Exploration of Grade 11 learners' mental constructions and difficulties in learning and solving trigonometric equations:

A Case of one school in Umlazi district

by

Njabulo Happyboy Dube

Student number: 210500469

Submitted in complete fulfilment of the academic requirements for the degree of

Doctor of Philosophy in Mathematics Education

School of Education

Faculty of Humanities

University of KwaZulu-Natal

Supervisor: Prof Zanele Ngcobo

November 2023

ABSTRACT

Trigonometry is a particularly challenging area of mathematics for high school learners. This study investigated South African Grade 11 learners' mental constructions and difficulties when learning and solving trigonometric equations. Participants were selected from an after-school mathematics programme that they attended on a voluntary basis. Data was collected using an activity sheet and semi-structured interviews. Baseline data was collected using an activity sheet from 17 learners Grade 10 learners; data was collected from the same learners a year later using another activity sheet with a different set of problems. Semi-structured interviews were conducted with 7 learners to probe their responses on the activity sheet. Most of the learners were found to be unable to make the necessary mental constructions to solve trigonometric problems at the Grade 11 level.

Dubinsky's (1991) constructivist APOS theory, which describes how learners construct their knowledge of mathematics concepts in stages characterised by action, process, object and schema, was used to analyse the mental constructions of learners. During both phases of data collection, most learners were found to rely on explicit step-by-step calculations to solve problems, indicating that they were operating at the action stage; a smaller number were able to do some of the steps mentally without writing them out, indicating that they had advanced to the process stage. No evidence was found of learners having advanced to the object or schema stages. Moreover, the findings showed that, while the learners perform procedures correctly, they applied rules without giving reasons.

Piaget and Garcia's (1989) triad mechanism were used to analyse the difficulties that hindered learners' mental construction of concepts. Learners' difficulties included incorrect conceptions of the equal sign, overgeneralization of rules, and failure to integrate algebra concepts into their construction of trigonometric concepts. Based on the findings, the study recommends that teachers reinforce basic algebraic skills—such as collecting like and unlike terms, using brackets, and addition and subtraction of algebraic terms—before introducing trigonometric concepts. Teachers are urged to explore different methods for teaching trigonometric equations to enable learners to construct knowledge effectively, such as collaborative learning and differentiated classroom activities.

Keywords: Trigonometric equations, Mental Construction, Difficulties, APOS theorem, Triad mechanics.

DECLARATION

I, Njabulo Happyboy Dube, declare that:

(i) The research reported in this dissertation, except where otherwise indicated, is my original work.

(ii) This dissertation has not been submitted for any degree or examination purposes to any other university.

(iii) This dissertation does not contain other persons' data, pictures, graphs or other information, unless specifically acknowledged as being sourced from other persons.

(iv) This dissertation does not contain other persons' writing, unless specifically acknowledged as being sourced from other researchers. Where other sources have been quoted, then:

a) Their words have been re-written, but the general information attributed to them has been referenced.

b) Where their exact words have been used, their writing has been placed inside quotation marks, and referenced.

(v) Where I have reproduced a publication of which I am an author, co-author or editor, I have indicated in detail which part of the publication was actually written by myself alone and have fully referenced such publications.

(vi) This dissertation does not contain text, graphics or tables copied from the Internet, unless specifically acknowledged, and the source detailed in the dissertation and in the References section.

Researcher: [redacted] 18 March 2024

Njabulo Happyboy Dube Date

SUPERVISION STATEMENT

The work done in this thesis was carried out in the School of Education, University of KwaZulu-Natal, from January 2020 to December 2023 under the supervision of Prof. Zanele Ngcobo.

This dissertation has been submitted with my approval.

[redacted]

25 March 2024

Date

ACKNOWLEDGEMENTS

The writing of this dissertation has been a challenging yet rewarding academic exercise. I would like to convey my gratitude to a number of people for playing an important role in my journey towards the completion of this study. I am greatly indebted to the following people:

Firstly, I thank God for giving the strength to complete this study. Without Him, this would have not been possible.

To my supervisor, Prof Zanele Ngcobo: thank you very much for the support and guidance you have given me. I have developed extremely well as an emerging academic researcher under your mentorship. I will forever be grateful to you for the difference you have made in my life.

To my family: I thank you most sincerely for always encouraging me and playing a vital role throughout my years of studying.

To my mother, Bawinile Dube: thank you so much for your support during the course of my studies.

To my lovely grandmother, Zodwa Elizabeth Dube: thanks for your support throughout my education journey – from Grade 1 through my doctorate.

To my strong father, Mbongeni Luthuli: for your words of encouragement, I thank you.

To my child, Kufeziwe Dube: for taking time to be with your father I thank you.

To Ndumiso Khuzwayo, Zanele Khanyile, John Cobongela, Dr Mandonsela, Nolwazi Mkhize, Kwanda Zondo, Xolani Madlala, Smanga Msomi, Akiena Ndlovu, Mandla Mweni, Wendy Talatala, Mohamed Tarig Mohamed Ahmed and Themba Malindi: for being there for me during the process of writing this thesis, I thank you.

To my teammates with whom I run the comrade marathon: I am grateful for supporting me every day.

To the KwaZulu-Natal Department of Education: I am grateful to you for allowing me access to schools so that I could conduct this research.

To the schools at which I conducted this research: thank you for making it possible for me to collect the data reported in this study.

To the individuals who participated in this research: your invaluable contributions and cooperation form the backbone of this study.

DEDICATION

I dedicate this study to

my grandmother, Zodwa Elizabeth Dube,

and my mother, Bawinile Dube,

who have always been a positive influence in my life,

and to my daughter, Kufeziwe Dube,

who always motivates me to work hard and keep going

so that I can be an example to her

of the value of education as she grows up.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

ABBREVIATIONS

APOS Action-Process-Object-Schema

CAPS Curriculum Assessment Policy Statement

DBE Department of Basic Education

DOE Department of Education

FET Further Education Training

FGD Focus Group Discussion

GD Genetic Decomposition

GET General Education and Training

LHS Left-Hand Side

NCTM National Council of Teachers of Mathematics

NRC National Research Council

OECD Organisation for Economic Co-operation and Development

PCK Pedagogical Content Knowledge

RHS Right-Hand Side

TIMSS Trends in International Mathematics and Science Study

CHAPTER ONE: THE RESEARCH PROBLEM

1.1 Introduction

Mathematics is generally viewed as an important school subject. It holds a central place in the curricula of most countries. However, the teaching of mathematics in South African schools has been found to yield some of worst results in the world. In 2011, the Trends in International Mathematics and Science Study (TIMSS) reported that of the 21 middle-income countries that participated, learners from South Africa had the lowest performance (Ndlovu & Mji, 2012). This study explores Grade 11 learners' mental constructions and difficulties when learning and solving the trigonometric equations through a case study conducted with 17 learners at a high school in Umlazi district in the South African province of KwaZulu-Natal.

1.2 Background to the study

This study investigates the mental constructions of Grade 11 learners and the difficulties they experienced learning and solving trigonometric equations. Traditionally, trigonometric equations were taught as mathematical rules, using a calculator. However, best practice in mathematics education currently indicates that teaching and learning must involve a process of developing learners' competencies to be able to conjecture, communicate, solve problems, and reason effectively, while developing a positive attitude towards mathematics (Mudrikah, 2016; Kaitera & Harmoinene, 2022). Conway and Sloane (2005) argue that the learning of mathematics does not involve only mastering a fixed set of facts and skills but requires the development of conjecturing, problem-solving and logical reasoning.

In South Africa, trigonometry is covered in the FET phase (Grades 10 – 12) of the curriculum. However, it is often introduced in Grade 9 where learners learn to work with triangles and name their sides and angles.

In Grade 10, learners work with similar triangles and special angles and learn to solve right-angled and trigonometric equations, define trigonometric functions and plot their graphs. In Grade 11, learners are introduced to reduction formula, special angles, trigonometric identities, trigonometric equations, and SINE-RULE, COSINE-RULE and AREA-RULE.

Table 1 Trigonometry curriculum in the FET phase

Trigonometry curriculum in the FET phase		
Grade 10	Grade 11	Grade 12
• Similarity of triangles • Defining the trigonometric ratios • Reciprocal ratios • Calculator skills • Special angles • Solving trigonometric equations • Defining ratios in the Cartesian plane	• Trigonometric identities • Reduction formula • Trigonometric equations • Area, sine, and cosine rules	• Compound angle identities • Double angle identities • Solving equations • Applications of trigonometric functions

The trigonometry content has a weighting of 40% in the Grade 10, Grade 11, and Grade 12 mathematics Paper 2 examination paper. According to the Department of Education (DBE, 2014), trigonometry is the section of Paper 2 for which learners score the lowest marks. Therefore, failure to understand trigonometry concepts affects the success rate of learners in mathematics.

1.2.1 Learning mathematics with understanding

Learning with understanding has increasingly received attention from educators and psychologists and has progressively been elevated to one of the most important goals for all learners in all subjects (Stylianides & Stylianides, 2007). The level of understanding is determined by the number of connections and the strength of connections (OECD, 2014).

Mathematics is a part of the broader family of sciences, which includes physical science and life science. Ngcobo et al. (2019) argue that the fundamental importance of mathematics to humans can be seen by examining the interrelationship between mathematics and the development of humans to advance the cause of humanity. Due to its fundamental importance and significance to humans and to their advancement and development, mathematics receives significant attention at both the policy level and in teaching and learning practice around the world (Brijlall & Maharaj, 2015).

The National Council of Teachers of Mathematics (NCTM, 2012) in the United States pinpoints the process standards of problem-solving, reasoning and proof, representation, communication, and connections` as ways to think about how learners should engage in learning mathematics content as they develop both procedural fluency and conceptual understanding. Kilpatrick et al. (2002) define procedural fluency as a skill in carrying out procedures flexibly, accurately, efficiently and appropriately. They further define conceptual understanding as comprehension of mathematical concepts, operations and relations. A good conceptual understanding may support the development of procedural fluency in multi-digit calculation and become a powerful device to solve mathematical problems (NRC, 2000). For learners to have a full understanding of mathematical concepts, they need to be well-versed in the skills and algorithms associated with a concept, the applications of the concept and its representations and metaphors, and the history of the concept (Usiskin, 2015).

According to Asmida et al. (2018), a study conducted in Indonesia found that the development of learners' conceptual understanding and procedural fluency had not been yet integrated into approaches to the teaching and learning of mathematics. Teachers used mathematics textbooks which do not address explicitly the development of learners' conceptual understanding and procedural fluency in solving the mathematical problem. To make sense of a problem, learners need to learn how to analyse the given information and the relationships in a problem so that they can understand the situation and identify possible ways to solve it (Lodge et al., 2018). While learners' struggles with learning mathematics are often seen in a negative light and viewed as a problem or learning difficulty, struggle is a natural part of the learning process: it is the learner's engagement of intellectual effort to expand their understanding of the mathematical concept (Permatasari, 2016). A learner is not a blank slate but a creator of their learning: to make sense of the problem they need to be able to construct their knowledge.

1.2.2 Teaching and learning of trigonometry in South African schools

The new South African curriculum, the Curriculum Assessment Policy Statement (DBE,2012), provides opportunities for educators and researchers to see mathematics in ways that present mathematics as a discipline. This allows the development of teaching strategies that help educators see teaching from new perspectives.

Trigonometry is a branch of mathematics that deals with the relationships of sides and angles in triangles (Orhun, 2010). Trigonometry is one of the important topics in the mathematics school curriculum which requires an integration of algebraic, geometric, and graphical reasoning (Stols, 2011; NCTM, 2012).

Trigonometry is an area of mathematics that learners often experience as particularly difficult and abstract compared with other topics in mathematics. In South Africa, Grade 11 learners are expected to demonstrate that they understand the concept of a ratio and that division is not commutative. They should be able to use a calculator to convert fractions, correct to two decimal places, measure the length of sides of a triangle and be able to draw a scale.

Several authors have called for more meaningful connections in the teaching and learning of trigonometry (e.g., Bressoud, 2010; Thompson, 2008; Mensah, 2017). The status quo concerning the teaching and learning of trigonometry in mathematics in South Africa is not up to standard (Taylor, 2021). According to Gernetzky (2012) a study conducted in South Africa found that mathematics teachers and learners were not confident in the teaching and learning of trigonometry in the classroom. This results in teachers spending less time in the classroom (OECD, 2016). Learners may also avoid attending classes.

1.2.3 Learners' construction of conceptual knowledge in mathematics

Brijlall and Ndlovu (2013) state that constructivism explains how learners learn. According to the National Research Council (2000), learners learn by connecting new knowledge with the knowledge and concepts that they already know, thereby constructing new meaning that is easy for them to understand. This study is aligned to this understanding of learning. Construction of knowledge, including mathematical concepts, develops through the formation of mental objects and associations between them (Robinson, 2013). Learners construct knowledge differently; as a result, a teacher needs to be aware of learners' different abilities to learn in the classroom. Constructing knowledge is a collaborative process that aims to produce new understanding (Glaés-Coutts & Nilsson, 2021).

In mathematics teaching, educators should embrace the importance of allowing learners to develop their representations and construct knowledge for themselves to enable them to solve high order problems. To facilitate learners' construction of mathematical knowledge in the area of trigonometry, it is important to help Grade 10 learners, when introduced to trigonometric equations. Traditionally, however, teaching approaches have tended to favour knowledge confirmation rather than knowledge construction. Mudrikah (2016) argues that, for constructivism learning theory to be applied to mathematical learning, it must pay attention to the meaning of the process of formation of new knowledge and its main elements to support the learning process that is expected. Mudrikah (2016) states that learners fail to do well in mathematics as a subject because of the lack of a sound foundation in the content from the previous grades. As a result, the learners fail to construct new knowledge because they do not have the necessary background. This results in learners learning new knowledge in each grade rather than constructing new knowledge using what they already know.

1.3 Statement of the problem

This study explores the mental constructions of Grade 11 learners and the difficulties they encountered in learning to solve trigonometric equations. In recent years, learners have performed poorly in the area of trigonometry (Chigonga, 2016).

The South African education system has achieved a significant large numbers of learners in the schools (Mlachila & Moeletsi, 2019). Also, the number of learners that are enrolling in mathematics and sciences subjects has been increasing (Jojo, 2019). However, research has shown that South African learners in the foundation phase do not have a solid foundation in numeracy and literacy (McGhie et al., 2020; Govender & Hugo, 2020). This affects the throughput of learners in the secondary and tertiary phases of education.

Furthermore, it has been reported that outdated teaching practices and lack of basic content knowledge have resulted in poor teaching standards (Mupa & Chinooneka, 2015) Trigonometry is one of the areas of mathematics in which learners perform very poorly, although it represents around 40% total mark out of 100% of the content in the mathematics paper 2 in the examination paper during the final three years of basic education. This reality motivated the researcher to investigate the reasons contributing to learners' low performance in trigonometry.

1.4 Motivation for the study

Teaching practice in mathematics at the secondary education level has been criticized for not developing and preparing learners for the expertise required in a real environment (Jojo, 2019). As a mathematics teacher, I have been teaching mathematics at the high school level for seven years. I have watched learners develop as they progress from Grade 8 to Grade 12 and seen the amount of learning that they need in order to be able to understand their expanding environment. My interest in research began when I became interested in how learners learn, how they construct the knowledge that enables them to advance in their understanding of mathematics, and what challenges them in their journey of learning mathematics. While teachers certainly play a role in this process, I have also observed how learners teach themselves by gathering the information they need and through their experience with the world around them.

This motivated me to investigate learners' mental constructions in the process of learning and solving trigonometric equations. With reference to APOS theory, I wondered how learners' mental construction of action, process, and object link with the preliminary genetic decompositions. Constructivism emphasizes the importance of the knowledge, beliefs and skills an individual brings to the experience of learning. Also, recognising the construction of new understanding by the learner, the combination of prior learning and readiness to learn is important.

To investigate this, this study was designed to allow learners solved trigonometric equations using different methods to determine the solution. Their responses were analysed to understand their mental constructions and the difficulties they experienced when they were learning and solving trigonometric equations.

It is my hope that this research will contribute to teachers' understanding of learners' mental construction of knowledge and learners' difficulties when they are learning and solving trigonometric equations.

1.5 Rationale for the study

Research is about systematically obtaining and analysing data to increase understanding and knowledge of a particular topic. The rationale for this study relates to three aspects. First, it addresses a gap in the literature on the teaching and learning of trigonometry. Second, the study generates new theoretical knowledge by applying the theoretical lenses of APOS theory and Piaget and Garcia (1989) triad mechanism to the teaching and learning of trigonometric equations. Finally, the study extends the researcher's own expertise to facilitate the construction of knowledge by high school learners in the area of trigonometry.

1.5.1 Addressing the gap

While much research has been done in the area of mathematics – including on the topic of trigonometry – little research has been done on the teaching and learning of the topic of trigonometry. This study does not explore mathematics as a subject or trigonometry as a topic, however, but focuses on trigonometric equations. Limited research has been done on trigonometric equations, both internationally and in South Africa. The research that has been conducted in about the poor performance in trigonometry. Therefore, this study explores mental constructions and difficulties when solving and learning trigonometric equations.

1.5.2 Generating new knowledge

This study explores the mental construction of learners when they are learning and solving trigonometric equations which are guided by Dubinsky's (1991) APOS theory, which focusses on the constructs Action, Process, Object and Schema (APOS). These constructs reflect how learners mentally construct knowledge: first from the action, where the learner makes internal constructions and processes the knowledge without external help; knowledge is then constructed as an object enabling the learner to perform transformations on the process. Finally, the learner combines the constructs of action, process, and object into a coherent schema in their mind (Dubinsky & McDonald, 2001).

This study contributes to understanding how learners construct knowledge related to the trigonometric equation. The researcher designed a preliminary genic decomposition that could be used in the teaching and learning of trigonometric equations to analyse the mental constructions displayed by Grade 11 learners. This can help teachers to prepare to teach learners trigonometric equations.

1.5.3 The researcher's experience

The researcher has been teaching mathematics for the past seven years in General Education and Training (GET) and Further Education and Training (FET) levels at different schools. Working with different learners from the different environments, the researcher became interested in knowing how learners learn and construct knowledge when they are solving and learning mathematics. During the years of teaching mathematics, the failure of learners to understand certain topics in mathematics was observed by the researcher. For learners to be allowed to choose mathematics and physical science as subjects from Grade 10 onward, they must obtain at least 50% for mathematics in Grade 9. Although mathematics learners in Grades 10 to 12 had met these criteria, there were some topics in which they were not doing well, including trigonometry and Euclidean geometry.

1.6 Aim and purpose of the study

The bright future of a country depends upon the educational system that builds morality and behaviours of its citizens which in depths requires attractive investment in education at a global scale. South African curriculum has been questioned in recent years and a lot of changes have been made in the curriculum as results of school's teachers are using CAPS (Ngobeni et al., 2023). Although has been no change in the results performance in some topics in the mathematics curriculum. For instance, learners have been performing poorly in trigonometry. However, the Department of Education provided different interventions to assist teachers and learners in trigonometry, as well as other topics that learners struggled with, such as Euclidean geometry. Jameel and Ali (2016) point out that the department of basic education and teachers can provide everything to assist learners, but if learners do not rehearse and work on a mathematical problem that coherently leads them towards poor achievement in trigonometry.

1.7 Objectives of the study

The study aimed to explore Grade 11 learners' mental constructions and difficulties in learning and solving trigonometric equations. The objectives of the study were:

1. to explore Grade 11 learners' mental construction of knowledge when learning and solving trigonometric equations.
2. to determine the difficulties that hinder the construction of knowledge among Grade 11 learners in learning and solving trigonometric equations.
3. to explore the extent to which the preliminary genetic decomposition explains learners' mental constructions of trigonometric equations.

1.8 Research questions

This study explores grade 11 learners' mental constructions and difficulties in learning and solving trigonometric equations. Therefore, the study was guided by the following questions:

1. What are learners' mental constructions of learning and solving trigonometric equations?
2. Why do learners succeed or fail to make the necessary mental constructions in learning and solving trigonometric equations?
3. How do learners' mental constructions of action, process, and object link with the preliminary genetic decompositions?

1.9 Location of the study

The study was conducted with Grade 11 mathematics learners at a high school in Kwasanti, a township located 10 kilometres from Pinetown in the Umlazi District of the province of KwaZulu-Natal, South Africa. Kwasanti is considered a residential area offering low-cost housing near to urban areas (Mampane & Bouwer, 2011). IsiZulu is the dominant language used in the community.

Township schools are known for their poor education, which is characterised by, among other things, behaviour problems among learners, limited school resources and poor sanitation (Mogashoa & Mboweni, 2017). Most learners at this school travel long distances to school. The school was started in 1993, but only opened officially in 2000. The school falls into Quintile 3. The school was chosen for this study because of it has been performing low in mathematics results.

1.10 Significance of the study

Few studies have explored learners' mental constructions and the difficulties they experience with regard to trigonometric equations at South African schools. However, trigonometry has been identified as one of the most challenging mathematics topics in Africa (Arhin & Hokor, 2021). Of the research that has been done, most has focused on trigonometry as a topic; limited research has been done exploring learners' difficulties with learning and solving trigonometric equations. This study seeks to address this gap.

It is important for teachers to understand learners' mental construction of the knowledge they are learning as well as the difficulties learners encounter when they are learning and solving trigonometric equations. The findings of this study may help teachers to understand how learners construct knowledge and also understand learners' difficulties when they are learning and solving trigonometric equations. This study will also help learners to understand how to solve trigonometric equations using a different method, which will help to prepare them for Grade 12.

1.11 Overview of the thesis

This this is organized into eight chapters, as follows.

Chapter One: Introduction and background of the study

This chapter has introduced the study topic and provided a background to the study. The problem that the study seeks to address was described, and the aim, purpose and rationale for doing the study were explained. The research questions and objectives were presented. The location of the study and its significance were also discussed.

Chapter Two: Literature review

Chapter Two provides a review of literature related to learners' mental construction of knowledge in mathematics and the difficulties they encounter in learning and solving trigonometric equations.

Chapter Three: Theoretical framework

This chapter presents the study's theoretical framework. APOS (Action, Process, Object and Schema) theory (Dubinsky,1991) was discussed, and the proposed genetic decomposition indicated, showing how this links to the framework for research and learners' mental construction of knowledge. Furthermore, the triad mechanism (Piaget and Garcia, 1989) was also discussed to explain learners' challenges in solving trigonometric equations.

Chapter Four: Research design

This chapter presents the methodology that is used in this study. The research paradigm, design and approach are presented. The methods used for sampling, data collection and data analysis are described. The issue of trustworthiness is discussed. The chapter also discusses ethical considerations and the limitations of the study.

Chapters Five – Seven: Findings

Chapter Five: Analysis of summative tasks

Chapter Five presents the data collected in Phase One of the study, which discussion the grade 10 task. Analysis of the data to understand the learners' mental construction of concepts in Grade 10 was informed by the development of the genetic decomposition informed by the teaching and learning process.

Chapter Six: Analysis of activity sheets and interviews

Chapter Six presents the findings from data collected in Phase Two, when the learners were in Grade 11. The data was collected using activity sheets written responses and interviews.

Chapter Seven: Analysis of difficulties of learners in solving trigonometric equations

Chapter Seven presents the analysis of the findings regarding the difficulties (errors and misconceptions) learners faced when leaning and solving equations. The data collected in Phase Two was obtained from both the written responses and from interviews.

Chapter Eight: Discussion, conclusion, and recommendations

Chapter Eight synthesizes the findings presented in Chapters Five, Six and Seven. It discusses the implication of the findings and provides conclusions and recommendations.

CHAPTER TWO: REVIEW OF LITERATURE

2.1 Introduction

The previous chapter provided a background to the study, stated the problem and explained the motivation for the study. It presented the rationale, aim and purpose of the study and articulated the research questions and objectives of the study. The teaching and learning of trigonometry and the construction of mathematical concepts was discussed.

This chapter presents the literature that informed this study. Section 2.2 discusses the importance of trigonometry in mathematics and presents the topics in the mathematics curriculum. Section 2.3 reviews recent and historical literature on the teaching and learning of trigonometry in schools. Section 2.4 explores the significance of how learners construct knowledge in mathematics and the role of teachers in helping learners construct knowledge. The importance of language and mathematical notation in the construction of mathematical concepts in trigonometry is discussed. Section 2.5 reviews the literature exploring misconceptions in mathematics particularly in the learning of trigonometric concepts. Section 2.6 presents factors identified in the literature that enable and hinder the learning of trigonometric concepts. Finally, Section 2.7 discusses APOS studies that focus on learners' knowledge construction with regard to trigonometric concepts.

2.2 The importance of trigonometry in mathematics education

Trigonometry is a branch of mathematics that deals with the relationships between the sides of a triangle and the angles formed at the vertices of those triangles (Saxena, 2015). It includes topics such as trigonometric ratios, trigonometric functions, trigonometric identities, trigonometric equations. In South Africa, trigonometric concepts and ideas continue to be an important component of the high school mathematics curriculum. Trigonometry is used not only in pure mathematical studies but also in physical application (Bornstein, 2017). Simons (2016) argues that real-world problems involving trigonometry are common in fields such as engineering, construction, design and physics. Therefore, it is important for learners who are interested in the scientific or engineering fields to develop basic trigonometric knowledge in high school to be able to pursue studies in these fields. Furthermore, a good understanding of trigonometry is crucial for learners when they solve advanced mathematics tasks, since it equips learners with a comprehensive knowledge of the necessary mathematical concepts (Koyunkaya, 2016).

During the era of the Bantu Education Act 1953 mathematics was prohibited from being included in the curriculum of the black education system. Trigonometry was brought into the mathematics curriculum for all South African learners in 1994. It is regarded as a key component of mathematics education, representing 50 out of 150 marks on the Grade 11 Examination Paper 2 in Curriculum Assessmment Policy (DBE, 2011). While trigonometry concepts are introduced from the beginning of high school, in the curriculum assessment policy statement (CAPS) document, the content progression for trigonometry from Grade 10 to Grade 12 is clearly outlined (Brijlall & Maharaj, 2014; Naidoo & Naidoo, 2016; Simons, 2016). The table below illustrates the weighting of trigonometry in the curriculum as stipulated in the CAPS curriculum.

Table 2 Weighting of trigonometry in CAPS curriculum (DBE, 2011; p. 57)

Weighting of Content Areas			
Description	**Grade 10**	**Grade 11**	**Grade. 12**
PAPER 1 (Grades 12:bookwork: maximum 6 marks)			
Algebra and Equations (and inequalities)	30 ± 3	45 ± 3	25 ± 3
Patterns and Sequences	15 ± 3	25 ± 3	25 ± 3
Finance and Growth	10 ± 3		
Finance, growth and decay		15 ± 3	15 ± 3
Functions and Graphs	30 ± 3	45 ± 3	35 ± 3
Differential Calculus			35 ± 3
Probability	15 ± 3	20 ± 3	15 ± 3
TOTAL	**100**	**150**	**150**
PAPER 2: Grades 11 and 12: theorems and/or trigonometric proofs: maximum 12 marks			
Description	**Grade 10**	**Grade 11**	**Grade 12**
Statistics	15 ± 3	20 ± 3	20 ± 3
Analytical Geometry	15 ± 3	30 ± 3	40 ± 3
Trigonometry	40 ± 3	50 ± 3	40 ± 3
Euclidean Geometry and Measurement	30 ± 3	50 ± 3	50 ± 3
TOTAL	**100**	**150**	**150**

In addition to this weighting, the CAPS document emphasises content progression, which means the content taught in a given grade is a prerequisite for the subsequent grade. For trigonometry, the content taught in Grade 11 is thus key to the content to be taught in Grade 12. The weighting of the content and content progression are key indicators of the importance of the topic in the curriculum. As illustrated in the table above, in Grades 10 and 12 the weighting of trigonometry is 40+, while in Grade 11 it is 50+ out of 150, indicating that the content taught for Paper 2 comes from trigonometry. Thus, for learners to achieve good grades in Paper 2 they need to be competent in trigonometry, among other topics.

Thompson (2008) argues that the challenges that high school learners experience with trigonometry result from incoherence in their foundational understanding developed at lower grades. These challenges need to be investigated and understood, given that trigonometry is so important in the South African curriculum and to the future of learners who want to continue their studies in maths or the sciences at the university level.

2.3 The teaching and learning of trigonometry in South African schools: past and present research

Trigonometry is an area of mathematics that learners have been found to experience as particularly difficult and abstract, compared with other mathematics topics (Gerhana et al., 2017). Challenger (2009) asserts that learners claim that trigonometry is difficult and dislike it because they see it as a complicated aspect of mathematics and are uncertain as to whether they should apply triangle trigonometry, circle trigonometry or analytic trigonometry in a given situation. The literature demonstrates that many learners do not develop a sound conceptual basis in trigonometry and that some use algebraic notation informally (Maharaj, 2008). Orhun (2015) studied the difficulties faced by learners when solving problems in trigonometry. The study discovered that learners did not have a strong foundation in trigonometry since they made mistakes while learning and solving trigonometric problems. In another study, Usman and Hussaini (2017) analysed the errors made by 80 randomly selected high school learners when solving problems while learning trigonometry. They found that the most frequent errors made by learners included comprehension errors, transformation errors and process skill errors. Drawing on some of the older studies, Jonassen (2000) suggests that learners develop a stronger conceptual understanding of trigonometry if they can inter-relate numerical and symbolic representations with their graphical outputs. This suggests that addressing the abstract nature of trigonometry using visual representation such as graphs, pictures and videos may assist learners to conceptualise trigonometric concepts.

Much of the literature on trigonometry has focused on trigonometric functions and identities. However, these are not the only concepts that learners struggle within trigonometry. The 2020 South African Department of Basic Education (DBE, 2020) moderators' report indicated that learners experienced difficulty understanding most trigonometric concepts, and especially trigonometric equations.

The literature reveals challenges in the teaching and learning of mathematics. In the South African context, this is evident in the findings of the Trends in International Mathematics and Science Study (TIMSS), which has shown that year after year South African learners perform below the 400 TIMSS point, indicating a lack of basic mathematical knowledge (Mullis et al., 2020). In addition, the South African Grade 12 pass rate in mathematics have been found to be poor over the years (Department of Basic Education, 2020). These findings have been substantiated by the research of other scholars. Bansilal et al. (2014) found that mathematics teachers' knowledge of the content they were supposed to teach was not adequate. Similar findings have been echoed by other studies (e.g., Stols, 2015; Pournara et al., 2015; Ngcobo et al., 2019).

Turning to research on the learning of mathematics, scholars such as Maharaj (2014) found that learners had not developed mental construction of mathematical concepts and argued that teachers need to encourage learners to engage with what they are writing in order to assist them to improve. Similarly, Ngcobo et al. (2019) and Madonsela et al. (2020), in their studies of learners' conceptualizations of trigonometric concepts, found that learners were operating at an action level, indicating that they had not conceptualized the content they were being taught. Chen et al. (2015). found that learners possessed poor imagining skills which are critical for the conceptualization of 3D images.

Researchers such as Brijlall and Maharaj (2015) attribute the problem of learners' underachievement to teachers employing poor strategies in the teaching and learning of mathematics and a lack of appropriate depth in their pedagogical content knowledge (Brijlall & Maharaj, 2015). Ndlovu et al. (2017) argue that, for effective teaching, pre-service mathematics must have a thorough grounding in their subject matter before exiting teacher training.

Some authors argue that to enable learners to perform at high competence levels in mathematics, they need to be taught the relevant mathematical knowledge and skills, coupled with effective teaching strategies that help them exercise their reasoning power and utilise their imagination in the solving of mathematical problems (Ngcobo et al., 2019).

Beyond South Africa's borders, similar concerns about the teaching and learning of mathematics prevail. Rohimah and Prabawonto (2019) report that, in a study conducted in Indonesia, three aspects of learners' difficulties in solving trigonometric equations were identified. The study found that learners had difficulty factoring in the form of trigonometric quadratic equations, deciphering the form of the problem and using the trigonometric equation. Chikiwa (2015), in a study on the teaching of trigonometry in a Grade 11 multilingual mathematics class, in Eastern Cape South Africa, found that the use of symbols and specialised language in trigonometry negatively impacts learners' understanding in the classroom. This indicates that it is important for teachers to unpack relevant symbols and terms in each content area.

It is in this context that the researcher elected to explore Grade 11 learners' mental construction of trigonometric equations, with the aim of understanding the mental constructions they use and possible barriers that hinder construction of the necessary mental constructions, as depicted in APOS (Action Process Object Schema) theory (Dubinsky, 1991).

2.4 Learners' construction of knowledge in mathematics

The content changes that have occurred within the domain of mathematics concepts have brought with them a wave of reform in mathematics education (Barnes & Venter, 2008). The construction of knowledge has been studied by psychologists and educational theorists for many years, including studies by Perkins (1993), Piaget (1995), Vygotsky (1998), Alagic (2003) and Garder (2011). Researchers have examined how learners learn and have highlighted the strategies and learning environment that enable learners' construction of knowledge to be most effective and meaningful (Vygotsky, 1978; Mevarech & Kramarski, 1997; Van de Walle et al., 2016; Kim & Baylor, 2006; Gardner, 2011; Moreeng & Du Toit, 2013; Darling-Hammond et al., 2020).

