

THE THEORY OF OPTION VALUATION

by

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PREFACE

The work described in this thesis was carried out in the Department of Mathematical Statistics, at the University of Natal in Durban, from January 1991 to December 1992, under the supervision of Doctor M. Murray.

These studies represent original work by the authoress and have not been submitted in any form to another University. Where use was made of the work of others, it has been duly acknowledged in the text.

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ABSTRACT

Although options have been traded for many centuries, it has remained a relatively thinly traded financial instrument. Paradoxically, the theory of option pricing has been studied extensively. This is due to the fact that many of the financial instruments that are traded in the market place have an option-like structure, and thus the development of a methodology for option-pricing may lead to a general methodology for the pricing of these derivative-assets.

This thesis will focus on the development of the theory of option pricing. Initially, a fundamental principle that underlies the theory of option valuation will be given. This will be followed by a discussion of the different types of option pricing models that are prevalent in the literature.

Special attention will then be given to a detailed derivation of both the Black-Scholes and the Binomial Option pricing models, which will be followed by a proof of the convergence of the Binomial pricing model to the Black-Scholes model.

The Black-Scholes model will be adapted to take into account the payment of dividends, the possibility of a changing interest rate and the possibility of a stochastic variance for the rate of return on the underlying asset. Several applications of the Black-Scholes model will finally be presented.

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CHAPTER 1

INTRODUCTION

Option trading, especially on stocks, has had a long and varied history. In fact, its origins can be traced back to the time of the ancient Greeks when the following quotation was made by Aristotle.

There is an anecdote of Thales the Milesian and his financial device, which involves a principle of universal application, but is attributed to him on account of his reputation for wisdom. He was reproached for his poverty, which was supposed to show that philosophy was of no use. According to the story, he knew by his skill in the stars while it was yet Winter that there would be a great harvest of olives in the coming year; so, having a little money, he gave deposits for the use of all the olive presses in Chios and Miletus, which he hired at a low price because no one bid against him. When the harvest time came, and many wanted them all at once and of a sudden, he let them out at any rate which he pleased, and made a quantity of money. Thus he showed the world that philosophers can easily be rich if they like

...

Aristotle's Politics, Book One,
Chapter eleven, Jowett translation.

The early seventeenth century, however, saw the first extensive use of options. Tulip bulb growers in Amsterdam wrote call option contracts which they then sold to tulip bulb dealers for a certain fee. The holder of such a contract then had the right to purchase, at some future date, the as of yet

unharvested tulip bulbs at a fixed price. The dealers, in turn, sold these bulbs, for future delivery, based on the value of these call option contracts. There were, however, many irregularities in the tulip bulb option market. For example, there were no financially sound option endorsers to guarantee that the writers would fulfil their contracts, and there were no margin requirements necessary to keep speculators from bankrupting themselves. As a result, the tulip bulb market collapsed in 1636, with these investors and speculators, who had gained some experience in Amsterdam, moving to England following the accession of William and Mary to the English throne in 1688. Although option trading was declared illegal by the Barnards Act of 1733, option trading continued until the financial crisis of 1931, when it was eventually banned.

In 1958, however, option trading resumed on a small scale. In an attempt to promote the development of an options market, a properly constituted options trading market (known as the Chicago Board Option Exchange (CBOE)) was set up in America in 1973. This brought into existence, for the first time, a market for option trading that contained standardized (fixed) expiry dates. The CBOE's lead was then followed by the American Stock Exchange, the Philadelphia Stock Exchange, the Midwest Stock Exchange and the Pacific Stock Exchange. Outside America, the European Options Exchange in Amsterdam, the London Stock Exchange and the London International Financial Futures Exchange also began promoting an options market with fixed expiry dates.

In South Africa, the development of an options trading market was primarily

influenced by the combined interest of the Dutch and British in South Africa as well as the development of the mining industry. At the time the following two mining houses, Anglo American and Johannesburg Consolidated Investments, were the most active in the writing of option contracts. Stockbroking companies, however, began to enter the options market in the late 1940's when two members of the JSE, Mr M.R. Johnson and Mr R.C.J. Anderson, began to sell options. In the early eighties, efforts were made to create a formalised exchange in Krugerrand futures and options. These efforts, however, failed due to insufficient financial backing and a lack of tradeability (market liquidity). In 1987, Eskom introduced the first standardized option contract to appear in South Africa. Known as the E168-11%-2008 option contract, it was written on the Eskom E168 stock which matures in the year 2008 and pays a coupon rate of 11 per cent. At the time of writing this thesis, exchange traded options are available on the following stocks:¹

E168-11%-2008,	R150-12%-2005,	R119-14%-1997,	UG55-15%- 2005,
R147-11,5%-2008,	R144-12,5%-1996,	E170-13,5%-2020,	E167- 12%-1996,
T004-7,5%-2008,	P001-10%-2008,	P002-10%-1993,	P005-12%-1998,

while over-the-counter options are available on a limited range of listed shares and futures, foreign currency and commodities.

While options have been traded for many centuries, the valuation of these contracts is relatively new, with the first attempt being recorded by Bachelier in 1900. There have since been numerous contributions to the theory of

¹The prefix E refers to an Eskom Loan, R to an RSA stock, UG to the Umgeni Waterboard, T to Transnet, P to Post and Telecommunications (Telkom).

option pricing. However, it was Fischer Black and Myron Scholes who, in 1973, presented in their paper entitled, “The Pricing of Options and Corporate Liabilities”, the first satisfactory option pricing model, accepted by both academics and market participants. The objective of this thesis will be to review the derivative theory of option valuation with specific attention being given to the derivation of the Black-Scholes and binomial option pricing models.

In the second chapter we will present some terminology and notation that will be employed in this thesis. Thereafter, a fundamental principle for option valuation will be given and the chapter will close with a discussion of some basic option trading strategies.

In chapter three, we will examine three types of option pricing approaches that have been adopted in the literature, namely, the discounted expected value, the recursive optimisation and the general equilibrium pricing approaches.

Two derivations of the Black-Scholes call option pricing model will then be presented in chapter four. The first derivation will use an Itô calculus approach, and the second derivation will be based upon the capital asset pricing framework of Sharpe and Lintner. Thereafter, we will show how the value of a European put option can be derived from the value of a call option, which is followed by the derivation of a valuation model for an American put option.

Our aim in chapter five will be to highlight the arbitrage pricing principle underlying option pricing theory. This will be done by deriving a two-state, discrete-time analogue to the Black-Scholes option pricing model, known as the Binomial option pricing model. We will then show, with the use of the De Moivre-Laplace theorem, that the Black-Scholes option pricing model can be derived as a special limiting case of the Binomial option pricing model.

Chapter six will deal with the relaxation of three of the underlying assumptions of the Black-Scholes model. By taking into account the payment of dividends and the possibility of a changing short term interest rate, we will show that the resulting option pricing model is only a slight modification of the Black-Scholes model. We will also derive, and solve, a partial differential equation for the price of an option where the variance of the rate of return on the underlying asset is assumed to be stochastic.

In the final chapter, the inherent flexibility of the Black-Scholes option pricing model will be illustrated where the valuation of the debt and equity of a firm, the valuation of convertible bonds, warrants, collateralised loans and the pricing of certain insurance contracts will be considered.

CHAPTER 2

BASIC OPTION CONCEPTS AND STRATEGIES

The aim of this chapter will be to introduce the terminology and notation that will be employed in this thesis. A fundamental principle that underlies option pricing theory will be given. Thereafter, four basic option trading strategies will be discussed.

2.1 TERMINOLOGY AND NOTATION

An *option contract* is a contract which, for a predefined period of time, gives the owner the right, but not the obligation, to trade a certain number of units of an underlying asset at a fixed price that is called the *exercise* or *strike* price. The price that is paid for the option is called the option *premium* while the date on which the option expires is called the *expiration* or *maturity* date.

Given the above definition, two basic types of option contracts can be identified, namely, a *call* option, which gives the purchaser the right to buy the underlying asset at the exercise price, and, a *put* option, which gives the purchaser the right to sell the underlying asset at the exercise price. The option holder is then said to have a *long position* in the contract as the option grants him the opportunity to exercise his option if he so wishes. The option writer, however, has a financial obligation to the option holder should he decide to exercise the option, and is thus said to have a *short position* in the contract. If the exercise price is set equal to the currently prevailing asset price, then the option is said to be trading *at-the-money*. If the exercise price is set below (above) the asset price, then the call (put) option is referred to as trading *in-the-money*, while, if the exercise price is set above (below) the asset price, then the call (put) option is referred to as trading *out-of-the-money*. An option that can be exercised at any time on, or before, the expiration date is called an *American* option, while one that can only be

exercised at maturity is called a *European option*.

In order to facilitate the discussion in the chapters to follow, the following notation will be employed, namely:

- t - current date,
- t^* - expiration date of the option,
- T - time to expiration ($t^* - t$),
- S_t - a random variable denoting the price of the underlying asset at time t ,
- S_{t^*} - a random variable denoting the price of the underlying asset at time t^* ,
- X - exercise price of the option,
- $C(s_t, t)$ - price of a call option at time t , based on an underlying asset with current price $S_t = s_t$,
- $C(s_{t^*}, t^*)$ - price of a call option at time t^* , based on an underlying asset with current price $S_{t^*} = s_{t^*}$,
- $P(s_t, t)$ - price of a put option at time t , based on an underlying asset with current price $S_t = s_t$,
- $P(s_{t^*}, t^*)$ - price of a put option at time t^* , based on an underlying asset with current price $S_{t^*} = s_{t^*}$,
- σ - standard deviation or volatility of return,
- r - risk-free interest rate,
- $N\{\cdot\}$ - the cumulative standard normal distribution,
- $n(\cdot)$ - the standard normal density function.

2.2 A FUNDAMENTAL PRINCIPLE FOR OPTION VALUATION

Since the exercising of an option is voluntary, the purchase price for a call option on an underlying asset, with current price, s_t , and time to expiration, T , can be given by:

$$C(s_t, t) = \max[0, s_t - X] . \quad (1)$$

This is often referred to as the option's *intrinsic value*. Similarly, the intrinsic value for a put option on an underlying asset with current price, s_t , and time to expiration, T , can be given by:

$$P(s_t, t) = \max[0, X - s_t] . \quad (2)$$

The above pricing principle will be used extensively throughout this thesis.

2.3 BASIC OPTION TRADING STRATEGIES

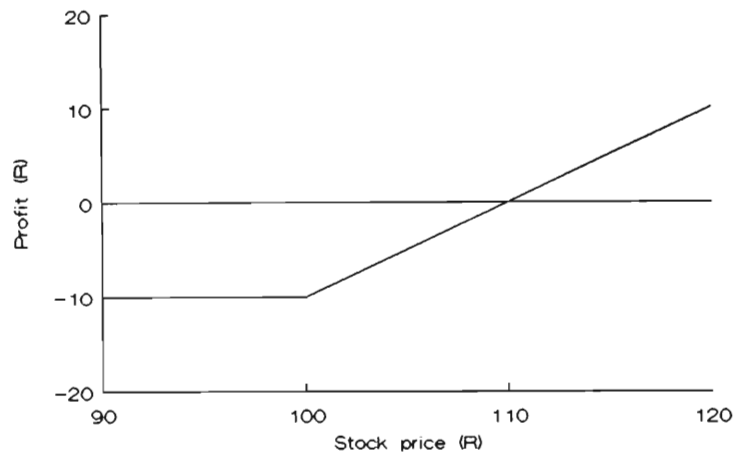
Before considering further the theory of option pricing, we will briefly focus our attention on a discussion of four basic option trading strategies that are being employed in the market. Our purpose for doing so will be to illustrate how certain risk-reward characteristics of the underlying asset can be obtained using call and put option instruments. In order to simplify the discussion, the following basic assumptions will be made, namely, that

- (1) the underlying asset on which the option is written costs R90, and that
- (2) put and call options, which expire in six months, are available.

The strike price of the option will be assumed to be R100 and the premium to be paid, R10.¹

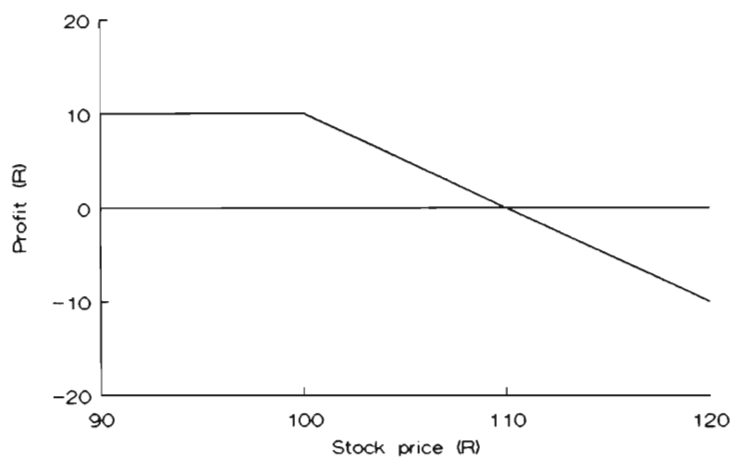
¹Although the price of both the call and put option is the same here, it does not presume that a call and put option with the same parameters will have the same value.

2.3.1 Purchase a call option



The profit-loss position of a call option holder, is illustrated above. The investor will lose the entire premium, if, by the expiration date, the underlying asset is still selling below the strike price of R100. He will only break even when the price of the asset equals the strike price plus the option premium paid for the call, in this case, R110 ($R100 + R10$). Thus only when the price of the asset rises above this break-even price of R110, does he come into profit.

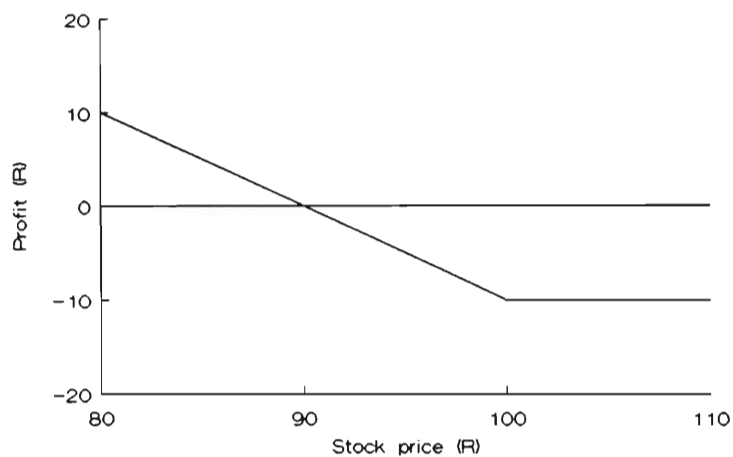
2.3.2 Sell or write a call option



The above graph illustrates the position of a “naked” or uncovered call option writer.² If the option holder does not exercise the option, the writer will get to keep the entire premium. Thus, this position is profitable so long as the price of the asset does not rise above that of the break-even price of $R100 + R10 = R110$. However, the option writer’s profit is limited only to that of the option premium that was paid, while his potential loss may be enormous.

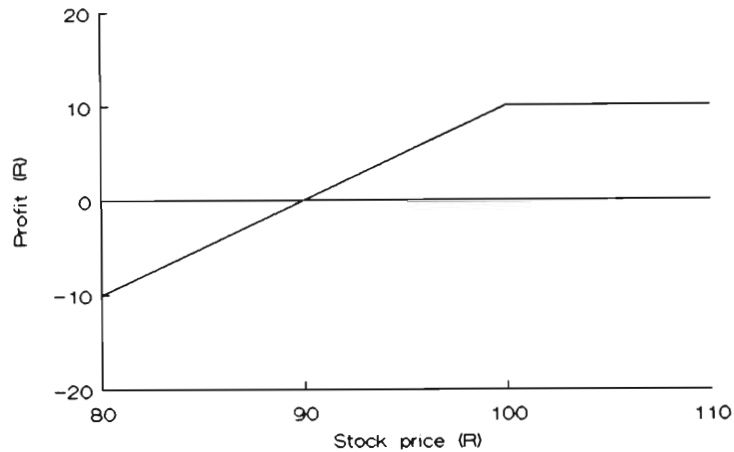
²A “Naked” or uncovered option writer is one who writes an option on a stock that he does not own.

2.3.3 Purchase a put option



As in the case of a call option holder, the put option holder's risk is limited to the premium paid for the option. The break-even point will then be given by the strike price less the premium paid, in this case, R90. If the stock price falls to R80, the put option buyer could realise a profit of R10 by buying the stock in the physical market at R80 and exercising his option to sell the stock at R100 to the option writer.

2.3.4 Sell or write a put option



Just as the writer of a “naked” call option receives the entire premium if, by the expiration date, the asset price is below the strike price, the writer of a “naked” put option receives the entire premium if the price of the asset is above the strike price.

In the case illustrated above, the writer will break even at the strike price less the premium paid for the option, in this case, R90. If the stock price falls below this price, the writer will begin to make a loss. If the stock price remains above this break-even price, the writer will make a profit that is limited to the premium paid.

Numerous other speculative strategies that involve options are also possible. Although it is not possible to explore every strategy here, the few that

we have explored do illustrate to some extent, the way in which options can be used to reduce an investor's exposure to market risk. It should, however, also be noted that even though the investor now has the opportunity to benefit from any favourable price movements in the underlying asset, this benefit comes at a cost, namely, that of the option premium.

CHAPTER 3

THEORIES OF OPTION PRICING

Broadly speaking, the following three types of option pricing models can be identified in the literature, namely

- (i) discounted expected-value models,
- (ii) recursive optimisation models, and
- (iii) general equilibrium models.

