \geq 2, \tag{2.1.39}
$$

where $y^{(k)} = \frac{d^k y}{dx^k}$, $k = 1, 2, \ldots, n$. If the ODE (2.1.39) is invariant under symmetry G, it follows from (2.1.23) that E must satisfies the following system of equations:

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dy'}{\eta' - y'\xi'} = \cdots = \frac{dy^{(n)}}{\eta_n}. \tag{2.1.40}
$$

These equations are called the associated Lagrange's system of G or the characteristic system [15]. By integrating the first equation of (2.1.40), we obtain a solution of the form

$$u(x,y) = r(x,y) = \text{ constant } = C_1. \tag{2.1.41}$$

The solution $u(x,y)$ is called the group invariant and solving the remaining equations respectively gives: the first differential invariant $v_1(x,y,y')$, the second differential invariant $v_2(x,y,y',y'')$, and so on. It is obvious that E must be a function of these invariants in order for it to admit (2.1.13) [15]. We can show that, in terms of the invariants $u(x,y)$ and $v_1(x,y,y') = v(x,y,y')$, the original equation E can be written as follows

$$N(u,v,v',\ldots,v^{(n-1)}) = v^{(n-1)} - g(u,v,v',\ldots,v^{(n-2)}),$$

where $'$s denote differentiation with respect to u [15]. Thus the variables for the reduction of order are obtained by requiring

$$G^{[1]}z = 0, \tag{2.1.42}$$

where $z = z(x,y,y')$ is an arbitary function of its arguments.

If the Lie bracket relation between two symmetries is

$$[G_1, G_2] = \lambda G_1 \tag{2.1.43}$$

with $\lambda = 0$ or a constant usually scaled to 1 [**?**], then reduction via G_1 results in $G_2^{[1]}$ being a point symmetry of the reduced equation. Reduction via G_2 will result in $G_1^{[1]}$ being an non-local symmetry of the reduced equation [**?**]. We refer the reader to [15] for further reading and more information about reduction of order.

Let us present the method of reduction of order by reducing the order of (2.1.24) using the Lie symmetry (2.1.32). In case of (2.1.32), the associated

Lagrange's system is

$$\frac{dx}{x^2} = \frac{dy}{xy} = \frac{dy'}{y - xy'}. \tag{2.1.44}$$

Taking the first and second equations into account we obtain

$$u = \frac{x}{y} \Rightarrow x = uy. \tag{2.1.45}$$

Taking the second and third equations into account, we have

$$\begin{aligned} \frac{dy}{xy} &= \frac{dy'}{y - xy'} \\ \frac{dy}{y} &= \frac{udy'}{1 - uy'} \end{aligned} \tag{2.1.46}$$

and integrating the last equation we obtain

$$v = y - xy'. \tag{2.1.47}$$

Taking the derivative of v and substituting from the original equation will give us the reduced equation. We can do this as follows

$$\begin{aligned} v' &= \frac{dv}{du} \\ &= \frac{dv}{dx}\Big/\frac{du}{dx} \\ &= \frac{-xy''}{\frac{y-xy'}{y^2}} \\ &= -\frac{x}{y}\frac{1}{y - xy'} \\ &= -\frac{u}{v}. \end{aligned} \tag{2.1.48}$$

We re-write (2.1.24) in terms of the new variables as

$$vv' = -u \tag{2.1.49}$$

and transform the symmetry G_2 as follows

$$\begin{aligned}
G_2^{[1]} &= x\partial_x + \frac{1}{2}y\partial_y - \frac{1}{2}y'\partial_y' \\
&= x\frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{1}{2}y\frac{\partial u}{\partial y}\frac{\partial}{\partial u} - \frac{1}{2}y'\frac{\partial u}{\partial y'}\frac{\partial}{\partial u} + x\frac{\partial v}{\partial x}\frac{\partial}{\partial v} + \frac{1}{2}y\frac{\partial v}{\partial y}\frac{\partial}{\partial v} - \frac{1}{2}y\frac{\partial v}{\partial y'}\frac{\partial}{\partial v} \\
&= \frac{1}{2}u\partial_u + \frac{1}{2}v\partial_v. \qquad (2.1.50)
\end{aligned}$$

Similarly for G_1 we obtain

$$G_1^{[1]} = \frac{1}{y}\partial_u + y'\partial_v. \qquad (2.1.51)$$

Note that G_2 is a point symmetry of the reduced equation as it can be written in terms of the new variables. G_1 can not be written solely in terms of the new variables and becomes a nonlocal symmetry.

2.2 Partial Differential Equations

In this section we will utilize the infinitesimal transformation that we already introduced in constructing solutions of PDEs, in a similar fashion to ODEs. The infinitesimal invariance for a system of PDEs is derived in the same manner as for ODEs. The invariant surfaces of the corresponding Lie group of point transformations lead to the invariance of the solution [40]. Note that in ODEs when we mention reduction of order, we meant the decreasing of order of a given ODE to an lower order ODE. However, the reduction of a PDE simply means decreasing the number of variables of the original PDE [40].

2.2.1 A Scalar PDE

Consider a scalar PDE given by

$$F(x, u, u^1, \dots, u^{(k)}) = 0, \tag{2.2.52}$$

where $x = (x_1, x_2, \dots, x_n)$ represents n independent variables, u denotes the co-ordinate corresponding to the dependent variable, u^j denotes the set of coordinates corresponding to j^{th} order partial derivatives of u with respect to x_j and the coordinates of u^j corresponding to $\frac{\partial^i u}{\partial x_{i_1} \partial x_{i_2} \dots \partial x_{i_j}}$ are denoted by $u_{i_1 i_2 \dots i_j}$, $i_j = 1, 2, \dots, n$ for $j = 1, 2, \dots, k$ [15]. A one-parameter Lie group of transformation is given by

$$\bar{x} = X(x, u; \varepsilon), \tag{2.2.53}$$

$$\bar{u} = U(x, u; \varepsilon), \tag{2.2.54}$$

where x and u are defined above.

Definition 2.2.1. *The group (2.2.53)–(2.2.54) leaves PDE (2.2.52) invariant if and only if its kth extension in $(x, u', \dots, u^{(k)})$-space, leaves the surface (2.2.52) invariant* [15].

A solution $u = \Theta(x)$ satisfies (2.2.52) with [15]

$$u_{i_1 i_2 \dots i_j} = \frac{\partial^i \Theta(x)}{\partial x_{i_1} \partial x_{i_2} \dots \partial x_{i_j}}, i_j = 1, 2, \dots, n \text{ for } j = 1, 2, \dots, k. \tag{2.2.55}$$

2.2.2 Extended Transformations of One Dependent and n independent variables

Let us consider the one-parameter Lie group of transformations

$$\bar{x}_i = x_i + \varepsilon \xi_i (x, u) + \mathrm{O}(\xi^2), \tag{2.2.56}$$

$$\bar{u} = u + \varepsilon \xi_i (x, u) + \mathrm{O}(\xi^2), \tag{2.2.57}$$

with the generator [15]

$$G = \xi_i (x, u) \partial_{x_i} + \eta (x, u) \partial_u. \tag{2.2.58}$$

The corresponding general extension is given by [15]

$$\begin{aligned} G^{[n]} &= \xi_i (x, u) \partial_{x_i} + \eta (x, u) \partial_u + \eta_i^{(1)} (x, u, u_1) \partial u_i + \cdots \\ &\quad + \eta^{(n)}_{i_1 i_2 \ldots i_n} (x, u, u_1, \ldots, u_n) \partial u_{i_1 i_2 \ldots i_n} \end{aligned} \tag{2.2.59}$$

with $n = 1, 2, \ldots$. The explicit formulae for the function $\eta^{(n)}$ is given by

$$\eta_i^{(1)} = D_i \eta - (D_i \xi_i) u_j, \qquad i = 1, 2, \ldots, n, \tag{2.2.60}$$

$$\eta^{(n)}_{i_1 i_2 \ldots i_n} = D_{i_n} \eta^{(n-1)}_{i_1 i_2 \ldots i_{n-1}} - (D_{i_n} \xi_i) u_{i_1 i_2 \ldots i_{n-1} j}, \tag{2.2.61}$$

where $i_l = 1, 2, \ldots, n$ for $l = 1, 2, \ldots, n$ with $n = 2, 3, \ldots$ and D_i is a total derivative [15] defined by

$$D_i = \frac{D}{D_{x_i}} = \partial x_i + u_i \partial_u + u_{ij} \partial u_j + \cdots + u_{i i_1 i_2 \ldots i_n} \partial u_{i_1 i_2 \ldots i_n} + \cdots .$$

In the case of one dependent variable u and two independent variables x_1 and x_2 for the extended one-parameter Lie group of transformations given by $\bar{x}_i$ and $\bar{u}_i$, we have

$$\begin{aligned} \bar{x}_i &= x_i + \varepsilon \xi_i (x_1, x_2, u) + 0(\varepsilon^2), \qquad i = 1, 2, \\ \bar{u} &= u + \varepsilon \eta (x_1, x_2, u) + 0(\varepsilon^2), \\ \bar{u}_i &= u_i + \varepsilon \eta^{(i)}_{ij} (x_i, x_2, u, u_1, u_2) + 0(\varepsilon^2). \end{aligned} \tag{2.2.62}$$

Then $\eta_{ij}^{(i)}$ is define by

$$
\begin{aligned}
\eta_1^{(1)} &= \frac{\partial\eta}{\partial x_1} + \left[\frac{\partial\eta}{\partial u} - \frac{\partial\xi_1}{\partial x_1}\right]u_1 - \frac{\partial\xi_2}{\partial x_1}u_2 - \frac{\partial\xi_1}{\partial u}(u_1)^2 \\
&\quad - \frac{\partial\xi_2}{\partial u}u_1u_2, \qquad (2.2.63)\\
\eta_2^{(1)} &= \frac{\partial\eta}{\partial x_2} + \left[\frac{\partial\eta}{\partial u} - \frac{\partial\xi_2}{\partial x_2}\right]u_2 - \frac{\partial\xi_1}{\partial x_1}u_1 - \frac{\partial\xi_2}{\partial u}(u_2)^2 \\
&\quad - \frac{\partial\xi_1}{\partial u}u_1u_2, \qquad (2.2.64)\\
\eta_{11}^{(2)} &= \frac{\partial^2\eta}{\partial x_1^2} + \left[2\frac{\partial^2\eta}{\partial x_1\partial u} - \frac{\partial^2\xi_1}{\partial x_1^2}\right]u_1 - \frac{\partial^2\xi_2}{\partial x_1^2}u_2 + \left[\frac{\partial\eta}{\partial u} - 2\frac{\partial\xi_1}{\partial x_1}\right]u_{11} \\
&\quad - 2\frac{\partial\xi_2}{\partial x_1}u_{12} + \left[\frac{\partial^2\eta}{\partial u^2} - 2\frac{\partial^2\xi_1}{\partial x_1\partial u}\right](u_1)^2 - 2\frac{\partial\xi_1}{\partial u}(u_1)^2 \\
&\quad - 2\frac{\partial^2\xi_2}{\partial x_1\partial u}u_1u_2 - \frac{\partial^2\xi_1}{\partial u^2}(u_2)^3 - \frac{\partial^2\xi_2}{\partial u^2}(u_1)^2u_2 - 3\frac{\partial\xi_1}{\partial u}u_1u_{11} \\
&\quad - \frac{\partial\xi_2}{\partial u}u_2u_{11} - 2\frac{\partial\xi_2}{\partial u}u_1u_{12}, \qquad (2.2.65)\\
\eta_{12}^{(2)} &= \frac{\partial^2\eta}{\partial x_1\partial x_2} + \left[\frac{\partial^2\eta}{\partial x_1\partial u} - \frac{\partial^2\xi_2}{\partial x_1\partial x_2}\right]u_2 + - \frac{\partial\xi_2}{\partial x_1}u_{22} \\
&\quad + \left[\frac{\partial^2\eta}{\partial x_2\partial u} - \frac{\partial^2\xi_2}{\partial x_1\partial x_2}\right]u_1 + \left[\frac{\partial\eta}{\partial u} - \frac{\partial\xi_1}{\partial x_1}\right]u_{11} \\
&\quad + \left[\frac{\partial\eta}{\partial u} - \frac{\partial\xi_1}{\partial x_1} - \frac{\partial\xi_2}{\partial x_2}\right]u_{12} - \frac{\partial^2 l\xi_2}{\partial x_1\partial u}(u_2)^2 \\
&\quad + \left[\frac{\partial^2\eta}{\partial u^2} - \frac{\partial^2\xi_1}{\partial x_1}\partial u - \frac{\partial^2\xi_2}{\partial x_2\partial u}\right]u_1u_2 - \frac{\partial\xi_1}{\partial x_2}u_{11} - \frac{\partial^2\xi_1}{\partial x_2\partial u}(u_1)^2 \\
&\quad - \frac{\partial^2\xi_2}{\partial^2 u}u_1(u_2)^2 - \frac{\partial^2\xi_1}{\partial^2 u}(u_1)^2u_2 - 2\frac{\partial\xi_2}{\partial u}u_2u_{12} - \frac{\partial\xi_1}{\partial u}u_1u_{12} \\
&\quad - \frac{\partial\xi_1}{\partial u}u_2u_{11} - \frac{\partial\xi_2}{\partial u}u_1u_{22}, \qquad (2.2.66)\\
\eta_{22}^{(2)} &= \frac{\partial^2\eta}{\partial x_2^2} + \left[2\frac{\partial^2\eta}{\partial x_2\partial u} - \frac{\partial^2\xi_2}{\partial x_2^2}\right]u_2 - \frac{\partial^2\xi_1}{\partial x_2^2}u_1 + \left[\frac{\partial\eta}{\partial u} - 2\frac{\partial\xi_2}{\partial x_2}\right]u_{22}
\end{aligned}
$$

