Analysis of nonlinear Benjamin equation posed on the real line

By

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DECLARATION

The work described by this thesis was carried out at the University of Kwazulu-Natal, School of Mathematics, Statistics and Computer Science, University of Kwazulu-Natal, Westville Campus, under the supervision of Dr. N. Parumasur and Prof. S. Shindin.

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Dedication

This thesis is dedicated to Almighty God, who guided me from the beginning of this program to the end. I return all the glory, honour and adoration to His holy name.

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Abstract

The thesis contains a comprehensive theoretical and numerical study of the nonlinear Benjamin equation posed in the real line. We explore wellposedness of the problem in weighted settings and provide a detailed study of existence, regularity and orbital stability of traveling wave solutions. Further, we present a comprehensive study of the Malmquist-Takenaka-Christov (MTC) computational basis and employ it for the numerical treatment of the nonstaionary and the stationary Benjamin equations.

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Outcome of Research Work - Publication

Details of publication that form part and/or include research presented in this thesis.

- Sergey Shindin, Nabendra Parumusur and Olabisi Aluko, Analysis of Malmquist-Takenaka-Christov rational approximation with applications to the nonlinear Benjamin equation. *Commun. Nonlinear Sci. Numer. Simul.*, **94** (2021), 105571.
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Chapter 1

Introduction

Evolution equations appear naturally in mathematical description of various phenomena arising in physics, engineering, technology, biology, social sciences, e.t.c. As a consequence, rigorous mathematical treatment of these equations (both qualitative and quantitative) constitutes a large body of pure and applied modern mathematical research. In the thesis, we focus on the analysis of one such problem, namely on the nonlinear Benjamin equation (NBE) that describes propagation of internal waves in a two fluid system [10]. In past two decades, noticeable consideration has been given to the NBE equation, mainly due to its mathematical complexity and potential applications in applied science and engineering. In particular, the solvability of Cauchy problem associated to the NBE is addressed in [29, 63, 64, 95]. Existence and orbital stability of traveling wave solutions is the main subject of [5, 10, 13, 74]. Numerical treatment of both stationary and non-stationary NBE can be found in [5, 26, 35, 36, 53]. Nevertheless, despite significant progress, rigorous theoretical and numerical analysis of the NBE model is far from being complete. In this Chapter, we provide a survey of the current state of art of theoretical and numerical results available for the NBE model and describe our contribution to each of these topics.

1.1 The nonlinear Benjamin equation

The nonlinear Benjamin equation was proposed by T.B. Benjamin in his study of long internal waves propagating in a two-fluid system [10, 11] (see also derivations

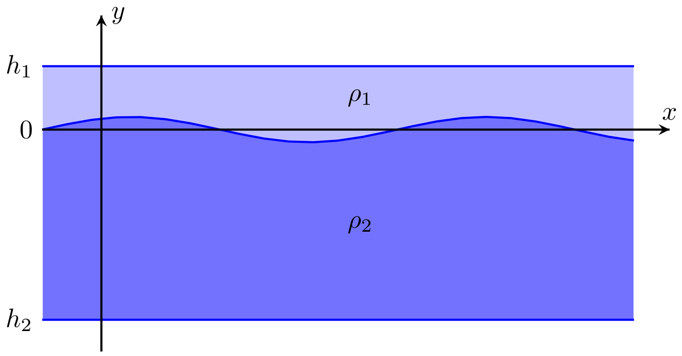

Figure 1.1: The sketch showing the wave propagating in a two-fluid system

in [5]) shown in Figure 1.1.

The system is made up of two fluids with densities $\rho_1 < \rho_2$ and depths $h_2 < 0 < h_1$, $|h_1| < |h_2|$, where the interface between two fluids is subject to the tension T. The system is bounded by the rigid planes $y = h_2$ and $y = h_1$, from below and above, respectively. The position of the interface at rest corresponds to the plane $y = 0$, see Fig. 1.1.

To derive the dispersion relation, connecting the wave number ξ and the frequency ω of a surface harmonic wave, we follow the exposition from [62]. Assuming that the velocity fields in each fluid are potential, we have

$$u_i(x, y, t) = C_i \cosh\big[\xi(y + h_i)\big] \cos(\xi x) e^{i\omega t}, \quad i = 1, 2. \tag{1.1.1}$$

Since the interface wave is harmonic, its vertical displacement reads

$$\eta = \kappa \cos(\xi x) e^{i\omega t}, \tag{1.1.2}$$

where κ is the wave amplitude. It is expected that the normal component of the fluid velocities at the interface surface are equal to the normal velocity of the surface itself. Hence,

$$\eta_t + u_{iy}\Big|_{y=0} = 0, \quad i = 1, 2,$$

and from(1.1.1) and (1.1.2), we obtain

$$C_i = -\frac{i\omega\kappa}{\xi \sinh(\xi h_i)}, \quad i = 1, 2. \tag{1.1.3}$$

The pressures are given by

$$p_i = \rho_i\big(u_{it} - g\kappa\big), \quad i = 1, 2.$$

In the presence of capillarity modeled by the surface tension T, the pressures must satisfy the following jump condition

$$p_1 - p_2 + T\eta_{xx}\Big|_{y=0} = 0.$$

Upon the substitution of (1.1.3) into the last formula, we deduce

$$\omega^2 = \frac{g\xi(\rho_2-\rho_1)+T\xi^3}{\rho_1\coth(\xi h_1)-\rho_2\coth(\xi h_2)},$$

which, when rewritten in terms of the wave speed $c^2 = \frac{\omega^2}{\xi^2}$, yields the following dispersion relation

$$c^2 = \frac{g(\rho_2-\rho_1)+T\xi^2}{\rho_1\xi\coth(\xi h_1)-\rho_2\xi\coth(\xi h_2)}.$$

Note that under the assumptions that $|\xi h_2|$ is extremely large and respectively $\coth(\xi h_2)\approx -\operatorname{sgn}\xi$, the above relation becomes

$$c(\xi)^2 = \frac{g(\rho_2-\rho_1)+T\xi^2}{\rho_1\xi\coth(\xi h_1)+\rho_2|\xi|}.$$

T.B. Benjamin considered the scenario, where the wave amplitude is much smaller as compared to h_1 and $|\xi|$ is close to zero (long waves). The upper fluid of the model usually has a vital influence on the dispersive properties of waves at small finite values of ξ, see [9]. Developing $c(\xi)$ in Taylor' series in powers of $|\xi|$, we have

$$c(\xi) = c(0)\Big[1 - \frac{1}{2}\frac{\rho_2}{\rho_1}h_1|\xi| + \frac{1}{2}\Big(\frac{T}{g(\rho_2-\rho_1)h_1^2} + \frac{1}{2}\frac{\rho_2^2}{\rho_1^2} - \frac{1}{3}\Big)h_1^2\xi^2 + \mathcal{O}(\xi^3)\Big],$$

provided the quantity $\frac{T}{g(\rho_2-\rho_1)h_1^2}$ is sufficiently large and $h_1|\xi|$ is small. After neglecting the $\mathcal{O}(\xi^3)$ terms, the approximate dimensionless dispersion relation reads

$$c(\xi) = \alpha - \beta|\xi| + \gamma\xi^2,$$

where

$$\alpha = \sqrt{\frac{gh_1(\rho_2-\rho_1)}{\rho 1}}, \quad \beta = \frac{\alpha}{2}\frac{\rho_2}{\rho_1}, \quad \gamma = \frac{\alpha}{2}\Big(\frac{T}{g(\rho_2-\rho_1)h_1^2} + \frac{1}{2}\frac{\rho_2^2}{\rho_1^2} - \frac{1}{3}\Big).$$

Hence, approximately the unidirectional propagation of internal long waves is governed by the dimensionless equation

$$u_t + \big(\alpha u - \beta\mathcal{H}[u_x] - \gamma u_{xx} + \delta(-u)^\ell\big)_x = 0, \qquad (1.1.4)$$

where $\ell \geq 2$ is an integer. The concrete meaning of operator $\mathcal{H}$ depends on the problem settings. In particular, for long waves in an unbounded channel ($x \in \mathbb{R}$) or for 2π-periodic waves ($x \in [-\pi, \pi]$), $\mathcal{H}$ is the standard Hilbert transform, i.e.

$$\mathcal{H}[u](x) = \tfrac{p.v.}{\pi} \int_{\mathbb{R}} \frac{u(y)}{x-y} dy, \quad x \in \mathbb{R},$$
$$\mathcal{H}[u](x) = \tfrac{p.v.}{2\pi} \int_{-\pi}^{\pi} u(x-y) \cot \tfrac{y}{2} dy, \quad x \in [-\pi, \pi],$$

where $p.v.$ stands for the principal value integral.

We note that for $\beta = 0$ and $\ell = 2$, (1.1.4) is equivalent to the classical Korteweg-de Vries equation (KdV), while letting $\gamma = 0$ we recover as special case the well known Benjamin-Ono (BO) equation. Analysis of both equations received significant attention in physical and mathematical literature, see e.g. [18, 31, 52, 57, 58, 76, 88, 93] and references therein.

1.2 Wellposedness

Study of Cauchy Problem (1.1.4) in various functional settings has generated a substantial body of modern research, see [14, 18, 56, 57, 58, 88]. Before we provide concrete details, we note that (1.1.4) is an extension of the classical KdV equation ($\beta = 0, \ell = 2$). Since the perturbation term $\mathcal{H}[u_{xx}]$ is of low order two, many classical wellposedness results available for KdV allow verbatim extension to the Benjamin equation (1.1.4). Among these results, we mention two [14, 56].

Bona and Smith in [14] demonstrated global wellposedness of the KdV equation on the real line for initial data $u_0 \in H^s(\mathbb{R})$, with $s \geq 2$. Their analysis makes use of a perturbation technique, coupled with use of the first integrals of the equation. As a consequence, they established existence of global (in time) classical solutions for initial data in $H^s(\mathbb{R})$, with $s \geq 3$, s is integer. The results were further extended to $s = 2$ using a weak compactness argument. The latter solutions (with $s = 2$) satisfy the KdV in the weak sense. Generalization of these results to fractional order Sobolev spaces $H^s(\mathbb{R})$, $s \geq 2$, is done in [15].

Independently, the wellposedness analysis of a generalized KdV model (which covers $\ell \geq 2$ as well) was done by T. Kato in [56]. Using the theory of quasi-linear evolution equations, he demonstrated existence of global mild and weak solution on the real line for initial data in $H^s(\mathbb{R})$, with $s > \frac{3}{2}$. The above results apply directly to (1.1.4) and can be formulated as follows:

Theorem 1.2.1. *For $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$ and any fixed $T > 0$, the pure initial value problem, associated with* (1.1.4)*, has a unique, globally defined weak solution of class*

$$u \in C([0,T); H^s(\mathbb{R}) \cap C^1((0,T); H^{s-3}).$$

The solution u is classical if $s \geq 3$.

The wellposedness results for (1.1.4) with rough initial data ($s \leq \frac{3}{2}$) were obtained relatively recently using methods different from the classical techniques of J. Bona, R. Smith and T. Kato. The studies of the Cauchy problem in the periodic and in the real line settings were done in parallel. We present the concrete details below.

1.2.1 The periodic settings

The first low regularity results for initial data in $L^2(\mathbb{T})$ (and in $L^2(\mathbb{R})$) were obtained in the paper by F. Linares [64], where the variation of constant formula associated to the group of unitary operators $\{e^{t\mathcal{A}}\}_{t\in\mathbb{R}}$, generated by the linear part of (1.1.4) (i.e. by the operator $\mathcal{A} := -\alpha\partial_x + \beta\mathcal{H}\partial_{xx} + \gamma\partial_{xxx}$) was employed. The solvability of the resulting abstract Volterra-type integral equation is done in the Bourgain space[1] $Y_{s,b}$, whose norm is defined in terms of classical trigonometric Fourier coefficients as follows

$$\|u\|_{Y_{s,b}} = \left(\sum_{n\neq 0}\int_{\mathbb{R}} |n|^{2s}(1+|\tau - \beta|n|n + \gamma n^2|)^{2b}|\hat{u}_n(\tau)|^2 d\tau\right)^{\frac{1}{2}}, \quad s,b \in \mathbb{R}. \quad (1.2.1)$$

The local solvability for initial data in $H^s(\mathbb{T})$, $s \geq 0$, is achieved by combining a bilinear estimates and the classical contraction mapping principle. The global result follows from the preservation of $L^2(\mathbb{T})$ norm of the local solutions. The main results of [64] can be summarized as follows:

[1]Since the seminal works of J. Bourgain [17, 18] on low regularity solutions for KdV and the nonlinear Schrödinger equations (NSE), the approach is standard in the analysis of semi- and quasi-linear evolution equations.

Theorem 1.2.2. *For $u_0 \in L^2(\mathbb{T})$, with $\int_{\mathbb{T}} u_0(x)dx = 0$ and any $T > 0$, the periodic initial value problem associated to equation* (1.1.4)*, with $\ell = 2$, has a unique mild solution u of class*

$$u \in C([0,T]; L^2(\mathbb{T})) \cap Y_{0,\frac{1}{2}}, \quad (u^2)_x \in Y_{0,-\frac{1}{2}}.$$

The solution map $u_0 \mapsto u(t)$ is locally Lipschitz continuous from $L^2(\mathbb{T})$ to $C([0,T]; L^2(\mathbb{T})) \cap Y_{0,\frac{1}{2}}$.

In [86] a consideration to small data initial value problem for (1.1.4) in $H^s(\mathbb{T})$, $s \geq -\frac{1}{2}$ is given. The authors improved the crucial bilinear estimate, employed in [64], and applied the classical Picard-Lindelöf iterations to extend F. Linares results from $s \geq 0$ to $s \geq -\frac{1}{2}$. The main results of their work reads:

Theorem 1.2.3. *For $s \geq -\frac{1}{2}$ and $u_0 \in H^s(\mathbb{T})$, there exist $T = T(\|u_0\|_{H^s(\mathbb{T})})$, such that the periodic initial value problem for* (1.1.4) *has a unique mild solution u of class*

$$u \in C([0,T], H^s(\mathbb{T})) \cap Y_{s,\frac{1}{2}} \cap L^1([0,T], H^s(\mathbb{T})).$$

The solution map $u_0 \mapsto u$ is analytic from $H^s(\mathbb{T})$ to $C([0,T], H^s(\mathbb{T})) \cap Y_{s,\frac{1}{2}} \cap L^1([0,T], H^s(\mathbb{T}))$.

Theorem 1.2.3 is purely local and its proof cannot be extended to the values of s below $-\frac{1}{2}$, for it is shown in the same paper that the bilinear estimate

$$\|u_x v\|_{Y_{s,-\frac{1}{2}}} \leq c\|u\|_{Y_{s,\frac{1}{2}}} \|v\|_{Y_{s,\frac{1}{2}}},$$

fails for $s < -\frac{1}{2}$ in general.

1.2.2 The real line settings

In the real line settings, a variety of approaches, ranging from bilinear estimates to I-method (see [29, 59, 63, 64]) were employed. As in the periodic case, one of the standard analytic tools here is the Bourgain spaces $X_{s,b}$, whose norm in the case of $\mathbb{R}$ is given by

$$\|u\|_{X_{s,b}} = \left(\int_{\mathbb{R}} \int_{\mathbb{R}} (1 + |\tau - \beta|\xi|\xi + \gamma\xi^2|)^{2b} (1 + |\xi|)^{2s} |\hat{u}(\xi,\tau)|^2 d\xi d\tau \right)^{\frac{1}{2}}.$$

The first low-regularity results in the real line case were obtained in the earlier cited paper [64]. The analysis of [64] is parallel to the periodic case, and yields the following result:

Theorem 1.2.4. *For $u_0 \in L^2(\mathbb{R})$, $b \in \left(\frac{1}{2}, \frac{5}{6}\right)$ and any $T > 0$, the initial value problem associated to equation* (1.1.4)*, with $\ell = 2$, has a unique mild solution u satisfying*

$$u \in C([0,T]; L^2(\mathbb{R})) \cap X_{0,b}, \quad (u^2)_x \in X_{0,b-1}.$$

The solution map $u_0 \mapsto u(t)$ is locally Lipschitz continuous from $L^2(\mathbb{R})$ to $C([0,T]; L^2(\mathbb{R})) \cap X_{0,b}$.

In the case of the real line, the original result of [64] was extended in [59] by observing that the bilinear estimate

$$\|(u^2)_x\|_{X_{s,b-1}} \leq c\|u\|_{X_{s,b}}\|v\|_{X_{s,b}}$$

holds for each $-\frac{3}{4} < s \leq 0$, with some $\frac{1}{2} < b = b_s < 1$. This yields the following refinement of Theorem 1.2.4:

Theorem 1.2.5. *For $u_0 \in H^s(\mathbb{R})$, $-\frac{3}{4} < s \leq 0$, there exists $T = T(\|u_0\|_{H^s(\mathbb{R})})$ ($T(a) \to \infty$, as $a \to 0^+$) and $\frac{1}{2} < b < 1$, such that the initial value problem associated to* (1.1.4)*, with $\ell = 2$, has a unique mild solution u of class*

$$u \in C([-T,T]; H^s(\mathbb{R})) \cap X_{s,b}, \quad (u^2)_x \in X_{s,b-1}.$$

The solution map $u_0 \mapsto u$ is locally Lipschitz continuous from $H^s(\mathbb{R})$ to $C([-T,T]; H^s(\mathbb{R})) \cap X_{s,b}$.

Theorems 1.2.4 and 1.2.5 are purely local. The first global results in $H^s(\mathbb{R})$, $-\frac{3}{4} < s < 0$, are obtained in [63]. The local analysis proceeds along the lines of [59, 64] but uses a refined bilinear estimate for a frequency cut-off operator I applied to the quadratic nonlinearity (uv). In addition to local solvability, this gives explicit estimates for the growth rate of mild solutions and on the lengths of their intervals of existence. These bounds allows to track the growth rate of the cut-off functional and, combined with the standard continuation argument, yield global wellposedness. The analysis of [63] can be summarized as follows

Theorem 1.2.6. *Let $u_0 \in H^s(\mathbb{R})$, with $s > -\frac{3}{4}$. For $\frac{1}{2} < b < 1$ and for any $T > 0$, there exist a unique mild solution u to (1.1.4), with $\ell = 2$, such that*

$$u \in X_{s,b}^T \subset C([0,T]; H^s(\mathbb{R})).$$

The solution map $u_0 \mapsto u$ is continuous from $H^s(\mathbb{R})$ to $X_{s,b}^T$.

The local and global analysis of [59, 63, 64] does not carry to the critical case of $H^{-\frac{3}{4}}(\mathbb{R})$. The main obstacle is that the bilinear estimate

$$\|(uv)_x\|_{X_{s,b-1}} \leq \|u\|_{X_{s,b}} \|v\|_{X_{s,b}},$$

employed in [59, 64], as well as its I analogue from [63], fail for $s \leq -\frac{3}{4}$. The critical case was settled recently in [29], where in place of the Bourgain $X_{s,b}$ space, a Besov-Bourgain space $\bar{F}^s$, equipped with the norm

$$\|u\|_{\bar{F}^s} = \left[\sum_{k \geq 0} 2^{2sk} \left(\sum_{j \geq 0} 2^{\frac{j}{2}} \left\| \eta_j(\xi) \left[\eta_k(\tau - \beta|\xi|\xi + \gamma\xi^2) * \hat{u}(\tau, \xi) \right] \right\|_{L^2(\mathbb{R}^2)} \right)^2 \right]^{\frac{1}{2}},$$

is employed (here η_0 is a compactly supported positive cut-off function and $\eta_k(\cdot) = \eta_0(\frac{\cdot}{2^k}) - \eta_0(\frac{\cdot}{k-1})$, $k \geq 1$). Long calculations presented in [29] demonstrate that an analogue of the bilinear estimate holds in $\bar{F}^{-\frac{3}{4}}$. This allows to settle positively the local well-posedness of (1.1.4) for initial data in $H^{-\frac{3}{4}}(\mathbb{R})$. The global analysis then proceeds along the lines of [63] and yields the following sharp result

Theorem 1.2.7. *For $u_0 \in H^s(\mathbb{R})$, with $s \geq -\frac{3}{4}$ and any $T > 0$, there exist a unique mild solution u to (1.1.4), with $\ell = 2$, such that*

$$u \in \bar{F}^{-\frac{3}{4}}(\mathbb{R}) \cap C([-T,T]; H^{-\frac{3}{4}}(\mathbb{R})).$$

The solution map is continuous from $H^s(\mathbb{R})$ into $C([-T,T]; H^s(\mathbb{R}))$ for all $s \geq -\frac{3}{4}$.

1.2.3 The weighted settings

Due to the presence of the Hilbert transform $\mathcal{H}$, the Fourier symbol of the group generated by the linear part of (1.1.4) has finite regularity at the origin. As a consequence, one cannot expect super-algebraic spatial decay of solutions even for the rapidly decreasing initial data u_0 from the Schwartz class $\mathcal{S}(\mathbb{R})$. Studies of the interplay between existence, regularity and asymptotic of global solutions to (1.1.4) are initiated in [95]. In particular, for the weighted Sobolev spaces

$$Z_{s,r} = H^s(\mathbb{R}) \cap L^2(\mathbb{R}, |x|^{2r}dx), \quad \mathring{Z}_{s,r} = \{u \in Z_{s,r} : \hat{u}(0) = 0\},$$

the following result was obtained:

Theorem 1.2.8. *Let $s \geq 1$, $0 \leq r \leq \frac{s}{2}$ and $r < \frac{5}{2}$. If $u_0 \in Z_{s,r}$ then a mild global solution (if it exists) must be of class $C([0,\infty); Z_{s,r})$. If $s > \frac{3}{2}$, then the initial value problem associated to* (1.1.4) *is globally well-posed in $Z_{s,r}$. In addition, if $\frac{5}{2} \leq r < \frac{7}{2}$ and $r \leq \frac{s}{2}$ the initial value problem is globally well-posed in $\mathring{Z}_{s,r}$.*

Theorem 1.2.8 indicates that in the presence of the Hilbert transform the Fourier symbol of the linear part of the Benjamin equation is irregular near the origin (its third order derivative develops jump discontinuity at the origin). As a consequence, one can expect only slow algebraic decay of solutions at infinity ($r < \frac{5}{2}$). The situation can be slightly improved in the weighted homogeneous Sobolev spaces $\mathring{\mathcal{Z}}_{s,r}$, where the propagation of the discontinuity in the Fourier image of the solution can be properly controlled for $r \leq \frac{s}{2}$, $r \in \left[\frac{5}{2}, \frac{7}{2}\right)$ (see the last statement in Theorem 1.2.8).

1.3 The stationary Benjamin equation

The main motivation of developing model (1.1.4) in works of Benjamin [11, 10] was to investigate existence and persistence of small amplitude internal waves that arise in a two fluid system under a capillarity effect. Such waves are governed by solutions of the form $u(x,t) = -\phi(x-ct)$, where ϕ describes the waves profile and c denotes the propagation speed. Substituting the ansatz into (1.1.4) and integrating over $(-\infty, x]$, we arrive at

$$C\phi - \beta\mathcal{H}[\phi'] - \gamma\phi'' = \delta\phi^\ell, \quad C = \alpha - c. \tag{1.3.1}$$

For $c < \alpha$, the latter formula is further simplified by letting $\phi(x) = \frac{C^{\frac{1}{\ell-1}}}{\delta^{\frac{1}{\ell+1}}}\varphi\left(\sqrt{\frac{C}{\gamma}}x\right)$ and $\mu = \frac{\beta}{2\sqrt{\gamma C}}$. In terms of φ, the traveling wave equation reads

$$\varphi - 2\mu\mathcal{H}[\varphi'] - \varphi'' = \varphi^\ell. \tag{1.3.2}$$

As observed in [10], the pseudo-differential equation (1.3.2) is solvable only when $\mu < 1$. Furthermore, the solutions (if they exist) are physically relevant only for μ close to 1. The solvability of the stationary Benjamin equation was examined in periodic and real line settings by a number of authors [5, 10, 74]. We list the concrete details below.

1.3.1 The periodic settings

The existence theory for (1.3.2) is quite simple and was carried out originally by T.B. Benjamin himself in [10]. His analysis uses a combination of the positive-operator method and the Leray-Schauder degree theory (applied in the spirit of M.A. Krasnoselskii [61]). In this approach, (1.3.2) is viewed as a fixed point problem

$$\hat{\varphi} = \mathcal{G}_\mu(\hat{\varphi}),$$

where $\hat{\varphi} = (\hat{\varphi}_n)_{n\in\mathbb{Z}} \in \ell^2$ is the vector of the discrete Fourier coefficients of φ,

$$\mathcal{G}_\mu(\varphi)_n = \tfrac{1}{\kappa_\mu(n)}(\hat{\varphi}^{*\ell})_n, \quad \kappa_\mu(n) = 1 - 2\mu|n| + n^2, \quad n \in \mathbb{Z}$$

and $*\ell$ represents the Fourier convolution power. Since the nonlinear operator $\mathcal{G}_\mu$ preserves the positive cone in ℓ^2 and is compact there, the following holds:

Theorem 1.3.1. *For each $\mu \in [0, 1)$, the nonlinear equation $\hat{\varphi} = \mathcal{G}_\mu(\hat{\varphi})$, with $\ell = 2$, (and hence the equation* (1.3.2)*, with $\ell = 2$) has periodic positive definite, classical solution $\hat{\varphi} \in \ell^s$ (for any $s \geq 0$) different from trivial solutions — $\varphi = C$, $C \in \mathbb{R}$.*

Though Theorem 1.3.1 is proven in [10] explicitly in the quadratic case only ($\ell = 2$), it extension to arbitrary $\ell \in \mathbb{N}$ is straightforward and in fact Theorem 1.3.1 holds for all integer powers $\ell \geq 2$.

1.3.2 The real line settings

We note that for $\mu = 0$, (1.3.2) is just an ODE, whose unique $L^2(\mathbb{R})$ solution can be found explicitly and is given by

$$\varphi_0(x) = \left[\tfrac{\ell+1}{2}\operatorname{sech}^2\!\left(\tfrac{\ell-1}{2}x\right)\right]^{\frac{1}{\ell-1}}. \tag{1.3.3}$$

Theoretically, for μ near zero the existence of solutions to (1.3.2) can be established via the standard perturbation argument. This program was executed in [5]. In particular, using the classical perturbation theory (see [56]) the authors demonstrated that the solution map $\mu \mapsto \varphi$ is analytic in a small neighborhood of 0, with values in $L^2(\mathbb{R})$. This yields the existence of smooth in μ branch of orbitally stable even traveling waves solutions emanating from $\varphi_0(x)$ of (1.3.3).

The analysis of [5] is purely local, furthermore, it is not very practical as we are interested in the solutions to (1.3.2) for μ near 1. The major difficulty in dealing with (1.3.2) posed on the real line is that in the fixed point problem

$$\hat{\varphi} = \hat{\mathcal{G}}_\mu(\hat{\varphi}), \quad \hat{\mathcal{G}}_\mu(\hat{\varphi}) = \tfrac{\alpha_\ell}{\kappa_\mu}\hat{\varphi}^{*\ell}, \quad \kappa_\mu = 1 - 2\mu|\xi| + \xi^2, \quad \alpha_\ell = (2\pi)^{\frac{\ell}{2}}, \quad \mu \in [0,1), \tag{1.3.4}$$

the nonlinear operator $\mathcal{G}_\mu := \mathcal{F}^{-1}\hat{\mathcal{G}}_\mu\mathcal{F} : H^s(\mathbb{R}) \to H^s(\mathbb{R})$, $s > \frac{1}{2}$, is no longer compact. In his treatment (see [10]), T.B. Benjamin used a modification of (1.3.3), which he compactified by adding a small viscosity term. Unfortunately, the analysis of [10] contains a gap. Indeed, the denominator $\langle \hat{\varphi}, \hat{\varphi}^{*2}\rangle$ of the "compactified map" employed in [10] is not separated from zero in the cone segment

$$\mathcal{C}_t = \left\{\hat{\varphi} \in L^2(\mathbb{R}) \,\middle|\, \hat{\varphi} \geq 0 \text{ a.e. in } \mathbb{R},\ \|\hat{\varphi}\|_{L^2(\mathbb{R})} \geq t\right\},$$

for any $t > 0$. The latter renders the "compactified map" unbounded in the positive cone and as a consequence the Schauder-Leray index theory is inapplicable.

The first successful global treatment of (1.3.2) is presented in the work of J.A. Pava [74]. His approach is based on the variational formulation of (1.3.2) in the physical space. Indeed, the original non-stationary model (1.1.4) has two constants of motion: the Hamiltonian

$$\mathcal{N}(u) = \tfrac{1}{2}\int_{\mathbb{R}} \Big[\varphi^2 - 2\mu\varphi\mathcal{H}[\varphi]_x + \varphi_x^2 - \tfrac{2}{\ell+1}(-u)^{\ell+1}\Big]\,dx \tag{1.3.5a}$$

and

$$\mathcal{I}(u) = \|u\|^2_{L^2(\mathbb{R})}. \tag{1.3.5b}$$

The equation (1.3.2) can be realized as an Euler-Lagrange equation associated to the variational problem of minimizing the total energy functional $\mathcal{N}(u)$ in $H^1(\mathbb{R})$, subject to the constrained total charge $\mathcal{I}(u) = \lambda^2$. As shown in [74], the difficulties related to the non-compactness of the embedding $H^1(\mathbb{R}) \hookrightarrow L^2(\mathbb{R})$ can be successfully treated via the classical concentration compactness argument of P.L. Lions [65, 66], yielding the following result

Theorem 1.3.2. *For each $\mu \in [0,1)$ and $\ell = 2$, the variational problem*

$$I_\lambda = \inf\bigl\{\mathcal{N}(\varphi)\,\big|\,\varphi \in H^1(\mathbb{R}),\ \mathcal{I}(\varphi) = \lambda^2\bigr\}, \tag{1.3.6}$$

attains its minimum at $\varphi_\mu \in H^1(\mathbb{R})$. The minimizer φ_μ (after rescaling) satisfies (1.3.2)*.*

An alternative approach to the problem (1.3.2) and its extensions is developed in the paper of H. Chen and J. L. Bona [13]. Similar to the work of J.A. Pava [74], the approach is variational and is based on the compactness alternative of P.L. Lions [65, 66]. However, this time the associated variational problem is formulated not in terms of first integrals $\mathcal{N}(u)$ and $\mathcal{I}(u)$, but in terms of functionals, associated to the left and the right-hand sides of (1.3.2). The summary of their results can be stated as follows:

Theorem 1.3.3. *For each $\mu \in [0,1)$ and $\ell = 2$, the variational problem*

$$\Theta_\lambda = \inf\bigl\{\|\varphi\|^2_{L^2(\mathbb{R})} - 2\mu\langle\varphi, \mathcal{H}[\varphi]_x\rangle_{L^2(\mathbb{R})} + \|\varphi_x\|^2_{L^2(\mathbb{R})}\,\big|\,\varphi \in H^1(\mathbb{R}),\ \langle\varphi^2, \varphi\rangle_{L^2(\mathbb{R})} = \lambda\bigr\}, \tag{1.3.7}$$

attains its minimum at $\varphi_\mu \in H^1(\mathbb{R})$. The minimizer φ_μ is Bochner positive definite ($\hat{\varphi}_\mu > 0$) and satisfies (1.3.2)*, with $\ell = 2$.*

To conclude this section, we mention that even a basic question of existence of a single traveling wave for μ near 1 is mathematically nontrivial. For that reason rigorous analysis of coexistence of a system of interacting Benjamin traveling waves (solitons) remains completely unexplored. Apart from numerical experiments, indicating that the interactions are non-elastic not much is published in the special literature.

1.3.3 Orbital stability

From physical perspective, existence of traveling wave solutions to idealized model (1.3.2) does not guarantee that such waves do really exist and can be observed experimentally. It might happen that small perturbations, which are unavoidable in realistic settings, totally destroy the wave in a very short period of time. Hence, there arises the question of stability of the wave. In the context of the Benjamin equation (1.1.4), traveling waves are orbits

$$\mathcal{O}_{\mathcal{T}}(\varphi_\mu) = \{\mathcal{T}_t\varphi_\mu \,|\, t \in \mathbb{R}\},$$

emanating from solutions φ_μ to (1.3.2) under the action of the translations group $\mathcal{T}_t\varphi(\cdot) = \varphi(\cdot + t)$. Hence, traveling wave solutions φ_μ make physical sense if and only if the trajectories of (1.1.4) starting at $t = 0$ near $\mathcal{O}_{\mathcal{T}}(\varphi_\mu)$ in the phase space of (1.1.4) stay close to this orbit for all $t > 0$.

The general theory of orbital stability as well as some practical stability criteria can be found in [45, 46]. Applications of this theory to the orbital stability of traveling waves generated by the KdV equation ($\beta = 0$, $\ell \geq 2$) and by the Benjamin-Ono equation ($\gamma = 0$, $\ell = 2$) can be found in [16] and [5], respectively. As shown by Albert et. al. [5], for small values of μ (equivalently, small β) and $\ell = 2$, the former results extend to the Benjamin equation (1.1.4) via the standard perturbation theory [56]. However, the results of [5] are local in nature and do not apply to the values of μ near unity as the classical perturbation theory does not allow to control multiplicity of the second eigenvalue $\lambda = 1$ associated to the eigenfunction $\partial_t\mathcal{T}_t\varphi_\mu|_{t=0}$ of the linearized operator $\mathcal{G}'_\mu(\varphi_\mu)[\cdot]$, when μ is far away from zero.

A weaker substitute for the orbital stability of the Benjamin traveling waves was obtained in the work of J.A. Pava [74] as a byproduct of his variational approach and the general stability considerations related to the concentration compactness argument noted by P.L. Lions [65, 66].

Theorem 1.3.4. *Assume $\mu \in [0, 1)$ and $U_\mu \subset H^1(\mathbb{R})$ is the set of nontrivial traveling waves obtained in Theorem 1.3.2. Then for every $\epsilon > 0$ there exist $\delta > 0$, such that*

$$\inf_{\varphi \in U_\mu} \|u(\cdot, t) - \varphi\|_{H^1(\mathbb{R})} < \epsilon, \quad \textit{provided} \quad \inf_{\varphi \in U_\mu} \|u_0 - \varphi\|_{H^1(\mathbb{R})} < \delta.$$

Theorem 1.3.4 implies that the solution set U_μ is a positive attractor for the evolution problem (1.1.4). However, the structure of U_μ can be very complicated and may lead potentially to a dynamical chaos.

The orbital stability analysis for pseudo-differential equations involving global operators is known to be very hard. There are very few results of this type known in the literature. Among them, we mention the stability analyses of the Benjamin-Ono traveling waves [6, 72] and of the fractional nonlinear Schrödinger traveling waves [41]. The problem of orbital stability for the Benjamin traveling waves remains open.

1.4 Numerical analysis

The Benjamin equation describes an interesting physical phenomenon (namely propagation of internal waves in a two fluid system driven by the capillarity effect), hence finding solution to NBE, either exact or numerical, has attracted significant attention in the special literature [5, 26, 35, 36, 53]. Unfortunately, the presence of global operator in the model, makes it impossible to construct closed form solutions. Hence, a significant effort is directed to solving (1.1.4) and its stationary counterpart (1.3.2) numerically. There are two major computational challenges arising in numerical treatment of the Benjamin equation. The first one is due to the non-locality of the Hilbert transform whose spatial semi-discretization yields, in general, dense matrices of large size. The second technical difficulty is connected with the unboundedness of spatial domain when (1.1.4) or (1.3.2) are treated globally in the real line. Below, we review several results related to the construction and analysis of computational schemes suitable for integrating the Benjamin equation numerically.

1.4.1 The non-stationary Benjamin equation

Several authors contributed to the numerical analysis of (1.1.4) in either periodic or real settings. We list their results in the chronological order.

In [53], H. Kalisch and J. L. Bona proposed a fully discrete numerical scheme which they use to study interaction of periodic traveling waves. The scheme employs Fourier-type spectral semi-discretization in space, which allow authors to treat the global operator $\partial_x \mathcal{H}$ efficiently (the associated matrix in the frequency space is diagonal). The semi-discretized solution is then advanced in time with the aid of two-step linearly implicit numerical scheme of order two. The second order convergence of the fully-discrete numerical procedure is demonstrated numerically.

An alternative numerical technique for the periodic Benjamin equation was developed in [26], where the finite-differences on uniform grids are used for approximation of spatial derivatives, while the Hilbert transform is recovered via the discrete Fourier transform. The solution is integrated in time using an explicit predictor-corrector time-stepping procedure. Unfortunately, paper [26] lack any sort of theoretical and/or numerical analysis of the scheme. However, the numerical evolution of the Benjamin traveling waves on very large spatial intervals is in a good agreement with earlier results obtained by other authors (see, in particular, [5]).

An extensive numerical treatment of (1.1.4) in context of the real line was undertaken recently in the papers of V. Dougalis et. al., see [35, 36]. In particular, some numerical analysis for the non-stationary Benjamin equation (1.1.4) posed in the real line is presented in [35]. To cope with the unboundedness of the spatial domain, the authors make use of a simple domain truncation technique – $\mathbb{R}$ is replaced with large interval $[-L, L]$ and periodic boundary conditions are imposed. Exact solutions of the truncated model are approximated in space using cubic finite-elements on uniform grids. The equation is then semi-discretized in space via the standard Galerkin technique, the semi-discretization of the non-local operator $\mathcal{H}$ is obtained by passing from the physical to the frequency space with the aid of the discrete Fourier transform. The resulting hybrid finite-element/spectral-Fourier model is integrated in time using 2-stage symmetric and symplectic Kuntzmann-Butcher Runge-Kutta method of classical order four. The proposed fully-discrete numerical method is implicit (in time) and requires Newton-type iterations at each time-integration step. Though the paper contains no formal theoretical convergence and stability analyses of the proposed numerical algorithm, it presents extensive numerical simulations, illustrating its accuracy and general computational efficiency.

1.4.2 Stationary Benjamin equation

The stationary equation (1.3.2) has received a significant attention of numerical community in both bounded or unbounded settings. The most commonly used computational techniques are based on continuation with respect to parameter $\mu \in$

$[0, 1)$. Below, we present brief review of the key results obtained in [5, 26, 35, 36].

The first numerical scheme for computing Benjamin traveling waves, with $\mu \in [0, 1)$, was proposed by Albert et. al. in [5]. In their approach, the originally unbounded spatial domain $\mathbb{R}$ is truncated to a large but bounded interval. Equation (1.3.2), equipped with the periodic boundary conditions, is then approximated in this interval using the standard spectral-Fourier technique. The spectral Fourier discretization yields a large nonlinear system of algebraic equations which, in addition, depends on the parameter μ. Note that for $\mu = 0$ (the KdV case) the exact solution to (1.3.2) is known. Using this solution as the initial guess, the numerical traveling wave at $\mu > 0$ is obtained by solving the associated algebraic system via Newton-type iterations. The procedure is repeated iteratively to produce a sequence of numerical traveling waves at prescribed grid points $\mu_i \in [0, 1)$.

It is common knowledge that the continuation technique of [5] works well, provided the totality of the exact solutions to (1.3.2) constitutes a smooth functions of parameter μ. The latter fact is unknown when μ is far away from the origin (see discussion in Section 1.3.3) and hence any rigorous theoretical analysis of the scheme is unavailable for large values of μ. Nevertheless, as indicated by the authors the decay of the numerical solutions for large values of x and $\mu \approx 1$ is in a good agreement with the asymptotic theory of Benjamin traveling waves, developed in [10].

Calvo et. al. [26] provide a numerical study of the even Benjamin interface waves. Authors proceed by discretizing spatial derivatives with fourth-order-accurate finite difference on evenly spaced grids. As in [5], the computation of the Hilbert transform was done via the discrete Fourier transform. Again, similarly to [5], using the known traveling wave at $\mu = 0$ as the initial guess, the solutions on a fixed μ-grid in $[0, 1)$ are recovered via incremental use of Newton iterations.

Dougalis et. al., in [36], present compilations of several continuation schemes for solving nonlinear Benjamin equation (1.3.2). The spatial discretization was accomplished in the Fourier space and is equivalent to the techniques used by Albert et. al. [5]. However, the continuation approach is different. Authors propose two types of continuation schemes: (i) a Square Operator Method (SOM) and its modification

(MSOM) where the problem of finding the stationary Benjamin solutions is reduced to the problem of finding an equilibrium of a specially constructed evolution equation; (ii) a quasi-Newtonian iterations, where linear systems arising at each iteration step are solved via the preconditioned Conjugate Gradient algorithm. Extensive numerical simulations, presented by the authors, indicate that the second approach is the most efficient. Some extensions of these results appeared in the recent work [37] by the same authors.

1.4.3 Malmquist-Takenaka-Christov (MTC) system

The Benjamin equation (1.1.4) (together with KdV posed on the real line, Benjamin-Ono and many others) belongs to a large family of quasi-linear hyperbolic equations. A large number of highly accurate numerical techniques, appropriate for solving this type of problems in unbounded domains are developed in the special literature. Among many, we mention methods based on the use of transparent boundary conditions [20, 27, 48] and methods involving various mapping techniques [20, 27, 48]. Unfortunately, due to the presence of non-local operator $\mathcal{H}$, these techniques are hard to apply in the context of the Benjamin equation. Precisely for that reason, all available numerical techniques mentioned above use a simple domain truncation strategy, where $\mathbb{R}$ is artificially reduced to a large interval $[-L, L]$ and periodic boundary conditions are imposed. Note however that the non-local term $\mathcal{H}$ leads to the jump discontinuity in the Fourier symbol associated to the linear part of the Benjamin equation. As a consequence, the exact solutions to either of the equations (1.1.4) or (1.3.2) decay at most algebraically for large values of x. This presents a serious computational challenge, as an accurate numerical approximation of such solutions requires very large values for the truncation parameter L.

In the thesis, we employ a different approach by seeking approximation to the numerical solution directly on the real line by applying a family of rational orthogonal functions proposed separately by F. Malmquist [68], S. Takenaka [92] and rediscovered, in context of spectral methods, by C.I. Christov [30]. The MTC system possesses a number of attractive computational features:

(i) the system provides a complete orthogonal basis in $L^2(\mathbb{R})$, while functions comprising the family are uniformly bounded in $\mathbb{R}$, [19, 30, 51, 96];

(ii) the associated differentiation matrices are skew-Hermitian and are tridiagonal/block tridiagonal, depending on the ordering of basis function, see [19, 30, 51, 96];

(iii) the MTC members are eigenfunctions of the Hilbert transform in $\mathbb{R}$, [96];

(iv) the MTC system is directly connected (via a simple substitution) to the classical trigonometric basis and hence computing the discrete MTC expansion coefficient can accomplished efficiently via the standard Fast Discrete Fourier Transform (FDFT), [19, 30, 51, 96];

(v) in fact, it is shown in [51] that the MTC system is the only system of rational functions in $\mathbb{R}$ that possesses properties (i)-(iv);

(vi) the MTC system behaves well with respect to the product of its members, [30].

Due to properties (i)-(vi), the MTC system was applied successfully to solve a wide range of practical problems arising in the approximation theory, signal processing, harmonic analysis, numerical analysis, e.t.c. (see for instance [21, 22, 23, 24, 39, 49, 73, 96, 98] and references therein).

Unfortunately, not much is known about the approximation properties of these functions. Results of this type are known for functions analytic everywhere in $\mathbb{C} \setminus \{-i, i\}$ only, see [19, 96] and the discussion in [51]. One of the main purposes of the present thesis is to establish a solid theoretical background for the MTC approximation theory that is suitable for numerical applications. We will demonstrate that, apart from properties (i)-(vi), the MTC system is particularly suitable for rapid approximation of functions whose Fourier images possess integrable singularities at the origin. Since the latter is a generic situation for the solutions of (1.1.4) and (1.3.2), the use of the MTC basis leads to fast and efficient numerical schemes for both the non-stationary and the stationary Benjamin equations.

1.5 Outline of the thesis

The thesis is organized as follows: Chapter 1 is introductory and contain review of recent theoretical and numerical results available for the Benjamin equation. Chapter 2 contains technical results pertinent for the subsequent theoretical and numerical analyses of the thesis. In Chapter 3, we present the wellposedness analysis of equation (1.1.4) in the scale of variable weight Sobolev spaces. Existence, regularity and orbital stability of Benjamin traveling waves are presented in Chapter 4. The Malmquist-Takenaka-Christov basis and its computational properties are discussed in Chapter 5. An MTC-type collocation scheme for the nonstationary Benjamin equation and its rigorous convergence and stability analyses are presented in Chapter 6. In Chapter 7, an MTC-type continuation scheme for the stationary Benjamin equation is discussed. Chapter 8 concludes the thesis.

Chapter 2

Preliminaries

In this chapter, we present some definitions and notations that will be useful throughout the thesis. Also, we provide detailed proofs of some technical interpolation results for a scale of variable weight Sobolev spaces for which we have no immediate references.[1] These spaces appear naturally in Chapter 5, in our study of the general approximation properties of the MTC system. Furthermore, these spaces provide a natural framework for the study of the interplay between regularity, asymptotic and existence of global solutions to (1.1.4) as compared to the scale of spaces employed in [95]. We begin our presentation by fixing notation and citing key results, that are used in the thesis.

2.1 Notation

2.1.1 Fourier transform

Throughout our work, symbols

$$\mathcal{F}[\varphi](\xi) = \hat{\varphi}(\xi) = \tfrac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi x} \varphi(x) dx,$$
$$\mathcal{F}^{-1}[\hat{\varphi}](x) = \varphi(x) = \tfrac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\xi x} \hat{\varphi}(\xi) d\xi,$$

denote the normalized Fourier transform and its inverse. Letters x and ξ are reserved for the physical and the frequency variables, respectively. Symbol $*$ denotes the

[1] The results of this chapter are extensions and adaptions of known results from the weighted interpolation theory, re-proven in the new context by the author of the thesis, see [89].

standard Fourier convolution. Letter c stands for a generic positive constants, whose particular value is irrelevant.

2.1.2 Weighted Lebesgue spaces

Let Ω be an open subset in $\mathbb{R}^n$ and $w \in L^{1,\mathrm{loc}}(\Omega)$ be almost everywhere positive in Ω. We employ the symbol

$$L^p_w(\Omega; B) := L^p_w(\Omega, wdx; B), \quad 1 \le p \le \infty,$$

to denote the weighted Lebesgue spaces with values in a Banach space B. We write shortly $L^p_w(\Omega)$, when B is either $\mathbb{R}$ or $\mathbb{C}$. Note that $L^p_w(\Omega)$ is a Hilbert space if $p = 2$. In the sequel, we deal with the power weights $w_{pr}(x) = x^{pr}$, $r > -\frac{1}{p}$. For such weights we use the shortcut $L^p_r(\mathbb{R}_+)$. When $r = 0$, we write simply $L^p(\Omega)$.

2.2 Complex Interpolation

There are different approaches in constructing interpolation spaces. In this thesis, we use the complex interpolation method, see e.g. the classical text [12].

We say that a couple $[X_0, X_1]$ of Banach spaces is compatible if X_0, X_1 are two subspaces of a larger topological vector space X. Following A. Calderon [12, 25], to such a couple we associate the class $\mathcal{F}[X_0, X_1]$ of vector valued functions $f : S \to X_0 + X_1$ that are uniformly bounded and continuous in the strip $S = \{z : 0 \le \operatorname{Re} z \le 1\}$ and holomorphic in its interior $S_0 = \{z : 0 < \operatorname{Re} z < 1\}$. By virtue of the Hadamard three-lines theorem, the expression

$$\|f\|_{\mathcal{F}[X_0,X_1]} = \max_{j=0,1} \sup_{t\in\mathbb{R}} \|f(j+it)\|_{X_j},$$

provides a norm in $\mathcal{F}[X_0, X_1]$, furthermore $\mathcal{F}[X_0, X_1]$, equipped with this norm is complete. For $\theta \in (0, 1)$, we let

$$[X_0, X_1]_\theta = \{x \in X_0 + X_1 | \|X\|_{[X_0,X_1]_\theta} < \infty\},$$

$$\|x\|_{[X_0,X_1]_\theta} = \inf_{f\in\mathcal{F}[X_0,X_1]: f(\theta)=x} \|f\|_{\mathcal{F}[X_0,X_1]}.$$

The new space $[X_0, X_1]_\theta$, equipped with the norm $\|\cdot\|_{[X_0,X_1]_\theta}$, is a Banach space that satisfies

$$X_0 \cap X_1 \hookrightarrow [X_0, X_1]_\theta \hookrightarrow X_0 + X_1.$$

Furthermore for any linear operator $T \in \mathcal{L}(X_0, Y_0) \cap \mathcal{L}(X_1, Y_1)$, where $[Y_0, Y_1]$ is another admissible interpolation pair, we have

$$\|T\|_{[X_0,X_1]_\theta \to [Y_0,Y_1]_\theta} \le \|T\|^{1-\theta}_{X_0 \to Y_0} \|T\|^{\theta}_{X_1 \to Y_1}, \quad \theta \in (0,1).$$

The latter implies that for each $\theta \in (0,1)$, $[X_0, X_1]_\theta$ is indeed an interpolation space and that the interpolation functor $[\cdot,\cdot]_\theta$ is exact, see [12].

Below, we quote several classical results from the complex interpolation theory which are pertinent for our analysis.

Theorem 2.2.1. *[A. Calderon, see [12, Theorem 4.4.1]] Let $[X_0^i, X_1^i]$, $i = 1, \dots, n$ and $[Y_0, Y_1]$ be compatible Banach couples. Let $T \in \mathcal{L}(X_0^1, \dots, X_0^n; Y_0) \cap \mathcal{L}(X_1^1, \dots, X_1^n; Y_1)$ be a bounded multilinear map such that*

$$\|T(x_1, \dots, x_n)\|_{Y_j} \le M_j \prod_{i=1}^{n} \|x_i\|_{X_j^i}, \quad j = 0, 1.$$

Then for any $\theta \in (0,1)$, we have $T \in \mathcal{L}([X_0^1, X_1^1]_\theta, \dots, [X_0^n, X_1^n]_\theta; [Y_0, Y_1]_\theta)$ and

$$\|T(x_1, \dots, x_n)\|_{[Y_0,Y_1]_\theta} \le M_0^{1-\theta} M_1^{\theta} \prod_{i=1}^{n} \|x_i\|_{[X_0^i, X_1^i]_\theta}.$$

Theorem 2.2.2. *[see, [12, Theorem 5.1.2]] Assume that X_0 and X_1 is a compatible couple of Banach spaces and that $p_0, p_1 \in [1, \infty)$ and $\theta \in (0,1)$. Then*

$$[L^{p_0}(\Omega, d\mu; X_0), L^{p_1}(\Omega, d\mu; X_1)]_\theta = L^p(\Omega, d\mu; [X_0, X_1]_\theta),$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. In addition, if $1 \le p_0 < \infty$, $p_0 \in [1, \infty)$, then

$$[L^{p_0}(\Omega, d\mu; X_0), L^{\infty}_0(\Omega, d\mu; X_1)]_\theta = L^p(\Omega, d\mu; [X_0, X_1]_\theta),$$

with $\frac{1}{p} = \frac{1-\theta}{p_0}$, where $L^\infty_0(\Omega, d\mu; X)$ is the completion in the uniform norm of the space of simple X-valued functions.

Theorem 2.2.3. *[Stein-Weiss, see [12, Theorem 5.5.3]] Assume that $p_i \in (0, \infty)$, for $i = 0, 1$. Then we have with equal norms*

$$[L^{p_0}(\Omega, w_0 d\mu), L^{p_1}(\Omega, w_1 d\mu)]_\theta = L^p(\Omega, w d\mu), \quad \theta \in (0,1),$$

where $w = w_0^{\frac{(1-\theta)p}{p_0}} w_1^{\frac{\theta p}{p_1}}$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

For $s \in \mathbb{R}$, $q \in [1,\infty]$ and a Banach space X, the symbol $\ell_s^q(X)$ denotes the space of vector valued sequences $\{x_j\}_0^\infty \subset X$, such that

$$\|x_j\|_{\ell_s^q(X)} = \Big(\sum_{j\geq 0}(2^{sj}\|x_j\|_X)^q\Big)^{\frac{1}{q}} < \infty.$$

Theorem 2.2.4. *[see, [12, Theorem 5.6.3]] With the equal norms, we have*

$$[\ell_{s_0}^{q_0}(X_0), \ell_{s_1}^{q_1}(X_1)]_\theta = \ell_s^q([X_0, X_1]_\theta), \quad \theta \in (0,1), \quad s \in \mathbb{R}, \quad q \in [1,\infty],$$

where $\frac{1}{q} = \frac{(1-\theta)q_1 + \theta q_0}{q_0 q_1}$ *and* $s = (1-\theta)s_0 + \theta s_1$.