In this chapter we will briefly examine each of the above-mentioned models with a view to gaining a useful insight into the development of the Black-Scholes model. Use will be made of a European call option since it is the simplest type of option that is traded.

3.1 DISCOUNTED EXPECTED-VALUE MODELS

Discounted expected-value models assume that a call option will only be exercised at maturity.¹ For every possible price S_{t^*} that the underlying asset might assume on the expiration date of the option, the following are calculated:

- (a) the probability that the stock will assume the price S_{t^*} , viz. $P(S_{t^*} = s_{t^*})$, and
- (b) the expected future price for the option, namely

$$\begin{aligned} E[C(S_{t^*}, t^*)] &= E\{\max[0, S_{t^*} - X]\} \\ &= \int_X^\infty (S_{t^*} - X)f(S_{t^*})dS_{t^*} , \end{aligned} \tag{3}$$

where $f(S_{t^*})$ denotes the density function of S_{t^*} .

Employing an appropriate discount rate θ , we can then express the present value of (3) as

$$C(s_t, t) = e^{-\theta T} \int_X^\infty (S_{t^*} - X)f(S_{t^*})dS_{t^*} . \tag{4}$$

¹A call option holder, upon exercising the option, will receive the maximum of zero or $s_t - X$ where s_t is the current price of the underlying asset. However, prior to the date of expiration, the value of the call option is worth at least the difference between the current asset price and the present value of the exercise price, that is $C(s_t, t) = \max(0, s_t - Xe^{-rT})$. Since $r, T > 0$, $Xe^{-rT} < X \Rightarrow s_t - Xe^{-rT} > s_t - X$ it follows that the profit that can be obtained from exercising the option prior to maturity, i.e. $\max(0, s_t - X)$ is less than the intrinsic value for the option. Thus it is never optimal to exercise an American call option on a non-dividend paying asset prior to expiration because the return from exercising this option would be less than the return that one would obtain from selling the option at its intrinsic value in the market place.

Examples of discounted expected-value models are those that have been developed by Sprenkle (1964), Boness (1964) and Samuelson (1965). A brief discussion of these models will now follow.

3.1.1 The Sprenkle Model

Sprenkle (1964) assumed that the price of the underlying stock has a lognormal distribution. The expected value of the option on maturing is then given by

$$E [C(S_{t^*}, t^*)] = \int_X^\infty (S_{t^*} - X)\Lambda(S_{t^*})dS_{t^*} , \quad (5)$$

where $\Lambda(S_{t^*})$ denotes the lognormal density function of the stock price.

The following theorem (Smith (1976)) can now be used to evaluate the above integral, viz.

Theorem

If S_{t^*} follows a lognormal distribution and

$$Q = \begin{cases} 0, & \text{if } S_{t^*} > \phi X, \\ \lambda S_{t^*} - \gamma X, & \text{if } \phi X \geq S_{t^*} > \psi X, \\ 0, & \text{if } S_{t^*} < \psi X, \end{cases}$$

denotes a random variable, then

$$\begin{aligned} E(Q) &= \int_{\psi X}^{\phi X} (\lambda S_{t^*} - \gamma X) \Lambda(S_{t^*}) dS_{t^*}, \\ &= e^{\rho T} \lambda s_t \left[N \left\{ \frac{\ln(s_t/\psi X) + [\rho + (\sigma^2/2)]T}{\sigma\sqrt{T}} \right\} - N \left\{ \frac{\ln(s_t/\phi X) + [\rho + (\sigma^2/2)]T}{\sigma\sqrt{T}} \right\} \right] \\ &\quad - \gamma X \left[N \left\{ \frac{\ln(s_t/\psi X) + [\rho - (\sigma^2/2)]T}{\sigma\sqrt{T}} \right\} - N \left\{ \frac{\ln(s_t/\phi X) + [\rho - (\sigma^2/2)]T}{\sigma\sqrt{T}} \right\} \right]. \end{aligned}$$

where ρ denotes the continuously compounded expected rate of growth in the stock,² ψ , ϕ , λ and γ denote arbitrary, but known, parameters $\Lambda(S_{t^*})$ the lognormal density for S_{t^*} and $N\{\cdot\}$ denotes the cumulative standard normal distribution.

Applying the above result with $\lambda = \gamma = \psi = 1$, and $\phi = \infty$, to (5), one can obtain

$$\begin{aligned} E[C(S_{t^*}, t^*)] &= e^{\rho T} s_t \left[N \left\{ \frac{\ln(s_t/X) + [\rho + (\sigma^2/2)]T}{\sigma\sqrt{T}} \right\} \right] \\ &\quad - X \left[N \left\{ \frac{\ln(s_t/X) + [\rho - (\sigma^2/2)]T}{\sigma\sqrt{T}} \right\} \right]. \end{aligned} \quad (6)$$

² $E(S_{t^*}/S_t) = e^{\rho T}$

3.1.2 The Boness Model

Boness (1964), also assuming a lognormal distribution for the stock price, derived an expression for the expected terminal price of a call option by using the following conditional expectation argument, viz.

$$\begin{aligned}
 E[C(S_{t^*}, t^*)] &= E[E(\max[0, S_{t^*} - X] | S_{t^*} > X)], & (7) \\
 &= [E(S_{t^*} | S_{t^*} > X) - E(X | S_{t^*} > X)] P(S_{t^*} > X), \\
 &= \left[\left(\int_X^\infty S_{t^*} \Lambda(S_{t^*}) dS_{t^*} \right) / \left(\int_X^\infty \Lambda(S_{t^*}) dS_{t^*} \right) - X \right] \int_X^\infty \Lambda(S_{t^*}) dS_{t^*}, \\
 &= \int_X^\infty (S_{t^*} - X) \Lambda(S_{t^*}) dS_{t^*}. & (8)
 \end{aligned}$$

Discounting (8) by the expected rate of return on the stock, ρ , he arrived at the following present value formula for the price of an option

$$C(s_t, t) = e^{-\rho T} \int_X^\infty (S_{t^*} - X) \Lambda(S_{t^*}) dS_{t^*}. \quad (9)$$

Using the theorem of Smith with $\lambda = \gamma = e^{-\rho T}$, $\psi = 1$ and $\phi = \infty$, (9) may then be solved to yield:

$$\begin{aligned}
 C(s_t, t) &= s_t N \left\{ \frac{\ln(s_t/X) + [\rho + (\sigma^2/2)]T}{\sigma\sqrt{T}} \right\} \\
 &\quad - e^{-\rho T} X N \left\{ \frac{\ln(s_t/X) + [\rho - (\sigma^2/2)]T}{\sigma\sqrt{T}} \right\}. & (10)
 \end{aligned}$$

3.1.3 The Samuelson Model

In Samuelson's (1965) approach he chose to distinguish between the different risk characteristics of the option and those of the underlying asset. Having assumed that the distribution of the terminal stock price is lognormal, he discounted the expected terminal call option value by β , the expected rate of return on the option, rather than by ρ , the expected rate of return on the underlying stock, to yield

$$C(s_t, t) = e^{-\beta T} \int_X^\infty (S_{t^*} - X) \Lambda(S_{t^*}) dS_{t^*} . \quad (11)$$

Using the theorem of Smith and letting $\lambda = \gamma = e^{-\beta T}$, $\psi = 1$ and $\phi = \infty$ we then obtain the result that

$$\begin{aligned} C(s_t, t) = & e^{-(\beta-\rho)T} s_t N \left\{ \frac{\ln(s_t/X) + [\rho + (\sigma^2/2)]T}{\sigma\sqrt{T}} \right\} \\ & - e^{-\beta T} X N \left\{ \frac{\ln(s_t/X) + [\rho - (\sigma^2/2)]T}{\sigma\sqrt{T}} \right\} . \end{aligned} \quad (12)$$

3.2 RECURSIVE OPTIMISATION MODELS

The possibility of exercising an American option before maturity is taken into account in the recursive optimization model. The method consists of dividing the life of an option into a series of fixed time periods. At the end of each period the call option-holder then has the choice of either

- (i) exercising the option, or
- (ii) holding onto the call option for one more period.

In order to find the value of a call option with strike price X , we will divide the life of the call option into n periods. Letting s_t denote the price of the underlying asset at time t and $C(s_t, t, j)$ denote the value of a call option at time t , with j periods remaining to maturity, then, in view of the above two choices, we find that $C(s_t, t, j)$ will be given by the maximum of

- (i) zero,
- (ii) $s_t - X$, the present intrinsic value of the option, or
- (iii) the expected value of the call option with j periods to expiration, given that the option holder has decided to hold the option for one more period, i.e.

$$e^{-\theta h} E[C(S_{t+h}, t+h, j-1) | S_t = s_t], \quad (13)$$

where h is defined to be equal to the length of each time period and θ denotes an appropriate discount rate.

Thus by determining all the possible values that the price of the underlying asset might assume $n - 1$ periods before maturity, and a suitable course of action for each of these possible values, the value of a call option at time t ,

with n periods remaining to maturity, can now be derived from the following recursive equation (Samuelson, 1965, p.158)

$$\begin{aligned}
 C(s_t, t, n) &= \max[0, s_t - X, e^{-\theta h} \int_0^{\infty} C(S_{t+h}, t+h, n-1) \\
 &\quad P(S_{t+h} \leq s_{t+h} | S_t = s_t) dS_{t+h}] , \\
 C(S_{t^*}, t^*, 0) &= \max[0, S_{t^*} - X] .
 \end{aligned} \tag{14}$$

The above recursive optimisation procedure will be used in the next chapter to find the value of an American put option.

3.3 GENERAL EQUILIBRIUM MODELS

The approach of the general equilibrium model is to attempt to create, using a suitable option strategy, a hedged position in a certain asset in such a manner that the expected rate of return on this hedged position equals the return on a riskless asset. Such an approach has led to the derivation of the following pricing formula for options:

$$C(s_t, t) = s_t N \left\{ \frac{\ln(s_t/X) + [r + (\sigma^2/2)]T}{\sigma\sqrt{T}} \right\} - X e^{-rT} N \left\{ \frac{\ln(s_t/X) + [r - (\sigma^2/2)]T}{\sigma\sqrt{T}} \right\} . \quad (15)$$

Known as the Black-Scholes pricing formula, the above formula has become widely used in the market place and will form the focus of our attention in the next chapter.

CHAPTER 4

THE BLACK-SCHOLES OPTION PRICING MODEL

In this chapter we will derive the Black-Scholes formula for pricing a European call option. Initially, an Itô calculus approach will be used, and then, for comparative purposes, a capital asset pricing framework approach will be presented. Thereafter a pricing formula for a European put option and an American put option will be presented.

4.1 DERIVATION OF THE BLACK-SCHOLES CALL OPTION VALUATION MODEL USING ITÔ'S LEMMA

Given the following assumptions, namely that

- (1) the risk-free interest rate, r , is known and assumed to be constant,
- (2) the stock price has a lognormal distribution with a constant variance rate of return,
- (3) there are no dividend payments on the stock,
- (4) the option is a European option,
- (5) there are zero transaction costs and taxes,
- (6) trading takes place continuously,
- (7) there are no penalties for short sales,

let us assume that the price of a call option is expressible as a function of t , the current point in time, and s_t , the price of the underlying asset at time t . Furthermore, assume that $C(s_t, t)$ represents a twice continuously differentiable function with respect to s_t , and that the dynamics of the asset price can be adequately described by a stochastic differential equation of the form

$$dS_t/s_t = \mu dt + \sigma dw , \quad (16)$$

where μ denotes the expected instantaneous rate of return on the underlying asset, σ^2 the instantaneous variance for that return, and $w(\cdot)$ a standardized Wiener process.

Black and Scholes (1973) then demonstrate that it is possible to create a riskless hedge by combining a single share of the underlying stock with an appropriate quantity of European call options. This portfolio, if adjusted continuously with changes in the underlying stock price should then, in equilibrium, earn a rate of return that is identical to that of the riskless interest rate r .

In order to create such a hedged position, the number k , of call options that should be sold short, against one share of the stock that is to be held

long should satisfy the equation

$$k [C(s_t + \Delta s_t, t + \Delta t) - C(s_t, t)] = \Delta s_t .$$

Since

$$\begin{aligned} C(s_t + \Delta s_t, t + \Delta t) &= C(s_t, t) + \frac{\partial C(s_t, t)}{\partial t} \Delta t + \frac{\partial C(s_t, t)}{\partial s_t} \Delta s_t + \\ &\frac{1}{2} \frac{\partial^2 C(s_t, t)}{\partial t^2} (\Delta t)^2 + \frac{1}{2} \frac{\partial^2 C(s_t, t)}{\partial s_t^2} (\Delta s_t)^2 + \frac{\partial^2 C(s_t, t)}{\partial s_t \partial t} \Delta s_t \Delta t + \dots, \end{aligned}$$

this implies that

$$C(s_t + \Delta s_t, t + \Delta t) - C(s_t, t) \simeq C_1(s_t, t) \Delta s_t \quad \text{for small } \Delta t$$

where $C_1(s_t, t)$ refers to the partial derivative of $C(s_t, t)$ with respect to s_t .

Therefore

$$k = \frac{1}{C_1(s_t, t)} .$$

Thus the value, at time t , of the hedged portfolio can be given by

$$s_t - \frac{C(s_t, t)}{C_1(s_t, t)}, \tag{17}$$

with a change in value of this investment position over a short time interval being given by

$$dS_t - \frac{dC(S_t, t)}{C_1(s_t, t)}. \tag{18}$$

Since $C(s_t, t)$ is twice continuously differentiable, Itô's Lemma (see Appendix A) can now be used to express $dC(S_t, t)$ as follows:

$$dC(S_t, t) = C_1(s_t, t)dS_t + C_2(s_t, t)dt + \frac{1}{2}C_{11}(s_t, t)(dS_t)^2 . \quad (19)$$

Substitution of $(dS_t)^2$ from (16) then yields:¹

$$\begin{aligned} dC(S_t, t) &= C_1(s_t, t)dS_t + C_2(s_t, t)dt + \frac{1}{2}C_{11}(s_t, t)(\mu s_t dt + \sigma s_t dw)^2, \\ &= C_1(s_t, t)dS_t + C_2(s_t, t)dt + \frac{1}{2}C_{11}(s_t, t)(\mu^2 s_t^2 (dt)^2 \\ &\quad + 2\mu s_t^2 \sigma dt dw + \sigma^2 s_t^2 (dw)^2), \\ &= C_1(s_t, t)dS_t + C_2(s_t, t)dt + \frac{1}{2}C_{11}(s_t, t)\sigma^2 s_t^2 (dw)^2, \\ &= C_1(s_t, t)dS_t + C_2(s_t, t)dt + \frac{1}{2}C_{11}(s_t, t)\sigma^2 s_t^2 dt. \end{aligned} \quad (20)$$

Substitution of $dC(S_t, t)$ in (18) then yields:

$$\begin{aligned} dS_t - \frac{1}{C_1(s_t, t)} \left[C_1(s_t, t)dS_t + C_2(s_t, t)dt + \frac{1}{2}C_{11}(s_t, t)\sigma^2 s_t^2 dt \right] , \\ = -\frac{1}{C_1(s_t, t)} \left[C_2(s_t, t) + \frac{1}{2}C_{11}(s_t, t)\sigma^2 s_t^2 \right] dt . \end{aligned} \quad (21)$$

Since the hedged position is riskless, it must earn a rate of return that is equal to the risk-free interest rate. This then implies that the change in value of the hedged position in (21) must be equal to the value of the initial

¹We obtain step three as a consequence of the following results which appear in Appendix A, namely:

$$(dt)^2 = o(dt) \text{ and } dt dw = 0 .$$

hedged position in (17) multiplied by rdt , i.e.

$$-\frac{1}{C_1(s_t, t)} \left[C_2(s_t, t) + \frac{1}{2} C_{11}(s_t, t) \sigma^2 s_t^2 \right] dt = \left[s_t - \frac{C(s_t, t)}{C_1(s_t, t)} \right] r dt,$$

$$\Rightarrow -C_2(s_t, t) - \frac{1}{2} C_{11}(s_t, t) \sigma^2 s_t^2 = r s_t C_1(s_t, t) - r C(s_t, t),$$

which yields a second order linear, partial differential equation for the value of an option of the form

$$C_2(s_t, t) = r C(s_t, t) - r s_t C_1(s_t, t) - \frac{1}{2} \sigma^2 s_t^2 C_{11}(s_t, t). \quad (22)$$

A suitable boundary value condition that is needed to solve the above differential equation can be given by

$$C(s_{t^*}, t^*) = \max[0, s_{t^*} - X]. \quad (23)$$

To obtain a solution to the above partial differential equation, Black and Scholes noted that (22) could be transformed into a familiar heat-transfer equation which has a solution that is given by

$$C(s_t, t) = s_t N(d_1) - e^{-rT} X N(d_2), \quad (24)$$

where

$$d_1 = \frac{\ln(s_t/X) + [r + (\sigma^2/2)]T}{\sigma\sqrt{T}}, \quad (25)$$

and

$$d_2 = \frac{\ln(s_t/X) + [r - (\sigma^2/2)]T}{\sigma\sqrt{T}}. \quad (26)$$

(24) may also be used to find the value of an American call option since it is never optimal to exercise an American call option before maturity.²

4.2 DERIVATION OF THE BLACK-SCHOLES CALL OPTION VALUATION MODEL USING A CAPITAL ASSET PRICING FRAMEWORK

Consider the following Sharpe-Lintner formulation of the capital asset pricing model which states that the instantaneous rate of return on the call option over and above the instantaneous risk free rate of return, takes the form:

$$E \left(\frac{dC(S_t, t)}{C(s_t, t)} \right) = r + \beta_c (\mu_m - r) , \quad (27)$$

where μ_m denotes the expected instantaneous rate of return on the market portfolio,

$$\beta_c = \frac{\text{Cov} \left(\frac{dC(S_t, t)}{C(s_t, t)}, r_{m_t} | S_t = s_t \right)}{\text{var}(r_{m_t})} \quad (28)$$

and r_{m_t} is a random variable denoting the return on the market portfolio.

