An analysis of Grade 12 learners' mental constructions and difficulties of differentiation and integration: A case study of one school in Umlazi District

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DECLARATION

I, Promise Sethembiso Madonsela, declare that this PhD thesis, titled: **An analysis of grade 12 learners' mental constructions and difficulties of Differentiation and Integration: A case study of one school in Umlazi District:**

(i) is my original and independent research work, except where otherwise indicated.

(ii) has not been submitted for any degree or examination at any other university.

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Signed [redacted] DATE: 02 September 2023

Promise Sethembiso Madonsela **(Researcher)**

I hereby declare that this thesis has been submitted for examination with my approval:

Sig[redacted] DATE: 02 September 2023

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Annatoria Zanele Ngcobo **(Supervisor)**

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ABSTRACT

This study is an analysis of Grade 12 learners' mental constructions and the difficulties they experienced with the calculus topics of differentiations and integration at a school in Umlazi District, South Africa. With the introduction of Technical Mathematics, which emphasises application, as a high school subject in South Africa in 2016, increasing attention has focussed on learners' mental constructions. Consequently, learners' understanding of differentiation and integration, which are key concepts taught in Grade 12, has come under scrutiny. It has therefore become imperative to investigate learners' mental construction and the nature of the difficulties they encounter in learning the concepts of differentiation and integration.

The study relied on the APOS (Action, Process, Object and Schema) theory of learning to explain the nature of learners' mental constructions. A qualitative approach was used. Data was collected using activity sheets with all participants and interviews with five participants. A pilot study preceded the main study. Purposive sampling was used to select ten Grade 12 learners for the pilot study and ten others for the main study. Analysis of data was both deductive and inductive.

The findings revealed that learners' conception of differentiation and integration was mainly at the action stage (with reference to the APOS model), with few learners operating at the process stage. Weak foundational mathematical knowledge was found to hinder learners' development of the necessary mental constructions 3.4. Concepts were overgeneralised from one domain to another and everyday language was used to explain mathematical concepts. Based on these findings, it is recommended that the teaching of differentiation and integration be integrated with algebraic concepts, which are the pre-requisite concepts needed for the schema development of differentiation and integration. It is also suggested that the genetic decomposition designed in this study be used by teachers as a tool to analyse their learners' mental constructions in order to design and plan appropriate alternative instructional activities to improve learners' development of differentiation and integration schema.

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1 INTRODUCTION TO THE STUDY

1.1 Introduction

Many stakeholders are concerned about how poorly South African children perform when using mathematics concepts such as differentiation and integration in secondary school and tertiary education (Makgato, 2007). Globally, the low performance of students in mathematics is regarded as the most significant academic issue. Due to the way it combines with so many other disciplines, including accounting, technology, physical science, and technical subjects, mathematics is regarded as a fundamental subject in the South African curricula. This suggests that there is a need for a pool of technologically skilled labour to meet the current millennial generation's economic and technological development goals, and that this can only be accomplished by giving careers in science, engineering, and technology the appropriate amount of emphasis (Ogbonnaya et al., 2011).

According to Brijlall and Maharaj (2015), the fact that mathematics is part of a larger family of sciences, such as physical science, has increased attention on this subject in both policy and teaching and learning practices around the world, due to its fundamental importance and significance to humans and their advancement and development. In the 2012 South African Curriculum and Assessment Policy Statement (CAPS) for mathematics (e.g., FET phase), the Department of Education (DOE) states: "The objective of Mathematics is the building of correct links between mathematics as a discipline and application of mathematics in the real-world contexts" (DOE, 2018 p. 9). This explains why policymakers and other relevant stakeholders are becoming increasingly interested in and engaged in discourse about how mathematics should be organized and taught in different school systems around the world and, particularly, how students construct and comprehend knowledge when completing mathematical tasks (Jojo, 2014). According to Madonsela et al., (2020), for students to perform at their optimal level in math, they must be taught the necessary mathematical information and abilities using efficient teaching methodologies that allow them to use their creativity and reason.

South Africa has performed worse than other African nations on several consecutive cross-national examinations of educational accomplishment (DOE, 2012; 2014; 2016). Brijlall

(2008) found that South African students had trouble understanding fundamental mathematical ideas. Every year when the Grade 12 results are released nationwide in South Africa, the poor performance of students is also on display, as the mathematics average is low when compared to other subjects, like physical science (DOE,2012).

Numerous studies on mathematics education have shown that it is difficult for university students and school students to conceptualize mathematical ideas (Bennett et al., 2011; Cardella & Atman, 2004; Huang et al., 2011). According to Aziz et al. (2010), the challenges in learning mathematics show up in a variety of ways, including poor application of mathematical concepts, low accomplishment in math, a lack of mathematical ability and ineffective mathematical problem-solving. According to Ndlovu and Brijlall (2015), the capacity for cognitive reasoning facilitates the mental constructions required to understand a given mathematical topic. In South Africa, the issue of students' low maths proficiency has been ascribed to the ineffective methods used in the subject's teaching and learning, as well as teachers' inadequate depth in pedagogical content understanding, among other things (Brijlall & Maharaj, 2015).

1.2 Background to the Study

The nature of teaching and learning mathematics is a topic of interest for researchers in the field of mathematics education. The better we understand the nature of knowledge and mental constructions, the better we can plan the teaching strategies and activities that will improve the learning of mathematics as a subject (Jojo, 2014; Ndlazi, 2015; Sinyosi, 2015).

Students generally struggle to conceptualize mathematical ideas, especially in calculus, according to research on the teaching and learning of mathematics (Lam, 2009). Differential calculus is taught to secondary school students around the world (Lam, 2009), including South Africa. Hassani (1998) reports that research on students' conceptual knowledge of foundational concepts of calculus has revealed significant gaps between their understanding of these ideas conceptually and their ability to carry out processes based on them. Differential calculus has been taught in Grade 12 in South Africa for a long time, yet students continue to struggle with problems based on fundamental principles and differentiation rules, as well as issues involving cubic functions and optimisation (DBE 2011; 2012; 2013; 2014; 2015; 2016).

According to Berggren (2016), calculus focuses on calculating instantaneous rates of change (differential calculus) and summing an infinite number of minor elements to derive a single whole (integral calculus). Tall (2006) claims that the focus of calculus is on mastering symbolic methods for differentiation and using these to address a variety of issues. The calculus concepts that students learn at school establish the groundwork for their future study of higher mathematics (Marrongele, 2012). Given the significance of this topic for academic comprehension, there is a need to understand learners' mental constructions and challenges with regard to both differentiation and integration. This study explores the nature of the challenges of teaching both differentiation and integration for both learners and educators.

Numerous studies conducted locally and abroad have revealed that pupils have difficulty with calculus (Berger, 2006; Brijlall & Bansilal, 2010; Habineza, 2010; Koepf & Ben-Israel, 1994). This is supported by Maharaj and Brijlall's (2013) finding that the exclusion of the integrated topic in school calculus leads to students entering universities unprepared. They also found that the poor quality of mathematics instruction at the school stage contributes to many of the difficulties that students exhibit when handling calculus. According to Hoffmeester (2015), there is concern about the quality of mathematics education in South Africa because of the detrimental effects that students' poor arithmetic proficiency has on the nation's progress. She argues that South Africa's aim for a sustainable democracy will not be realized unless it raises the quality and quantity of learners who can serve as the nation's engineers, scientists, and technical professionals.

The nature of mathematics teaching and learning is a topic of research in mathematical education. The majority of research on the teaching and learning of mathematics has shown that students struggle to conceptualize mathematical ideas, and the poor quality of mathematics instruction at the school stage contributes to many of the difficulties students display when handling this aspect of the subject (Jojo, Maharaj, & Brijlall, 2013).

At the secondary education level, calculus is introduced in Grade 12 and is called differential calculus; as learners progress to tertiary education, integral calculus is introduced. The exclusion of the topic of integration from calculus at the secondary level results in students entering universities inadequately prepared (Ndlazi, 2015). This study analyses learners' mental constructions and difficulties with regard to differentiation and integration which are currently taught in Technical Mathematics at the secondary level.

1.3 Problem Statement

Technical Maths was piloted in a few schools in 2016 and the first cohort wrote matric in 2018. It was then introduced to other technical schools and the results reflected that learners are struggling with differentiation and integration (DOE, 2019). However, diagnostic reports paint a bleak picture regarding the performance of learners in the topics taught in technical maths, as shown in

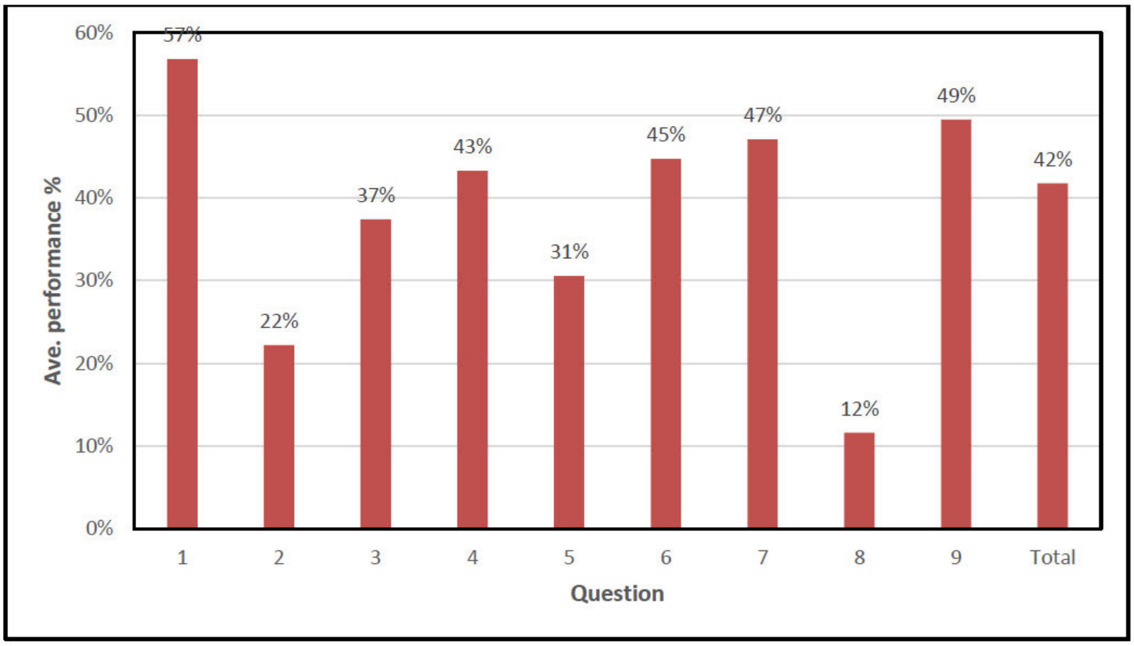
Table 1**.1**, reflecting the qualitative analysis done by moderators.

Table 1.1 Overall achievement rates in Technical Mathematics (DOE, 2018; 2019; 2020; 2021).

Year	Number who wrote	% who achieved 30% and above
2018	10 025	50,7
2019	9 670	42,7
2020	10 731	32,4
2021	13 403	60, 1

As is evident from Table 1.1, over the past years more than 50% of learners performed below 30%, thus failing Technical Mathematics in grade 12 results. In 2021, approximately 40% of learners failed Technical Mathematics. While learner performance appeared to improve in 2021, it was stated by the moderators that this was due to the inclusion of a practical component, suggesting that moderators did not see improvement in learners' demonstration of their conceptual knowledge (DOE, 2018; 2021).

Furthermore, an analysis per topic showed that, in the majority of the topics, learners performed below 50%; this included differentiation and integration, as shown in Figure 1.1.



Q	Topic	Q	Topic
1	Equations, inequalities & binary numbers	6	Differential calculus (differentiation)
2	Nature of the roots of quadratic equations	7	Differential calculus (Cubic function)
3	Exponents & surds, logarithms & complex numbers	8	Differential calculus (Maxima and Minima)
4	Functions and graphs	9	Integration
5	Finance, growth & decay		

Figure 1.1: Analysis of 2021 matric maths results, per topic (DBE, 2021)

For calculus concepts, while learners' performance was above 40%, moderators' comments revealed that learners struggling to apply concepts:

> *Although candidates performed well in questions that required lower order thinking skills, many learners performed poorly in questions that demanded analytical, evaluative and problem-solving skills (DBE, 2021, p.3).*

> *Many candidates demonstrated limited knowledge of the relationship between the gradient of the tangent and the derivative of the curve at the point of contact (DBE, 2021, p.17).*

> *Many candidates failed to equate the area of the shaded part to the area bounded by the curve (DBE, 2021, p.20).*

Ndlazi (2015) notes that while students tend to excel in other topics they tend to struggle with integration due to it being embedded in application. Serhan (2015) found that students possessed procedural knowledge of the integration concept, meaning that it was memorisation driven. Extending the argument on procedural knowledge, Nafia et al., (2022) posit that students generally lack understanding of integration techniques.

This analysis of learners' performance since 2018 as reported in the literature confirms that calculus concepts – particularly differentiation and integration – are challenging for the majority of learners, as well as for students in higher education. While a number of studies have been conducted at the university level to investigate students' understanding of these concepts, a limited number of studies have been conducted at the secondary school level that focus specifically on understanding learners' mental constructions and difficulties. This study contributes to filling this gap.

1.4 Purpose of the Study

The main aim of the study was to analyse Grade 12 learners' mental constructions of differentiation and integration using APOS (Action-Process-Object-Schema), which attempts to describe the mental structures that deal with the nature of mathematical concepts and possible ways that learners construct certain concepts in mathematics (Jojo et al.,2013). According to Dubinsky and McDonald (2001), the APOS can be used as a tool to objectively explain why students struggle with a wide range of mathematical topics and to provide ways for them to master those concepts. Researchers have highlighted that the exclusion of integration at the secondary school stage resulted in students entering universities underprepared for it (Jojo et al., 2013; Ndlazi, 2015). In this study, participants were learners who were taking Technical Mathematics, which included differentiation and integration. The study endeavours to arrive at recommendations that could contribute to an improvement in the learning of calculus at the Grade 12 level.

1.5 Rationale of the Study

Dubinsky (1997) emphasises that mathematics topics or concepts that learners find difficult to comprehend should be analysed by means of research before designing alternative instructional strategies. Concurring with Dubinsky, Ndlovu (2014) argues that analysing one's learning assists in understanding any barriers that might hinder the learning process and thus help in identifying appropriate alternative instructional strategies.

It has been found that learners find concepts such as differentiation and integration difficult to comprehend, thus identifying the need for analysis by means of research to ascertain the learners' mental constructions as well as the difficulties learners have with differentiation and integration. Diagnostic reports by the Department of Basic Education (DBE) have highlighted the need for this, as well. For example, DBE (2021) reports have indicated that, while learners can answer questions that require recall of facts, they struggle with the application of facts to solve questions requiring higher-order thinking. The most observable trend is the mismatch between rules of differentiation and of integration, which suggests a lack of conceptualisation of these concepts. In response to this, this study was conceptualised based on my personal experience as a mathematics teacher and the evidence of learners' difficulties with differentiation and integration reported in the literature and in the DBE's reports.

1.5.1 Researcher's experience with learners' learning differentiation and integration

The researcher has been a mathematics teacher in the General Education and Training (GET) and Further Education and Training (FET) phase, especially for Grade 12 for the past 25 years; she has also taught at a university as a part-time lecturer for a few years. During her teaching at both institutions, the failure of learners and students to conceptualise mathematical concepts has been observed to be a consistent problem – particularly with regard to differentiation and integration. The poor performance by learners for these two concepts culminated in an interest, on the part of the researcher, to find out how learners construct mathematical knowledge when learning differentiation and integration in calculus.

The researcher's observation of learners' results on different assessments of these topics revealed that learners do well on tasks that require procedural knowledge, but when some of these tasks are rephrased, requiring them to explain their thinking and discuss the problem without applying the rules, learners have some difficulties. Even if their answers are correct, their explanation displays a lack of conceptualisation of the learnt concepts. Conceptual knowledge, as defined by Hapasaalo (2004, as cited in Jojo, 2014), denotes knowledge of and a skilful "drive" along particular networks, the elements of which can be concepts, rules and even problems given in various forms of representation. This led to an interest in how learners conceptualise differentiation and integration and how they learn these concepts effectively. Hence the study aims to reveal the nature of learners' mental constructions and the difficulties that hinder the constructions of the necessary mental constructions purported in APOS theory. As argued by Dubinsky (1997) and Ndlovu (2015), understanding the nature of learners' mental constructions and the difficulties they experience assists in identifying or designing appropriate alternative instructional strategies. However, the design of alternative instructional strategies is beyond the scope of this study.

1.5.2 Learners' experience with learning differentiation and integration

This study was conceptualised after Technical Mathematics was piloted in some technical schools in South Africa in 2016. After the piloting stage, it was escalated and implemented in all schools offering technical subjects in South Africa. However, the challenges observed during the pilot stage seemed to persist and still prevail, suggesting that learners continue to experience difficulties with many topics taught in technical mathematics, such as differentiation and integration.

In the Curriculum and Assessment Policy Statement (CAPS) (Department of Basic Education (DBE), 2014), differentiation and integration form a third of the concept assessed in Grade 12, which shows that they are critical topics that learners need to understand in order to pass. Thus, learners' poor performance on these topics means that they are struggling with a topic that forms a significant component of their learning. Research is thus needed to investigate and analyse the nature of learners' mental constructions and the difficulties they experience.

1.5.3 Bridging the gap: making a contribution to the knowledge field

It is true that there is a plethora of research focusing on learners' understanding of calculus concepts (e.g., Brijlall & Ndlovu, 2013; Bansilal & Pillay, 2014; Parrot & Leong; 2015). While the aforementioned studies focused on how learners solved problems (Bansilal & Pillay, 2014) or the teaching of calculus concepts (Parrot & Leong, 2015), a limited number of studies have focused on analysing learners' mental constructions (Brijlall & Ndlovu, 2013). While studies such as Brijlall and Ndlovu (2013) focus on the analysis of learners' mental constructions of optimisation problems, the study did not extend the argument to explain the difficulties that potentially hinder the constructions of the necessary mental construction. The current study, thus, by analysing the nature of learners' mental constructions, further extends the argument to analyse learners' difficulties. In the context of South Africa, before the introduction of technical mathematics, integration was not part of the school curriculum; therefore, while there are studies focusing on learners' understanding of differentiation, there is a dearth when it comes to integration. Most of the research on integration has been done at the tertiary level (for example: Ndlazi, 2018; Borji & Font; 2019; Bresock, 2022). However, findings from such studies reveal that students' difficulties with integration emanate from the basic concepts that should have been taught at school. As the current study focuses on learners' mental constructions of the concepts taught at school, it addresses this gap in the literature.

In addition, the study contributes to the knowledge field as the researcher used genetic decomposition to analyse learners' mental constructions. Ndlovu (2014), drawing on the work of Dubinsky (1991), posits that to reveal the nature of one's mental construction for a particular concept, a genetic decomposition ought to be designed to serve as an analytic tool. With reference to this study, the researcher designed a genetic decomposition to analyse learners' mental construction of differentiation and integration, which is a contribution to the field of knowledge in the teaching and learning of differentiation and integration concepts. A detailed discussion of the genetic decomposition is presented in Chapter 3 under 'theoretical framework'.

This study is significant as it contributes to the knowledge and literature on the mental constructions of Grade 12 learners in the learning of differentiation and integration. The study is important also as it focuses on learners´ difficulties in the form of the errors and misconceptions that learners encounter when learning differentiation and integration. This might assist mathematics educators to be more effective in their teaching as, by using learners´ errors, they may better help learners to overcome these errors. It is the hope of the researcher that the results of this study will establish a basis for thinking about possible intervention strategies to improve learners' understanding of calculus, especially differentiation and integration concepts. This will benefit all educators in the field of mathematics and the knowledge of these two concepts. This can also assist in proposing programmes or strategies that may assist in reducing the failure rate in calculus. The framework used in this study is APOS theory, which proved a useful tool in analysing learners' work together, with genetic decomposition.

1.6 Aims and Objectives of the Study

The study's aim was to examine the mental constructions of integration and differentiation challenges faced by calculus learners in Grade 12. To do this, the study concentrated on how calculus learners in Grade 12 construct their mathematical knowledge of differentiation and integration. Hence the study uses Apos (Action-Process-Object-Schema), which attempts to characterize the mental structures that deal with the nature of mathematical concepts, to analyze Grade 12 learners' mental constructions and challenges regarding differentiation and integration concepts (Brijlall, 2008),

The following objectives directed the investigation to achieve this aim:

- To explore Grade 12 learners' mental construction of differentiation and integration:
- To explore how learners' mental constructions evolve in relation to the designed genetic decomposition; and
- To explore what hinders or enhances the evolution of students' mental constructions needed in the conceptualization of differentiation and integration.

1.7 Research Questions

This study sought to provide an answer to the following key question:

How can calculus learners create meaning when learning differentiation and integration?

Aligning with the purpose of the study, the following questions were used to unpack the key question:

1. What are Grade 12 learners' mental constructions of differentiation and integration?
2. How have Grade 12 learners' mental constructions of differentiation and integration evolved in relation to the genetic decomposition?
3. Why have learners' mental constructions needed in the conceptualisation of differentiation and integration evolved, or failed to evolve?

1.8 Significance of the Study

Many studies have found that learners at the secondary school level are struggling with differential calculus, especially in finding the derivative. According to the Department of

Basic Education (DBE; 2012; 2014; 2016) the learning of differential calculus still poses difficulties for South African Grade 12 mathematics learners. According to Ndlazi (2015), because integration is not covered in school calculus, students who enter universities are ill-prepared for it. Additionally, the teaching of differentiation and integration at the school level, which seems insufficient, contributes to the excessive number of difficulties that learners display when managing this part of the subject (Jojo et al., 2013). This research focused on learners' mental constructions for the calculus concepts of differentiation and integration. Arnon et al. (2014) write that "research into students' mathematical learning helps to forecast what they may learn about a certain mathematical subject and the conditions by which that learning takes place" (p. 27). The researcher selected two concepts that are interlinked and which are found at both the high school and the university levels: differentiation and integration. APOS theory was used, and genetic decomposition was formulated in order to help educators analyse the knowledge constructed by learners when solving differentiation and integration. The findings of the study can inform the design of teaching materials and approaches to improve learning and promote the development of learners' mental constructs (Ndlazi, 2015).

1.9 Overview of Research Design and Methodology

This section presents a brief description of the research methodology. A detailed description, along with the rationale for the methodological choices in this study, is presented in Chapter 4.

1.9.1 Research paradigm

The interpretive paradigm serves as the foundation for this investigation. An interpretive paradigm attempts to comprehend the subjective realm of human experience and attempts to comprehend and interpret each aspect of the phenomenon being studied (Cohen et al., 2017). In interpretivist research, the researcher and the participants interact with the goal of comprehending the event from the participant's perspective, according to Strasheim (2011). In this study, the researcher and participants interacted through activity sheets and semi-structured interviews.

1.9.2 Research design

The study employed a qualitative methodology, which is an interpretive approach to research that focuses on people in specific contexts and makes use of a variety of subjective data (Grace, 2009). Qualitative research focuses on data rooted in human experience, based on situations or contexts (Maxwell, 2017; Norris et al., 2019). Devetak and Vogrinc (2013) claim that qualitative research employs an exploratory methodology that emphasizes the use of open-ended questions and probes that provide participants the chance to provide their own, original responses. In order to conduct an in-depth examination, a select set of participants in this study received special attention.

1.9.3 Research style

A case study is a type of research where a person, group of individuals, or organization is described in depth and analysed in one or more situations (Stake, 2005). Stake (2005) asserts that a case study offers a singular illustration of actual individuals in actual circumstances, allowing readers to comprehend the events more clearly than if only abstract theories were presented. Understanding learners' mental constructs requires an in-depth analysis of their written and verbal responses; therefore, a case study was deemed appropriate for this study.

1.9.4 Data collection methods

In this study, data was collected using the following instruments activity sheet, audio-visual recording and semi-structured interviews.

1.9.4.1 Activity sheet

Tasks were given to each learner to complete (see Appendix B). These tasks involved problems focusing on either differentiation or integration. Structured activity worksheets, according to Brijlall & Maharaj (2014), provided a model for how effective mathematics instruction could be prepared with the goal of simultaneously addressing the cognitive and affective domains as students answer problems. The goals of differentiation and integration are to increase students' grasp of calculus and how to apply it in practical situations. In order to understand the mental constructions that students use to build knowledge of differentiation and integration and how these mental constructions relate to the hypothesized genetic decomposition, the researcher used activity worksheets to obtain the necessary data.

1.9.4.2 Semi-structured interviews

The aim of the interview was to get a clear understanding of the mental constructions that the learners are able, or unable, to make. The learners' responses to each task on the activity worksheets were categorized according to the mental constructions they used and two participants from each category were selected for the interviews. The reason for choosing two learners per category was to allow for the possibility of one learner withdrawing. The interviews made it possible to understand more about how students conceptualized concepts through the process of cognitive production. To determine the extent of their comprehension of differentiation and integration, learners were asked to respond to open-ended questions. The stage of questioning and the kinds of questions to be asked were determined by their responses on activity sheets. The information from the activity sheet was used to produce the interview questions, which were not pre-set.

1.10 Definitions of Key Terms Used in the Study

Calculus, according to Berggren (2012), is a field of mathematics that focuses on calculating the instantaneous rate of change (differential calculus) and adding an infinite number of minor elements to a single whole (integral calculus). Differential and integral calculus are the two branches of calculus, as previously mentioned. In calculus, limits, functions, derivatives, integrals, and infinite series are key. The rules of differentiation and integration, along with APOS Theory and genetic decomposition, are covered in this study along with the topics.

APOS Theory. APOS theory posits that the key mental structures are actions, processes, objects, and schema (Dubinsky, 2002). The theory explains the mental frameworks that deal with a mathematical concept's nature and how it emerges in a person's mind. It describes a potential method for learning a topic to build the necessary knowledge. The theory further uses the model of what might be going on in the mind of an individual to evaluate students' successes and failures in dealing with mathematical problems (Dubinsky, 2002). APOS is discussed in more detail in Chapter 3.

Genetic decomposition. This refers to a structured set of mental constructs which describe how a particular concept can develop in the mind of an individual (Jojo, 2011). It is regarded as a diagnostic tool which provides the investigator with insight into how a learner can develop a concept throughout the different stages of Action, Process, Object and Schema. At the end, the investigator must be able to come up with a suitable activity that will enable learners to develop a better understanding of the concept that is being taught (Jojo, 2011).

1.11 Structure of the Thesis

This thesis consists of eight chapters. The first four chapters provide an overview of the research project. The second four chapters present the findings, analysis, conclusions and recommendations.

Chapter 1: Introduction to the Study

This chapter has introduced the study by providing the background, the research problem, and the research questions for the study.

Chapter 2: Literature Review

A survey of the literature on how students construct meaning when studying mathematics is found in this chapter. It represents general discussions on how calculus is taught and learned. This chapter also discusses how different mathematical concepts are meant to be understood, with an emphasis on the two notions of differentiation and integration. Additionally, it reviews procedural and conceptual learning styles as well as the concepts of concept images and definitions.

Chapter 3: Theoretical Framework

This chapter discusses the theoretical framework used in the generation and the analysis of the data for the study. The concepts of differential and integral calculus, a framework for the research in mathematics education, APOS theory and genetic decomposition of analysing learners' difficulties are discussed. The initial genetic decomposition for both differentiation and integration are suggested.

Chapter 4: Research Methodology

The research techniques and methodology used for this study are described in this chapter. Research paradigms are discussed, positioning this work within an interpretivist paradigm. The qualitative case study served as inspiration for this chapter's appropriate approach analysis. In-depth reviews of the instrument, data analysis, and research methodologies utilized to gather data are also included.

The presentation and analysis of the study's data are the main topics of the following chapters.

Chapter 5: Analysis of Written Responses from Activity Sheet (Phase 1)

This chapter focuses on the validation of the research instrument used in Phase 1 of the study. The findings of this preliminary study are outlined in detail as well as how these influenced the initial genetic decomposition.

Chapter 6: Analysis of Written Responses from Activity Sheet and Interviews on Differentiation (Phase 2)

This chapter presents the analysis of students' responses from activity sheets and interviews on differentiation. Data was presented and analysed and findings and meaning learners attached to differentiation.

Chapter 7: Analysis of Written Responses from Activity Sheet and Interviews on Integration (Phase 2)

This chapter reports on data obtained through the research instrument (activity worksheet) and on semi-structured interviews. Discussions on the suggested concept definition and suggested concept image of integration ensues.

Chapter 8: Synthesis of the Findings, Conclusions, and Recommendations

This final chapter revisits the research questions that guided the study and synthesises the findings. The limitations of the study are discussed and recommendations are made for teaching practice as well as for further research.

The next chapter presents the literature review.

2 LITERATURE REVIEW

2.1 Introduction

The previous chapter outlined the background, problem statement, purpose, rationale and overview of the study. In this chapter, the literature on relevant studies about the learning and teaching of differentiation and integration is discussed. Studies exploring the learning of mathematics concepts, conceptualisation of calculus concepts and teaching of mathematics for understanding differentiation and integration concepts, are reviewed. The factors associated with poor conceptualisation of calculus concepts, knowledge construction in mathematics and learners' difficulties with differentiation and integration are explored. In particular, studies using APOS theory are reviewed.

2.2 Learning Mathematics with Understanding

The understanding of mathematical concepts is vital in the process of teaching and learning mathematics. One of the main learning principles put forward by the National Council of Teachers of Mathematics (NCTM) (2000) is that students must learn mathematics *with understanding*. Sierpinska (1994) defines understanding as being able to explain and justify finding evidence and examples, generalizing, applying, analogizing and representing a topic in a new way. On the other hand, Usiskin (2012) asserts that a learner has a full understanding of mathematical concepts if they can deal effectively with the skills.

In response to a high failure rate in mathematics in the past, the South African government called for renewed efforts to improve the teaching and learning of mathematics (DBE,2012). Numerous adjustments have been made as a result of the implementation of the Curriculum and Assessment Policy Statement (CAPS) for Grades 10 through 12 in an effort to improve mathematics instruction (DBE, 2012). For learners aiming to enter technical fields, the Department of Basic Education launched Technical Mathematics as a subject for Grades 10 through 12 in 2016. Technical mathematics works with concrete objects to demonstrate properties and theorems, whereas pure mathematics works with abstract objects to demonstrate properties and theorems (Usiskin, 2015). Taking Technical Mathematics as a subject in high school does not enable students to enrol at a university, but it does allow them to continue their education at a Technical and Vocational Training college(DBE2018)., where they can continue with an N4 qualification after they complete Grade 12.

With regard to the two key areas of calculus, differentiation and integration, differentiation was already part of the Pure Mathematics curriculum, while integration was not. The introduction of Technical Mathematics into the high school curriculum led to the inclusion of integration in the school curriculum. Learners who take Technical Mathematics thus learn both differentiation and integration, unlike those taking Pure Mathematics. However, since the introduction of Technical Mathematics in 2016, diagnostic reports from moderators indicated that learners who are doing Technical Mathematics have performed poorly with regard to these two concepts (DBE, 2018).

Studies have shown that a teacher's lack of subject-matter expertise in calculus is one of the factors contributing to students' poor academic performance in the subject (Likwambe, 2006; Lam, 2009). Likwambe (2006) found that teachers being unqualified or underqualified to teach mathematics has contributed to the cycle of poverty in mathematics education by depriving students of the necessary knowledge to form accurate conceptual images of the concepts that have been taught. Similarly, Lam (2009) examined the calculus content knowledge of 27 in-service mathematics instructors; the findings showed a lack of familiarity with several differential calculus concepts.

On the other hand, other studies have found that the challenges learners have in mathematics are due to the fact that learners have not learnt to use the necessary mental constructions and thus cannot construct schema for particular concepts (Siyepu, 2013; Maharaj, 2013; Ndlovu, 2014; Ndlazi, 2015). Brijlall and Ndlovu (2013) explored learners’ mental constructions of optimisation and found that learners were operating at the action stage, suggesting that they had not conceptualised the concepts. Siyepu (2013) conducted a study with first-year university students exploring their mental constructions of differentiation rules. The findings revealed that, due to misconceptions the students had, they struggled to construct the necessary mental constructions, which then hinder their conceptualisation of the concepts. Maharaj (2013) found that students did not engage with what they wrote, suggesting that while they could perform procedures, these were performed without being interiorised to form a coherent schema in the mind of the learner. These studies, with the exception of Brijlall and

Ndlovu (2013), focus on undergraduate university students' mental construction of particular calculus concepts (Ndlazi, 2015). There is paucity in the research to understand learners' mental constructions of the concepts and the errors that potentially hinder the construction of the necessary mental construction for learners to develop a coherent schema of these concepts. Given that integration was not taught as part of the mathematics curriculum in South African schools before 2016, there is no history of research exploring learners' mental constructions for integration concepts at the high school level – although studies involving university students have shown that they experience difficulty comprehending these concepts (e.g. Maharaj, 2013; Ndlazi, 2015).

Ndlazi (2015) states that the exclusion of the topic of integration from high school calculus has resulted in students entering universities being highly underprepared as they have not constructed the required knowledge around integration to cope with calculus at the tertiary level. This lack of knowledge construction was evident in a study by Maharaj (2013) that explored university students' mental construction of integration concepts in natural sciences. The findings revealed that students had difficulty applying the rules of integration due to a lack of appropriate mental structures. The study revealed a lack of appropriate mental structures needed to help students conceptualise the content taught at the university level. As at that time integration was not taught at the secondary level, there was no literature exploring the mental structures needed for the conceptualisation of integration at the secondary stage that would be needed for the development of the concept at the tertiary stage. Tall (2008) states that, whichever way calculus is approached, there seem to be inherently difficult concepts which seem to cause problems no matter how they are taught, and that learners failed to answer conceptual questions correctly.

As mentioned above, since the introduction of Technical Mathematics in the high school curriculum, there has been evidence in moderators' reports that learners are having difficulties with conceptualising differentiation and integration; the dearth of literature in this field propelled the researcher to conduct this study. The aim is not only to analyse learners' mental constructions of differentiation and integration, but to go further to explore difficulties that might hinder the construction of the necessary mental constructions.

As purported by Dubinsky (1997, as cited in Ndlovu & Brijlall, 2015) topics that learners find difficult should be analysed by means of research to plan appropriate alternative instructional strategies. Dubinsky (1997) put much of the blame for such difficulties on the pedagogical approaches used for concept development, arguing that traditionally lectures simply instruct students without giving them opportunities to experiment with different types of problems; students thus have a passive role in their own learning, which doesn't equip them to deal with problems beyond those they encountered in the classroom. Dubinsky (1997) also blames the students themselves for their lack of prior knowledge of the concepts that are crucial to the learning of calculus - such as the function concept; curriculum planners are also blamed for not requiring that pre-requisite concepts be taught before embarking on calculus concepts. In this context, this study aims to contribute to the field by designing a genetic decomposition that could help in analysing learners' mental construction of the concepts so that appropriate alternative instructional strategies could be planned to enhance learners' mental constructions of concepts.

2.3 Factors Associated with Poor Conceptualisation of Calculus Concepts

Siyepu (2015) asserts that several researchers have found that the errors and misconceptions displayed by students in their attempts to solve mathematical problems perpetuate their poor performance in their learning of mathematics. The findings of his study revealed that students commit different types of errors, such as interpretation, arbitrary procedural, linear extrapolation, and conceptual errors. Molefe and Brodie (2010) argue that errors are signs of underlying misconceptions. Similar findings have been echoed by other studies (Tall, 2008; Thompson, 2008; Ndlovu & Brijlall, 2016). Poor performance of students in mathematics in higher education is also ascribed to the misconceptions and errors that they bring to higher education from the secondary level (Molefe & Brodie, 2010). Luneta and Makonye (2010) state that "students' performance in calculus is undermined by weak basic algebraic skills of factorisation, handling operations in directed numbers, solving equations and poor understanding of indices" (p. 167).

This study does not duplicate Siyepu's (2015) study but rather addresses the remaining gap, as this study focuses on Grade 12 learners at the secondary school level. In his study, Siyepu (2015) mentions that university students struggle with calculus concepts as they enter underprepared. For that reason, the researcher decided to design the study for the secondary school level, to understand learners' mental constructions as well as the difficulties, in terms of errors and misconceptions, that hinder their schema development. According to Ndlazi (2015), learners entering universities were significantly underprepared for calculus due to the removal of integration from high school calculus. Many of the difficulties that learners have while managing this part of the topic are a result of the mathematics instruction that is provided at the school stage, which appears to be insufficient (Maharaj, 2010).

Tall (2008) suggests that at all stages students experience difficulties with conceptualisation concepts taught in calculus. Along those lines of thought, Dubinsky (1997) posits that there is a need to explore students' difficulties by means of research in order to explore alternative instructional strategies to address the identified difficulties. It is within these parameters that, in this study, the researcher explored Grade 12 learners' mental construction of differentiation and integration and further explored the errors evident in their construction of knowledge. This study also contributed to mathematics pedagogy by designing the genetic decomposition that was used to analyse learners' mental construction of differentiation and integration.

2.4 Knowledge Construction in Mathematics

Mathematical knowledge is constructed in many ways, but the whole process takes place cognitively. Asiala et al. (1997) describe knowledge construction as "[a]n individual's tendency to respond to perceived mathematical problem situations by reflecting on problems and their solutions in a social context and by constructing or reconstructing mathematical actions, processes and objects and organizing these in schemas to use in dealing with the situations" (p. 5). Extending this point of contention, Ndlovu (2014) asserts that knowledge is cognitively constructed by each learner based on their previous experiences. This means that the current mental constructs of a learner may be either positively or negatively affected by their previous knowledge (2008). For a learner to solve mathematical problems, there is a need to recall and reconstruct examples cognitively that will make sense mathematically.

A constructivist understanding of learning sees learners as active participants in the development of their own understanding. Rosken and Rolka (2007) extend this argument by not only placing emphasis on previously learnt concepts but also emphasising that constructing new knowledge around new concepts requires a learner to form a comprehensive concept image. Constructing mathematical knowledge is thus a process in which a learner attempts to make sense of the information by trying to connect prior knowledge structures with emerging structures. It is therefore critical that learners' mental constructs are analysed to scaffold them accordingly. While recent studies are concerned with learners' poor performance in mathematics, the focus on how learners construct knowledge dates back to scholars such as Piaget (1978), Vygotsky (1979) and Tall (2008).

2.5 Studies of Learners' Difficulties in Calculus

Mathematics is considered to be a difficult subject by many students due to adverse teaching styles, difficulty following the instruction, difficulty understanding the subject, and difficulty remembering equations and methods for solving problems (Gafoor & Kurukkan, 2015) The researcher has observed that many of the difficulties students experience at the tertiary level have been inherited from their secondary schooling – especially in calculus. It is very important to lay a firm foundation that will remain with them even if they are learning new things, Many of the errors that learners make display a lack of fundamental mathematics. According to Gafoor and Kurukkan (2015), mathematics has certain inherent challenges because it is abstract and cumulative. As a result, learners need a strong foundation because they might not be able to learn new information without prior knowledge.

Some studies have concentrated specifically on the challenges that learners face when learning calculus. For instance, Donaldson (1963) highlighted three different sorts of mistakes that learners made when learning mathematics. The mistakes were (a) structural, because the learners failed to understand the relationships present in the problem; (b) arbitrary, because the learners failed to consider the limitations set forth in the material; and (c) executive, because they failed to carry out manipulations despite understanding the underlying principle. In a study involving 60 high school learners and 50 college students, Orton (1983) conducted research utilizing Donaldson's (1963) system of classification of errors. He developed a technique for conducting clinical interviews to find out how well students understood basic calculus. The students' responses to activities involving integration and boundaries were carefully examined. Students' stages of understanding, as well as frequent mistakes and misunderstandings, were discovered from the data collected. The students generally struggled to comprehend integration as the upper limit of a sum and the connection between definite integrals and areas under the curve. Orton (1983) claims that many teachers have come to terms with the idea that integration cannot be made simple and have reacted in various ways.

Some teachers have implemented integration as a rule or as an anti-differentiation strategy, while others have tried to build up students' and learners' understanding of limits and background algebra before introducing integration.

Thomas and Ye (2017) conducted a study on the processes and conceptions of learners that were related to integration. Their research looked into how learners thought about, and formed misconceptions about, the Riemann integral. They discovered that learners' comprehension of key concepts was hampered by their lack of specific conceptual understanding and frequent use of an instrumental, process-oriented mode of thought.

Kiat (2005) studied how challenging it was for learners to solve integration problems. He administered a six-question test, followed by interviews with a few learners. Kiat's study does not specifically mention this, but it appears that he modified Orton's (1983) classification of errors. The present study thus takes this line of investigation further by analysing errors committed by Grade 12 learners when solving differentiation and integration. Furthermore, this study attempts to identify approaches that will alleviate the poor attainment of results in mathematics. The researcher believes that any intervention designed to improve learners' performance must be informed by research which can provide important information that can "lead to better targeted interventions" (Makgato, 2007, p.67).

Calculus has evolved into a potent tool for resolving significant issues in the fields of science and mathematics and is arguably one of the greatest inventions of human thought (Weber, 2012). A significant portion of contemporary mathematics education is made up of this subject. The fundamental theorem of calculus connects its two main divisions, differential and integral calculus. In the same way that geometry is the study of shape and algebra is the study of operations and how to use them to solve equations, calculus is the study of change. Calculus represents the first time in which the learner is confronted with the *limit* concept, involving calculations that are no longer performed by simple arithmetic and algebra, and infinite processes that can only be carried out by indirect arguments. Teachers often attempt to circumvent the problems by using an "informal" approach, playing down the technicalities. However, whatever method is used, general dissatisfaction with the calculus course has emerged in various countries around the world in the past (Tall, 1992).

According to Lam (2009), secondary school students in a number of different nations have difficulties when taught differential calculus. It has been noted that South African Grade 12 learners still struggle with differential calculus, and they also struggle to answer differentiation problems (DBE, 2012; 2014; 2016). Differential calculus is an important subject that can be applied to anything that moves, changes, or has a shape, according to Rohde (2012). As such, if its fundamentals are not well understood, real-life problems cannot be solved without difficulty.

Brijlall and Maharaj (2014) allude that the quality of mathematics instruction at the school stage appears to be inadequate and is a factor in the numerous challenges that learners face while dealing with the integration topics. Ndlazi (2015), who asserts that the sequencing used in schools mostly encourages instrumental understanding rather than conceptual comprehension, backs up this claim. Surface learning and deep learning are two opposing theories of learning that have been studied in depth (Cano & Berbén, 2009). Numerous research on the teaching and learning of integration have found that learners have trouble comprehending integral calculus (Brijlall & Bansilal, 2011; Habineza, 2016; Ndlazi, 2015). Although differentiation and integration are important concepts in calculus, learners generally find them difficult to cope with and encounter various difficulties while solving integration problems (Seah, 2016). It is within these parameters that this study also analyses the nature of difficulties learners encounter when learning integration and differentiation. The researcher observed that learners experience difficulties in calculus especially when dealing with differentiation and integration when she was marking Technical Mathematics senior certificate examination papers.

For the purpose of this study, the researcher focused on how learners construct meaning when solving differentiation and integration in mathematics. Stroud & Booth (2009) state: "When differentiating we start with an expression and proceed to find the derivative, but when we are integrating, we start with derivative and then find the expression from which it has been derived" (p. 335). The indefinite integral is defined by $f(x)dx = F(x) + C$, where C is any arbitrary constant, according to the definition put forward by Smith and Minton (2003). An indefinite integral of a function f in an interval [a, b] was defined by Koepf and Ben-Israel (1994) in 1994, as an antiderivative or a primal of a function, respectively. As a result, a function F that satisfies the equation $F^1(x) = f(x)$ at all points x in the interval $[a, b]$ is said to be an indefinite integral of $f(x)$.

The researcher observed that the learners usually failed to recognise an appropriate formula to use when dealing with differentiation and integration. This concurs with the findings of Maharaj (2008), who reports that students had difficulties recognizing equivalent equations and making decisions as to which transformations are permissible and should be made in the context of the given equation. Kiat (2005) found that only a small number of students remembered the required formula. Most common among difficulties or operation breakdown was the inability of learners to identify and recall appropriate formulas and technical errors, which is due to a lack of content knowledge.

All the difficulties learners experience emanate from a basic lack of knowledge of the rules of calculus and algebra. Even when solving the definite integral, they fail even to know that the integers $a\ and\ b$ are referred to as the limits of integration, whereby the number a represents the lower limit, and the upper limit is represented by the number b (DBE, 2018; 2019). If $f(x)$ is the integral of $f(x)$, then $\int_a^b f(x)\, d(x) = f(b) - f(a)$. According to Stroud and Booth (2009), the typical method for introducing a definite integral is to compute the area under a function's graph by dividing the area into strips. Although most students could integrate, Rasslan and Tall (2002) found that they were "unable (or unwilling) to describe the meaning of a definite integral" (p. 7).

2.6 Learners' Difficulties with Differentiation and Integration

Differential calculus is a field of mathematics that calculates instantaneous rates of change, while integral calculus is a branch of mathematics that determines some whole by adding an unlimited number of little factors (Berggren, 2012). As alluded by Zhang, (2003), the aim of teaching calculus is to assist learners to construct mathematical ideas and to think logically. Therefore, in the process of teaching calculus concepts, the focus should be on helping learners construct the correct schema of the taught concept. However, a plethora of studies has argued that the learning that has taken place has not yielded the construction of schema (e.g. Ndlovu and Brijlall, 2013; Jojo, 2013; Ndlanzi, 2015; Borji, 2019). Although Tall (2008) posits that calculus instruction has evolved over the years, this has not alleviated learners' difficulties. As noted in some of the studies, learners' understanding of calculus concepts is primarily procedural. For example, Luneta's (2014) study found that the procedural expertise of university students in routine differentiation was sufficient, but they lacked a conceptual comprehension of the derivative. Luneta (2014) argued that when students try to construct mathematical meaning, difficulties arise to due errors and misconceptions in their prior knowledge constructions. Extending this argument, Masingila et al. (2012) found that, although students can distinguish between different strategies and identify their limits, they frequently are unaware of the underlying mathematical concepts that serve as the foundation for these techniques.

According to Ferrini-Mundy and Findell (2010), most students misunderstand the ideas that lead up to basic calculus. Porter and Masingila (2000) also noted that many students in university calculus programs had only a cursory comprehension of fundamental calculus principles, which is something that most teachers might not be aware of. According to Tall and Vinner (1981), one challenge to students understanding differential calculus is that they transition abruptly from studying the discrete and the finite to studying the continuous and infinite. This concurs with the findings of Artique (1998) in his study conducted with French students, where he found that some students were unable to understand that 0.999 999 999... (a limit process) really equals an object. The argument raised by the above authors coincides with the comments raised by moderators in South Africa that while learners can recall facts they are unable to apply those facts to solve questions involving higher-order thinking skills (DBE, 2021). It is in the context of these fundamental findings that the researcher decided to conduct a study focusing on analysing learners' mental constructions and identifying the difficulties that hinder the construction of the necessary mental constructions in differentiation and integration.