2.3 Weighted Bessel potential spaces in half line

The interpolation theory of Sobolev spaces provides an indispensable tool in the analysis of partial differential equations. Multiple fundamental results related to existence and optimal regularity and asymptotic of solutions to such problems are formulated directly in the framework of this theory. In the classical settings of Euclidean domains equipped with the Lebesgue measure, the theory took its more or less complete shape at around early 70-s, see e.g. classical texts [3, 12]. However, it was realized quickly that the direct extensions of the theory to more general weighted settings is not so straightforward as the underlying constructions such as Paley-Littlewood decomposition, Mihlin's multiplier theorem e.t.c, involve passage to the frequency (Fourier) space, which at that time was hard to justify in the weighted settings.

Notable progress was achieved in late 70-s early 80-s and is connected with the rapid development of real methods in the classical Fourier analysis, see e.g. [42, 43, 44, 90, 91], and the discovery of A_p weights by B. Muckenhoupt in his work on weighted inequalities for the classical Hardy-Littlewood maximal function [71]. Since the latter function controls a large class of operators arising in the classical Fourier analysis, technical methods of the unweighted interpolation theory extend directly to the A_p-weighted context. In particular, it turns out that for weighted Sobolev spaces over regular Euclidean domains the following complex interpolation identity holds

$$
\begin{aligned}
&[W^{s_0,p}(\mathbb{R}^d, w_0 dx), W^{s_1,p}(\mathbb{R}^d, w_1 dx)]_\theta = W^{(1-\theta)s_0+\theta s_1,p}(\mathbb{R}^d, (w_0^{1-\theta} w_1^{\theta}) dx),\\
&s_0, s_1 \in \mathbb{R}, \quad 1 < p < \infty, \quad w_0, w_1 \in A_p, \quad \theta \in (0,1).
\end{aligned}
\tag{2.3.1}
$$

It is worth to mention that the A_p weight in $\mathbb{R}^d$ may have very rough local behaviour, but have to satisfy the following growth and integrability restrictions

$$
A_p = \{w \in L^{1,\text{loc}}(\mathbb{R}^d) \,|\, [w]_p < \infty\}, \quad 1 \le p < \infty, \quad A_\infty = \bigcup_{p \ge 1} A_p, \tag{2.3.2a}
$$

$$
[w]_1 = \operatorname*{ess\,sup}_{\mathbb{R}^d} \frac{\mathcal{M}[w]}{w}, \quad [w]_p = \operatorname*{ess\,sup}_{I \in \mathbb{R}^d} \frac{w(I)^{\frac{1}{p}} \bar{w}_p(I)^{\frac{1}{p'}}}{\lambda(I)}, \quad 1 < p < \infty, \tag{2.3.2b}
$$

where $\mathcal{M}[w](x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |w(x)| dx$, $B \subset \mathbb{R}$ is open and connected, is the standard Hardy-Littlewood maximal function, $\bar{w}_p = w^{-\frac{p'}{p}}$ and I is a d-dimensional cube with sides parallel to the coordinate planes. In particular, (2.3.2b) implies that

(i) A_p weights grow at most algebraically at infinity;

(ii) the local decay is restricted by the integrability exponent p.

For instance, when $d = 1$, $w_\alpha := |x|^\alpha \in A_p$ if and only if $-1 < \alpha < \frac{p}{p'}$. Since, in general, the A_p condition is sufficient but not necessary for (2.3.1) to hold, a significant efforts were undertaken to overcome the indicated limitations.

The first problem was resolved successfully in early 2000-s by V.S. Rychkov [80]. His idea was to restrict measures of Euclidean cubes, appearing in the ess sup-expression of formula (2.3.2b). This gives a local A_p^{loc} version of the original Muckenhoupt classes and allows for the exponentially growing weights. The new weight class, combined with an appropriate reproducing formula (see Section 2.3.5 for details) yields a convenient Littlewood-Paley characterization of the underlying Sobolev spaces and leads directly to the A_p^{loc} version of the interpolation identity (2.3.1). Even more general, A_∞^{loc} real and complex interpolation identities for weighted Triebel-Lizorkin and Besov spaces were established.

The second limitation is more subtle as the local integrability for both the original weight w and for its Hölder conjugate $\bar{w}_p$ is embedded directly into the condition (2.3.2b). The requirement seems to be superfluous, the lack of integrability of the conjugate weights $\bar{w}_p$ shall not affect the basic interpolation properties of the underlying function spaces. For the basic weighted Lebesgue spaces this is indeed the case — the identity (2.3.1), with $s_0 = s_1 = 0$, is known to hold for all locally integrable weights w, without any restrictions on $\bar{w}_p$, see e.g. Theorem 2.2.3 quoted earlier. Unfortunately, in context of Sobolev spaces such results are unavailable (mainly due to the lack of suitable weighted Littlewood-Paley theory). In the thesis, we provide a construction which allow to relax the local integrability restriction for the Bessel potential spaces in the positive half-line. In particular, we show that for these spaces the complex interpolation identity (2.3.1) holds for all locally integrable power weights $w_\alpha = x^\alpha$, $\alpha > -1$.

2.3.1 One-sided local maximal functions

In our exposition, we combine the notion of one-sided $A_{\pm,p}$ classes of [82] with the localization ideas of [80].

2.3.1.1 Definition and basic properties

Let X be an open connected subset of $\mathbb{R}$, equipped with a regular, positive Borel measure μ. For $x \in X$, we define the restricted one-sided Hardy-Littlewood maximal functions by means of the identities

$$\mathcal{M}^{+}_{\mu,t}[f](x) = \sup_{0<\mu((x,b)\cap X)<t} \frac{1}{\mu((x,b)\cap X)} \int_{(x,b)\cap X} |f| d\mu, \tag{2.3.3a}$$

$$\mathcal{M}^{-}_{\mu,t}[f](x) = \sup_{0<\mu((a,x)\cap X)<t} \frac{1}{\mu((a,x)\cap X)} \int_{(a,x)\cap X} |f| d\mu. \tag{2.3.3b}$$

The global version ($t = \infty$) of these operators was introduced originally by E. Sawyer [82] in his study of one-weight weak- and strong-type weighted inequalities. Extensions of his original results, including general two-weight estimates, were obtained by F.J. Martin-Reyes in [69].

Basic properties of operators $\mathcal{M}^{\pm}_{\mu,t}$ are quite similar to those of the classical Hardy-Littlewood maximal function.

Lemma 2.3.1. *Operators $M^{\pm}_{\mu,t}$ satisfy*

$$\|\mathcal{M}^{\pm}_{\mu,t}[f]\|_{L^{1,\infty}(X,d\mu)} \le c_1^{\pm}\|f\|_{L^1(X,d\mu)}, \tag{2.3.4a}$$

$$\|\mathcal{M}^{\pm}_{\mu,t}[f]\|_{L^{p}(X,d\mu)} \le c_p^{\pm}\|f\|_{L^p(X,d\mu)}, \quad 1 < p \le \infty, \tag{2.3.4b}$$

with some $c_p^{\pm} > 0$ independent on f and t.[2]

Proof. The proof is standard and goes along the same lines as for the classical Hardy-Littlewood maximal operator, see e.g. [44, 90]. For the sake of brevity, we consider $\mathcal{M}^{+}_{\mu,t}$ only, the case of $\mathcal{M}^{-}_{\mu,t}$ is identical.

(a) To begin, we show that $\mathcal{M}^{+}_{\mu,t}[f](x) < \infty$ is μ-measurable. Indeed, consider $f \in L^{1,\mathrm{loc}}(X, d\mu)$ and assume that for some $x \in X$, $\mathcal{M}^{+}_{\mu,t}[f](x) < \infty$. Then, for $\varepsilon > 0$ and some $(x, b) \subset X$, with $0 < \mu((x, b)) < t$, we have

$$\mathcal{M}^{+}_{\mu,t}[f](x) - \varepsilon \le \frac{1}{\mu((x,b))}\int_{(x,b)} |f| d\mu.$$

Elementary calculations show

$$\mathcal{M}^{+}_{\mu,t}[f](x) - \varepsilon \le \begin{cases} \mathcal{M}^{+}_{\mu,t}[f](x+\delta) + \frac{\mu((x,x+\delta))}{\mu((x,b))}\mathcal{M}^{+}_{\mu,t}[f](x), & 0 < \delta < b - x, \\ \left[1 + \frac{\mu((x-\delta,x))}{\mu((x,b))}\right]\mathcal{M}^{+}_{\mu,t}[f](x-\delta), & 0 < \mu((x-\delta, b)) < t. \end{cases}$$

Since μ is regular, $\mu((x-\delta, x+\delta)) \to 0$ as $\delta \to 0$. We conclude that

$$\mathcal{M}^{+}_{\mu,t}[f](x) - c\varepsilon \le \mathcal{M}^{+}_{\mu,t}[f](y), \quad |x - y| < \delta,$$

with some $\delta > 0$ and an absolute constant $c > 1$. The case $\mathcal{M}^{+}_{\mu,t}[f](x) = \infty$ is treated similarly. The calculations indicate that $\mathcal{M}^{+}_{\mu,t}[f]$ is lower-semicontinuous. Hence, the level sets $E_\xi = \{\mathcal{M}^{+}_{\mu,t}[f] > \xi\}$, $\xi > 0$ are open and μ-measurable.

(b) To establish estimate (2.3.4a), for a given cut-off parameters $n, m > 0$, we define $\mathcal{M}^{+,n}_{\mu,t}[f](x) = \sup\left\{\frac{1}{\mu((x,b))}\int_x^b |f| d\mu \,\middle|\, 0 < \mu((x,b)) < t,\, 0 < b - x < n\right\}$. As in part (a) of the proof, we conclude that the sets $E_{\xi,n} = \{\mathcal{M}^{+,n}_{\mu,t}[f](x) > \xi\}$ and $E_{\xi,n,m} = E_{\xi,n} \cap (-m, m)$ are open. Now, for each $x \in E_{\xi,n,m}$, we pick an interval $(x, b_x) \subset X$ with $0 < \mu((x, b_x)) < t$ and $0 < b_x - x < n$, so that

$$\xi < \frac{1}{\mu((x,b_x))}\int_x^{b_x} |f| d\mu.$$

[2] In (2.3.4a), $L^{p,\infty}(X, d\mu)$ denotes the classical Lorentz space equipped with the norm $\|f\|_{L^{p,\infty}(X,d\mu)} = \sup_{t>0} t^{\frac{1}{p}}\mu\{|f| > t\}$.

The family of open intervals $\mathcal{I}_{E_{\xi,n,m}} = \big\{(x,b_x)\big\}_{x\in E_{\xi,n,m}}$ covers $E_{\xi,n,m}$. Using the Besicovitch covering theorem (see [44, 90]), we extract a finite number (say ℓ) of countable disjoint subcollections $\mathcal{I}_j = \{(x_{ji}, b_{x_{ji}})\}_{i\geq 0}$, $1 \leq j \leq \ell$, that cover $E_{\xi,n,m}$. Direct summation gives:

$$\mu(E_{\xi,n,m}) \leq \sum_{j=1}^{\ell}\sum_{i}\mu((x_{ji}, b_{x_{ji}})) \leq \frac{1}{\xi}\sum_{j=1}^{\ell}\sum_{i}\int_{x_{ji}}^{b_{x_{ji}}}|f|d\mu \leq \frac{\ell}{\xi}\|f\|_{L^1(X,d\mu)}.$$

Since the cut-off parameters n and m are arbitrary, we arrive at (2.3.4a). To complete the proof, we note that (2.3.4b), with $p=\infty$, holds trivially. Hence, (2.3.4b), with $1<p<\infty$, follows directly form the classical Marcinkiewicz interpolation theorem. □

2.3.1.2 Weighted weak-type inequalities

In the next step, we study weighted inequalities of the form

$$\|\mathcal{M}^{+}_{\mu,t}[f]\|_{L^{p,\infty}(X,d\nu)} \leq \kappa^{+}_{p}(t)\|f\|_{L^{p}(X,d\nu)}, \quad 1\leq p<\infty, \tag{2.3.5a}$$

$$\|\mathcal{M}^{-}_{\mu,t}[f]\|_{L^{p,\infty}(X,d\nu)} \leq \kappa^{-}_{p}(t)\|f\|_{L^{p}(X,d\nu)}, \quad 1\leq p<\infty, \tag{2.3.5b}$$

where μ and ν are two regular, positive Borel measures in X. As we shall see shortly, for a fixed μ inequalities (2.3.5) impose some restrictions on admissible measures ν. In particular, we have

Lemma 2.3.2. *Assume* (2.3.5) *holds. Then μ and ν are mutually absolutely continuous.*

Proof. The proof follows closely the line of arguments from [91, Chapters I-II].

(a) As before, we consider operator $\mathcal{M}^{+}_{\mu,t}$ only. Assume initially that (2.3.5a) is satisfied. Let $(a,b)\subset X$, $0<\mu((a,b))<t$, $x\in(a,b)$, $E\subset(x,b)$ be open and χ_E be the indicator function of E. Elementary calculations show that $\mathcal{M}^{+}_{\mu,t}[\chi_E](y)$ strictly increases in (a,x) and $\mathcal{M}^{+}_{\mu,t}[\chi_E](y)=1$, $y\in E$. Therefore, for each $y\in(a,x)$ we have

$$(y,x)\cup E\subset\Big\{\mathcal{M}^{+}_{\mu,t}[\chi_{(x,b)}] > \tfrac{\mu(E)}{\mu((y,b))}\Big\},$$

and, in view of (2.3.5a),

$$\Big(\frac{\mu(E)}{\mu((a,b))}\Big)^{p} \leq \kappa^{+}_{p}(t)\frac{\nu(E)}{\nu((a,x)\cup E)},\ x\in(a,b),\ 0<\mu((a,b))<t,\ E\subset(x,b). \tag{2.3.6a}$$

Similarly, (2.3.5b) implies

$$\Big(\frac{\mu(E)}{\mu((a,b))}\Big)^p \le \kappa_p^-(t)\frac{\nu(E)}{\nu(E\cup(x,b))},\ x\in(a,b),\ 0<\mu((a,b))<t,\ E\subset(a,x). \tag{2.3.6b}$$

Since μ and ν are assumed to be outer regular, inequalities (2.3.6) show that μ is absolutely continuous with respect to ν.

Conversely, assuming that (2.3.6a) holds with $E=(x,b)$, it can be verified that (2.3.5a) holds for simple functions. Since the latter function class is dense in $L^p(X,d\nu)$, $1\le p<\infty$, it follows that (2.3.5a) and (2.3.6a) are equivalent.

(b) To prove that ν is absolutely continuous with respect to μ, we introduce the following notation: We say that $\nu\preccurlyeq_+\mu$, if there exists $0<\alpha_+(t),\beta_+(t)<1$, such that $\mu((x,b))\le\alpha_+(t)\mu((a,b))$ yields $\nu((x,b))\le\beta_+(t)\nu((a,b))$, for all $(a,b)\subset X$ with $0<\mu((a,b))\le t$. Similarly, we say that $\nu\preccurlyeq_-\mu$, if there exists $0<\alpha_-(t),\beta_-(t)<1$, such that $\mu((a,x))\le\alpha_-(t)\mu((a,b))$ yields $\nu((a,x))\le\beta_-(t)\nu((a,b))$, for all $(a,b)\subset X$ with $0<\mu((a,b))\le t$.

We note that $\mu\preccurlyeq_\pm\nu$ and $\nu\preccurlyeq_\mp\mu$ are equivalent. Indeed, assume, for instance, $\mu\preccurlyeq_+\nu$. Then $\mu((a,x))\le\frac{1-\alpha_+(t)}{2}\mu((a,b))$ implies $\alpha_+(t)\mu((a,b))<\frac{1+\alpha_+(t)}{2}\mu((a,b))\le\mu((x,b))$. Consequently, $\beta_+(t)\nu((a,b))<\nu((x,b))$ and $\nu((a,x))<(1-\beta_+(t))\nu((a,b))$.

Next, we observe that conditions (2.3.6a) and (2.3.6b) imply $\mu\preccurlyeq_+\nu$ and $\mu\preccurlyeq_-\nu$ (or $\nu\preccurlyeq_-\mu$ and $\nu\preccurlyeq_+\mu$), respectively. Our next aim is to show that either of the implications is reversible.

(c) Let $\mu\preccurlyeq_+\nu$, i.e. $\nu((x,b))\le\beta_+(t)\nu((a,b))$ yields $\mu((x,b))\le\alpha_+(t)\mu((a,b))$. We consider $(a,b)\subset X$, with $0<\mu((a,b))<t$ and choose $x\in(a,b)$ so that

$$\frac{\nu((x,b))}{\nu((a,b))}=\delta^n,$$

with some $n\ge1$ and $0<\delta<\beta_+(t)$. For $y\in[a,b]$, we define $f(y)=\frac{\nu((y,b))}{\nu((a,b))}$. By our assumptions, $f(y)$ is continuous and strictly monotone decreasing in $[a,x]$. Since $f(a)=1$ and $f(b)=0$, there exists a monotone sequence $a<x_{n-1}<\cdots<x_1<x_0=x<b$, such that $f(x_i)=\delta^{n-i}$. For the intervals $I_i=(x_i,b)$, we have

$$\nu(I_0)=\delta\nu(I_1)=\cdots=\delta^{n-1}\nu(I_{n-1})=\delta^n\nu((a,b)).$$

Therefore, when $\delta < \beta_+(t)$ the condition $\mu \preccurlyeq_+ \nu$ implies

$$\mu(I_0) \leq \alpha_+(t)\mu(I_1) \leq \cdots \leq \alpha_+^{n-1}(t)\mu(I_{n-1}) \leq \alpha_+^n(t)\mu((a,b)).$$

Consequently,

$$\frac{\mu((x,b))}{\mu((a,b))} \leq \alpha_+^n(t) \leq \Big(\frac{\nu((x,b))}{\nu((a,b))}\Big)^{\frac{1}{p}},$$

with $p = \max\{1, \frac{\ln \beta_+(t)}{\ln \alpha_+(t)}\}$. By virtue of the last estimate, we conclude that for any $x \in (a,b)$, condition $\mu \preccurlyeq_+ \nu$ yields (2.3.6a), with $\kappa_p^+(t) \leq \beta_+^{-1}(t)$ and $1 \leq p = \max\{1, \frac{\ln \beta_+(t)}{\ln \alpha_+(t)}\} < \infty$.

(d) By virtue of parts (b) and (c) of the proof, conditions (2.3.6a) and (2.3.6b) imply that, for some $1 \leq q_1, q_2 < \infty$, the reverse inequalities hold:

$$\Big(\frac{\nu((a,x))}{\nu((a,b))}\Big)^{q_1} \leq c_{q_1}^-(t)\frac{\mu((a,x))}{\mu((a,b))}, \quad x \in (a,b), \quad 0 < \mu((a,b)) < t, \tag{2.3.7a}$$

$$\Big(\frac{\nu((x,b))}{\nu((a,b))}\Big)^{q_2} \leq c_{q_2}^+(t)\frac{\mu((x,b))}{\mu((a,b))}, \quad x \in (a,b), \quad 0 < \mu((a,b)) < t. \tag{2.3.7b}$$

By part (a) of the proof, (2.3.7) are equivalent to weak-type estimates similar to (2.3.5), but with μ and ν interchanged. Hence, repeating the arguments from part (a) of the proof, we conclude that ν is absolutely continuous with respect to μ. $\square$

2.3.1.3 $A_{p,\pm}^{\mathbf{loc}}(\mu)$ weights

By virtue of Lemma 2.3.2 and the Radon-Nikodym Theorem [78] , the measures ν, appearing in (2.3.5a), are weighted measures, i.e. $\frac{d\nu}{d\mu} = w$ for some μ-a.e. positive, measurable (weight) function w. In the sequel, we use symbol w to denote both: a weight function and its associated measure, so that $w((a,b)) = \int_a^b w d\mu$. Further, with a given weight w and parameter $1 < p < \infty$, we associate new weight $\bar{w}_p = w^{-\frac{p'}{p}}$. Using this notation, we introduce the quantities

$$[w]_{1,t}^{\pm} = \operatorname*{ess\,sup}_{x \in X} \frac{\mathcal{M}_{\mu,t}^{\mp}[w](x)}{w(x)}, \tag{2.3.8a}$$

$$[w]_{p,t}^{+} = \sup_{0<\mu((a,b))<t,\, x\in(a,b)} \frac{w^{\frac{1}{p}}((a,x))\bar{w}_p^{\frac{1}{p'}}((x,b))}{\mu((a,b))}, \quad 1 < p < \infty, \tag{2.3.8b}$$

$$[w]_{p,t}^{-} = \sup_{0<\mu((a,b))<t,\, x\in(a,b)} \frac{w^{\frac{1}{p}}((x,b))\bar{w}_p^{\frac{1}{p'}}((a,x))}{\mu((a,b))}, \quad 1 < p < \infty. \tag{2.3.8c}$$

Lemma 2.3.3. *Assume* $1 \le p < \infty$ *and* $[w]^{\pm}_{p,t_0} < \infty$ *for some fixed* $0 < t_0 < \infty$. *Then* $[w]^{\pm}_{p,t} < \infty$ *for any finite* $t_0 < t < \infty$.

Proof. (a) As before, we consider the case of $[w]^{+}_{p,t_0} < \infty$ only. Let $p = 1$ and $x \in X$ be the Lebesgue point[3] of the quotient $\frac{\mathcal{M}^{\mp}_{\mu,t}[w](x)}{w(x)}$. Consider an interval $(a, x) \subset X$, with $0 < \mu((a, x)) < \frac{3}{2}t_0$. If $a < x_1 < x_2 < x$ are chosen so that $\mu(a, x_1) = \mu(x_1, x_2) = \mu(x_2, x)$, then

$$\frac{w((a,x))}{\mu((a,x))} \le [w]^{+}_{1,t_0}(w(y) + w(x)),$$

for μ-a.e. $y \in (x_1, x_2)$. We integrate the last inequality over (x_1, x_2) to obtain

$$\begin{aligned}\frac{w((a,x))}{\mu((a,x))} &\le [w]^{+}_{1,t_0}\Big(\frac{w((x_1,x_2))}{\mu((x_1,x_2))} + w(x)\Big)\\ &\le [w]^{+}_{1,t_0}\Big(2\frac{w((x_1,x))}{\mu((x_1,x))} + w(x)\Big) \le [w]^{+}_{1,t_0}(1 + 2[w]^{+}_{1,t_0})w(x).\end{aligned}$$

Hence, $[w]^{+}_{1,\frac{3}{2}t_0} \le [w]^{+}_{p,t_0}(1 + 2[w]^{+}_{1,t_0}) < \infty$.

(b) Assume $1 < p < \infty$. In that case for any $(a, b) \subset X$, with $0 < \mu((a, b)) < t_0$ and any $x \in (a, b)$, formula (2.3.8b) (combined with the Hölder inequality) implies

$$\mu^p((x,b))w((a,x)) \le ([w]^{+}_{p,t_0})^p \mu^p((a,b))w((x,b)).$$

This allow us to conclude that w satisfies (2.3.6a), with $\kappa^{+}_{p}(t_0) = ([w]^{+}_{p,t_0})^p + 1$. Similarly, we verify that $\bar{w}_p$ satisfies (2.3.6b), with $\kappa^{-}_{p'}(t_0) = ([w]^{-}_{p,t_0})^{p'} + 1$. We consider now an interval $(a', b') \subset X$, $0 < \mu((a', b')) < 2t_0$ and choose $a' < a < x_1 < b < b'$ so that $\mu((a', a)) = \mu((a, x_1)) = \mu((x_1, b)) = \mu((b, b'))$. By virtue of (2.3.6) and (2.3.8b), we have the estimate

$$w^{\frac{1}{p}}((a', x_1))\bar{w}_p^{\frac{1}{p'}}((x_1, b')) \le c\mu((a', b')),$$

with $c = 2[w]^{+}_{p,t_0}\big(([w]^{+}_{p,t_0})^p + 1\big)^{\frac{1}{p}}\big(([w]^{+}_{p,t_0})^{p'} + 1\big)^{\frac{1}{p'}}$. When $x \in (a', b')$ is arbitrary, the last inequality yields either

$$w^{\frac{1}{p}}((a',x))\bar{w}_p^{\frac{1}{p'}}((x,b')) \le w^{\frac{1}{p}}((a',x))\bar{w}_p^{\frac{1}{p'}}((x,x_1)) + w^{\frac{1}{p}}((a',x_1))\bar{w}_p^{\frac{1}{p'}}((x_1,b')),$$

[3]Given a regular Borel measure μ in Ω and $f \in L^{1,\mathrm{loc}}(\Omega, d\mu)$, we say that $x \in \Omega$ is a Lebesgue point of f if $\lim_{\mu(B)\to 0} \frac{1}{\mu(B)} \int_B |f - f(x)| d\mu 0$, where $B \subset \Omega$ is μ-measurable and $x \in B$.

for $a' < x < x_1$, or

$$w^{\frac{1}{p}}((a',x))\bar{w}_p^{\frac{1}{p'}}((x,b')) \leq w^{\frac{1}{p}}((x_1,x))\bar{w}_p^{\frac{1}{p'}}((x_1,b')) + w^{\frac{1}{p}}((a',x_1))\bar{w}_p^{\frac{1}{p'}}((x_1,b')),$$

for $x_1 \leq x < b'$. In both cases, we obtain

$$w^{\frac{1}{p}}((a',x))\bar{w}_p^{\frac{1}{p'}}((x,b')) \leq c'\mu((a',b')),$$

where $c' = 2[w]^+_{p,t_0}\left(1 + \left(([w]^+_{p,t_0})^p + 1\right)^{\frac{1}{p}}\left(([w]^+_{p,t_0})^{p'} + 1\right)^{\frac{1}{p'}}\right)$. Therefore, using formula (2.3.8b), we obtain $[w]^+_{p,2t_0} \leq c' < \infty$. □

In view of Lemma 2.3.3, the following weight classes

$$A^{\mathrm{loc}}_{p,\pm}(\mu) = \{w \in L^{1,\mathrm{loc}}(X, d\mu) \,|\, [w]^\pm_{p,t} < \infty \text{ for some fixed } t > 0\}, \quad 1 \leq p < \infty,$$

are well defined. Directly from the definition, it follows that $w \in A^{\mathrm{loc}}_{p,\pm}(\mu)$ if and only if $\bar{w}_p \in A^{\mathrm{loc}}_{p',\mp}(\mu)$. Furthermore, it is easy to verify that $A^{\mathrm{loc}}_{p,\pm}(\mu) \subseteq A^{\mathrm{loc}}_{q,\pm}(\mu)$, whenever $1 \leq p \leq q < \infty$.

We remark that the $A^{\mathrm{loc}}_{p,\pm}(\mu)$ class defined above is essentially a local version of $A_{\pm,p}$ weights of E. Sawyer, introduced in connection with one-sided Hardy-Littlewood maximal functions in [69, 82]. The idea of localizing the A_p condition is due to V.S. Rychkov, see [80]. Lemma 2.3.3 can be viewed as a one-sided analogue of [80, Lemma 1.2], note however, that the periodic extension argument employed in [80] does not apply in the one-sided context due to the asymmetry of conditions (2.3.8).

Theorem 2.3.4. *The weak-type inequalities* (2.3.5) *hold if and only if* $w \in A^{loc}_{p,\pm}(\mu)$.

Proof. (a) Necessity. Assume initially that $p = 1$. Then (2.3.5) implies (2.3.6). Since $x \in (a,b)$ is arbitrary, we conclude that $\mathcal{M}^\mp_{\mu,t}[w](x) \leq \kappa^\pm_p(t)w(x)$ at each Lebesgue point of w. Hence, $[w]^\pm_{1,t} \leq \kappa^\pm_p(t)$.When $1 < p < \infty$, inequalities (2.3.5a) and (2.3.5b) applied to $f^+ = \bar{w}_p\chi_{(x,b)}$ and $f^- = \bar{w}_p\chi_{(a,x)}$, respectively, yield the estimate $[w]^\pm_{p,t} \leq \kappa^\pm_p(t)$.

(b) Sufficiency. To begin, we establish (2.3.5a) for functions supported in intervals with finite μ-measure. Let $f \in L^p(w)$ be supported in $(a'',b') \subset X$ with $0 < \mu((a'',b')) < t$. In that case, all level sets $E_\xi = \{\mathcal{M}^+_{\mu,t}[f] > \xi\}$ are contained in some $(a',b')\subset X$, with $0<\mu((a',b'))<2t$ and μ-measures of connected components of E_ξ are uniformly bounded by $2t$.

Let (a,b) be a connected component of E_ξ. Then

$$\xi\mu((x,b))\le\int_x^b|f|d\mu,$$

for all $x\in(a,b)$. We let $x_0=a$ and choose a monotone increasing sequence $\{x_i\}_{i\ge 0}$, $x_i<b$, so that

$$\int_{x_0}^b|f|d\mu=2\int_{x_1}^b|f|d\mu=\cdots=2^i\int_{x_i}^b|f|d\mu=\cdots.$$

For a fixed index $i\ge 0$, we obtain

$$\xi w^{\frac{1}{p}}((x_i,x_{i+1}))\le\frac{w^{\frac{1}{p}}((x_i,x_{i+1}))}{\mu((x_i,b))}\int_{x_i}^b|f|d\mu=4\frac{w^{\frac{1}{p}}((x_i,x_{i+1}))}{\mu((x_i,b))}\int_{x_{i+1}}^{x_{i+2}}|f|d\mu.$$

The last inequality, combined with (2.3.6a), yields

$$\begin{aligned}\xi w((x_i,x_{i+1}))&\le 4\int_{x_{i+1}}^{x_{i+2}}\frac{w((x_i,x))}{\mu((x_i,x))}|f|(x)d\mu(x)\\&\le 4\int_{x_{i+1}}^{x_{i+2}}\mathcal{M}^-_{\mu,2t}[w](x)|f|(x)d\mu(x)\le 4[w]^+_{1,2t}\|f\chi_{(x_{i+1},x_{i+2})}\|_{L^1(w)},\end{aligned}$$

when $p=1$. Similarly, using Hölder's inequality and (2.3.8b), we obtain

$$\begin{aligned}\xi w^{\frac{1}{p}}((x_i,x_{i+1}))&\le 4\frac{w^{\frac{1}{p}}((x_i,x_{i+1}))\bar{w}_p^{\frac{1}{p'}}((x_{i+1},x_{i+2}))}{\mu((x_i,b))}\|f\chi_{(x_{i+1},x_{i+2})}\|_{L^p(w)}\\&\le 4[w]^+_{p,2t}\|f\chi_{(x_{i+1},x_{i+2})}\|_{L^p(w)},\end{aligned}$$

when $1<p<\infty$. In both cases, direct summation over i gives the inequality

$$\xi w^{\frac{1}{p}}((a,b))\le 4[w]^+_{p,2t}\|f\chi_{(a,b)}\|_{L^p(w)},$$

for each connected component of E_ξ. Adding inequalities for all connected components of E_ξ, we obtain (2.3.5a), with $\kappa_p^+(t)=4[w]^+_{p,2t}$.

To complete the proof, we note that the set X can be cut into at most countable number of disjoint intervals $I_i=(a_i,a_{i+1})$, such that $\mu(I_i)=t-\varepsilon$, $0<\varepsilon<\frac{t}{2}$. We let $f_i=f\chi_{I_i}$. By construction, each interval I_i meets the level sets of at most three maximal functions $\mathcal{M}^+_{\mu,t}[f_j]$, $j=i,i+1,i+2$. Consequently,

$$\begin{aligned}\xi^p w(\{\mathcal{M}^+_{\mu,t}[f]>\xi\})&\le 3\xi^p\sum_i w\big(\{\mathcal{M}^+_{\mu,t}[f_i]>\tfrac{\xi}{3}\}\big)\\&\le 3^{p-1}(4[w]^+_{p,2t})^p\sum_i\|f_i\|^p_{L^p(w)}=3^{p-1}(4[w]^+_{p,2t})^p\|f\|^p_{L^p(w)},\end{aligned}$$

and we arrive at (2.3.5a), with $\kappa_p^+(t) = 3^{\frac{1}{p'}} 4[w]_{p,2t}^+$. $\square$

2.3.2 Structure of $A_{p,\pm}^{loc}(\mu)$ weights

The internal structure of $A_{p,\pm}^{\mathrm{loc}}(\mu)$ classes closely resembles that of A_p and A_p^{loc} weights, see e.g. [43, 90, 91]. Almost all results (excluding most notably the reverse Hölder inequality, see the discussion in [82, 91]) have their analogues in the one-sided settings.

2.3.2.1 Structure of $A_{1,\pm}^{\mathbf{loc}}(\mu)$ class

Our presentation follows closely the ideas of [69, 82, 91].

Lemma 2.3.5. *Let $w \in A_{p,\pm}^{loc}(\mu)$, $1 < p < \infty$. Then $v = \mathcal{M}_{\mu,t}^{\mp}[w] \in A_{1,\pm}^{loc}(\mu)$.*

Proof. (a) Assume $w \in A_{p,+}^{\mathrm{loc}}(\mu)$ for some $1 < p < \infty$. Consider $I = (a, b) \subset X$, with $0 < \mu(I) < t$. There are two options: either (i) there exists an interval $I' = (a', b) \subset X$ with $\mu(I') = 2\mu(I)$ and such that $X \cap (-\infty, b) \setminus I' \neq \emptyset$ or (ii) there exists an interval $I' = (a', b) \subset X$ with $\mu(I') \leq 2\mu(I)$ and such that $X \cap (-\infty, b) \setminus I' = \emptyset$. In either case, we denote $w_1 = \chi_{I'} w$ and $w_2 = \chi_{X\cap(-\infty,b)\setminus I'} w$. Then

$$\int_I \mathcal{M}_{\mu,2t}^{-}[w] d\mu \leq \int_I \mathcal{M}_{\mu,2t}^{-}[w_1] d\mu + \int_I \mathcal{M}_{\mu,2t}^{-}[w_2] d\mu.$$

We estimate each term separately.

(b) Consider w_1. For each $x \in I$ there exists $a' < a'' < x$ such that

$$\frac{3}{4}\mathcal{M}_{\mu,2t}^{-}[w_1](x) \leq \frac{w_1((a'', x))}{\mu((a'', x))}.$$

We choose $a''' \in (a'', x)$ so that $\mu((a'', a''')) = \mu((a''', x))$, then

$$\frac{w_1((a''', x))}{\mu((a'', x))} \leq \frac{1}{2}\mathcal{M}_{\mu,2t}^{-}[w_1](x).$$

By virtue of the assumption $w \in A_{p,+}^{\mathrm{loc}}(\mu)$, these inequalities, combined together, yield

$$\begin{aligned}\mathcal{M}_{\mu,2t}^{-}[w_1](x) &\leq 4\frac{w_1((a'', a'''))}{\mu((a'', x))} \leq 4([w]_{p,2t}^+)^p \Big(\frac{\mu((a'', x))}{\bar{w}_p((a''', x))}\Big)^{p-1} \\ &\leq 2(2[w]_{p,2t}^+)^p \mathcal{M}_{\bar{w}_p,2t}^{-}[\chi_{I'}\bar{w}_p^{-1}](x)^{p-1}.\end{aligned}$$

Hence with the aid of Lemma 2.3.1, we obtain

$$\int_I \mathcal{M}^-_{\mu,2t}[w_1]d\mu \leq 2(2[w]^+_{p,2t})^p \int_I \bar{w}_p^{-1}\mathcal{M}^-_{\bar{w}_p,2t}[\chi_{I'}\bar{w}_p^{-1}]^{p-1}\bar{w}_p d\mu$$
$$\leq 2(2[w]^+_{p,2t})^p \int_I \mathcal{M}^-_{\bar{w}_p,2t}[\chi_{I'}\bar{w}_p^{-1}]^p \bar{w}_p d\mu \leq 2c(2[w]^+_{p,2t})^p \int_{I'} w d\mu,$$

where $c > 0$ is an absolute constant. Since, $\mu(I') \leq 2\mu(I)$, we arrive at the inequality

$$\int_I \mathcal{M}^-_{\mu,2t}[w_1]d\mu \leq 4c(2[w]^+_{p,2t})^p \mathcal{M}^-_{\mu,2t}[w](b)\mu(I). \tag{2.3.9}$$

(c) The quantities involving w_2 vanish when $X \cap (-\infty, b)/I' = \emptyset$. Therefore, we assume $\mu(I') = 2\mu(I)$ and such that $X \cap (-\infty, b)/I' \neq \emptyset$. Let $x \in I$, we estimate the quotient $\frac{w((y,a'))}{\mu((y,x))}$, where $y < a'$ is such that $0 < \mu((y,x)) < 2t$. If $0 < \mu((y,b)) < 2t$, we immediately obtain

$$\frac{w((y,a'))}{\mu((y,x))} = \frac{w((y,a'))}{\mu((y,b))}\Big(1 + \frac{\mu((x,b))}{\mu((y,x))}\Big) \leq \frac{w((y,a'))}{\mu((y,b))}\Big(1 + \frac{\mu(I)}{\mu(I)}\Big) \leq 2\mathcal{M}^-_{\mu,2t}[w](b).$$

It remains to consider the case when $2t \leq \mu((y,b)) < \mu(I) + 3t$. We choose $y < y' < a'$ so that $\mu((y',a')) = t - \mu(I)$. Then, in view of (2.3.6a),

$$\frac{w((y,a'))}{\mu((y,x))} \leq \frac{w((y,x))}{\mu^p((y,x))}\mu^{p-1}((y,x)) \leq [\kappa_p^+(2t)]^p \frac{w((y',x))}{\mu((y',x))}\Big(\frac{\mu((y,x))}{\mu((y',x))}\Big)^{p-1}.$$

Since $\mu((y,x)) < 2t$, $\mu((y',x)) = \mu((y',a')) + \mu((a',x)) \geq t - \mu(I) + \mu(I) = t$ and since $\mu((y',b)) = t + \mu(I) < 2t$, we conclude

$$\frac{w((y,a'))}{\mu((y,x))} \leq [2\kappa_p^+(2t)]^p \mathcal{M}^-_{\mu,2t}[w](b).$$

From the proof of Theorem 2.3.4, we know that $\kappa_{p,+}(2t) \leq 4[w]^+_{p,4t}$, hence,

$$\int_I \mathcal{M}^-_{\mu,2t}[w_2]d\mu \leq (8[w]^+_{p,4t})^p \mathcal{M}^-_{\mu,2t}[w](b)\mu(I). \tag{2.3.10}$$

Finally, we combine (2.3.9) and (2.3.10) together to obtain

$$\mathcal{M}^-_{\mu,t}[\mathcal{M}^-_{\mu,2t}[w]] \leq 8c(8[w]^+_{p,4t})^p \mathcal{M}^-_{\mu,2t}[w],$$

μ-a.e. in X. This concludes the proof. □

Lemma 2.3.6. *Assume $w \in A^{loc}_{1,\pm}(\mu)$. Then $w^r \in A^{loc}_{1,\pm}(\mu)$ for some $r > 1$.*

Proof. Consider $w \in A_{1,+}^{\mathrm{loc}}(\mu)$ and let $I = (a, x) \subset X$, where $0 < \mu((a, x)) < t$. For each $\xi > 0$, we have

$$\begin{aligned}
\int_I w^r d\mu &= \Big(\int_{I\cap\{w\le\xi\}} + \int_{I\cap\{w>\xi\}}\Big) w^r d\mu \\
&\le \mu(I)\xi^r + (r-1)\Big(\int_0^\xi + \int_\xi^\infty\Big) s^{r-2} w(I\cap\{w>s\}\cap\{w>\xi\})ds \\
&\le \mu(I)\xi^r + w(I)\xi^{r-1} + (r-1)\int_\xi^\infty s^{r-2} w(I\cap\{w>s\})ds \\
&\le \mu(I)\xi^r + w(I)\xi^{r-1} + (r-1)\int_\xi^\infty s^{r-2} w(I\cap\{\mathcal{M}_{\mu,t}^-[w\chi_I]>s\})ds.
\end{aligned}$$

We observe that $E_s = \{\mathcal{M}_{\mu,t}^-[w\chi_I] > s\} \subset I$, provided that $\xi \le \mathcal{M}_{\mu,t}^-[w](x)$. Further, in each connected component $I_j = (a_j, b_j) \subset E_s$ we have $s\mu(I_j) \le w(I_j)$ and $s\mu(I_j) \ge w(I_j)$ as $b_j \notin E_s$ and $\mu(I_j) < t$. It follows that $s\mu(I_j) = w(I_j)$ and upon summation over j, we have $s\mu(E_s) = w(E_s)$. With the aid of the last identity and the assumption $w \in A_{1,+}^{\mathrm{loc}}(\mu)$, we infer

$$\begin{aligned}
\int_I w^r d\mu &\le \mu(I)\xi^r + w(I)\xi^{r-1} + (r-1)\int_\xi^\infty s^{r-1}\mu(\{\mathcal{M}_{\mu,t}^-[w\chi_I]>s\})ds \\
&\le \mu(I)\xi^r + w(I)\xi^{r-1} + \frac{1}{r'}\int_I \big(\mathcal{M}_{\mu,t}^-[w\chi_I]\big)^r d\mu \\
&\le \mu(I)\xi^r + w(I)\xi^{r-1} + \frac{[w]_{1,t}^+}{r'}\int_I w^r d\mu.
\end{aligned}$$

Hence, $w^r \in A_{1,+}^{\mathrm{loc}}(\mu)$, provided $1 < r < \frac{[w]_{1,t}^+}{[w]_{1,t}^+ - 1}$. □

2.3.2.2 The open end property

Lemmas 2.3.5 and 2.3.6 yield the following fundamental result:

Theorem 2.3.7. *Assume $w \in A_{p,\pm}^{loc}(\mu)$, $1 < p < \infty$. Then $w \in A_{q,\pm}^{loc}(\mu)$ for some $1 < q < p$.*

Proof. Assume that $w \in A_{p,+}^{\mathrm{loc}}(\mu)$, $1 < p < \infty$ and $x \in (a, b) \subset X$, with $0 < \mu((a, b)) < t$. For $k > 0$, we choose $a \le \ldots < x_1 < x_0 = x$ so that $\bar{w}_q((x_k, b)) = 2^k \bar{w}_q((x, b))$. If the sequence $\{x_k\}_{k\ge0}$ is finite ($k \le N-1$), we let $x_N = a$. In that case, $2^{N-1} \leq \frac{\bar{w}_q((a,b))}{\bar{w}_q((x,b))} \leq 2^N$. In either case, we have

$$\begin{aligned} w((a,x))\bar{w}_q^q((x,b)) &\leq \sum_{k\geq 0} 2^{-qk} w((x_{k+1},x_k))\bar{w}_q^q((x_k,b)) \\ &\leq \sum_{k\geq 0} 2^{-qk} \int_{x_{k+1}}^{x_k} \Big(\int_y^b \mathcal{M}^+_{\mu,t}[\chi_{(x_{k+1},b)}\bar{w}_p]^{\frac{p-1}{q-1}} d\mu\Big)^q w(y)d\mu(y) \\ &\leq \mu^q((a,b)) \sum_{k\geq 0} 2^{-qk} \int_{x_{k+1}}^{x_k} \mathcal{M}^+_{\mu,t}\big[\mathcal{M}^+_{\mu,t}[\chi_{(x_{k+1},b)}\bar{w}_p]^{\frac{p-1}{q-1}}\big]^q w d\mu. \end{aligned}$$

Since $\bar{w}_p \in A^{\text{loc}}_{p',-}(\mu,(x_{k+1},b))$, Lemma 2.3.5 implies $v = \mathcal{M}^+_{\mu,t}[\chi_{(x_{k+1},b)}\bar{w}_p] \in A^{\text{loc}}_{1,-}(\mu,(x_{k+1},b))$. Therefore, when $1 < q < p$ is close to p, we apply Lemma 2.3.6 (with X replaced by (x_{k+1},b)) to obtain

$$\mathcal{M}^+_{\mu,t}\big[\mathcal{M}^+_{\mu,t}[\chi_{(x_{k+1},b)}\bar{w}_p]^{\frac{p-1}{q-1}}\big] \leq c_1(t)\mathcal{M}^+_{\mu,t}[\chi_{(x_{k+1},b)}\bar{w}_p]^{\frac{p-1}{q-1}},$$

with $c_1(t) > 0$ depending on $[w]^+_{p,t}$ only. Further, the argument similar to that used in part (b) of Lemma 2.3.5, indicates that μ-a.e. in (x_{k+1},b)

$$\mathcal{M}^+_{\mu,t}[\chi_{(x_{k+1},b)}\bar{w}_p](x) \leq c_2(t)\mathcal{M}^+_{w,t}[\chi_{(x_{k+1},b)}w^{-1}](x)^{p'-1},$$

with $c_2(t) > 0$ depending on $[w]^+_{p,t}$ only. The last two inequalities, combined with Lemma 2.3.1, allow us to conclude that

$$\begin{aligned} w((a,x))\bar{w}_q^q((x,b)) &\leq c_3(t)\mu^q((a,b)) \sum_{k\geq 0} 2^{-qk} \int_{x_{k+1}}^{x_k} \mathcal{M}^+_{w,t}[\chi_{(x_{k+1},b)}w^{-1}]^{q'} w d\mu \\ &\leq c^+_{q'} c_3(t)\mu^q((a,b)) \sum_{k\geq 0} 2^{-qk}\bar{w}_q((x_{k+1},b)) \\ &\leq 2c^+_{q'}c_3(t)\Big(\sum_{k\geq 0} 2^{-(q-1)k}\Big)\mu^q((a,b))\bar{w}_q((x,b)), \end{aligned}$$

where $c_3(t) = c_1(t)c_2(t)$ depends only on $[w]^+_{p,t}$ and $c^+_{q'}$ is the absolute constant from Lemma 2.3.1. Hence, $[w]^+_{q,t} < \infty$ and the proof is complete. □

2.3.2.3 Some consequences

Theorem 2.3.7 has a number of important consequences, we mention two, which are of direct importance for our applications.

Corollary 2.3.8. *The strong type inequalities*

$$\|\mathcal{M}^{\pm}_{\mu,t}[f]\|_{L^p_w(\mathbb{R}_+)} \leq c^{\pm}_p(t)\|f\|_{L^p_w(\mathbb{R}_+)}, \quad 1 < p < \infty, \tag{2.3.11}$$

hold if and only if $w \in A^{loc}_{p,\pm}(\mu)$.

Proof. By definition of the classes $A^{\rm loc}_{p,\pm}(\mu)$ and in view of Theorem 2.3.7, $w \in A^{\rm loc}_{q,\pm}(\mu)$ for all $p-\varepsilon < q < \infty$ and some $\varepsilon > 0$. Hence, (2.3.11) follow directly from the weak type inequalities (2.3.5) (see Theorem 2.3.4) and the Marcinkiewicz interpolation theorem. □

Corollary 2.3.9. *Let* $1 < p < \infty$. *Then* $A^{loc}_{p,\pm}(\mu) = A^{loc}_{1,\pm}(\mu)\big[A^{loc}_{1,\mp}(\mu)\big]^{1-p}$.

Proof. The inclusion $A^{\rm loc}_{1,\pm}(\mu)\big[A^{\rm loc}_{1,\mp}(\mu)\big]^{1-p} \subset A^{\rm loc}_{p,\pm}(\mu)$ is trivial. We show that any $w \in A^{\rm loc}_{p,\pm}(\mu)$ can be factorized as $w = w_1 w_2^{p-1}$, with some $w_1 \in A^{\rm loc}_{1,\pm}(\mu)$ and $w_2 \in A^{\rm loc}_{1,\mp}(\mu)$. For this, we let

$$T_1[\cdot] = w^{-1}\mathcal{M}^{\mp}_{\mu,t}[w|\cdot|], \quad T_2[\cdot] = \mathcal{M}^{\pm}_{\mu,t}[|\cdot|^{\frac{p'}{p}}]^{\frac{p}{p'}}.$$

By Corollary 2.3.8,

$$\|T_1[f]\|_{L^{p'}_w(\mathbb{R}_+)} \le c^{\mp}_{p'}(t)\|f\|_{L^{p'}_w(\mathbb{R}_+)}, \quad \|T_2[f]\|_{L^{p'}_w(\mathbb{R}_+)} \le [c^{\pm}_p(t)]^{\frac{p}{p'}}\|f\|_{L^{p'}_w(\mathbb{R}_+)}.$$

Hence, the operator

$$T[\cdot] = \sum_{k\ge 0} \frac{(T_1+T_2)^k[\cdot]}{2^k\|T_1+T_2\|^k_{L^{p'}_w(\mathbb{R}_+)\to L^{p'}_w(\mathbb{R}_+)}}$$

is bounded from $L^{p'}_w(\mathbb{R}_+)$ to $L^{p'}_w(\mathbb{R}_+)$. For arbitrary $u \in L^{p'}_w(\mathbb{R}_+)$, we have

$$\mathcal{M}^{\mp}_{\mu,t}[wT[u]] = wT_1[T[u]] \le w(T_1+T_2)[T[u]] \le 2\|T_1+T_2\|_{L^{p'}_w(\mathbb{R}_+)\to L^{p'}_w(\mathbb{R}_+)} wT[u].$$

Similarly, we obtain

$$\mathcal{M}^{\pm}_{\mu,t}[T[u]^{\frac{p'}{p}}]^{\frac{p}{p'}} = T_2[T[u]] \le (T_1+T_2)[T[u]] \le 2\|T_1+T_2\|_{L^{p'}(w)\to L^{p'}(w)}T[u].$$

These calculations show that $w_1 = wT[u] \in A^{\rm loc}_{1,\pm}(\mu)$ and $w_2 = T[u]^{\frac{p'}{p}} \in A^{\rm loc}_{1,\mp}(\mu)$, with $[w_1]^{\pm}_{p,t}$ and $[w_2]^{\mp}_{p,t}$ depending on $[w]^{\pm}_{p,t}$ only. □

To conclude, we note that Corollary 2.3.9 is the direct one-sided analogue analogue of the P. Jones factorization theorem. In particular, using Corollary 2.3.9 as a starting point one can build a one-sided version of the operator extrapolation theory in the spirit of Rubio de Francia (see e.g. the development in [33, 43] in the context of A_p weights).

2.3.3 One sided singular integrals

In what follows, we apply the results of Sections 2.3.1 and 2.3.2 to the analysis of a class of vector valued regular singular integrals posed in the open half line $X = \mathbb{R}_+$, equipped with the standard Lebesgue measure λ. In these settings, we abbreviate

$$\mathcal{M}_t^{\pm}[\cdot] = \mathcal{M}_{\lambda,t}^{\pm}[\cdot].$$

Further, for vector valued functions $f : \mathbb{R}_+ \to B$, with values in a Banach space B, we let $\mathcal{M}_t^{\pm}[f] = \mathcal{M}_t^{\pm}[\|f\|_B]$. Our exposition here is laconic and follows closely the line of arguments from [43, 90].

Consider vector valued convolution operators of the form

$$\mathcal{T}_{\kappa^{\pm}}[f](x) = (\kappa^{\pm} * f)(x), \quad x \in \mathbb{R}_+, \tag{2.3.12a}$$

where B_0, B_1 are two given Banach spaces, $f : \mathbb{R}_+ \to B_0$ and $\kappa^{\pm}(x) \in \mathcal{L}(B_0, B_1)$. As in the classical theory (see [43, 90]), we assume

$$\mathcal{T}_{\kappa^{\pm}} \in \mathcal{L}(L^2(\mathbb{R}_+; B_0), L^2(\mathbb{R}_+; B_1)). \tag{2.3.12b}$$

In view of our applications, we consider compactly supported kernels only, i.e. kernels with $\operatorname{supp} \kappa^{\pm} \subset (-t, t) \cap \mathbb{R}_{\pm}$, which for all $x, y, \bar{y} \in \operatorname{supp} \kappa^{\pm}$, with $|x| > 0$ and $|y - \bar{y}| \leq \frac{1}{2}|x - y|$, satisfy

$$\|\kappa^{\pm}(x)\|_{B_0 \to B_1} \leq \frac{c}{|x|}, \tag{2.3.12c}$$

$$\|\kappa^{\pm}(x - y) - \kappa^{\pm}(x - \bar{y})\|_{B_0 \to B_1} \leq c\frac{|y - \bar{y}|}{|x - y|^2}. \tag{2.3.12d}$$

In connection with $\mathcal{T}_{\kappa^{\pm}}$, we define

$$\mathcal{M}_{\kappa^{\pm}}[\cdot] = \sup_{\varepsilon > 0} \|\mathcal{T}_{\chi_{|x|>\varepsilon}\kappa^{\pm}}[\cdot]\|_{B_1}. \tag{2.3.13}$$

The following result is an adaptation of the classical "good-λ inequality" to the one-sided settings, see e.g. [90, Proposition 6, Section V.4.4] or [43, Theorem 9.4.3].