$$- 2\frac{\partial \xi_1}{\partial x_2}u_{12} + \left[\frac{\partial^2 \eta}{\partial u^2} - 2\frac{\partial^2 \xi_2}{\partial x_2 \partial u}\right](u_2)^2 - 2\frac{\partial^2 \xi_1}{\partial x_2 \partial u}u_1 u_2 - \frac{\partial^2 \xi_2}{\partial u^2}(u_2)^3$$
$$- \frac{\partial^2 \xi_1}{\partial^2 u}u_1(u_2)^2 - 3\frac{\partial \xi_2}{\partial u}u_2 u_{22} - \frac{\partial \xi_1}{\partial u}u_2 u_{22} - 2\frac{\partial \xi_1}{\partial u}u_2 u_{12}. \quad (2.2.67)$$

Using the same method, we can further extend the generators for different values of i [15].

Theorem 2.2.1. *Let*

$$G = \xi_i\,(x,u)\,\partial_{x_i} + \eta\,(x,u)\,\partial_u \quad (2.2.68)$$

be the infinitesimal generator of (2.2.53)–(2.2.54). Let

$$\begin{aligned} G^{[k]} &= \xi_i\,(x,u)\,\partial_{x_i} + \eta\,(x,u)\,\partial_u + \eta_i^{(1)}\left(x,u,u^1\right)\partial u_i \\ &\quad + \cdots + \eta^{(k)}_{i_1 i_2 \ldots i_k}\left(x,u,u^1,\ldots,u^{(k)}\right)\partial u_{i_1 i_2 \ldots i_k} \end{aligned} \quad (2.2.69)$$

be the kth extended infinitesimal generator of (2.2.68) where $\eta_i^{(i)}$ are given by (2.2.60) and $\eta^{(k)}_{i_1 i_2 \ldots i_k}$ are given by (2.2.61), $i_j = 1,2,\ldots,n$ for $j = 1,2,\ldots,k$, in terms of $(\xi(x,u),\eta(x,u))$ and $\xi(x,u)$ denote $(\xi_1(x,u),\xi_2(x,u),\ldots,\xi_k(x,u))$. Then (2.2.53)–(2.2.54) is admitted by PDE (2.2.52) if and only if [15]

$$G^{[k]}F(x,u,u^1,\ldots,u^{(k)}) = 0 \quad (2.2.70)$$

where

$$F(x,u,u^1,\ldots,u^{(k)}) = 0.$$

Consider the heat equation [15]

$$\frac{\partial^2 u}{\partial x_1^2} = \frac{\partial u}{\partial x_2} \quad (2.2.71)$$

or, equivalently

$$u_{11} - u_2 = F\left(x_1,x_2,u,u_1,u_2,u_{12},u_{11},u_{22}\right). \quad (2.2.72)$$

Equation (2.2.72) admits an infinitesimal generator of the form

$$G = \xi^1(x_1, x_2, u)\partial_{x_1} + \xi^2(x_1, x_2, u)\partial_{x_2} + \eta(x_1, x_2, u)\partial_u. \tag{2.2.73}$$

According to the Theorem 2.2.1, we have to consider the second extension of G:

$$G^{[2]} = G + \eta_1^{(1)}\partial_{u_1} + \eta_1^{(1)}\partial_{u_2} + \eta_{12}^{(2)}\partial_{u_{12}} + \eta_{11}^{(2)}\partial_{u_{11}} + \eta_{22}^{(2)}\partial_{u_{22}}. \tag{2.2.74}$$

The satisfaction of the invariant condition $G^{[2]}F = 0$ yields

$$\begin{aligned} \xi^1 F_{x_1} + \xi^2 F_{x_2} + \eta_1^{(1)} F_{u_1} + \eta_2^{(1)} F_{u_2} & \\ + \eta_{12}^{(2)} F_{u_{12}} + \eta_{11}^{(2)} F u_{11} + \eta_{22}^{(2)} F_{u_{22}} &= 0 \end{aligned} \tag{2.2.75}$$

in general. In the case of the heat equation this expression reduces to

$$\eta_2^{(1)} - \eta_{11}^{(2)} = 0. \tag{2.2.76}$$

Substituting from (2.2.64)–(2.2.67) we obtain

$$\begin{aligned} \eta_2^{(1)} - \eta_{11}^{(2)} &= \eta_{x_2} + u_2\eta_u - u_1\xi^1_{x_2} - u_2^2\xi^1_u - u_2\xi^2_{x_2} + u_1u_2\xi^1_{x_1} \\ & - \eta_{x_1x_1} - 2u_1\eta_{x_1u} + u_1\xi^2_{x_1x_1} + u_2\xi^2_{x_1x_1} + u_{11}\eta_2 \\ & + 2u_{11}\xi^1_{x_1} + 2u_{12}\xi^2_{x_1} - u_1^2\eta_{uu} + 2u_1^2\xi^1_{x_1u} + 2u_1^2u_2\xi^1_{x_1u} \\ & + u_2^3\xi_{uu} + 2u_1^2u_2\xi_{uu} + 3u_1u_{11}\xi^1_u + u_2u_{11}\xi^2_u + 2u_1u_{12}\xi^2_u \\ &= 0. \end{aligned} \tag{2.2.77}$$

We replace u_{11} by u_2 in (2.2.77) to obtain

$$\begin{aligned} \eta_2^{(1)} - \eta_{11}^{(2)} &= (\eta_{x_2} - \eta_{x_1x_1}) + 2u_1u_{12}\left(\xi^2_u\right) + 2u_1u_2\left(\xi^1_u + \xi^2_{xu}\right) \\ & + u_1^2\left(2\xi^1_{xu} - \eta_{uu}\right) + u_1^2u_2\left(\xi^2_{uu}\right) + u_2^3\left(\xi^1_{uu}\right) \\ & + 2u_{12}\left(\xi^2_{x_1}\right) + u_2\left(\xi_{x_1x_1} + 2\xi_{x_1} - \xi^2_{x_2}\right) + u_2^2\left(\xi^2_u - \xi^2_u\right) \\ & + u_1\left(\xi^1_{x_1x_1} - \xi^1_{x_2} - 2\eta^2_{x_1x_2}\right) \\ &= 0. \end{aligned} \tag{2.2.78}$$

Equation (2.2.78) must be an identity for all derivatives of (u_1, u_1, u_{12}) as η, ξ^1 and ξ^2 only depend on (x_1, x_2, u). We thus obtain the reduced determining equations

$$
\begin{aligned}
2\xi^2_u &= 0 && (2.2.79)\\
\xi^1_u + \xi^2_{x_1 u} &= 0 && (2.2.80)\\
2\xi^1_{x_1 u} - \eta_{uu} &= 0 && (2.2.81)\\
\xi^2_{uu} &= 0 && (2.2.82)\\
\xi^1_{uu} &= 0 && (2.2.83)\\
2\xi^2_{x_1} &= 0 && (2.2.84)\\
\xi^2_{x_1 x_1} + 2\xi^1_{x_1} - \xi^2_{x_2} &= 0 && (2.2.85)\\
\xi^1_{x_1 x_1} - \xi^1_{x_2} - 2\eta_{x_1 u} &= 0 && (2.2.86)\\
\eta_{x_2} - \eta_{x_1 x_1} &= 0. && (2.2.87)
\end{aligned}
$$

From (2.2.79) and (2.2.84) we observe that ξ^2 is a function of x_2

$$\xi^2(x_2) = a(x_2). \tag{2.2.88}$$

Equation (2.2.80) requires ξ^1 to be a function of x_1 and x_2 and substituting by ξ^2 into (2.2.85) we obtain

$$\xi^1(x_1, x_2) = \tfrac{1}{2} a_{x_2} x_1 - b(x_2). \tag{2.2.89}$$

Substituting (2.2.89) into (2.2.81) yields

$$\eta(x_1, x_2, u) = c(x_1, x_2)u + d(x_1, x_2). \tag{2.2.90}$$

Equation (2.2.86) requires

$$c_{x_1} = -\tfrac{1}{4} a_{x_2 x_2} x_1 - \tfrac{1}{2} b_{x_2}$$

solving this we obtain

$$c(x_1, x_2) = -\tfrac{1}{8} a_{x_2 x_2} x_1^2 - \tfrac{1}{2} b_{x_2} x_1 + e(x_2). \tag{2.2.91}$$

Taking equation (2.2.87) into account, we obtain

$$\Big(c_{x_2} - c_{x_1 x_1}\Big) u - \Big(d_{x_2} - d_{x_1 x_1}\Big) = 0. \tag{2.2.92}$$

Equating the coefficients of u to zero, we obtain

$$c_{x_2} = c_{x_1 x_1} \tag{2.2.93}$$

$$d_{x_2} = d_{x_1 x_1}. \tag{2.2.94}$$

Substituting (2.2.91) into (2.2.93) we obtain

$$-\tfrac{1}{8} x_1^2 - \tfrac{1}{2} b_{x_2 x_2} x_1 + e_{x_2} = \tfrac{1}{4} a_{x_2 x_2} \tag{2.2.95}$$

and comparing the coefficients yield

$$a_{x_2 x_2 x_2} = 0, \qquad b_{x_2 x_2} = 0, \qquad e_{x_2} = -\tfrac{1}{4} a_{x_2 x_2}. \tag{2.2.96}$$

Solving the above equations, we obtain

$$\begin{aligned} a(x_2) &= \tfrac{1}{2} k_1 x_2^2 + k_2 x_2 + k_3 \\ b(x_2) &= k_4 x_2 + k_5 \\ e(x_1, x_2) &= -\tfrac{1}{4} k_1 x_2 + k_6. \end{aligned}$$

Now, we have satisfied all the reduced determining equations, then we can re-write our infinitesimals as

$$\begin{aligned} \xi^1(x_1, x_2) &= \tfrac{1}{2} \left(k_1 x_2 + k_2\right) x_1 - k_4 x_2 - k_5 \\ \xi^2(x_2) &= \tfrac{1}{2} k_1 x_2^2 + k_2 x_2 + k_3 \\ \eta(x_1, x_2, u) &= \left(-\tfrac{1}{8} k_1 x_1^2 - \tfrac{1}{2} k_4 x_1 - \tfrac{1}{4} k_1 x_2 + k_6\right) u + d(x_1, x_2) \end{aligned}$$

where $k_1, \ldots, k_6$ are arbitrary constants and $d(x_1, x_2)$ satisfies the heat equation. The heat equation (2.2.72) admits a six-parameter Lie group of transformations with the infinitesimal generators

$$\begin{aligned}
X_1 &= \partial_{x_1} && (2.2.97)\\
X_2 &= \partial_{x_2} && (2.2.98)\\
X_3 &= x_1\partial_{x_1} + 2x_2\partial_{x_2} && (2.2.99)\\
X_4 &= u\partial_u && (2.2.100)\\
X_5 &= x_1x_2\partial_{x_1} + x_2^2\partial_{x_2} - \left(\tfrac{1}{4}x_1^2 + \tfrac{1}{2}x_2\right)u\partial_u && (2.2.101)\\
X_6 &= x_2\partial x_1 - \tfrac{1}{2}x_1u\partial_u && (2.2.102)
\end{aligned}$$

and the infinite-dimensional subalgebra

$$X_d = d\left(x_1, x_2\right)\partial_u, \qquad (2.2.103)$$

where α satisfies the heat equation. The commutation relation among (2.2.97)–(2.2.102) and (2.2.103) is given in Table 2.1, where

$$\begin{aligned}
d' &= x_1d_{x_1} + 2x_2d_{x_2}\\
d'' &= x_2d_{x_1} + x_1d\\
d''' &= 4x_1x_2d_{x_1} + 4x_2^2d_{x_2} + \left(x_1^2 + 2x_2\right)d.
\end{aligned}$$

We refer the reader to [15] for more details of finding the infinitesimal generators of PDEs.