In recent years, researchers such as Jojo et al. (2013), Maharaj (2014), Brijlall and Maharaj (2015), Bansilal et al., 2017 and Ndlovu et al. (2020) have explored learners' and undergraduate students' construction of mathematical concepts. Most of their studies found that construction of knowledge is a process involving different stages that an individual passes through to construct the relevant schema. According to Tall (2008), many concepts that we use are not formally defined; however, we learn to recognise them through our experience with them being used in appropriate contexts. Learners learning mathematics may use a range of different processes depending on the context and make different errors depending on the specific problem under consideration. For instance, a learner may correctly solve $\sin\theta = \frac{1}{2}$ but may make an error solving $\tan\theta + 2 = 0$. The inclusion of the numeral in the equation using tan might cause cognitive conflict for a learner who has not constructed the knowledge necessary to solve trigonometric equations but uses processes based on the problem given.

A learner demonstrates understanding by performing a variety of actions that rely on critical thinking: explaining, applying, generalising, representing in new ways, and making analogies and metaphors (Halpern, 2013). The teacher needs to create opportunities for learners to demonstrate their understanding and also demonstrate a variety of approaches during teaching and learning to ensure that learners are successful in constructing knowledge. For learners to construct knowledge, a teacher must encourage effective learning and foster an environment that supports deeper understanding (Pitsoe & Maila, 2012).

As learners need to construct their own knowledge for each mathematical concept, the primary role of the teacher is not to lecture, explain, or otherwise attempt to transmit mathematical knowledge, but to create situations for learners that facilitate them making the necessary mental constructions. Hitt and Kieran (2009), in a study conducted in Canada, found that learners experienced difficulty in their construction of concepts and their application of these concepts to solve problems. A study conducted by Anderson (2007) with 54 rural high school learners revealed that the learning of mathematics concepts was a complex process that involved the learner developing new ideas while transforming their ways of doing, thinking and building skills through algorithms and following certain procedures, as well constructing or acquiring mathematical concepts.

Learning mathematics concepts involves the development of the identity of each learner as a member of a mathematics classroom. Through relationships and experiences with their peers, teachers, family and community, learners come to know who they are relative to mathematics

(Ishimaru etal., 2015). Anderson (2007) submits that learning mathematics concepts and skills also involves how mathematics fits in with learners' other current and future activities.

2.4.1 The importance of language and mathematical notation in the construction of mathematical concepts

Language and mathematical notation play an important role in knowledge construction in mathematics. Mathematics language teaching and learning has been highlighted by researchers as a key factor contributing to the poor performance of learners in mathematics (Mji & Makgato, 2006; Nath & Veneesha, 2009; Kiwanuka et al., 2015; Robertson & Graven, 2019).

Research shows that mathematics language is key to success in mathematics (Seethler et al., 2011; Erath et al., 2018), and that a learner's general knowledge and understanding of mathematical language can predict mathematical performance (Van der Walt, 2009). Several studies have indicated that language plays an important role in learner performance in mathematics. Mji and Makgato (2006) define mathematics language as an instrument of inclusion and exclusion, including in mathematics teaching. Robertson and Graven (2019) explored the power of language to either include or exclude certain groups of learners from the opportunities for mathematical sense-making in a study conducted in South Africa, where English is the language of teaching and learning – including assessment – for mathematics but is not the home language of most learners. The study found that most of the learners did not understand the content taught if the teachers used English for teaching and learning; in light of this, teachers need to ensure principled use of learners' home language in mathematics classrooms to aid learners' understanding.

In South Africa, English is the medium through which most learners learn in the classroom. However, mathematics, as a subject, has its own language that learners need to master in order to construct a concept. For example, in mathematics the word 'limit' carries a different meaning and is used in a different way than it is used in everyday English. This can result in a confusion in the learner's knowledge construction of the concept of 'limit' as it is taught.

For learners to achieve their best in mathematics, they need a good understanding of the language of mathematics. A study conducted by Prediger et al. (2018) found that learners with low mathematics language proficiency might be hindered not only by difficulty reading during tests, but also seem to be constrained during the whole learning and teaching process, especially when engaging with topics with higher cognitive demands. For a learner to understand and be able to construct knowledge, good communication is important. According to Rohid (2019), good mathematical communication refers to the ability to arrange and link mathematical ideas through communication and use mathematical language to express mathematical ideas correctly. Communicating their mathematical ideas helps learners clarify and solidify their understanding of mathematics. By sharing their mathematical understanding in written or oral form with their classmates, learners develop confidence in themselves and their peers as mathematics learners and enable teachers to better monitor their process. However, in the case of trigonometric concepts, learners are not taught the language of mathematics in school, which can hinder their construction of knowledge (Biyela et al., 2016; Chikiwa & Schafer, 2018; Kersaint et al., 2013).

The role that language plays in the teaching and learning of mathematics is given importance in the current literature on mathematics education. Mulwa (2014) indicates that the primary function of language in mathematics training is to allow both the instructor and the learner to communicate mathematical knowledge precisely. Mulwa (2014) emphasises that, in order to achieve the goals of mathematics instruction, teachers and textbook authors must employ a language whose structure, meaning, technical terminology, and symbols can be grasped by learners at their particular level. For example, in the earlier years of high school, learners are taught the concept 'pattern'; in the later years, however, the terms 'sequence' and 'series' are introduced to refer to the same thing. Using the terms 'simplify', 'prove' and 'proof' would be difficult for learners in the lower grades because of the embedded language making concepts difficult to comprehend.

In the same vein, the mathematical notation used should be appropriate for the level of the learner. What a learner at the Grade 12 level would be able to understand would not be true of a learner in a lower grade. Furthermore, various conventions used in mathematics need to be understood by learners so that they can use this aspect of mathematical language and interpret letters correctly in context. This should be taught explicitly to learners during the building of their knowledge of trigonometry, e.g., Sin θ, Cos θ, and Tan θ. Mulwa (2015) states that high failure rates in examinations are attributed, in part, to the extent to which the meaning of some mathematical terms is not understood by learners for whom English is a second language. Mulwa found that learners had difficulty using mathematical terms and related concepts, which resulted in challenges in learning mathematics. The language challenges that learners face in mathematics learning contribute to the difficulties and challenges learners face in mathematics classrooms, while not receiving adequate attention. These challenges cannot be overcome without giving attention to the aspect of language – through which they enter the domain of mathematics and through which they are assessed (Gafoor & Sarabi, 2015). A similar study conducted by Mbugua et al. (2012) found that achievement in mathematics is strongly correlated to learners' understanding of mathematical language. Therefore, mathematics language is the most critically important factor that affects learners' understanding. Issues around language thus need to be addressed in the teaching and learning of mathematics.

2.4.2 The construction of knowledge in trigonometry

Kepceoglu (2016) describes trigonometry as a branch of physical mathematics that deals with the understanding of concepts and their applications. Trigonometry is an important area of mathematics that links algebraic, geometric and graphic reasoning. Kamber and Takaci (2018) argue that, as trigonometry connects algebraic and geometric ways of thinking, if learners do not understand algebra, they are likely to perform poorly in trigonometry. Many high school learners are not familiar with these types of reasoning; thus, trigonometry presents a challenge for these learners (Naidoo & Govender, 2014).

In the South African context, the following aspects of trigonometry are covered in the curriculum:

> Grade 10: Similarity of triangles; defining trigonometric ratios; reciprocal ratios; calculator skills; special angles; solving trigonometric equations; and defining ratios in the Cartesian plane.
>
> Grade 11: Trigonometric identities; reduction formula; trigonometric equations; and area, sine, and cosine rules.
>
> Grade 12: Compound angle identities; double angle identities; solving equations; and applications of trigonometric functions.

As purported by Demir et al. (2012) and evidenced in the CAPS curriculum, trigonometry equations are an important topic in secondary school mathematics that requires integration of different algebraic, geometric and graphic reasoning skills. Thus, for learners to understand trigonometric equations they need to have a sound knowledge of algebraic, geometric and graphic reasoning. Construction of knowledge related to trigonometric equations, therefore, is premised, to some extent, on the construction of knowledge related to other prerequisite concepts.

Mathematics, as a discipline, and trigonometry, as a topic within it, are embedded in language and notation; thus, in learning mathematics and understanding the concepts involved in trigonometry equations, it is important to use the correct mathematics language (Kissane & Kemp, 2014). Kissane and Kemp (2014) state that trigonometric equations have two expressions which are universally equivalent, which normally requires mathematical proof.

In their study with pre-service mathematics teachers concerning the knowledge construction of trigonometric concepts such as trigonometric equations, Nabie et al. (2018) found that constructing knowledge to deal with the abstract nature of mathematics helped to enhance participants' understanding of trigonometric equations and other trigonometric concepts. While many studies have explored learners' understanding of trigonometric functions, limited attention has been given to trigonometric equations. Nabie et al. (2018) note that the abstract nature of trigonometric concepts proved to be a challenge that hinders knowledge construction. It is therefore imperative to explore the mental constructions used by learners to solve trigonometric equations, so that when alternative teaching strategies are implemented, as recommended by Nabie et al. (2018), they are designed to address and enhance learners' development of relevant mental constructs.

2.5 Misconceptions in mathematics

The child's perspective, it is a reasonable and viable conception based on their experience in different contexts or their daily life activities (Fujii, 2020). A 'misconception' can be defined as associating an incorrect meaning to a concept, choose the wrong concept to use, or a relationship between incorrect concepts (Maryati & Priatna, 2018). Nearly any concept, regardless of how well it is taught, can potentially be misunderstood. Teachers often encounter misconceptions and errors in learners' work and thinking in mathematics. Investigating where the misconception began is vital as this lends insight into the learner's mathematical development and what may have gone astray in their thinking and deductions.

Ojose (2015) states that learners of all grade levels commonly have misconceptions regarding various concepts in mathematics. The misconceptions learners that may have from previous inadequate teaching, informal thinking, or poor remembrance may lead to very serious learning difficulties in mathematics (Njoroge, 2022). Suparno (2013) argues, however, that a misconception is not necessarily a 'wrong' idea but may be an embryonic understanding of the concept or generalisation that the learner has made; misconceptions thus can be a natural stage of development. They may, however, give rise to further errors. Since misconceptions are cognitively constructed: they are more than a simple arithmetic error or lack of accuracy, however, simple math problem can be a result of the underlying misconception. For example, in the example $2^2 = 4$, a learner who conceives this as requiring the base to be multiplied with the exponent will arrive at the correct answer. Encountering other sums, such as $2^3 = 6$ and $2^4 = 8$, the learner may become aware of their underlying misconception. However, if a learner makes an error on one sum involving this concept but continues to solve similar sums correctly, this could be attributed to inaccuracy or simple arithmetic error – what Olivier (1989, cited in Ndlovu et al., 2017) refers to as 'slip'. As misconceptions give rise to errors that may hinder further knowledge construction, they should be taken seriously and addressed.

Chiconga (2016) explored learners' errors when solving trigonometric equations to identify which concepts were problematic. The study revealed that learners misinterpreted sine, cosine, and tangent of the angle; when their values were negative, learners failed to identify relevant quadrants. The study also found that, while learners made errors while solving trigonometric equations, their teachers were observed to also experience difficulty teaching the same content.

A study conducted by Sujarwo et al. (2020) analysed the misconceptions of learners while learning mathematics. The findings indicated that learners' misconceptions may have resulted from a lack of mastery of prerequisite concepts, inadequate reasoning ability, and learners having errors in their basic knowledge of the concept in that area of mathematics.

Misconceptions do not occur only among learners with basic mathematics skills or poor competence in mathematical thinking, but also among learners with higher competence in mathematical creative thinking (Rafiah & Ekawati, 2017). As a result, it is crucial to clarify the misconceptions that learners with high competence in mathematical problem-solving experience – especially in trigonometry, which is often overlooked.

Several factors that cause learners' misconceptions are identified in the literature. Mohyuddin and Khalil (2016) highlight two factors, in particular, that contribute to misconception in mathematics: interpreting new experiences or new concepts incorrectly; and the emotional and intellectual misunderstanding that has been attached to learners' mathematics knowledge, making it difficult for learners to accept new concepts that contradict their existing understanding.

Misconceptions have been identified as one of the barriers learners encounter when they are learning mathematics (Ay, 2017). According to Ojose (2015), a misconception is a misunderstanding arising from incorrect meanings. In a study conducted by Kaczmarczyk et al. (2010) to identifying learners' misconceptions about programming, it was found that if the learner understood a concept as fundamentally different from its scientific meaning, they were likely to construct a misconception. Other studies, such as those conducted by Mishra, (2020); Lodge et al., (2018); and Mutambara and Bansilal, (2022), concur with Kaczmarczyk et al. (2010) that misconceptions emerge as a result of individual experiences and wrong beliefs of individuals about mathematics concept. A learner holding a misconception from a previous topic in mathematics may develop a new misconception because of the previous one (Alreshidi, 2023). As learners build new mathematics concepts on concepts previously learnt, it is important to identify and address misconceptions carried forward through their previous construction of knowledge in mathematics.

2.5.1 Misconceptions in the learning of trigonometric concepts: What has been discovered?

Trigonometric concepts have been found to be challenging for learners; as result, teachers have reported that misconceptions frequently are formed by learners during the teaching and learning of trigonometry in the FET phase (Chigonga, 2016). A preliminary survey conducted for the Quality Teaching and Learning Enhancement in Mathematics, Science, and Technology project, carried out in schools in Mankweng circuit in Polokwane, found that teachers encountered problems when teaching trigonometry (Mavhungo et al., 2015) due to learners' misconceptions. Orhun (2013) investigated learners' level of learning, errors and misconceptions in trigonometry and found that errors tended to be systematic: they corresponded with knowledge gaps in previously learnt concepts. For example, after being taught identities, a learner used that knowledge incorrectly in the context of $(\sin\theta + \cos\theta)^2$ to say the answer was equal to 1, indicating that the learner's misconceptions did not originate from the new knowledge but from previously learnt concepts; as a result they do not know the different between $(sin^2\theta + cos^2\theta)\ and\ (\sin\theta + \cos\theta)^2$. Findings by Orhun (2013) revealed that new concepts learnt can also be the cause for misconception. For example, when introducing trigonometric equation with the domain $360° \leq \theta \leq 360°$ in mathematic problems without using the general equation might be constructed by learners as that the solution will always be restricted in the given domain. Constructing such knowledge might give rise to errors when introducing negative angles.

Suparno (2013) assert that misconceptions in trigonometry can be caused by learners, teachers, learning contexts, teaching styles and textbooks. Brijlall & Maharaj (2014) argue that teachers' lack of appropriate depth in pedagogical content knowledge (PCK) contributes towards students' underachievement in certain areas of mathematics, such as trigonometry. Walida and Hasana (2020) also investigated the types of misconceptions experienced by learners and the factors that contributed to these misconceptions. In a study involving 17 learners, data was collected from their responses to a test and from interviews. Their findings indicated that learners had difficulty interpreting symbols using notation, generalizing and applying rules.

Table 3 Example of learners' errors (Walida & Hasana, 2020)

Learner's error	Correct working
$1 + \sin\theta = cos^2\theta$	$1 + \sin\theta = cos^2\theta$
$1 + \sin\theta = 1 - sin^2\theta$	$1 + \sin\theta = 1 + sin^2\theta$
$\sin\theta = sin^2\theta$	$\sin\theta - sin^2\theta = 0$
$\frac{\sin\theta}{\sin\theta} = \frac{sin^2\theta}{sin\,\theta}$	$\sin\theta(1 - \sin\theta) = 0$
$1 = Sin\ \theta$	$\sin\theta = 0$ or $\sin\theta = -1$
$\theta = sin^{-1}(1)$	$\theta = sin^{-1}(0) or\ \theta = sin^{-1}(1)$
$\theta = 90$	$\theta = 0\ or\ \theta = 90$

They found that factors that contributed to learners' misconceptions included learners' lack of ability to understand the concept; their lack of willingness to study and to solve mathematics problems, hindering their cognitive development; and their lack of motivation.

A focus on learners' errors as indicators of their mathematical thinking has been shown to help teachers better understand learners' thinking, modify how they interact with learners in the classroom, and revise their teaching strategy (Brodie, 2014; Venkat & Adler, 2012). To help learners better comprehend the subject and prevent mistakes from learners, educators must concentrate on the misunderstandings that learners have about trigonometry.

Winarso (2017) reports that a study conducted in Indonesia investigating the level of misconceptions experienced by Grade 12 in mathematics found that 13% of learners fell into the low category, 40% did not understand the concept and 47% did understand the concept. Misconceptions are related to a concept that produces a mathematical object and symbol: for example, if we look at sine, cosine is the concept and the symbols of trigonometric functions. Hence, if a learner is unable to assimilate and accommodation, this creates a gap in their learning of the concept which may lead to mathematical errors or misconceptions (Sarwadi et al., 2014). In addition, many misconceptions are related to the learner's ability to use operations (Makonye & Fakude, 2016). For example, presenting the result of the calculation of $Sin = \frac{1}{2}$ and the value of $Sin\ 30°$ are not the same and they do not give the same answer.

2.6 Factors that enable or hinder the learning of trigonometric concepts

Research conducted on learning difficulties in mathematics by Telima (2011) indicates that difficulties in teaching and learning mathematics have resulted in the mass failure of learners in examinations. Learners' difficulties in learning mathematics are a concern worldwide, and studies have documented several reasons. The teaching and learning of mathematics, like any other subject, requires both the teacher and learner to communicate effectively (Mulwa, 2014). The National Research Council (2012) submits that learners learn mathematics well only when they actively construct mathematical knowledge; to do this, learners must examine, represent, transform, solve, apply and communicate as they learn.

Acharya (2017), in a study on diversity in mathematics education, identified some of the factors that may affect the learning of mathematics. Acharya (2017) found that learners' anxiety about mathematics, limited prior knowledge, lack of effort, and a lack of parental support were factors that contributed to learners not doing well in mathematics. Learners' achievement of knowledge depends on their needs, interest, practices, and seriousness about the subject matter (Darling-Hammond et al., 2020). Learners tend to understand more if are interested in the topic and are given time to practice, rather than teacher teaching all the time. A solid foundational knowledge of mathematics established at the primary education level has been found to be the key factor determining high performance by learners at the secondary education level (Acharya, 2017; Kiwanuka et al., 2015).

Despite the relative importance of mathematics, learners' performance in the subject in Africa has remained consistently poor (Kapasa et al., 2015). Learners' mathematics results are typically poor compared to other subjects (Musonda, 2017). Reid O'Connor& Norton, (2022) found that learners face many difficulties in learning mathematics, including poor arithmetic skills, difficulty with the abstract content associated with algebra and trigonometry, and an inability to apply correct methods to a new problem. Brijlall and Ndlovu (2013) also indicate that the resource materials that are used encourage an instrumental, rather than a relational, understanding of concepts. However, mathematics educators have put up self-sacrificing and spirited efforts aimed at identifying the major problems associated with the learning of mathematics in schools (DBE, 2014).

Trigonometry is a key area of mathematics. It helps learners develop cognitive strategies, such as problem-solving, by engaging the reasoning and proofing capabilities of the learners (Phonapichat et al., 2014). Learning trigonometric concepts can only be achieved when the real-life applications of trigonometry and its importance are shared with learners (Madonsela et al., 2020). Solving trigonometric problems uses skills such as identifying problems, defining a problem, exploring solutions, acting on strategies, and looking back and evaluating (Permata et al., 2018). According to Ernaningsih and Wicasari (2017) and May and Courtney (2016), problem-solving in trigonometry may be very difficult for high school learners and teachers to master. Aminudin et al. (2019) identified critical thinking skills as a key aspect that learners need to develop continuously throughout their learning to be able to solve challenging problems.

Critical thinking helps learners to identify which information is important and which is not. Ennis (2018) defines critical thinking as a logical and reflective thinking process used to make decisions about what to believe and what to do. A recent study by Aminudin et al. (2019) aimed to determine the causes of difficulties experienced by learners while solving trigonometry problems. They identified four causes: a demotivated to do trigonometry; learning of trigonometry that did not emphasise critical thinking; ignorance about critical thinking; and motivation for solving mathematics problems. Aizikovitsh-Udi and Cheng (2015) concur with Aminudin et al.'s (2019) findings that critical thinking is lacking among learners. Aizikovitsh-Udi and Cheng (2015) argue that, to foster the development of critical thinking skills in learners, these skills must first be defined, and the mental processes involved in critical thinking must be understood.

Another challenge that has been identified is that learners may not have developed clear concepts for trigonometry and may use algebraic notation as an informal and inconsistent personal shorthand (Siyepu, 2012; Cangelosi et al., 2013). Nurmeidina and Rafidiayan (2019) also indicate that learners may have difficulty understanding the information given to solve problems. Learners need to be taught using simple trigonometry problems at first; when they have developed a sound understanding they can be challenged with critical questions. This will help them develop the skills to think deeply and critically when solving trigonometric problems.

2.7 APOS studies on learners' knowledge constructions of trigonometric concepts

To date, only a few studies have researched learners' mental construction of trigonometry concepts with reference to APOS theory (Ngcobo et al., 2019). Martinz-Planell and Delgado (2016) analysed the mental construction made by learners in developing a unit circle approach to Sine and Cosine and their corresponding inverse trigonometric functions. Their findings indicated that the conjecture mental construction is useful in describing a learner's behaviour in problem-solving situations. Their results suggest that learners who have a process conception of the conjectured mental constructions can perform better in problem-solving activities. Ngcobo et al. (2019); Madonsela et al. 2020) explored the mental constructions made by Grade 12 learners when solving for the unknown properties of triangles in trigonometry. The study was carried out in a school in KwaZulu-Natal and involved 17 learners. Data was collected from the written responses of learners using a structured activity sheet and was subsequently analysed using APOS theory. The findings indicated that most of the participants were still operating at the action stage.

Siyepu (2015) analysed the errors made with trigonometric functions by a group of 30 learners. Her results revealed conceptual and procedural errors. Conceptual errors showed a failure to grasp the concepts used in a problem and a failure to appreciate the relationships in a problem. Procedural errors occurred when learners failed to carry out manipulations, even if concepts were understood. Errors of interpretation and linear extrapolation were also made. Interpretation errors occurred when learners wrongly interpreted a concept due to over-generalisation of their existing schema. The findings of these studies indicate that more work is needed to explore the usefulness of APOS for analysing the mental constructions that are made by learners as they engage with various concepts in trigonometry. Many other studies have been done that employed APOS theory to understand individuals' mental constructions with regard to other areas of mathematics. However, less research has been done using APOS theory to understand learners' mental constructions in the area of trigonometry, even though it is considered one of the most challenging topics in mathematics.

2.8 Conclusion

This chapter has reviewed various studies related to mathematics and, specifically, trigonometry. The chapter covered the importance of trigonometry in mathematics, the teaching and learning of trigonometry in schools and misconceptions in mathematics and trigonometry as a topic. The importance of language in teaching and learning mathematics was also explored. The next chapter presents the theoretical framework that was adopted for the study.

CHAPTER THREE: THEORETICAL FRAMEWORK

3.1 Introduction

The previous chapter reviewed the literature related to the teaching and learning of trigonometry in schools and discussed the construction of knowledge. This chapter presents the theoretical framework within which this study is located. The study explores Grade 11 learners' mental constructions and misconceptions in learning and solving trigonometric. The study employs two related theories to understand learners' mental construction and misconception when learning and solving the trigonometric equation. The study is underpinned by the APOS (Action, Process, Object, Schema) theory (Dubinsky, 1991), which deals with the teaching and learning of mathematics. APOS theory, as a theory of learning, pays serious attention to how learners come to understand mathematical concepts and how this should inform pedagogic interventions (Inglis, 2015). However, it does not account for what might hinder a learner from making the necessary mental constructions. To address this, the study draws on Piaget's triad mechanism theory (Piaget & Garcia, 1989), which is closely linked to APOS theory. This chapter justifies the use of APOS theory and the triad mechanism in this study and explains their application. The key principles of APOS theory are presented and APOS theory is linked to Piaget's triad mechanism theory. As both APOS theory and Piaget's triad mechanism theory are grounded in constructivism, the chapter begins with a discussion of constructivism.

3.2 Value and role of a theoretical framework

Constructing a theoretical framework is one of the most important aspects of the research process (Osanloo & Grant, 2016). Anfara and Mertz (2015) define a theoretical framework as any empirical or quasi-empirical theory of social and psychological processes that can be applied to the understanding of the phenomenon. The theoretical framework guides the researcher's choice of research design and approach and selection of procedures for research inquiry analytical tools (Dickson et al., 2018). The theoretical framework serves as a guide to developing the study philosophically, epistemologically, methodologically and analytically. Osanloo and Grant (2016) stated that the theoretical framework serves as the structure and support for the rationale, the problem statement, the purpose, the significance, and the research questions. Ravitch and Carl (2016) also indicate that the theoretical framework assists the researchers in situating and contextualizing formal theories into their studies as a guide. Furthermore, the theoretical framework provides a grounding base for the literature review and, most importantly, the methods and analysis.

According to Akintoye (2015), the theoretical framework makes research findings more meaningful and generalizable. Imenda (2014) posits that research without a theoretical framework lacks proper suitable guidance to the research of approved literature and academic discussions of the research findings.

The selection of theories to be used depends on the discipline or field of research; even within a particular discipline, a specific theory or theories that resonate with the area of inquiry must be selected (Kivunja, 2018).

3.3 Constructivism: the foundation of APOS theory and Piaget's triad mechanism

Several learning theories, such as Piaget's theory of constructivism (Piaget, 1967), Vygotsky's theory of scaffolding (Vygotsky, 1962), and Skinner's theory of behavioural learning (Stones,1970), focus on teaching and learning. Constructivism posits that learners construct their knowledge and understanding through their own experiences and reflections upon these experiences (Dennick, 2016). It engages with learners' level of understanding and shows that the understanding can increase and change to higher-level thinking (Mvududu & Thiel-Burgess, 2012). Constructivism models indicate the way learners make sense of new concepts and also how content can be taught and presented effectively (Amineh & Asl, 2015).

Constructivism, and other broad learning theories, are general and do not specifically refer to the learning of mathematical concepts. This study is underpinned by APOS theory and also draws on Piaget's triad theory – both of which have their roots in constructivist theory but engage specifically with mathematics. The focus of this study is not only to understand learners' mental constructions but also to explore learners' misconceptions that potentially hinder learners from constructing the necessary mental constructions when they are learning and solving trigonometric equations, as purported by APOS theory. APOS theory is a framework for the process of learning mathematics that pertains specifically to learning more complex mathematical concepts (Weyer, 2010). This section presents a brief overview of constructivism, as it is the broader learning theory which underpins the more specific theories that guide this study.

Constructivism is premised on the understanding that learners construct knowledge for themselves, rather than just passively taking in information. As learners experience the world and reflect upon those experiences, they build representations and incorporate new information into their pre-existing knowledge (schemas). Assimilation and accommodation are elements of this process. Assimilation refers to the process of taking new information and fitting it into an existing schema; accommodation refers to using newly acquired information to revise and redevelop an existing schema (Piaget, 1967). Figure 1 (adapted from Narayan et al., 2013) illustrates the fundamental concepts within constructivism and the process of knowledge construction.

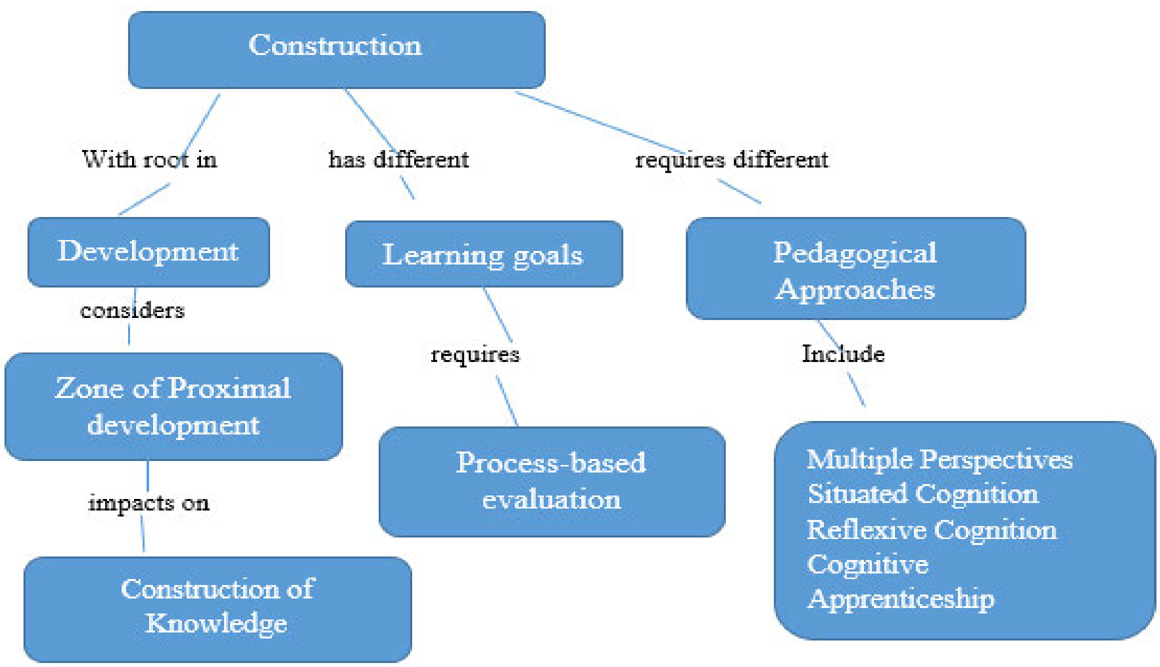


Figure 1 The elements of constructivism (adapted from Narayan et al., 2013)

Constructivist theory has several implications for teaching and learning. First, it posits that learners will learn best when they are engaged in learning experiences rather than passively receiving information. When the teacher provides clear answers to the task given to learners, they are restricting learners from engaging in their own construction of knowledge. As knowledge cannot be imparted directly to learners, but must be constructed by the learners themselves, the goal of teaching is to provide experiences that facilitate learners' construction of knowledge (Chuang, 2021). Second, it implies that learning is inherently a social process because it is embedded within a social context where learners and teachers work together to build knowledge.

Darling-Hammond (2016) posit that understanding the difference between the world out there (the society) and the learner's own world helps a teacher to decide on the type of pedagogy to follow to create a constructivist classroom. If the teacher lacks an understanding of these two worlds – the world out there and their own world – it puts the teacher in a dilemma of whether to present the knowledge or help learners to construct their own knowledge. Vygotsky (1978), who is often considered the 'father' of social constructivism, argues that a learner arrives at school already having constructed some knowledge on the topic from their interaction with society through their community. Therefore, society plays an important role in learners' development. Vygotsky (1978) views knowledge construction as starting with the social interaction of people: interactions that involve sharing, comparing and debating among learners and mentors. Learners are members of the community who receive support and supervision from mentors as they help the learners to gain skills to survive in difficult situations, developing comprehension, independence and competence (James et al., 2001). Therefore, in the process of teaching and learning, the knowledge that a learner has already constructed forms the foundation on which new knowledge will be constructed.

Constructivism provides the foundation for the study's theoretical framework, described in the next section.

3.3.1 Mental construction

Mental construction is different for each learner in the classroom during teaching and learning. Research on such cognitive construction was pioneered by Piaget (1967). According to the theory of cognitive constructivism that has resulted, ideas are constructed by individuals through a personal process (Brau, 2020).

Social construction is the theory of knowledge that examines the development of a jointly constructed understanding of the world that forms the basis for shared assumptions about reality. Social constructivism posits that a student's ideas are constructed through interactions with the teacher and other students (McLeod, 2016). Amineh and Asl (2015) define social constructivism as a theory of knowledge rooted in sociology and communication theory that examines the knowledge and understandings of the world that are developed jointly by individuals. Social construction presumes that understanding, significance, and meaning are developed in collaboration with other human beings (Dennick, 2016). The theory centres on the notion that meanings are developed through coordination with others, rather than separately and individually. These phenomena, as defined by the authors, contribute to learners' mental construction of knowledge in the mathematics classroom.

Another aspect of construction that links to mental construction is metacognition, which is discussed below.

3.3.2 Metacognition

Metacognition is commonly defined as 'thinking about thinking' (Lai, 2011). According to Zulkiply (2009), metacognition refers to the knowledge that people have about their thought processes. It is an ability to think about, understand and manage one's learning. Metacognition includes knowledge about learning and about oneself as a learner and the skills of monitoring and regulating one's cognitive process. Jaleel (2016) defines metacognition as a system that helps a person understand and control their cognitive performance. This system allows people to take charge of their learning. Flavell (1979) defines metacognition as knowledge about cognition and cognitive phenomena. According to Kuzle (2013), metacognition helps the learners to recognise the presence of a problem that needs to be solved, discern what exactly the problem is, and understand how to reach a goal.

Mental construction occurs when the learner process is activated during the problem-solving process (Ndlovu & Brijlall, 2015). Problem-solving requires more than just arithmetic and calculation skills. Zawojeski et al. (2013) argues that, during the process of solving a mathematical problem, cognitive and metacognitive processes are parallel and interactive, rather than sequential. People who show high performance during dynamic, complex problem-solving tasks can also make errors. However, during problem-solving, the flexibility of the thought process and adaptability of the previous knowledge to a new context shows cognitive growth.

Metacognition consists of two levels (Flavell, 1979), as shown in Figure 2: the object level and the meta level.