Lemma 2.3.10. *Assume $f \in L^1(\mathbb{R}_+)$ satisfy $\operatorname{supp} f \subset \cup_j I_j$, where $|I_j| < t$ and $\operatorname{dist}(I_j, I_k) \geq 2t$, $j \neq k$. For $\kappa^{\pm}$ as above and $w \in A_\infty^{loc}$ (see [80]), there exists*

$0 < \alpha_w < 1$ *such that for any* $0 < \beta < 1$ *one can find* $\gamma > 0$ *so that the following holds*

$$w(\{\mathcal{M}_{\kappa^\pm}[f] > \xi\} \cap \{\mathcal{M}^{\mp}_{4t}[f] \leq \gamma\xi\}) \leq \alpha_w w(\{\mathcal{M}_{\kappa^\pm}[f] > \beta\xi\}), \tag{2.3.14}$$

for all $\xi > 0$.

Proof. (a) We consider the right-sided operators $\mathcal{M}_{\kappa^-}$, $\mathcal{M}^+_{\lambda,t}$ only. The proof in the left-sided case is identical. Standard arguments (see [43, 90]) indicate that under our assumptions, the level set $E_{\beta\xi}(f) = \{\mathcal{M}_{\kappa^-}[f] > \beta\xi\}$ is open. The support assumption guarantee that every open connected component I of $E_{\beta\xi}(f)$ satisfies $|I| < 2t$. It is sufficient to establish (2.3.14) for a single component $I = (a, b)$, the general result follows by summation.

(b) The set $\hat{I} = (I \setminus \{\mathcal{M}^+_{4t}[f] > \gamma\xi\})$ is closed in the relative topology of I. If the Lebesgue measure $|\hat{I}|$ of $\hat{I}$ is zero, (2.3.14) holds trivially. So assume $|\hat{I}| > 0$, let $x = \min \overline{\hat{I}}$, $\hat{x} = b + (b - x)$, $f_1 = \chi_{[x,\hat{x}]} f$ and $f_2 = (1 - \chi_{[x,\hat{x}]})f$ and observe that

$$w(E_\xi(f) \cap I) \leq w(E_{\tau\xi}(f_1) \cap I) + w(E_{(1-\tau)\xi}(f_2) \cap I), \quad 0 < \tau < 1.$$

We estimate each term separately.

To bound $w(E_{\tau\xi}(f_1) \cap I)$, we employ the standard weak-type inequality (see e.g. [90, Corollary 2, Section I.7.3]) to obtain initially

$$\lambda(E_{\tau\xi}(f_1) \cap I) \leq \tfrac{c}{\tau\xi} \int_x^{\hat{x}} \|f\|_{B_0} dy \leq \tfrac{2c}{\tau\xi} |I| \mathcal{M}^+_{4t}[f](x) \leq \tfrac{2c\gamma}{\tau} |I|,$$

and then, using the inclusion $w \in A^{\text{loc}}_\infty$,

$$w(E_{\tau\xi}(f_1) \cap I) \leq \alpha_w w(I),$$

with $0 < \alpha_w < 1$, provided $1 < \gamma < 0$ is sufficiently small.

(c) To bound $w(E_{\tau\xi}(f_1) \cap I)$, we note that in view of (2.3.12c) and (2.3.12d), for $y \in (x, b)$, we have

$$\left\| \mathcal{T}_{\chi_{|x|>\varepsilon}\kappa^-}[f_2](y) - \mathcal{T}_{\chi_{|x|>\varepsilon}\kappa^-}[f_2](b) \right\|_{B_1} = 0,$$

if $\hat{x} \geq b + t$, or

$$\begin{aligned}
&\big\|\mathcal{T}_{\chi_{|x|>\varepsilon}\kappa^-}[f_2](y) - \mathcal{T}_{\chi_{|x|>\varepsilon}\kappa^-}[f_2](b)\big\|_{B_1} \\
&\qquad \leq \int_{\hat{x}+\varepsilon}^{b+t} \|f(z)\|_{B_0}\|\kappa^-(y-z) - \kappa^-(b-z)\|_{B_0\to B_1} dz \\
&\qquad \leq c|b-y| \int_{\hat{x}}^{b+t} \|f(z)\|_{B_0} \tfrac{dz}{(z-y)^2} \\
&\qquad \leq c \sum_{j\geq 0} \tfrac{|b-y|}{|\hat{x}-y+(2^j-1)(b-y)|^2} \int_{\hat{x}+(2^j-1)(b-y)}^{\hat{x}+(2^{j+1}-1)(b-y)} \|f\|_{B_0}\chi_{[\hat{x},b+t]} dz \\
&\qquad \leq 4c \sum_{j\geq 0} \tfrac{2^{-j}}{|\hat{x}-x+(2^{j+1}-1)(b-y)|} \int_{x}^{\hat{x}+(2^{j+1}-1)(b-y)} \|f\|_{B_0}\chi_{[x,b+t]} dz \\
&\qquad \leq 8c\mathcal{M}^+_{3t}[f](x) \leq 8c\mathcal{M}^+_{4t}[f](x) \leq 8c\gamma\xi,
\end{aligned}$$

when $\hat{x} < b + t$. In either case, since $b \notin E_{\beta\xi}(f)$, taking supremum over $\varepsilon > 0$, we obtain

$$\mathcal{M}_{\kappa^-}[f_2](y) \leq (\beta + 8c\gamma)\xi \leq (1-\tau)\xi, \quad y \in (x, b),$$

provided $0 < \gamma < 1$ is small and $0 < \tau < 1$ is chosen appropriately. Hence, $w(E_{(1-\tau)\xi}(f_2) \cap I) = 0$ and we conclude that (2.3.14) holds. □

Once the basic estimate (2.3.14) is settled, the boundedness of $\mathcal{M}_{\kappa^\pm}$ and $\mathcal{T}_{\kappa^\pm}$ follows almost immediately see e.g. [90].

Corollary 2.3.11. *For $\kappa^\pm$ as above and $w \in A^{loc}_{p,\mp}(\lambda) \cap A^{loc}_\infty$, $1 < p < \infty$, the following holds*

$$\big\|\mathcal{M}_{\kappa^\pm}\big\|_{L^p_w(\mathbb{R}_+;B_0)\to L^p_w(\mathbb{R}_+)} < \infty, \tag{2.3.15a}$$

$$\big\|\mathcal{T}_{\kappa^\pm}\big\|_{L^p_w(\mathbb{R}_+;B_0)\to L^p_w(\mathbb{R}_+;B_1)} < \infty. \tag{2.3.15b}$$

Proof. (a) Consider $f \in C^\infty_0(\mathbb{R}_+; B_0)$ initially. Without loss of generality, we may assume that f satisfies the support condition of Lemma 2.3.10 (for any function in $\mathbb{R}$ is a sum of at most four functions satisfying this condition). By our assumptions, $g_\varepsilon = \mathcal{T}_{\chi_{|x|>\varepsilon}\kappa^\pm}[f]$ is compactly supported and smooth, with $\|g_\varepsilon\|_{L^\infty(\mathbb{R},B_1)}$ bounded independently of $\varepsilon > 0$. Since $w \in L^{1,\mathrm{loc}}(\mathbb{R}_+)$, we conclude that $\|\mathcal{M}_{\kappa^\pm}[f]\|_{L^p_w(\mathbb{R}_+)} < \infty$. Once this fact is established, we make use of Lemma 2.3.10 and Corollary 2.3.8 to obtain

$$\big\|\mathcal{M}_{\kappa^\pm}[f]\big\|_{L^p_w(\mathbb{R}_+)} \leq c\big\|\mathcal{M}^\mp_{4t}[f]\big\|_{L^p_w(\mathbb{R}_+)} \leq c'\|f\|_{L^p_w(\mathbb{R}_+;B_0)},$$

for all $f \in C_0^\infty(\mathbb{R}_+; B_0)$. The standard density argument allows one to pass from $C_0^\infty(\mathbb{R}_+; B_0)$ to $L^p_w(\mathbb{R}_+; B_0)$. Hence, the bound (2.3.15a) is settled.

(b) Estimate (2.3.15b) is the direct consequence of (2.3.15a), as

$$\|\mathcal{T}_{\kappa^\pm}[f]\|_{B_1} \leq \mathcal{M}_{\kappa^\pm}[f] + c\|f\|_{B_0},$$

a.e. in $\mathbb{R}_+$, see e.g. [90, Section I.7.4]. $\square$

Note that the classical version of "good-λ inequality" is based on the Whitney decomposition of the level set $E_{\beta\xi}(f)$. Due to the asymmetry of the operators, the one-sided analogue of Lemma 2.3.10 requires revision of admissible geometrical arguments. The proof of Lemma 2.3.10 as it stands seems hard to extend to general $\mathbb{R}^n$ domains. A possible alternative route would be the use of sharp one-sided maximal functions in the spirit of [42].

2.3.4 Bessel potential spaces with $A^{loc}_{p,\pm}(\lambda)$ weights

Now we turn our attention to Bessel potential spaces in $\mathbb{R}_+$. These are defined as images of weighted Lebesgue spaces $L^p(w)$ under the action of the Fourier multiplier

$$\mathcal{J}^s_-[f] = \mathcal{F}^{-1}[\hat{\kappa}^-_s] * f, \quad \hat{\kappa}^-_s = \frac{(i-\xi)^s}{\sqrt{2\pi}}, \quad s \geq 0,$$

restricted to $\mathbb{R}_+$, i.e. $L^{s,p}_w(\mathbb{R}_+) = \mathcal{J}^s_-[L^p_w(\mathbb{R}_+)]$, (see [8]). In particular, for $1 < p < \infty$ and $s \geq 0$, $L^{s,p}(\mathbb{R}_+)$ agree with the complex interpolation version of the classical Sobolev spaces $W^{s,p}(\mathbb{R}_+)$ as defined in [3].

For a fixed $t > 0$ and $\varphi^\pm \in C_0^\infty(\mathbb{R})$, with $\operatorname{supp}\varphi^\pm \subset \mathbb{R}_\pm \cap (-t, t)$, we define

$$\mathcal{T}_{\varphi^\mp}[f](x) = \varphi^\mp * f, \quad x \in \mathbb{R}_+.$$

Trivially, we have

$$|\mathcal{T}_{\varphi^\mp}[f]| \leq t\|\varphi^\mp\|_\infty \mathcal{M}^\pm_t[f], \quad x \in \mathbb{R}_+,$$

for special cut-off functions $\varphi^\pm$, the following one-sided analogue of the Proposition in [90, Section II.2.1] holds.

Lemma 2.3.12. *Let $\varphi \in C_0^\infty(-\frac{t}{2}, \frac{t}{2})$ be radially non-increasing. Assume $\varphi = const$, $x \in (\frac{t}{4}, -\frac{t}{4})$, $\int_{\mathbb{R}} \varphi dx = 1$ and define $\varphi^\pm(\cdot) = \varphi(\pm\frac{t}{2} + \cdot)$. Then*

$$\|\mathcal{T}_{\varphi^\mp}\|_{L^p_w(\mathbb{R}_+) \to L^p_w(\mathbb{R}_+)} < \infty, \tag{2.3.16}$$

provided $1 < p < \infty$ and $w \in A^{loc}_{p,\pm}(\lambda)$.

Proof. (a) Under our assumptions, we have

$$|\mathcal{T}_{\varphi^\mp}[f]| \leq 2\mathcal{M}_t^\pm[f], \quad x \in \mathbb{R}_+. \tag{2.3.17}$$

Indeed, any function φ that satisfy the above conditions is uniformly approximated from the above by step functions $\varphi_n = \sum_{i=0}^n a_i \chi_{(-t_i, t_i)}$, where $0 < a_i$, $t < 4t_i < 2t$, and $\int_{\mathbb{R}} \varphi_n dx = 1$. For such functions, we have

$$\begin{aligned}
|\mathcal{T}_{\varphi^\mp}[f]|(x) &\leq \sum_{i=0}^n a_i \int_{\mp\frac{t}{2} - t_i}^{\mp\frac{t}{2} + t_i} |f|(x - \tau) d\tau \\
&\leq \sum_{i=0}^n a_i 2t_i \frac{t + 2t_i}{4t_i} \mathcal{M}_t^\pm[f](x) \leq 2\mathcal{M}_t^\pm[f](x).
\end{aligned}$$

(b) In view of (2.3.17) and the inclusion $w \in A^{\mathrm{loc}}_{p,\pm}(\lambda)$, the assertion is the direct consequence of Corollary 2.3.8. $\square$

Operator $\mathcal{J}^s_-$, $s \geq 0$, is known to be invertible in the class of smooth functions restricted to $\mathbb{R}_+$, [81]. We denote the associated inverses by $\bar{\mathcal{J}}^{-s}_-$. For $0 < \varepsilon < 1$ and $\varphi^\mp$ from Lemma 2.3.12, let $\varphi^\mp_\varepsilon(\cdot) = \varepsilon^{-1}\varphi^\mp(\varepsilon^{-1}\cdot)$, $\mathcal{J}^{-s}_{\varepsilon,-} = \bar{\mathcal{J}}^{-s}_- \mathcal{T}_{\varphi^-_\varepsilon}$ and

$$\mathcal{J}^{-s}_-[f] = \lim_{\varepsilon \to 0} \mathcal{J}^{-s}_{\varepsilon,-}[f].$$

Lemma 2.3.13. *Operator $\mathcal{J}^{-s}_- : L^{s,p}_w(\mathbb{R}_+) \to L^p_w(\mathbb{R}_+)$, $1 < p < \infty$, $s \geq 0$, is one-to-one, provided $w \in A^{loc}_{p,+}(\lambda)$.*

Proof. Straightforward calculations show that $\mathcal{T}_{\varphi^-_\varepsilon, t}$ and $\mathcal{J}^s_-$ commute (note that $\operatorname{supp} \mathcal{F}^{-1}[\hat{\kappa}^-_s]$ and $\operatorname{supp} \varphi^- \subset \mathbb{R}_-$). Therefore, for each $f \in L^{s,p}_w(\mathbb{R}_+)$ (by definition $f = \mathcal{J}^s_-[\phi]$ for some $\phi \in L^p_w(\mathbb{R}_+)$), we have

$$\|\mathcal{J}^{-s}_{\varepsilon,-}[f] - \phi\|_{L^p_w(\mathbb{R}_+)} = \|\mathcal{T}_{\varphi^\mp_\varepsilon, t}[\phi] - \phi\|_{L^p_w(\mathbb{R}_+)} \to 0, \quad \text{as} \quad \varepsilon \to 0.$$

The conclusion follows from the uniqueness of strong limits. $\square$

Lemma 2.3.13 indicates that $\mathcal{J}_-^s$, $s \geq 0$, are isomorphisms of the scales $L_w^p(\mathbb{R}_+)$ and $L_w^{s,p}(\mathbb{R}_+)$, $1 < p < \infty$, $w \in A_{p,\pm}^{\mathrm{loc}}(\lambda)$. Hence, $L_w^{s,p}(\mathbb{R}_+)$, $1 < p < \infty$, equipped with the norms $\|\cdot\|_{L_w^{s,p}(\mathbb{R}_+)} := \|\mathcal{J}_-^{-s}[\cdot]\|_{L_w^p(\mathbb{R}_+)}$, are Banach spaces.

2.3.5 Interpolation

2.3.5.1 Square function characterization

To proceed further, we employ the following local reproducing formula of V.S. Rychkov (see [80] for the details)

$$\delta = \sum_{j\geq 0} \varphi_j^\pm * \psi_j^\pm, \tag{2.3.18a}$$

where $\varphi_0^\pm, \psi_0^\pm \in C_0^\infty(\mathbb{R})$, with $\operatorname{supp}\varphi_0^\pm, \operatorname{supp}\psi_0^\pm \subset (-t,t)\cap\mathbb{R}_\pm$ for some $t > 0$, have non vanishing zeroth moment; $\varphi^\pm(\cdot) = \varphi_0^\pm(\cdot) - 2^{-1}\varphi_0^\pm(2^{-1}\cdot)$, $\psi^\pm(\cdot) = \psi_0^\pm(\cdot) - 2^{-1}\psi_0^\pm(2^{-1}\cdot)$ and $\varphi_j^\pm(\cdot) = 2^j\varphi^\pm(2^j\cdot)$, $\psi_j^\pm(\cdot) = 2^j\psi^\pm(2^j\cdot)$, $j \geq 1$. Furthermore, both $\varphi_0^\pm$ and $\psi_0^\pm$ can be chosen so that

$$\int_{\mathbb{R}} x^k\varphi^\pm dx = \int_{\mathbb{R}} x^k\psi^\pm dx = 0, \quad 0 \leq k \leq m, \tag{2.3.18b}$$

for any given positive integer $m > 0$ (in the sequel, we employ symbol $\{\varphi\}_m$ to denote the number of vanishing moments of function φ).

For $\varphi_0^\pm$ as above, with $\{\varphi^\pm\}_m \geq \max\{0, s\}$, $s \in \mathbb{R}$, we define

$$\mathcal{S}_{\varphi^\pm}^s[f] = \Big(\sum_{j\geq 0} 2^{2js}|\varphi_j^\pm * f|^2\Big)^{\frac{1}{2}}, \quad x \in \mathbb{R}_+.$$

Theorem 2.3.14. *For $1 < p < \infty$, $s \geq 0$ and $w \in A_{p,+}^{loc}(\lambda) \cap A_\infty^{loc}$, we have*

$$\|f\|_{L_w^{s,p}(\mathbb{R}_+)} \approx \|\mathcal{S}_{\varphi^-}^s[f]\|_{L_w^p(\mathbb{R}_+)}, \tag{2.3.19}$$

where $\approx$ means the bilateral estimate.

Proof. (a) To begin, we show that

$$\|f\|_{L_w^p(\mathbb{R}_+)} \approx \|\mathcal{S}_{\varphi^\pm}^0[f]\|_{L_w^p(\mathbb{R}_+)}, \tag{2.3.20}$$

provided $w \in A_{p,\mp}^{\mathrm{loc}}(\lambda)\cap A_\infty^{\mathrm{loc}}$. The proof is identical to that of [80, Theorem 1.10], with the exception that, instead of [90, Theorem 2, Section V.4.2] and its [90, Corollary, Section V.4.2], we invoke Corollary 2.3.11.

Define $\kappa^\pm : \mathbb{R}_+ \to \ell_2$ by means of the formulas $\kappa^\pm(\cdot) = \{\varphi_j^\pm(\cdot)\}_{j\ge 0}$. Operator $\mathcal{T}_{\kappa^\pm} : L^p_w(\mathbb{R}_+) \to L^p_w(\mathbb{R}_+;\ell_2)$ fells in the scope of Corollary 2.3.11. Hence,

$$\|\mathcal{S}^0_{\varphi^\pm}[f]\|_{L^p_w(\mathbb{R}_+)} = \|\mathcal{T}_{\kappa^\pm} f\|_{L^p_w(\mathbb{R}_+;\ell_2)} \le c\|f\|_{L^p_w(\mathbb{R}_+)},$$

provided $w \in A^{\mathrm{loc}}_{p,\mp}(\lambda) \cap A^{\mathrm{loc}}_\infty$.

The converse inequality follows from the standard duality argument and the local reproducing formula (2.3.18a). Indeed, for $g \in L^{p'}_{\bar{w}_p}(\mathbb{R}_+)$ supported in $\mathbb{R}_+$, we let $g^\pm(\cdot) = g(\pm\cdot)$ and note that $(g^\pm)^- = g^\mp$. Then

$$|\langle f, g\rangle| = |f * g^-|(0) = \Big|\sum_{j\ge 0}(\varphi_j^\pm * f) * (\psi_j^\pm * g^-)\Big|(0) \le \int_{\mathbb{R}_+} \mathcal{S}^0_{\varphi^\pm}[f]\mathcal{S}^0_{\psi^\mp}[g]d\tau$$
$$\le \|\mathcal{S}^0_{\varphi^\pm}[f]\|_{L^p_w(\mathbb{R}_+)}\|\mathcal{S}^0_{\psi^\mp}[g]\|_{L^{p'}_{\bar{w}_p}(\mathbb{R}_+)} \le c\|\mathcal{S}^0_{\varphi^\pm}[f]\|_{L^p_w(\mathbb{R}_+)}\|g\|_{L^{p'}_{\bar{w}_p}(\mathbb{R}_+)}.$$

Hence, (2.3.20) is settled.

(b) The general result follows from [80, Theorem 2.18], invertibility of $\mathcal{J}^s_-$, Lemma 2.3.13 and (2.3.20). Indeed, letting

$$\bar{L}^{s,p}_w(\mathbb{R}_+) = \{f \mid \operatorname{supp} f \in \mathbb{R}_+, \quad \|\mathcal{S}^s_{\varphi^-}\|_{L^p_w(\mathbb{R}_+)} < \infty\},$$

from [80, Theorem 2.18] we infer that $\mathcal{J}^s_-\big[\bar{L}^{0,p}_w(\mathbb{R}_+)\big] \subset \bar{L}^{s,p}_w(\mathbb{R}_+)$, $\beta \ge 0$ and the map is onto, as $\mathcal{J}^s_-$ is invertible. Hence, by (2.3.20) and Lemma 2.3.13, $\bar{L}^{s,p}_w(\mathbb{R}_+) = \mathcal{J}^s_-[\bar{L}^{0,p}_w(\mathbb{R}_+)] = \mathcal{J}^s_-[L^p_w(\mathbb{R}_+)] = L^{s,p}_w(\mathbb{R}_+)$ as Banach spaces. □

The key feature of Theorem 2.3.14, as compared to its analogue [80, Theorem 1.10], is the use of condition $w \in A^{\mathrm{loc}}_{p,+}(\lambda) \cap A^{\mathrm{loc}}_\infty$, instead of $w \in A^{\mathrm{loc}}_p$. The former class is significantly larger than the latter one. For instance, $w_\alpha(x) = |x|^\alpha \in A^{\mathrm{loc}}_p$ if and only if $-1 < \alpha < \frac{p}{p'}$, while $w_\alpha(x) \in A^{\mathrm{loc}}_{p,+}(\lambda) \cap A^{\mathrm{loc}}_\infty$, for all $\alpha > -1$.

2.3.5.2 An interpolation identity

In view of Theorem 2.3.14, we have

Corollary 2.3.15. *Assume $1 < p < \infty$ and $w_0, w_1 \in A^{loc}_{p,+}(\lambda) \cap A^{loc}_\infty$. Then*

$$[L^{s_0,p}_{w_0}(\mathbb{R}_+), L^{s_1,p}_{w_1}(\mathbb{R}_+)]_\theta = L^{(1-\theta)s_0+\theta s_1,p}_{w_0^{1-\theta}w_1^\theta}(\mathbb{R}_+),\ s_0, s_1 \ge 0,\ \theta \in (0,1), \tag{2.3.21}$$

where $[\cdot,\cdot]_\theta$ denotes the standard complex interpolation functor of A. Calderon [12, 25].

Proof. Directly from (2.3.8) and the definition of $A^{loc}_{p,+}(\lambda)$ and $A^{\rm loc}_{\infty}$ weights (see [80]), it follows that $w_0^{1-\theta}w_1^{\theta} \in A^{\rm loc}_{p,+}(\lambda) \cap A^{\rm loc}_{\infty}$, while, in view of Theorem 2.3.14, $L^{s,p}_w(\mathbb{R}_+)$ is a retracts of $L^p_w(\mathbb{R}_+; \ell^2_s)$. Hence, combining the arguments of Theorems 2.2.2 and 2.2.3, we have the desired result. □

2.4 Variable weight Sobolev spaces

2.4.1 Definition

We define the variable weight Sobolev space $H^s_r(\mathbb{R})$ of real valued functions by

$$H^s_r(\mathbb{R}) = \{f \in \operatorname{Re}\mathcal{S}' \mid \|f\|_{H^s_r(\mathbb{R})} < \infty\}, \quad s > -\tfrac{1}{2}, \quad r \geq 0, \tag{2.4.1a}$$

$$\|f\|^2_{H^s_r(\mathbb{R})} = \tfrac{1}{2}\|f\|^2_{L^2(\mathbb{R})} + \|\mathcal{P}_{\pm}[\kappa^{\pm}_{r,\ell} f]\|^2_{\mathring{H}^s(\mathbb{R})}, \tag{2.4.1b}$$

where $\operatorname{Re}\mathcal{S}'$ is the space of real valued tempered distributions, $\mathcal{P}_{\pm}$ are Fourier multipliers (projectors) associated with the Heaviside functions $\hat{h}^{\pm}(\xi) = \frac{1\pm\operatorname{sgn}(\xi)}{2}$, $\kappa^{\pm}_{r,\ell}(\cdot) = \frac{1}{\sqrt{2\pi}}(i\ell \pm \cdot)^r$ and $\mathring{H}^s(\mathbb{R})$ is the standard homogeneous Sobolev space of order s, see e.g. [12]. Note that for $s \in \mathbb{N}$, the $H^s_r(\mathbb{R})$-norm is equivalent to $\|f\|_{L^2(\mathbb{R})} + \sum_{m=0}^{s} \|\kappa^{\pm}_{r-s+m,\ell} f^{(m)}\|_{L^2(\mathbb{R})}$, i.e. $\|\cdot\|_{H^s_r(\mathbb{R})}$ is a Sobolev-like norm, where weak derivatives of different orders are integrated against different weights.

2.4.2 Properties

Basic properties of the variable weighted Sobolev space $H^s_r(\mathbb{R})$ are listed below

Lemma 2.4.1. *$H^s_r(\mathbb{R})$, with $s > -\frac{1}{2}$ and $r \geq 0$, are Hilbert spaces. The embeddings*

$$H^{s_0}_{r_0}(\mathbb{R}) \hookrightarrow H^{s_1}_{r_1}(\mathbb{R}), \tag{2.4.2a}$$

are dense and continuous, provided

$$-\frac{1}{2} < s_1 \leq s_0 \leq s_1 + r_0 - r_1, \quad 0 \leq r_1 \leq r_0. \tag{2.4.2b}$$

In addition, the embedding

$$H^{s_0}_{r_0}(\mathbb{R}) \hookrightarrow H^{s_1}(\mathbb{R}), \quad 0 \leq s_1 < s_0 < r_0, \quad r_0 > 0, \tag{2.4.2c}$$

is dense and compact. Finally , for $s_0, s_1 > -\frac{1}{2}, r_0, r_1 \geq 0$ and $\theta \in (0,1)$, we have

$$[H^{s_0}_{r_0}(\mathbb{R}), H^{s_1}_{r_1}(\mathbb{R})]_\theta = H^{(1-\theta)s_0+\theta s_1}_{(1-\theta)r_0+\theta r_1}(\mathbb{R}), \tag{2.4.3}$$

where $[\cdot,\cdot]_\theta$ denotes the standard complex interpolation functor of A. Calderon [12, 25].

Proof. (a) In terms of Fourier images, (2.4.2b) reads

$$\|f\|^2_{H^s_r(\mathbb{R})} = \|\hat{f}\|^2_{L^2(\mathbb{R}_\pm)} + \|\mathcal{J}^{-r}_{\mp}[\hat{f}]\|^2_{L^2_s(\mathbb{R}_\pm)}, \quad \mathcal{J}^{-r}_{\mp}[\hat{f}] = \sqrt{2\pi}\big(\hat{\kappa}^{\pm}_{r,\ell} * \hat{f}\big). \tag{2.4.4}$$

Note that $\operatorname{supp}\hat{\kappa}^{\pm}_{r,\ell} \subset \mathbb{R}_\mp$. Hence, for real valued distributions (whose Fourier images are Hermitian, i.e. $\hat{f}(\xi) = -\overline{\hat{f}(-\xi)}$) the choice of sign in (2.4.1b), (2.4.4) is irrelevant.

It was shown in subsection 2.3.4 that $L^{r,2}_s(\mathbb{R}_\pm) = \mathcal{J}^r_\mp[L^2_s(\mathbb{R}_\pm)]$ are Banach spaces for $w \in A^{loc}_{2,\pm}(\lambda)$. Since, $|\xi|^s \in A^{loc}_{2,+}(\lambda)$ in $\mathbb{R}_+$ and $|\xi|^s \in A^{loc}_{2,-}(\lambda)$ in $\mathbb{R}_-$, for any $s > -\frac{1}{2}$ and since $L^2(\mathbb{R}) \cap \operatorname{Re}\mathcal{S}'$ distributions are regular, we conclude that the quantity $\|\cdot\|_{H^s_r(\mathbb{R})}$ is a norm in $H^s_r(\mathbb{R})$. The completeness of $H^s_r(\mathbb{R})$ follows from the completeness of $L^{r,2}_s(\mathbb{R}_\pm) \cap L^2(\mathbb{R}_\pm)$. In view of (2.4.4), the bilinear form

$$\langle f, g\rangle_{H^s_r(\mathbb{R})} = \langle \hat{f}, \hat{g}\rangle_{L^2(\mathbb{R}_+)} + \langle \mathcal{J}^{-r}_\mp[\hat{f}], \mathcal{J}^{-r}_\mp[\hat{g}]\rangle_{L^2_s(\mathbb{R}_+)}$$

is an inner product in $H^s_r(\mathbb{R})$. Hence, the first claim of Lemma 2.4.1 is settled.

(b) The denseness and continuity of the embeddings (2.4.2a)-(2.4.2b) is the direct consequence of (2.4.4) and the embedding inequality (21) from [8].

To show compactness of (2.4.2c), consider $u \in H^{s_0}_{r_0}(\mathbb{R})$, with $\|u\|_{H^{s_0}_{r_0}(\mathbb{R})} \leq 1$. In terms of Fourier images, for $s_1 < s_0$, we have

$$\begin{aligned}\int_{|\zeta|>|\xi|} \big(1+|\zeta|^2\big)^{s_1} |\hat{u}|^2(\zeta)d\zeta &\leq \sup_{|\zeta|>|\xi|} \big(1+|\zeta|^2\big)^{s_1-s_0} \|u\|^2_{H^{s_0}(\mathbb{R})} \\ &\leq c\big(1+|\xi|^2\big)^{s_1-s_0}, \quad \xi \in \mathbb{R},\end{aligned}$$

uniformly for $\|u\|^2_{W^s_r(\mathbb{R})} \leq 1$.

Since u is assumed to be real valued, its Fourier image $\hat{u}$ is Hermitian. Therefore, for $h > 0$, we have

$$\int_{\mathbb{R}} (1+|\xi|^2)^{s_1} \big|\hat{u}(\xi+h) - \hat{u}(\xi)\big|^2 d\xi \leq 2\int_{\mathbb{R}_+} (1+|\xi|^2)^{s_1} \big|\hat{u}(\xi+h) - \hat{u}(\xi)\big|^2 d\xi =: 2I_1.$$

To bound I_1, for $0 \le r' \le r_0$, we let $\hat{u} = \mathcal{J}_-^{+r'}[\hat{v}_{r'}]$. The analysis presented in [8] indicates that

$$\hat{u} = \mathcal{J}_-^{+r'}[\hat{v}_{r'}] \in L^2_{s'(\mathbb{R}_+)},\ -\tfrac{1}{2} < s' \le s_0 \le s' + r_0 - r',\ 0 \le r' \le r_0, \tag{2.4.5a}$$

$$\|\hat{v}_{r'}\|_{L^2_{s'}(\mathbb{R})} \le c\|\hat{v}_{r_0}\|_{L^2_{s_0}(\mathbb{R})},\ -\tfrac{1}{2} < s' \le s_0 \le s' + r_0 - r',\ 0 \le r' \le r_0. \tag{2.4.5b}$$

Using these facts, the inclusion $\operatorname{supp}\hat{\kappa}^+_{-r',\ell} \subset \mathbb{R}_-$, and the standard Minkowski inequality, we obtain initially

$$\begin{aligned} I_1 &= 2\pi \int_{\mathbb{R}_+} (1+|\xi|^2)^{s_1} \big|(\hat{\kappa}^+_{-r',\ell} * \hat{v}_{r'})(\xi+h) - (\hat{\kappa}^+_{-r',\ell} * \hat{v}_{r'})(\xi)\big|^2 d\xi \\ &\le 2\pi \left[\int_{\mathbb{R}} \big|\hat{\kappa}^+_{-r',\ell}(\zeta-h) - \hat{\kappa}^+_{-r',\ell}(\zeta)\big| d\zeta \left(\int_{\mathbb{R}+} |\hat{v}_{r'}|^2(\xi)\big(1+|\xi-\zeta|^2\big)^{s_1} d\xi\right)^{\frac{1}{2}}\right]^2 \end{aligned}$$

and then, using the elementary inequality $\big(1+|\xi-\zeta|^2)^{s_1} \le c[(1+|\zeta|^2)^{s_1} + |\xi|^{2s_1}]$ (that holds uniformly for $\xi, \zeta \in \mathbb{R}$, with some constant $c > 0$) and formulas (2.4.5), (2.4.1b),

$$\begin{aligned} I_1 &\le c\left[\int_{\mathbb{R}+} \big(1+|\zeta|^2\big)^{s_1} \big|\hat{\kappa}^+_{-r',\ell}(\zeta-h) - \hat{\kappa}^+_{-r',\ell}(\zeta)\big| d\xi\right] \big(\|\hat{v}_{r'}\|^2_{L^2(\mathbb{R}+)} + \|\hat{v}_{r'}\|^2_{L^2_{s_1}(\mathbb{R}+)}\big) \\ &\le c\left[\int_{\mathbb{R}+} \big(1+|\zeta|^2\big)^{s_1} \big|\hat{\kappa}^+_{-r',\ell}(\zeta-h) - \hat{\kappa}^+_{-r',\ell}(\zeta)\big| d\xi\right] \|u\|^2_{H^{s_0}_{r_0}(\mathbb{R})}, \end{aligned} \tag{2.4.6}$$

provided $0 \le s_0 \le r_0 - r'$ and $-\frac{1}{2} < s_1 \le s_0 \le s_1 + r_0 - r'$. We note that the right hand side of (2.4.6) tends to zero, as $h \to 0$, as $\hat{\kappa}^+_{-r',\ell} \in L^1(\mathbb{R}+, |\xi|^s d\xi)$, for all $s \ge 0$, provided $r' > 0$.

The calculations, presented above, indicate that the the unit ball of $H^{s_0}_{r_0}(\mathbb{R})$ is equibounded, equitight and equicontinuous in $H^{s_1}(\mathbb{R})$, provided $s_1 < s_0$ and $0 \le s_0 \le r_0 - r'$, $-\frac{1}{2} < s_1 \le s_0 \le s_1 + r_0 - r'$, for some $0 < r' \le r$. Hence, on the account of the Kolmogorov-Riesz theorem, the embedding (2.4.2c) is indeed compact.

(c) Interpolation identity (2.4.3) follow from Theorem 2.2.4 and formula (2.3.21), if we view $H^s_r(\mathbb{R})$ as a retract of the vector valued Banach space $\tilde{H}^s_r(\mathbb{R}) = \{(u,v) | u \in L^2(\mathbb{R}),\ \hat{v} \in L^{2,r}_s(\mathbb{R}_-) \cap L^{2,r}_s(\mathbb{R}_+)\}$. □

To conclude this section, we note that $H^s_0(\mathbb{R}) = H^s(\mathbb{R})$, where $H^s(\mathbb{R})$ is the standard Sobolev spaces, as defined in [3]. When $s > -\frac{1}{2}$, the latter is known to be a Banach algebra. As shown below, the property extends to $H^s_r(\mathbb{R})$, with $s > \frac{1}{2}$ and $r \geq 0$, this fact is crucial for the analysis of Chapters 3 and 6.

Lemma 2.4.2. *Assume $s > \frac{1}{2}$ and $r \geq 0$. Then $H^s_r(\mathbb{R})$ is a Banach algebra, i.e. for any $f, g \in H^s_r(\mathbb{R})$*

$$\|fg\|_{H^s_r(\mathbb{R})} \leq c\|f\|_{H^s_r(\mathbb{R})}\|g\|_{H^s_r(\mathbb{R})}, \tag{2.4.7}$$

with $c > 0$ independent of f and g.

Proof. (a) Using the elementary estimate $|\xi_0 + \xi_1|^s \leq c\big(|\xi_0|^s + |\xi_1|^s\big)$ (which holds for all $\xi_0, \xi_1 \in \mathbb{R}$ and $s > -1$, with an absolute constant $c > 0$ controlled by s only) combined with the standard convolution Young inequality, for any two Hermitian functions

$$\hat{f}, \hat{g} \in L^2_s(\mathbb{R}_\pm) \cap L^2(\mathbb{R}_\pm) = L^2(\mathbb{R}_\pm, (1 + |\xi|^{2s})d\xi) =: \bar{L}^2_s(\mathbb{R}_\pm),$$

we have

$$\|\hat{f} * \hat{g}\|_{L^2_s(\mathbb{R}_\pm)} \leq c\big(\|\hat{f}\|_{L^1(\mathbb{R}_\pm)}\|g\|_{L^2_s(\mathbb{R}_\pm)} + \|f\|_{L^2_s(\mathbb{R}_\pm)}\|\hat{g}\|_{L^1(\mathbb{R}_\pm)}\big).$$

By our assumption $s > \frac{1}{2}$, hence the direct application of the Cauchy-Schwarz inequality yields

$$\|\hat{f}\|_{L^1(\mathbb{R}_\pm)} \leq \|(1 + |\xi|^{2s})^{-\frac{1}{2}}\|_{L^2(\mathbb{R}_+)}\|\hat{f}\|_{\bar{L}^2_s(\mathbb{R}_\pm)} \leq c\|\hat{f}\|_{\bar{L}^2_s(\mathbb{R}_\pm)}$$

and we conclude

$$\|\hat{f} * \hat{g}\|_{L^2_s(\mathbb{R}_\pm)} \leq c\|\hat{f}\|_{\bar{L}^2_s(\mathbb{R}_\pm)}\|\hat{g}\|_{\bar{L}^2_s(\mathbb{R}_\pm)}.$$

(b) We let

$$\bar{L}^{r,2}_s(\mathbb{R}_\pm) := \mathcal{J}^r_\mp[\bar{L}^2_s(\mathbb{R}_\pm)] = L^{r,2}_s(\mathbb{R}_\pm) \cap L^{r,2}_0(\mathbb{R}_\pm)$$

and observe that $\bar{L}^{r_1,2}_s(\mathbb{R}_\pm) \hookrightarrow \bar{L}^{r_0,2}_s(\mathbb{R}_\pm)$, whenever $0 \leq r_0 \leq r_1$ (see [8, formula (21)]). By definition, $\mathcal{P}_+ + \mathcal{P}_- = \mathcal{I}$, where $\mathcal{I}$ is the identity operator. Therefore,

$$\mathcal{P}_+[\kappa^-_{r,\ell}fg] = \mathcal{P}_+[\kappa^-_{\frac{r}{2},\ell}f]\mathcal{P}_+[\kappa^-_{\frac{r}{2},\ell}g] + \mathcal{P}_+[\kappa^-_{\frac{r}{2},\ell}f]\mathcal{P}_-[\kappa^-_{\frac{r}{2},\ell}g] + \mathcal{P}_-[\kappa^-_{\frac{r}{2},\ell}f]\mathcal{P}_+[\kappa^-_{\frac{r}{2},\ell}g].$$

Finally, $\kappa^-_{\frac{r}{2},\ell} = \sum_{i=0}^{\frac{r}{2}} \binom{r/2}{i}(2i\ell)^{\frac{r}{2}-i}\kappa^+_{i,\ell}$, provided $\frac{r}{2}$ is a positive integer. These facts, combined with part (a) of the proof, yield the bound

$$\begin{aligned}\|\hat{f} * \hat{g}\|_{L^{r,2}_\alpha(\mathbb{R}_\pm)} &\leq c\sum_{i,j=0}^{\frac{r}{2}} \|\hat{f}\|_{\bar{L}^{i,2}_s(\mathbb{R}_\pm)}\|\hat{g}\|_{\bar{L}^{j,2}_s(\mathbb{R}_\pm)} \\ &\leq c\|\hat{f}\|_{\bar{L}^{\frac{r}{2},2}_s(\mathbb{R}_\pm)}\|\hat{g}\|_{\bar{L}^{\frac{r}{2},2}_s(\mathbb{R}_\pm)}, \quad \frac{r}{2} \in \mathbb{N}.\end{aligned} \tag{2.4.8}$$

(c) We note that for any $s > -\frac{1}{2}$, $w = 1 + |\xi|^{2s} \in A^{\text{loc}}_{+,2}(\lambda) \cap A^{\text{loc}}_{\infty}$. Hence, by Corollary 2.3.9,

$$[\bar{L}^{2,r_0}_s(\mathbb{R}_+), \bar{L}^{2,r_1}_s(\mathbb{R}_+)]_\theta = \bar{L}^{2,(1-\theta)r_0+\theta r_1}_s(\mathbb{R}_+),$$

$\theta \in (0,1)$, $r_0, r_1 \geq 0$, $s > -\frac{1}{2}$. Viewing the convolution product in the Fourier space as a bilinear map from $\bar{L}^{\frac{r}{2},2}_s(\mathbb{R}_+) \times \bar{L}^{\frac{r}{2},2}_s(\mathbb{R}_+)$ to $L^{r,2}_s(\mathbb{R}_+)$, $s > \frac{1}{2}$, $r \geq 2$ and making use of the classical multilinear complex interpolation theorem of A. Calderon (see Theorem 2.2.1), we infer from (2.4.8)

$$\|\hat{f} * \hat{g}\|_{L^{r,2}_s(\mathbb{R}_+)} \leq c\|\hat{f}\|_{\bar{L}^{\frac{r}{2},2}_s(\mathbb{R}_+)}\|\hat{g}\|_{\bar{L}^{\frac{r}{2},2}_s(\mathbb{R}_+)}, \quad s > \tfrac{1}{2}, \quad r \geq 2.$$

By virtue of [8, formula (21)],

$$\|\hat{f}\|_{\bar{L}^{\frac{r}{2},r}_s(\mathbb{R}_\pm)} \leq \|\hat{f}\|_{L^{\frac{r}{2},2}_0(\mathbb{R}_\pm)} + \|\hat{f}\|_{L^{\frac{r}{2},2}_s(\mathbb{R}_\pm)} \leq c\|\hat{f}\|_{L^{r,2}_s(\mathbb{R}_\pm)}, \quad 0 \leq s \leq \tfrac{r}{2},$$

while the direct application of the convolution Young inequality in the Fourier space, followed by [8, formula (21)], for all $s > -\frac{1}{2}$ and $r \geq 0$ gives

$$\begin{aligned}\|fg\|_{L^2(\mathbb{R})} &\leq c\big(\|f\|_{L^2(\mathbb{R})}\|\hat{g}\|_{\bar{L}^2_s(\mathbb{R}_\pm)} + \|\hat{f}\|_{\bar{L}^2_s(\mathbb{R}_\pm)}\|g\|_{L^2(\mathbb{R})}\big) \\ &\leq c\big(\|f\|_{L^2(\mathbb{R})}\|g\|_{L^2(\mathbb{R})} + \|\hat{f}\|_{L^{r,2}_s(\mathbb{R}_\pm)}\|\hat{g}\|_{L^{r,2}_s(\mathbb{R}_\pm)}\big).\end{aligned}$$

Combining the last three inequalities, we conclude that (2.4.7) holds, with $\frac{1}{2} < s \leq \frac{r}{2}$ and $r \geq 2$.

(d) To complete the proof, we remark that in the standard non-weighted Sobolev settings ($r = 0$), (2.4.7) holds for any $s > \frac{1}{2}$, see [3]. Hence, the interpolation identity (2.4.3) and part (c) of the proof, combined together, yield (2.4.7) for any $r \geq 0$. $\square$

2.5 Auxiliary function spaces

In addition to the variable weigh Sobolev spaces defined in Section 2.4, the existence analysis of Chapter 4 makes use of the scale $H^{s,r}_{\mu,\varepsilon}(\mathbb{R})$. The latter are Hilbert spaces of real valued functions, under the inner product

$$\langle \varphi, \psi \rangle_{H^{s,r}_{\mu,\varepsilon}} = \int_{\mathbb{R}} \kappa^s_\mu \hat{\varphi}\overline{\hat{\psi}} d\xi + \varepsilon^2 \int_{\mathbb{R}} \partial^r_\xi \hat{\varphi} \overline{\partial^r_\xi \hat{\psi}} d\xi, \quad \varepsilon > 0, \quad 0 \leq \gamma < 1, \quad s, r \geq 0,$$

where the Fourier symbol $\kappa_\mu(\xi) = 1 - 2\mu|\xi| + \xi^2$, is defined in (1.3.4).

In the case of $\varepsilon = 0$ or $r = 0$, we write shortly $H^s_\mu(\mathbb{R})$.[4] Note that $(1-\mu)\kappa_0 \leq \kappa_\mu \leq \kappa_0$ if $\xi \in \mathbb{R}$ and $0 \leq \mu < 1$, so that $H^s_\mu(\mathbb{R}) = H^s(\mathbb{R})$ as Banach spaces. In particular, $H^s_\mu(\mathbb{R})$, with $s > \frac{1}{2}$ and $0 \leq \mu < 1$, are Banach algebras [3]. Observe also, $H^0_{\mu,0}(\mathbb{R}) = L^2(\mathbb{R})$. In the latter case, we omit all subscripts in the inner product and the induced norm.

Most of the analysis of Chapter 4 is carried out in the frequency domain $\mathcal{F}[H^{s,r}_{\mu,\varepsilon}(\mathbb{R})]$. We let $\hat{H}^{s,r}_{\gamma,\varepsilon}(\mathbb{R}) = \mathcal{F}[H^{s,r}_{\mu,\varepsilon}(\mathbb{R})]$. The symbols $\operatorname{Re}\hat{H}^{s,r}_{\mu,\varepsilon}(\mathbb{R})$ and $\operatorname{Im}\hat{H}^{s,r}_{\mu,\varepsilon}(\mathbb{R})$ denote the subspaces of even and odd $\hat{H}^{s,r}_{\mu,\varepsilon}(\mathbb{R})$ functions, respectively. $\hat{H}^s_\mu(\mathbb{R})$ is naturally identified with the weighted Lebesgue space $L^2(\mathbb{R}, \kappa^s_\mu d\xi)$.

Section 4.2 deals with the regularity of traveling waves. There, $C_0(\mathbb{R})$ denotes the space of continuous functions that vanish at infinity. Symbol $W^{p,s}_\rho(\Omega)$ is reserved for the scale of the exponentially weighted Sobolev space, equipped with the norm

$$\|\varphi\|_{W^{p,s}_\rho} = \sum_{r=0}^{s} \|e^{\frac{\rho|\xi|}{p}} \partial^r_\xi \varphi\|_{L^p(\Omega)}, \quad 1 \leq p \leq \infty, \quad \rho > 0, \quad s \geq 0.$$

In the case of $s = 0$, we write shortly $L^p_\rho(\Omega)$. Also, by definition $L^\infty_\rho(\Omega) = L^\infty(\Omega)$.

[4]We use Greek subscripts to distinguish these spaces from the variable weight Sobolev spaces $H^s_r(\mathbb{R})$ of Section 2.4.

Chapter 3

Wellposedness in weighted settings

In this Chapter, we discuss the wellposedness of the non-linear Benjamin equation in the settings of the variable weight Sobolev spaces defined in Section 2.4. In context of the Benjamin equation, these spaces arise naturally and provide direct control of the behavior of solutions and their weak derivatives for large values of x. Our result complements and extends recent research on the global wellposedness of the Benjamin equation in weighted Sobolev-like spaces and provides a theoretical foundation for building robust numerical schemes.

3.1 Technical estimates

The wellposedness analysis of (1.1.4) relies on the properties of the scale $H_r^s(\mathbb{R})$ established in Chapter 2 and on two technical estimates allowing us to control linear and nonlinear quantities appearing in the Benjamin equation. We begin our analysis with the linear part of (1.1.4).

Let

$$\mathcal{A} = -\alpha\partial_x + \beta\mathcal{H}\partial_{xx} + \gamma\partial_{xxx}$$

and let H be a Hilbert space obtained by completion of the Schwartz space $\operatorname{Re} S$ with respect to the inner product $\langle\cdot,\cdot\rangle_H$. To the couple $\mathcal{A}$ and H, we associate the bilinear form

$$\mathcal{Q}_H(u,v) = \langle\mathcal{A}u, v\rangle_H, \quad u,v \in \operatorname{Re} S.$$

Passing to the Fourier images and using the Cauchy-Schwarz inequality, it is not difficult to verify that

$$\mathcal{Q}_{H^{s_0}(\mathbb{R})}(u,v) = -\mathcal{Q}_{H^{s_0}(\mathbb{R})}(v,u), \tag{3.1.1a}$$

$$\big|\mathcal{Q}_{H^{s_0}(\mathbb{R})}(u,v)\big| \leq \|u\|_{H^{s_1}(\mathbb{R})}\|v\|_{H^{2s_0-3-s_1}(\mathbb{R})}, \quad s_0, s_1 \in \mathbb{R}. \tag{3.1.1b}$$

Furthermore, we have

Lemma 3.1.1. *For $s > -\frac{1}{2}$, $r > 0$, the bilinear form $\mathcal{Q}_{H^s_r(\mathbb{R})}(\cdot,\cdot)$ satisfies*

$$\big|\mathcal{Q}_{H^s_r(\mathbb{R})}(u,u)\big| \leq r(1+r^2)c_s\big(\|u\|^2_{H^s_r(\mathbb{R})} + \|u\|^2_{H^{s+2r}(\mathbb{R})}\big), \tag{3.1.2}$$

with $c_s > 0$ controlled by $s > -\frac{1}{2}$ and the coefficients of $\mathcal{A}$ only.

Proof. (a) Assume initially $u \in \operatorname{Re}\mathcal{S}$. Then, (2.4.1b) and (3.1.1a) yield

$$\big|\mathcal{Q}_{H^s_r(\mathbb{R})}(u,u)\big| = \Big|\langle \mathcal{P}_+\big[[\kappa^+_{r,\ell},\mathcal{A}]u\big], \mathcal{P}_+[\kappa^+_{r,\ell}u]\rangle_{\mathring{H}^s(\mathbb{R})}\Big|,$$

where $[\kappa^+_{r,\ell},\mathcal{A}] = \kappa^+_{r,\ell}\mathcal{A} - \mathcal{A}\kappa^+_{r,\ell}$ is the commutator of $\mathcal{A}$ and $\kappa^+_{r,\ell}$.