2.3 Group Invariant Solutions

Let G be a local group of transformations acting on an open set. A transformation group G is called a symmetry group of the system of PDEs if each

Table 2.1: The commutator table for Lie algebra arising from the infinitesimal generators (2.2.102)

	X_1	X_2	X_3	X_4	X_5	X_6	X_d
X_1	0	0	X_1	X_5	$-\frac{1}{2}X_6$	0	$X_{d_{x_1}}$
X_2	0	0	$2X_2$	$X_3+\frac{1}{2}X_6$	X_1	0	$X_{d_{x_2}}$
X_3	$-X_1$	$-2X_2$	0	$2X_4$	X_5	0	$-X_d$
X_4	$-X_5$	$-X_3-\frac{1}{2}X_6$	$-2X_4$	0	0	0	$X_{d'}$
X_5	$\frac{1}{2}X_6$	$-X_1$	$-X_5$	0	0	0	$X_{d''}$
X_6	0	0	0	0	0	0	$X_{d'''}$
X_d	$-X_{d_{x_1}}$	$-X_{d_{x_2}}$	X_d	$-X_{d'}$	$-X_{d''}$	$-X_{d'''}$	0

group element $g \in G$ transforms solutions of a system of PDEs to other solutions of the systems of PDEs [40]. A solution of a system of PDEs is called G-invariant if all transformations in G leaves it unchanged. We will utilise the Lie point symmetries to generate the new solutions. These solutions will be invariant under the group generated to find them and are called group invariant solutions [40].

It may be difficult if not impossible to classify these group invariant solutions, since the reduction may include any linear combination of the symmetries that are admitted. The basic idea of group invariant solutions is that the solutions which are invariant under a given r-parameter symmetry group of a system of PDEs, can all be obtained by solving a PDE with fewer independent variables than the original PDE [40].

Definition 2.3.1. *We say a solution $u = (x, t)$ is invariant under the group* *generated by*

$$X = \xi\partial_x + \tau\partial_t + \eta\partial_u \tag{2.3.104}$$

if and only if the characteristic vanishes on the solution. This means that every invariant solution satisfies the invariant surface condition [32]

$$Q \equiv \eta - \xi u_x - \tau u_t = 0. \tag{2.3.105}$$

Let us consider the heat equation to illustrate group invariant solutions [40]

$$u_t = u_{xx}. \tag{2.3.106}$$

We have already computed the symmetry group of (2.3.106) in the preceding section. Taking

$$Y = c\partial_x + \partial_t, \tag{2.3.107}$$

where c is a fixed constant, we generate a translation group of the form [40]

$$(x, t, u) \mapsto (x + c\varepsilon, t + \varepsilon, u), \qquad \varepsilon \in \Re. \tag{2.3.108}$$

From the definition 2.3.1., we deduce that the group generated by (2.3.107) has the characteristic

$$Q = -cu_x - u_t. \tag{2.3.109}$$

Equating the invariant surface condition to zero results in the global invariants

$$y = x - ct, \qquad w = u. \tag{2.3.110}$$

Treating w as a function of y (ie. $w = h(y)$), this implies that $u = h(x - ct)$. Using the chain rule to obtain the derivatives of u with respect to x and t in terms of the derivatives of w with respect to y, we obtain

$$u_t = -cw_y, \qquad u_{xx} = w_{yy}. \tag{2.3.111}$$

Substituting these expressions into (2.2.68) results in

$$-cw_y = w_{yy}. \tag{2.3.112}$$

Equation (2.3.112) is a simple ODE for travelling wave solutions to (2.3.106). We can integrate (2.3.112) to obtain the general solution

$$w(y) = \beta e^{-cy} + \lambda, \tag{2.3.113}$$

where β and λ are arbitrary constants of integration. If we consider the global invariants (2.3.110), we can then re-write (2.3.113) as

$$w(x,t) = \beta e^{-c(x-ct)} + \lambda. \tag{2.3.114}$$

Equation (2.3.114) is the most general travelling wave solution to the heat equation [40].

Chapter 3

Hidden Symmetries for ODEs

3.1 Introduction

In chapter 2 we introduced the concept of reduction of order and invariants of symmetries of ODEs. Based upon the Lie algebra of symmetries we expect a certain number of symmetries after the reduction of order. However, it has been observed that, for ODEs, after reduction, we may sometimes obtain more symmetries than we expected [2]. This also occurs when we increase the order of an ODE. These symmetries were called *hidden* symmetries and were systematically investigated by Abraham-Shrauner and Guo [2].

The symmetries that are lost after reduction of order are called Type I hidden symmetries [11]. We refer to those symmetries that are gained after reduction of order as Type II hidden symmetries [11]. In this chapter we focus on the origins of hidden symmetries for ODEs.

3.2 Non-local Symmetries

Here we explore the origin of hidden symmetries as nonlocal symmetries of the original ODE. We will also introduce a suitable approach to determine nonlocal symmetries of ODEs directly. In general, few equations admit the required number of point symmetries to allow reduction to quadratures [19]. However, these nonlocal symmetries will allow us to reduce an ODE to quadratures despite the lack of a suitable number of point symmetries [26]. Although it is not a straight forward task to calculate nonlocal symmetries directly, some ideas to determine nonlocal symmetries were presented in [26].

For now, let us consider the Emden-Fowler equation [27]

$$2yy'''' + 5y'y''' = 0. \tag{3.2.1}$$

From SYM [47], we observe that equation (3.2.1) possesses the three symmetries

$$\begin{aligned} G_1 &= \partial_x && (3.2.2)\\ G_2 &= x\partial_x && (3.2.3)\\ G_3 &= y\partial_x. && (3.2.4)\end{aligned}$$

Note that these symmetries are not sufficient to reduce (3.2.1) to a quadrature. Taking the Lie bracket of the symmetries of (3.2.1), we have

$$\begin{aligned} [G_1, G_2] &= \partial_x (x\partial_x) - x\partial_x (\partial_x)\\ &= \partial_x - 0 = G_1 \end{aligned}$$

and

$$[G_1, G_3] = \partial_x (y\partial_x) - y\partial_x (\partial_x) = 0.$$

Using G_1 we obtain the transformation

$$u = y \qquad v = y' \tag{3.2.5}$$

which we use to reduce equation (3.2.1) to

$$2u(v^2v''' + 4vv'v'' + v'^3) + 5(v^2v'' + vv'^2) = 0. \tag{3.2.6}$$

The Lie bracket relations above guarantee that (3.2.6) has two symmetries under (3.2.5): G_2 becomes

$$X_2 = v\partial v \tag{3.2.7}$$

and G_3 becomes

$$X_3 = u\partial_u + v\partial v. \tag{3.2.8}$$

Surprisingly with SYM [47] we find a third point symmetry of (3.2.6), *viz.*

$$X_4 = 2u^2\partial_u + uv\partial_v. \tag{3.2.9}$$

To investigate the origin of (3.2.9) we write it in terms of the original variables as

$$X_4 = 2y^2\partial_y + yy'\partial_{y'}.$$

The origin of X_4 must be a symmetry of the form

$$G_4 = \xi(x,y)\partial_y + 2y^2\partial_y + yy'\partial_{y'}.$$

Since

$$\eta = 2y^2$$

and

$$\eta' - y'\xi' = yy'$$

we have

$$4yy' - y'\xi' = yy' \tag{3.2.10}$$

$$\Rightarrow \qquad \frac{d\xi}{dx} = 3y'. \tag{3.2.11}$$

Solving this equation we obtain

$$\xi = 3\int y dx$$

and G_4 becomes

$$G_4 = 3\left(\int y dx\right)\partial_x + 2y^2\partial_y, \tag{3.2.12}$$

which is a nonlocal symmetry. Hence, one origin of hidden symmetries is nonlocal symmetries. More hidden symmetries can be discovered in this problem. Under the transformation

$$t = vu^{-\frac{1}{2}} \qquad w = \tfrac{1}{2}\left(v'u^{\frac{3}{2}} - \tfrac{1}{2}vu^{-\frac{1}{2}}\right), \tag{3.2.13}$$

(generated via X_4) we reduce (3.2.6) to the linear equation

$$w'' + 3w' + 2w = 0 \tag{3.2.14}$$

which can be easily solved. From SYM [47] we obtain eight point symmetries of (3.2.14)

$$W_1 = \partial_t \tag{3.2.15}$$
$$W_2 = e^{-t}\partial_w \tag{3.2.16}$$
$$W_3 = e^{-2t}\partial_w \tag{3.2.17}$$
$$W_4 = -3t\partial_t + 4t\partial_w \tag{3.2.18}$$
$$W_5 = e^{-t}\partial_t + e^t w\partial_w \tag{3.2.19}$$
$$W_6 = e^t w\partial_t - 2e^t w^2\partial_w \tag{3.2.20}$$
$$W_7 = e^{-t}\partial_t - 2e^{-t}w\partial_w \tag{3.2.21}$$
$$W_8 = e^{2t}w\partial_t - e^{2t}w^2\partial_w. \tag{3.2.22}$$

The inherited symmetries are $W_3 \leftarrow X_2$ and $W_5 \leftarrow X_3$ from (3.2.6). This equation has collected another six Type II hidden symmetries [25]. It is interesting to note that the hidden symmetry X_4 was the correct symmetry to reduce (3.2.6) further. This is an important use of hidden symmetries.

We now attempt to determine nonlocal symmetries of ODEs that reduce to hidden symmetries in a systematic manner. For the purpose of this section, we call a set of infinitesimal transformations

$$\bar{x} = x + \varepsilon\xi \tag{3.2.23}$$

$$\bar{y} = y + \varepsilon\eta \tag{3.2.24}$$

$$\bar{I} = I + \xi\gamma, \tag{3.2.25}$$

where $I = \int f(x,y)dx$, a first order one-parameter Lie group of nonlocal infinitesimal transformations [25]. This nonlocal transformation has the generator

$$G = \xi\,(x,y,I)\,\partial_x + \gamma\,(x,y,I)\,\partial_I + \eta\,(x,y,I)\,\partial_y, \tag{3.2.26}$$

where

$$\frac{\partial\gamma}{\partial y} - y\frac{\partial\xi}{\partial_y} = 0. \tag{3.2.27}$$

Equation (3.2.27) removes the possibility of derivatives in η [25]. In general, we could require ξ, γ, η to depend on x, y, y' and $I = \int f(x,y)dx$ but that is not necessary here. We can ignore η in (3.2.26) because it can be considered the first extension of

$$\bar{G} = \xi\,(x,y,I)\,\partial_x + \gamma\,(x,y,I)\,\partial_I, \tag{3.2.28}$$

with

$$\eta = \frac{d\gamma}{dx} - y\frac{d\xi}{dx}. \tag{3.2.29}$$

Now we find nonlocal symmetries of the form (3.2.26). Let us analyse the general second order equation

$$E(y,y',y'') = y'' - g(y,y') = 0, \tag{3.2.30}$$

with Lie point symmetry

$$G_1 = \partial_x, \tag{3.2.31}$$

for the existence of nonlocal symmetries [25]. If we consider the case whereby (3.2.30) possesses two point symmetries, the Lie bracket relation is given by (where λ is a constant either 0 or 1)

$$[G_1, G_2]_{LB} = \lambda G_1. \tag{3.2.32}$$

This guarantees that the reduction via G_1 results in G_2 being a point symmetry of the reduced equation. If G_1 is in the form of (3.2.31), then G_2 must be of the form [26]

$$G_2 = (\lambda x + c(y))\partial_x + a(y)\partial_y. \tag{3.2.33}$$

Using G_1 we obtain the transformation

$$u = y, \qquad v = y' \tag{3.2.34}$$

which we use to reduce equation (3.2.30) to

$$vv' = g(u, v). \tag{3.2.35}$$

Taking the first extension of (3.2.33) and (3.2.34) G_2 becomes

$$Y_2 = a(u)\partial_u + v(a'(u) - \big(\lambda + c'(u)\big)v)\partial_v. \tag{3.2.36}$$

Utilising (3.2.26), we obtain the structure of the nonlocal symmetry as

$$G_{nl} = (\lambda x + I)\partial_x + a(y)\partial_y, \tag{3.2.37}$$

where

$$I = \int c(y)dx.$$

Note that G_{nl} is an example of what is termed a useful nonlocal symmetry [26], since it becomes local in the reduced ODE.