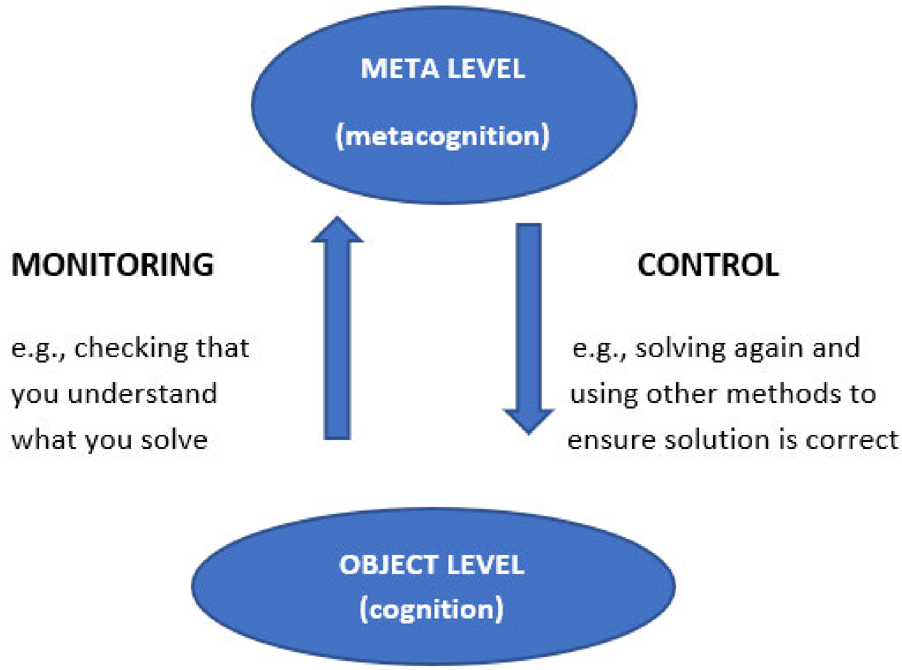


Figure 2 Model of metacognition (Flavell, 1979)

The object level is where cognitive processes – one's thinking – occurs. At the object level, cognitive strategies are used to help the learner achieve a particular goal (understanding the method of the problem). For example, when the learner is given a problem to solve, thinking needs to occur in their mind about how to attempt the problem and solve it. Without any thinking, no attempt will occur. At the object level, cognitive strategies are used to help the learner achieve a particular goal. Solving $\sin\,(180 - \theta) = \frac{3}{4}$, to reach a solution thinking needs to occur; in this case, a learner needs to think of the Cartesian plane and which quadrant (180- θ) belongs to, then determine whether $\sin \theta$ is negative or positive in that quadrant.

The meta level is where thinking about thinking takes place. At this higher-order level, metacognitive strategies are used to make sure that the learner reaches the goal they have set. For example, the learner thinks about how well they have understood the method of solving a trigonometric equation, or the learner chooses to do more problems to enhance their understanding.

To continue with the example above: to solve $\sin(180 - \theta) = \frac{3}{4}$, the learner would begin with thinking about how well they understood the topic of trigonometry equations during teaching and learning in the classroom. If they feel they have an adequate understanding of the problem, they will attempt to solve the problem. If not, they will ask for help from peers or review their textbook to understand better.

3.3.3 Mathematical knowledge and its construction

According to Arnawa and Nita (2020), the teaching and learning of mathematics at the school level should be approached sequentially: first, understanding definitions; second, understanding theorems; and third, practicing solving mathematical problems using definitions and theorems. In practice, this means that before learners are given practice exercises to do, they should first be grounded in the definitions and theoretical aspects of the content taught. As mentioned by Ndlovu (2012), concept definition is the preceding factor for learners to construct the appropriate concept image. This means understanding the definition of the concept is critical for learners to be able to make the necessary mental construction. Allowing learners to engage in doing exercises assists learners to apply the theoretical aspects that have been taught. What is key is that, as emphasized in constructivism theory, the learner's interaction with the content is key – meaning that, for knowledge construction learners, are actively involved in the learning process. For a learner to become a reliable problem solver in mathematics, creativity, intuition and experience are required (Bishara, 2016). This can be obtained through adequate practice in the classroom. For this to be achieved, the teacher needs to ensure that learners are given time to think and time to do problem-solving. As this study explores learners' mental construction of key knowledge when learning and solving trigonometric equations, it is necessary to have a sound understanding of what scholars mean by the term 'mental construction'.

3.4 APOS theory

APOS theory is based on the principle that an individual learner learns mathematics by applying certain mental mechanisms to build specific mental structures meaning the way the learner will think, which the learner then uses to deal with a problem connected or related to the corresponding situation (Chagwiza et al., 2021). In this study, APOS theory is used to understand learners' mental construction when learning trigonometry concepts, such as trigonometric equations problems. Previous studies (e.g., Ngcobo et al, 2019; Madonsela et al., 2020) demonstrate the usefulness of APOS theory for understanding learners' mental construction of trigonometric concepts.

Tall (1999) questions the applicability of APOS to areas of mathematics other than algebra (specifically, advanced mathematics and geometry), while Inglis (2015) questions its application to arithmetic. Both researchers, however, agree that APOS is a valuable theory for understanding and explaining knowledge construction.

APOS theory was determined to be suitable for this study because the aim of this study is not to compare learners' performance but to explore their construction of knowledge when learning and solving trigonometric equations. APOS theory emphasises that the teaching of mathematics should be based on helping learners construct mental structures and build new ones to have powerful structures for handling more advanced mathematics (Voskoglou, 2013). For example, when learners construct appropriate structures to solve trigonometric equations at the school level, it might help them to deal with advanced trigonometric concepts at the undergraduate level. According to Dubinsky (2010), the APOS theory and its application to teaching practice are based on the following assumptions:

The assumption of mathematical knowledge: An individual's mathematical knowledge defines how the learner can respond to perceived mathematical problem situations and their solutions by reflecting on them in a social context and constructing or reconstructing mental structures to use for dealing with the situations.

Hypothesis of learning: An individual does not learn mathematical concepts directly: they apply mental structures to make sense of a concept (Piaget, 1964). Learning is facilitated if the individual possesses mental structures appropriate for a given mathematical concept. If appropriate mental structures are not present, then learning the concept is almost impossible.

These assumptions indicate that teaching must employ strategies that help learners build appropriate mental structures and guide them to apply these structures to construct their understanding of mathematics concepts.

3.4.1 Key mental constructs in APOS theory

The key mental structures identified in APOS theory are action, process, object and schema (APOS). The figure below illustrates the application of the Preliminary genetic decomposition of solving trigonometric equations. using APOS theory.

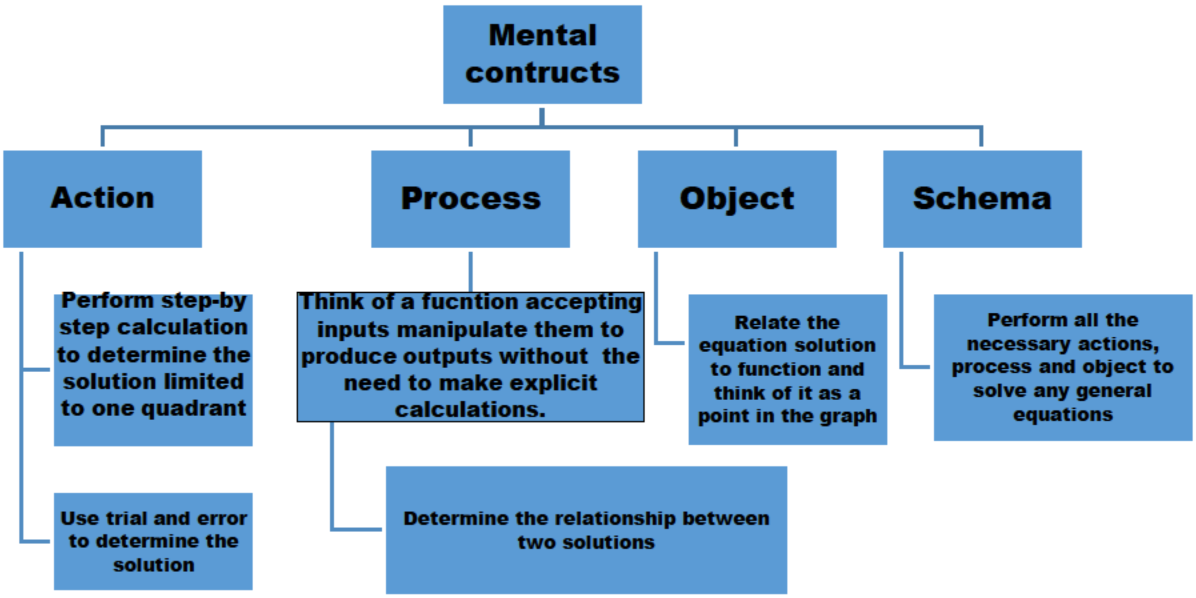


Figure 3 Preliminary genetic decomposition of solving trigonometric equations

3.4.1.1 Action

An action is external in the sense that each step of the transformation needs to be performed explicitly and guided by external instructions; additionally, each step prompts the next – that is, subsequent steps of the action cannot yet be imagined, and none can be skipped (Arnon et al., 2014). An action is a repeatable physical or mental manipulation that transforms objects (Brijlall & Maharaj, 2010). Action is based on rules and algorithms, where a rule is practiced repeatedly until it becomes routine and takes place without conscious thought. Weller et al. (2009) state that action is a reaction to stimuli that an individual perceives as external. For example, given $\tan\theta = 5\ and\theta\epsilon[0°; 360°]$, at an action level, learners can use trial and error to substitute and determine the value θ. In another scenario, a learner would carry out all the steps to solve for θ but would only consider one solution in the first quadrant. For example, for the $sin\ \theta$ of a positive angle, the learner might indicate only one solution because sin is positive in the first quadrant, failing to look at the second quadrant or be given interval such as $[0°; 180°]$.

3.4.1.2 The process

In APOS theory, 'process' refers to when an individual repeats and reflects on an action and the action becomes interiorised into a mental process (Weller et al., 2009). Thus, a process is a structure that performs the same operation as the action but is undertaken completely in the mind of the individual: the individual can imagine performing the transformation without having to execute each step explicitly. For example, at the process stage, a learner would know that there would be two solutions – one in the first and the second in the third quadrants – to make a statement true. At the process stage, the learner is able to skip some steps and still determine the solution and understand the question.

3.4.1.3 Object

Dubinsky and McDonald (2001) explain that an 'object', in APOS theory, is constructed from the process when the individual becomes aware of the process as a totality and realises that transformations can act on it. Weller et al. (2009) also indicate that it is when one becomes aware of a process as a totality that an object can be constructed in such transformations. Therefore, we can say that the individual has encapsulated the process into a cognitive object. An example is when a learner becomes aware of the equation as a representation of a function. Another example would be seeing $\tan\theta = 5$ in totality in which actions can be applied and the learner can visualize the solution graphically. A learner who has constructed concept definition and concept image of $\tan\theta = 5$ as a function is considered to be at the object stage.

3.4.1.4 The schema

A schema is characterized by its action and its continuous reconstruction as determined by the mathematical activity (Dubinsky, 1991). This indicates that a particular mathematical concept of a single collection made up of actions, processes, facilities, and other schemas that are connected through some form of general principles and framework (Voskoglou, 2013). However, the schema must be coherent in the sense that it provides – explicitly or implicitly – a method for determining which phenomena fall within the schema and which do not.

Figure 4 illustrates the relationship between the four mental constructs identified in APOS theory.

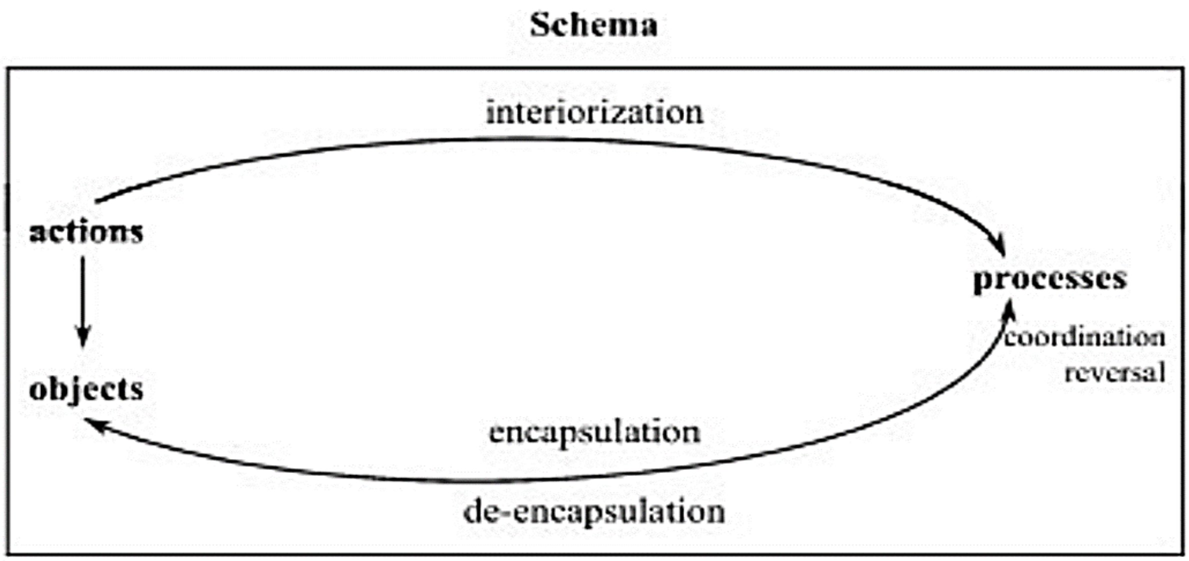


Figure 4 Mental structures and mechanisms for the construction of mathematical knowledge in APOS theory (Arnon et al., 2014)

Dubinsky (1991) characterizes the overall relationship among these elements as a circular feedback system. However, in the construction of mathematical knowledge, learners may be inconsistent because of a lack of understanding of trigonometric equations. For example, they must have a deep understanding of algebra: a lack of knowledge in algebra might hinder the learner's mental construction of trigonometric equations. The depth and complexity of an individual's understanding of a concept depends on their ability to form connections among the mental structures that constitute it. These connections from the basis of a schema whose coherence is crucial to an individual's ability to make sense of mathematical situations related to the concept (Arnon et al, 2014).

3.4.2 Genetic decomposition

According to Brijlall & Ndlovu (2013), genetic decomposition is an organised set of mental constructs that represents how a concept develops in the mind of a person. The genetic decomposition was designed to help teachers and researchers to accommodate their experiences with particular concepts in mathematics. The analysis of results in this study used two theories:

APOS theory, which was extended with the use of Piaget's triad mechanism to explain the difficulties that hinders the construction of knowledge of trigonometric equations.

When using APOS theory, mathematics researchers first make a description of the model that might explain the path that learners might take to make the proposed constructions. This model is known as the 'genetic decomposition' and consists of mental constructions (action, process, objects, and schemas) and mental mechanisms (such as assimilation, interiorisation, encapsulation, de-encapsulation, coordination) put together in a way to explain the learning of the concept in question.

An individual's mathematical knowledge is their capacity to respond to perceived mathematical problem situations by reflecting on problems and their solutions in a social context and by constructing and reconstructing mathematical actions, processes, and objects and organising these into schemas to use in dealing with the situations (Asiala et al., 2004, cited in Jojo et al., 2013).

Dubinsky and McDonald (2001) argue that understanding a mathematical concept begins with manipulating previously constructed mental or physical objects to form actions. Actions are then interiorised to form processes which are then encapsulated to form objects. These objects could be de-encapsulated back to the processes from which they were formed and, finally, organised as schema.

3.5 Piaget's triad mechanism theory

APOS theory is premised on how concepts are constructed in the mind of an individual. It does not, however, explain the barriers that can hinder the development of mental constructs. To address this, Piaget and Garcia's (1989) triad mechanism was used in conjunction with APOS theory to explain the misconceptions that may have hindered learners' mental constructions, as illustrated in the genetic decomposition below. Like APOS theory, the triad mechanism explains learners' learning of mathematics; unlike APOS theory, the triad mechanism explains challenges that can hinder the construction of a concept.

The triad mechanism, introduced by Piaget and Garcia (1989), is used to show the development of a schema. The triad mechanism identifies three stages explaining other constructions in the mind implicating mental representations and transformations in the analysis of schema formations.

These stages are the intra-stage, which focuses on a single entity; followed by the inter-stage, which is the study of transformations between objects; and the trans-stage, at which schema are developed by connecting action, processes and objects (Jojo et al., 2010).

The triad mechanism consists of three stages, referred to as intra, inter, and trans in the development of connections. The intra stage focuses on a single 'object'. This implies that everything is constructed as isolated facts. The learner has a collection of rules but at this moment cannot lead them to form coherent thoughts. The inter-stage focuses on transformations between objects (Ndlovu, 2014). At this stage, the learner can make connections between concepts but cannot explain the underlying features. Trans, noted as schema development, is about the connection of actions, processes, and objects. At this stage, the learner can construct the structure of a mathematical concept and explain its underlying features. Clark et al., (1997) (cited in Ndlovu, 2014) assert that at the trans level the elements of the schema must go beyond being described essentially by a list to being described by a single rule. Dubinsky (1991) believes that an individual at the trans stage cannot construct various systems of transformations. Jojo (2011) illustrate the link between APOS theory and Reflective abstraction using diagram as shown below in figure 5.

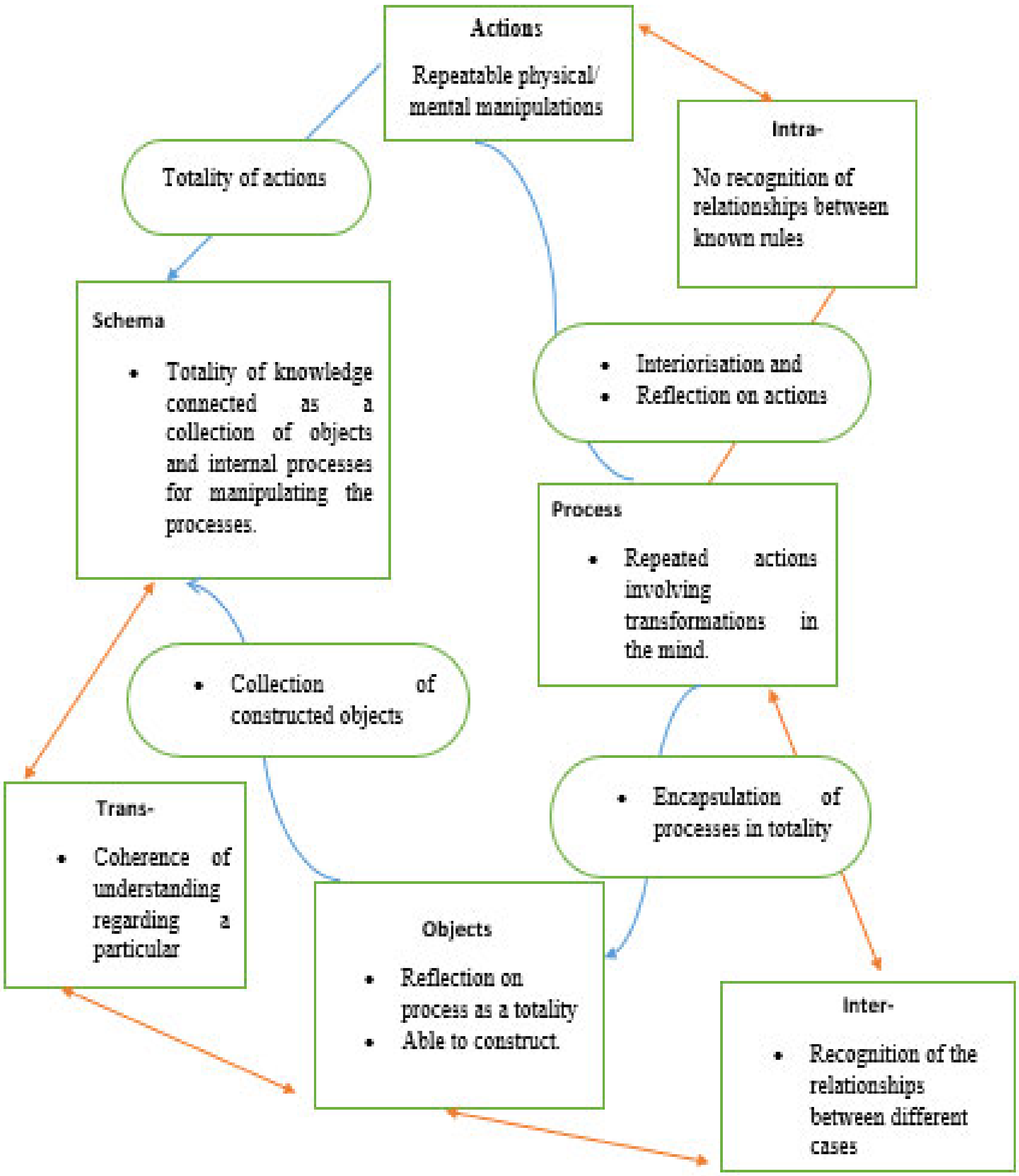


Figure 5 APOS theory extended (Jojo, 2011)

As shown above, the intra stage link with Action stage in the sense that at the action stage learning is more grounded on carrying procedures triggers by physical cues meaning that learning is merely instrumental. Once stage action is interiorised recognition of relationship begin to be constructed.

Once the process is encapsulated leading to lead to coherent understanding which link to trans stage. Below the explanation of each stage is discussed

3.5.1 Stages of the Triad Mechanism

3.5.3.1 Intra-stage

In the intra-stage, a learner focuses on a stage object in isolation of any other action, process, or object. For example, a learner might have a different way of solving given expressions, such as $\sin x = \frac{1}{2}$, using learned rules for finding the solution. A learner who does not form mental relations with such entities is said to be operating at the intra-stage of schema development. The learner has a collection of rules for solving the trigonometric equation but no recognition of the solution because of the misconceptions in the concept. Learners at this stage are able to understand different rules to solve trigonometric equations, however, while others are still at the lower level. In this stage, learners can solve some of the problems by applying trigonometry rules that have been memorised and most of the solutions calculated are incorrect.

3.5.3.2 Inter-stage

When a first step is wrong, it hard to understand what operations are involved or coordinate with other more or less similar ones for the creation of systems that involve certain transformations (Fuentealba et al., 2018). In this stage, learners show the ability to begin to able to collect all of the different rules and recognize the relationships between them. Leaners in the inter-stage show evidence of having collected some or all the rules of trigonometric functions in the Cartesian plane and can identify which trigonometric ratio is positive and negative in which quadrant.

3.5.3.3 Trans-stage

At the trans-stage, a learner has constructed the underlying structure of the chain rule (Jojo, 2015). They have linked the composition and decomposition of trigonometric ratios and their functions and recognized various forms of the chain rule as linked – in the sense that they follow from the Cartesian plan to the trigonometric functions to solve the trigonometric equation. It is only at this stage of development that the underlying structure of the chain rule schema is constructed through reflection on relationships between the various objects from the previous stage. A learner who displays a coherent understanding of the Cartesian plane rules of trigonometric ratios and an understanding of trigonometric functions as a schema has reached the trans-stage of development.

3.6 Conclusion

The genetic decomposition of the chain rule concept may be learned. A learner is said to understand the chain rule once their collection of trigonometric ratios in the Cartesian plane and their understanding of the trigonometric function functions composition is capable of operating on mental constructions acquired and are able to reflect on the explicit structure of the chain rule which these constructions are implicitly containing. The learner's mathematics knowledge of the chain rule depends on their relationship between mental constructs together with the interconnections that the learner uses to understand the concept or rule, and the way in which they use, or fail to use, that concept in problem situations.

This chapter has presented the theoretical framework used for this study. APOS theory and Piaget's triad mechanism were presented, along with a discussion of constructivism, in which they are both rooted. APOS theory will be applied to the analysis of learners' mental constructions in Chapters Five and Six, while the triad mechanism will be applied to the analysis of learner's errors and misconceptions in Chapter Seven. The following chapter presents the methodology used in this study, describes the participants, and sets out the limitations of the study.

CHAPTER FOUR: RESEARCH DESIGN AND METHODOLOGY

4.1 Introduction

The previous chapter presented the theoretical framework used for this study, providing an overview of APOS theory and the triad mechanism. This chapter presents the methodological processes followed in the study. The chapter first discusses different paradigms and how this research is positioned in relation to them. Next, the chapter discusses the qualitative design that was used to engage with learners' mental constructions and misconceptions when learning and solving trigonometric equations. The methods used for sampling and for data collection and analysis are discussed. Finally, the chapter addresses the trustworthiness of the findings, ethical considerations and the limitations of the study.

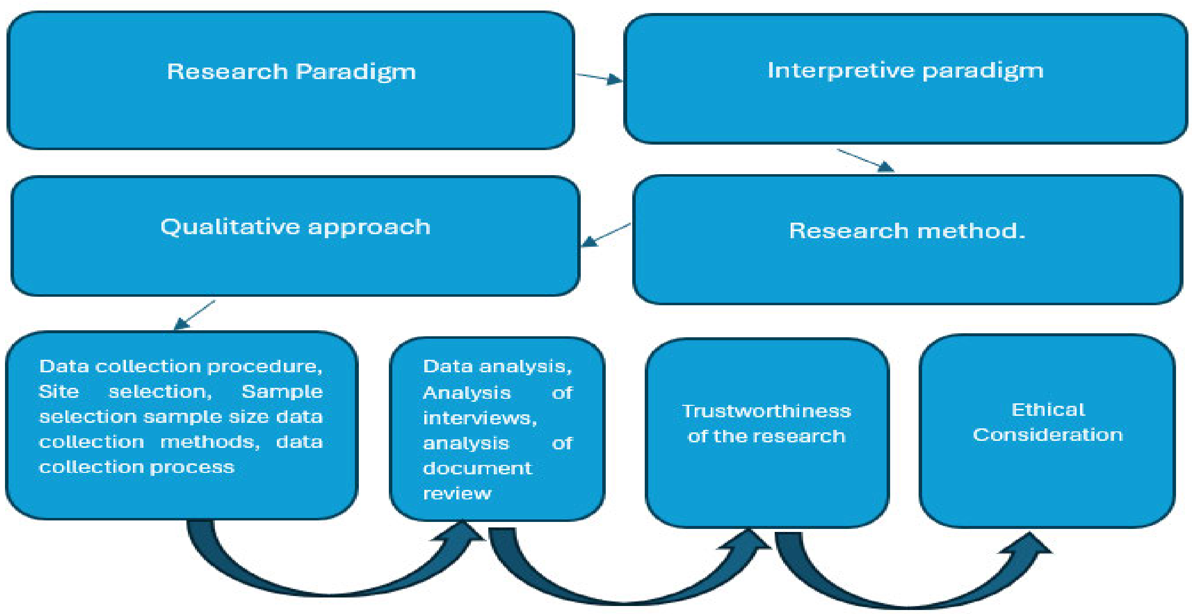


Figure 6 Visual summary of Chapter Four

The heart of this chapter focuses on methodological design. As purported by Plomp (2013), research methodology is a theory of how an inquiry should proceed. Extending this philosophical point of view, Creswell and Creswell (2017) posit that methodology can be defined as a way of systematically explaining the research problem and may be understood as the science of studying how research is done scientifically. It involves an analysis of the assumptions, principles and procedures used in a particular approach to inquiry. This study is aligned with this understanding of methodology. In addition, Corbin and Strauss (2008) define methodology as a way of thinking about and studying social phenomena. This is quite pertinent to this study, which aims to understand learners' learning, which is a social phenomenon.

While Chapter One defined the phenomenon under investigation, this chapter provides a systematic description of how the inquiry was undertaken, framed by the following research questions:

- What are the learners' mental constructions of learning and solving trigonometric equations?
- Why do learners succeed in making, or fail to make, the necessary mental constructions when learning and solving trigonometric equations?
- How do learners' mental constructions of action, process, and object align with the preliminary genetic decompositions?

4.2 Research paradigm

The term 'paradigm' has been broadly defined by many scholars. While different scholars have different ways of defining it, all agreed on the same philosophical point: that it is a way of seeing the world; in the context of research, it is the lens that frames the research process.

Hughes (2010) states that a paradigm is a way of seeing the world that frames a research topic and influences the way that researchers think about the topic. This view can be traced back to the work of Fraser and Robinson (2004), Denzin and Lincoln (2011), Flick (2009) and others. For example, Fraser and Robinson (2004) posit that a paradigm is a set of beliefs about the way in which problems exist and a set of agreements on how such problems can be investigated – a point reiterated by Denzin and Lincoln (2011) and Flick (2009). Denzin and Lincoln (2011) extend their argument beyond the idea that a paradigm is the way one perceives the world, to be a human construction that deals with first principles or indicates where the researcher is coming from so as to construct meaning embedded in data. Paradigms are, thus, important because they provide beliefs and dictates that, for scholars in a particular discipline, influence what should be studied, how it should be studied, and how the results of the study should be interpreted. The same philosophical perspectives are advocated in the work of Kivunja and Kuyini (2017). In summary, a paradigm can be said to be a way of looking at a phenomenon which helps to guide the researcher on how to make decisions and conduct research (Taylor & Medina, 2011).

While in agreement with the other scholars, Chilisa and Kawulich (2012) add another dimension to the theoretical understanding of a paradigm. They moot that every researcher has a particular understanding of what constitutes knowledge and truth, and this understanding shapes the researcher's thoughts and views about themselves and other people as much as their thought and views about the world. Therefore, while they agree that a paradigm depicts one's view of seeing the world, this view is influenced by one's understanding of what is considered to be the truth. Through interacting with the world, we each constructs our reality, and our perspective shapes our understanding of how things are connected (Creswell, 2007; Cohen et al., 2011).

Applying this to the context of this research, this study explores Grade 11 learners' mental construction and misconceptions when learning and solving trigonometric equations. The researcher's interaction with the world of teaching and experiences of teaching mathematics shape how he perceives how the learning of these concepts should take place (as illustrated in the genetic decomposition presented in Chapter 3). Our learning is informed by multiple realities; thus, there is no single truth. As the realities of a learner's construction of knowledge when learning is subjective, an interpretive paradigm was identified as the most suitable paradigm to frame this study.

4.2.1 The interpretive paradigm

This study was concerned with mental construction in the learning of trigonometric equations. The main aim was to explore the mental constructions and misconceptions that learners make when learning and solving trigonometric equations. This study is guided by the interpretative paradigm because it is aligned with constructivist theory, which understands each person creates their own knowledge from their own experiences, resulting in multiple realities or views of truth. it is thus critical, in research, to attempt to understand participants' experience through their own world view. Interpretivism rejects the notion that a single, verifiable reality exists independent of our senses (Rehman & Alharthi, 2016). It refuses to adopt any permanent unvarying standards by which truth can be universally known (Lincoln & Guba, 2005). Instead, interpretivists believe in socially constructed, multiple realities. Truth and reality are created, not discovered. It is not possible to know reality as it is because it is always mediated by our senses.

In contrast to the positivist paradigm, which believe that knowledge is neutral and may be assessed via observation of action, the interpretive paradigm is concerned with understanding the reality of the participants in their natural settings. While the critical paradigm emphasises emancipation, interpretivists strive to interpret the world through the eyes of the participants. The interpretive paradigm pays attention to reality and values what people say, do and feel and how they make meaning of the phenomena under study; it thus allows the researcher to view the world through the perceptions and experiences of participants (Thanh & Thanh, 2015)**.**

4.2.2 Suitability of the interpretive paradigm for this study

This study aimed to reveal the nature of learners' mental constructions and to explore how they came to construct their understanding of the learnt concepts in the classroom-based situation, as well as to understand the misconceptions they had developed with regard to trigonometric equations. The interpretive paradigm was found to be appropriate for this study because the goal of the interpretive paradigm is to develop an understanding of social life and to discover how people construct meaning in a natural setting (Chowdhury, 2014). Human activity cannot be understood from some external reality. Interpretivism, therefore, focuses on people's subjective experiences, on how people construct the social world by sharing meanings, and on how they interact with or relate to each other. Social constructions such as language (including text and symbols), consciousness and shared meanings are used to gain access to and understand reality (Hiller, 2016).

In this study, an interpretive paradigm helped the researcher contend with subjective meaning, as the researcher sought to recognise individual interpretations and understandings of trigonometric equations. In this context, it was of importance to analyse learners' responses to written tasks in order to reveal their mathematical thinking in the context of trigonometric equations. It was also of significance to use interviews to understand how participants constructed meaning, and to draw out embedded meanings. This was done with the hope that engaging in dialogue would shed light on the mental constructions made, in relation to the proposed genetic decomposition. This kind of knowledge makes sense only when drawn from the participants themselves because different contexts might yield different results. Through the interpretive paradigm, the researcher was able to observe different approaches to solving problems and use multiple ways to understand how learners constructed meaning.

4.3 Research design and style

A research design can be thought of as the logic, or master plan, of the research that drives how the study will be conducted. Yin (2017) states that a research design is an action plan for getting from here to there, where 'here' may be defined as the initial set of questions to be answered and 'there' as the set of answers (conclusions).

Bertram and Christiansen (2014) explain that research designs are either quantitative, qualitative or mixed methods. A research design should be selected that is aligned with the research paradigm.

Informed by the literature, the qualitative research design was deemed suitable for this study. The qualitative approach provides multiple ways to explore the inherent complexity and variability of human behaviour and experience (Grace et al., 2009). This study explored Grade 11 learners' mental constructions and misconceptions when learning and solving trigonometric equations. A learning process is a complex process; thus, learners will construct multiple, different realities through the process. Therefore, a qualitative design was deemed to be appropriate as it guided the researcher to understand the construction of knowledge related to trigonometric equations from the participants' perspectives.