Direct calculations give

$$\mathcal{P}_+\big[[\kappa^+_{r,\ell},\mathcal{A}]u\big] = \mathcal{P}_+\Big[(\mathcal{A}_0\kappa^+_{r,\ell})\cdot u + \partial_x\big[(\mathcal{A}_1\kappa^+_{r,\ell})\cdot u\big] + \partial_{xx}\big[(\mathcal{A}_2\kappa^+_{r,\ell})\cdot u\big]\Big],$$

with

$$\mathcal{A}_0 = -\alpha\partial_x - \beta\mathcal{H}\partial_{xx} + 4\gamma\partial_{xxx},\ \mathcal{A}_1 = 2\beta\mathcal{H}\partial_x - 3\gamma\partial_{xx},\ \mathcal{A}_2 = 3\gamma\partial_x.$$

The last formula, combined with (2.4.5b) and the standard interpolation inequality[1]

$$\|u\|_{H^{s_0(1-\theta)+s_1\theta}_{r_0(1-\theta)+r_1\theta}(\mathbb{R})} \leq \|u\|^{1-\theta}_{H^{s_0}_{r_0}(\mathbb{R})}\|u\|^{\theta}_{H^{s_1}_{r_1}(\mathbb{R})},\ s_0,s_1 > -\tfrac{1}{2},\ r_0,r_1 \geq 0,\ \theta \in (0,1),$$

yields

$$\begin{aligned}
\big\|\mathcal{P}_+\big[[\kappa^+_{r,\ell},\mathcal{A}]u\big]\big\|_{\mathring{H}^s(\mathbb{R})} &\leq r(1+r^2)c'_s\sum_{j=0}^{2}\big\|\mathcal{P}_+\big[\kappa^+_{r-1,\ell}u\big]\big\|_{\mathring{H}^{s+j}(\mathbb{R})}\\
&\leq r(1+r^2)c_s\Big(\big\|\mathcal{P}_+\big[\kappa^+_{r,\ell}u\big]\big\|_{\mathring{H}^s(\mathbb{R})}+\big\|\mathcal{P}_+\big[\kappa^+_{r-1,\ell}u\big]\big\|_{\mathring{H}^{s+2}(\mathbb{R})}\Big)\\
&\leq r(1+r^2)c_s\big(\|u\|_{H^s_r(\mathbb{R})} + \|u\|_{H^{s+2r}(\mathbb{R})}\big),
\end{aligned}$$

[1] The inequality follows directly from (2.4.3), as the complex interpolation functor $[\cdot,\cdot]_\theta$ is exact of exponent $\theta \in (0,1)$, see [12].

where the generic constants $c_s, c'_s > 0$ depends on $s > -\frac{1}{2}$ and the coefficients α, β and γ of the operator $\mathcal{A}$ only. The last bound, together with the standard Cauchy-Schwarz inequality, completes the proof. $\square$

Let H be a Hilbert space employed in the definition of $\mathcal{Q}_H(\cdot,\cdot)$. To control the quadratic nonlinearity $(u^2)_x$, we define

$$\mathcal{T}_H(u,v,w) = \tfrac{1}{2}\langle u, wv_x\rangle_H + \tfrac{1}{2}\langle v, wu_x\rangle_H, \quad u,v,w \in \operatorname{Re}\mathcal{S}.$$

Partial integration, combined with the standard Gagliardo-Nirenberg inequality, indicates that

$$\big|\mathcal{T}_{L^2(\mathbb{R})}(u,v,w)\big| \le \|u\|_{L^2(\mathbb{R})}\|v\|_{L^2(\mathbb{R})}\|w\|_{H^s(\mathbb{R})}, \quad s > \tfrac{3}{2}. \tag{3.1.3}$$

In addition to (3.1.3), we have

Lemma 3.1.2. *For $s > \frac{1}{2}$, $r \ge 0$, $\mathcal{T}_{H^s_r(\mathbb{R})}(\cdot,\cdot,\cdot)$ extends to a bounded trilinear form in $H^s_r(\mathbb{R})^2 \times H^{s+1}(\mathbb{R})$. In particular, we have*

$$\big|\mathcal{T}_{H^s_r(\mathbb{R})}(u,v,w)\big| \le c_{s,r}\|u\|_{H^s_r(\mathbb{R})}\|v\|_{H^s_r(\mathbb{R})}\|w\|_{H^{s+1}(\mathbb{R})}, \tag{3.1.4}$$

with $c_{s,r} > 0$ depending on s and r only.

Proof. (a) To begin, we assume that $r = 0$. For sufficiently regular distributions $u, v, w \in \operatorname{Re}\mathcal{S}$, direct calculations give

$$\begin{aligned}\big|\mathcal{T}_{H^s(\mathbb{R})}(u,v,w)\big| \le{}& \tfrac{1}{2}\big|\langle uv, w_x\rangle_{L^2(\mathbb{R})}\big| \\ &+ \tfrac{1}{2}\int_{\mathbb{R}} |\hat{w}(\xi)|d\xi \int_{\mathbb{R}} \big||\zeta|^{2s-1} - |\zeta-\xi|^{2s-1}\big| \cdot |\zeta-\xi| \cdot |\zeta| \\ &\qquad \cdot \big[|\hat{u}(\zeta)\hat{v}(\xi-\zeta)| + |\hat{v}(\zeta)\hat{u}(\xi-\zeta)|\big]d\zeta.\end{aligned}$$

Using the elementary estimate $|\xi_1^{2s-1} - \xi_2^{2s-1}| \le (2s-1)|\xi_1 - \xi_2|(\xi_1^{2(s-1)} + \xi_2^{2(s-1)})$ (that holds uniformly for all $\xi_1, \xi_2 \in \mathbb{R}_+$ and $s > \frac{1}{2}$), followed by Young's inequality (with exponents $2s > 1$, $\frac{2s}{2s-1} > 1$), and changing the order of integration, we infer

$$\begin{aligned}\big|\mathcal{T}_{H^s(\mathbb{R})}(u,v,w)\big| \le{}& \tfrac{1}{2}\big|\langle uv, w_x\rangle_{L^2(\mathbb{R})}\big| \\ &+ \tfrac{2s-1}{2}\int_{\mathbb{R}} |\hat{u}(\xi)||\xi|^{2s}d\xi \int_{\mathbb{R}} |\zeta\hat{w}(\zeta)| \cdot |v(\xi-\zeta)|d\zeta \\ &+ \tfrac{2s-1}{2}\int_{\mathbb{R}} |\hat{v}(\xi)||\xi|^{2s}d\xi \int_{\mathbb{R}} |\zeta\hat{w}(\zeta)| \cdot |u(\xi-\zeta)|d\zeta.\end{aligned}$$

This bound, together with the elementary inequality $(|\xi_1|+|\xi_2|)^s \le 2^s(|\xi_1|^s+|\xi_2|^s)$, yields

$$
\begin{aligned}
\big|\mathcal{T}_{H^s(\mathbb{R})}(u,v,w)\big| \le{}& \tfrac{1}{2}\big|\langle uv, w_x\rangle_{L^2(\mathbb{R})}\big| \\
&+ (2s-1)2^s \int_{\mathbb{R}} |\xi\hat{w}(\xi)|d\xi \int_{\mathbb{R}} |\zeta|^s|\hat{u}(\zeta)|\cdot|\xi-\zeta|^s|v(\xi-\zeta)|d\zeta \\
&+ (2s-1)2^{s-1} \int_{\mathbb{R}} |\hat{u}(\xi)|d\xi \int_{\mathbb{R}} |\zeta|^{s+1}|\hat{w}(\zeta)|\cdot|\xi-\zeta|^s|v(\xi-\zeta)|d\zeta \\
&+ (2s-1)2^{s-1} \int_{\mathbb{R}} |\hat{v}(\xi)|d\xi \int_{\mathbb{R}} |\zeta|^{s+1}|\hat{w}(\zeta)|\cdot|\xi-\zeta|^s|u(\xi-\zeta)|d\zeta.
\end{aligned}
$$

On the account of the embedding $H^s(\mathbb{R}) \hookrightarrow L^2(\mathbb{R})$, the last inequality implies

$$
\big|\mathcal{T}_{H^s(\mathbb{R})}(u,v,w)\big| \le c_{s,0}\|u\|_{H^s(\mathbb{R})}\|v\|_{H^s(\mathbb{R})}\|w\|_{H^{s+1}(\mathbb{R})}, \quad s > \tfrac{1}{2}, \tag{3.1.5}
$$

where the generic constant $c_{s,0} > 0$ depends on $s > \frac{1}{2}$ only. The estimate (3.1.5) shows that the trilinear form $\mathcal{T}_{H^s(\mathbb{R})}(u,v,w)$ extends continuously to $H^s(\mathbb{R})^2 \times H^{s+1}(\mathbb{R})$ and the case of $r = 0$ is settled.

(b) Next, we let $s = 1$ and $r > 0$. Using the identities

$$
\begin{aligned}
&\mathcal{P}_+[uv] = \mathcal{P}_+[u]\mathcal{P}_+[v] + \mathcal{P}_+\big[\mathcal{P}_+[u]\mathcal{P}_-[v] + \mathcal{P}_-[u]\mathcal{P}_+[v]\big], \\
&\langle \mathcal{P}_\pm[u], \mathcal{P}_\mp[v]\rangle_{\mathring{H}^s(\mathbb{R})} = 0, \quad u,v \in \mathring{H}^s(\mathbb{R}), \quad s \in \mathbb{R},
\end{aligned}
$$

and the commutativity of Fourier multipliers $\mathcal{P}_\pm$ and ∂_x, we infer

$$
\begin{aligned}
\mathcal{T}_{H^1_r(\mathbb{R})}(u,v,w) ={}& -\tfrac{1}{2}\langle w_x, uv\rangle_{L^2(\mathbb{R})} \\
&+ \Big[\langle \mathcal{P}_+[\kappa^+_{r,\ell}u], \mathcal{P}_+[w]\mathcal{P}_+[\kappa^+_{r,\ell}v]_x\rangle_{\mathring{H}^1(\mathbb{R})} \\
&+ \langle \mathcal{P}_+[\kappa^+_{r,\ell}v], \mathcal{P}_+[w]\mathcal{P}_+[\kappa^+_{r,\ell}u]_x\rangle_{\mathring{H}^1(\mathbb{R})}\Big] \\
&- r\Big[\langle \mathcal{P}_+[\kappa^+_{r,\ell}u], \mathcal{P}_+[w]\mathcal{P}_+[\kappa^+_{r-1,\ell}v]\rangle_{\mathring{H}^1(\mathbb{R})} \\
&+ \langle \mathcal{P}_+[\kappa^+_{r,\ell}v], \mathcal{P}_+[w]\mathcal{P}_+[\kappa^+_{r-1,\ell}u]\rangle_{\mathring{H}^1(\mathbb{R})}\Big] := I_1 + I_2 - rI_3.
\end{aligned}
$$

For $s > \frac{1}{2}$, $r > 0$, the embedding $H^s_r(\mathbb{R}) \hookrightarrow H^s(\mathbb{R})$ gives

$$
|I_1| \le c\|u\|_{H^1_r(\mathbb{R})}\|v\|_{H^1_r(\mathbb{R})}\|w\|_{H^2(\mathbb{R})},
$$

with $c > 0$ depending on $r > 0$ only.

To bound I_2, we pass to the frequency space. Calculations similar to those of part (a) above, yield

$$|I_2| \le c\|u\|_{H^1_r(\mathbb{R})}\|v\|_{H^1_r(\mathbb{R})}\|w\|_{H^2(\mathbb{R})},$$

with an absolute constant $c > 0$.

It remains to estimate I_3. Passing to the Fourier images, changing the order of summation and using the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}|I_3| &\le c\|u\|_{H^1_r(\mathbb{R})}\big(\|\mathcal{J}^{1-r}_-[\hat{v}]\|_{L^2_1(\mathbb{R})}\|w\|_{H^1(\mathbb{R})} + \|\mathcal{J}^{1-r}_-[\hat{v}]\|_{L^2(\mathbb{R})}\|w\|_{H^2(\mathbb{R})}\big)\\ &+ c\|v\|_{H^1_r(\mathbb{R})}\big(\|\mathcal{J}^{1-r}_-[\hat{u}]\|_{L^2_1(\mathbb{R})}\|w\|_{H^1(\mathbb{R})} + \|\mathcal{J}^{1-r}_-[\hat{u}]\|_{L^2(\mathbb{R})}\|w\|_{H^2(\mathbb{R})}\big),\end{aligned}$$

with an absolute constant $c > 0$. On the account of (2.4.5b), we have

$$\|\mathcal{J}^{1-r}_-[\hat{u}]\|_{L^2_s\mathbb{R})} \le c_s\|\mathcal{J}^{-r}_-[\hat{u}]\|_{L^2_s(\mathbb{R})}, \quad s > -\tfrac{1}{2},$$

with a constant $c_s > 0$. Hence, the standard density argument, combined with our estimates, gives

$$\big|\mathcal{T}_{H^1_r(\mathbb{R})}(u,v,w)\big| \le c^1_r\|u\|_{H^1_r(\mathbb{R})}\|v\|_{H^1_r(\mathbb{R})}\|w\|_{H^2(\mathbb{R})}, \quad r \ge 0. \tag{3.1.6}$$

(c) The proof of the general case $s > \frac{1}{2}$, $r \ge 0$, is based on the following straightforward modification of the standard interpolation argument:
Given a regular compatible interpolation pair (H_0, H_1) of Hilbert spaces with $H := H_0 \cap H_1$ being dense in H_i, $i = 0, 1$; a regular compatible interpolation pair of Banach spaces (B_0, B_1), with H being dense in $B := B_0 \cap B_1$; and the family of trilinear forms

$$\begin{aligned}&\mathcal{T}_\theta(u,v,w) = \langle u, T(v,w)\rangle_{[H_0,H_1]_\theta} + \langle v, T(u,w)\rangle_{[H_0,H_1]_\theta}, \quad \theta \in (0,1),\\ &T \in \mathcal{L}(H^2_i, H_i), \quad i = 0, 1,\end{aligned}$$

defined initially in H^3 and satisfying

$$|\mathcal{T}_i(u,v,w)| \le c_i\|u\|_{H_i}\|v\|_{H_i}\|w\|_{B_i}, \quad i = 0, 1.$$

We claim that $\mathcal{T}_\theta$ extends to a bounded trilinear form in $[H_0, H_1]^2_\theta \times [B_0, B_1]_\theta$ and

$$|\mathcal{T}_\theta(u,v,w)| \le c_0^{1-\theta}c_1^\theta\|u\|_{[H_0,H_1]_\theta}\|v\|_{[H_0,H_1]_\theta}\|w\|_{[B_0,B_1]_\theta}, \quad \theta \in (0,1). \tag{3.1.7}$$

To see that (3.1.7) holds, it is sufficient to observe that H_i, $i = 0, 1$, can be realized as completions of H with respect to the inner products $\langle\cdot,\cdot\rangle_{H_0} = \langle\cdot,\mathcal{M}\cdot\rangle_H$ and $\langle\cdot,\cdot\rangle_{H_1} = \langle\cdot,\mathcal{I}-\mathcal{M}\cdot\rangle_H$, where $\mathcal{M}$, $\mathcal{I}-\mathcal{M}$ are positive definite bounded and selfadjoint maps in H. Since the complex interpolation functor $[\cdot,\cdot]_\theta$ is exact of exponent $\theta \in (0,1)$, the theory of exact interpolation of Hilbert spaces (see the classical paper [34]) implies that $\langle\cdot,\cdot\rangle_{[H_0,H_1]_\theta} = \langle\cdot,\mathcal{M}^{1-\theta}(\mathcal{I}-\mathcal{M})^\theta\cdot\rangle_H$, $\theta \in (0,1)$.

Further, the map

$$z \mapsto \mathcal{M}^{1-z}(\mathcal{I}-\mathcal{M})^z u,$$

$u \in H$, is analytic in the strip $S := \{0 < \operatorname{Re} z < 1\}$ and is continuous and bounded in its closure $\overline{S}$ (see e.g., [79] for the standard results on spectral resolution of bounded normal operators and the associated functional calculus). Hence, for $f_u(z), f_v(z) \in \mathcal{F}[H_0,H_1]$ and $f_w(z) \in \mathcal{F}[B_0,B_1]$[2], with values in H the function

$$\begin{aligned} g(z) &= c_0^{z-1}c_1^{-z}\langle f_u(z), \mathcal{M}^{1-z}(\mathcal{I}-\mathcal{M})^z T(f_v(z), f_w(z))\rangle_H \\ &+ c_0^{z-1}c_1^{-z}\langle f_v(z), \mathcal{M}^{1-z}(\mathcal{I}-\mathcal{M})^z T(f_u(z), f_w(z))\rangle_H \end{aligned}$$

is well defined, analytic in S and continuous and bounded in $\overline{S}$. The last assertion allows to repeat verbatim the standard interpolation and density arguments (see e.g. [12, Theorem 4.1.2, p.88, or 4.4.2, p. 97]) and thus (3.1.7) is settled.

(d) To complete the proof of (3.1.4), we note that the scale of Hilbert spaces $H^s_r(\mathbb{R})$ and the family $\mathcal{T}_{H^s_r(\mathbb{R})}(\cdot,\cdot,\cdot)$, $s > \frac{1}{2}$, $r \geq 0$, of the trilinear forms fall in the scope of the interpolation argument discussed above. Indeed, any pair $(s,r) \in (\frac{1}{2},\infty) \times \overline{\mathbb{R}_+}$ can be realized as a convex combination of two points laying in the lines $\ell_0 = (\frac{1}{2},\infty) \times \{0\}$ and $\ell_1 = \{1\} \times \overline{\mathbb{R}_+}$, respectively. Furthermore, the endpoint spaces, chosen this way, satisfy all the density constraints imposed in item (c) above.[3] Hence, the interpolation identity (2.4.3), combined with (3.1.5), (3.1.6) and (3.1.7), settles the claim. $\square$

[2]The notation $\mathcal{F}[H_0,H_1]$ and $\mathcal{F}[B_0,B_1]$ for the spaces of vector valued analytic functions, with boundary values in H_i and B_i, $i = 0, 1$, respectively, is standard, see Section 2.2.

[3]The claim follows immediately from denseness of embeddings in (2.4.2a).

3.2 Wellposedness

The main result of this Chapter is the following theorem

Theorem 3.2.1. *Assume* $u_0 \in H^s_r(\mathbb{R}) \cap H^{s+2r+1}(\mathbb{R})$*, with* $\frac{1}{2} < s < r$ *and* $s+2r \geq 3$*. Then for any finite value of* $0 < T < \infty$*, the unique global weak solution* u *to* (1.1.4) *satisfies*

$$u \in L^\infty([0,T]; H^s_r(\mathbb{R}) \cap H^{s+2r+1}(\mathbb{R})). \tag{3.2.1}$$

Proof. (a) It is well known, that for the initial data $u_0 \in H^s(\mathbb{R})$, $s \geq 3$, the Benjamin equation is classically globally wellposed, see the discussion in Section 1.2. In particular, for such input data and any finite value of $0 < T < \infty$, we have $u \in L^\infty([0,T], H^s(\mathbb{R}))$. Furthermore, for any $v \in L^2(\mathbb{R})$

$$\langle u_t, v\rangle_{L^2} = \mathcal{Q}_{L^2(\mathbb{R})}(u,v) - 2\delta\langle uu_x, v\rangle_{L^2(\mathbb{R})}, \quad t \in (0,T], \tag{3.2.2a}$$

$$u(0) = u_0, \quad n \geq 0. \tag{3.2.2b}$$

Hence, to complete the proof of Theorem 3.2.1, it remains to show that under the assumption $u_0 \in H^s_r(\mathbb{R}) \cap H^{s+2r+1}(\mathbb{R})$, the inclusion (3.2.1) holds.

(b) For the sake of brevity, for $s > -\frac{1}{2}$, $r \geq 0$, we let

$$W^s_r(\mathbb{R}) = H^s_r(\mathbb{R}) \cap H^{s+2r}(\mathbb{R}), \quad \langle \cdot,\cdot\rangle_{W^s_r(\mathbb{R})} := \langle \cdot,\cdot\rangle_{H^s_r(\mathbb{R})} + \langle \cdot,\cdot\rangle_{H^{s+2r}(\mathbb{R})}.$$

On the account of Lemma 2.4.1, the embedding

$$W^s_r(\mathbb{R}) \hookrightarrow L^2(\mathbb{R}), \quad 0 < s < r, \tag{3.2.3}$$

is dense and compact. Hence, for $0 < s < r$, $L^2(\mathbb{R})$ can be realized as a completion of $W^s_r(\mathbb{R})$ with respect to the inner product $\langle \cdot,\cdot\rangle_{L^2(\mathbb{R})} = \langle \cdot, \mathcal{M}\cdot\rangle_{W^s_r(\mathbb{R})}$, with some positive definite bounded selfadjoint map $\mathcal{M} \in \mathcal{L}(W^s_r(\mathbb{R}))$. It is not difficult to verify that $\mathcal{M}$ extends uniquely to a selfadjoint positive definite bounded linear map $\mathcal{M}^s_r \in \mathcal{L}(L^2(\mathbb{R}))$ and that $\mathcal{M}^s_r L^2(\mathbb{R}) \subset W^s_r(\mathbb{R})$. By virtue of (3.2.3), the last inclusion indicates that $\mathcal{M}^s_r$, with $0 < s < r$, is compact.

(c) By part (b) of the proof, the spectrum of $\mathcal{M}^s_r$ is discrete, while the collection of associated eigenfunctions $\{\varphi_k\}_{k\geq 0}$ is a complete orthogonal basis in both $L^2(\mathbb{R})$ and $W^s_r(\mathbb{R})$, $0 < s < r$.

We denote $\mathcal{P}_n = \text{span}\{\varphi_k\}_{k=0}^n$ and let $\mathcal{P}_n : L^2(\mathbb{R}) \to \mathbb{P}_n$ be the associated orthogonal projector. By the definition of $W_r^s(\mathbb{R})$, for $s+2r \geq \frac{3}{2}$, we have $\{\varphi_k\}_{k\geq 0} \subset H^{\frac{3}{2}}(\mathbb{R})$. Hence, the sequence of linear Galerkin approximations

$$\langle u_{nt}, v\rangle_{L^2(\mathbb{R})} = \mathcal{Q}_{L^2(\mathbb{R})}(u_n, v) - 2\delta\langle uu_{n_x}, v\rangle_{L^2(\mathbb{R})}, \quad v \in \mathbb{P}_n, \tag{3.2.4a}$$

$$u_n(0) = \mathcal{P}_n[u_0], \quad n \geq 0, \tag{3.2.4b}$$

is well defined.

The special choice of the orthogonal basis indicates that

$$\langle u_{nt}, v\rangle_{W_r^s(\mathbb{R})} = \mathcal{Q}_{W_r^s(\mathbb{R})}(u_n, v) - 2\delta\langle uu_{nx}, v\rangle_{W_r^s(\mathbb{R})}, \quad v \in \mathbb{P}_n. \tag{3.2.5}$$

Letting $v = u_n$ in (3.2.5) and using Lemmas 3.1.1-3.1.2, for $\frac{1}{2} < s < r$, we obtain

$$\tfrac{d}{dt}\|u_n\|^2_{W_r^s(\mathbb{R})} \leq 2\big[r(1+r^2)c_s + 2\delta c_{s,r}\|u\|_{H^{s+2r+1}(\mathbb{R})}\big]\|u_n\|^2_{W_r^s(\mathbb{R})}. \tag{3.2.6}$$

Bound (3.2.6), combined with the standard Gronwall's inequality, indicates that

$$\{u_n\}_{n\geq 0} \in L^\infty([0,T]; W_r^s(\mathbb{R})), \quad \tfrac{1}{2} < s < r, \quad s+2r \geq \tfrac{3}{2}, \tag{3.2.7a}$$

uniformly in $n \geq 0$, for every finite fixed value of $0 < T < \infty$. In turn, for $s+2r \geq 3$, using (3.1.1b), (3.1.3), (3.2.7a), (3.2.3), integrating over the interval $[0, T]$ and using the Cauchy-Schwarz inequality, we obtain

$$\|u_{nt}\|_{L^2([0,T]\times\mathbb{R})} \leq cT^{\frac{1}{2}}\|u\|_{L^\infty([0,T];H^{s+2r+1}(\mathbb{R}))}\|u_n\|_{L^\infty([0,T];H^{s+2r}(\mathbb{R}))},$$

where the generic constant $c > 0$ depends on the coefficients α, β, γ and δ of (1.1.4) only. From the last bound it follows that

$$\{u_{nt}\}_{n\geq 0} \in L^2([0,T] \times \mathbb{R}), \tag{3.2.7b}$$

uniformly in $n \geq 0$.

(c) The rest of the proof is standard. Using the uniform inclusions (3.2.7) and passing to subsequences, we conclude that: (i) $\{u_n\}_{n\geq 0}$ and $\{u_{nt}\}_{n\geq 0}$ converge weak-$*$ to some w and w_t in $L^\infty([0,T]; W_r^s(\mathbb{R}))$ and $L^2([0,T] \times \mathbb{R})$, respectively; (ii) $u_n(0)$ converges weakly to u_0 in $L^2(\mathbb{R})$; and (iii) in view of (3.2.3), $\{u_n\}_{n\geq 0}$, converges strongly in $L^2([0,T]\times\mathbb{R})$. Passing to the weak limit in (3.2.4), we see that for $s+2r\geq 3$,

$$\langle w_t, v\rangle_{L^2(\mathbb{R})} = \mathcal{Q}_{L^2(\mathbb{R})}(w,v) - 2\delta\langle uw_x, v\rangle_{L^2(\mathbb{R})}, \quad v\in L^2(\mathbb{R}), \tag{3.2.8a}$$

$$w(0) = u_0. \tag{3.2.8b}$$

Subtracting (3.2.2), (3.2.8) and letting $v = w-u$, in view of (3.1.1a) and (3.1.3), we obtain

$$\tfrac{d}{dt}\|u-w\|^2_{L^2(\mathbb{R})} \leq 2\delta\|u\|_{H^{s+2r+1}(\mathbb{R})}\|u-w\|^2_{L^2(\mathbb{R})}.$$

Since $\|u(0)-w(0)\|_{L^2(\mathbb{R})} = 0$, Gronwall's inequality gives $\|u-w\|_{L^\infty([0,T];L^2(\mathbb{R}))} = 0$ and since $w\in L^\infty([0,T];W^s_r(\mathbb{R}))$, (3.2.1) follows. The proof is complete. $\square$

Theorem 3.2.1 significantly extends the wellposedness results of [95], discussed in Section 1.2.3. Indeed, for $s\in\mathbb{N}$, we have (see [89] and discussion in Section 2.4.1)

$$\|u\|_{H^s_r(\mathbb{R})} \approx \|(1+|\cdot|)^{r-s}u\|_{L^2(\mathbb{R})} + \|(1+|\cdot|)^r u^{(s)}\|_{L^2(\mathbb{R})}, \quad s\in\mathbb{N}, \quad r>0.$$

Consequently, for large values of x, the solutions to (1.1.4) behave as $\mathcal{O}(|x|^{-p})$, $x\to\infty$, for any given value of $0<p$, provided $p\leq r-s$ and the asymptotic of the input data u_0 and $u_0^{(s)}$ is properly controlled at infinity (i.e., if $u_0\in\mathcal{Z}_{s+2r+1,r-s}$ and $u_0^{(s)}\in\mathcal{Z}_{0,r}$, $\frac{1}{2}<s<r$, $s+2r\geq 3$, in the notation of [95]).

Chapter 4

Traveling waves

In this Chapter, the problem of existence of Benjamin traveling waves is settled globally for $0 \leq \mu < 1$ via a combination of concentration-compactness and a small viscosity limit techniques. Further, a detailed study of large ξ-asymptotic and regularity of traveling wave solutions is provided. Finally, the orbitally stability of quadratic waves ($\ell = 2$) for small values of the wavespeed parameter μ is theoretically confirmed.

4.1 Existence of solitary waves

4.1.1 A small-viscosity variational problem

Given $\lambda, \varepsilon > 0$, we study the following perturbed/small-viscosity variational problem:

$$I_\lambda^\varepsilon = \sup\{\mathcal{G}_\varepsilon(\varphi) \mid \|\varphi\|_{H^1_\mu}^2 = \lambda, \varphi \in H^{1,1}_{\mu,\varepsilon}(\mathbb{R}), \varphi(x) = \varphi(-x)\}, \tag{4.1.1a}$$

$$\mathcal{G}_\varepsilon(\varphi) = \tfrac{1}{\ell+1}\langle \varphi, \varphi^\ell \rangle - \tfrac{\varepsilon^2}{2}\|x\varphi\|^2, \ \ (\text{or } \hat{\mathcal{G}}_\varepsilon(\hat{\varphi}) = \tfrac{\alpha_\ell}{\ell+1}\langle \hat{\varphi}, \hat{\varphi}^{*\ell} \rangle - \tfrac{\varepsilon^2}{2}\|\hat{\varphi}_\xi\|^2), \tag{4.1.1b}$$

where $\ell \geq 2$ is an integer and $[\cdot]^{*\ell}$ denotes the Fourier convolution power. Below, we show that the supremum in (4.1.1a) is attained for some $\varphi_{\mu,\varepsilon} \in H^{1,1}_{\mu,\varepsilon}(\mathbb{R})$.

We begin with several elementary observations. First, in equation (1.3.2) for a fixed $0 \leq \mu < 1$, the quantities I_λ^ε are uniformly bounded independently of ε, as $H^1_\mu(\mathbb{R})$ are Banach algebras. Second, for a fixed $\lambda > 0$ the supremum I_λ^ε is strictly positive, provided ε is sufficiently small (e.g. if $\varepsilon < c\lambda^{\frac{\ell-1}{4}}$, with some absolute constant $c > 0$). The claim is easy to verify by fixing a function $\varphi \in H^{1,1}_{\mu,\varepsilon}(\mathbb{R})$ with nonnegative Fourier image $\hat{\varphi}$ (take for instance $\hat{\varphi} = e^{-\xi^2}$) and then choosing ε so that $\hat{\mathcal{G}}_\varepsilon(\hat{\varphi}) > 0$. Finally, formula (4.1.1b) implies that any maximizing sequence of (4.1.1a) (i.e. a sequence $\{\varphi_n \,|\, \|\varphi_n\|^2_{H^1_\mu} = \lambda, \varphi_n(x) = \varphi_n(-x)\}_{n\geq 0} \subset H^{1,1}_{\mu,\varepsilon}(\mathbb{R})$ that satisfies $\lim_{n\to\infty} \mathcal{G}_\varepsilon(\varphi_n) = I^\varepsilon_\lambda$) is uniformly bounded in $H^{1,1}_{\mu,\varepsilon}(\mathbb{R})$.

The preceding observations ensure that variational problem (4.1.1) is well posed. To obtain the maximizer $\varphi_{\mu,\varepsilon}$, we employ the classical concentration-compactness argument of P. L. Lions [65, 66]. The first ingredient of the method is super-additivity of I^ε_λ.

Lemma 4.1.1. *For each $\lambda > 0$ and any $\alpha \in (0, \lambda)$, we have*

$$I^\varepsilon_\lambda > I^\varepsilon_\alpha + I^\varepsilon_{\lambda-\alpha}. \tag{4.1.2}$$

Proof. The supremum I^ε_λ is super-linear. Indeed, let $\{\varphi_n \,|\, \|\varphi_n\|^2_{H^1_\mu} = \lambda, \varphi_n(x) = \varphi_n(-x)\}_{n\geq 0}$ be a maximizing sequence for (4.1.1). Then for any $\theta > 1$, we have

$$\theta I^\varepsilon_\lambda = \theta \lim_{n\to\infty} \mathcal{G}_\varepsilon(\varphi_n) \leq \theta^{\frac{1-\ell}{2}} \limsup_{n\to\infty} \mathcal{G}_\varepsilon(\theta^{\frac{1}{2}}\varphi_n) \leq \theta^{\frac{1-\ell}{2}} I^\varepsilon_{\theta\lambda} < I^\varepsilon_{\theta\lambda}.$$

Inequality (4.1.2) follows directly from this estimate, see [66, Lemma II.1]. □

The second ingredient is the following adaption of the classical concentration-compactness principle of P. L. Lions [65, Lemma III.1] to the present settings.

Lemma 4.1.2. *Let $\{\varphi_n\}_{n\geq 0}$ be a sequence of even functions in $H^1_\mu(\mathbb{R})$, that satisfies $\|\varphi_n\|^2_{H^1_\mu} = \lambda$ and $\varepsilon^2\|\hat{\varphi}_{n\xi}\| \leq M$, for all $n \geq 0$ and some fixed $\lambda > 0$, $0 \leq \mu < 1$ and $M > 0$. There exists a subsequence, still denoted by $\{\varphi_n\}_{n\geq 0}$, that satisfies one of the following three alternatives:*

(a) (Compactness) For any $\delta > 0$, one can find $\rho > 0$ so that

$$\|\chi_{[-\rho,\rho]}\hat{\varphi}_n\|^2_{\hat{H}^1_\mu} \geq \lambda - \delta, \quad \textit{for all } n \geq 0. \tag{4.1.3}$$

(b) (Vanishing) For each fixed value of $\rho > 0$, the sequence satisfies

$$\lim_{n\to\infty} \sup_\xi \|\chi_{[\xi-\rho,\xi+\rho]}\hat{\varphi}_n\|^2_{\hat{H}^1_\mu} = 0. \tag{4.1.4}$$

(c) (Dichotomy) There exist two sequences $\{\varphi_{n,i}\}_{n\geq 0}$*,* $i = 1, 2$*, and* $\alpha \in (0, \lambda)$*, such that*

$$\lim_{n\to\infty} \|\hat{\varphi}_{n,1} + \hat{\varphi}_{n,2} - \hat{\varphi}_n\|_{\hat{H}^1_\mu} = 0, \tag{4.1.5a}$$

$$\lim_{n\to\infty} \|\hat{\varphi}_{n,1}\|^2_{\hat{H}^1_\mu} = \alpha, \quad \lim_{n\to\infty} \|\hat{\varphi}_{n,2}\|^2_{\hat{H}^1_\mu} = \lambda - \alpha, \tag{4.1.5b}$$

$$\lim_{n\to\infty} \mathrm{dist}(\mathrm{supp}\,\hat{\varphi}_{n,1}, \mathrm{supp}\,\hat{\varphi}_{n,2}) = \infty, \tag{4.1.5c}$$

$$\limsup_{n\to\infty} \varepsilon^2\big(\|\hat{\varphi}_{n,1\xi}\|^2 + \|\hat{\varphi}_{n,2\xi}\|^2 - \|\hat{\varphi}_{n\xi}\|^2\big) \leq 0. \tag{4.1.5d}$$

Proof. The proof of parts (b) and (c) is identical to that of [65, Lemma III.1]. To prove part (a), we observe that by the same [65, Lemma III.1], there exists a sequence $\{\xi_n\}_{n\geq 0}$, such that for any $\delta > 0$, we can find $0 < \rho' < \infty$, so that

$$\|\chi_{[\xi_n - \rho', \xi_n + \rho']}\hat{\varphi}_n\|^2_{\hat{H}^1_\mu} \geq \lambda - \delta.$$

Since $\hat{\varphi}_n$ are even, it follows that $\limsup_{n\to\infty} |\xi_n| < \infty$. Otherwise, we are able to choose a subsequence $\{\xi_{n_k}\}_{k\geq 0}$, so that $[\xi_{n_k} - \rho', \xi_{n_k} + \rho'] \cap [-\xi_{n_k} - \rho', -\xi_{n_k} + \rho'] = \emptyset$ and then obtain $\lambda = \|\varphi_{n_k}\|^2_{H^1_\mu} \geq 2\lambda - 2\delta$. Which is obviously impossible when δ is small. Hence, letting $\rho = \rho' + \limsup_{n\to\infty} |\xi_n|$, we arrive at (4.1.3). $\square$

Lemmas 4.1.1 and 4.1.2, combined together, yield existence of a maximizer.

Theorem 4.1.3. *Assume* ε *is sufficiently small so that* $I^\varepsilon_\lambda > 0$*. Then any maximizing sequence* $\{\varphi_n \,|\, \|\varphi_n\|^2_{H^1_\mu} = \lambda, \varphi_n(x) = \varphi_n(-x)\}_{n\geq 0}$ *of* (4.1.1) *contains a subsequence, still labeled by* $\{\varphi_n\}_{n\geq 0}$ *that converges strongly in* $H^{1,1}_{\mu,\varepsilon}(\mathbb{R})$ *to a maximizer* $\varphi_{\mu,\varepsilon}$ *of* (4.1.1).

Proof. (a) The proof is standard and consists in ruling out of vanishing and dichotomy. So assume the maximizing sequence is vanishing. Then, for $\rho > 0$ fixed, we have

$$(1-\mu)^{\frac{1}{2}}\|\chi_{[\xi-\rho,\xi+\rho]}\hat{\varphi}_n\| \leq \|\chi_{[\xi-\rho,\xi+\rho]}\hat{\varphi}_n\|_{\hat{H}^1_\mu} < \delta_n,$$

with $\lim_{n\to\infty} \delta_n = 0$, uniformly in ξ. Let $I_j = [(j-1/2)\rho, (j+1/2)\rho]$, $\hat{\varphi}_{n,j} = \chi_{I_j}\hat{\varphi}_n$,

$j \in \mathbb{Z}$ and $\xi \in I_k$. Classical Cauchy-Schwartz inequality implies initially

$$\begin{aligned}|\hat{\varphi}_n * \hat{\varphi}_n|(\xi) &\le \sum_{i\in\mathbb{Z}}\sum_{j=-1}^{1} |\hat{\varphi}_{n,i} * \hat{\varphi}_{n,k+j-i}|(\xi) \\ &\le \sum_{i\in\mathbb{Z}}\sum_{j=-1}^{1} \|\hat{\varphi}_{n,i}\|\|\hat{\varphi}_{n,k+j-i}\| \le \frac{3\delta_n}{1-\mu}\sum_{i\in\mathbb{Z}} \frac{\|\hat{\varphi}_{n,i}\|_{\hat{H}^1_\mu}}{\inf\{\kappa_0^{\frac{1}{2}} \,|\, \xi \in I_i\}} \\ &\le \frac{3\delta_n}{1-\mu}\Big(\sum_{i\in\mathbb{Z}} \sup\{\kappa_0^{-1} \,|\, \xi \in I_i\}\Big)^{1/2} \|\hat{\varphi}_n\|_{\hat{H}^1_\mu} =: c(\gamma,\rho)\lambda^{\frac{1}{2}}\delta_n,\end{aligned}$$

and then

$$\begin{aligned}\langle \hat{\varphi}_n, \hat{\varphi}_n^{*\ell}\rangle = \langle \hat{\varphi}_n^{*2}, \hat{\varphi}_n^{*(\ell-1)}\rangle &\le \|\hat{\varphi}_n^{*2}\|_{L^\infty}\|\hat{\varphi}_n\|_{L^1}^{\ell-1} \\ &\le c(\mu,\rho)\|\kappa_\mu^{-\frac{1}{2}}\|^{\ell-1}\lambda^{\frac{\ell}{2}}\delta_n.\end{aligned}$$

Passing to the limit in the last inequality, we conclude that $0 < I^\varepsilon_\lambda \le 0$, which is impossible.

(b) Assume the dichotomy occurs. We define two sequences $\{\beta_{n,i}\}_{n\ge 0}$, $i = 1, 2$, so that $\|\beta_{n,1}\hat{\varphi}_{n,1}\|^2_{\hat{H}^1_\mu} = \alpha$ and $\|\beta_{n,2}\hat{\varphi}_{n,2}\|^2_{\hat{H}^1_\mu} = \lambda-\alpha$. In view of (4.1.5a) and (4.1.5b), $\beta_{n,i} \to 1$, $i = 1, 2$, and $\delta_n := \|\beta_{n,1}\hat{\varphi}_{n,1} + \beta_{n,2}\hat{\varphi}_{n,2} - \hat{\varphi}_n\|_{\hat{H}^1_\mu} \to 0$, as $n \to \infty$. Further, by virtue of (4.1.5c), at least one of the quantities $|\zeta|$, $|\xi - \zeta|$, where $\zeta \in \operatorname{supp}\hat{\varphi}_{n,1}$ and $\xi - \zeta \in \operatorname{supp}\hat{\varphi}_{n,2}$, increases indefinitely with n. Consequently,

$$\lim_{n\to\infty}\delta'_n := \lim_{n\to\infty}\sup\{\kappa_\mu^{-\frac{1}{2}}(\zeta)\kappa_\mu^{-\frac{1}{2}}(\xi-\zeta) \,|\, \zeta \in \operatorname{supp}\hat{\varphi}_{n,1}, \xi-\zeta \in \operatorname{supp}\hat{\varphi}_{n,2}\} = 0,$$

and $\delta''_n := \|\hat{\varphi}_{n,1} * \hat{\varphi}_{n,2}\|_{L^\infty} \to 0$ as $n \to \infty$. Using these facts and the Cauchy-Schwartz inequality, it is not difficult to verify that

$$\langle \hat{\varphi}_n, \hat{\varphi}_n^{*\ell}\rangle \le \langle(\beta_{n,1}\hat{\varphi}_{n,1}), (\beta_{n,1}\hat{\varphi}_{n,1})^{*\ell}\rangle + \langle(\beta_{n,2}\hat{\varphi}_{n,2}), (\beta_{n,2}\hat{\varphi}_{n,2})^{*\ell}\rangle + \delta'''_n,$$

with $\delta'''_n \le c(\alpha,\mu,\lambda)(\delta_n + \delta''_n)$. The last estimate, combined with (4.1.5d), implies

$$\begin{aligned}I^\varepsilon_\lambda \le \limsup_{n\to\infty}&\hat{\mathcal{G}}_\varepsilon(\beta_{n,1}\hat{\varphi}_{n,1}) + \limsup_{n\to\infty}\hat{\mathcal{G}}_\varepsilon(\beta_{n,2}\hat{\varphi}_{n,2}) \\ &+ \limsup_{n\to\infty}\tfrac{\varepsilon^2}{2}(\|\hat{\varphi}_{n,1\xi}\|^2 + \|\hat{\varphi}_{n,2\xi}\|^2 - \|\hat{\varphi}_{n\xi}\|^2) \le I^\varepsilon_\alpha + I^\varepsilon_{\lambda-\alpha},\end{aligned}$$

and we conclude that the dichotomy does not occur, for the last inequality contradicts Lemma 4.1.1.

(c) In view of (a) and (b) above, it follows that any maximizing sequence $\{\hat{\varphi}_n\}_{n\geq 0}$ shall satisfy the compactness alternative (4.1.3) of Lemma 4.1.2. Furthermore, since κ_μ, κ_μ^{-1} are bounded on compact intervals, any such sequence, restricted to a nonempty interval $[-\rho, \rho]$, is precompact in the weighted space $L^2([-\rho,\rho], \kappa_\mu d\xi)$, see [3]. Therefore, letting $\delta_n = \frac{1}{n+1}$, $n \geq 0$, and using (4.1.3), combined with the standard diagonal argument, we extract a subsequence, still labeled by $\{\varphi_n\}_{n\geq 0}$, that converges strongly to some $\varphi_{\mu,\varepsilon} \in H^1_\mu(\mathbb{R})$. Straightforward calculations show that

$$\left|\langle \hat{\varphi}_n, \hat{\varphi}_n^{*\ell}\rangle - \langle \hat{\varphi}_{\mu,\varepsilon}, \hat{\varphi}_{\mu,\varepsilon}^{*\ell}\rangle\right| \leq \frac{\|\varphi_n - \varphi_{\mu,\varepsilon}\|_{H^1_\mu}}{\sqrt{1-\mu^2}}\Big(\|\hat{\varphi}_n^{*\ell}\| + \sum_{j=1}^{\ell}\|\hat{\varphi}_{\mu,\varepsilon}^{*j} * \hat{\varphi}_n^{*(\ell-j)}\|\Big),$$

and, since the expression in the round brackets is uniformly bounded,[1], we have

$$\lim_{n\to\infty}\langle \hat{\varphi}_n, \hat{\varphi}_n^{*\ell}\rangle = \langle \hat{\varphi}_{\mu,\varepsilon}, \hat{\varphi}_{\mu,\varepsilon}^{*\ell}\rangle.$$

The sequence $\{\varphi_n\}_{n\geq 0}$ is uniformly bounded in $H^{1,1}_{\mu,\varepsilon}(\mathbb{R})$ and hence has a subsequence that converges weakly to the same limit $\varphi_{\mu,\varepsilon}$ in $H^{1,1}_{\mu,\varepsilon}(\mathbb{R})$. Since $\|\hat{\varphi}_{\mu,\varepsilon\xi}\| \leq \liminf_{n\to\infty}\|\hat{\varphi}_{n\xi}\|$, we have $\mathcal{G}_\varepsilon(\varphi_{\mu,\varepsilon}) \geq I^\varepsilon_\lambda$. Thus, $\mathcal{G}_\varepsilon(\varphi_{\mu,\varepsilon}) = I^\varepsilon_\lambda$ and $\varphi_{\mu,\varepsilon}$ maximizes $\mathcal{G}_\varepsilon(\cdot)$. Finally, the identities $\hat{\mathcal{G}}_\varepsilon(\hat{\varphi}_{\mu,\varepsilon}) = I^\varepsilon_\lambda$ and $\lim_{n\to\infty}\langle \hat{\varphi}_n, \hat{\varphi}_n^{*\ell}\rangle = \langle \hat{\varphi}_{\mu,\varepsilon}, \hat{\varphi}_{\mu,\varepsilon}^{*\ell}\rangle$ imply $\lim_{n\to\infty}\|\hat{\varphi}_{n\xi}\| = \|\hat{\varphi}_{\mu,\varepsilon\xi}\|$ and $\lim_{n\to\infty}\|\varphi_n\|_{H^{1,1}_{\mu,\varepsilon}} = \|\varphi_{\mu,\varepsilon}\|_{H^{1,1}_{\mu,\varepsilon}}$. Hence, the convergence is strong in $H^{1,1}_{\mu,\varepsilon}(\mathbb{R})$. □

4.1.2 The small-viscosity limit

Let $\varphi_{\mu,\varepsilon}$ be a nontrivial maximizer obtained in Theorem 4.1.3. In view of the elementary inequality $\hat{\mathcal{G}}_\varepsilon(\hat{\varphi}) \leq \hat{\mathcal{G}}_\varepsilon(|\hat{\varphi}|)$, it follows that maximizers $\varphi_{\mu,\varepsilon}$ are translates of positive definite,[2] even functions $\mathcal{F}^{-1}[|\hat{\varphi}_{\mu,\varepsilon}|]$. Below, we study the limit behavior of *even positive definite solutions* $\varphi_{\mu,\varepsilon}$. We start with two elementary observations.

Lemma 4.1.4. *The supremum I^ε_λ is a strictly decreasing function of $|\varepsilon|$. The limit quantity $I_\lambda = \lim_{\varepsilon\to 0} I^\varepsilon_\lambda$ is super-linear and, hence, is super-additive. Furthermore,*

$$I_\lambda = \sup\{\mathcal{G}(\varphi) \mid \|\varphi\|^2_{H^1_\mu} = \lambda, \varphi \in H^1_\mu(\mathbb{R}), \varphi(x) = \varphi(-x)\}, \quad \mathcal{G} := \mathcal{G}_0. \tag{4.1.6}$$

[1] Recall that $H^1_\mu(\mathbb{R})$, with $0 \leq \mu < 1$, are Banach algebras.

[2] The classical result of S. Bochner asserts that φ is positive definite iff $\hat{\varphi} \geq 0$.

Proof. (a) For $0 < \varepsilon_1 < \varepsilon_2$, we have

$$I_\lambda^{\varepsilon_2} = \mathcal{G}_{\varepsilon_2}(\varphi_{\mu,\varepsilon_2}) < \mathcal{G}_{\varepsilon_1}(\varphi_{\mu,\varepsilon_2}) \leq I_\lambda^{\varepsilon_1}.$$

This settles the first claim. Further, since I_λ^ε are uniformly bounded by a constant independent of ε, it follows that the quantity $I_\lambda = \lim_{\varepsilon\to 0} I_\lambda^\varepsilon = \sup_{\varepsilon>0} I_\lambda^\varepsilon$ is well defined. The super-linearity and super-additivity of I_λ are verified in the same way as in Lemma 4.1.1.

(b) To prove the second assertion, we let

$$\bar{I}_\lambda = \sup\{\mathcal{G}(\varphi) \,|\, \|\varphi\|_{H_\mu^1}^2 = \lambda, \varphi \in H_\mu^1(\mathbb{R}), \varphi(x) = \varphi(-x)\}.$$

Since $\mathcal{G}_\varepsilon(\varphi) \leq \mathcal{G}(\varphi)$, for all $\varepsilon > 0$ and $\varphi \in H_{\mu,\varepsilon}^{1,1}$, we have $I_\lambda \leq \bar{I}_\lambda$. Assume the strict inequality holds. By definition of $\bar{I}_\lambda$, for any $\delta_1 > 0$ there exists $\varphi \in H_\mu^1(\mathbb{R})$, such that $\mathcal{G}(\varphi) > \bar{I}_\lambda - \delta_1$. Further, since $H_{\mu,1}^{1,1}(\mathbb{R})$ is dense in $H_\mu^1(\mathbb{R})$, for any $\delta_2 > 0$, there exists an even $H_{\mu,1}^{1,1}(\mathbb{R})$ function ψ, such that $\|\psi\|_{H_\mu^1}^2 = \lambda$ and $\|\varphi - \psi\|_{H_\mu^1} \leq \delta_2$. Note also that $H_\mu^1(\mathbb{R})$ is a Banach algebra, therefore $|\mathcal{G}(\varphi) - \mathcal{G}(\psi)| \leq c(\lambda,\ell)\|\varphi - \psi\|_{H_\mu^1}$, with $c(\lambda,\ell) > 0$ that depends on λ and ℓ only. Combining all the inequalities together, we infer

$$\begin{aligned}
\mathcal{G}_\varepsilon(\psi) = \mathcal{G}(\psi) - \varepsilon^2\|x\psi\|^2 &> \mathcal{G}(\varphi) - \left[c(\lambda,\ell)\delta_2 + \varepsilon^2\|x\psi\|^2\right] \\
&> \bar{I}_\lambda - \left[\delta_1 + c(\lambda,\ell)\delta_2 + \varepsilon^2\|x\psi\|^2\right] > I_\lambda,
\end{aligned}$$

provided δ_1, δ_2 and ε are small. However, by definition of I_λ, $\mathcal{G}_\varepsilon(\psi) \leq I_\lambda$. Hence, $I_\lambda = \bar{I}_\lambda$, i.e. (4.1.6) holds. $\square$

Lemma 4.1.5. *Any sequence $\{\varphi_{\mu,\varepsilon_n}\}_{n\geq 0}$, with $\varepsilon_n \to 0$, contains a subsequence that satisfies the compactness alternative of Lemma 4.1.2.*

Proof. The result follows from Lemma 4.1.4 in the same way as parts (a), (b) of Theorem 4.1.3 follow from Lemma 4.1.1. $\square$

As evident from the proof of Lemma 4.1.4, any sequence $\{\varphi_{\mu,\varepsilon_n}\}_{n\geq 0}$, with $\varepsilon_n \to 0$ is maximizing for the variational problem (4.1.6). Unfortunately, the arguments employed in part (c) of Theorem 4.1.3 are not applicable in the present situation as we do not have uniform a priori estimates for $\|x\varphi_{\mu,\varepsilon_n}\|$. Nevertheless, the compactness alternative still allows us to extract some useful information.

Lemma 4.1.6. *Let $\{\varphi_n\}_{n\geq 0}$ be uniformly bounded in $H^1_\mu(\mathbb{R})$ (say $\|\varphi_n\|_{H^1_\mu} \leq M$) and $\{\psi_n\}_{n\geq 0}$ satisfy the compactness alternative of Lemma 4.1.2. Then there exist subsequences, still labeled by $\{\varphi_n\}_{n\geq 0}$ and $\{\psi_n\}_{n\geq 0}$, such that $\varphi_n \to [\{\varphi_n\}]$, $\psi_n \to [\{\psi_n\}]$ and $\varphi_n\psi_n \to [\{\varphi_n\psi_n\}]$ weakly in $H^1_\mu(\mathbb{R})$,*[3] *and*

$$[\{\varphi_n\psi_n\}] = [\{\varphi_n\}][\{\psi_n\}]. \tag{4.1.7}$$

Proof. By the assumptions $\{\varphi_n\}_{n\geq 0}$, $\{\psi_n\}_{n\geq 0}$ are uniformly bounded in $H^1_\mu(\mathbb{R})$, hence the sequence of products $\{\varphi_n\psi_n\}_{n\geq 0}$ is also uniformly bounded and all three are weakly compact in $H^1_\mu(\mathbb{R})$. Passing to the subsequences, if necessary, we conclude that $\varphi_n \to [\{\varphi_n\}]$, $\psi_n \to [\{\psi_n\}]$ and $\varphi_n\psi_n \to [\{\varphi_n\psi_n\}]$ weakly in $H^1_\mu(\mathbb{R})$. Therefore, for a fixed $h \in L^2(\mathbb{R})$, we have[4]

$$\langle \hat{h}, \hat{\varphi}_n * \hat{\psi}_n\rangle = \langle \hat{h} * [\{\hat{\varphi}_n\}], \hat{\psi}_n\rangle + \langle \hat{h} * (\hat{\varphi}_n - [\{\hat{\varphi}_n\}]), \hat{\psi}_n\rangle.$$

As $n \to \infty$, the first term approaches $\langle h, [\{\varphi_n\}][\{\psi_n\}]\rangle$, we have to show that the second term vanishes.