This then implies that

$$E \left(\frac{dC(S_t, t)}{C(s_t, t)} \right) = r dt + (\mu_m - r) \beta_c dt . \quad (29)$$

Similarly,

$$E \left(\frac{dS_t}{s_t} \right) = r dt + (\mu_m - r) \beta_s dt , \quad (30)$$

²See Section 3.1, footnote 1.

where

$$\beta_s = \frac{\text{Cov}\left(\frac{dS_t}{s_t}; r_{m_t}\right)}{\text{Var}(r_{m_t})}.$$

From (20) we can obtain the result that

$$\begin{aligned} \beta_c &= \frac{\text{Cov}\left(\frac{dC(S_t, t)}{C(s_t, t)}; r_{m_t} \middle| S_t = s_t\right)}{\text{Var}(r_{m_t})} \\ &= \frac{\text{Cov}\left(\frac{C_1(s_t, t)dS_t}{C(s_t, t)} + \frac{C_2(s_t, t)dt}{C(s_t, t)} + \frac{\frac{1}{2}C_{11}(s_t, t)\sigma^2 s_t^2 dt}{C(s_t, t)}; r_{m_t} \middle| S_t = s_t\right)}{\text{Var}(r_{m_t})} \\ &= \frac{\text{Cov}\left(\frac{C_1(s_t, t)dS_t}{C(s_t, t)}; r_{m_t} \middle| S_t = s_t\right)}{\text{Var}(r_{m_t})} \\ &= \frac{\frac{s_t C_1(s_t, t)}{C(s_t, t)} \text{Cov}\left(\frac{dS_t}{s_t}; r_m \middle| S_t = s_t\right)}{\text{var}(r_{m_t})} \\ &= \frac{s_t C_1(s_t, t)}{C(s_t, t)} \beta_s. \end{aligned} \tag{31}$$

Multiplying (29) by $C(s_t, t)$, and substituting for β_c from (31), one can then obtain the result that

$$E(dC(S_t, t)) = rC(s_t, t)dt + (\mu_m - r)s_t C_1(s_t, t)\beta_s dt. \tag{32}$$

Taking the expected value of (20) then yields:

$$E(dC(S_t, t)) = C_1(s_t, t)E(dS_t) + C_2(s_t, t)dt + \frac{1}{2}\sigma^2 s_t^2 C_{11}(s_t, t)dt, \tag{33}$$

which upon substitution for $E(dS_t)$ from (30) yields:

$$\begin{aligned}
 E(dC(S_t, t)) &= r s_t C_1(s_t, t) dt + (\mu_m - r) s_t C_1(s_t, t) \beta_s dt \\
 &\quad + C_2(s_t, t) dt + \frac{1}{2} \sigma^2 s_t^2 C_{11}(s_t, t) dt .
 \end{aligned} \tag{34}$$

Combining (32) and (34) then yields the following partial differential equation for pricing a call option, namely

$$\begin{aligned}
 rC(s_t, t) dt + (\mu_m - r) s_t C_1(s_t, t) \beta_s dt &= r s_t C_1(s_t, t) dt \\
 + (\mu_m - r) s_t C_1(s_t, t) \beta_s dt + C_2(s_t, t) dt + \frac{1}{2} \sigma^2 s_t^2 C_{11}(s_t, t) dt .
 \end{aligned}$$

This then implies that

$$C_2(s_t, t) = rC(s_t, t) - r s_t C_1(s_t, t) - \frac{1}{2} \sigma^2 s_t^2 C_{11}(s_t, t) ,$$

which is exactly the same pricing equation as is given in (22).

4.3 A VALUATION MODEL FOR A EUROPEAN PUT OPTION

In this section we will show how the value of a European put option on an underlying asset with exercise price X , and maturity date t^* , can be derived from the value of a European call option on the same underlying asset, with the same exercise price and maturity date.

Consider the following two portfolios:

Portfolio A: one call option, with exercise price X , maturing at time t^* , on an underlying asset with current price s_t , and one discount bond that will be worth X at time t^* .

Portfolio B: one put option with the same characteristics as those given for the call option in portfolio A, and one share of the underlying asset.

On maturing, portfolio A will be worth

$$\max[0, S_{t^*} - X] + X = \max[X, S_{t^*}],$$

while portfolio B will be worth

$$\max[0, X - S_{t^*}] + S_{t^*} = \max[S_{t^*}, X].$$

At time t the value of portfolio A will be

$$C(s_t, t) + X e^{-r(t^*-t)}$$

while the value of portfolio B at time t will be

$$P(s_t, t) + s_t.$$

The options, being of the European type, cannot be exercised prior to maturity. Since they have the same value at maturity they must therefore have

the same value at time t . This implies that a European put option must be priced so that,

$$P(s_t, t) = C(s_t, t) - s_t + X e^{-r(t^* - t)}. \quad (35)$$

This is known as the put-call parity.

Substitution of the Black-Scholes formula for $C(s_t, t)$ from (24) then yields:

$$\begin{aligned} P(s_t, t) &= s_t \cdot N\{d_1\} - X e^{-rT} \cdot N\{d_2\} \\ &= -s_t[1 - N\{d_1\}] + X e^{-rT} [1 - N\{d_2\}] \\ &= -s_t N(-d_1) + X e^{-rT} N(-d_2), \end{aligned} \quad (36)$$

where d_1 and d_2 are as given in (25) and (26) respectively.

4.4 A VALUATION MODEL FOR AN AMERICAN PUT OPTION

Since it might be optimal to exercise an American put option prior to the expiry date, the valuation model for a European put option that was given in the previous section, cannot be used to price an American put option. As a result the recursive optimisation procedure, outlined in section 3.2, will be used to derive the value of an American put option.

Consider the following two portfolios:

Portfolio A: One American put option, with strike price X and time to expiration, $T = t^* - t$, plus one share of the underlying asset.

Portfolio B: A discount bond that will be worth X at time t^* .

If the option is exercised at time $t < t^*$, the value of portfolio A will be

$$X - s_t + s_t = X,$$

while portfolio B will be worth

$$Xe^{-r(t^*-t)}.$$

At expiration (time t^*) portfolio A will be worth

$$\max[X - S_{t^*}, 0] + S_{t^*} = \max[X, S_{t^*}],$$

while portfolio B will be worth X .

Therefore portfolio A is always worth at least as much as portfolio B regardless of whether the option is exercised prior to expiration.

Thus, should one expect the asset price at expiration, s_{t^*} , to be less than the exercise price, X , it might be preferable to receive X , at time t , rather than at some later date (time t^*).

This being the case, to derive a valuation model for an American put option, it is necessary to take into account the possibility of exercising the option prior to the expiration date, prompting our use of the recursive optimisation procedure of section 3.2.

If the life of an American put option with strike price X is divided into n time periods, each of length h , then the value of this put option, at time t , with n periods remaining to maturity, on an underlying asset with price s_t , which is assumed to have a lognormal distribution can be given by:

$$\begin{aligned}
P(s_t, t, n) &= \max[0, X - s_t, e^{-\rho h} E[P_{n-1}(S_{t+h}, t + h) | S_t = s_t]], \\
&= \max[0, X - s_t, e^{-\rho h} \int_0^\infty P_{n-1}(S_{t+h}, t + h) P(S_{t+h} \leq s_{t+h} | S_t = s_t) dS_{t+h}], \\
&= \max[0, X - s_t, e^{-\rho h} \int_0^X (X - S_{t+h}) \Lambda(S_{t+h}) dS_{t+h}]. \tag{37}
\end{aligned}$$

If we let $\psi = 0$, $\phi = 1$, $\lambda = -e^{-\rho h}$, $\gamma = -e^{-\rho h}$ and $\rho = r$ in the theorem of Smith, we find that

$$\begin{aligned}
P(s_t, t, n) &= \max \left[0, X - s_t, -s_t N \left\{ \frac{-\ln(s_t/X) - (r + \frac{\sigma^2}{2})h}{\sigma\sqrt{h}} \right\} \right. \\
&\quad \left. + e^{-rh} X N \left\{ \frac{-\ln(s_t/X) - (r - \frac{\sigma^2}{2})h}{\sigma\sqrt{h}} \right\} \right]. \tag{38}
\end{aligned}$$

Thus, by determining a suitable course of action for each of the possible values that the underlying asset might assume $n - 1$ periods before maturity, the value of an American put option, at time t , with n periods remaining to maturity, can now be derived from (38) where r denotes the risk-free interest rate and σ^2 , the variance of the rate of return on the underlying asset.

CHAPTER 5

THE BINOMIAL OPTION PRICING MODEL

Our aim, in this chapter, will be to highlight the arbitrage pricing principle that underlies option pricing theory. This will be done by deriving an option pricing model which is set in a discrete time framework, known as the Binomial Option pricing model. We will also show how the model derived in Chapter 4, namely, the Black-Scholes model, which is set in a continuous-time framework, can be derived as a special limiting case of this binomial pricing model.

5.1 DERIVATION OF THE BINOMIAL OPTION PRICING MODEL

Consider dividing the time to expiration, T , of a call option into n periods each of length $h = \frac{T}{n}$. Suppose, furthermore, that the price of the asset at the end of each time period, $[t, t + h)$, will either increase to us_t , with probability q , or decrease to vs_t , with probability $1 - q$.

To avoid the possibility of making a riskless profit, any portfolio that contains the above stock and options on the above stock will require that $v < R < u$, where¹

$$R = (1 + r)^{T/n} . \quad (39)$$

¹Since $R \geq 1$, $v < 1$ and $u > 1$, it follows that $v < R$ and $v < u$. We may now consider the case where $v < u < R$ and $v < R < u$. If $v < u < R$ then a riskless arbitrage opportunity is created by lending the proceeds from a short sale at the risk-free interest rate R . Therefore we must have $v < R < u$ for no risk free arbitrage opportunities to arise.

In order to construct a portfolio that is risk free, a hedged position is created by writing one call option against τ shares of the underlying asset such that a gain (loss) in holding the underlying asset is offset by a loss (gain) in holding the option. The cost of this investment position is then given by the cost of buying the shares less the premium received for the option that is sold short, i.e.

$$\text{initial cost} = \tau s_t - C_n(s_t, t) . \quad (40)$$

At the end of the current period, this investment position will be worth either

$$\tau u s_t - C_{n-1}(u s_t, t + h) , \quad (41)$$

with probability q , if the asset price rises to $u s_t$, or

$$\tau v s_t - C_{n-1}(v s_t, t + h) , \quad (42)$$

with probability $1 - q$, if the asset price decreases to $v s_t$.

A condition for no arbitrage opportunities would now imply that,

$$\tau u s_t - C_{n-1}(u s_t, t + h) = \tau v s_t - C_{n-1}(v s_t, t + h) . \quad (43)$$

This then implies that the following choice for τ needs to be made:

$$\tau = \frac{C_{n-1}(u s_t, t + h) - C_{n-1}(v s_t, t + h)}{s_t(u - v)} . \quad (44)$$

Since the portfolio constructed was risk-free it must earn the risk-free rate of interest, thus requiring that

$$\tau s_t - C_n(s_t, t) = \frac{1}{R}[\tau u s_t - C_{n-1}(u s_t, t + h)]. \quad (45)$$

Substitution of τ from (44), then yields

$$\begin{aligned} & \frac{C_{n-1}(u s_t, t + h) - C_{n-1}(v s_t, t + h)}{u - v} - C_n(s_t, t) \\ &= \frac{1}{R} \left[\frac{u C_{n-1}(u s_t, t + h) - u C_{n-1}(v s_t, t + h)}{u - v} - C_{n-1}(u s_t, t + h) \right], \\ &= \frac{1}{R} \left[\frac{d C_{n-1}(u s_t, t + h) - u C_{n-1}(v s_t, t + h)}{u - v} \right], \\ \Rightarrow C_n(s_t, t) &= \frac{1}{R} \left[\frac{R - v}{u - v} C_{n-1}(u s_t, t + h) + \frac{u - R}{u - v} C_{n-1}(v s_t, t + h) \right]. \quad (46) \end{aligned}$$

On defining²

$$p_2 = \frac{R - v}{u - v},$$

(46) may now be written as:

$$\begin{aligned} C_n(s_t, t) &= \frac{1}{R} [p_2 C_{n-1}(u s_t, t + h) + (1 - p_2) C_{n-1}(v s_t, t + h)], \\ &= R^{-1} [p_2 \max[0, u s_t - X] + (1 - p_2) \max[0, v s_t - X]], \quad (47) \end{aligned}$$

²For no arbitrage opportunities to occur we require that $v < R < u$, which implies that $0 < \frac{R - v}{u - v} < 1$.

which yields a single period pricing formula for the option under consideration.

In order to extend (47) to a multiperiod framework, let us define the following random variable:

$I =$ the number of times that the stock price rises in the n time periods remaining to maturity.

Given the above definition, we may then argue that if the stock price rises k times (and falls $n - k$ times) in the n time periods left to maturity, we will have the following probability distribution for I , namely

$$P(I = k) = \binom{n}{k} q^k (1 - q)^{n-k} .$$

Since in equilibrium, we require that,

$$\begin{aligned} q(us_t) + (1 - q)(vs_t) &\stackrel{e}{=} Rs_t , \\ \Rightarrow q &\stackrel{e}{=} \frac{R - v}{u - v} = p_2 , \end{aligned}$$

we can obtain the following result

$$P(I = k) \stackrel{e}{=} \binom{n}{k} p_2^k (1 - p_2)^{n-k} .$$

Thus, conditional on the assumption that k rises in the stock price occur, the price of the call option, at maturity, can be given by:

$$C_n(s_t, t | I = k) = \binom{n}{k} p_2^k (1 - p_2)^{n-k} \max[0, u^k v^{n-k} s_t - X] . \quad (48)$$

Discounting the sum of all the possible terminal option value outcomes by R^{-n} , the pricing formula,

$$C_n(s_t, t) = R^{-n} \sum_{k=0}^n \binom{n}{k} p_2^k (1 - p_2)^{n-k} \max[0, u^k v^{n-k} s_t - X] \quad (49)$$

can be obtained for a call option which expires in n time periods, and where the underlying asset price follows a binomial process.

In order to simplify the expression that is given in (49) let

$i =$ the minimum number of asset price rises over the n time periods that is required for the call option to finish in-the-money.

This then implies that i is the smallest, non-negative integer satisfying:

$$u^i v^{n-i} s_t > X ,$$

$$\Rightarrow i \ln u + (n - i) \log v > \ln X - \ln s_t ,$$

$$\Rightarrow i \ln \frac{u}{v} > \ln \frac{X}{s_t} - n \ln v ,$$

$$\Rightarrow i > \frac{\ln \frac{X}{s_t} - n \ln v}{\ln \frac{u}{v}} . \quad (50)$$

Thus, we can write (50) as

$$i = \frac{\ln \frac{X}{s_t} - n \ln v}{\ln \frac{u}{v}} + \epsilon , \quad (51)$$

where, $0 \leq \epsilon < 1$, is introduced so as to make i an integer.

Hence, (49) may be written as

$$C_n(s_t, t) = R^{-n} \sum_{k=i}^n \binom{n}{k} p_2^k (1 - p_2)^{n-k} (u^k v^{n-k} s_t - X) . \quad (52)$$

Splitting (52) into two terms then results in the following discrete-time formula for a call option that has n periods remaining to maturity:

$$\begin{aligned} C_n(s_t, t) &= s_t \left[\sum_{k=i}^n \binom{n}{k} p_2^k (1 - p_2)^{n-k} \left(\frac{u^k v^{n-k}}{R^n} \right) \right] \\ &\quad - X R^{-n} \left[\sum_{k=i}^n \binom{n}{k} p_2^k (1 - p_2)^{n-k} \right] , \end{aligned} \quad (53)$$

$$= s_t B_1 - X R^{-n} B_2 , \quad (54)$$

where

$$B_1 = \sum_{k=i}^n \binom{n}{k} p_1^k (1 - p_1)^{n-k} , \quad (55)$$

$$B_2 = \sum_{k=i}^n \binom{n}{k} p_2^k (1 - p_2)^{n-k} , \quad (56)$$

$$p_1 = \left(\frac{u}{R} \right) \left(\frac{R - v}{u - v} \right) , \quad (57)$$

$$p_2 = \frac{R - v}{u - v} . \quad (58)$$

5.2 CONVERGENCE OF THE BINOMIAL OPTION PRICING FORMULA TO THE BLACK-SCHOLES OPTION PRICING FORMULA

Having derived a Binomial option pricing model, we will, in this section, attempt to show how the above model contains the Black-Scholes model as a special limiting case. The discussion hereafter will be based on a general convergence procedure that was developed by Hsia (1983).