Creswell and Clark (2017) explain that, in addition to the selection of a research design, the researcher must choose the type of study they will conduct. In line with the interpretive paradigm and qualitative research design, a case study was deemed suitable to be a suitable research style for this study. The qualitative design and case study design are discussed next.

4.3.1 Qualitative research

Qualitative research can be understood as "an approach for exploring and understanding the meaning individuals or groups ascribe to a social or human problem" (Creswell, 2014, p. 32). According to Cropley (2015), qualitative research examines the way people make sense of their own concrete real-life experiences in their minds and through their words. This study aimed to develop an in-depth understanding of learners' learning and misconceptions when learning and solving trigonometric equations. A qualitative approach was used to gain an understanding of the underlying reasons, opinions and motivations of learners. The qualitative approach also provided insights into the situation and contributed to the development of ideas (Aspers & Corte, 2019).

Qualitative research is often concerned with achieving an in-depth understanding of a phenomenon (Dworkin, 2012). Creswell and Poth (2016) state that the qualitative research approach focuses on understanding the meaning that an individual or a group ascribe to a social or human problem and generating 'words' for data analysis – rather than 'numbers' (as is the case in quantitative research). Social science research asks questions about why people behave the way they do, how opinions and attitudes are formed, how people are affected by the events that go on around them, and how and why cultures and practices have developed in the way they have. Therefore, a qualitative approach is used in this study because the aim was not to find out how much the learners knew but to understand learners' mental constructions and the misconceptions they encountered in learning and solving trigonometric equations.

According to Thorogood and Green (2018), qualitative research has a particular role to play in helping the researcher to generate useful knowledge about the topic at a range of levels: from that of individual perceptions through how the systems work. The process of research in the qualitative design involves emerging questions and procedures, data typically collected in the participants' settings, data analysis conducted inductively – building from particular or general themes, and the researcher making interpretations of the meaning of the data.

This study was located within the broad category of qualitative research design to understand leaners' mental constructions when learning and solving trigonometric equations and learners' difficulties about trigonometric equations.

Qualitative research provides an opportunity to understand peoples' perception in their natural settings and directly speaks to the participants, seeing how they behave and act within their context. According to Austin and Sutton (2014), qualitative researchers have a desire to step beyond the known and enter the world of participants, to see the world from their perspective. In doing so, they make discoveries that contribute to the development of empirical knowledge. This study aimed at exploring the learner's construction of knowledge when learning and solving trigonometric equations and learners' mental constructions of action, process, and object link with the preliminary genetic decompositions. As mentioned earlier, these variables cannot be measured but can only be interpreted from the world view of the participants. By understanding their world view, the researcher hoped to gain insight into leaners' mental constructions of knowledge related to trigonometric equations that could be useful for teaching trigonometric equations in schools. Furthermore, the researcher also hoped to gain an understanding of learners' misconceptions that could also be helpful for teachers.

By its nature, a qualitative research methodology allows one to use different research strategies to collect data. Merrian (2002, as cited in Ndlovu, 2014), describes four qualities of qualitative research: (1) qualitative research elicits participation accounts of meanings, experience or perception about concepts; (2) it produces descriptive data; (3) qualitative approaches allow for more diversity in responses as well as capacity to adapt to new development or issues; and (4) in qualitative methods, forms of data collected can include interviews, group discussions, observations, various texts, pictures and other materials.

This study makes use of a variety of methods to collect data as it used learners' response to trigonometric equations from Grade 10 and 11 on trigonometric questions and primary data collected by means of interviews. Using a variety of resources in this study collected over a long period of time allows for an in-depth understanding of the phenomena interpreted from different stand points but in the same contexts, in this case the school. In qualitative research, the idea is to discover patterns of behaviour or thoughts in a set of texts (Cohen et al, 2011). Creswell (2012) concurs with this idea, and states that the researcher establishes patterns and searches out correspondence between two or more categories.

Since the study was based on a qualitative approach, inductive analyses were used. Firstly, written responses to Grade 10 and 11 tasks from trigonometric equations were analysed using the APOS theory and triad mechanism. Then, the interview responses were coded to determine patterns and categories. The trends that emerged were analysed in line with the framework of the APOS theory, which emphasises the importance of understanding individual action, process, object, and schema.

4.3.2 Case study

In line with the interpretive paradigm and qualitative research design, a case study was deemed to be a suitable research style for this study. A case study aims to describe what it is like to be in a particular situation. In doing so it aims to capture the reality of the participant and their thoughts about the particular situation (Cohen, 2018).

According to Crowe et al. (2011), the case study approach allows in-depth, multi-faceted explorations of complex issues in real-life settings. This ensures that the issue is not explored through one lens, but a variety of lenses that allow multiple facets of the phenomenon to be revealed and understood. Moreover, a qualitative case study facilitates the exploration of a phenomenon within its context, using a variety of data sources (Baxter & Jack, 2008). In this study, the case study enabled the researcher to explain, describe and interpret the data on learners' mental constructions of knowledge as well as the difficulties they developed around learning and solving trigonometric equations.

In this study the case study was used to analyse a phenomenon, to generate hypotheses, and to validate a method use to collect the data (Teegavarapu et al., 2008). A case study method enables a researcher to closely examine the data within a specific context (Zainal, 2007). Case studies, in the true sense, explore and investigate contemporary real-life phenomena through detailed contextual analysis of a limited number of events or conditions, and their relationships (Starman, 2013). A case study is a particular strategy for qualitative empirical research that allows an in-depth investigation of a contemporary phenomenon within its real-life context (Yin, 2018). Krusenvik (2016) states that case study research is often described as a flexible but challenging methodology that is most used in social science research. According to Baxter and Jack (2008), a qualitative case study is an approach to research that facilitates exploration of a phenomenon within its context through the use of a variety of data sources. This ensures that the issue is not explored through one lens but rather a variety of lenses, which allows for multiples facets of the phenomenon to be revealed and understood. In this study, the researcher used a variety of data sources to allow for the understanding and interpretation of learners' mental construction of trigonometric concepts as well as misconceptions. Different people can learn the same concept but construct the meaning differently; thus, using a case study allowed the researcher to explore the multiple facets of knowledge constructions using APOS theory.

Yin (2003, p.3), stated that a case study design should be considered when the focus of the study is to answer the "how" and "why" questions; "when you cannot manipulate the behaviour of those involved in the study; when you want to cover contextual conditions because you believe they are relevant to the phenomenon under study; and when the boundaries are not clear between the phenomenon and context". The advantage of the case study is that it can close in on a real-life situations and test views directly in relation to phenomena as they unfold in practice (Flyvbjerg, 2006). Josefsson (2016) states that the most obvious advantage is that the case study provides a detailed analysis in the individual case. The case study can also offer important evidence to complement experiments. Yin (2009) argues that in all the fields, the need for case studies comes from the desire of understanding complex social phenomena. Therefore, in this study the researcher explores Grade 11 learners' mental construction and difficulties when they are solving and learning trigonometric equations. The case study also provides insight and illuminates meaning that expands the readers' experiences.

Most critics indicate that it is the case study theory, reliability, and validity that are at issue; thus, the very status of the case study as a scientific method is questioned (Flyvbjerg, 2006). Case studies provide very little basis for scientific generalization since they use a small number of subjects, some conducted with only one subject (Zainal, 2007). De Massis and Kotlar (2014) conclude that the findings and recommendations that case studies provide can be either confirmed or denied in terms of utility and veracity, because of the nature of the case study. Yin (2009) argues that the case study is seen as only a preliminary research method and cannot be used to describe or test propositions. Since the aim for this study is to understand Grade 11 leaners mental construction and difficulties when learning and solving trigonometric equations with no intention to generalize them, these limitations are not applicable to this study.

Thus, by using a qualitative case study research approach, the researcher aimed to gain an understanding of learners' mental constructions of learning and solving trigonometric equations.

4.4 Sample and Sampling

Sampling is an element of data collection and is defined by Berndt (2020) as the fragment or section of the population that is selected for the research process. Sampling involves making decisions about which people, settings, events or behaviours to include in the study and deciding how many people, individuals, groups or objects will be observed. Fagerholt et al. (2010) maintain that sampling involves specifying what precisely will be scrutinized in a particular study. Thus, the researcher needs to come to a decision about the number of individuals to interview and research sites to work within the study.

Being able to select a reasonable number of cases and materials to study makes the research more manageable. This study focused on examining the learners' mental constructions and difficulties when learning and solving trigonometric equation. Since it was underpinned by the APOS theory, the sample was selected on the basis that it is in Grade 11 that trigonometric equations are taught in different forms with restrictions to prepare learners for Grade 12. Since the aim is not to achieve findings that can be generalised, one Grade 11 class at one school was selected.

4.4.1 Context of the study

This study was conducted at one school which is situated in a township (KWAS). The school is categorised as Quintile 3 by the Department of Basic Education. Quintile 3 schools do not charge school fees and receive substantial government funding. Most of the learners identify as African. The school is not well-resourced and does not have sufficient facilities to make teaching and learning possible. The school offers academic learning areas in line with the Curriculum Assessment Policy Statement. The school's pass rate is between 65% to 89%. Enrolment is at 1400 learners, with 48 teachers, including four mathematics teachers. The school has structured after-school programmes for Grade 12 Mathematics and Science learners; participation is optional for learners in lower grades.

4.4.1 Sampling

The researcher is a teacher in the school and teaches Grade 12 learners and offers an optional after-school programme to Grade 10 and 11 learners. The sample taken for this study consisted of 17 Grade 11 learners that participated in the after-school programme. Trigonometry in South African schools is introduced at Grade 10 level; thus, the selected learners were the same learners who had been at the school in Grade 10 when trigonometric equations were introduced.

4.4.1.1 Purposive Sampling

The sampling method used for this study was purposive sampling. This method was deemed appropriate for this study because the researcher made a specific choice about participants to include in the sample of the study. According to Palinkas et al. (2015), purposive sampling is used in qualitative research for the identification and selection of information-rich cases related to the phenomenon of interest. Furthermore, Crossman and Nick (2018) indicate that purposive sampling is useful when a researcher is studying a phenomenon or trend that relates to what are considered average members of the affected population. Therefore, this study explored Grade 11 learners' mental construction and difficulties when learning and solving trigonometric equations. Purposive sampling was suitable for the study because it was intended to understand the mental constructions and difficulties of Grade 11 learners.

The researcher specifically selected participants for this study on the basis that trigonometric equations have been a problem to learners who are doing mathematics. Cohen et al. (2011), however, suggests that in purposive sampling researchers chose participants that have knowledge of the phenomena being studied. It is for these two reasons the researcher chose Grade 11. Firstly, over the years Grade 10 learners have had difficulties with trigonometric equations. Secondly, learners continue to experience those difficulties in Grade 11. As a result, learners find it hard to understand Grade 11 and Grade 12 trigonometric equations because the issues have continued from Grade 10. Therefore, Grade 11 was chosen because, having learnt key concepts in Grade 10, they have a knowledge of the concept being researched; for this reason, the researcher assumed they would provide rich data to explore.

To avoid interfering with the school programme and coercing learners to participate in this study, at the beginning of learners' Grade 11 the researcher gave learners flyers about this study and the after-school programme. Those who signed up for the after-school programme were recruited to participate in this study. Over the years, approximately 30 learners have participated in the after-school programme. Therefore, in this study purposive sampling was adopted, since the researcher purposefully selected the participants.

4.4.1.2 Participants in this study

Seventeen learners participated in this study. These learners were involved in the optional after-school programme After the analysis of written responses, the responses were categorized using the constructs of APOS theory: ACTION-PROCESS-OBJECT-SCHEMA. Two learners in each category (in total, seven learners) were interviewed. The selected participants had been involved in the programme since Grade 10; therefore, they were presumed to have in-depth knowledge of the phenomena being studied. Furthermore, the researcher targeted one group knowing that 17 learners did not represent the wider population, but only itself. The researcher was quite aware that there were many learners doing trigonometry equation at other schools to in other districts and that the selected learners were by no means representative of this whole cohort of learners. However, the findings from this study speak to what is probably happening to other learners when they are studying and solving trigonometry equations as it shown that it is one of the challenging topics in South African schools.

4.5 Data collection

This study aims to explore Grade 11 learners' mental constructions and difficulties when learning and solving trigonometric equations. Since mental constructions are evident during the learning and assessment process, it was during teaching and engagement with activities that data was collected. Data was collected using two methods: document analysis and interviews. Document analysis involved analysing learners' responses to classroom tasks done in the lesson. To interrogate learners written responses and thinking process, interviews were conducted. In addition, focus group discussions, in the form of whole class discussions, were used. To avoid interruptions, school guidelines were adhered to: learners worked independently on the tasks instead of in groups and whole class discussion was used and video recorded. In addition, interviews were conducted with selected individual learners.

During the collection of date the limitation was that the learners were not confident enough to appear before the video camera to enable every moment of teaching and learning in the class to be captured, as the researcher was teaching more than 30 learners in the after-school programme and 17 learners agreed to participate in the study. Therefore, notes were taken to serve as a backup in case the researcher encountered problems. Since the researcher was also a teacher at the school, issues of power dynamics and bias were a challenge.

It is within these parameters that the study was conducted with learners in the after-school programme rather during the normal teaching time. The after-school programme is an optional programme where learners are free of any sense of needing to impress the teacher as the activities they engage in at the after-school programme do not contribute to their marks. The after-school programme was designed to provide a space for learners to discuss their learning challenges with the teacher or with peers. Thus, the teacher facilitating the after-school programme (who was also the researcher conducting this study) also did not face pressure to complete the curriculum or to push learners for correct answers, which otherwise could have resulted in bias.

4.5.1 Document analysis

Document analysis is a systematic procedure for critiquing or evaluating documents (Rapley & Reece, 2018). Rapley and Reece (2018) states that document analysis requires that data must be examined and explicated in order to elicit meaning to gain understanding and develop empirical knowledge. As this study was conducted with Grade 11 learners, their responses to summative Grade 10 assessment tasks were analysed to understand their mental constructions of the concepts at Grade 10 level. Morgan (2022) states that document analysis is a form of qualitative research in which documents are interpreted by the researcher to give voice and meaning around the assessment topic. Hence the analysis of Grade 10 tasks was used to inform the development of the genetic decomposition that was used to guide the analysis of the mental constructions in Grade 11 and inform the teaching and learning process.

4.5.2 Administration of structured activity sheets

Structured activity worksheets modelled how meaningful mathematics teaching could be planned with the aim of simultaneously addressing the cognitive and effective domains when students solve problems (Brijlall & Maharaj, 2009). For the purpose of the study, structured activity sheets were administered to explore Grade 11 learners' mental constructions of trigonometric equations. APOS theory is theory of learning that advocates that knowledge construction is hierarchical, meaning that the construction of knowledge of the next level depends on the preceding level (Arnon et al, 2014), thus the structured activity sheet was designed to explore the hierarchical construction of trigonometric equation among the Grade 11 mathematics learners.

Adhering to school guidelines regarding social interactions, individual learners were given one activity sheet with all the tasks to be done in class and as homework. A second set of tasks were done in class, only, in the form of an assessment. The aim was not to do pre- and post-tests but, instead, to explore how the learners' mental construction of solving trigonometric equations evolved over the course of teaching. Learners were given five lessons on trigonometric equations and homework after a lesson and before the assessment. The lessons were conducted for one week, including the weekend.

4.5.3 Semi-structured interviews

After the tasks were analysed using the preliminary genetic decomposition, learner's responses were categorized according to the ACTION-PROCESS-OBJECT mental constructions, as describe in Figure 2, and the preliminary genetic decomposition that was designed after analysing their responses in the previous summative tasks in Grade 10. A total of 7 learners was interviewed. The selection of learners per category was informed by their responses, as learners could display the same mental constructs but be at different levels of thinking; this also catered for withdrawals during the interview process. Semi-structured interviews provide the opportunity for the interviewer to probe and expand an interviewee's responses. According to Trigueros et al. (2017), a semi-structured interview based on a flexible topic guide provides a loose structure of open-ended questions to explore knowledge. In this study, semi-structured interviews were used to probe learners' written responses, as the literature indicates that to understand the level of individual mental constructions it is important to probe the thinking, because having the correct answer does not always mean that one understands the concept (Maharaj, 2014; Ndlovu & Brijlall, 2013; 2015; 2016).

According to Easwaromoorthy and Zarinpoush (2006), semi-structured interviews are useful when the researcher is systematically collecting in-depth information from several respondents. Therefore, the semi-structured interview assisted the researcher in obtaining in-depth information from learners concerning their mental constructions and difficulties in learning and solving trigonometric equations. To ensure that the researcher capture the detailed interview, interviews were done using WhatsApp call and were audio-recorded on the phone as the conversation was taking place.

4.5.4 Focus group discussion/ whole class discussion

Focus group discussion (FGD) is a qualitative research method and data collection technique in which a selected group of people discusses a given topic (Van Eeuwijk & Angehrin, 2017). The focus group is frequently used as a qualitative approach to gain an in-depth understanding of social issues (Nyumba et al., 2018). The method of focus group discussion aims to obtain data from a purposely selected group of individuals. As this study aimed to explore the learners' mental construction and difficulties when learning and solving and trigonometric equations in Grade 11, the learners were divided in groups of four by the researcher (who was also the teacher), who guided the groups on a predetermined topic and created an environment that encouraged participants to share their perceptions.

Observation is one of the most important research instruments that was used and at the same time one of the most diverse. The term includes several types, techniques and approaches, which may be difficult to compare in terms of enactment and anticipated results (Ciesielska et al., 2018). While learners engaged in the learning activities, video recording was also used. The aim of using video recording was to capture learners' activities while in the lesson. As the researcher was also teacher, it was not possible to teach and also observe learners to capture the nuances of their activities. In addition, in ACE teaching style, which is aligned with the framework of the study, observing the nuances of classroom activities is critically important. Furthermore, this is a qualitative study; therefore, open-ended observation was important to allow participants to be themselves while being observed; therefore, the use of the video recording assisted the researcher and the participants to continue with the activities in the classroom. The video recorder was used to observe even non-visible behaviours among the learners. The focus was not on their appearance, but on what they said and how they said it. The class discussion was transcribed.

Table 4 Summary of data generating techniques

Critical Research questions	Participants	Data Collection Technique and Instruments
1. What are the learners' mental constructions of learning and solving trigonometric equations?	Learners	Documents analysis, e.g., summative tasks and activity sheets and video recordings
2. Why do learners succeed/ fail to make the necessary mental constructions in learning and solving trigonometric equations?	Learners	Semi-structured interviews and video recordings
3. How do learners' mental constructions of action, process and object link with the preliminary genetic decompositions?	Learners	Document analysis of activity sheets, interviews and video recordings

4.6 Data analysis

Data analysis was informed by the two theories that comprised the theoretical framework for this study: the APOS theory (Dubinsky 1991) and the triad mechanism developed by Piaget and Garcia (1989). The preliminary genetic decomposition informed by the APOS theory and mental constructs (shown in Figure 3) was used to analyse the mental constructions made by learners. The triad mechanism, comprising intra-, inter- and trans- stages, was used to analyse the difficulties learners revealed.

Data analysis was undertaken using both deductive and inductive approaches. The deductive analysis involved using the categories already established in APOS theory to explain the mental constructions, while the categories defined in the triad mechanism were used to analyse learners' difficulties. For the deductive analysis, patterns that emerged that explained how learners constructed knowledge were analysed inductively, informed by the construction of knowledge discussed in the literature review.

Video and audio recordings made during interviews were transcribed. From the transcriptions, the researcher created narratives a methodology described by Denzin et al. (2023) learning moments, themes, interactions, and the overall cadence of the discussion. The researcher was presented with raw data from learners and the transcriber's initial impressions. The aim was to understanding learners' learning trajectories through the lenses of the theories underpinning the study.

4.7 Trustworthiness of the findings

Quality concerns play a central role in the research process, from the inception of a research study, through data collection to the analysis and presentation of research findings (Ali & Yosuf, 2011). Quality demands that the research be accurate and correct. In qualitative research, questions of validity and reliability are key to assuring quality. Ensuring that qualitative research is accurate and correct can be more complex. Trustworthiness is thus the criterion typically used to ensure quality in qualitative research, rather than validity and reliability. Lincoln and Guba (1985) identify the key features of trustworthiness as credibility, dependability and confirmability, which need to be taken into consideration when conducting qualitative research.

4.7.1 Validity and reliability

Validity in qualitative research refers to the appropriateness of the tools and processes used, and the data gathered (Leung, 2015). Validity is broadly defined by Sireci (2016) as the state of a study being bounded or justifiable, relevant, meaningful, logical and conforming to accepted principles, or the quality of being sound, just and well-founded. It involves ascertaining whether the research question is valid for the desired outcome, the choice of methodology is appropriate for answering the research question, the design is valid for the methodology, and the sampling and data analysis methods are appropriate. According to Creswell and Miller (2000), validity is affected by the researcher's perception of validity in the study and choice of paradigm.

Reliability refers to the exact replicability of the processes and the results (Leung, 2015). It is an explanation idea, whereas the qualitative study's quality notion aims to generate understanding (Stebacko, 2001).

According to Healy and Perry (2000), reliability and validity are more relevant to a quantitative study than they are to a qualitative study. However, Patton (1990) states that validity and reliability are two factors that any qualitative researcher should be concerned about while designing a study, analysing results, and judging the quality of the study.

4.7.2 Trustworthiness

Trustworthiness is the quality, authenticity and truthfulness of the findings of qualitative research. Trustworthiness addressed strategies for ensuring that the study process was carried out correctly. (Gunawan, 2015). Trustworthy studies not only demand the integration of multiple sources of evidence but also must continually take place over time (Cope, 2014). In this study, as indicated in the data collection section above, the learner's responses to an activity were first analysed, then substantiated using semi-structured interviews. This allowed the voice of the learner to come out. Thus, the data collection process wase carried over time using multiple methods.

4.7.2.1 Credibility

Credibility refers to establishing that the results of the research are believable (Hays et al., 2016). Forero et al. (2018) also indicate that the purpose of credibility is to establish confidence that the results are true and believable. Credibility depends more on the richness, rather than the amount, of data gathered and thus corresponds to a qualitative, rather than a quantitative, approach. To ensure credibility in this study, data triangulation was used. Data was collected using the written test and the written response from learners was taken and will be submitted in the university. Furthermore, interviews were conducted, which were recorded and saved.

4.7.2.2 Dependability

Dependability ensures that research findings are consistent and can be repeated (Noble & Smith, 2015). This is measured by the standard of how the study was conducted and the data analysed and presented. Lincoln and Guba (1985) state that the purpose of qualitative inquiry is to assess the repeatability of results if the inquiry occurred within the same cohort of participants. To ensure dependability in this study, the researcher developed detailed drafts of the study would be conducted throughout the study and developed a detailed record of the data collection process. To ensure the dependability of the data, a number of issues were taken into account. For instance, during the data collection and analysis in the first phase, data was collected over a period of three months, and, during the second phase, the data was collected over the period of the year. While the data was being analysed, when some of the recordings were unclear, participants were called in to confirm or explain what they had said during the interview. Data auditing was done to ensure the dependability of the study.

The consent letter was provided to the learners, and it was read together to ensure that participant understood it before they signed. The participants were told that the data obtained would not be disclosed to anybody other than the university structures, and that it would be held safely by the institution. Also, it was explained that their real names would not be used. The learners were informed that if they would like to read the results of the study before they were made public, they had the right to do so.

4.7.2.3 Confirmability

Confirmability refers to how the research findings are supported by the data collected, to establish whether the researcher has been biased during the study. Krostjens and Moser (2018) state that confirmability is the degree to which the findings of the research study could be concerned with establishing that the interpretations of the findings were not figments of the inquirer's imagination but were derived from the data. To ensure confirmability in this study, the researcher checked and rechecked the data during the entire research. Coding was used during the analyses of results.

4.8 Ethical considerations

To ensure that all ethical issues were appropriately addressed, a letter outlining the nature, process and purpose of the study was given to the Department of Education and to the principal of the school, seeking permission to conduct the study at the school. The ethical clearance reference number is HSSREC/00003000/2021 (see Appendix B). Letters of informed consent also were given to all the participants to read and sign (see Appendix A). In the letter, it was clearly stated that participation is voluntary, and that participants could withdraw anytime they wanted to, only needing to inform the researcher if they wished to do so. Participants were made aware of their rights as participants when they read and signed the statement (Choy, 2014). Before the commencement of the study, the researcher explained and emphasised these issues to participants. To ensure confidentiality, pseudo names were used for the research site and the participants.

4.9 Conclusion

This chapter has highlighted the major methodological considerations of the study. The research was presented, and the interpretive research paradigm used in this study was discussed. The chapter considered the qualitative research paradigm, the methods of data capturing, ethical issues and trustworthiness in line with qualitative research approaches. Chapter Five summarizes the findings obtained from analysed data on the trigonometric equations test questions and the interviews.

CHAPTER FIVE: LEARNERS' MENTAL CONSTRUCTIONS FOR TRIGONOMETRIC EQUATIONS: ANALYSIS OF WRITTEN RESPONSES FOR PHASE ONE

5.1 Introduction

The previous chapter focused on the research design and the methodology of the study. Limitations and issues of the trustworthiness of the study were explained in detail. While this study explored Grade 11 learners' mental constructions and difficulties when learning and solving trigonometric equations, it was also an aim of the study to explore how learners' mental constructions evolved over a period of two years while learning trigonometry. In the South African context, learners start to learn trigonometric equations in Grade 10. Thus, Phase One of the study was conducted while the participants were still in Grade 10.

Grade 10 trigonometric equations involve finding the lengths of the sides and angles of right-angled triangles, as well as more general trigonometric equations. In Grade 11, learners must calculate the angle given the equation, and trigonometric equations that involving the interval that satisfy the equation. The research questions (presented in Chapter One) were as follows:

1. What are the learners' mental constructions when learning and solving trigonometric equations?
2. Why do learners succeed/fail to make the necessary mental constructions in learning and solving trigonometric equations?
3. How do learners' mental constructions of action, process and object link with the preliminary genetic decompositions?

For Phase One, qualitative methods were used, and the first set of data was collected by administering an assessment task to a group of mathematics learners (n=17). The task was designed to provide insight into their mental constructions when solving trigonometric equations.

The data generated from the written tasks administered in Phase One corresponded to the first research question, which guides the discussion in this chapter:

What are the learners' mental construction when learning and solving trigonometric equations?

This data also contributed to the full data set used to answer the third research question, which will be discussed in the final chapter.

5.2 Analysis of learners' written responses to tasks

The task was designed to provide insight into learners' mental constructions when solving and learning trigonometric equations. The design of the task was guided by the understanding of the mental constructions the learners made when learning mathematics concepts in the classroom which led to improved instructional methods and curriculum development (Ndlovu & Brijlall, 2015). The task was chosen to allow the learners to make the necessary mental constructions.

The data were analysed to assess the levels of APOS theory learners were operating at with respect to trigonometric equations. Understanding the level learners are operating at assists in understanding learners' conceptual development and thus their development of a schema – in this case, a schema for trigonometric equations. The task consisted of six items involving trigonometric equations. The questions were categorized using the four cognitive levels stipulated in the curriculum: knowledge, routine tasks, complex procedures, and problem-solving. For the purpose of this study, questions at the same cognitive level were grouped together to form an item. Thus, the ten problems were grouped into six items.

The first part of the data analysis involved categorising participants into three categories based on their responses. The first category was for learners who had made the necessary mental constructions to solve the problem, although their solutions may have revealed different levels of conceptual development. The second category was for learners who had attempted to solve the problems but had failed to come to the correct solution, with no evidence of mental constructions. The third category was for learners who had not attempted to solve the problem. Table 5, adopted from Ndlovu and Brijlall (2016), was used to categorise learners' responses during the first part of the analysis of written responses.

Table 5 Categories for analysis of learners' written responses (adopted from Ndlovu & Brijlall, 2016)

Category	1	2	3
Indicator	Made the necessary mental constructions	No evidence of making the mental constructions	No attempt
Number of responses in each category			

This analysis was undertaken for each item.

The second part of the analysis focused on learners' mental construction to ascertain the level learners were operating at. Genetic decomposition is a hypothetical model that describes the mental structures and mechanisms that learners might need to construct in order to learn a specific mathematical concept (Arnon et al., 2014). APOS theory guided the analysis of the mental construction made. Thus, deductive analysis was used in this chapter. Table 6 shows a sample of the tables used to analyse learners' mental constructions, also adopted from Ndlovu and Brijlall (2016).

Table 6 Categories for analysis of learners' mental constructions

Mental constructions made	Action	Process	Object	Coherent schema
Number of responses in each category				

While APOS theory has been found to be useful for determining the nature of mental constructions, it does not necessarily aid in identifying the difficulties that may have hindered the development of the necessary mental constructions. In light of this, APOS theory was used to determine the nature of the mental constructions made when solving each item.

5.2.1 Analysis of learners' responses to Item 1 (equations of the form $a\sin x; \cos x; \tan x = b$ where $a \geq 1$)

Item 1 was designed to provide insight into learners' mental constructions when solving questions that require recalling knowledge and facts. In mathematics, solving one mathematical problem requires knowledge of the concepts learned before or learned in other topics – knowledge which Tall (2008) refers to as 'met before'. To solve trigonometric equations, knowledge of how to solve an algebraic equation is critical, as is knowledge of trigonometric ratio.

Two questions were given to learners, as shown below. The function of sine and cosine are continuous, while a tangent function is discrete. Thus, using one question – using either sine or cosine – was considered adequate to gain insight into the learner's mental constructions.

Solve for x

1.1 $3\sin x = 2$

1.2 $7\tan x = 10$

Figure 7 Item 1

As illustrated in Table 7, all learners (17) attempted to solve Item 1 and presented a solution to Item 1.1. However, only 16 learners provided a solution for Item 1.2.

Table 7 Analysis of learners' written responses to Item 1

Category	**1**	**2**	**3**
Indicator	Made the necessary mental constructions	No evidence of making the necessary mental constructions	No attempt made
No of responses for Item 1.1	17	0	0
No of response for Item 1.2	16	1	0

Based on the categorization of learners' responses, it appears that most of the learners knew the necessary facts to solve trigonometric equations of the form $\boldsymbol{a sinx; cosx; tanx = b \text{ where } a \geq 1 \text{ and } b \geq 1}$.

Next, the levels the learners were operating at were identified. These are presented in Table 8

Table 8 Mental constructions made by learners for Item 1

Level of mental construction	**Action**	**Process**	**Object**	**Coherent schema**
Number of responses for Item 1.1	14	3	0	0
Number of responses for Item 1.2	16	0	0	0

Learners operating at the action level when solving trigonometric equations provide a step-by-step solution (this is discussed further in Chapter Three, under genetic decomposition). As purported by Arnon et al. (2014), at this level each step is performed explicitly; no step can be skipped, meaning that the learner at this stage cannot visualise a certain step in their mind. For an individual to transition to the next step, each procedure needs to be performed explicitly. Therefore, the analysis of participants' written responses to Item 1.1 reveals that 14 were operating at the action level and while 3 learners were operating in the process stage. For Item 1.2, 16 participants were operating at the action level, carrying out all procedures step by step. This is illustrated by Fihlela's responses to Items 1.1 and 1.2, shown in Figures 8 and 9.

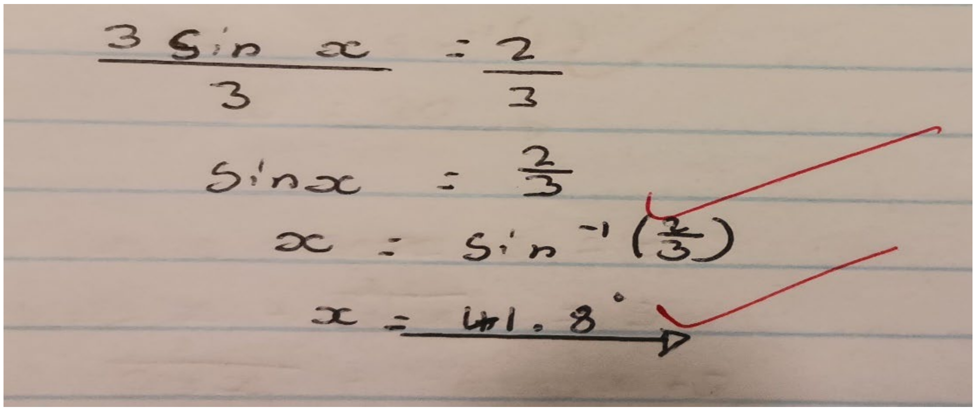


Figure 8 Fihlela’s written response to Item 1.1, illustrating action level

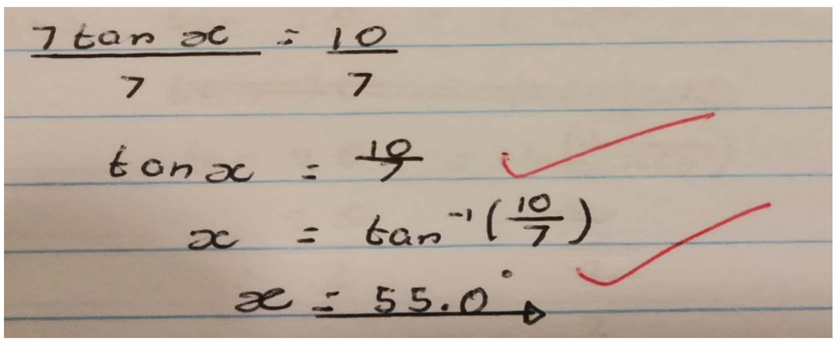


Figure 9 Fihlela’s written response to Item 1.2, illustrating action level

Fihlela’s responses were allocated to Category 1 (See Table 7); i.e., categorized with those that provided the correct solution. In both Items 1.1 and 1.2, Fihlela performed all the steps needed to solve for the variable x. While he carried out the necessary procedures, the final demonstrated that he knew the procedure to solve the problem, and that he had constructed the meaning of the solution as he answer was given in degrees. Unlike in algebraic equations, where variable x represents an unknown number, in the trigonometric equations it represents an angle. Fihlela, along with other learners categorized as operating at an action level, displayed knowledge of the procedure but did not correctly construct the meaning of the concept.