Fix $\delta > 0$ and $\rho > 0$, so that $\|\chi_{[-\rho,\rho]}\hat{\psi}_n\|^2_{\hat{H}^1_\mu} \geq \lambda - \delta$, for all $n \geq 0$, and split the second term as follows

$$\begin{aligned}\langle \hat{h} * (\hat{\varphi}_n - [\{\hat{\varphi}_n\}]), \hat{\psi}_n\rangle &= \langle \hat{h} * (\hat{\varphi}_n - [\{\hat{\varphi}_n\}]), \chi_{\mathbb{R}/[-\rho,\rho]}\hat{\psi}_n\rangle \\ &\quad + \langle \hat{h} * (\hat{\varphi}_n - [\{\hat{\varphi}_n\}]), \chi_{[-\rho,\rho]}\hat{\psi}_n\rangle =: A_{1,n} + A_{2,n}.\end{aligned}$$

The first quantity is easy to estimate. Straightforward calculations yield

$$|A_{1,n}| \leq \|\hat{h} * (\hat{\varphi}_n - [\{\hat{\varphi}_n\}])\|_{L^\infty}\|\chi_{\mathbb{R}/[-\rho,\rho]}\hat{\psi}_n\|_{L^1} \leq \frac{\|\kappa_\mu^{-\frac{1}{2}}\|}{\sqrt{1-\mu^2}} 2M\|h\|\sqrt{\delta}.$$

To bound $A_{2,n}$, we let $\mathcal{T}_t\hat{h}(\cdot) = \hat{h}(t-\cdot)$ and $g_n(t) = \hat{h} * (\hat{\varphi}_n - [\{\hat{\varphi}_n\}])(t) = \langle \mathcal{T}_t\hat{h}, \hat{\varphi}_n - [\{\hat{\varphi}_n\}]\rangle$. Then

$$|A_{2,n}| \leq \|\kappa_\mu^{-\frac{1}{2}}\|\sqrt{\lambda}\|g_n\|_{L^\infty([-\rho,\rho])}.$$

Since $\hat{\varphi}_n \to [\hat{\varphi}]$ weakly, it follows that $g_n \to 0$ pointwise in $[-\rho,\rho]$. Furthermore,

$$\|g_n\|_{L^\infty([-\rho,\rho])} \leq 2M\|\kappa_\mu^{-\frac{1}{2}}\|\|h\|, \quad |g_n(t) - g_n(\tau)| \leq 2M\|\kappa_\mu^{-\frac{1}{2}}\|\|\mathcal{T}_{t-\tau}\hat{h} - \hat{h}\|.$$

[3] Here and everywhere below, we use square brackets to denote the operation of taking weak limits.

[4] Note that the operations $[\cdot]$ and $\hat{\cdot}$ commute.

The group of translations $\{\mathcal{T}_t\}_{t\in\mathbb{R}}$ is continuous in $L^2(\mathbb{R})$, and we conclude that the sequence $\{g_n\}_{n\geq 0}$ is equibounded and equicontinuous in $C[-\rho,\rho]$. The classical Arzela-Ascoli theorem [78, Theorem 11.28], combined with the pointwise convergence mentioned earlier on, implies that $g_n \to 0$ in the strong topology of $C[-\rho,\rho]$. Hence,

$$\lim_{n\to\infty} |A_{2,n}| \leq \|\kappa_\mu^{-\frac{1}{2}}\|\sqrt{\lambda} \lim_{n\to\infty} \|g_n\|_{L^\infty[-\rho,\rho]} = 0,$$

and

$$\lim_{n\to\infty} |\langle h, \varphi_n\psi_n\rangle - \langle h, [\{\varphi_n\}][\{\psi_n\}]\rangle| \leq \frac{\|\kappa_\mu^{-\frac{1}{2}}\|}{\sqrt{1-\mu^2}} 2M\|h\|\sqrt{\delta}.$$

Since $\delta > 0$ is arbitrary, we conclude that $\varphi_n\psi_n \to [\{\varphi_n\}][\{\psi_n\}]$ in the weak topology of $L^2(\mathbb{R})$. Note that $[\{\varphi_n\}], [\{\psi_n\}] \in H^1_\mu(\mathbb{R})$ and so is the product $[\{\varphi_n\}][\{\psi_n\}]$, therefore density of $L^2(\mathbb{R})$ in $H^{-1}(\mathbb{R})$ indicates that (4.1.7) holds in $H^1_\mu(\mathbb{R})$. □

Lemmas 4.1.4-4.1.6 yield the main result of the section.

Theorem 4.1.7. *For each fixed value of* $0 \leq \mu < 1$, *there exists a nontrivial even positive definite solution* $\varphi_\mu \in H^1_\mu(\mathbb{R})$ *of* (1.3.4) *that delivers the supremum to* (4.1.6), *i.e.* $\mathcal{G}(\varphi_\mu) = I_{\|\varphi_\mu\|^2_{H^1_\mu}}$.

Proof. (a) Our arguments are based on the following observation: in (4.1.1) both the objective functional $\mathcal{G}_\varepsilon(\cdot)$ and the constraint are of class C^1 (in fact analytic). Since the constraint is regular, the classical principle of Lagrange implies

$$\langle h, \varphi^\ell_{\mu,\varepsilon}\rangle - \varepsilon^2\langle xh, x\varphi_{\mu,\varepsilon}\rangle = 2\tau\langle h, \varphi_{\mu,\varepsilon}\rangle_{H^1_\mu}. \tag{4.1.8}$$

Identity (4.1.8) holds for all $h \in H^{1,1}_{\mu,\varepsilon}(\mathbb{R})$, hence

$$\tau = \tfrac{1}{2\lambda}\big((\ell+1)I^\varepsilon_\lambda + (\ell-1)\tfrac{\varepsilon^2}{2}\|x\varphi_{\mu,\varepsilon}\|^2\big) > \tfrac{(\ell+1)}{2\lambda} I^{\varepsilon_0}_\lambda > 0.$$

In view of Lemma 4.1.4, the inequality holds for all $0 < \varepsilon < \varepsilon_0$ and some fixed ε_0.

Let $\theta \in C^\infty_0(\mathbb{R})$ be a smooth nonnegative cut-off function with $\operatorname{supp}\theta = [-2,2]$, that satisfies $\theta(x) = 1$, $|x| \leq 1$. We denote $\theta_\rho(\cdot) = \theta(\cdot/\rho)$, $\rho > 0$, and let $h(x) = x^2\theta_\rho(x)\varphi_{\mu,\varepsilon}(x)$ in (4.1.8), to obtain

$$\begin{aligned}
&\tfrac{1}{2\tau}\langle \theta_\rho(x\varphi_{\mu,\varepsilon})^2, \varphi^{\ell-1}_{\mu,\varepsilon}\rangle - \tfrac{\varepsilon^2}{2\tau}\langle \theta_\rho x^2\varphi_{\mu,\varepsilon}, x^2\varphi_{\mu,\varepsilon}\rangle \\
&\quad = \langle \theta_\rho(x\varphi_{\mu,\varepsilon}), (x\varphi_{\mu,\varepsilon}) - 2\mu\mathcal{H}[(x\varphi_{\mu,\varepsilon})']\rangle + \langle \theta_\rho(x\varphi_{\mu,\varepsilon})', (x\varphi_{\mu,\varepsilon})'\rangle \\
&\quad + 2\mu\langle \theta_\rho(x\varphi_{\mu,\varepsilon}), \mathcal{H}[\varphi_{\mu,\varepsilon}]\rangle - \frac{1}{2}\langle (2\theta_\rho + 2x\theta'_\rho + x^2\theta''_\rho)\varphi_{\mu,\varepsilon}, \varphi_{\mu,\varepsilon}\rangle.
\end{aligned}$$

Rearranging the terms and sending ρ to infinity, we arrive at

$$\|x\varphi_{\mu,\varepsilon}\|^2_{H^{1,1}_{\mu,\bar{\varepsilon}}} = \|\varphi_{\mu,\varepsilon}\|^2 + \tfrac{1}{2\tau}\langle x^2\varphi^2_{\mu,\varepsilon}, \varphi^{\ell-1}_{\mu,\varepsilon}\rangle - \mu\hat{\varphi}_{\mu,\varepsilon}(0)^2, \quad \bar{\varepsilon} = \tfrac{\varepsilon}{\sqrt{\tau}}. \tag{4.1.9}$$

(b) Identities (4.1.8) and (4.1.9) have two important consequences. First, for each fixed value of $\varepsilon > 0$, $x\varphi_{\mu,\varepsilon} \in H^{1,1}_{\mu,\bar{\varepsilon}}(\mathbb{R})$, so that every even nonnegative maximizer $\hat{\varphi}_{\mu,\varepsilon}$ satisfies

$$\hat{\varphi}_{\mu,\varepsilon} - \tfrac{\bar{\varepsilon}^2}{2\kappa_\mu}\hat{\varphi}_{\mu,\varepsilon\xi\xi} = \tfrac{\alpha_\ell}{2\tau\kappa_\mu}\hat{\varphi}^{*\ell}_{\mu,\varepsilon}, \tag{4.1.10}$$

in the weak sense of $H^1_\mu(\mathbb{R})$. Second, it follows that each sequence $\{\varphi_{\mu,\varepsilon_n}\}_{n\geq 0}$, with $\varepsilon_n \to 0$, contains nontrivial weak limit points.

Indeed, passing to subsequences if necessary and using Lemma 4.1.5, we may assume that the sequence $\{\varphi_{\mu,\varepsilon_n}\}_{n\geq 0}$ converges weakly to $[\varphi_\mu] \in H^1_\mu(\mathbb{R})$ and in addition, satisfies the compactness alternative of Lemma 4.1.2. That is for any $\delta > 0$ there exists $\rho > 0$, such that $\|\chi_{[-\rho,\rho]}\hat{\varphi}_{\mu,\varepsilon_n}\|^2_{H^1_\mu} \geq \lambda - \delta$, for all $n \geq 0$. Taking into account that $0 < 1-\mu^2 \leq \kappa_\mu$ and $\hat{\varphi}_{\mu,\varepsilon_n} \geq 0$, and using the identity $\lim_{|\xi|\to\infty} |\hat{\varphi}_{\mu,\xi_n}\xi| = 0$,[5] we multiply both sides of (4.1.10) by κ_μ and integrate over $\mathbb{R}$ to obtain

$$0 < \big(\tfrac{(1-\mu^2)(\ell+1)}{\alpha_\ell\lambda} I^{\varepsilon_0}_\lambda\big)^{\frac{1}{\ell-1}} < \big(\tfrac{1-\mu^2}{\alpha_\ell} 2\tau_n\big)^{\frac{1}{\ell-1}} \leq \|\hat{\varphi}_{\mu,\varepsilon_n}\|_{L^1}.$$

The Cauchy-Schwartz inequality applied to the quantity $\|\chi_{\mathbb{R}/[-\rho,\rho]}\hat{\varphi}_{\mu,\varepsilon_n}\|_{L^1}$, yields the bound

$$\langle\chi_{[-\rho,\rho]}, \hat{\varphi}_{\mu,\varepsilon_n}\rangle > \big(\tfrac{(1-\mu^2)(\ell+1)}{\alpha_\ell\lambda} I^{\varepsilon_0}_\lambda\big)^{\frac{1}{\ell-1}} - \|\kappa_\mu^{-\frac{1}{2}}\|\delta^{\frac{1}{2}} > 0,$$

provided $\delta > 0$ is sufficiently small. Sending n to infinity, we conclude that the duality pairing $\langle\chi_{[-\rho,\rho]}, [\hat{\varphi}_\mu]\rangle$ is strictly positive and hence $[\varphi_\mu]$ does not vanish.

(c) Obviously, we can choose the sequence $\{\varphi_{\mu,\varepsilon_n}\}_{n\geq 0}$ so that, in addition to the assumptions employed in part (b), each $\{\varphi^m_{\mu,\varepsilon_n}\}_{n\geq 0}$, $1 \leq m \leq \ell$, converges weakly in $H^1_\mu(\mathbb{R})$ and $\frac{\varepsilon_n^2}{2}\|\hat{\varphi}_{\mu,\varepsilon_n\xi}\|^2$ converges to some $0 \leq \sigma < \infty$.[6] Then Lemma 4.1.6, applied successively to the pairs $(\{\varphi_{\mu,\varepsilon_n}\}_{n\geq 0}, \{\varphi^{m-1}_{\mu,\varepsilon_n}\}_{n\geq 0})$, $2 \leq m \leq \ell$, implies that $[\varphi^m_\mu] = [\varphi_\mu]^m$, $2 \leq m \leq \ell$. Hence, letting $h \in H^{1,2}_{\mu,1}(\mathbb{R})$ in (4.1.8), integrating by parts

[5] The identity follows from the classical Sobolev embedding $H^1(\mathbb{R}) \subset C_0(\mathbb{R})$ (see [3]), the inclusion $\hat{\varphi}_{\gamma,\varepsilon\xi} \in H^{1,1}_{\gamma,\bar{\varepsilon}}(\mathbb{R})$ and our definition of the spaces $H^{s,r}_{\gamma,\varepsilon}(\mathbb{R})$.

[6] The latter follows from the inequality $0 < \mathcal{G}_\varepsilon(\varphi_{\mu,\varepsilon}) < I_\lambda$, indicating that $\{\varepsilon^2\|\hat{\varphi}_{\mu,\varepsilon\xi}\|^2\}_{n\geq 0}$ is uniformly bounded in $\mathbb{R}$, as $\varepsilon \to 0$.

and passing to the limit, we arrive at the identity

$$\langle h, [\varphi_\mu]\rangle_{H^1_\mu} = M\langle h, [\varphi_\mu]^\ell\rangle, \quad M = \tfrac{\lambda}{(\ell+1)I_\lambda+(\ell-1)\sigma}. \tag{4.1.11}$$

Note that $H^{1,2}_{\mu,1}(\mathbb{R})$ is dense in $H^1_\mu(\mathbb{R})$ and, by virtue of the standard approximation argument, it follows that (4.1.11) holds for all $h \in H^1_\mu(\mathbb{R})$.

(d) To complete the proof, we observe that

$$I_\lambda \geq \lim_{n\to\infty} \mathcal{G}_0(\varphi_{\mu,\varepsilon_n}) = \lim_{n\to\infty} \mathcal{G}_{\varepsilon_n}(\varphi_{\mu,\varepsilon_n}) + \lim_{n\to\infty} \varepsilon_n^2 \|\hat{\varphi}_{\mu,\varepsilon_n\xi}\|^2 = I_\lambda + \sigma,$$

which gives $\sigma = 0$ and $M = \frac{\lambda}{(\ell+1)I_\lambda}$. By construction, $\|[\varphi_\mu]\|^2_{H^1_\mu} \leq \lambda$. On another hand, letting $h = [\varphi_\mu]$ in (4.1.11), we obtain

$$\mathcal{G}([\varphi_\mu]) = \tfrac{I_\lambda}{\lambda}\|[\varphi_\mu]\|^2_{H^1_\mu}.$$

while, in view of (4.1.6), for $\psi = \frac{\sqrt{\lambda}}{\|[\varphi_\mu]\|_{H^1_\mu}}[\varphi_\mu]$, we have

$$\mathcal{G}(\psi) = \big(\tfrac{\lambda}{\|[\varphi_\mu]\|^2_{H^1_\mu}}\big)^{\frac{\ell+1}{2}}\mathcal{G}([\varphi_\mu]) \leq I_\lambda.$$

The last two formulas give the reverse inequality $\lambda \leq \|[\varphi]\|^2_{H^1_\mu}$. It follows now that: (i) $\|[\varphi_\mu]\|^2_{H^1_\mu} = \lambda$; (ii) the the sequence $\{\varphi_{\gamma,\varepsilon_n}\}_{n\geq 0}$, defined in part (c) of the proof, converges strongly in $H^1_\mu(\mathbb{R})$; and (iii) the limit $[\varphi_\mu] \in H^1_\mu(\mathbb{R})$ maximizes $\mathcal{G}(\cdot)$ subject to the constraint, listed in (4.1.6). Theorem 4.1.7 is a byproduct of assertions (i)–(iii); for even, nonzero, nonnegative function $\hat{\varphi}_\mu := \big(\frac{\lambda}{(\ell+1)I_\lambda}\big)^{\frac{1}{\ell-1}}[\hat{\varphi}_\mu]$ satisfies (1.3.4) in the sense of $H^1_\mu(\mathbb{R})$. □

Few remarks are in place here. As mentioned in Section 1.3, the global existence ($0 \leq \gamma < 1$ and $\ell = 2$) of traveling waves for (1.3.4) is established by several authors.

In particular, the variational approach was pursued in [13, 74]. Both papers treat the problem in physical space (in variable x), but differ in setting of the associated variational problem: in [13], the first integrals of (1.1.4) are used while in [74] the variational problem is similar to (4.1.6), but with the objective functional and the constraint interchanged. In either case, it is hard to establish a direct link between variational problem and eigenstructure of the linearized operator associated with (1.3.4). Formulation (4.1.6) is free from this drawback — below we show that (4.1.6), combined with the positivity of $\hat{\varphi}_\mu$, yields simplicity of the first two eigenvalues of the linearized operator for small values of μ. The information has important physical consequences as it controls the orbital stability of the wave. In addition, it implies local regularity of the solitary waves as functions of the wavespeed parameter μ, which is a standard assumption in a rigorous analysis of continuation numerical schemes.

4.2 Regularity

In this section, we study regularity and asymptotic behaviour of the traveling waves obtained in Theorem 4.1.7. The results form a theoretical foundation for the numerical analysis of Chapter 7.

4.2.1 Large $|\xi|$ asymptotic of $\hat{\varphi}_\mu$

Let φ_μ, $0 \le \mu < 1$, be a solution obtained in Theorem 4.1.7. Directly from (1.3.4), it follows that $\varphi_\mu \in H^s_\mu(\mathbb{R})$, for all $s \ge 1$. A more precise characterization is given by

Lemma 4.2.1. *For any* $0 \le \mu < 1$*, there exists* $\rho > 0$ *such that* $\hat{\varphi}_\mu \in L^p_\rho(\mathbb{R}) \cap C_0(\mathbb{R})$*,* $1 \le p \le \infty$*. Furthermore,* $e^{\rho|\xi|}\hat{\varphi}_\mu \in C_0(\mathbb{R})$*.*

Proof. (a) In view of the inclusion $\varphi_\mu \in H^1_\mu(\mathbb{R})$, we have $\hat{\varphi}_\mu \in L^1(\mathbb{R})$. Elementary calculations show that $\frac{1+|\xi|}{\kappa_\mu(\xi)} \le \frac{3}{2(1-\mu)} =: \beta_\mu$ uniformly in $\mathbb{R}$. Consequently,

$$\begin{aligned}\|(1+|\xi|)^n \hat{\varphi}_\mu\|_{L^1} &\le \beta_\mu \|(1+|\xi|)^{n-1} \hat{\varphi}_\mu^{*\ell}\|_{L^1} \\ &\le \beta_\mu \sum_{i_1+\cdots+i_\ell=n-1} \binom{n-1}{i_1,\ldots,i_\ell} \prod_{k=1}^{\ell} \|(1+|\xi|)^{i_k} \hat{\varphi}_\mu\|_{L^1}.\end{aligned}$$

We let $b_0 = \|\hat{\varphi}_\mu\|_{L^1}$ and

$$b_n = \beta_\mu \sum_{i_1+\cdots+i_\ell=n-1} \binom{n-1}{i_1,\ldots,i_\ell} \prod_{k=1}^{\ell} b_{i_k}, \quad n \ge 1.$$

Induction on n shows that $\|(1+|\xi|)^n \hat{\varphi}_\mu\|_{L^1} \le b_n$ for all $n \ge 1$. Straightforward calculations give,[7]

$$b_n = b_0\big[(\ell-1)\beta_\mu b_0^{\ell-1}\big]^n\big(\tfrac{1}{\ell-1}\big)_n, \quad n \geq 1.$$

Hence, we conclude that

$$\begin{aligned}\|\hat{\varphi}_\mu\|_{L^1_\rho} &\leq \sum_{n\geq 0}\tfrac{\rho^n}{n!}\|(1+|\xi|)^n\hat{\varphi}_\mu\|_{L^1} \leq b_0\sum_{n\geq 0}\big[2(\ell-1)\beta_\mu b_0^{\ell-1}\rho\big]^n \\ &= \tfrac{b_0}{1-(\ell-1)\beta_\mu b_0^{\ell-1}\rho},\end{aligned}$$

provided that $\rho < \frac{1}{(\ell-1)\beta_\mu b_0^{\ell-1}}$. Further, in view of (1.3.4), the inclusion $\hat{\varphi}_\mu \in L^1(\mathbb{R})$ and the continuity of $L^1(\mathbb{R})$ norm, we infer that $\hat{\varphi}_\mu \in C_0(\mathbb{R})$ and $\|\hat{\varphi}_\mu\|_{L^\infty} \leq \frac{1}{2(1-\mu)}\|\hat{\varphi}_\mu\|_{L^1}^{\ell-2}\|\hat{\varphi}_\mu\|_{L^2}^2 < \infty$. The above estimates settles the first claim of the Lemma, as $\|\hat{\varphi}_\mu\|_{L^p_\rho} \leq \|\hat{\varphi}_\mu\|_{L^\infty}^{\frac{1}{p'}}\|\hat{\varphi}_\mu\|_{L^1_\rho}^{\frac{1}{p}} < \infty$, when $1 < p < \infty$.

(b) The second assertion follows directly from part (a) of the proof, for the inclusion $\hat{\varphi}_\mu \in L^1_\rho(\mathbb{R}) \cap C_0(\mathbb{R})$ implies $e^{\rho|\xi|}\hat{\varphi}_\mu \in C(\mathbb{R})$ and $e^{\rho|\xi|}\hat{\varphi}_\mu \to 0$, as $|\xi| \to \infty$. □

4.2.2 Regularity

We present two regularity results. The first one characterizes solutions on the real line and is similar to the results obtained earlier in [74]. The second one deals with the regularity of $\hat{\varphi}_\mu$ restricted to the positive half line and is relevant for the numerical analysis of the problem. We begin with the following technical Lemma (see also [74]).

Lemma 4.2.2. *The operator $\hat{\mathcal{G}}_\mu(\hat{\varphi}) = \frac{\alpha_\ell}{\kappa_\mu}\hat{\varphi}^{*\ell}$, viewed as a map from $L^1_\rho(\mathbb{R}) \cap L^\infty(\mathbb{R})$ to $L^1_\rho(\mathbb{R}) \cap C_0(\mathbb{R})$, is Frechet differentiable. The differential $\hat{\mathcal{G}}'_\mu(\hat{\varphi})$, viewed as a map form $L^1_\rho(\mathbb{R}) \cap L^\infty(\mathbb{R})$ to $L^1_\rho(\mathbb{R}) \cap C_0(\mathbb{R})$, is compact positive and irreducible. Its dominant eigenvalue ℓ is simple and the associated eigenfunction is given by $\hat{\varphi}_\mu$.*

Proof. (a) Elementary pointwise estimate $e^{\rho|\xi|}|\hat{\varphi}^{*\ell}| \leq |e^{\rho|\xi|}\hat{\varphi}|^{*\ell}$, combined with Young's convolution inequality, shows that $\hat{\mathcal{G}}_\mu(\hat{\varphi})$ is Frechet differentiable and $\hat{\mathcal{G}}'_\mu(\hat{\varphi})[\cdot] =$

[7] The identity can be verified either directly (e.g. using induction) or deduced with the aid of the generating function $g(z) = \sum_{n\geq 0}\frac{b_n}{n!}z^n$.

$\frac{\ell}{\kappa_\mu}\hat{\varphi}^{*(\ell-1)} * [\cdot] : L^1_\rho(\mathbb{R}) \cap L^\infty(\mathbb{R}) \to L^1_\rho(\mathbb{R}) \cap C_0(\mathbb{R})$ is continuous. Furthermore, for any $\hat{\psi} \in L^1_\rho(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $\delta \in \mathbb{R}$, we have

$$\begin{aligned} e^{\rho|\xi|}\big|\hat{\mathcal{G}}'_\mu(\hat{\varphi})[\hat{\psi}]\big|(\xi) &\le |\tfrac{\alpha_\ell}{\kappa_\mu(\xi)}\hat{\varphi}^{*(\ell-2)}||e^{\rho|\xi|}\hat{\varphi}||\hat{\psi}|(\xi) \\ &\le \tfrac{\alpha_\ell}{\kappa_\mu(\xi)}\|\hat{\varphi}\|^{\ell-2}_{L^1_\rho}\|e^{\rho|\xi|}\hat{\varphi}\|_{L^\infty}\|\hat{\psi}\|_{L^1_\rho}, \end{aligned}$$

and

$$\begin{aligned} \|(I-\mathcal{T}_\delta)\hat{\mathcal{G}}'_\mu(\hat{\varphi})[\hat{\psi}]\|_{L^i_\rho} &\le \ell\alpha_\ell\|(I-\mathcal{T}_\delta)[\kappa_\mu^{-1}]\kappa_\mu^{-1}\hat{\varphi}^{*(\ell-1)} * \hat{\psi}\|_{L^i_\rho} \\ &\quad + \ell\alpha_\ell\|\kappa_\mu^{-1}(I-\mathcal{T}_\delta)[\hat{\varphi}^{*(\ell-1)} * \hat{\psi}]\|_{L^i_\rho} \\ &\le \ell\alpha_\ell\|(I-\mathcal{T}_\delta)[\kappa_\mu^{-1}]\|_{L^\infty}\|\hat{\varphi}\|^{\ell-1}_{L^1_\rho}\|\hat{\psi}\|_{L^i_\rho} \\ &\quad + \tfrac{\ell\alpha_\ell}{2(1-\mu)}\|\hat{\varphi}\|^{\ell-2}_{L^1_\rho}\|\hat{\psi}\|_{L^1_\rho}\|(I-\mathcal{T}_\delta)\hat{\varphi}\|_{L^i_\rho}, \quad i = 1, \infty. \end{aligned}$$

Taking into account that κ_μ^{-1} is uniformly continuous in $\mathbb{R}$ and that the group of translations $\{\mathcal{T}_\delta\}_{\delta\in\mathbb{R}}$ is strongly continuous in either of the spaces $L^1_\rho(\mathbb{R})$, $C_0(\mathbb{R})$, we infer from the classical Kolmogorov-Riesz and Arzela-Ascoli theorems that, in fact, $\hat{\mathcal{G}}'_\mu(\hat{\varphi})$ is compact.

(b) In view of (1.3.4) and the inclusion $\hat{\varphi}_\mu \in C_0(\mathbb{R})$, $\hat{\varphi}_\mu > 0$ in $\mathbb{R}$. Consequently, the integral operator $\hat{\mathcal{G}}'_\mu(\hat{\varphi}_\mu)$ is irreducible, positive and compact. Since $L^1_\rho(\mathbb{R}) \cap L^\infty(\mathbb{R})$ is a Banach lattice, we infer from the Perron-Frobenius theory of such operators [85, Chapter V, Theorem 5.2] that the derivative $\hat{\mathcal{G}}'_\mu(\hat{\varphi}_\mu)$ has a unique (up to scaling) positive eigenvector, that corresponds to a simple, dominant positive eigenvalue. It follows directly from (1.3.4) that $(\ell, \hat{\varphi}_\mu)$ is the dominant positive eigenpair. □

Lemma 4.2.3. *Let $\rho > 0$ be as in Lemma 4.2.1, then $\hat{\varphi}_\mu \in W^{p,1}_\rho(\mathbb{R})$, $1 \le p \le \infty$, and $\hat{\varphi}_\mu \in C^1_0(\mathbb{R}_+)$. Furthermore, $\hat{\varphi}_\mu$ can be extended to a continuous function on the closed half-line $\bar{\mathbb{R}}_+$.*

Proof. (a) We observe that the subspaces of even and odd $L^1_\rho(\mathbb{R}) \cap L^\infty(\mathbb{R})$ functions are invariant under the action of $\hat{\mathcal{G}}'_\mu(\hat{\varphi}_\mu)$. In view of Lemma 4.2.2, the dominant eigenpair is given by $(\ell, \hat{\varphi}_\mu)$, where the eigenfunction $\hat{\varphi}_\mu$ is even. Hence, the classical Fredholm alternative for compact operators implies that the map $\hat{\mathcal{B}} = I - \frac{1}{\ell}\hat{\mathcal{G}}'_\mu(\hat{\varphi}_\mu)$ is invertible in the subspace of odd $L^1_\rho(\mathbb{R}) \cap L^\infty(\mathbb{R})$ functions.

(b) Note that κ_μ, $\hat\varphi_\mu$ are even, consequently, $\hat\psi = -\hat{\mathcal{B}}^{-1}(\frac{\kappa_{\mu\xi}}{\kappa_\mu}\hat\varphi_\mu) \in L^1_\rho(\mathbb{R}) \cap L^\infty(\mathbb{R}) \subset L^p_\rho(\mathbb{R})$, $1 \le p \le \infty$. Let $\hat h$ be a compactly supported $C_0^{(\infty)}(\mathbb{R})$ function. If $\hat h$ is even, we have $\langle \hat\psi, \hat h\rangle = -\langle \hat\varphi_\mu, \hat h_\xi\rangle = 0$. For $\hat h$ odd, we let $\hat{\mathcal{B}}\hat v = \frac{\hat h}{\kappa_\mu}$. Then, $\big(\frac{\hat h}{\kappa_\mu}\big)_\xi = \hat{\mathcal{B}}\hat v_\xi + \frac{\kappa_{\mu\xi}}{\kappa_\mu}\big(\hat v - \frac{\hat h}{\kappa_\mu}\big)$, and we infer

$$
\begin{aligned}
\langle \hat\psi, \hat h\rangle &= \big\langle \hat\psi, \tfrac{\hat h}{\kappa_\mu}\big\rangle_{\hat H^1_\mu} = -\big\langle \tfrac{\kappa_{\mu\xi}}{\kappa_\mu}\hat\varphi_\mu, \hat{\mathcal{B}}^{-1}\tfrac{\hat h}{\kappa_\mu}\big\rangle_{\hat H^1_\mu} = -\big\langle \hat\varphi_\mu, \tfrac{\kappa_{\mu\xi}}{\kappa_\mu}\hat v\big\rangle_{\hat H^1_\mu} \\
&= -\big\langle \hat\varphi_\mu, \big(\tfrac{\hat h}{\kappa_\mu}\big)_\xi + \tfrac{\kappa_{\mu\xi}}{\kappa_\mu^2}\hat h\big\rangle_{\hat H^1_\mu} + \langle \hat{\mathcal{B}}\hat\varphi_\mu, \hat v_\xi\rangle_{\hat H^1_\mu} = -\langle \hat\varphi_\mu, \hat h_\xi\rangle.
\end{aligned}
$$

Hence, $\hat\varphi_{\mu\xi} = \hat\psi$ in the sense of distributions and the first claim of the Lemma is settled.

(c) The inclusion $\hat\varphi_\mu \in C^1_0(\mathbb{R}_+)$ follows from the identity

$$
\hat\varphi_{\mu\xi} = \tfrac{\kappa_{\mu\xi}}{\kappa_\mu}\hat\varphi_\mu - \tfrac{1}{\ell}\hat{\mathcal{G}}'_\mu(\hat\varphi_\mu)[\hat\varphi_{\mu\xi}],
$$

where $\frac{1}{\ell}\hat{\mathcal{G}}'_\mu(\hat\varphi_\mu)[\hat\varphi_{\mu\xi}] \in C_0(\mathbb{R})$[8] and $\frac{\kappa_{\mu\xi}}{\kappa_\mu}\hat\varphi_\mu \in C_0(\mathbb{R}/\{0\})$, has a simple jump discontinuity at the origin. In particular, it follows that $\hat\varphi_{\mu\xi}$ can be extended to a continuous function in the closed positive half-line $\bar{\mathbb{R}}_+$. □

Lemma 4.2.4. *Let $\rho > 0$ be as in Lemma 4.2.1, then $\hat\varphi_\mu \in W^{p,n}_\rho(\mathbb{R}_+) \cap C_0^{(n)}(\mathbb{R}_+)$, $1 \le p \le \infty$, for all $n \ge 1$. Furthermore,*

$$
\|\partial_\xi^n \hat\varphi_\mu\|_{L^p_\rho} \le n! r_\mu^{n+1}, \quad 1 \le p \le \infty, \quad n \ge 0, \tag{4.2.1}
$$

where $r_\mu > 0$ depends on $0 \le \mu < 1$ only.

Proof. (a) We let $\hat\psi_1 = \hat\varphi_{\mu\xi}$ and $\hat\psi_m = \hat\varphi_\mu^{*(m-1)} * \hat\varphi_{\mu\xi}$, $m \ge 2$. Using this notation, Lemma 4.2.3 and parity of $\hat\varphi_\mu$, for $\xi \in \mathbb{R}_+$, we write

$$
\partial_\xi\big(\kappa_\mu\hat\varphi_\mu\big) = \alpha_\ell \int_{\mathbb{R}_+} \big[\hat\varphi_\mu(|\xi-\zeta|) - \hat\varphi_\mu(\xi+\zeta)\big]\hat\psi_{\ell-1}(\zeta)d\zeta, \tag{4.2.2a}
$$

$$
\hat\psi_m = \int_{\mathbb{R}_+} \big[\hat\varphi_\mu(|\xi-\zeta|) - \hat\varphi_\mu(\xi+\zeta)\big]\hat\psi_{m-1}(\zeta)d\zeta, \quad m \ge 2. \tag{4.2.2b}
$$

In view of Lemma 4.2.3, the quantities $\hat\varphi_{\mu\xi}$ and $\hat\psi_m$, $1 \le m \le \ell-1$, are regular $\mathcal{D}'(\mathbb{R}_+)$ distributions. Throughout the proof, the quantities $\partial_\xi^k\hat\varphi_\mu(0)$ are understood

[8] See part (a) of the proof.

as right-handed limits. Using (4.2.2) and integrating by parts, we obtain *formal* identities

$$\begin{aligned}\partial_\xi^{n+1}\hat{\varphi}_\mu &= -\binom{n+1}{2}\frac{\kappa_\mu''}{\kappa_\mu}\partial_\xi^{n-1}\hat{\varphi}_\mu - \binom{n+1}{1}\frac{\kappa_\mu'}{\kappa_\mu}\partial_\xi^{n}\hat{\varphi}_\mu \\ &\quad + \tfrac{\alpha_\ell}{\kappa_\mu}\int_{\mathbb{R}_+}\big[\operatorname{sgn}^n(\xi-\zeta)\partial_\xi^n\hat{\varphi}_\mu(|\xi-\zeta|) - \partial_\xi^n\hat{\varphi}_\mu(\xi+\zeta)\big]\hat{\psi}_{\ell-1}(\zeta)d\zeta \\ &\quad + \tfrac{\alpha_\ell}{\kappa_\mu}\sum_{k=0}^{n-1}[1-(-1)^k]\partial_\xi^k\hat{\varphi}_\mu(0)\partial_\xi^{n-1-k}\hat{\psi}_{\ell-1}, \quad n\geq 1, \end{aligned} \tag{4.2.3a}$$

and

$$\begin{aligned}\partial_\xi^n\hat{\psi}_m &= \int_{\mathbb{R}_+}\big[\operatorname{sgn}^n(\xi-\zeta)\partial_\xi^n\hat{\varphi}_\mu(|\xi-\zeta|) - \partial_\xi^n\hat{\varphi}_\mu(\xi+\zeta)\big]\hat{\psi}_{m-1}(\zeta)d\zeta \\ &\quad + \sum_{k=0}^{n-1}[1-(-1)^k]\partial_\xi^k\hat{\varphi}_\mu(0)\partial_\xi^{n-1-k}\hat{\psi}_{m-1}, \quad n\geq 1, \quad m\geq 2. \end{aligned} \tag{4.2.3b}$$

By virtue of Lemma 4.2.3, $\partial_\xi^n\hat{\varphi}_\mu, \hat{\psi}_m \in L^p_\rho(\mathbb{R}_+)\cap C_0(\mathbb{R}_+)$, $1\leq p<\infty$, $n=0,1$. This fact, combined with (4.2.3) and induction on n, allow us to conclude that $\partial_\xi^n\hat{\varphi}_\mu, \partial_\xi^n\hat{\psi}_m \in L^p_\rho(\mathbb{R}_+)\cap C_0(\mathbb{R}_+)$, for all $1\leq p<\infty$ and $n,m\geq 1$. Furthermore, for all $1\leq p<\infty$ and $n,m\geq 1$, each of the quantities $\partial_\xi^n\hat{\varphi}_\mu$ and $\partial_\xi^n\hat{\psi}_m$ can be extended to a continuous function defined on the closed positive half-line $\bar{\mathbb{R}}_+$. Hence, the first assertion of the Lemma is settled.

(b) To obtain $L^\infty(\mathbb{R}_+)$ estimates, we let

$$\beta = \max_{1\leq m\leq \ell-1}\|\hat{\psi}_m\|_{L^1_\rho}, \quad \alpha = \max\{\|\tfrac{\kappa_\mu''}{2\kappa_\mu}\|_{L^\infty}, \|\tfrac{\kappa_\mu'}{\kappa_\mu}\|_{L^\infty}, \|\tfrac{1}{\kappa_\mu}\|_{L^\infty}\},$$

$\|\partial_\xi^n\hat{\varphi}_\mu\|_{L^\infty} = n!a_n$, $n=0,1$, $b_{1,n}=a_{n+1}$ and

$$a_{n+1} = \alpha a_{n-1} + \alpha(1+2\beta)a_n + \frac{2\alpha}{n+1}\sum_{k=0}^{n-1}a_k b_{\ell-1,n-1-k}, \quad n\geq 1, \tag{4.2.4a}$$

$$b_{m,n} = 2\beta a_n + \frac{2}{n+1}\sum_{k=0}^{n-1}a_k b_{m-1,n-1-k}, \quad n\geq 0, \quad m\geq 2. \tag{4.2.4b}$$

Comparing (4.2.3) with (4.2.4) and using induction on n, we conclude that $\|\partial_\xi^n\hat{\varphi}_\mu\|_{L^\infty}\leq n!a_n$ and $\|\partial_\xi^n\hat{\psi}_m\|_{L^\infty}\leq (n+1)!b_{m,n}$, for all $n\geq 1$ and $1\leq m\leq \ell-1$. In addition, defining

$$r_\infty = \max\{\|\hat{\varphi}_\mu\|_{L^\infty}, \|\hat{\varphi}_{\mu\xi}\|_{L^\infty}, 2\alpha(1+2^\ell\beta)\},$$

and using induction on n one more time, we infer directly from (4.2.4) that $a_n \le r_\infty^{n+1}$, $n \ge 0$, and that (4.2.1) holds in $L^\infty(\mathbb{R}_+)$ if r_μ is replaced with r_∞.

(c) $L^1_\rho(\mathbb{R}_+)$ estimates are obtained in a similar manner. Here, we let $\|\partial_\xi^n \hat{\varphi}_\mu\|_{L^1_\rho} = n!\bar{a}_n$, for $n = 0, 1$, $\bar{b}_{1,n} = \bar{a}_{n+1}$,

$$\bar{a}_{n+1} = \alpha \bar{a}_{n-1} + \alpha(1+2\beta)a_n + \frac{2\alpha}{n+1}\sum_{k=0}^{n-1} r_\infty^{k+1}\bar{b}_{\ell-1,n-1-k}, \quad n \ge 1, \tag{4.2.5a}$$

$$\bar{b}_{m,n} = 2\beta \bar{a}_n + \frac{2}{n+1}\sum_{k=0}^{n-1} r_\infty^{k+1}\bar{b}_{m-1,n-1-k}, \quad n \ge 0, \quad m \ge 2, \tag{4.2.5b}$$

and $r_\mu = \max\{r_\infty, \|\hat{\varphi}_\mu\|_{L^1_\rho}, \|\hat{\varphi}_{\mu\xi}\|_{L^1_\rho}\}$, to infer (using induction on n) that (4.2.1) holds in $L^1_\rho(\mathbb{R}_+)$. The elementary interpolation inequality $\|f\|_{L^p_\rho} \le \|f\|_{L^\infty}^{\frac{1}{p'}} \|f\|_{L^1_\rho}^{\frac{1}{p}}$ settles (4.2.1) for the remaining values of the exponent p. □

4.3 Orbital stability

4.3.1 Local regularity for $\mu \approx 0$

Equation (1.3.4) is translation invariant and hence cannot have isolated and/or unique solutions. In Section 4.1, we singled out a particular solution class by restricting problem (1.3.4) to the subspace of even $H^1_\mu(\mathbb{R})$ functions. The trick ensures minimal regularity of the solution set $\mathcal{S}(\mu_0, \mu_1) = \{(\mu, \varphi_\mu)\}_{\mu_0 \le \mu \le \mu_1} \subset \mathbb{R} \times H^1_\mu(\mathbb{R})$, (or equivalently $\hat{\mathcal{S}}(\mu_0, \mu_1) = \{(\mu, \hat{\varphi}_\mu)\}_{\mu_0 \le \mu \le \mu_1} \subset \mathbb{R} \times \operatorname{Re} \hat{H}^1_\mu(\mathbb{R})$) for small values of $\mu_1 > \mu_0$, and can be exploited in numerical simulations.

As mentioned earlier, the local structure of $\mathcal{S}(\mu_0, \mu_1)$ is controlled by the eigenstructure of $\mathcal{G}'_\mu(\varphi_\mu)$. In the case of even positive solutions $\hat{\varphi}_\mu$, the principle part of the spectrum can be computed explicitly.

Lemma 4.3.1. *Let φ_μ be an even, positive definite traveling wave obtained in Theorem 4.1.7. Then the spectrum of $\mathcal{G}'_\mu(\varphi_\mu) : H^1_\mu(\mathbb{R}) \to H^1_\mu(\mathbb{R})$, $0 \le \mu < 1$, is real and discrete. The eigenpair pair (ℓ, φ_μ) is simple and dominant. Any other eigenvalue λ of $\mathcal{G}'_\mu(\varphi_\mu)$ satisfies $\lambda \le 1$. In addition, if $(1, \eta)$ is an eigenpair of $\mathcal{G}'_\mu(\varphi_\mu)$, then*

$$\langle \varphi_\mu^{\ell-2}\eta^2, \eta\rangle = 0. \tag{4.3.1}$$

Proof. (a) Linear operator $\hat{\mathcal{G}}'_\mu(\hat{\varphi}_\mu)$, viewed as a map from the weighted Lebesgue space $\hat{H}^1_\mu(\mathbb{R})$ space to itself, is evidently selfadjoint. The proof of compactness, positivity and irreducibility goes along the same lines as in Lemma 4.2.2. In particular, it follows that the spectrum is real and discrete and the dominant eigenpair $(\ell, \hat{\varphi}_\mu)$ is simple.

For a non-dominant eigenpair (λ, η) of $\mathcal{G}'_\mu(\varphi_\mu)$, with $\|\eta\|^2_{H^1_\mu} = \|\varphi_\mu\|^2_{H^1_\mu}$, we let $\psi = \frac{\varphi_\mu + t\eta}{\sqrt{1+t^2}}$, $t \in \mathbb{R}$. Since φ_μ is a constrained maximizer of $\mathcal{G}(\varphi)$ in $H^1_\mu(\mathbb{R})$ and since $\langle \varphi_\mu, \psi \rangle_{H^1_\mu} = 0$, for small values of t we have

$$\begin{aligned} 0 &\leq (1+t^2)^{\frac{\ell+1}{2}} \left[\mathcal{G}(\varphi_\mu) - \mathcal{G}(\psi)\right] \\ &= \mathcal{G}(\varphi_\mu) \left[\tfrac{\ell+1}{2}(1-\lambda)t^2 + \tbinom{\ell+1}{3} \tfrac{\langle \varphi_\mu^{\ell-2}\eta^2, \eta\rangle}{\mathcal{G}(\varphi_\mu)} t^3\right] + \mathcal{O}(t^4). \end{aligned} \tag{4.3.2}$$

From (4.3.2) it follows that $\lambda \leq 1$. If $\lambda = 1$, the quadratic term in the right-hand side of (4.3.2) vanishes, hence (4.3.1) holds. □

Next, we turn our attention to (1.3.4), with $\mu = 0$. As mentioned in Section 1.3, in this case the solution is given explicitly by formula (1.3.3). Furthermore, the following holds

Lemma 4.3.2. *The spectrum $\sigma\big(\mathcal{G}'_0(\varphi_0)\big)$ of the operator $\mathcal{G}'_0(\varphi_0) : H^1(\mathbb{R}) \to H^1(\mathbb{R})$ is given by*

$$\sigma\big(\mathcal{G}'_0(\varphi_0)\big) = \{0\} \cup \left\{\frac{\delta^2 \ell(\ell+1)}{2(n+\delta)(n+\delta+1)}\right\}_{n \geq 0}, \quad \delta = \frac{2}{\ell - 1}, \tag{4.3.3}$$

eigenvalues $\lambda_n = \frac{\delta^2\ell(\ell+1)}{2(n+\delta)(n+\delta+1)}$, $n \geq 0$, are simple and the associated eigenfunctions $\eta_n \in H^1(\mathbb{R})$ satisfy $\eta_n(-x) = (-1)^n \eta_n(x)$, $n \geq 0$.

Proof. In view of Lemma 4.3.1, the spectrum $\sigma\big(\mathcal{G}'_0(\varphi_0)\big)$ is real and discrete. By virtue of (1.3.3), eigenpairs $(\lambda, \eta_\lambda) \in \mathbb{C} \times H^1(\mathbb{R})$ satisfy

$$\eta_\lambda - \eta''_\lambda = \frac{\ell(\ell+1)}{2\lambda} \operatorname{sech}^2\Big(\frac{\ell-1}{2} x\Big) \eta_\lambda.$$

Upon the substitution $t = \tanh\Big(\frac{\ell-1}{2}x\Big)$, the latter equation reads

$$(1-t^2)\eta''_\lambda - 2t\eta'_\lambda + \Big(\nu(1+\nu) - \frac{\delta^2}{(1-t^2)}\Big)\eta_\lambda = 0,$$

where $\delta = \frac{2}{\ell-1}$ and $\nu(1+\nu) = \frac{\delta^2\ell(\ell+1)}{2\lambda}$. The nontrivial $H^1(\mathbb{R})$ solutions are given explicitly by the following family of the associated Legendre functions (see [2])

$$\bar{P}^{-\delta}_{n+\delta}(t) = \tfrac{1}{2}\Big[e^{\frac{\delta\pi i}{2}}P^{-\delta}_{n+\delta}(t-i0)+e^{-\frac{\delta\pi i}{2}}P^{-\delta}_{n+\delta}(t+i0)\Big], \quad t\in[-1,1], \quad \nu = n+\delta, \quad n\geq 0,$$

and (4.3.3) is settled. Further, since $\mathcal{G}_0'(\varphi_0)$ is selfadjoint, each $\lambda_n = \frac{\delta^2\ell(\ell+1)}{2(n+\delta)(n+\delta+1)}$, $n\geq 0$, is simple. The pairity of the associated eigenfunctions

$$\eta_n(x) = \bar{P}^{-\delta}_{n+\delta}\Big(\tanh\Big(\frac{\ell-1}{2}x\Big)\Big), \quad n\geq 0,$$

follows from the properties of $P^{-\delta}_{n+\delta}(t)$, $t\in(-1,1)$, see [2]. □

Lemmas 4.3.1 and 4.3.2 yield the following local result

Lemma 4.3.3. *For small values of* $|\mu_0|, |\mu_1|$, $\mu_0 < 0 < \mu_1 < 1$, *there exists an isolated and analytic in* μ *branch* $\hat{\mathcal{S}}(\mu_0,\mu_1) = \{(\mu,\hat{\varphi}_\mu)\}_{\mu_0\leq\mu\leq\mu_1} \subset \mathbb{R}\times\operatorname{Re}\hat{H}^1(\mathbb{R})$ *of even, positive definite solutions to* (1.3.4). *For each* $\mu\in(\mu_0,\mu_1)$, φ_μ *satisfy the variational problem* (4.1.6).

Proof. (a) We let $\mathcal{N}_\mu(\varphi) = \varphi - \mathcal{G}_\mu(\varphi)$. It is not difficult to verify that the subspace $\operatorname{Re}\hat{H}^1_\mu(\mathbb{R})$ of even $\hat{H}^1_\mu(\mathbb{R})$ functions is invariant under the action of $\hat{\mathcal{G}}_\mu(\hat{\varphi})$ and $\hat{\mathcal{N}}_\mu(\hat{\varphi}) = \hat{\varphi} - \hat{\mathcal{G}}_\mu(\hat{\varphi})$. On the account of Lemma 4.3.2,

$$0\notin\sigma\Big(\hat{\mathcal{N}}_0'(\hat{\varphi}_0)\Big|_{\operatorname{Re}\hat{H}^1(\mathbb{R})}\Big),$$

so that the restriction of $\hat{\mathcal{N}}_0'(\hat{\varphi}_0)$ to $\operatorname{Re}\hat{H}^1(\mathbb{R})$ is invertible. Further, $\hat{\mathcal{N}}_\mu(\hat{\varphi})$ is analytic in $\hat{\varphi}\in H^1(\mathbb{R})$ and in μ, when $\operatorname{Re}\mu < 1$. Hence, the first claim follows from the standard Implicit Function Theorem.

(b) From the compactness of operators $\hat{\mathcal{G}}_\mu'(\hat{\varphi}_\mu) : \operatorname{Re}\hat{H}^1_\mu(\mathbb{R}) \to \operatorname{Re}\hat{H}^1_\mu(\mathbb{R})$, $\mu<1$, it follows that the even, positive definite variational solutions $\hat{\varphi}_\mu$ of Theorem 4.1.7 satisfy $\lim_{\mu\to 0}\|\hat{\varphi}_\mu - \hat{\varphi}_0\|_{\operatorname{Re}\hat{H}^1} = 0$, where φ_0 is given by (1.3.3). Since the solution branch $\hat{\mathcal{S}}(\mu_0,\mu_1) = \{(\mu,\hat{\varphi}_\mu)\}_{\mu_0\leq\mu\leq\mu_1}$ is isolated in $\operatorname{Re}\hat{H}^1(\mathbb{R})$, for sufficiently small values of $|\mu|$ each element of $\hat{\mathcal{S}}(\mu_0,\mu_1)$ necessarily satisfies (4.1.6). □

To conclude this section, we remark that the Fourier symbol κ_μ, $\mu\leq 0$ is radially decreasing. In this situation, the ideas of [41] apply and one can show that Corollary 4.3.3 holds for $\mu\in(-\infty,\mu_1)$ and some small $0<\mu_1<1$. However, as mentioned earlier in Section 1.3, traveling waves with $\mu<0$ are physically irrelevant and hence, we refrain ourselves from going into any further technical details.

4.3.2 Orbital stability

Let $0 \le \mu < 1$ be fixed and let φ_μ be the even, positive definite traveling wave, constructed in Theorem 4.1.7, let

$$u_c(x) = -\frac{C^{\frac{1}{\ell-1}}}{\delta^{\frac{1}{\ell+1}}}\varphi_\mu\Big(\sqrt{\tfrac{C}{\gamma}}x\Big), \quad C = \alpha - c,$$

be the associated traveling wave of (1.1.4) and let $\{\mathcal{T}_t\}_{t\in\mathbb{R}}$ be the group of translations. We define open ε-neighborhood of the orbit $\mathcal{O}_c = \{\mathcal{T}_t u_c\}_{t\in\mathbb{R}}$ by means of the identity

$$\mathcal{U}_{\varepsilon,c} = \{v \in H^1(\mathbb{R}) \mid \inf_{t\in\mathbb{R}} \|v - \mathcal{T}_t u_c\|_{H^1} < \varepsilon\},$$

and say [45, 46] that $\mathcal{O}_c$ is *orbitally stable* if any other solution to (1.1.4) that starts near $\mathcal{O}_{\mathcal{T}}(\varphi_\mu)$ stay close to it for all $t \in \mathbb{R}$.

It is worth to mention that general criteria of orbital stability/instability for a large class of nonlinear wave equations (including KdV-type evolution problems) are well known, see e.g. [45, 46] and references therein. In particular, if we define

$$\mathcal{E}(u) = \int_{\mathbb{R}} \Big[\tfrac{\alpha}{2}u^2 + \tfrac{\gamma}{2}u_x^2 - \tfrac{\beta}{2}u\mathcal{H}[u_x] - \tfrac{\delta}{\ell+1}(-u)^{\ell+1}\Big]dx, \tag{4.3.4a}$$

$$\mathcal{Q}(u) = \tfrac{1}{2}\int_{\mathbb{R}} u^2 dx, \tag{4.3.4b}$$

then the orbital stability is guaranteed under

(i) a suitable restriction on the eigenstructure of the operator $\mathcal{E}''(u_c) - c\mathcal{Q}''(u_c)$ (see Assumption 2 in [45]);

(ii) positivity of the quantity $d''(c)$, where $d(c) = \mathcal{E}(u_c) - c\mathcal{Q}(u_c)$.