Using the results that for small r

$$\ln(1 + r) \simeq r \tag{59}$$

and

$$e^{-rT} \simeq (1 + r)^{-T}, \tag{60}$$

we may, for small r , write the Black-Scholes option pricing formula in (24) as:

$$C(s_t, t) = s_t N(d_1) - \frac{X}{(1 + r)^T} N(d_2), \tag{61}$$

where

$$d_1 = \frac{\ln(s_t/X) + [\ln(1 + r) + (\sigma^2/2)]T}{\sigma\sqrt{T}}, \tag{62}$$

and

$$d_2 = \frac{\ln(s_t/X) + [\ln(1 + r) - (\sigma^2/2)]T}{\sigma\sqrt{T}}. \tag{63}$$

On comparison of (61) with (54) and recalling that

$$R = (1 + r)^{T/n},$$

where T denotes the time to expiration, and n the number of periods into which the life of a call option has been divided; the proof of the convergence of the Binomial option pricing model to the Black-Scholes option pricing model will be complete if it can be shown that $B_1 \rightarrow N(d_1)$ and $B_2 \rightarrow N(d_2)$ as $n \rightarrow \infty$.

To this end, use will be made of the De Moivre-Laplace theorem (Rahman, 1968) which states that a binomial distribution converges to a normal distribution if $np \rightarrow \infty$ as $n \rightarrow \infty$.

From (55) and (56)

$$B_j \stackrel{r}{=} \sum_{k=1}^n P(I_j = k), \quad j = 1, 2, \dots,$$

where I_j denotes a binomial random variable with parameters n and p_j .³

Since B_j is related to a cumulative distribution function of a random variable having a binomial distribution with parameters n and p_j and since $p_j \in (0, 1)$, by the De-Moivre Laplace Theorem we need only show that $B_j \rightarrow N(d_j)$ as $n \rightarrow \infty$.

³The symbol $\stackrel{r}{=}$ means that B_j is related to the cumulative distribution function of the random variable I_j .

Now

$$B_j \rightarrow \int_i^\infty f(t)dt = \int_{\frac{i-E(I_j)}{\sqrt{\text{Var}(I_j)}}}^\infty n(z)dz = N(x_j), \quad j = 1, 2, \quad (64)$$

where

$$x_j = \frac{E(I_j) - i}{\sqrt{\text{Var}(I_j)}}, \quad (65)$$

- I_j denotes the random number of asset price rises in the n time periods,
- i denotes the minimum number of rises in the asset price over the n time periods that is required for the call option to finish in-the-money,
- $f(\cdot)$ denotes a normal density function,
- $n(\cdot)$ denotes a standard normal density function,
- $N(\cdot)$ denotes the cumulative standard normal distribution.

In order to obtain an expression for $E(I_j)$ and $\text{Var}(I_j)$, recall that the price of the underlying asset at expiration (time t^*) is given by:

$$S_{t^*} = u^{I_j} v^{n-I_j} s_t.$$

Hence,

$$\frac{S_{t^*}}{s_t} = u^{I_j} v^{n-I_j}.$$

This implies that

$$\begin{aligned} \ln \left(\frac{S_{t^*}}{s_t} \right) &= I_j \ln u + (n - I_j) \ln v, \\ &= I_j \ln \left(\frac{u}{v} \right) + n \ln v, \end{aligned}$$

and thus that,

$$I_j = \frac{\ln(S_{t^*}/s_t) - n \ln v}{\ln(u/v)}. \quad (66)$$

Using the properties of expectations, we may write the mean and variance of I_j respectively as:

$$E(I_j) = \frac{E[\ln(S_{t^*}/s_t)] - n \ln v}{\ln(u/v)}, \quad (67)$$

and

$$\text{Var}(I_j) = \frac{\text{Var}[\ln(S_{t^*}/s_t)]}{[\ln(u/v)]^2}. \quad (68)$$

Substituting (51) and the above expressions for the mean and variance of I_j into (65) one can obtain the result that

$$\begin{aligned} x_j &= \frac{\frac{E[\ln(S_{t^*}/s_t)] - n \ln v}{\ln(u/v)} - \frac{\ln(X/s_t) - n \ln v}{\ln(u/v)} - \epsilon}{\frac{\sqrt{\text{Var}[\ln(S_{t^*}/s_t)]}}{\ln(u/v)}}, \\ &= \frac{E[\ln(S_{t^*}/s_t)] - n \ln v - \ln(X/s_t) + n \ln v}{\sqrt{\text{Var}[\ln(S_{t^*}/s_t)]}} \\ &\quad - \frac{\epsilon}{\frac{\sqrt{\text{Var}[\ln(S_{t^*}/s_t)]}}{\ln(u/v)}}, \\ &= \frac{E[\ln(S_{t^*}/s_t)] + \ln(s_t/X)}{\sqrt{\text{Var}[\ln(S_{t^*}/s_t)]}} - \frac{\epsilon}{\frac{\sqrt{\text{Var}[\ln(S_{t^*}/s_t)]}}{\ln(u/v)}}. \end{aligned} \quad (69)$$

Since the stock price rises with probability p_j , and thus falls with probability $1 - p_j$,

$$\text{Var}(I_j) = \frac{\text{Var}[\ln(S_{t^*}/s_t)]}{[\ln(u/v)]^2} = np_j(1 - p_j) .$$

Substituting this value for $\text{Var}(I_j)$ in (69) one can obtain the result that:

$$\begin{aligned} x_j &= \frac{\ln(s_t/X) + E[\ln(S_{t^*}/s_t)]}{\sqrt{\text{Var}[\ln(S_{t^*}/s_t)]}} - \frac{\epsilon}{\sqrt{np_j(1 - p_j)}} , \\ &= \frac{\ln(s_t/X) + E[\ln(S_{t^*}/s_t)]}{\sqrt{\text{Var}[\ln(S_{t^*}/s_t)]}} - \frac{\epsilon}{\sqrt{n}} \frac{1}{\sqrt{p_j(1 - p_j)}} . \end{aligned} \quad (70)$$

Noting that $\frac{\epsilon}{\sqrt{n}} \frac{1}{\sqrt{p_j(1 - p_j)}}$ tends to zero as $n \rightarrow \infty$,

$$x_j \rightarrow \frac{\ln(s_t/X) + E[\ln(S_{t^*}/s_t)]}{\sqrt{\text{Var}[\ln(S_{t^*}/s_t)]}} \text{ as } n \rightarrow \infty \quad (71)$$

In order to simplify the denominator of x , note that, as $n \rightarrow \infty$, the underlying asset's price dynamics can be shown to converge to a geometric Brownian motion process of the form:

$$dS_t = \mu s_t dt + \sigma s_t dw ,$$

where $w(\cdot)$ denotes a standard Wiener process.

We can now use this result to find the variance of $\ln(S_{t^*}/s_t)$ which can be done by letting $Y = \ln S_t$.

Application of Itô's lemma then yields

$$dY = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dw ,$$

and thus,

$$\begin{aligned} Y(t^*) &= Y(t) + \int_t^{t^*} \left(\mu - \frac{1}{2}\sigma^2\right) dt + \int_t^{t^*} \sigma dw , \\ &= Y(t) + \left(\mu - \frac{1}{2}\sigma^2\right) (t^* - t) + \sigma[w(t^*) - w(t)] . \end{aligned}$$

Now

$$\begin{aligned} \ln(S_{t^*}/s_t) &= Y(t^*) - Y(t), \\ &= \left(\mu - \frac{1}{2}\sigma^2\right) (t^* - t) + \sigma[w(t^*) - w(t)], \\ &= \left(\mu - \frac{1}{2}\sigma^2\right) T + \sigma w(T) . \end{aligned}$$

This then implies that

$$\ln(S_{t^*}/s_t) \sim N \left[\left(\mu - \frac{1}{2}\sigma^2\right) T; \sigma^2 T \right] , \quad (72)$$

and thus substituting for $\text{Var}[\ln(S_{t^*}/s_t)] = \sigma^2 T$ in (71), we have:

$$x_j = \frac{\ln(s_t/X) + E[\ln(S_{t^*}/s_t)]}{\sigma\sqrt{T}} . \quad (73)$$

Examination of the value for x_j in (73), and the values for d_1 in (62) and d_2 in (63), show that in order to prove the convergence of the Binomial option

pricing model to the Black-Scholes option pricing model, we need only show that, as $n \rightarrow \infty$,

$$E[\ln(S_{t^*}/s_t)] = \begin{cases} [\ln(1+r) + (\sigma^2/2)]T & , \text{ for } j = 1, \\ [\ln(1+r) - (\sigma^2/2)]T & , \text{ for } j = 2 \end{cases} \quad (74)$$

$$(75)$$

To prove (74), recall that, from (57),

$$p_1 = \left(\frac{u}{R}\right) \left(\frac{R-v}{u-v}\right),$$

which implies that,⁴

$$R = \left(p_1 \frac{1}{u} + (1-p_1) \frac{1}{v}\right)^{-1},$$

and thus that

$$R^n = (1+r)^T = \left[p_1 \left(\frac{1}{u}\right) + (1-p_1) \left(\frac{1}{v}\right)\right]^{-n}. \quad (76)$$

Dividing the life of the call option into n periods and letting S_j denote the price of the asset during the time period j , we may write:

$$s_t/S_{t^*} = (s_0/S_n) = (s_0/s_1)(s_1/s_2) \cdots (s_{n-1}/S_n) = \prod_{j=1}^n \frac{s_{j-1}}{S_j}.$$

4

$$\begin{aligned} (57) \text{ implies } \quad R p_1 &= u \left(\frac{R-v}{u-v}\right), \\ &= \frac{uR}{u-v} - \frac{uv}{u-v}, \\ \Rightarrow \quad R p_1 - \frac{uR}{u-v} &= -\frac{uv}{u-v}, \end{aligned}$$

As the future price of the asset depends only on the current price of the asset and not on its past history of prices, we may use the properties of expectations to write:⁵

$$E(s_t/S_{t^*}) = \prod_{j=1}^n E\left(\frac{s_{j-1}}{S_j}\right). \quad (77)$$

Since

$$S_j = \begin{cases} (s_{j-1})u & , \text{ with probability } p_1, \text{ and} \\ (s_{j-1})v & , \text{ with probability } 1 - p_1, \end{cases}$$

one can obtain the result that

$$E\left(\frac{s_{j-1}}{S_j}\right) = p_1 \left(\frac{1}{u}\right) + (1 - p_1) \left(\frac{1}{v}\right). \quad (78)$$

Thus (77) may be written as:

$$[E(s_t/S_{t^*})]^{-1} = \left[p_1 \left(\frac{1}{u}\right) + (1 - p_1) \left(\frac{1}{v}\right) \right]^{-n}. \quad (79)$$

$$\begin{aligned} \Rightarrow R \left[p_1 - \frac{u}{u-v} \right] &= -\frac{uv}{u-v}, \\ \Rightarrow R &= \left(-\frac{uv}{u-v} \right) \left(\frac{u-v}{p_1(u-v) - u} \right), \\ &= \frac{-uv}{u(p_1 - 1) - p_1v}, \\ &= \left(\frac{-p_1v - u(1 - p_1)}{-uv} \right)^{-1}, \\ &= \left(p_1 \frac{1}{u} + (1 - p_1) \frac{1}{v} \right)^{-1}. \end{aligned}$$

⁵ $E(XY) = E(X)E(Y)$ if X and Y are independent random variables.

Combining (76) and (79) then results in

$$\begin{aligned}
(1+r)^T &= [E(s_t/S_{t^*})]^{-1}, \\
\Rightarrow (1+r)^{-T} &= E(s_t/S_{t^*}), \\
\Rightarrow -T \ln(1+r) &= \ln[E(s_t/S_{t^*})], \tag{80}
\end{aligned}$$

when $P(S_j = s_{j-1}u) = p_1$.

From the result that is given in (72), namely, that (S_{t^*}/s_t) has a lognormal distribution, we may deduce that (s_t/S_{t^*}) has a lognormal distribution.⁶

The properties of the lognormal distribution can now be employed to simplify (80) as follows:

$$\begin{aligned}
-T \ln(1+r) &= \ln[E(s_t/S_{t^*})], \\
&= E[\ln(s_t/S_{t^*})] + \frac{1}{2} \text{Var}[\ln(s_t/S_{t^*})],^7 \\
&= E[-\ln(S_{t^*}/s_t)] + \frac{1}{2} \text{Var}[-\ln(S_{t^*}/s_t)],^8 \\
&= -E[\ln(S_{t^*}/s_t)] + \frac{1}{2} \text{Var}[\ln(S_{t^*}/s_t)].
\end{aligned}$$

⁶Aitchison and Brown (1957, p.11). See (B1)

⁷Aitchison and Brown (1957, p.8). See (B2)

⁸ $\log(s_t/S_{t^*}) = -\log(S_{t^*}/s_t)$

Therefore,⁹

$$\begin{aligned} E[\ln(S_{t^*}/s_t)] &= T \ln(1+r) + \frac{1}{2} \text{Var}[\ln(S_{t^*}/s_t)], \\ &= [\ln(1+r) + \sigma^2/2]T, \end{aligned}$$

and (74) is proved.

Following a similar argument, one can also show that (75) holds when $P(S_j = s_{j-1}u) = p_2$. In order to see this note that, from (58),

$$\begin{aligned} p_2 &= \frac{R-v}{u-v}, \\ \Rightarrow R^n &= (1+r)^T = [p_2u + (1-p_2)v]^n. \end{aligned} \quad (81)$$

Now,

$$E(S_{t^*}/s_t) = \prod_{j=1}^n E(S_j/s_{j-1}) = [p_2u + (1-p_2)v]^n. \quad (82)$$

(81) and (82) together imply that

$$(1+r)^T = E(S_{t^*}/s_t).$$

Thus

$$\begin{aligned} T \ln(1+r) &= \ln[E(S_{t^*}/s_t)], \\ &= E[\ln(S_{t^*}/s_t)] + \frac{1}{2} \text{Var}[\ln(S_{t^*}/s_t)]. \\ \Rightarrow E[\ln(S_{t^*}/s_t)] &= T \ln(1+r) - \frac{1}{2} \text{Var}[\ln(S_{t^*}/s_t)], \\ &= [\ln(1+r) - \sigma^2/2]T, \end{aligned}$$

⁹ $\text{Var}[\ln(S_{t^*}/s_t)] = \sigma^2 T$. See (72).

and (75) is also true.

With the use of the De Moivre-Laplace theorem we have proved that $B_1 \xrightarrow[n \rightarrow \infty]{} N(d_1)$ and $B_2 \xrightarrow[n \rightarrow \infty]{} N(d_2)$, and so the convergence of the Binomial option pricing model to the Black-Scholes option pricing model is complete.

It should be noted that because p_1 and p_2 are fixed constants between zero and one, one need only require that $n \rightarrow \infty$ for the above proof to hold true. Previous attempts at proving the above convergence result have required far more stringent conditions. For example, Rendleman and Barter (1979) show in the appendix to their paper that the convergence of their binomial option pricing model to the Black-Scholes option pricing model depends on the relation:¹⁰

$$\lim_{n \rightarrow \infty} p_1 = q .$$

Hsia (1983) shows that Rendleman and Barter's condition implies that as $n \rightarrow \infty$, $\psi = \phi = \theta$ and that this is possible if and only if $\psi = \phi = \theta = \frac{1}{2}$ as $n \rightarrow \infty$.¹¹

¹⁰ Rendleman and Barter (1979, p.1109), p_1, p_2 and q are equivalent to ψ, ϕ and θ respectively in Rendleman and Barter.

¹¹Hsia (1983, p.46).

Cox, Ross and Rubinstein (1979) show that the convergence of their binomial option pricing model to the Black-Scholes option pricing model holds only for the case where¹²

$$u = \exp(\sigma\sqrt{T/n}),$$

$$v = u^{-1}, \text{ and}$$

$$p = \frac{1}{2} + \frac{1}{2} \left(\frac{u}{\sigma} \right) \sqrt{T/n}.$$

Hsia's convergence proof however imposes no restrictions on u , d and p and is therefore a more general proof of the convergence of the binomial option pricing model to the Black-Scholes model.

¹² Cox, Ross and Rubinstein (1979, p.249).

CHAPTER 6

MODIFICATIONS OF THE BLACK-SCHOLES MODEL

The derivation of the Black-Scholes formula is based on the fulfillment of certain “ideal conditions”. In this chapter attention will be given to the relaxation of the following three assumptions, namely that

- (i) there are no dividend payments on the asset,
- (ii) the short-term interest rate is known and is constant, and
- (iii) the variance of the rate of return on the asset is constant.

In the first section of this chapter, we will examine the effect that a dividend payment has on the Black-Scholes formula for a European call option. We will show that with a slight modification of the Black-Scholes formula dividend payments can be taken into account.

In section two we will incorporate a time varying interest rate into the Black-Scholes model, and, in section three, we will derive and solve a partial differential equation for the price of a call option on an underlying asset where the variance of the rate of return on the asset is assumed to be stochastic. All the other assumptions that have been made by Black and Scholes (1973) will however be maintained throughout the chapter.

6.1 THE EFFECT OF DIVIDENDS

To analyse the effect of making a dividend payment on a European call option, let us denote by D , the dividend payment that is to be made per share on an underlying asset that has a current price of s_t . Furthermore, let us assume that the dividend payments are made continuously so that the dividend yield, which we will denote by $\delta = D/s_t$, is constant.

