RELATIVISTIC SPHERICAL STARS

by

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Abstract

In this thesis we study spherically symmetric spacetimes which are static with a perfect fluid source. The Einstein field equations, in a number of equivalent forms, are derived in detail. The physical properties of a relativistic star are briefly reviewed. We specify two particular choices for one of the gravitational potentials. The behaviour of the remaining gravitational potential is governed by a second order differential equation. This equation has solutions in terms of elementary functions for some cases. The differential equation, in other cases, may be expressed as Bessel, confluent hypergeometric and hypergeometric equations. In such instances the solution is given in terms of special functions. A number of solutions to the Einstein field equations are generated. We believe that these solutions may be used to model realistic stars. Many of the solutions found are new and have not been published previously. In some cases our solutions are generalisations of cases considered previously. For some choices of the gravitational potential our solutions are equivalent to well–known results documented in the literature; in these cases we explicitly relate our solutions to those published previously. We have utilised the computer package MATHEMATICA Version 2.0 (Wolfram 1991) to assist with calculations, and to produce figures to describe the gravitational field. In addition, we briefly investigate the approach of specifying an equation of state relating the energy density and the pressure. The solution of the Einstein field equations, for a linear equation of state, is reduced to integrating Abel's equation of the second kind.

To my family

especially my wife Nompumelelo and my mother Dorothy

and all my friends

for being pillars of support and encouragement.

Preface

The study described in this thesis was carried out in the Department of Mathematics and Applied Mathematics, University of Natal, Durban, during the period January 1992 to December 1993. This thesis was completed under the excellent supervision of Dr. S. D. Maharaj.

This study represents original work by the author. It has not been submitted in any form to another University nor has it been published previously. Where use was made of the work of others it has been duly acknowledged in the text.

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Contents

1 Introduction

General relativity is a relativistic theory of gravitation and has been successfully applied to describe phenomena in astrophysics and cosmology (Misner *et al.* 1973). It reduces to the Newtonian theory for weak gravitational fields. In general relativity the gravitational field of a body is contained in the curvature of spacetime which is described by the Riemann tensor. The matter content is represented by the symmetric energy-momentum tensor for charged and uncharged matter. The energy-momentum tensor is related to the curvature of the manifold via the Einstein field equations which satisfy the conservation laws, namely the Bianchi identities. The Einstein field equations are a set of highly nonlinear partial differential equations which are difficult to integrate without simplifying assumptions. In order to solve the Einstein field equations of general relativity it is sometimes assumed that spacetime admits a particular symmetry in the hope that the field equations are simplified (Maharaj *et al.* 1991). Another approach is to attempt to integrate the field equations directly without *ab initio* specifying a spacetime symmetry, e.g. a conformal Killing vector. The latter is the approach that we follow in this thesis.

It is important to find explicit solutions to the Einstein field equations for astrophysical and cosmological applications. Our interest here is relativistic astrophysics, in particular spherically symmetric stars which are static. Exact solutions to the Einstein field equations are very important as they throw light on the qualitative features of these gravitational fields. They facilitate the investigation and discussion of the physical properties of relativistic stars (Schutz 1985, Shapiro and Teukolsky 1983). It is difficult to study these features in the general Einstein field equations; in an exact solution a physical analysis is easier. In addition explicit calculations, e.g. surface redshift of stars, are possible to describe physical properties of stars. We should emphasise that an exact solution is only the first step in the modelling of stars. For a realistic stellar model we require additional physical constraints (Glass and Goldman 1978).

There exist many solutions to the Einstein field equations in the literature. Only some of the solutions are treated seriously as a large number of the known solutions are not physically acceptable. Some of the famous exact solutions, applicable to relativistic astrophysics, are:

(i) *The Schwarzschild exterior solution* describes the exterior gravitational field to a static, spherically symmetric body. In fact we can show that every spherically symmetric exterior solution is static (and therefore given by the Schwarzschild exterior line element) even if the interior solution is nonstatic. This general result is called Birkhoff's theorem. It is the Schwarzschild exterior solution that is utilised in the classical tests of general relativity: bending of light, perihelion advance, spectral shift and time delay in radar signals.

(ii) *The Schwarzschild interior solution* is valid for the interior of the star where the energy density is taken to be constant. The Schwarzschild interior and the Schwarzschild exterior solutions match smoothly at the boundary of the star. The Schwarzschild interior solution may be used to model relativistic stars for which the variations in the energy density are small, and is a good approximation for small stars in which the pressures are not too large.

(iii) *The Reissner–Nordström solution* represents the exterior gravitational field for a static, spherically symmetric charged body. In practice astrophysical bodies are uncharged and consequently the influence of the electromagnetic field may be neglected. However this solution is important as a simple example of an exact solution of the Einstein–Maxwell system of equations and may be utilised as a first approximation in some physical situations.

(iv) *The Kerr solution* describes the exterior gravitational field of a rotating, axially symmetric gravitating body. The Kerr solution reduces to the Schwarzschild exterior solution in the appropriate limit. We should point out that an interior solution that matches smoothly to the exterior Kerr line element has not yet been found (Stephani 1990).

For a more detailed exposition to the exact solutions of the Einstein field equations the reader is referred to Kramer *et al.* (1980).

In this thesis we investigate static, spherically symmetric spacetimes with a perfect fluid source. These assumptions are usually made in the study of relativistic stars and lead to forms of the field equations which are generalisations of the corresponding Newtonian equations (Schutz 1985). Our objective is to find new solutions to the Einstein field equations that may be applied to relativistic stars. We believe that the solutions presented in this thesis are physically reasonable and may be utilised to model realistic stars.

In chapter 2 we briefly consider only those aspects of differential geometry and general relativity necessary and relevant for this thesis. We begin by introducing the metric tensor field, the metric connection and the covariant derivative on the manifold. These are used to define the Riemann tensor, the Ricci tensor, the Ricci scalar and the Einstein tensor. The matter content is described by the energy–momentum tensor. We are then in a position to motivate the Einstein field equations. The field equations are derived in detail for a static, spherically symmetric line spacetime. Two other equivalent forms of the field equations are obtained which for some applications simplifies the solution of the field equations. In the first form of the field equations it is easier to compare with the corresponding Newtonian equations (Schutz 1985). The second form of the field equations utilises the transformations of Durgapal and Bannerji (1983), and this form is utilised in later chapters. The Schwarzschild exterior solution and the Schwarzschild interior solution with constant energy density are reviewed. A brief outline of some of the physical properties required of interior solutions to the field equations, for a realistic relativistic stellar model, is discussed.

In chapter 3 we consider a particular class of solutions to the Einstein field equations. We choose a form for the metric function which generalises that of Finch and Skea (1989). A mistake in the transformation presented by Finch and Skea (1989) is corrected and we provide details of the derivation missing in their treatment. Then we consider a case which is related to the confluent hypergeometric differential equation via a complex transformation. As this solution is difficult to interpret analytically the behaviour of the metric functions are presented graphically. Two other solutions are found and are related to existing solutions in the literature by appropriate choices of the constants of integration. Finally we point out that our method may be applied to find further new solutions.

In chapter 4 we specify a form for one of the gravitational potentials, different to that used in chapter 3, in the hope of generating a class of new solutions to the Einstein field equations. This form of the metric function generalises that of Durgapal and Bannerji (1983). We generate a new solution to the Einstein field equations which reduces, as a particular case, to the Durgapal and Bannerji (1983) solution. The solution of the field equations, for the general form of the gravitational potential chosen, is reduced to finding a solution of the hypergeometric equation which depends on one parameter. The solution to this equation is given in terms of special functions. We regain our generalisation of the Durgapal and Bannerji (1983) solution as a special case of the hypergeometric equation. In addition a new solution is obtained, in terms of elementary functions, as a special case of the general hypergeometric solution. The package MATHEMATICA Version 2.0 (Wolfram 1991) has been utilised to assist in obtaining explicit forms of the hypergeometric function for particular values of the parameter.

We briefly investigate another approach of finding solutions to the Einstein field equations in chapter 5. In this approach we assume a linear equation of state relating the energy density and pressure. This approach was used by Ibanẽz and Sanz (1982) to obtain a new solution. The paper of Ibanẽz and Sanz (1982) is briefly reviewed and their field equations analysed. Their line element and field equations are related to the standard literature by a coordinate transformation. Then the solution of the Einstein field equations is reduced to obtaining a solution to a first order differential equation. This is Abel's equation of the second kind which is difficult to integrate. An equivalent second order differential equation is generated which for some cases may be more easily integrated.

The results obtained in this thesis are summarised in the conclusion. Some areas of future research emanating from the results in this thesis and other related topics are pointed out. We believe that many of the results obtained in this thesis are original and have not been previously documented.

2 Static Spherically Symmetric Spacetimes

2.1 Introduction

In this chapter we briefly introduce those aspects of differential geometry and general relativity relevant for this thesis. For detailed expositions on differentiable manifolds and tensor analysis the reader is referred to Bishop and Goldberg (1968), Hawking and Ellis (1973), Misner *et al.* (1973) and Wald (1984). In §2.2 we introduce the metric tensor field, the metric connection and the covariant derivative on the spacetime manifold. This makes it possible to define the Riemann tensor, the Ricci tensor, the Ricci scalar and the Einstein tensor. The general energy–momentum tensor, for uncharged matter, is defined and related to the curvature of the manifold via the Einstein field equations. In §2.3 we study the spacetime geometry of static, spherically symmetric spacetimes. The field equations are derived in detail in Schwarzschild coordinates. A second form of the field equations is obtained by the introduction of the mass function which is proportional to the total mass contained within a sphere. A third form of the field equations is also obtained by the introduction of a new coordinate and redefining two metric functions. This third form of the field equations was first utilised by Durgapal and Bannerji (1983) and helps in the search to find new solutions. In §2.4 we briefly review the Schwarzschild exterior solution and the Schwarzschild interior solution with a constant energy density. We consider conditions that should be imposed on the solutions of the field equations for a realistic relativistic stellar model in §2.5. Finally we consider the Buchdahl limit which provides a general limit on the radius of relativistic stars.

2.2 Spacetime Geometry

We take spacetime M to be a 4–dimensional differentiable manifold endowed with a symmetric, nondegenerate metric field $\mathbf{g}$ of signature $(- + + +)$. A manifold with an indefinite metric tensor field, as is the case in general relativity, is termed a pseudo–Riemannian manifold. Points in the manifold may be labelled by the real coordinates $(x^a) = (x^0, x^1, x^2, x^3)$ where x^0 is timelike, and x^1, x^2, x^3 are spacelike. The metric tensor field $\mathbf{g}$ is important for the discussion of metrical properties in a manifold and is necessary for the definition of the length of a curve in M. If the curve is given by $x^a(u)$, $u_1 \leq u \leq u_2$, then the length is defined as the integral

$$s = \int_{u_1}^{u_2} \left| g_{ab} \frac{dx^a}{du} \frac{dx^b}{du} \right|^{1/2} du$$

Equivalently we may write

$$ds^2 = g_{ab} dx^a dx^b \tag{2.1}$$

where we have dropped the modulus signs, without loss of generality, to obtain the socalled line element (2.1) or the fundamental metric form. The metric tensor field $\mathbf{g}$ may be associated with the metric connection $\mathbf{\Gamma}$. The fundamental theorem of Riemannian geometry guarantees the existence of a unique symmetric connection preserving inner products under parallel transport. This connection is called the metric connection $\mathbf{\Gamma}$ and may be expressed in terms of the components of the metric tensor g and its derivatives

$$\Gamma^a{}_{bc} = \frac{1}{2} g^{ad}(g_{cd,b} + g_{db,c} - g_{bc,d}) \tag{2.2}$$

where we utilise the notation that commas denote partial differentiation.