In the present case, the first group of conditions is satisfied automatically for small values of μ on the account of Lemmas 4.3.1-4.3.2 and the classical perturbation theory of compact selfadjoint operators, see [55]. The difficulty arises with (ii). Elementary manipulations show that

$$d''(c) = -\sqrt{\gamma}C^{\frac{7-3\ell}{2(\ell-1)}}\langle\varphi_\mu, \psi_\mu\rangle = -\sqrt{\gamma}C^{\frac{7-3\ell}{2(\ell-1)}}\big\langle\big(I - \mathcal{G}'_\mu(\varphi_\mu)\big)\psi_\mu, \psi_\gamma\big\rangle_{H^1_\gamma}, \tag{4.3.5a}$$

$$\hat{\psi}_\mu = \hat{\mathcal{G}}'_\mu(\hat{\varphi}_\mu)[\hat{\psi}_\mu] + \tfrac{1}{\kappa_\mu}\hat{\varphi}_\mu, \tag{4.3.5b}$$

and since the selfadjoint map $I - \mathcal{G}'_\mu(\varphi_\mu)$ is sign indefinite, the positivity of $d''(\gamma)$ is not straightforward. As an alternative, one can use part (C) of [45, Theorem 3], which asserts that condition $d''(c) > 0$ holds if and only if u_c realizes a constrained minimum of $\mathcal{E}(u)$ in $H^1(\mathbb{R})$, subject to $\mathcal{Q}(u) = \mathcal{Q}(u_c)$.

In context of (1.1.4), it is sufficient to verify that φ_μ solves the following variational problem

$$J_\lambda = \sup\big\{\mathcal{G}(\varphi) - \tfrac{1}{2}\|\varphi\|^2_{H^1_\mu} \,\big|\, \|\varphi\|^2 = \lambda,\, \varphi \in H^1(\mathbb{R})\big\}. \tag{4.3.6}$$

The next result show that this is the case, provided (4.3.6) has nontrivial solutions ψ.

Lemma 4.3.4. *Assume that for some $\lambda > 0$, (4.3.6) has a nontrivial solution $\psi \in H^1(\mathbb{R})$. Then φ_μ solves (4.3.6), with $\lambda = \|\varphi_\mu\|^2$. Converse is also true. Function ψ realizes maximum of $\mathcal{G}(\varphi)$ over $\varphi \in H^1_\mu(\mathbb{R})$, with $\|\varphi\|_{H^1_\mu} = \|\psi\|_{H^1_\mu}$.*

Proof. (a) To begin, we make two observations. First, the variational solution φ_μ maximizes $\mathcal{G}(\varphi)$ over the entire space $H^1_\mu(\mathbb{R})$, subject to the constraint $\|\varphi\|^2_{H^1_\mu} = \|\varphi_\mu\|^2_{H^1_\mu} = \lambda_1$. Second, brief inspection of the objective functional in (4.3.6) indicates that under assumptions of Lemma 4.3.4, variational problem (4.3.6) has nontrivial solutions for all $\lambda > 0$. In particular, it is possible to choose $\lambda > 0$ so that $(\ell+1)\mathcal{G}(\psi) = \|\psi\|^2_{H^1_\mu}$.

(b) Let ψ be the solution to (4.3.6) that satisfies $\mathcal{G}(\psi) = \|\psi\|^2_{H^1_\mu}$. Then

$$\begin{aligned}
-\tfrac{\ell-1}{2(\ell+1)}\|\varphi_\mu\|^2_{H^1_\mu} &= \mathcal{G}(\varphi_\mu) - \tfrac{1}{2}\|\varphi_\mu\|^2_{H^1_\mu} \\
&\geq \Big(\tfrac{\|\varphi_\mu\|_{H^1_\mu}}{\|\psi\|_{H^1_\mu}}\Big)^{\ell+1}\mathcal{G}(\psi) - \tfrac{1}{2}\Big(\tfrac{\|\varphi_\mu\|_{H^1_\mu}}{\|\psi\|_{H^1_\mu}}\Big)^2\|\psi\|^2_{H^1_\mu} \\
&= \left[\tfrac{1}{\ell+1}\Big(\tfrac{\|\varphi_\mu\|_{H^1_\mu}}{\|\psi\|_{H^1_\mu}}\Big)^{\ell+1} - \tfrac{1}{2}\Big(\tfrac{\|\varphi_\mu\|_{H^1_\mu}}{\|\psi\|_{H^1_\mu}}\Big)^2\right]\|\psi\|^2_{H^1_\mu},
\end{aligned}$$

and we conclude that

$$\|\varphi_\mu\|^2_{H^1_\mu} \leq \|\psi\|^2_{H^1_\mu}. \tag{4.3.7a}$$

On another hand

$$
\begin{aligned}
-\tfrac{\ell-1}{2(\ell+1)}\|\psi\|^2_{H^1_\mu} &= \mathcal{G}(\psi) - \tfrac{1}{2}\|\psi\|^2_{H^1_\mu} \\
&\geq \left(\tfrac{\|\psi\|}{\|\varphi_\mu\|}\right)^{\ell+1}\mathcal{G}(\varphi_\mu) - \tfrac{1}{2}\left(\tfrac{\|\psi\|}{\|\varphi_\mu\|}\right)^2\|\varphi_\mu\|^2_{H^1_\mu} \\
&= \left[\tfrac{1}{\ell+1}\left(\tfrac{\|\psi\|}{\|\varphi_\mu\|}\right)^{\ell+1} - \tfrac{1}{2}\left(\tfrac{\|\psi\|}{\|\varphi_\mu\|}\right)^2\right]\|\varphi_\mu\|^2_{H^1_\mu},
\end{aligned}
$$

and then

$$
\tfrac{1}{\ell+1}\left(\tfrac{\|\psi\|}{\|\varphi_\mu\|}\right)^{\ell+1} - \tfrac{1}{2}\left(\tfrac{\|\psi\|}{\|\varphi_\mu\|}\right)^2 \leq -\tfrac{\ell-1}{2(\ell+1)}\left(\tfrac{\|\psi\|_{H^1_\mu}}{\|\varphi_\mu\|_{H^1_\mu}}\right)^2. \tag{4.3.7b}
$$

It is easy to verify that in the positive half-line $\mathbb{R}_+$, function $f(t) = \frac{t^{\ell+1}}{\ell+1} - \frac{t^2}{2}$ is bounded from below by $f(1) = -\frac{\ell-1}{2(\ell+1)}$. Hence, inequalities (4.3.7) are consistent if and only if $\|\varphi_\mu\|_{H^1_\mu} = \|\psi\|_{H^1_\mu}$ and $\|\varphi_\mu\| = \|\psi\|$. Hence, $\mathcal{G}(\varphi_\mu) = \mathcal{G}(\psi)$, $\mathcal{G}(\varphi_\mu) - \frac{1}{2}\|\varphi_\mu\|^2_{H^1_\mu} = \mathcal{F}(\psi) - \frac{1}{2}\|\psi\|^2_{H^1_\mu}$ and the conclusion of Lemma 4.3.4 follows. □

Solvability of (4.3.6) for $\ell = 2$, together with a weak form of stability of the resulting traveling waves, is established in [74].[9] This result, combined with Lemmas 4.3.1–4.3.4 allow us to settle the question of orbital stability for small values of μ affirmatively.

Corollary 4.3.5. *For $\ell = 2$ and μ small, the solitary waves constructed in Theorem 4.1.7 are orbitally stable.*

Unfortunately, analysis of (4.3.6) for $\ell > 2$ is not an elementary exercise and fells outside the scope of our work. As illustrated by the inequality below

$$
\mathcal{G}(\varphi) - \tfrac{1}{2}\|\varphi\|^2_{H^1_\mu} \leq \tfrac{\|\kappa_\mu^{-1}\|^{\ell-1}_{L^2}}{\ell+1}\|\varphi\|^2_{L^2}\|\varphi\|^{\ell-1}_{H^1_\mu} - \tfrac{1}{2}\|\varphi\|^2_{H^1_\mu}, \tag{4.3.8}
$$

possible unboudedness of the objective functional in (4.3.6) for $\ell > 3$ is one of the potential difficulties that may arise, thought the cases $\ell = 2$ and $\ell = 3$ are seems to be identical. As a partial substitute for the lack of rigorous analysis, in Chapter 7 we compute the quantity $d''(\mu) = -\langle\varphi_\mu, \psi_\mu\rangle$ (see (4.3.5)) numerically.

[9] The author formulation is different from ours but the result is equivalent to the solvability of (4.3.6).

Chapter 5

Malmquist-Takenaka-Christov basis

5.1 Introduction

Spectral methods are special techniques for expressing solutions of differential equations as finite linear combination of orthogonal basis functions, where we choose coefficients in the sum to closely satisfy the differential equations. Spectral methods also refer to as global methods, where necessary information for the computation at any given point depend on neighboring points and the entire domain. There are three main types of spectral methods, these are spectral-collocation, Galerkin and the Tau method. Among these, the most widely used is the spectral collocation method because in context of nonlinear partial differential equations (PDEs) its implementation is the easiest and it returns suitable results for a sufficiently smooth problems [20, 27, 40, 48, 84, 94]. Pure spectral-Galerkin methods are used mostly in context of linear PDEs with simple boundary conditions and with either constant or polynomial coefficient. In this situation spectral-Galerkin semidiscretization yields linear systems with sparse matrices, provided basis functions are chosen appropriately [83]. For more complicated boundary conditions (BCs), the spectral-Tau method is the most efficient.

As mentioned in the Subsection 1.4.3, finding solution to the Benjamin equation (1.1.4) or (1.3.2) is computationally challenging as the problem is posed on unbounded domain and involves global operator $\mathcal{H}$. In this chapter, we employ the Malmquist-Takenaka-Christov (MTC) system.

The MTC functions $\{\phi_n\}_{n\geq 0}$ are defined as Fourier preimages of the classical Laguerre functions, (see [51, 96, 97] and references therein for an alternative definition and historical remarks). That is, for $k \geq 0$, we have

$$\mathcal{F}[\phi_{2k}](\xi) = \hat{\phi}_{2k}(\xi) = \tfrac{\sqrt{\ell}}{\sqrt{2}}\varphi_k^{0,\ell}(\xi), \quad k \geq 0, \tag{5.1.1a}$$

$$\mathcal{F}[\phi_{2k+1}](\xi) = \hat{\phi}_{2k+1}(\xi) = -i\tfrac{\sqrt{\ell}}{\sqrt{2}}\operatorname{sgn}(\xi)\varphi_k^{0,\ell}(\xi), \quad k \geq 0, \tag{5.1.1b}$$

where

$$\varphi_k^{s,\ell}(\xi) = e^{-\frac{\ell|\xi|}{2}} L_k^{(s)}(\ell|\xi|), \quad k \geq 0, \quad \ell > 0 \tag{5.1.2a}$$

and $L_k^{(s)}(\cdot)$ are the standard generalized Laguerre polynomials see [2]. Note that for $s > -1$, the collection $\{\varphi_k^{s,\ell}\}_{k\geq 0}$ provides a complete orthogonal basis in the weighted space $L^2_{\frac{s}{2}}(\mathbb{R}_+)$. In particular,

$$\langle \varphi_k^{s,\ell}, \varphi_m^{s,\ell}\rangle_{L^2_{\frac{s}{2}}(\mathbb{R}_+)} = \int_{\mathbb{R}_+} \varphi_k^{s,\ell}(\xi)\varphi_m^{s,\ell}(\xi)\xi^s d\xi = \tfrac{1}{\ell^{s+1}}\tfrac{\Gamma(n+s+1)}{\Gamma(n+1)}\delta_{km}, \quad k, m \geq 0. \tag{5.1.2b}$$

Straightforward calculations show that

$$\phi_{2k}(x) = 2\sqrt{\tfrac{\ell}{\pi}}\operatorname{Im}\tfrac{(2x+i\ell)^k}{(2x-i\ell)^{k+1}} = \tfrac{2}{\sqrt{\pi\ell}}\sin\tfrac{(2k+1)\theta}{2}\sin\tfrac{\theta}{2}, \tag{5.1.3a}$$

$$\phi_{2k+1}(x) = 2\sqrt{\tfrac{\ell}{\pi}}\operatorname{Re}\tfrac{(2x+i\ell)^k}{(2x-i\ell)^{k+1}} = \tfrac{2}{\sqrt{\pi\ell}}\cos\tfrac{(2k+1)\theta}{2}\sin\tfrac{\theta}{2}, \tag{5.1.3b}$$

where $x = \frac{\ell}{2}\cot\frac{\theta}{2}$, $\theta \in (0, 2\pi)$ and $\ell > 0$. As evident from (5.1.1) and (5.1.2), the system $\{\phi_n\}_{n\geq 0}$ is a complete orthonormal basis in $L^2(\mathbb{R})$ and

$$\langle \phi_k, \phi_m\rangle_{L^2(\mathbb{R})} = \delta_{km}, \quad k, m \geq 0.$$

In context of spectral methods, functions ϕ_n, $n \geq 0$, were discovered by C. I. Christov [30] in an attempt to obtain a computational basis that behaves well with respect to the product of its members. In particular, the following holds

$$\phi_{2k}\phi_{2m} = \tfrac{1}{2\sqrt{\pi\ell}}\big(\phi_{2(k+m)} - \phi_{2(k+m)+2} + \phi_{2(m-k)} - \phi_{2(m-k)-2}\big), \tag{5.1.4a}$$

$$\phi_{2k+1}\phi_{2m+1} = \tfrac{1}{2\sqrt{\pi\ell}}\big(-\phi_{2(k+m)} + \phi_{2(k+m)+2} + \phi_{2(m-k)} - \phi_{2(m-k)-2}\big), \tag{5.1.4b}$$

$$\phi_{2k}\phi_{2m+1} = \tfrac{1}{2\sqrt{\pi\ell}}\big(\phi_{2(k+m)+1} - \phi_{2(k+m)+3} + \phi_{2(m-k)+1} - \phi_{2(m-k)-1}\big). \tag{5.1.4c}$$

The system $\{\phi_n\}_{n\geq 0}$ has a number of attractive computational features, e.g. in view of (5.1.3), the MTC functions are connected with the trigonometric basis and hence direct and inverse spectral transforms can be computed efficiently via Fast Fourier Transform (FFT) algorithm [19, 30, 51, 96, 97]. Differentiation and computing of the Hilbert transform are also easy [51, 96, 97]

$$\tfrac{d}{dx}\phi_{2k} = \tfrac{k+1}{\ell}\phi_{2k+3} - \tfrac{2k+1}{\ell}\phi_{2k+1} + \tfrac{k}{\ell}\phi_{2k-1}, \tag{5.1.5a}$$

$$\tfrac{d}{dx}\phi_{2k+1} = -\tfrac{k+1}{\ell}\phi_{2k+2} + \tfrac{2k+1}{\ell}\phi_{2k} - \tfrac{k}{\ell}\phi_{2k-2}, \tag{5.1.5b}$$

$$\mathcal{H}[\phi_{2k}] = \phi_{2k+1}, \quad \mathcal{H}[\phi_{2k+1}] = -\phi_{2k}. \tag{5.1.5c}$$

In the context of the Benjamin equation, identity (5.1.5c) is particularly important. This property was explicitly used for the closely related Benjamin-Ono equation in [21, 22].

As far as we are aware, the only rigorous approximation result related to the MTC basis is the geometric convergence rate of the continuous MTC-Fourier series for functions analytic in the exterior of a neighborhood of $\{i, -i\}$ in $\mathbb{C}$ (see [19, 97], the discussion in [51] and references therein). Unfortunately, in context of differential equations (and in particular of (1.1.4)) the result is not very informative. In the sequel, we derive several alternative error bounds directly in $H^s_r(\mathbb{R})$ settings. The estimates form a necessary theoretical background for the convergence analysis of an MTC pseudo-spectral scheme, presented in Chapter 6.

5.2 Projection errors

Let n be a positive integer, $\mathbb{P}_n$ be the finite dimensional linear space spanned by $\{\phi_k(x)\}_{k=0}^n$, $x \in \mathbb{R}$ and $\hat{\mathbb{P}}_n$ be the finite dimensional space spanned by $\left\{e^{-\frac{\ell|\xi|}{2}} x^{\xi}\right\}_{k=0}^n$, $\xi \geq 0$. In connection with $\mathbb{P}_n$ and $\hat{\mathbb{P}}_n$, we define two families of orthogonal projectors $\mathcal{P}_n : L^2(\mathbb{R}) \to \mathbb{P}_n$ and $\hat{\mathcal{P}}^s_n : L^2_{\frac{s}{2}}(\mathbb{R}_+) \to \hat{\mathbb{P}}_n$, $s > -1$, $n > 0$:

$$\mathcal{P}_n[f] = \sum_{k=0}^{n} \phi_k \hat{f}_k, \quad \hat{f}_k = \langle f, \phi_k \rangle_{L^2(\mathbb{R})},$$

$$\hat{\mathcal{P}}^s_n[f] = \sum_{k=0}^{n} \tfrac{\ell^{s+1}\Gamma(k+1)}{\Gamma(k+s+1)} \varphi^{s,\ell}_k \hat{f}^{s,\ell}_k, \quad \hat{f}^{s,\ell}_k = \langle f, \varphi^{s,\ell}_k \rangle_{L^2_{\frac{s}{2}}(\mathbb{R}_+)}.$$

By virtue of (2.4.4) and (5.1.1), for real valued functions we have

$$\|(\mathcal{I} - \mathcal{P}_n)[f]\|^2_{H^s_r(\mathbb{R})} = \left\|(\mathcal{I} - \hat{\mathcal{P}}^0_{\lceil \frac{n}{2} \rceil})[\hat{f}]\right\|^2_{L^2(\mathbb{R}_+)} + \left\|(\mathcal{I} - \hat{\mathcal{P}}^0_{\lceil \frac{n}{2} \rceil})[\hat{f}]\right\|^2_{L^{2,r}_s(\mathbb{R}_+)}. \tag{5.2.1}$$

A comprehensive discussion of the Laguerre-type projectors $\hat{\mathcal{P}}_n^s$, $s > -1$, is found in [8].[1] In particular, for $s_0, s_1 > -\frac{1}{2}$ and $r_0, r_1 \geq 0$, Theorems 1 and 2 of [8] give the bounds,

$$\|(\mathcal{I} - \hat{\mathcal{P}}_n^{s_0})[\hat{f}]\|_{L^{2,r_0}_{\frac{s_1}{2}}(\mathbb{R}_+)} \leq c(\ell n)^{r_0 + \frac{s_0 - s_1 - r_1}{2}} \|\hat{f}\|_{L^{2,r_1}_{\frac{s_0+r_1}{2}}(\mathbb{R}_+)}, \tag{5.2.2a}$$

$$s_1 \leq s_0 + r_0, \quad r_1 \geq s_0 - s_1 + 2r_0 \tag{5.2.2b}$$

and

$$\|(\mathcal{I} - \hat{\mathcal{P}}_n^{s_0})[\hat{f}]\|_{L^{2,r_0}_{\frac{s_1}{2}}(\mathbb{R}_+)} \leq c(\ell n)^{\frac{s_1 - s_0 - r_1}{2}} \|\hat{f}\|_{L^{2,r_1}_{\frac{s_0+r_1}{2}}(\mathbb{R}_+)}, \tag{5.2.3a}$$

$$s_1 \geq s_0 + r_0, \quad r_1 \geq s_1 - s_0. \tag{5.2.3b}$$

Combining (5.2.1), (5.2.2), (5.2.3) and (2.4.1), we have

Lemma 5.2.1. *Assume* $s > -1$ *and* $r_0, r_1 \geq 0$. *Then*

$$\|(\mathcal{I} - \mathcal{P}_n)[f]\|_{H^s_{r_0}(\mathbb{R})} \leq c\left(\tfrac{\ell n}{2}\right)^{r_0 - r_1 - s} \|f\|_{H^{r_1}_{2r_1}(\mathbb{R})}, \quad -\tfrac{1}{2} < s \leq \tfrac{r_0}{2} \leq \tfrac{s+r_1}{2}, \tag{5.2.4a}$$

$$\|(\mathcal{I} - \mathcal{P}_n)[f]\|_{H^s_{r_0}(\mathbb{R})} \leq c\left(\tfrac{\ell n}{2}\right)^{s - r_1} \|f\|_{H^{r_1}_{2r_1}(\mathbb{R})}, \quad 0 \leq \tfrac{r_0}{2} \leq s \leq r_1. \tag{5.2.4b}$$

with a constant $c > 0$ *independent of* n *and/or* f.

Lemma 5.2.1 provides a complete description of the MTC projection errors in $H^s_r(\mathbb{R})$ settings. In particular, it explains a peculiar disparity in the asymptotic of the MTC-Fourier coefficients of closely related holomorphic functions $f(x)$, $g(x) = e^{i\xi_0 x} f(x)$, $\xi_0 \in \mathbb{R}$, see examples and discussion in [51, 96, 97].

Indeed, by virtue of Lemma 5.2.1, $|\hat{f}_k| \to 0$ spectrally (faster than any inverse power of k), provided $\hat{f}(\xi)$ is smooth in $\mathbb{R}_\pm$ and decreases faster than any inverse power of $|\xi|$ at infinity. Since $\hat{g}(\xi) = \hat{f}(\xi - \xi_0)$, the latter condition is violated if $\hat{f}(\xi)$ has an integrable singularity at the origin. This is particularly the case when $f(x)$ is rational, with poles in the upper and lower complex half planes.

[1] The case of $\ell = 1$ is treated in [8] explicitly. However, trivial modification of arguments yield (5.2.2), (5.2.3) for any $\ell > 0$.

5.3 Interpolation errors

Operators $\mathcal{P}_n$ are hard to use in practice as the integrals of the form $\langle f, \phi_n \rangle_{L^2(\mathbb{R})}$ are impossible to compute in most realistic applications. The practical approach consists in replacing the inner products with quadratures. In the no-boundaries setting of the real line $\mathbb{R}$, it is natural to use Gaussian quadratures. The quadrature approximation leads to a rational interpolation process, whose properties are briefly discussed below.

For $n = 2p - 1$, we let

$$\langle f, \phi_k \rangle \approx \bar{f}_k = \tfrac{\pi}{4\ell p} \sum_{m=0}^{2p-1} (\ell^2 + 4x_m^2) \phi_k(x_m) f(x_m), \tag{5.3.1a}$$

$$x_m = \tfrac{\ell}{2} \cot\left(\tfrac{2m+1}{4p}\pi\right), \quad 0 \le m \le 2p - 1. \tag{5.3.1b}$$

The discrete inner product (5.3.1) is exact, provided $f \in \mathbb{P}_n$. In practice, we use the discrete spectral coefficients $\bar{f}_k$ and approximate f by

$$\mathcal{I}_n[f] = \sum_{k=0}^{n} \bar{f}_k \phi_k. \tag{5.3.2}$$

Directly from (5.1.3), (5.3.1) and (5.3.2), it follows that

$$\mathcal{I}_n[f](x_m) = f(x_m), \quad 0 \le m \le 2p - 1, \tag{5.3.3}$$

i.e. $\mathcal{I}_n[\cdot]$ is an interpolation operator.

Computational properties of $\mathcal{I}_n$ are very similar to those of rational Gauss-Chebyshev interpolants, discussed in [87] and the generalized Gauss-Laguerre interpolants of [8]. In particular, we have

Lemma 5.3.1. *Assume $f \in \mathbb{P}_n$, $s > -\frac{1}{2}$ and $r \ge 0$. Then*

$$\|f\|_{H_r^s(\mathbb{R})} \le c\left(\tfrac{n}{2\ell}\right)^{r+|s|-\min\{0,2s\}} \|f\|_{L^2(\mathbb{R})}, \tag{5.3.4}$$

with a constant $c > 0$ independent of n and/or f.

Proof. Since $f \in \mathbb{P}_n$, $n = 2p - 1$, we have $\hat{f} \in \hat{\mathbb{P}}_p$, $\xi \in \mathbb{R}_+$. In [8, Lemma 6], it is shown that for such functions

$$\|\hat{f}\|_{L_s^{2,r}(\mathbb{R}_+)} \le c(\ell p)^{r-\min\{0,2s\}} \|\hat{f}\|_{L_s^2(\mathbb{R}_+)},$$
$$\|\hat{f}\|_{L_s^2(\mathbb{R}_+)} \le c(\ell p)^{|s|} \|\hat{f}\|_{L^2(\mathbb{R}_+)}.$$

In view of (2.4.4), these inequalities imply (5.3.4). $\square$

Lemma 5.3.2. *Assume $s > \frac{1}{2}$. Then*

$$\|\mathcal{I}_n\|_{H^s(\mathbb{R})\to L^2(\mathbb{R})} \leq c\left(\tfrac{\ell n}{2}\right), \tag{5.3.5}$$

with $c > 0$ independent of n.

Proof. Since the discrete inner product (5.3.1) is exact for $f \in \mathbb{P}_n$, we have

$$\|\mathcal{I}_n[f]\|^2_{L^2(\mathbb{R})} = \tfrac{\pi}{4\ell p}\sum_{m=0}^{2p-1}(\ell^2 + 4x_m^2)f^2(x_m).$$

In view of the classical Sobolev embedding [3], we have $\|f\|_{L^\infty(\mathbb{R})} \leq c_s\|f\|_{H^s(\mathbb{R})}$, provided that $s > \frac{1}{2}$. Consequently,

$$\begin{aligned}\|\mathcal{I}_n[f]\|^2_{L^2(\mathbb{R})} &\leq \tfrac{c_s\pi}{4\ell p}\Big[\sum_{m=0}^{2p-1}(\ell^2 + 4x_m^2)\Big]\|f\|^2_{H^s(\mathbb{R})} \\ &= c\Big[\sum_{m=0}^{2p-1}\frac{\ell^2}{\sin^2\left(\frac{(2m+1)\pi}{4p}\right)}\Big]\|f\|^2_{H^s(\mathbb{R})} = cS_n^2\|f\|^2_{H^s(\mathbb{R})}.\end{aligned}$$

In [87, Lemma 4] it is shown $S_n^2 = 2(2\ell p)^2$. Hence, (5.3.5) is settled. $\square$

The interpolation error bounds are obtained combining Lemmas 5.2.1, 5.3.1, 5.3.2.

Corollary 5.3.3. *Let $s > -\frac{1}{2}$, $r_0 \geq 0$, $\varepsilon > 0$ and $r_1 \geq r_0 + |s|$. Then,*

$$\|(\mathcal{I} - \mathcal{I}_n)[f]\|_{H^s_{r_0}(\mathbb{R})} \leq c\left(\tfrac{\ell n}{2}\right)^{\frac{3}{2}+\varepsilon+r_0+|s|-\max\{0,2s\}-r_1}\|f\|_{H^{r_1}_{2r_1}(\mathbb{R})}, \tag{5.3.6}$$

with a constant $c > 0$ independent of n and/or f.

Proof. In view of Lemmas 5.3.1 and 5.3.2, we have

$$\begin{aligned}\|(\mathcal{I} - \mathcal{I}_n)[f]\|_{H^s_{r_0}(\mathbb{R})} &= \|(\mathcal{I} - \mathcal{P}_n) + (\mathcal{I} - \mathcal{P}_n)[f]\|_{H^s_{r_0}(\mathbb{R})} \\ &\leq \|(\mathcal{I} - \mathcal{P}_n)[f]\|_{H^s_{r_0}(\mathbb{R})} + \|(\mathcal{I} - \mathcal{P}_n)\|_{H^s_{r_0}(\mathbb{R})} \\ &\leq \|(\mathcal{I} - \mathcal{P}_n)[f]\|_{H^s_{r_0}(\mathbb{R})} + \|\mathcal{I}_n(\mathcal{I} - \mathcal{P}_n)[f]\|_{H^s_{r_0}(\mathbb{R})} \\ &\leq \|(\mathcal{I} - \mathcal{P}_n)[f]\|_{H^s_{r_0}(\mathbb{R})} + c\left(\tfrac{\ell n}{2}\right)^{r_0+|s|-\min\{0,2s\}}\|\mathcal{I}_n(\mathcal{I} - \mathcal{P}_n)[f]\|_{L^2(\mathbb{R})} \\ &\leq \|(\mathcal{I} - \mathcal{P}_n)[f]\|_{H^s_{r_0}(\mathbb{R})} + c\left(\tfrac{\ell n}{2}\right)^{1+r_0+|s|-\min\{0,2s\}}\|(\mathcal{I} - \mathcal{P}_n)[f]\|_{H^{\frac{1}{2}+\varepsilon}(\mathbb{R})}.\end{aligned}$$

Hence, (5.3.6) is the direct consequence of Lemma 5.2.1. $\square$

Chapter 6

An MTC-type collocation scheme: The non-stationary Benjamin equation

In this Chapter, we shift our focus to the numerical analysis of nonstationary Benjamin equation posed on the real line. The main purpose here is to develop an MTC pseudo-spectral scheme and study its stability, consistency and convergence.

6.1 An MTC collocation scheme

To obtain a spatial semi-discretization, for a given $n = 2p - 1$, $p \in \mathbb{N}$, we approximate the automorphism $\mathcal{J}$ by the finite dimensional skew symmetric map $\mathcal{J}_n = -\mathcal{P}_n \partial_x \mathcal{P}_n : \mathbb{P}_n \to \mathbb{P}_n$ and replace (1.1.4) with

$$\bar{u}_t = \mathcal{J}_n \nabla_{\bar{u}} \mathcal{G}_n(\bar{u}), \quad \bar{u}(0) = \mathcal{I}_n[u_0], \tag{6.1.1a}$$

$$\mathcal{G}_n(\bar{u}) = \tfrac{1}{2} \int_{\mathbb{R}} \Big(\alpha |\bar{u}|^2 - \beta \bar{u} \mathcal{H}[\mathcal{J}_n \bar{u}] + \gamma |\mathcal{J}_n \bar{u}|^2 + \tfrac{2\delta}{3} \bar{u} \mathcal{I}_n[\bar{u}^2] \Big) dx, \tag{6.1.1b}$$

where $\bar{u} \in \mathbb{P}_n$. Note that if $n = 2p - 1$, the operator $\mathcal{J}_n$ is non-degenerate. This follows from identities (5.1.5a)-(5.1.5b) and the fact that the eigenvalues of the differentiation matrix $-\mathcal{J}_n$ are given explicitly by $\pm i \frac{\xi_n}{\ell}$, $1 \leq k \leq p$, where ξ_k are roots of the classical Laguerre polynomial $L_p(x)$ (see the proof of Lemma 6.1.1 below and [96]). As a consequence, the finite dimensional semi-discrete system (6.1.1) of ODEs is again Hamiltonian.

By construction, the semi-discrete vector field $\nabla\mathcal{G}_n(\bar{u})$ is smooth and hence the initial value problem (6.1.1a) is locally well-posed. Unfortunately (and in contrast to the classical solution to (1.1.4), where apart from the Hamiltonian its $L^2(\mathbb{R})$ norm is preserved), the only conserved semi-discrete quantity $\mathcal{G}_n(\bar{u})$ is sign indefinite. As a consequence, we have insufficient amount of a priori information to establish uniform global bounds on the growth rate of the numerical solution $\bar{u}$. To alleviate the problem, we proceed indirectly. Instead of estimating $\bar{u}$, we compare it to the reference solution $\tilde{u} = \mathcal{P}_n[u] \in \mathbb{P}_n$, where u (the exact classical solution to (1.1.4)) is assumed to be globally defined and regular. Our approach is based on the elementary observation that in the Cauchy problem $y' = y^s$, $y(0) = y_0$, $s > 1$, the blow up time is inverse proportional to the size of the input data. This observation is widely used in numerical analysis and, in particular, in the context of spectral methods, see e.g. [67].

6.1.1 Auxiliary estimates

In our analysis, we make use of three technical estimates. The first one is a discrete analogue of the classical Gagliardo-Nirenberg inequality, the second is used to estimate discrete power nonlinearities and the last one is an extension of the classical Gronwall's Lemma.

Lemma 6.1.1. *Let $u \in \mathbb{P}_n$, $n = 2p-1$. Then*

$$\|u_x\|_{L^2(\mathbb{R})} \le c\left(\tfrac{\ell n}{2}\right)^{\frac{1}{2}} \|\mathcal{J}_n u\|_{L^2(\mathbb{R})}, \tag{6.1.2a}$$

$$\|u\|_{L^\infty(\mathbb{R})} \le c\left(\tfrac{\ell n}{2}\right)^{\frac{1}{4}} \|u\|^{\frac{1}{2}}_{L^2(\mathbb{R})} \|\mathcal{J}_n u\|^{\frac{1}{2}}_{L^2(\mathbb{R})}, \tag{6.1.2b}$$

where $c > 0$ is an absolute constant.

Proof. Identities (5.1.5) imply

$$\|u_x\|^2_{L^2(\mathbb{R})} = \|\mathcal{J}_n u\|^2_{L^2(\mathbb{R})} + \tfrac{p^2}{\ell^2}\left[|\hat{u}_{2(p-1)}|^2 + |\hat{u}_{2p-1}|^2\right],$$

where $\hat{u}_k = \langle u, \phi_k\rangle_{L^2(\mathbb{R})}$, $0 \le k \le n$. Our main task is to bound the sum $|\hat{u}_{2(p-1)}|^2 + |\hat{u}_{2p-1}|^2$.

Let $\hat{u}_e = (\hat{u}_0, \ldots, \hat{u}_{2(p-1)})^T$ and $\hat{u}_o = (\hat{u}_1, \ldots, \hat{u}_{2p-1})^T$ be $\mathbb{R}^p$ vectors that contain the even and the odd MTC-Fourier coefficients of $u \in \mathbb{P}_n$. Then, by virtue of (5.1.5), the even and the odd MTC-Fourier coefficients of $-\mathcal{J}_n u$ are given by $\frac{1}{\ell} D\hat{u}_o$ and $-\frac{1}{\ell} D\hat{u}_e$, respectively, where $D = (d_{ij}) \in \mathbb{R}^{p\times p}$ is the symmetric three-diagonal matrix, whose entries are given by $d_{ii} = -2i-1$, $d_{i,i+1} = d_{i+1,i} = i$, $0 \le i \le p$. Using the three-term recurrence formula for the classical Laguerre polynomials $L_n(x)$ (see [2, 96]), we find that

$$D = Q\Lambda Q^T, \quad \Lambda = \operatorname{diag}(\xi_0, \ldots, \xi_{p-1}), \quad Q_{ij} = \tfrac{\sqrt{\xi_j} L_i(\xi_j)}{p|L_{p-1}(\xi_j)|},$$

where ξ_i, $0 \le i \le p-1$, are the (strictly positive) roots of $L_p(x)$ and that matrix $Q \in \mathbb{R}^p$ is orthogonal.

Let e_p be the standard unit vector in $\mathbb{R}^p$ and $|\cdot|$, $\cdot$ denote the usual Euclidean norm and the inner product in $\mathbb{R}^p$. With this notation, we obtain

$$\begin{aligned} |\hat{u}_{2(p-1)}|^2 + |\hat{u}_{2p-1}|^2 &= |e_p \cdot \hat{u}_e|^2 + |e_p \cdot \hat{u}_o|^2 \\ &\le \ell^2 |\Lambda^{-1} Q^T e_p|^2 \|\mathcal{J}_n u\|^2_{L^2(\mathbb{R})}. \end{aligned}$$

Note that

$$c\xi_i \le \tfrac{(i+1)^2}{p} \le C\xi_i, \quad 0 \le i \le p-1,$$

for some absolute constants $c, C > 0$ (see e.g. [70, formula (2.3.50), p. 141]). Hence,

$$|\Lambda^{-1} Q^T e_p|^2 = \tfrac{1}{p^2} \sum_{i=0}^{p-1} \tfrac{1}{\xi_i} \le \tfrac{c}{p}$$

and (6.1.2a) follows. Bound (6.1.2b) follows from (6.1.2a) and the standard Gagliardo-Nirenberg inequality. $\square$

Lemma 6.1.2. *Assume $v \in \mathbb{P}_n$, $m > 0$, $2 \le k \le 5$ and $1 \le r \le 2$. Then*

$$\left|\langle \mathcal{I}_n[\tilde{u}^m], \mathcal{I}_n[v^k] \rangle_{L^2(\mathbb{R})}\right| \le c\left(\tfrac{\ell n}{2\varepsilon}\right)^{\frac{k-2}{6-k}} \|v\|_{L^2(\mathbb{R})}^{\frac{2(k+2)}{6-k}} + \varepsilon(k-2)\|\mathcal{J}_n v\|^2_{L^2(\mathbb{R})}, \tag{6.1.3a}$$

$$\left|\langle \mathcal{I}_n[\tilde{u}^m], \mathcal{I}_n[v^r \mathcal{J}_n v] \rangle_{L^2(\mathbb{R})}\right| \le c\varepsilon^{-\frac{2}{3-r}} \left(\tfrac{\ell n}{2\varepsilon}\right)^{\frac{r-1}{3-r}} \|v\|_{L^2(\mathbb{R})}^{\frac{2(r+1)}{3-r}} + \varepsilon\|\mathcal{J}_n v\|^2_{L^2(\mathbb{R})}, \tag{6.1.3b}$$

where $\varepsilon > 0$ is arbitrary and $c > 0$ depends on k and $\|\tilde{u}\|_{L^\infty(\mathbb{R})}$ only.

Proof. Let $w_i = \frac{\pi}{4\ell p}(\ell^2 + 4x_i^2)$, $0 \le i \le 2p-1$, where x_i is defined in (5.3.1b). Since quadrature (5.3.1a) is exact in $\mathbb{P}_n$ and in view of Lemma 6.1.1, we have

$$
\begin{aligned}
\big|\langle \mathcal{I}_n[\tilde{u}^m], \mathcal{I}_n[v^k]\rangle_{L^2(\mathbb{R})}\big| &\le \sum_{i=0}^{n} w_i |\tilde{u}(x_i)|^m |v(x_i)|^k \\
&\le \|\tilde{u}\|^m_{L^\infty(\mathbb{R})} \|v\|^{k-2}_{L^\infty(\mathbb{R})} \|v\|^2_{L^2(\mathbb{R})} \\
&\le c\|\tilde{u}\|^m_{L^\infty(\mathbb{R})} \|v_x\|^{\frac{k-2}{2}}_{L^\infty(\mathbb{R})} \|v\|^{\frac{k+2}{2}}_{L^2(\mathbb{R})} \\
&\le c\big(\tfrac{\ell n}{2}\big)^{\frac{k-2}{4}} \|\tilde{u}\|^m_{L^\infty(\mathbb{R})} \|\mathcal{J}_n v\|^{\frac{k-2}{2}}_{L^2(\mathbb{R})} \|v\|^{\frac{k+2}{2}}_{L^2(\mathbb{R})}.
\end{aligned}
$$

Hence, Young's inequality, with exponents $\frac{4}{k-2}$ and $\frac{4}{6-k}$, yields (6.1.3a). The proof of (6.1.3b) is identical. □

Lemma 6.1.3. *Let $u \in C[0,T]$ be non-negative. Assume that*

$$
u(t) \le f(t) + a\int_0^t u(s)ds + b\int_0^t (t-s)u(s)ds, \quad t \in [0,T], \tag{6.1.4a}
$$

where $a, b > 0$ and $f(t)$ is integrable and non-negative. Then

$$
u(t) \le f(0) + e^{\frac{a+\sqrt{a^2+4b}}{2}t} \int_0^t \big[1 + \tfrac{b}{2}(t-s)^2\big] f(s)ds, \quad t \in [0,T]. \tag{6.1.4b}
$$

Proof. Let $U(t) = e^{\lambda t}\int_0^t (t-s)u(s)ds$, where $\lambda = -\frac{a+\sqrt{a^2+4b}}{2}$ is the negative root of the quadratic equation $\lambda^2 + a\lambda - b = 0$. Then (6.1.4a) is equivalent to

$$
U''(t) \le (a + 2\lambda)U'(t) + f(t), \quad U(0) = U'(0) = 0, \quad t \in [0,T].
$$

Since $a + 2\lambda < 0$, integrating twice, we obtain

$$
e^{-\lambda t}U(t) = \int_0^t (t-s)u(s)ds \le e^{-\lambda t}\int_0^t (t-s)f(s)ds.
$$

Upon substitution into (6.1.4a), we have

$$
u(t) \le f(t) + e^{-\lambda t} b\int_0^t (t-s)f(s)ds + a\int_0^t u(s)ds,
$$

which, combined with the standard Gronwall's inequality, gives (6.1.4b). □

6.1.2 Stability

Now, we turn to the study of the numerical error $e = \tilde{u} - \bar{u}$. Applying operator $\mathcal{P}_n$ to both sides of (1.1.4), subtracting (6.1.1) and passing to the quadrature (as we did in Lemma 6.1.2), we infer

$$e_t = \mathcal{J}_n \nabla_e \big[\mathcal{G}_n(-e) + \mathcal{E}_n(e,t) + \mathcal{D}_n(e,t)\big], \quad e(0) = e_0. \tag{6.1.5a}$$

$$\mathcal{E}_n(e,t) = 2\delta \big\langle \tilde{u}(t), \mathcal{I}_n[e^2]\big\rangle_{L^2(\mathbb{R})}, \tag{6.1.5b}$$

$$\begin{aligned} \mathcal{D}_n(e,t) = \big\langle e, \alpha(\mathcal{I} - \mathcal{P}_n)\big[u\big](t) + \beta \mathcal{H}\big[(\partial_x + \mathcal{J}_n)u\big](t), \\ - \gamma(\partial_{xx} - \mathcal{J}_n^2)[u](t) + \delta\big(u^2(t) - \mathcal{I}_n\big[\tilde{u}^2\big](t)\big)\big\rangle_{L^2(\mathbb{R})}, \end{aligned} \tag{6.1.5c}$$

where ∇_e denotes the gradient with respect to variable e. Equation (6.1.5a) is not Hamiltonian. Nevertheless, differentiating and using the skew-symmetry of the discrete automorphism $\mathcal{J}_n$, we obtain

$$\tfrac{d}{dt}\big[\mathcal{G}_n(-e) + \mathcal{E}_n(e,t) + \mathcal{D}_n(e,t)\big] = \partial_t[\mathcal{E}_n(e,t) + \mathcal{D}_n(e,t)],$$

which, after integration in time, gives

$$\begin{aligned} |\mathcal{G}_n(-e)| \le{}& |\mathcal{G}_n(-e_0)| + |\mathcal{E}_n(e_0,0)| + |\mathcal{D}_n(e_0,0)| \\ &+ |\mathcal{E}_n(e,t)| + |\mathcal{D}_n(e,t)| \\ &+ \int_0^t \big|\partial_t[\mathcal{E}_n(e(s),s) + \mathcal{D}_n(e(s),s)]\big| ds. \end{aligned} \tag{6.1.6}$$

We use (6.1.6) to control the $L^2(\mathbb{R})$ norm of $\mathcal{J}_n e$.

Lemma 6.1.4. *Let $\gamma > 0$, $0 < \varepsilon < \frac{\gamma}{4}$ and*

$$\|e\|_{L^2(\mathbb{R})} \le \big(\tfrac{\ell n}{2\varepsilon}\big)^{-\frac{1}{4}}, \tag{6.1.7a}$$

in some interval $[0,T]$. Then for each $t \in [0,T]$, we have

$$\begin{aligned} \|\mathcal{J}_n e\|^2_{L^2(\mathbb{R})} \le c\Big(&|\mathcal{G}_n(-e_0)| + |\mathcal{E}_n(e_0,0)| + |\mathcal{D}_n(e_0,0)| \\ &+ \|e\|^2_{L^2_n(\mathbb{R})} + \int_0^t \|e\|^2_{L^2_n(\mathbb{R})} ds \\ &+ \|\nabla_e \mathcal{D}_n(e,t)\|^2_{L^2(\mathbb{R})} + \int_0^t \|\nabla_e \partial_t \mathcal{D}_n(e,t)\|^2_{L^2(\mathbb{R})} ds\Big), \end{aligned} \tag{6.1.7b}$$

where $c > 0$ depends on α, β, γ, δ, $\|\tilde{u}\|_{L^\infty(\mathbb{R})}$ and $\|\tilde{u}_t\|_{L^\infty(\mathbb{R})}$ only.

Proof. We bound each term in (6.1.6) separately. First, we use the Cauchy-Schwarz inequality, unitarity of $\mathcal{H}$ and Lemma 6.1.2 to obtain

$$\mathcal{G}_n(-e) \ge \big(\tfrac{\gamma}{2} - \varepsilon\big)\|\mathcal{J}_n e\|^2_{L^2(\mathbb{R})} - c_1\|e\|^2_{L^2(\mathbb{R})} - c_1\big(\tfrac{\ell n}{2\varepsilon}\big)^{\frac{1}{3}}\|e\|^{\frac{10}{3}},$$

with $c_1 > 0$ that depends on the parameters α, β, γ and δ only. Using Lemma 6.1.2, we have also

$$
\begin{aligned}
|\mathcal{E}_n(e,t)| &\le c_2 \|e\|^2_{L^2(\mathbb{R})},\\
|\partial_t \mathcal{E}_n(e,t)| &\le c_3 \|e\|^2_{L^2(\mathbb{R})},
\end{aligned}
$$

where $c_2 > 0$ depends on δ and $\|\tilde{u}\|_{L^\infty(\mathbb{R})}$ and $c_3 > 0$ depends on δ and $\|\tilde{u}_t\|_{L^\infty(\mathbb{R})}$ only. The quantity $\mathcal{D}_n(e,t)$ is linear in e and by the Minkowski inequality,

$$
\begin{aligned}
2|\mathcal{D}_n(e,t)| &\le \|e\|^2_{L^2(\mathbb{R})} + \|\nabla_e \mathcal{D}_n(e,t)\|^2_{L^2(\mathbb{R})},\\
2|\partial_t \mathcal{D}_n(e,t)| &\le \|e\|^2_{L^2(\mathbb{R})} + \|\nabla_e \partial_t \mathcal{D}_n(e,t)\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

Hence (6.1.7b) is the direct consequence of the above bounds, (6.1.6) and assumption (6.1.7a). $\square$

We remark that the assumption $\gamma > 0$ appearing in Lemma 6.1.4 is not restrictive, for if $\gamma < 0$ one can use $-\mathcal{G}_n(\cdot)$ instead of $\mathcal{G}_n(\cdot)$.

Theorem 6.1.5 (Stability)**.** *Assume that for some fixed $C > 0$, $T > 0$ and $\epsilon > 0$,*

$$
\max\{\|\tilde{u}\|_{L^\infty([0,T]\times\mathbb{R})}, \|\tilde{u}_t\|_{L^\infty([0,T]\times\mathbb{R})}\} < C, \tag{6.1.8a}
$$

$$
\begin{aligned}
&\|e_0\|_{L^2(\mathbb{R})} + |\mathcal{G}_n(-e_0)|^{\frac{1}{2}} + |\mathcal{E}_n(e_0,0)|^{\frac{1}{2}} + |\mathcal{D}_n(e_0,0)|^{\frac{1}{2}}\\
&\qquad + \|\nabla_e \mathcal{D}_n(e,t)\|_{L^2([0,T]\times\mathbb{R})} + \|\partial_t \nabla_e \mathcal{D}_n(e,t)\|_{L^2([0,T]\times\mathbb{R})}\\
&\qquad = \mathcal{O}\Big(\big(\tfrac{\ell n}{2}\big)^{-\frac{1+\epsilon}{4}}\Big),
\end{aligned} \tag{6.1.8b}
$$

uniformly for large values of $n = 2p - 1 > 0$. Then there exists $c > 0$, that depends on C, T and parameters α, β, γ and δ of (1.1.4) *only, such that*

$$
\begin{aligned}
\|e\|_{C([0,T],L^2(\mathbb{R}))} \le c\Big(&\|e_0\|_{L^2(\mathbb{R})} + |\mathcal{G}_n(-e_0)|^{\frac{1}{2}} + |\mathcal{E}_n(e_0,0)|^{\frac{1}{2}} + |\mathcal{D}_n(e_0,0)|^{\frac{1}{2}}\\
&+ \|\nabla_e \mathcal{D}_n(e,t)\|_{L^2([0,T]\times\mathbb{R})} + \|\nabla_e \partial_t \mathcal{D}_n(e,t)\|_{L^2([0,T]\times\mathbb{R})}\Big),
\end{aligned} \tag{6.1.8c}
$$

for all sufficiently large values of $n > 0$.

Proof. (a) We multiply both sides of (6.1.4a) by e, integrate with respect to x over $\mathbb{R}$ and take into account the skew-symmetry of the automorphism $\mathcal{J}_n$. This gives

$$
\begin{aligned}
\tfrac{1}{2}\tfrac{d}{dt}\|e\|^2_{L^2(\mathbb{R})} &= -\langle \mathcal{J}_n[e], \mathcal{I}_n[e^2]\rangle_{L^2(\mathbb{R})}\\
&\qquad - \langle \mathcal{J}_n[e], \nabla_e \mathcal{E}_n(e,t)\rangle_{L^2(\mathbb{R})} - \langle \mathcal{J}_n[e], \nabla_e \mathcal{D}_n(e,t)\rangle_{L^2(\mathbb{R})}.
\end{aligned}
$$

Lemma 6.1.2 and the Cauchy-Schwarz inequality give the bounds

$$
\begin{aligned}
|\langle \mathcal{J}_n[e], \mathcal{I}_n[e^2]\rangle_{L^2(\mathbb{R})}| &\le \varepsilon \|\mathcal{J}_n e\|^2_{L^2(\mathbb{R})} + c_1\big(\tfrac{\ell n}{2\varepsilon}\big)\varepsilon^{-2}\|e\|^6_{L^2(\mathbb{R})}, \\
|\langle \mathcal{J}_n[e], \nabla_e \mathcal{E}_n(e,t)\rangle_{L^2(\mathbb{R})}| &\le \|\mathcal{J}_n e\|^2_{L^2(\mathbb{R})} + c_2\|e\|^2_{L^2(\mathbb{R})}, \\
|\langle \mathcal{J}_n[e], \nabla_e \mathcal{D}_n(e,t)\rangle_{L^2(\mathbb{R})}| &\le \|\mathcal{J}_n e\|^2_{L^2(\mathbb{R})} + c_3\|\nabla_e \mathcal{D}_n(e,t)\|^2_{L^2(\mathbb{R})},
\end{aligned}
$$

where $c_1, c_3 > 0$ are absolute constants and $c_2 > 0$ depends on $\|\tilde{u}\|_{L^\infty([0,T]\times\mathbb{R})}$ only.

(b) In view of (6.1.8b) and local continuity of $\|e\|^2_{L^2(\mathbb{R})}$, we see that (6.1.7a) holds locally in some nonempty closed interval $[0,\tau_0]$, $0 < \tau_0 \le T$. Therefore, combining our estimates from part (a) of the proof and using (6.1.7a) with $\varepsilon = \mathcal{O}(1)$, $0 < \varepsilon < \frac{\gamma}{2}$, we conclude that the following holds

$$
\begin{aligned}
\tfrac{1}{2}\tfrac{d}{dt}\|e\|^2_{L^2(\mathbb{R})} \le{}& \big(c_1 + c_2\big)\|e\|^2_{L^2(\mathbb{R})} \\
&+ (2+\varepsilon)\|\mathcal{J}_n e\|_{L^2(\mathbb{R})} + c_3\|\nabla_e \mathcal{D}_n(e,t)\|^2_{L^2(\mathbb{R})},
\end{aligned}
$$

uniformly in $[0,\tau_0]$. Integrating the last formula with respect to time and combining the result with Lemma 6.1.4, we obtain

$$
\begin{aligned}
\|e\|^2_{L^2(\mathbb{R})} \le{}& \|e_0\|^2_{L^2(\mathbb{R})} + c_4 t\big(|\mathcal{G}_n(-e_0)| + |\mathcal{E}_n(e_0,0)| + |\mathcal{D}_n(e_0,0)|\big) \\
&+ c_4\int_0^t (1+t-s)\|e\|^2_{L^2(\mathbb{R})}ds + c_4\int_0^t \|\nabla_e \mathcal{D}_n(e,t)\|^2_{L^2(\mathbb{R})}ds \\
&+ c_4\int_0^t (t-s)\|\nabla_e \partial_t \mathcal{D}_n(e,t)\|^2_{L^2(\mathbb{R})}ds,
\end{aligned}
\tag{6.1.9}
$$

where $t \in [0,\tau_0]$ and $c > 0$ depends on $C > 0$ and parameters α, β, γ and δ of the model (1.1.4) only.