Weller et al. (2009) state that when a person is operating at the process level they perform the same operation as at the action level, but they carry this out in their mind without having to execute each step explicitly. Using this explanation of the process level, learners in this study were identified as operating at the process level if they skipped some steps during the process of determining the solution. Ndeo was one learner who did not perform all the steps to solve for the variable x, as shown in extract 5.3 below.

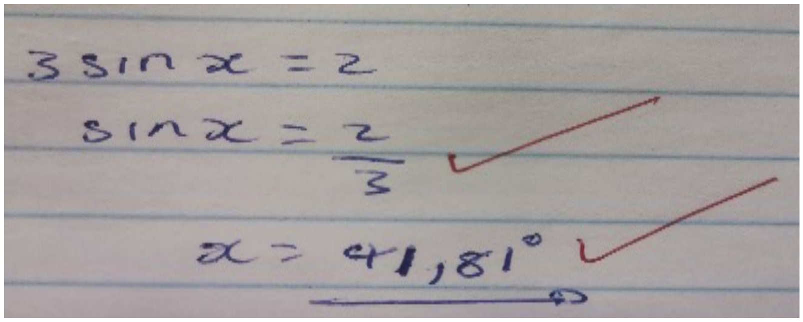


Figure 10 Ndeo's written response to Item 1, illustrating the process level

Ndeo carried out the necessary procedures to the final solution and showed that she understood that the solution was an angle, because she gave the answer in degrees. Only 3 learners demonstrated that they were operating at the process level because they did not write down every step to find the solution. None of the learners were found to be operating at the object level or had formulated a coherent schema for either Items 1.1 or 1.2. Their inability to formulate a coherent schema suggests that, although some of the learners could solve the trigonometric equations, their conceptual knowledge was still at a basic level: while they were able to recall the necessary facts and carry out the necessary procedures, their understanding of the concept had not yet been cognitively constructed, as demonstrated by Fihlela's failure to use a degree symbol, suggesting that he did not understand the solution was an angle. Ndlovu's (2014) study found that learners operating at the action level had not made sense of the importance of terminology or notation in mathematics and had not memorised, or fully understood, these notations. As was found in this study, the learners operating at the action level were just performing computations.

5.2.2 Analysis of learners' responses to Item 2 ($\boldsymbol{equations\ of\ the\ form\ asinx; cosx; tanx \pm b = 0, where\ a \geq 1\ and\ b > or < 0}$)

Item 2 was designed to provide insight into learners' mental constructions when solving questions that require routine tasks. A routine problem demands the application of a similar or previously learned formula to a new situation, which usually involves the use of arithmetic skills (Kablan & Uğur, 2021). In Item 1, only one step is required to make the function the subject of the formula, while in Item 2 multiple steps are required before one can solve the given equation. Learners were given two questions to solve for x, as shown in Figure 11.

Solve for x,

2.1 $4 \sin x - 3 = 0$

2.2 $5 \cos x + 4 = 0$

Figure 11 Item 2

For these items, learners were required to understand the rules of signs. The difference between Item 1 and Item 2 is that in Item 2 the equations have two terms on the left-hand side, that are equal to zero on the right-hand side, while in Item 1 there is only one term on the left-hand side, equal to a number greater than zero on the right-hand side. Unlike Item 1, Item 2 requires multiple steps to determine the value of the angle. Table 9 below summarises the analysis of the learners' responses.

Table 9 Analysis of learners' written responses to Item 2

Category	**1**	**2**	**3**
Indicator	Made the necessary mental constructions	No evidence of making the necessary mental constructions	No attempt made
No of responses for Item 2.1	15	2	0
No of responses for Item 2.2	16	1	0

As shown in Table 9, 15 of 17 learners for Item 2.1 and 16 of 17 learners for Item 2.2 displayed some level of understanding of how to solve trigonometric equations that require the use of procedures involving multiple steps. The learners in Category 1 gave a complete response to all aspects of the item for both Items 2.1 and 2.2. In Item 2.1, 2 learners failed to make the necessary mental constructions; however, 1 of the 2 learners was able to solve Item 2.2. To interrogate further the mental constructions made the learners, Category 1 responses were further categorized in terms of the mental constructions identified by APOS theory. A summary of the analysis of mental constructions is presented in Table 10 below.

Table 10 Mental construction made by learners for Item 2

Mental constructions made	**Action**	**Process**	**Object**	**Coherent schema**
Number of responses for Item 2.1	15	0	0	0
Number of responses for Item 2.2	15	1	0	0

In line with APOS, the learners' responses revealed that, for both Items 2.1 and 2.2, 15 learners were found to be operating at the action level. An individual with an action conception would need to have explicit expressions for each function and could only think about the composition for specific values, they would perform step-by-step procedures, as shown in the example in Figure 12.

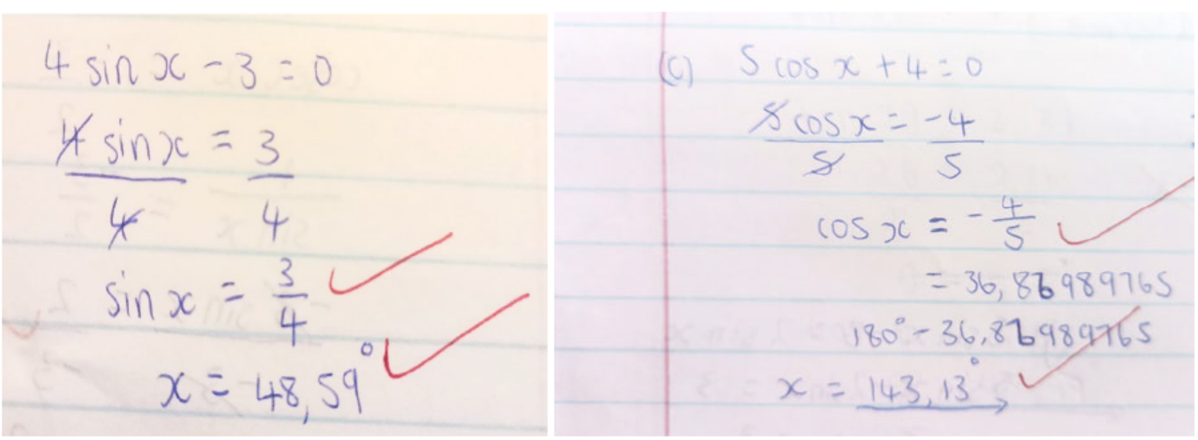


Figure 12 Makwanza's written responses to Item 2.1 (left) and Item 2.2 (right), illustrating performance at the action level

As evident in the figure above, Makwanza explicitly performed all the steps to determine the angle represent by x. Beyond carrying out procedures in her response, she carried out all the steps in the final solution. The external cue that triggered Makwanza's solution to each problem was the sign attached to the numerical value. For example, in Item 2.1, after making the function the subject of the formula the equation was $\sin x = \frac{3}{4}$, where three-quarters were positive while in Item 2.2 $\cos x = -\frac{4}{5}$,where four-fifths were negative, thus guiding her to find where the solution is located. However, her understanding was limited to one solution; she lacked the understanding that the function was either positive or negative and, thus, the solution lay in more than one quadrant.

When solving equations, the focus is on the value(s) that make the statement to be true. In dealing with linear equations, there is one solution to the equation; however, in trigonometric equations, there is more than one solution to the equation because the focus is on determining whether the function is either positive or negative, not on the numerical value. However, as shown in Makwanza's solution, for learners operating at the action level construction is limited to finding the value only, ignoring the aspect of the position of a function. Therefore, the 15 learners categorised as operating at the action level based on their responses to Items 2.1 and 2.2 had not yet constructed an understanding of the meaning of the solution of trigonometric functions; their understanding was limited to a knowledge of how to carry out procedures, not how to make meaning.

Dlamini, in contrast, was categorized as operating at the process level for Item 2.2. Like the learners categorized as working at the action level, he arrived at a correct solution to the equation; however, he was able to perform some of the steps mentally, rather than writing them out, as shown in the extract below.

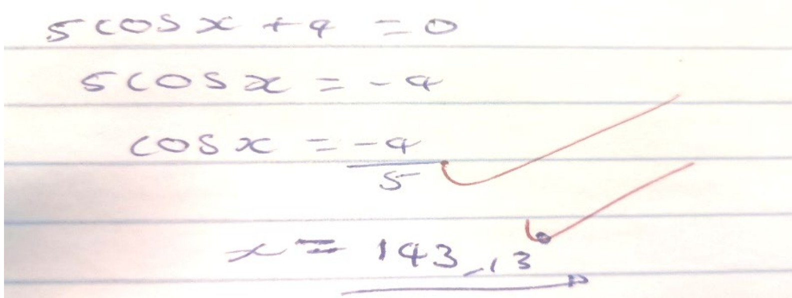


Figure 13 Extract from Dlamini's response to Item 2.2 demonstrating process level

As noted above, the steps involved with making the function the subject of the formula are performed in the learners' minds at the process level according to APOS theory. After making the trigonometric function the subject of the formula, Dlamini determined the quadrant where the solution lies mentally, and then used the reference angle to determine the solution. However, his inability to construct the relevant schema hindered his ability to apply the relevant actions and processes to determine the other solution to make the statement true.

5.2.3 Analysis of learners' responses to Item 3 ($\boldsymbol{equation\ of\ the\ form\ asinx; cosx; tanx = b; where\ a\ is\ a\ variable}$)

To solve Item 3, an understanding of special angles and algebra is required. Wilson (2016) defines a complex procedure is one where a procedure involving conceptual understanding is used; the procedure is thus carried out with sense-making. According to this understanding, Item 3 requires a complex procedure.

$\boldsymbol{Solve\ for\ x}$

3.1 $x\ cos\ 30° = 2$

Figure 14 Item 3

Unlike in Items 1 and 2, where the unknown was the angle, in Item 3 the unknown is the coefficient of a function. In this instance, learners were expected to find x, the co-efficient of the function. Learners' understanding of the meaning of the variable in the equation was thus critical to solving the equation because it determined the procedures to be followed to solve for the unknown. Table 11 below summarised the analysis of learners' responses to Item 3.

Table 11 Analysis of learners' written responses to Item 3

Category	**1**	**2**	**3**
Indicator	Made the necessary mental constructions	No evidence of making the necessary mental constructions	No attempt made
No. of responses	5	9	3

The table above indicates that learners experienced difficulty working with trigonometric equations involving special angles. Of the 17 learners who attempted Item 3, only 5 made the necessary mental constructions, as reflected in Category 1.

Table 12 below shows the analysis of mental constructions made by learners.

Table 12 Mental construction made by learners for Item 3

Mental constructions made	**Action**	**Process**	**Object**	**Coherent schema**
No, of responses in each category	5	0	0	0

The analysis of learners' responses revealed that of the 5 learners who made the necessary mental constructions, all were operating at the action stage, because they could only arrive at the correct answer after carrying out all the procedural steps explicitly, showing that they had not yet interiorized action as a process. This is illustrated by Shozi 'response below.

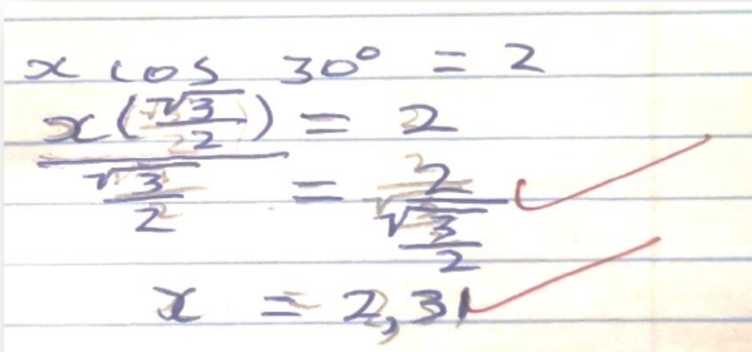


Figure 15 Shozi's response to Item 3, illustrating that action has not been interiorized as process

Three learners did not attempt to solve the equation in Item 3. It was observed that learners were unable to apply the concept of special angles to solve the equation, when the variable to be solved was the co-efficient, in the same way they would solve a linear equation. function. It is possible learners are not used to solving trigonometric equations where the unknown represented the angle and thus constructed the solution to a trigonometric equation that is different to that of a linear equation. Their concept of a trigonometric equation was evidence that they had memorised it but not understand it. Some of the 9 learners who were found not to have made the necessary mental constructions, even though the variable to be solved was the co-efficient, treated it as an angle: they changed the equation to 30 $\cos x = 2$ and followed the same procedures as in Item 1. Shozi, however, was one of the 5 learners who was categorized as operating at the action level as he displayed an understanding of the variable to be solved and carried out the necessary procedures.

5.2.4 Analysis of learners' responses to Item 4 (equation of the form $\boldsymbol{acosecx; sec\ x; cotx \pm b = 0\ where\ a \geq 1; b < 0}$)

Item 4 aimed to explore learners' mental constructions when solving trigonometric equations involving a reciprocal. At the Grade 10 level, this is considered to be higher order, or problem solving. It is not part of the Grade 10 curriculum. However, learners are expected to use their knowledge of solving equations involving cos, sine, and tan to solve equations involving reciprocals. Cahyani et al. (2021) assert that the purpose of learning mathematics is to enable learners to develop the ability and knowledge to solve mathematical problems in an unfamiliar context. Ndlovu (2014), in her study, argued that learners' understanding of concepts is strongest when they are able to use known concepts to solve unfamiliar problems. In this study, for learners to be able to solve Item 4, a knowledge of how to solve trig ratios and trig equations involving the cos, sine, and tan was critical.

Item 4 was designed to explore learners' insight into performing the necessary actions and processes to solve equations involving reciprocals. The function sine, cosine, as mentioned in Item 1, is continuous; thus, their reciprocal functions are continuous, while the tangent function is discrete and has an inverse function of cotangent.

Solve for x

4.1 $\cot x - 2 = 0$

Figure 16 Item 4

The results shown in Table 13 below indicate that all of the learners attempted Item 4. However, only 12 learners provided the correct answer, showing some level of understanding. The other 5 learners attempted to solve the problem but failed to provide the correct answer. They made algebraic errors in their solutions and were found to lack the necessary knowledge to relate the reciprocal cotangent to tangent.

Table 13 Analysis of learners' written responses to Item 4

Category	**1**	**2**	**3**
Indicator	Made the necessary mental constructions	No evidence of making the necessary mental constructions	No attempt made
No. of responses	12	5	0

The 12 responses in which learners made the necessary mental constructions were analysed to identify the APOS levels at which learners were working. The results are shown in Table 14 below.

Table 14 Analysis of mental constructions made by learners for Item 4

Level of mental constructions	**Action**	**Process**	**Object**	**Coherent schema**
No of responses	12	0	0	0

The analysis of learners' written responses revealed that all 12 learners were operating at the action level because their responses were written step-by-step. The genetic decomposition (discussed in Chapter Three) states that learners at the action level will show all the steps in their responses. Mandonsela et al. (2020) state that action is based on rules. Therefore, learners followed the rules in solving Item 4. Although Item 4 aimed to explore the encapsulation of the process into an object, learners' responses showed that they were actually operating at the action level, because only one solution was provided – showing that their focus was on carrying out the necessary procedures and not meaning making, as the solution to the item was found in two quadrants. Makhanya's response illustrates the carrying out of procedures without constructing the meaning of the solution to the problem.

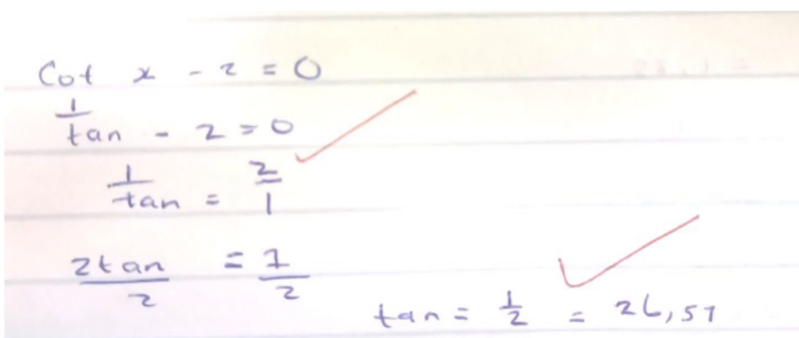


Figure 17 Makhanya's response to Item 4, showing that meaning has not been constructed

As evident in the response above, Makhanya made the connection between function tan and its reciprocal cot, thus representing cot as $\frac{1}{tan}$. However, in representing the cot as tan, she ignored the angle and went on to perform the calculation without indicating the correct notation for the angle being solved. Ndlovu and Brijlall (2015) term this 'number grabbing'. Makhanya correctly performed the procedures for making tan the subject of the formula, but by dropping the angle she treated the function as the variable. Also, she gave her solution in numeral form, which indicated that she was approaching it as a routine calculation – carrying out the action of finding the solution as a procedure, without making sense of what exactly was being calculated. The positive sign indicated that the solution was positive but no understanding of how many solutions made the statement to be true was demonstrated. Makhanya's written response indicated that she had not yet interiorised the solution of triangles involving reciprocals as a process, as she explicitly performed all the steps with no meaning-making. Dubinsky (1991, cited in Ndlovu & Brijlall, 2015) posits that the stages of APOS theory are hierarchical, with progress to the next level dependent on the construction of the previous level; thus, failure to construct the process level would result in the learner being unable to encapsulate the process into an object.

5.2.5 Analysis of learners' responses to Item 5 (equations of the form $\boldsymbol{a cosec x; sec\ x; cot x \pm b = 0\ where\ a \geq 1; b < 0\ or\ b > 0}$)

Item 5 was designed to investigate learners' mental constructions for solving reciprocals and assess their problem-solving skills. Item 5 requires the same skills and knowledge of reciprocals that is required in Item 4; however, while in Item 4 learners solved reciprocals of tan, in Item 5 learners solved reciprocals of sine and cosine. Faulkner et al. (2021) state that to solve a problem efficiently one has to acquire new information, select relevant information to employ to solve the given problem. To solve Items 5.1 and 5.2, knowledge of algebra and of reciprocals was key.

Items 5.1 and 5.2 were grouped together because they test the same knowledge and are reciprocals of sin θ *and* cos θ, which are continuous functions.

Solve for x

5.1 $2\ cosec\ x + 3 = 0$

5.2 $4 \sec x - 12 = 0$

Figure 18 Item 5

For Item 5.1, 16 learners attempted to solve the problem, but only 11 were categorized as making the necessary mental constructions. Five attempted but failed to solve the problem, suggesting that they had not constructed an understanding of how to solve trigonometric equations involving reciprocals. A larger number of learners (14) were able to solve the problem in Item 5.2.

Table 15 Analysis of learners' written responses to Item 5

Category	**1**	**2**	**3**
Indicator	Made the necessary mental constructions	No evidence of making the necessary mental constructions	No attempt made
No of responses for 5.1	11	5	1
No of responses for 5.2	14	2	1

Further analysis of the data was done to ascertain the levels at which the learners were operating, as shown in Table 16.

Table 16 Analysis of mental construction made by learners for Item 5

Mental constructions made	**Action**	**Process**	**Object**	**Coherent schema**
Number of responses for Item 5.1	11	0	0	0
Number of responses for Item 5.2	14	0	0	0

Learners' written responses revealed that 11 learners operated at the action stage of APOS, in terms of decomposing the given trigonometric equation in Item 5.1, and 14 learners in Item 5.2. Analysis of participants' written responses revealed that none of the learners had operated at the process or object levels in their solutions to the Item 5 problems.

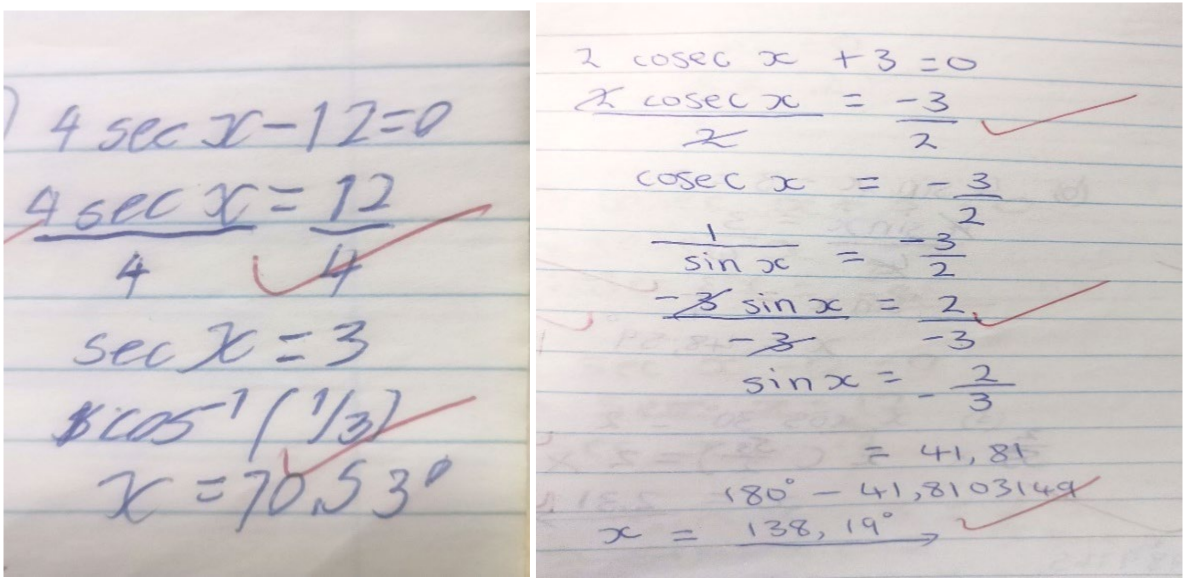


Figure 19 Zulu's response for Item 5.1 (left) and Makhwanza's response for Item 5.2 (right)

As indicated above in Table 16, for Item 5.1 11 learners were categorised in Category 1 for Item 5.1 and 14 learners for Item 5.2. Learners displayed a mathematical understanding of the concept of the reciprocals of cosec and sec. The responses indicate that learners identified the co-efficient and that they needed to divide with it on both sides and they understood the inverse of $\sec x \; that \; is \; equal \; to \; \frac{1}{\cos x} \; and \; cosec \; x = \frac{1}{\sin x}$. Learners in Category 1 were found to be able to make the necessary mental constructions and were operating at the action stage. In the extract shown in Figure 19, Zulu performed the required procedures step by step to answer Item 5.1; Makhwanza also carried out step-by-step procedures to determine the answer for Item 5.2. Some of the learners did not use the correct notation to indicate that they were calculating an angle, however. While they performed the procedure, it was evident that they did this with no meaning-making. For example, Zulu, when writing sec as cos^{-1}, did not include the angle but indicated in the final answer that he was determining the value of the angle. If a learner views a procedure as an isolated fact, it suggests that they are viewing the procedure as an externally directed transformation of solving any other equation as in algebra where the power of the first variable triggers the number of solutions. Thus, in trigonometric functions the power is viewed as 1; thus, the learner cannot think of the equation having two solutions. In addition, the meaning of the notation is not yet interiorised.

5.2.6 Analysis of learners' responses to Item 6

Item 6 was designed to provide insight into learners' mental constructions when solving a challenging trigonometric equation. An understanding of algebra and of trigonometric ratios was required to solve this problem. Özreçberoğlu and Çağanağa (2018) argue that understanding a problem is as important as solving it to developing an understanding of the meaning of mathematics.

Item 6 required learners to solve a challenging trigonometric equation that involved problem-solving skills:

Solve for x

6.1 $5\,sin\,x - 3 = 2\sin x$

Figure 20 Item 6

Item 6 was attempted and solved by 14 of the 17 learners. Only 13 learners showed the development of mental constructions in their responses. Only 1 learner gave an incorrect answer; they showed no evidence of understanding the problem. Three learners did not attempt to answer Item 6.

Table 17 Analysis of learners' written responses to Item 6

Category	**1**	**2**	**3**
Indicator	Made the necessary mental constructions	No evidence of making the necessary mental constructions	No attempt made
No. of responses	13	1	3

Table 18 indicates the level at which learners were found to be functioning in their responses to Item 6, with reference to APOS theory.

Table 18 Level of mental constructions made by learners for Item 6

Mental constructions made	Action	Process	Object	Coherent schema
No. of responses	12	1	0	0

For Item 6, of the 17 learners, 12 were found to have an action conception of the trigonometric equation; they interpreted the question correctly and applied the correct rules to solve it. 1 learner operated in the process level stage. Learners' written responses revealed that most of the learners who made mental construction developed action conception in item 6.

Figure 21 shows Ntaka's written response to Item 6. Ntaka was categorised in Category 1 (Table 17) together with 12 other learners.

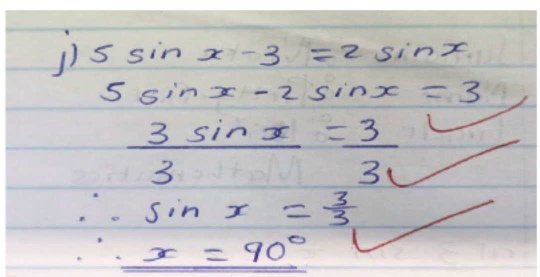


Figure 21 Ntaka's response to Item 6

Ntaka demonstrated that she knew the procedure to solve the problem, and that she had constructed the meaning of the solution as her answer was given in degrees. In terms of APOS theory, she had developed an process conception of the concept. In terms of genetic decomposition, the understanding of the structure of the trigonometric equation was accommodated. Ntaka's response indicated she was able to identify the like terms as she transposed 2 sin x to the left and subtracted both of them, ending up having 3 sin x, as she continued to divide both sides by 3 and solve for the angle. She jumped the second last step as she divided both sides with 3 which is equal to 1.

However, Most of the learners in Category 1 demonstrated that they were operating at the action level, with minor errors by some. It was also observed that one of the learners operated at the process and object levels in their responses to Item 6.

5.3 Summary of findings on learners' mental constructions

An analysis of learners' responses during Phase One revealed that they were capable of carrying out procedures, which suggests they had developed procedural knowledge. Hurrel (2021) notes that, while procedural knowledge is important, when procedures are performed without making sense of the concepts it can lead to peculiar and unreasonable solutions. The findings from Phase One of this study showed that, while learners were able to perform the procedures correctly most of the time, they were applying rules without reason, simply as centred around recollection of facts, meaning that they were solving using the rules but not understanding how to solve the items. This was made evident when they failed to use correct notation to indicate that their solution represented an angle. Also, when solving trigonometric equations involving reciprocals, they demonstrated knowledge of the fact that cot is the same as $\frac{1}{tan}$, but arrived at unreasonable solutions because they did not demonstrate the unknown being calculated and indicate what the answer was. Thus, most learners displayed an action conception of how to solve trigonometric equations, where the action was not yet interiorised. External cues triggered their knowledge of the correct procedures to be followed but this was not internalized.

5.4 Conclusion

This chapter has reported the findings with regard to the mental constructions made for solving trigonometric equations during Phase One of this study, when the learners were still in Grade 10. APOS theory was used to analyse learners' mental constructions for trigonometric equations as revealed in their written responses to problems.

The next chapter presents the findings with regard to the learners' mental constructions, made while solving trigonometric equations once they were in Grade 11. Learners' responses to interview questions are also presented.

CHAPTER 6: LEARNERS' MENTAL CONSTRUCTION OF TRIGONOMETRIC EQUATIONS

6.1 Introduction

Chapter Five presented an analysis of the data collected in Phase One. The data for Phase One was collected while the learners were still in Grade 10 to enable observation of the development of learners' mental constructions related to solving trigonometric equations across Grades 10 and 11. This chapter analyses data collected during Phase Two of the study once the learners had progressed to Grade 11 related to learners' mental constructions and their development over a period of 1 year and 6 months. Learners' errors and the misconceptions that were found to hinder the development of their mental constructions around trigonometry in Phase Two are discussed in the next chapter.

Three research questions guided this study, as presented in Chapter One. The collection of data using written tasks in Phase Two was designed to answer the third research question:

> **How do learners' mental constructions of action, process, and object align with the preliminary genetic decompositions?**

Question 2 will be discussed in the next chapter.

6.2 Collection and analysis of data

In Phase Two, the same 17 learners were given an activity sheet with three tasks, each containing two or three problems to solve (called 'items'). Some of the learners were selected for interviews with the researcher based on their responses to the activity sheet.

6.2.1 Activity sheet

Tests and exams are important evaluation instruments in the teaching-learning process as they are used primarily to measure and improve student learning; for this reason, they should be well-designed (De Guzman & Adamos, 2020). The design of the questions for Phase Two was guided by the level of learners' mental constructions demonstrated during Phase One (presented in Chapter Five).

Seventeen learners who had attended an afterschool programme since Grade 10 participated. According to Bassey et al (2022), test findings serve as a crucial decision base for ascertaining the knowledge, understanding, and skills of different learners. Therefore, the tasks chosen in the second phase were those that the researcher identified as suitable for allowing learners to make the necessary mental constructions, as recommended by Ndlovu and Brijlall (2015).

Brijlall and Ndlovu (2013) note that structured worksheet can model how meaningful mathematics can be constructed. For this study, tasks were structured to model the developments of concepts in the solution of trigonometric equations. The tasks on the activity sheet (shown in Appendix G) were structured as follows:

Task One consisted of 3 items:

- Item 1 explores learners' mental constructions when it comes to understanding the quadrant where trigonometric functions have a solution.
- Item two explores learners' mental construction of the relationship between the quadrant in the cartesian plan and value of the function.
- Item 3 explored learners' mental constructions of understanding when the function is defined or not.

Task 2 consisted of three items; of the items explored learners' mental constructions of the general solution of trigonometric equations.

Task 3 consist of two items which explored learners' mental constructions of the application of restrictions when solving trigonometric equations.

For analysis, learners' responses were coded using the same categories presented in Chapter Five.

6.2.2 Interviews

Seven learners were chosen for interviews based on the evidence of their development since Phase One demonstrated in their written responses during both phases. One learner from each category was selected for the interview. The interviews were conducted to engage learners about how they had approached solving the problems, as recommended by Dubinsky (1997), Maharaj (2014), and Ndlovu and Brijlall (2015). The selected learners were asked various questions to understand how they constructed various mental structures. The aim was to verify learners' written responses and to engage learners with what they wrote as alluded by Maharaj (2014) as well as to interrogate their thinking processes to ascertain their mental constructions.

6.3 Analysis of learners' responses to Task 1 (equations of the form $\boldsymbol{asinx}$; $\boldsymbol{cosx}$; $\boldsymbol{tanx} = \boldsymbol{b\ where\ a \geq 1}$)

Task 1 consists of three items designed to reveal learners' mental constructions of trigonometric functions and the quadrant they belong to. Moreover, the focus was to ascertain learners' mental constructions when determining the quadrant in which the function has a solution. The Cartesian plane which defines the trigonometric ratios in terms of where they are positive and negative is shown in Figure 22.

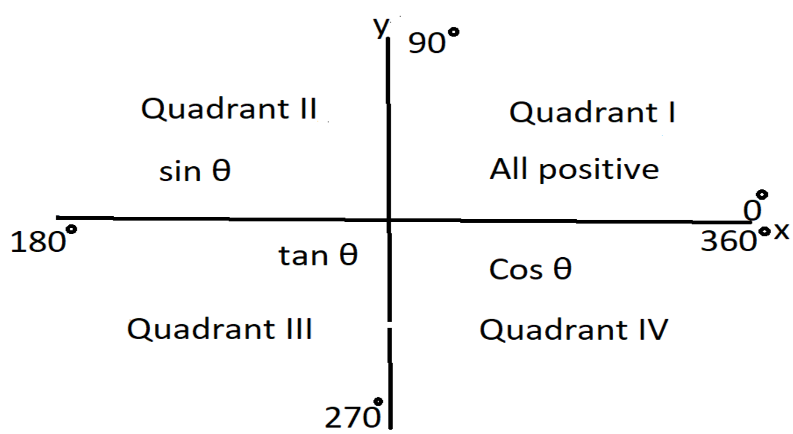


Figure 22 Quadrants of the trigonometric ratios

6.3.1 Item 1: Determining the quadrant where the function has a solution.

Item 1 was designed at a Level 1 in terms of the cognitive levels specified in the CAPS document, as it required recall of facts:

1. Given tan θ = -2.5

1.1 in which quadrant would the function have a solution? explain

Figure 23 Item 1 (Task 1)

All seventeen learners gave correct responses to Item 1. Learners were considered to have made all the necessary mental constructions for Item 1 if they could correctly identify the quadrants where the given function had a solution. As shown in Table 19, all learners identified the correct quadrant and provided the explanation. Although not all provided the accurate explanation but were able to provide justification of the quadrant where the given function has a solution.

Table 19 Analysis of learners' responses to Item 1 (Task 1)

Category	**1**	**2**	**3**
Indicator	Made the necessary mental constructions	No evidence of making the necessary mental constructions	No attempt made
No. responses	17	0	0

In terms of APOS theory item, Item 1 was design to explore whether learners had constructed the action-level concept of sign needed to determine the quadrant in which a solution lies. The sign (+ or –) acts as a visual cue to trigger recall of which quadrant the function solution lies, because the learners understood which trigonometric ratio is positive and negative to which quadrant.