2.6.1 Learners' difficulties with differential calculus

For South African Grade 12 mathematics learners, mastering differential calculus presents challenges because many of them still struggle to answer problems based on differentiation rules (DBE, 2012; 2014; 2016). Calculus is abstract and involves complicated ideas, which contributes to the challenges encountered in teaching and studying, as well as the fact that many learners do not understand essential calculus concepts (Artique, 2009). Therefore, the focus of this study was to analyse learners' mental constructions and the difficulties they experience in the application of differentiation rules and integration.

According to Zakaria and Salleh (2015), learners' difficulties in solving differentiation problems are a result of the emphasis placed throughout calculus instruction on procedural comprehension rather than conceptual understanding. A vicious cycle is put in motion because the teacher is aware that conceptual inquiries are rarely answered correctly. According to research comparing the outcomes of students taking advanced placement calculus courses in school, there is evidence that students who take an initial calculus course based on elementary procedures may experience unanticipated limitations in terms of their attitudes when they take a more challenging course later (Habineza, 2016).

When it comes to learning differentiation principles, learners who want to study advanced mathematics must pay close attention to derivatives of distinct functions. The derivative is a concept that is constructed from other concepts (Naidoo, 2009). In addition, the derivative can be viewed as a function, or as a number when calculated at a particular moment, the upper bound of a series of secant slopes, or a rate of change. Naidoo (2009) states that it is essential that students comprehend the meaning of a derivative and how to differentiate between different functions.

Calculus calls for a high degree of mental comprehension, but many students find differentiation to be confusing (Parameswaran, 2010). According to Parameswaran (2010), students should not be taught differentiation rules until they have a firm grasp of what a derivative is and are familiar with the connection between a function and its derivative; students should be required to research methods for locating and examining the derivatives of a range of various functions. This argues that instructors at the school stage should create tasks that let learners investigate fundamental differentiating ideas. Even at the university stage, lectures must first ensure that students comprehend basic ideas before introducing ideas like integration or the chain rule.

Student errors in elementary analysis were the subject of a case study by Luneta and Makonye (2010) involving a Grade 12 class in South Africa. The goal of the study was to classify errors that students made in response to calculus questions, explore the errors that students made in differential calculus, and show how students' calculus errors were related to their misconceptions (Luneta & Makonye, 2010). In the summary of their findings, they draw the conclusion that students' inadequate comprehension of calculus is caused by inadequacies in their understanding of algebra (Luneta & Makonye, 2010). They also contend that language issues and a poor grasp of fundamental calculus concepts – in which calculus procedures like rules of differentiation are instrumental – are also to blame for students' poor understanding of calculus. This shows that these students struggled with calculus conceptual understanding. They also noted that several students showed a lack of understanding of calculus nomenclature; for example, mistaking turning points for axial intercepts (Luneta & Makonye, 2010). This shows that it is important to focus on connecting learners' grasp of pertinent ideas and pertinent procedures that are covered in calculus sessions when instructing.

Jojo et al. (2013) claim that the complexity of calculus concepts deserves exploration because students struggle, while it is an important concept for both high school learners and undergraduate students. Siyepu (2013) looked into learners' mistakes that were visible in the derivatives of trigonometric functions. Data from 30 students enrolled in mechanical engineering at a university of technology in South Africa was gathered using a qualitative case study approach. The data gathered showed that learners' grasp of distinction was hampered by poor conceptualization. Siyepu (2013) urges educators to design classroom interactions with a focus on understanding mathematical symbols, rules and formulas.

Therefore, taking cognisance of the studies that have been done in this field and their recommendations, this present study undertook to explore learners' mental constructions of differentiation and integration as well as errors that hinder the construction of the necessary mental constructions.

2.6.2 Learners' difficulties with integral calculus

Calculations of the area bounded by curves, the volume of a solid in a revolution, and other functions employ integral calculus to determine, characterize, and apply the integral of functions (Steward et al., 2008). An increasing number of studies have revealed that learners struggle to comprehend the idea of integral calculus (Tall, 1983; Kiat, 2005; Yee & Lam, 2008; Mahir, 2009; Usman, 2012; Salazar, 2014; Zakaria & Salleh, 2015; Ndlazi, 2015). Zakaria and Salleh (2015) observed low scores for university students' calculus proficiency, notably in the area of integral calculus. Many of the students found calculus to be exceedingly challenging and abstract, which prevented them from developing a deeper grasp. They argue that, unless a strong foundation in its pre-requisite skills is achieved, students' performance in integral calculus will still be low. As this concept has now been introduced into high school mathematics, and responding to the recommendations in the literature, the researcher was prompted to explore and analyse learners’ mental constructions of differentiation and integral calculus and to also analyse the difficulties that hinder the construction of necessary mental constructions. Not objecting to the call by Zakaria and Salleh (2015), the researcher is of the opinion that if the aim is to enhance teaching and learning, research should go beyond analysing performance and investigate learners’ mental constructions because, by understanding the stage learners are operating at and the difficulties they encounter, alternative instructional strategies can be developed aimed at helping learners bridge the gap.

Kiat (2005) identifies three types of errors: conceptual errors, where the learner fails to grasp the concepts and relationships within a problem; procedural errors, caused by failure to carry out manipulations or algorithms even though the concepts in the problem are understood; and technical errors, resulting from a lack of mathematical content knowledge in other topics. Kiat (2005) found that students tend to confuse procedures for integration and differentiation. Integration is the reversal of differentiation, which means learners or students who have not grasped differentiation are more likely to struggle with integration; therefore, a lack of content knowledge of differentiation among other topics would negatively affect learners’ understanding of integration. Using the categories specified by Kiat (2005), such difficulties can be attributed to conceptual and technical errors.

In a study by Mahir (2009) that looked at how well students performed conceptually and procedurally on integration, it was discovered that students' conceptual grasp of integration was insufficient. In this case, the research group was made up of college students who had successfully finished calculus at one university. Researchers looked at how well students understood the fundamental theorem of calculus, the integral-area relation, and the integral as an algebraic sum. When teaching the idea of integration, Mahir (2009) argued in favour of using a variety of contexts and against assessments of learners that encourage memorization.

Huang et al. (2011) conducted a quasi-experimental study, yielding different recommendations than Mahir’s (2009) strategy. A class of learners was divided into two groups: one group received education based on procedures, while the other received training based on concepts. The results showed that whereas the process group had inadequate conceptual comprehension of the idea of integration, the conceptual group fared well in both knowledge classifications (Huang et al., 2011). Participants took an examination in indefinite integral calculus that required the use of integration formulas, to investigate students' learning challenges in integral calculus. The quantitative data showed that students struggled with integrals, and the qualitative data showed that many errors were more related to the students' trigonometry skills than to the integration itself. It was further found, as was the case in earlier studies, that the students' integration issues were clearly attributable to the underlying mathematical knowledge and abilities they had acquired in fundamental mathematics (Huang et al., 2011).

Ndlazi (2015) carried out a study to examine how first-year engineering students at a South African university of technology developed the idea of integration. First, a pilot phase was included as Phase 1 of the study to identify difficulties to be investigated further when the study was made available to a broader group of students. Seven students participated in the pilot phase and were interviewed after completing an activity sheet. In the next phase, 22 first-year students received a revised activity sheet. The aim was to offer a thorough analysis of students' integral calculus concept formation. In order to study new mental constructions that emerged as students discussed concepts, focus groups were also used. The research's conclusions showed that, for integral calculus, learners mostly used an action stage of cognition.

As evident from the literature reviewed here, there are many studies on students' understanding of integration concepts at the university level, dating back to the 1980s (e.g. Orton, 1983b; Rösken and Rolka; 2007; Pettersson and Scheja, 2008; Mahir 2009; Huang et al., 2011; and Habineza 2010; Maharaj, 2013 and Ndlazi, 2015). While the literature has echoed that students' difficulties at the tertiary level emanate from a lack of background knowledge established at the secondary level (Maharaj, 2013; Ndlazi, 2015) there has been limited research focused on the secondary schooling level (e.g. Brijlall & Ndlovu, 2013; Ndlovu, 2014). Furthermore, due to integration not being part of the South African school curriculum, there has been no research done to explore learners' mental construction of integration. It is within these parameters that the researcher in this study decided to focus on these concepts as they are now taught at the secondary level. Since there is no previous study that has looked at differentiation and integration at the secondary level, this study attempts to address this gap.

2.7 Learning of Mathematical Concepts

There are different perspectives on how learners learn mathematical concepts; however, the common ground is that learning of mathematical concepts takes place once one has constructed a coherent schema of the concepts. Scholars that foreground their research to explain how learners come to learn and understand mathematics date back to the work of Piaget (1978), Dubinsky (1991), Tall (2014) and others. While there are new developments in the field, the concept of how learning takes place is still foregrounded in the work of previous scholars. For example, APOS theory is an extension of Piaget's (1978) work on reflective abstraction. Therefore, to understand learners' learning of differentiation and integration, the researcher deemed it necessary to review the literature that seeks to explain how learners come to learn mathematical concepts. While this study focuses on the mental constructions which define, learning as purported in APOS theory, the review of literature reviews other frameworks to present a broader perspective.

The cognitive components of learners' mathematical reasoning have been examined from many angles and with various emphasises. Nevertheless, these approaches all share a constructivist perspective on students' learning and, specifically, include students' intuitive thinking as a key component to obtaining new knowledge (Presmeg, 2006). It is commonly known that concepts and formal statements are frequently associated, in a person's mind, with some specific instances. Engelbrecht et al. (2010) mooted that, in the learning process, understanding does not happen at a particular stage but is an ongoing process. While it converges to full understanding it does not reach the limit of full understanding. This means that understanding is dynamic and continuously evolves. It is not static: as one comes into contact with new knowledge it is assimilated into the existing schema and knowledge is extended to accommodate new schema.

Fischbein (2001) researched students' individual concept development and elaborated on how intuitive models influence the acquisition of mathematics. The author posits that the possibility that such specific models could, for that person, become universal representatives of the relevant concepts and propositions and hence acquire the heuristic properties of models, is typically overlooked. For example, in the case of learning differentiation and integration, learners encounter a vast variety of material and their prior knowledge or experience with the concept or related concepts will influence how they decide to incorporate new mathematical ideas and build concepts.

Tall and Vinner (1981) dealt with the creative processes in students' learning of mathematics. Their concept image and concept definition paradigm make it possible to examine how students represent mathematical ideas. They defined an individual's concept image for a given concept as "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (Tall & Vinner, 1981, p. 152); they argue that this concept image "is built up over years..." (p. 152) and changes "as the individual meets new stimuli and matures" (p. 152). As such, a concept image is embedded in a network of different experiences and concepts with diverse relations between them (Rösken & Rolka, 2007). Tall and Vinner (1981) introduced the term "evoked concept image" to indicate that "portion of the concept image which is activated at a particular time" (p.160). Thus, the evoked concept image is subject to the possessed concept image together with time and the way an individual is prompted to demonstrate the concept image (Habineza, 2010; Vinner, 1991).

On the other hand, the words employed to define a specific concept are thought of as concept definition. A concept definition might be formal or personal, the latter being one that is accepted by the mathematical community (Tall & Vinner, 1981). In mathematics, specifically, it is typically considered essential to consider all facets of a term. For instance, when asked to consider the relative extrema of a continuous (real-valued) function on a closed interval, students frequently overlook the extrema at the interval's endpoints. In that case, learners' thinking is dominated by the concept image of the symbol $\frac{dy}{dx}$, which denotes the gradient of a curve $with\ respect\ to\ x$, and they do not consider the relationship of y with respect to x in the given equation. Vinner (1991) explains why students' misconceptions, like the one mentioned before, occur in learning situations. He shows that during the process of concept formation, the relationship between the concept image and concept definition is reciprocal. Therefore, if the concept definition is false, probably the concept image constructed will be incorrect and vice versa.

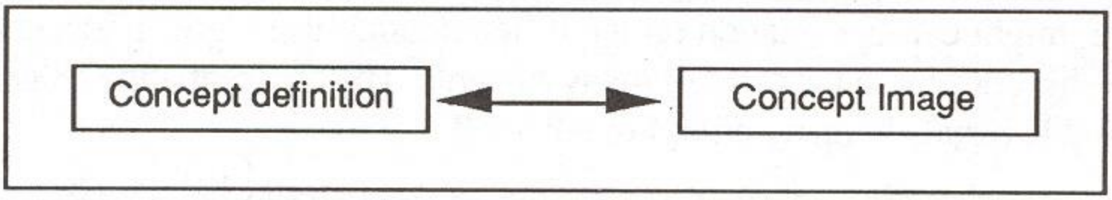


Figure 2.1: Relationship between concept image and concept definition during concept formation (Vinner, 1991, p. 70).

As mentioned above, this current study is framed with the process of learning as defined in APOS theory, which considers learning as "an individual tendency to deal with perceived mathematical problems situations by constructing action, process and objects and organised them in schemas to make sense of the situation and solved problems" (Dubinsky & MacDonald, 2001, p.2). For example, in the case of differentiation and integration, to form a coherent understanding of the concept and be able to solve the problems learners need to have constructed action, process and object of the concept. APOS theory posits that the formation of a concept begins at the action stage, where physical entities are interiorised into a process, then encapsulated into an object. This means that, when learners are at the action stage, the physical interactions which are then repeated are meant to help the learner construct the concept definition, which then will be interiorised into a process and the physical entities performed solely in the mind. It is at that stage the learner is constructing the concept image which can then be encapsulated into an object. However, the object can also be de-encapsulated into a process by applying the processes and object meaning that the concept image formed in reverse, to inform the concept definition.

2.8 Learners' Conceptualisation of Calculus Concepts: Notations and Terminology

According to Dubinsky and MacDonald (2001) a mathematical concept is first formed as an action, which is an externally directed transformation of a previously conceived object (or objects). This means that the concept is first conceived as a physical entity. For example, the notation f' is a physical entity for a derivative; therefore, the learner needs to conceive this notation which will trigger the actions to be performed in finding the derivative. Therefore, as indicated by Ndlovu (2012), understanding mathematical concepts is predicated on understanding mathematical symbols and notations. Notations and symbols form an important characteristic of mathematics, and studies have shown that, in many cases, students use notation incorrectly (Jojo, 2011; Ndlovu, 2013; Siyepu, 2013; Ndlovu, 2014). Ndlovu (2012) noted that learners were failing to link mathematical symbols and formulae with suitable procedures to be applied, especially when finding the derivative using the rules of differentiation; instead, they use the first principle formula. Siyepu (2013) reported a study to investigate inaccuracies seen in the derivatives of trigonometric functions. Data from 30 students enrolled in mechanical engineering at a university of technology in South Africa were gathered using a qualitative case study approach. The study found that students' grasp of distinction was hampered by poor conceptualization. Siyepu (2013) argued for structuring classroom interactions so that the emphasis is on understanding mathematical symbols, mathematical laws, and mathematical formulae. Such a strategy might help learners achieve meaningful mathematical knowledge.

According to Findel (2006), learning mathematics necessitates not only the creation of concepts but also the acquisition of their standard names, notations, and verbal and mathematical grammar for use in mathematical discourse. According to Naidoo and Naidoo (2007), alternative teaching strategies are necessary to improve calculus instruction and comprehension. According to Ndlovu (2013), it is critical to grasp notation, because employing the improper notation could result in the solution having a false sense of meaning. and it is crucial to be able to communicate mathematical ideas in words.

In this study, Grade 12 learners of Technical Mathematics should be able to use the fundamental principles of differentiation and integration in this course. The focus of this study is how Grade 12 learners construct the meaning of these two concepts and the difficulties which hinder their mental constructions. As many studies show that learners struggle with the abstract notation used in calculus, in this study the understanding of notation is important, because using the wrong notation may result in an incorrect solution.

One of the notations in calculus that learners tend to have difficulty with is Leibniz notation. The main problem is with the use of the Leibniz notation $\frac{dy}{dx}$, whether it is a fraction or a single indivisible symbol. The lack of understanding of the relationship between the dx in $\frac{dy}{dx}$ and the dx in $f(x)\,dx$ seems to be the source of the problem.

2.9 Teaching Mathematics for Understanding Differentiation and Integration

There are many perspectives on teaching mathematics for understanding and different terms used to refer to mathematical understanding. However, the key idea about teaching mathematics for understanding is that teachers should go beyond emphasising rules and memorisation but rather emphasise making connections between concepts (Ndlovu, 2012). While the current study is not focusing on teaching mathematics for understanding, as argued by Dubinsky (1997) topics that learners find challenging should be analysed so that alternative instructional strategies can be identified to enhance learners' understanding. Thus, it is imperative to understand what literature is deemed to address teaching mathematics for understanding. According to Sierspinska (1994, as cited in Ndlovu, 2014), teaching for understanding means helping students to make connections between concepts. Hasenbank (2006) defines this as 'conceptual understanding', while Skemp (1992) defines it as 'relational understanding'. However, in the context of this study, framed within APOS theory that emphasises the construction of the schema, teaching for understanding should be understood as assisting learners to construct the correct schema of the taught concept. While different terminology is used to describe understanding in the teaching and learning of mathematics, two broad views form the cornerstones when one describes teaching and understanding in mathematics: that is, procedural and conceptual understanding.

2.9.1 Conceptual understanding and procedural understanding

Siegler and Alibali (2001) argue procedural understanding and conceptual understanding are both important because the advances in one type support the advance in another type. For example, learners need to understand facts before they can be able to apply them in another context. However, when a procedure is routinised it cannot be applied to another similar context. It is purported in APOS that a concept is first conceived as an action. Therefore, teaching for understanding should not be premised on the memorisation of rules but on understanding the procedure so that it can be applied to another context(Ndlovu, 2012). In her study with Grade 12 learners, Ndlovu (2012) argued that to enhance understanding instruction should be directed at helping learners make connections between concepts. Her argument was based on her findings that learners solving optimisation problems demonstrated a knowledge of memorisation rules without meaning-making. For example, even when finding the derivative of the equation not presented in terms of x and y, learners used the notation $\frac{dy}{dx}$, suggesting that the procedure was carried out without formulating an understanding of the concept. Extending the argument, Jojo (2014) posits that teaching for understanding should foreground unearthing learners' insight and comprehension of the concept.

Herbert and Pierce (2011), after conducting a study with first-year students taking introductory calculus, rate and hence the derivative of functions where they observed that first-year students possessed the procedural knowledge of the concepts, suggested that when teaching introductory calculus teachers should start with area and integration, rather than the rate and derivative. While not objecting to the recommendation by Herbert and Pierce (2011), instead of specifying topics this study adopts the view of Tall (2008, as cited in Brijlall and Ndlovu, 2013) and Jojo (2014) that to enhance learners' conceptual understanding teachers should help learners make connections between concepts building from what they know to the unknown. For instance, learners in Grade 12 might comprehend differentiation as an operation on polynomials that adheres to specific criteria and is carried out in a particular order. The learner will come to see differentiation as a more all-encompassing process that is not just a collection of rules applied to specific activities as they encounter it more frequently. Some studies have suggested that teachers must interchange methods or teaching strategies to help learners get a better understanding of a certain concept. This is supported by Naidoo and Naidoo (2007), who suggest that there should be a change in the way students learn mathematics and that there is a need for alternative methods of instruction so as to enhance teaching and learning for understanding.

2.10 Studies Using APOS Theory

Several studies have used APOS (Action, Process, Object and Schema) theory, both internationally and in South Africa, which has heightened the importance of the APOS theoretical framework. This is echoed by Arnon et al. (2014), who note that APOS theory has been used in many studies as a strictly developmental tool, as a strictly analytical evaluation tool, or as both. An increasing number of studies both locally and internationally have revealed that many students have difficulty in understanding calculus (Usman, 2012; Mahir, 2009).

2.10.1 South African studies using APOS theory

APOS theory is a theory of learning mathematics. The researcher elaborates on APOS theory in the next chapter. Several studies have been conducted exploring learners' mental construction of mathematical concepts. For example, Brijlall and Ndlovu (2013) investigated high school learners' mental construction of optimization problems in calculus. The study findings revealed that learners were operating at the action stage, which means they were limited to knowing procedures but struggled to apply the concepts learnt. Brijlall and Ndlovu (2013) formulated itemized genetic decomposition to analyse learners' mental constructions of optimization problems. However, since the study focused on optimization, the genetic decomposition was only applicable to optimization problems and could no be used to analyse other calculus concepts, like differentiation and integration.

Brijlall and Maharaj (2011) conducted a study to develop insight into pre-service mathematics students' mathematical reasoning about aspects of the derivative concept. The study attempted to answer the following query: Does the continuity idea become a complete mathematical entity for each of the pre-service mathematics students or how is the continuity concept interpreted by them? For the study to be more qualitative, concepts like student-centeredness, collaborative learning, and self-discovery were used. The study's goal was to determine if pre-service mathematics students correctly and successfully apply various interpretations of the concept of continuity in their reasoning. The study was guided by the APOS theory. APOS was employed particularly to determine whether the students could create a coherent understanding of continuity or if they simply presented several independent interpretations as separate pieces of knowledge.

According to Brijlall and Maharaj (2011), the APOS theory gave the researchers direction, served as the foundation for generalization, and allowed them to construct a genetic decomposition of the continuity idea. The learners were assisted in developing mental constructs using pedagogy based on worksheets with collaborative instructional design. They go on to say that ensuring that students have the essential subject knowledge of the derivative idea is a crucial part of preparing student instructors for teaching mathematics. Additionally, they discovered that the educational approach and techniques employed in the study had a favourable impact on the development of the student teachers' derived schema. According to Maharaj (2010, p. 104), the clear pedagogic technique made sure that most student teachers understood that they needed to "dissect" the visual problems presented on the design worksheets and apply previously learned concepts in little "packets" to find a solution. According to the findings of their study, when given the opportunity to work in small groups and use their own tools for thinking, the participants had the ability to understand the notion of continuity.

Brijlall and Maharaj (2011) found that the structured worksheets promoted group work, which created a setting favourable to reflective abstractions. In South Africa, this phenomenon spans the secondary and tertiary education levels. Calculus forms an integral part of the Technical Mathematics curriculum in South Africa. This concept of Technical Mathematics was introduced in the year 2016 with the idea of introducing a completely different mathematics curriculum solely in the technical and vocational pathway of the schooling system (Jojo, 2019). This section of mathematics is important to many students at the tertiary level because of its widespread use in science, engineering, economics, business, medicine, industry and many other fields to understand and apply the concept of change and motion. According to Zakaria and Salleh (2015), students' performance in calculus at the university is low. Similarly, Bezuidenhout (2001) confirms that most first-year students have a very weak conception of the concepts in calculus and cited a lack of background knowledge from the secondary level. In addition, Salazar (2014) noted that students find calculus to be an abstract concept and hard to learn. However, considering the importance of calculus as introduced into the Technical Mathematics curriculum and its role in other academics, there is the need to investigate learners’ mental constructions and difficulties in differentiation and integration which are aspects of calculus and an area needed to be studied by learners doing Technical Mathematics.

In a different context, but also with first-year university students, Siyepu (2013) conducted a study using APOS to explore errors that are displayed by students when learning derivatives of trigonometric functions in an extended curriculum programme. The findings of this study revealed that students committed interpretation, arbitrary, procedural, linear extrapolation and conceptual errors and these errors hindered their attempt to construct the necessary mental constructions. In his study, Siyepu (2013) found that students made interpretation mistakes when they incorrectly perceived the nature of the problem as a result of the over-generalization of some mathematical rules. When learners acted irrationally and disregard the limitations imposed by the material, they committed arbitrary errors. Although they comprehend the concepts in the challenge, students can make procedural mistakes when they don't carry out manipulations or algorithms. Errors in linear extrapolation arise when the property $f(a+b) = f(a) + f(b)$, which only holds true when f is a linear function, is overgeneralised. Failure to understand the concepts involved in the problem or to recognize the linkages involved in the problem can lead to conceptual errors. According to Siyepu's (2013) research, students tend to overgeneralize specific mathematical techniques, algorithms, and differentiation rules in their solutions, which is why they make mistakes.

Siyepu's (2013) results also showed that the lecturer could spot and correct mistakes that students made when studying derivatives of trigonometric functions by using written assignments, classroom audio and video recordings, and learning activities. In their interviews, the students asserted that they profited from class discussions because they received prompt feedback from their classmates and the instructor. Additionally, they stated that when they continued to practice with the lecturer's guidance and the assistance of more experienced students, their performances got better. Siyepu (2013) supports the view from other literature that the identification of errors has immense potential to address students' poor understanding of derivatives of trigonometric functions.

APOS theory was employed in integral calculus by Brijlall and Bansilal (2011) in a study documenting the growth in knowledge of the Riemann Sum. At one university in South Africa, they collaborated with teacher candidates to teach high school mathematics. Brijlall and Bansilal (2011) presented a genetic decomposition of a Riemann Sum; however, their study revealed students had only "limited knowledge in the early phases of constructing the notion" (p. 137). Only at the action stage were the students able to estimate the area of a region under a graph using the upper and lower sums. At higher stages of construction, conceptual reasoning was not demonstrated.

Ndlazi (2015) carried out a study to investigate how first-year engineering students at a South African university of technology developed a concept of integration. The research's conclusions showed that, for integral calculus, learners mostly used an action stage of cognition. Their definition of an integral was limited to the idea of locating an integral without any connection to the region below a function's graph. Students typically thought of an integral as the opposite of a derivative. Regarding integration strategies, students relied on rules and algorithms without considering the objects and processes that the rules and algorithms contained. Despite noteworthy capabilities in abilities, like finishing a square and breaking down fractions into partial fractions, there was little comprehension of the underlying concepts.

Ndlovu (2019) examined a different idea and used APOS theory to characterize the kind of mental constructions pre-service instructors made when they learned about matrix algebra. The nature of the mental constructions formed by these pre-service instructors was examined using the preliminary genetic decompositions for matrix algebra ideas in order to comprehend and explain the mental constructions made or not made. The results of this study showed that pre-service teachers' mental constructs generally agreed with the initial genetic decompositions. They also showed that many students primarily operated at the action and process stages, while only a small number of pre-service teachers did so.

Since learning linear algebra is widely considered to be challenging for average students, the study developed a modified itemised genetic decomposition that is expected to aid in the teaching and understanding of concepts related to matrix algebra. The purpose of making the modified genetic decomposition available is to aid in the teaching and learning of advanced mathematics by allowing lecturers to examine students' mental constructs as they acquire the ideas of matrix algebra. The modified genetic decomposition is a contribution to APOS theory since it demonstrates how it may be applied to various mathematical concepts in different contexts, in addition to helping with the teaching and learning of particular mathematical concepts.

While many studies have explored mental constructions of calculus concepts, only Brijlall and Ndlovu (2013) have studied this with high school learners and their focus was on optimisation problems. Even among the studies conducted at the university level, none have explored the two concepts together and provided a genetic decomposition that could be used to analyse learners' mental constructions. This study, therefore, aimed to bridge that gap as there is evidence that learners at the secondary level and students at the tertiary level find these two topics challenging to comprehend.

2.10.2 International studies using APOS theory

Studies exploring mental constructions have also been conducted internationally, and APOS theory has various applications in mathematics education research. It has been used in many studies as a strictly developmental tool, a strictly analytical evaluative tool, or as both (Arnon et al., 2014; see De Vries & Arnon, 2004; Parraguez & Oktaç, 2010; Asiala et al., 1997; Kabael, 2011). In a study carried out with 10 undergraduate mathematics students at an American university, Parraguez and Oktac (2010) employed APOS theory. They concentrated on students' potential conceptualization of the vector space concept. Parraguez and Oktaç's findings (2010) led to pedagogical recommendations for how to teach the concept of vector space. The first recommendation was that flexibility in thinking about algebraic structures should be encouraged during education in order to help students establish the required schema for vector spaces. Second, it was noted that it was important to emphasize the connection between the two vector space operations (Parraguez & Oktaç, 2010). If the APOS theory is included in teaching and pedagogical methodologies, this potential instruction enhancement is achievable.

Asiala, et al. (1997) conducted interviews on derivatives with 41 engineering, science and mathematics students who had completed two semesters of calculus. These students were taken through an instructional treatment that used a strategy called the ACE Teaching Cycle. According to Asiala et al. (1997), the Ace cycle is an instructional strategy consisting of Activities, Class Discussion and Exercises. From this study, it was observed that students relied on formulae to evaluate a function. Reliance on a formula was also displayed when answering a question that required students to relate the slope of a tangent to the derivate. Although the given tangent line had two points on it, some students started by finding the equation in the form of $y = mx + c$, then differentiated it in order to determine its gradient. Asiala et al. (1997) provided recommendations regarding the pedagogical strategies used. The use of the ACE teaching cycle with carefully designed computer activities was reasonably effective in assisting students to "develop a relatively strong process conception in the understanding of the $f(x)$ notation and in interpreting the relationship between the derivative, its graph, and the graph of the function" (Asiala et al., 1997, p. 5). This is an example of using APOS theory both as an analytical evaluative and a developmental tool.

DeVries and Arnon (2004) reporting, on their study of students' conceptualisation of a solution of system equations, exhibit the dual usage of the APOS theory. They interviewed 12 students at a teaching college shortly after finishing a one-semester linear algebra course. The focus of the interviews was to explore students' ideas about what a solution to a system of equations means. DeVries and Arnon (2004) conceded that their instrument was not adequate for probing for deeper insight into their research questions; however, certain observations regarding students' conceptualisation could still be made. For example, some students relied on memorised rules without proper understanding, rather than reason, to answer questions about a solution. According to DeVries and Arnon (2004), such students' conception of solutions developed out of using algorithms like the Gaussian method to solve the equation or system of equations.

The findings of DeVries and Arnon's (2004) study also resulted in a formulation of an initial genetic decomposition for a solution to systems of linear equations. At an action stage, DeVries and Arnon (2004) recommended that students should be enabled to identify the two functions, their common domain and co-domain, and a solution as that element of the domain which produces true equality when substituted. The Process stage of development should involve students being assisted in identifying functions, domains, and co-domains without substituting values into equations.

Kabael (2011) reported on the use of APOS theory to analyse how students, in a course in the mathematics education programme at a university in Turkey, generalised the function notion from single-variable to two-variable function concepts. Kabael (2011) conducted interviews with six students whose conceptual stages were perceived as processes for both single and two-variable functions. A student at the process stage of conception was expected to be able to convert between the various representations of a function; namely, graphical representation, algebraic and table representations (Dubinsky,1991). The findings showed that students who had a schema conception of single-variable functions demonstrated a good understanding of the notion of two-variable functions. Those students whose understanding of a function concept was either at an action or process stage showed a weak process conceptual stage of the two-variable function. Kabael (2011) concluded that there was a direct relationship between students' construction of the concept of two-variable functions and their conceptual stages of a general function concept. Several other studies have heightened the importance of

APOS as a theoretical framework and how a corresponding genetic decomposition informs teaching and improves learning.

Additionally, the genetic breakdown of the concept would likely be aided by studying students' responses, which might lead to a deeper comprehension of how students develop their knowledge. There is a notable lack of studies that examine how learners conceptualize difference and integration, especially on a global scale.

Mulligan and Michelmore (2018) also noted students' weaknesses and acknowledged their concerns about the large number of students who depended on memorizing. Brown et al., (2013) support this by mentioning that, in their research, students showed a type of learning approach which they refer to as 'surface approach'. In the surface approach, the main feature is described as memorizing; students merely use disconnected parts of the course or module content without linking them. Boaler (2013) points out that in various homes, schools, and even universities, students' main goal is to do well on tests and examinations; as a result, they concentrate on memorizing mathematical procedures so that they can reproduce these on tests or examinations. Penglase (2004) also attests to this by mentioning how disturbing it is to notice that research results from various researchers continue to confirm that higher education students will learn for what they think might be tested in exams at the cost of properly understanding the concepts.

It has been acknowledged that background knowledge in calculus and in mathematics, in general, is a prerequisite for the schema development of new knowledge (Brijlall & Ndlovu, 2013; Ndlovu, 2014; Maharaj, 2013). Therefore, there is a need to understand learners' mental constructions of mathematical concepts to scaffold accordingly their needs since "research into students learning of mathematics helps predict what they may learn about a specific mathematical concept and conditions by which that learning takes place" (Arnon et al., 2014, p. 123).

There is a growing concern about learner performance in mathematics all over the world and South Africa is not an exception (Department of Education, 2012). Shole (2019) states that mathematics is a challenging subject to master for students from the primary education level to the tertiary level. A plethora of studies have been done on learners' mathematics performance in the context of South Africa. Based on learners' poor performance in mathematics, there is a need to understand how learners construct knowledge in order to understand the root cause of their difficulties. In the area of calculus, there is an inadequate number of studies on learners' knowledge at the secondary stage. However, a growing number of these studies have focused on university students and findings have revealed that most of the difficulties students have with university mathematics emanate from their lack of background knowledge at the high school stage (Ndlazi, 2015). Therefore, there is a need to explore learners' mental constructions of school mathematics concepts that seem to hinder their understanding of concepts learnt later at the university stage.

2.11 Implications of the Literature Review for this Study

The concepts of differentiation and integration are key to calculus. As noted above, in South Africa, integration was not part of the school curriculum until the introduction of Technical Mathematics in the school curriculum in 2016. Therefore, while some research has been conducted on learners' understanding of differentiation at the secondary level (e.g. Luneta & Makonye, 2010; Brijlall & Ndlovu, 2013; Makgakga &Makwakwa, 2014), research on integration has focussed on the tertiary level since the concept of integration was only taught at that level. However, as pointed out by Ndlanzi (2015), the challenges learners display at the university level reveal what should have been taught at secondary school level (for example, definition of the concepts of definite and indefinite integral). The studies that have focussed on differential calculus have mainly focussed on the application of rules, with limited attention given to learners' understanding of concept definition. Tall (2008) posits that correct concept definition helps one to construct the correct concept image. Therefore, the studies reviewed here concerning learners' understanding of differentiation and integration have paid limited attention to understanding learners' mental construction of concept definition and most of the studies have been conducted at the tertiary level, while the basic knowledge needed is taught at secondary level. As mentioned, in the context of South Africa, no studies were found that explored the nature of learners' mental construction of differentiation and integration at the school level.

2.12 Conclusion

The aim of the study is to analyse Grade 12 learners' mental constructions of differentiation and integration. In this chapter, the researcher has reviewed the literature to understand what other scholars have argued concerning the phenomena being studied. This chapter has reviewed the literature concerning factors that cause poor performance on calculus concepts, especially in differentiation and integration. Learners' challenges with the conceptualisation of calculus concepts can be identified, especially in differentiation, integration, and the use of calculus notations. Jojo (2011) states that it is imperative, however, that procedures learnt with meaning are linked to conceptual knowledge-concept image and definition, which is everything associated in somebody's mind related to mental pictures, properties, mental representation, contexts of applications and even statements. From all that has been discussed in this chapter, it is evident that there is more to be researched in calculus. The following chapter focuses on the theoretical framework for this study.

3 THEORETICAL FRAMEWORK

3.1 Introduction

This chapter discusses the theoretical framework within which the study is located. The study uses APOS (Action-Process-Object-Schema) theory to analyze Grade 12 learners' mental constructions and difficulties with differentiation and integration concepts. However, the study does not only investigate the nature of learners' mental constructions and difficulties, it also focuses on analyzing these difficulties to understand the underlying misconceptions that lead to these difficulties. To achieve this, Olivier's (1989) framework for the categorisation of errors is used.

The chapter begins with exploring instructional approaches which could assist learners in making mental constructions, followed by a discussion of the notion of mental constructions. Piaget's notion of reflective abstraction and the origins and development of APOS theory are discussed. Next, the theoretical orientation of this study is presented.

3.2 Mental Constructions

Ferriri-Mundy and Graham (1991, cited in Ndlovu, 2014) describe the learning of mathematics as a constructive process in which a student attempts to make sense of the presented information by evaluating, connecting and organising it to link prior knowledge to new knowledge. Therefore, in the process of forming mental constructions of new concepts, one's prior knowledge acts as a foundation. While some researchers speak of 'cognitive constructions' and others refer to 'metacognition', all of these refer to knowledge constructions in the learning of mathematics. The inference is that learning occurs when a person actively participates in the building of meaning by interacting with the ideas and experiences that are being offered and integrating these with previously learned concepts.

In the context of the APOS theory which frames this study, mental constructions relate to an individual process of making sense of mathematical concepts by building and using certain mental structures, through 'stages' of development: action, process, object and schema. (Anorn et al, 2014, p.17).

3.3 Piaget's Notion of Reflective Abstraction

Piaget introduced the concept of 'reflective abstraction' to describe the construction of logico-mathematical structures by individuals during cognitive development, which he saw as the primary process for mental constructions in the formation of thought and the mental mechanism by which all logical and mathematical structures arise in a person's mind (Arnon et al., 2014). According to Dubinsky (1991), Piaget believed that reflective abstraction is a powerful tool that is required for the formation of more complex mathematical notions. Piaget frequently reaffirmed this viewpoint. For instance, with regard to the development of thought, Piaget argued that "the development of cognitive structures is due to reflective abstraction" (Piaget, 1978, p. 143).

Piaget makes two important observations while examining how reflective abstraction leads to the construction of logico-mathematical structures. The first is that reflective abstraction does not have an absolute beginning but is present at the earliest ages in the coordination of sensory-motor structures(Piaget, 1978). The implication here was that an individual cannot determine the time at which a child starts to develop logical thinking. The second is that reflective abstraction continues up through higher mathematics to the extent that the entire history of the development of mathematics from antiquity to the present day may be considered an example (Jojo et al., 2013).

Piaget (1978) identified two kinds of reflective abstraction. One form is reflection: this takes the form of consciousness, quiet contemplation, and so-called 'operations' on the content, as well as reflecting operations and content from a lower to a higher stage of cognition (i.e. from processes to objects). The other form is the reconstruction and reorganization of the content and operations at this higher stage, which turns the operations into content to which new operations can be applied. Dubinsky (1991) posits that the second form is close to certain mathematical ideas; – for example, the mathematical concept function – and explains that functions are first constructed as operations that transform elements in a set called 'domain' into elements in a set called 'the range'.

According to Piaget (1978), reflective abstraction is a personal activity and therefore can be initiated by learners. Extending on Piaget's definition, Dubinsky (1991) defines the reflective abstraction construct as follows:

- Interiorization, where a student follows a procedure's steps and then defines a notion by thinking about the procedure. While someone can develop internal processes without following step-by-step procedures when solving sums, they are not dependent on outside cues to solve mathematical issues.
- Coordination: When analysing a mathematical idea, a learner studies two distinct processes and combines them into one coordinated process.
- Encapsulation: A concept is encapsulated by the learner by creating their own personal meaning. Personifying a concept is the act of encapsulation. An abstract idea or a group of abstract ideas take on meaning for a particular person.
- Generalization: After a concept has been fully understood, it is expanded upon and applied to a larger group of mathematical puzzles.
- Reversal: A learner constructs a new mathematical notion by reversing the steps of the original notion.

The theory of reflective abstraction explains the construction of logical thinking (Ndlovu, 2014). In extending this theory, Dubinsky (1991) extracted certain features of reflective abstraction, reconstructed them and extended their applicability to the learning of advanced mathematics to form a coherent theory of mathematical knowledge and constructions: APOS theory, which frames this study. Drawing from Piaget's definition and the reflective abstraction construct defined by Dubinsky, it can be argued that reflective abstraction is a powerful tool to understand the individual knowledge construction process on which APOS theory is built. APOS theory is, thus, an extension of Piaget's theory of reflective abstraction (Dubinsky, 1991).

According to Piaget (1978) development of knowledge about an object – either mental or physical – requires both objects and a subject acting on the object. The subject (knower) and the object cannot be dissociated (Arnon et al., 2014). In other words, one cannot speak of a subject without talking about objects. The theory of learning in mathematics education focuses on the development of conceptual understanding in the learning of mathematics. There are many theories in mathematics education that are related to the aspects of learning mathematics. According to Dubinsky and McDonald (2001), a theory is an effort to comprehend how mathematics can be learned and has programs to support mathematical learning. They argue that a theory ought to be able to account for specific student achievements and failures in the study of mathematics. According to Weyer (2010), learning theory should: (a) have the ability to explain phenomena; (b) be adaptable to a wider variety of experiences; (c) facilitate the organization of thoughts concerning learning events; and d) offer a language for discussing learning. It is a framework for the process of learning mathematics that specifically relates to the learning of difficult mathematical concepts. As this study is concerned with understanding learners' mental constructions and the difficulties they encounter with this, in the learning of differentiation and integration therefore it is premised on APOS theory, which is a theory of learning mathematical concepts.

3.4 Research Framework

This study is based on the specific framework developed by Asiala et al., (1997) for research and curriculum development in mathematics education. This framework for research and curriculum creation serves as the foundation for APOS theory, which emphasizes the cognitive development of a learner striving to build the necessary knowledge when learning mathematical concepts. Therefore, to understand mathematical concepts one needs to construct the necessary mental constructions for action, process, and object, and organise these to form coherent schema. According to Maharaj (2013), to improve the development of these mental structures, it is necessary to identify them and then design learning activities that will be appropriate for their growth. According to research based on this theory, it is necessary to identify the mental structures associated with a given notion before creating appropriate learning activities to aid in the development of these mental structures (Arnon et al., 2014).

In this study, the researcher incorporates the key components of the framework for research and curriculum development in mathematics education, as recommended by Asiala et al., (1997), which focuses on students' cognitive development as they construct mathematical knowledge, in accordance with APOS theory. According to Dubinsky and McDonald (2001), APOS theory-based research should align with a framework made up of three parts: a theoretical analysis, a design and implementation of the teaching, and observations and evaluations. This is shown in Figure 3.1.

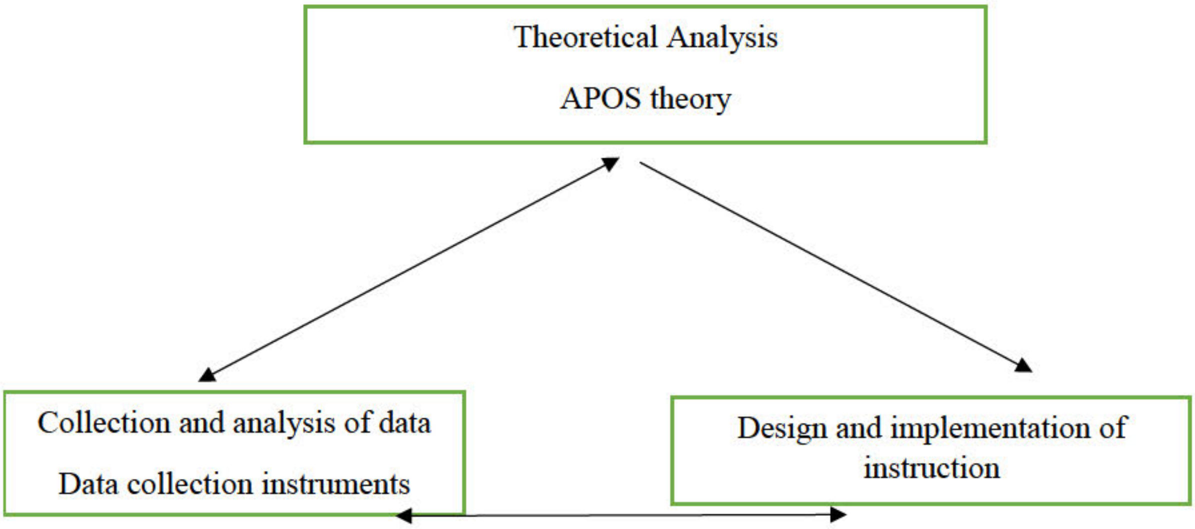


Figure 3.1: The Research Framework (adapted from Asiala et al., 1997)

In this study, the researcher uses APOS theory to analyse, or reveal, the nature of learners' mental construction of differentiation and integration. (APOS theory is discussed in detail in the next section.) Moreover, to conduct the theoretical analysis of the differentiation and integration concept relating to specific mental constructions that the learners made to explain the understanding of differentiation and integration, the researcher designed the genetic decomposition of the concept under study. (Genetic decomposition is discussed in detail in Section 3.7). To ascertain the extent to which any of the mental constructions indicated in the genetic decomposition are constructed by the learner, a model for instructional design, in the form of a structured activity sheet, was developed. Brijlall and Maharaj (2015) explain that structured activity sheets model how meaningful mathematics teaching or learning could be planned with the aim to attend to cognitive and affective domains when learners solve problems. The final step engaged with the implementation of the design, leading to the collection of data, which then was analysed through the lens of the theoretical perspective.

In summary, the first step is to adopt the theoretical underpinning of the study – which in this case is APOS theory – and to design the genetic decomposition to articulate specific mental constructions for the particular concepts. The second step is to design instruction – in the case of this study, a structured activity sheet. And the last is data collection, to analyse the mental construction made/not made. The theoretical analysis informs the design of the instruction which informs observation and data collection through the lens of the theory adopted. A detailed description of the research framework used in this study can be found in Asiala et al, (2004) or Arnon et al. (2014).

3.5 APOS Theory

This study uses APOS theory, which postulates that the mental constructions needed in the learning and conceptualisation of mathematical concepts are action, process and object; these are organised to form a coherent schema (Asiala et al., 1997). Actions are exterior changes that require guidance and explicit cues to carry out tasks. Actions get interiorized into processes that may be done, and even reversed, mentally when they are repeated and thought about. Processes are said to have been encapsulated into objects when people can observe them as a whole and perform changes on them. A schema of a particular mathematical notion is made up of actions, processes, and objects, as well as their relationships and any other linking schema (Asiala et al., 1997; Dubinsky & McDonald, 2001; Maharaj, 2010). (A detailed discussion of the mental constructions as depicted in APOS theory is presented in Section 3.7.1, together with the constructions relating to differentiation and integration indicated in the genetic decomposition.)

Arnon et al. (2014, p. 10) provide a visual representation of the mental architecture and mental mechanisms of APOS, shown in Figure 3.2.

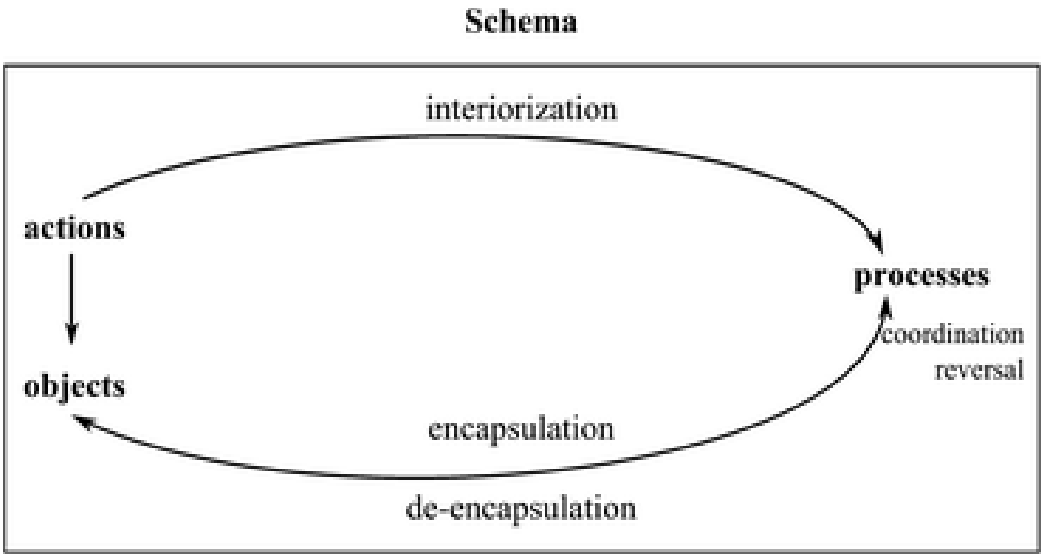


Figure 3.2: APOS Theory (Arnon, et al., 2014, p. 10)

Dubinsky (2010) notes that two key presumptions about mathematical knowledge and theories about how math is learned form the foundation of APOS theory and its application to classroom instruction:

Mathematical knowledge presumption: The ability of a person to respond to apparent mathematical problems and their solutions by 1) reflecting on them in a social context and 2) creating or rebuilding mental structures to use in handling the situations, is referred to as mathematical knowledge.