(c) Inequality (6.1.9) falls in the scope of Lemma 6.1.3, hence, definitely (6.1.8c) holds in the small interval $[0,\tau_0]$. Furthermore, from the same Lemma 6.1.3, it follows that the constant $c > 0$ in (6.1.8c) behaves like $c'(1+\tau_0^{\frac{3}{2}})e^{c''\tau_0}$, where $c', c'' > 0$ are independent of $n > 0$ and τ_0. The observation implies that $\|e(\tau_0)\|^2_{L^2(\mathbb{R})} = \mathcal{O}\Big(\big(\frac{\ell n}{2}\big)^{-\frac{1+\epsilon}{4}}\Big)$, i.e. for $n > 0$ sufficiently large, (6.1.7a) is satisfied at the endpoint τ_0. In view of the last fact and by continuity of $\|e\|^2_{L^2(\mathbb{R})}$, we conclude that (6.1.9) can be extended to a larger interval $[0,\tau_1]$, $0 < \tau_0 < \tau_1 \le T$, without increasing the size of the constant $c_4 > 0$.

The assertion of Theorem 6.1.5 follows from the standard continuation argument. Repeating the continuation step described above inductively, we construct an ascending sequence $0 < \tau_0 < \tau_1 < \cdots \leq T$, such that (6.1.9) (with $c_4 > 0$ being fixed) holds in each $[0, \tau_i]$, $i \geq 0$. Assuming $\tau^* = \sup \tau_i < T$, we arrive at the contradiction; for if $\tau^* < T$, the continuation step extends (6.1.9) beyond the maximal interval $[0, \tau^*]$. $\square$

6.1.3 Consistency and convergence

In what follows, we use the results of Chapters 2 and 5 to demonstrate that assumptions (6.1.8a), (6.1.8b) are satisfied, provided the exact solution u is sufficiently regular. We begin with (6.1.8a).

Lemma 6.1.6. *Assume $u, u_t \in L^\infty([0,T], H^s_{2s}(\mathbb{R}))$, $s > 1$. Then*

$$\|\tilde{u}\|_{L^\infty([0,T]\times\mathbb{R})} \leq c\big[1 + \left(\tfrac{\ell n}{2}\right)^{1-s}\big] \|u\|_{L^\infty([0,T],H^s_{2s}(\mathbb{R}))}, \tag{6.1.10a}$$

$$\|\tilde{u}_t\|_{L^\infty([0,T]\times\mathbb{R})} \leq c\big[1 + \left(\tfrac{\ell n}{2}\right)^{1-s}\big] \|u_t\|_{L^\infty([0,T],H^s_{2s}(\mathbb{R}))}, \tag{6.1.10b}$$

where $c > 0$ is an absolute constant.

Proof. By the standard Gagliardo-Nirenberg inequality,

$$\|\tilde{u}\|^2_{L^\infty(\mathbb{R})} \leq c\|\tilde{u}\|_{L^2(\mathbb{R})}\|\tilde{u}_x\|_{L^2(\mathbb{R})}.$$

Since the MTC functions form a complete orthogonal basis in $L^2(\mathbb{R})$, we have $\|\tilde{u}\|_{L^2(\mathbb{R})} = \|\mathcal{P}_n[u]\|_{L^2(\mathbb{R})} \leq \|u\|_{L^2(\mathbb{R})}$. To bound the norm of $\tilde{u}_x$, we write

$$\|\tilde{u}_x\|_{L^2(\mathbb{R})} \leq \|u_x\|_{L^2(\mathbb{R})} + \|(\mathcal{I} - \mathcal{P}_n)[u]\|_{H^1(\mathbb{R})},$$

and apply Lemma 5.2.1. This gives (6.1.10a). Bound (6.1.10b) follows along the same lines. $\square$

Next, we show that each term in (6.1.8b) is small.

Lemma 6.1.7. *Assume $u, u_t \in L^\infty([0,T], H^s_{2s}(\mathbb{R}))$ and $s \geq 2$ and $\epsilon > 0$. Then*

$$\|e_0\|_{L^2(\mathbb{R})} \leq c\Big(\frac{\ell n}{2}\Big)^{\frac{3}{2}+\epsilon-s}\|u_0\|_{H^s_{2s}(\mathbb{R})}, \tag{6.1.11a}$$

$$|\mathcal{G}_n(-e_0)|^{\frac{1}{2}} \leq c\Big(\frac{\ell n}{2}\Big)^{\frac{5}{2}+\epsilon-s}\Big(\|u_0\|_{H^s_{2s}(\mathbb{R})} + \|u_0\|^{\frac{5}{3}}_{H^s_{2s}(\mathbb{R})}\Big), \tag{6.1.11b}$$

$$|\mathcal{E}_n(e_0,0)|^{\frac{1}{2}} \leq c\Big(\frac{\ell n}{2}\Big)^{\frac{3}{2}+\epsilon-s}\|u_0\|^{\frac{3}{2}}_{H^s_{2s}(\mathbb{R})}, \tag{6.1.11c}$$

$$|\mathcal{D}_n(e_0,0)|^{\frac{1}{2}} \leq c\Big(\frac{\ell n}{2}\Big)^{\frac{7+2\epsilon}{4}-s}\Big(\|u_0\|_{H^s_{2s}(\mathbb{R})} + \|u_0\|^{\frac{3}{2}}_{H^s_{2s}(\mathbb{R})}\Big), \tag{6.1.11d}$$

$$\|\nabla_e \mathcal{D}_n(e,t)\|_{L^2([0,T]\times\mathbb{R})} \leq c\Big(\frac{\ell n}{2}\Big)^{2-s} \Big(\|u\|_{L^2([0,T],H^s_{2s}(\mathbb{R}))} + \|u\|^2_{L^4([0,T],H^s_{2s}(\mathbb{R}))}\Big), \tag{6.1.11e}$$

$$\|\partial_t\nabla_e \mathcal{D}_n(e,t)\|_{L^2([0,T]\times\mathbb{R})} \leq c\Big(\frac{\ell n}{2}\Big)^{2-s} \Big(\|u_t\|_{L^2([0,T],H^s_{2s}(\mathbb{R}))} + \|u_t\|^2_{L^4([0,T],H^s_{2s}(\mathbb{R}))}\Big). \tag{6.1.11f}$$

In each inequality the generic constant $c > 0$ is independent of u, u_0, $T > 0$ and $n > 0$.

Proof. (a) Since $e_0 = \mathcal{I}_n\big[(\mathcal{I} - \mathcal{P}_n)[u_0]\big]$, as in the proof of Corollary 5.3.3, we obtain (6.1.11a).

(b) We employ the Cauchy-Schwarz inequality and Lemma 6.1.2 (with $\varepsilon = 1$) to obtain

$$\begin{aligned}
2|\mathcal{G}_n(-e_0)| &\leq c\|e_0\|^2_{L^2(\mathbb{R})} + c\|\mathcal{J}_n e_0\|^2_{L^2(\mathbb{R})} + c\Big(\frac{\ell n}{2}\Big)^{\frac{1}{3}}\|e_0\|^{\frac{10}{3}} \\
&\leq c\|(\mathcal{I} - \mathcal{I}_n)[u_0]\|^2_{H^1(\mathbb{R})} + c\|(\mathcal{I} - \mathcal{P}_n)[u_0]\|^2_{H^1(\mathbb{R})} + c\Big(\frac{\ell n}{2}\Big)^{\frac{1}{3}}\|e_0\|^{\frac{10}{3}},
\end{aligned}$$

with $c > 0$, depending on parameters α, β, γ and δ only. Hence, (6.1.11a) and Corollary 5.3.3 imply (6.1.11b).

(c) From the definition of $\mathcal{E}_n(e,t)$ and Lemma 6.1.2, we have

$$|\mathcal{E}_n(e_0,0)| \leq 2|\delta|\|\tilde{u}_0\|_{L^\infty(\mathbb{R})}\|e_0\|^2_{L^2(\mathbb{R})}$$

and (6.1.11c) is a consequence of Lemma 6.1.6 and (6.1.11a).

(d) The functional $\mathcal{D}_n(e_0,0)$ is linear in e_0. Consequently,

$$|\mathcal{D}_n(e_0,0)| \leq \|e_0\|_{L^2(\mathbb{R})}\|\nabla_e\mathcal{D}_n(e_0,0)\|_{L^2(\mathbb{R})}$$

and (6.1.11d) follows directly from (6.1.11a) and the proof of (6.1.11e) below.

(e) From (6.1.5c) we have

$$\begin{aligned}\nabla_e \mathcal{D}_n(e,t) &= \alpha(\mathcal{I}-\mathcal{P}_n)[u] + \beta\mathcal{H}\big[(\partial_x+\mathcal{J}_n)[u]\big]\\ &\quad - \gamma(\partial_x^2-\mathcal{J}_n^2)[u] + \delta\big(u^2-\mathcal{I}_n\big[\mathcal{P}_n[u]^2\big]\big)\\ &= \alpha E_1 + \beta E_2 + \gamma E_3 + \delta E_4.\end{aligned}$$

We bound each term separately. First of all, by Lemma 5.2.1,

$$\|E_1\|_{L^2([0,T]\times\mathbb{R})} \le c\big(\tfrac{\ell n}{2}\big)^{-s}\|u\|_{L^2([0,T],H^s_{2s}(\mathbb{R}))}.$$

Further, using the definition of $\mathcal{J}_n$, we obtain

$$\begin{aligned}\|E_2\|^2_{L^2(\mathbb{R})} &= \|\partial_x(\mathcal{I}-\mathcal{P}_n)[u]\|^2_{L^2(\mathbb{R})} + \tfrac{n+1}{2\ell}\big(|\hat{u}_{n-1}|^2+|\hat{u}_n|^2\big)\\ &\le \|(\mathcal{I}-\mathcal{P}_n)[u]\|^2_{H^1(\mathbb{R})} + c\big(\tfrac{\ell n}{2}\big)\|(\mathcal{I}-\mathcal{P}_{n-2})[u]\|^2_{L^2(\mathbb{R})},\end{aligned}$$

so that by Lemma 5.2.1,

$$\|E_2\|_{L^2([0,T]\times\mathbb{R})} \le c\big(\tfrac{\ell n}{2}\big)^{1-s}\|u\|_{L^2([0,T],H^s_{2s}(\mathbb{R}))}.$$

Similar calculations give also

$$\|E_3\|_{L^2(\mathbb{R})} \le \|(\mathcal{I}-\mathcal{P}_n)[u]\|_{H^2(\mathbb{R})} + \big(\tfrac{\ell n}{2}\big)\|(\mathcal{I}-\mathcal{P}_{n-2})[u]\|_{L^2(\mathbb{R})}$$

and

$$\|E_3\|_{L^2([0,T]\times\mathbb{R})} \le c\big(\tfrac{\ell n}{2}\big)^{2-s}\|u\|_{L^2([0,T],H^s_{2s}(\mathbb{R}))}.$$

Finally,

$$E_4 = (\mathcal{I}-\mathcal{I}_n)[u^2] + \mathcal{I}_n[u^2-\mathcal{P}_n[u]^2] = E_{41}+E_{42}.$$

First, we employ Lemma 2.4.2 and Corollary 5.3.3 to obtain

$$\|E_{41}\|_{L^2([0,T]\times\mathbb{R})} \le c\big(\tfrac{\ell n}{2}\big)^{\frac{3}{2}+\epsilon-s}\|u\|^2_{L^4([0,T],H^s_{2s}(\mathbb{R}))}.$$

Second, from Lemmas 2.4.2 and 5.3.1, we infer

$$\begin{aligned}\|E_{42}\|_{L^2(\mathbb{R})} &\le c(\tfrac{\ell n}{2})\|(I-\mathcal{P}_n)[u](I+\mathcal{P}_n)[u]\|_{H^{\frac{1}{2}+\epsilon}(\mathbb{R})}\\ &\le c(\tfrac{\ell n}{2})\Big(2\|u\|_{H^{\frac{1}{2}+\epsilon}([R])} + \|(I-\mathcal{P}_n)[u]\|_{H^{\frac{1}{2}+\epsilon}(\mathbb{R})}\Big)\|(I-\mathcal{P}_n)[u]\|_{H^{\frac{1}{2}+\epsilon}(\mathbb{R})}.\end{aligned}$$

The last bound and Lemma 5.2.1 yield

$$\|E_{42}\|_{L^2([0,T]\times\mathbb{R})} \le c\left(\tfrac{\ell n}{2}\right)^{\frac{3}{2}+\epsilon-s}\|u\|^2_{L^4([0,T],H^s_{2s}(\mathbb{R}))}.$$

Combining all our estimates together, we arrive at (6.1.11e). To obtain (6.1.11f), replace u with u_t. □

Combining Theorem 6.1.5 and Lemmas 6.1.6, 6.1.7, we obtain

Corollary 6.1.8 (Convergence)**.** *Assume* $u, u_t \in L^\infty([0,T], H^s_{2s}(\mathbb{R}))$*,* $s > \frac{11}{4}$ *and* $\epsilon > 0$*. Then the numerical solution* $\bar{u}$ *satisfies*

$$\|u-\bar{u}\|_{L^\infty([0,T],L^2(\mathbb{R}))} \le c\left(\tfrac{\ell n}{2}\right)^{\frac{5}{2}+\epsilon-s} \left(\|u_0\|_{H^s_{2s}(\mathbb{R})} + \|u_0\|^{\frac{5}{3}}_{H^s_{2s}(\mathbb{R})} + \|u\|_{H^1([0,T],H^s_{2s}(\mathbb{R}))}\right), \tag{6.1.12}$$

uniformly for large values of $n > 0$*, with* $c > 0$ *that depends on the terminal time* $T > 0$*, parameters* α*,* β*,* γ *and* δ *of the model* (6.1.1) *and on the regularity of the exact solution* u *only.*

Proof. Note that

$$\|u-\bar{u}\|_{L^2(\mathbb{R})} \le \|e\|_{L^2(\mathbb{R})} + \|(\mathcal{I}-\mathcal{P}_n)[u]\|_{L^2(\mathbb{R})}.$$

Hence (6.1.12) follows from Lemma 5.2.1 and the fact that under the assumption $s > \frac{11}{4}$, the numerical error $e = \tilde{u} - \bar{u}$ fells in the scope of Theorem 6.1.5. □

To conclude this section, we remark that if $u, u_t \in L^\infty([0,T], H^s_{2s}(\mathbb{R}))$, for any $s > \frac{11}{4}$, then, according to Corollary 6.1.8, the convergence rate is spectral, i.e. the semi-discretization error $\|u-\bar{u}\|_{L^\infty([0,T],L^2(\mathbb{R}))}$ decreases faster than any inverse power of $n > 0$.

6.1.4 Implementation and simulations

The semi-discretization (6.1.1) leads to a finite dimensional system of ODEs whose solution is not known explicitly and itself requires an appropriate numerical treatment. Below, we discuss briefly a suitable time-stepping algorithm and then switch to simulations.

6.1.4.1 Implementation

The semi-discretization (6.1.1) can be written in the form

$$Y' = (\alpha J + \beta H J^2 - \gamma J^3)Y + JF(Y) = DY + JF(Y), \quad Y(0) = Y_0, \qquad (6.1.13)$$

where, the neutral symbol $Y \in \mathbb{R}^{n+1}$, $n = 2p - 1$, represents either the vector

$$Y = (\bar{u}_0, \ldots, \bar{u}_{2p-1})^T,$$

of the discrete MTC-Fourier coefficients or the vector of physical values

$$Y = (\bar{u}(x_0), \ldots, \bar{u}(x_n))^T,$$

computed at the nodal points x_m, $0 \leq m \leq n$. The square skew-symmetric matrices $J, H \in \mathbb{R}^{(n+1)\times(n+1)}$ provide suitable realizations of the discrete operators $\mathcal{J}_n$ and $\mathcal{P}_n\mathcal{H}\mathcal{P}_n$, respectively. The concrete form of J and H depends on the particular representation of Y. For instance, in the MTC-Fourier (frequency) space J and H have simple two-by-two block structure with nonzero three-diagonal, respectively diagonal, blocks in the reverse block diagonal (see identities (5.1.5)), while both matrices are dense in the physical space. The nonlinearity $F(Y)$, representing $\mathcal{I}_n[\bar{u}^2]$, is given explicitly by

$$F(Y) = \delta(\bar{u}^2(x_0), \ldots, \bar{u}^2(x_n))^2,$$

in the physical space.

Time-stepping The spectrum of operator J, computed explicitly in the proof of Lemma 6.1.1 (see also [96]), indicates that (6.1.13) is stiff, and hence fully explicit time-stepping schemes cannot be unconditionally stable. Furthermore, since the nonlinearity $F(Y)$ is multiplied by J, the semi-implicit splitting-type schemes that separate stiff and nonstiff components of the vector field (see e.g. discussion in [87], in connection with the nonlinear Schrödinger equation) are also not plausible here. From the prospective of numerical stability, we are forced to use fully implicit A-stable algorithms.

In our simulations, we make use of the implicit 4-stage 8-order Gauss-type Runge-Kutta method (IRK8 in the sequel)

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array}, \quad b, c \in \mathbb{R}^4, \quad A \in \mathbb{R}^{4\times 4},$$

of J. Kuntzmann and J. Butcher, (for the concrete values of the coefficients A, b and c see [47, Table 7.5, p. 209]). A single IRK8 time step of length τ, applied to (6.1.13), reads

$$Z_1 = \big(I - \tau[A \otimes D]\big)^{-1}\Big(\mathbb{1} \otimes Y_0 + \tau[A \otimes J]G(Z_1)\Big), \tag{6.1.14a}$$

$$Z_1 = (Y_{1,1}^T, \ldots, Y_{1,4}^T)^T, \quad G(Z_1) = (F(Y_{1,1})^T, \ldots, F(Y_{1,4})^T)^T, \tag{6.1.14b}$$

$$Y_1 = Y_0 + \tau \sum_{i=1}^{4} b_i F(Y_{1,i}), \tag{6.1.14c}$$

where $\mathbb{1} = (1, 1, 1, 1)^T$ and $\otimes$ is the standard Kronecker product. We observe that the spectrum of A contains two pairs of complex conjugate eigenvalues with nontrivial real parts and therefore, from Lemma 4.1.1, we deduce that the spectrum of matrix $\big(I - \tau[A \otimes D]\big)^{-1}[A \otimes J]$ is uniformly bounded with respect to the space discretization parameter $n > 0$. Further, the theory of Section 6.1.1 indicates that for smooth exact solutions of the Benjamin equation (1.1.4), the semi-discrete nonlinearity $F(Y)$ is bounded uniformly in $n > 0$ along the trajectories of (6.1.13). Hence, the fully discrete scheme (6.1.14) is unconditionally stable. Moreover, it follows that for a fixed Y_0, moderately small values of time step τ and independently of $n > 0$, the nonlinear map, defined by the right-hand side of (6.1.14a), is a contraction. As a consequence, the nonlinear equation (6.1.14a) can be solved efficiently via basic fixed point iterations. The observation is important from practical point of view as Newton-type iterations are prohibitively expensive for large values of $n > 0$. We note also that the exact flow φ_t, generated by (6.1.13), is symplectic. The IRK8 scheme is known to be symmetric and symplectic [47], hence, the discrete flow of (6.1.14) preserves this property automatically.

Computational complexity A single fixed point iteration, applied to (6.1.14a), involves: solving linear systems with matrix $I - \tau[A \otimes D]$; the matrix-vector multiplication with matrices D and J and finally; computing the nonlinearity $F(Y)$. In view of the special structure of J and H, in the Fourier-Christov space each matrix-vector operation requires $\mathcal{O}(n)$ flops, while computing of $F(Y)$ involves the use of the discrete direct and inverse MTC-Fourier transforms (see formulas (5.3.1a) and (5.3.2), (5.3.3), respectively). Because of (5.1.3) and (5.3.1b), both operations can be accomplished in $\mathcal{O}(n \log_2 n)$ flops via the direct and inverse discrete Fast Fourier Transforms [19, 30, 51, 96, 97] and the cost of a single iteration is $\mathcal{O}(n \log_2 n)$. As noted earlier, for any given tolerance ε the total number of such iterations is finite and depends on the time step τ only. Hence, the overall complexity of a single time step of (6.1.14) is $\mathcal{O}(n \log_2 n)$.

6.1.4.2 Simulations

Below, we provide several simulations illustrating the accuracy of (6.1.1) in several computational scenarios.

6.1.4.3 Slowly decreasing solutions

We begin with the generic situation where, due to the nature of the Fourier symbol in the linear part of (1.1.4), solutions decay at most algebraically.

Example 1 First, we simulate (1.1.4) in time interval $[0, 2]$, with $\alpha = \beta = \gamma = \delta = 1$. Since for these values of the model parameters, analytic formulas for solutions are not available, we augment (1.1.4) with a source term $f(x, t)$. The latter is chosen so that the exact solution reads

$$u(x,t) = \sum_{k=1}^{3} \tfrac{r_k}{a_k^2 + (x - x_{k,0} - c_k t)^2},$$

$$r_1 = 2, \quad r_2 = 1, \quad r_3 = 3, \quad a_1 = 1, \quad a_2 = 1, \quad a_3 = 2,$$

$$c_1 = 1, \quad c_2 = -2, \quad c_3 = 0, \quad x_{1,0} = -1, \quad x_{2,0} = 1, \quad x_{3,0} = 0.$$

Note that $u(x, t)$ is smooth (in fact $u \in H^s(\mathbb{R})$, $s \in \mathbb{R}$), but has a polynomial decay rate at infinity ($u = \mathcal{O}(|x|^{-2})$ at $|x| \to \infty$). In view of this fact, accurate approximation of such functions with the aid of standard trigonometric basis requires huge number of spatial grid points. Nevertheless, straightforward calculations show

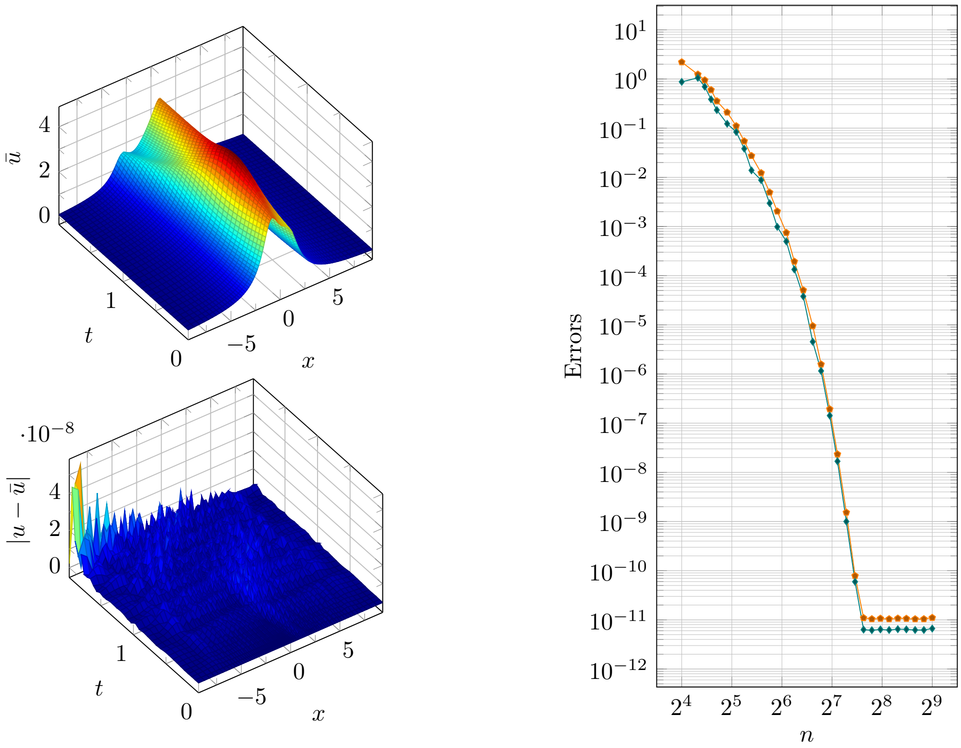

Figure 6.1: The left diagrams (top to bottom): the numerical solution of the Benjamin equation (1.1.4) $\bar{u}$ and the pointwise error $|u-\bar{u}|$, with $n = 2^7 - 1$. The right diagram: $\|u-\bar{u}\|_{L^\infty([0,T];L^2(\mathbb{R}))}$ (orange, pentagon), $\|u-\bar{u}\|_{L^\infty([0,T]\times\mathbb{R})}$ (teal, diamond).

that the quantity $\mathcal{FP}_+[u]$ is smooth and decreases exponentially in the positive half line $\mathbb{R}_+$. Hence, u falls in the scope of the theory presented in Chapter 5 and earlier part of Chapter 6 and we expect rapid error decay already for moderate values of $n > 0$.

The numerical results, for $2^4 - 1 \leq n \leq 2^9 - 1$, $\ell = 2^3$ and $\tau = 2 \cdot 10^{-2}$, are plotted in the right diagram of Fig. 6.1. Both $\|\cdot\|_{L^\infty([0,T],L^2(\mathbb{R}))}$ and $\|\cdot\|_{L^\infty([0,T]\times\mathbb{R})}$ errors decrease spectrally (note that both curves are concave) as n increases. For $n > 2^7$, the numerical errors settle near 10^{-11}. This is a consequence of the inexact time-stepping procedure employed in our calculations. Simulations, not reported here, indicate that for $n > 2^7$ the error can be further reduced by choosing smaller time integration steps.

To illustrate the quality of the approximation, we plot the numerical solution $\bar{u}$ and the associated pointwise error $|u-\bar{u}|$, obtained with $n = 2^7 - 1$, in the two left

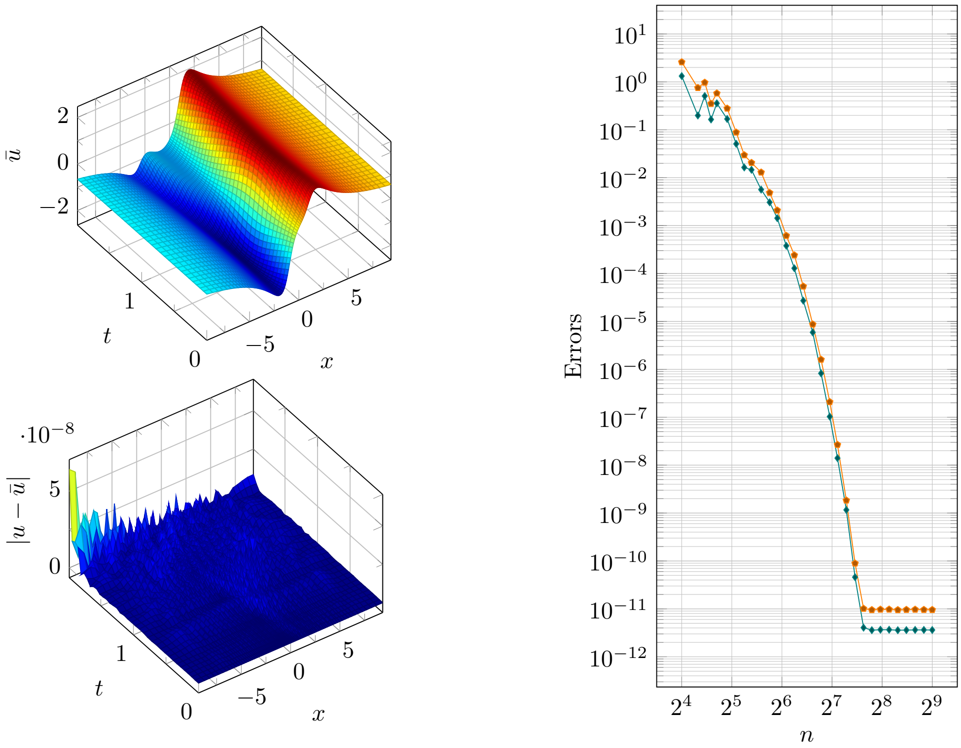

Figure 6.2: The left diagrams (top to bottom): the numerical solution of the Benjamin equation (1.1.4) $\bar{u}$ and the pointwise error $|u-\bar{u}|$, with $n = 2^7 - 1$. The right diagram: $\|u-\bar{u}\|_{L^\infty([0,T];L^2(\mathbb{R}))}$ (orange, pentagon), $\|u-\bar{u}\|_{L^\infty([0,T]\times\mathbb{R})}$ (teal, diamond).

diagrams of Fig. 6.1. It is clearly visible that the pointwise error does not exceed the magnitude of $5 \cdot 10^{-8}$ uniformly in the computational domain.

Example 2 In our second simulation, we keep the numerical parameters of Example 1 unchanged, but make use of another source term which gives the following exact solution

$$u(x,t) = \sum_{k=1}^{3} \frac{r_k(x-x_{k,0}-c_k t)}{a_k^2+(x-x_{k,0}-c_k t)^2}.$$

In this settings $u(x,t) = \mathcal{O}(|x|^{-1})$, as $|x| \to \infty$. Nevertheless, the truncated Fourier image $\mathcal{FP}_+[u]$ has exactly the same qualitative features as in Example 1 and the resulting convergence rate is spectral (see the left diagram in Fig 6.2). In the particular case of $n = 2^7 - 1$, the numerical solution and the pointwise error are shown in the top- and bottom-left diagrams of Fig. 6.2, respectively. The observed behavior is very much alike to the one, reported in Example 1.

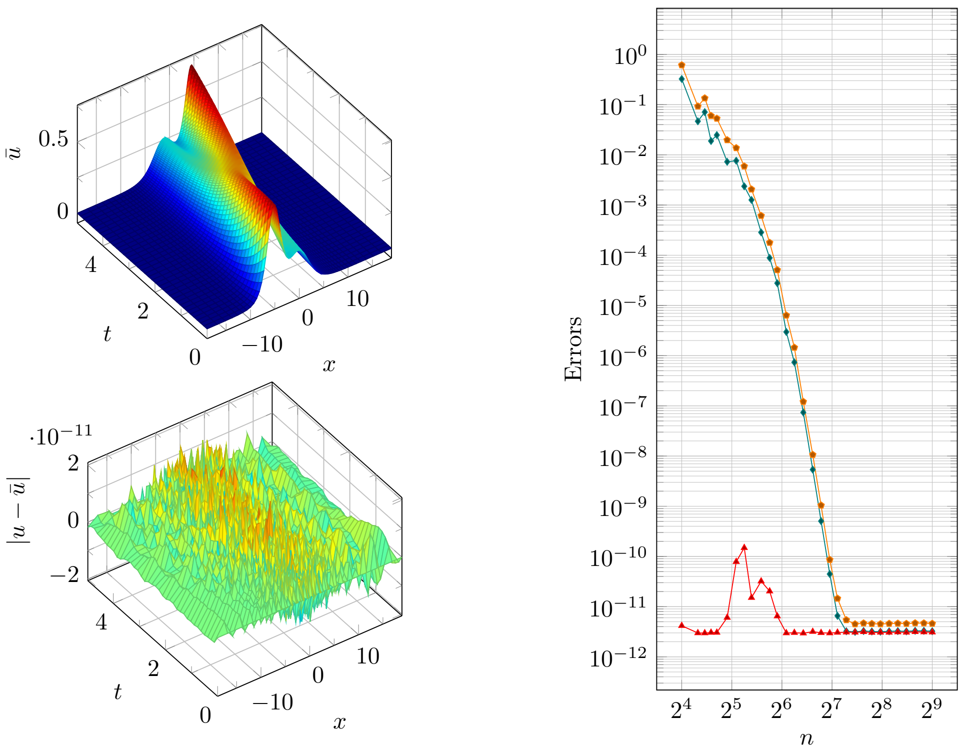

Figure 6.3: The left diagrams (top to bottom): the numerical solution of the Benjamin equation (1.1.4) $\bar{u}$ and the pointwise error $|u-\bar{u}|$, with $n = 2^7 - 1$. The right diagram: $\|u-\bar{u}\|_{L^\infty([0,T];L^2(\mathbb{R}))}$ (orange, pentagon), $\|u-\bar{u}\|_{L^\infty([0,T]\times\mathbb{R})}$ (teal, diamond).

6.1.4.4 The Korteweg-de Vries scenario

The Benjamin equation (1.1.4) contains two special case $\gamma = 0$ and $\beta = 0$, which are of independent interest. The first one corresponds to the Benjamin-Ono equation, and is not considered here. In the second case, we have the classical Korteweg-de Vries (KdV) equation. The latter is known to be completely integrable and possesses a large number of special solutions. For instance, when

$$\alpha = \beta = 0, \quad \gamma = -1, \quad \delta = -3,$$

the inverse scattering transform yields the so called N-solitons (see e.g. [1])

$$u(x,t) = -2\partial_{xx} \ln \det(I + A(x,t)), \tag{6.1.15a}$$

$$A(x,t) = \left(b_i e^{8\lambda_i^3 t} \frac{e^{-\lambda_i x - \lambda_j x}}{\lambda_i + \lambda_j}, 1 \le i, j \le N\right), \tag{6.1.15b}$$

$$\lambda_i = \tfrac{1}{2}\sqrt{v_i} \quad b_i = 2\lambda_i e^{2\phi_i \lambda_i}, \quad 1 \le i \le N, \tag{6.1.15c}$$

which describe evolution of N traveling waves, whose velocities and the phases are controlled by v_i and ϕ_i, respectively. Directly from (6.1.15), it follows that N-solitons are smooth and decay exponentially to zero as $|x|$ increases. Hence, such solutions fall in the scope of the theory developed in Chapters 5 and 6.

Example 3 To illustrate the above statement, in (6.1.15) we let

$$v_1 = \tfrac{3}{2}, \quad v_2 = \tfrac{1}{2}, \quad \phi_1 = -3, \quad \phi_2 = 0,$$

choose u_0 according to (6.1.15), take $\ell = 2^3$, $\tau = 10^{-2}$ and integrate (6.1.1) numerically in time interval $[0, 5]$. The results of simulations (see in Fig. 6.3) are qualitatively similar to those obtained in Examples 1 and 2. In particular, the plots of $\|\cdot\|_{L^\infty([0,T],L^2(\mathbb{R}))}$ and $\|\cdot\|_{L^\infty([0,T]\times\mathbb{R})}$ errors indicate that the convergence rate is spectral. Note however that in the bottom-left diagram of Fig. 6.3 the pointwise error is smaller than in the two previous Examples. This is connected with the exponential decay of the 2-soliton at infinity (its accurate spatial resolution requires fewer grid points than in Examples 1 and 2).

By construction, the scheme (6.1.1) is conservative and the semi-discrete Hamiltonian $\mathcal{G}_n(\bar{u})$ remains constant along the exact trajectories of (6.1.1). In order to test the conservation properties of the fully discrete scheme, in the right diagram of Fig. 6.3, we added the plot of the quantity $\max_{t\in[0,T]} |\mathcal{G}_n(\bar{u}_0) - \mathcal{G}_n(\bar{u})|$, measuring the largest deviation in the Hamiltonian. We observe that the deviation remains several orders of magnitude smaller than either of the $\|\cdot\|_{L^\infty([0,T],L^2(\mathbb{R}))}$ and $\|\cdot\|_{L^\infty([0,T]\times\mathbb{R})}$ errors, until the latter settle near 10^{-11}.

Example 4 We repeat calculations of Example 3, but this time with

$$v_1 = 1, \quad v_2 = 1, \quad v_3 = \tfrac{1}{2}, \quad \phi_1 = -4, \quad \phi_2 = -2, \quad \phi_3 = 0.$$

This scenario describes an elastic collision of three traveling waves, see the top-left diagram in Fig. 6.4. The exact 3-soliton has exactly the same qualitative features as the 2-soliton of Example 3, with the exception that now the exponential decay rate is slightly slower. This manifests in larger numerical errors, see the bottom-left diagram in Fig. 6.4.

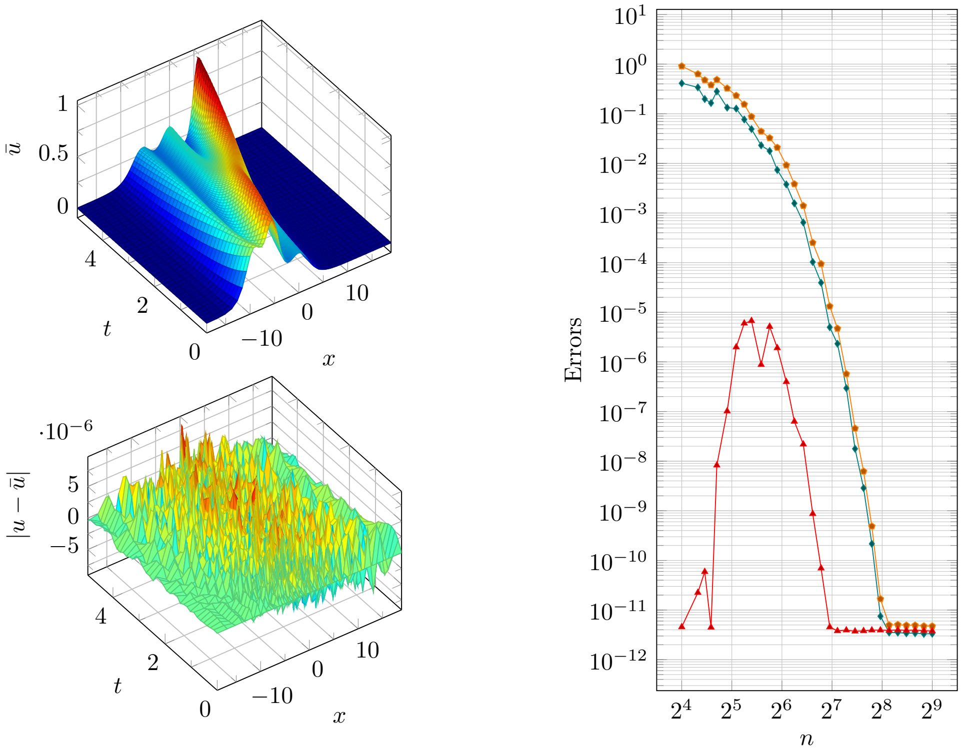

Figure 6.4: The left diagrams (top to bottom): the numerical solution of the Benjamin equation (1.1.4) $\bar{u}$ and the pointwise error $|u-\bar{u}|$, with $n = 2^7 - 1$. The right diagram: $\|u-\bar{u}\|_{L^\infty([0,T];L^2(\mathbb{R}))}$ (orange, pentagon), $\|u-\bar{u}\|_{L^\infty([0,T]\times\mathbb{R})}$ (teal, diamond).

6.1.4.5 Traveling waves

In our last two simulations, we model an interaction of traveling waves. In the context of the Benjamin equation (1.1.4), the traveling wave solutions are given by $u(x,t) = v_\sigma(x-ct)$, where v_σ satisfies

$$v_\sigma - 2\sigma\sqrt{\tfrac{\gamma}{\alpha-c}}\mathcal{H}[\partial_x v_\sigma] - \tfrac{\gamma}{\alpha-c}\partial_{xx}v_\sigma + \tfrac{\delta}{\alpha-c}v_\sigma^2 = 0, \quad x \in \mathbb{R}, \tag{6.1.16a}$$

$$\sigma = \frac{\beta}{2\sqrt{\gamma(\alpha-c)}}, \quad \gamma, \delta, \nu > 0, \quad c < \alpha. \tag{6.1.16b}$$

For a rigorous treatment of (6.1.16), see [5, 10, 26, 35, 36, 53, 74] and references therein.

The exact solutions to (6.1.16), apart from the trivial case of $\alpha = 0$, are not available. In our simulations, the *even* traveling waves are constructed numerically, see Chapter 7 also. We employ a simple continuation scheme, which works as follows:

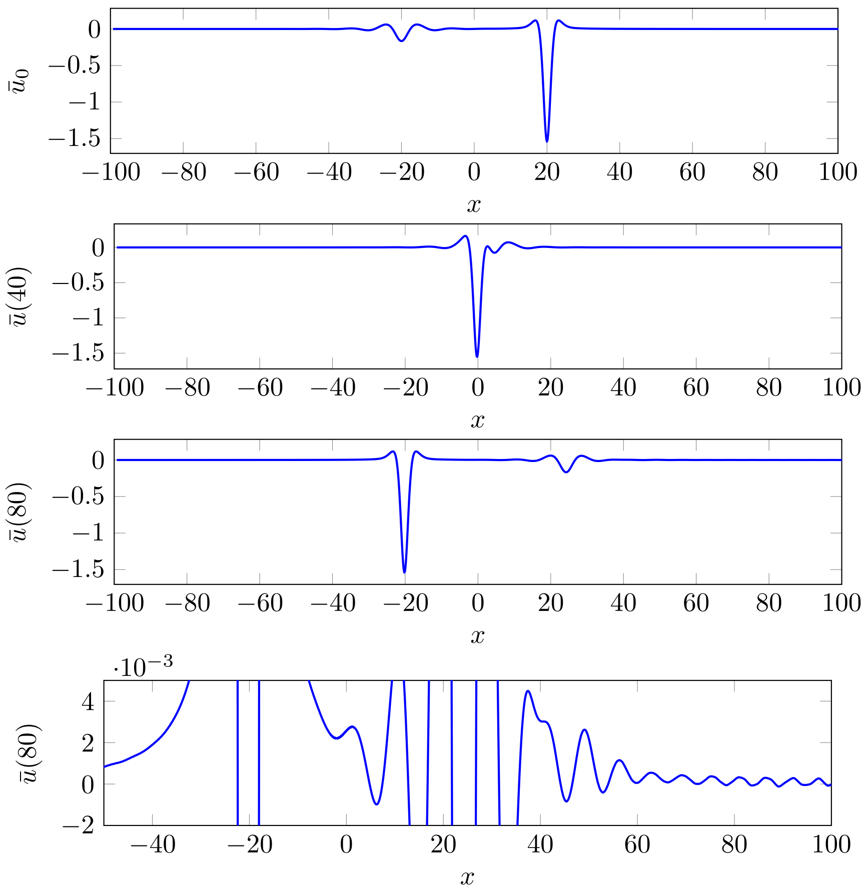

Figure 6.5: Inelastic collision of two traveling waves, Example 5.

for a given α, β, γ, δ and c, that satisfy (6.1.16b) and $0 \le \sigma < 1$; (i) we let

$$\bar{v}_0 = \mathcal{I}_n[v_0], \quad v_0(x) = -\tfrac{3(\alpha-c)}{2\delta}\,\mathrm{sech}\Big(\sqrt{\tfrac{\alpha-c}{4\gamma}}x\Big)^2;$$

(ii) introduce a continuation grid $0 < \sigma_1 < \ldots < \sigma_N = \sigma$ and (iii) apply simplified Newton's iterations to the sequence of the discrete nonlinear problems

$$\bar{v}_{\sigma_j} + 2\sigma_j\sqrt{\tfrac{\gamma}{\alpha-c}}\mathcal{H}[\mathcal{J}_n\bar{v}_{\sigma_j}] - \tfrac{\gamma}{\alpha-c}\mathcal{J}_n^2\bar{v}_{\sigma_j} + \tfrac{\delta}{\alpha-c}\mathcal{I}_n[\bar{v}_{\sigma_j}^2] = 0, \quad 1 \le j \le N, \tag{6.1.17}$$

where for each j, $\bar{v}_{\sigma_j}$ is restricted to be even. The iterations terminate when the $L^2(\mathbb{R})$-norm of the defect in (6.1.17) drops below the accuracy threshold of $\varepsilon_n = 10^{-12}\sqrt{\frac{2(1-\sigma)}{n}}$. We mention only that in all our simulations the simplified Newton's process converges rapidly to the discrete solutions $\bar{v}_{\sigma_j}$ but, as observed by many authors, the number of iterations increases when σ approaches its upper bound of 1.

Example 5 We let $n = 2^{12} - 1$, $\ell = 2^3$, $\alpha = \gamma = \delta = 1$, $c_1 = \frac{1}{2}$, $c_2 = -\frac{1}{2}$, $\sigma_1 = 0.95$, $\beta = \sigma_1\sqrt{4\gamma(\alpha - c_1)}$ and $\sigma_2 = \frac{\beta}{\sqrt{4\gamma(\alpha - c_2)}}$. As an initial condition, we take the sum of two translated traveling waves

$$\bar{u}_0(x) = \bar{v}_{\sigma_1}(x + 20) + \bar{v}_{\sigma_2}(x - 20)$$

and integrate (6.1.1) numerically in time interval $[0, 80]$. With this settings, the solution describes a collision of two traveling waves moving towards each other. The collision occurs near $t = 40$, past that time the waves continue to move in the opposite directions. The initial profile of the numerical solution and its profiles near the collision time and at the terminal time are shown in the top three diagrams of Fig. 6.5. As observed in [35, 53], an interaction of the Benjamin traveling waves is inelastic — after collisions, numerical solutions develop a persistent small amplitude oscillating tail. In agreement with these observations, the latter is clearly visible in the bottom diagram of Fig. 6.5, where the magnified view of the terminal profile is presented.

Example 6 In our last example, we use $n = 2^{12} - 1$, $\ell = 2^3$, $\alpha = \gamma = \delta = 1$, $c_1 = \frac{3}{4}$, $c_2 = \frac{1}{10}$, $\sigma_1 = 0.95$, $\beta = \sigma_1\sqrt{4\gamma(\alpha - c_1)}$, $\sigma_2 = \frac{\beta}{\sqrt{4\gamma(\alpha - c_2)}}$,

$$\bar{u}_0(x) = \bar{v}_{\sigma_1}(x + 30) + \bar{v}_{\sigma_2}(x + 4)$$

and $[0, 80]$ for the time integration interval. The scenario describes propagation of two traveling waves moving in the same direction and colliding near $t = 40$. The numerical results are shown in Fig. 6.6, where as before, the top three diagrams contain the solution profiles at the initial, near collision and the terminal times, while the bottom diagram contains a magnified view of the solution at terminal time $t = 80$. Once again, the small dispersive tails (of the amplitudes $\approx 10^{-4}$ before the slow wave and $\approx 10^{-3}$ after the fast wave) are clearly visible.

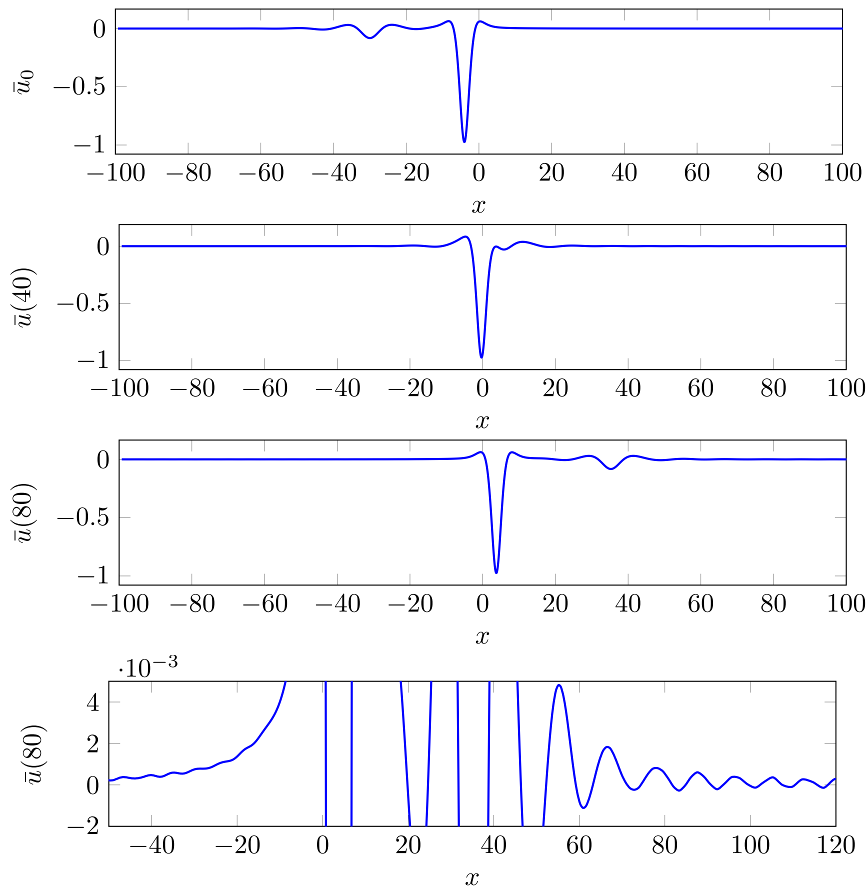

Figure 6.6: Inelastic collision of two traveling waves, Example 6.

Chapter 7

An MTC-type continuation scheme: The stationary Benjamin equation

Numerical analysis of (1.3.2) has generated a considerable body of research in both physical and numerical literature [5, 26, 35, 36]. The general idea consists in advancing known solution at $\mu = 0$ using a continuation technique (see, [5, 26, 35, 36] and references therein). Unfortunately, in most cases the numerical analysis is very informal. The main purpose of this Chapter is to fill in this gap. Below, we present a complete analysis of a spectral-type numerical scheme based on the use of MTC functions. We mention, that similar approach was used successfully in context of closely related Benjamin-Ono equation, see [21] and references therein.

7.1 An MTC continuation scheme

7.1.1 Spatial discretization

Let $\mathcal{A}_\mu$ be the operator associated with symbol κ_μ, and let $\mathcal{P}_n : L^2(\mathbb{R}) \to \mathbb{P}_n$ be the MTC-projector, defined in Section 5.2. We denote $\mathcal{A}_{\mu,n} = \mathcal{P}_n \mathcal{A}_\mu \mathcal{P}_n$. For computational purposes, we replace (1.3.2) with

$$\mathcal{A}_{\mu,n}\varphi_n = \mathcal{P}_n[\varphi_n^\ell], \quad \varphi_n \in \mathbb{P}_n, \quad 0 \leq \mu < 1. \tag{7.1.1}$$

For a fixed value of n problem (7.1.1) is a finite dimensional system of nonlinear equations that has to be solved iteratively. In view of Lemmas 4.3.1-4.3.3, it is natural to expect that the Jacobi matrix of the system is non-degenerate near $\mathcal{P}_n[\varphi_\mu]$, for small values of μ, where φ_μ is the variational, even, positive definite solution of equation (1.3.2) obtained in Theorem 4.1.7. In these settings, Newton's iterations are numerically feasible. The detailed analysis of the scheme (7.1.1) is provided below.

7.1.2 Error analysis

In our analysis, we make use of two technical results from Chapter 5, see [8] also. The first one is the inverse inequality

$$\|\hat{f}\|_{L^2_s(\mathbb{R}_+)} \le cn^{|s|}\|\hat{f}\|_{L^2(\mathbb{R})_+}, \quad \hat{f} \in \hat{\mathbb{P}}_n, \quad s > -\tfrac{1}{2}, \tag{7.1.2a}$$

that holds with $c_s > 0$ independent on $\hat{f}$ and n, see [8, formula (37b)] and the proof of Lemma 5.3.1. The second is the following error estimate

$$\|(I - \hat{\mathcal{P}}_n)\hat{f}\|_{L^2(\mathbb{R}_+)} \le cn^{-\frac{s}{2}}\|\hat{f}\|_{L^{2,s}_{\frac{s}{2}}(\mathbb{R}_+)}, \quad s \ge 0, \tag{7.1.2b}$$

which is a particular case of (5.2.3). It is important to note that (7.1.2) holds with an absolute constant $c > 0$.

In context of problem (1.3.4), bounds (7.1.2) combined with Lemma 4.2.4 yield the following basic estimates:

Lemma 7.1.1. *If $n > 0$ is sufficiently large, $0 \le \mu < 1$ and $p > 0$ is a fixed number, then*

$$\|\mathcal{P}_n[\varphi_\mu]\|_{H^1_\mu} \le \tfrac{cn}{\sqrt{1-\mu}}, \tag{7.1.3a}$$

$$n^p\|(I - \mathcal{P}_n)[\varphi_\mu]\| \le cr_\mu \exp\big\{-c_\mu n^{\frac{1}{3}}\big\}, \tag{7.1.3b}$$

where $c > 0$ is an absolute constant, $c_\mu = 3\big(\frac{\rho}{8r_\mu^2}\big)^{\frac{1}{3}}$ and the quantities ρ and r_μ are defined in Lemmas 4.2.1 and 4.2.4, respectively.