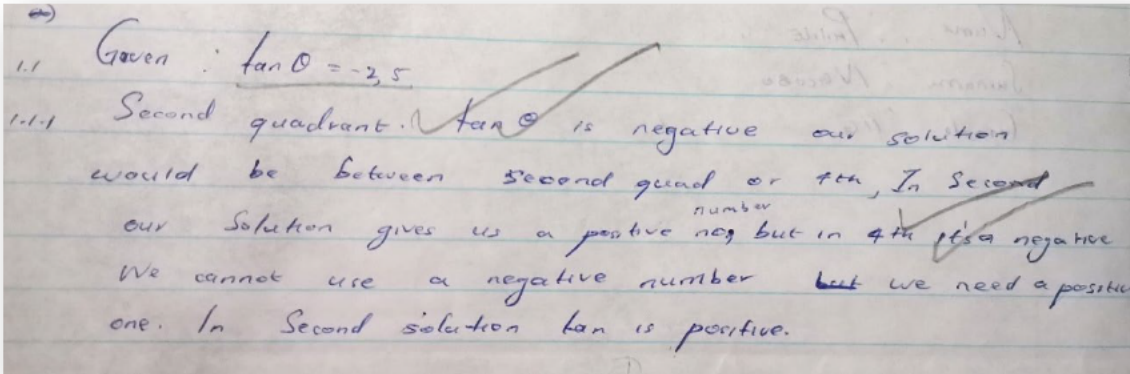

1.1 Given: tan θ = -2,5
1.1.1 Second quadrant. tan θ is negative our solution would be between second quad or 4th, In Second our Solution gives us a positive number neg but in 4th it's a negative We cannot use a negative number but we need a positive one. In Second solution tan is positive.

Figure 24 Extract from Shozi's written response to Item 1

The findings showed that all 17 learners had constructed an action-level concept of the relationship between the sign of the function and the quadrant in which the solution lies. The extract above provides an example of the responses given by the learners for this item.

The response indicates that Shozi knew that the solution belongs in the second and fourth quadrants. Her explanation indicates she understood that the sign (-) indicates where the given function will have a solution; however, her explanation shows that she confused the meaning of the sign (-) to also imply that the answer would be negative. While Shozi was able to identify the correct quadrant when the given function had a solution, the meaning of the sign (-) in this context was misunderstood. During the interview, the following transpired:

> Researcher: *In this question, you were given* $\tan\theta = -2.5$. *The question asks in which quadrant would the function have a solution and explain. Therefore, how did you solve this question?*
>
> Shozi: *Firstly, I said that* $\tan\theta = -2.5$ *Okay. I checked the sign of tan, and then in which quadrant would tan be negative, and it will be 2nd and 4th.*
>
> *Researcher: In your explanation in the first question, you said also tan is positive in the second quadrant because it gives us a positive solution. Do you want to explain how?*
>
> Shozi: *Sir… eish… I made a mistake in my explanation. Because the reference angle is negative, the solution will be in quadrant 2 and 4.*

Researcher: *Is there any other method you could have used?*

Shozi: *I would have used a calculator if I wanted to know exactly what the reference angle is and wanted to find the exact value of tan in quadrant 2 and 4.*

Shozi's responses throughout the interview demonstrated that she comprehended the idea of trigonometric ratios and the quadrants in which each trig ratio is positive and negative and was able to rectify the error made in her written response about the solution being negative.

In the same category, Zulu also demonstrated an understanding of the meaning of the signs ($\pm$) in determining the quadrant where the function would have a solution, as shown below in his written response.

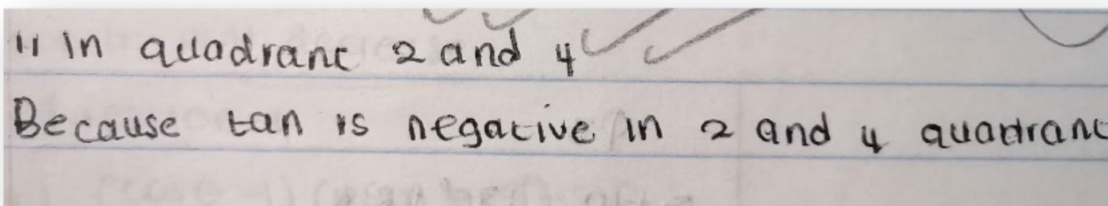


Figure 25 Zulu's written response to Item 1 (Task 1)

In the interview, the researcher asked Zulu to explain his response:

Researcher: *In 1.1 you were given tan θ = -2.5. The question says, "In which quadrant would the function have a solution" and you said quadrant 2nd and 4th. Why?*

Zulu: *I said that the function will have a solution in quadrants 2nd and 4th because tan is a negative in this quadrant as I was given the negative value of y. According to the question, it says, "Which quadrant would the function have a solution", which is why I said on the 2nd and 4th quadrants – because we are given the value of y as negative. Uyabona [you see], Sir in the Cartesian plane tan is negative in the 2nd and 4th quadrant kodwa ke the [but] where the y value is negative is in the 4th quadrant, if I was plotting the coordinates.*

Researcher: *Then tell me, while we are still in this question: is there any other method you could have used to solve this question?*

Zulu: *Yes, sir.*

Researcher: *How?*

Zulu: *If I wanted the exact solutions, I would have calculated the reference angle at the end I would say* 180°*reference angle, then finds the answer and* 360 *reference angle. Therefore, that will tell which quadrant my answer belongs to.*

During the interview, Zulu also showed that he understood that the value -2.5 was the y coordinate and related his explanation to plotting coordinates, which showed that he understood in relation to function. While both Shozi and Zulu's response showed that the external cue (-) triggered their identification of the quadrant where the solution lay – which indicated they were operating at the action stage, in terms of APOS theory – Zulu's explanation during the interviews showed that he had interiorised the action into a process, as he was able to relate the position where the solution lay to a graphical representation in the Cartesian plane.

For Item 1, the findings showed that most of the learners were operating at the action stage with only a few, such as Zulu, displaying process conception. While most of the learners were still operating at the action stage in their response to this item, just as they had been in Phase One, some showed development of their mental constructions as they displayed process conception.

6.3.2 Item 2: Determining the value of the angle

While in Item 1, learners were asked to identify the quadrant where the solution lay, in Item 2 they were required to show their understanding of the meaning of the reference angle in determining the value of the function.

1. Given $\tan\theta = -2.5$

1.2 Is the value of $180 < \theta$ *or* $\theta < 180$

Figure 26 Item 2 (Task 1)

The analysis of learners' responses to Item 2 is presented in Table 20 below.

Table 20 Analysis of learners' responses to Item 2 (Task 1)

Category	**1**	**2**	**3**
Indicator	Made the necessary mental constructions	No evidence of making the necessary mental constructions	No attempt made
No. of responses	6	9	2

In contrast to Item 1, while learners were able to identify where the solution lay, they struggled to predict the value of $tan\theta = -2.5$. As shown in Table 20 above, only 6 learners made the necessary mental constructions. Learners who first did step-by-step calculations before determining the value were categorized as operating at the action stage; those who were able to identify the value without performing step-by-step calculations were considered to have constructed the process conception; and those who had made the connection of where the solution lay, and the value of the function were deemed to have encapsulated the process into an object. Table 21 below show the learners' mental constructions evident in Item 2.

Table 21 Analysis of learners' mental constructions for Item 2 (Task 1)

Mental constructions made	**Action**	**Process**	**Object**	**Coherent schema**
No. of responses	0	6	0	0

The 6 learners who demonstrated the necessary mental constructions were categorized as operating at the process stage because they were able to predict that the value of θ was less than 180° and also greater than 180° without performing step-by-step calculations, as illustrated by Makwanza response, below.

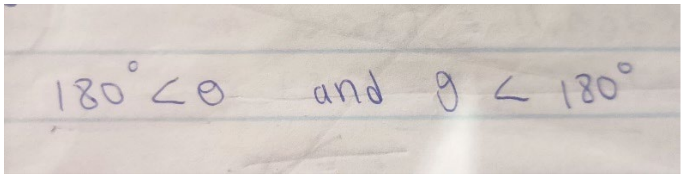


Figure 27 Extract from Makwanza's written response to Item 2 (Task 1)

In the interview, the researcher asked her to describe how she approached the problem:

> *Researcher: I see you choose two answers. Explain how you solved the question.*
>
> *Makwanza: I wrote two answers to this question because in 2nd quadrant* θ *is less than* $180°$*, because I will subtract the reference angle, and in 4th quadrant* θ *is greater than* $180°$*, because it will be* $360°$ *subtract the reference angle.*
>
> *Researcher: Ok. So, can you tell me what is the connection between the quadrant where the solution lies and the value of the angle?*
>
> *Makwanza: Mhhh…. angazi [I don't know], sir.*

While Makwanza could predict the value without first attempting to perform calculations, she could not make the connection between the position where the solution lay and the value of θ; she could only explain the procedural process that, since tan is positive in the second quadrant, the value will be less than $180°$ due to subtracting the reference angle, and greater in the fourth quadrant. Although she was able to predict the value and explain the procedural parts, she could not make connection between the concepts. This supports the claim by Roberts et al. (2022) that students' responses are not necessarily indicative of a conceptual understanding.

6.3.3 Item 3: Determining whether the function is defined or not

Item 3 was designed to provide insight into whether the learners understood whether the function was defined or not. This item also assessed learners' mental constructions of recalling of facts and rules to solve related questions. Recall questions may involve facts, definitions, terms, or basic instructions, as well as running a quick algorithm or using a formula (Singh et al., 2022). In this item, for the learner to be classified as making the necessary mental constructions, they must state whether the function is defined or not and provide the correct explanation or calculations.

> *1. Given tan θ = -2.5*
>
> *1.3 would you say this function is defined or not? Explain*

Figure 28 Item 3 (Task 1)

The results showed that 8 learners out of 17 displayed an understanding of the question. Out of 15 learners who said the function is defined, 8 learners provided a correct justification. The 7 in category 2 failed to justify their answers.

Table 22 Analysis of learners' responses to Item 3 (Task 1)

Category	**1**	**2**	**3**
Indicator	Made the necessary mental constructions	No evidence of making the necessary mental constructions	No attempt made
No. of responses	8	7	2

Eight of the learners were found to have made the necessary mental constructions, with evidence of operating at the action stage. This is illustrated in Ntaka's response:

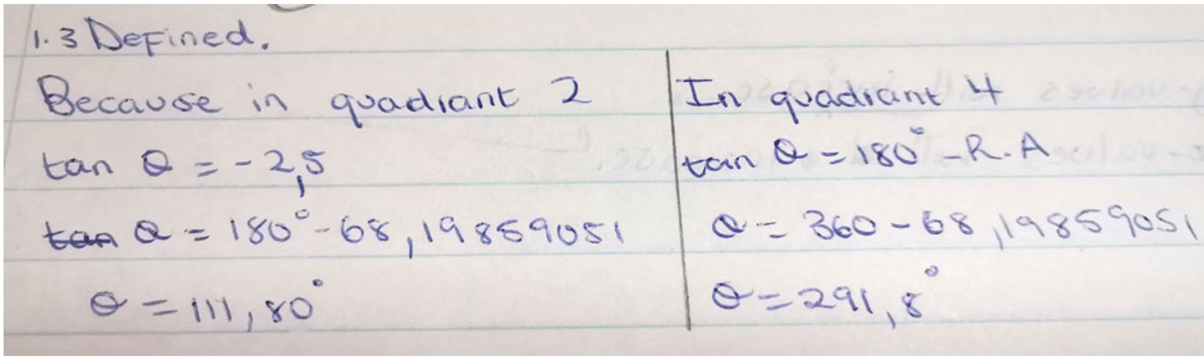


Figure 29 Ntaka's written response to Item 3 (Task 1)

All of these 8 learners explained why they were saying it was defined by performing the calculation and solving for $tan\theta = -2.5$. Their failure to explain why the function was defined suggests that they had not interiorised the action of finding the solution to understand what it means when the function is defined. This inability to interiorise the action into a process was evident in Ntaka's response during the interview, as shown below:

> *Researcher: When you were answering this question, you said that the function is defined. Why are you saying it is defined?*
>
> *Ntaka: I said it defined.*
>
> *Researcher: Why?*
>
> *Ntaka: Because when I calculate the value of θ, I got reference angle subtracted from 180 degrees and 360 degrees and got the answer. So, it is defined.*
>
> *Researcher: What do we mean when we say the function is defined?*
>
> *Ntaka: It means the answer is known. If you punch the values in the calculator and got an error, it means it does not have an answer, so it's undefined, njenga masidvider ngo 0 [as we are diving by 0].*

Ntaka indicated that she understood the function being defined as meaning calculating and getting the answer.

The findings from Task 1 showed that the learners constructed concept as isolated facts. For example, while all of the learners were able to determine the quadrants where the given function had a solution, not all were able to use that knowledge to predict the value of the solution and to explain if the function was defined. While the learners showed development of their mental constructions for some concepts – such as where some learners showed interiorization of the action to process – the majority were still found to be operating at the action stage. In contrast to the findings from Phase One (Chapter Five), where most of the learners showed no evidence of being able to make the necessary mental constructions, the data generated in Phase Two shows gradual development of learner's mental constructions. This suggests that some learners need an extended period of time to construct knowledge of a concept.

6.4 Analysis of learners' responses to Task 2 (equations of the form $\boldsymbol{sin(\theta + a) = b, cos(\theta + a) = b, tan(\theta + a) = b}$)

Task 2 focused on learners' mental constructions when solving trigonometric equations that required finding a general solution. Task 2 was comprised of three items.

6.4.1 Item 1: Solving a trigonometric equation of the form $\boldsymbol{cos(\theta + a) = b}$

Item 1 of Task 2 explored learners' mental constructions when determining a general solution for trigonometric equations involving horizontal shift. For this item, those learners who were able to determine the general solution were considered to have made the necessary mental constructions. Learners who attempted the problem but were unable to determine the general solution were considered to have not made the necessary mental constructions.

2 Find the general solution of the following equation

2.1 $\cos(\theta + 10°) = -0.67$

Figure 30 Item 1 (Task 2)

The analysis of learners written responses is presented in Table 23.

Table 23 Analysis of learners' responses to Item 1 (Task 2)

Category	**1**	**2**	**3**
Indicator	Made the necessary mental constructions	No evidence of making the necessary mental constructions	No attempt made
No. of responses	10	6	1

As shown in Table 23, 16 of the 17 learners attempted to solve Item 1. Only 10, however, were able to determine the general solution. In terms of the learners' mental constructions, as these 10 learners were able to determine the general solution they were categorised as operating at the action stage. This was confirmed by their use of step-by-step calculations to determine the solution. Arnon et al. (2014) notes that performing each step explicitly is characteristic of the action stage, in terms of APOS theory. APOS theory states that, during the action phase, a person transforms themselves under the influence of environmental stimulus while applying a mathematical notion in relation to an explicit algorithm (Umam & Susandi, 2022). The following extract provides an example of the approach taken by a learner who was at the action level.

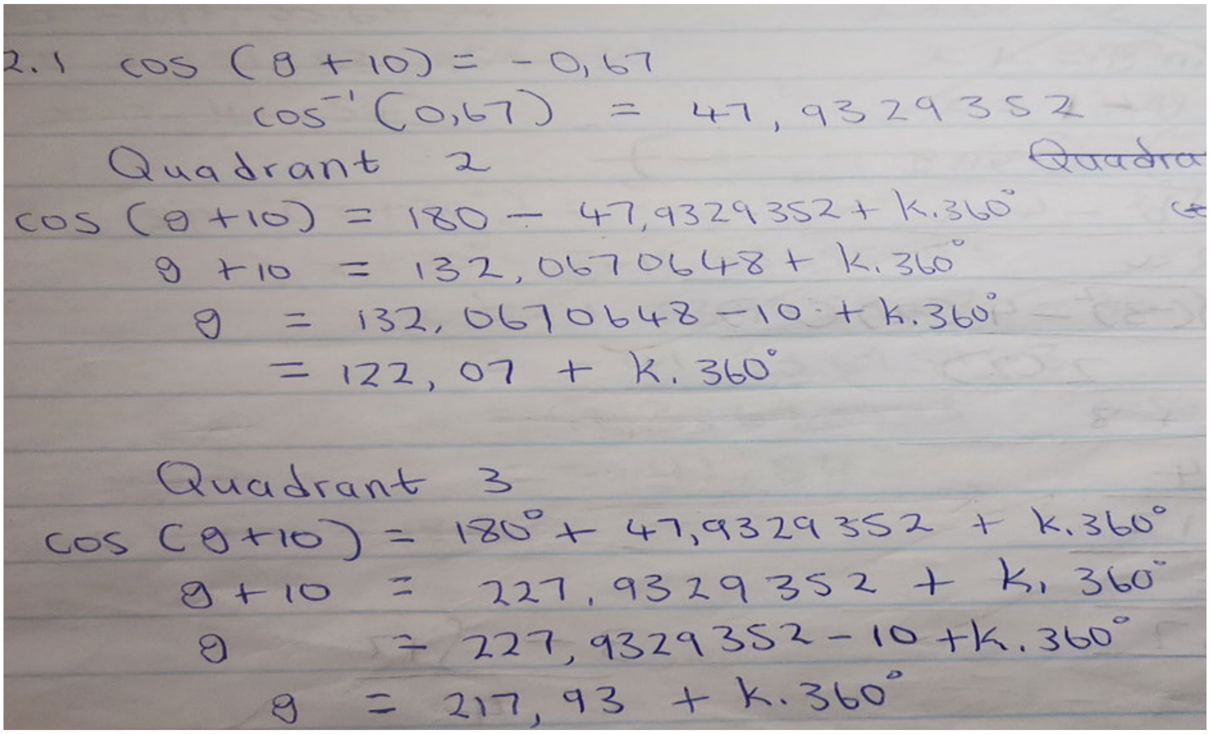


Figure 31 Makwanza's written response to Item 1 (Task 2)

Makwanza was one of the 10 learners who gave a correct and complete response for Item 1. She was thus considered to have developed the concept of determining the general solution as an action. While she carried out the necessary procedures, it was noted that she has not constructed the meaning of the solution as she did not use notation to indicate that she was dealing with angles. The incorrect use of notation, or ignorance of the importance of notation, suggests that a learner has not developed a conceptual understanding of the concept. While such a learner may perform step-by-step calculations, when the concepts are constructed as a discrete body of facts, it causes a challenge in the application (when solving) of the problems (Ndlovu & Brijlall, 2015). Although Makwanza did not use degree notation in her written response, during the interview she indicated that she knew she was calculating the angle, and repeatedly referenced the correct notation:

Researcher: Explain how you solve this equation.

Makwanza: I used the calculator to calculate the reference angle. Then I checked in which quadrant where cos is negative – because I was given -0,67, therefore the sign before the number tells me where the solution should be. Ngiyazi ukuthi [I know that] cos is negative in the second quadrant. Then I used 180 degrees – reference angle. After that, I had θ *+10* *equal to 132,0670648 degrees +K 360 degrees where K is the element of real numbers. Since I am solving for θ, I transposed 10 degrees to the right then I had 132,0670648 – 10 degrees +K 360, where K is the element of real numbers. My answer was 122, 07 degrees +K 360, where K is the element of real numbers. Also, cos in negative in the 3rd quadrant I used 180 + reference angle. After that, I had θ +10 equal to 227.9329352 degrees +K 360degrees where K is the element of real numbers. Transposed 10 degrees to the right, then I had 227.9329352– 10 degrees +K 360, where K is the element of real numbers. My answer was 217.93 degrees +K 360 degrees, where K is the element of real numbers.*

Researcher: Are there any other methods you could have used to solve the question?

Makwanza: No, this is the only one I know.

Researcher: Will the question always have a solution?

Makwanza: Mmm… yes.

Researcher: How will you tell if the equation has a solution or not?

Makwanza: Yes, I think I can be able by doing it in my head, but I can't see ngoba kunzima [it's difficult].

The explanation given by Makwanza indicates that she understood how to determine the general solutions. In the interview she explicitly articulated each and every step as she was relying on external cues such as sign (-) to trigger the position of the solution; each step triggered the next step. The same was evident in the other learners' written responses, thus they showed no evidence of having interiorized the action into a process. In the interview, it was evident that Makwanza knew that she was calculating an angle as she repeatedly referred to 'degrees' in her explanation; however, she did not use the notation for degrees in her written response. But the focus was on the correct answer provided, that why she was categorised as one of the 10 learners who made mental construction.

6.4.2 Item 2: Solving a trigonometric equation of the form $\pm acosa = \pm bsin\ \alpha$

To solve Item 2, an understanding of trigonometric identity, algebraic equations and quotient identity was needed. Learners who were able to correctly determine the general solution were categorized as having made the necessary mental constructions.

2 Find the general solution of the following equation

2.2 $\sqrt{3}\cos\ \theta - 3\sin\ \theta = 0$

Figure 32 Item 2 (Task 2)

Table 24 Analysis of learners' responses to Item 2 (Task 2)

Category	**1**	**2**	**3**
Indicator	Made the necessary mental constructions	No evidence of making the necessary mental constructions	No attempt made
No. of responses	9	7	1

As shown in the table above, only 9 learners provided the correct answer and were thus categorised as having made the necessary mental constructions. However, the analysis of the learners' mental constructions showed that all 9 were operating at the action stage. As these learners were also found to be operating at the action stage in Item 1 of Task 2, it appears clear that these learners had not progressed beyond the action stage for determining the general solution when solving trigonometric equations. It was observed, however, that although the learners were operating at the action stage, they were able to use the correct or necessary knowledge to determine the general solution, as illustrated in Makhanya's response below.

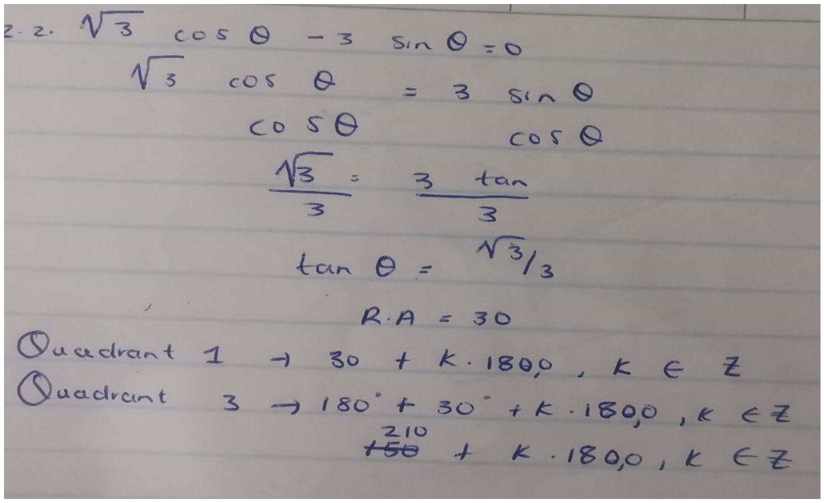


Figure 33 Makhanya's written response to Item 2 (Task 2)

As shown in the response above, in step 3 Makhanya divided both sides of the equation by $cos\theta$; however, she did not use the correct notation to show that she was dividing. This is considered to be a slip as, in the next step, when diving by 3, she used the correct notation. In addition, even if, in step 3, she did not use the division notation it can be assumed that she was aware that she was dividing by $cos\theta$ because she was able to relate to quotient identity that $\frac{\sin\theta}{\cos\theta} = tan\theta$. Although there is evidence to suggest that Makhanya understood the notation even if she failed to use it, she documented each step, suggesting that each step triggered the next step which suggests that she was operating at the action stage. This was confirmed during the interview:

Researcher: Explain how you solve 2.2.

Makhanya: I solve this equation by taking the square root of 3 sin θ *the subject of the formula by manipulating 3 sin* θ *and changing to positive sin* θ *by dividing with cos* θ *which became tan* θ *and dividing by square root of three both sides to remove the square root. And I was left with tan* θ *as a subject of the formula. Then I got a reference angle. After that, I went to the 1st and 3rd quadrant, because it is where the tan is positive, to find the answers.*

Researcher: Is there any other method you could have used? Explain.

Makhanya: No, I only know this method.

Researcher: Will the equation always have a solution?

Makhanya: Yes, because if you can make trig ratio a subject of the formula then you be able to find the solution using quadrants.

Researcher: How would you tell if the equation has a solution or not?

Makhanya: No, I can't tell until I solve the equation.

In the interview, Makhanya indicated that she only knew of one method to solve such a problem and would not be able to tell if a solution exists without doing a calculation. Also, her explanation of what it means to say the equation has a solution suggests that, while she could perform the procedure, the concept has not been encapsulated into an object. In order to determine the existence of a solution, learners need to encapsulate the general solution as process. Based on the learners' responses, none of the learners that were considered to have made the necessary mental construction had constructed the general solution as the process. Instead, all had displayed the action conception of the general solution.

6.4.3 Item 3: Solving a trigonometric equation of the form $\pm a\sin^2\theta = b \pm c\sin\theta$.

Item 3 explored learners' mental constructions when solving trigonometric equations that require them to apply their knowledge of algebraic equations. This question qualifies as a Level 3 question in terms of cognitive demand. Level 3 questions, due to their complexity, require deep understanding employing reasoning, planning, and the use of evidence (Greene, 2020). For learners to be able to solve this trigonometric equation, a knowledge of how to solve trinomials was critical.

2 Find the general solution of the following equation

2.3 $3sin^2\,\theta - 3\sin\theta = 1$

Figure 34 Item 3 (Task 2)

Table 25 Analysis of learners' responses to Item 2 (Task 2)

Category	**1**	**2**	**3**
Indicator	Made the necessary mental constructions	No evidence of making the necessary mental constructions	No attempt made
No. of responses	1	12	4

Only one learner was able to correctly determine the general solution in Item 3. While learners' difficulties are discussed in the next chapter, it was noted that learners were unable to relate this equation with quadratic equation as suggesting that learners were unable to generalise their knowledge around solving quadratic equations to trigonometric equations. Most learners identified the reference angle as $90°$, suggesting that they were treating the whole equation as $sin\theta = 1$. Their inability to make connections between concepts became a barrier to solving the trigonometric equation of the quadratic form. Only Makwanza, as shown in the extract below, was able to determine the general solution of the trigonometric equation in the form of a quadratic equation.

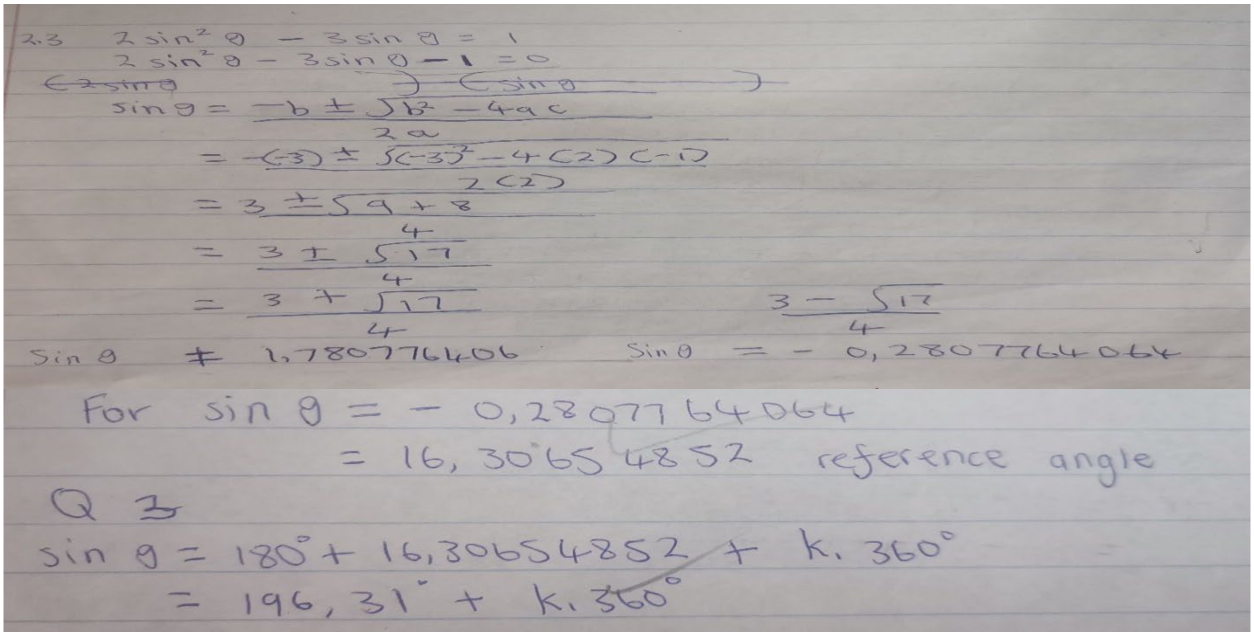


Figure 35 Makwanza's written response to Item 3 (Task 2)

Makwanza was able to factorise the equation and identify the value that had solution. However, in determining the general solution, she considered the third quadrant to be the only quadrant where the solution lay. Her responses were investigated through an interview.

Researcher: Explain how you solved the problem.

Makwanza: I transposed 1 to the left side, then I was left with a trinomial which is $2sin\theta - 3sin\theta - 1 = 0$. *Ok, I tried to look for factors and I didn't get them. After that I used the quadratic formula, I did a substitution of a=2, b=3, and c=-1. Then I calculate to find the reference angle and I could get a solution to the first one, which was undefined, and the second one got the solution.*

Researcher: Why do you say the other solution was undefined?

Makwanza: When I tried to calculate the reference angle of 1,78 it shows undefined.

Researcher: What does undefined mean?

Makwanza: Mhhh…. [Pause] There is no solution.

Researcher: Why do you think the other coordinates has no solution?

Makwanza: Yoh! [Pause]. Sengiyabona. [Now I see]. When I factorise it, my answer was $\sin\theta = 1.78$. *This is not possible, Sir. No solution.*

Researcher: Why?

Makwanza: Angithi [Isn't], Sir, the graph of sin turns ku 1 on the y value when x is $90°$, *so it cannot go higher than that unless the graph was shifted up vertically? This is normal graph of sin, so the maximum value is 1, yingakho (1,78) enganayo I -x coordinate [That is why there is no x coordinate for 1,78].*

Researcher: Okay, that is good that you can pick up that. Now tell me why you only used the 3rd quadrant to determine the solution?

Makwanza: Eish, sinzima lesisibalo [the sum is difficult], but I see now I should have also included the 4th quadrant, because sin is negative in the 4th quadrant.

Although Makwanza's written response suggested that she was operating at the action stage, during the interview she demonstrated encapsulation of the solution of a trigonometric equation into an object as she was able to understand $sin\theta = \pm b$ as a static structure, construct the process for solving the equation as a totality, imagine the graph of $sin\theta$ and apply the transformation to it, on which value will yield a solution. The development of Makwanza's mental constructions suggest that when learners are exposed to learning a concept over an extended time, they gradually develop an accurate concept image. Furthermore, as alluded by Maharaj (2014) and Ndlovu and Brijlall (2015), engaging learners with their thought processes assists them in the development of the necessary mental constructions. This was evident in the interview with Makwanza.

6.5 Analysis of learners' responses to Task 3 (equations of the form $sin\alpha = sin\beta, cos\,\alpha = cos\,\beta, \tan\alpha = tan\,\beta$)

6.5.1 Item 1: Solving trigonometric equations of the form $a.b = 0$ using special angles

Item 1 was designed to provide insight into the learners' mental constructions when solving complex Level 4 questions, which involve high cognitive demand (Khudhair & Jasim, 2021).

Learners were asked to make links between several concepts within the subject area or choose one of several choices as the solution to the problem (Greene, 2020). To solve this trigonometric equation, familiarity with solving algebraic equations was vital.

3. Solve for θ without the use of a calculator if

3.1 $(\cos\theta - 1)(2\sin\theta + \sqrt{3}) = 0$ *and* $\theta \in [0°; 360°]$

Figure 36 Item 1 (Task 3)

Those learners able to provide correct and complete answers solving for θ were categorised as having made the necessary mental constructions. As shown in the table below, only 8 learners were considered to have made the necessary mental constructions.

Table 26 Analysis of learners' responses to Item 1 (Task 3)

Category	**1**	**2**	**3**
Indicator	Made the necessary mental constructions	No evidence of making the necessary mental constructions	No attempt made
No. of responses	8	8	1

The analysis of learners' mental constructions revealed that all 8 learners who had made the necessary mental constructions were operating at the action stage. According to the preliminary genetic decomposition, presented in Chapter Three, learners functioning at the action level must solve a trigonometric equation step by step. Therefore, of the 16 learners who attempted to solve Item 1, the 8 who demonstrated mental construction were working at the action level in Item 1. During the analysis of the task and the interview, it was observed that the learners had mastered algebraic concepts, as they all used the same method to solve this item. During the interviews, Fihlela, Makhanya and Shozi indicate that they knew only one method for solving this kind of equation, which is to factorise. x. The written responses of Fihlela and Makhanya are shown below.