The instantaneous return on the dividend-paying asset can therefore be given by

$$dS_t + \delta s_t dt .$$

Consider now a portfolio that is formed by selling $\frac{1}{C_1(s_t, t)}$ European call options short against one share of stock that is held long.¹

The cost of creating such a portfolio will be given by

$$\text{initial value} = s_t - \frac{C(s_t, t)}{C_1(s_t, t)} , \quad (83)$$

and thus the instantaneous change in value of this portfolio, will take the form:

$$\text{change in value} = (dS_t + \delta s_t dt) - \frac{dC(S_t, t)}{C_1(s_t, t)} . \quad (84)$$

¹The subscript in $C_i(s_t, t)$ refers to the partial derivative of $C(s_t, t)$ with respect to its i th argument. This notation will be employed throughout the chapter.

Substituting for $dC(S_t, t)$ from (20) then yields

$$\begin{aligned}
\text{change in value} &= (dS_t + \delta s_t dt) - \frac{1}{C_1(s_t, t)} [C_1(s_t, t)dS_t + C_2(s_t, t)dt \\
&\quad + \frac{1}{2}C_{11}(s_t, t)\sigma^2 s_t^2 dt] , \\
&= \delta s_t dt - \frac{1}{C_1(s_t, t)} [C_2(s_t, t)dt + \frac{1}{2}C_{11}(s_t, t)\sigma^2 s_t^2 dt] . \quad (85)
\end{aligned}$$

In order to assume that the portfolio we created is perfectly hedged, it must, for no arbitrage opportunities to occur, earn the risk-free interest rate. Thus, the change in value of the portfolio in (85) must be equal to the initial value of the portfolio in (83) multiplied by the risk-free interest rate, r , i.e.

$$\begin{aligned}
C_1(s_t, t)\delta s_t dt - C_2(s_t, t)dt - \frac{1}{2}C_{11}(s_t, t)\sigma^2 s_t^2 dt &= [s_t C_1(s_t, t) - C(s_t, t)]r dt, \\
\Rightarrow C_1(s_t, t)\delta s_t - C_2(s_t, t) - \frac{1}{2}C_{11}(s_t, t)\sigma^2 s_t^2 &= r s_t C_1(s_t, t) - r C(s_t, t) . \\
\Rightarrow C_2(s_t, t) &= r C(s_t, t) - r s_t C_1(s_t, t) + C_1(s_t, t)\delta s_t - \frac{1}{2}C_{11}(s_t, t)\sigma^2 s_t^2, \\
&= r C(s_t, t) - s_t C_1(s_t, t)[r - \delta] - \frac{1}{2}\sigma^2 s_t^2 C_{11}(s_t, t) . \quad (86)
\end{aligned}$$

A boundary condition needed to solve the above second order partial differential equation can be given by

$$C(s_{t^*}, t^*) = \max[0, s_{t^*} - X] . \quad (87)$$

It can be established by substitution that the solution to (86) subject to the boundary condition in (87) is given by:

$$C(s_t, t) = e^{-\delta T} s_t N \left\{ \frac{\ln(s_t/X) + [r - \delta + (\sigma^2/2)]T}{\sigma\sqrt{T}} \right\} - e^{-rT} X \cdot N \left\{ \frac{\ln(s_t/X) + [r - \delta - (\sigma^2/2)]T}{\sigma\sqrt{T}} \right\}. \quad (88)$$

Thus the price of a European call option on an underlying asset, which pays a dividend continuously at a rate δ , is given by the above modification of the Black-Scholes formula.

To modify the European put option formula to account for dividend payments, one simply substitutes the modified solution for $C(s_t, t)$ in (88) into the relation obtained in (35), namely,

$$P(s_t, t) = C(s_t, t) - s_t + X e^{-r(t^*-t)}.$$

It should be noted that (88) does not hold for an American call option on a dividend paying asset, as it can be shown that, by prematurely exercising the option a riskless profit opportunity may occur. To see this consider the following two portfolios:

Portfolio A: The purchase of one American call option, with exercise price X , maturing at time t^* , on an underlying asset with current price s_t , which pays a certain dividend D at time t^* , and one discount bond that will be worth X at time t^* .

Portfolio B: One share of the underlying asset on which the option in portfolio A is written.

At maturity, portfolio A will be worth

$$\max[0, S_{t^*} - X] + X + D = \max[X + D, S_{t^*} + D]$$

and portfolio B will be worth

$$S_{t^*} + D.$$

Thus at maturity the value of portfolio A will be greater than or equal to that of portfolio B. At time $t < t^*$, however,

$$\text{value of portfolio A} = \max[0, s_t - X] + (X + D)e^{-r(t^*-t)},$$

and

$$\text{value of portfolio B} = s_t + De^{-r(t^*-t)}.$$

Thus, when $s_t < X$, the value of portfolio A is not always greater than or equal to the value of portfolio B. Thus it might be advantageous to exercise an American call option on a dividend paying asset prior to maturity.

6.2 THE EFFECT OF A TIME VARYING INTEREST RATE

To analyse the effect of a time varying interest rate on the value of a call option we will follow the approach of Merton (1973a), where he assumed that the price of a call option can be expressed as a function not only of the underlying stock's price and time to maturity, but also of the rate of return on a pure discount bond.

Assuming the following price dynamics for the underlying asset and the discount bond

$$dS_t = \mu S_t dt + \sigma S_t dw, \quad (89)$$

$$dB_t = \alpha B_t dt + \delta B_t dq, \quad (90)$$

where

μ and α denote the instantaneous expected returns on the stock and bond respectively,

σ^2 and δ^2 denote the instantaneous variance of returns on the stock and bond respectively,

w and q denote standard Wiener processes, and

B_t is the price of a pure discount bond that pays R1, T years from now,

Merton proceeded to create a hedged portfolio consisting of the following investment in the call option, the underlying asset and the discount bond:

$$H_t = Q_C C(S_t, t) + Q_S S_t + W_B B_t, \quad (91)$$

with

$$W_C = Q_C C(S_t, t), \quad (92a)$$

$$W_S = Q_S S_t, \text{ and} \quad (92b)$$

$$W_B = Q_B B_t \quad (92c)$$

denoting the total amount invested in the option, the underlying asset and the discount bond respectively, where the total investment is zero, i.e.² the value of the hedged portfolio can be given by

$$H_t = W_C + W_S + W_B = 0,$$

Under the assumption that the price of the call option is a function of the price of the underlying asset, the bond price and time, we may apply Itô's lemma to express the change in the value of the call option as follows:³

$$\begin{aligned}
dC(S_t, B_t, t) &= C_1(s_t, b_t, t)(\mu s_t dt + \sigma s_t dw) + C_2(s_t, b_t, t)(\alpha b_t dt + \delta b_t dq) \\
&\quad + C_3(s_t, b_t, t)dt + \frac{1}{2}C_{11}(s_t, b_t, t)\sigma^2 s_t^2 dt + C_{12}(s_t, b_t, t)\rho\sigma\delta s_t b_t dt \\
&\quad + \frac{1}{2}C_{22}(s_t, b_t, t)\delta^2 b_t^2 dt, \\
&= \{[\mu s_t C_1(s_t, b_t, t) + \alpha b_t C_2(s_t, b_t, t) + C_3(s_t, b_t, t) \\
&\quad + \frac{1}{2}C_{11}(s_t, b_t, t)\sigma^2 s_t^2 + C_{12}(s_t, b_t, t)\rho\sigma\delta s_t b_t \\
&\quad + \frac{1}{2}C_{22}(s_t, b_t, t)\delta^2 b_t^2]/C(s_t, b_t, t)\} C(s_t, b_t, t)dt \\
&\quad + \{\sigma s_t C_1(s_t, b_t, t)/C(s_t, b_t, t)\} C(s_t, b_t, t)dw \\
&\quad + \{\delta b_t C_2(s_t, b_t, t)/C(s_t, b_t, t)\} C(s_t, b_t, t)dq, \\
&= \beta C(s_t, b_t, t)dt + \gamma C(s_t, b_t, t)dw + \eta C(s_t, b_t, t)dq,
\end{aligned} \tag{93}$$

² This may be achieved by financing long positions with proceeds from short sales and/or by borrowing.

³ $dS_t dB_t = \rho\sigma\delta s_t b_t dt$ where ρ is the instantaneous correlation coefficient between the asset and the bond. See (A12)

where

$$\begin{aligned} \beta = & [\mu s_t C_1(s_t, b_t, t) + \alpha b_t C_2(s_t, b_t, t) + C_3(s_t, b_t, t) \\ & + \frac{1}{2} \sigma^2 s_t^2 C_{11}(s_t, b_t, t) + \rho \sigma \delta s_t b_t C_{12}(s_t, b_t, t) \\ & + \frac{1}{2} \delta^2 b_t^2 C_{22}(s_t, b_t, t)] / C(s_t, b_t, t), \end{aligned} \quad (94)$$

$$\gamma = \sigma s_t C_1(s_t, b_t, t) / C(s_t, b_t, t), \text{ and} \quad (95)$$

$$\eta = \delta b_t C_2(s_t, b_t, t) / C(s_t, b_t, t). \quad (96)$$

On substituting Q_C , Q_S and W_B from (92a), (92b) and (92c) respectively into the following expression for the instantaneous change in value of the hedged position, namely,

$$dH_t = Q_C dC(S_t, b_t, t) + Q_S dS_t + W_B dB_t,$$

then yields:

$$\begin{aligned} dH_t &= W_C \frac{dC(S_t, b_t, t)}{C(s_t, b_t, t)} + W_S \frac{dS_t}{s_t} + W_B \frac{dB_t}{b_t}, \\ &= W_C(\beta dt + \gamma dw + \eta dq) + W_S(\mu dt + \sigma dw) \\ &\quad + W_B(\alpha dt + \delta dq). \end{aligned}$$

Since $W_C + W_S + W_B = 0$, we may substitute $W_B = -(W_C + W_S)$ to obtain:

$$\begin{aligned}
 dH_t &= W_C(\beta dt + \gamma dw + \eta dq) + W_S(\mu dt + \sigma dw) & (97) \\
 &\quad - (W_C + W_S)(\alpha dt + \delta dq), \\
 &= [W_S(\mu - \alpha) + W_C(\beta - \alpha)]dt + [W_C\gamma + W_S\sigma]dw \\
 &\quad + [W_C\eta - (W_C + W_S)\delta]dq.
 \end{aligned}$$

In order to get a return that is certain, suppose we choose an investment strategy where the coefficients of dw and dq in (97) are always zero. Also, since our initial investment was zero, our return from the hedged position in equilibrium must also be zero to avoid arbitrage opportunities. These conditions may be stated as follows:

$$\begin{aligned}
 (\mu - \alpha)W_S + (\beta - \alpha)W_C &= 0, \\
 \sigma W_S + \gamma W_C &= 0, \\
 -\delta W_S + (\eta - \delta)W_C &= 0.
 \end{aligned}$$

So a non-trivial solution to the above system of equations exists iff

$$\frac{\beta - \alpha}{\mu - \alpha} = \frac{\gamma}{\sigma} = \frac{\delta - \eta}{\delta} . \quad (98)$$

Thus

$$\begin{aligned} (\mu - \alpha)W_S + (\beta - \alpha)W_C &= \sigma W_S + \gamma W_C, \\ \Rightarrow \mu - \alpha &= \sigma \end{aligned}$$

and

$$\beta - \alpha = \gamma .$$

This implies that

$$\frac{\beta - \alpha}{\mu - \alpha} = \frac{\gamma}{\sigma} .$$

It can be shown similarly that

$$\frac{\delta - \eta}{\delta} = \frac{\gamma}{\sigma} .$$

If (98) is true, then

$$\frac{\gamma}{\sigma} = 1 - \frac{\eta}{\delta}$$

and thus from the definition of γ and η in (95) and (96) respectively, we can obtain

$$\begin{aligned} s_t C_1(s_t, b_t, t) / C(s_t, b_t, t) &= 1 - b_t C_2(s_t, b_t, t) / C(s_t, b_t, t), \\ \Rightarrow C(s_t, b_t, t) &= s_t C_1(s_t, b_t, t) + b_t C_2(s_t, b_t, t) . \end{aligned} \quad (99)$$

(98) also implies that

$$\beta - \alpha = \gamma(\mu - \alpha)/\sigma .$$

Substituting for β and γ from (94) and (95) respectively, we can obtain:

$$\begin{aligned} & \mu s_t C_1(s_t, b_t, t) + \alpha b_t C_2(s_t, b_t, t) + C_3(s_t, b_t, t) + \frac{1}{2} \sigma^2 s_t^2 C_{11}(s_t, b_t, t) \\ & + \rho \sigma \delta s_t b_t C_{12}(s_t, b_t, t) + \frac{1}{2} \delta^2 b_t^2 C_{22}(s_t, b_t, t) - \alpha C(s_t, b_t, t) \\ & = s_t C_1(s_t, b_t, t)(\mu - \alpha), \end{aligned}$$

or

$$\begin{aligned} & \alpha b_t C_2(s_t, b_t, t) + \alpha s_t C_1(s_t, b_t, t) + C_3(s_t, b_t, t) + \frac{1}{2} \sigma^2 s_t^2 C_{11}(s_t, b_t, t) \\ & + \rho \sigma \delta s_t b_t C_{12}(s_t, b_t, t) + \frac{1}{2} \delta^2 b_t^2 C_{22}(s_t, b_t, t) - \alpha C(s_t, b_t, t) = 0 . \end{aligned}$$

Substituting for $C(s_t, b_t, t)$ from (99) then yields:

$$\begin{aligned} & \frac{1}{2} [\sigma^2 s_t^2 C_{11}(s_t, b_t, t) + 2\rho\sigma\delta s_t b_t C_{12}(s_t, b_t, t) + \delta^2 b_t^2 C_{22}(s_t, b_t, t)] \\ & + C_3(s_t, b_t, t) = 0 , \end{aligned} \tag{100}$$

which is a second-order, linear partial differential equation for the value of a call option when the interest rate is time varying.

The following boundary conditions can then be specified to solve the above partial differential equation

$$\begin{aligned} C(s_{t^*}, 1, t^*) &= \max[0, s_{t^*} - X] , \text{ and} \\ C(0, b_t, t) &= 0 . \end{aligned}$$

It can be verified by substitution that the solution to (100) is given by

$$C(s_t, b_t, t) = s_t N \left\{ \frac{\ln(s_t/X) - \ln b_t + (\widehat{\sigma}^2/2)T}{\widehat{\sigma}\sqrt{T}} \right\} - b_t \cdot X \cdot N \left\{ \frac{\ln(s_t/X) - \ln b_t - (\widehat{\sigma}^2/2)T}{\widehat{\sigma}\sqrt{T}} \right\}, \quad (101)$$

where

$$\widehat{\sigma}^2 T = \int_0^T [\sigma^2 + \delta^2 - 2\rho\sigma\delta] dt \Rightarrow \widehat{\sigma}^2 = \sigma^2 + \delta^2 - 2\rho\sigma\delta$$

is the instantaneous variance arising from the asset and the discount bond.

Note that since there are no dividend payments (101) can also be used to value an American call option and a European put option using the relationship given in (35).

6.3 THE EFFECT OF A CHANGING VARIANCE ON THE RATE OF RETURN OF THE UNDERLYING ASSET

In this section we will assume that the underlying asset on which a European call option is written, has a non-constant variance for its instantaneous rate of return. To examine the effect of this on the price of the European call option, we will propose that a continuous time diffusion process be used to describe both, the return on the underlying asset, and the standard deviation of that return. Upon deriving a suitable partial differential equation for the price of a call option, we will find that the incorporation of a changing variance assumption into the model will introduce two sources of risk that need to be eliminated in a hedged portfolio. Three approaches for eliminating this risk will then be presented.

The first approach will attempt to diversify away the random term in the portfolio by forming a hedged position consisting of a short position in the underlying asset and a long position of $\frac{1}{C_1(s_t, \sigma_t, t)}$ call options. The second approach will assume that there exists another asset with exactly the same price dynamics as that which is given for the standard deviation of the underlying asset. To diversify away the random term in a portfolio, a hedge will be created by purchasing one share of the underlying asset long, m shares of this new asset long, and selling $\frac{1}{C_1(s_t, \sigma_t, t)}$ call options short.

In the third approach, a portfolio consisting of a long position of w_1 shares in the underlying asset, a short position of one call option with expiration date t_1^* , and w_2 call options with expiration date t_2^* , ($t_2^* \neq t_1^*$) will be hedged

against those two sources of risk. A solution to the partial differential equation governing the price of the call option under consideration will then follow.

6.3.1 The model for a random variance

Let us assume that the price dynamics of the underlying asset, and the variance of the rate of return on the underlying asset are given respectively by the following stochastic processes:

$$dS_t = \mu_s s_t dt + \sigma_t s_t^\alpha dw, \quad (\alpha \geq 0), \quad (102)$$

$$d\sigma_t = \mu_\sigma \sigma_t dt + \delta_t \sigma_t^\beta dq, \quad (\beta \geq 0), \quad (103)$$

where

s_t denotes the price of the underlying asset at time t ,

μ_s denotes the expected return on the underlying asset,

σ_t denotes the variance of the return on the underlying asset at time t ,

μ_σ denotes the expected change in the volatility of return,

δ_t denotes the variance of the volatility of the return on the underlying asset at time t , and

$w(\cdot), q(\cdot)$ denote standard Wiener processes with $(dw)(dq) = \rho_t dt$, where ρ_t denotes the instantaneous correlation coefficient between the returns on the underlying asset and the volatility of the returns.