Proof. (a) Direct application of (7.1.2a) yields

$$\begin{aligned}
\|\mathcal{P}_n[\varphi_\mu]\|_{H^1_\mu} &= \|\hat{\mathcal{P}}_n[\hat{\varphi}_\mu]\|_{\hat{H}^1_\mu} \\
&\le \sqrt{2}\big(\|\hat{\mathcal{P}}_n[\hat{\varphi}_\mu]\|_{L^2(\mathbb{R}_+)} + 2\mu\|\hat{\mathcal{P}}_n[\hat{\varphi}_\mu]\|_{L^{2,0}_1(\mathbb{R}_+)} + \|\hat{\mathcal{P}}_n[\hat{\varphi}_\mu]\|_{L^{2,0}_2(\mathbb{R}_+)}\big) \\
&\le cn\|\hat{\mathcal{P}}_n[\hat{\varphi}_\mu]\|_{L^2(\mathbb{R}_+)} \le cn\|\hat{\varphi}_\mu\|_{L^2(\mathbb{R}_+)} \le \tfrac{cn}{\sqrt{1-\mu}}\|\hat{\varphi}_\mu\|_{H^1_\mu}.
\end{aligned}$$

It is not difficult to verify that

$$\tfrac{d}{d\mu}\|\hat{\varphi}_\mu\|^2_{L^{2,0}_{\kappa_\mu}(\mathbb{R}_+)} = -2\|\hat{\varphi}_\mu\|^2_{L^{2,0}_1(\mathbb{R}_+)} < 0.$$

Consequently, for $0 \le \mu < 1$ the quantity $\|\varphi_\mu\|_{H^1_\mu}$ is uniformly bounded from the above by the absolute constant $\|\varphi_0\|_{H^1}$. Hence, (7.1.3a) follows.

(b) We employ (7.1.2b), the identity $\mathcal{J}_-^{-s}[\hat{\varphi}_\mu] = (-1)^s e^{\frac{\xi}{2}}\partial^s_\xi\big[e^{-\frac{\xi}{2}}\hat{\varphi}_\mu\big]$ and Lemma 4.2.4 to obtain

$$\begin{aligned}
n^p\|(I-\mathcal{P}_n)[\varphi_\mu]\|_{L^2(\mathbb{R})} &= \sqrt{2}n^p\|(I-\hat{\mathcal{P}}_n)[\hat{\varphi}_\mu]\|_{L^2(\mathbb{R}_+)} \le cn^{p-m}\|\hat{\varphi}_\mu\|_{L^{2,2m}_{2m}(\mathbb{R}_+)}\\
&\le \tfrac{cn^{p-m}}{2^{2m}}\left(\tfrac{2m}{e\rho}\right)^m \sum_{r=0}^{2m}\tbinom{2m}{r}2^r\|\partial^r_\xi\hat{\varphi}_\mu\|_{L^2_\rho(\mathbb{R}_+)}\\
&\le cr_\mu n^{p-m}(2m)!\big(\tfrac{2mr_\mu^2}{e\rho}\big)^m.
\end{aligned}$$

Since $m > 0$ is arbitrary, for $n > \frac{8p^3r_\mu^2}{\rho}$, we let $m \le \big(\frac{n\rho}{8r_\mu^2}\big)^{\frac{1}{3}}$. Hence (7.1.3b) is completely settled. □

Below, we employ the classical Newton-Kantorovich theorem [54] to show that the discrete problem (7.1.1) is solvable.

Lemma 7.1.2. *Operator $\mathcal{A}_{\mu,n}$ is invertible and*

$$\|\mathcal{A}^{-1}_{\mu,n}\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} \le \tfrac{1}{\sqrt{1-\mu}}, \quad 0 \le \mu < 1. \tag{7.1.4}$$

Proof. The finite dimensional bilinear form $A_n(\cdot,\cdot) = \langle\mathcal{A}_{\mu,n}\cdot,\cdot\rangle_{L^2(\mathbb{R})} : \mathbb{P}_n\times\mathbb{P}_n \to \mathbb{R}$ is clearly bounded in $L^2(\mathbb{R})$ norm. Further, for any $\varphi \in \mathcal{P}_n$

$$A_n(\varphi,\varphi) = \|\varphi\|^2_{H^1_\mu} \ge (1-\mu^2)\|\varphi\|_{L^2(\mathbb{R})} \ge (1-\mu)\|\varphi\|_{L^2(\mathbb{R})}.$$

Hence, (7.1.4) is a straightforward consequence of the classical Lax-Milgram lemma. □

In view of Lemma 7.1.2, the discrete problem (7.1.1) can be rewritten in the form

$$\mathcal{N}_{\mu,n}(\varphi_n) = \varphi_n - \mathcal{G}_{\mu,n}(\varphi_n) = 0, \quad \mathcal{G}_{\mu,n}(\varphi_n) = \mathcal{A}^{-1}_{\mu,n}\mathcal{P}_n[\varphi^\ell_n].$$

Lemma 7.1.3. *For $n > 0$ sufficiently large and for small values of μ, the derivative $\mathcal{N}'_{\mu,n}(\mathcal{P}_n[\varphi_\mu]) : \mathbb{P}_n \to \mathbb{P}_n$ is invertible. Furthermore,*

$$\|\mathcal{N}'_{\mu,n}(\mathcal{P}_n[\varphi_\mu])^{-1}\|_{H^1_\mu \to H^1_\mu} \leq \tfrac{3\ell-1}{2(\ell-1)-c(|\mu|+r_\mu(1-\mu)^{\frac{\ell-1}{2}}\exp\{-c_\mu n^{\frac{1}{3}}\})}, \tag{7.1.5}$$

where $c > 0$ is an absolute constant and the quantities r_μ, $c_\mu > 0$ are defined in Lemmas 4.2.4 and 7.1.1, respectively.

Proof. (a) For $\varphi \in \mathbb{P}_n$, consider the linear operator $\tilde{\mathcal{N}}'_{\mu,n}[\varphi] = I - \tilde{\mathcal{G}}_{n,\mu}[\varphi]$, with $\tilde{\mathcal{G}}_{n,\mu}[\varphi] = \ell\mathcal{A}^{-1}_{\mu,n}\mathcal{P}_n[\varphi_\mu^{\ell-1}\varphi]$. Using Lemmas 4.3.1-4.3.2 (see in particular formula (4.3.3)) and the classical perturbation theory of compact selfadjoint operators [55], for small values of μ, we have

$$\begin{aligned}\inf_{\varphi\in\mathbb{P}_n,\, \|\varphi\|_{H^1_\mu}=1} \big|\langle \varphi, \tilde{\mathcal{N}}'_{\mu,n}[\varphi]\rangle_{H^1_\mu}\big| &\geq \inf_{\varphi\in H^1_\mu(\mathbb{R}),\, \|\varphi\|_{H^1_\mu}=1} \big|\langle \varphi, \big(I - \mathcal{G}'_\mu(\varphi_\mu)\big)[\varphi]\rangle_{H^1_\mu}\big| \\ &\geq \tfrac{2(\ell-1)-c|\mu|}{3\ell-1},\end{aligned}$$

with some uniform constant $c > 0$. Since $\tilde{\mathcal{N}}'_{\mu,n}[\varphi] : H^1_\mu(\mathbb{R}) \cap \mathbb{P}_n \to H^1_\mu(\mathbb{R}) \cap \mathbb{P}_n$ is selfadjoint, it follows that

$$\|(\tilde{\mathcal{N}}'_{\mu,n})^{-1}\|_{H^1_\mu \to H^1_\mu} \leq \tfrac{3\ell-1}{2(\ell-1)-c|\mu|}. \tag{7.1.6}$$

(b) For $\varphi \in \mathbb{P}_n$, we have

$$\begin{aligned}\big|\langle \varphi, \big(\tilde{\mathcal{N}}'_{\mu,n} - \mathcal{N}'_{\mu,n}(\mathcal{P}_n[\varphi_\mu])\big)[\varphi]\rangle_{H^1_\mu}\big| &\leq \ell\big|\langle \varphi, (\varphi_{\mu,n}^{\ell-1} - \mathcal{P}_n[\varphi_\mu]^{\ell-1})\varphi\rangle_{L^2(\mathbb{R})}\big| \\ &\leq \ell\|\kappa_\mu^{-\frac{1}{2}}\|^{\ell-1}_{L^2(\mathbb{R})}\|\varphi\|^2_{H^1_\mu}\left[\sum_{k=0}^{\ell-2}\|\mathcal{P}_n[\varphi_\mu]\|^k_{H^1_\mu}\|\varphi_\mu\|^{\ell-2-k}_{H^1_\mu}\right]\|(I-\mathcal{P}_n)[\varphi_\mu]\|_{L^2(\mathbb{R})}\end{aligned}$$

Since $\|\kappa_\mu^{-\frac{1}{2}}\| \leq (1-\mu)^{-\frac{1}{2}}$, estimates (7.1.3) give

$$\|\tilde{\mathcal{N}}'_{\mu,n} - \mathcal{N}'_{\mu,n}(\mathcal{P}_n[\varphi_\mu])\|_{H^1_\mu \to H^1_\mu} \leq c r_\mu (1-\mu)^{\frac{1-\ell}{2}} \exp\{-c_\mu n^{\frac{1}{3}}\}.$$

The latter bound, combined with (7.1.6) and the classical Neumann lemma, yields the result. □

Lemma 7.1.4. *The derivative $\mathcal{N}'_{\mu,n}(\cdot) : \mathbb{P}_n \to \mathbb{P}_n$ is locally Lipschitz continuous. Specifically, for any $\psi_i \in \mathbb{P}_n$ with $\|\mathcal{P}_n[\varphi_\mu] - \psi_i\|_{H^1_\mu} \leq \delta \leq 1$, $i = 0, 1$, we have*

$$\|\mathcal{N}'_{\mu,n}(\psi_0) - \mathcal{N}'_{\mu,n}(\psi_1)\|_{H^1_\mu \to H^1_\mu} \leq c(1-\mu)^{1-\ell} n^{\ell-2}\|\psi_0 - \psi_1\|_{H^1_\mu}, \quad 0 \leq \mu < 1, \tag{7.1.7}$$

where $c > 0$ is an absolute constant.

Proof. Elementary calculations yield

$$\begin{aligned}\|\mathcal{N}'_{\mu,n}(\psi_0)-\mathcal{N}'_{\mu,n}(\psi_1)\|_{H^1_\mu\to H^1_\mu} &\le \ell\|\psi_0^{\ell-1}-\psi_1^{\ell-1}\| \\ &\le \ell\|\kappa_\mu^{-\frac{1}{2}}\|^{\ell-1}\left[\sum_{k=0}^{\ell-2}\|\psi_0\|^k_{H^1_\mu}\|\psi_1\|^{\ell-2-k}_{H^1_\mu}\right]\|\psi_0-\psi_1\|_{H^1_\mu}.\end{aligned}$$

Hence, (7.1.7) is a direct consequence of (7.1.3a). □

Lemma 7.1.5. *For $n>0$ sufficiently large, the defect $\mathcal{N}_{\mu,n}(\mathcal{P}_n[\varphi_\mu])$ satisfies*

$$n^{\ell-1}\|\mathcal{N}_{\mu,n}(\mathcal{P}_n[\varphi_\mu])\|_{H^1_\mu}\le cr_\mu(1-\mu)^{-\frac{\ell}{2}}\exp\big\{-c_\mu n^{\frac{1}{3}}\big\}, \tag{7.1.8}$$

where $c>0$ is an absolute constant and the quantities r_μ, $c_\mu>0$ are defined in Lemmas 4.2.4 and 7.1.1, respectively.

Proof. The exact solution satisfies

$$\mathcal{P}_n[\varphi_\mu]+\mathcal{A}^{-1}_{\mu,n}\big[\mathcal{P}_n\mathcal{A}_\mu(I-\mathcal{P}_n)[\varphi_\mu]-\mathcal{P}_n[\varphi_\mu^\ell]\big]=0.$$

Hence,

$$\mathcal{N}_{\mu,n}(\mathcal{P}_n[\varphi_\mu])=\mathcal{A}^{-1}_{\mu,n}\big[\mathcal{P}_n[\varphi_\mu^\ell-\mathcal{P}_n[\varphi_\mu]^\ell]-\mathcal{P}_n\mathcal{A}_\mu(I-\mathcal{P}_n)[\varphi_\mu]\big].$$

Taking the $H^1_\mu(\mathbb{R})$ inner product with $\psi\in\mathbb{P}_n$, $\|\psi\|_{H^1_\mu}=1$, we infer

$$\big|\langle\mathcal{N}_{\mu,n}(\mathcal{P}_n[\varphi_\mu]),\psi\rangle_{H^1_\mu}\big|\le\tfrac{1}{\sqrt{1-\mu}}\|\varphi_\mu^\ell-\mathcal{P}_n[\varphi_\mu]^\ell\|_{L^2(\mathbb{R})}+\sqrt{2}\|\kappa_\mu\hat{\psi}\|_{L^2(\mathbb{R}_+)}\|(I-\mathcal{P}_n)[\varphi_\mu]\|_{L^2(\mathbb{R})}.$$

Derivations similar to those used in the proof of (7.1.3a) yield

$$\|\kappa_\mu\hat{\psi}\|_{L^2(\mathbb{R}_+)}\le\tfrac{cn}{\sqrt{1-\mu}}.$$

Consequently,

$$\begin{aligned}&n^{\ell-1}\|\mathcal{N}_{\mu,n}(\mathcal{P}_n[\varphi_\mu])\|_{H^1_\mu}\\ &\qquad\le\tfrac{n^{\ell-1}}{\sqrt{1-\mu}}\left[\|\kappa_\mu^{-\frac{1}{2}}\|^{\ell-1}\sum_{k=0}^{\ell-1}\|\mathcal{P}_n[\varphi_\mu]\|^k_{H^1_\mu}\|\varphi_\mu\|^{\ell-1-k}_{H^1_\mu}+cn\right]\|(I-\mathcal{P}_n)[\varphi_\mu]\|_{L^2(\mathbb{R})}\\ &\qquad\le c(1-\mu)^{-\frac{\ell}{2}}n^{2\ell-2}\|(I-\mathcal{P}_n)[\varphi_\mu]\|_{L^2(\mathbb{R})},\end{aligned}$$

and (7.1.8) follows from (7.1.3b). □

Lemmas 7.1.3–7.1.5 combined together yield the main result of this section.

Theorem 7.1.6. *For $n > 0$ sufficiently large and for small values of μ, the discrete problem* (7.1.1) *has an isolated solution $\varphi_{\mu,n} \in \mathbb{P}_n$, that satisfies*

$$\|\varphi_\mu - \varphi_{\mu,n}\|_{H^1_\mu} \leq cr_\mu(1-\mu)^{-\frac{\ell}{2}} \exp\big\{-\tfrac{c_\mu}{2} n^{\frac{1}{3}}\big\}, \tag{7.1.9}$$

where $c > 0$ is an absolute constant and the quantities r_μ, $c_\mu > 0$ are defined in Lemmas 4.2.4 and 7.1.1, respectively. Further, for small values of $\mu_0 \leq \mu < \mu_1 < 1$, the numerical trajectory $\mathcal{S}_n(\mu_0, \mu_1) = \{(\mu, \varphi_{\mu,n})\}_{\mu_0 \leq \mu \leq \mu_1}$ is a smooth parametric curve in $\mathbb{R} \times \mathbb{P}_n$ that does not meet any other solution branches of (7.1.1).

Proof. (a) We let

$$\begin{aligned}
\bar{\alpha} &= \|\mathcal{N}'_{\mu,n}(\mathcal{P}_n[\varphi_\mu])^{-1}\mathcal{N}'_{\mu,n}(\mathcal{P}_n[\varphi_\mu])\|_{H^1_\mu},\\
\bar{\beta} &= \|\mathcal{N}'_{\mu,n}(\mathcal{P}_n[\varphi_\mu])^{-1}\|_{H^1_\mu \to H^1_\mu},\\
\bar{\sigma} &= \sup\left\{\frac{\|\mathcal{N}'_{\mu,n}(\mathcal{P}_n[\varphi_\mu])[\psi_0] - \mathcal{N}'_{\mu,n}(\mathcal{P}_n[\varphi_\mu])[\psi_1]\|_{H^1_\mu}}{\|\psi_0 - \psi_1\|_{H^1_\mu}} \,\middle|\, \psi_i \in \mathbb{P}_n, \|\mathcal{P}_n[\varphi_\mu] - \psi_i\|_{H^1_\mu} \leq 1\right\}.
\end{aligned}$$

Choosing $n > 0$ sufficiently large and using Lemmas 7.1.3–7.1.5, we have

$$\bar{\alpha}\bar{\beta}\bar{\sigma} \leq cr_\mu(1-\mu)^{\frac{2-3\ell}{2}} \exp\big\{-c_\mu n^{\frac{1}{3}}\big\} < \tfrac{1}{2},$$

uniformly, for small values of μ. Hence, problem (7.1.1) fells in the scope of the classical Newton-Kantorovich theorem [54]. It follows that the Newton iterations

$$\varphi_{\mu,n,j+1} = \varphi_{\mu,n,j} - \mathcal{N}'_{\mu,n}(\varphi_{\mu,n,j})^{-1}[\mathcal{N}_{\mu,n}(\varphi_{\mu,n,j})], \quad j \geq 0,$$

with $\varphi_{\mu,n,0} = \mathcal{P}_n[\varphi_\mu]$, converge to the unique zero $\varphi_{\mu,n}$ of (7.1.1) in the open ball

$$B(\mathcal{P}_n[\varphi_\mu], \delta) = \big\{\varphi \in \mathbb{P}_n \,\big|\, \|\varphi - \mathcal{P}_n[\varphi_\mu]\|_{H^1_\mu} < \bar{\delta}\big\} \subset \mathbb{P}_n, \quad \delta = \tfrac{1}{\bar{\beta}\bar{\sigma}} = \mathcal{O}\big(\tfrac{(1-\mu)^{\ell-1}}{n^{\ell-2}}\big),$$

and

$$\|\mathcal{P}_n[\varphi_\mu] - \varphi_{\mu,n}\|_{H^1_\mu} \leq 2\bar{\alpha}\bar{\beta} \leq cr_\mu(1-\mu)^{-\frac{\ell}{2}} \exp\big\{-c_\mu n^{\frac{1}{3}}\big\}. \tag{7.1.10}$$

In addition, we observe that

$$\|(I - \mathcal{P}_n)[\varphi_\mu]\|_{H^1_\mu} \leq \big[\|\kappa_\mu \hat{\varphi}_\mu\|_{L^2(\mathbb{R})} + \|\kappa_\mu \hat{\mathcal{P}}_n[\hat{\varphi}_\mu]\|_{L^2(\mathbb{R})}\big]^{\frac{1}{2}} \|(I - \mathcal{P}_n)[\varphi_\mu]\|^{\frac{1}{2}}_{L^2(\mathbb{R})}, \tag{7.1.11}$$

where $\|\kappa_\mu \hat{\varphi}_\mu\| \le \rho r_\mu$ and $\|\kappa_\mu \hat{\mathcal{P}}[\hat{\varphi}_\mu]\| \le \frac{cn^2}{\sqrt{1-\mu}}$, by virtue of Lemma 4.2.4 and formula (7.1.2a), respectively. We conclude that the error bound (7.1.9) follows directly from (7.1.10), (7.1.11) and Lemma 7.1.1.

(b) The nonlinear map $\mathcal{N}_{\mu,n}(\varphi)$ is analytic in $\varphi \in \mathbb{P}_n$ and μ. Lemma 7.1.3, estimate (7.1.10) and the standard perturbation theory of linear operators, indicate that $\mathcal{N}'_{\mu,n}(\varphi_{\mu,n}) : \mathbb{P}_n \to \mathbb{P}_n$ is invertible. Hence, for each fixed value of $\mu \in [\mu_0, \mu_1]$ the analytic version of the Implicit Function theorem applies. Locally, $\varphi_{\mu,n}$ gives rise to an analytic branch of isolated numerical solutions $\varphi_{\mu,n}(t)$, $|t-\mu| < \delta'$. In view of part (a) of the proof $\varphi_{\mu,n}(t) = \varphi_{t,n}$, when δ' is small. Standard covering argument shows that local solution branches, glued together, make up an analytic parametric curve $\mathcal{S}_n(\mu_0, \mu_1) \subset \mathbb{R} \times \mathbb{P}_n$, for μ_0, μ_1 small. □

Theorem 7.1.6 assures that the numerical scheme (7.1.1) is robust in the sense that the associated nonlinear equation does not generate spurious numerical branches bifurcating out of the physically meaningful numerical trajectory $\mathcal{S}_n(\mu_0, \mu_1)$, when μ_0, μ_1 are small. Estimate (7.1.9) indicates that numerical solutions do converge and the convergence rate is subgeometric in $H^1_\mu(\mathbb{R})$. In fact, the same is true in the uniform topology of $C_0(\mathbb{R})$, for $H^1_\mu(\mathbb{R}) \subset C_0(\mathbb{R})$. Further, careful inspection of Lemmas 4.2.1 and 4.2.4 indicates that $\rho = \mathcal{O}\big((1-\mu)^\ell\big)$ and $r_\mu = \mathcal{O}\big((1-\mu)^{\frac{1-\ell}{2}}\big)$. Hence, (7.1.9) is equivalent to

$$\|\varphi_\mu - \varphi_{\mu,n}\|_{H^1_\mu} \le c(1-\mu)^{\frac{1-2\ell}{2}} \exp\big\{ -c'[(1-\mu)^{2\ell-1} n]^{\frac{1}{3}} \big\}, \tag{7.1.12}$$

where $c, c' > 0$ are some absolute constants. In view of (7.1.12), we expect the accuracy of (7.1.1) to deteriorate when μ is away from zero, and $n > 0$ being fixed. The latter phenomenon is observed by a number of authors, see [5, 10, 74, 13].

To conclude this section, we remark that Theorem 7.1.6 is local in nature. At present, we are unable to control the multiplicity of the second eigenvalue $\lambda_1 = 1$ of $\mathcal{G}'_\mu(\varphi_\mu)$, for all values of $0 \le \mu < 1$. In view of Lemma 4.3.1, the spectral gap between the dominant $\lambda_0 = \ell$ and the second eigenvalue $\lambda_1 = 1$ of $\mathcal{G}'_\mu(\varphi_\mu)$ persists and is precisely equal to $\ell - 1$. By virtue of Lemma 4.3.2 the spectrum of $\mathcal{G}'_0(\varphi_0)$ is simple and hence λ_1 remain simple for small values of μ. However, we cannot control the behavior of eigenvalues λ_n, $n \ge 2$, for $0 \le \mu < 1$ away from zero. It may happen that for some $0 < \mu' < 1$ a finite number of them reaches λ_1 and the local analysis, presented above, fails. The numerical simulations, presented below, indicate that the second eigenvalue remain simple for all values of $0 \leq \mu < 1$, however we do not have rigorous proof of this fact.

7.2 Simulations

In our simulations, the following basic continuation scheme is employed: we assume that $\varphi_{\mu,n}$ is known, choose $\delta > 0$ so that $\mu + \delta < 1$ and solve (7.1.1) numerically using Newton's iterations:

$$\varphi_{\mu+\delta,n,0} = \varphi_{\mu,n}, \tag{7.2.1a}$$

$$\varphi_{\mu+\delta,n,j+1} = \varphi_{\mu+\delta,n,j} - \mathcal{N}'_{\mu+\delta,n}(\varphi_{\mu,n+\delta,j})^{-1}[\mathcal{N}_{\mu+\delta,n}(\varphi_{\mu+\delta,n,j})], \quad j \geq 0. \tag{7.2.1b}$$

Computations terminate as soon as

$$\|\varphi_{\mu+\delta,n,j} - \varphi_{\mu+\delta,n,j+1}\|_{L^2(\mathbb{R})} \leq \tfrac{\varepsilon}{\sqrt{1-\mu}}, \tag{7.2.1c}$$

where $\varepsilon > 0$ is a user set tolerance. More sophisticated initial guess and/or stopping criterion can be used, but we do not pursue such a generality here. For $\mu = 0$ the exact solution is known (see formula (1.3.3)) and is used to initialize the continuation process. After each continuation step, the identity (5.1.1a) is applied to reconstruct the Fourier image $\hat{\varphi}_{\mu,n}$.

Note that solving (1.3.2) directly, when $\mu \to 1$, requires very large values of n, as oscillations in the exact solutions cannot be properly captured on rough grids. To circumvent the problem, the original equation (1.3.2) is rescaled. In our simulations, we solve for the dilate $\varphi_\mu(\tau x)$, with $\tau = 2^5$.

7.2.1 Numerical solutions

We let $\varepsilon = 10^{-10}$ and solve (7.1.1) for $0 \leq \mu \leq 0.9999$ and $\ell = 2, 3, 4, 5, 6$. The numerical solutions $\varphi_{\mu,2^9}$ and corresponding snapshots for selected values of μ are plotted Fig. 7.1 and Fig. 7.2, respectively. One can see that independently on a particular value of ℓ and for large values of μ, all solutions develop oscillations:

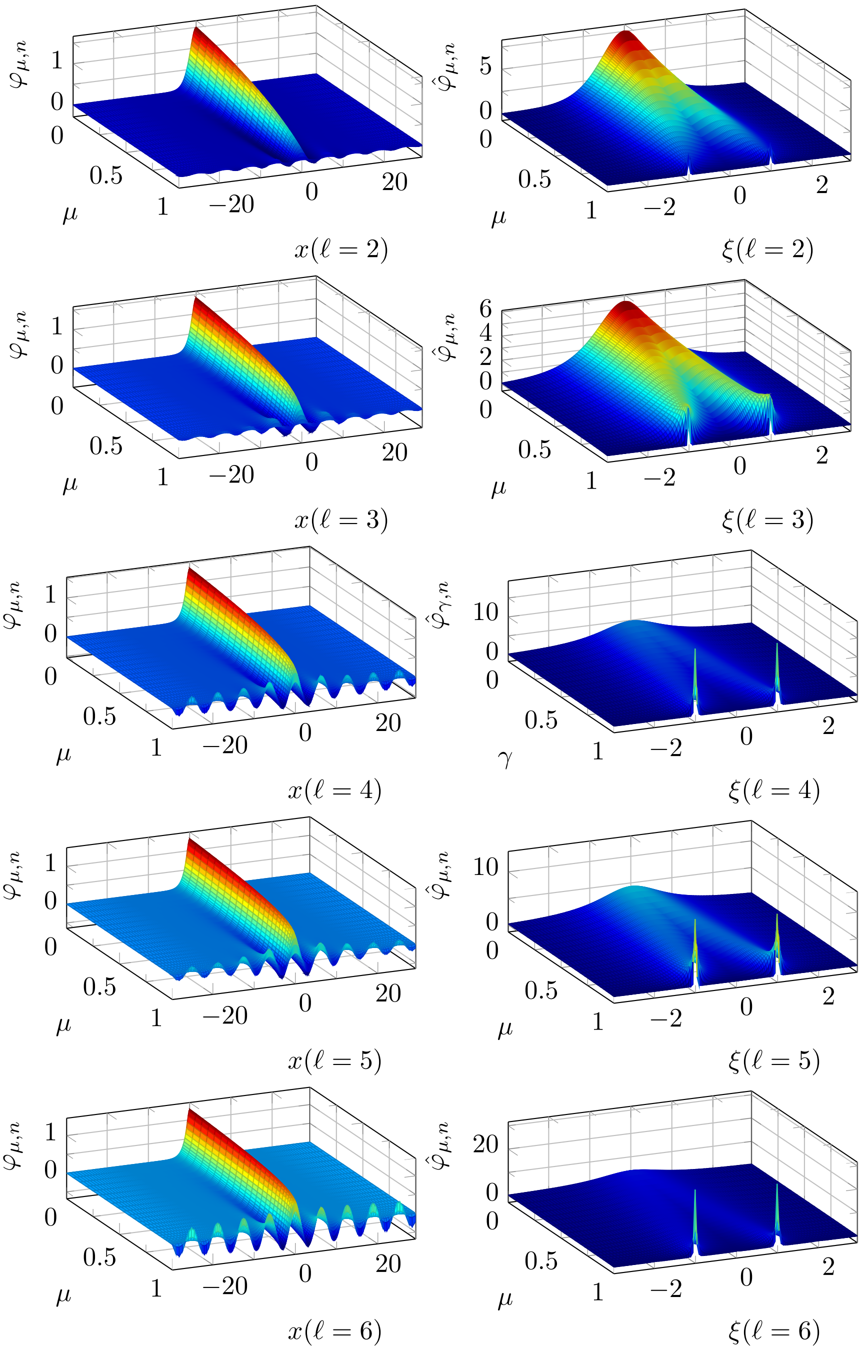

Figure 7.1: The numerical solutions $\varphi_{\mu,n}$ and $\hat{\varphi}_{\mu,n}$, $\mu \in [0, 0.9999]$, $n = 2^9$.

in each case, the number of oscillations is finite; the number of oscillations and their amplitude increases together with μ. In the case of $\ell = 2$, similar qualitative behavior was predicted theoretically and/or observed numerically by a number of authors [5, 13, 74]. In particular, our numerical data agrees with computations

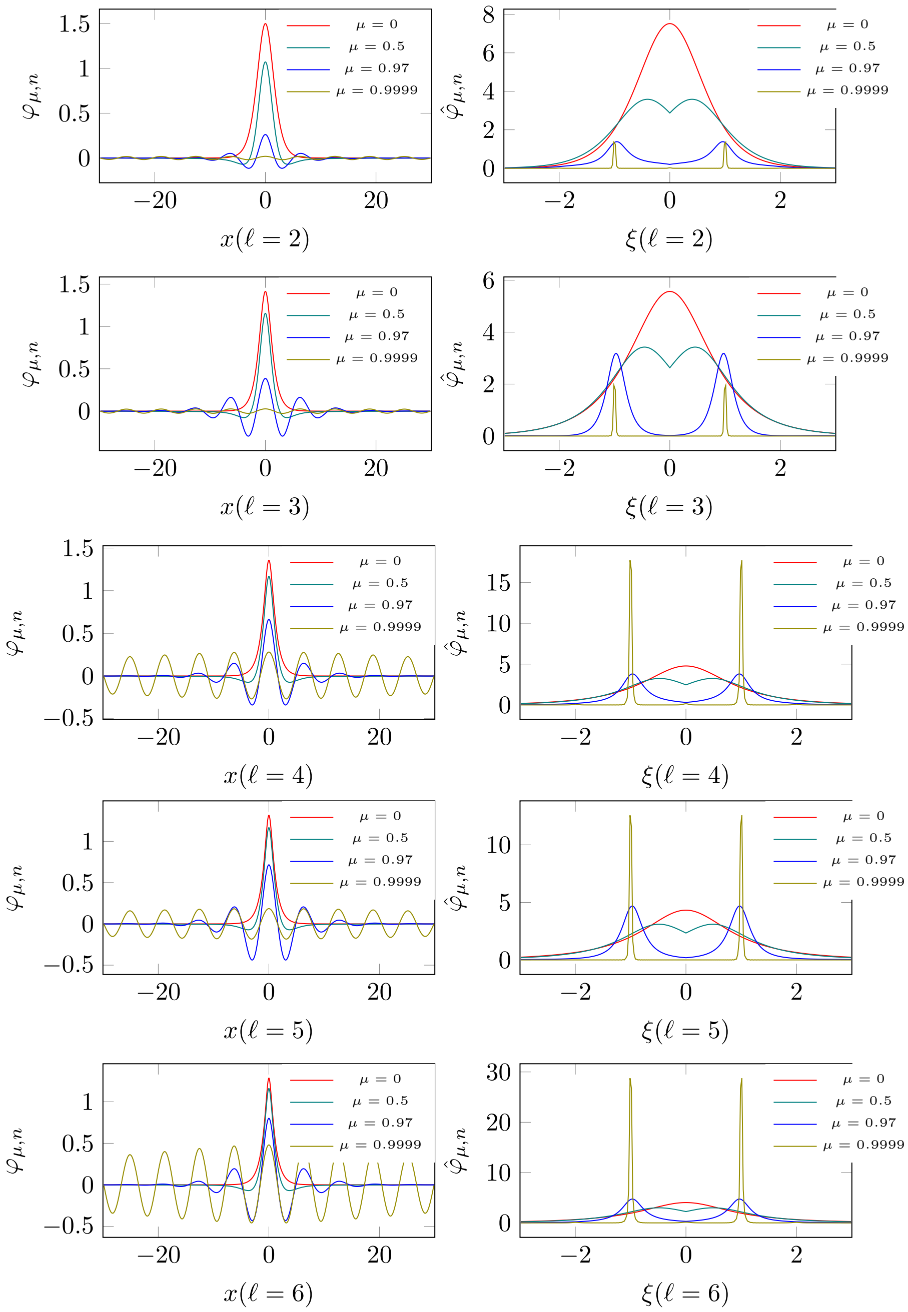

Figure 7.2: The snapshots of $\varphi_{\mu,n}$ for selected values of μ, $n = 2^9$.

reported in [36].

The plots of $\hat{\varphi}_{\mu,n}$ provide further illustration of this phenomenon. When μ is near 1, the graphs of $\hat{\varphi}_{\mu,n}$ resemble superposition of two δ-like functions supported near $\xi = \pm 1$ (see right diagrams in Fig. 7.2). There is however a significant difference in the behavior of the amplitudes of the delta-peaks for different values of the parameter

ℓ. For $\ell = 2$ and $\ell = 3$ the delta-peak amplitudes, after some short transition stage, behave as non-increasing functions of μ (see right diagrams in Fig. 7.1), while for $\ell \geq 4$ the amplitudes increase rapidly as $\mu \to 1$. This observation is intimately connected with the orbital stability of the computed waves, we discuss this issue in some details below.

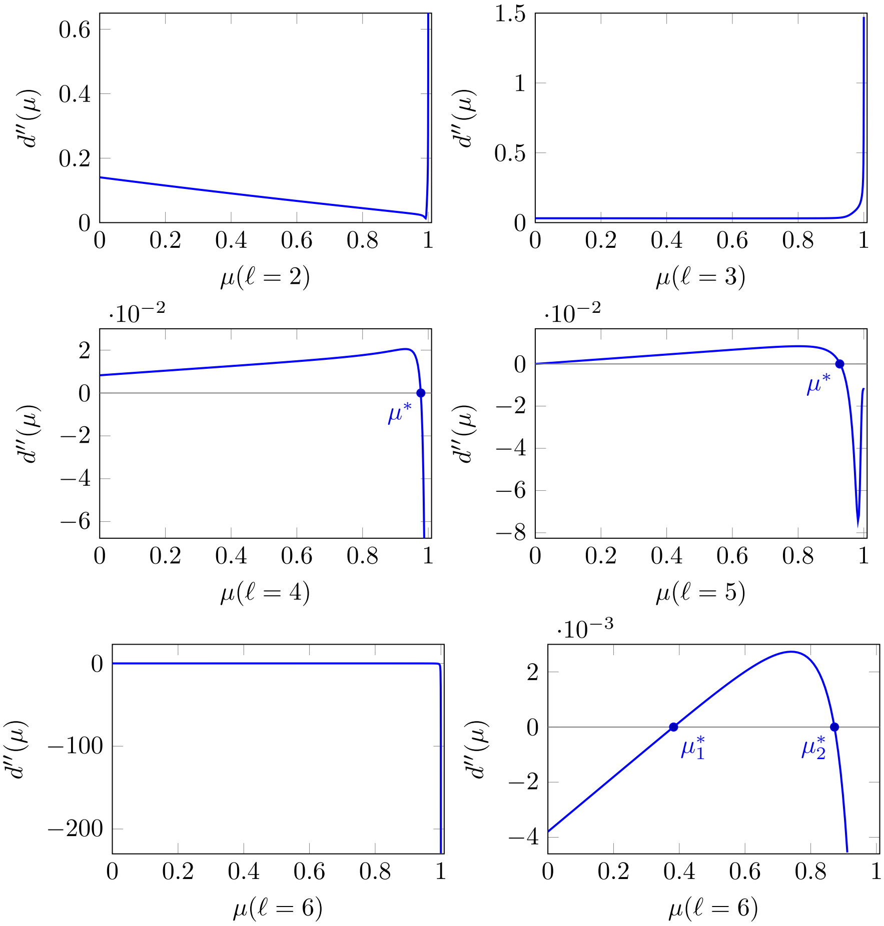

Figure 7.3: The stability indicator $d''(\mu)$, $n = 2^{14}$.

7.2.2 Orbital Stability

For $\ell = 2, 3, 4, 5, 6$, the plots of the stability indicator $d''(\mu)$ (see Section 4.3.2) are shown in Fig. 7.3. The top-left diagram serves as an illustration to Corollary 4.3.5, which asserts that for $\ell = 2$, traveling waves are orbitally stable for small values of $\mu \in [0, 1)$. In fact, the numerically computed quantity $d''(\mu)$ remains positive for all

μ in the computational domain.

We do not have an analogue of Corollary 4.3.5 for $\ell \geq 3$, however, when $\beta = 0$, (1.1.4) reduces to classical KdV equation and stability theory, developed in [16], applies. In particular, traveling waves with $\beta = 0$ (equivalently $\mu = 0$) are stable, provided that $\ell < 5$. Using standard perturbation theory, we expect traveling waves with $\ell = 3$ and $\ell = 4$ to be orbitally stable near $\mu = 0$. Similarly, in the case of $\ell \geq 6$, we expect instability near the origin. The case of $\ell = 5$ is special, for then $d''(0) = 0$ and the simple perturbation argument does not apply. We discuss each particular case separately.

For $\ell = 3$, the plot of $d''(\mu)$ is shown in the top-right diagram of Fig. 7.3. As expected $d''(\mu)$ is positive near the origin, interestingly and similarly to the case of $\ell = 2$, it remains positive for all values of $\mu \in [0, 0.9999]$. Taking into account (4.3.8), we suspect that traveling waves remain stable for all values of $\mu \in [0, 1)$, it would be interesting to have a rigorous proof of this assertion.

The case of $\ell = 4$ is qualitatively different from the previous two (see the left diagram in the second row of Fig. 7.3). The quantity $d''(\mu)$ is positive near the origin but crosses μ-axis as μ increases. The change in sign occurs at $\mu^* \approx 0.977472$, passing this point the curve $d''(\mu)$ rapidly goes down, indicating that associated traveling waves lose their stability. We cannot claim that the instability persists in $[\mu^*, 1)$, but in view of (4.3.8),[1] it is a reasonable assumption.

The question of stability is particularly interesting when $\ell = 5$, for in this case $d''(0) = 0$ (see the right diagram in the second row of Fig. 7.3). Numerical simulations show that small perturbations in μ yield orbitally stable waves. The stabilizing effect persists for $\mu \in (0, \mu^*)$, $\mu^* \approx 0.926735$. In the interval $[\mu^*, 0.9999]$, the solutions lose their stability. Here again the remark made above applies, it is natural to expect that the waves φ_μ remain unstable for $\mu \in [\mu^*, 0)$.

For the sake of completeness, we repeat calculations with $\ell = 6$. Here, $d''(0) < 0$ [16], and we expect traveling waves to be unstable near the origin. The two bottom diagrams in Fig. 7.3 show that this is indeed the case. However, as μ increases the graph of $d''(\mu)$ crosses μ-axis twice (see the bottom-right diagram in Fig. 7.1), which

[1]For (4.3.8) indicates that variational problem (4.3.6) might be unbounded.

yields nontrivial stability interval (μ_1^*, μ_2^*), with $\mu_1^* \approx 0.382645$ and $\mu_2^* \approx 0.872382$ (see the magnified plot in the bottom-right diagram of Fig. 7.3). As in two previous examples, the stability is lost once μ passes the threshold value μ_2^*.

We remark that calculations presented above for $\ell \geq 3$ shall be taken with some caution as the magnitudes of $d''(\mu)$ over large portions of stability regions are very small (especially for $\ell = 6$) and may be easily affected by a numerical noise (caused by the approximation and the round off errors). Nevertheless, when producing the above diagrams, we repeated computations with increasing values of n until the plots stabilize. For $2 \leq \ell \leq 6$, this occurs when $n \geq 2^{12}$. The diagrams displayed in Fig. 7.3 are obtained with $n = 2^{14}$.

7.2.3 Accuracy

In view of Theorem 7.1.6 and estimate (7.1.12), for small fixed values of $\mu \in [0, 1)$, we expect at least sub-geometric convergence rate provided n is sufficiently large. To illustrate the assertion, we present some accuracy tests for the numerical solutions $\varphi_{\mu,n}$ with $\mu \in [0, 0.9999]$ and $2^6 \leq n \leq 2^{12}$. Since the exact solutions are unavailable, as the reference solution we take $\varphi_{\mu,n}$ with $n = 2^{14}$. The work/precision diagrams, containing L^2-errors $\|\varphi_{\mu,n} - \varphi_{\mu,2^{14}}\|$, are presented in Fig. 7.4. We observe that for each value of ℓ and for each fixed value of μ, the errors decrease at least sub-geometrically (the surfaces are concave in the direction of n-axis). Further, for moderate values of parameter μ the decay is very rapid. In fact, for the large range of μ and n, the errors are comparable with the machine epsilon. In complete agreement with (7.1.12), the accuracy deteriorates as $\mu \to 1$ and/or as ℓ increases.

Quantity $\|\varphi_{\mu,n} - \varphi_{\mu,2^{14}}\|$ is not an ideal measure for the numerical accuracy, for $\varphi_{\mu,2^{14}}$ itself is not exact. Nevertheless, when n is large, we have

$$\|\varphi_{\mu,n} - \varphi_\mu\| \leq \|\mathcal{N}_\mu'(\varphi_\mu)^{-1}\|_{L^2\to L^2}\|\mathcal{N}_\mu(\varphi_{\mu,n})\| + \mathcal{O}(\|\varphi_{\mu,n} - \varphi_\mu\|^2),$$

i.e the quantities (L^2-defects) $\|\mathcal{N}_\mu(\varphi_{\mu,n})\|$ might serve as an adequate reflection of true numerical errors, see Fig. 7.5. The diagrams shown in Fig. 7.5 are closely resembling those presented in Fig. 7.4 and hence supporting our theoretical conclusions about accuracy of scheme (7.1.1) and the reliability of presented computational

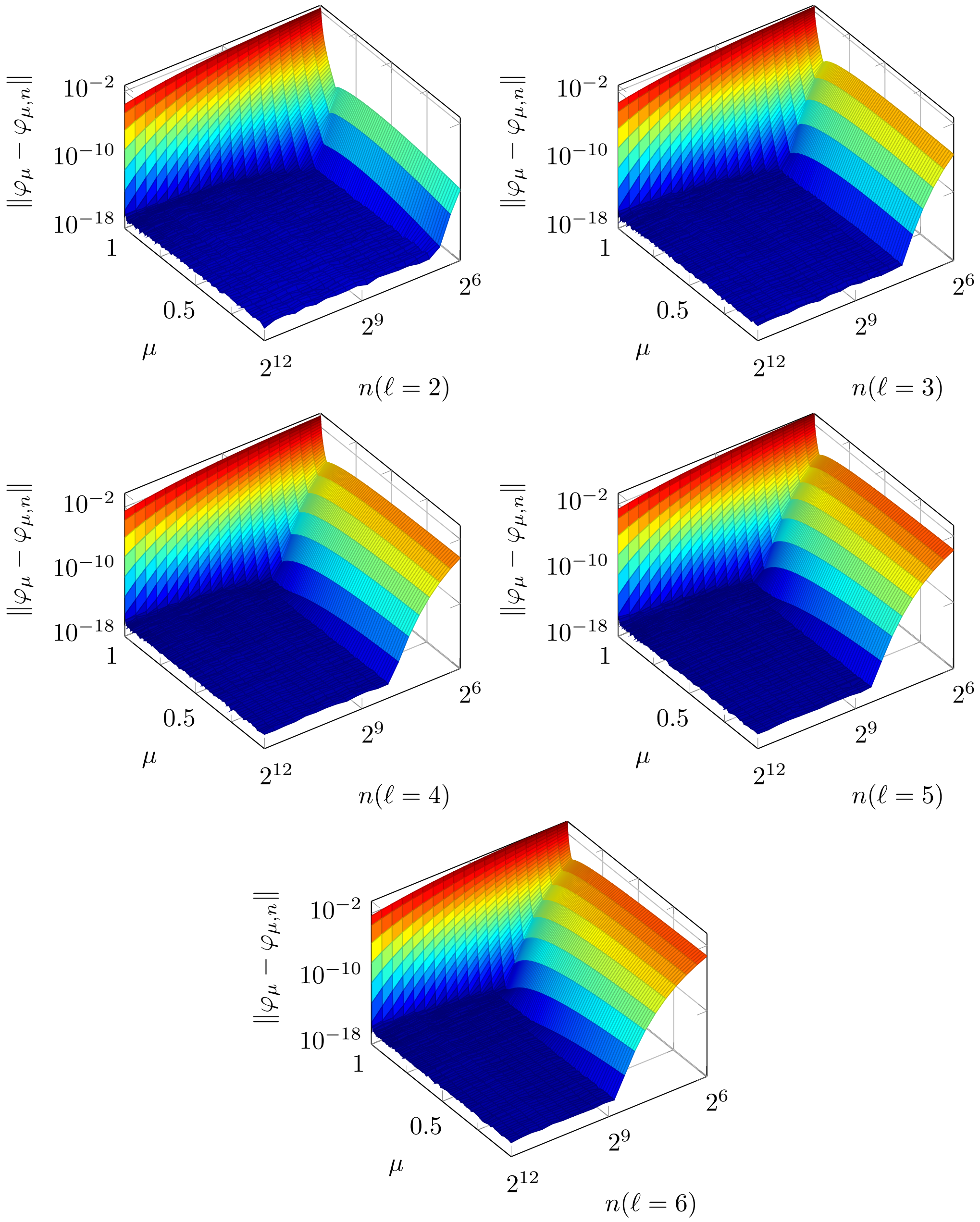

Figure 7.4: The work/precision diagrams: L^2-errors.

results.

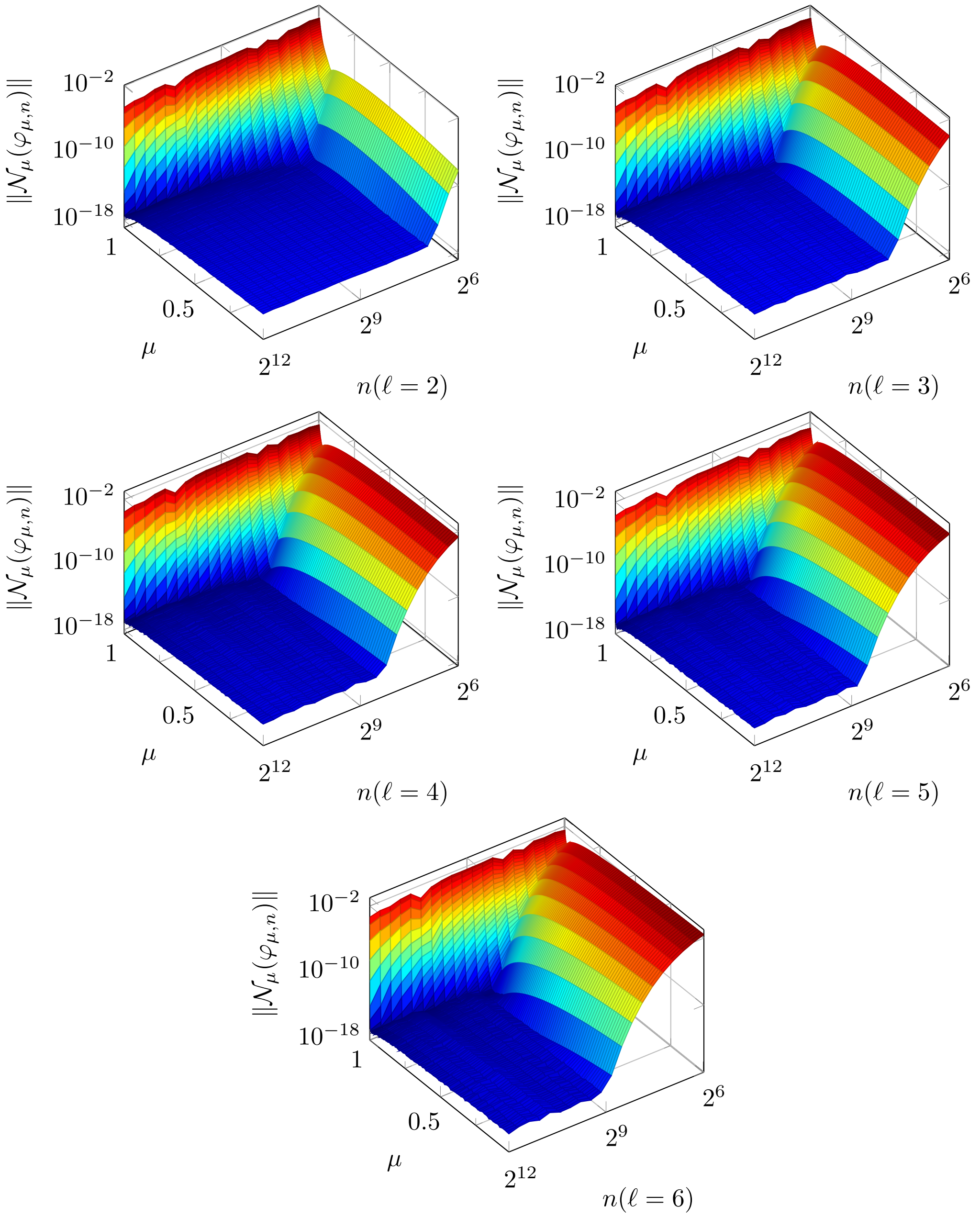

Figure 7.5: The work/precision diagrams: L^2-defects.

Chapter 8

Conclusion

In this thesis, we presented theoretical and numerical analyses of the stationary and non-stationary Benjamin equation posed on the real line.

Chapter 2 is introductory in nature. Here, we provided a detailed proof of an interpolation identity for a scale of variable weight Sobolev spaces for which we have no immediate references. This result is new and extends some recent investigations in the theory of weighted function spaces. In the context of nonstationary Benjamin equation, the variable weight Sobolev spaces appear naturally and play fundamental role in the analysis of Chapters 3, 5 and 6.

Chapter 3 contains wellposedness analysis of the non-linear Benjamin equation in the settings of the variable weight Sobolev spaces defined in Chapter 2. Our result is new, extends significantly recent research on the global wellposedness of the Benjamin equation in weighted Sobolev-like spaces and provides a theoretical foundation for building robust numerical schemes.

The existence, regularity and orbital stability of traveling waves are considered in Chapter 4. In particular, the problem of existence is settled globally for $0 \leq \mu < 1$ via a combination of concentration-compactness and a small viscosity limit techniques. A detailed asymptotic and regularity analyses (carried out in Section 4.2) are new and extend earlier results of [5, 10, 13, 74]. Further, it is shown that the quadratic waves $\ell = 2$ are orbitally stable for small values of wavespeed parameter μ.

The Malmquist-Takenaka-Christov basis and its computational properties are discussed in Chapter 5. Rigorous theoretical analysis indicates that the MTC approximations converge rapidly, provided the Fourier images of the approximated functions are regular away from the origin and decay rapidly at infinity. This makes them particularly suitable for solving semi- and quasi-linear problems containing Fourier multipliers, whose symbols are not smooth at the origin.

An MTC-type collocation scheme for the nonstationary Benjamin equation is presented in Chapter 6. It is shown that the convergence rate of the scheme is controlled solely by the regularity and asymptotic of the Fourier images of solutions in $\mathbb{R} \setminus \{0\}$, while allowing square integrable singularities at the origin. As a consequence, and in contrast to the Hermite or algebraically mapped Chebyshev bases in $L^2(\mathbb{R})$, the MTC-type approximations converge spectrally under very mild restrictions on the solutions decay at infinity and admit an efficient practical implementation, comparable to the best spectral-Fourier and hybrid spectral-Fourier/finite-element methods, described in the literature.

In the final Chapter 7, an MTC-type continuation scheme for the stationary Benjamin equation is presented. Rigorous error analysis of the scheme indicates that for small values of the wavespeed parameter μ, the numerical approximations converge subgeometrically and hence yield very accurate numerical results for moderate values of the discretization parameter n. The computational scheme is used to study orbital stability of the Benjamin traveling waves numerically. Simulations suggest that the quadratic ($\ell = 2$) and the cubic ($\ell = 3$) waves are orbitally stable for all values of $0 \leq \mu < 1$, however, a rigorous proof of this fact is not available.

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