We now determine the general form of (3.2.30) that admits (3.2.31) as a symmetry. We require

$$G_{nl}^{[2]} E|_{E=0} = 0, \tag{3.2.38}$$

in order to determine the coefficient functions in (3.2.37) ie.

$$-a\frac{\partial g}{\partial y} + ((\lambda + c)y' - y'a')\,\frac{\partial g}{\partial y'} + ((a' - 2(\lambda + c))g + y'^2(a'' - c')) = 0. \quad (3.2.39)$$

The solution of the associated Lagrange's system is reduced to that of two first order ODEs which we solve for g to obtain [25]

$$g = \exp\left(-\int \phi dy\right)\left[\int \psi \exp\left(\int \phi dy\right) + L(u)\right], \tag{3.2.40}$$

where

$$\phi = \frac{a' - 2(l + c)}{a}, \tag{3.2.41}$$

$$\psi = \frac{a(c' - a'')}{u^2}\exp\left(-2\int \frac{l + c}{a}dy\right), \tag{3.2.42}$$

$$u = \frac{a}{y'}\exp\left(-\int \frac{\lambda + c}{a}dy\right). \tag{3.2.43}$$

In this fashion, we can find nonlocal symmetries of any given ODE.

3.3 Contact Symmetries

Firstly, let us demonstrate the method of finding contact symmetries of ODEs. We have studied Lie point symmetries of the form

$$G = \xi\partial_x + \eta\partial_y.$$

It is also natural to consider the symmetries of the form

$$G = \xi(x, y, y', \ldots)\partial_x + \sum_{i=0}^{n} \eta_i\,(x, y, y', \ldots)\,\partial_{y^{(i)}}. \tag{3.3.44}$$

These symmetries are called *generalized* symmetries and the procedure to obtain them is the same as the one for point symmetries. A subset of generalised symmetries called contact symmetries, is of great use in the study of higher order ODEs. For third-order ODEs, consider the infinitesimal transformation given by group generator of the form [45]

$$G = \xi\,(x, y, y')\,\partial_x + \eta_j\,(x, y, y')\,\partial_y + \eta_j'\,(x, y, y')\,\partial_{y'} \tag{3.3.45}$$

where

$$\eta_j' = \frac{d\eta_j}{dx} - \frac{d\xi_j}{dy'}. \tag{3.3.46}$$

The conditions in (3.3.46) are replaced by the coefficient functions in terms of a generating function $\Omega_j(x, y, y')$ which is [6]

$$\frac{\partial \eta_j}{\partial y'} = \frac{\partial \xi_j}{\partial y'} y'. \tag{3.3.47}$$

If it happens that the coefficient functions ξ_j and η_j depends only on x and y, then the contact symmetry becomes a point symmetry. Contact symmetries that are not identical to point symmetries are those that have at least one of ξ_j or η_j depending on y' as an intrinsic contact symmetry [6].

One can compute contact symmetries of an ODE using a similar approach to the Lie classical method for point symmetries. The contact symmetries for a second order are difficult to compute since there are infinite number of contact symmetries where the determining equation is a single PDE. Contact symmetries of third-order ODEs are found by separating the determining equations to a set of differential equations. The systematic technique of finding contact symmetries of a third-order ODE prevents the introduction of the generating function and can be implemented via Program Lie [6, 29]. For contact symmetries of a third-order ODE of the form

$$y''' = f(x, y, y', y''), \tag{3.3.48}$$

one can let $v = y'$, implying

$$v' = y''. \tag{3.3.49}$$

Equation (3.3.48) can be written as

$$v'' = f(x, y, v, v') \tag{3.3.50}$$

where primes denotes differentiation with respect to x. The coefficient functions are found by computer programs such as Program Lie and SYM [29, 47] and they are functions of (x, y, v). The coefficient functions can then be expressed as functions of (x, y, y') for the contact symmetries of the original third-order ODE [6].

Let us consider

$$y''' = 0. \tag{3.3.51}$$

Using SYM [47] we obtain ten contact symmetries of (3.3.51) as follows

$$\begin{aligned}
G_1 &= \partial_y && (3.3.52)\\
G_2 &= x\partial_y + \partial_{y'} && (3.3.53)\\
G_3 &= x^2\partial_y + 2x\partial_{y'} && (3.3.54)\\
G_4 &= \partial_x && (3.3.55)\\
G_5 &= x\partial_x + y\partial_y && (3.3.56)\\
G_6 &= x^2\partial_x + 2xy\partial_y + 2y\partial_{y'} && (3.3.57)\\
G_7 &= y\partial_y + y'\partial_{y'} && (3.3.58)\\
G_8 &= 2y'\partial_x + y'^2\partial_y && (3.3.59)\\
G_9 &= 2\left(xy' - y\right)\partial_x + xy'^2\partial_y + y'^2\partial_{y'} && (3.3.60)\\
G_{10} &= \left(x^2y' - 2xy\right)\partial_x + \left(\tfrac{1}{2}x^2y'^2 - 2y^2\right)\partial_y \\
&\quad + \left(xy'^2 - 2yy'\right)\partial_{y'}. && (3.3.61)
\end{aligned}$$

These contact symmetries form a representation of the Lie algebra $sp(4)$. Equations (3.3.52)–(3.3.58) are all point symmetries and (3.3.59)–(3.3.61) are called intrinsic contact symmetries. Note that ten is the maximum number of contact symmetries for any third order ODE.

We can reduce (3.3.51) to quadratures using the Abelian subalgebras of G_1, G_2 and G_3 or G_8, G_9 and G_{10} in any order. Note that we can not use the symmetries of the group generators G_4, G_5 and G_6 to reduce (3.3.51) to quadratures since these group generators represent the subgroup which is not solvable [6]. Let us use the symmetry (3.3.52) with the invariants $u = x$ and $v = y'$ to reduce the order of (3.3.51), we obtain

$$\ddot{v} = 0 \tag{3.3.62}$$

where the overdot presents differentiation with respect to u. The group generators of (3.3.62) are given by

$$V_j = G_j(u)\partial_u + G_j(v)\partial_v, \qquad j = 2, 3, \cdots, 10. \tag{3.3.63}$$

We can then rewrite the contact symmetries of (3.3.51) in terms of new variables as

$$V_2 = \partial_v \tag{3.3.64}$$
$$V_3 = 2v\partial_v \tag{3.3.65}$$
$$V_4 = \partial_u \tag{3.3.66}$$
$$V_5 = u\partial_u \tag{3.3.67}$$
$$V_6 = u^2\partial_x + 2\int vdu\partial_u \tag{3.3.68}$$
$$V_7 = v\partial v \tag{3.3.69}$$
$$V_8 = v\partial_u \tag{3.3.70}$$
$$V_9 = 2\Big(uv - \int vdu\Big)\partial_u + v^2\partial_u \tag{3.3.71}$$
$$V_{10} = 7\Big(u^2v - 2u\int vdu\Big)\partial_u + \Big(u^2v - 2v\int vdu\Big)\partial_u \tag{3.3.72}$$

One can notice that in this reduction, the symmetries G_6, G_9 and G_{10} are nonlocal and symmetries G_2, G_3, G_4, G_5, G_7 and G_8 are local. Equation

(3.3.51) has seven point symmetries. One of those point symmetry is used up in the reduction of order, one point symmetry became a nonlocal symmetry, and one contact symmetry became a point symmetry. Thus we are only left with six point symmetries of the ODE (3.3.62) [6]. However, in principle we know that a second order ODE possess eight point symmetries. We can conclude that there are two Type II hidden symmetries. Their origin lies in nonlocal contact symmetries. We have thus shown that hidden symmetries can arise from contact and nonlocal contact symmetries.

3.4 Point Symmetries

We now show that hidden symmetries can also arise from point symmetries. Let us take an equation arising from the analysis of shear-free spherically symmetric spacetimes (in cosmology) [39]

$$yy'''' + \tfrac{5}{2}y'y''' = y^{-3}. \tag{3.4.73}$$

Equation (3.4.73) has two Lie point Symmetries

$$\begin{aligned} \Gamma_1 &= \partial_x \\ \Gamma_2 &= x\partial_x + \tfrac{4}{5}\partial_y. \end{aligned}$$

Taking the Lie bracket relation

$$\begin{aligned} \left[\Gamma_1, \Gamma_2\right] &= \partial_x \left(x\partial_x + \tfrac{4}{5}\partial_y\right) - \left(x\partial_x + \tfrac{4}{5}\partial_y\right)(\partial_x) \\ &= \partial_x + 0 = \Gamma_1. \end{aligned}$$

Using Γ_1 and the transformation $u = y$ and $v = y'$, we reduce (3.4.73) to a third order ODE

$$u(v^3v''' + 4v^2v'v'' + vv'^3) + \tfrac{5}{2}v(v^2v'' + vv'^2) = u^{-3}. \tag{3.4.74}$$

Equation (3.4.74) has also two point symmetries

$$\begin{aligned}\Delta_2 &= \tfrac{4}{5}u\partial_u - \tfrac{1}{5}v\partial_v \\ \Delta_3 &= 2u^2\partial_u - uv\partial_v\end{aligned}$$

Δ_2 is an inherited symmetry, since it is the same as Γ_2. Δ_3 is a Type II hidden symmetry. We could conclude that it arose from the nonlocal symmetry

$$G_3 = 5\left(\int y dx\right)\partial_x + 2y^2\partial_y \tag{3.4.75}$$

of (3.4.73). However, our purpose here is to investigate whether it could arise from a point symmetry. If it does, it will have to be a point symmetry of a different fourth-order equation than can be reduced to (3.4.74). We note that the Lie bracket of Δ_2 and Δ_3 is

$$[\Delta_2, \Delta_3] = \tfrac{4}{5}\Delta_3. \tag{3.4.76}$$

We seek a fourth-order ODE with Σ_1, Σ_2 and Σ_3 as symmetries such that if one reduces the order by Σ_1, then the Lie bracket relation of Σ_2 and Σ_3 will be same as (3.4.76) ie. the Lie algebra A_2.

Looking at Table 3.1, we have two realisation of A_2: there is one with connected symmetries and one with unconnected symmetries. Two symmetries of the higher order ODE are connected if the two symmetries of the lower order ODE are connected [11]. It is clear that we will use the realisation with unconnected symmetries as Δ_2 and Δ_3 are unconnected [11]. Then, if we start with the three point symmetries of the Lie algebra $A_1 \oplus A_2$, we realise that this will be the most direct path to the Lie algebra A_2. From Table 3.1, we obtain

$$\Sigma_1 = t\partial_q \tag{3.4.77}$$

$$\Sigma_2 = -t\partial_t - q\partial_q \tag{3.4.78}$$

$$\Sigma_3 = \partial_q. \tag{3.4.79}$$

Taking the Lie algebra of the above symmetries, we have

$$\begin{aligned}\left[\Sigma_1, \Sigma_2\right] &= (t\partial_q)(-t\partial_t - q\partial_q) - (-t\partial_t - q\partial_q)(t\partial_q) \\ &= -t\partial_q + t\partial_q = 0, \qquad (3.4.80)\\ \left[\Sigma_1, \Sigma_3\right] &= (t\partial_q)(\partial_q) - (\partial_q)(t\partial_q) = 0, \qquad (3.4.81)\\ \left[\Sigma_2, \Sigma_3\right] &= (-t\partial_t - q\partial_q)(\partial_q) - (\partial_q)(-t\partial_t - q\partial_q) \\ &= \partial_q = \Sigma_3. \qquad (3.4.82)\end{aligned}$$

A general fourth-order ODE is given by

$$\ddot{\ddot{q}} = f(t, q, \dot{q}, \ddot{q}, \dddot{q}). \qquad (3.4.83)$$

If equation (3.4.83) is invariant under Σ_3, then Σ_3 will transform (3.4.83) to an ODE without the variable q, ie.