Let **Y** be a covariant vector field. Then the covariant derivative of **Y** is given by

$$Y_{a;b} = Y_{a,b} - \Gamma^d{}_{ab} Y_d$$

where the semicolon denotes covariant differentiation. The covariant derivative is the modification of the partial derivative such that when operating on a (r, s) tensor field it produces a $(r, s+1)$ tensor field on M. On covariantly differentiating for a second time we obtain

$$Y_{a;bc} - Y_{a;cb} = \left(\Gamma^d{}_{ac,b} - \Gamma^d{}_{ab,c} + \Gamma^e{}_{ac}\Gamma^d{}_{eb} - \Gamma^e{}_{ab}\Gamma^d{}_{ec}\right) Y_d$$

$$\equiv R^d{}_{abc} Y_d$$

The quantity $R^d{}_{abc}$ is a $(1,3)$ tensor field and is called the Riemann tensor or the curvature tensor. The Riemann tensor provides a measure of the curvature of a manifold, i.e. it provides a measure of deviation from flatness. In flat Minkowski spacetime we have that $R^a{}_{bcd} = 0$ and for a curved spacetime $R^a{}_{bcd}$ is nonvanishing in general . From the above it is clear that the nonvanishing of the Riemann tensor arises from the noncommutativity of the covariant derivative. Upon contraction of the Riemann tensor

$$R^a{}_{bcd} = \Gamma^a{}_{bd,c} - \Gamma^a{}_{bc,d} + \Gamma^a{}_{ec}\Gamma^e{}_{bd} - \Gamma^a{}_{ed}\Gamma^e{}_{bc} \tag{2.3}$$

we obtain

$$R_{ab} = R^c{}_{acb}$$

$$= \Gamma^{d}{}_{ab,d} - \Gamma^{d}{}_{ad,b} + \Gamma^{e}{}_{ab}\Gamma^{d}{}_{ed} - \Gamma^{e}{}_{ad}\Gamma^{d}{}_{eb} \tag{2.4}$$

where R_{ab} is the Ricci tensor. A contraction of the Ricci tensor (2.4), i.e. a second contraction of the Riemann tensor (2.3), yields

$$R = R^{a}{}_{a}$$

$$= g^{ab} R_{ab} \tag{2.5}$$

where R is the Ricci scalar. We construct the Einstein tensor $\mathbf{G}$, in terms of the Ricci tensor (2.4) and the Ricci scalar (2.5), as follows

$$G^{ab} = R^{ab} - \frac{1}{2} R g^{ab} \tag{2.6}$$

A distinguishing characteristic of the Einstein tensor is that it has zero divergence

$$G^{ab}{}_{;b} = 0 \tag{2.7}$$

which follows from definition. This property of the Einstein tensor is sometimes called the Bianchi identity.

In general relativity the matter distribution is described by the symmetric energy–momentum tensor $\mathbf{T}$ given by

$$T^{ab} = (\rho + p)u^a u^b + p g^{ab} + q^a u^b + q^b u^a + \pi^{ab} \tag{2.8}$$

where ρ is the energy density, p is the isotropic (kinetic) pressure, q^a is the heat flow vector and π^{ab} is the anisotropic (stress) pressure tensor. These quantities are measured relative to a fluid 4–velocity $\mathbf{u}$. We will only consider perfect fluids for which there are no heat conduction and stress terms:

$$q^a = 0 \qquad \pi^{ab} = 0$$

For a perfect fluid energy–momentum tensor equation (2.8) takes the simplified form

$$T^{ab} = (\rho + p)u^a u^b + pg^{ab} \tag{2.9}$$

The perfect fluid form (2.9) is applicable in many situations in relativistic astrophysics and cosmology (Misner *et al.* 1973). The energy–momentum tensor (2.8) is coupled to the Einstein tensor (2.6) via the Einstein field equations

$$G^{ab} = T^{ab} \tag{2.10}$$

in suitable units. The field equations (2.10) give the relationship between the curvature of the manifold M and the matter distribution in spacetime. For further information on the motivation and derivation of the Einstein field equations see Felice and Clarke (1990), Misner *et al.* (1973) and Stephani (1990).

2.3 The Field Equations

The field equations (2.10) are highly nonlinear and to find solutions we require simplifying assumptions. The standard approach is to impose a symmetry requirement on the spacetime manifold (Kramer *et al.* 1980). In this thesis we are concerned with solutions to the Einstein field equations which are static and spherically symmetric. This means we are considering that class of spacetime, admitting a Lie algebra spanned by four Killing vectors, invariant under rotations. Such solutions are applicable in relativistic astrophysics (Shapiro and Teukolsky 1983). In standard coordinates $(x^a) = (t, r, \theta, \phi)$, the generic form of the line element is given by

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.11}$$

For the line element (2.11) the nonvanishing connection coefficients (2.2) are given by

$$\Gamma^0{}_{01} = \nu' \qquad \Gamma^2{}_{12} = \frac{1}{r}$$

$$\Gamma^1{}_{00} = \nu' e^{2(\nu-\lambda)} \qquad \Gamma^2{}_{33} = -\sin\theta\cos\theta$$

$$\Gamma^1{}_{11} = \lambda' \qquad \Gamma^3{}_{13} = \frac{1}{r}$$

$$\Gamma^1{}_{22} = -re^{-2\lambda} \qquad \Gamma^3{}_{23} = \cot\theta$$

$$\Gamma^1{}_{33} = -re^{-2\lambda}\sin^2\theta$$

where primes denote differentiation with respect to r. Substituting the above connection coefficients in the Ricci tensor (2.4) for the line element (2.11) we obtain the nonvanishing components of the Ricci tensor (2.4):

$$R_{00} = \left[\nu'' + \nu'^2 - \nu'\lambda' + \frac{2\nu'}{r}\right] e^{2\nu-2\lambda} \tag{2.12a}$$

$$R_{11} = -\left[\nu'' + \nu'^2 - \nu'\lambda' - \frac{2\lambda'}{r}\right] \tag{2.12b}$$

$$R_{22} = 1 - [1 + r(\nu' - \lambda')]\, e^{-2\lambda} \tag{2.12c}$$

$$R_{33} = \sin^2\theta R_{22} \tag{2.12d}$$

Then the Ricci tensor components (2.12) and the definition (2.5) yield the following form for the Ricci scalar

$$R = 2\left[\frac{1}{r^2} - \left(\nu'' + \nu'^2 - \nu'\lambda' + \frac{2\nu'}{r} - \frac{2\lambda'}{r} + \frac{1}{r^2}\right)e^{-2\lambda}\right] \tag{2.13}$$

for a static, spherically symmetric spacetime. The Ricci tensor components (2.12) and the Ricci scalar (2.13) generate the corresponding nonvanishing components of the Einstein tensor (2.6). These are given by

$$G_{00} = \frac{1}{r^2}e^{2\nu}\left[r\left(1 - e^{-2\lambda}\right)\right]' \tag{2.14a}$$

$$G_{11} = -\frac{1}{r^2}e^{2\lambda}\left(1 - e^{-2\lambda}\right) + \frac{2\nu'}{r} \tag{2.14b}$$

$$G_{22} = r^2 e^{-2\lambda}\left[\nu'' + \nu'^2 + \frac{\nu'}{r} - \nu'\lambda' - \frac{\lambda'}{r}\right] \tag{2.14c}$$

$$G_{33} = \sin^2\theta G_{22} \tag{2.14d}$$

for the line element (2.11).

We formulate the field equations for the case of a perfect fluid energy–momentum tensor. The energy–momentum tensor (2.9) and the Einstein tensor components (2.14) for a comoving fluid velocity vector

$$u^a = e^{-\nu}\delta^a{}_0 \qquad u^a u_a = -1$$

generate the Einstein field equations

$$\frac{1}{r^2}\left[r\left(1 - e^{-2\lambda}\right)\right]' = \rho \tag{2.15a}$$

$$-\frac{1}{r^2}\left(1-e^{-2\lambda}\right)+\frac{2\nu'}{r}e^{-2\lambda}=p \tag{2.15b}$$

$$e^{-2\lambda}\left[\nu''+\nu'^2+\frac{\nu'}{r}-\nu'\lambda'-\frac{\lambda'}{r}\right]=p \tag{2.15c}$$

for a static, spherically symmetric spacetime. The conservation laws $T^{ab}{}_{;b}=0$, obtained from (2.7) and (2.10), reduce to the equation

$$\frac{dp}{dr}=-(\rho+p)\frac{d\nu}{dr} \tag{2.16}$$

Equation (2.16) is a direct consequence of the field equations and may be used in place of one of the field equations in (2.15). The field equations (2.15) are three equations with four unknowns and consequently we need an additional condition to find a solution. Sometimes it is convenient to assume an equation of state of the form

$$p=p(\rho)$$

which will have different functional forms for different fluids. The relationship between equations of state and solutions to the field equations is pursued in chapter 5. The field equations (2.15) may be expressed in a variety of equivalent forms which for some applications make the integration process simpler. In the following we present two equivalent forms of (2.15).

A second form of the field equations is obtained by introducing the "mass" function $m(r)$. From equation (2.15a) it follows that

$$\frac{1}{2}r(1-e^{-2\lambda})=\frac{1}{2}\int_0^r \rho(\tilde{u})\tilde{u}^2d\tilde{u}+k$$

where k is a constant. This suggests that we define a "mass" function $m(r)$ as

$$m(r)\equiv\frac{1}{2}\int_0^r \rho(\tilde{u})\tilde{u}^2d\tilde{u} \tag{2.17}$$

We may interpret the quantity $m(r)$ as being proportional to the total mass contained within a sphere of radius r. However note that here r is the coordinate radius (the true radius of the sphere is given as

$$R(r) = \int_0^r e^{\lambda(\tilde{u})} d\tilde{u}$$

by Stephani (1990)). On utilising (2.17) the field equation (2.15a) can be written as

$$m(r) = \frac{1}{2} r \left(1 - e^{-2\lambda}\right) \tag{2.18}$$

where we have set $k = 0$ so that $e^{-2\lambda}$ remains finite at $r = 0$. Essentially in (2.18) the metric variable λ has been replaced by the new function $m(r)$. In terms of the mass function (2.17) we can express the field equations (2.15) as the following system of differential equations

$$\frac{d}{dr} m(r) = \frac{1}{2} r^2 \rho \tag{2.19a}$$

$$\frac{d\nu}{dr} = \frac{\frac{1}{2} p\, r^3 + m(r)}{r\,[r - 2m(r)]} \tag{2.19b}$$

$$\nu'' + \nu'^2 + \left[\frac{r - [r\, m(r)]'}{r[r - 2m(r)]}\right] \nu' - \frac{[r\, m'(r) - m(r)]}{r^2[r - 2m(r)]} = \frac{p\, r}{[r - 2m(r)]} \tag{2.19c}$$

In the above we have three equations in the four unknowns ρ (or m), p, ν and λ. Once λ is specified then $m(r)$ and ρ can be found from (2.18) and (2.19a). The remaining unknowns p and ν are then determined by (2.19b) and (2.19c) in principle. In practice this is not an easy matter as the resulting equations are highly nonlinear. It is interesting to observe that we can express the pressure gradient in terms of m, ρ and p. On substituting (2.19b) into (2.16) we obtain the result

$$\frac{dp}{dr} = -\frac{(\rho + p)[m(r) + \frac{1}{2} p r^3]}{r[r - 2m(r)]} \tag{2.20}$$

known as the Oppenheimer–Volkoff equation (Oppenheimer and Volkoff 1939). In the Newtonian limit

$$p << \rho \qquad\qquad m << r$$

the pressure gradient (2.20) becomes

$$\frac{dp}{dr} = -\frac{\rho m(r)}{r^2} \tag{2.21}$$

(Schutz 1985). This is exactly the equation for hydrostatic equilibrium for Newtonian stars (Chandrasekar 1939). A comparision of the relativistic equation (2.20) and the Newtonian equation (2.21) reveals that the relativistic correction tends to steepen the pressure gradients relative to the Newtonian gradients. This means that for a fluid to remain static it must have stronger internal forces in general relativity than in Newtonian theory.

We now consider a third form of the field equations that we utilise in chapters 3 and 4 to obtain new solutions. In this case it is convenient to introduce a new coordinate x and two metric functions $y(x)$ and $Z(x)$. The appropriate transformation is given by

$$x = Cr^2$$

$$Z(x) = e^{-2\lambda(r)}$$

$$A^2 y^2(x) = e^{2\nu(r)}$$

where A and C are constants. For this transformation the Einstein field equations (2.15) take the form

$$\frac{1-Z}{x} - 2\dot{Z} = \frac{\rho}{C} \tag{2.22a}$$

$$\frac{4Z\dot{y}}{y} + \frac{Z-1}{x} = \frac{p}{C} \tag{2.22b}$$

$$4Zx^2\ddot{y} + 2\dot{Z}x^2\dot{y} + (\dot{Z}x - Z + 1)y = 0 \tag{2.22c}$$

where dots represent differentiation with respect to x. The metric functions y and Z are now dependent on the new coordinate x. This form of the field equations has been used by Durgapal and Bannerji (1983), Durgapal and Fuloria (1985) and Finch and Skea (1989) to generate new solutions.