Assuming that the price of the call option is a function of the underlying asset price, the changing variance and time, we have, by Itô's lemma, the

result that a change in the price of a call option takes the form:

$$\begin{aligned}
dC(S_t, \sigma_t, t) &= C_1(s_t, \sigma_t, t)dS_t + C_2(s_t, \sigma_t, t)d\sigma_t + C_3(s_t, \sigma_t, t)dt \\
&\quad + \frac{1}{2}C_{11}(s_t, \sigma_t, t)(dS_t)^2 + \frac{1}{2}C_{22}(s_t, \sigma_t, t)d\sigma_t^2 + C_{12}(s_t, \sigma_t, t)dS_t d\sigma_t, \\
&= C_1(s_t, \sigma_t, t)dS_t + C_2(s_t, \sigma_t, t)d\sigma_t + C_3(s_t, \sigma_t, t)dt \\
&\quad + \frac{1}{2}C_{11}(s_t, \sigma_t, t)\sigma_t^2 s_t^{2\alpha} dt + \frac{1}{2}C_{22}(s_t, \sigma_t, t)\delta_t^2 \sigma_t^{2\beta} dt \\
&\quad + C_{12}(s_t, \sigma_t, t)\delta_t \sigma_t^{1+\beta} s_t^\alpha \rho_t dt .
\end{aligned} \tag{104}$$

The initial value of a hedged position which consists of a long position in the stock and a short position of $\frac{1}{C_1(s_t, \sigma_t, t)}$ call options is then given by

$$H_t = s_t - \frac{C(s_t, \sigma_t, t)}{C_1(s_t, \sigma_t, t)}, \tag{105}$$

with the instantaneous change in the value of the hedged position being given by:

$$dH_t = dS_t - \frac{1}{C_1(s_t, \sigma_t, t)} dC(S_t, \sigma_t, t). \tag{106}$$

Upon substitution of $dC(S_t, \sigma_t, t)$ from (104), we may then develop (106) as follows:

$$\begin{aligned}
& dS_t - \frac{1}{C_1(s_t, \sigma_t, t)} dC(S_t, \sigma_t, t) = dS_t \\
& - \frac{1}{C_1(s_t, \sigma_t, t)} [C_1(s_t, \sigma_t, t) dS_t + C_2(s_t, \sigma_t, t) d\sigma_t + C_3(s_t, \sigma_t, t) dt \\
& + \frac{1}{2} C_{11}(s_t, \sigma_t, t) \sigma_t^2 s_t^{2\alpha} dt + \frac{1}{2} C_{22}(s_t, \sigma_t, t) \delta_t^2 \sigma_t^{2\beta} dt \\
& + C_{12}(s_t, \sigma_t, t) \delta_t \sigma_t^{1+\beta} s_t^\alpha \rho_t dt] , \\
& = - \frac{1}{C_1(s_t, \sigma_t, t)} [C_3(s_t, \sigma_t, t) + \frac{1}{2} \sigma_t^2 s_t^{2\alpha} C_{11}(s_t, \sigma_t, t) + \frac{1}{2} \delta_t^2 \sigma_t^{2\beta} C_{22}(s_t, \sigma_t, t) \\
& + \delta_t \sigma_t^{1+\beta} s_t^\alpha \rho_t C_{12}(s_t, \sigma_t, t)] dt - \frac{1}{C_1(s_t, \sigma_t, t)} C_2(s_t, \sigma_t, t) d\sigma_t , \\
& \Rightarrow dC(S_t, \sigma_t, t) = C_1(s_t, \sigma_t, t) dS_t + \eta dt + \delta_t \sigma_t^\beta C_2(s_t, \sigma_t, t) dq , \quad (107)
\end{aligned}$$

where we have substituted for $d\sigma_t$ from (103), and

$$\begin{aligned}
\eta & = \mu_\sigma \sigma_t C_2(s_t, \sigma_t, t) + C_3(s_t, \sigma_t, t) + \frac{1}{2} \sigma_t^2 s_t^{2\alpha} C_{11}(s_t, \sigma_t, t) \\
& + \frac{1}{2} \delta_t^2 \sigma_t^{2\beta} C_{22}(s_t, \sigma_t, t) + \delta_t \sigma_t^{1+\beta} s_t^\alpha \rho_t C_{12}(s_t, \sigma_t, t) . \quad (108)
\end{aligned}$$

Thus the change in value of the hedged position in (106) may be written as:

$$dH_t = - \frac{1}{C_1(s_t, \sigma_t, t)} [\eta dt + \delta_t \sigma_t^\beta C_2(s_t, \sigma_t, t) dq] . \quad (109)$$

In order to derive a suitable partial differential equation that will govern the price of an option in equilibrium, it is necessary to eliminate the random term dq in (109).

6.3.2 Three procedures to eliminate the random term dq in dH_t

A. By assuming (in equilibrium), that price fluctuations due to the random term in the variance are completely diversifiable, the change in the value of the hedged position must be equal to the initial value of the hedge multiplied by $r dt$, i.e.

$$-\frac{1}{C_1(s_t, \sigma_t, t)} \eta dt = r \left[s_t - \frac{C(s_t, \sigma_t, t)}{C_1(s_t, \sigma_t, t)} \right] dt . \quad (110)$$

Substitution of η from (108), (110) may be developed as follows:

$$\begin{aligned} & -\frac{1}{C_1(s_t, \sigma_t, t)} [\mu_\sigma \sigma_t C_2(s_t, \sigma_t, t) + C_3(s_t, \sigma_t, t) + \frac{1}{2} \sigma_t^2 s_t^{2\alpha} C_{11}(s_t, \sigma_t, t) \\ & + \frac{1}{2} \delta_t^2 \sigma_t^{2\beta} C_{22}(s_t, \sigma_t, t) + \delta_t \sigma_t^{1+\beta} s_t^\alpha \rho_t C_{12}(s_t, \sigma_t, t)] = r s_t - \frac{r C(s_t, \sigma_t, t)}{C_1(s_t, \sigma_t, t)}, \\ & \Rightarrow -[\mu_\sigma \sigma_t C_2(s_t, \sigma_t, t) + C_3(s_t, \sigma_t, t) + \frac{1}{2} \sigma_t^2 s_t^{2\alpha} C_{11}(s_t, \sigma_t, t) \\ & + \frac{1}{2} \delta_t^2 \sigma_t^{2\beta} C_{22}(s_t, \sigma_t, t) + \delta_t \sigma_t^{1+\beta} s_t^\alpha \rho_t C_{12}(s_t, \sigma_t, t)] = r s_t C_1(s_t, \sigma_t, t) - r C(s_t, \sigma_t, t), \\ & \Rightarrow C_3(s_t, \sigma_t, t) = r C(s_t, \sigma_t, t) - r s_t C_1(s_t, \sigma_t, t) - \frac{1}{2} \sigma_t^2 s_t^{2\alpha} C_{11}(s_t, \sigma_t, t) \\ & - \mu_\sigma \sigma_t C_2(s_t, \sigma_t, t) - \frac{1}{2} \delta_t^2 \sigma_t^{2\beta} C_{22}(s_t, \sigma_t, t) - \rho_t \delta_t \sigma_t^{1+\beta} s_t^\alpha C_{12}(s_t, \sigma_t, t), \quad (111) \end{aligned}$$

which is a second order, linear partial differential equation.

Subject to the following conditions:

$$C(S_{t^*}, \sigma_{t^*}, t^*) = \max[0, S_{t^*} - X] ,$$

$$C(s_t, \sigma_t, t) \rightarrow 0 \text{ as } s_t \rightarrow 0, \text{ and}$$

$$C(s_t, \sigma_t, t) = \max[0, s_t - X e^{-rT}] \text{ for } \sigma_t = 0 ,$$

(111) may then be solved to yield the price of a European call option on an underlying asset which has a changing variance rate of return.

B. A second approach is to assume that there exists an asset with the same random term as the variance of the underlying asset. Suppose this asset has the following price dynamics:

$$dP_t = \mu_p p_t dt + \lambda_t p_t^\beta dq . \quad (112)$$

A hedge can now be created by purchasing one share of the underlying asset long, selling $\frac{1}{C_1(s_t, \sigma_t, t)}$ call options short, and purchasing m shares of the asset, P , long.

The initial value of this position is given by:

$$H_t = s_t + m p_t - \frac{C(s_t, \sigma_t, t)}{C_1(s_t, \sigma_t, t)} . \quad (113)$$

Thus the instantaneous change in value of this hedged position is given by:

$$dH_t = dS_t + m dP_t - \frac{1}{C_1(s_t, \sigma_t, t)} dC(S_t, \sigma_t, t) . \quad (114)$$

Substitution of dP_t from (112), and for $dC(S_t, \sigma_t, t)$ from (107), then yields:

$$\begin{aligned}
dH_t &= dS_t + m(\mu_p p_t dt + \lambda_t p_t^\beta dq) - \frac{1}{C_1(s_t, \sigma_t, t)} [C_1(s_t, \sigma_t, t) dS_t \\
&\quad + \eta dt + \delta_t \sigma_t^\beta C_2(s_t, \sigma_t, t) dq], \\
&= -\frac{1}{C_1(s_t, \sigma_t, t)} \eta dt + m \mu_p p_t dt \\
&\quad + \left[m \lambda_t p_t^\beta - \frac{1}{C_1(s_t, \sigma_t, t)} \delta_t \sigma_t^\beta C_2(s_t, \sigma_t, t) \right] dq. \quad (115)
\end{aligned}$$

To eliminate the random term dq , we need to set

$$\begin{aligned}
m \lambda_t p_t^\beta - \frac{1}{C_1(s_t, \sigma_t, t)} \delta_t \sigma_t^\beta C_2(s_t, \sigma_t, t) &= 0. \\
\therefore m &= \frac{\delta_t \sigma_t^\beta}{\lambda_t p_t^\beta} \frac{C_2(s_t, \sigma_t, t)}{C_1(s_t, \sigma_t, t)}. \quad (116)
\end{aligned}$$

Substitution of (116) into (113) then yields

$$H_t = s_t - \frac{1}{C_1(s_t, \sigma_t, t)} \left[C(s_t, \sigma_t, t) - \frac{\delta_t \sigma_t^\beta}{\lambda_t p_t^{\beta-1}} C_2(s_t, \sigma_t, t) \right]. \quad (117)$$

Thus, in equilibrium, we must have:

$$\begin{aligned}
&-\frac{1}{C_1(s_t, \sigma_t, t)} \left[\eta - \mu_p \frac{\delta_t \sigma_t^\beta}{\lambda_t p_t^{\beta-1}} C_2(s_t, \sigma_t, t) \right] dt \\
&= r s_t dt - \frac{1}{C_1(s_t, \sigma_t, t)} [C(s_t, \sigma_t, t) - \frac{\delta_t \sigma_t^\beta}{\lambda_t p_t^{\beta-1}} C_2(s_t, \sigma_t, t)] r dt,
\end{aligned}$$

$$\begin{aligned}
& \Rightarrow - \left[\eta - \mu_p \frac{\delta_t \sigma_t^\beta}{\lambda_t p_t^{\beta-1}} C_2(s_t, \sigma_t, t) \right] dt \\
& = \left[r s_t C_1(s_t, \sigma_t, t) - r C(s_t, \sigma_t, t) + r \frac{\delta_t \sigma_t^\beta}{\lambda_t p_t^{\beta-1}} C_2(s_t, \sigma_t, t) \right] dt. \quad (118)
\end{aligned}$$

Upon substitution for η from (108), we can obtain:

$$\begin{aligned}
& -\mu_\sigma \sigma_t C_2(s_t, \sigma_t, t) - C_3(s_t, \sigma_t, t) - \frac{1}{2} \sigma_t^2 s_t^{2\alpha} C_{11}(s_t, \sigma_t, t) - \frac{1}{2} \delta_t^2 \sigma_t^{2\beta} C_{22}(s_t, \sigma_t, t) \\
& -\delta_t \sigma_t^{1+\beta} s_t^\alpha \rho_t C_{12}(s_t, \sigma_t, t) + \mu_p \frac{\delta_t \sigma_t^\beta}{\lambda_t p_t^{\beta-1}} C_2(s_t, \sigma_t, t) \\
& = r s_t C_1(s_t, \sigma_t, t) - r C(s_t, \sigma_t, t) + \frac{r \delta_t \sigma_t^\beta}{\lambda_t p_t^{\beta-1}} C_2(s_t, \sigma_t, t) ,
\end{aligned}$$

$$\begin{aligned}
\Rightarrow C_3(s_t, \sigma_t, t) & = r C(s_t, \sigma_t, t) - r s_t C_1(s_t, \sigma_t, t) - \frac{1}{2} \sigma_t^2 s_t^{2\alpha} C_{11}(s_t, \sigma_t, t) \\
& - \frac{1}{2} \delta_t^2 \sigma_t^{2\beta} C_{22}(s_t, \sigma_t, t) - \rho_t \delta_t \sigma_t^{1+\beta} s_t^\alpha C_{12}(s_t, \sigma_t, t) \\
& - \left[\mu_\sigma \sigma_t + \frac{\delta_t \sigma_t^\beta}{\lambda_t p_t^{\beta-1}} (r - \mu_p) \right] C_2(s_t, \sigma_t, t) , \quad (119)
\end{aligned}$$

which is equivalent to the partial differential equation given in (111) with $\mu_\sigma \sigma_t$ replaced by

$$\mu_\sigma \sigma_t + \frac{\delta_t \sigma_t^\beta}{\lambda_t p_t^{\beta-1}} (r - \mu_p) .$$

C. A third approach is to form a hedged position by purchasing w_1 shares of the underlying asset long, selling one call option short, $C(s_t, \sigma_t, t_1)$, with expiration date, t_1^* , and w_2 call options short, $C(s_t, \sigma_t, t_2)$, with expiration date, t_2^* .

The initial value of this hedged position is given by

$$H_t = w_1 s_t - C(s_t, \sigma_t, t_1) - w_2 C(s_t, \sigma_t, t_2) . \quad (120)$$

Thus the instantaneous change in the value of this hedged position is given by

$$dH_t = w_1 dS_t - dC(S_t, \sigma_t, t_1) - w_2 dC(S_t, \sigma_t, t_2) . \quad (121)$$

Substituting for dS_t from (102), and for $dC(S_t, \sigma_t, t)$ from (107), we may develop (121) as follows:

$$\begin{aligned} dH_t &= w_1 [\mu_s s_t dt + \sigma_t s_t^\alpha dw] - C_1(s_t, \sigma_t, t_1) dS_t - \eta dt - \delta_t \sigma_t^\beta C_2(s_t, \sigma_t, t_1) dq \\ &\quad - w_2 [C_1(s_t, \sigma_t, t_2) dS_t + \eta dt + \delta_t \sigma_t^\beta C_2(s_t, \sigma_t, t_2) dq] , \\ &= w_1 [\mu_s s_t dt + \sigma_t s_t^\alpha dw] - C_1(s_t, \sigma_t, t_1) [\mu_s s_t dt + \sigma_t s_t^\alpha dw] - \eta dt \\ &\quad - \delta_t \sigma_t^\beta C_2(s_t, \sigma_t, t_1) dq - w_2 [C_1(s_t, \sigma_t, t_2) [\mu_s s_t dt + \sigma_t s_t^\alpha dw] \\ &\quad + \eta dt + \delta_t \sigma_t^\beta C_2(s_t, \sigma_t, t_2) dq] , \\ &= [w_1 \mu_s s_t - C_1(s_t, \sigma_t, t_1) \mu_s s_t - \eta - w_2 C_1(s_t, \sigma_t, t_2) \mu_s s_t - w_2 \eta] dt \\ &\quad + [w_1 \sigma_t s_t^\alpha - C_1(s_t, \sigma_t, t_1) \sigma_t s_t^\alpha - w_2 C_1(s_t, \sigma_t, t_2) \sigma_t s_t^\alpha] dw \\ &\quad + [-\delta_t \sigma_t^\beta C_2(s_t, \sigma_t, t_1) - w_2 \delta_t \sigma_t^\beta C_2(s_t, \sigma_t, t_2)] dq . \quad (122) \end{aligned}$$

To eliminate the random terms dw and dq , we set

$$[w_1 - C_1(s_t, \sigma_t, t_1) - w_2 C_1(s_t, \sigma_t, t_2)] \sigma_t s_t^\alpha = 0 ,$$

$$\Rightarrow w_1 - C_1(s_t, \sigma_t, t_1) - w_2 C_1(s_t, \sigma_t, t_2) = 0 , \quad (123)$$

and

$$[-C_2(s_t, \sigma_t, t_1) - w_2 C_2(s_t, \sigma_t, t_2)] \delta_t \sigma_t^\beta = 0 ,$$

$$\Rightarrow w_2 = \frac{-C_2(s_t, \sigma_t, t_1)}{C_2(s_t, \sigma_t, t_2)} . \quad (124)$$

Substituting (124) into (123) we then find that

$$w_1 = C_1(s_t, \sigma_t, t_1) - \frac{C_2(s_t, \sigma_t, t_1)}{C_2(s_t, \sigma_t, t_2)} C_1(s_t, \sigma_t, t_2) . \quad (125)$$

Since the hedge position created is now riskless, the change in the value of the hedge position in (122) must, in an equilibrium market, be equal to the initial value of the hedge in (120) multiplied by $r dt$, i.e.

$$\begin{aligned} & [w_1 \mu_s s_t - C_1(s_t, \sigma_t, t_1) \mu_s s_t - \eta - w_2 C_1(s_t, \sigma_t, t_2) \mu_s s_t - w_2 \eta] dt \\ & = r [w_1 s_t - C(s_t, \sigma_t, t_1) - w_2 C(s_t, \sigma_t, t_2)] dt . \end{aligned} \quad (126)$$