$$\begin{aligned} E = \ddot{\ddot{q}} - f(t, q, \dot{q}, \ddot{q}, \dddot{q}) &= 0 \\ \Sigma_3^{[4]} E\big|_{E=0} &= 0 \\ -\frac{\partial f}{\partial x} &= 0 \end{aligned}$$

which implies that

$$\ddot{\ddot{q}} = f(t, \dot{q}, \ddot{q}, \dddot{q}). \qquad (3.4.84)$$

Equation (3.4.84) is the family of equations that are invariant under Σ_3. If equation (3.4.84) is invariant under Σ_1, we have

$$\ddot{\ddot{q}} = f(t, \ddot{q}, \dddot{q}). \qquad (3.4.85)$$

Finally, if equation (3.4.85) is invariant under Σ_2 we have

$$\ddot{\ddot{q}} = t^{-3} f(t\ddot{q}, t^2\dddot{q}).$$

We notice that if we reduce the fourth-order ODE by Σ_1, then the Lagrange's system associated with this reduction is

$$\frac{dt}{0} = \frac{dq}{t} = \frac{d\dot{q}}{1}.$$

Solving this results in the reduction variables

$$t = x, \qquad y = t\dot{q} - q.$$

We represent the derivatives as follows

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d(t\dot{q} - q)}{d(t)} = t\ddot{q}, \\
\frac{d^2y}{dx^2} &= \frac{d}{dx}\frac{d}{dx} = t\dddot{q} + \ddot{q}, \\
\frac{d^3y}{dx^3} &= \frac{d}{dx}\frac{d^2y}{dx^2} = t\ddddot{q} + 2\dddot{q}.
\end{aligned}$$

As a result the structure of the third-order ODE is given by [11]

$$x^2y''' - 2(xy'' - y') = f(y', xy'' - y'). \tag{3.4.86}$$

The symmetries Σ_2 and Σ_3 can be transformed as follows

$$\begin{aligned}
\Sigma_2 &= -t\frac{\partial}{\partial t} - q\frac{\partial}{\partial q} + 0\frac{\partial}{\partial \dot{q}} \\
&= -t\frac{\partial x}{\partial t}\partial_x - q\frac{\partial x}{\partial q}\partial_x - t\frac{\partial y}{\partial t}\partial_y - q\frac{\partial y}{\partial t}\partial_y \\
&= -x\partial_x - y\partial_y
\end{aligned} \tag{3.4.87}$$

and

$$\begin{aligned}
\Sigma_3 &= 0\frac{\partial}{\partial t} + \frac{\partial}{\partial q} + 0\frac{\partial}{\partial \dot{q}} \\
&= \frac{\partial x}{\partial q}\partial_x + \frac{\partial y}{\partial q}\partial_y \\
&= -\partial_y.
\end{aligned} \tag{3.4.88}$$

We observe that both Σ_2 and Σ_3 can be written solely in terms of the new variables. They are also point symmetries of (3.4.86). Taking the Lie bracket of Σ_2 and Σ_3 we have

$$\begin{aligned}
[\Sigma_2, \Sigma_3] &= (-x\partial_x - y\partial_y)(-\partial_y) - (-\partial_y)(-x\partial_x - y\partial_y) \\
&= 0 - \partial_y = \Sigma_3,
\end{aligned}$$

which satisfies the property (3.4.76)[11]. We note that the form of Σ_2 and Σ_3 is not that of Δ_2 and Δ_3. These symmetries can be written as

$$\Delta_2 = u\partial_u - \tfrac{1}{4}v\partial_v$$

and

$$\Delta_3 = 2u^2\partial_u - uv\partial_v.$$

We let

$$x = F(u,v), \qquad y = G(u,v),$$

to obtain the transformation between Σ_2 and Δ_2 as well as Σ_3 and Δ_3. Operating on x and y with Δ_2 and Δ_3, we have

$$\begin{aligned} u\left(\frac{\partial x}{\partial u}\right)\left(\partial_x\right) - \tfrac{1}{4}v\left(\frac{\partial x}{\partial v}\right)\left(\partial_x\right) + u\left(\frac{\partial y}{\partial u}\right)\left(\partial_y\right) - \tfrac{1}{4}v\left(\frac{\partial y}{\partial v}\right)\left(\partial_y\right) \\ = \left(uF_u - \tfrac{1}{4}vF_v\right)\partial_x + \left(uG_u - \tfrac{1}{4}vG_v\right)\partial_y, \end{aligned} \tag{3.4.89}$$

and

$$\begin{aligned} 2u^2\left(\frac{\partial x}{\partial u}\right)\left(\partial_x\right) + uv\left(\frac{\partial x}{\partial v}\right)\left(\partial_x\right) + 2u^2\left(\frac{\partial y}{\partial u}\right)\left(\partial_y\right) + uv\left(\frac{\partial y}{\partial v}\right)\left(\partial_y\right) \\ = \left(2u^2F_u + uvF_v\right)\partial_x + \left(2u^2G_u + uvG_v\right)\partial_y, \end{aligned} \tag{3.4.90}$$

respectively. These expressions are required to have the same form as Σ_2 and Σ_3 . By comparing the coefficients of ∂_x and ∂_y we obtain

$$uF_u - \tfrac{1}{4}vF_v = -F \tag{3.4.91}$$

$$uG_u - \tfrac{1}{4}vG_v = -G \tag{3.4.92}$$

$$2u^2F_u + uvF_v = 0 \tag{3.4.93}$$

$$2u^2G_u + uvG_v = -1. \tag{3.4.94}$$

Equations (3.4.91)–(3.4.94) can be easily solved to obtain

$$F(u,v) = x = u^{-2/3}v^{4/3}$$

and

$$G(u,v) = y = \frac{1}{2u}.$$

If one compares (3.4.74) and (3.4.86), its clear that f in (3.4.86) must be of the form

$$f(y', xy'' - y') = f(\alpha, \beta) = \frac{32\alpha^4}{3} + \frac{3\beta^2}{\alpha} + \frac{7\alpha}{8} + \frac{7\beta}{8}. \qquad (3.4.95)$$

We summarise our results as follows. Using symmetry Γ_1 and transformation $u = y$ and $v = y'$ we reduced (3.4.73) to a third order equation (3.4.74). The reduced equation (3.4.74) admitted Γ_3 as a Type II hidden symmetry. We have found an equation (3.4.95) which can also be reduced to (3.4.74). Equation (3.4.95) admits three point symmetries Σ_1, Σ_2 and Σ_3. It appears that Γ_3 arises from the point symmetry Σ_3 of (3.4.95) [11]. We have thus shown that hidden symmetries of ODEs also arise from point symmetries of another equation of the same order as the original equation.

Table 3.1: Realisations of three dimensional Lie algebras

Algebra	Symmetries	Invariant Equation
$A_{3,2}^{I}$	$\partial_y,\ \partial_x,\ x\partial_x + (x+y)\partial_y$	$y_{xx} = A\exp[-y_x]$
$A_{3,2}^{II}$	$\partial_Y,\ -\log X\partial_Y,\ X\partial_X + Y\partial_Y$	$XY_{XX} = -Y_X + \bar{A}$
$A_1 \oplus {A_2}^{I}$	$\partial_x,\ \partial_y,\ x\partial_x$	$y_{xx} = Ay_x^2$
$A_1 \oplus {A_2}^{II}$	$\partial_Y,\ X\partial_Y,\ X\partial_X + Y\partial_Y$	$Y_{XX} = \bar{A}X^{-1}$
$A_{3,3}^{I}$	$\partial_x,\ \partial_y,\ x\partial_x + y\partial_y$	$y_{xx} = 0$
$A_{3,3}^{II}$	$\partial_Y,\ X\partial_Y,\ Y\partial_Y$	$Y_{XX} = 0$
$A_{3,4}^{I}$	$\partial_x,\ \partial_y,\ x\partial_x + y\partial_y$	$y_{xx} = Ay_x^{\frac{3}{2}}$
$A_{3,4}^{II}$	$\partial_Y,\ X\partial_Y,\ Y\partial_Y$	$Y_{XX} = \bar{A}X^{\frac{-3}{2}}$
${A_{3,5}^{a}}^{I}(0 < \vert a\vert < 1)$	$\partial_x,\ \partial_y,\ x\partial_x + y\partial_y$	$y_{xx} = Ay_x^{\frac{a-2}{a-1}}$
${A_{3,5}^{a}}^{II}(0 < \vert a\vert < 1)$	$\partial_Y,\ X\partial_Y,\ (1-a)X\partial_X + Y\partial_Y$	$Y_{XX} = \bar{A}X^{\frac{1-2a}{a-1}}$
$A_{3,6}^{I}$	$\partial_x,\ \partial_y,\ y\partial_x - x\partial_y$	$y_{xx} = A(1+y_x^2)^{\frac{3}{2}}$
$A_{3,6}^{II}$	$X\partial_Y,\ \partial_Y,\ (1+X^2)\partial_X + XY\partial_Y$	$Y_{XX} = \bar{A}(1+X^2)^{-\frac{3}{2}}$
${A_{3,7}^{b}}^{I}(b > 0)$	$\partial_x,\ \partial_y,\ (bx+y)\partial_x + (by-x)\partial_y$	$y_{xx} = A(1+y_x^2)^{\frac{3}{2}}\exp[b\arctan y_x]$
${A_{3,7}^{b}}^{II}(b > 0)$	$X\partial_Y,\ \partial_Y,\ (1+X^2)\partial_X + (XY+bY)\partial_Y$	$Y_{XX} = \bar{A}(1+X^2)^{-\frac{3}{2}}\exp[b\arctan X]$

Chapter 4

Hidden Symmetries for PDEs

4.1 Introduction

PDEs are commonly used to describe many scientific problems in different fields. In general, finding solutions of PDEs is not always a straight forward task especial for nonlinear PDEs [4]. However, using Lie's idea of symmetry we can determine group invariant solutions to many PDEs. Unlike for ODEs, the reduction of a PDE simply refers to the decrease in the number of independent and dependent variables of a given PDE invariant under a Lie group [4]. It is possible to further reduce the number of variables of a PDE eventually resulting in an ODE. However, this depends on the structure of the associated Lie algebra. In chapter 3 we introduced the concept of hidden symmetries of ODEs. In a similar fashion hidden symmetries of PDEs were identified by Abraham-Shrauner and co-workers [1, 3, 4, 5, 22]. In this chapter the origins of these hidden symmetries of PDEs is the focus of our study.

4.2 Point Symmetries: PDE $\rightarrow$ ODE reduction

Initially [3, 4], the only source of hidden symmetries for PDEs was thought to be point symmetries of another PDE. Let us illustrate this idea via the model equation [4]

$$u_{xxx} + u(u_t + cu_x) = 0, \tag{4.2.1}$$

where c is a constant [4]. From SYM [47], we obtain the four group generators of (4.2.1) as

$$\begin{aligned} U_1 &= \partial_t & (4.2.2) \\ U_2 &= \partial_x & (4.2.3) \\ U_3 &= 3t\partial_t + (x + 2ct)\partial_x & (4.2.4) \\ U_4 &= t\partial_t + ct\partial_x + u\partial_u. & (4.2.5) \end{aligned}$$

We choose the combination

$$V_a = cU_2 + U_1 \tag{4.2.6}$$

and so reduce (4.2.1) via the transformation

$$w = u, \qquad y = x - ct. \tag{4.2.7}$$

Substituting (4.2.7) into (4.2.1) results in

$$w_{yyy} = 0. \tag{4.2.8}$$

Equation (4.2.8) is a third order ODE. From SYM [47] we obtain seven Lie group generators of (4.2.8):

$$
\begin{aligned}
X_1 &= \partial_y && (4.2.9)\\
X_2 &= \partial_w && (4.2.10)\\
X_3 &= y^2\partial_w && (4.2.11)\\
X_4 &= y\partial_y && (4.2.12)\\
X_5 &= y\partial_w && (4.2.13)\\
X_6 &= w\partial_w && (4.2.14)\\
X_7 &= \tfrac{1}{2}y^2\partial_y + yw\partial_w. && (4.2.15)
\end{aligned}
$$

Considering the Lie point symmetries of (4.2.1), we observe that (4.2.8) inherits three symmetries:

$$X_1 \leftarrow U_1, \qquad X_4 \leftarrow U_3, \qquad X_6 \leftarrow U_4.$$

The remaining four symmetries are Type II hidden symmetries [4]. It was observed by Abraham-Shrauner [1] that these Type II hidden symmetries do not arise from contact or nonlocal symmetries, since (4.2.7) only contains variables. From this observation, we can can conclude that the origin of these hidden symmetries might be point symmetries of another PDE. We require those PDEs to have the same independent and dependent variable as (4.2.8) and that they can be reduced to (4.2.8) [4].