Theorizing about learning: Mathematical concepts are not directly taught to people. Individuals use mental structures to interpret a concept. If a person possesses the mental structures necessary for a specific mathematical notion, learning is made easier. Learning the concept is nearly impossible if the necessary mental structures are missing.

This suggests that mathematical instruction should employ methods for 1) assisting learners construct the necessary mental structures, and 2) assisting learners in using these structures to construct mathematical knowledge of the taught mathematical topics.

These mental structures, according to APOS theory, are actions, processes, objects, and schemas. Instances of reflective abstraction, which according to APOS theory include mental processes including interiorization, encapsulation, coordination, reversal, and generalization, give rise to these mental constructs (Dubinsky & McDonald, 2001). Dubinsky and McDonald (2001) outline five kinds of construction in reflective abstraction:

1. **Interiorisation:** Actions are interiorised into a system of operations. During this phase, a learner becomes familiar with a process and can carry it through mental representations.

2. **Encapsulation** is the capacity to apply an action to a process and view the process as a whole. This is the point at which the development of mathematical understanding progresses from one stage to another, where new forms of the process are created by drawing on earlier ones to create an object.

3. **Coordination**: Two or more processes are coordinated to form a new process; e.g. differentiation and integration require coordination of the composition of functions and derivatives (Jojo, 2013).

4. **Reversal:** the ability to reverse thought processes used in previously-interiorised processes. A new process can be constructed by reversing the existing one. For example, $\int in\ 2x^2\, dx = \frac{2x^{2+1}}{2+1} + C = \frac{2x^3}{3} + C$, the process of differentiation is reversed.

5. **Generalisation**: the ability to apply existing schema to a wider range of contexts.

3.6 Genetic Decomposition

A genetic decomposition, according to Arnon et al., (2014), is a fictitious model that depicts the mental frameworks and processes that a learner might need to develop to understand a certain mathematical subject. Until it is experimentally tested, genetic decomposition is referred to as 'preliminary genetic decomposition' (Arnon et al., 2014). According to Ndlovu (2013), a genetic breakdown is frequently presented in a linear format, but learning is not linear and how people cognitively create relevant knowledge of a given notion is crucial. The comprehension of any mathematical concept by a student depends on their preceding experiences. This foundational knowledge establishes a link between the previously known and the new mathematical topic that will be examined. (Ndlovu, 2013). The principles of preliminary genetic decomposition for differentiation and integration are summarized below.

3.6.1 Preliminary genetic decomposition of differentiation

It is essential to understand how individuals cognitively construct appropriate knowledge of a given concept. The section below discusses the components of APOS theory and their relevance to this study. Figure 3.2 presents the preliminary genetic decomposition designed to analyse the learners' mental constructions of differentiation and integration concepts.

Action stage

The action stage, in APOS theory, is where each step of the learning process (transformation) needs to be performed clearly and guided by external instructions (Arnon et al., 2014) According to Brijlall and Ndlovu (2013), action is based on rules and algorithms, where a rule is practised repeatedly until it becomes routine and takes place without conscious thinking. For the concept of differentiation, the action involves connecting notation to the concept of differentiation to generate a definition; connecting two points to define and determine and to determine the slope; and performing step-by-step calculations to determine derivatives. Also, at this stage, the learner follows the rules that govern each problem. For example, when evaluating $f(x) = x^2 + 2x + 3$, the learner will follow the rule which says if $f(x) = x^n$ then $f'(x) = nx^{n-1}$, which is the power rule. Then the solution would be elaborated, and all steps would be displayed as follows:

$y = x^2 + 2x + 3, if\ f(x) = y\ then\ \frac{dy}{dx}(x^2) + \frac{dy}{dx}(2x) + \frac{dy}{dx}(3) - - - - Step1$

$=2x^{2-1} + 2x^{1-1} + 0 - - - - - - - - - - - - - - - - - Step2$

$=2x + 2 - - - - - - - - Step3$

At this stage, the learner relies on the algorithm for differentiation. The mathematical notation f*' or* $\frac{dy}{dx}$ will act as the external cue for the learner to know that they need to find the derivative, then apply the step-by-step procedures to determine the derivative. The learner, at this stage, cannot imagine the next step without completing the previous step and, secondly, cannot imagine the answer. In addition, the learner, at this stage, cannot imagine if the answer makes sense in relation to the question. The rules are applied disjointedly without making connections between concepts. The clear definitions serve as external cues for the whole solution. According to Moore (2012), a learner at the action stage is unlikely to be able to solve a situational problem of a function without being given a formula.

Process stage

The repetition and reflection phases of the APOS theory are when an individual transitions from depending on outside stimuli to exercising internal control over their behaviours (Arnon et al., 2014). The ability to envisage performing the steps without having to perform each one explicitly and the capacity to skip and reverse stages are characteristics of this stage. At the process stage, learners are still following steps but display stages of understanding. Cooley (2001) states that, through reflections and internal operation for the derivative, differentiation is interiorised. At the process stage when working on differentiation, as a result of interiorised actions, the steps to determine the derivative are performed in the mind and the solution can be determined without physically presenting all the steps. The interiorisation of the action of linking notation to definition is performed without being presented visually with the notation for the derivative. The action to calculate slope is interiorised into a process to construct meaning of the derivative and to find the derivative at any point with any domain given.

Moreover, the action of determining the derivate can be reversed to its original function. According to Dubinsky and McDonald (2001), a process conception entails a type of knowing that produces an entirely new object without the aid of outside inputs while carrying out the same operation as the action. Dubinsky and McDonald (2001) stated that such repeated actions are interiorised into a mental process.

Object stage

During the object stage, the individual becomes aware of the process as a totality and realises that transformations can act on it (Dubinsky & McDonald, 2001). At the object stage, the learner can differentiate using different notations or advanced techniques foe differentiation. A learner would be deemed to possess a schema for differentiation when they display a coherent set of knowledge for the differentiation concept. For example, when a learner notices appropriate actions, processes and applies procedures as a whole and understands that transformations can be performed on it, then the learner is at the object stage (Mphuti, 2011). For example, given $f(x) = x^2 + 2x + 3$, the learner applies the procedures to determine the derivative, that is $f^1(x) = 2x + 2$. Also, at this stage the learner conceptualises $f(x) = x^2 + 2x + 3$ as the function and compares objects arising from the same process. According to Arnon et al. (2014), de-encapsulating an object back to the process that created it is only necessary in certain circumstances. This claim makes it clear how crucial it is for a person to be able to switch back and forth between an object and a process. Encapsulation, according to Weyer (2010), is the mental transformation of a process into a cognitive object that can be seen.

Schema Stage

Schema development is the process through which numerous events, procedures, and things are related in a person's mind to create a comprehensive framework. These linkages help someone make choices when faced with a mathematical difficulty (Dubinsky & McDonald, 2001). According to Maharaj (2013), an APOS study is only able to describe the potential thinking abilities of an individual: having a particular mental structure does not guarantee that that person will use it in a particular circumstance; other aspects must be taken into account. Arnon et al., (2014), asserts that a schema for a certain mathematical concept is comprised of an individual's collection of actions, processes, objects and other schemas linked consciously or unconsciously in a coherent framework in the individual's mind The process of finding a derivative is coordinated with other processes; for example, working with exponents and integers through the recognition that any number raised to power $\mathbf{0}$ is thus being able to understand that $\boldsymbol{2x}$ becomes a constant. The need to perform the action of comparing results obtained in the previous process makes it possible to encapsulate the process into an object (Trigueros et al., 2019).

According to Jean Piaget (1978), genetic epistemology was determined to be appropriate for this investigation. Based on this epistemology, an initial genetic decomposition integrating model was created for this study. It entails an algebra schema, a function schema, and the derivative of a quadratic equation: the derivative is de-encapsulated into its original function.

Prerequisite knowledge

- Already constructed knowledge of the definition of the concept differentiation
- Have a working knowledge of the gradient concept
- Have constructed the knowledge and understanding of the function concept (i.e. well-developed concept image of the function schema) (Asiala et al, 2001)

As part of their differentiation schema, a learner should:

- define in their own words the meaning of differentiation as a concept
- perform the necessary calculations to determine the gradient of a curve at any given point; and
- perform the necessary procedures to manipulate symbols to differentiate a given function.

As part of the derivative schema, a learner should be able to:

- differentiate the function using the rules of differentiation;
- recognise a suitable method to be used in a given function, even if it is not mentioned;
- remove brackets before finding the derivative (for example: $if\ f(x) = (x+2)^2$); and
- find the gradient and the tangent of a curve at a given point, (for example: if $f(x) = x^2 + 2x + 3\ then\ find\ f'(3)$).

Figure 3.3 summarises the preliminary genetic decomposition for the application of differentiation rules. For example, if given a function of the form: $y = x^2 + 2x + 3$:

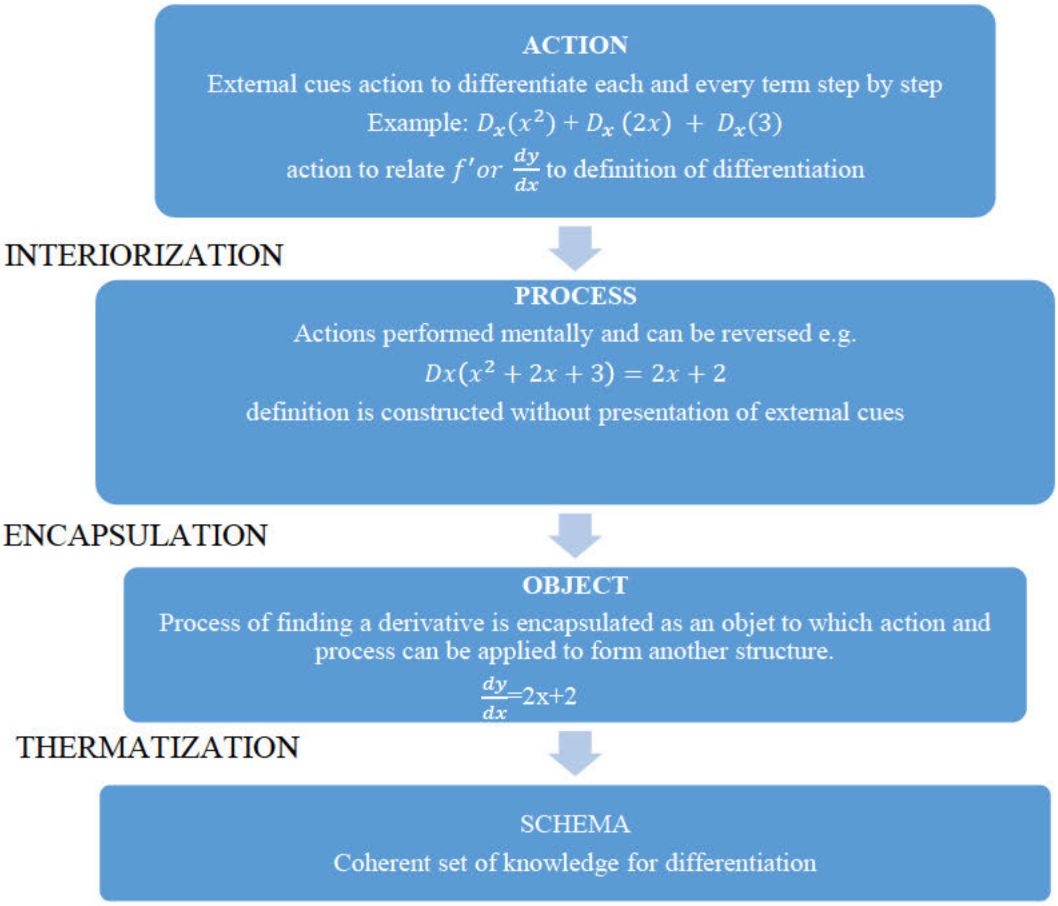


Figure 3.3: Preliminary genetic decomposition for differentiation

3.6.2 Preliminary genetic decomposition of integration

Both differentiation and integration are included in the high school curriculum for Technical Mathematics. Integration is not included in the curriculum for Pure Mathematics. Thus, before Technical Mathematics was added as a subject, integration was taught only at the tertiary level. Studies such as Maharaj (2013) and Ndlazi (2015) report that students find these concepts difficult to comprehend and suggest that such difficulties need to be analysed by means of research. This study explores both differentiation and integration, since both of these topics are now taught at the highschool level. This section describes the preliminary genetic decomposition of integration concepts.

Preliminary genetic decomposition suggests that, for students to succeed in integration, they must have developed the ability to explain, to recognise in other contexts and to derive consequences, for the function and the derivative schemas (Duffin & Simpson, 2000). Functions and derivatives are the building blocks of integral calculus, as indicated by Jojo

(2011), who states that "definitions of derivatives, integral functions, the relationships between average and instantaneous rates of change, and many other topics in calculus all require students to have a clear understanding of the concept of a function" (p. 45).

Action stage

At this stage, the action to evaluate integrals is performed explicitly, step by step. Therefore, at this stage a learner knows how to evaluate integrals by following explicit algorithmic steps (Ndlanzi, 2015). Learners employ explicit definitions of integral at this stage and these explicit definitions serve as external cues for the whole sum, when evaluating a given integral (Ndlanzi, 2015). For example, the solution of the integral $\int (2x+2)dx$ will be elaborated and steps will be shown as follows:

$\int 2xdx + \int 2dx$ -----------------------step1

$= \frac{2x^{1+1}}{1+1} + C_1 + \frac{2x^{0+1}}{0+1} + C_2$-----------------step 2

$= x^2 + 2x + C$--------------------------step3

Explicit definitions serve as external cues for the whole solution to be produced (Ndlanzi, 2015).

Process stage

At this stage, the repeated actions have been interiorised and can be performed internally without explicitly writing down all the steps. Some steps can be performed cognitively. The process of differentiation can be reversed. Therefore, the action of constructing antiderivative is interiorised as the reversal of differentiation. As indicated by Tarr and Maharaj (2021), a learner at the process stage can omit some steps and arrive at the solution; for example, if $\int (2x+2)dx\ then\ the\ answer\ will\ be\ x^2 + 2x + C.$

Object stage

As mentioned, at the object stage a process is viewed as a totality and is encapsulated into mental objects. At this stage, the learner is able to perform other actions, such as integration. For example, the process of integrating a function is encapsulated into an object where the derivative, $f'(x)$, is viewed as a total entity to which action and process to be performed to determine $f(x)$ or encapsulate a given function, such as $\int 2xdx + \int 2dx$ is a whole entity to be split into separate functions before evaluating. Learners at this stage can distinguish between objects arising from similar processes and can argue why is it necessary to split the function into different functions. According to Mahir (2009), a good understanding of the rules of differentiation is essential for solving integrals.

Schema stage

When a learner exhibits a coherent body of knowledge for the notion, that person is said to have a complete schema for integration. Despite the fact that the four stages of action, process, object, and schema are organized hierarchically, it is likely that learners don't always develop constructions in a straight line (Dubinsky & McDonald, 2001). Constructions of other diverse mathematical notions, according to Dubinsky and McDonald (2001), become more dialectical than linear concepts. According to Jojo (2011), functions and derivatives are the building blocks in integral calculus; she states that "definitions of derivatives, integral functions, the relationships between average and instantaneous rates of change….and many other topics in calculus all require students to have a clear understanding of the of the concept of a function" (p. 45).

According to Maharaj (2010, p.41) a " genetic decomposition of a concept is a structured set of mental constructs which might describe how the concept can develop in the mind of an individual". A genetic decomposition integrating model was created for this study. It entails an algebra schema, a function schema, and the derivative of a quadratic equation: the derivative is de-encapsulated into its original function.

Pre-requisite knowledge

- Already constructed knowledge of finding the derivative
- Algebraic procedures for solving equations are already constructed
- Understanding of formulae for calculating the area of geometric shapes

As part of their integration schema, a learner should be able to:

- construct the anti-derivative;
- conceptualise integration as anti-derivative; and
- encapsulate the procedure of finding the area under the curve.

The action of finding the integral is performed step-by-step without making connections between the two concepts (differentiation and integration). The step-by-step procedure acts as an external cue since the learner cannot imagine the next step without performing the previous one. The actions are repeated until they become interiorised into a process. The interiorised process can be performed mentally. The process of solving integrals is coordinated with the process of differentiation to construct definitions of the concepts of differentiation and integration. Applying action and processes to $\int(2x+2)dx$ results in the encapsulation of the concepts as an object. The given integral is seen in totality and is understood as the area under the curve and the learner is able to explain the relationship between definite and indefinite integrals.

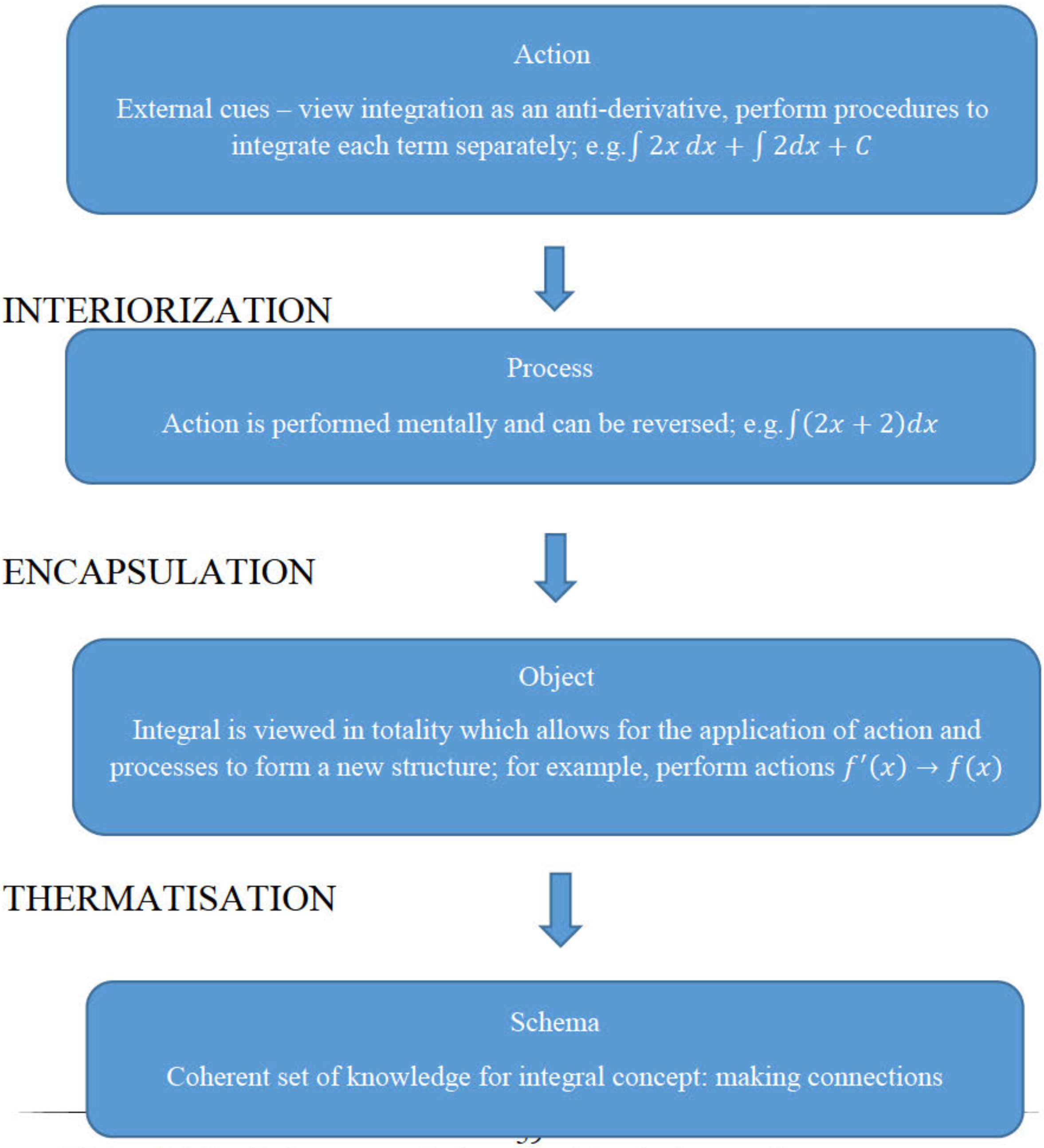


Figure 3.4: Preliminary genetic decomposition for integration

3.7 Analysis of Learners' Difficulties and Errors

Exploring mental constructions focuses on the construction of knowledge articulating the necessary mental construction an individual needs to make in order to have a coherent schema. However, the inability to make the required mental constructions suggest that the learner has made certain errors that hindered the knowledge construction process, therefore while exploring the necessary mental construction it is equally important to analyse the errors that potentially hinder the construction of the required mental construction.

This study does not only explore the mental constructions of Grade 12 learners when learning differentiation and integration, it also analyses the possible errors that might hinder the knowledge construction process. In view of the challenges faced in the teaching and learning of mathematics, Olivier (1989) advocates for the application of clearly articulated theories to overcome these challenges. Some studies have paid attention to understanding undergraduate students' difficulties when learning calculus concepts (e.g. Siyepu, 2013; Jojo, 2013; Maharaj, 2013; Ndlazi, 2015). Studies have tended to focus on either exploring learners' mental constructions or their errors. This study explores both mental constructions and errors because errors are the potential stumbling block that might hinder the construction of the necessary mental constructions. According to Luneta and Makonye (2010), an error in learners' written work results in learners not obtaining a perfect score in routine assessments or on examinations; while some of these errors occur spontaneously due to carelessness, others are structural and are caused by incorrect underlying mental structures –misconceptions. Luneta and Makonye (2010) state that it is easy to correct an error, but very difficult to eradicate a misconception because it usually becomes embedded in the learner's conceptual schema.

To unpack and interpret the errors that learners make which potentially hinder their construction of the required mental constructions, this study adopted Olivier's (1989) conceptual framework for handling learners' misconceptions. Olivier (1989) states that learners demonstrate different mistakes which arise from different reasons. He groups these as 'slips', 'errors' and 'misconceptions. Slips are wrong answers owing to processing; they are not systematic but are carelessly made by both experts and novices, and they are easily detected and quickly corrected. For example, in $\int 2(3+4x)^4 = \int(6+8x)^4 = [\frac{(6+8x)^5}{5}] + C$, a technical mistake was made by wrongly multiplying the constant of 2 into the binomial before integrating. 2. Errors are incorrect responses as a result of planning; they are systematic in that they occur frequently under the same conditions. Errors are symptoms of the conceptual structures that underlie them and are their root causes. Errors result from past knowledge that learners have acquired, either in the mathematics classroom or from their interactions with the real world (Smith & Hagan, 1993). According to Davis (1984) and Olivier (1989), errors occur when a concept is overly generalized from one domain to another. According to Olivier (1989), students frequently mix up their knowledge of the commutative property in relation to the four fundamental operations of addition, subtraction, multiplication, and division. For example, given that in addition $6 + 4 = 4 + 6$, students tend to assume that even in subtraction commutative property holds, but it is not true that $6 - 4 = 4 - 6$ because $6 - 4 = 2$ whilst $4 - 6 = -2$. He further explains that this arises as teachers introduce integers, since learners develop the understanding that they only subtract smaller numbers from bigger numbers and their answers are always positive.

Another contributing factor that may influence learners is that the commutative property holds under addition and multiplication. As a result, they over-generalise the rules for operations. Olivier (1989) suggests that errors should be addressed in teaching and learning as they generate further errors if left unattended. He further explains that key errors are those that, if left unresolved, have the potential to block or impede further progress. Errors demonstrate how learners are thinking and applying prior information to novel situations (Brodie & Berger, 2010).

Students' poor arithmetic performance is partly attributed to misconceptions and mistakes they carry over from lower-stage schooling to higher education (Brodie & Berger, 2010; Kiat, 2005; and Luneta & Makonye, 2010).

For the purpose of this study, APOS theory – which is an extension of Piaget’s notion of reflective abstraction – is used as the lens to understand the mental constructions learners possess in the learning of differentiation and integration. To understand the mistakes that might hinder the constructions of required mental constructions the combination of two frameworks was vital for this study. These two frameworks complement each other, because APOS theory explains the knowledge construction process and Olivier's (1989) theory is premised on understanding the errors that hinder the cognitive processes involved with making the required mental constructions.

3.9 Conclusion

This chapter has presented the theoretical framework of the study. APOS theory as purported by Asiala et al. (1997), underpins the study. APOS theory of learning focuses on how learning a mathematical concept might take place. APOS can help in understanding learners' mental constructions made in the learning of mathematical concepts. In attempting to show how differentiation and integration concepts can be conceptualised, the genetic decomposition for both concepts was discussed, to aid interpretation of the mental constructions made using APOS theory. In this study, APOS theory was linked with Olivier's (1989) theory to analyse learners' difficulties that hindered learners in developing a schema.

The next chapter presents the methodology and research design of the study.

4 RESEARCH METHODOLOGY

4.1 Introduction

The APOS (Action-Process-Objects-Schema) theory, which serves as the main part of the study's theoretical framework, was introduced in the previous chapter. APOS attempts to characterize the mental structures that deal with the nature of mathematical concepts; in this study, it was used to analyze Grade 12 students' mental constructions and challenges with differentiation and integration concepts. This chapter presents the critical research questions, interpretive research paradigm and qualitative research approach used in this study. In addition, the chapter discusses the important research procedures and methods used to conduct this study. The chapter also discusses the authenticity and trustworthiness of the study, triangulation and limitations of the study.

4.2 Critical Research Questions

In order to describe the Grade 12 students' mental constructs in the learning of differentiation and integration concepts, the study used the APOS (Action, Process, Object, Schema) approach. The study also aimed to clarify the challenges that learners encounter when understanding differentiation and integration principles. It used genetic decomposition to determine whether learners could create the necessary mental structures when learning calculus concepts. The following research questions were posed in order to comprehend the phenomena supporting this study:

- What are Grade 12 learners' mental constructions of differentiation and integration?
- How have Grade 12 learners' mental constructions of differentiation and integration evolved in relation to the genetic decomposition?
- Why have the mental constructions needed in the conceptualisation of differentiation and integration evolved or not evolved?

4.3 Interpretive Research Paradigm

This study is concerned with the knowledge constructions used in the learning of differentiation and integration. The main aim was to analyse Grade 12 learners' mental constructions when learning differentiation and integration concepts**.** In line with the aim of the study, the interpretive paradigm was deemed appropriate for this study. According to Antwi and Hamza (2015), the interpretivist paradigm is concerned with how well a person understands their surroundings. According to Cohen et al. (2017), the interpretive paradigm is characterized by modelling the world from the perspective of a human being. According to these authors, this is in contrast to the way that humans are studied through observation, in which the observer's perspective is reflected. For interpretive researchers who seek to understand how individuals perceive the world, the individuals are the starting point (Cohen et al., 2017). Cohen et al. (2017) point out that interpretive researchers are motivated to look at circumstances through the participants' eyes rather than their own.

The interpretive paradigm underpins this study. An interpretive paradigm attempts to comprehend the subjective realm of human experience and attempts to comprehend and interpret each aspect of the phenomenon being studied (Cohen, et al., 2017). In interpretive research, a researcher interacts with participants to comprehend the event from the participant's point of view (Strasheim & Eiselen, 2011, as cited in Ndlazi, 2015). This study adopted the interpretivist paradigm because it focuses on understanding learners' internalized conceptions of differentiation and integration. The interpretivist paradigm, according to Guba and Lincoln (2005), supports a perspective of various, equally legitimate, and socially produced realities. Therefore, reality is contingent upon the unique individuals or groups creating it, both in terms of shape and content. Realities are thus socially and experientially based, and local and specific in nature (Guba & Lincoln, 2005).

Angen (2000) asserts that interpretivism assumes that the researcher's values are present throughout the entire interview process and that the truth is negotiated as part of this process. He notes common characteristics of interpretive approaches: naturalistic techniques or methodologies, such as interviews, observations, and text analysis, are extensively emphasized; these techniques guarantee an appropriate exchange of ideas between the researchers and those they work with; and from the study process, general meanings are derived. According to Cohen et al. (2017), the interpretative inquiry uncovers and interprets the viewpoints of the study's participants, and the questions' outcomes are essentially influenced by the context. The interpretive paradigm was appropriate for this study since it used semi-structured interviews, observations, and written replies from students to activities as data collection methods.

Cohen et al. (2017) state that the interpretative paradigm relies primarily on observations, interviews, and written test analysis. They note that these methods provide an effective exchange of ideas between the researcher and those with whom they collaborate in order to create a meaningful reality and derive meaning from the research process. It was crucial to this study to understand how students build differentiation and integration knowledge. Cohen et al. (2017) also state that the goal of the interpretive paradigm is to develop an understanding of social interaction and to discover how people construct meaning in a natural setting. The goal of this study was to analyse the nature of mental constructions and to analyse learners' mathematical thinking on differentiation and integration concepts, with the aim of identifying the difficulties and misconceptions learners have.

Learners were given tasks to solve problems involving differentiation and integration individually, and it was of importance to analyse learners' responses to written activity sheets. This was done in order to identify the level of a learner's understanding or to expose their mathematical thinking regarding differentiation and integration. The researcher used multiple methods to understand how learners construct meaning through the interpretive paradigm. This included semi-structured interviews which were used to understand the meaning constructed. According to Denzin (2001), interpretive studies examine how problematic, turning point experiences are organised, constructed, and given meaning. The implication here is that the researcher is set to ask and answer crucial questions that are coming up in the study. The style of questioning in this study reflects the interpretive research design, and it was important to see how learners constructed knowledge of differentiation and integration.

4.4 Context of the study

The study was conducted in a South African high school that offers Technical Mathematics as a subject. The school admits only 10 learners each year in Grade 12 to do technical subjects, of which technical mathematics is one of these subjects. A learner who wants to go to a technical college must have taken technical mathematics and achieved marks above 50%. In Technical Mathematics, two calculus concepts have been found to be difficult for learners: differentiation and integration. These concepts have been researched widely at the tertiary level of education; lecturers have reported difficulties in teaching students these concepts and have noted that students have progressed from the secondary phase without an adequate foundation in these concepts. This research explores the issue at the secondary level, with the aim of understanding learners' mental constructions and difficulties with regard to differentiation and integration.

4.5 Qualitative Research Approach

This study used qualitative methods to answer the research questions. According to Creswell (2014), qualitative research entails using the perspectives of participants to explore and grasp the meaning of an event. The qualitative method, according to Higgs et al.,(2009), offers numerous approaches to comprehend the intrinsic complexity and diversity of human behaviour and experience. According to Fouché and Delport (2002), a method is considered qualitative if no statistics are utilized to categorize, organize, and analyse the pertinent data gathered. They add that qualitative methods allow for the collection of data through the use of interviews, focus groups, photographs, and other materials. Denzin and Lincoln (2011) identify four characteristics of qualitative research: participant accounts of meanings, experiences, or perceptions about concepts are elicited; descriptive data are produced; qualitative approaches allow for a greater variety of responses as well as the ability to adapt to new developments or issues; interviews, group discussions, observations, various texts, pictures, and other materials can be used to collect data.

A qualitative approach was chosen for this study because the aim is to analyse Grade 12 learners' mental construction of mathematical knowledge for differentiation and integration. The researcher used inductive and deductive analysis in this study. This was accomplished by categorizing each participant's written comments; first; codes were identified and these codes were grouped into categories. The data began to reveal themes and patterns, and these topics were then examined using genetic decomposition. The genetic decomposition indicated the mental constructions that the learners used in order to understand differentiation and integration concepts.

4.6 Case Study Design

To answer the research questions, the researcher employed a case study research strategy. A case study, according to Cohen et al. (2017), offers a singular illustration of actual individuals in actual circumstances, allowing the reader to comprehend the events more vividly than merely providing them with abstract notions. An in-depth description and analysis of one or more situations, such as a person, group of people, or organization, are given in a case study, which is a type of research style (Stake, 2005). Understanding one's mental constructions requires an in-depth analysis of their responses, both written and verbal and therefore; a case study thus was deemed appropriate for this study.

A case study should immerse the reader in the circumstances and experience of research participants, according to Patton (2014). The case study design permits the use of various data collection techniques for triangulation purposes in order to obtain supporting evidence (Johansson, 2007; Lincoln & Guba, 1990; Yin, 2018). Case studies typically concentrate on one or two key themes that are essential to comprehending the system under study (Tellis, 1997). Data were gathered in two stages, during the pilot study and the main study. Since the primary goals of this study were to address "why" and "how" questions, a case study methodology seemed a suitable approach (Oancea & Punch, 2014). Because the researcher in this study did not aim for the results to be generalized, careful consideration was given when choosing the participants.

The main aim of conducting a pilot study was to test the appropriateness of the research instruments, research questions, theoretical framework, methods of data collection and data analysis. The activity sheets were administered and data were collected from learners' responses to differential and integral problems. The responses were examined, and semi-structured interviews were used to confirm and elucidate the learners' comprehension of the ideas in differential and integral calculus.

4.7 Participants in the Study

The participants in this study were Grade 12 learners taking Technical Mathematics at one of the schools in Umlazi District in KwaZulu-Natal, South Africa. The study was conducted in two phases: a pilot study, followed by the main study. As only ten learners are accepted for Technical Mathematics every year, the researcher decided to involve all ten Grade 12 learners in both phases of the study. However, as the pilot study was done in 2020 and the main study was done in 2021, a different group of ten Grade 12 learners participated in each. Activity worksheets were administered to both groups of Grade 12 learners on separate occasions. Both activity sheets are presented in Appendix 1.

As the researcher is the Technical Mathematics teacher at this school, it was easy to select participants. The researcher explained to all learners the aim of the study through Zoom meetings. The researcher first discussed with learners what needed to be done; it was explained that they had to solve problems individually and at a later stage their solutions to their problems would be discussed in a Zoom meeting.

4.7.1 Purposive sampling

The study used purposive sampling. According to Cohen et al. (2017), purposive sampling is a feature of qualitative research whereby researchers select cases that are accessible and have in-depth knowledge about a particular issue. Purposive sampling, according to Yin (2018), is the process of choosing participants or data sources based on how rich and pertinent the information they are expected to provide is to the research issue. Purposive sampling, according to Bertram et al., (2015), is when a researcher specifically chooses which individuals to include.

The selection of participants in qualitative research is solely based on relevance to the research issue rather than representativeness (Neuman & Celano, 2006). Neuman and Celano (2006) explain that qualitative researchers may choose examples gradually and according to a particular case's content.

All ten students taking Technical Mathematics were chosen to make up the study sample in order to generate rich data for the study. Purposive sampling was determined to be appropriate for this study because the researcher did not attempt to generalize the results.

4.8 Research Procedures

The study was conducted in two phases. The first was carried out at the start of 2020, just before the South African government placed the nation under lockdown because of the Covid-19 pandemic. The purpose of Phase 1 was to test the research tools and validate the activity sheets. The only criteria for participation were to volunteer and be willing. Participants were given activity sheets which they completed at home. The analysis in this phase of the study was based only on learners' responses; due to Covid-19 regulations and time constraints, there were no interviews. A full discussion of Phase 1 is presented in Chapter 5.

Phase 2, conducted in 2021 with the new Grade 12 learners, was done virtually on WhatsApp due to Covid-19 restrictions. Some changes were made to the activity sheet based on Phase 1 findings. Some concepts were added, such as finding the area under the curve using integration rules. The main aim of Phase 2 was to explore learners' cognitive construction of the concepts taught relating to differentiation and integration. A recruitment letter was scanned and sent to all learners via WhatsApp. Those who agreed to participate in the study were given consent letters to be signed by both parents and assent letters to sign themselves. The tasks were designed to require learners to construct concepts, rather than use rote memory. Dhlamini and Mogari (2012) emphasise that knowledge is constructed, rather than created. Learners were requested to solve the tasks independently. The researcher provided all learners with data for their phones, in order to be able to send back solutions via WhatsApp.

Using the preliminary genetic decomposition, learners' responses were analysed in order to understand the level at which they were operating. After that, in cases where learners had difficulty making the necessary mental constructions, the challenges that hindered their constructions were analysed. The following five-point rubric, which was derived from Jojo et al. (2011), was used to analyse, code, and score responses to the activity sheet.

Table 4.1: Codes used to analyse learners' written responses (adapted from Jojo, Brijlall & Maharaj, 2011)

Score	Assessment criteria	Description of mental action
5	A thorough solution that addresses every facet of the question and shows thorough mathematical comprehension of the idea being tested.	Made all mental structures in accordance with the genetic breakdown recommendations
4	A response that is only partially complete and has a few small computational errors but nevertheless conveys the problem's primary notion.	Concept understanding is primarily conceptual
3	Inadequate reasoning and incomplete responses to all facets of the topic	Displays few mental constructions, conceptual understanding at a minimal stage.
2	No reasoning to justify written response	Displays few mental constructions, but at a procedural stage.
1	No written response or complete principle misapplication	No mental construction of a concept

In the second phase, data collection was also collected using semi-structured interviews. Due to Covid-19, interviews were done telephonically with selected participants. Each interview took about 30 minutes. The interviews aimed to understand the extent to which learners had made the necessary mental constructions. They also aimed to engage with learners' thinking to understand their mental constructions and the difficulties that had hindered them from making the necessary mental constructions. Interviews, according to Rubin and Rubin (2011), are dialogues in which the interviewer gently prompts the subject to elaborate on the topic under inquiry. They add that, in qualitative interviews, questions should be related to what each respondent is aware of and willing to offer.

Learners' responses to the activity sheet were categorised in terms of their mental constructions, as explained in APOS theory. Two learners in each category were purposefully selected, based on their responses, to be interviewed. The reason for choosing two learners per category was to allow for the possibility that some learners would decide to withdraw or would not be willing to participate in the interviews. Due to the Covid-19 pandemic, the researcher was only able to secure interviews with five learners instead of eight learners, however. As stated before, the main aim of the interviews was to obtain information on how learners have cognitively constructed the necessary mental constructions to conceptualise the concepts. Learners were asked to answer open-ended questions to ascertain their stage of understanding of differentiation and integration. Their responses to activity sheets determined the stage of questioning and the types of questions to be asked. The interview questions were not set before-hand, but were generated based on their responses on the activity sheet.

Qualitative interviews may be negatively impacted by power dynamics between research and participant. In this study, the researcher was the participants' teacher, thus there was a significant power differential. This was addressed, to some extent, by letting the participants choose the time of the interview. Brooks et al. (2018) state that the interview environment must be comfortable, private and quiet. All five participants were interviewed from the comfort of their homes, since the interviews were virtual (video call on WhatsApp) with the consent of the participants. The researcher started by giving a short background about the importance of the interview. The main aim was to identify their stage of understanding using their answers from activity sheets. The interview questions focused on the rules of differentiation and integration and integral as an area, in line with the activity sheet. Cohen et al. (2017) state that interviews enable participants to discuss their interpretation of the world and to express how they regard the situation from their own point of view. In this study, the interviews provided for greater clarity and detail to participants' written responses.

In addition to semi-structured individual interviews, focus group interviews were conducted using WhatsApp video calls. Focus groups are a type of group interview, but they differ from traditional group interviews in that the emphasis is placed on interaction within the group as it discusses a subject that the researcher provides (Morgan et al., 2017). According to Gibbs (2013), this technique entails interviewing several people at once, with the emphasis being placed on how the participants interact with one another. Smaller groups, according to Smithson (2008), produce pertinent information and give everyone the chance to express themselves. In this study, the focus group discussions were intended to enable the learners to help each other build the essential mental models. The researcher was only able to engage two learners, due to Covid-19 restrictions. The discussion was conducted via Zoom. The researcher asked probing questions and provided some explanations where necessary, as recommended by Smithson (2008). While the group size was very small, both learners were actively involved, and that allowed space for them both to express themselves. The researcher was able to analyse the stage of their mental constructions using APOS. APOS theory is not only premised on knowing the stage the learners are operating at but also helps them to make the necessary mental construction to learn a particular concept.

4.8.1 Primary research instrument: Activity sheet for learners

The main aim of the study was to analyse Grade 12 learners' mental constructions of differentiation and integration. To gather such data two instruments were used, which were activity sheets with questions on differentiation and integration and semi-structured interviews. Activity sheets guide responses to questions and help students build the necessary mental models (Ndlovu & Mji, 2012). According to Brijlall and Maharaj (2014), structured activity sheets serve as an example of how effective mathematics instruction could be planned with the intention of concurrently targeting the cognitive and affective domains when students solve problems.

The activity sheet which constituted the main research instrument in this study is provided in Appendix B. It consists of 3 items, each with sub-items; for example, item 1 has sub-item 1.1; 1.2; and 1.3. Items 1 and 2 explored learners' understanding of the techniques of differentiation, whereas item 3 was about integration. This instrument was designed in order to get meaning from learners for both concepts (differentiation and integration). The instrument was administered after the learners had been taught these two concepts in the classroom. For both concepts, learners were required to state in their own words the difference between differentiation and integration and the meaning of symbols. Due to the Covid 19 epidemic, activity papers were delivered to students via WhatsApp as was previously described in this study.

4.8.2 Secondary research instrument: Semi-structured interviews

Semi-structured interviews were used in the study. According to Kvale and Brinkmann (2009), the researcher prepares a series of questions in advance for this type of interview in order to learn more about particular difficulties. According to Creswell (2014), semi-structured interviews give respondents the flexibility and freedom to choose what needs to be described or argued as well as how much justification to provide. He adds that one-on-one interviews are beneficial since interviewees have the opportunity to ask questions.

After students had completed the activity sheets, their replies were categorized, and five students were interviewed via WhatsApp video chat. The primary goal of the interviews was to clearly comprehend the stage at which the learners functioned. According to Bertram et al. (2015), conducting interviews is a useful method for gathering information on a person's knowledge. Participants were able to explore their worldview and describe how they viewed various events during interviews (Cohen et al., 2017). Patton (2014) reaffirmed this by stating that interviews enable us to see things from another person's perspective. Unstructured interviews, according to Denzin & Lincoln (2011), offer more depth with the main objective of comprehending the phenomenon. Unstructured interviews enable the interviewer to delve further if necessary (Cohen et al., 2017). In this study, learners' responses to each activity were categorized based on the mental models they used, and one participant was randomly chosen from each category for the interviews. Therefore, since this study aimed to explore learners' experiences in creating the knowledge to solve differentiation and integration, interviews were an appropriate technique for data collection. The amount of questioning and the kinds of questions asked were based on their responses to the activity sheet. Consequently, the information from the activity sheet was used to produce the interview questions.

Interviews were recorded on WhatsApp video call to ensure that nothing was missing from participants' responses. In this study, it was vital to analyse everything the participants said, hence recording was deemed important. According to Cohen et al. (2017), a respondent may feel restricted in their responses if they are recorded. However, according to Hoepfl (1997), recordings offer the advantage of more faithfully collecting data than hastily made notes may and can help the researcher concentrate on the interview. Patton (2014) points out that an audio recording enables the interviewer to pay closer attention and improves the correctness of the data, as it captures what the subject actually said.

As this study required a thorough analysis of data, it took significant time to conduct the interviews and four weeks to complete the analysis. Some of the questions were prepared in advance, and others were brief and to the point in order to save learners' data while still generating rich data from the time spent. The following chapters present the data analysis that was done on the transcripts of the interviews.

4.9 Ethical Issues

Cohen et al. (2018) assert that qualitative analysis typically focuses on particular situations and exceptional circumstances and may involve delicate and private issues. It brings up the issues of personal identity, secrecy, and privacy. The researcher conducted the study with her own learners at the school where she teaches. A researcher has an ethical duty to consider the *primum non nocere* first: to do no harm to participants. The researcher was given permission to proceed with the study by the university, with ethical clearance HSSREC/00002126/2020, which is attached as Appendix C. Additionally, assent and consent were obtained from learners and parents/guardians for their voluntary participation in the study without compensation. The KwaZulu-Natal Department of Education was contacted to obtain permission to do the study at the school, and the gatekeeper's consent from the school principal was obtained. The decision to participate was entirely voluntary, and it had no effect on the evaluation or assessment of the learners in any studies or course while at school.

Before conducting the study, the researcher provided the Grade 12 learners doing Technical Mathematics with a letter stating the reasons and the purpose of the study. The letter defined informed consent, the right to withdraw, and confidentiality. Each participant was requested to sign the assent letter. The researcher explained the procedures to be followed during the process of the research, the anticipated timeframes, and the contact details of the university.

Pseudonyms (made-up names) were given to participants in place of their real names on the transcripts. The respondents' identities were kept completely private. The only usage of the data was the study, and it was all kept secure. Participants were free to leave the study whenever they wanted. The fundamental principle of anonymity, according to Cohen et al. (2018), is that no personal information submitted by participants should be revealed; someone is considered anonymous when neither they nor anybody else can determine their identity from the information they have provided. For the purposes of this study, the participant identities were provided in private, and their anonymity was ensured. In addition, Cohen et al. (2018) state that the best method to preserve a participant's right to privacy is to promise secrecy and not to share any information about them in any way that would make it possible to identify them. In this study, the researcher explained to the participants the meaning of confidentiality. The consent letter explained in detail the step taken to ensure confidentiality; data was stored in a secure place.

The study was conducted in a South African school with learners that were doing Grade 12 level Technical Mathematics. The school had one class of ten learners that were doing Technical Mathematics. The researcher decided to use the Technical Mathematics class because the two concepts of differentiation and integration are only found in the Technical Mathematics curriculum. In this research, the researcher played two roles: educator and researcher. This brought with it both opportunities and pitfalls. Ball (2000) stated that this approach "offer[s] the researcher a role in creating the phenomenon to be investigated, coupled with the capacity to examine it from the inside, to learn that which is less visible" (p. 388). One of the opportunities this afforded the researcher was to be able to guide the learners as they worked individually. In that way, the researcher was able to learn about the learners' thinking processes, which helped the researcher to be able to analyse learners' responses. The challenge that arose was that of power dynamics. Ball (2000) states that students might have some reservations as to what they should or should not say. In this study, the researcher was able to clearly explain the purpose of the study and also state that the research activities had no bearing on learners' assessment.