2.4 Schwarzschild Solutions

Schwarzschild (1916*a*) was the first person to obtain an exact solution to the Einstein field equations. This solution represents the gravitational field exterior to a static, spherically symmetric body. Later he obtained a second solution describing the gravitational field inside the spherically symmetric body for a constant energy density source. These two solutions match smoothly at the surface of the body. In chapters 3 and 4 we will find new solutions for other forms of the energy–momentum tensor. Such solutions are important in astrophysics and may be used to model relativistic stars (Shapiro and Teukolsky 1983).

(a) *Schwarzschild exterior solution*

We provide only an outline of the derivation of the Schwarzschild exterior solution as this is well documented in the literature (Stephani 1990). In the region outside the star both the energy density and pressure vanish

$$\rho = 0 \qquad p = 0$$

Integrating the field equations (2.19) we obtain the metric functions

$$e^{\nu} = \left(1 - \frac{2M}{r}\right)^{1/2} \qquad e^{\lambda} = \left(1 - \frac{2M}{r}\right)^{-1/2}$$

where we have specified the constants of integration by utilising the boundary conditions. Here the constant M represents the energy density of the star. The Schwarzschild exterior solution may then be written as

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{2.23}$$

One can describe the local geometry of spacetime in the solar system to a good approximation by the Schwarzschild solution (2.23). The Schwarzschild exterior solution is essential for a discussion of the classical tests of general relativity: the bending of light, perihelion advance of Mercury, gravitational red shift and the time delay in radar propagation. For a thorough treatment of these classical tests see D'Inverno (1992), Wald (1984) and Will (1981). For the exterior Schwarzschild solution (2.23) the metric components become singular when $r = 0$ and $r = 2M$. The singularity at $r = 2M$ is not a true singularity of the spacetime structure but represents a breakdown of the coordinates that have been used to obtain the general form of the line element (2.11). We may avoid this coordinate singularity by utilising the Kruskal–Szekeres coordinates which cover all of spacetime. For a detailed treatment of the relationship between the Schwarzchild coordinates and Kruskal–Szekeres coordinates the reader is referred to Felice and Clarke (1990), Misner *et al.* (1973) and Stephani (1990).

We have derived the exterior Schwarzschild solution corresponding to an interior static gravitating body. However, even if the interior is nonstatic the exterior solution is given by the Schwarzschild line element (2.23). This result is called the Birkhoff's theorem: every spherically symmetric exterior solution is static. This is a remarkable result and depends only on spherical symmetry of the interior source. Note that Birkhoff's theorem may be interpreted as the analogue of the corresponding result in electrodynamics, i.e. a spherically symmetric distribution of charge does not radiate (Stephani 1990).

(b) *Schwarzschild interior solution*

In order to obtain an interior solution it is necessary to fix an equation of state. For the Schwarzchild interior solution (Schwarzschild 1916*b*) we suppose that the energy density is uniform. Let us consider the region inside the star with

$$\rho = \rho_0, \qquad r \leq R$$

where ρ_0 is constant and R is the radius of the star. On integrating (2.19a) we obtain

$$m(r) = \frac{1}{6}\rho_0 r^3 \tag{2.24}$$

where we have set the constant of integration to be zero so that at $r = 0$ we have $m = 0$. Upon substituting (2.24) into (2.20) and integrating we obtain

$$\frac{\rho_0 + 3p}{\rho_0 + p} = \left(\frac{\rho_0 + 3p_c}{\rho_0 + p_c}\right)\left(1 - \frac{2m}{r}\right)^{1/2} \tag{2.25}$$

where $(\rho_0 + 3p_c)/(\rho_0 + p_c)$ is the constant of integration in which $p_c = p(0)$ is the central pressure. We obtain the following explicit form of the central pressure

$$p_c = \rho_0 \left[\frac{1 - (1 - 2M/R)^{1/2}}{3\,(1 - 2M/R)^{1/2} - 1}\right]$$

from (2.25) and where we have set $m(R) = M$. Replacing p_c in (2.25) we obtain the pressure

$$p = \rho_0 \left[\frac{(1 - 2Mr^2/R^3)^{1/2} - (1 - 2M/R)^{1/2}}{3\,(1 - 2M/R)^{1/2} - (1 - 2Mr^2/R^3)^{1/2}}\right] \tag{2.26}$$

Now on substituting (2.26) into (2.16) and integrating we obtain the first metric function ν as

$$e^{\nu} = \frac{3}{2}(1 - 2M/R)^{1/2} - \frac{1}{2}\left(1 - 2Mr^2/R^3\right)^{1/2} \tag{2.27}$$

where we have chosen the constant of integration such that the interior solution joins continuously to the exterior solution at the surface of the star. On using (2.18) and (2.24) the remaining metric function λ is given by

$$e^{\lambda} = \left[1 - \frac{1}{3}\rho_0 r^2\right]^{-1/2} \tag{2.28}$$

The Schwarzschild interior solution may then be written as

$$ds^2 = -\left[\frac{3}{2}(1 - 2M/R)^{1/2} - \frac{1}{2}\left(1 - 2Mr^2/R^3\right)^{1/2}\right]^2 dt^2 + \left[1 - \frac{1}{3}\rho_0 r^2\right]^{-1} dr^2$$

$$+ r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.29}$$

The solution (2.29) may be used to model relativistic stars for which the variation in ρ is small. It is a good approximation for small stars in which the pressures are not too large.

2.5 Physical Properties

The Einstein field equations (2.19) admit a variety of solutions. However many of these solutions do not correspond to a physical matter distribution. For a realistic relativistic stellar model we need to impose conditions on the solutions of the field equations. This greatly reduces the number of possibilities allowed by the field equations. Restricting the model to physically acceptable matter we generate the following conditions:

(a) The pressure and energy density should be positive and finite everywhere in the interior of the star including the origin:

$$0 < p < \infty \qquad 0 < \rho < \infty$$

(b) The pressure and energy density should be monotonic decreasing functions of r outward to the surface of the star. At the boundary the pressure must vanish:

$$\frac{dp}{dr} \leq 0 \qquad \frac{d\rho}{dr} \leq 0 \qquad p(R) = 0$$

(c) The interior line element should be joined continuously to the exterior Schwarzschild line element at the boundary of the star.

$$e^{\nu(R)} = \left(1 - \frac{2M}{R}\right)^{1/2} \qquad e^{\lambda(R)} = \left(1 - \frac{2M}{R}\right)^{-1/2}$$

(d) The speed of sound should be everywhere less than the speed of light:

$$0 \leq \frac{dp}{d\rho} \leq 1$$

For a complete analysis of the conditions (a)–(d) for a relativistic star the reader is referred to Knutsen's analysis (Knutsen 1989) of the analytic solution of Durgapal and Fuloria (1985). It should be pointed out that most solutions do not satisfy all the conditions (a)–(d) throughout the interior of the star and may be valid only for some regions of spacetime. For example the Tolman solutions (Tolman 1939) become singular at the center. Such solutions have to be treated as an envelope of the core of the star and have to be matched to some other solution valid for the core.

Some of the conditions (a)–(d) may in fact be too stringent and one has to be careful not to neglect solutions that are physically reasonable. For example condition (b) requiring that the pressure and energy density be strictly decreasing outwards to the surface may be too restrictive for some physical applications (Maharaj and Maartens 1989). We should also point out that our model utilises a perfect fluid energy–momentum tensor. Some analyses suggest that an anisotropic energy–momentum tensor may produce a realistic solution to model high surface redshifts (Bowers and Liang 1974). For an exact solution to the field equations, with a nonvanishing anisotropic pressure terms, see Maharaj and Maartens (1989). There is also the possibility of having a nonzero electromagnetic field in which case the energy–momentum tensor (2.9) has to be modified. For the physical relevance of solutions representing charged stars in astrophysics the reader is referred to Herrera and Ponce de Leon (1985), Maartens *et al.* (1986) and Maartens and Maharaj (1990).

For the Schwarzschild interior solution we observe that

$$p_c \to \infty \quad \text{as} \quad M/R \to 4/9$$

Thus there are no uniform density stars with radii smaller than $9M/4$: to support such stars in a static configuration requires pressures larger than infinite. In fact this restriction on the radius is a limit on more general stars. It is possible to obtain a condition on the maximum possible mass M for a given radius R from the condition that ρ should decrease outwards ($p'(r) < 0$) on the grounds of stability . The pressure p must vanish on the surface of the star and we require that ρ and p are finite in the interior. Then a spherically symmetric star can only exist in a state of stable equilibrium if M and R satisfy the inequality

$$\frac{M}{R} < \frac{4}{9}$$

This limit on the radius of the star was found by Buchdahl (1959) and is sometimes referred to as Buchdahl's theorem.

3 A generalisation of Finch and Skea

3.1 Introduction

In this chapter we consider a class of solutions to the Einstein field equations. The field equations are reduced to a system of three equations in three unknowns by assuming an explicit form for one of the gravitational potentials. The metric function depends on the parameter n which may take on any real value. Then the differential equation governing the remaining metric function is obtained. This treatment of the gravitational potentials is done in §3.2. We first consider the case $n = -1$, which was also analysed by Duorah and Ray (1987) and Finch and Skea (1989), in §3.3. Mistakes in their paper are pointed out and we provide the details of the appropriate transformation missing in their analysis. In §3.4 we consider $n = -2$ and show that, in fact, this case is related to the confluent hypergeometric differential equation via a complex transformation. A graphical representation of the behaviour of the gravitational potentials is presented as an illustration. We find the solution of the Einstein field equations for $n = 1$, in §3.5, and provide the appropriate transformations that relate this result to the Schwarzschild interior solution. The case $n = 2$ is analysed in §3.6 and the general solution is related to that of Mehra (1966). Finally in §3.7 we consider some other cases of n which illustrate the difficulty of finding further new solutions.

3.2 Choice of the metric function Z

The field equations (2.22) comprise a system of three equations in the four unknowns ρ, p, y and Z. Clearly we require an additional condition to find a solution to this system. One option is to assume an equation of state relating ρ and p. We investigate this possibility in chapter 5. Another possibility is to specify a form of one of ρ, p, y and Z. Finch and Skea (1989) choose a simple form for the metric function $Z(x)$ which leads to a new solution of the field equations. Here we assume a form for $Z(x)$ that generalises that of Finch and Skea (1989) in the anticipation of finding other new solutions to the Einstein field equations (2.22).

The metric function $Z(x)$ is chosen to have the form

$$Z = (1 + x)^n \tag{3.1}$$

where we take $n \neq 0$. If $n = 0$ then the metric function $e^{\lambda} = 1$ and furthermore ρ vanishes from (2.22a), which is not acceptable. Upon substituting (3.1) into the field equation (2.22c) we obtain

$$4x^2(1+x)^n\ddot{y} + 2nx^2(1+x)^{n-1}\dot{y} + \left[nx(1+x)^{n-1} - (1+x)^n + 1\right] y = 0 \tag{3.2}$$

which is a differential equation governing the behaviour of $y(x)$. Note that we have essentially reduced the solution of the field equations to integrating (3.2). Once a solution $y(x)$ is found the energy density ρ and pressure p are obtained from (2.22a) and (2.22b) respectively. In the following sections of this chapter we consider different values of n and seek solutions from the ordinary differential equations that result from (3.2). We do not undertake an analysis of the physical properties of the solutions as this falls outside the scope of this thesis. Our intention is to find exact solutions to the Einstein field equations.

3.3 The case $n = -1$

This case was first investigated by Duorah and Ray (1987). However their solution is incorrect because of an elementary mistake that arises in the integration process. This mistake was pointed out by Finch and Skea (1989) in their thorough investigation of the solution of the field equations for this case. With $n = -1$ equation (3.1) gives the metric function

$$Z = \frac{1}{1+x} \tag{3.3}$$

and the second metric function $y(x)$ is governed by

$$4(1+x)\ddot{y} - 2\dot{y} + y = 0 \tag{3.4}$$

from (3.2).