We will demonstrate two methods to find possible PDEs. The first method utilises an educated "guess" of a PDE. The symmetries of this PDE are then checked to see if they reduce to the group generators of (4.2.8). If this holds, our PDE is correct.

The PDEs that are likely to reduce to (4.2.8) under (4.2.7) may be of the form

$$u_{xxx}=0, \qquad u_{ttt}=0, \qquad u_{xxt}=0, \qquad u_{xtt}=0, \tag{4.2.16}$$

where $u=u(x,t)$ [4]. If we take the third equation from (4.2.16) we observe that it has eight symmetries [47]

$$\begin{aligned}
U_1 &= \partial_x && (4.2.17)\\
U_2 &= x\partial_x && (4.2.18)\\
U_3 &= u\partial_u && (4.2.19)\\
U_4 &= F_1(x)\partial_u && (4.2.20)\\
U_5 &= F_2(t)\partial_u && (4.2.21)\\
U_6 &= F_3(t)\partial_t && (4.2.22)\\
U_7 &= F_4(t)x\partial_u && (4.2.23)\\
U_8 &= x^2\partial_x+2ux\partial_u. && (4.2.24)
\end{aligned}$$

Here $F_1(x)$ and $F_j(t), j=2,\ldots,4$ represent arbitrary functions. Since we need to recover (4.2.9)–(4.2.15), we have to choose the polynomials in x for $F_1(x)$ and t for $F_j(t)$ as well as the combinations in a systematic manner. By different choices of $F_1(x)$ and $F_j(t)$ the Lie group generators (4.2.17)–(4.2.24) become

$$
\begin{aligned}
V_1 &= \partial_x && (4.2.25)\\
V_2 &= \partial_u && (4.2.26)\\
V_3 &= \partial_t && (4.2.27)\\
V_4 &= (x-ct)^2\partial_u && (4.2.28)\\
V_5 &= (x-ct)\partial_x && (4.2.29)\\
V_6 &= (x-ct)\partial_u && (4.2.30)\\
V_7 &= u\partial_u && (4.2.31)\\
V_8 &= \tfrac{1}{2}(x-ct)^2\partial_x + u(x-ct)\partial_u. && (4.2.32)
\end{aligned}
$$

One can notice that these generators reduce to the seven generators in (4.2.9)–(4.2.15) [4]. Now it is clear that the hidden symmetries in (4.2.9)–(4.2.15) have no connection with the original PDE (4.2.1). They are inherited point symmetries of another PDE which reduces to (4.2.7) [4]. These hidden symmetries are inherited as

$$
V_1 \to X_2, \qquad V_4 \to X_3, \qquad V_6 \to X_5, \qquad V_8 \to X_7.
$$

In this fashion we can consider the other equations in (4.2.16).

The second method involves reverse transformations. We seek a Lie point generator and we use this generator to determine PDEs. These PDEs ensure that the Type II hidden symmetries of (4.2.7) are inherited from them [5]. Consider a group generator

$$
U_a = \xi^x(x,t,u)\partial_x + \xi^t(x,t,u)\partial_t + \eta(x,t,u)\partial_u \tag{4.2.33}
$$

with the assumption that it is a function of independent variables x, t and dependent variable $u(x,t)$. It is necessary that the group generator U_a must reduce to one of the group generators of (4.2.8) (say X_7 for simplicity). Taking this condition into account we obtain

$$
\begin{aligned}
U_a(u) &= \eta = yw = (x-ct)u && (4.2.34)\\
U_a(y) &= \xi^x - c\xi^t = \frac{1}{2}y^2 = \frac{1}{2}(x-ct)^2. && (4.2.35)
\end{aligned}
$$

In order to solve for ξ^x and ξ^t we require an additional condition. To add this condition, we take the traveling wave symmetry

$$U_c = c\partial_x + \partial_t. \tag{4.2.36}$$

We insist that U_a and U_c must satisfy the commutator

$$\big[U_c, U_a\big] = A_a U_c \tag{4.2.37}$$

where A_a is constant. Taking (4.2.35) and (4.2.37) and comparing the coefficients yields

$$
\begin{aligned}
\xi^x &= \frac{1}{2}(x-ct)^2 + A_a ct + cf^a(x-ct) && (4.2.38)\\
\xi^t &= A_a t + f^a(x-ct) && (4.2.39)
\end{aligned}
$$

where f^a is an arbitrary function of its argument. Now, we have the freedom of choosing different values of A_a and f^a and which leads to different group generators. If we set A_a and f^a both to zero, we have

$$U_a = \tfrac{1}{2}(x-ct)^2\partial_x + (x-ct)u\partial_u$$

which is exactly V_8. From (4.2.26)–(4.2.32) we find the common invariants that reduced to $u_{xxx} = 0$ to be u_{xxx} for V_1, V_2, V_3, V_4, V_6, $u_x^3 u_{xxx}$ for V_5, u_{xxx}/u for V_7 and $u_x^2 u_{xxx}$ for V_8. Since all invariants are set to zero to generate the PDE, the PDE that we are looking can be given by

$$u_{xxx} = 0.$$

This PDE is the source of Type II hidden symmetries of (4.2.8). In general, we can conclude that Type II hidden symmetries of PDEs may be inherited from point symmetries of other PDEs that also reduce to it [4].

4.3 Point symmetries: PDE $\rightarrow$ PDE reduction

In the previous section, we showed that hidden symmetries occurring in the reduction of a PDE to an ODE can arise from point symmetries of the original PDE. Here we show the same result but in the case of reducing a PDE to another PDE.

To illustrate this idea, let us consider [5]

$$u_{xxx} + u(u_t + cu_x) + u_x u_{xx} = 0. \tag{4.3.40}$$

Equation (4.3.40) possesses the three Lie point symmetries

$$U_1 = \partial_t \tag{4.3.41}$$

$$U_2 = \partial_x \tag{4.3.42}$$

$$U_3 = (x + 2ct)\,\partial_x + 3t\partial_t. \tag{4.3.43}$$

We take the combination

$$X_c = U_1 + cU_2, \tag{4.3.44}$$

which generates the transformation

$$y = x - ct, \qquad w = u \tag{4.3.45}$$

to reduce (4.3.40) to

$$w_{yyy} + w_y w_{yy} = 0. \tag{4.3.46}$$

From SYM [47] we obtain three Lie group generators of (4.3.46):

$$V_1 = \partial_y \tag{4.3.47}$$

$$V_2 = y\partial_y \tag{4.3.48}$$

$$V_3 = \partial_w. \tag{4.3.49}$$

Considering the Lie point symmetries of (4.3.40), we observe that (4.3.46) inherits two symmetries

$$U_1 \to V_1, \qquad U_2 \to V_2.$$

We conclude that V_3 is a Type II hidden symmetry [5]. To find the origin of V_3, we will determine the Lie point group generators using the reverse method for finding PDEs from which the symmetries of (4.3.46) are inherited [5].

Consider the group generator X_a of a PDE with similar assumptions as in the previous section:

$$X_a = \xi^x(x,t,u)\partial_x + \xi^t(x,t,u)\partial_t + \eta(x,t,u)\partial_u. \tag{4.3.50}$$

We require X_a to reduce to one of the group generators of (4.3.46) (say V_3 for simplicity). Taking this condition into account we obtain

$$X_a(y) = \xi^x - c\xi^t = 0 \tag{4.3.51}$$

$$X_a(u) = \eta = C_a, \tag{4.3.52}$$

where C_a represents a constant usually scaled to 1. We also insist that the symmetries used to reduce (4.3.40) to (4.3.46) must satisfy the commutator

$$\left[X_c, X_a\right] = A_a X_c \tag{4.3.53}$$

for the Lie algebra to close. Taking (4.3.51) and (4.3.53) and comparing the coefficients yields

$$\xi^t = \left(A_a - bc\right)t + bx + f^a\left(x - ct\right) \tag{4.3.54}$$

where f^a is an arbitrary function of its argument. We assume that ξ^t is independent of u and we also exclude the linear part of f^a to avoid an over-counting of symmetries [5]. Note that the solution of (4.3.51), (4.3.52) and (4.3.53) leads to the infinite Lie algebra

$$X_1 = \partial_t \tag{4.3.55}$$

$$X_2 = \partial_x \tag{4.3.56}$$

$$X_3 = (x+2ct)\partial_x + 3t\partial_t \tag{4.3.57}$$

$$X_4 = \partial_u \tag{4.3.58}$$

$$X_5 = t\,(c\partial_x + \partial_t) \tag{4.3.59}$$

$$X_6 = f(x-ct)\,(c\partial_x) \tag{4.3.60}$$

From these Lie symmetries we compute the differential invariants and thereafter construct the possible PDEs. The differential invariants are

$$X_1 : \quad (x, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, u_{xxx}, u_{xxt}, u_{xtt}, u_{ttt}),$$

$$X_2 : \quad (t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, u_{xxx}, u_{xxt}, u_{xtt}, u_{ttt}),$$

$$X_3 : \quad \left(\frac{x-ct}{t^{1/3}}, u, \frac{u_t + cu_x}{u_x^3}, \frac{u_{xx}}{u_x^2}, \frac{u_{tt}+cu_{xt}}{u_{xx}^3}, \frac{u_{xt}+cu_{xx}}{u_{xx}^2}, \frac{u_{xxx}}{u_x^3}, \frac{u_{xxt}+cu_{xxx}}{(u_{xxx})^{5/3}}, \right.$$
$$\left. \frac{u_{xtt}+cu_{xxt}+c^2u_{xxx}}{(u_{xxx})^{7/3}}, \frac{u_{ttt}+3cu_{xtt}+3c^2u_{xxt}+c^3u_{xxx}}{u_{xxx}^3}\right),$$

$$X_4 : \quad (t, x, u_x, u_t, u_{xx}, u_{xt}, u_{xxx}, u_{xxt}, u_{ttt})$$

$$X_5 : \quad \big[x-ct, u, u_t, t(u_t - cu_x), u_{xx}, t(u_{xt}+cu_{xx}), t^2(u_{tt}+2cu_{xt}+c^2uxx),$$
$$u_{xxx}, t(u_{xxt}+cu_{xxx}), t^2(u_{xtt}+2cu_{xxt}+c^2u_{xxx}),$$
$$t^3(u_{ttt}3cu_{xtt}+4c^2u_{xxt}+2c^3u_{xxx})\big],$$

$$X_6 : \quad (x-ct, u, u_t + cu_x, u_{tt}+2cu_{xt}+c^2u_{xx}, u_{ttt}+3cu_{xtt}$$
$$+3c^2u_{xxt}+c^3u_{xxx}).$$

The PDE that we are looking for must admit at least X_c, X_3 and X_4. This ensures that the Type II hidden symmetry is inherited from it [5]. The general form of this PDE is defined as

$$\begin{aligned}
&F\left(\frac{u_t + cu_x}{u_x^3}, \frac{u_{xx}}{u_x^2}, \frac{u_{tt} + cu_{xt}}{u_{xx}^3}, \frac{u_{xt} + cu_{xx}}{u_{xx}^2}, \frac{u_{xxx}}{u_x^3}, \frac{u_{xxt} + cu_{xxx}}{u_{xxx}^{5/3}},\right.\\
&\quad\left.\frac{u_{xtt} + cu_{xxt} + c^2u_{xxx}}{u_{xxx}^{7/3}}, \frac{u_{ttt} + 3cu_{xtt} + 3c^2u_{xxt} + c^3u_{xxx}}{u_{xxx}^3}\right) \quad = \quad 0.
\end{aligned}$$

We can further restrict the above equation in order to ensure that it reduces to (4.3.46) via X_c to

$$\begin{aligned}
u_{xxx} + u_xu_{xx} \quad = \quad & u_x^3\overline{F}\left(\frac{u_t + cu_x}{u_x^3}, \frac{u_{xx}}{u_x^2}, \frac{u_{tt} + cu_{xt}}{u_{xx}^3}, \frac{u_{xt} + cu_{xx}}{u_{xx}^2}, \frac{u_{xxx}}{u_x^3},\right.\\
&\frac{u_{xxt} + cu_{xxx}}{u_{xxx}^{5/3}}, \frac{u_{xtt} + cu_{xxt} + c^2u_{xxx}}{u_{xxx}^{7/3}},\\
&\left.\frac{u_{ttt} + 3cu_{xtt} + 3c^2u_{xxt} + c^3u_{xxx}}{u_{xxx}^3}\right)\\
&\times G\left(\frac{u_t + cu_x}{u_x^3}, \frac{u_{tt} + cu_{xt}}{u_{xx}^3}, \frac{u_{xt} + cu_{xx}}{u_{xx}^2}, \frac{u_{xxt} + cu_{xxx}}{u_{xxx}^{5/3}},\right.\\
&\frac{u_{xtt} + cu_{xxt} + c^2u_{xxx}}{u_{xxx}^{7/3}},\\
&\left.\frac{u_{ttt} + 3cu_{xtt} + 3c^2u_{xxt} + c^3u_{xxx}}{u_{xxx}^3}\right) \qquad (4.3.61)
\end{aligned}$$

such that $G(0, 0, 0, 0, 0, 0) = 0$. Now we can conclude that (4.3.61) is a master PDE that reduces to the target PDE and it is the source of the Type II hidden symmetry of (4.3.46).