4.10 Authenticity and Trustworthiness of the Study

Given that the study was qualitative in nature, issues of validity and reliability were considered. Credibility, transferability, dependability, and confirmability are all aspects of validity in qualitative designs (Creswell, 2014; Guba & Lincoln, 1994). Cohen et al. (2017) state: "Reliability in qualitative research can be regarded as between what researchers record as data and what actually occurs in the natural setting that is being researched" (p. 149). According to Creswell (2014), there are numerous strategies to address dependability in qualitative research, including taking thorough field notes and using high-quality recording tools that are simple to use. Instead of using validity and reliability in this study, authenticity and trustworthiness were adopted suggested by Guba and Lincoln (1994). In this study, the issue of authenticity of this study is discussed under triangulation.

4.11 Triangulation

Different researchers have given different definitions for triangulation. Triangulation refers to the use of multiple perspectives against which to check one's own position (Kelly, 2006). Cohen et al. (2017) assert that triangulation "attempts to map out, or explain more fully, the richness and complexity of human behaviour by studying it from more than one standpoint" (p. 254). Cohen et al. (2017) also describe triangulation "as the use of two or more methods of data collection in the study of some aspect of human behaviour" (p. 143). In this study, the data from activity sheets, semi-structured interviews and video recordings were triangulated which ensured the credibility and trustworthiness of the study. In addition, the accuracy of the responses was not the main purpose of the study; rather, the focus was on identifying learners' mental constructions for differentiation and integration. More so, the videos were analyzed using visualization and audio-transcription to ensure that files were converted into thematic text and codes that aligned with the themes generated from the semi-structured interviews and activity sheets.

The study, which examined mental construction with a group of Grade 12 learners studying Technical Mathematics, employed structured activity worksheets, video recordings of group discussions and semi-structured interviews as techniques for data collection. Methodological triangulation, which involves using many approaches to examine the same subject, was thus used in this study (Cohen et al., 2017). The Phase 1 study was carried out to determine whether the instruments would produce the necessary data and to ensure the authenticity and dependability of the data. During Phase 1, data were collected over six months in order to ensure the reliability of the research instrument. The second phase took eight months. The learners were given the assurance that the information would only be used for the study's objectives and not for ongoing evaluation. Naturalists have claimed that dependability and trustworthiness are vital to ensure validity, rigor, quality, and trustworthiness (Cohen at al., 2017). The Guba and Lincoln (1985) criteria were used to ensure the study's rigour, as shown in

Table 4**.2** below.

Table 4.2: Criteria to enable trustworthiness in the study (adapted from Guba and Lincoln, 1985)

Quality Criterion	Criteria	Possible provision made by the researcher
Credibility	Prolonged field experience Triangulation Interview technique	Triangulation. The researcher used different methods of data collection, such as activity sheets and interviews. The semi-structured interviews were audio and video recorded. Pseudonyms were used throughout the research. This was done to ensure that data analysis was believable and trustworthy.
Transferability	Dense description	Description of a phenomenon involves quotes from interviews and learners (participants) written responses from activity worksheets on calculus concepts which are differentiation and integration.
Dependability	Dependability Audit Triangulation	Methodological descriptions were given; transcripts of interviews, analysis of learners' written response and description of the coding system used were given. This was done at each stage of data collection and data analysis.
Conformability	Conformability Triangulation	The use of pseudonyms made it possible to go back and confirm findings without compromising confidentiality. To reduce the researcher's bias, at any point she was able to go back and confirm results from semi-structured interviews and written responses. The researcher can trace back the original data.

4.12 Limitations of the Study

Because this was a case study involving a single school with only ten learners enrolled in Grade 12 Technical Mathematics, the sample size was small. As a result, the findings cannot be extrapolated to other situations. The researcher believes that the results are robust enough to be applied to other contexts. The study's conclusion may be impacted by the problem of power dynamics between the researcher and the participants.

In order to prevent this from obstructing the study process, several safeguards were established. The researcher clarified that, because the activity was not part of the school's annual assessment program, it would in no way impact their continuous assessment. Learners were provided with an assessment plan to ensure that they understood that participation in the study comprised no part of their assessment. During interviews, which were conducted using video calls, learners were allowed to speak in English or isiZulu as these two languages are commonly used at the school. Due to the Covid-19 pandemic, participants conducted activities for the study from their homes, which helped to reduce the anxiety associated with face-to-face interaction.

4.13 Conclusion

Methodological concerns pertaining to the study were covered in this chapter. Also discussed were the crucial research topics and the research tools. The chapter addressed the theoretical framework and qualitative paradigm used in this study. The procedures used to acquire the data were consistent with the qualitative approach. Compliance, triangulation, and involvement were addressed as ways to guarantee validity and reliability. The outcomes of Phase 1 of the study, which sought to validate the activity sheet as the primary research instrument, are presented in the following chapter.

5 ANALYSIS OF WRITTEN RESPONSES FROM ACTIVITY SHEET (PHASE 1)

5.1 Introduction

The previous chapter presented the research design, methodology, and data collection methods of the study. This chapter presents the analysis of participants' written responses from the pilot study (Phase 1) that was conducted to identify areas for deeper investigation when the study proper was conducted, as recommended by Arnon et al. (2014). Definitions, symbols of differentiation and application of differentiation rule are discussed, followed by application of the power rule to differentiate a function and finding the area under a curve. The errors made by participants are analysed.

5.2 Collection of Preliminary Data for the Pilot Study (Phase 1)

For the Phase 1 pilot study, activity sheets were designed in order to explore learners' mental constructions of differentiation and integration and to identify any difficulties that hindered learners' ability to make the necessary mental constructions. Ndlovu (2014) states that understanding the mental constructions that learners make when learning mathematical concepts can enable improved instructional methods and curriculum development. The questions from the activity sheet used for Phase 1 is presented in Appendix A.

The activity sheets consisted of 4 questions called 'items' for the purpose of this study. These items had sub-items; for example, item 1 had sub-items 1.1 and 1.2. Items 1 and 2 focused on solving problems in differentiation and items 3 and 4 focused on integration problems. In total, the instrument consisted of 10 items which were graded from easy to difficult. Items 1 and 3 consisted of recall questions requiring respondents to define terms; learners were expected to understand the rules of differentiation and integration concepts. These questions are stage one questions that required learners' procedural understanding. Later items on the activity sheet required higher-order thinking, and learners were expected to apply their knowledge or problem-solving skills to find solutions. The following section presents the analysis of learners' responses for each item and includes extracts from learners' responses.

The activity sheets were administered to ten Grade 12 learners in February 2020 before the school was closed due to Covid -19 pandemic. The selection of learners was done in accordance with the categories discussed in Chapter 4. At the school where the study was conducted, only ten learners are accepted each year to do Technical Mathematics. Learners were given the activity sheets to complete individually at home. Four items were completed over a period of three days. Thereafter, the data were analysed to assess learners' understanding of differentiation and integration.

5.3 Symbols and Rules used in Differentiation

This section was designed to explore learners' mental construction of the definitions and symbols used in differentiation. Learners were expected to define 'differentiation' and state the symbol used when finding the derivative. Item 1 consisted of three sub-items: item 1.1 on the meaning of differentiation; item 1.2 on power rule notation; and 1.3 for symbols of differentiation.

5.3.1 Item 1: Definition and rules of differentiation

Item 1 was about the rules of differentiation, and the stage of difficulty was also considered from sub-item 1.1 to sub-item 1.2. In Item 1.1, learners were expected to define in their own words the meaning of 'differentiation', whereas in sub-item 1.2 learners were expected to understand the meaning of 'the gradient of a curve' at any given point. In this instance, the learners were expected to state the name used in finding the gradient of the curve. In item 1.3 learners were expected to know the symbols of differentiation. All learners attempted item 1, however, the majority failed to provide a complete response to all three sub-items. The main trend of the learners was to define 'differentiation' using everyday, rather than mathematical, language.

Item 1.1 (definition of differentiation)

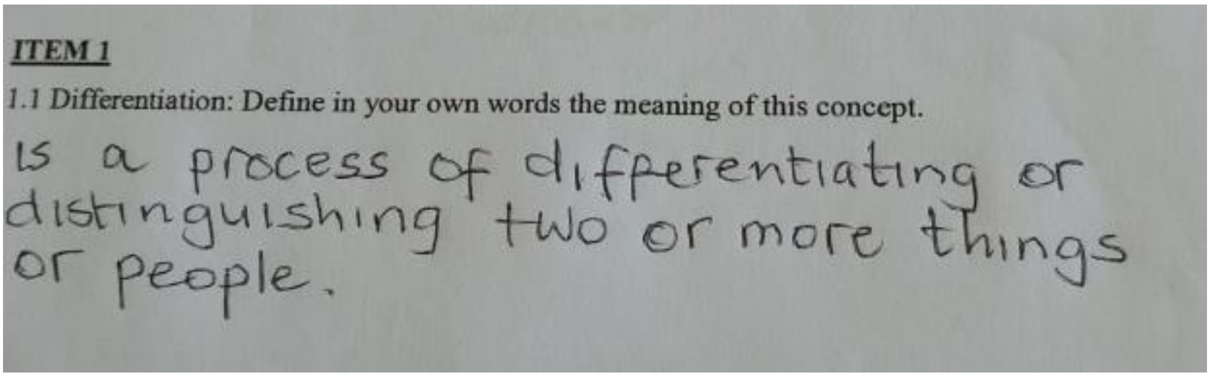
Answer the following questions 1.1 Differentiation: Define in your own words the meaning of this concept.

Item 1.1 was intended to provide insight into whether the learners had developed the concept of differentiation. Learners' responses to the activity sheet were categorised, analysed, coded, and scored according to the five-point rubric adapted from Jojo et al. (2011). The researcher modified these categories to suit this study by using selected elements, as displayed in Table 5.1 below.

Table 5.1: Learners' responses to item 1.1: meaning of 'differentiation'

Category	**1**	**2**	**3**
Indicator	Not Attempted	Not Achieved Incomplete or incorrect responses	Achieved Learners who made some or all mental constructions suggested by the genetic decomposition
No. of responses	0	8	2

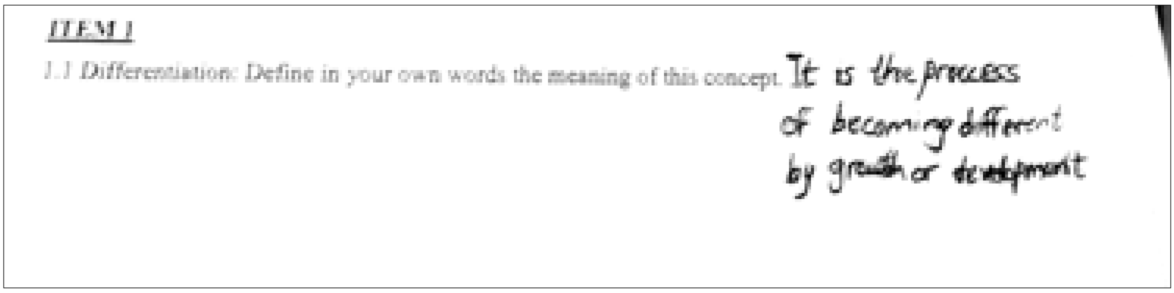
All ten learners attempted item 1.1, where they were asked to define the meaning of the differentiation concept. Category 2 represents all learners with incorrect written responses. Those responses showed different types of errors between steps. In this category, learners provided different definitions for the differentiation concept. Eight learners in Category 2 made different types of mistakes. In defining the term differentiation, four learners used everyday language in their attempt to define the concept differentiation. For example, one wrote: "it is a process of becoming different by growth or development". Three learners stated that "differentiation is a process of distinguishing two or more things or people" (see Figure 5.1 and Figure 5.2 below).


ITEM 1

1.1 Differentiation: Define in your own words the meaning of this concept.

is a process of differentiating or distinguishing two or more things or people.

Figure 5.1: Pretty's written response for item 1.1


ITEM 1

1.1 Differentiation: Define in your own words the meaning of this concept. It is the process of becoming different by growth or development

Figure 5.2: Akhona's written response for item 1.1

Failure to use mathematical language to define the term 'differentiation' was an indication that these learners had not constructed an understanding of the concept. These learners focused on the verb 'differ' to explain the meaning of 'differentiation'.

Pretty used the spoken language of 'being different' to define differentiation. Ndlovu (2013; 2014) emphasises the importance of terminology in the development of mathematical understanding of a taught concept. This was evident in Akhona's response; she stated that differentiation is a process of becoming different through growth or development. In defining the term 'differentiation', Pretty and Akhona responded to recognition of the everyday word 'different' within the term and used the meaning of 'different' to attempt to construct a definition for the term 'differentiation'. From the above responses, it can be noted that the meaning of the everyday use of the term 'different' was confused with the concept of 'differentiation'. Tall (2008, as cited in Ndlovu, 2014) alludes to the fact that the everyday use of spoken language can to some extent be a source of difficulty to learners when the same or similar terms are encountered in mathematics, as was evident with the learners in the study. Similarly, Luneta and Makonye (2010) state that some students display insufficient knowledge of calculus terminologies such as confusing 'turning points' and 'axial intercepts'. They suggested that, when teaching, the emphasis should be made on linking students' understanding of appropriate concepts and appropriate procedures that are taught in calculus lessons.

A different scenario ensued with learners in Category 3. These learners were found to have made some or all of the required mental constructions suggested by the genetic decomposition because they were able to state that differentiation is a process of finding the gradient of a curve at any given point. In addition, they mentioned that one differentiates a function concerning x. Beyond defining the term, they further explained the procedural steps followed when differentiating a function in terms of mental constructions. Learners in category 3 were classified as operating at the action stage of APOS. Their responses relied on external cues to construct meaning; such responses place them at the action stage. They had managed to give a step-by-step method defining the meaning of the differentiation concept.

5.3.2 Misconceptions in defining the concept of differentiation

In this section, the researcher observed that learners committed different types of errors. In item 1.1, some learners failed to distinguish the calculus concept from similar terms in everyday English. Tall (2008, as cited in Ndlovu, 2013) notes that lack of concept definition hinders the construction of the concept image, having found that learners tended to confuse the use of everyday language with mathematical language – for example, confusing the meaning of the word ‘limit’ in mathematics with the everyday meaning of the word ‘limit’ (i.e. ‘cannot go beyond’). As evident in the responses of the learners in Category 2, the use of the everyday word ‘different’ was confused with the meaning of the concept term ‘differentiation’. In the next section, the researcher discusses the learners’ mental constructions and errors that hindered the construction of the necessary mental construction.

Item 1.2 aimed to explore learners’ understanding of the power rule when differentiating. The power rule is one of the rules, in addition to differentiation by the first principle, that learners apply when differentiating, thus their understanding of the rule – beyond procedural knowledge – is critical in the learning of the concept.

Item 1.2 (finding derivative of a function)

1.2 Complete the following: if y= x^n then $\frac{dy}{dx}=$ ---

As with item 1.1**,** all ten learners responded to item 1.2. Two trends were evident in their responses. Most learners generated their own terms and symbols. The first trend was that some learners explained the symbol for differentiation in their own words. The second trend was that some learners used incorrect notations or made errors in their use of exponents.

Table 5**.2** illustrates the analysis of the learners' mental construction of defining the power rule.

Table 5.2: Learners' response to item 1.2(use of power rule if $y = x^n$)

Category	**1**	**2**	**3**
Indicator	Not Attempted	Not Achieved Incomplete or incorrect responses	Achieved Learners who made some or all mental constructions suggested by the genetic decomposition
No. of responses	0	7	3

Seven learners in Category 2 focused on explaining the meaning of the symbol as they stated that the aim of using the symbol $\frac{dy}{dx}$ is to find the gradient of a function, which is the change in 'y' over the change in 'x'. When illustrating the application of the power rule they committed several mistakes, such as adding 1 instead in the exponent after multiplying the co-efficient by *n*, as illustrated in Notha's response in Figure 5.3 below.

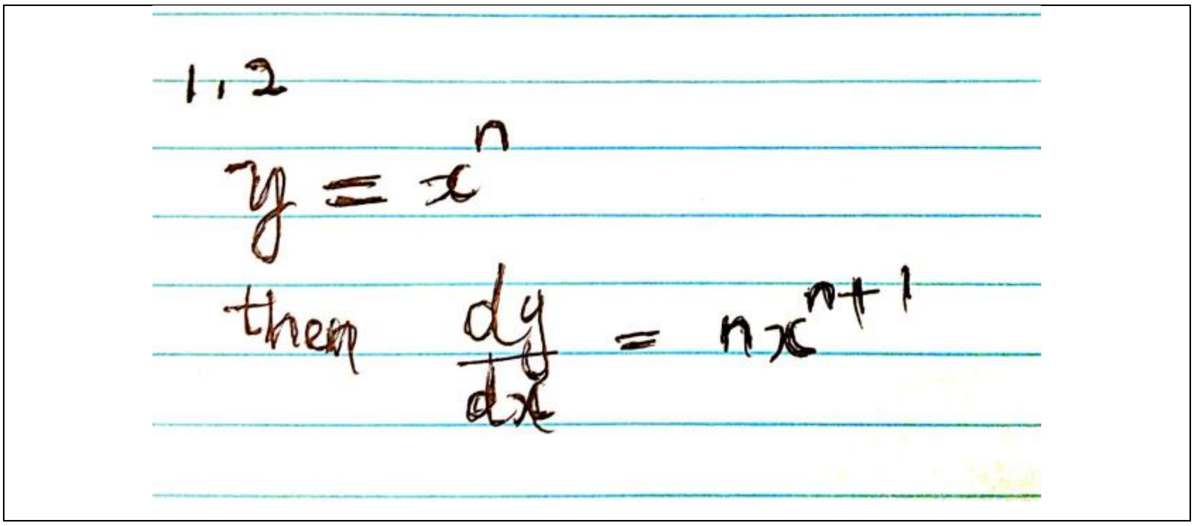


Figure 5.3: Notha's written response to item 1.2

The learners in Category 2 made some procedural errors, as shown in Notha's response. He knew that the first process needed was to multiply the coefficient by n, but he made a mistake using the exponent. He added 1 instead of subtracting it from n. This shows that he lacked an understanding of the power rule. This could be considered to be a structural error, since he knew the power rule, but failed to use it properly. This means that his action conception of the power rule was not fully developed.

Three learners in Category 3 correctly completed the number sentence as $y = x^n \; then \; nx^{n-1}$, and demonstrated an understanding of the concepts and applied the procedures correctly. The learners in Category 3 were considered to have interiorised the action into the process as they were able to conduct the steps in their minds and present the answer using correct notation showing understanding of the application of the power rule in the absence of numerals, but utilising notational symbols only. The evidence of their process conception was that, instead of writing a step-by-step procedure when completing the number sentence, they simply presented the answer in notational symbols, indicating that the steps were performed in their minds.

5.3.3 Misconceptions that hinder learners' mental constructions

In item 1.2, learners' misconceptions followed two trends. First, some learners in Category 2, after multiplying the co-efficient by n, added 1 to n $(n + 1)$ instead of subtracting 1 from n $(n - 1)$, thus confusing the process of differentiation with that of integration by adding 1 to the exponent instead of subtracting it. Some did not multiply the coefficient by n but only subtracted 1 from n: ($\frac{dy}{dx} = x^{n-1}$). What is evident in these responses is the lack of understanding of the meaning of differentiation as finding the rate of change, as well as the lack of procedural fluency. As mooted by Kiat (2005), such procedural errors are due to the confusion between the differentiation and integration processes while the technical errors reveal that these learners were struggling with poor mathematical skills and carelessness. This is due primarily to rules being memorised without understanding, as when the learners used the power rule to differentiate but, when given symbolical notation with no numerals, failed to illustrate the process of differentiation.

5.4 Application of Power Rule to Differentiate a Function

Item 2 aimed at exploring the learners' mental construction of using the power rule to differentiate the function. Item 2 was made up of three sub-items. Learners were given four sums to apply the power rule in finding the derivative. Item 2.1 focuses on differentiating a quadratic function, where b and c are equal to zero. In Item 2.2, learners were required to differentiate a quadratic function where a, b, c $\neq 0$. In item 2.3, learners were required to differentiate a quadratic function involving parenthesis, e.g., $f(x) = (\ ax \pm b)^2$.

5.4.1 Differentiating function of $\boldsymbol{b, c = 0}$

Item 2.1 (differentiating using power rule)

2.1 Differentiate using the power rule:

$$f(x) = 3x^2$$

In item 2.1, learners were expected to use the power rule to differentiate the function. A learner who has developed an action conception of the concept of differentiation would be expected to use a step-by-step procedure when solving a problem; for example, if $f(x) = 6x^3$, then the answer will be $f'(x) = \frac{6x^{3-1}}{3-1} = \frac{6x^2}{2} = 3x^2$. A learner who has developed a process conception of the concept, in contrast, would write the answer without explicitly writing down the steps. At the object stage, a learner can reflect on the operations applied and be aware of the process as a totality. That means a learner at this stage can use one method to find a solution and verify the answer by using another method. Table **5.3** below presents the analysis of the learners’ responses to item 2.1.

Table 5.3: Learners' responses to item 2.1: Differentiating using the power rule

Category	1	2	3
Indicators	**Not Attempted**	**Not Achieved**: Incomplete or incorrect responses	**Achieved:** Learners who made some or all mental constructions proposed by genetic decomposition
No. of responses	0	6	4

The analysis of learners’ responses to sub-item 2.1 revealed that all ten learners had attempted this question. As is revealed in Table 5.3, four learners provided the correct response to item 2.1. Out of the six learners in Category 2, four learners provided the correct answer however used the first principle instead of the power rule for finding the derivative. Since the steps they followed to get the answer did not align with the item, which required the application of the power rule, they were also categorised as ‘not achieved’. Two learners in Category 2 confused integration procedures with those of differentiation, as shown in Figure 5.4 below.

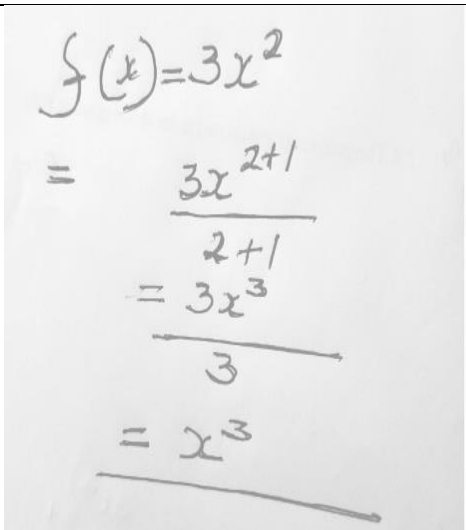


Figure 5.4: Zinhle’s written response to item 2.1

Looking at Zinhle's written response, certain errors were evident. Instead of multiplying the coefficient by 2, she divided the expression by 2 and added 1 to both the exponent and the denominators, which resembles the process of integration. Secondly, incorrect notation was used to represent the derivative, as she left the answer as $f(x)$. Her answer was left as a function, not a derivative, which could be interpreted as saying $f(x) = 3x^2$, and also equal to x^3.

Xolo, who also provided an incorrect response, made an error similar to Zinhle's, adding 1 in the exponents instead of subtracting, as shown in Figure 5.5 below.

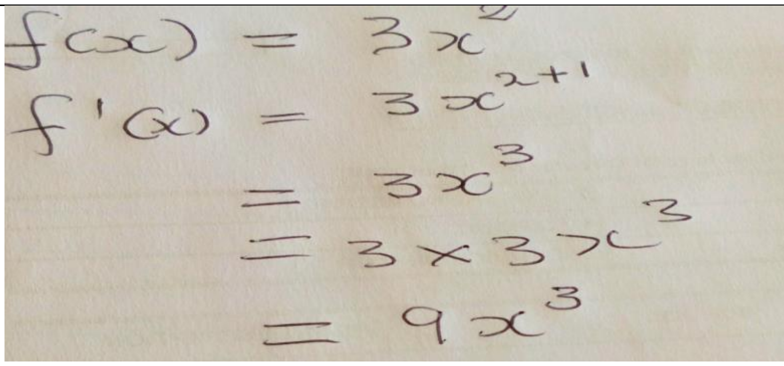


Figure 5.5: Xolo's written response to item 2.1

Contrary to what Zinhle did, Xolo used the correct notation to indicate that he was finding the derivative, but his response showed a lack of knowledge of how to find the derivative or apply the power rule. It could be argued that Xolo memorised the rule that, when differentiating, there is multiplication involved. Like Zinhle, there was evidence that the rules had been memorised with no understanding. Comparing the responses of Xolo and Zinhle, they committed different types of mistakes. Xolo added 1 and multiplied the coefficient of $x\ by$ 3, while in Zinhle's case, 1 was also added in the exponent, but the coefficient was divided by 3. Their responses followed two similar trends: either using integration approaches or adding 1 in the exponent. Ndlazi (2015), similarly, found that although students knew the rules and how to apply them, there may be no construction of meaning attached; this can hinder the presentation of a correct solution.

The four learners in Category 3 were considered to have made some of the mental constructions of applying the power rule to find the derivative. Two learners managed to arrive at the correct solution by following a step-by-step approach, indicating that they were operating at the action stage. The other two learners were in the process stage, as they expressed the answer of $f'(x) = 6x^2$ without explicitly writing down all the steps.

5.4.2 Misconceptions that hindered learners' mental constructions for item 2.1

Muzangwa and Chifamba (2012) suggest that a lack of conceptual knowledge in calculus is a challenge that can limit learners when learning related science applications. The responses of learners in Category 2 revealed that learners failed to distinguish between integration differentiation procedures. One of the underlying misconceptions evident in the learners' responses was that of seeing the equal sign as the operator symbol instead of meaning equivalence. Kazunga and Bansilal (2017) found that seeing an equal sign as an operation or an indication of something other than equivalence is an underlying misconception evident in many incorrect mathematical solutions provided by students. Zinhle displayed a similar misconception in her solution. Ndlovu (2014) argues that learners do not pay attention to the meaning of notation when differentiating: their focus is on performing the calculations routinely, with no meaning attached. Similarly, Zinhle ignored writing down the notation to show that she was finding the derivative, suggesting that her focus was on trying to manipulate the numbers given.

Building from item 2.1, which explored the use of the power rule to differentiate, item 2.2 focused on exploring learners' mental constructions of differentiation of the sum and the difference of the function, where learners were expected to differentiate a term by term.

Item 2.2 Differentiate $f(x) = x^2 + 2x - 3$ using the rules of differentiation (power rule)

In this question, learners were expected to use the power rule to determine the derivative of the given function. As in item 2.1, a person with an objective understanding knows the relationship between the function and its derivative. For example, the derivative of a quadratic function is linear. The power rule is defined as follows: if $f(x) = ax^n$ and n is a real number and a is a constant, then $f\ f'^{(x)} = anx^{n-1}$. A learner who has developed an action conception of the concept of differentiation would be expected to find the derivative of each term in a step-by-step method. A learner who has developed a process conception of the differentiation concept would be expected to give the correct answer without a step-by-step procedure or method. For example, if $f(x) = x^2 + 2x - 3,$ then a learner at the process stage will produce: $f\,'(x) = 2x + 2.$ At the process stage, action is being interiorised into a process and the learner can think of the process without following a step-by-step procedure. In this case, the learner gave the final answer without following step-by-step procedures of differentiation.

Table 5.4: Learners' written responses to item 2.2 Differentiate $f(x) = x^2 + 2x - 3$

Category	1	2	3
Indicator	Not attempted.	Not Achieved. Learners with incomplete responses and made errors in between steps	Achieved. Learners who made some or all mental constructions suggested by the genetic decomposition
No. of responses	0	7	3

Seven learners in Category 2 attempted the question but with incomplete responses and with errors between steps. Three errors were evident among learners whose responses were categorised as 'incorrect': first, using the incorrect rule to differentiate; second, only differentiating the terms with variable x; and third, confusing finding differentiation with integration. Examples of using incorrect rules were displayed by Zola who, like Zinhle in item 2.1, first added 1 in the exponent; but, unlike Zinhle in item 2.1, Zola multiplied the co-efficient with the new exponent as follows:

$f(x) = \; x^2 + 2x - 3,\; her\; answer\; was\; f'(x) = 3.2x^{2+1} + 2.2x^{1+1} \; = 6x^3 + 4x^2.$

Zola displayed a lack of procedural fluency in applying the power rule. Another common error committed by learners in Category 2 was not differentiating the constant. The learners differentiated the terms with variable x only. This was probably due to memorisation of the power rule which, in general, is given as $f(x) = an^x$ without the constant term. This illustrates that learning mathematical concepts instrumentally can hinder the learning process.

One example of a learner confusing integration rules with differentiation rules is displayed below. Although Akhona did not apply the integration rules correctly, adding 1 in the exponent was an indication that she was attempting to integrate the function instead of finding the derivative (see Figure 5.6 below).

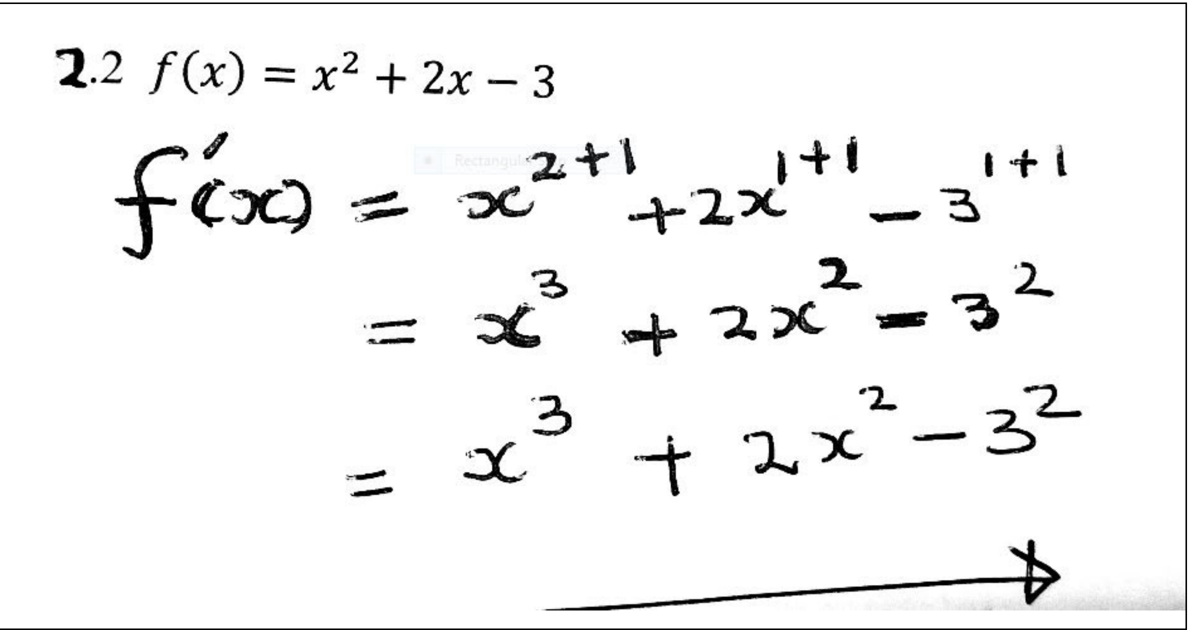


Figure 5.6: Akhona's written response to sub-item 2.2

Akhona added 1 to the exponent instead of subtracting it. Learners are first taught differentiation, followed by integration – which is the reverse of differentiation. Tall (2008, as cited in Ndlovu, 2014) terms such errors as 'met after', where, due to a lack of understanding, learners confuse the new knowledge learned or a new concept with a previous concept. Looking more closely at Akhona's response, it is clear that she did not understand the meaning of the notation $f'(x)$. Secondly, she understood the equal sign to be an operator symbol, rather than indicating equivalence. Siyepu (2013) posits that such errors persist due to surface-stage procedures, where an individual acquires knowledge by rote learning without engaging with its meaning.

A different pattern was found with learners in Category 3, who displayed construction of the application of power rule to differentiate a function, but were found to operate at different stages. For example, Lihle performed step-by-step procedures to determine the derivative (see Figure 5.7) below.

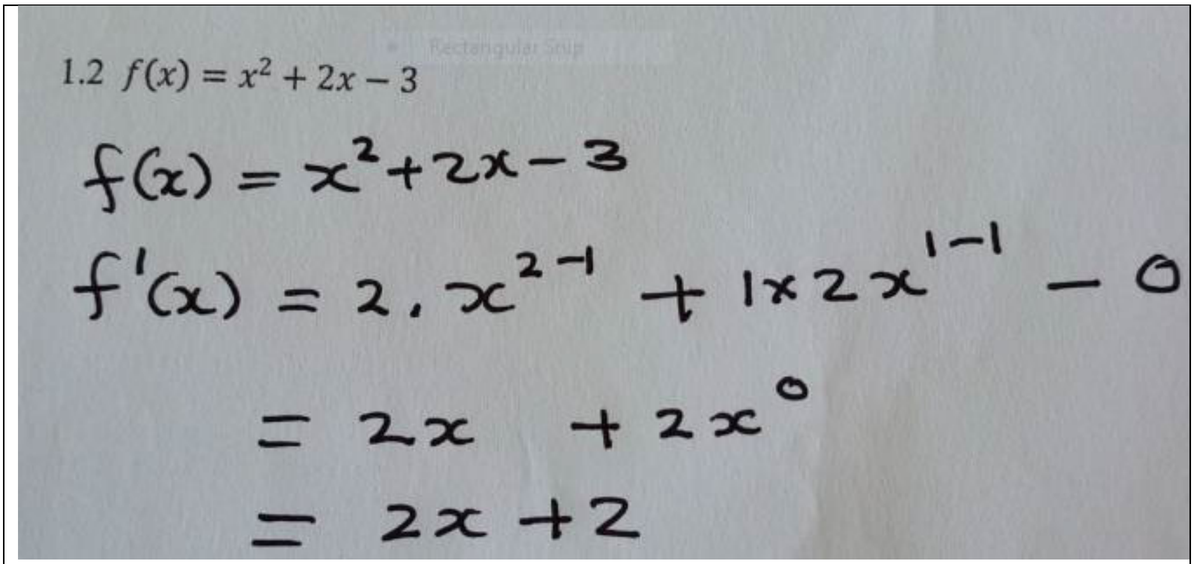


Figure 5.7: Lihle's written response to sub-item 2.2

Lihle's written response indicated that she applied the correct procedures for differentiation. She has constructed procedural knowledge for finding the derivative using the rules and thus displayed the action conception of the application of power rule to differentiate the function. This is in line with Jojo et al. (2013), who stated that students operating at the action stage view a mathematical procedure as a series of individual steps. The other two learners displayed a process conception, as they could present the answer without carrying out all the steps, thus showing that the action has been interiorised into a process.

5.4.3 Misconceptions that hindered learners' mental constructions for item 2.2

As stated above, the common error identified among learners in Category 2 resulted from concepts being learned as isolated facts. For example, the relationship between integration and differentiation had not been constructed and the relationship between the function and its derivative had not been constructed. A learner who knows that the derivative of a quadratic function is a linear function would know that adding 1 to the exponents of the first term would result in a cubic function, which is not the derivative of a quadratic function. The lack of constructing the correct meaning for the equal sign was the major misconception displayed by these learners. For them, the equal sign implied the second, third or fourth step, rather than equivalence. For learners to be able to construct the schema for differentiation, it is critical that the focus is on helping them make connections between the concepts, rather than learning them as isolated facts. As purported by Olivier (1989, as cited in Ndlovu et al., 2017), a learner may fail to solve problems due to not possessing the schema needed for the concepts.

It appeared that the seven learners in Category 2 did not possess the schema needed for differentiation.

Item 2.3 was designed to explore learners' mental construction of the application of the power rule to differentiate a function with parentheses. In the South African school curriculum, the use of the chain rule is not taught. Learners are taught to first simplify the expression and remove the brackets before applying the power rule. Instead of having a singular term after simplifying the expression, it would be the sum and difference of the function, and thus the same processes would be applied to differentiate item 2.2.

Item 2.3 Finding the derivative of brackets

2.3 Differentiate

$$f(x) = (2x + 1)^2$$

Table 5.5: Learners' written responses to item 2.3: Finding the derivative of brackets

Category	1	2	3
Indicator	Not attempted.	Not achieved. Incomplete or incorrect response	Achieved. Learners who made some or all mental constructions
No. of responses	0	7	3

The analysis of learners' responses in Category 2 revealed two trends. On the one hand, learners attempted to differentiate without simplifying the expression: e.g. removing the brackets first. On the other hand, as in item 2.2, learners failed to carry out procedures of differentiation correctly. In Category 2, four learners attempted to differentiate the expression without removing the brackets and, as a result, committed errors when applying the differentiation procedures. Nolly was one learner in Category 2 who differentiated the function without brackets removed (see Figure 5.8).

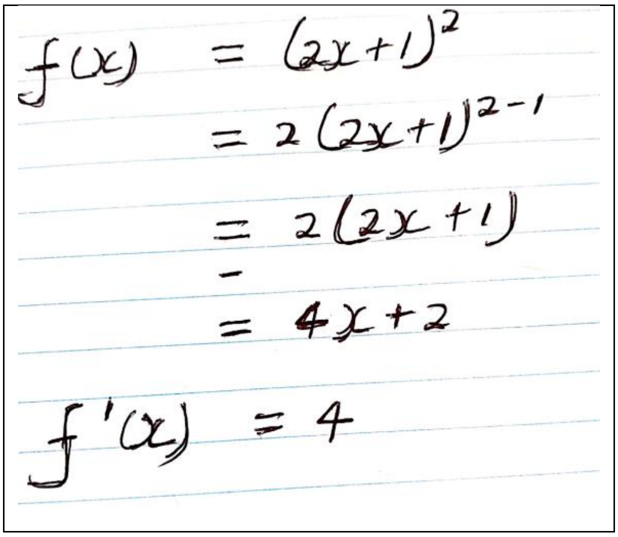


Figure 5.8: Nolly's written response to sub-item 2.3

As noted above, Nolly applied the rules to differentiate term by term. Instead of simplifying the expression, Nolly multiplied the whole term in the brackets by the exponent 2, treating the function as $f(x) = ax^n$. Instead of removing the brackets, she applied the power rule to differentiate the function. After applying the power rule, she applied the distributive property to remove the brackets and then applied the power rule to find the second derivative. Two main errors are evident in this response. By applying the power rule in step one, she was finding the derivative; however, she used the notation $f(x), not\ f'(x),$ meaning that although she has applied the power rule the resulting answer is still considered a function, not a derivative. This shows that she had not constructed the meaning of notation. In addition, the relationship between the derivative and the function had not been constructed. It appears that Nolly was performing procedures without an understanding of what exactly she was determining. This is one example of rules that are memorised with no understanding, as it was noted that, in item 2.2, she was able to perform the step-by-step procedure to find the derivative. Similarly, in item 2.3, she performed the steps but with no understanding.

Ndlovu (2013) moots that student tends to use notations loosely due to memorisation. In the case of differentiation, it was evident that learners were not sure when to use the symbol of differentiation. Some learners differentiated without using the symbol, which indicated a lack of both procedural and conceptual understanding. According to Siyepu (2015), conceptual errors are made because of not grasping the concepts involved in the problem or because of not understanding the relationships involved in the problem. In this instance, learners failed to realize the need to remove brackets, which was the prerogative of differentiation. While the sample of responses used shows the different errors learners made, it is evident that the learners in Category 2 had not yet conceptualised the relationship between the function and its derivative. They were trying to carry out procedures with no meaning, indicating that they had memorised rules but not understood them. On the other hand, Xolo knew that she must first remove the brackets, as shown in Figure 5.9 below.

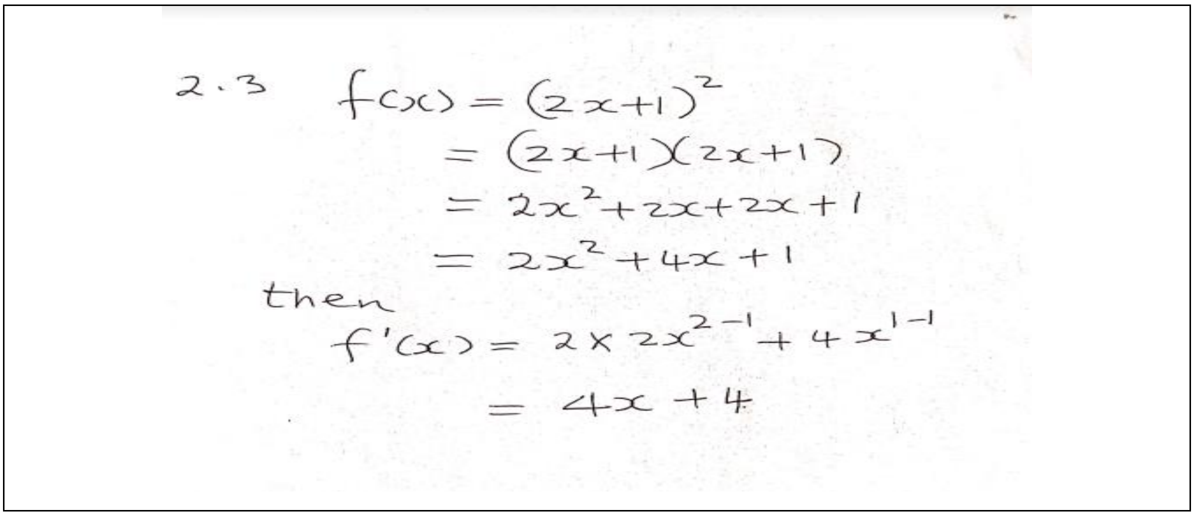


Figure 5.9: Xolo's written response to item 2.3

Although Xolo removed the brackets, due to computational errors her response was categorised as 'incorrect'; she arrived at $2x^2$ instead of $4x^2$. Although her simplification of the expression when removing brackets was incorrect, the rules used to differentiate the functions were applied correctly. In item 2.2, Xolo was categorised as operating at the process stage as she was able to express the answer without carrying out all the steps. However, in item 2.3, although she failed to simplify the expression she was able to perform step-by-step to differentiate the function, indicating that she has developed an action conception of differentiation. What is noticeable is that, while she was categorised as being at the process stage when differentiating functions with one or more terms separated by ±, she struggled with the function given in a different format. She struggled to solve a binomial expression, thus indicating that a lack of basic knowledge hindered her development of the new schema. This aligns with Muzangwa and Chifamba (2012) who state that students' errors and misconceptions in calculus are caused by knowledge gaps in basic algebra. Brodie and Berger (2010), also, argue that many students' errors are produced by misapplications of standard algorithms. In this study, Xolo failed to simplify the expression, thus arriving at an incorrect answer.

Two learners in Category 3 were operating at the action stage as they followed a step-by-step technique for removing brackets. Siwe's response is shown below.

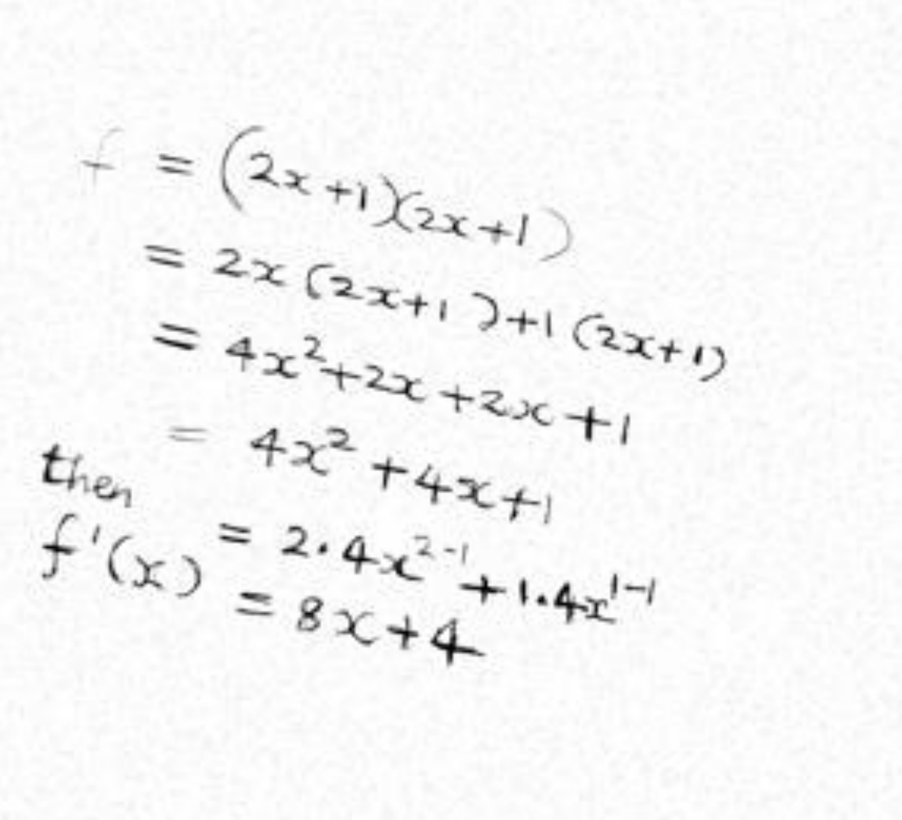


Figure 5.10: Siwe's written response for item 2.2

The exponent 2 triggered an understanding that it means there are two bases and distributive property was used to simplify the expression, then step-by-step procedures were used to find the derivative of the given function. A different case ensued with Joy, who, without breaking down the brackets was able to write down the answer for the function without brackets, suggesting that the action of simplifying binomials had been interiorised. Following that she expressed the derivative without doing the step-by-step application of the power rule, this also indicates interiorisation of the power rule. In addition, she used notation correctly throughout.

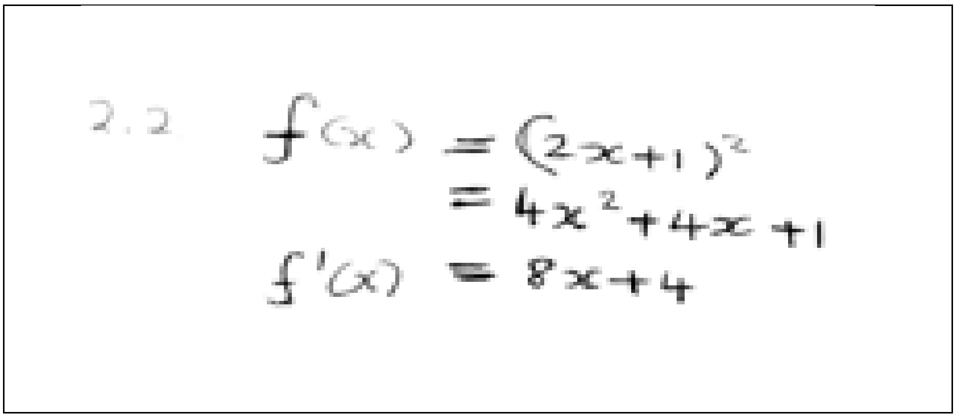


Figure 5.11: Joy's written response for item 2.3

As it is evident in the responses above, two learners in Category 3 displayed action conception for differentiating functions with parentheses and one displayed process conception. What is evident is that learners operating at the action stage in item 2.1 either continuously operated at the action stage in sub-items 2.2 and 2.3 or displayed pre-action in sub-items 2.1 and 2.3. This indicates that, although learners operating at an action stage are considered to have made the basic mental construction needed for the concept, if that construction is built on memorisation without understanding, when encountering problems that required multiple procedures they either failed to perform those or carried on operating at the basic a basic level. Their understanding of the concepts was still at the surface stage.