Finch and Skea (1989) state that the above equation (3.4) may be transformed into a Bessel equation of order $\frac{3}{2}$. Even though the solution given by Finch and Skea (1989) is correct the intermediate transformation

$$u(v) = y(x)x^{3/4}$$

$$v = \sqrt{1+x}$$

leading to the Bessel equation is not valid. This may be easily verified by substituting the above transformation into (3.4) to obtain

$$v^2\frac{d^2u}{dv^2} + \left[\frac{2-5v^2}{v^2-1}\right]v\frac{du}{dv} + \left[v^2 + \frac{3v^2}{2(v^2-1)} + \frac{21v^4}{4(v^2-1)^{7/4}}\right]u = 0$$

which does not admit Bessel functions of order $\frac{3}{2}$ as a solution. In this section we rederive the Finch and Skea (1989) solution, correcting the mistakes in their derivation and providing details of the transformations involved. We should point out that the details of the solution presented by Finch and Skea (1989) are sketchy and it is difficult to follow their derivation. We attempt to provide the details missing from their arguments in addition to correcting mistakes in their transformations. Their solution is important in relativistic astrophysics and we believe that our analysis makes their results more accessible.

It is convenient to transform (3.4) by introducing the new variable

$$X = 1 + x$$

Then (3.4) becomes

$$4X\frac{d^2Y}{dX^2} - 2\frac{dY}{dX} + Y = 0 \tag{3.5}$$

where $y(X) = Y$. We now introduce a new function $u(X)$ such that

$$Y(X) = u(X)X^m$$

where $m \epsilon R$. Then the ordinary differential equation (3.5) becomes

$$4X^2\frac{d^2u}{dX^2} + (8m-2)X\frac{du}{dX} + (4m^2 - 6m + X)u = 0$$

If we define

$$\mathcal{X} = X^\alpha$$

where $\alpha \epsilon R$ then this differential equation takes the form

$$4\alpha^2 X^{2\alpha}\frac{d^2u}{d\mathcal{X}^2} + [4\alpha(\alpha-1)X^\alpha + 2\alpha(4m-1)X^\alpha]\frac{du}{d\mathcal{X}} + [4m^2 - 6m + X]u = 0 \tag{3.6}$$

We need to replace X with $\mathcal{X}$ in (3.6) such that a Bessel equation is obtained. On choosing $\alpha = \frac{1}{2}$ and $m = \frac{3}{4}$ in equation (3.6) we obtain

$$\mathcal{X}^2\frac{d^2u}{d\mathcal{X}^2} + \mathcal{X}\frac{du}{d\mathcal{X}} + \left[\mathcal{X}^2 - \left(\tfrac{3}{2}\right)^2\right]u = 0$$

which is a Bessel equation of order $\frac{3}{2}$. The desired transformation for (3.4) is then given by

$$u(v) = y(x)\,(1+x)^{-3/4}$$

$$v = \sqrt{1+x}$$

which corrects the transformation of Finch and Skea (1989).

Now on using the above transformation in the differential equation (3.4) we obtain the Bessel equation

$$v^2\frac{d^2u}{dv^2} + v\frac{du}{dv} + \left[v^2 - \left(\tfrac{3}{2}\right)^2\right]u = 0$$

where we have utilised the same variables as Finch and Skea (1989). The solution, in the variables u and v, of this Bessel equation is given by

$$u(v) = a\,J_{3/2}(v) + b\,J_{-3/2}(v) \tag{3.7}$$

In the above solution a and b are constants and $J_{3/2}$ and $J_{-3/2}$ are linearly independent Bessel functions of fractional order. It is possible to express the solution (3.7) in terms of elementary functions by using the following identities

$$J_{3/2}(v) = \sqrt{\frac{2}{\pi v}}\,\frac{\sin v}{v} - \sqrt{\frac{2}{\pi v}}\,\cos v$$

$$J_{-3/2}(v) = -\sqrt{\frac{2}{\pi v}}\,\frac{\cos v}{v} - \sqrt{\frac{2}{\pi v}}\,\sin v$$

Then the general solution of (3.4) can be written as

$$y(x) = \left(c_1 + c_2\sqrt{x+1}\,\right)\sin\sqrt{x+1} + \left(c_2 - c_1\sqrt{x+1}\,\right)\cos\sqrt{x+1} \tag{3.8}$$

where we have redefined the constants

$$c_1 = a\sqrt{\frac{2}{\pi}} \qquad\qquad c_2 = -b\sqrt{\frac{2}{\pi}}$$

The metric function $y(x)$, given by (3.8), is now in the form presented by Finch and Skea (1989).

The metric functions y and Z are given by (3.8) and (3.3) respectively. The quantities ρ and p may be calculated from (2.22a) and (2.22b). The solution to the field equations (2.22), in terms of the radial coordinate r, is then given by the system

$$\frac{\rho}{C} = \frac{3 + Cr^2}{(1 + Cr^2)^2} \tag{3.9a}$$

$$\frac{p}{C} = -\frac{1}{1 + Cr^2}\frac{(\beta\sqrt{1 + Cr^2} + 1) + (\beta - \sqrt{1 + Cr^2}\,)\tan\sqrt{1 + Cr^2}}{(\beta\sqrt{1 + Cr^2} - 1) - (\beta + \sqrt{1 + Cr^2}\,)\tan\sqrt{1 + Cr^2}} \tag{3.9b}$$

$$e^{\lambda} = \sqrt{1 + Cr^2} \tag{3.9c}$$

$$e^{\nu} = A\Big[\,(\,c_1 + c_2\sqrt{1 + Cr^2}\,)\,\sin\sqrt{1 + Cr^2}$$

$$+(\,c_2 - c_1\sqrt{1 + Cr^2}\,)\,\cos\sqrt{1 + Cr^2}\,\Big] \tag{3.9d}$$

where we have set $\beta = c_1/c_2$. We have utilised the same notation, in this section, as Finch and Skea (1989) to facilitate comparison with their results. For a detailed analysis of the physical features of the solution (3.9) the reader is referred to Finch and Skea (1989). We should point out that this solution is regular in the interior of the relativistic star and matches smoothly to the Schwarzschild exterior at the boundary.

3.4 The case $n = -2$

Upon substituting $n = -2$ into (3.1) we obtain

$$Z = \frac{1}{(1+x)^2} \tag{3.10}$$

and equation (3.2) becomes

$$4(1+x)\ddot{y} - 4\dot{y} + (x+3)y = 0$$

We transform the above differential equation into a more convenient form by introducing a new variable

$$X = 1 + x$$

to obtain

$$4X\frac{d^2Y}{dX^2} - 4\frac{dY}{dX} + (X+2)Y = 0 \tag{3.11}$$

where $y(X) = Y$. We have not succeeded in integrating this equation in terms of elementary functions.

However we can transform (3.11) into a well-known ordinary differential equation by letting

$$y = e^{-\frac{1}{2}z} X^2 F(z) \qquad\qquad z = iX$$

On substituting this into (3.11) and simplifying we obtain

$$z\frac{d^2F}{dz^2} + (3-z)\frac{dF}{dz} - \left(\frac{3}{2} + \frac{i}{2}\right)F = 0$$

which is of the form

$$z\frac{d^2F}{dz^2} + (b-z)\frac{dF}{dz} - aF = 0 \tag{3.12}$$

where a and b are constants. Equation (3.12) is the confluent hypergeometric differential equation. The general solution is given in terms of the Kummer functions

$M(a;b;z)$ and $U(a;b;z)$. For our differential equation in this case we have

$$a = \frac{3}{2} + \frac{i}{2} \qquad\qquad b = 3$$

For the differential equation (3.11) the general solution is then given by

$$Y = e^{-\frac{1}{2}iX}\, X^2 \left[c_1\, M\left(\frac{3}{2} + \frac{i}{2}; 3; iX\right) + c_2\, U\left(\frac{3}{2} + \frac{i}{2}; 3; iX\right)\right] \tag{3.13}$$

where c_1 and c_2 are constants. In terms of the metric function $y(x)$ we have

$$y(x) = e^{-\frac{1}{2}i(1+x)}\,(1+x)^2 \left[c_1\, M\left(\frac{3}{2} + \frac{i}{2}; 3; i(1+x)\right)\right.$$

$$\left. + c_2\, U\left(\frac{3}{2} + \frac{i}{2}; 3; i(1+x)\right)\right] \tag{3.14}$$

The properties of the confluent hypergeometric equation and the Kummer functions M and U are analysed in detail by Abramowitz and Stegun (1972). As far as we are aware solutions of this type, for spherically symmetric gravitational fields, have not been considered previously. We believe that the Kummer functions (3.14) are new solutions to the Einstein field equations for a static, spherically symmetric gravitational field.

The metric functions $y(x)$ and $Z(x)$ are given by (3.14) and (3.10) respectively. The matter variables ρ and p can be calculated from (2.22a) and (2.22b) respectively. Thus we generate the general solution of the field equations for $n = -2$. Note that $i = \sqrt{-1}$ appears in the right hand side of (3.14). In principle we can reexpress the solution in terms of only the real variable x. However this is difficult in practice as i appears as part of the parameters and the argument of the special functions M and U. This case is the object of ongoing research. Note that it is possible to obtain a graphical representation of the solution. For example with $c_1 = 1$,

$c_2 = 1$ and $0.67 \leq x \leq 6.16$ we generate curves for $y(x)$ and $Z(x)$. These curves are presented in figures 1 and 2 respectively:

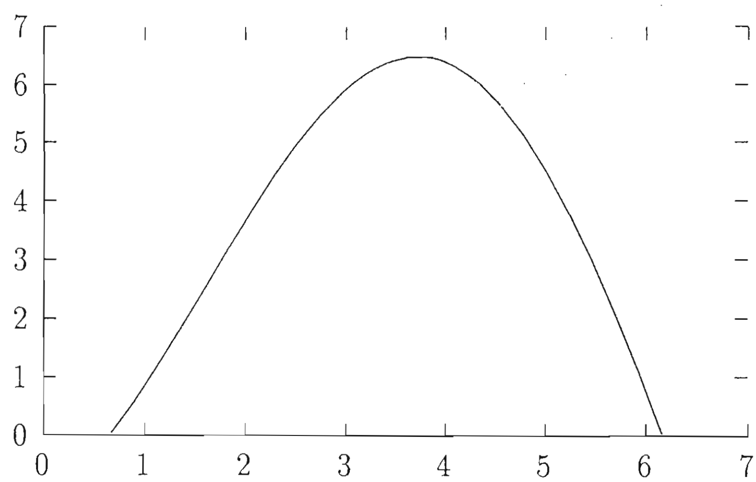

Fig 1: The Function $y(x)$

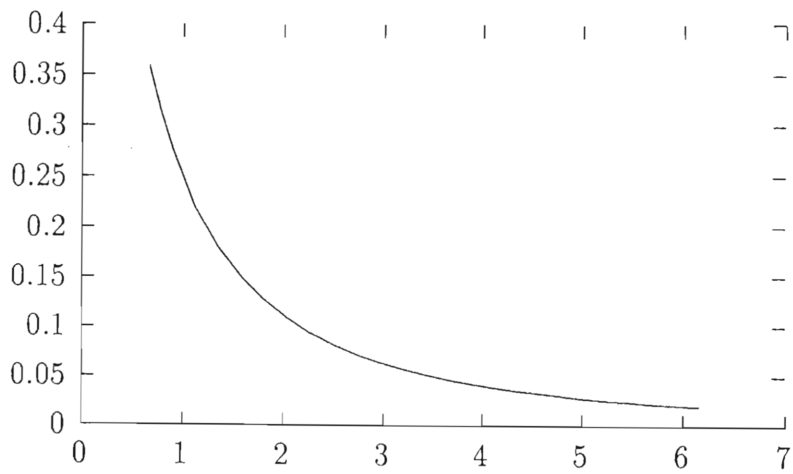

Fig 2: The Function $Z(x)$

In generating these curves we have utilised the computer package MATHEMATICA Version 2.0 (Wolfram 1991). We observe from figures 1 and 2 that the gravitational functions $y(x)$ and $Z(x)$ are well–behaved, continuous functions in the interval $0.67 \leq x \leq 6.16$. The functions $y(x)$ and $Z(x)$ remain finite in the interval considered. This suggests that the solutions of Einstein equations, expressible in terms of the confluent hypergeometric functions, lead to physically reasonable models in relativistic astrophysics.