4.4 Contact and Non-local Symmetries

In this section we will show that it is also possible for hidden symmetries of PDEs to arise from contact and potential (nonlocal) symmetries of the original PDE [22]. We will use the modified Monge-Ampere equation and the nonlinear wave equation to demonstrate contact and nonlocal symmetries as the origins of hidden symmetries as studied by Govinder and Abraham-Shrauner [22].

4.4.1 Hidden Symmetries from Contact Symmetries

Contact symmetries of PDEs can either be intrinsic or point extended contact symmetries. By point extended contact symmetries we refer to those symmetries that are just point symmetries [22]. Intrinsic contact symmetries are those symmetries where the coefficient functions contain at least one occurrence of u_t and/or u_x [22]. The intrinsic contact symmetries can be further distinguished into homogeneous and heterogeneous symmetries. If the coefficient function contains only first derivatives we refer to those intrinsic contact symmetries as homogeneous [22]. When the coefficient functions contain a mixture of variables and derivatives we refer to those as heterogeneous [22].

Now let us consider the "modified" Monge-Ampere equation [43]

$$u_{tt}(u_{xx} - \lambda) - u_{xt}^2 = 0. \tag{4.4.62}$$

If $\lambda = 0$, under the transformation

$$\overline{u} = u - \tfrac{1}{2},$$

we can recover Monge-Ampere equation

$$u_{tt}u_{xx} - u_{xt}^2 = 0. \tag{4.4.63}$$

The Monge-Ampere equation has fifteen point symmetries [33] and an infinite number of intrinsic contact symmetries. It has a characteristic function defined as

$$\begin{aligned} W &= F(u - tu_t - xu_x, u_t, u_x) + tG(u - tu_t - xu_x, u_t, u_x) \\ &\quad + xH(u - tu_t - xu_x, u_t, u_x) + uK(u - tu_t - xu_x, u_t, u_x). \end{aligned}$$

The symmetry generator of (4.4.63) is given by

$$Y_i = \xi^1\partial_t + \xi^2\partial_x + \eta\partial_u + \zeta^1\partial_{u_t} + \zeta^2\partial_{u_x} \tag{4.4.64}$$

with the coefficient functions

$$
\begin{aligned}
\xi^1 &= -W_{u_t} \\
\xi^2 &= -W_{u_x} \\
\eta &= W - u_t W_{u_t} - u_x W_{u_x} \\
\zeta^1 &= W_t + u_t W_t \\
\zeta^2 &= W_x + u_x W_u.
\end{aligned} \tag{4.4.65}
$$

We also note that the "modified" Monge-Ampere equation has fifteen point symmetries

$$
\begin{aligned}
Z_1 &= \partial_x \\
Z_2 &= \partial_t \\
Z_3 &= \partial_u \\
Z_4 &= x\partial_t \\
Z_5 &= t\partial_t \\
Z_6 &= t\partial_u \\
Z_7 &= x\partial_x + (2u)\partial_u \\
Z_8 &= (2u - x^2\lambda)\partial_t \\
Z_9 &= \tfrac{1}{\lambda}\partial_x + (2x)\partial_u \\
Z_{10} &= \tfrac{x}{\lambda}\partial_x + x^2\partial_u \\
Z_{11} &= \tfrac{t}{\lambda}\partial_x + (tx)\partial_u \\
Z_{12} &= \left(-x^2 + \tfrac{2u}{\lambda}\right)\partial_x + \left(2u - x^2\lambda\right)\partial_u \\
Z_{13} &= (2tx)\partial_x + (2t^2)\partial_t + (2tu + tx^2\lambda)\partial_u \\
Z_{14} &= \left(3x^2 - \tfrac{2u}{\lambda}\right)\partial_x + (2tx)\partial_t + (2x^3\lambda)\partial_u \\
Z_{15} &= (4ux - 2x^3\lambda)\partial_x + (4tu - 2tx^2)\partial_t + (4u^2 - x^4\lambda^2)\partial_u.
\end{aligned}
$$

It has a characteristic function defined by

$$\begin{aligned} W &= F\Bigg(u + \tfrac{1}{2}\lambda x^2 - tu_t - xu_x, u_t, u_x - \lambda x\Bigg) \\ &\quad + tG\Bigg(u + \tfrac{1}{2}\lambda x^2 - tu_t - xu_x, u_t, u_x - \lambda x\Bigg) \\ &\quad + xH\Bigg(u + \tfrac{1}{2}\lambda x^2 - tu_t - xu_x, u_t, u_x - \lambda x\Bigg) \\ &\quad + \Bigg(u - \tfrac{1}{2}\Bigg)K\Bigg(u + \tfrac{1}{2}\lambda x^2 - tu_t - xu_x, u_t, u_x - \lambda x\Bigg). \end{aligned} \tag{4.4.66}$$

We want to reduce (4.4.62) to a PDE with fewer variables. We use the translation in x to reduce (4.4.62) ie.

$$r = t, \qquad s = u. \tag{4.4.67}$$

Substituting (4.4.67) into (4.4.62) results in

$$s_{rr} = 0. \tag{4.4.68}$$

Equation (4.4.68) is a second order ODE that can be easily solved. It is well known that any second order ODE has eight point symmetries. Utilising SYM [47], we obtain

$$\begin{aligned} X_1 &= \partial_r \\ X_2 &= s\partial_r \\ X_3 &= r\partial_r \\ X_4 &= \partial_s \\ X_5 &= s\partial_s \\ X_6 &= r\partial_s \\ X_7 &= (rs)\partial_r + s^2\partial_s \\ X_8 &= r^2\partial_r + (rs)\partial_s. \end{aligned}$$

If we merely look at these point symmetries, we observe that five symmetries are inherited from the point symmetries of equation (4.4.62):

$$X_1 \leftarrow Z_2, \qquad X_4 \leftarrow Z_3, \qquad X_3 \leftarrow Z_5, \qquad X_6 \leftarrow Z_6, \qquad X_7 \leftarrow Z_{15}.$$

It is clear that X_2, X_5 and X_8 are type II hidden symmetries, since they do not rise from point symmetries of (4.4.62) [22]. However, we know that (4.4.68) posses an infinite number of contact symmetries. They could be a source of these hidden symmetries. We need to analyse these contact symmetries more closely. Let us consider a special case where $F = H = K = 0$ and

$$\begin{aligned} G &= \lambda\big(u + \tfrac{1}{2}\lambda x^2 - tu_t - xu_x\big) - \tfrac{1}{2}(u_x - \lambda x)^2 \\ &= \lambda(u - tu_t) - \tfrac{1}{2}u_x^2. \end{aligned} \tag{4.4.69}$$

The characteristics function of (4.4.62) becomes

$$W = t\Big(\lambda(u - tu_t) - \tfrac{1}{2}u_x^2\Big). \tag{4.4.70}$$

Utilising the symmetry generator (4.4.64) with

$$\begin{aligned} \xi^1 &= \lambda t^2, \\ \xi^2 &= tu_x, \\ \eta &= t\Big((\lambda(u - tu_t) - \tfrac{1}{2}u_x^2 + \lambda tu_t + u_x^2\Big) = t(\lambda u + \tfrac{1}{2}u_x^2), \\ \zeta^1 &= \lambda(u - tu_t) - \tfrac{1}{2}u_x^2 - t\lambda u_t + t\lambda u_t, \\ \zeta^2 &= t\lambda u_x \end{aligned}$$

we obtain the heterogeneous intrinsic contact symmetry

$$z = \lambda t^2\partial_t + tu_x\partial_x + \tfrac{1}{2}t\Big(u_x^2 + 2\lambda u\Big)\partial_u + \Big(\lambda(u - tu_t) - \tfrac{1}{2}\Big)\partial_{u_t} + \lambda tu_x\partial_{u_x}. \tag{4.4.71}$$

Substituting (4.4.67) into (4.4.71) and setting $\lambda = 1$, we obtain

$$L_1 = r^2\partial_r + (rs)\partial_s. \tag{4.4.72}$$

One can observe that $X_8 \leftarrow L_1$ is a symmetry of (4.4.68) which does not come from point symmetries of the Monge-Ampere equation but rather from a contact symmetry. We can continue in this fashion to determine other contact symmetries (note that depending on choices of F, H, K and G it may be difficult to determine them). Now, we can conclude that Type II hidden symmetries of (4.4.68) can arise from contact symmetries of the original equation. This is similar to the case of hidden symmetries of ODEs. However, unlike in the ODE case, we note that in the case of contact symmetries, it is only heterogeneous intrinsic contact symmetries that manifest themselves as the source of hidden symmetries [22].

4.4.2 Nonlocal/Potential Symmetries

In this section we will give more details about the classes of symmetries of differential equations by considering the nonlocal symmetries whose infinitesimals at any point x depends on a global behavior of $u(x)$ in some neighborhood of x [15].

Consider a system of PDEs $R\{x, u\}$. We can write one or more PDEs of this system in a conserved form with respect to some choice of its variables [15]. Naturally, a conserved form will lead to an auxiliary dependent variable v which is called a potential variable and to an auxiliary system of PDEs $S\{x, u, v\}$ [15].

Definition 4.4.1. *$R\{x, u\}$ is embedded in $S\{x, u, v\}$: Any solution $(u(x), v(x))$ of $S\{x, u, v\}$ will define a solution $u(x)$ of $R\{x, u\}$ and to any solution $u(x)$ of $R\{x, u\}$ there corresponds a function $v(x)$ such that ($u(x)$, $v(x)$) defines a solution of $S\{x, u, v\}$* [15].

Definition 4.4.2. *A local symmetry in G_s (where a group G_s is defined by* *a local symmetries) will induce a nonlocal symmetry admitted by $R\{x, u\}$ if and only if the infinitesimals of variables (x, u) of $S\{x, u, v\}$ depend explicitly on the potential variables v. This nonlocal symmetry is called a potential symmetry of $R\{x, u\}$* [15].

Theorem 4.4.1. *If $R\{x, u\}$ is a scalar evolution equation with two independent variables $x = (x_1, x_2)$ written in conserved form*

$$D_2 u - D_1 f(x_1, u, u_1, \ldots, u_{k-1}), \tag{4.4.73}$$

where

$$D_i = \partial_{x_i} + u_i \partial_u + \ldots + u_{i_1 i_2 \ldots i_{k-1}} \partial_{u_{i_1 i_2 \ldots i_{k-1}}}$$

and $u_p = \frac{\partial^p u}{\partial x_1^p}$, $p = 1, 2 \ldots, k-1$, with associated auxiliary symmetry $S\{x, u, v\}$ given by

$$\begin{aligned} v_{x_1} &= u, \\ v_{x_2} &= f(x, u, u_1, \ldots, u_{k-1}), \end{aligned}$$

then a solution $(u(x), v(x))$ of $S\{x, u, v\}$ leads to solution $v(x)$ of the evolution equation $T\{x, v\}$ given by

$$v_{x_2} = f(x, u, u_1, \ldots, u_k).$$

For further details about potential symmetries we refer the reader to [15].