5.4.4 Misconceptions that hindered learners' mental constructions for item 2.3

Luneta and Makonye (2010) found that "errors demonstrated by the students in Calculus emanate from prior knowledge as students attempt to construct mathematical meanings" (p. 40). This was evident in the responses by learners in Category 2 that demonstrated that they lacked algebraic skills such as simplifying; in addition, they displayed incorrect application of the power rule to differentiate the function, as well as incorrect use of notation.

5.5 Definition of Integration and Application of Integration Rules

In this section, the focus was on learners' understanding and application of integration rules. Item 3, as presented below, focused on exploring learners' mental construction of concept definition on integration. This item also intended to provide insight into whether the learners had developed the mental constructions for defining the symbols of integration.

Item 3 (Defining the meaning of integrals)

In your own understanding define the meaning of the following symbols 3.1 $\int f(x)dx$ -- 3.2 $\int_a^b f(x)dx$---

These first two items (items 3.1 and 3.2) were designed to explore learners' understanding of concept definitions. Jojo (2011) states that, for procedures to be learned with meaning, they should be linked to the concept image and concept definition under study. In this study, procedures for differentiation and integration are explored and thus the construction of the meaning of the procedures is embedded in the learners' understanding of the definition associated with the concepts. The analysis of learners' responses to item 3 is illustrated in the table below, followed by the discussion.

Table 5.6: Learners' responses on items 3.1 and 3.2 on the definition of integral symbols

Category	**1**	**2**	**3**
Indicator	**Not Attempted**	**Not Achieved** Incomplete or incorrect response	**Achieved** Learners who made some or all mental constructions
No. of responses on $\int f(x)dx$-----	2	6	2
No. of responses on $\int_a^b f(x)dx$------	2	8	0

5.5.1 Learners' definitions of $\int f(x)dx$

Learners in Categories 2 and 3 had different explanations for $\int f(x)dx$. Four learners stated that $\int f(x)dx$ is the indefinite integral of $f(x)$ *with respect to x where* $f(x) = y$. Two learners mentioned that this is a process of finding the anti-derivative. One learner, who was categorised as ‘not achieved’, said: “we are reversing the process of differentiation but there is no limit”.

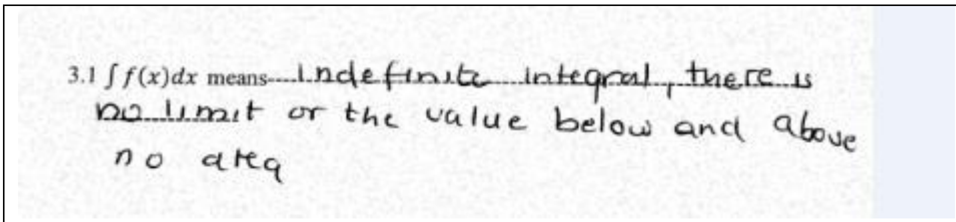

3.1 $\int f(x)dx$ means indefinite integral, there is no limit or the value below and above no area

Figure 5.12: Zola’s written response to indefinite integral $\int f(x)dx$ (item 3.1)

Instead of differentiating the differences between the symbols, Zola focused on explaining the symbols, saying indefinite integral has no limit, no values below and above, and no area. It seems she had some idea that it involved the area but could not articulate it. One learner stated that $f(x)$ stands for derivative, but $\int f(x)\, dx$ stands for anti-derivative. This learner failed to give the meaning behind the integral symbols. Instead of defining $\int f(x)\, dx$, the learner attempted to explain the relationship between the symbol for derivative and anti-derivative.

Two learners in Category 3 made a satisfactory attempt at defining indefinite integral. One learner stated that $\int f(x)dx$ is the integral of $f(x)$ with respect to x; she further stated that you need to reverse the process of differentiation first. The other learner in Category 3 showed an understanding of indefinite integral by saying “when you are integrating you need to add 1, but when you are differentiating you subtract 1 from the exponent”. He further stated this is the inverse of differentiation. These two learners used the symbol $\int f(x)$ as an external cue to define indefinite integral and thus were categorised as operating at the action stage in their understanding of the concept definition of indefinite integral.

5.5.2 The meaning of $\int_a^b f(x)dx$

This section focused on the meaning of definite integrals and the learners were expected to give the meaning of a and b. As shown in the above table, two learners did not attempt sub-item 3.2. Various incorrect responses were evident among the learners in Category 2. Two learners in Category 2 stated that the function was definite and that they needed to substitute the values of a and b, while another two learners focused on explaining the procedure; for example, saying that one needs to integrate the function first and, thereafter, substitute the value of a and b. They failed to give a precise explanation and the purpose of a and b in the function. The other two learners stated that this formula is used when finding the area of a function $f(x)$ between the values of $x = a \; and \; x = b$, which was an incomplete statement of the meaning of $\int_a^b f(x)dx$. The other two stated that the symbol is for finding the integration with the upper limit and lower limit, as shown in Figure 5.13 below. Learners' responses indicated that they tended to link definite integral with the area between a and b without proper meaning, as illustrated in Zinhle's response below.

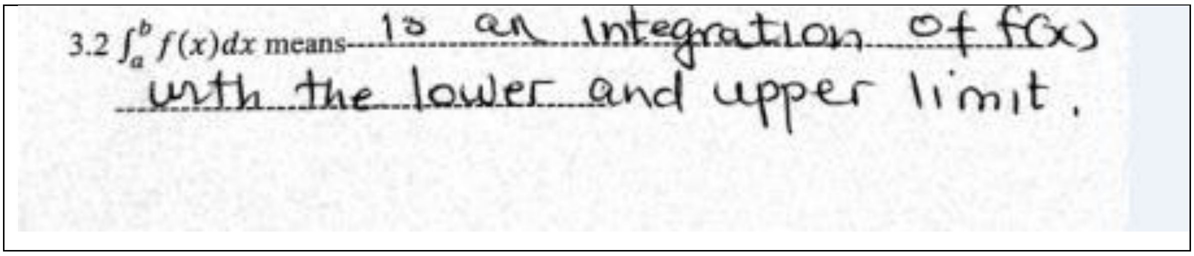

3.2 $\int_a^b f(x)dx$ means-- is an integration of f(x) with the lower and upper limit.

Figure 5.13: Zinhle written response to sub-item 3.2

Zinhle knew the value of a and b, but her explanation lacked a proper understanding of the definite integral. She could not link the relationship between the upper limit and lower limit with the processes of integration. What was noticeable in Zinhle's response and those of other learners in Category 2, was the lack of understanding of the meaning of the notational symbol used when finding definite and indefinite integral.

5.5.3 Solutions for the indefinite in item 3.3

An understanding of notational symbols was critical for the application of procedures in item 3.3. The focus of this item was on the techniques of integration when finding the indefinite integral. The aim was to explore learners' insight on solving Grade 12 mathematical problems on indefinite integral. This item also intended to provide insight into whether learners had a conceptual understanding of indefinite integral.

Determine the following

3.3 $\int 2x \; dx$

In this item, learners were expected to integrate an expression. Exploring learners' understanding of the application of the integration rule is critical in ascertaining their understanding.

Table 5.7: Learners' written responses to item 3.3 Finding $\int 2x dx$

Category	1	2	3
Indicator	**Not Attempted**	**Not Achieved** Incomplete or incorrect response	**Achieved** Learners who made some or all mental constructions
No. of responses	0	7	3

All ten learners attempted item 3.3, but seven experienced difficulty applying integral procedures. Out of these seven learners, four learners seemed unsure of the method for finding the integration, and the remaining three learners in Category 2 made errors between steps. Some learners subtracted 1 in the exponent instead of adding 1, while others, after adding 1 in the exponent, did not divide the expression with the new exponent. Since the integral is the reverse process of the derivative, which means finding the original function before it was differentiated, other learners failed to determine the constant term. As in items 1 and 2, which focused on differentiation, learners displayed confusion about the procedures of differentiation and integration, thus showing that they had not grasped which rule applied to which concept and when to apply it.

Olivier (1989, as cited in Ndlovu et al., 2017) found that the retrieval of incorrect schema was the main barrier because, when differentiating a function, some applied the rules of integration and, when integrating a function, they applied rules for differentiation – indicating the retrieval of incorrect schema, thus finding the incorrect variables when trying to find the solution. As illustrated in Joy's response below, the schema retrieved is flawed. For example, when attempting to integrate the function, Joy subtracted 1 instead of adding, which she confused with finding the derivative. By subtracting 1, the exponent turned to 0, which means the base should be 1 as per the laws of exponents; however, she continued to divide by 0. Both procedures for integration and differentiation were retrieved and applied incorrectly.

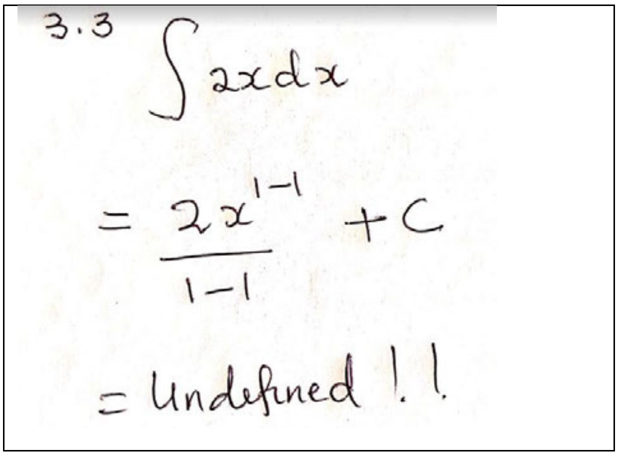


Figure 5.14: Lihle's written response to sub-item 3.3

Lile had not constructed the meaning of the relationship between the process of differentiation and integration because, by saying it is undefined, it means there was no function that was derived. If he understood that integration is the reversal of derivatives, he would then recognise that his answer was not making sense. His lack of conceptual understanding of the process of differentiation and integration resulted in an inability to make a necessary connection between the learned concepts. Knowledge gaps in integration were prevalent in most of the learners' responses. This is in line with Ndlazi's (2015) findings that knowledge gaps in differentiation, symbolic notation and integration were found to influence students' success in solving integrals.

On the other hand, learners in Category 2 were operating at the action stage in relation to their understanding of applying integration procedures. This is illustrated by the response shown in Figure 5.15 below.

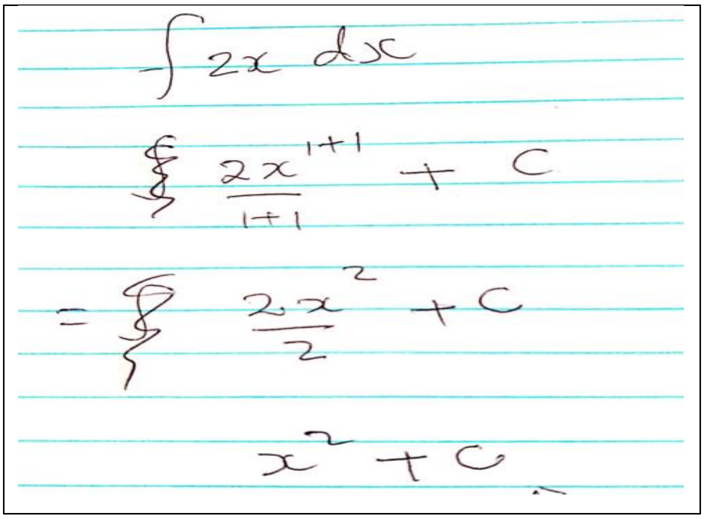


Figure 5.15: Nolly's response to sub-item 3.3

Nolly performed the step-by-step procedures to integrate the given function. The notation for the integral acted as the external cue that triggered the procedures to be followed.

5.6 Finding the Area Under the Curve

In this section, the main focus was to explore learners' understanding of definite integral in the form of $\int_a^b f(x)dx$, where a and b stand for the upper limit and lower limits, respectively. Learners were also expected to use the knowledge learned to find the definite integral as well as the value of the area under the curve. Learners were expected to integrate the functions and plot the graphs on a Cartesian plane and, finally, find the area under the curve, between the values of $x = a \text{ and } x = b$. Item 4 is made up of two sub-items, 4.1 and 4.2, as reflected below.

Item 4 (on curve sketching)

4.1 Draw the graph of $y = x^2$ and find the area under the curve using integration methods between $x = 0 \text{ and } x = 3$ and shade the region.

4.2 Draw the graph of $y = 2x + 1$ and find the area bounded by $x = 1 \text{ and } x = 4$ using integration methods. Shade the region.

In both sub-items 4.1. and 4.2, the use of any methods to sketch the graphs indicates the action stage; however, the ability to sketch the graph without doing calculation steps indicates the process stage. For example, in sub-item 4.1, a learner at the process stage would recognise that it is a quadratic function with a turning at the origin and it is positive. Without doing any calculations, they would be able to sketch the graphs. In item 4.2, a learner at the process stage would recognise that the slope is positive, passing the $y - \text{intercept}$ at 1, and only focus on calculating the $x - \text{intercept}$. A learner at the object stage could carry out the necessary actions and processes to determine the area under the curve.

5.6.1 Definite integral on the area under the curve item 4.1

Item 4.1 was designed to explore learners' understanding of the definite integral on the curve and the area bounded by the values of x. Learners needed to have a conceptual understanding of how to find the integration of the function. Learners were expected to use integration procedures in curve sketching and also find the area under the curve. The analysis of the responses revealed two main trends. First, learners tended to substitute the values of a and b without integrating the function. Second, learners drew a curve without mentioning the area. Table 5.8 illustrates the analyses of learners' responses to item 4.1.

Table 5.8: Learners' written responses to item 4.1 Drawing graph of $y = x^2$

Category	1	2	3
Indicator	**Not Attempted**	**Not Achieved** Incomplete or incorrect response	**Achieved (**Learners who made some or all mental constructions. Learners were able to integrate and draw the graph, finally finding the area.
No. of responses	0	6	4

Learners in Category 2 attempted to calculate the area under the curve without having sketched the graph. In finding the area under the curve, they substituted the values of the upper and lower limits in the given function and then found the difference between the two as the area under the curve, as illustrated in Notha's response below.

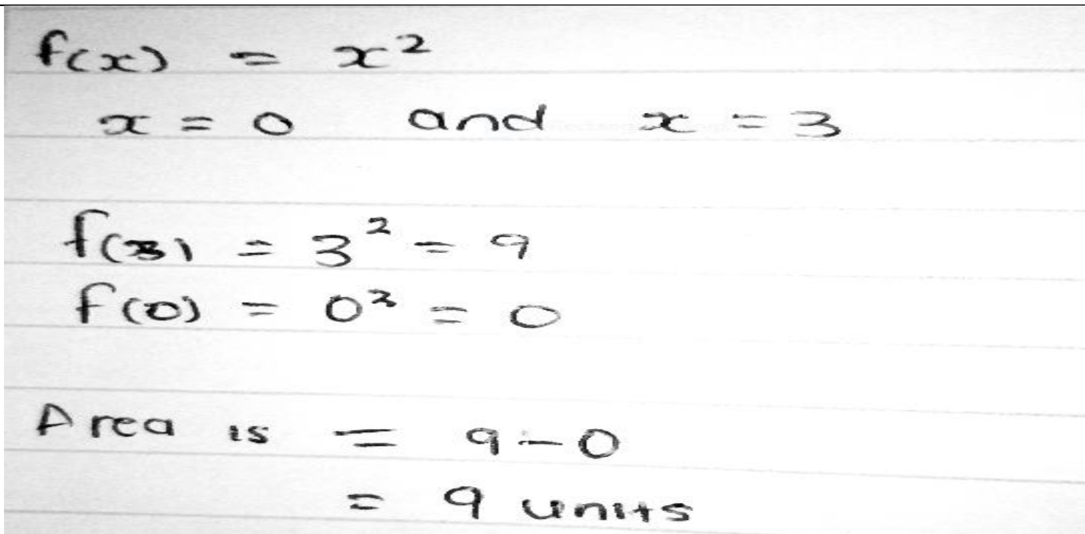


Figure 5.16: Notha's written response to item 4.1

As shown in the above response, Notha substituted the values of the upper and lower limits in the given function. Notha had not yet grasped the concept principles of integration. Notha and other learners in Category 2 seemed to be guessing, as their responses revealed that they were not sure which procedure to use. As shown in Table 5.8, all six learners in Category 2 had not constructed the mental construction of integration.

Among the four learners in Category 3, only one learner displayed an object-stage conception of the concept. This was evident because they provided a complete and correct response to item 4.1. They correctly carried out the procedure necessary to show the area under the curve and illustrated the area under the curve in the graph, as shown in Zola's response below. His response demonstrated a clear understanding of the concept of a definite integral, as he managed to perform all the necessary calculations to determine the area under the curve.

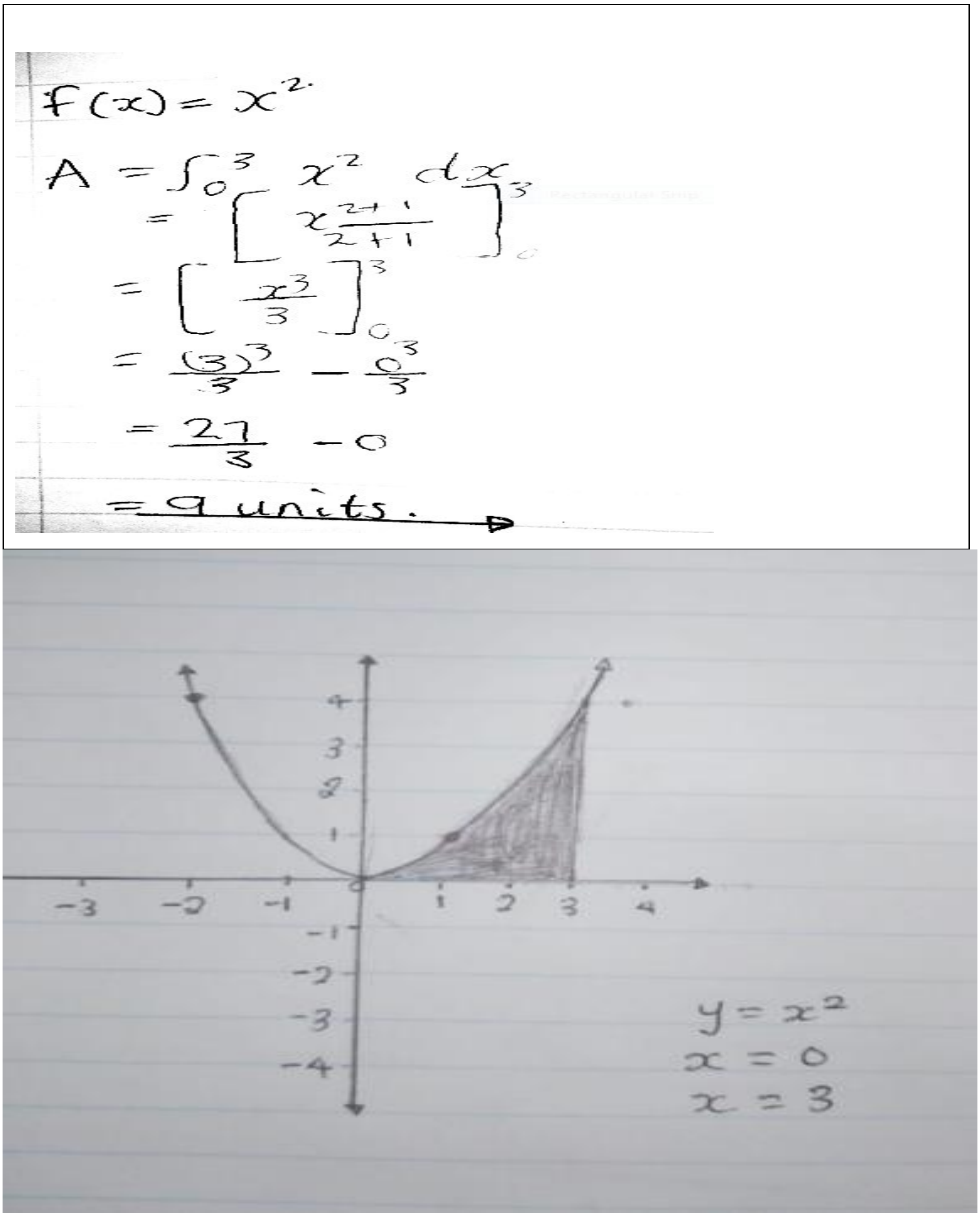


Figure 5.17: Zola's written response to item 4.1

Learners were expected to calculate the value of the area and also to plot the graph in order to indicate the area under the curve. Zola was one learner in Category 3 who managed to draw the graph and indicate the area by shading the required area. Zola displayed a complete understanding of the meaning of the concept of the area. In his response, he managed to illustrate the bounded area between $x = 0 \, and \, x = 3$ by shading the required area. He provided no explanation but indicated on the graph the shaded region. The other three learners were categorised as operating at the action stage because they carried out the necessary procedures to determine the answers and for graph sketching however, they did not indicate the area bounded by the curve.

5.6.2 Definite integral of the straight line

Item 4.2 aimed to explore learners' knowledge of drawing a straight line and finding the bounded area using integration. Learners were given a graph of $y = 2x + 1$ to draw and find the area bounded by $x = 1\ and\ x = 4$ using integration methods. The focus of sub-item 4.2 was on exploring learners' understanding of the definite integral of the straight line. This item included both definite integral and the area bounded by the values of x. Learners attempted the question but seven of them struggled to find the definite integral on a straight-line function. The table below presents the categories of learners' responses to item 4.2

Table 5.9: Learners' written responses to item 4.2 Finding the area under the curve

Category	**1**	**2**	**3**
Indicator	**Not attempted:**	**Not Achieved:** Incomplete or incorrect response	**Achieved:** Learners who made some or all mental constructions. Learners were able to integrate and draw the graph, finally finding the area.
No. of responses	0	7	3

The learners in Category 2 attempted to solve the questions; however, as in item 4.1, instead of using integration rules to find the area under the curve they substituted the upper and lower limits in the original function of $y = 2x + 1$ and then finding the difference between the value which they considered as the area, as illustrated by Lihle's response, below.

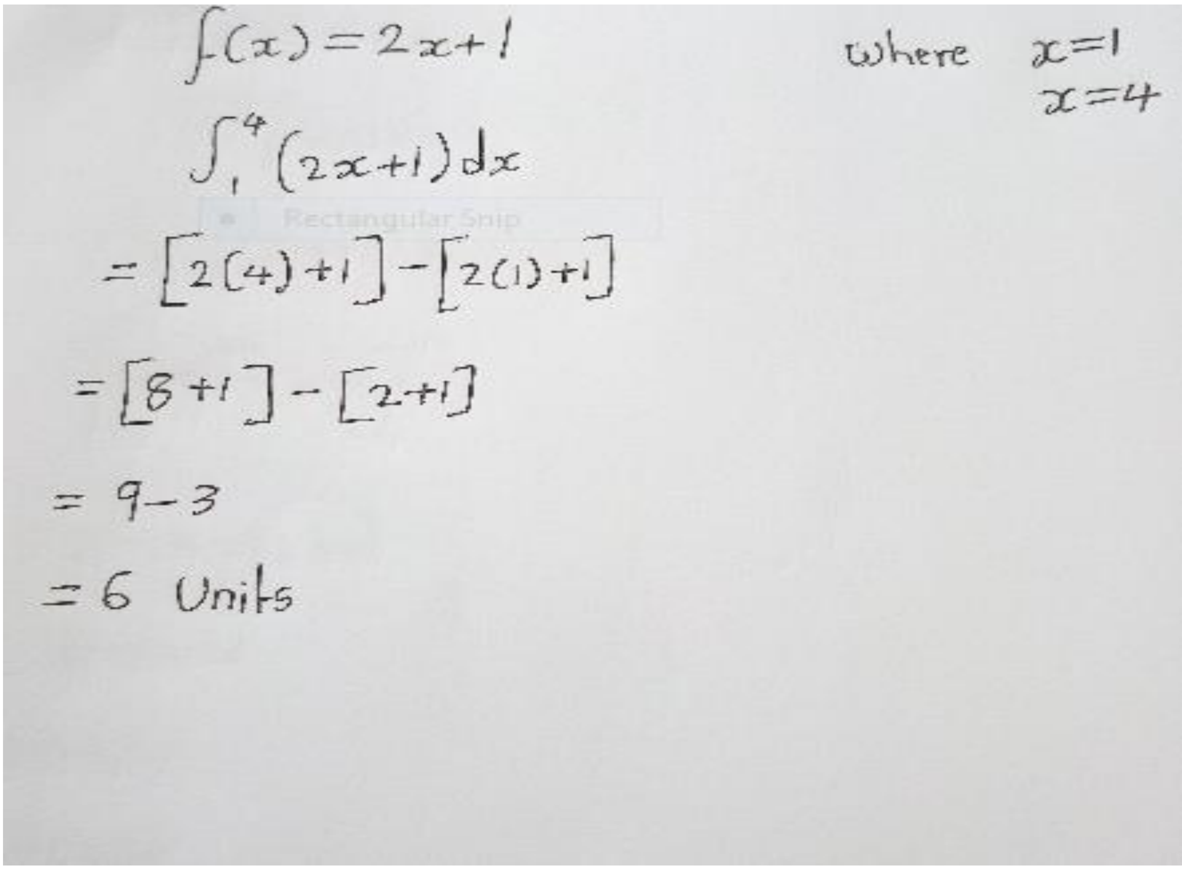


Figure 5.18: Lihle's written response to item 4.2

Lihle substituted the values of the upper and lower limits in the original function. Such a response indicates a weak conceptualisation of the concept of the definite integral and the area. Thomas et al. (2017) posit that conceptual understanding which is strictly related to the goal of becoming mathematically competent is evident when one is able to solve problems in an unfamiliar situation. In this instance, Lihle was aware that, in order to get the area, he needed to substitute the upper and lower limits, but made the mistake of substituting in the wrong function. In many instances, learners showed an inadequate conception of definite integral as an area and the use of algorithms. Six of these learners had made the same error in sub-item 4.1; the 7th was among those categorised as operating at the action stage, as they solved the problem and sketched the correct graph – although they did not indicate or explain the area bounded by the curve. In this item, the learner failed to carry out the necessary procedures to determine the area under the curve.

In interrogating the responses in Category 3, the analysis shows that all three learners were able to integrate the function and indicated the area bounded by the curve in the graph, as illustrated in the response in Figure 5.19 below.

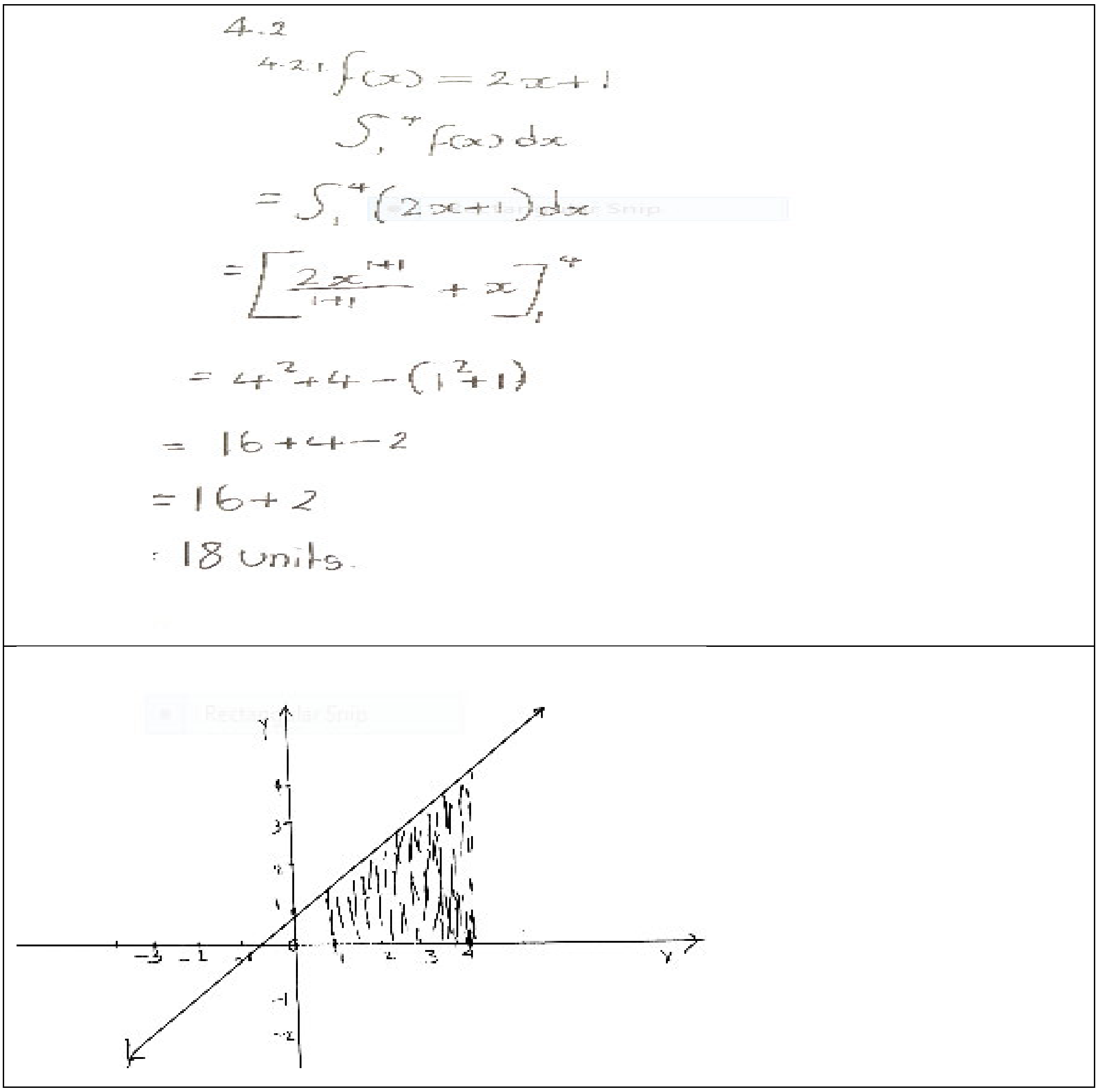


Figure 5.19: Pretty's written response to item 4.2

Pretty performed the action of finding the area and also combined the action and processes to sketch the graph and correctly identify the area under the curve. Her response indicated that she had encapsulated the process of determining the area under the curve. Both Pretty and Zola demonstrated object conception in items 4.1 and 4.2.

5.6.3 Mental constructions depicted by learners for items 4.1 and 4.2

Learners' demonstrated understanding of integrating the function and curve sketching placed them at the action stage of APOS theory on both items. All 3 learners in Category 3 managed to sketch the graph and indicated the bounded area without following a step-by-step procedure. At this stage of comprehension, these learners were placed at the action-process stage of APOS theory. In summary, learners struggled to recover the knowledge required on these two items. The researcher also observed that learners had not interiorised the action of integrating the function or the action of finding the area under the curve.

5.7 Errors Committed by Learners in the Application of Integration Rules

Many learners committed errors where they integrated indefinite integral without adding a constant C. The analysis revealed that learners were performing procedures without making sense of why they needed to carry out that procedure. A failure to make a connection between concepts – for example, between differentiation and integration – was the main issue that led to confusion about the rules. These results indicate that the learners lacked conceptual and procedural understanding of both concepts and this hindered their construction of the necessary mental constructions. Kilpatrick et al. (2001) state that both conceptual and procedural knowledge are needed for mathematical proficiency.

The researcher noticed that participants in the pilot study (Phase 1) seemed to confuse procedures; especially when the item had many sub-items, they answered one question correctly and gave incomplete or wrong responses for others. In finding the area under the curve, the most pertinent error was the substitution of the values of the upper and lower limits without first integrating the given function. When finding the limit of a function, learners are taught that when given a function with $x \to a$ they must treat it as $x = a$ and they need to substitute the value in the given function. As mentioned above, rules memorised with no understanding lead to the retrieval of a flawed schema. In addition, the construction of an equal sign signifying an operator rather than equivalence was a significant hindrance for these learners. As alluded to by Aygor and Ozdag (2012), learners' misconceptions are a source of difficulty and hinder learning. Similarly, Ndlovu and Brijlall (2015) state that errors made by students could be the result of their failure to conceptualise concepts, leading to poor interpretation of the nature of the problem. It was also evident with these participants that a lack of conceptualisation of concepts and failure to interpret the nature of the problem led to failure to solve the problems. Finally, errors made by learners could also be the result of them not having an adequate schema to distinguish between the different procedures used in differentiation and integration.

5.8 Conclusion

The chapter has presented the analysis of data generated in Phase 1 of the study, which was conducted to validate the research instrument. The research instrument used was a reliable tool to reveal the nature of mental constructions learners made. The results of the Phase 1 study revealed that the majority of the learners were operating below the action stage for all questions, indicating that they had not constructed the necessary mental constructions for differentiation and integration. In addition, the errors learners committed hindered their construction of the necessary mental constructions. This implies that teaching should focus on the application of procedures and interrogate what learners write in teaching and learning settings. This could help them to recognise the difference between differentiation (derivatives) and integration (antiderivatives) and to know that these two are related. Further, during teaching and learning the focus must be on the inverse relationship between derivatives and antiderivatives.

For the main study (Phase 2), the activity sheet was restructured to include questions related to average gradient to yield data on the mental constructions made by learners, and interviews were used to engage learners about their written responses, as recommended by Maharaj (2014).

In the next chapter, the researcher provides the analysis of learners' responses to the activity sheet and interviews with selected learners during Phase 2 of the study.

6 ANALYSIS OF WRITTEN RESPONSES FROM ACTIVITY SHEETS AND INTERVIEWS ON DIFFERENTIATION (PHASE 2)

6.1 Introduction

In the previous chapter, learners' written responses during the pilot study (Phase 1) were analysed. This chapter focuses on the analysis of activity sheets and interviews during the study proper (Phase 2). Learners' understandings of the concepts and rules used for differentiation are discussed and their mental constructions are addressed in line with genetic decomposition, as explained in Chapter 3. The written responses of the ten participants were categorised and coded and five learners were chosen for interviews to discuss their responses on the activity sheets in more depth. This chapter also discusses learners' understanding of removing brackets and square root signs for differentiation.

6.2 Analysis of Activity Sheets and Interviews on Differentiation

The activity sheets consisted of four sets of questions which were administered to ten learners that were taking Technical Mathematics. The main aim of the activity sheet was to understand the mental constructions that learners make of differentiation and the difficulties that hindered learners in making the necessary mental constructions. For the purpose of this study, the questions will be referred to as 'items', each of which has 'sub-items. For example, Item 1 has sub-items 1.1 and 1.2. Items 1 and 2 focused on solving problems on differentiation, whereas Items 3 and 4 focused on integration problems. The analysis of integration will be discussed in the next chapter.

In total, the instrument consisted of 11 questions which were graded using four cognitive stages as stipulated in the Curriculum and Assessment Policy Statement (CAPS). As purported by Dubinsky (1991), learning of mathematical concepts is hierarchical, thus a structured worksheet was designed to model how meaningful teaching could be planned to enhance learners' cognitive domains while solving problems (Ndlovu, 2014).

Learners' responses to the activity sheet and extracts from interviews were analysed, coded, and scored according to the following five-point rubric adapted from Jojo (2011). These categories were modified to suit this study and the activity sheet that was piloted in Phase 1 was improved for Phase 2. All items of differentiation were coded (scored) as per the scoring rubric shown in Table 6.1 below.

Table 6.1: Codes used to categorise learners' written responses

Category	Assessment Criteria	Description of mental action
5	A complete response to all aspects of the item and indicating a complete mathematical understanding of the concept assessed.	Made a schema mental construction as suggested in the genetic decomposition
4	Partially complete response with . minor computational errors demonstrating understanding of the main idea of the problem	Displayed object conceptions of the application of differentiation
3	A complete response to all aspects of the concept without following a step-by-step method and incomplete reasoning	Displaying Process of mental constructions, and conceptual understanding at a minimal stage
2	Follow step-by-step procedures but no reasoning to justify written responses. Displaying errors that hindered them from interiorising action into process stage understanding of the concept	Displaying Action mental constructions, but at a procedural stage
1	No written response or complete principle error	No mental construction of the concept

6.3 Learners' Understanding of Differentiation Symbols

Learners were expected to know, and be able to provide, the mathematical definition of differentiation. In the following section, the researcher discusses in detail the mental constructions learners made for items 1.1 and 1.2, which focus on exploring their understanding of differentiation. Beyond exploring their mental constructions, difficulties that hindered learners in making the necessary mental constructions are also discussed.

Interviews were conducted in order to verify what had emerged in the learners' written responses and to get greater clarity on the written responses and how learners arrived at their solutions. The interview was prepared by the researcher based on learners' responses on activity sheets and focused on understanding the mental constructions made or not made. The questions were similar for all the learners. They were asked to justify their responses to questions and explain strategies used to solve problems. During the interviews, the learners were allowed to use the language of their choice to express themselves. The researcher translated their responses from their vernacular language into English.

The main aim was to identify the barriers that might have hindered the learners from making the necessary mental constructions. It was important to identify the stage at which the learners were operating in terms of APOS theory. The analysis of learners' responses to the activity sheets, along with extracts from the interviews, is presented next.

6.3.1 Learners' understanding of the concept of differentiation

Item 1 aimed to explore learners' understanding of the rules of differentiation. Item 1 contained questions that were designed to evoke learners' mental images and definitions related to this concept. Learners were expected to define differentiation as a concept and the power rule symbols. Rasslan and Tall (2002) state that all mathematical concepts, other than primitive ones, have proper definitions. This is affirmed by Jojo (2011) who states that for procedures to be learned with meaning, they should be linked to the concept image and concept definition of the concept under study. In this study, it was necessary to determine learners' understanding of the differentiation concept before using the rules to solve problems.

In item 1.1, learners were expected to define the meaning of differentiation, while in item 1.2 learners were expected to know the procedures when solving differentiation problems using the power rule. Questions were asked based on the categories discovered in their responses to activity sheets in order to elicit learners' conceptual understanding of the concepts.

Item 1 (on the meaning of differentiation)

1.1 Define using your understanding of the meaning of differentiation----------------------
1.2 The power rule is defined as follows, if $f(x) = ax^n$ then $f^1(x)$ =------------------------ --

Item 1 was intended to provide insight into whether or not learners had developed an action conception of the concept of differentiation. In item 1.1, learners were expected to state the meaning of the differentiation concept, and for item 1.2 learners were expected to define the power rule formula. As part of the schema of defining the meaning of differentiation, a learner should:

- be able to state that differentiation is a method of finding the derivative of a function; and
- be able to state that differentiation means the rate of change of one quantity with respect to another.

For 1.2, a learner should:

- be able to write the formula for the power rule, i.e.

$$f^1 = nax^{n-1}$$

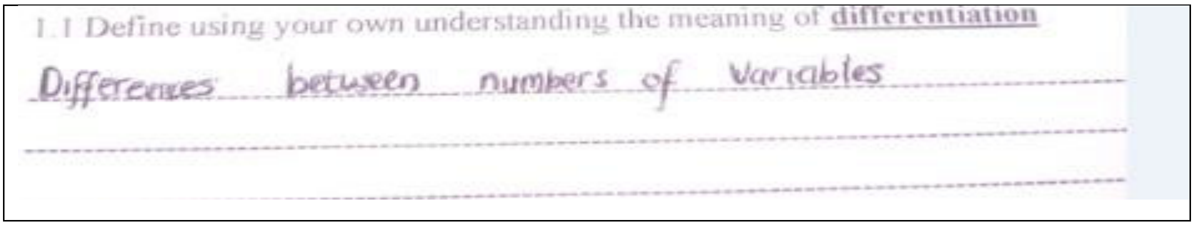
Table 6.2 illustrates the analysis of learners' mental constructions about the meaning of differentiation and the definition of the power rule.

Table 6.2: Categories of learners' written responses to item 1

Category	**1**	**2**	**3**
Indicator	No written response * No mental constructions	Displayed Action mental constructions. Followed a step-by-step method.	Displayed Process mental constructions
Number of learners in each category for item 1.1	6	3	1
Number of learners for item 1.2	5	3	2

As illustrated in Category 1 in Table 6.2 above, six learners for item 1.1 and five learners for 1.2 were categorised as unable to make adequate mental constructions. Three learners, for both items, were categorised as operating at the action stage, as shown in category 2. One learner for item 1.1 and two learners for item 1.2 operated at the process stage. This is evident as no learner seems to have constructed the object understanding of both concepts.

The discussion now focuses on each sub-item under item 1. For item 1.1, six learners in category 1 provided incorrect and disjointed explanations of differentiation, as they stated that the meaning of differentiation is not to be the same when you compare two or more items. Four of the learners, instead of defining differentiation, attempted to compare the word ‘differentiation’ to the word difference and also tried to make a link between numerals and variables by saying that differentiation is when you find the differences between numbers and variables. Nkosy was one of the six learners who articulated the incorrect definition, as shown in Figure 6. 1.


1.1 Define using your own understanding the meaning of **differentiation**

Differences between numbers of Variables

Figure 6.1: Nkosy’s written response to sub-item 1.1

For Category 1, all six learners failed to use a mathematical definition of differentiation; their written responses revealed that they lacked an understanding of the concept of differentiation. The lack of concept definition of the correct mathematical definition of differentiation hindered Nkosy from constructing the necessary mental construction. The excerpt below is extracted from an interview that took place between the researcher and Nkosy:

Extract 1: Nkosy’s interview response to item 1
Researcher: Can you explain your statement – it is not clear?
Nkosy: Here, ma’am, I was explaining the process of differentiation.
Researcher: Tell me exactly what are you saying here: what process?
Nkosy*:* Madam, you know, ukuthi [when] you are differentiating you forget about the number and focus on variables.

Researcher: Please explain further what you really mean when you are saying differences between numbers and variables.
Nkosy: I was supposed to explain the process, madam; now I realise that this is a mistake.
Researcher: Now that you have realised that you have made a mistake, can you continue to explain further?
Nkosy: As I have explained, madam, when you are doing differentiation, you differentiate the variable – which is x – not the number.

As can be noted in the above extract, Nkosy was trying to define the processes of finding the gradient of the curve, instead of defining the differentiation concept. In his written response, he talked about the number and the variable. During the interview session, when he was engaged with what he wrote he started to realise what should have been the focus of his response, as he alluded that, when we differentiate the function, the focus must be on the variables and not on the number. Maharaj (2014) and Ndlovu et al. (2017) argue that it is important to engage learners with what they have written in order to assist them to make the necessary mental constructions. Although in the written response it seems as if Nkosy has not even constructed the action stage, in the interview session he was able to use mathematical language to define the concept and make references to the procedures necessary for differentiation. That seemed to move him to the action stage.

A different situation arose with Gabbi who, instead of defining differentiation focused on the word “different” and used everyday terminology to extend her definition (see Figure 6.2 below).

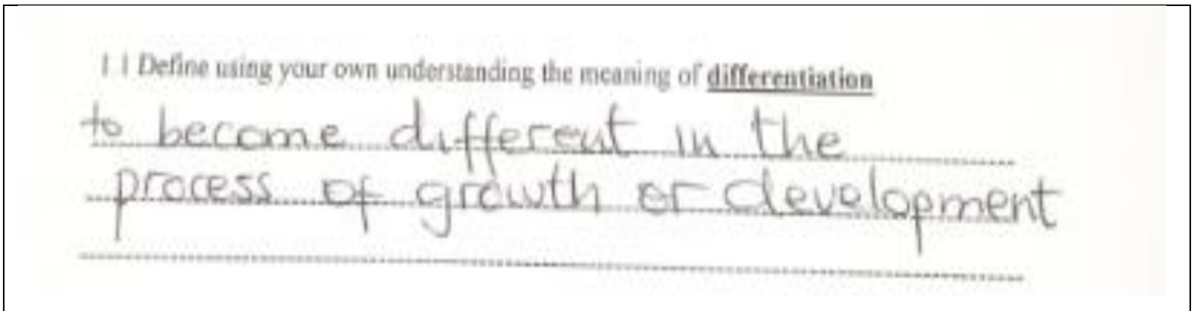


Figure 6.2: Gabbi’s written response to item 1.1

Gabbi’s concept definition for differentiation is incorrect. She defined the term as if she was referring to the growth of a person. The answer she provided was not mathematically relevant; this was an indication that she had a misunderstanding of the concept. The misunderstanding was caused by the word ‘different’ which was found in the word ‘differentiation’. During the interview, the following transpired:

Extract 2: Gabbi’s interview response to item 1.1

Researcher: Can you please explain your answer on the meaning of differentiation?
Gabbi: Madam, this is difficult but *ngizochaza ngendlela yami* [I will explain using my understanding]. *Kodwa ngicabanga ukwehluka kwezi* number. [But I think this is about the differences in numbers].
Researcher: In your answer, you mentioned the word ‘process’. Can you explain further?
Gabbi: Madam, I think this is a process of growth of numbers in a graph.
Researcher: This is not clear. You said in your answer this is a process of growth of numbers. Which are those numbers?
Gabbi: Numbers in the graph are not the same, *uma ikhula noma yehla madam* [when the graph increases or decreases, madam].
Researcher: Now you are mentioning the numbers in the graph. Which are those numbers?
Gabbi: Eyi ma’am, I am now confused. My statement is incorrect. *Angisazi ngithini, mina* [I don’t know what to say]. I can see my answer is not right.
Researcher: Have you used the dictionary to get the meaning of differentiation?
Gabbi: In a way, madam.

Gabbi seemed quite confused about the definition of differentiation; her response revealed that she did not know the concept of differentiation. This was evident in her answer during the interview, when she said “I don’t know what to say, ma’am”, when asked about numbers in a graph. Her responses during the interview, even after being probed, confirmed that she had not constructed the meaning of the differentiation concept. Her inability to provide the correct definition indicates that learners, like Gabbi, in category 1 have not made the necessary mental constructions, as stated on the preliminary genetic decomposition. This demonstrates that the knowledge necessary for defining differentiation was lacking, as most of them used the English language exclusively to explain the concept.

Learners at the action stage are expected to use correct mathematical terminology when defining differentiation or stating the procedures of differentiation. As shown in the analysis, three learners used correct mathematical language when providing their definitions, indicating that they were operated at the action stage of APOS.

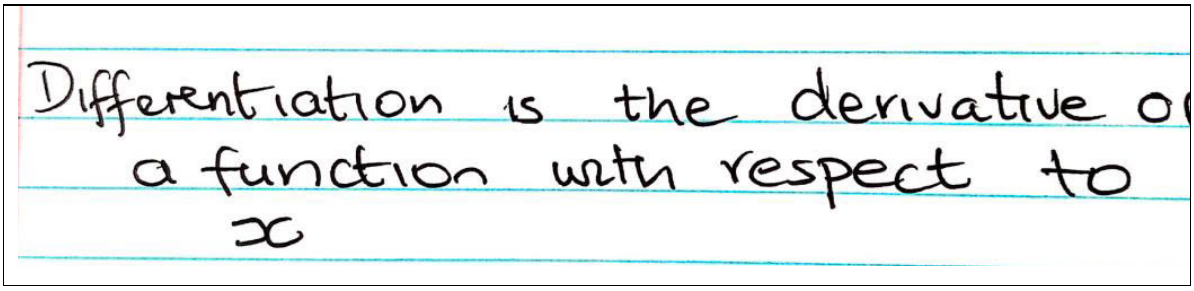


Figure 6.3: Lukho's response to item 1.1

Lukho operated at an action stage as she used the term 'derivative of a function', and it seemed as if she had an understanding of the concept. Her response indicated that she has constructed the necessary mental constructions regarding the meaning of differentiation, as indicated in the preliminary genetic decomposition.