3.5 The case $n = 1$

On substituting $n = 1$ into (3.1) we obtain the metric function

$$Z = 1 + x \tag{3.15}$$

The remaining metric function $y(x)$ can be calculated from

$$4(1 + x)\ddot{y} + 2\dot{y} = 0 \tag{3.16}$$

which follows from (3.2). Upon integrating (3.16) we obtain the solution

$$y(x) = c_1\sqrt{1 + x} + c_2 \tag{3.17}$$

where c_1 and c_2 are constants of integration.

We can now obtain a general solution of the field equations by calculating ρ and p via the equations (2.22a) and (2.22b) respectively. The general solution is then given by the system

$$\frac{\rho}{C} = -3 \tag{3.18a}$$

$$\frac{p}{C} = \frac{3c_1\sqrt{1+Cr^2}+c_2}{c_1\sqrt{1+Cr^2}+c_2} \tag{3.18b}$$

$$e^{2\lambda} = \frac{1}{1+Cr^2} \tag{3.18c}$$

$$e^{2\nu} = A^2\left[\, c_1^2(1+Cr^2) + 2c_1c_2\sqrt{1+Cr^2} + c_2^2 \,\right] \tag{3.18d}$$

The general solution (3.18) is not new and in fact it is equivalent to the Schwarzschild interior solution (Schwarzschild 1916*b*). The solutions (2.29) and (3.18) may be shown to be equivalent by the following transformation

$$c_1 = -1$$

$$c_2 = 3\sqrt{1-2M/R}$$

$$A = \frac{1}{2}$$

$$C = -\frac{1}{3}\rho_0$$

$$= -\frac{2M}{R^3}$$

Even though the solution (3.18) is the same as the Schwarzschild interior solution note that our solution was obtained using a different approach. In the Schwarzchild interior solution ρ is assumed to be constant whereas in our case we assumed a form for the metric function Z. Thus we have proved that the choice of the gravitational potential Z given by (3.15) is equivalent to assuming that the energy density ρ is constant.

3.6 The case $n = 2$

With $n = 2$ equation (3.1) gives the gravitational potential

$$Z = (1 + x)^2 \tag{3.19}$$

Equation (3.2) becomes

$$4(1 + x)^2 \ddot{y} + 4(1 + x)\dot{y} + y = 0 \tag{3.20}$$

which is of Euler–Cauchy form. The general solution to (3.20) is given by

$$y(x) = \tilde{c}_1 \cos\left(\ln \sqrt{1 + x}\,\right) + \tilde{c}_2 \sin\left(\ln \sqrt{1 + x}\,\right)$$

where $\tilde{c}_1$ and $\tilde{c}_2$ are constants. We may redefine the constants $\tilde{c}_1$ and $\tilde{c}_2$ as

$$\tilde{c}_1 = c_1 \cos c_2 \qquad\qquad \tilde{c}_2 = c_1 \sin c_2$$

where c_1 and c_2 are new constants. Then $y(x)$ has the compact form

$$y(x) = c_1 \cos\left(c_2 - \ln \sqrt{1 + x}\,\right) \tag{3.21}$$

which is equivalent to the form given above.

The general solution to the field equations is then given by the system

$$\frac{\rho}{C} = -6 - 5Cr^2 \tag{3.22a}$$

$$\frac{p}{C} = 2(1 + Cr^2)\tan\left(c_2 - \ln \sqrt{1 + Cr^2}\,\right) + (2 + Cr^2) \tag{3.22b}$$

$$e^{2\lambda} = \frac{1}{(1+Cr^2)^2} \tag{3.22c}$$

$$e^{2\nu} = A^2 c_1^2 \cos^2\left(c_2 - \ln\sqrt{1+Cr^2}\,\right) \tag{3.22d}$$

The solution (3.22) was found previously by Mehra (1966) in a very different form. This case was also later analysed by Kuchowicz (1967). However note that Mehra (1966) assumed a form for the energy density ρ. We have used a different avenue to obtain our general solution: we specified a form for the gravitational potential Z.

We now establish the transformation relating our solution(3.22) to that of Mehra (1966). The solution of Mehra (1966), adapted to our coordinates, is given by the system

$$\rho = \rho_c\left(1 - \frac{r^2}{a^2}\right) \tag{3.23a}$$

$$p = \frac{2}{a}\sqrt{\frac{\rho_c}{5}}\sqrt{1 - \frac{\rho_c}{15}\left(5r^2 - 3\frac{r^4}{a^2}\right)}\left[\frac{d_2 - d_1\tan z/2}{d_1 + d_2\tan z/2}\right]$$

$$-\frac{\rho_c}{15}\left(5 - 3\frac{r^2}{a^2}\right) \tag{3.23b}$$

$$e^{2\lambda} = \left[1 - \frac{\rho_c}{15}\left(5r^2 - 3\frac{r^4}{a^2}\right)\right]^{-1} \tag{3.23c}$$

$$e^{2\nu} = \left[\sqrt{1 - \frac{2a^2\rho_c}{15}}\cos\left(\frac{z_1 - z}{2}\right) - \frac{a}{3}\sqrt{\frac{\rho_c}{5}}\sin\left(\frac{z_1 - z}{2}\right)\right]^2 \tag{3.23d}$$

where ρ_c, a, d_1, d_2 are constants. The quantity z is given by

$$z = \ln\left[\frac{r^2}{a^2} - \frac{5}{6} + \sqrt{\frac{r^4}{a^4} - \frac{5r^2}{3a^2} + \frac{5}{a^2\rho_c}}\right]$$

and the constant z_1 has the form

$$z_1 = \ln\left[\frac{1}{6} + \sqrt{\frac{5}{a^2\rho_c} - \frac{2}{3}}\right]$$

Even though the Mehra (1966) solution has a complicated form it is the same as our solution. Our solution (3.22) and the Mehra solution (3.23) may be shown to be equivalent by the relationships

$$\rho_c = -6C$$

$$a^2 = -\frac{6}{5C}$$

$$z_1 = \ln\frac{1}{3}$$

$$\frac{d_2}{d_1} = \tan\left(c_2 + \ln\sqrt{\frac{5}{3}}\right)$$

$$\frac{1}{5} = A^2{c_1}^2$$

$$\cos^{-1}\left(\frac{1}{\sqrt{5}}\right) - \ln\sqrt{5} = c_2$$

Note that the variable r in the solution (3.22) and the variable z in the solution (3.23) are related by

$$z = \ln\left[\frac{5}{3}(1 + Cr^2)\right]$$

We have studied the Mehra (1966) solution in particular because it is often referred to in the literature and is listed by Kramer *et al.* (1980). The form of the solution (3.23) to the field equations as presented by Mehra (1966) is complicated and difficult to utilise in applications. Our equivalent form (3.22) is simpler and more compact and we believe that this form should make the analysis of the physical features of a relativistic star easier.

3.7 Other cases

It is possible to generate a variety of differential equations governing the metric function $y(x)$ from (3.1) for different values of n. These will correspond to new solutions of the Einstein field equations (2.22). However the differential equations resulting from (3.2) are extremely complicated and we have, as yet, not succeeded in integrating them. For example in the case of $n = -3$ we obtain the metric function

$$Z = \frac{1}{(1+x)^3}$$

and $y(x)$ is governed by the equation

$$4(1+x)\ddot{y} - 6\dot{y} + (x^2 + 4x + 6)y = 0$$

We have been unable to solve this differential equation. As another example we consider $n = 3$. Then $Z(x)$ is given by

$$Z = (1+x)^3$$

and $y(x)$ has to satisfy the differential equation

$$4(1+x)^3\ddot{y} + 6(1+x)^2\dot{y} + (2x+3)y = 0$$

As yet we have not found a solution for this differential equation. The analysis of these equations, and other cases of n, will be the subject of future investigation. Power series solutions may be generated utilising the method of Frobenius. The utilisation of numerical methods is also a possibility that will be pursued.

4 A generalisation of Durgapal and Bannerji

4.1 Introduction

In this chapter we specify a particular form for one of the gravitational potentials. Our intention is to generate new classes of solutions to the Einstein field equations. The gravitational potential chosen depends on the parameter k which may take on any real value. Then the second order ordinary differential equation governing the other metric function is obtained in §4.2. We first consider the case $k = -\frac{1}{2}$, which was initially analysed by Durgapal and Bannerji (1983), in §4.3. However their result is only a particular solution for this case. A general solution for $k = -\frac{1}{2}$, which we believe is new, is obtained. In §4.4 we consider the general case for arbitrary k. We find that the solutions are related to the hypergeometric differential equation. A brief discussion of the hypergeometric function, which is a special function, is presented. This class of solutions reduces to elementary and special functions for particular values of the parameter k. We regain, from the hypergeometric solution, the generalisation of the Durgapal and Bannerji (1983) solution discussed earlier in §4.3. Another special case of the general hypergeometric solution is considered in detail in §4.5. This turns out be a new solution to the Einstein field equations and is given in terms of elementary functions.

4.2 Another choice of the metric function Z

In chapter 3 the particular choice of the metric function (3.1) for $Z(x)$ generated a number of new solutions to the field equations. Here we choose a different functional form of the metric function $Z(x)$. Durgapal and Bannerji (1983) used a form for $Z(x)$ which leads to a new solution of the field equations. In this chapter we shall assume a functional form, which reduces to that of Durgapal and Bannerji (1983), in the hope of finding other new solutions to the Einstein field equations (2.22). We suppose that the metric function $Z(x)$ is given by

$$Z = \frac{1+kx}{1+x} \tag{4.1}$$

where we take $k \neq 1$. If $k = 1$ then the metric function $e^{\lambda} = 1$ and the energy density ρ vanishes. The other case excluded here is $k = 0$ in which case we regain the results of §3.3. We believe that the general form of the metric function (4.1) has not been postulated previously. It has the advantage of producing some well known models found before for particular values of k, in addition to generating new solutions.

Upon substituting equation (4.1) into the field equation (2.22c) we obtain the condition

$$4(1+x)(1+kx)\ddot{y} + 2(k-1)\dot{y} + (1-k)y = 0 \tag{4.2}$$

which is a differential equation that governs the behaviour of the gravitational potential $y(x)$. Solutions to equation (4.2) are crucial in the generation of new solutions to the field equations. Once (4.2) has been integrated then the energy density ρ and pressure p are obtained via (2.22a) and (2.22b) respectively. We attempt to integrate (4.2) for the particular case of $k = -\frac{1}{2}$ and for arbitrary values of k in subsequent sections.

4.3 The case $k = -\frac{1}{2}$

This case was first considered by Durgapal and Bannerji (1983). We first briefly outline their procedure and then generalise their solution. Durgapal and Bannerji (1983) assumed that the energy density was given by

$$\rho = \frac{3C(3 + Cr^2)}{2(1 + Cr^2)^2}$$

and then substituted ρ in equation (2.15a). On integrating the resulting differential equation they obtained the metric function

$$e^{-2\lambda} = \frac{2 - Cr^2}{2(1 + Cr^2)}$$

Thereafter they used the transformation

$$Cr^2 = x \qquad e^{-2\lambda} = Z \qquad e^{2\nu} = A^2 y^2$$

to simplify the subsequent calculations. From this transformation they immediately obtained

$$Z(x) = \frac{(2 - x)}{2(1 + x)} \tag{4.3}$$

Then substituting (4.3) in (2.22c) Durgapal and Bannerji (1983) obtained the equation

$$2(2 - x)(1 + x)\ddot{y} - 3\dot{y} + \frac{3}{2}y = 0 \tag{4.4}$$

which governs the behaviour of $y(x)$. This analysis is equivalent to substituting $k = -\frac{1}{2}$ in (4.2). Durgapal and Bannerji (1983) state that a particular solution for $y(x)$, from (4.4), is given by

$$y = (1 + x)^{3/2} \tag{4.5}$$

Their solution of (2.22) is then given by the system

$$\frac{\rho}{C} = \frac{3(3+Cr^2)}{2(1+Cr^2)^2} \tag{4.6a}$$

$$\frac{p}{C} = \frac{9(1-Cr^2)}{2(1+Cr^2)^2} \tag{4.6b}$$

$$e^{2\lambda} = \frac{2(1+Cr^2)}{2-Cr^2} \tag{4.6c}$$

$$e^{2\nu} = A^2(1+Cr^2)^3 \tag{4.6d}$$

Equations (4.6) describe the solution of the field equations (2.22) for the function (4.3). However note that this is not the general solution for the gravitational potential (4.3) as $y(x)$, given by (4.5), is only a particular solution of (4.4). It is possible to find the general solution of (4.4) and therefore obtain the general solution of the Einstein field equations for the metric function $Z(x)$ with $k = -\frac{1}{2}$.