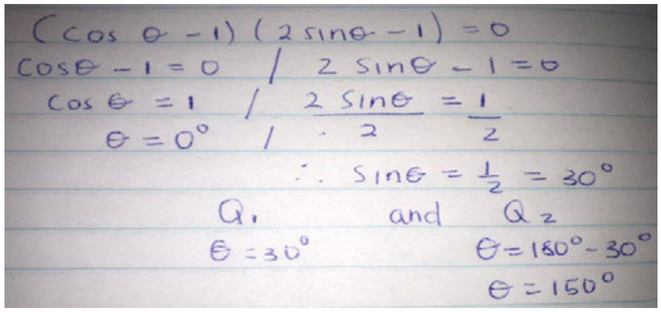


Figure 37 Makhanya's written response to Item 1 (Task 3)

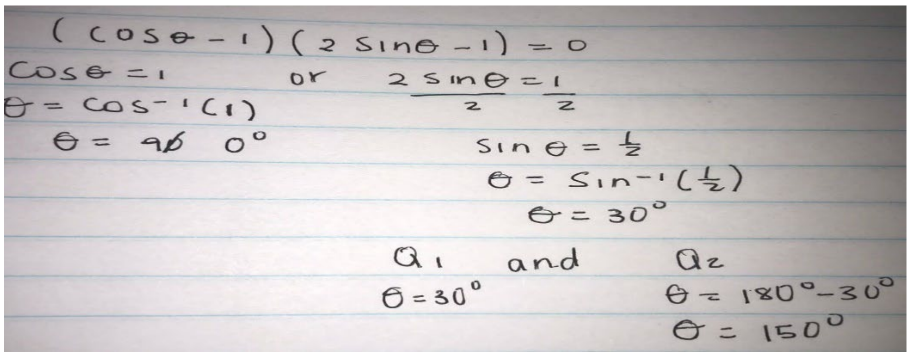


Figure 38 Fihlela's written response to Item 1 (Task 3)

In the interview with Fihlela, he explained the process he had used as follows:

Researcher: Explain how you solve the question.

Fihlela: What I did here, I took $\cos\theta - 1 = 0$ *and* $2\sin\theta - 1 = 0$ *. In this equation I apply my knowledge of solving the algebraic equation on question 1 when we solve x. Separated by the two equations, first, I solve* $\cos\theta - 1$ *by transposing. Then I had cos* θ *is equal to 1. The value of* θ *was equal to 0 degrees. On the other side, I had 2sin* θ *and transposed 1 to the right and divided by 2 into both sides. I was left with sin* θ *equal to* $\frac{1}{2}$ *and my theta was 30 degrees.*

Researcher: Is there any other method you could have used? Explain.

Fihlela: No.

Researcher: Will the equation always have a solution?

Fihlela: Yes.

Researcher: How would you tell if the equation has a solution or not.

Fihlela: If you know about solving algebraic equations in paper 1 of mathematics, it is easy to see how to solve this equation.

Fihlela shows a good understanding of the concept of solving trigonometric equations as he showed good knowledge of algebraic concepts in his solution. He was able to apply it with reasonable explanations and displayed a full understanding of the concept. His response was categorised as having made the necessary mental constructions. According to Thanheiser and Melhuish (2023), learners' correct answers do not always demonstrate that they grasp the subject. However, Fihlela displayed a good understanding of the concept.

6.5.2 Item 2: Solving trigonometric equations involving special angles (equations of the form $\boldsymbol{sin\alpha = sin\beta, cos\ \alpha = cos\ \beta, \tan\alpha = tan\ \beta}$)

This question requires application of different heuristics to solve the problem and draws on knowledge of various concepts (Ndlovu, 2014). For this item, learners were required to draw on their knowledge of solving algebraic equations, application of special angles, and consider the constraints in determining the solution.

3. Solve for θ without the use of a calculator if

3.2 $\sin\theta\left(2\cos\theta + \sqrt{3}\right) = 0$ *and* $\theta \in [0°; 360°]$

Figure 39 Item 2 (Task 3)

Out of 17 learners, only 2 made the necessary mental constructions and gave a correct response to Item 2. The focus of this item required learners to write all the answers under the given interval [0°; 360°]. If a learner could not determine the answer, they were categorised as 'no evidence of making the necessary mental constructions. As shown in the table below, 9 learners did not make the necessary mental constructions. However, of these 9 learners, 5 learners got part of the answer correct, but they did not follow the instruction given.

Table 27 Analysis of learners' responses to Item 2 (Task 3)

Category	**1**	**2**	**3**
Indicator	Made the necessary mental constructions	No evidence of making the necessary mental constructions	No attempt made
No. of responses	2	9	6

The analysis of learners' responses indicates that the 2 learners that made the necessary mental constructions had developed the action conception of the solution of solving trigonometric equations using special angles. This was observed as learners were able to imagine some of steps that needed to be performed; not all of the steps were explicitly shown, suggesting that some steps were performed in the mind. For example, looking at Makhanya's response (which showed the necessary mental constructions), in step 2 of her solution she made $cos\theta$ the subject of the formula without explicitly performing the division process and, further, determined the special mentally and perform the step $180° - 30°$ mentally to determine θ.

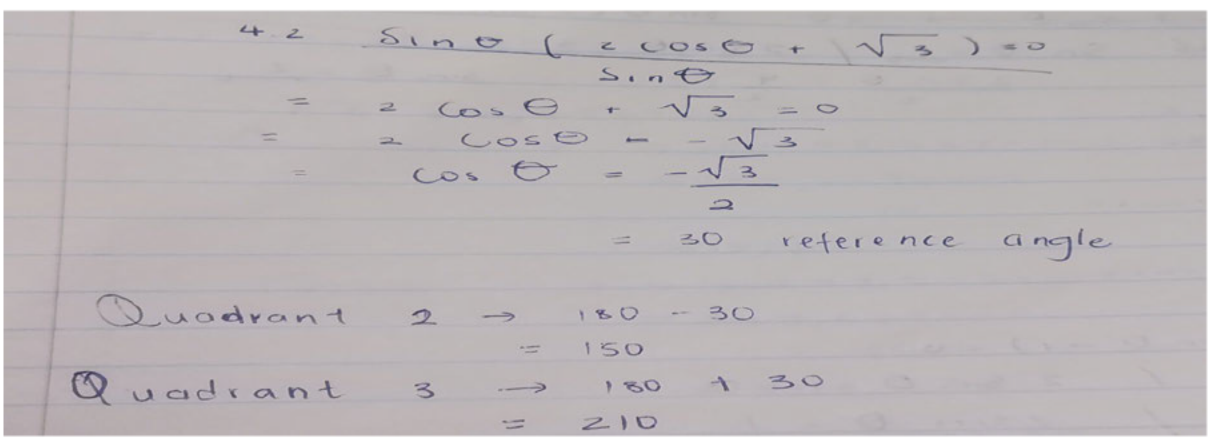


Figure 40 Makhanya's written response to Item 2 (Task 3)

Makhanya was one of the learners whose response was placed in category 1 and who was found to be operating at the action level. It was observed in her response she understood algebraic concepts as she used them in the solution. However, her final solution shows that she had not constructed the meaning of the solution as she left the answer as a numeral instead of using the degree notation, indicating that she did not understand that she was finding an angle. The same was observed with most of the learners. Also, most of the learners lacked an understanding of the interval. Makhanya, however, displayed a good understanding of the interval. In the interview, she explained her process as follows:

> Researcher: *Ndoe you were given* $\sin\theta\left(2\cos\theta+\sqrt{3}\right)=0$. *So how did you solve the equation?*
>
> Makhanya: *I answered this question separating* $\sin\theta=0$ *and* $2\cos\theta+\sqrt{3}=0$. *Yabona [you see], Sir, mina [I] what I did is that I transported* $\sqrt{3}$ *to the left and became negative* $\sqrt{3}$ *and I was left with* $2\cos\theta$ *and I divide by 2 both sides. Then I was left with* $\cos\theta=\frac{-\sqrt{3}}{2}$ *. After that, I used the calculator – I do not want to lie sir – and said the shift* $coc^{-1}\left(\frac{\sqrt{3}}{2}\right)$ *and the value of θ= 150. Also,* $\sin\theta=0$ *again I used the calculate the answer was* θ= 0.
>
> *Researcher: Makhanya, is there any other method you could use to solve it?*
>
> *Makhanya: Ayi, no sir.*

Makhanya's responses reveals that her action conception has been interposed into a process. This was also observed as she clearly explained how she had solved this item. This shows that she worked step-by-step in answering the question. Comparing her responses to those she gave in Phase One of the data collection, significant development of her conceptual understanding of solving and learning trigonometric equations. The next section summarises the findings on learners' mental constructions presented in this chapter.

6.6 Summary of findings on learners' mental constructions

The findings for Phase Two showed that, while the majority of the learners' responses displayed that they were unable to make the necessary mental constructions, it was evident that most of those who were able to operate at the action stage. In fewer instances, learners displayed process conception. While some of the written responses revealed that learners were operating at the action stage, when engaging learners in interviews about how they had solved the problems gradual development of their mental constructions could be observed: while the findings for Phase One showed that learners who made the necessary mental constructions were operating at the action stage, the findings for Phase Two showed development in the mental constructions of some of the learners (for example, Makhwanza and Shozi).

6.7 Conclusion

This chapter has discussed the findings from Phase Two of the study pertaining to Grade 11 learners' mental constructions of trigonometric equations. APOS theory was used to explain the nature of learners' mental constructions of the trigonometric equations. The findings showed that the majority of those learners who had made the necessary mental constructions had an action conception of the solution of trigonometric equations. The next chapter discusses learners' solutions who were unable to construct the concepts because they experienced difficulties when solving trigonometric equations.

CHAPTER SEVEN: DIFFICULTIES THAT HINDER LEARNERS' MENTAL CONSTRUCTION OF TRIGONOMETRIC EQUATIONS

7.1 Introduction

The previous chapter presented findings on the learners' mental constructions when solving trigonometric equations. The findings revealed whether learners were able to make the necessary mental constructions for different types of trigonometric problems or not. A failure to make the necessary mental constructions implies that the learner experienced challenges with particular parts of the trigonometric equation. The focus of this chapter is the analysis of these challenges. While APOS theory was useful for analysing learners' mental constructions, it cannot be used to analyse the errors and misconceptions that hindered learners' mental constructions. For this, the triad mechanism was used. As discussed in Chapter Three, the triad mechanism uses three stages: the Intra-stage, which focuses on a single entity, followed by the Inter stage, which is the study of transformations between objects; and finally, the Trans-stage, which refers to schema development that makes connections between actions, processes, and objects, which was used in category 1 (Jojo et al., 2013).

This study was guided by three research questions, presented in Chapter One. The findings in this chapter answer the second research question:

> **Why do learners succeed/fail to make the necessary mental constructions when learning and solving trigonometric equations?**

Globally, mathematics education researchers have been concerned with learners' difficulties related to trigonometry topics at the high school and university levels (see Siyepu, 2015; Chigonga, 2016; Nurmeidina & Rafidiyah, 2019; Fahrudin & Pramudya, 2019; Hamzah, et al., 2021). The aim of this chapter is to extend this discourse into the area of the challenges experienced by learners related to trigonometric equations.

7.2 Analysis of learners' conceptual errors when solving trigonometric equations

This chapter focuses on the errors and misconceptions that learners manifest in learning and solving trigonometric equations. Misconceptions are one of the main reasons for students' poor performance in mathematics (Aygor & Ozdag, 2012). Drews et al. (2020) state that a misconception is when a learner's conception conflicts with the acceptable meaning and understanding of mathematical concepts. Understanding the learner's misconceptions helps the teacher to understand what interventions should be provided for the learner for that concept.

Drews et al. (2020) posit that errors that occur at the surface level of knowledge, are mostly procedural and can easily be corrected by teachers or learners themselves. The literature (Sirkiä & Sorva, 2012; Mutodi et al., 2023). notes that errors are symptoms of underlying misconceptions, meaning that learners commit errors because there is an underlying misconception which is the source of the errors. For example, if a learner believes that division results in a smaller quotient, that may expect that even when diving with a fraction the quotient will be smaller than the dividend.

Arnawa and Nita (2019) posit that a misconception arises when a student believes a false concept is a true concept. For example, some learners believe that adjacent sides and opposite sides are fixed irrespective of the angle used. Due to such a misconception, errors such as incorrect naming of the sides may arise. Therefore, misconceptions manifest in errors and errors hinder the construction of the correct understanding of the taught concept.

In the analysis of learners' mental constructions in Chapters Five and Six, three categories were used to categorise learners' responses. The first category was for responses that demonstrated that learners had made the necessary mental constructions, whether at the action stage, process or object stage, in terms of APOS. The second category was for responses that showed no evidence of learners making the necessary mental constructions, while the third category was for learners who did not attempt to solve the problem. As this chapter focuses on the errors and misconceptions that hindered learners' mental constructions, it deals with responses that fell into the second category: where learners who attempted to solve the question gave incorrect responses. These responses are examined to shed light on the misconceptions that learners had that hindered them from making the necessary mental constructions.

The analysis of learners' responses revealed two things that hindered the learners in making the necessary mental constructions. First, some learners lacked an understanding of concepts, which was most evident in the use of incorrect rules or the incorrect application of facts when solving a sum. Second, some learners showed evidence of knowing the concept, but their responses showed a lack of understanding of the inter-relationship between concepts. Learners' errors were categorized using the themes that emerged during the analysis process, as shown in Table 28.

Table 28 Categories for analysis of difficulties hindering learners' mental constructions.

Theme indicator	**Lacked understanding of the concept (incorrect rules/facts used to determine the answer)**	**Made connections between concepts but no understanding of the underlying features (answer is correct but not connected to the problem or vice versa: steps are correct, answer is wrong)**
No. of responses		

The triad mechanism assisted in explaining the challenges that arose, resulting in errors. The following sections present the analysis and findings related to Tasks 1, 2 and 3 in Phase Two.

7.3 Analysis of learners' difficulties hindering the necessary mental constructions in Task 1

As shown in Chapter Six, Task 1 had three sub-questions, referred to as items. Since all of the learners solved Item 1 of Task 1, this section analyses the errors for Items 2 and 3, where some learners failed to make the necessary mental constructions.

In Item 2, learners were required to indicate whether the value would be $180 < \theta or\ \theta < 180$. See the figure below.

1. Given tan θ = -2.5

1.2 Is the value of 180 < θ or θ < 180

Figure 41 Item 2 (Task 1)

Table 29 shows the analysis of learners' difficulties in line with the themes discussed above.

Table 29 Table Analysis of learners' difficulties in Item 2 (Task 1)

Theme indicator	Lacks understanding of the concept (incorrect rules/facts used to determine the answer	Made connections between concepts but no understanding of the underlying features (answer is correct but not connected to the problem or vice versa steps are correct answer is wrong
1.2	9	0

As discussed in Chapter Six, of the 17 learners that participated in the study, 6 made the necessary mental constructions, 9 experienced challenges, and 2 did not attempt to solve Item 2. As shown in Table 29, all 9 learners failed to make the necessary mental constructions due to a lack of understanding of the concept. It was evident that learners were guessing because they either chose greater or lesser and, when asked during the interview how they had determined the answer, some confirmed that they had guessed, and others were unable to explain their process. For example, Ntaka wrote $\theta < 180°$, as shown in Figure 42. When asked why she wrote this, her response was *"I do not know, I just guess because* $\tan\theta$ *is equal to a negative number"*.

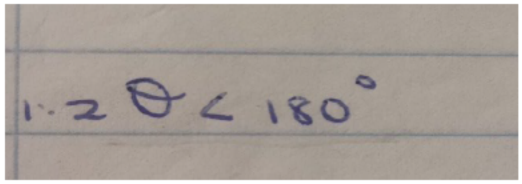


Figure 42 Ntaka's written response

It is evident in this response that Ntaka lacked the conceptual understanding that, in a general Cartesian plane without any restrictions, trig functions have more than one solution. In this case, $tan\theta$ is negative in the second and fourth quadrants. However, most learners that chose less than $180°$ only considered the reference angle, not where the solution lay.

In the triad mechanism, the intra-stage concepts are constructed as isolated facts. At the intra-stage, while the learner might have learned a collection of rules, these are not linked to form coherent understanding. In Ntaka's response, and in those of the other learners, she only focused on the negative side to decide that $\theta < 180°$. This suggests that, to Ntaka, a negative sign means 'less' – an underlying misconception which gave rise to the error of thinking that $tan\theta$ would have one solution.

In Item 3, learners were required to state whether the function was defined or not and provide the reason. The learners who were able to state the correct answer and provide the correct reason were regarded as having constructed the knowledge. Seven learners who attempted this item were unable to provide the correct answer. The table below illustrates the errors and misconceptions evident in their responses.

1. Given tan θ = -2.5

1.3 would you say this function is defined or not? Explain

Figure 43 Item 3 (Task 1)

Table 30 Analysis of learner's difficulties in Item 3 (Task 1)

Theme indicator	Lacks understanding of the concept (incorrect rules/facts used to determine the answer	Made connections between concepts but no understanding of the underlying features (answer is correct but not connected to the problem or vice versa steps are correct answer is wrong
No. of responses	7	0

As shown in Table 30, 7 learners showed a lack of understanding of the concept. This was evident in the lack of justification of their answers as shown in the extract below.

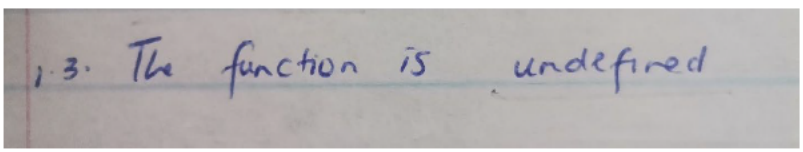


Figure 44 Makhanya's written response to Item 3 (Task 1)

Similar to responses for Item 2, the learners who provided the answer only, without an explanation, indicated that they showed a lack of understanding of the concept. Their responses revealed that they were guessing. During the interviews, they could not justify their answers, as illustrated in the interview discussion with Makhanya below.

Researcher: Explain why you said that the function is undefined.

Makhanya: It is undefined because the denominator is not zero. But, sir, we are not given the denominator, but I know it is 1.

Researcher: What happens if you divide by 1?

Makhanya: The answer is the same as what you were dividing.

Researcher: So, here you said the denominator is 1. So, what it is that makes the function undefined if the denominator is 1?

Makhanya: Eish, sir, these things are confusing. I just thought it's undefined. Angazi nami [I don't know why I said that].

Makhanya's remark reveals a lack of understanding of the concept of definition and vagueness around its meaning. However, it was found that even learners who answered "yes" (indicating that it was defined) were unable to justify their answers. This suggests they were guessing. Others said it was undefined because the denominator was 0, which is not true. This suggests that they were generalising the concept of division by 0 to a context where it did not apply.

Siyepu (2015) found that one of the difficulties students had with trigonometric concepts was the overgeneralization of rules, arising from a lack of conceptual understanding. In this case, the rules of divisibility by zero were overgeneralized to explain any concept that was deemed to be undefined. From the learner's responses it could be concluded that, while the learners had learned the collection of rules, these were not linked to form coherent understanding which had hindered their mental construction of the meaning of 'defined' and 'undefined', in the context of trigonometric equations.

Another learner, Zulu, responded "yes" – that it was defined. When his response was probed during the interview, he explained his thinking:

Zulu: Sir, I said 'yes' because if I try to solve, I can get the answer, but I don't know how to explain.

Although Zulu could not articulate the meaning of being defined, he understood that if the equation has a solution, it means it is defined. This suggests that he had made a connection between the rules of solving equations and the concept of being defined or undefined. He was categorized as not having made the necessary constructions for this item, because instead of providing a clear definition he just solved the equation and found the values of θ. Accordioning to triad mechanism their responses revealed that they are operating in the intra-stage as they lack the understanding of the concept.

7.4 Analysis of learners' difficulties hindering the necessary mental constructions in Task 2

For Task 2, Item 1, learners were required to find the general solution given for $\cos(\theta + 10) = -0.67$.

2 Find the general solution of the following equation

2.1 $\cos(\theta + 10°) = -0.67$

Figure 45 Item 1 (Task 2)

Table 31 Analysis of learner's difficulties in Item 1 (Task 2)

Theme indicator	**Lacks understanding of the concept (incorrect rules/facts used to determine the answer**	**Made connections between concepts but no understanding of the underlying features (answer is correct but not connected to the problem or vice versa steps are correct answer is wrong**
No. of responses	6	0

The analysis revealed that 6 learners had failed to make the necessary mental constructions due to their lack of understanding of basic algebraic concepts that they should have already constructed: i.e., incomplete prerequisite knowledge and procedural knowledge. For example, manipulating the equation to make θ the subject of the formula. Six learners were also not cognizant of the correct use of notation, as in most cases there was a distinction when referring to a degree or numeral. In Ndoe's response, below, the reference angle is expressed as numeral which is subtracted from a degree: $\theta + 10° = 180° - 47{,}93$. This suggests that learners consider a numeral and a degree to be the same thing or to have the same value.

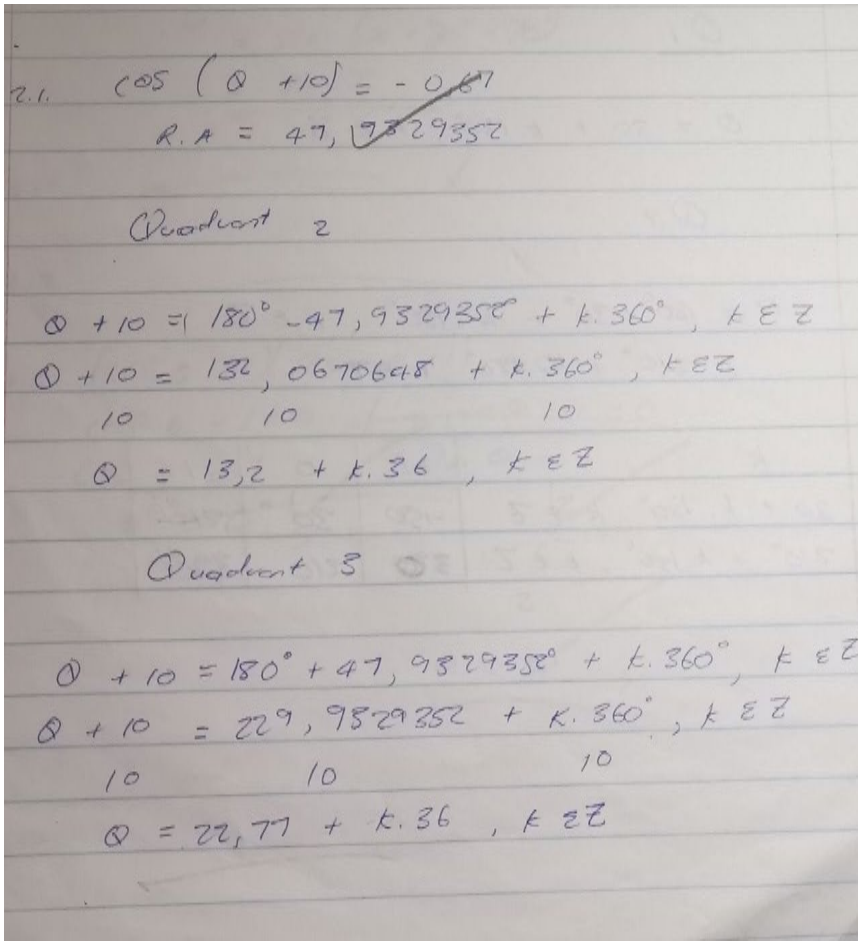


Figure 46 Ndoe's written response to Item 1 (Task 2)

In making θ the subject of the formula, Ndoe divided by 10 instead of subtracting 10°, which is basic knowledge for solving algebraic equations. According to Ngu & Phan (2020), the placement of the pronumeral (e.g., x), whether it is a numerator $(\text{e.g.} \sin 30° = \frac{x}{5})$ or a denominator $(\text{e.g.} \sin 30° = \frac{5}{x})$ requires algebraic skills. However, the learners in this study struggled to answer despite having correctly solved a similar problem $(\frac{x}{4} = 3)$. This suggests that the learners had failed to integrate what they had learnt in algebra when solving trigonometric problems. As noted above, when learners learn concepts as isolated facts, they experience challenges when they need to integrate concepts. A failure to transition to inter-stage or trans-stage seemed to be the main source of learners' difficulties, hindering them from making connection between concepts.

For Item 2, learners also were required to solve general problems given $\sqrt{3}cos\theta - 3\sin\theta = 0$ (see Figure 47 below). This item tested learners' understanding of solving using the concept of trigonometric identities. Again, learners' prerequisite knowledge of how to manipulate algebraic equations was key.

2 Find the general solution of the following equation

2.2 $\sqrt{3}\,cos\ \theta - 3\ sin\ \theta = 0$

Figure 47 Item 2 (Task 2)

The table below presents the analysis of learners' difficulties for Task 2, Item 2.

Table 32 Analysis of learner's difficulties in Item 2 (Task 2)

Theme indicator	**Lacks understanding of the concept (incorrect rules/facts used to determine the answer**	**Made connections between concepts but no understanding of the underlying features (answer is correct but not connected to the problem or vice versa steps are correct answer is wrong**
No. of responses	6	1

Sixteen of the 17 learners attempted to solve this problem. Only 7 learners showed no evidence of mental construction as they provided incorrect answers. The analysis found that 6 of the 7 learners failed to make necessary mental constructions because they lacked an understanding of the concept; 1 learner demonstrated an understanding of the concept but gave an incorrect answer. Learners frequently make errors when dividing. Similar to Item 2, they also lack the application of basic concepts which Tall (2008) terms "met before", which hindered the development of learners' mental constructions of the trigonometric equation, as illustrated in the extract below.

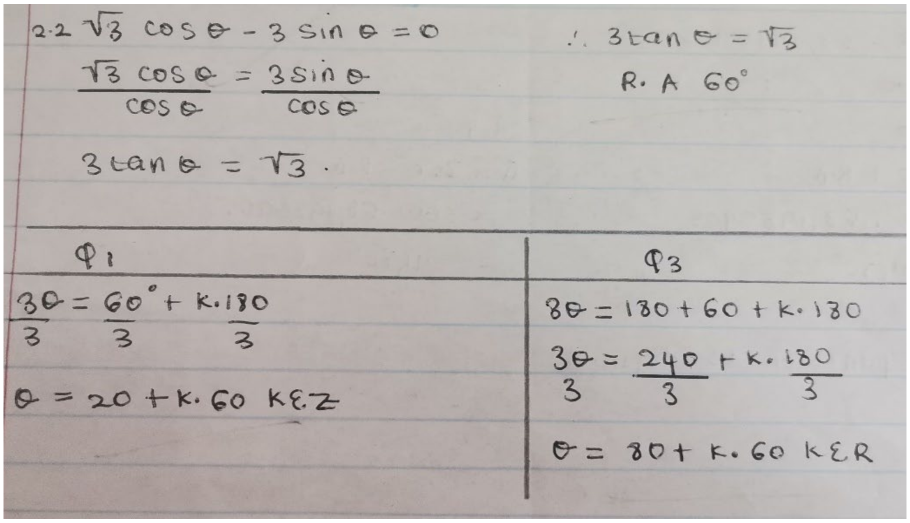


Figure 48 Zulu's written response to Item 2 (Task 2)

Zulu seems to have understood what he needed to do, because the method he used was correct; however, he made an error when dividing. He transposed $3sin\ \theta$ to the right and divided by $cos\ \theta$ on both sides. As a result, he obtained $3tan = \sqrt{3}$, which was correct, but the reference angle was incorrect as he gave the reference angle as 60. The error made by Zulu in this item was that, instead of dividing by 3 on both sides he divided by $\sqrt{3}$; as a result, the answer was incorrect. This error was caused by skipping the step of dividing, where he would have realized that the answer was incorrect. Polya (1957, as cited by Ortiz, 2016) proposes two versions of a model: (1) define the problem, develop a plan, implement the plan, and evaluate; or (2) plan, do, and check. Zulu's response indicates that he planned and did but did not check. During the interview, he explained his process as follows:

Researcher: How did you find the answer?

Zulu: I divided by 3 on both sides to find the reference angle and I used shift tan of $\frac{\sqrt{3}}{3}$.

Researcher: Can you please do it again, using a calculator?

Zulu: Yes, Sir. [He does it again using a calculator.] Eish, sir: I made a mistake yazi (you know) I divide with $\sqrt{3}$, *not* 3, *both sides.*

According to the triad mechanism, Zulu is operating in the trans-stage, because he demonstrated a knowledge of the rules that were required to solve the problem yet did not recognize the solution because of an error that he made.

Item 3 was classified as a Level 3 question in Chapter 6. A knowledge of algebraic concepts as well as a knowledge of solving quadratic equations was needed to solve this trigonometric equation.

2 Find the general solution of the following equation

2.3 $2sin^2\theta - 3\sin\theta = 1$

Figure 49 Item 3 (Task 2)

Table 33 Analysis of learner's difficulties for Item 3 (Task 2)

Theme indicator	**Lacks understanding of the concept (incorrect rules/facts used to determine the answer**	**Made connections between concepts but no understanding of the underlying features (answer is correct but not connected to the problem or vice versa); steps are correct answer is wrong**
No. of responses	12	0

In their solutions for this item, 12 learners showed no evidence of making the necessary mental constructions. Instead of applying their knowledge of solving quadratic equation by first equating the left-hand side (LHS) to 0, the learners attempted to factorize the LHS, as shown in the extracts below.

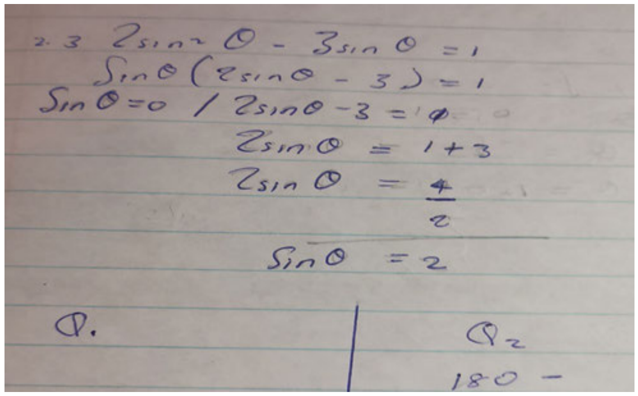


Figure 50 Dlamini's written response to Item 3 (Task 2)

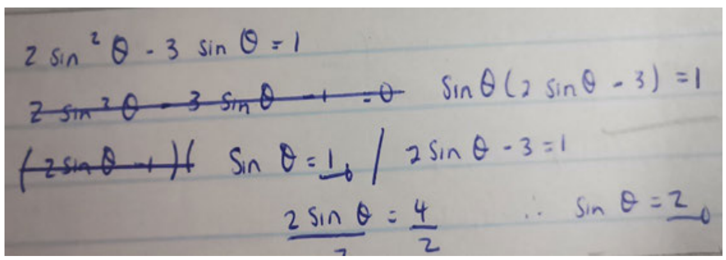


Figure 51 Makhanya's written response to Item 3 (Task 2)

As noted in Dlamini's response, only one factor ($3sin\theta$) was equated to the value on the RHS and $sin\theta$ equated to 0. It appears that, while Dlamini recalled some of the facts (i.e., that $ab = 0$ therefore $a = o$ or $b = 0$) this was incorrectly applied, in this instance. In addition, algebraically, the power 2 in the equation gives an indication that there are two solutions; however, as shown in their responses, the learners failed to recognise that they have common factor. Also, some learners treated the equal sign as an operator (i.e., to indicate the next step), rather than to mean equivalence. Kazunga and Bansilal (2018) found, similarly, that an incorrect conception of the equal sign was the main source of learners' misconceptions. Mbewe (2013) argues that a lack of understanding of structural features in the conceptual area leads learners to use many incorrect rules. In this study, in Item 3 learners incorrectly applied the rules they knew. As shown in the responses above, learners recalled many rules that were not applicable to the problem; as a result, they were unable to determine the solution because they treated $sin\theta$ as a variable to solve for in an algebraic equation, instead of determining where the solution lay for the given function. Even during the interviews, learners were only able to explain the procedure and were unable to identify the errors in their response, as shown in Makhanya's response:

Researcher: What was your understanding of the question? In short: what you were expected to do?

Makhanya: To solve the equation.

Researcher: Explain how you were supposed to approach the problem.

Makhanya: I solved the equation by taking out the common factor, which is $sin\,\theta$. *Then I was left with* $sin\,\theta = 1\ and\ 2\,sin\,\theta - 3 = 1$. *After that, I had* $2\sin\theta = 4$. *Divide both sides with 2. Then I had* $\sin\theta = 2$ *and I found the answer.*

Researcher: If this was algebraic equation, how were you going to approach it?

Makhanya: This is not algebra, sir. In algebra, we have x and y. *Here we have words sin, cos. So it's not the same.*

Learners such as Makhanya demonstrated that they had not transitioned to the stage of making connections between concepts and thus could not draw from their previously learnt knowledge to apply it to a new context.

7.5 Analysis of learners' difficulties hindering the necessary mental constructions in Task 3

Item 1 of Task 3 required complex procedures to be carried out; as with the previous items, a sound knowledge of algebra was key.

3. Solve for θ without the use of a calculator if

3.1 $(cos\,\theta - 1)\left(2sin\,\theta + \sqrt{3}\,\right) = 0\ and\ \theta \in [0°; 360°]$

Figure 52 Item 1 (Task 3)

Table 34 shows the analysis of learners' difficulties in solving Item 1.

Table 34 Analysis of learners' difficulties in Item 1 (Task 3)

Theme indicator	**Lacks understanding of the concept (incorrect rules/facts used to determine the answer**	**Made connections between concepts but no understanding of the underlying features (the answer is correct but not connected to the problem or vice versa steps are the correct answer is wrong**
3.1	8	0

Eight learners displayed a lack of understanding of the concept relating to solving a trigonometric equation of the form $ab = 0$. In some cases, this was by their revoking incorrect schema; others were found to have developed flawed schema, a problem noted by Olivier (1989). An incorrect conception of the meaning of the equal sign was found to be the main source of learners' errors. Some learners attempted to use the foil method; others moved one expression to the RHS, as shown in the examples below.