Although the learners in Category 2 attempted to answer the question, the analysis showed that they had interpreted the concept using different descriptions of differentiation. Some used mathematical terminology and others used notational symbols, as they stated that differentiation is an average gradient and it is written as $\frac{\Delta y}{\Delta x}$ or $\frac{y^2-y^1}{x^2-x^1}$. While their answers were correct, it showed that they had a procedural understanding of what differentiation is.

John was the only learner who was categorised at the process stage in terms of having a conception of an average gradient. He correctly stated that differentiation is a process of finding the average gradient. He must have done this several times and had interiorised this into a process as the action happened in his mind. His response indicated that he had constructed the necessary mental constructions regarding the meaning of an average gradient, as indicated in the preliminary genetic decomposition. During the interview with John, this is what transpired:

Extract 3 John's interview response to item 1.1
Researcher: Can you clarify your response?
John: This is the process used to find the gradient.
Researcher: Ok. Can you clarify your answer? What is the gradient?

John: The gradient of a line passing through the curve.

Researcher: What is the relationship between the line and the curve?

John: The gradient of the line is the same as the gradient of a curve where they meet. *Kodwa eye curve ibizwa ngokuthi average gradient noma I derivative.* [But the gradient of a curve is called an average gradient or the derivative].

As can be noted in Extract 3, above, John attempted to explain the meaning of an average gradient as his response displayed some level of understanding of an average gradient. As stated in genetic decomposition, a learner at the process stage is able to make connections between an average gradient of a curve and the gradient of a straight line, and this is what the learners in category 2 did. When action has been interiorised into a process, a learner can give answers in more than one way and verify them using another way.

Item 1.2 was designed to explore learners' understanding of the power rule. Learners were expected to show their conceptual understanding of the rule. The analysis showed that while learners knew the procedures involved in using the power rule, some experienced some difficulties defining it, which was an indication that the rule had been memorised without constructing the meaning. The majority of the learners failed to manipulate the symbols without using numbers. Five learners in Category 1 failed to illustrate the power rule. While attempting to carry out the procedures, they could not execute them correctly. Some, instead of subtracting 1 in the exponent, added it. Some learners failed to multiply the coefficient by the exponent, as the rule stated that if $y = x^n$ then $\frac{dy}{dx} = nx^{n-1}$. Ntando used the addition sign instead of using subtraction on the exponent, as shown in Figure 6.4.

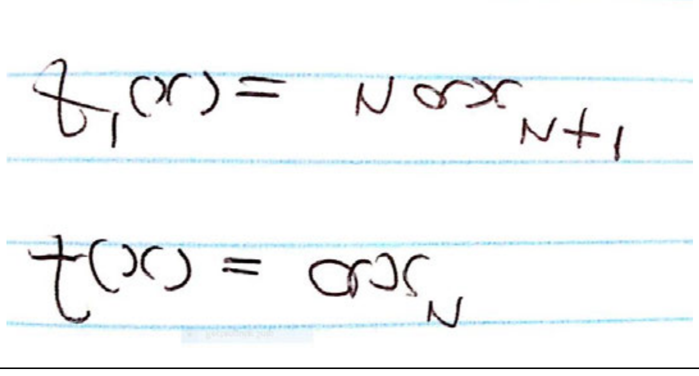


Figure 6.4: Ntando's written responses to item 1.2

What was evident is that Ntando had confused the differentiation rules with those of integration. Similar findings were evident in a study conducted by Howe et al. (2017) with 70 Computer Science diploma students during their advanced calculus examinations. The study revealed that the students made procedural errors as a result of confusing the differentiation and integration processes. Such cognitive conflict may occur when rules are memorized without a deep understanding of the concept. To gain further insight into Ntando's understanding of power rules, an interview was conducted.

Extract 4: Ntando's interview response to item 1.2

Ntando: Power rule, madam: *indlela emfishane yokuthola iderivative.* [It is the short method of finding the derivative.]

Researcher: What is the meaning of the derivative?

Ntando: Mem, usuke uthola igradient yecurve, ebizwa nge average gradient *(Ma'am, here we are finding the gradient of the curve, which is called an average gradient).*

Researcher: I can see, in your exponent, you have added on. Can you further explain what happened?

Ntando: Ma'am, first of all, uthatha iexponent uyibeke ngaphambili bese uthi add one on it,

(You take the exponent and put it in front and add one on it afterwards)

Researcher: Are you sure of the process of differentiation?

Ntando: Ma'am, these two methods are confusing but…. ungathi ngenze iphutha kumele ngithi subtract one *(It seemed as if I have made a mistake, I was supposed to subtract one)*

Ntando displayed some degree of confusion in the explanation of her action. She indicated that, in her mind, it was clear that something must be done with the exponent. This was evident in her response as she indicated during the interview that these methods are confusing. Misconceptions were displayed in her written response as she added '1' instead of subtracting it and this showed a lack of knowledge of construction for the power rule. This was a problem for all learners in Category 1, as they kept on adding the exponent, showing confusion between the procedures of differentiation and integration. The other common error that was prevalent in their responses related to multiplication and division by the exponents. This is in line with the findings of Ndlazi (2015), who found that her students used both integration and differentiation terms interchangeably.

In a different scenario, three learners displayed an action conception of the concept of differentiation, because they followed step-by-step procedures in solving the problem.

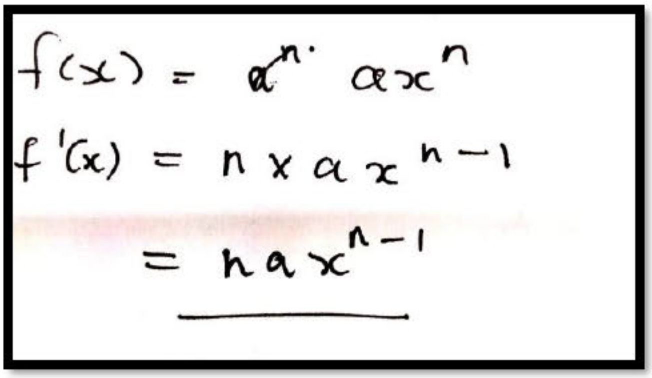


Figure 6.5: Nozi's written response in item 1.2

It was noted that Nozi applied the rules of differentiation correctly, where 'n' multiplied the coefficient at the same time that '1' was subtracted from the exponent. All three learners presented correct responses, which was an indication that they understood the concept very well. Jojo et al. (2013) state that students operating at the action stage view a mathematical procedure as a series of individual steps. On the other hand, Sethu was classified as operated at the process stage of APOS, and it seemed as if the action took place internally in her mind.

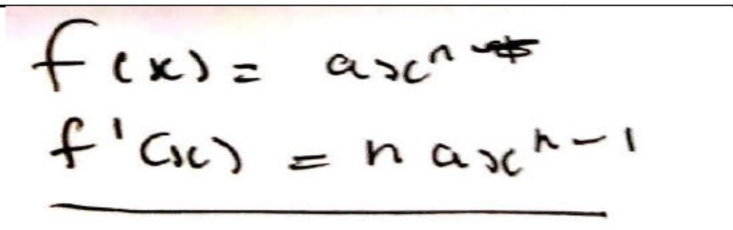


Figure 6.6: Sethu's written response to item 1.2

Sethu's response indicated that she knew the rules of differentiation, and a conceptual understanding of the rule was evident even though the correct steps were not followed. This is in line with Dubinsky and MacDonald (2001, cited in Ngcobo et al., 2021, p 68) that an individual who has a process conception "can think of performing a process without actually doing it". It seemed as if Sethu had interiorised an action into the process. The focus of this study is primarily on the mental constructions and difficulties that hinder learners in making the necessary mental constructions advocated in genetic decomposition. With both items (i.e., 1.1 and 1.2), it was evident that none of the six learners operated at an object stage. Errors made by learners were characterised by confusion between the procedures used for differentiation and integration, and an inability to provide a proper definition of differentiation.

6.3.2 Mental constructions depicted for item 1

Interview questions were asked based on the categories discovered in learners' responses to activity sheets, in order to elicit learners' conceptual understanding of differentiation. Symbols and notations play a vital part in the development of action conception in calculus. In item 1, the analysis showed that most of the learners were operating at the action stage, with some below the action stage. Only a few demonstrated process conception. While APOS theory does not include a stage for learners operating below the action stage, scholars such as Steward et al. (2008) and Vilakazi (2021) have introduced a pre-action stage. In this study, the learners who did not possess action conception could be categorised as operating at the pre-action stage. For the purpose of this study, these learners were allocated to Category 1.

6.3.3 Errors hindering the construction of the concept of differentiation

Item 1 consisted of sub-items that needed learners to give the real meaning of differentiation. It was found that the learners had difficulty giving an accurate meaning for the concept of differentiation. From the analysis of written responses, a number of errors were identified which hindered the construction of coherent schema using the categories of mistakes purported by Olivier (1989, as cited in Ngcobo et al., 2021). The learners' mistakes were categorised as errors, indicating underlying misconceptions. Four learners stated that differentiation is about the difference between numbers and variables. This was an indication that these learners were confused between the concept of differentiation and the word 'difference'. This was possibly due to the fact that the concept is new to learners since calculus is only introduced in Grade 12. Learners demonstrated limited knowledge of calculus concepts.

In the interview sessions, learners continued to demonstrate confusion about the meaning of differentiation, as some stated that the meaning of differentiation is to become different in the process of growth and development. In many instances, learners used the everyday understanding of being different, which showed a poor understanding of differentiation as a calculus concept. In the same manner, learners demonstrated limited knowledge of the power rule. They also confused the procedures of differentiation with those of integration. These errors are summarised below.

6.3.4 Summary of errors that hindered the construction of schema in item 1

- Learners stated that differentiation is about the difference in numbers
- Learners saw differentiation as becoming different in the process of growth and development
- Learners demonstrated confusion between procedures of differentiation and those of integration, using both procedures interchangeably in one sum.

Based on the analysis, the learners' lack of a conceptual understanding of differentiation and power rule concepts revealed a lack of procedural fluency, which led to learners failing to construct the necessary constructions of the concept. These errors hindered learners' mental constructions of the defining terms related to differentiation. Ndlovu (2014) argues that language plays a vital role in the construction of understanding of mathematical concepts thus the incorrect use of mathematical language presented a barrier to learners' understanding. Secondly, incorrect concept definition led to the construction of a distorted concept image of concept differentiation. Thirdly 'met after' hindered the procedural knowledge of the already taught concept. The overarching barrier was their lack of construction of meaningful understandings of the concept. Ndlovu (2014) states that, when concepts are learned as isolated facts, they become a barrier to learning. This was evident among the participants in this study. The findings of this study also suggest that learners need more clarification from teachers and there is a need to interrogate what they write and say pertaining to the concept in question. According to Ndlazi (2015), this could help them to recognize and refine their mental structures and schema.

6.4 Application of Differentiation Rules

Item 2 explored learners' understanding and application of the power rule in differentiation. Differentiation involves working with equations involving numerals, variables, and exponents, therefore concepts met before such as exponential and surd laws are important to the development of an understanding of differentiation rules. To apply the rules of differentiation, learners needed a strong background in basic algebraic operations such as factorisation, converting surds to exponential forms, and simplification of algebraic fractions.

6.4.1 Analysis of learners' responses to item 2

As illustrated in the preliminary genetic decomposition presented in Chapter 3, providing a step-by-step solution corresponds to the action stage; however, the ability to perform the step mentally and present the correct solution is an indication that the action has been interiorised into a process. When actions and processes can be combined to transform an entity, the process has become encapsulated into an object. For this item, no participant displayed an object and schema mental construction. The aim was to gain insight into whether learners had developed a schema conception of the differentiation concept. As part of the schema of the power rule, a learner should:

- be able to find the derivative using the power rule;
- follow a step-by-step procedure at the action stage; and
- give the correct answer without following all the steps at the process stage.

Item 2 (on the use of power rule)

Differentiate the following using the power rule
2.1 $f(x) = 3x^2$ --

2.2 $f(x) = 2x^2 + 4x - 3$ --

Item 2 intended to provide insight into whether or not learners had developed an action and process conception of the power rule in differentiation. Learners were expected to differentiate the expression with one and more terms using the power rule.

Table 6.3: Learners' written responses on item 2: On the use of the power rule

Category	1	2	3
Indicator	No written response * No mental constructions	Displayed an action mental construction followed step-by-step procedures some with few errors	Displayed a process mental construction No steps followed only the answer given, some with solutions with errors
No. of learners in each category on item 2.1	5	4	1
No. of learners on item 2.2	4	4	2

As illustrated in

Table 6.3, all of the learners attempted both sub-items. Five learners for item 2.1 and four learners for item 2.2 displayed no mental constructions, as their responses presented different types of errors. Selected responses with errors from Category 1 were analysed and follow-up interviews were conducted to identify learners' stage of understanding and the difficulties that hindered them in making necessary mental constructions. Four learners, for both items, fell into Category 2 since they demonstrated the action stage of understanding as they followed step-by-step procedures. One learner in item 2.1, and two learners in item 2.2, were placed in Category 2 as they had interiorised action into a process stage. A process is perceived as internal, in the sense that it has meaningful connections to an individual's other mathematical knowledge. Arnon et al. (2014) state that meaningful connections give the individual the capability to perform conversions among different representations; they also allow an individual to justify the process.

One learner, Zibu, follow step-by-step procedures when using the power rule on sub-item 2.1, as suggested in the preliminary genetic decomposition, revealing that they were operated at the action stage.

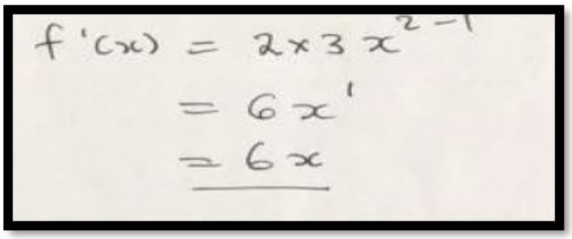


Figure 6.7: Zibu's response to item 2.1

Zibu's response displayed that she fell into Category 2 as she followed the procedures correctly, subtracted 1 from the exponent and managed to multiply the coefficient by the exponent.

The study also focused on those learners who had some difficulties applying the power rules when solving problems on differentiation. Three learners demonstrated misconceptions when applying the power rule to find the derivative. These learners with incorrect responses indicated that they had not made the necessary mental construction, as expected in the preliminary genetic decomposition. The researcher interviewed two learners who had difficulty applying the rules of differentiation.

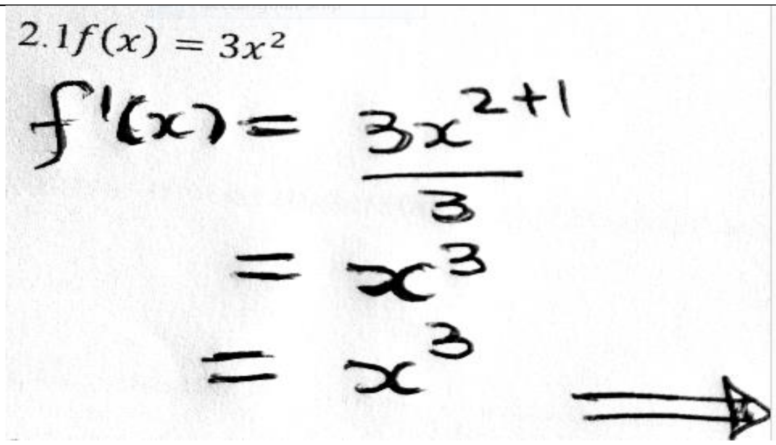


Figure 6.8: Ntando's written response to 2.1

Ntando's response revealed that she was confused by the procedures of differentiation with that of integration. She added a 1 to the exponent and also divided by the exponents. During the interview, this is what transpired.

Extract 5: Ntando's interview response to item 2.1

Researcher: In item 2.1, can you explain what you were trying to do?

Ntando: Ma'am, I was finding the derivative, kumele sibheke I exponent ka x (we *must focus on the exponent of* x*)* we must differentiate with the respect to x, asiyithinti inumber engaphambili (we *don't touch the number in front).*

Researcher: Yes, I can see, but can you explain what you have done to the exponent?

Ntando: Mina-ke ngibheke u x^2 ngayiyeka inumber ewu 3 (What *I did was to focus on* x^2 *not the number 3).*

Researcher: Please explain clearly, what have you done to x^2 ?

Ntando: Mina ngihlanganise u 2 and 1 kwaphuma u 3 (I *have added 1 to 2 and the answer I got was 3). Ngase* ngi divider ngaye futhi lowo 3 (*Then I have divided by 3*).

Researcher: Think carefully. Remember that this is differentiation. What is the correct procedure when you are differentiating?

Ntando: Madam, angisakhumbuli kahle umehluko phakathi (I *don't remember the difference between)* differentiation and integration.

Researcher: Think carefully, when to add 1 and when to subtract one.

Ntando: Madam, I have made a mistake here. I was supposed to subtract 1 bese ngithi (*then*) multiply by exponent. Ngeze iphutha ngadivider nge exponent (I *made a mistake by dividing with an exponent).*

As can be seen in the extract above, Ntando seemed to be confused about the procedures of differentiation and integration. She acknowledged this by confirming that she didn't remember which method to be used when finding the derivative. She lacked the conceptual understanding of the power rule concept as she failed to apply the rule correctly. As the researcher asked her about the addition and subtraction of 1 in the exponent, Ntando stated that the correct procedure was to subtract 1 and, at the same time, one needs to multiply by the exponent. Most learners tended to confuse the procedures of these two concepts, but when engaged during interviews, they seemed to understand the processes or the methods of differentiation better.

Extract 6: Ntando's interview response to item 2.1 (continuation)

Researcher: Do you understand, now, how to differentiate?

Ntando: Mina bengicabanga ukuthi ziyafana lezizindlela *(I think these methods are similar).*

Researcher: Do you see the difference now, between differentiation and integration?

Ntando: Yes, madam, but kuyadida (but *it confuses*) especially ukuthi kumele kwenzekeni ku exponent (*what needs to be done to the exponent).*

Researcher: Since you understand now, what can you say as your answer?

Ntando: I think the correct answer is 6.

Researcher: Yes, you are correct now.

It was clear that, in the second part of the interview, Ntando knew the process of differentiation but got confused along the way by integration methods. The lack of understanding was revealed when she was asked the difference between these two concepts, which were differentiation and integration. The interview confirmed that her process conception of differentiation had not yet developed. This was observed when she stated that she was confused by these procedures since they are similar. The researcher observed that learners in Category 1 were not certain about which method to be used on these two concepts (differentiation and integration). This was an indication that they had not constructed the necessary mental constructions needed to develop the understanding and application of the power rule. These learners experienced difficulties in adding or subtracting 1 in the exponent. The extract from Ntando reveals the gaps in the knowledge she has constructed. She knew that this was about doing something to the exponent but was confused by which operation to use. She stated that the focus must be on x as a variable, but failed to link it with the context of the question. This meant that she understood what was needed by the question, but her responses in the interview revealed that the concept had not fully developed.

Figure 6.9 presents the written response of John, who displayed inconsistency in using the power rule. He managed to multiply the coefficient by the exponent but made the mistake of adding 1 to the exponent. As a result, the answer was incorrect,

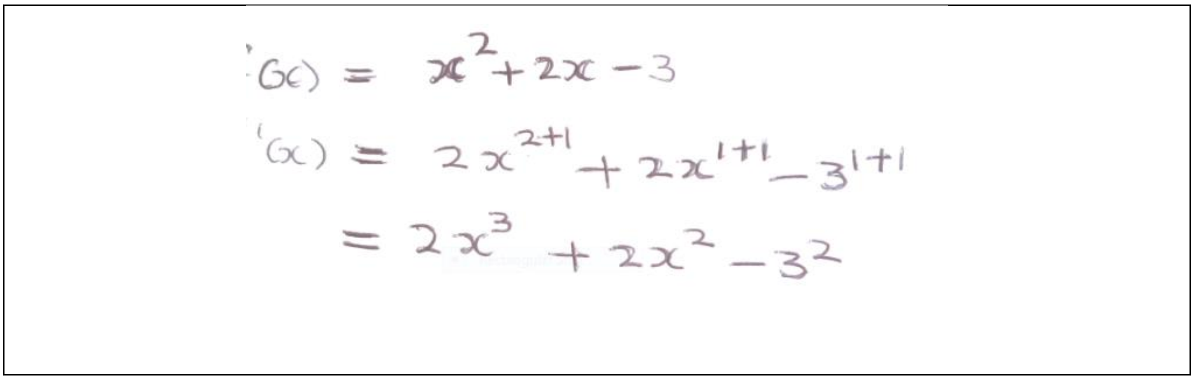


Figure 6.9: John's written response to 2.2

As evident in the above response, John displayed a lack of knowledge construction of the rules of differentiation as he added 1 to the exponents instead of subtracting it. Secondly, he differentiated a constant term that is not permissible in calculus. The researcher also noticed that John confused the procedures as he multiplied the coefficient by the exponent correctly but, with the exponent, the procedure was incorrectly done. He thus displayed no mental constructions. John's response revealed that he knew the rules but he experienced difficulties in using them properly. Six learners attempted the problem and gave incorrect responses that indicated that they lacked knowledge, as expected in the preliminary genetic decomposition. For item 2.2, two learners in Category 3 had interiorised the concept of differentiation into a process. Their solutions were: $f(x) = 2x^2 + 4x - 3$ and $f^1(x) = 4x + 4$. While they missed some steps, they were able to determine the correct answer and showed that they had interiorised an action to process.

The researcher noted that most of the learners used 1 incorrectly in the exponent: they added 1, instead of subtracting it. Their errors ranged from adding 1 to the exponent instead of subtracting it, differentiating the constant term (which was 3 in this item) and forgetting that the derivative of the constant is 0, and not using the correct notation, which was $f^1\ (x)$ for this item. The researcher interviewed John about his written response to item 2.2. The following is an excerpt:

Extract 7: John's interview response to item 2.2

Researcher: Can you explain your solution to me? What happened with your exponents?

John: Ma'am, ngiyadifferentiator la ngibuka u (x) ayi inumber *(Ma'am I am differentiating here, and I am focusing on* $x, not\ the\ number$*).*

Researcher: I can see you have added 1 through in all the terms on the exponents. Why?

John: Phela ma'am, kumele sikwenze lokhu kwi *(Ma'am, this must be done)* term by term.

Researcher: Can you explain what you have done in your answer? Please clarify.

John: Ngi adder u one (1) kuwo wonke ama exponents ka x **(**I add one into all exponents of x).

Researcher: Please explain to me further because the number 3 has no $x - value$, but you have added 1 on it, why?

John: Mina ma'am ngibheka ama exponents bese ngi adder u (1) kuphela *(I focused on exponents only and I just add one).*

Researcher: Remember, you have said earlier you just focus on x only, but what happened now?

John: What do you mean, ma'am?

Researcher: In your response, I can see that you have added 1 on a constant term, which is 3. Can you explain why?

John: Oh no, I think I have made a mistake here: this must be zero.

Researcher: How come?

John: Not sure, ma'am, ngicabanga ukuthi nakhona kumele ngithi *(I think even there I must)* add one (1). Kwi term ngayinye *(in all the terms).*

Researcher: There is 2 in front of the first term now, what happened there?

John: I don't know what happened there.

Researcher: Are you sure of your answer, do you add 1 or subtract it?

John: Sengiyadideka men, angisazi kahle, sengicabanga I integration manje *(I am confused now, I am thinking of integration now).*

Researcher: Thinking of integration now, why?

John: Sengididekile manje (*Now I am confused)* ma'am.

Researcher: What is confusing you now?

John: Mhmm ma'am lezi zindlela ziyadida *(These methods are confusing).*

Researcher: Please tell me where your confusion is.

John: Angazi ngenze njani *(I don't know what to do)* but I think I need to subtract one.

John's response during interviews revealed a lack of understanding of basic techniques for differentiation. In his response, it seemed as if he lacked the procedural understanding of differentiation and this created a barrier to constructing the necessary knowledge and understanding of the concept. When probed by the researcher, his explanation revealed that he had some deep confusion about the methods of differentiation. His response revealed many misconceptions about adding 1 to the exponent. When asked to provide clarity on a constant term, he displayed a lack of understanding of the rules of differentiation. Another misconception revealed in John's response was the lack of understanding of the co-efficient x. He knew that you need to multiply the coefficient by the exponent first before subtracting one from the exponent. But John failed to recognise that, with differentiation, 1 is being subtracted from the exponent; instead, he introduced an error by adding 1 instead. When asked about 2 in front of the first term, he failed to explain what happened. His responses revealed that he had not made the necessary mental constructions. Most of the learners were not sure which method to use when finding the derivative. Dubinsky (2002) states that, in many cases, the difficulty does not depend on the nature of the formal expressions but rather on the loss of connections between the expressions and the situational instructions. Jojo (2011) confirms that students perceive differentiation as a separate entity and even the rules applied cannot be remembered correctly.

Only one learner (Sandy) managed to differentiate the expression without following the step-by-step method correctly; this was an indication that the learner had made a necessary mental construction of the differentiation concept. He managed to apply the basic algebraic rules on both items.

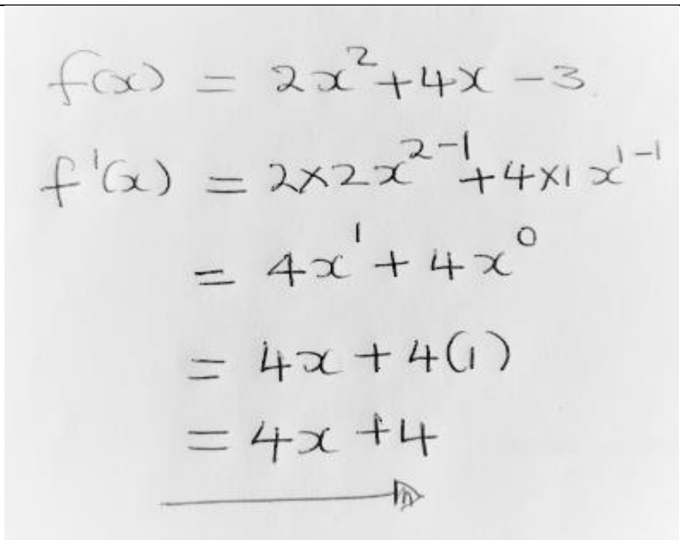


Figure 6.10: Sandy's written response to item 2.2

Sandy's response revealed that he made some mental constructions, as expected by the preliminary genetic decomposition. This can be observed in his first step when using a power rule correctly, where he displayed a coherent understanding of how the rule is used. Sandy managed to subtract 1 from the exponent and at the same time was able to multiply the coefficient by the exponent. He displayed a coherent understanding of the relationship of the power rule to the exponential laws, which was an indication that he could relate to the previous knowledge of finding the exponents. Sandy's response indicated that the action conception of differentiation concept had been developed as he followed a step-by-step procedure, operating at the action stage.

6.4.2 Errors and misconceptions for items 2.1 and 2.2

For these items, learners' written responses displayed a poor level of understanding of the rules of differentiation. In many instances, the learners failed to apply the rules of differentiation correctly, as they continually confused differentiation procedures with those of integration. Common errors occurred when learners added 1 to the exponent instead of subtracting it. Secondly, learners seemed to forget to multiply the coefficient of x by the exponent. The researcher observed four learners' responses which were full of computational errors between steps; one example was a learner who added 1 to 2 and produced an answer of 2, again. It appeared that the learner was using multiplication rules when doing addition. The learner made another mistake when multiplying $(3)(3)$, and got an answer as 6. These mistakes were not expected from Grade 12 learners. This shows that learners were being affected by the ‘met before’ as Tall (2008) argues that ‘met befores’ always causes problems as they have a tendency of impending generalisations at the end and can cause confusion.

This was evident in the two examples given above, as learners confused addition with multiplication. In an expression with more than one term, two learners failed to differentiate a constant term, they subtracted 1 from the exponent of the constant term, which is not the correct procedure. The rules of differentiation state that the derivative of the constant term is 0. Learners used differentiation and integration interchangeably; this revealed some gaps in their knowledge constructions, especially in basic algebra and in using the power rule. According to de Lima and Tall (2006), teaching and learning algebra has long been seen as a source of difficulty in the mathematics community.

6.4.3 Summary of errors that hindered the construction of schema for items 2.1 and 2.2

- Adding one on the exponent instead of subtracting it e.g., $2x^{1+1}$
- Dividing by the exponent instead of multiplying by it
- Differentiating the constant term

Based on the analysis, learners’ lack of a conceptual understanding of differentiation and power rule concepts and lack of procedural fluency resulted in them being unable to construct the necessary mental constructions.

6.4.4 Mental constructions depicted for items 2.1 and 2.2

If a learner was able to solve both items correctly they would, according to APOS theory, be demonstrating an action conception of differentiation using the power rule. A learner who used step-by-step procedures to arrive at the correct answer would be working at the action stage of APOS. The item targeted the process stage, as stated in the genetic decomposition in Chapter 3. Six learners, for both items, demonstrated the action stage of APOS since they followed step-by-step procedures to differentiate the function using the power rule. In this section, only three learners in Category 3 had interiorized the concept of differentiation at the process stage.

6.5 Learners' Understanding of Bracket Removal in Differentiation

Item 2.3 was designed to investigate whether the learner had developed a process conception of brackets removal when finding the derivative.

Item 2.3 Finding $\boldsymbol{f^1\ if\ f(x) = (2x+1)^2}$

2.3 Differentiate $f(x) = (2x+1)^2$ Using the rules of differentiation.

As shown in **Error! Not a valid bookmark self-reference.**, four learners displayed no mental constructions when solving item 2.3. Five learners displayed an action conception of the item since they followed step-by-step differentiation procedures.

Table 6.4: Allocation of scores for item 2.3: Finding the derivative $\boldsymbol{of\ (2x+1)^2}$

Category	**1**	**2**	**3**
Indicator	No written response * No mental construction	Displayed an action mental construction	Displayed Process mental constructions
No. of learners in each category	4	5	1

Item 2.3 seemed to be difficult for the learners, this was evident in the responses of the four learners in Category 1, who made different types of mistakes and failed to provide a completely correct answer. The learners seemed to apply the rules of differentiation without understanding. This confirms the findings of Ndlovu (2019), who reports that, in many instances, students applied algorithms without conceptualising the concepts involved. For item 2.3, the learners failed to use the distributive law to remove brackets, which was an indication that they lacked an adequate foundation in basic algebra. For this item, it was imperative to see that they had interiorised the action into a process by being able to use the rules of differentiation properly. Some learners gave wrong answers, and their mistakes ranged from failing to remove brackets properly using the distributive law to differentiating the function without bracket removal. Learners applied the rules of differentiation without understanding, as was found by Siyepu (2013), who reports that students applied the

derivative rules without understanding the meaning of the notation used. The difficulties displayed by learners indicated that they had not constructed the meaning of the rules of differentiation and terminology used when learning the concept.

Gabbi was one of the learners in Category 1 who tried to do the differentiation procedures but failed to follow the rule properly and did not remove the brackets. Gabbi's response for item 2.3 indicated a number of gaps in her understanding of the basic rules of differentiation, as she differentiated the function without removing brackets. (See Figure 6.11 below.)

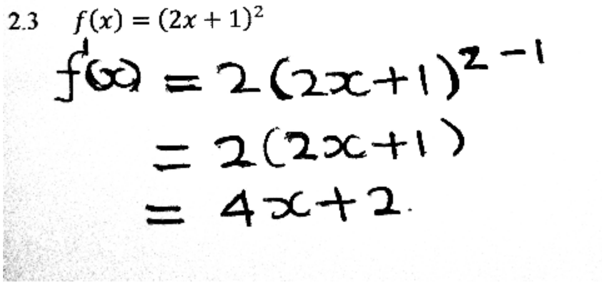


Figure 6.11: Gabbi's written response to item 2.3

Gabbi's response revealed that she did not understand the rules of differentiation where there are brackets. This error was common across most of the learners' responses in Category 2; this is what Siyepu (2013) refers to as conceptual errors which arise due to "students' failure to grasp the concept involved or to appreciate the relationship involved in the problem" (p. 458). Gabbi failed to remove the brackets when there was a 2 as an exponent and continued to place 2 in front of the brackets. She was able to use the procedure of differentiation, where she subtracted 1 from the exponent and, at the same time, she managed to multiply by the exponent. When asked to explain her response during the interview, she stated the following:

Extract 8: Gabbi's Interview response to item 2.3

Researcher: What is your understanding of differentiation?
Gabbi: Mina ngicabanga izinto ezimbili *(I think of two things*) if you subtract one from exponent kumele futhi uthi (*and we must also*) multiply by the exponent in front. Lokho kumele ukwenze kanyekanye (*This must be done at the same time*).
Researcher: Now, there are brackets in this problem. Is there anything you can do here?
Gabbi: Yes, Madam. Basically, in mathematics we remove brackets first, kodwa la Thisha (but *here Teacher)* no, I don't think so.

Researcher: What is your understanding about Bodmas rule? Does the rule apply in differentiation? **Gabbi:** Mina ngibona singekho isidingo (I *see no need*) since my focus is on exponents, I need to subtract 1 from the exponents, and also multiply by the exponent in front. **Researcher:** What if the sum has exponents inside the brackets: how are you going to differentiate? **Gabbi:** I think ngeke ngenze lutho ngaphakathi *(I think I would do nothing inside the brackets);* there is no need to differentiate ngaphakathi *(inside).*

From the interview, it can be observed that Gabbi had a poor understanding of the importance of bracket removal, which is the golden rule in differentiation and in mathematics as a whole. Her focus was not on the removal of brackets but on the process of differentiation. As the researcher probed her understanding of the Bodmas rule, she stated that there is no need to remove brackets when you are differentiating. This revealed a lack of understanding of basic algebra, which can be traced back to Grade 8 mathematics, where there is heavy emphasis on brackets removal. Another misconception revealed by Gabbi's response was the differentiation of the terms inside brackets; this was based on her response when she stated that there is no need to differentiate terms inside brackets. In most cases, her explanation focused on explaining the procedures done on the exponents which are outside the brackets. Her response on the activity sheet indicated that her action conception for brackets removal before differentiation was not fully developed.

6.6 Learners' Understanding of Square Root Sign Removal

Item 2.4 intended to provide insight into whether or not learners had developed the process conception of the square root removal before differentiation. In this item, learners were expected to apply the basic techniques of square root removal before they found the derivative.

Item 2.4: Finding the derivative of the square root sign

2.4 Differentiate the following $$f(x) = \sqrt{x^6}$$

.

Table **6.5** provides the allocation of scores for item 2.4.

Table 6.5: Allocation of scores for item 2.4: Finding the derivative of $\sqrt{x^6}$

Category	**1**	**2**	**3**
Indicator	No written response *No mental constructions	Displayed an action mental construction. Followed step-by-step procedures with some computational errors	Displayed a process of mental constructions. Gave answers without following steps
No. of learners in each category	5	5	0

In this section, two out of five learners in Category 1 failed to attempt the question, and the remaining three made different types of errors. Some of these errors were computational errors that could be easily corrected. This is in line with Olivier (1989) who differentiated the types of errors that students have when solving mathematical problems as follows:

(1) Slips: wrong answers owing to processing; they are not systematic but are carelessly made by both experts and novices. They can be easily detected and quickly corrected.
(2) Errors: wrong answers owing to planning; they are systematic in that they are applied regularly in the same circumstances. Errors are the symptoms of the underlying conceptual structures that are the cause of errors.
(3) Misconceptions: underlying beliefs and principles in the cognitive structures that are the cause of systematic conceptual errors.

The learners knew that one needs to remove the square root sign before performing differentiation, but their problem seemed to be on the exponent side. When removing the square root sign, their focus was on the exponent: they confused the exponent; they often used 2 instead of $-$ 12; and they incorrectly removed the square root sign. These can be considered to be conceptual errors, in line with Siyepu (2013), who found that students committed conceptual errors owing to poor interpretation of errors such as $6x^{\frac{1}{2}} = \sqrt{6x}$. Learners seemed to have difficulty grasping the concept of differentiation when the function has a square root sign on it. This also suggests that educators or teachers need to put more emphasis on how to differentiate the sum from the square root sign.

The written responses of Lukho and Sethu revealed that the action conception of differentiating the sum with the square root sign has not been developed. Learners had some difficulty with this item; they could not apply the rules correctly and made different types of mistakes. The extracts below show examples of two learners in Category 1 who made different types of mistakes (see Extracts 5 and 6).

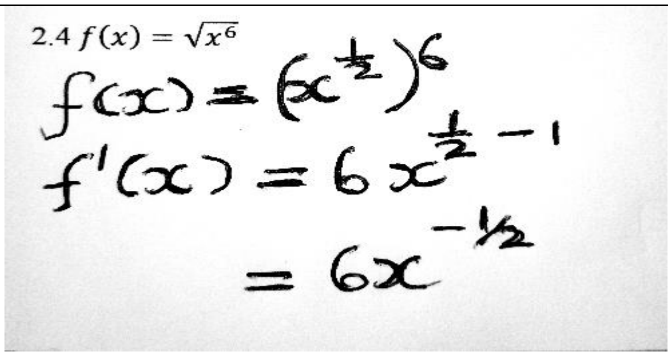


Figure 6.12: Lukho's written response for item 2.4

The written response of Lukho revealed that the action conception of differentiation of the square root concept was not yet developed. She did not apply the rule correctly; instead, she made a serious mistake by using $\frac{1}{2}$ incorrectly. She was supposed to put exponent 6 inside the brackets and put $\frac{1}{2}$ outside. This error can be considered a conceptual error. According to Siyepu (2013), arbitrary errors originate from conceptual errors as students change the nature of the problem owing to poor interpretation errors, such as $6x^{\frac{1}{2}} = \sqrt{6x}$. Learners seemed to be confused and had some difficulties in solving this problem. Lukho's response showed that she had a poor understanding of square root removal before differentiation.

This suggests that educators need to put more emphasis on finding the derivative of a function with the square root sign. These learners had some understanding that they were supposed to remove the square root sign before differentiation but failed to use the correct procedure. All ten learners' responses revealed that they experienced difficulties with the removal of the square root sign. Lukho's response revealed that she had not made the necessary mental constructions, as suggested in the preliminary genetic decomposition. Lukho was interviewed to probe her responses on the activity sheet. During interviews this is what transpired:

Extract 9: Lukho's interview response to item 2.4

Researcher: In your answer, I see $\frac{1}{2}$. Can you explain what happened there?

Lukho: Inhloso yami ukususa *(My main aim is to remove)* the square root sign.

Researcher: Tell me, how to remove the square root sign?

Lukho: Madam kumele ususe I square root bese sifaka u half *(You need to remove the square root sign and replace it with half).*

Researcher: Please clarify your statement, because I see brackets in your sum. Why putting $\frac{1}{2}$ inside brackets?

Lukho: Ungamfaka noma kuphi kuyefana *Yyou can put $\frac{1}{2}$ anywhere, it is still the same).*

Researcher: Where are the brackets now, in your second step?

Lukho: Ngiwasusile ngase ngiya differentiator *(I have removed them and then differentiated the function).*

Researcher: How? Now I see $\frac{1}{2}$ and 6, and $\frac{1}{2} - 1$ what happened there? Please explain.

Lukho: Oh, madam, ngenze iphutha la *(I have committed a big mistake here).*

Researcher: Where?

Lukho: Madam bekumele ngisuse amabrackects kuqala bese ngithola x^3 *(I was supposed to remove brackets first in order to get x^3).*

The aim of interviews is to correlate participants' verbal representation and written representation of mathematical statements (Maharaj, 2014). Maharaj (2014) cautions that students may not really engage with what they write. During the interviews, Lukho's mistakes or errors were found to have resulted from not engaging with what she had written. She was aware that, in order to remove a square root sign, you need to put $\frac{1}{2}$, but was not sure of the position. This suggested a weak understanding of the removal of a square root sign when performing differentiation. This showed that the learners could carry out the procedures without constructing the meaning of the concept and, as Maharaj (2014) points out, may not interrogate what they write. During the interview session, Lukho recognised that she had committed an error by mistakenly putting $\frac{1}{2}$ in the wrong position; as a result, the differentiation was incorrect. Her response revealed that the action conception of brackets removal when doing differentiation was not fully developed.

It appeared that the learners were not certain which exponent to use to remove the square root sign; this confirms that the learners found it difficult to understand the concept of square root removal when doing differentiation. Maharaj (2014) suggests that more time is needed or should be devoted to helping students develop the mental structures at the process and object stage and that teaching should focus on unpacking the structures given in symbolic form. The errors made by the learners for this item were the result of not interrogating the meaning of what they were writing. Maharaj (2014) states that students need to interrogate what they write and say in the context of a problem. In terms of APOS theory, the extract reveals that the learner did not have the necessary mental constructions, as expected in the preliminary genetic decomposition.

Sethu, another learner in Category 1, made a different mistake (see Figure 6.13 below).

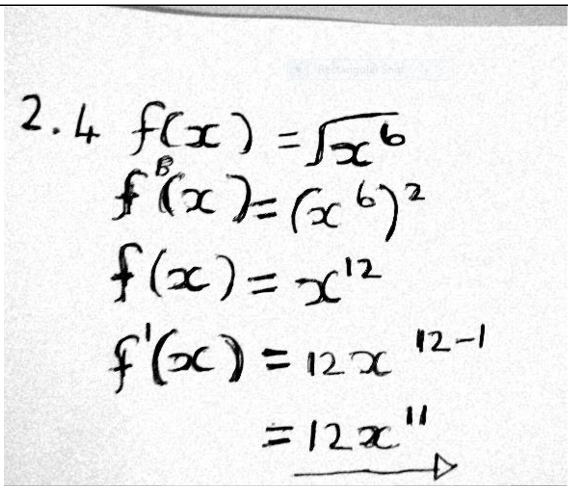


Figure 6.13: Sethu's written response for item 2.4

Sethu seemed to know the rules for removing a square root sign but provided responses with inaccuracies in their workings. As shown in the above figure, the learner committed the error of using 2 as an exponent to remove the square root sign instead of $\frac{1}{2}$. This was an indication that the learner knew the rules of differentiation but struggled with the exponent part of the solution. This might have been a careless mistake or an indication of a misconception. The solution was incorrect, and this revealed that the learner did not have a mental construction of the concept of differentiation when there is a square root sign. Most of the learners committed computational errors in their manipulations and could not provide a completely correct answer. Brijlall and Maharaj (2011) state that poor performance is a result of students not having a good grasp of the concepts that are expressed in symbolic form.

The issue of the removal of the square root sign indicated that learners lacked experience working with exponents. Their difficulties were revealed when they were working with fractions, as many failed even to subtract 1 from $\frac{1}{2}$. Tall (2008) states that students' previous knowledge plays a major role in constructing new knowledge. It was evident that these learners had a problem with basic algebraic procedures, especially those that involve brackets and fractions. The interviews provided a deeper understanding of the learners' thought processes and helped in identifying whether their misconceptions are persistent or not. Luneta and Makonye (2010) state that students' performance in calculus is undermined by weak basic algebraic skills of factorisation, handling operations and poor understanding of indices. They further state that algebraic incompetence has a direct impact on learning calculus.

Five learners in Category 2 showed that they knew the process of removing a square root sign before finding the derivative. They knew that when removing a square root sign, you need to replace it with $\frac{1}{2}$ to get the correct answer. Elliot was one of the learners who followed the correct procedure of removing a square root sign, as he managed to use $\frac{1}{2}$ properly (see Figure 6.14 below).

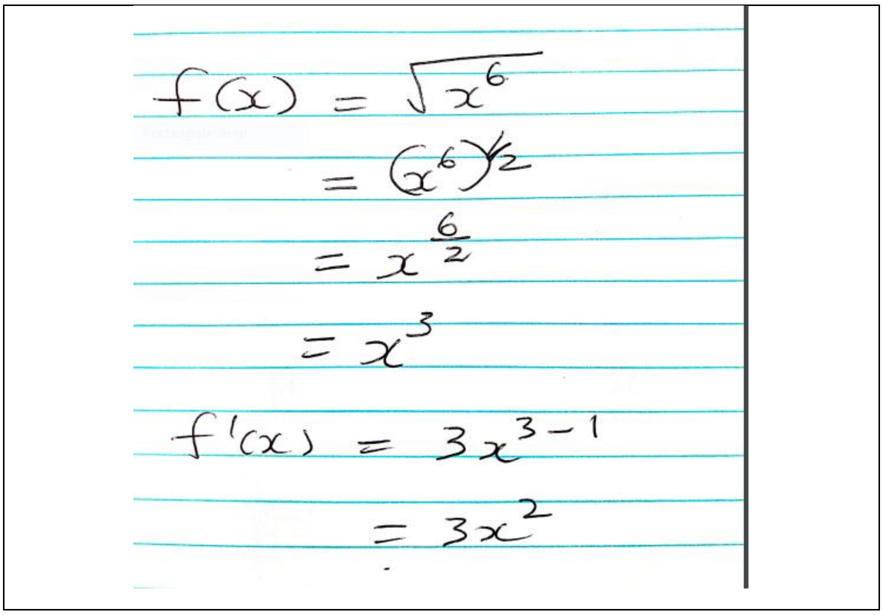


Figure 6.14: Elliot's written response to item 2.4

Elliot made the necessary mental constructions, as advocated in the preliminary genetic decomposition. His response indicated that he understood the procedure for removing the square root sign before differentiation. He displayed an understanding of the use of $\frac{1}{2}$ as an exponent when removing the square root sign. In terms of APOS, he had developed the action conception of differentiation, and followed a step-by-step procedure, as indicated in the genetic decomposition.

6.6.1 Mental constructions depicted for items 2.3 and 2.4

Learners struggled on both items as they had difficulty removing brackets and the square root sign before differentiation. In terms of APOS theory, five learners displayed an action conception for both items as they used step-by-step procedures to get the correct answer. These two items targeted the process conception as per the genetic decomposition, but none of the learners was able to interiorise action into process.

6.6.2 Errors hindering the construction of the necessary mental constructions for items 2.3 and 2.4

Learners displayed different kinds of errors as they tried to remove brackets and the square root sign before embarking on differentiation. The requirement for differentiation procedures is to remove brackets and square root signs before finding the derivative of a function. Learners lacked both conceptual and procedural understanding of the concept as many learners failed to use basic algebra concepts correctly – especially distributive and exponential laws. For both items 2.3 and 2.4, removal of brackets and the square root sign was considered to be the major stumbling block that hindered learners' mental construction of the concept of differentiation.