We provide detailed calculations leading to the general solution of the differential equation (4.4). It is well known that if

$$y_1(x) = (1+x)^{3/2}$$

is one solution of a linear, homogeneous differential equation then

$$y_2(x) = y_1(x)v(x) \tag{4.7}$$

is a linearly independent second solution. It remains to explicitly determine $v(x)$ from equation (4.4). On substituting equation (4.7) into equation (4.4) we obtain, after simplication,

$$2(2-x)(1+x)y_1\ddot{v} + [4(2-x)(1+x)\dot{y}_1 - 3y_1]\,\dot{v}$$

$$+\left[2(2-x)(1+x)\dddot{y}_1 - 3\ddot{y}_1 + \frac{3}{2}y_1\right] v = 0 \tag{4.8}$$

Since y_1 is a solution to equation (4.4) we have that equation (4.8) reduces to

$$2(2-x)(1+x)y_1\ddot{v} + [4(2-x)(1+x)\dot{y}_1 - 3y_1]\,\dot{v} = 0 \tag{4.9}$$

Upon substituting $y_1(x)$ into equation (4.9) and rearranging we have that

$$\frac{\ddot{v}}{\dot{v}} = \frac{-9+6x}{2(2-x)(1+x)}$$

$$= \frac{1}{2}\left[\frac{1}{2-x} + \frac{-5}{1+x}\right]$$

which may be integrated by partial fractions to yield

$$\dot{v} = \tilde{c}_1 \frac{1}{(2-x)^{1/2}(1+x)^{5/2}}$$

where $\tilde{c}_1$ is the first constant of integration. The variables v and x in this differential equation separate and we have

$$v = \tilde{c}_1 \int \frac{dx}{(2-x)^{1/2}(1+x)^{5/2}} + \tilde{c}_2 \tag{4.10}$$

where $\tilde{c}_2$ is a second constant of integration. The integral in (4.10) may be simplified if we introduce the new variable

$$u = (2-x)^{1/2}$$

Then equation (4.10) becomes

$$v = -2\tilde{c}_1 \int \frac{du}{(3-u^2)^{5/2}} + \tilde{c}_2 \tag{4.11}$$

which is a simpler form to integrate than (4.10). The integral may be evaluated using elementary trigonometric substitution and we obtain the solution for v as

$$v(x) = \frac{-2\tilde{c}_1(2-x)^{1/2}\,(2x+5)}{27(1+x)^{3/2}} + \tilde{c}_2 \tag{4.12}$$

in terms of x. On substituting (4.12) into (4.7) we obtain

$$y_2(x) = \frac{-2\tilde{c}_1}{27}(2-x)^{1/2}\,(2x+5) + \tilde{c}_2(1+x)^{3/2} \tag{4.13}$$

which is a second solution to the differential equation (4.4). Clearly $y_2(x)$ is linearly independent of $y_1(x)$. The general solution to equation (4.4) is then given by

$$y(x) = \alpha_1 y_1(x) + \alpha_2 y_2(x)$$

where α_1 and α_2 are constants. This general solution has the explicit form

$$y(x) = c_1(1+x)^{3/2} + c_2(2-x)^{1/2}\,(2x+5) \tag{4.14}$$

where the new constants

$$c_1 = \alpha_1 + \alpha_2\tilde{c}_2 \qquad\qquad c_2 = \frac{-2\alpha_2\tilde{c}_1}{27}$$

have been introduced for simplicity.

The quantities ρ and p can be calculated from (2.22a) and (2.22b) respectively. The metric function $y(x)$ is given by (4.14), and the function $Z(x)$ is given by (4.3). The general solution to the Einstein field equations (2.22) is now given by the system

$$\frac{\rho}{C} = \frac{3(3+Cr^2)}{2(1+Cr^2)^2} \tag{4.15a}$$

$$\frac{p}{C} = \frac{9c_1\,(1+Cr^2)^{1/2}(1-Cr^2) - c_2(2-Cr^2)^{1/2}(10Cr^2+13)}{2c_1(1+Cr^2)^{5/2} + 2c_2(2-Cr^2)^{1/2}(1+Cr^2)(2Cr^2+5)} \tag{4.15b}$$

$$e^{2\lambda} = \frac{2(1+Cr^2)}{2-Cr^2} \tag{4.15c}$$

$$e^{2\nu} = A^2\left[c_1(1+Cr^2)^{3/2} + c_2(2-Cr^2)^{1/2}(2Cr^2+5)\right]^2 \tag{4.15d}$$

in terms of the original variable r. The equations (4.15) are the general solution of the Einstein field equations corresponding to $Z(x)$ given by (4.3). We believe that this solution is new and has not been published in the literature previously. We note that on setting

$$c_1 = 1 \qquad\qquad c_2 = 0$$

in equations (4.15) we regain the solution (4.6) of Durgapal and Bannerji (1983) as a special case. This verifies that our solution is indeed a generalisation of the solution given by Durgapal and Bannerji (1983).

4.4 The general case

In this section we consider the metric function Z, as given by (4.1), for arbitrary k. We shall use this form together with (4.2), which generates $y(x)$, in the anticipation of finding new solutions. Our intention in this section is to find the general solution of (4.2) and then consider some simple subcases that may arise. It is convenient to introduce the new variable X by

$$X = 1 + x$$

Then equation (4.2) becomes

$$4X(X-K)\frac{d^2Y}{dX^2} + 2K\frac{dY}{dX} - KY = 0$$

where $y(X) = Y$ and we have defined

$$K = \frac{k-1}{k} \qquad k \neq 0, 1$$

as a new constant. If we now define the variable $\mathcal{X}$ such that

$$X = K\mathcal{X}$$

then the above differential equation becomes

$$\mathcal{X}(1-\mathcal{X})\frac{d^2\mathcal{Y}}{d\mathcal{X}^2} - \frac{1}{2}\frac{d\mathcal{Y}}{d\mathcal{X}} + \frac{1}{4}K\mathcal{Y} = 0 \tag{4.16}$$

where $\mathcal{Y} = Y(\mathcal{X})$. Equation (4.16) is a special case of the hypergeometric differential equation.

The general hypergeometric differential equation in the standard form

$$z(1-z)\frac{d^2w}{dz^2} + [c - (a+b+1)z]\,\frac{dw}{dz} - ab\,z = 0 \tag{4.17}$$

where a, b and c are constants, is given by Abramowitz and Stegun (1972). The solutions to this equation are given in terms of the hypergeometric function

$$F(a,\ b;\ c;\ z)$$

The solutions of (4.17) are categorised by the three regular singular points

$$z\ =\ 0,\ 1,\ \infty$$

of the equation. The general theory of differential equations distinguishes between the following six cases

(i) None of the numbers c, $c-a-b$, $a-b$ is an integer.

(ii) One of the numbers a, b, $c-a$, $c-b$ is an integer.

(iii) $c - a - b$ is an integer but c is not an integer.

(iv) $c = 1$.

(v) $c = m + 1$, where m is a natural number.

(vi) $c = 1 - m$, where m is a natural number.

For the general properties of the solutions for each of the six cases given above the reader is referred to Abramowitz and Stegun (1972).

On comparing (4.16) and (4.17) we find that

$$a = \frac{-1 \pm \sqrt{1+K}}{2} \qquad b = \frac{-1 \mp \sqrt{1+K}}{2} \qquad c = -\frac{1}{2}$$

in our case. We note that the solution will be guaranteed to be real provided $1+K \geq 0$ which is equivalent to

$$k < 0 \qquad \text{or} \qquad k \geq \frac{1}{2}$$

It is easy to see, from the above for our values of a, b and c, that the cases (iii)–(vi) are not applicable to (4.16). Real solutions may only be contained in cases (i) and (ii). As $c \neq 0, -1, -2, \cdots$, the first solution of equation (4.16) is given as a hypergeometric series

$$\mathcal{Y}_1 = 1 + \frac{ab}{1!\,c}\mathcal{X} + \frac{a(a+1)\,b(b+1)}{2!\,c(c+1)}\mathcal{X}^2$$

$$+ \frac{a(a+1)\,(a+2)\,b(b+1)\,(b+2)}{3!\,c(c+1)\,(c+2)}\mathcal{X}^3 + \cdots$$

$$= F(a,\, b;\, c;\, \mathcal{X}) \tag{4.18}$$

Also since $c \neq 2, 3, 4, \cdots$, the second solution is given by

$$\mathcal{Y}_2 = \mathcal{X}^{1-c}\left[1 + \frac{(a-c+1)\,(b-c+1)}{1!\,(-c+2)}\mathcal{X}\right.$$

$$\left. + \frac{(a-c+1)\,(a-c+2)\,(b-c+1)\,(b-c+2)}{2!\,(-c+2)\,(-c+3)}\mathcal{X}^2 + \cdots\right]$$

$$= \mathcal{X}^{1-c}\,F(a-c+1,\ b-c+1;\ 2-c;\ \mathcal{X}) \qquad (4.19)$$

The hypergeometric series solutions (4.18) and (4.19) have been obtained from Kreyzig (1972). The general solution to equation (4.16) may be written as the combination

$$\mathcal{Y} = c_1\,F(a,\ b;\ c;\ \mathcal{X}) + c_2\,\mathcal{X}^{1-c}\,F(a-c+1,\ b-c+1;\ 2-c;\ \mathcal{X}) \qquad (4.20)$$

where c_1 and c_2 are constants.

We note that a variety of new solutions, in terms of elementary and special functions, may be obtained from (4.20) for particular values of a and b (with $c = -\frac{1}{2}$). Some values of K may reduce (4.20) to solutions that have been previously found. We consider one such case as an example. On choosing $K = 3$ which is equivalent to $k = -\frac{1}{2}$, equation (4.16) becomes

$$\mathcal{X}(1-\mathcal{X})\frac{d^2\mathcal{Y}}{d\mathcal{X}^2} - \frac{1}{2}\frac{d\mathcal{Y}}{d\mathcal{X}} + \frac{3}{4}\mathcal{Y} = 0$$

The first solution to this differential equation is given, from (4.18), in terms of the hypergeometric function F as

$$\mathcal{Y}_1 = F\left(\frac{1}{2},\ -\frac{3}{2};\ -\frac{1}{2};\ \mathcal{X}\right)$$

The explicit form for this hypergeometric function is obtainable from MATHEMATICA Version 2.0 (Wolfram 1991) in the form

$$\mathcal{Y}_1 = \sqrt{1-\mathcal{X}}\,(1+2\mathcal{X})$$

which reduces to

$$y_1(x) = \frac{1}{\sqrt{27}}(2-x)^{1/2}\,(2x+5)$$

in terms of the variable x. The second linearly independent solution can be obtained from (4.19) as

$$\mathcal{Y}_2 = \mathcal{X}^{3/2}\,F\left(2,\,0;\,\frac{5}{2};\,\mathcal{X}\right)$$

On using MATHEMATICA Version 2.0 (Wolfram 1991) we obtain the explicit form of this solution and we write it in terms of the variable x as

$$y_2(x) = \frac{1}{\sqrt{27}}\,(1+x)^{3/2}$$

Then the general solution is given by

$$y(x) = \tilde{c}_1 y_1(x) + \tilde{c}_2 y_2(x)$$

where $\tilde{c}_1$ and $\tilde{c}_2$ are constants. We may reexpress this solution as

$$y(x) = c_1(2-x)^{1/2}(2x+5) + c_2(1+x)^{3/2} \tag{4.21}$$

where we have defined the new constants

$$c_1 = \frac{\tilde{c}_1}{\sqrt{27}} \qquad\qquad c_2 = \frac{\tilde{c}_2}{\sqrt{27}}$$

Observe that this solution is equivalent to our generalisation of the Durgapal and Bannerji (1983) solution discussed in §4.3. Thus we have demonstrated that the solution (4.21) is in fact a special case of the hypergeometric function. We note that there may exist a variety of solutions, previously found in the literature, that may be regained from our general case. However, we should point out that some solutions may not be contained in this class of solutions. For example, the Finch and Skea (1989) solution discussed in chapter 3 is excluded as $k \neq 0$. It would be interesting to find all cases of (4.20) which produce solutions in terms of elementary functions.