As an example we consider the nonlinear wave equation [22]

$$u_{tt} = \left[f(u) u_x\right]_x. \tag{4.4.74}$$

Using SYM [47] we obtain three point symmetries of (4.4.74):

$$\begin{aligned} U_1 &= \partial_x & (4.4.75) \\ U_2 &= \partial_t & (4.4.76) \\ U_3 &= t\partial_t + x\partial_x. & (4.4.77) \end{aligned}$$

We note that equation (4.4.74) is already in a conserved form and from theorem 4.4.1, we can write the natural potential form of (4.4.74) as [22, 7]

$$\begin{aligned} v_x &= u_t && (4.4.78) \\ v_t &= f(u)u_x. && (4.4.79) \end{aligned}$$

If we differentiate (4.4.78) with respect to t we obtain

$$v_{xt} = u_{tt}. \tag{4.4.80}$$

Differentiate (4.4.79) with respect to x we obtain

$$v_{tx} = f_x(u)u_x^2 + f(u)u_{xx} = \big[f(u)u_x\big]_x. \tag{4.4.81}$$

Equating v_{xt} and v_{tx} yields (4.4.74). If we assume that $f(u)$ is arbitrary and analyse (4.4.78)–(4.4.79) for point symmetries we obtain (4.4.76)–(4.4.77) and the additional symmetries

$$\begin{aligned} U_4 &= \partial_v && (4.4.82) \\ U_5 &= u\partial_t + v\partial_x. && (4.4.83) \end{aligned}$$

U_4 is expected from the definition of (4.4.78)–(4.4.79). However, U_5 is new and is called a potential symmetry of (4.4.74). We can reduce (4.4.74) using U_1 with the transformation

$$p = t, \qquad q = u \tag{4.4.84}$$

to obtain

$$q_{pp} = 0. \tag{4.4.85}$$

Equation (4.4.85) is a second order ODE with eight Lie point symmetries

$$Z_1 = \partial_p \tag{4.4.86}$$

$$Z_2 = q\partial_p \tag{4.4.87}$$

$$Z_3 = p\partial_p \tag{4.4.88}$$

$$Z_4 = \partial_q \tag{4.4.89}$$

$$Z_5 = p\partial_q \tag{4.4.90}$$

$$Z_6 = q\partial_p \tag{4.4.91}$$

$$Z_7 = pq\partial_p + q^2\partial_q \tag{4.4.92}$$

$$Z_8 = p^2\partial_p + pq\partial_q. \tag{4.4.93}$$

Two of these point symmetries are inherited from point symmetries of (4.4.74) viz.

$$U_2 \to Z_1, \qquad U_3 \to Z_3.$$

It appears that the remaining six symmetries are Type II hidden symmetries. However, one of these six hidden symmetries is inherited symmetry from a nonlocal symmetry ie.

$$Z_2 \leftarrow U_5.$$

From this observation we conclude that hidden symmetries of a PDE they also arise from nonlocal symmetries of the original PDE [22]. Unlike the case of ODEs, these nonlocal symmetries must be potential symmetries of the original equation.

Chapter 5

Conclusion

5.1 Introductory Material

The main focus of this dissertation was to provide an overview of hidden symmetries of differential equations. We achieved this goal by using Lie's ideas of symmetry analysis. We now provide a brief summary of the dissertation by giving the main results achieved during the course of our investigations.

In chapter two, we introduced and briefly discussed the concept of Lie symmetry analysis of differential equations. We indicated how to find the symmetries of an ODE by considering a second order equation

$$y'' = \frac{1}{y^3}.$$

We found three Lie point symmetries and used them to demonstrate the idea of Lie brackets. We also indicated how to reduce the order of an ODE and we were guided by the Lie bracket relations of these point symmetries to choose a proper reduction symmetry. In general, this ensures that an optimal number of symmetries remain point symmetries to allow further reductions

[21]. We presented something similar for PDEs and the heat equation was used as an example. Instead of reduction of order, we outlined the concept of group invariant solutions for PDEs.

5.2 ODEs

In chapter three, we introduced the concept of hidden symmetries of ODEs and we focused on their origins. We showed that in the reduction of the Emden-Fowler equation [27]

$$2yy'''' + 5y'y''' = 0 \tag{5.2.1}$$

via the transformation $u = y$ and $v = y'$ to the third-oder ODE

$$2u(v^2v''' + 4vv'v'' + v'^3) + 5(v^2v'' + vv'^2) = 0 \tag{5.2.2}$$

a Type II hidden symmetry X_4 appeared and its origin was the nonlocal symmetry defined by

$$G_4 = 3\left(\int ydx\right)\partial_x + 2y^2\partial_y. \tag{5.2.3}$$

We used X_4 to reduce (5.2.2) further to

$$w'' + 3w' + 2w = 0. \tag{5.2.4}$$

More Type II hidden symmetries were discovered from this reduction.

We presented a systematic approach of finding nonlocal symmetries of any second order ODE by analysing a general equation

$$E(y, y', y'') = y'' - g(y, y') = 0, \tag{5.2.5}$$

which admitted the point symmetry $G_1 = \partial_x$. We noted that this method could be applied even for higher ODEs and we established that nonlocal symmetries can be one of the sources of hidden symmetries.

We also introduced and discussed the method of finding contact symmetries of ODEs. We showed that contact symmetries may also give rise to hidden symmetries and illustrated this by considering the third-order equation

$$y''' = 0 \tag{5.2.6}$$

which possessed ten symmetries. We reduced (5.2.6) to a second-order ODE and upon the reduction Type II hidden symmetries were obtained. Their origin was in contact symmetries and nonlocal contact symmetries.

We also noticed that hidden symmetries of ODEs could originate from point symmetries. This was shown by considering the fourth-order ODE [39]

$$yy'''' + \tfrac{5}{2}y'y''' = y^{-3} \tag{5.2.7}$$

which admitted the symmetries Γ_1 and Γ_2. Equation (5.2.7) was reduced to

$$u(v^3v''' + 4v^2v'v'' + vv'^3) + \tfrac{5}{2}v(v^2v'' + vv'^2) = u^{-3} \tag{5.2.8}$$

via Γ_1. Equation (5.2.8) admitted the Type II hidden symmetry

$$\Delta_3 = 2u^2\partial_u - uv\partial_v. \tag{5.2.9}$$

We noticed that one could conclude that the origin of Δ_3 was a nonlocal symmetry

$$G_3 = 5\left(\int y dx\right)\partial_x + 2y^2\partial_y. \tag{5.2.10}$$

However, we managed to find the fourth-order equation

$$\ddddot{q} = t^{-3}f(t\ddot{q}, t^2\dddot{q}),$$

where

$$f(\alpha, \beta) = \frac{32\alpha^4}{3} + \frac{3\beta^2}{\alpha} + \frac{7\alpha}{8} + \frac{7\beta}{8} \tag{5.2.11}$$

which can be reduced to (5.2.8). The symmetries of (5.2.11) reduced directly to symmetries of (5.2.8), thus providing a point source of hidden symmetries.

From our investigations, we now conclude that hidden symmetries of ODEs can arise from nonlocal symmetries of the original equation, contact symmetries of the original symmetries, as well as point symmetries of another equation of the same order as the original equation.

5.3 PDEs

In chapter four we extended our study by considering hidden symmetries of PDEs. The idea was similar to that of ODEs. In the past [3, 4, 5] the origin of hidden symmetries of PDEs was interpreted to be the point symmetries of another PDE. However, more recent results [22] have reported that contact and nonlocal symmetries may also give rise to hidden symmetries.

We started by investigating the case of point symmetries. We demonstrated this using two examples. Firstly, we considered the reduction of a PDE

$$u_{xxx} + u(u_t + cu_x) = 0, \tag{5.3.12}$$

to the ODE

$$w_{yyy} = 0. \tag{5.3.13}$$

Equation (5.3.13) had five Type II hidden symmetries and we assumed that their origin was the point symmetries of another PDE in the same independent and dependent variables. We obtained the possible PDEs via two methods. The first method used an educated "guess" of a PDE. We established that

$$u_{xxx} = 0$$

was one of the possible PDEs that can be reduced to (5.3.13) and it appeared to be the origin of the Type II hidden symmetries of (5.3.13). The second method involved reverse transformations. We considered the group generator

$$U_a = \xi^x(x,t,u)\partial_x + \xi^t(x,t,u)\partial_t + \eta(x,t,u)\partial_u \tag{5.3.14}$$

and used it to determine the general class of PDEs from which the Type II hidden symmetries of (5.3.13) were inherited as point symmetries of these PDEs. We used the second example to explore our second method but for the reduction of a PDE to another PDE. From the group generator we computed the differential invariants and we restricted those invariants to obtain a general form of a PDE that admitted the relevant symmetries. With further restrictions we obtained the master PDE

$$\begin{aligned}
u_{xxx} + u_x u_{xx} \quad = \quad & u_x^3 \overline{F}\Bigg(\frac{u_t + cu_x}{u_x^3}, \frac{u_{xx}}{u_x^2}, \frac{u_{tt} + cu_{xt}}{u_{xx}^3}, \frac{u_{xt} + cu_{xx}}{u_{xx}^2}, \frac{u_{xxx}}{u_x^3}, \\
& \frac{u_{xxt} + cu_{xxx}}{u_{xxx}^{5/3}}, \frac{u_{xtt} + cu_{xxt} + c^2 u_{xxx}}{u_{xxx}^{7/3}}, \\
& \frac{u_{ttt} + 3cu_{xtt} + 3c^2 u_{xxt} + c^3 u_{xxx}}{u_{xxx}^3}\Bigg) \\
& \times G\Bigg(\frac{u_t + cu_x}{u_x^3}, \frac{u_{tt} + cu_{xt}}{u_{xx}^3}, \frac{u_{xt} + cu_{xx}}{u_{xx}^2}, \frac{u_{xxt} + cu_{xxx}}{u_{xxx}^{5/3}}, \\
& \frac{u_{xtt} + cu_{xxt} + c^2 u_{xxx}}{u_{xxx}^{7/3}}, \\
& \frac{u_{ttt} + 3cu_{xtt} + 3c^2 u_{xxt} + c^3 u_{xxx}}{u_{xxx}^3}\Bigg)
\end{aligned} \tag{5.3.15}$$

which can be reduced to (5.3.13). It appears that the master PDE supplies the inherited Type II hidden symmetries of (5.3.13). We concluded that hidden symmetries of PDEs could arise from the point symmetries of other

PDEs that reduce to the target equation. We also note that more symmetries may be found by the second method. However, it is not always possible to compute some choices of the group generators [5].

We also demonstrated the case of contact and nonlocal symmetries as the sources of hidden symmetries of PDEs [22]. We considered the "modified" Monge-Ampere equation [43]

$$u_{tt}(u_{xx} - \lambda) - u_{xt}^2 = 0 \tag{5.3.16}$$

(which admitted fifteen point symmetries). We reduced (5.3.16) using the transformation $r = t$, $s = u$ to

$$s_{rr} = 0 \tag{5.3.17}$$

which has eight point symmetries. We observed that five of these point symmetries were inherited from the point symmetries of (5.3.16) and the remaining three were Type II hidden symmetries. Their origin was found to be the heterogeneous intrinsic contact symmetries of the original equation.

We also considered the nonlinear wave equation [22]

$$u_{tt} = \left[f(u)u_x\right]_x \tag{5.3.18}$$

(which possessed three point symmetries). However, by considering the natural (potential) form of (5.3.18) we obtained the two additional symmetries

$$U_4 = \partial_v \tag{5.3.19}$$

$$U_5 = u\partial_t + v\partial_x \tag{5.3.20}$$

where U_5 was found to be a nonlocal or potential symmetry. We reduced (5.3.18) to

$$q_{pp} = 0 \tag{5.3.21}$$

which has eight point symmetries. Only two symmetries of (5.3.21) were inherited from the point symmetries of (5.3.18) and the remaining six symmetries were Type II hidden symmetries. However, for those Type II hidden symmetries U_5 was found to be one origin. From this observation, we conclude that hidden symmetries of PDEs also arise from nonlocal symmetries of the original PDE.

We have established that hidden symmetries of PDEs can arise from point symmetries of other PDEs that can be reduced to the target PDE. They also arise from contact symmetries of the original PDE and we note that it is only heterogeneous intrinsic contact symmetries that manifest themselves as the sources of hidden symmetries. In the case of nonlocal symmetries we only managed to show that they are also the origin of hidden symmetries via potential symmetries.

5.4 Future Work

The main outstanding issue here is to investigate nonlocal symmetries of PDEs as the origin of Type II hidden symmetries more broadly. In particular, it is of interest to consider non-potential nonlocal symmetries. Such an approach may follow the work of Gandarias [20].

In the case of ODEs, it was shown that Type II hidden symmetries could be predicted by taking a geometric approach [31]. It would be of interest to apply a similar geometric analysis for PDEs.

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