6.6.3 Summary of errors that hindered the construction of schema for items 2.3 and 2.4:

- Differentiating without removing brackets on item 2.3
- Failing to use the distributive law to remove brackets
- Used 2 instead of using $\frac{1}{2}$ in order to remove the square root sign in item 2.4

Based on the analysis, a lack of conceptual understanding of the integration concepts and a lack of procedural fluency led to learners failing to construct the necessary mental constructions. Moreover, the analysis revealed that learners lacked basic algebra competency, as the majority had difficulty with the removal of brackets and square roots. The difficulties that were displayed in learners' responses revealed their lack of knowledge construction which hindered them from making the necessary mental construction of differentiation.

6.7 Conclusion

In this chapter, the researcher presented the data analysis from the activity sheets and interviews for the differentiation section. Most of the responses revealed various types of errors. Learners struggled to use the rules of differentiations – the power rule; removal of brackets before differentiation; and removal of square roots – correctly. The analysis revealed a knowledge gap with regard to the rules of differentiation and basic algebra. Data analysis in this chapter revealed that process conception within algebra was a barrier for learners to interiorise action into process. The next chapter presents the analysis of learners' written responses and interviews for the integration section.

7 ANALYSIS OF WRITTEN RESPONSES AND INTERVIEWS ON INTEGRATION (PHASE 2)

7.1 Introduction

This chapter presents the analysis of learners' responses to activity sheets and transcripts from the interviews relating to integration, with an aim to identify the stages of their conceptual development and mental constructions and analyse their errors. The chapter explores learners' understanding of integral symbols and ability to apply integration techniques, as well as their understanding of the area under a curve and the definite integral.

7.2 Analysis Activity Sheets and Interviews on Integration

The activity sheet consisted of two sets of questions referred to in this thesis as items 3 and 4 (See Appendix A2). Item 3 comprises 4 sub-items (3.1; 3.2; 3.3 and 3.4) that focus on the definition of terms and finding the indefinite integral and definite integral using the basic rules of integration. Item 4 is made up of two sub-items (4.1 and 4.2) and focuses on finding the area included by the curve using integration methods. These items are similar to those used for the pilot study (Phase 1), discussed in Chapter 5; they were refined (Phase 2). In addition, different participants participated in Phase 2.

Learners’ responses to the activity sheet and extracts from interviews were presented, analysed, coded, and scored according to the following five-point rubric adapted from Jojo et al. (2013). These categories were modified to suit this study and all six items related to integration were coded (scored) as per the rubric in Table 7.1.

Table 7.1: Codes used to categorise learners written responses for integration

Score	Assessment Criteria	Description of mental construction
5	A complete response to all aspects of the item and indicating a complete mathematical understanding of the concept assessed	Made a schema mental construction as suggested in the genetic decomposition
4	Partially complete response with minor computational errors demonstrating understanding of the main idea of the problem	Displayed object conception of the application of integration
3	A complete response to all aspects of the concept without following a step-by-step method and incomplete reasoning	Displaying a process of mental constructions, and conceptual understanding at a minimal stage
2	Follow step-by-step procedures but no reasoning to justify written responses. Displayed errors that hindered them from interiorising action into a process stage	Displaying Action mental constructions but at a procedural stage
1	No written response or complete principle error	No mental constructions of the concept

7.3 Learners' Understanding of the Definition of Integral Symbols

Learners were expected to know and be able to provide the mathematical definition of the definite and indefinite integral. In the following section, the researcher discusses in detail the mental construction learners made in items 3.1 to 3.4 which focused on exploring their understanding of definite and indefinite integral. Beyond exploring their mental constructions, the difficulties that hindered learners in making the necessary mental constructions are also discussed. The discussion is presented per item.

7.3.1 Learners' definitions of the indefinite integral $\int f(x)dx$

Item 3.1 explored learners' understanding of the symbols used for integration. It was intended to provide insight into whether or not the learners had developed a process conception for the correct use of the integration symbols. Rosken and Rolka (2007) state that definitions play a role in students' learning, whereby intuition inherent in concept images dominates conceptual learning. The focus of this approach to defining and analysing learners' learning of definite and indefinite integral calculus follows Tall and Vinner's (1981) model of concept image and concept definition.

All ten learners attempted item 3.1; however, the majority of learners focused on explaining the process of integration instead of explaining the meaning of the symbol.
Item: 3.1 on the meaning of the symbol $\int f(x)dx$

3.1 What is the meaning of this symbol $\int f(x)dx$?

For item 3.1, learners were expected to state the meaning of the indefinite integral. The researcher used the preliminary genetic decomposition in Chapter 3 to explain learners' mental constructions. As part of the schema of defining the meaning of the indefinite integral, a learner should:

- be able to state that an indefinite integral is a function that takes the antiderivative of another function;
- be able to state that indefinite integral is the reverse function of differentiation function; and
- be able to state that indefinite integrals are expressed without limits and contain an arbitrary constant.

Table 7.2 illustrates the analysis of learners' mental constructions about the meaning of the symbol$\int\ f(x)dx$.

Table 7.2: Learners' responses to item 3.1 (the meaning of the symbol of $\int f(x)dx$)

Score	1	2	3	4	5
Indicator	Showed no mental constructions	Displayed the action conception of the definition	Displayed the process conception of the definition	Had the object conception of the definition	Displayed a schema conception. Made mental constructions proposed in the genetic decomposition
No. of learners	0	6	4	0	0

As per the table above, Category 1 refers to learners who either did not attempt to solve the problem or provided an incorrect response. As illustrated in Category 1 of Table 7.2, no learner was categorised as failing to make a mental construction. Several of the learners were categorised as operating at the action stage, as shown in Category 2, with four operating at the process stage. As shown in Table 7.2, no learners formed the coherent schema necessary to be able to provide a coherent definition of the integration symbols expected in item 3.1. Learners in Category 2 were categorised as operating at the action stage because they defined the integral as an anti-derivative. Although they did not use the correct mathematical term, saying that the function was differentiated before and the symbol means the process needs to be reversed suggested that they understood that it is an anti-derivative. Below is a sample of the responses provided by learners in Category 2.

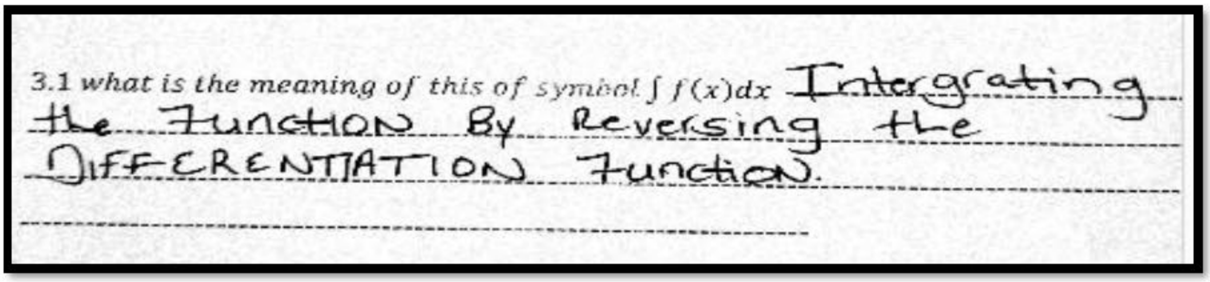

3.1 what is the meaning of this of symbol $\int f(x)dx$ Intergrating the Function By Reversing the DIFFERENTIATION Function.

Figure 7.1: Gabbi's response to item 3.1

Gabbi and the other five learners in Category 2 failed to use proper mathematical terminology; instead, they focused on the process of reversing the function. Their responses showed that they had not constructed a meaningful understanding of the notional symbol of integration.

These learners in the action stage lacked the conceptual understanding of the concept but their responses were partially correct. The stage of the learners' understanding hindered them from achieving a process stage understanding of the concept. Many of the learners at the action stage stated that $\int f dx(x)$ was differentiated before, and could be brought back by reversing the process. One of the learners explained by saying the function was differentiated before and now must be brought back to its original status, without explaining the process.

The learners struggled with the conceptualisation of integration as a limit. Orton (1983) found that students struggled with conceptualising integration as a limit of a sum or area. In this study, learners defined $\int f(x)dx$ as the inverse of differentiation, One learner stated that "integrating the function you need to reverse the differentiation function". It was observed that the learners had an idea of the meaning of indefinite, as they said it was the reversal of differentiation, but could not explain the real meaning of indefinite integrals. This was also observed by Habineza (2010), who found that the understanding displayed by his first-year mathematics student teachers did not include underlying concepts of integrals but demonstrated orientation towards the anti-derivative.

Tall (2008, as cited in Ndlovu, 2014) posits that to formulate a concept image one needs to have the correct concept definition; thus, the lack of a correct concept definition means the concept is not yet constructed in a learner's mind. What transpired with the definition of integral among the learners in this study categorised as operating at the action stage was also apparent when defining differentiation, as discussed in Chapter 6. They also failed to use mathematical terminology when defining concepts.

To explore further their insight into the definition of integration, interviews were conducted with some of the learners. The following is an excerpt from the interview that took place with Gabbi.

Extract 10: Interview with Gabbi about item 3.1

Researcher: What is the meaning of the symbol here?

Gabbi: Imele I integration ma'am *(stands for integration, ma'am).*

Researcher: What is integration? Can you explain further.

Gabbi: Integration isho ukubuyela emva kwidifferentiation *(Integration means you need to go back to differentiation.*

Researcher: What do you mean?

Gabbi: Kumele ubuyele emuva bese ureveser yonke into oyenzile (*you need to go back and reverse the whole process.)*

Researcher: How to go back? Tell me more about this.

Gabbi: Ma'am, kumele uthi *(you need to)* integrate $f(x)$ with respect to x.

Researcher: How? Please explain further: clarify your statement.

Gabbi: Ma'am, kumele siyi reverse yonke lento ibuyele ekuqaleni ingaka differentiatwa *(we need to reverse everything as it was before differentiated).*

Researcher: Yes, I get that. Now I need you to tell me that process – how to reverse the process?

Gabbi: Ma'am, these methods are the same, maybe uma udifferentiator you subtract 1 kanti uma u integrator you add 1 ku exponent *(when differentiating you need to subtract one but when integrating you add 1).*

Researcher: What is the meaning of the sign used here?

Gabbi: Ma'am, sengiyadideka manje *(Ma'am now I am confused).*

As can be noted in the extract above, Gabbi attempted to define the integration process but failed to state the real meaning of the indefinite integral. As in the written response, she focused on describing the procedure when doing integration instead of defining the meaning of indefinite integral.

In a different scenario, four learners in Category 3 displayed a process conception because they did not focus on explaining the integration procedure but rather described it in relation to the original function, as shown in Nkosy's response below. As stated in the preliminary genetic decomposition presented in Chapter 3, a learner at the process stage can make the connection between differentiation and integration by linking it to its original function, and this is what the learners in Category 3 did. Moreover, when an action has been interiorised as a process, it is possible to find solutions in more than one way and verify them using another way. Nkosy and the other two learners showed that they had constructed a meaningful understanding of the concept. Nkosy's response is shown in Figure 7.2 below.


ANTIDERIVATIVE OF
ANOTHER FUNCTION

Figure 7.2: Nkosy's response to item 3.1

Nkosy mentioned the key concept, which is anti-derivative, showing that he had made the connection between the notational symbol and definition. Nkosy mentioned this key concept without trying to describe the procedure for integration, but rather by making connections to the function. However, as observed in the response, it was clear that the process was not yet encapsulated to form the object. In contrast, Nozi, who was also in Category 3, used the notational symbol to give the meaning of the indefinite integral. Nozi's answer was $\int f(x)dx = F(x) + C$. Although the answer was correct, Nozi missed the point by failing to give the proper definition of an indefinite integral.
In the next section, the researcher discusses the errors evident in learners' responses which hindered their construction of the appropriate schema.

7.3.2 Errors hindering learners' construction of the necessary mental constructions of indefinite integral

From the analysis of learners' written responses, a number of errors were identified which hindered the construction of the coherent schema using the categories of mistakes purported by Olivier (1989, as cited in Ngcobo et al., 2021). The learners' mistakes were categorised as errors, which is an indication of underlying misconceptions. For example, failure to provide a meaningful definition of an indefinite integral suggests that the knowledge constructed has gaps. Six learners in Category 2 (Table 7.2) made no mention of the constant C while describing the procedure. Confusion between differentiation and integration procedures was also apparent in some responses, as seen in Zibu's response below.

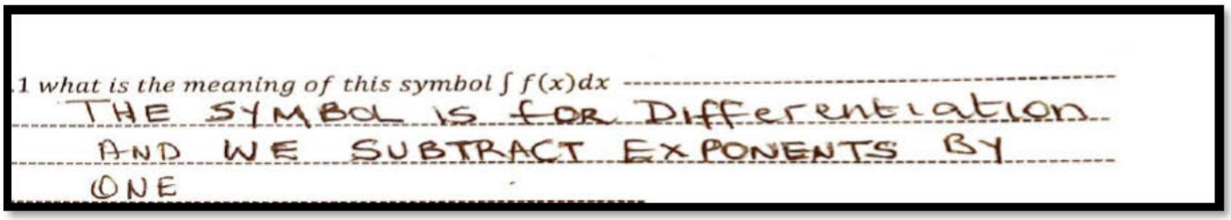

.1 what is the meaning of this symbol $\int f(x)dx$
THE SYMBOL IS FOR Differentiation
AND WE SUBTRACT EXPONENTS BY
ONE

Figure 7.3: Zibu's written response to item 3.1

Similar findings were evident in a study by Howe et al. (2017) involving 70 computer science diploma students that found that the students made procedural errors in their advanced calculus examinations due to their confusion between differentiation and integration processes. Habineza (2010), after conducting a study with student-teachers, argued that a lack of understanding of the underlying concepts hindered participants' development of the integral concepts. Tall (1992) established that failing to give a satisfactory coherent meaning leads to cognitive conflict which is usually resolved by keeping the various meaning of differential in separate compartments ($\frac{dy}{dx} = \lim_{n\to\infty} f(x)dx$) in differentiation and dx means "with respect to x" in integration.

7.3.3 Summary of errors that hindered learners' construction of schema

- Concept definition of indefinite integral
- Ignoring the constant term when integrating the function
- Using incorrect procedures when integrating the function.

Based on the analysis, learners' lack of conceptual understanding of the integration concepts and lack of procedural fluency left them unable to construct the necessary mental constructions. When defining the indefinite integral, they focused on trying to explain the relationship between differentiation and integration, as they stated that the function had been differentiated before and the process was reversed. Moreover, the analysis revealed that an incorrect concept definition of the indefinite integral hindered the construction of the concept image.

In the next section, the mental constructions learners made for the definite integral, as well as the difficulties that hindered them from making the necessary mental constructions, are discussed.

7.3.4 Learners' understandings of the definite integral ($\int_a^b f(x)dx$)

All ten learners attempted item 3.2; however, the majority tended to explain the values of the limits of integration, instead of focusing on giving the meaning of the definite integral. This is in line with Orton's (1983) finding that students had problems understanding integration as a limit of the sum and the relationship between the definite integral and the area under the curve.

The aim of item 3.2 was to gain insight into whether learners had developed a schema conception of the definite integral. The learner should:

- be able to state that the difference between the values of the integral of a given function $f(x)$ for the upper value of b and the lower value a of the independent variable x; *and*
- be able to state that a definite integral is an area under a curve between two fixed limits.

Item 3.2: on the meaning of the symbol $\int_a^b f(x)dx$

3.2 $\int_a^b f(x)dx$ *means*

The analysis of learners' responses revealing their mental construction of definite integral is shown in Table 7.3 below:

Table 7.3: Learners' responses to item 3.2 (meaning of $\int_a^b f(x)dx$)

Score	1	2	3	4	5
Indicator	Showed no mental constructions	Displayed Action mental constructions.	Displayed Process mental constructions	Displayed Object conception of the meaning of the definite integral.	Made schema conceptions as proposed in the genetic decomposition
No. of learners	0	5	5	0	0

As shown in Table 7.3, no learner was categorised as failing to make mental constructions (Category 1). Category 2 represented five learners that displayed an action conception because they defined the definite integral by explaining the meaning of values of a and b as a limit of a function. Although they didn't use the correct definition of the concept, they demonstrated some stage of understanding as they mentioned two important values of x as a limit of integration. This is in line with Ndlazi (2015), who found in her study of first-year engineering students that they tended to link a definite integral with the notion of area, without mention of the orientation. These five learners were also categorised as operating at the action stage in item 3.1. Below is the sample of responses provided by learners in Category 2.

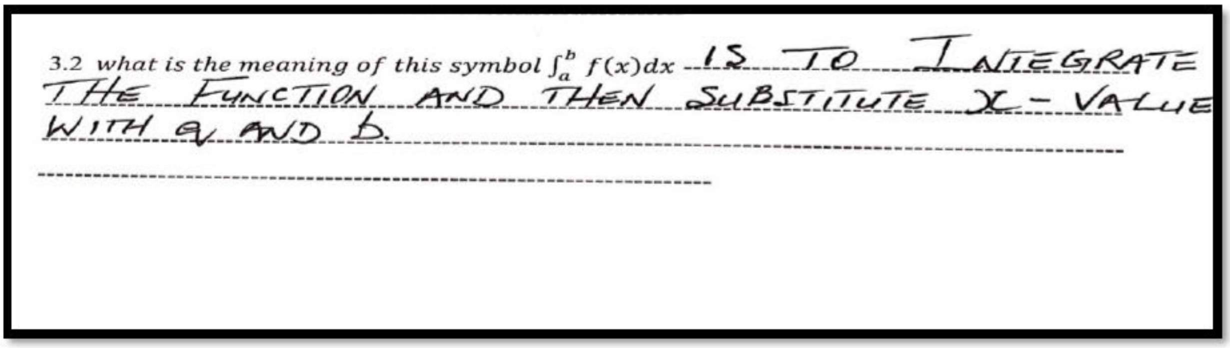

3.2 what is the meaning of this symbol $\int_a^b f(x)dx$ IS TO INTEGRATE THE FUNCTION AND THEN SUBSTITUTE X - VALUE WITH a AND b.

Figure 7.4: John's written response to item 3.2

John and the other four learners missed the point by stating that when integrating a function, you need to substitute the values of x and y respectively. Focusing on substitution showed that his thinking was guided by the physical steps that are involved in solving definite integral, thus thinking of the actual values to be substituted in the function. His response revealed that he had not constructed a meaningful understanding of the symbol of the definite integral. John relied mostly on an algorithmic integral to state the meaning of a definite integral, as he only focused on explaining the values of the upper and lower limits. His conception of definite integral relied on external cues, and this suggests that he lacked an understanding of the mathematical meaning of the symbol. To gain further insight into his understanding of integration, John was interviewed. The following is an excerpt from the interview with John.

Extract 11: Interview with John about item 3.2

Researcher: What is your understanding of $\int_a^b f(x)$?

John: Ngicabanga ukuthi u *a umele i lower limit and b umele i upper limit (I think a stands for the lower limit and b stands for upper limit).*

Researcher: What is the meaning of the limit?

John: Okokuqala ma'am uyayiyeka ilimit bese uya integrator *(firstly you need to ignore the limit and start by integrating the function).*

Researcher: Then after you have done that, what will be your next step?

John: Bese uya sustituter madam la kukhona u $x\ ufaka\ ivalue\ ka\ a\ and\ b$) (then you need to substitute the value of x with the values of a and b).

Researcher: How? Please explain.

John: Eyi madam ngiyadideka manje *(Now, madam, I am confused).*

Researcher: Since we have one value for x, please tell me, how are you going to substitute?

John: I think you substitute both of them, kodwa angazi uqala ngayiphi *(but I don't know which one you substitute first).*

As can be noted in the extract above, John lacked a conceptual understanding of a definite integral as he was unable to state the meaning of a definite integral. As in the written response, he focused on describing the upper and lower limits of integration instead of defining the meaning behind the symbol of the definite integral. In many instances, responses from interviews revealed that learners had some difficulty in explaining the approaches they had used. No learner in this section mentioned the notion of finding the area bounded by a and b, which was an indication that they lacked a conceptual understanding of the concept. Probing questions revealed that John lacked a deeper understanding of the rules of integration, especially for the definite integral. John also confused the procedures of differentiation with those of integration, as discussed in Chapter 6.

Five learners in Category 3 displayed a process conception because they did not focus on explaining the limits of integration; instead, they defined integral as a limit of a sum without explaining the step-by-step process of the definite integral. As presented in the preliminary genetic decomposition in Chapter 3, a learner at the process stage can state that $\int_a^b f(x)dx$ means the difference between an upper value b and a, the lower value or the area under the curve bounded by the limits. Moreover, when the action has been interiorised into a process, a learner can give answers in more than one way and verify them using another way.

7.3.5 Errors that hindered the construction of necessary mental construction for the definite integral

The learners displayed different kinds of errors as they tried to define definite integral. The researcher observed that the learners tended to confuse differentiation and integration, as they mentioned the term ‘anti-derivative’. This displayed a lack of conceptual understanding of the meaning of definite integral. As in item, 3.1 learners' incorrect concept definition of the definite integral was found to be the major stumbling block that hindered learners’ understanding.

7.3.6 Summary of errors made by learners

- The concept definite of the definite integral is flawed
- The upper limit and lower limit are understood to stand for values to be substituted

Ndlazi (2015) found that students failed to give the real meaning of the definite integral and most students viewed an integral as an anti-derivative with limited extension to the area concept. The same was evident with the participants in this study. Furthermore, the schema retrieved for the definite integral was flawed, thus leading learners to focus on external cues such as the lower limit and upper limit when trying to construct a definition. Ndlovu and Brijlall (2015) argue that learning rules by memorisation is a hindrance for many learners because the focus is not on understanding the concept but rather on performing the procedure.

7.4 Learners' Understanding of the Application of Integration Techniques

Items 3.3 and 3.4 investigated learners’ knowledge of applying the rules of integration. The aim was to reveal the nature of their mental constructions and the difficulties that hindered their construction of the necessary mental construction. In integral calculus, integration techniques and formulae are the keys to carrying out integration successfully. In item 3.3, learners’ responses reflected the following trends:

(1) application of differentiation methods instead of integration methods;
(2) learners ignored the constant term when integrating; and
(3) learners used notation incorrectly; e.g., the integrated function still contained the symbol for integration.

Item 3.3 explored learners' knowledge of the basic integration techniques, especially for solving indefinite integral problems. The item was intended to reveal learners' insights into how to find an indefinite integral and to identify their stage of understanding of the concept. The preliminary genetic decomposition in Chapter 3 was used to explain learners' mental constructions. Brijlall and Ndlazi (2019) have suggested developing the genetic decomposition of a concept so that instructors may organise their teaching of the concept for learning to take such a route. As part of the schema of solving the indefinite integral, a learner:

- in the action stage will follow a step-by-step procedure in applying the rules of the integration concept;
- in the process stage will perform the same action in the mind without following step-by-step procedures and yield the same results; and
- with an object conception, will become aware of a process as a totality, having encapsulated the process into a cognitive object.

Item 3.3: On the use of integration rules

3.3 Use integration rules to determine the original function $$\int 6x dx$$

Learners were expected to find the indefinite integral using the rules of integration. The main aim of this item was to explore the learners' mental constructions for using integration rules when integrating the given function. The item focused only on the fundamental procedures of integration, since the study was conducted at the secondary education level. The table below presents the analysis of the learners' responses and evidence of their mental constructions.

Table 7.4: Learners' responses to item 3.3 (on finding $\int 6x dx$ using integration rules)

Scores	1	2	3	4	5
Indicator	Showed no mental constructions at all	The displayed action stage of mental constructions Followed step-by-step procedures	Displayed the Process stage of mental constructions –answered without steps	Displayed the object stage of the mental constructions	Displayed the schema stage of mental constructions.
No. of learners	6	2	2	0	0

As shown in Category 1 of

Table 7.4, no learner was categorised as failing to make mental constructions. Two learners displayed an action conception of the indefinite integral concept, as shown in Category 2, because they followed a step-by-step procedure in applying the rules of integration, as can be seen in the extract below.

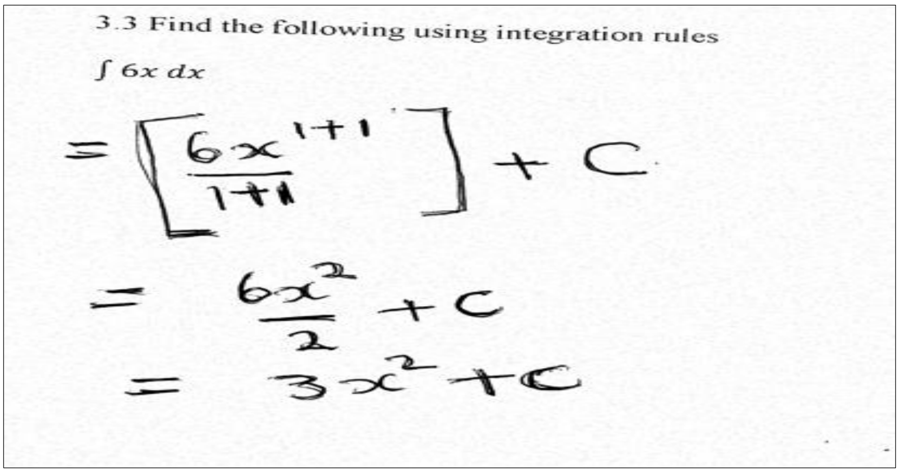


Figure 7.5: Zibu's written response to item 3.3

As evident in the above response, Zibu carried out the necessary procedures step-by-step to determine the integrated function. The integral notation acted as the external cue which triggered the procedures to be followed. The other two learners, who were considered to be operating at the process stage, showed interiorisation of the action into the process as they were able to determine the original function without having to explicitly carry out the step-by-step procedures. These learners were able to skip some steps and still arrive at the correct answer. For example, without showing the addition of 1 to the exponent and diving by it, they could present the answer as $\frac{6x^2}{2} + C$ and simplify the answer to be $3x^2 + c$. The learners in Category 1 failed to make the necessary mental constructions due to failing to apply the integration rules. Various errors were evident in the learners' responses; for example, incorrect use of notation and lack of accuracy when carrying out the procedures, as illustrated in Ntando's response below.

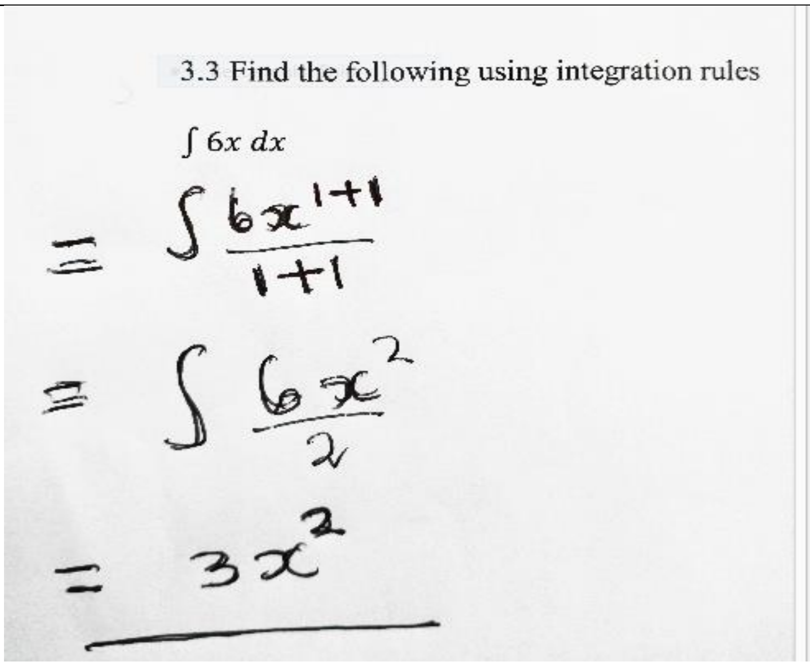


Figure 7.6: Ntando's written response to item 3.3

Ntando carried out some of the procedures, such as adding 1 to the exponent and dividing the expression by the new exponent. However, she did not find the constant term and used incorrect notation to represent the integrated function. Her lack of understanding of the concept was also evident during the interview, as illustrated in the extract below.

Extract 12: Interview with Ntando about item 3.3

Researcher: Could you please explain what happened here, and why 1 is added?
Ntando: I was integrating the function, madam.
Researcher: Can you explain the process in your solution?
Ntando: Madam, kumele sithi add one ku exponent, bese futhi sidivider ngaleyo answer. *(We need to add one to the exponent and also divide by that answer).*
Researcher: Yes, I can see, but the integral sign is still here. Why?
Ntando: Uma udifferentiator noma uIntegrator uyisusa ekugcineni isign. (*When differentiating or integrating, the signs are removed in your final answer).*
Researcher: Really? Even with integration?
Ntando: Yes, madam, kuyafana *(it is the same).*
Researcher: Where is the value of the constant?
Ntando: Madam, I don’t know kahle kodwa I think siyifaka ekugcineni *(but I think we need to put it in the final answer).*
Researcher: But there is no C in your final answer. Why?
Ntando: I made a mistake, madam.

It was evident from Ntando's interview response that she had memorised the rules, as she said that differentiation and integration procedures are more or less the same. Moreover, Ntando's response showed that this was not linked to the procedures that need to be carried out. Also, what was evident in the responses of learners in Category 1 was that the previously learned concepts, if not understood, hindered the construction of new knowledge.

Differentiation is introduced before integration; however, the two concepts are interlinked, thus the knowledge learned in differentiation can either enhance or hinder the development of the schema of integration. In the case of the learners in this category, it hindered the evolvement of their schema of integration. This was evident as the learners, instead of integrating the function, differentiated the differentiated function from the exponent, illustrating that the cues provided by the notation had been ignored. Errors emanating from the failure to consider the constraints of the problem are categorised as arbitrary errors by Orton (1983). In this study, this extended to the failure to take into account the constraints indicated by the notation. The symbol $\int x$ dx illustrates an integral, thus it triggers the procedure and constraints of the problem. However, some learners in Category 1 ignored such constraints in their attempt to find the solution.

7.4.1 Application of definite integral $\int_a^b f(x)dx$

Item 3.4 aimed at exploring learners' knowledge of understanding the definite integral procedures to find the area bounded by a and b. It was intended to provide insight into learners' mental constructions of calculating indefinite integral. Learners were expected to find the definite integral using the basic rules of integration procedures. Learners were given the values of a and b to determine the bounded area of the function.

Item 3.4: on using integration rules

3.4 Find the following using integration rules $\int_1^2 4x\,dx$

All ten learners attempted item 3.4; however, the majority failed to display the necessary mental constructions as they did not apply the procedures correctly. Table 7.5 illustrates the analysis of the responses to item 3.4.

Table 7.5: Learners' responses to item 3.4 (on finding $\int_1^2 4xdx$ using integration rules)

Scores	1	2	3	4	5
Indicator	Showed no mental constructions at all	Displayed the action conception of the definite integral	Displayed the process conception of the definite integral	Displayed the object conception of the definite integral	Constructed the schema
No. of learners	7	3	0	0	0

For this item, no learner displayed interiorisation of the action into a process. This is cause for concern, as it suggests that for the few learners who had constructed some mental construction, i.e., action conception, their understanding of the concept was at the surface stage giving them a limited ability to carry out procedures.

The focus of this item was on the basic rules for the definite integral, but the learners found it difficult to follow the procedures of integration, especially for the definite integral, as they employed incorrect integration procedures. The errors made by the learners included not using the correct integration procedures, where 1 was subtracted instead of adding it, and some forgot to divide by the exponents. Most learners substituted the limits incorrectly and, in many instances, brackets were not incorporated. Below is a sample of the responses provided by learners in Category 1.

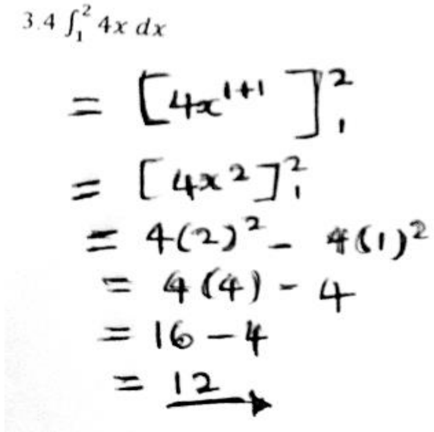


Figure 7.7: Lukho's written response to item 3.4

Lukho and the other six learners in Category 1 failed to integrate the function; instead, they substituted the limits of integration in the wrong function. Their responses showed that they had not constructed a meaningful understanding of the concept of a definite integral. The learners demonstrated some gaps in their knowledge of definite integral as their focus was on the limits of integration rather than finding the definite integral. Lukho had not conceptualised the fundamental difference between differentiation and integration. As purported by Tall (1992), students have challenges understanding and using symbols correctly; for example, learners confuse the techniques for definite integral with those for indefinite integral, where the constant 'C' can be used for both functions. During the interview with Lukcho, this is what transpired:

Extract 13: Lukho's interview response for item 3.4

Researcher: Could you please explain your steps in your sum?

Lukho: Ngiqale nga integrated the function, madam *(I have started by integrating the function, madam).*

Researcher: Then what should you do? Explain all the steps.

Lukho: I have to find the integral kuqala *(firstly*) and then substitute the values of the limits.

Researcher: Yeah, I can see that, but I am missing something here. Can you please explain a step-by-step procedure of integration.

Lukho: Madam, mina ngi add u one ku exponent kuphela ngakhohlwa to multiply *(Madam I added 1 on the exponent and forgot to multiply by the exponent).*

Researcher: Do you need to multiply by the exponent?

Lukho: Yes, madam, I think so.

Researcher: Mhmm, you are not too sure?

Lukho: Ngempela madam ziyadida lezizindlela (These methods are confusing madam).

Researcher: Which methods now?

Lukho: Angazi kumele ngidivide nini futhi ngimultiply kuphi nge exponent. *(I don't know when to divide or multiply by the exponent).*

Although Lukho said he had integrated the function, when asked about the procedure for integration he admitted that he did not understand when to apply which rule, meaning that he had memorised the rules but did not understand the concept, and thus was confused as to when to retrieve the correct schema to solve the problem. This focus on substitution of limits was also observed in Sandy's response, below. Unlike Lukho, who did attempt to perform some procedures of integration, Sandy simply substituted the limit in the given function without integrating it first, as shown in her response below.

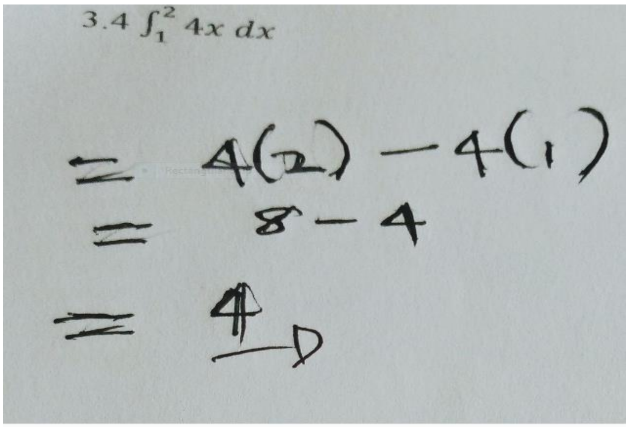


Figure 7.8: Sandy's written response to item 3.4

What is observed is that the learners did not pay attention to sense-making when solving the sums or interrogate if their responses made sense in relation to the question, showing a lack of computational fluency. Ndlovu and Brijlall (2015) found the same in their study of first-year pre-service teachers at a South African university. In this study, a lack of sense-making was evident among the learners when solving differentiation and integration problems. Siyepu (2013), too, found that when students lacked procedural knowledge, they tended to overgeneralise the rules applicable to other concepts to new concepts learned. This was also evident in this study when the learners, instead of integrating the function, used rules of differentiation instead.

The three students were considered to have the action conception as they carried out procedures to determine the bounded area; however, there were some slips in their responses, such as ignoring the inclusion of brackets with the upper and lower limits, but they managed to determine the correct answer, as illustrated in the extract below.

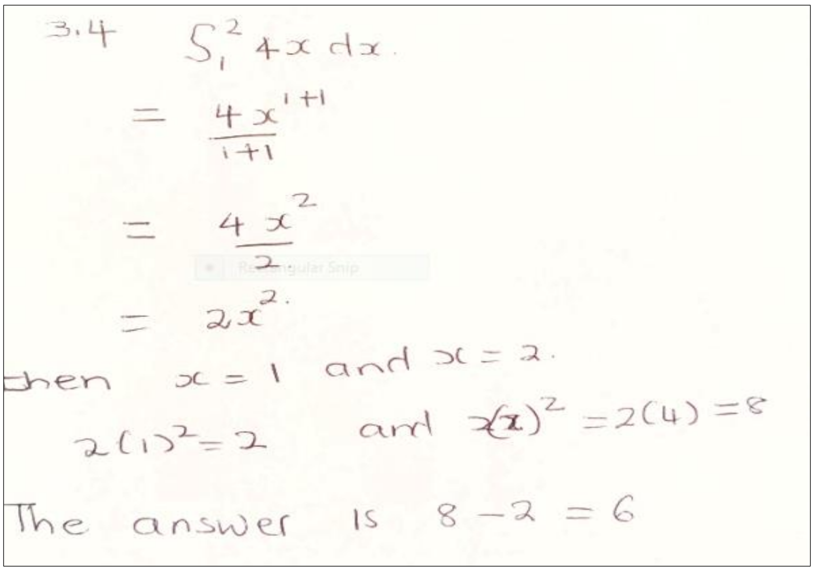


Figure 7.9: Elliot's written response to item 3.4

The analysis of learners' responses revealed several errors that hindered the construction of schema. For example, retrieval of the incorrect schema, learners using differentiation procedures for integration, incorrect use of notation – such as not removing the integral notation even when the function has been integrated – and, lastly, substitution of limits before integrating the function, as well as ignoring the constant when integrating, suggesting that procedures had been memorised.

7.5 Learners' Understanding of the Area Under the Curve

Learners were expected to show their understanding that the area between the function and the $x -$ axis can be calculated using integration methods. In this section, integration is applied to determine the area bounded by $x = b$ (upper limit) and $x = a$ (lower limit), where a and b are the element of integers. By using the definite integral, we can calculate the exact area enclosed by the curve, $f(x)$, and the x-values. The analysis of learners' responses displayed certain trends. First, learners removed the square root sign before finding the integral and substituted the values of x. Second, some learners used the graphical presentation in finding the bounded area between the values of x or used incorrect formulas to calculate the bounded area.

Item 4.1 investigated learners' mental construction of the area bounded by the curve. As part of the schema of finding the area under the curve, a learner should:

- be able to remove the square root sign using exponential laws before integration; and

- be able to illustrate the area bounded by the function using a graphical representation.

Item 4.1 on using integration methods to draw graphs

4.1 Find the bounded area by drawing the graph $y = \sqrt{9 - x^2}$ using integration methods between $x = 0\ and\ x = 3$

The analysis found that three learners did not attempt item 4.1 and five learners, although attempting 4.1, failed to produce a correct answer to the problem. Even those who were able to apply integrations to determine the bounded area did not sketch the graph to indicate the bounded area. As can be seen in the table below, learners' mental constructions of the area bounded by the curve was limited to action conception.

Table 7.6: Learners' responses to item 4.1 (on drawing graph of $y = \sqrt{9 - x^2}$ using integration methods)

Scores	**1**	**2**	**3**	**4**	**5**
Indicator	Showed no mental constructions at all* No written response	Displayed action mental constructions	Displayed process mental constructions	Displayed object and pieces of mental constructions of the concept	Made schema of mental constructions as proposed in genetic decomposition
No. of learners	8	2	0	0	0

As mentioned above, of the eight learners categorised as not making the necessary mental constructions (Category 1), only five attempted item 4.1 but failed to solve it due to various errors which are elaborated below. In determining the area bounded by the curve, two learners used the formula for calculating the area of a triangle, as illustrated by one response below.

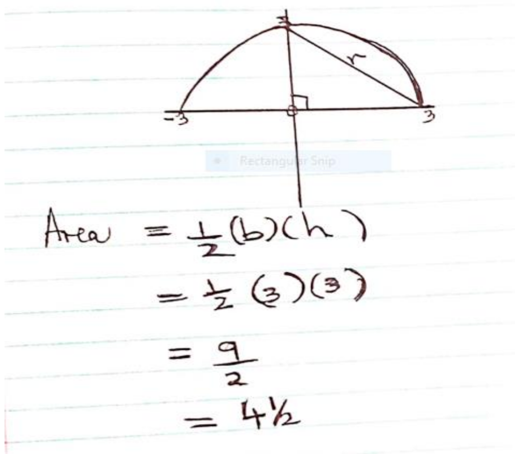


Figure 7.10: Sethu's written response to item 4.1

Sethu was one of the two learners who used the area formula for finding the area of a triangle. As shown in her response in Figure 7.10, she started by drawing the triangle inside the curve, showing all the sides of a right-angled triangle. She applied the formula for finding the area of a triangle in different settings. In this case, the learner used a formula for finding the area that is correct in a different setting, but not correct for the problem at hand. According to Olivier (1989, as cited in Siyepu, 2013), the source of misconception is often an overgeneralisation of previous knowledge that applies to a different domain.

Another learner in Category 1 substituted the limit values without integrating the function. The method this learner used was: $y = \sqrt{9 - x^2}$ $=\sqrt{9 - 3^2}$ $=\sqrt{0}$ =0. This learner had made the same error in item 3.4. Knowledge of basic algebra laws, such as removing the square root, and laws of exponents were critical in the solution of this problem; thus, learners who failed to remove the root sign and simply substituted the limits appeared to have failed to see the interrelationship between concepts and relied on memorisation of the rules. The errors revealed that the concept image of the concept constructed by these five learners was flawed. The response below illustrates some of the errors made by learners in Category 1.

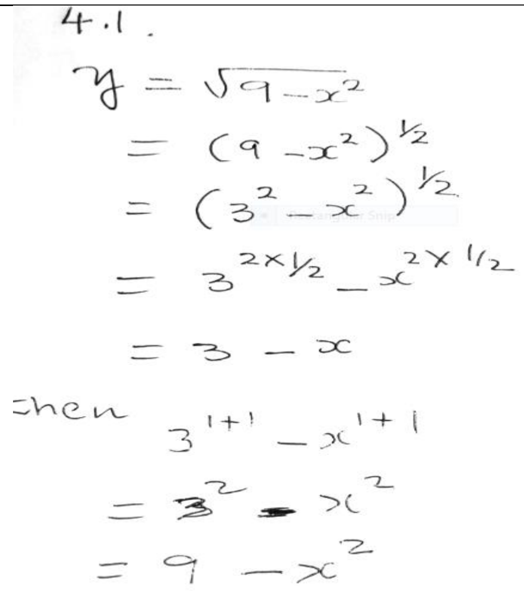


Figure 7.11: Nkosy's written response to item 4. 1

Nkosy applied the correct procedure to remove the root sign; however, to simplify the expression he incorrectly applied the law of exponents that states that the exponent outside the brackets must be multiplied by the exponent of each base inside the brackets. Such laws are applied when the operation inside the brackets is multiplication. Thus, Nkosy overgeneralised the rules of exponents to binomial expressions. Furthermore, he failed to apply the integration rules to integrate the function so that he could determine the area bounded by the curve. For both integration and differentiation, it is imperative for a learner to understand basic algebra rules so that they can apply them to solutions involving differentiation and integration concepts.

The following is an excerpt from the interview with Nkosy.

Extract 14: Nkosy's interview response to item 4.1

Researcher: Can you please explain your first step?

Nkosy: Ngisuse isquare root ngafaka uhalf esikhundleni saso *(I have removed the square root sign and I have replaced it by half).*

Researcher: Why did you do that?

Nkosy: It is what I was taught to do: if I want to remove the square root, I should change it to $\frac{1}{2}$

Researcher: Okay. Explain the next step?

Nkosy: Ngisuse ama brackets *(I have removed brackets).*

Researcher: Is this the correct procedure of brackets removal?

Nkosy: Kunzima uma kuno half, madam *(it is difficult if there is half as an exponent).*

Researcher: What makes it difficult?

Nkosy: Eish, ma'am.

Researcher: If the exponent was 2, what you would have done?

Nkosy: Ma'am bengizosebenzisa ifoil method (I would use the foil method).

Researcher: If the exponent was 3, what you would do?

Nkosy: Eish, ma'am, kusazomele ngicabange (I still need to think).

Researcher: Okay, after simplifying the expression what did you do?

Nkosy: Uma wenza integration kumele sithi add one *(we need to add 1 when you are finding integration).*

Researcher: Is that all?

Nkosy: I don't understand integration, madam.

Researcher: Which part of integration you do not understand?

Nkosy: Eish, ma'am, lamarules maning ayaconfuse (*there are too many rules and they are confusing).*

It became apparent during the interview with Nkosy that he was relying on memorisation of rules. For example, when asked to explain why he used $\frac{1}{2}$ when removing the square root, he could not explain except to say that that is how he had been taught to do it. It was clear that he had memorised the foil method, also, as he could only apply it to expressions with a power of 2. He further alluded to the fact that since he had memorised so many rules, he could not recall how to apply them all correctly. This is evidence that when rules are memorised the conceptual schema cannot be constructed, thus when solving problems learners retrieved the incorrect schema. There is evidence from the analysis of learners' responses to item 4.1 that a lack of basic concepts hindered the development of their differentiation and integration schema. This aligns with Dubinsky's (2002) observation that the lack of basic knowledge of other related concepts is the main source of students' difficulties in conceptualising advanced concepts.

The two learners in Category 2 used step-by-step procedure to determine the function to be integrated. They further applied the correct procedures when integrating the function and determining the solution, but did not sketch the graph to illustrate the bounded area between limits. It is possible that they were constrained by the absence of the formula for the area. Although John did not draw the graph, his written response revealed that integration procedures were correctly applied.

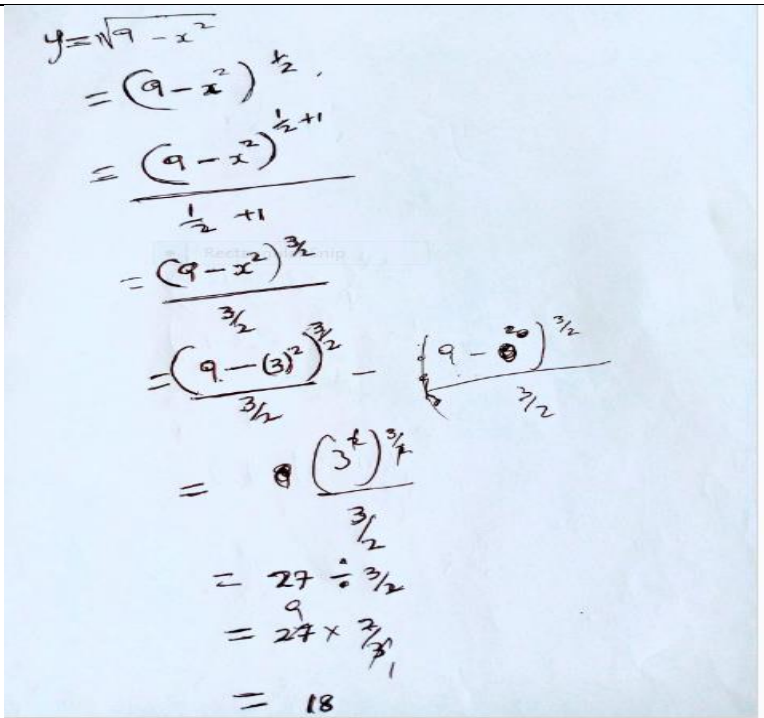


Figure 7.12: John's written response to item 4.1

Although the two learners in Category 2 carried out the procedures accurately, their failure to sketch the graph suggests that they had not conceptualised the concept of the area bounded by the curve. As carrying out a procedure is a physical entity but being able to connect the procedure to graph sketching requires one to be operating at the object stage, it is evident in the analysis of learners' solutions that most learners were operating below the action stage; for those operating at the action stage, their understanding of the concept was still at the surface level. This was also evident during the interviews, where learners could not justify their responses but rather tried to list procedural steps. This concurs with the findings of Ndlazi (2015), who reported that, in most cases, written responses and interviews revealed that students were operating at a procedural stage and experienced some difficulty in justifying the approaches they had used.