4.5 A new solution

There may exist a variety of solutions in the class (4.20) which reduces to elementary or special functions. In this section we consider one such special case from the general solution obtained in §4.4. We believe that this case is in fact a new solution to the Einstein field equations which has not been published previously. We seek solutions such that

$$c - a = m$$

where m is an integer. On using $a = \dfrac{-1 \pm \sqrt{1+K}}{2}$ and $c = -\frac{1}{2}$ we have that

$$1 + K = 4m^2$$

which is equivalent to

$$k = \frac{1}{2 - 4m^2}$$

Observe that if $m = \pm\frac{1}{2}$ then $k = 1$ which is not admissible as we have observed earlier. We consider the simple case $m = 0$ which is equivalent to $k = \frac{1}{2}$. Then equation (4.16) becomes

$$\mathcal{X}(1-\mathcal{X})\frac{d^2\mathcal{Y}}{d\mathcal{X}^2} - \frac{1}{2}\frac{d\mathcal{Y}}{d\mathcal{X}} - \frac{1}{4}\mathcal{Y} = 0$$

for $K = -1$.

One solution, with $K = -1$, to this hypergeometric differential equation is given by

$$F\left(-\frac{1}{2},\ -\frac{1}{2};\ -\frac{1}{2};\ \mathcal{X}\right)$$

which corresponds to (4.18) with $a = b = c = -\frac{1}{2}$. The explicit form of this special function, generated by MATHEMATICA Version 2.0 (Wolfram 1991), is given explicitly by

$$\mathcal{Y}_1 = \sqrt{1-\mathcal{X}}$$

in terms of the variable $\mathcal{X}$. In terms of the variable x we have

$$y_1(x) = (2+x)^{1/2} \tag{4.22}$$

This is one solution of

$$2(1+x)(2+x)\ddot{y} - \dot{y} + \frac{1}{2}y = 0 \tag{4.23}$$

which is generated from (4.2) with $k = \frac{1}{2}$. Unfortunately the second solution (4.19) given in terms of the hypergeometric function

$$\mathcal{X}^{3/2}\, F\left(1,\ 1;\ \frac{5}{2};\ \mathcal{X}\right)$$

is not given in a usable form by MATHEMATICA Version 2.0 (Wolfram 1991). We have to utilise a more direct method to find the second solution.

The linearly independent second solution to equation (4.23) is given by

$$y_2(x) = y_1(x)v(x) \tag{4.24}$$

and we have to find the arbitrary function $v(x)$. On substituting equation (4.24) into (4.23) we obtain

$$2(1+x)(2+x)y_1\ddot{v} + [4(1+x)(2+x)\dot{y}_1 - y_1]\dot{v}$$

$$+\left[2(1+x)(2+x)\ddot{y}_1 - \dot{y}_1 + \frac{1}{2}y_1\right]v = 0$$

which reduces to

$$2(1+x)(2+x)y_1\ddot{v} + [4(1+x)(2+x)\dot{y}_1 - y_1]\dot{v} = 0 \tag{4.25}$$

since y_1, given by (4.22), is a solution to equation (4.23). Upon substituting (4.22) into the differential equation (4.25) we have that

$$\frac{\ddot{v}}{\dot{v}} = \frac{-1-2x}{2(1+x)(2+x)}$$

$$= \frac{1}{2}\left[\frac{1}{1+x} - \frac{3}{2+x}\right]$$

which may be integrated by partial fractions to yield

$$\dot{v} = \tilde{c}_1\frac{(1+x)^{1/2}}{(2+x)^{3/2}}$$

where $\tilde{c}_1$ is the first constant of integration. Since the variables v and x in this differential equation separate we have

$$v = \tilde{c}_1\int\frac{(1+x)^{1/2}}{(2+x)^{3/2}}dx \; + \; \tilde{c}_2 \tag{4.26}$$

where $\tilde{c}_2$ is the second constant of integration. On introducing the new variable

$$u = (1+x)^{1/2}$$

then the integral in equation (4.26) may be simplified as

$$v = 2\tilde{c}_1\int\frac{u^2du}{(1+u)^{3/2}} \; + \; \tilde{c}_2$$

$$= -\,2\tilde{c}_1\int ud(1+u^2)^{-1/2} \; + \; \tilde{c}_2 \tag{4.27}$$

Upon integrating equation (4.27) by parts we find that

$$v = -\,2\tilde{c}_1\left[\frac{u}{(1+u^2)^{1/2}} - \int\frac{du}{(1+u^2)^{1/2}}\right] + \tilde{c}_2$$

The remaining integral in the above is a standard integral and we can write the solution for v as

$$v(x) = -\,2\tilde{c}_1\frac{(1+x)^{1/2}}{(2+x)^{1/2}} + \tilde{c}_1\ln\left[(1+x)^{1/2}+(2+x)^{1/2}\right]^2 + \tilde{c}_2 \qquad (4.28)$$

in terms of x. From equations (4.28) and (4.24) we obtain

$$y_2(x) = -\,2\tilde{c}_1(1+x)^{1/2} + \tilde{c}_1(2+x)^{1/2}\ln\left[(1+x)^{1/2}+(2+x)^{1/2}\right]^2$$

$$+\,\tilde{c}_2(2+x)^{1/2} \qquad (4.29)$$

which is a second solution to the differential equation (4.23).

The general solution to (4.23) is then given by

$$y(x) = \beta_1 y_1(x) + \beta_2 y_2(x)$$

where β_1 and β_2 are constants. We write this solution explicitly as

$$y(x) = c_1(2+x)^{1/2} + c_2\left\{(2+x)^{1/2}\ln\left[(1+x)^{1/2}+(2+x)^{1/2}\right]^2\right.$$

$$\left.-\,2(1+x)^{1/2}\right\} \qquad (4.30)$$

where we have defined

$$c_1 = \beta_1 + \beta_2\tilde{c}_2 \qquad\qquad c_2 = \beta_2\tilde{c}_1$$

as new constants. Thus we have demonstrated that the solutions of the hypergeometric equation (4.16) for $K = -1$ (i.e. $k = \frac{1}{2}$) may be expressed in terms of elementary functions.

The general solution to the Einstein field equations (2.22) is then given by the system

$$\frac{\rho}{C} = \frac{3 + Cr^2}{2(1 + Cr^2)^2} \tag{4.31a}$$

$$\frac{p}{C} = \Big[c_1(2 + Cr^2)^{1/2} \; + \; 2c_2(1 + Cr^2)^{1/2}$$

$$+ \, c_2(2 + Cr^2)^{1/2} \ln \Big[(1 + Cr^2)^{1/2} \; + \; (2 + Cr^2)^{1/2}\Big]^2\Big] \Big/$$

$$\Big[2(1 + Cr^2)\Big\{c_1(2 + Cr^2)^{1/2} \; - \; 2c_2(1 + Cr^2)^{1/2}$$

$$+ \, c_2(2 + Cr^2)^{1/2} \ln \Big[(1 + Cr^2)^{1/2} \; + \; (2 + Cr^2)^{1/2}\Big]^2\Big\}\Big] \tag{4.31b}$$

$$e^{2\lambda} = \frac{2(1 + Cr^2)}{2 + Cr^2} \tag{4.31c}$$

$$e^{2\nu} = A^2 \Big[c_1(2 + Cr^2)^{1/2} \; + \; c_2 \Big\{(2 + Cr^2)^{1/2} \ln \Big[(1 + Cr^2)^{1/2} \; + \; (2 + Cr^2)^{1/2}\Big]^2$$

$$- \, 2(1 + Cr^2)^{1/2}\Big\}\Big]^2 \tag{4.31d}$$

in terms of the original variable r. The system (4.31) represents the general solution for the gravitational potential $Z(x)$ given by (4.1) with $k = \frac{1}{2}$. We believe that this is a new solution to the Einstein field equations. It is interesting to observe that the solutions for $k = \frac{1}{2}$ and $k = -\frac{1}{2}$ are closely related, but different. In particular the energy density ρ is found to be similar on comparison of (4.15) and (4.31). We believe that it is possible to find further new solutions by choosing other values of the parameter k. However, this will not be pursued in this thesis and will be the subject of future research.

5 An equation of state

5.1 Introduction

In chapters 3 and 4 we studied classes of solutions to the Einstein field equations which were obtained by assuming a form for one of the gravitational potentials. However this approach does not guarantee a physical equation of state. Another approach in seeking a solution is to *ab initio* assume an equation of state relating the energy density to the pressure. By assuming an equation of state a new solution was recently found by Buchdahl (1981) which provides the general relativistic generalisation of the $n = 1$ Newtonian polytrope. Ibanẽz and Sanz (1982) assumed a linear relationship between the energy density and pressure, and presented a new solution. It seems that Ibanẽz and Sanz (1982) are one of the few authors to have specified a linear equation of state in an attempt to find a new solution. Our intention in this chapter is to briefly review the paper of Ibanẽz and Sanz (1982), and analyse their resulting field equations. The paper of Ibanẽz and Sanz (1982) utilises unusual coordinates and we relate their line element and field equations, in §5.2, to our results from chapter 2. This makes comparison with results of the standard literature and this thesis easier. In §5.3 we reduce the solution of the Einstein field equations to obtaining a solution of Abel's equation of the second kind. This equation is highly nonlinear and is not easily integrable.

5.2 The basic equations

We briefly review the paper of Ibanẽz and Sanz (1982) in this section and relate their results to ours. In order to facilitate comparision we first show that our line element (2.11) and the field equations (2.15) are the same as those utilised by Ibanẽz and Sanz (1982). Their line element, in coordinates $(\tilde{t}, \tilde{r}, \tilde{\theta}, \tilde{\phi})$, is given by

$$ds^2 = -e^{2\tilde{\nu}}d\tilde{t}^2 + e^{2\tilde{r}+2\tilde{\beta}}d\tilde{r}^2 + e^{2\tilde{r}}\left(d\tilde{\theta}^2 + \sin^2\tilde{\theta}d\tilde{\phi}^2\right) \tag{5.1}$$

where $\tilde{\nu}$ and $\tilde{\beta}$ are functions of $\tilde{r}$. The field equations, corresponding to the line element (5.1), are given by

$$\rho = e^{-2(\tilde{r}+\tilde{\beta})}\left[2\frac{d\tilde{\beta}}{d\tilde{r}} + e^{2\tilde{\beta}} - 1\right] \tag{5.2a}$$

$$p = e^{-2(\tilde{r}+\tilde{\beta})}\left[2\frac{d\tilde{\nu}}{d\tilde{r}} - e^{2\tilde{\beta}} + 1\right] \tag{5.2b}$$

$$\frac{d^2\tilde{\nu}}{d\tilde{r}^2} + \left(\frac{d\tilde{\nu}}{d\tilde{r}}\right)^2 - \left[2 + \frac{d\tilde{\beta}}{d\tilde{r}}\right]\frac{d\tilde{\nu}}{d\tilde{r}} = 1 + \frac{d\tilde{\beta}}{d\tilde{r}} - e^{2\tilde{\beta}} \tag{5.2c}$$

for the metric functions $\tilde{\nu}$ and $\tilde{\beta}$.

The line element (2.11) is given in coordinates (t, r, θ, ϕ). We can relate (2.11) and (5.1) if we have

$$\tilde{t} = t \tag{5.3a}$$

$$e^{\tilde{r}} = r \tag{5.3b}$$

$$\tilde{\theta} = \theta \tag{5.3c}$$

$$\tilde{\phi} = \phi \tag{5.3d}$$

for the two sets of coordinates. The two metric functions must be related by

$$\tilde{\nu}(\tilde{r}) = \nu(r) \tag{5.4a}$$

$$\tilde{\beta}(\tilde{r}) = \lambda(r) \tag{5.4b}$$

The transformations (5.3) relate the coordinates $(\tilde{t}, \tilde{r}, \tilde{\theta}, \tilde{\phi})$ to (t, r, θ, ϕ), and the equations (5.4) relate the functions $\tilde{\nu}$, $\tilde{\beta}$ to ν, λ. With the assistance of these transformations we can easily verify that the field equations (5.2) are equivalent to (2.15). Note that (5.2a) is equivalent to (2.15a), (5.2b) is equivalent to (2.15b), and (5.2c) is generated by a linear combination of (2.15b) and (2.15c).