Upon substitution of η from (108) we can obtain:

$$\begin{aligned}
& w_1\mu_s s_t - C_1(s_t, \sigma_t, t_1)\mu_s s_t - C_3(s_t, \sigma_t, t_1) - \mu_\sigma \sigma_t C_2(s_t, \sigma_t, t_1) \\
& - \frac{1}{2}\sigma_t^2 s_t^{2\alpha} C_{11}(s_t, \sigma_t, t_1) - \frac{1}{2}\delta_t^2 \sigma_t^{2\beta} C_{22}(s_t, \sigma_t, t_1) - \delta_t \sigma_t^{1+\beta} s_t^\alpha \rho_t C_{12}(s_t, \sigma_t, t_1) \\
& - w_2\mu_s s_t C_1(s_t, \sigma_t, t_2) - w_2 C_3(s_t, \sigma_t, t_2) - w_2\mu_\sigma \sigma_t C_2(s_t, \sigma_t, t_2) \\
& - \frac{1}{2}w_2\sigma_t^2 s_t^{2\alpha} C_{11}(s_t, \sigma_t, t_2) - \frac{1}{2}w_2\delta_t^2 \sigma_t^{2\beta} C_{22}(s_t, \sigma_t, t_2) \\
& - w_2\delta_t \sigma_t^{1+\beta} s_t^\alpha \rho_t C_{12}(s_t, \sigma_t, t_2) = r w_1 s_t - r C(s_t, \sigma_t, t_1) - r w_2 C(s_t, \sigma_t, t_2). \quad (127)
\end{aligned}$$

Substituting for w_1 and w_2 from (125) and (124) respectively, we can obtain the following second-order linear partial differential equation for the price of a call option on an underlying asset, which has a rate of return variance that is changing over time:

$$\begin{aligned}
C_3(s_t, \sigma_t, t_1) &= r C(s_t, \sigma_t, t_1) - r s_t C_1(s_t, \sigma_t, t_1) - \frac{1}{2}\sigma_t^2 s_t^{2\alpha} C_{11}(s_t, \sigma_t, t_1) \\
& - \frac{1}{2}\delta_t^2 \sigma_t^{2\beta} C_{22}(s_t, \sigma_t, t_1) - \delta_t \sigma_t^{1+\beta} s_t^\alpha \rho_t C_{12}(s_t, \sigma_t, t_1) \\
& + \frac{C_2(s_t, \sigma_t, t_1)}{C_2(s_t, \sigma_t, t_2)} \left[C_3(s_t, \sigma_t, t_2) + \frac{1}{2}\sigma_t^2 s_t^{2\alpha} C_{11}(s_t, \sigma_t, t_2) \right. \\
& + \frac{1}{2}\delta_t^2 \sigma_t^{2\beta} C_{22}(s_t, \sigma_t, t_2) + \delta_t \sigma_t^{1+\beta} s_t^\alpha \rho_t C_{12}(s_t, \sigma_t, t_2) \\
& \left. + r s_t C_1(s_t, \sigma_t, t_2) - r C(s_t, \sigma_t, t_2) \right]. \quad (128)
\end{aligned}$$

6.3.3 A solution to the stochastic volatility problem

We will, in this section, attempt to find a solution to the partial differential equation in (111) using the method of Hull and White (1987). It will be assumed that $\rho_t = 0$ and that the volatility is uncorrelated with the asset price.

Since neither (111), nor the boundary conditions, depend upon investor risk preferences, we will assume that investors are risk neutral. Thus, it can be verified, by substitution, that the price of the call option, at time t , can be given by:

$$C(s_t, \sigma_t, t) = e^{-r(t^* - t)} \int \max[0, S_{t^*} - X] f(S_{t^*} | S_t, \sigma_t^2) dS_{t^*}, \quad (129)$$

where

$f(S_{t^*} | S_t, \sigma_t^2)$ denotes the conditional density function of S_{t^*} given the asset price and variance at time t .

By making the following substitution, namely:⁴

$$f(S_{t^*} | S_t, \sigma_t^2) = \int g(S_{t^*} | S_t, \bar{\sigma}^2) h(\bar{\sigma}^2 | \sigma_t^2) d\bar{\sigma}^2,$$

where $\bar{\sigma}^2$ is defined as the mean of the variance of the rate of return on the underlying asset over the life of the option, i.e.

$$\bar{\sigma}^2 = \frac{1}{t^* - t} \int_t^{t^*} \sigma_t^2 dt,$$

⁴for any 3 related random variables x, y and z ,

$$f(x|y) = \int g(x|y, z)h(z|y)dz.$$

(129) may be simplified as follows:

$$\begin{aligned} C(s_t, \sigma_t, t) &= e^{-r(t^*-t)} \int \int \max[0, S_{t^*} - X] g(S_{t^*} | S_t, \bar{\sigma}^2) h(\bar{\sigma}^2 | \sigma_t^2) dS_{t^*} d\bar{\sigma}^2, \\ &= \int \left[e^{-r(t^*-t)} \int \max[0, S_{t^*} - X] g(S_{t^*} | S_t, \bar{\sigma}^2) dS_{t^*} \right] h(\bar{\sigma}^2 | \sigma_t^2) d\bar{\sigma}^2. \end{aligned} \quad (130)$$

In order to simplify (130) further, the following lemma (Hull and White, 1987), is needed.

Lemma:

Suppose that, in a risk-neutral world, a stock price s_t and its instantaneous variance σ_t^2 follow the stochastic processes

$$dS_t = r s_t dt + \sigma_t s_t dw, \quad (a)$$

and

$$d\sigma_t = \mu_\sigma \sigma_t dt + \delta_t \sigma_t^2 dq \quad (b)$$

respectively, where r , the risk-free rate is assumed constant, μ_σ and δ_t are independent of s_t , and w and q are independent Wiener processes. Let $\bar{\sigma}^2$ be the mean variance over some time interval $[0, t^*]$ defined by

$$\bar{\sigma}^2 = \frac{1}{t^*} \int_0^{t^*} \sigma_t^2 dt. \quad (c)$$

Given (a), (b) and (c), then

$$\ln \left(\frac{S(t^*)}{s(0)} \right) \Big|_{\bar{\sigma}^2} \sim N \left(r t^* - \frac{\bar{\sigma}^2 t^*}{2}; \bar{\sigma}^2 t^* \right). \quad (131)$$

Using the above lemma,

$$e^{-r(t^*-t)} \int \max[0, S_{t^*} - X] g(S_{t^*} | S_t, \bar{\sigma}^2) dS_{t^*} = C(\bar{\sigma}^2), \quad (132)$$

where $C(\bar{\sigma}^2)$ denotes the Black-Scholes price for a call option on an asset with mean variance $\bar{\sigma}^2$.

Thus, employing the theorem of Smith (chapter 3, p.17) with $\psi = 1$, $\phi = \infty$, $\lambda = e^{-rT}$, $\gamma = e^{-rT}$, $\rho = r$ and $\sigma^2 = \bar{\sigma}^2$ we may rewrite (132) as follows:

$$C(\bar{\sigma}^2) = s_t N(d_1) - e^{-rT} N(d_2), \quad (133)$$

where

$$d_1 = \frac{\ln(s_t/X) + (r + \bar{\sigma}^2/2)T}{\sqrt{\bar{\sigma}^2 T}} \quad (134)$$

and

$$d_2 = \frac{\ln(s_t/X) + (r - \bar{\sigma}^2/2)T}{\sqrt{\bar{\sigma}^2 T}}. \quad (135)$$

Thus the value of the option in (130) can be given by

$$C(s_t, \sigma_t, t) = \int C(\bar{\sigma}^2) h(\bar{\sigma}^2 | \sigma_t^2) d\bar{\sigma}^2,$$

where $C(\bar{\sigma}^2)$ is given by (133).

Thus, the price of an option on an underlying asset, with a stochastic variance for its rate of return, can be given by the above Black-Scholes price for the option integrated over the distribution of its mean volatility.

CHAPTER 7

APPLICATION OF OPTION PRICING TECHNIQUES

In this chapter several applications of the option-pricing techniques of Black and Scholes (1973) will be given. In particular, a technique for valuing the debt and equity of a firm will be developed, and the pricing of convertible bonds, warrants, collateralised loans and insurance contracts will be considered.

7.1 PRICING OF THE DEBT AND EQUITY OF A FIRM

According to Smith (1979), the equity of a levered firm can be valued using the Black-Scholes formula if one makes the following assumptions:

- (a) the capital structure of the firm does not affect the total value of the firm,
- (b) the dynamics of the value of the firm's assets follow a lognormal distribution with a constant variance on the rate of return,
- (c) the risk-free interest rate, r , is known and is assumed to be constant, and
- (d) the firm issues pure discount bonds. At maturity, the bondholders receive the face-value of the bonds. The company is restricted to paying out the dividends only after the bonds have been paid off.

Under the above assumptions, the equity (E) of a firm can now be viewed as representing a call option on the face value of the bonds because the issuing of the pure discount bonds is equivalent to selling the value of the assets of the firm (v_t) to the bond holders, for the proceeds of the bond issue, plus a (call) option to buy back the assets of the firm from the bondholders when the bonds mature, at an exercise price that is equivalent to the face value of the bonds, namely X . The Black-Scholes formula may therefore be used to value the equity of a firm as follows:

$$E = v_t \cdot N \left\{ \frac{\ln(v_t/X) + [r + (\sigma_V^2/2)]T}{\sigma_V \sqrt{T}} \right\} - e^{-rT} X \cdot N \left\{ \frac{\ln(v_t/X) + [r - (\sigma_V^2/2)]T}{\sigma_V \sqrt{T}} \right\}, \quad (136)$$

where

E is the equity or total value of the stock,

V_t is a random variable denoting the total value of the assets of the firm,

v_t is the realisation of the random variable V_t at time t ,

X is the total face value of the bonds or the face value of the debt of the firm, and

σ_V^2 is the variance rate on the total value of the firm, v_t .

Similarly the value of the debt of the firm may be given by

$$\begin{aligned}
D &= v_t - E, \\
&= v_t - v_t N \left\{ \frac{\ln(v_t/X) + (r + (\sigma_V^2/2))T}{\sigma_V \sqrt{T}} \right\} \\
&\quad + e^{-rT} X \cdot N \left\{ \frac{\ln(v_t/X) + (r - (\sigma_V^2/2))T}{\sigma_V \sqrt{T}} \right\}, \\
&= v_t \left(1 - N \left\{ \frac{\ln(v_t/X) + (r + (\sigma_V^2/2))T}{\sigma_V \sqrt{T}} \right\} \right) \\
&\quad + e^{-rT} X \cdot N \left\{ \frac{\ln(v_t/X) + (r - (\sigma_V^2/2))T}{\sigma_V \sqrt{T}} \right\}, \\
&= v_t N \left\{ \frac{-\ln(v_t/X) - (r + (\sigma_V^2/2))T}{\sigma_V \sqrt{T}} \right\} \\
&\quad + e^{-rT} X \cdot N \left\{ \frac{\ln(v_t/X) + (r - (\sigma_V^2/2))T}{\sigma_V \sqrt{T}} \right\}. \tag{137}
\end{aligned}$$

7.2 PRICING OF CONVERTIBLE BONDS

Consider a situation where a convertible bondholder has the option, upon maturity of the bond issue, of either receiving the face value of the bonds, which we will denote by X , or a quantity of new shares that are set equal to a fraction of the firm's value, say αV_{t^*} , where $0 < \alpha < 1$. The maturity value of the convertible bond will therefore be given by

$$B_{t^*} = \min[V_{t^*}, \max[X, \alpha V_{t^*}]] ,$$

and thus, in equilibrium, the current value on the bond must be equal to the expected terminal value of the convertible bond, discounted at the appropriate expected rate of return on the firm; that is¹

$$\begin{aligned} b_t &= e^{-rT} E(B_{t^*}), \\ &= e^{-rT} \left[\int_0^X V_{t^*} \Lambda(V_{t^*}) dV_{t^*} + \int_{X/\alpha}^{X/\alpha} X \Lambda(V_{t^*}) dV_{t^*} + \int_{X/\alpha}^{\infty} \alpha V_{t^*} \Lambda(V_{t^*}) dV_{t^*} \right] , \\ &= e^{-rT} \left[\int_0^X V_{t^*} \Lambda(V_{t^*}) dV_{t^*} + \int_X^{\infty} X \Lambda(V_{t^*}) dV_{t^*} + \int_{X/\alpha}^{\infty} (\alpha V_{t^*} - X) \Lambda(V_{t^*}) dV_{t^*} \right] . \end{aligned}$$

Using the theorem of Smith (chapter 3, p.17) with $\psi = 0$, $\phi = 1$, $\lambda = e^{-rT}$, $\gamma = 0$ and $\rho = r$ for the first integral, and $\psi = 1$, $\phi = \infty$, $\lambda = 0$, $-\gamma = e^{-rT}$ and $\rho = r$ for the second integral, and $\psi = \frac{1}{\alpha}$, $\phi = \infty$, $\lambda = \alpha e^{-rT}$, $\gamma = e^{-rT}$ and $\rho = r$ for the third integral, one can obtain the result that the value of

¹ Λ is a lognormal density function.

a discount bond is given by

$$\begin{aligned}
b_t &= v_t \left[-N \left\{ \frac{\ln(v_t/X) + (r + (\sigma_V^2/2))T}{\sigma_V \sqrt{T}} \right\} \right] \\
&\quad + e^{-rT} X \cdot N \left\{ \frac{\ln(v_t/X) + (r - (\sigma_V^2/2))T}{\sigma_V \sqrt{T}} \right\} \\
&\quad + \alpha v_t N \left\{ \frac{\ln(\alpha v_t/X) + (r + (\sigma_V^2/2))T}{\sigma_V \sqrt{T}} \right\} \\
&\quad - e^{-rT} X \cdot N \left\{ \frac{\ln(\alpha v_t/X) + (r - (\sigma_V^2/2))T}{\sigma_V \sqrt{T}} \right\}, \\
&= v_t \cdot N \left\{ \frac{-\ln(v_t/X) - (r + (\sigma_V^2/2))T}{\sigma_V \sqrt{T}} \right\} + \\
&\quad e^{-rT} X \cdot N \left\{ \frac{\ln(v_t/X) + (r - (\sigma_V^2/2))T}{\sigma_V \sqrt{T}} \right\} \\
&\quad + \alpha v_t \cdot N \left\{ \frac{\ln(\alpha v_t/X) + (r + (\sigma_V^2/2))T}{\sigma_V \sqrt{T}} \right\} \\
&\quad - e^{-rT} X \cdot N \left\{ \frac{\ln(\alpha v_t/X) + (r - (\sigma_V^2/2))T}{\sigma_V \sqrt{T}} \right\} - v_t. \tag{138}
\end{aligned}$$

7.3 THE PRICING OF WARRANTS

The primary difference between a call option and a *warrant* is that a warrant is issued by a corporation against its own stock, while a call option is issued by a private individual against any stock. Warrants usually have maturities of several years or longer and are substantially out-of-the-money when issued.

Smith (1979, p.99) used the Black-Scholes option pricing formula to derive the equilibrium value of a warrant under the following assumptions:

- (a) The firm issues pure discount bonds. The bondholders receive the face-value of the bonds at maturity and the company is restricted to paying out dividends only after the bonds are paid off.
- (b) The capital structure of the firm does not affect the total value of the firm.
- (c) The dynamics of the value of the firm's assets follow a lognormal distribution with a constant variance on the rate of return.
- (d) The risk-free interest rate, r , is known and is assumed to be constant.
- (e) The only liabilities issued by the firm are its bonds and the warrants.
- (f) The total proceeds if the warrants are exercised is X (the exercise price per share times the total number of shares sold through the rights issue). The warrants expire after T time periods. If the warrants are exercised, the shares sold through the offering will be a fraction, α , of the total number of shares outstanding ($\alpha = \frac{Q_w}{Q_w + Q_s}$, where Q_w is the number of shares sold through the warrant issue and Q_s is the

existing number of shares). Any assets acquired with the proceeds of the warrant issue are acquired at competitive prices.

The value of the warrant at maturity (time t^*) will then be given by:

$$W_{t^*} = \max[0, \alpha(V_{t^*} + X) - X] ,$$

where

W_{t^*} is a random variable denoting the value of the warrant at maturity, X is the total proceeds if the warrants are exercised, and

$$\alpha = \frac{Q_w}{Q_s + Q_w} .$$

In equilibrium, the current value of the warrant will be equal to the expected terminal value of the warrant, discounted at the expected rate of return on the firm, i.e.

$$\begin{aligned} w_t &= e^{-rT} E(W_{t^*}), \\ &= e^{-rT} \int_{\left[\frac{1-\alpha}{\alpha}\right]X}^{\infty} [\alpha V_{t^*} - (1 - \alpha)X] \Lambda(V_{t^*}) dV_{t^*} . \end{aligned} \quad (139)$$

Using Smith's result (chapter 3, p.17) with $\psi = \frac{1 - \alpha}{\alpha}$, $\phi = \infty$, $\lambda = e^{-rT}\alpha$, $\gamma = (1 - \alpha)e^{-rT}$ and $\rho = r$, (132) may be solved to yield:

$$w_t = \alpha v_t N \left\{ \frac{\ln(\alpha v_t / (1 - \alpha)X) + (r + (\sigma_V^2/2))T}{\sigma_V \sqrt{T}} \right\} - (1 - \alpha)e^{-rT} X N \left\{ \frac{\ln(\alpha v_t / (1 - \alpha)X) + (r - (\sigma_V^2/2))T}{\sigma_V \sqrt{T}} \right\}, \quad (140)$$

which is equivalent to a call option on an asset with current price αv_t and exercise price $(1 - \alpha)X$.