Item 4.2 was designed to explore learners' understanding of the procedures of curve sketching using integration procedures. It was intended to provide insight into whether the learners had a conceptual understanding of the notion of an area under the curve using integration. In this item, learners were asked to integrate $\int x^2 dx$ and shade the bounded region.

Item 4.2: On definite integral

4.2 Find the definite integral of $\int_0^3 x^2 dx$ and draw the curve to indicate the bounded area

Table 7.7: Learners' responses to item 4.2 (on drawing the curve of $\int_0^3 x^2 dx$ using integration methods)

Scores	1	2	3	4	5
Indicator	Showed no mental constructions. No written responses	Displayed action mental constructions	Displayed process mental constructions	Displayed object mental constructions proposed in genetic decomposition	Made schema conceptions as proposed in genetic decomposition
No. of learners	10	0	0	0	0

Nine out of ten learners did not attempt item 4.2, while one learner attempted to solve the problem but failed. The analysis thus found that none of the learners had constructed the mental construction needed to solve this problem. The one learner who had attempted to solve the problem had attempted to sketch the curve instead of first finding the integral. The external cue of the quadratic function prompted the learner to sketch the graph. Looking at the sketch of the graph, it is apparent that, while the learner recognised the given function as quadratic, they could not mentally visualise the orientation of the graph and thus drew an incorrect graph, as shown below.

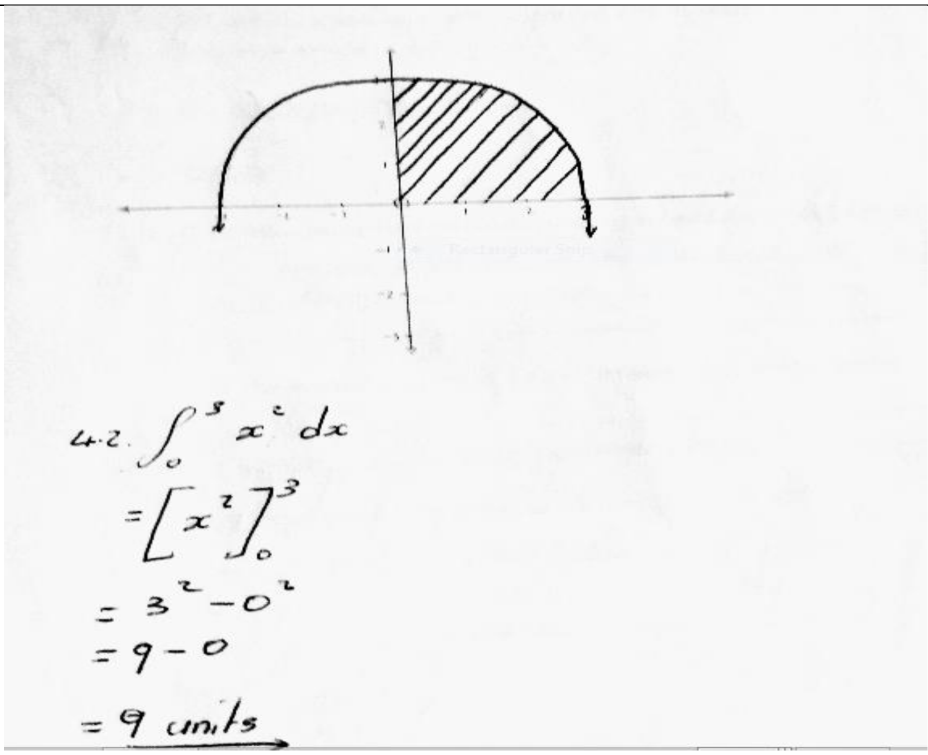


Figure 7.13: Nozi's written response to item 4.2

It is also possible that the external cue "bounded" prompted the learner to sketch the graph facing downward because bounded is considered closed, as seen in the shaded area. Tall (2008) and Ndlovu (2014), in their studies, found that learners' use of everyday language sometimes hindered their understanding of the same or similar words when used in a mathematics context. The graph sketching and solving of the equations were not seen to be connected. As with item 3.4, Lukho substituted the values of the limit into the given function, not the integrated function. The same errors evident in the previous item were also evident in this response.

7.6 Conclusion

In this chapter, the researcher has presented the data analysis of learners' responses to the integration section of the activity sheet and follow-up interviews. Analysis of learners' responses was based on their mental constructions using APOS theory and the difficulties they encountered as they attempted to solve integration problems. Most of the learners' written work and interviews indicated they had not constructed the necessary mental construction of integration; only a few learners displayed an action conception, and their understanding was found to be at the surface level The most significant factor contributing to learners being unable to construct the necessary mental construction appeared to be that they had merely memorised the concepts or learned isolated facts. Failure to construct definitions of concepts hindered learners' ability to make the necessary mental construction needed for schema development. The next, and final, chapter synthesises the study's findings and presents conclusions and recommendations.

8 SYNTHESIS OF THE FINDINGS, CONCLUSIONS AND RECOMMENDATIONS

8.1 Introduction

Chapters 6 and 7 have presented data and discussed the findings related to learners' mental constructions and misconceptions in differentiation and integration. This chapter revisits the research questions and synthesises the findings about learners' mental constructions of differentiation and integration, as well as the difficulties experienced by learners, with reference to APOS theory. It also summarises the difficulties learners experienced that hindered their ability to construct the mental constructions required to solve differentiation and integration problems. The chapter addresses the study's limitations, discusses the implications of the study for teaching and learning and makes recommendations for further research based on the findings.

The main aim of the study was to analyse Grade 12 learners' mental constructions and the difficulties they experienced working with differentiation and integration. The study is underpinned by APOS theory, a constructivist theory of learning advocated by Dubinsky (1991). APOS theory assisted in analysing learners' mental constructions in differentiation and integration.

To analyse learners' mental constructions, a genetic decomposition was formulated by the researcher to postulate the construction that a Grade 12 learner should make when solving differentiation and integration. According to Maharaj (2010, p. 41) a "genetic decomposition of a concept is a structured set of mental constructs which might describe how the concept can develop in the mind of an individual". The analysis was based on ten learners' written responses to activity sheets and transcribed interviews conducted with five participants. The interviews helped the researcher to gain a deeper understanding of the written responses given by learners. To analyse the difficulties that hindered learners' constructions of the necessary mental constructions explained in the genetic decomposition, learners' mistakes were categorised using Olivier's (1989) framework, which yielded some patterns that were analysed inductively to understand the key difficulties displayed by the learners.

8.2 Primary Research Questions

The primary question that guided the study was: “How do grade 12 learners construct mathematical knowledge of differentiation and integration in calculus?” To provide an answer to this main question, the following sub-questions were formulated.

- What are Grade 12 leaners’ mental constructions of differentiation and integration?
- How have Grade 12 learners’ mental constructions of differentiation and integration evolved in relation to the genetic decomposition?
- Why has the mental construction needed for the conceptualisation of differentiation and integration evolved or not evolved?

After analysing learners’ responses, the researcher identified emerging patterns that explained why certain mental constructions were made or not made. The main themes were identified and synthesised to answer the research questions.

8.3 Findings on Learners’ Mental Constructions for Differentiation

This section synthesises the findings of the study related to learners’ mental constructions working with the concept of differentiation and the errors that hindered their construction of the necessary mental constructions. To explore learners’ mental constructions of the differentiation concept, two aspects of differentiation were explored: defining concepts and rules and application of differentiation rules.

8.3.1 Learners' mental constructions of definitions for differentiation concepts

According to Tall and Vinner (1981), the definition of a concept refers to words that an individual uses to clarify a particular concept. The study found that learners displayed limited knowledge in terms of being able to define differentiation. Instead of using mathematical terminology, most of the learners attempted to define the concept using everyday language and confused the process of differentiating functions with the English term ‘differentiate’. For example, one learner wrote: “differentiation is when you find the difference between numbers and variables”; others said that differentiation is to be different in terms of growth. Learners tended to focus on the part of the term that they recognised from everyday language. As a result, they provided incorrect and disjoined explanations of differentiation.

Based on the findings, the researcher concluded that the majority of the learners had not made the necessary mental construction to be able to define the concept of differentiation. Even when learners were probed further during the interviews, instead of providing a definition, they attempted to explain the rules of differentiation. This was an indication that the learners had memorised the rules without constructing a concept image of differentiation.

With regard to learners' mental constructions of the power rule, five of the learners' responses showed a lack of knowledge construction for the power rule. This was evident in their written responses where, instead of defining the power rule, they attempt to illustrate procedures to differentiate. Even in illustrating these procedures, they were unable to execute this correctly; for example, one learner wrote that $f(x) = an^x$ and indicated the application of the power rule as $f'(x) = nax^{n+1}$. This means that even an understanding of the procedures to use the power rule had not been constructed, as they were unable to execute correctly even when there were no numerals involved.

Although five learners were unable to make the necessary mental constructions, the other five displayed some understanding of the power rule. Three of these displayed an action conception of the power rule as they illustrated the definition by articulating the step-by-step procedures of $f'(x) = n \times ax^{n-1}$, while the other two showed interiorisation of the action into a process and expressed the final answer as $f'(x) = nax^{n-1}$, showing the whole process.

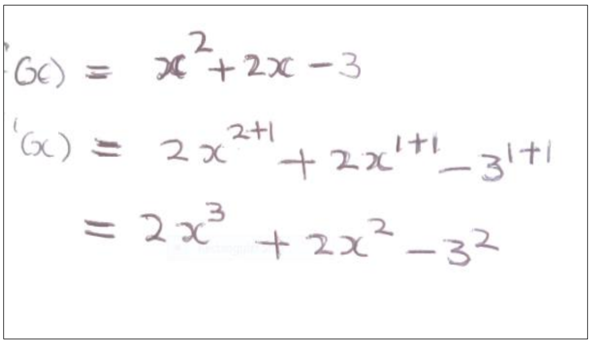
8.3.2 Learners' mental constructions for the application of the power rule to differentiate functions

To explore learners' mental construction of the application of the power rule, they were asked to solve a range of problems –for example, problems involving quadratic equations where b and $c = 0$ or a, b and $c > or < o$ or in the form $(f(x) = (ax^n \pm b)^n$ or $f(x) = \sqrt{ax^n}$. The findings revealed that learners were able to differentiate functions where a, b and c are either greater or less than 0, but not where functions were $b\ and\ c = 0$. Based on these findings, the researcher concluded that the learners had constructed their knowledge based on numerals and not on the concept itself, and thus focused on executing procedures and not meaning-making. The numeral triggered the procedure to be carried out thus it was concluded that the majority were operating at the action stage, in terms of the APOS theory, when it comes to the application of the power rule to differentiate functions in the form $ax^2 + bx + c$.

The findings further revealed that learners encountered difficulty differentiating the function of the form $f(x) = (ax^n \pm b)^n$ because they treat this function the same as $f(x) = ax^n$. Thus, instead of first simplifying the expression to remove brackets, they applied the power rule of nax^{n-1}. A similar scenario was noted with the differentiation function of the form $f(x) = \sqrt{ax^n}$. Learners' main difficulty was with removing the root sign or applying the laws of exponents after removing the root sign. Only six learners were found to have made a mental construction of some sort of differentiation of a function of the form $f(x) = (ax^n \pm b)^n$, with five operating at the action stage and one displaying a process conception of the concept. However, for the function of the form $f(x) = \sqrt{ax^n}$, only five learners displayed the action conception as they were able to accurately carry out the procedures for removing the root sign, apply the laws of exponents and differentiate the function. It was observed that, while the learners knew the rules of differentiation, these had been constructed as isolated facts. Thus, when a function required the application of other procedures before differentiating, the learners struggled, suggesting a lack of prerequisite knowledge of the previously learnt concept.

8.3.3 Misconceptions hindering mental constructions for differentiation

The learners who were classified as operating below the action stage were those who did not demonstrate that they had made the necessary mental constructions. The findings indicated that learners' misconceptions about differentiation impacted negatively on their understanding of differentiation concepts. Olivier (1989, as cited in Ndlovu et al., 2017) argues that misconceptions were the origin of errors. In this study, the findings showed that misconceptions were, indeed, the cause of many errors made by the learners and hindered their construction of the necessary mental constructions. One of the common errors was overgeneralisation of rules. When applying the power rule to differentiate functions, learners confused the rules of differentiation with those of integration. This was observed when, after multiplying the co-efficient with the exponent instead of subtracting 1, learners added 1, as shown in the response below.



It was interesting to note that some learners, when probed during the interviews, were able to identify the errors they had made and correct the procedure as needed. These findings support those of Maharaj (2014) and Ndlovu and Brijlall, (2015), who found that learners did not engage with what they wrote, thus needed to be probed further so that they reflected on the concept. In this study, three learners who failed to apply the rules of differentiation due to their overgeneralisation of the rules of integration were able to correct their thinking processes during the interview.

In another scenario, the rules for differentiating a function of form $f(x) = ax^n$ were overgeneralised arriving at $f(x) = (ax^n \pm b)^n$. In the teaching of differentiation, learners are generally taught first to differentiate the function of the $f(x) = ax^n$ before learning how to differentiate other types of functions. The findings show that when previously-learnt knowledge was not adequately developed, it hindered new learning. In this vein, Tall (2008) and Ndlovu (2014) posit that the previous knowledge learnt can have either positive or negative effects on the construction of knowledge for new concepts. In this study, it had negative effects because it hindered the construction of new learning. A lack of previously learnt knowledge was also evident in the differentiation function of the form $f(x) = (ax^n \pm b)^n$ and $f(x) = \sqrt{ax^n}$. For example, a solid foundation in the rules of simplifying binomials and the distributive property, as well as the laws of exponents, was critical for solving the given function.

The findings showed that learners' lack of procedural knowledge and conceptual understanding was the main source of their errors, because they performed a procedure but failed to explain the procedure or why it was the correct procedure. As has been reported in numerous studies (Jojo et al., 2013; Brijlall & Ndlovu, 2013; Maharaj, 2014) a lack of basic knowledge in mathematics is a fundamental hindrance to learners' construction of the necessary mental constructions. The findings of this study demonstrate this, as some learners' errors emanated from failing to apply basic procedures – such as removing brackets in the functions of the form $f(x) = (ax^n \pm b)^n$ and removing the root sign in functions of the form $f(x) = \sqrt{ax^n}$. Difficulty with the root sign was also evident in a study by Kazunga et al. (2017) that found that even teachers had difficulty with the square root function.

With regard to learners' ability to define concepts, it was found that the everyday use of terms similar to mathematics terminology hindered their understanding of the concept. For example, for differentiation, learners related the term to the everyday meanings of 'different' or explained differentiation as meaning the action of distinguishing between two or more things. Tall (2008) alluded to the fact that mathematical terms that also appear as words in the everyday use of language can hinder learning if learners have not constructed a mathematical concept definition of the term. Thus, Ndlovu (2014) emphasises the importance of teaching mathematical terminology to learners to assist with their construction of the concept image.

Learners in this study were able to correct their thinking when probed further during the interviews, suggesting that the errors were rooted in their cognitive structures and thus were conceptual in nature. Therefore, as also mooted by Riccomini (2005), necessary and appropriate instructions must be provided to rectify students' misconceptions and errors. However, to identify appropriate teaching approaches requires an understanding of the errors made and their causes. Thus, as stated by Brodie and Berger (2010) and Ngcobo et al. (2021), it is important for teachers to understand learners' errors and to take these into account in their planning of teaching approaches.

8.4 Findings on Learners' Mental Constructions for Integration

To explore the learners' mental constructions of the integration concepts and misconceptions hindering the construction of the necessary mental constructions, the following concepts were explored: (i) meaning of the symbols of integration $\int f(x)dx$ and $\int_a^b f(x)dx$; application of both definite integral and indefinite integral; and understanding of the bounded area under the curve. To understand the nature of the mental constructions made by learners and the difficulties they experienced, it was imperative to analyse learners' responses for all the items.

8.4.1 Learners' mental constructions of integration concepts

The researcher used APOS theory to analyse learners' mental constructions of integration, focusing on the basic rules use in Grade 12 Technical Mathematics. The first item focused on the meaning attached to symbol $\int f(x)dx$ (indefinite integral) and $\int_a^b f(x)dx$ (definite integral). The learners were asked to define these concepts using symbols, with the aim of identifying at which APOS stage they were operating. According to Tall and Vinner (1981), the definition of a concept refers to the words that an individual uses to explain a particular concept. They further state that the definition of an integral includes notions of an integral as an area, as a continuous summation and as an anti-derivative. In this study, it was important for learners to understand the language and the meaning of the terms used in differentiation and integration. Findell (2006) states that learning advanced mathematics involves learning concepts, language and notations. Understanding the language and the meaning of the concepts enables learners to construct the meaning of differentiation and integration.

Regarding the learners' mental construction of the meaning of these symbols, the findings revealed that the learners were operating at either the action stage or the process stage. Although the learners made some mental constructions, it was evident that they struggled to use the correct mathematical terminology to define integral concepts such as indefinite and definite integral; instead, they attempted to explain the rules involved in finding the integral. This was evident when defining definite integral: the learners focused on explaining that after integrating the function one needs to substitute values of x, with no mention of the area bounded by the curve.

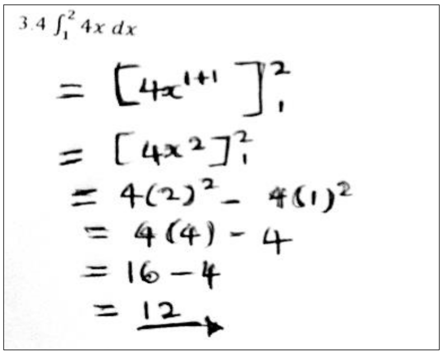
8.4.2 Learners' mental constructions of the application of integration rules

This section discusses the learners' mental constructions of integration rules according to the findings. Previous studies (Brijlall & Ndlovu, 2013; Ndlazi, 2015; Maharaj, 2013) have found that the challenges learners faced were due to not having constructed the necessary mental constructions to be able to construct schema for particular concepts. This may be due to misconceptions that the learners had developed related to the concepts. This study found that the learners were operating at the action stage, suggesting that they had not conceptualised the concept. This implies that, due to the misconceptions that the learners had, they struggled to construct the necessary mental constructions, which then hindered their conceptualisation of the concepts. This study found that some of the learners did not engage with what they wrote, suggesting that while they could perform procedures, these were performed without being interiorised to form a coherent schema in the mind of the learner. The findings thus showed that, although learners knew the rules and how to apply them, there was no construction of meaning attached. This can hinder the presentation of a correct solution.

8.4.3 Misconceptions hindering mental constructions for integration

Concerning the meaning of the symbols of integration, the learners made some of the necessary mental constructions. Even though they showed some stage of understanding, there were gaps identified in their responses, especially with those categorised as operating at the action stage. For example, with the indefinite integral $(\int f(x), the$ learners had some difficulty explaining the meaning of the symbol. A common trend displayed by the learners was to confuse definite integral with indefinite integral; some even stated that $\int f(x)$ has no end. Commonly, learners failed to give the mathematical meaning attached to these symbols, and they tended to say it is the reversal of differentiation. The gaps identified in the learners' use of correct mathematical knowledge hindered their interiorisation of the action into a process. This is similar to the findings of Ndlovu (2014), who reports that learners' lack of understanding of notation in mathematics hinders their development of schema. In this study, it was observed that a lack of understanding of notation hindered learners' interiorisation of the action into a process.

In the application of integration rules, 'met befores' as advocated by Dubinsky (1991) were considered the main hindrance to learners' development of mental constructions. Learners confused the procedures of integration with those of differentiation; for example, multiplying the co-efficient by the exponent instead of dividing by the exponent, or subtracting the exponent instead of adding as shown: $\int 6xdx = \frac{6x^{1-1}}{1-1}$. Conceptual understanding refers to making connections between concepts. In this study, it was evident that learners' conceptual understanding of differentiation and integration had not developed as they verbalised that integration is the reversal of integration, but when integrating the function, they ignored the incorporation of the constant 'C'. Another fundamental misconception of learners evident in this study was that of understanding the equal sign to be an operation symbol or to imply the next step, and not understanding it to indicate equivalence, as shown in the example below.



The misuse of the equal sign was also evident in a study conducted by Kazunga and Bansilal (2017), who found that even teachers had the tendency to use the equal sign as an operator or to mean something symbolic.

Misconceptions about notation were also observed. For example, some learners performed the procedure for integration without removing the integration symbols, which indicated their lack of knowledge of the rules of integration. Other learners were substituting the values of a and b in the function of the definite integral; for example, when given $\int_1^2 4xdx$, learners substitute the limits into the function instead of finding the integral.

The next section summarises the study’s response to the research questions that guided the study.

8.5 Summary of Findings on Learners' Mental Constructions for Differentiation and Integration

Although many of the learners in this study appeared to be operating below the action stage, especially in their construction of concepts that rely on higher cognitive functions, such as finding the area under the curve, some learners displayed an action conception of the concepts of differentiation and integration with a smaller number demonstrating interiorisation of the action into a process. It is of concern though that none of the learners had encapsulated the process into an object, meaning that they had not developed coherent schema for differentiation and integration concepts.

8.6 Development of Learners' Mental Constructions for Differentiation and Integration with Reference to APOS Theory

Based on the findings of this study, it was evident that the development of learners' mental constructions was limited to process conception, as none of the learners displayed encapsulation of a process into an object for any of the items. This finding suggests that, while the learners knew the relevant procedures, they had not constructed the conceptual meaning of the procedures, meaning that their understanding of the relationship that exists between the concepts had not yet solidified and they still understood the concepts as isolated facts. The learners had thus constructed a collection of rules, but these were not integrated into a coherent schema. This was demonstrated primarily in their confusion about whether to apply differentiation rules or integration rules in different situations.

8.7 Factors Hindering Learners' Development of Mental Constructions

The findings revealed some factors that inhibited learners' development of the necessary mental constructions. Learners' underlying misconceptions around concepts learned previously were found to be the source of many errors. For example, simplification of algebraic expressions of binomial terms is critical to solving differentiation functions of the form $f(x) = (ax \pm b)^n$. The rules for removing a square root and the laws of exponents are critical for differentiating functions of the form$f(x) = \sqrt{ax^n}$. Thus, a knowledge of basic algebra is needed for the conceptualisation of calculus concepts; a lack of basic algebra knowledge hindered learners' development of the necessary schema for differentiation. Tall (1981, as cited in Ndlovu, 2014) emphasises the development of concept definition for learners to construct the correct concept image. It was found in this study that the learners could not define differentiation and integration; thus, the fact that they had not constructed the necessary concepts hindered their further development. It was found that learners who failed to explain the power rule correctly when asked to define it also failed to apply it when differentiating functions. Similarly, learners who were unable to define definite and indefinite integrals struggled to determine the integral and calculate the area under the curve.

8.8 Implications of the Findings for Teaching and Learning

Analysis of learners' responses related to the concepts of differentiation and integration shows that it is imperative to highlight the implications for teaching and learning. Given that the teaching of differentiation and integration was introduced to learners in Grade 12 stage, it is imperative that, before these topics are taught, pre-requisite concepts are taught to enhance learners' readiness to construct knowledge about differentiation and integration. These concepts include exponents, solving fractions, surds and graphical representations. Reviewing these earlier topics will assist learners so that they will not be hindered in carrying out the procedure to differentiate functions, when encountering questions that involve simplification of expression.

This study further proposes that learners should be able to move from action to process of solving exponents. This could help learners to develop their conceptual understanding beyond the action stage. It is recommended that learners understand graph representation and the area bounded by those graphs. Teachers should introduce diverse pedagogical strategies to help learners grasp all of the concepts in calculus. This can be achieved by working in collaboration learning which will assist learners to perform better in these two concepts.

Consequently, for the learners to understand integration and differentiation, educators need to start by revising algebraic concepts such as exponents and brackets, removal, addition and subtraction of fractions, graphs, and properties of function. Emphasis should be placed on the correct notation for determining the derivative when using the rules. Educators should explain the need for brackets when determining the derivatives and the simplification of brackets. This means that to apply the rules of differentiation, learners need a strong background in algebraic operations; for example, factorisations by converting surds to exponential forms and simplifications of algebraic functions. These skills should be revised before learners are expected to differentiate examples that contain surds or when algebraic manipulations are required. Learners should be assisted in understanding the meaning of symbols in mathematics before using those symbols. This implies that teachers must recognise the language difficulties of learners and emphasis should be placed on the use of the correct notation when determining the derivative when using rules.

The study found that in the classroom there is a need to use pedagogical strategies that explain the need for brackets when determining the derivatives. As a result, learners need to have a strong background in basic algebraic operations, e.g., factorizations, converting surds to exponential form, and the simplification of algebraic functions. These skills should be revised before learners are expected to differentiate examples that contain surds or when algebraic manipulations are required. Hence, teachers should ensure that there is enough time for learners to understand the application of differentiation rules fully. When a problem is given, if there is any symbol like x, it means that the variable a has been defined. In the case of y, it implies that the variable b is defined. Sometimes the problem contains three variables; in this case, the learners must look out for the coefficient of a particular variable and the coefficient of the remaining two variables. If the problem has two variables, one with its coefficient and the other with its coefficient, then the learner can differentiate the other variable for it. If the problem has four variables, then it will have the coefficient of each variable.

Lastly, teachers need to understand that graphs are useful for showing the relationship between two variables. There are different types of graphs. The first is a linear graph that is often found in the basic equation $2abc$. Another kind of graph is a quadratic graph, which is used when the basic equation $2abc$ involves the square of a variable. The third is a cubic graph that shows the relationship between two variables when the basic equation is cubic in one variable.

8.9 Recommendations for Further Research

This research involved a single case study conducted at one school. As a result, the findings of this study cannot be generalised. However, the study yielded important insights concerning learners' mental construction of differentiation and integration concepts and the misconceptions hindering their construction of the necessary mental constructions. The APOS theory provided a valuable framework for examining Grade 12 learners' understanding of differentiation and integration. Most of the learners worked without difficulty at the action stage of understanding; however, several learners struggled with the part of the question requiring an object stage of understanding.

Most of those learners who had difficulty merely followed procedural steps without demonstrating an understanding of the concepts. This suggests that the learners' engagement with the concepts was not well-grounded. The study yielded valuable insights into learners' mental constructions of differentiation and integration. Reflecting on teaching this concept, more consideration should be given to the graphic representation of regions and how the choice of integration, first in one direction and then in the other, depends on the region of integration and hence implies the choice of limits of integration. This study has not only provided valuable insights into learners' mental constructions of differentiation and integration but has also illustrated the potential of the APOS framework to be used in future research and to influence the teaching and learning of differentiations and integration concepts. This aligns with the study's intention to present implications for the teaching and learning of differentiation and integration concepts to promote deeper conceptual understanding.

The present study has investigated Grade 12 learners' understanding of differentiation and integration at one South African high school. The researcher believes that, in addition to the factors identified in this study, there are other significant factors responsible for learners' errors and misunderstandings. Hence, it is recommended that further studies attempt to identify these factors through multiple case studies and longer time spent in the field at different schools. While some in-school factors were identified, it would be valuable to examine in-school and out-of-school factors in more depth to not only identify factors that inhibit, but also those that promote and enhance, understanding of the concepts of differentiation and integration effectively. Additionally, since this study was conducted in one district of KwaZulu-Natal, South Africa, it would be valuable to carry out similar studies in other districts and provinces for greater impact on policy and school practices.

8.10 Limitations of the Study

A study may be limited by obstacles that were encountered or gaps in what it was able to achieve. For this case study, data was collected at only one school, which was under lockdown at the time of the study due to the unprecedented impact of the Covid-19 pandemic. Data was collected from ten learners through their written responses to activity sheets and five of the research participants participated in a follow-up interview. Data generated from one research site may not provide rich and accurate findings. If this study were to be replicated, it would be beneficial for a broader range of research instruments to be used to enrich the findings.

8.11 Conclusion

This chapter revisited the aims, objectives and research questions that drove this study. It synthesised the findings concerning learners' mental constructions with regard to differentiation and integration and the misconceptions that hindered learners' construction of the necessary mental constructions. APOS theory was used to analyse the data. Based on the findings, the chapter provided recommendations for addressing the issues that impeded learners' construction of key mental constructions through improved pedagogical strategies, as well as recommendations for further research. The chapter concluded by outlining the limitations of the study and highlighting that its findings are not generalisable given that it is a case study.

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APPENDIX A1: DATA COLLECTION INSTRUMENTS

ACTIVITY WORKSHEET FOR PHASE ONE

STUDENT NAME ---

DATE --
--

DURATION 1 HOUR 30 MINUTES

Notes for the participants

1. Please answer all questions properly.
2. This activity worksheet does not form part of your assessment.
3. You may be invited for an interview based on your responses to this activity worksheet.

DIFFERENTIATION SECTION

ITEM 1

Answer the following questions:

1.1 Differentiation: Define in your own words the meaning of this concept____

____

1.2 Complete the following $y = x^n then \ \frac{dy}{dx}$ =--
----

ITEM 2

Differentiate using the power rule of differentiation.

$$2.1 f(x) = 3x^2$$

Differentiate

2.2 $f(x) = x^2 + 2x - 3$ using the rules of differentiation (Power rule)

2.3 $f(x) = (2x + 1)^2$

INTEGRATION SECTION

ITEM 3:

In your own understanding define the meaning of the following symbols.

3.1 $\int f(x)dx$ means --

3.2 $\int_a^b f(x)dx$ means --

Determine the following.

3.3 $\int 2x\, dx$--

ITEM 4:

4.1 Draw the graph of $y = x^2$ and find the area under the curve using integration methods between $x = 0$ and $x = 3$ and shade the region.

4.2 Draw the graph of $y = 2x + 1$ and find the area bounded by $x = 1$ and $x = 4$ using integration methods and shade the region.

APPENDIX A2: DATA COLLECTION INSTRUMENT FOR PHASE 2

LEARNER'S NAME: -------------------------------

DURATION: 2 HOURS------------------------------

Notes for the participants

1. Please answer all questions properly
2. This activity worksheet does not form part of your assessment
3. You may be invited for an interview based on your responses to this activity worksheet.
4. This activity worksheet is divided into two sections, which are differential calculus and integration section.

DIFFERENTIATION SECTION

ITEM1

1.1 Define using your own understanding the meaning of **<u>differentiation</u>**

1.2 The average gradient of a curve between two points is called

----

--

1.3 The power rule is defined as follows, if $f(x) = ax^n$

Then $f^1(x)$= ---

<u>ITEM 2</u>

Calculate the derivative using the power rule

2.1 $f(x) = 3x^2$

2.2 $f(x) = x^2 + 2x - 3$

2.3 $f(x) = (2x + 1)^2$

2.4 $f(x) = \sqrt{x^6}$

INTEGRATION SECTION

ITEM 3

Please explain in your understanding the difference between $\int f(x)dx$ and $\int_a^b f(x)dx$

3.1 $\int f(x)dx$ means--

3.2 $\int_a^b f(x)dx$ means ---

3.3 Find the following using integration rules

$\int 6x\ dx$

3.4 $\int_1^2 4x\ dx$

ITEM 4

4.1 Find the bounded area by drawing the graph of $f(x) = \sqrt{9 - x^2}$ and $between\ x = 1\ and\ x = 4$ using integration methods.

4.2 Find the definite integral $\int_0^3 x^2 dx$ and draw the curve to indicate the bounded area.

APPENDIX B: CONSENT FORMS/ LETTERS

UNIVERSITY OF KWAZULU-NATAL

INYUVESI YAKWAZULU-NATALI

INFORMED CONSENT LETTER TO PARTICIPANTS.

Research Project: An analysis of grade 12 learners'mental constructions and difficulties of Differentiation and Integration: A case study of one school in Umlazi District.

My name is Mrs Promise Sethembiso Madonsela. I am a PhD student in the School of Education at the University of KwaZulu–Natal in the discipline of Mathematics Education.

I am interested in analysing grade 12 mental constructions of differentiation and integration in one school in Umlazi District. To gather information, I am interested in asking you some questions.

Please note that:

- Your confidentiality is guaranteed as your inputs will not be attributed to you in person, but reported only as a population member opinion.
- The research will be done online because of COVID19 pandemic, activity sheets will be sent you through WhatsApp and time will be monitored (1h 30) and after finishing answers will be scanned back to me through your phone.
- The interviews will be on Zoom and may last for about 30 minutes and may be split depending on your preference.
- Any information given by you cannot be used against you, and the collected data will be used for purposes of this research only.
- Data will be stored in secure storage and destroyed after 5 years.
- You have a choice to participate, not participate or stop participating in the research. You will not be penalized for taking such an action.
- The research aims at knowing the challenges or difficulties and enabling factors that affect you when solving differentiation and integration. Your involvement is purely for academic purposes only, and there are no financial benefits involved.

- If you are willing to be interviewed, please indicate (by ticking as applicable) whether or r you are willing to allow the interview to be recorded by the following equipment:

	Willing	Not willing
Participate in the interview	✓	
Audio recorded	✓	
Video equipment	✓	

I can be contacted at:

E-mail: [redacted]

Landline: 031 9069194

Cell: [redacted]

My supervisor is Prof Zanele Ndlovu who is a lecturer at the School of Education, Edgewood campus of the University of KwaZulu-Natal.

She can be contacted at:

E-mail: Ndlovuz3@ukzn.ac.za

Phone number: [redacted]

You may also contact the Research Office through:

Ms Mbalenhle Ngcobo (HSSREC Research Office)

Tel: 031 260 3436

Email: mgcobom4@ukzn.ac.za)

Thank you for your contribution to this research.

DECLARATION

I........Sethu Khwela.. (full names of participant) hereby confirm that I understand the contents of this document and the nature of the research project, and I consent to participating in the research project.

I understand that I am at liberty to withdraw from the project at any time, should I so desire.

SIGNATURE OF PARTICIPANT **DATE**

..........[redacted]..................... 07/07/2020..............................

INCWADI YOMZALI

[redacted]

Resevoir Hills

4091

Mzali

Ngiyakubingelela, ngiwuthisha ofundisa eMakhumbuza High School. Ngenza ucwaningo mayelana nesifundo sezibalo (Mathematics). Abantwana abenzi kahle kulesisifundo iminyaka eminingi. Manje ucwaningo luzobheka izimbangela zalemiphumela emibi ngakangaka.

Umntwana wakho uyacelwa ukuba abe yingxenye yalolucwaningo. Igama lakhe lizovikeleka, futhi ngeke aphazamiseke ezifundweni zakhe. Konke kuzobayimfihlo, futhi awukho umholo ozotholakala. Imiphumela yocwaningo izosiza abaphethe ezemfundo ukeze ekugcineni kusizakale umntwana.

Ozithobayo

Promise S Madonsela

Signatur[redacted]......

Date.....07/09/2020.......

NB Khetha okukodwa (Tick one)

Ngiyavuma (Agree)	✓
Angivumi (Disagree)	

Umzali [redacted]

Usuku 17 — 09 — 2020

MAKHUMBUZA HIGH SCHOOL

DEPARTMENT OF BASIC EDUCATION

PROVINCE OF KWAZULU NATAL

ENQUIRIES: MR B DLOMO
TEL: 031-906-9194
FAX: 031-906-9194
[redacted]

POSTAL ADDRESS: P.O BOX 54400
PHYSICAL ADDRESS: SECTION D896
UMLAZI
4031
[redacted]

REGION: ETHEKWINI
DISTRICT: UMLAZI
CIRCUIT: PHUMELELA
EMIS:500194842

Promise Sethembiso Madonsela

LETTER TO THE PRINCIPAL

Dear Mr Dlomo

Request for permission to do data collection at your school

I am an educator at Makhumbuza High Schools currently registered for a PHD program with the University of KwaZulu –Natal doing a research study. My aim is to analyze grade 12 learner's mental construction and difficulties of differentiation and integration in calculus. The main aim of the study is to also investigate poor performance in Mathematics as a subject.

Due to the Covid-19 pandemic there will no physical contact with your learners. I will be communicating with them through online learning (WhatsApp and video calls). Please allow me to use your learners without using their real names for the study. A letter asking both permission and consent form to parents will follow soon. This exercise will help your school to achieve better results in mathematics

[redacted]

Researchers Signature

07/09/2020

Date

DECLARATION

I Sibusiso E. Dladla .... (Name) the principal of Makhumbuza High School on this day --7th---------Month-September-Year--2022---------------hereby grant permission to the researcher to go ahead with the research in the above-mentioned school following terms of reference noted in this request letter.

[redacted] ------------ 07/09/2020

Date

APPENDIX C: ETHICAL CLEARANCE CERTIFICATE

UNIVERSITY OF KWAZULU-NATAL™
INYUVESI YAKWAZULU-NATALI

05 January 2021

Mrs Promise Sethembiso Madonsela (200301752)
School of Education
Edgewood Campus

Dear Mrs Madonsela,

Protocol reference number: HSSREC/00002126/2020
Project title: An analysis of grade 12 learners' mental constructions and difficulties of differentiation and integration. A case study of one school in Umlazi District.
Degree: PhD

Approval Notification – Expedited Application

This letter serves to notify you that your application received on 15 October 2020 in connection with the above, was reviewed by the Humanities and Social Sciences Research Ethics Committee (HSSREC) and the protocol has been granted **FULL APPROVAL.**

Any alteration/s to the approved research protocol i.e. Questionnaire/Interview Schedule, Informed Consent Form, Title of the Project, Location of the Study, Research Approach and Methods must be reviewed and approved through the amendment/modification prior to its implementation. In case you have further queries, please quote the above reference number. PLEASE NOTE: Research data should be securely stored in the discipline/department for a period of 5 years.

This approval is valid until 07 January 2022.
To ensure uninterrupted approval of this study beyond the approval expiry date, a progress report must be submitted to the Research Office on the appropriate form 2 - 3 months before the expiry date. A close-out report to be submitted when study is finished.

All research conducted during the COVID-19 period must adhere to the national and UKZN guidelines.

HSSREC is registered with the South African National Research Ethics Council (REC-040414-040).

Yours sincerely,

[redacted]

Professor Dipane Hlalele (Chair)

/ms

Humanities and Social Sciences Research Ethics Committee
Postal Address: Private Bag X54001, Durban, 4000, South Africa
Telephone: +27 (0)31 260 8350/4557/3587 **Email:** hssrec@ukzn.ac.za **Website:** http://research.ukzn.ac.za/Research-Ethics

Founding Campuses: Edgewood Howard College Medical School Pietermaritzburg Westville

INSPIRING GREATNESS

KWAZULU-NATAL PROVINCE
EDUCATION
REPUBLIC OF SOUTH AFRICA

OFFICE OF THE HEAD OF DEPARTMENT

Private Bag X9137, PIETERMARITZBURG, 3200
Anton Lembede Building, 247 Burger Street, Pietermaritzburg, 3201
Tel: 033 392 1063

Email: Phindile.duma@kzndoe.gov.za

Enquiries: Phindile Duma

Ref.:2/4/8/41175

Mrs PS Madonsela
[redacted]
Reservoir Hills
DURBAN
4091

Dear Mrs Madonsela

PERMISSION TO CONDUCT RESEARCH IN THE KZN DoE INSTITUTIONS

Your application to conduct research entitled: **"AN ANALYSIS OF GRADE 12 LEARNERS' MENTAL CONSTRUCTIONS AND DIFFICULTIES OF DIFFERENTIATION AND INTEGRATION. A CASE OF ONE SCHOOL IN UMLAZI DISTRICT"**, in the KwaZulu-Natal Department of Education Institutions has been approved. The conditions of the approval are as follows:

1. The researcher will make all the arrangements concerning the research and interviews.
2. The researcher must ensure that Educator and learning programmes are not interrupted.
3. Interviews are not conducted during the time of writing examinations in schools.
4. Learners, Educators, Schools and Institutions are not identifiable in any way from the results of the research.
5. A copy of this letter is submitted to District Managers, Principals and Heads of Institutions where the Intended research and interviews are to be conducted.
6. The period of investigation is limited to the period from 21 November 2022 to 31 October 2025.
7. Your research and interviews will be limited to the schools you have proposed and approved by the Head of Department. Please note that Principals, Educators, Departmental Officials and Learners are under no obligation to participate or assist you in your investigation.
8. Should you wish to extend the period of your survey at the school(s), please contact Miss Phindile Duma at the contact numbers above.
9. Upon completion of the research, a brief summary of the findings, recommendations or a full report/dissertation/thesis must be submitted to the research office of the Department. Please address it to The Office of the HOD, Private Bag X9137, Pietermaritzburg, 3200.
10. Please note that your research and interviews will be limited to schools and institutions in KwaZulu-Natal Department of Education.

UMLAZI DISTRICT

[redacted]

Mr GN Ngcobo
Head of Department: Education
Date: 24 November 2022

GROWING KWAZULU-NATAL TOGETHER

APPENDIX D: FIRST EDITOR'S CERTIFICATE

[redacted]
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E-mail:
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Dr Saths Govender

27 November 2022

TO WHOM IT MAY CONCERN

LANGUAGE CLEARANCE CERTIFICATE

This serves to inform that I have read the final version of the thesis titled:

An analysis of grade 12 learners' mental constructions and difficulties of Differentiation and Integration: A case study of one school in Umlazi District, by Promise Sethembiso Madonsela, student no. 200301752.

To the best of my knowledge, all the proposed amendments have been effected and the work is free of spelling and grammatical errors. I am of the view that the quality of language used meets generally accepted academic standards.

Yours faithfully

[redacted]

DR S. GOVENDER
B Paed. (Arts), B.A. (Hons), B Ed.
Cambridge Certificate for English Medium Teachers
MPA, D. Admin.

APPENDIX E: SECOND EDITOR'S CERTIFICATE

CERTIFICATE OF PROFESSIONAL EDITING

I, Barbara L. Louton, declare that I am a professional editor with a Bachelor of Arts in Professional Writing and seventeen years of experience as an editor, researcher and writer.

I declare that I have proofread the following doctoral thesis:

An analysis of Grade 12 learners' mental constructions and difficulties of differentiation and integration: A case study of one school in Umlazi District

by

Promise Sethembiso Madonsela (Student number 200301752)
School of Education, College of Humanities, University of KwaZulu-Natal
(Supervisor: Prof. Annatoria Zanele Ngcobo)

Disclaimer:
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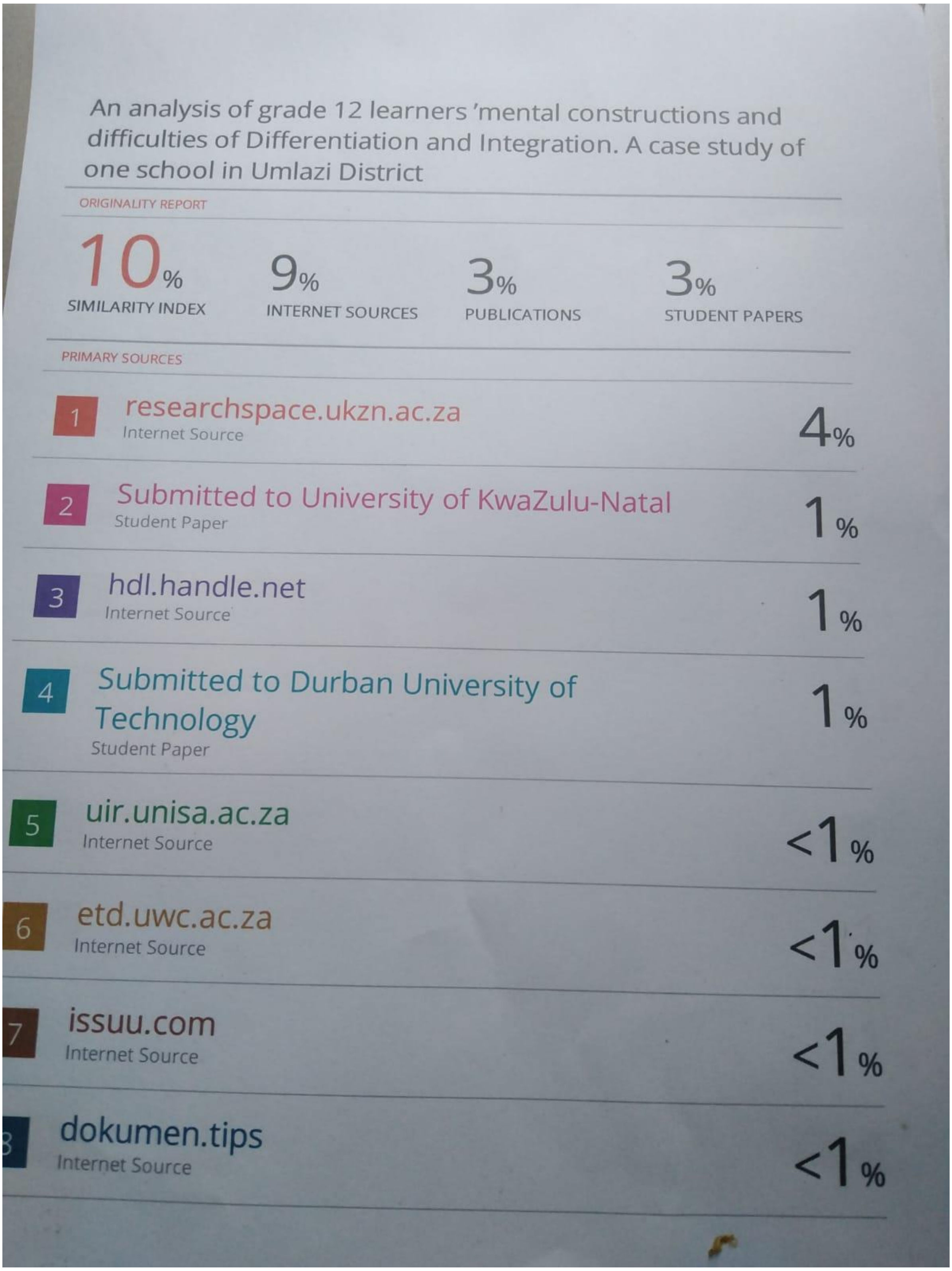
I can be contacted at:
Cell: [redacted]
Email: [redacted]

Barbara L Louton	[redacted]	8 July 2023
Name	**Signed**	**Date**

APPENDIX F: TURN IT IN REPORT



An analysis of grade 12 learners 'mental constructions and difficulties of Differentiation and Integration. A case study of one school in Umlazi District

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