Thus we have established the equivalence of the approach of Ibanẽz and Sanz (1982) and that of the standard literature. Henceforth we shall drop the tildes and utilise the notation of Ibanẽz and Sanz (1982). That is, the line element is given by

$$ds^2 = -e^{2\nu}dt^2 + e^{2r+2\beta}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{5.5}$$

and the field equations comprise the system

$$\rho = e^{-2(r+\beta)}\left(2\beta' + e^{2\beta} - 1\right) \tag{5.6a}$$

$$p = e^{-2(r+\beta)}\left(2\nu' - e^{2\beta} + 1\right) \tag{5.6b}$$

$$\nu'' + \nu'^2 - (2 + \beta')\nu' = 1 + \beta' - e^{2\beta} \tag{5.6c}$$

from (5.2).

For supermassive stars the barotropic equation of state

$$p = p(\rho)$$

may be utilised (Shapiro and Teukolsky 1983). This equation of state is used, for example, in the modelling of neutron stars. The simplest case of the above is a linear relationship between the energy density and pressure

$$p = n\rho \tag{5.7}$$

used by Ibanẽz and Sanz (1982) where $n \in [0, 1]$ for a physical equation of state.

(a) $n = 0$:

The case $n = 0$ corresponds to vanishing pressure and also the energy density vanishes from the field equations (5.6). Thus this case is neglected.

(b) $n \in (0, 1]$:

Ibanẽz and Sanz (1982) found a solution for the interval $n \in (0, 1]$ by assuming a form for the metric function ν. Their solution is given by the system

$$\rho = \left[1 + \frac{(1+n)^2}{4n}\right]^{-1} e^{-2r} \tag{5.8a}$$

$$p = n\left[1 + \frac{(1+n)^2}{4n}\right]^{-1} e^{-2r} \tag{5.8b}$$

$$e^{2\beta} = 1 + \frac{4n}{(1+n)^2} \tag{5.8c}$$

$$e^{2\nu} = e^{4nr/(1+n)} \tag{5.8d}$$

The class of solutions (5.8) has the line element

$$ds^2 = -e^{4nr/(1+n)}dt^2 + \left[1 + \frac{4n}{(1+n)^2}\right] e^{2r}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

We can show that this solution violates the condition of regularity at the centre by transforming to Schwarzschild coordinates (Ibanẽz and Sanz 1982). Thus these solutions may only be used locally to represent certain regions of the star.

From equations (5.6) we have that

$$p + \rho = e^{-2(r+\beta)}\,(2\beta' + 2\nu')$$

and on utilising equation (5.7) we may express the above equation as

$$\rho = \frac{2}{1+n} e^{-2(r+\beta)}(\beta' + \nu') \tag{5.9}$$

On equating (5.6a) and (5.9) we obtain, after simplification,

$$\nu' = n\beta' + \frac{1+n}{2}(e^{2\beta} - 1) \tag{5.10}$$

In equation (5.10) we have eliminated the energy density ρ and the pressure p. In addition (5.10) has the advantage of being a first order equation. We observe that the problem of finding a solution to the Einstein field equations given by the system (5.6) with the linear equation of state (5.7), is now reduced to obtaining a solution for the functions ν and β satisfying the differential equations (5.6c) and (5.10). Once ν and β are obtained then the energy density ρ and the pressure p can be calculated via equations (5.9) and (5.7) respectively.

We should point out that there are many other possible approaches to studying relativistic stars with physically valid energy density–pressure configurations. Glass and Goldman (1978) reformulated the field equations to obtain an equation connecting the energy density and the pressure which is free of the metric functions. This equation must be related to the Abel equation of the second kind that we derive in §5.3. This equation is difficult to solve but Glass and Goldman (1978) obtain some analytic solutions applicable to adiabatically stable stars.

5.3 Abel's equation

In this section we analyse the equations (5.6c) and (5.10) governing the behaviour of the potentials ν and β. We eliminate ν by substituting equation (5.10) in equation (5.6c) to obtain

$$n\beta'' + \frac{1}{2}(2n^2 + 3n + 1)\beta' e^{2\beta} + (n^2 - n)\beta'^2 - \frac{1}{2}(2n^2 + 5n + 1)\beta'$$

$$-\frac{1}{2}(n^2 + 4n + 1)e^{2\beta} + \frac{1}{4}(1 + n)^2 e^{4\beta} + \frac{1}{4}(n^2 + 6n + 1) = 0 \qquad (5.11)$$

which is a second order differential equation in β. We need to find a solution β to (5.11). Then (5.10) generates ν and the solution to (5.6) follows for the equation of state $p = n\rho$ in principle. However equation (5.11) is highly nonlinear and we have not yet found a solution in closed form. We can reduce (5.11) to a simpler form by introducing the new variable

$$B(r) = e^{-2\beta}$$

Then equation (5.11) becomes

$$BB'' - \frac{(1+n)}{2}B'^2 - \frac{(2n^2 + 5n + 1)}{2n}BB' + \frac{(2n + 1)(n + 1)}{2n}B'$$

$$-\frac{(n^2 + 6n + 1)}{2n}B^2 + \frac{(n^2 + 4n + 1)}{n}B - \frac{(1 + n)^2}{2n} = 0 \qquad (5.12)$$

Equation (5.12) is a second order nonlinear differential equation free of the exponential functions.

We now reduce equation (5.12) to a first order differential equation. On using the transformations

$$\mathcal{B} = B(r) \qquad\qquad \mathcal{A} = B'$$

then we have

$$\frac{d\mathcal{A}}{d\mathcal{B}} = \frac{B''}{B'}$$

so that equation (5.12) reduces to the form

$$\mathcal{B}\mathcal{A}\frac{d\mathcal{A}}{d\mathcal{B}} - \frac{(n+1)}{2}\mathcal{A}^2 - \frac{(2n^2+5n+1)}{2n}\mathcal{B}\mathcal{A} + \frac{(2n+1)(n+1)}{2n}\mathcal{A}$$

$$-\frac{(n^2+6n+1)}{2n}\mathcal{B}^2 + \frac{(n^2+4n+1)}{n}\mathcal{B} - \frac{(1+n)^2}{2n} = 0 \qquad (5.13)$$

which is Abel's equation of the second kind (Zwillinger 1989). There are few solutions, in closed form, known to Abel's equation and it seems that (5.13) does not fall in any category that has been studied previously (Kamke 1971). We have been unable to find a solution to this equation as yet. Equation (5.13) will be studied further in future.

Perhaps we should point out that (5.11) may be put into an equivalent form which may be useful in finding a solution. We define another variable w such that

$$w = B^{(1-n)/2}$$

Then using $B = e^{-2\beta}$ and $w = B^{(1-n)/2}$ we find that (5.12) may be written as

$$w'' - \frac{(2n^2+5n+1)}{2n}w' - \frac{(n^2+6n+1)(1-n)}{4n}w - \frac{(2n+1)(1-n)}{2n}\left(w^{(-1-n)/(1-n)}\right)'$$

$$+ \frac{(n^2+4n+1)(1-n)}{2n}w^{(-1-n)/1-n} - \frac{(1+n)^2(1-n)}{4n}w^{(-3+n)/(1-n)} = 0 \qquad (5.14)$$

The differential equation (5.14) is equivalent to (5.12) and for some approaches may be more easily integrated. Choices for specific values of n may reduce (5.14) to a simpler form.

6 Conclusion

Our objective in this thesis was to find new solutions to the Einstein field equations that may be applied to relativistic stars. These solutions have extensive applications in relativistic astrophysics as they may be used to model realistic stars. The spherically symmetric spacetimes, with a perfect fluid source, investigated are static and provide a good model for many stars, e.g. neutron stars (Shapiro and Teukolsky 1983). A number of new solutions to the Einstein field equations, which we believe to be physically reasonable, were obtained explicitly.

We now provide a broad outline of the work carried out in this thesis with special attention given to solutions to the Einstein field equations that were found in our investigation:

- We obtained the Einstein field equations in three equivalent forms for static, spherically symmetric gravitational fields.
- A brief review of the Schwarzschild exterior solution and the Schwarzschild interior solution with constant energy density was carried out. In this context Birkhoff's theorem and Buchdahl's theorem were introduced.
- The physical properties required of the interior solutions to the Einstein field equations, for a realistic relativistic stellar model, were discussed.

- We first considered the gravitational potential

$$Z = \frac{1}{1+x}$$

which has been previously analysed by Finch and Skea (1989). Mistakes in their transformation were corrected, and many of the details missing in their treatment were also provided.

- For the case

$$Z = \frac{1}{(1+x)^2}$$

the solution was related to the confluent hypergeometric differential equation via a complex transformation. The solution was subsequently given in terms of special functions, the Kummer functions. As the solution was difficult to interpret analytically the behaviour of the gravitational potentials was represented graphically with the help of MATHEMATICA Version 2.0 (Wolfram 1991). We believe that this is a new solution to the Einstein field equations.

- Two solutions were obtained for the functions

$$Z = 1 + x \qquad\qquad Z = (1+x)^2$$

These resulting solutions are not new and were explicitly related to the Schwarzschild interior solution (Schwarzschild 1916*b*) and the Mehra solution (Mehra 1966) by appropriate transformations. Note that we specified the function Z to obtain these solutions which is different from the approach normally utilised to generate the Schwarzschild interior and the Mehra solution. Thus we have established the equivalence of our approach with that of the classical literature. Moreover the Mehra(1966) solution has a complicated form which is difficult to follow and apply. Our solution has a more canonical form and is easier to apply in the analysis of the physical properties.

- The choice of the metric function

$$Z = \frac{1 - \frac{1}{2}x}{1 + x}$$

generates a new solution to the Einstein field equations. This case generalises the Durgapal and Bannerji (1983) solution.

- For the case

$$Z = \frac{1 + kx}{1 + x}$$

where k is arbitrary, the solution is related to the hypergeometric differential equation. The general solution is given in terms of the hypergeometric function. We believe that this is a new class of solutions to the Einstein field equations. As a special case of the hypergeometric equation we regained our generalisation of the Durgapal and Bannerji (1983) solution. Even though this is a large class, it should be noted that some solutions may not be regained from the hypergeometric equation, e.g. the Finch and Skea (1989) solution.

- Another new solution was obtained as a special case of the hypergeometric equation by choosing

$$Z = \frac{1 + \frac{1}{2}x}{1 + x}$$

This solution was expressed in terms of elementary functions with the assistance of MATHEMATICA Version 2.0 (Wolfram 1991).

- Finally we considered the equation of state

$$p = n\rho$$

where $n \epsilon [0, 1]$, relating the energy density and the pressure. We briefly reviewed the paper of Ibanẽz and Sanz (1982) who found a new solution for this equation of state. The solution to the Einstein field equations was then reduced to finding a solution to Abel's equation of the second kind.

In the above we have highlighted our results in this thesis.

A class of solutions was obtained for the choice of the metric function generalising the one used by Finch and Skea (1989). It would be interesting to study and analyse other values of the parameter n for the function

$$Z = (1+x)^n$$

which may result in the generation of further new solutions for this metric function. In particular an analysis of the solution given in terms of the Kummer functions for $n = -2$ should be performed in the future. We also believe that it is possible to choose other values of the parameter k for the metric function

$$Z = \frac{1+kx}{1+x}$$

which will generate other new solutions in closed form. In addition other special cases of the hypergeometric equation, resulting from this form of Z, should be studied further. The solution to Abel's equation of the second kind should be pursued.

Furthermore it is interesting to observe that there may exist other forms of the metric functions that we may utilise in generating new solutions. In particular there is the possibility of generalising the solutions of Durgapal *et al.* (1984) and Durgapal and Fuloria (1985). The fact that these solutions are physically acceptable is demonstrated by Knutsen (1989) in his treatment of the physical properties of the model of Durgapal and Fuloria (1985). An analysis of the physical properties of the new solutions found in this thesis will be another avenue to explore (Knutsen 1988). A solution–generating technique may be utilised in producing physically valid energy density–pressure configurations for adiabatically stable stars (Glass and Goldman 1978). We may also investigate anisotropic energy–momentum tensors in future (Maharaj and Maartens 1989, Ponce de Leon 1987 and Ponce de Leon 1988). Charged stars with nonzero electromagnetic tensor fields (Herrera and Ponce de Leon 1985, Maartens and Maharaj 1990) are other avenues worth exploring.

This thesis represents an attempt to finding solutions to Einstein field equations that may be applied to relativistic stars. We hope that we have demonstrated that the study of relativistic spherical stars is a fertile area of research. Clearly further investigation of other solutions that may be used to model realistic stars should be pursued.

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