7.4 THE PRICING OF COLLATERALISED LOANS

A collateralised loan is the sale of an asset (the collateral) to a lender so that he can use the asset over a period of time that is defined in the contract. In return, the borrower gets to keep the proceeds of the loan, which we will denote by X , and has the option to repurchase the asset at the maturity of the loan, namely time t^* , at an exercise price that is set equal to the amount of the original loan.

In order to value such a collateralised loan, Smith (1979) made use of the following assumptions; namely

- (1) the price dynamics of the collateral follow a lognormal distribution with a constant variance rate of return,
- (2) the net value of the flow of services which the collateral provides the lender, S , is a constant fraction, s , of the market value of the assets, i.e.

$$s = S/v_t ,$$

- (3) the dynamic behaviour of the asset value is independent of the probability of bankruptcy,
- (4) the cost to voluntary liquidation or bankruptcy is zero,
- (5) capital markets and the market for the collateral are perfect. Transaction costs or taxes are non-existent. All available information is freely accessible to all market participants. Participants are price takers,
- (6) the riskless rate of interest, r , is known and is assumed to be constant.

Given the above assumptions, the value of the collateralized loan, at time t^* , must be given by

$$D_{t^*} = \min[V_{t^*}, X] .$$

In equilibrium, therefore, we must have that

$$\begin{aligned} d_t &= e^{-rT} E(D_{t^*}) , \\ &= e^{-rT} \left[\int_0^X V_{t^*} \Lambda(V_{t^*}) dV_{t^*} + \int_X^\infty X \Lambda(V_{t^*}) dV_{t^*} \right] . \end{aligned} \quad (141)$$

Using the theorem of Smith with $\psi = 0$, $\phi = 1$, $\lambda = e^{-rT}$, $\gamma = 0$ in the first integral, and $\psi = 1$, $\phi = \infty$, $\lambda = 0$, $\gamma = -e^{-rT}$ in the second integral, and noting that the total return on the collateral is $r = \rho + s$ ($\rho = r - s$), (134) may be solved to yield:

$$\begin{aligned} d_t &= e^{-sT} v_t N \left\{ \frac{-\ln(v_t/X) - (r - s + (\sigma_V^2/2))T}{\sigma_V \sqrt{T}} \right\} \\ &+ e^{-rT} X N \left\{ \frac{\ln(v_t/X) + (r - s - (\sigma_V^2/2))T}{\sigma_V \sqrt{T}} \right\} . \end{aligned} \quad (142)$$

7.5 THE PRICING OF INSURANCE CONTRACTS

The insurance contract that we will consider is one that insures against the depreciation of an asset. It specifies a premium, p_t , to be paid at the current date, t . On the expiration date of the contract, t^* , the policy holder will receive the difference between the insured value, X , and the market value of the insured asset, V_{t^*} , if $X > V_{t^*}$, and will receive no payment if the market value of the insured asset, V_{t^*} , is greater than its insured value.

Thus, at time t^* , the value of the insured contract, P_{t^*} , must satisfy

$$P_{t^*} = \max[X - V_{t^*}, 0] .$$

This insurance contract may therefore be viewed as being equivalent to a European put option on the insured asset, with current price, v_t , and exercise price that is set equal to the insured value of the asset, X .

By making the following assumptions, namely

- (1) the price dynamics of the insured asset follow a lognormal distribution with a constant variance rate of return,
- (2) the riskless interest rate, r , is known, constant and the same for both borrowers and lenders,
- (3) capital markets are perfect,
- (4) trading takes place continuously, price changes are continuous and assets infinitely divisible,
- (5) the insured asset generates no pecuniary or non-pecuniary flows,

the price of the insurance contract may be expressed using the Black-Scholes put pricing solution in (37) with $S_t = v_t$ as follows:

$$\begin{aligned}
 p_t = & -v_t N \left\{ \frac{-\ln(v_t/X) - (r + (\sigma_V^2/2))T}{\sigma_V \sqrt{T}} \right\} \\
 & + X e^{-rT} N \left\{ \frac{-\ln(v_t/X) - (r - (\sigma_V^2/2))T}{\sigma_V \sqrt{T}} \right\}, \quad (143)
 \end{aligned}$$

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APPENDIX A

RESULTS FROM STOCHASTIC CALCULUS

In this Appendix several important results, which are frequently applied in this thesis, are presented.

Definition 1

A Wiener (or Brownian motion) process $\{Z_t; t \in [0, \infty]\}$ is a stochastic process on a probability space (Ω, \mathcal{F}, P) that satisfies the following properties:

(i) $Z_0(\omega) = 0$.

(ii) For $H \subset \mathbb{R}$, $P[Z_{t_i} - Z_{t_{i-1}} \in H, i \leq n] = \prod_{i \leq n} P[Z_{t_i} - Z_{t_{i-1}} \in H]$ where $0 \leq t_0 \leq t_1 \leq \dots \leq t_n$ denote points in time.

(iii) $P[Z_t - Z_s \in H] = \frac{1}{\sqrt{2\pi(t-s)}_H} \int \exp\left[-\frac{x^2}{2(t-s)}\right] dx,$

for $0 \leq s < t$.

(iv) For $t \geq 0$ and for each $\omega \in \Omega$, $Z_t(\omega)$ is continuous in t .

On letting

$$dZ(t) = \lim_{h \rightarrow 0} Z(t+h) - Z(t), \quad h > 0 \quad (\text{A1})$$

denote the Itô differential of a Wiener process, the following results can be shown to hold true:

Proposition A1

Let $dZ(t)$ be defined as in (A1), then

$$E[dZ(t)] = 0, \text{ and} \quad (\text{A2})$$

$$E[dZ(t)^2] = dt. \quad (\text{A3})$$

Proof: $Z(t)$, being a Wiener process, we have that

$$Z(t) - Z(s) \sim N(0; (t - s))$$

for $t > s$. Thus for $dt > 0$,

$$dZ(t) = Z(t + dt) - Z(t) \sim N(0; dt).$$

Hence

$$E[dZ(t)] = 0$$

and

$$E[dZ(t)^2] = dt.$$

Proposition A2

Let $dZ(t)$ be defined as in (A1), then

$$dZ(t)^2 \simeq dt . \tag{A4}$$

for small dt .

Proof:

$$\begin{aligned} \text{var}[dZ(t)^2] &= E[dZ(t)^4] - [E(dZ(t)^2)]^2 , \\ &= 3(dt)^2 - (dt)^2 , \\ &= 2(dt)^2 , \\ &= o(dt) . \end{aligned}$$

We also have, for small dt , the result that

$$E[dZ(t)^2] = dt$$

and thus it follows that $dZ(t)^2 \simeq dt$

Proposition A3

Let $dZ(t)$ be defined as in (A1), then

$$\text{var}(dZ(t)dt) = o(dt) . \tag{A5}$$

Proof:

$$\begin{aligned} \text{var}[dZ(t)dt] &= (dt)^2 \text{var}[dZ(t)] , \\ &= (dt)^3 = o(dt) . \end{aligned}$$

We also have for small dt the result that

$$E[dZ(t)dt] = dt E[dZ(t)] = 0 ,$$

thus, for small dt ,

$$dZ(t)dt \simeq 0$$

Proposition A4

Let $t > s$ and let ds be a real number such that $0 < ds < t - s + dt$. Then

$$E[dZ(t)dZ(s)] = 0 . \quad (\text{A6})$$

Proof:

$$\begin{aligned} E[dZ(t)dZ(s)] &= E[(Z(t + dt) - Z(t))(Z(s + ds) - Z(s))], \\ &= E[Z(t + dt)Z(s + ds)] - E[Z(t + dt)Z(s)] \\ &\quad - E[Z(s + ds)Z(t)] + E[Z(s)Z(t)] , \\ &= \min[t + dt, s + ds] \\ &\quad - \min[t + dt, s] - \min[s + ds, t] + \min[s, t] ,^1 \\ &= s + ds - s - s - ds + s , \\ &= 0 . \end{aligned}$$

¹ For $t > s$

$$\begin{aligned} E[Z(s)Z(t)] &= E[Z(s)Z(t) - Z(s)Z(s) + Z(s)Z(s)] , \\ &= E [Z(s)[Z(t) - Z(s)] + [Z(s)]^2] , \\ &= E(Z(s)[Z(t) - Z(s)]) + E[Z(s)]^2 , \\ &= E[Z(s)]E[Z(t) - Z(s)] + E[Z(s)]^2 , \\ &= E[Z(s)]^2 = s . \end{aligned}$$

For $s > t$ it can also be shown that

$$E(Z(s)Z(t)) = t .$$

Thus

$$E(Z(s)Z(t)) = \min(t, s)$$

Proposition A5

If $Z_1(t)$ and $Z_2(t)$ are standard Wiener processes and $dZ_1(t)$ and $dZ_2(t)$ are defined as in A1, then

$$dZ_1(t)dZ_2(t) \simeq \rho_t dt , \quad (\text{A7})$$

where ρ_t denotes the correlation coefficient between $dZ_1(t)$ and $dZ_2(t)$.

Proof:

$$\begin{aligned} E[dZ_1(t)dZ_2(t)] &= E[dZ_1(t)]E[dZ_2(t)] + \text{Cov}[dZ_1(t), dZ_2(t)] , \\ &= 0 + \text{cov}[dZ_1(t), dZ_2(t)] , \\ &= \rho_t \sqrt{\text{Var}[dZ_1(t)]} \sqrt{\text{Var}[dZ_2(t)]} , \\ &= \rho_t \sqrt{dt} \sqrt{dt} , \\ &= \rho_t dt .^2 \end{aligned}$$

²Step 2 is a consequence of (A2), namely that

$$E[dZ_1(t)] = E[dZ_2(t)] = 0 .$$

Step 3 obtains by making the substitution

$$\rho_t = \frac{\text{Cov}[dZ_1(t), dZ_2(t)]}{\sqrt{\text{Var}[dZ_1(t)]} \sqrt{\text{Var}[dZ_2(t)]}} .$$

As a result of (A2) and (A3) we have $\text{Var}[dZ_i(t)] = dt$.

Now

$$\begin{aligned}\text{Var}[dZ_1(t)dZ_2(t)] &= E[dZ_1(t)^2dZ_2(t)^2] - [E[dZ_1(t)dZ_2(t)]]^2, \\ &= E[dZ_1(t)^2]E[dZ_2(t)^2] - [E[dZ_1(t)dZ_2(t)]]^2, \\ &= (dt)^2 - \rho_t^2(dt)^2, \\ &= (1 - \rho_t^2)(dt)^2 = o(dt) .\end{aligned}$$

This then implies that for small dt

$$dZ_1(t)dZ_2(t) \simeq \rho_t dt .$$

Proposition A6

If $X(t)$ defines a non-standard Wiener process such that

$$dX(t) = \mu dt + \sigma dZ(t) , \quad (\text{A8})$$

where $Z(t)$ denotes a standard Wiener process, then

$$E[dX(t)] = \mu dt , \quad (\text{A9})$$

and

$$\text{Var}[dX(t)] = \sigma^2 dt . \quad (\text{A10})$$

Proof:

$$\begin{aligned} E[dX(t)] &= \mu dt + \sigma E[dZ(t)] , \\ &= \mu dt ,^3 \end{aligned}$$

This proves (A9).

$$\begin{aligned} \text{Var}[dX(t)] &= \sigma^2 \text{Var}(dZ(t)) , \\ &= \sigma^2 [E(dZ(t)^2) - [E(dZ(t))]^2] , \\ &= \sigma^2 dt ,^4 \end{aligned}$$

which proves (A10).

³From (A2), $E[dZ(t)] = 0$.

⁴From (A3), $E[dZ(t)^2] = dt$.

Proposition A7

If $X_1(t)$ and $X_2(t)$ denote two non-standard Wiener processes that satisfy the following stochastic differential equations

$$dX_j(t) = \mu_j dt + \sigma_j dZ, \quad j = 1, 2,$$

then

$$dX_1(t)dX_2(t) \simeq \rho_t \sigma_1 \sigma_2 dt. \quad (\text{A11})$$

Proof:

$$\begin{aligned} E[dX_1(t)dX_2(t)] &= E[dX_1(t)]E[dX_2(t)] + \text{Cov}[dX_1(t), dX_2(t)], \\ &= \mu_1 \mu_2 (dt)^2 + \text{Cov}[dX_1(t), dX_2(t)], \\ &\simeq \text{Cov}[dX_1(t), dX_2(t)], \\ &= \rho_t \sqrt{\sigma_1^2 dt} \sqrt{\sigma_2^2 dt}, \\ &= \rho_t \sigma_1 \sigma_2 dt. \end{aligned} \quad (\text{A12})$$

⁵The first step equals the second step because $E[dX_1(t)] = \mu_1 dt$ and $E[dX_2(t)] = \mu_2 dt$, (from A9). Substituting $(dt)^2 = o(dt)$ results in step 3, and we obtain step 4 by making the substitution that

$$\rho_t = \frac{\text{Cov}[dX_1(t), dX_2(t)]}{\sqrt{\text{Var}(dX_1(t))} \sqrt{\text{Var}(dX_2(t))}}.$$

$$\begin{aligned}
\text{Var}[dX_1(t)dX_2(t)] &= E[dX_1(t)^2dX_2(t)^2] - E[dX_1(t)dX_2(t)^2], \\
&= (\sigma_1^2dt + \mu_1^2dt^2)(\sigma_2^2dt + \mu_2^2dt^2) - (\rho_t\sigma_1\sigma_2dt)^2, \\
&= o(dt).^6
\end{aligned}$$

For small dt we must have that

$$\begin{aligned}
dX_1(t)dX_2(t) &\simeq E[dX_1(t)dX_2(t)], \\
&= \rho_t\sigma_1\sigma_2dt.
\end{aligned}$$

Thus (A11) is proved.

Itô's Lemma: Let $F(t, X_1(t), X_2(t))$ denote a twice differentiable function of t , and two stochastic processes $X_1(t)$ and $X_2(t)$. If

$$dX_j(t) = \mu_j dt + \sigma_j dZ_j, \quad j = 1, 2,$$

then

$$\begin{aligned}
dF[t, X_1(t), X_2(t)] &= \left[\frac{\partial F}{\partial t} + \frac{1}{2} \frac{\partial^2 F}{\partial X_1^2} \sigma_1^2 \right. \\
&\quad \left. + \frac{\partial^2 F}{\partial X_1 \partial X_2} \rho_t \sigma_1 \sigma_2 + \frac{1}{2} \frac{\partial^2 F}{\partial X_2^2} \sigma_2^2 \right] dt \\
&\quad + \frac{\partial F}{\partial X_1} dX_1(t) + \frac{\partial F}{\partial X_2} dX_2(t). \tag{A13}
\end{aligned}$$

⁶Substituting $E[dX_i(t)^2] = \text{Var}[dX_i(t)] + [E[dX_i(t)]]^2$, $E[dX_i(t)] = \mu_i dt$ from (A9), $\text{Var}[dX_i(t)] = \sigma_i^2 dt$ from (A10) and $E[dX_1(t)dX_2(t)] = \rho_t\sigma_1\sigma_2dt$ from (A11) results in step 2.

Result A3

Consider the random process $\{X(t), t \geq 0\}$ that satisfies the following stochastic equation

$$dX = aX dt + bX dZ,$$

for constants a and b , then $\frac{X(t)}{X(0)}$ has a lognormal distribution with mean $(a - \frac{1}{2}b^2)$ and variance b^2t .

Proof: let $Y = \log X$. Using Itô's lemma, we may write

$$dX = aX dt + bX dZ$$

as

$$dY = (a - \frac{1}{2}b^2) dt + b dZ ,$$

so that

$$\begin{aligned} Y(t) &= Y(0) + \int_0^t (a - \frac{1}{2}b^2) dt + \int_0^t b dZ , \\ &= Y(0) + (a - \frac{1}{2}b^2)t + b[Z(t) - Z(0)] , \\ &= Y(0) + (a - \frac{1}{2}b^2)t + b[Z(t)]. \\ \Rightarrow \log \left[\frac{X(t)}{X(0)} \right] &= \log X(t) - \log X(0) , \\ &= Y(t) - Y(0) , \\ &= (a - \frac{1}{2}b^2)t + b[Z(t)] . \end{aligned}$$

Thus,

$$\log \left[\frac{X(t)}{X(0)} \right] \sim N \left[\left(a - \frac{1}{2}b^2 \right) t, b^2 t \right].$$

Therefore $\frac{X(t)}{X(0)}$ has a lognormal distribution with mean, $\left(a - \frac{1}{2}b^2 \right) t$, and variance, $b^2 t$.

APPENDIX B

B.1 GENERAL PROPERTIES OF THE LOGNORMAL DISTRIBUTION

1.1 Theorem (Aitchison and Brown (1957, p.11))

If $X \sim \Lambda(\mu; \sigma^2)$ and b and c are constants, where $c > 0$ (say $c = e^a$) then

$$cX^b \text{ is } \Lambda(a + b\mu, b^2\sigma^2) .^1 \tag{B1}$$

1.2 If $X \sim \Lambda(\mu, \sigma^2)$, then

$$E(X^r) = \exp\left(r\mu + \frac{1}{2}r^2\sigma^2\right) . \tag{B2}$$

¹If X has a lognormal distribution with parameters μ and σ^2 , we write X is $\Lambda(\mu; \sigma^2)$.