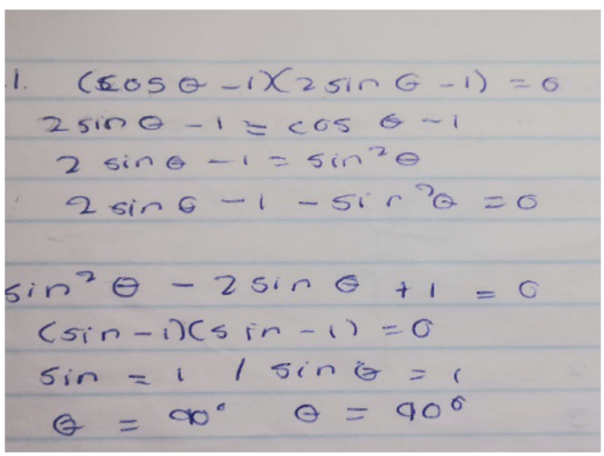


Figure 53 Shozi's written response to Item 1 (Task 3)

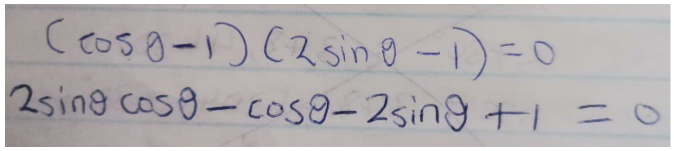


Figure 54 Ndoe's written response to Item 1 (Task 3)

Analysing the tendency of learners to move one factor to the RHS, confusing the process due to incorrect schema of transposing (as Shozi did, in the example in Figure 53), revealed that the equal sign was viewed by these learners as an operator symbol. In addition, they had constructed the meaning of identity incorrectly: while identity should hold true for all values in the equation, the learners' failure to understand that resulted in them overgeneralizing square identities, such as $sin^2\theta + cos^2\theta = 1$, to situations such as $sin\theta + \cos\theta = 1$, as noted in step 2 where $\cos\theta - 1$ is changed to be equal to $sin^2\theta$. Thereafter, the rules of solving quadratic equations were applied. During the interview, Shozi confirmed that she had considered the square identities to be applicable in every expression as long as there was $sin\theta, cos\theta\ and$ 1:

> *Sir, I thought that* $\cos\theta - 1$ *ulingana [is equal] to* $\sin^2\theta$, *because there was* $\cos\ \theta\ with - 1$ *. As a result, I had trinomials to solve by finding the factors.*

Learners given the expressions enclosed in two brackets revoked the schema of using foil method among some learners, as illustrated in Ndoe's response in Figure 54. As a result, these learners were unable to procced to solve the equations. Applying the rules of algebra when they were not applicable was thus another source of learners' errors that hindered them from making the necessary mental constructions. Ndoe, in her response during the interview, indicated that she was confused about which rule to use, as she spoke about transposing, while she had used the foil method:

> *I tried to transpose* $2\sin\theta - 1$, *but my answer was wrong. I removed the bracket, trying to eliminate some terms, but I was stuck again. Ay, sir: I did not know how to solve this one.*

Both Shozi and Ndoe indicated during their interviews that they did not fully comprehend the concept and thus had failed to solve the problem.

Item 2 used the algebraic form $x(x-1)=0$; however, as it was trigonometric equation, the learners needed to be cognizant of the constraints that applied in determining the solution.

3. Solve for θ without the use of a calculator if

3.2 $\sin\theta(2\cos\theta+\sqrt{3})=0$ *and* $\theta\in[0°;360°]$

Figure 55 Item 2 (Task 3)

Table 35 Analysis of learner's difficulties in Item 2 (Task 3)

Theme indicator	**Lacks understanding of the concept (incorrect rules/facts used to determine the answer**	**Made connections between concepts but no understanding of the underlying features (answer is correct but not connected to the problem or vice versa steps are correct answer is wrong**
3.2	9	0

Nine learners showed no evidence of making the necessary mental constructions for Item 2. The analysis of the results to identify the difficulties learners experienced revealed errors, such as the incorrect use of notation and failure to observe the constraints when determining the solution. This is shown in the extract below:

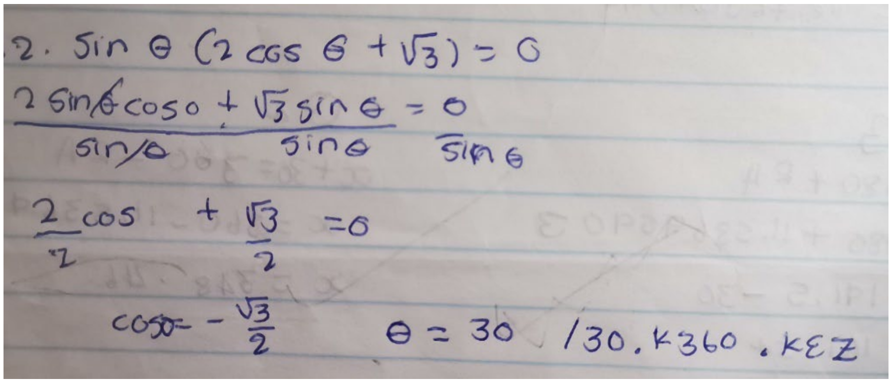


Figure 56 Ndoe's written response to Item 2 (Task 3)

In the previous item (Item 1) Ndoe had used the foil method to simplify the brackets, making the equation more complex, suggesting that while she had developed the schema, the retrieval mechanism was not located as she did not indicate that she needed to equate the two factors to 0. In this item (Item 2), while she was able to carry out the procedure to determine the reference angle, the notation use to indicate the angle was incorrect, meaning that she considered 30 to be a numeral, not a degree. Also, the notation she used to give the general solution was flawed, indicating that her schema for the general solution was flawed. This was confirmed by her response during the interview.

> *Sir, mina [I] what I know is that I need to have one trig ratio. So, I removed the bracket with* $\sin\theta$ *and divide it with* $\sin\theta$ *throughout.*

Ndoe's explanation of her process reveals the misconception that there should be one trig ratio. This misconception among learners led to them trying to eliminate one trig ratio, even in an equation that is in the quadratic form. Similarly, the misconception of having one trig ratio was confused with having one solution; thus, when learners determined the reference, they considered that to be the final answer, without consideration for the constraints given.

7.6 Summary of learners' difficulties hindering mental construction

The analysis revealed several difficulties that hindered learners from making the necessary mental constructions. First, it was evident that an incorrect conception of the equal sign as an operator symbol hindered learners' development of the necessary mental constructions. Second, overgeneralization of rules, or retrieval of incorrect rules, resulted in errors. Third, failure to integrate algebra into the learning of trigonometry created difficulties for learners. Conversely, however, learners' application of the rules of algebra where they did not apply make solving of the problem more difficult. It was therefore observed that, while the learners' recalled procedures or rules, they did not possess the correct schema needed to solve the equation.

It is noted that some of the short methods that learners are taught, or short explanations they are given, may contribute to these difficulties. For example, learners are taught always to simplify into one trig ratio; thus, instead on focusing on the concept, learners retrieve incorrect rules as they try to eliminate the other trig ratio. They may overgeneralize this to mean one trig ratio results in one solution, regardless of the constraints given. As evident in the analysis, learners' conceptualization of concepts as isolated facts hindered their development of the concept to be able to make the necessary mental construction, as most learners who failed to make the necessary mental constructions displayed elements of being at the intra stage, in terms of the triad mechanism.

7.7 Conclusion

This chapter has discussed the difficulties that hindered learners from making the necessary mental constructions when solving trigonometric equations. The triad mechanism was used to examine the errors and misconceptions that learners make when learning and solving trigonometric equations (Borji & Font, 2019). The data gathered from learners' written work and interviews provided sufficient information to identify learners' difficulties when solving trigonometric equations and the sources of their errors. The following chapter discuss the findings, recommendations and conclusion.

CHAPTER 8: DISCUSSION, CONCLUSION, AND RECOMMENDATIONS

8.1 Introduction

The preceding chapter presented the study's findings on the Grade 11 learners' mental constructions and difficulties when learning and solving trigonometric equations. This chapter seeks to confirm if the critical questions guiding this study have been answered as well as establish if the research aim has been accomplished.

The aim of the study was to explore Grade 11 learners' mental construction of trigonometric equations and the difficulties they experienced that hindered their ability to make the necessary mental construction.

The objectives of the research were:

1. to explore Grade 11 learners' mental construction of knowledge when learning and solving trigonometric equations;
2. to explore the enablers or challenges that enhance/ hinder the development of the learners' mental constructions in trigonometric equations; and
3. to explore the extent to which learners' mental constructions align with the preliminary genetic decomposition of trigonometric equations.

The corresponding research questions were:

1. What are the learners' mental constructions of trigonometric equations?
2. Why do learners succeed/ fail to make the necessary mental constructions in learning and solving trigonometric equations?
3. How do learners' mental constructions of action, process, and object align with the preliminary genetic decompositions?

This chapter articulates the key conclusions drawn from the study and makes recommendations anchored in the literature as well as the data that has been presented in this thesis: the analysis of learners' written responses administered in Phase One of the study to ascertain the mental constructions in trigonometric equations (Chapter Five): the analysis of learners' responses on activity sheets and semi-structured interviews collected in Phase Two of the study when learners had progressed to Grade 11 (Chapter 6); and the analysis of the Grade 11 learners' difficulties (errors and misconceptions) that hindered their ability to make the necessary mental constructions (Chapter 7).

8.2 Synthesis of the findings

The findings and conclusions on the three main questions address learners' mental constructions of trigonometric equations and aimed to unearth challenges that hinder the constructions of the necessary mental constructions. In responding to the research questions, the findings are discussed in line with themes that emerged during data analysis, which is learners' mental constructions of trigonometric equations, alignment of learners' mental constructions to the preliminary generic decomposition and errors and misconceptions evident to hinder learners' mental constructions.

8.2.1 Learners' mental constructions of trigonometric equations

This section consolidates the findings that emerged when interrogating learners' mental constructions of trigonometric equations. The study focused on learners' mental constructions for understanding the meaning of a trigonometric function and the quadrant where the solution lay. The study also explored learners' mental constructions for determining the general solution and solving trigonometric equations. Item 1 and Item 2 of Task 1 explored learners' construction of meaning about the relationship between the quadrant and the solution of a trigonometric equation. It also focused on identifying the value of the solution that was the angle without solving the equation. Identifying where the solution lay without performing any calculation physically was deemed to be an illustration of the process conception. The findings showed that while all 17 learners were able to tell whether the equation had a solution or not and were able to locate where the solution lay, the majority were still operating at the action stage. This was demonstrated by the fact that, before they could tell where the solution lay, they first needed to calculate the reference angle. Only then were they able to tell where $\tan\theta = -2.5$ would have a solution. The positive or negative sign (+ or –) acts as a physical cue to trigger one's memory of which quadrant the function solution lies. The findings showed that all 17 learners had constructed an action conception of the relationship between the sign of the function and the quadrant where the solution lay. Further interrogation during the interviews found that fewer learners displayed an evolution of their mental constructions to the next level of interiorization of the action to a process, where they could reflect on the physical steps to explain in which quadrant the solution would lie. This is illustrated in the following response by a participant:

> *According to the question, it says, "Which quadrant would the function have a solution?" Which is why I said on the 2nd and 4th quadrants because we are given the value of y as negative. Uyabona [You see], Sir, in the Cartesian plane tan is negative in the 2nd and 4th quadrant, kodwa ke the [but] where the y value is negative is in the 4th quadrant, if I was plotting the coordinates. If I wanted the exact solutions, I would have calculated the reference angle at the end, I would have said* $180°$ *minus reference angle then finds the answer and* $180°$ minus *eference angle. Therefore, that will tell which quadrant my answer belongs to.* ***(Zulu)***

The findings showed that, in certain cases where learners were working on a solution, they did not engage deeply with their thinking process; when they were asked to explain their thought process, they then reflected only on the concept they had constructed. Concurring with Maharaj (2014) and Ndlovu and Brijlall (2015) It became clear that when teachers encourage learners to discuss their writing, it helps them improve their mental constructions. However, as the majority were still operating at the action stage, it was evident – as was found by Madonsela et al (2020). The majority of the learners were still operating at the action stage in terms of their ability to work with trigonometric concepts.

While the learners could identify where the solution lay, the majority could not explain whether the solution was defined or not, in Item 3 of Task 1. This confirmed that the concept of making meaning about the solution of trigonometric equation and quadrant was still constructed at the action stage, because a person with a conceptual understand would be able to make a connection between identifying the quadrant where the solution lay and whether the solution was defined or not. Once they have identified the quadrant where the solution lies, it means that the trigonometric function is defined in that quadrant. While in this item all 7 learners identified where the solution lay but could not explain, only 8 learners could explain whether the solution was defined or not. As mooted by Ngcobo et al (2019), when concepts are learnt as isolated facts it hinders the learner's ability to make the necessary mental constructions.

8.2.1.1. Learners' mental constructions for determining the general solution or determining the angle

Three items focused on exploring learners' mental constructions for determining a general solution under Task 2: Item 1 was of the form $cos(\theta + a) = \pm b$; Item 2 was of the form $\pm acos\alpha \pm bsin\,\alpha = 0$; and Item 3 was of the form $\pm asin^2\theta = b \pm csin\theta$. With tasks where learners were required to determine the general solution, it was found that learners were more successful in solving questions with the trigonometric function on one side and only a number on the other. For example, for Item 1, 10 learners provided a correct response where the cos of an angle was on the left-hand side and the value was on the right-hand side. Although the majority were able to determine the general solution, in terms of their mental constructions it was evident that they were operating at the action stage as all 10 learners performed step-by-step calculations to determine the general solution. They were unable to skip any of the steps, meaning the preceding step was needed as a trigger for the next step. While learners were able to determine the general solution when given straight forward trigonometric equations such as Item 1 of Task 2, or solving equations that required recalling routine procedures such as $asinx = b$, or knowledge of reciprocals such as $cot\theta = b$, it was noted that the learners had to carry out step-by-step procedures to arrive at the solution, which indicated they were operating at the action stage. Also, it was noted that, while the learners had solved the questions, they were unable to tell whether the answer made. The action stage is the most basic stage of construction in APOS theory; for coherent understanding learners needs to progress through the process and object stages and ultimately form a coherent schema.

The learners struggled with questions that required the application of knowledge or previously learnt concepts, like the application of special angles. In the interviews, some learners indicated that they could not recognise that they were solving trigonometric equations that involved special angles. Similarly, they struggled with solving trigonometric equations requiring the integration of algebra to solving the equations or determine the general solution. For example, to determine the general solution of $3sin^2\theta - 3sin\theta = 1$, only one learner was able to apply their knowledge of how to solve quadratic equations to determine the correct value of $sin\theta$. Although the learner worked step by step to determine the value to be used to determine the reference angle, the action had been interiorized into a process as the learner was able to identify which of the solutions made sense in relation to a sine function – as the maximum value of a sine graph is one, therefore a value above 1 does not have an x coordinate for a sine graph where value of $a = 1$. During the interview, the learner showed she has encapsulated the process into an object as she was able to solve $sin\theta = \pm b$ as a static structure, conceived the process of solving equation as a totality, imagined the graph of $sin\theta$ and applied the transformation to it which values will yield a solution. Furthermore, she was able to imagine the relationship between the reference angle and the value of y, understanding that these actually represented coordinates (for a point on the graph) as illustrated in her response below.

> *Angithi [isn't it so], Sir: the graph of sin turns ku[from] 1 on the y value when x is* 90° *so it cannot go higher than that unless the graph was shifted up vertically. This is normal graph of sin, so the maximum value is 1; yingakho [that's why] 1,78 enganayo [does not have] I -x coordinate.*

The findings revealed that while some learners displayed process or object conceptions of determining the general solution for trigonometric equations, it was evident that the majority were still operating at the action stage as they relied on external cues to solve the problems.

8.2.1.2 Learners' mental constructions of solving trigonometric equations bounded by an interval

It was found that the learners could proceed step by step to determine the general solution. The next step was to explore their mental constructions to solving trigonometric equations bounded by an interval, for example $(\cos\theta - 1)(2\sin\theta + \sqrt{3}) = 0,\ \ \epsilon(0°; 360°)$. The findings showed that most of the learners' experienced difficulty calculate the angle when they were given the interval without using a calculator. While eight learners could solve the problem, it was noted that the majority were still operating at the action stage when dealing with solving trigonometric equations bounded by the interval. However, a few learners' responses show the development of an object conceptions as they demonstrated that they could apply the actions and processes of solving algebraic equations of the form $ab = 0$ to solve trigonometric equations and showed understanding of how to apply the constraints indicated in the interval. This is illustrated in the following example:

> *What I did here: took $\cos\theta - 1 = 0$ and $2\sin\theta - 1 = 0$. In this equation, I apply my knowledge of solving the algebraic equation on Question 1 when we solve x. Separated the two equations, first, I solve $\cos\theta - 1$. But transposing 1 then I had cos θ is equal to 1. The value of θ was equal to 0 degrees. On the other side, I had 2sin θ and transposed 1 to the right and divided by 2 into both sides. I was left with sin θ equal to $\frac{1}{2}$ and my θ was 30 degrees. The intervals tells where the solution should lie so I know sine is positive in the 1st and 2nd quadrant so my other value $\theta = 150°$.* ***(Fihlela)***

It was noticeable that the learners' mental constructions were gradually evolving; however, they experienced challenges applying the knowledge of special angles to solve trigonometric equations. Without using a calculator, learners could not identify the value of $cos\theta = -\frac{\sqrt{3}}{2}$.

In response to the research question about learners' mental constructions for solving trigonometric equations, the findings revealed that the majority of learners were operating at the action stage. However, there was evidence of evolution of the mental constructions of some learners as they displayed process- or object- level conceptions. However, none of the learners showed the development of trigonometric equations schema. This suggests that alternative strategies need to be developed to assist learners to construct the schema conception of trigonometric equations. While Madonsela et al. (2020), in their study about solving three dimensional problems in trigonometry, found that Grade 12 learners showed a lack of development of their mental constructions, the findings of the current reveal that learners had a basic knowledge of how to solve trigonometric equations but struggled to solve trigonometric equations that required the integration of algebraic concepts.

The next section synthesises the findings regarding factors displayed by learners which assist in the evolution of their mental constructions.

8.3 Factors enhancing the evolution of learners' mental constructions of solving trigonometric equations

Drawing on the findings, it was evident that learners with procedural knowledge were able to carry out step-by-step calculation to solve trigonometric equations. Rittle-Johnson and Schneider (2015) define procedure as a series of steps or actions taken to complete a goal; thus, knowledge of procedure assists one to know how to perform an algorithm. This procedural knowledge assisted learners in this study to construct the action conception for solving trigonometric equations. Furthermore, it was noted that where learners could recall the required facts and knowledge it assisted them with constructing the action conceptions. Although knowing the procedure is not enough to enable one to construct the schema of a concept, it assisted the learners in this study to understand the basic procedures needed.

The next section discusses the findings about the factors that hindered the evolution of the learners' mental constructions.

8.4. Factors hindered the development of learners' mental constructions

The findings revealed that learners' errors when solving trigonometric equations hindered their ability to make the necessary mental constructions. Olivier (1989, cited in Ndlovu et al., 2017) posits that errors are symptoms of underlying misconceptions. Thus, underlying misconceptions lead to the construction of an incorrect concept image. As discussed in Chapter 7, the findings revealed several errors made by learners that hindered the evolution of their mental constructions. It was evident that in certain instances learners invoked incorrect rules to solve the problem or overgeneralized a rule. For example, when solving for θ in the question $(\cos\theta - 1)(2\sin\theta + \sqrt{3}) = 0,\ \epsilon(0°; 360°)$, learners invoked the foil method of removing the brackets and thus could not proceed with solving the sum.

8.4.1 Wrong method of factoring/ misapplication of rules

Out of the 12 learners that showed no evidence of mental construction for Item 3 of Task 2 (see Table 32, learners used the wrong method for factorizing the trigonometric equation. The analysis found that these learners had misapplied the rules of factorization. For example, when given $3\sin^2\theta - 3\sin\theta = 1$, learners tried to find the common factor on the left side: $\sin\theta(2\sin\theta -$

$3) = 1$. Learners' use of the wrong method revealed their lack of understanding of the rules of factorizing and a failure to understand that they were solving trinomials. In a study of student' errors while solving mathematical problems, Tong and Loc (2017) identified the mistakes learners made related to not understanding the problem, not using proper calculation procedures and not writing the final answer correctly because of mistakes made during calculations. This was observed in the current study: for example, when Zulu was asked about Item 2 of Task 2, where he did everything correctly but arrived at an incorrect answer, he replied, "I made a mistake when I was using the calculation". Rashidov (2022) states that learners' lack of knowledge of mathematics problems may be a major reason why they cannot solve certain problems correctly and consistently.

8.4.2 Lack of understanding of the concept

The findings for Items 2 and 3 of Task 1 suggest that a lack of understanding of the concept was the main cause of the learners' misconception with regard to indicating whether the value of $\tan\theta = -2.5$ would be $180 < \theta \; or \; \theta < 180$. The analysis found that of the 17 learners, 9 had a misconception about these items. Gafoor and Kurukkan (2015) found that learners who felt that a mathematics concept was very difficult tend to forget it faster and made less effort to learn the concept than was the case with those who felt the concept was easy. Learners may tend to memorise the concept, rather than taking the time to properly understand it (Ahmad et al., 2018).

8.4.3 Lack of conceptualization of previously learnt concepts

Previous knowledge is critical in the conceptualization of new concepts; its existence or absence can thus either enhance or hinder the construction of new knowledge (Tall, 2008). In this study, findings showed that learners' lack of schema of algebraic concepts hindered their constructions for solving trigonometric equations. It was noted that learners struggled to solve problems that required the integration of algebra. For example, a knowledge of how to solve quadratic equations is critical to solve the equation $3sin^2\theta - 3sin\theta = 1$. Usman and Hussain (2017) argue that learners make errors when solving questions related to trigonometry because as that have not previously learnt basic concepts, they misinterpret what the question requires them to do.

Thus, in answering the research questions about factors that enhance or hinder learners' mental constructions, the findings revealed that learners' development of procedural knowledge assisted them to construct the action conception for solving trigonometric equation; however, errors and misconceptions hindered the evolution of their mental constructions.

8.5 Alignment of learners' mental constructions to the preliminary genetic decomposition

The preliminary genetic decomposition used in this study is represented in Figure 57 below. The findings revealed that learners' construction of the action stage aligned with the constructs presented in the preliminary genetic decomposition, as it was noted that learners at the action stage performed step-by-step calculations to solve the trigonometric equations. However, it was noted that learners did not use trial and error to determine the solution but, in questions that required them to show construction of concept definition – such as understanding of the quadrants where the solution was or was not defined – learners first **determined the reference angle** and **used the reference angle as the external cue** to determine whether the equation had a solution or not. Similarly, to determine the value of an angle, learners must first perform the calculation of a reference angle. Although few learners displayed the process and conception stage, the findings showed that for those who interiorized the action into a process, their mental construction aligned with the constructs illustrated in the preliminary genetic decomposition.

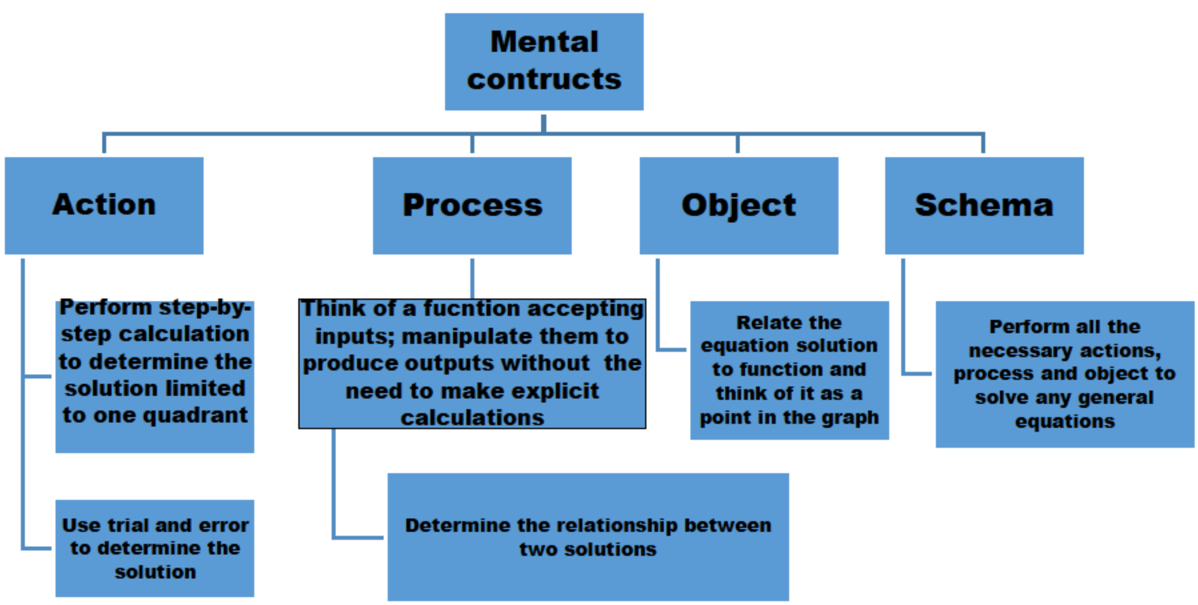


Figure 57 Preliminary genetic decomposition for solving trigonometric equations

The preliminary genetic decomposition was shown to be useful for analysing learners' mental constructions for trigonometric equations. However, it was found that some of the responses given by learners could not be explained using the preliminary genetic decomposition. To accommodate the findings, the genetic decomposition was modified based on learners' responses. For instance, in Item 1 of Task 1 (discussed in Chapter 6), the learner is asked to identify the quadrants in which a solution could be found and give reasons. However, a learner who **provided the correct answer but was unable to explain how they found the answer** was also considered to be operating at the action level. For instance, Zulu could not explain his answer while solving the problem but, during the interview, he indicated that he understood the concept. For Task 2, learners were regarded to be operating at the action level if they provided the correct answer step-by-step for Items 1, 2 and 3. However, some learners **provided the solution step by step but could not provide the answers in the degrees (°).** The genetic decomposition was modified to regard a learner who provided the answer as numerical as operating at the action level because, during the interviews, they were able to refer to the answer as the angle – for instance, saying,"30 degrees".

The next section discusses the contribution of the study to the field.

8.6 Contribution of this study to the field

The preliminary genetic decomposition (Figure 57) was designed to explain learners' mental constructions of trigonometric equations. Analysis of the data generated in this study revealed that some answers provided by learners included in the preliminary genetic decomposition. While using trial and error to determine the solution can still be considered as the mental construction to show action conception, the findings showed that learners tend to base their knowledge of finding or calculating the solution on determining the reference angle. The reference angle thus acts as an external cue that triggers the steps to be performed to determine the solution, to determine the quadrant where the solution lies, or to determine whether the equation is defined or not. To accommodate this, the genetic decomposition model has been revised, as shown in Figure 58, to allow for a more complete data analysis of learners' mental constructions of trigonometric equation. As stated by Arnon et al. (2014), preliminary genetic decomposition can be modified. In this study, the preliminary genetic decomposition was modified to align with the learners' mental constructions. The modified genetic decomposition is presented in Figure 58 below.

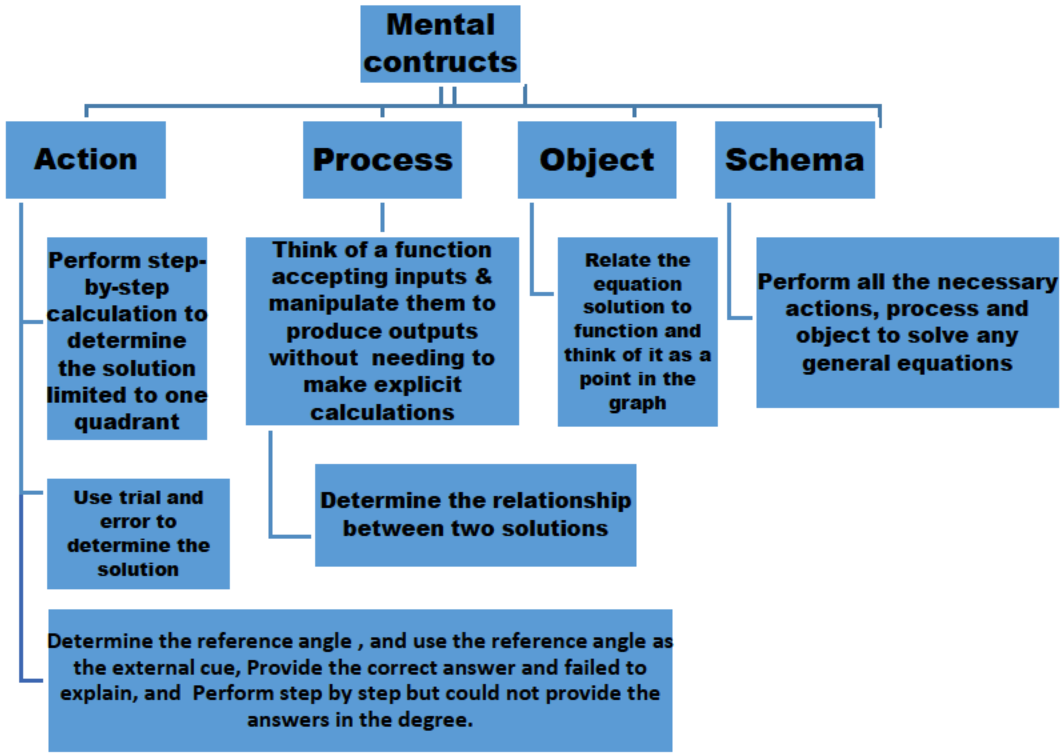


Figure 58 The modified preliminary genetic decomposition

The recommendations emanating from this study are presented next.

8.7 Recommendations

The study aimed to explore Grade 11 learners' mental construction and difficulties when solving and learning trigonometric equations. A sample of 17 learners wrote the test and 6 learners were chosen for interviews. The findings cannot be generalized to the Grade 11 that did not participate in the study. Therefore, the recommendations are based on the findings during the analysis of test results and interviews in this study conducted at one school in Umlazi District.

8.7.1 Recommendation for teaching and learning

The findings indicate that learners lacked basic algebra skills. To address this, it is recommended that, before teachers start teaching trigonometric equations, they assist their learners to bridge the gap by ensuring that they have grasped basic algebraic skills such as collecting like and unlike terms, using brackets, and the addition and subtraction of algebraic terms. A sound knowledge of the basic skills and concepts of algebra will support learners to be able to understand trigonometric equations. Teachers should, thus, take into consideration the importance of ensuring learners have foundational skills and knowledge before they teach new concepts and skills (Munna & Kalam, 2021).

It is also recommended that the Department of Basic Education – in particular, its mathematics advisors – emphasise the importance of topic integration in mathematics. During analysis of the findings, it was observed that if the learners lacked basic skills, they failed to understand and correctly answer trigonometric equations. This could be one of the factors that contributes to the poor performance of learners in trigonometry in Grade 12.

This study strongly supports and subscribes to the idea of using many different ways to teach trigonometric equations. Several researchers have advocated for the use of multiple approaches (JoJo, 2015) to achieve various benefits. The use of multiple methods to solve trigonometric equations encourages learners to better understand the connection between different methodologies for conceptual development. Teaching methods should allow learners to explain their answers. Teachers should pay close attention to learners' explanations so that they can identify misconceptions that have arisen and devise effective ways to build their learners' understanding of trigonometric equations. Learners should be given more and different trigonometric equations to simplify. Learners should be encouraged to share their successes and difficulties in trigonometric equations to help those who are struggling. Grouping learners according to their abilities can be an effective strategy for addressing their needs in the topic of trigonometric equations.

8.7.2. Recommendations for further research

This study was conducted with 17 learners in one school in the Umlazi District, which is one of the biggest districts in the province of KwaZulu-Natal. because of the small sample, a generalisation cannot be made to all learners at this school or other schools. Future research could include a large sample of participants from different schools to increase the validity of the findings.

In addition, trigonometry is a very broad topic, and this study focused on only one aspect: trigonometric equations. Learners' performance in this area, as documented in this study, thus cannot be seen to be representative of their overall performance for trigonometry. It is recommended that future research studies thus include other trigonometry topics.

As it was observed that a foundational knowledge of algebra is important for a learner to be able to understand trigonometric equations, it would also be beneficial for further research to explore learners' understanding of algebraic equations.

Finally, this study has focused on *learners* in the learning and solving trigonometric equations. It may be useful for future studies to explore *teachers'* understanding of trigonometric equations.

8.8 Limitations of the study

This study explores Grade 11 learners' mental constructions and difficulties in learning and solving trigonometric equations in the limited context of a small sample from one school in one district of one province of South Africa. Thus, the small size and arbitrary nature of the sample represents a limitation to the study, as it could not produce generalizable inferences about Grade 11 learners' mental construction and difficulties when learning and solving trigonometric equations.

Another limitation arose from the fact that participants understood that tests taken in the after-school programme (which were used to generate data for the study) do not contribute to their summative marks. This may have impacted the seriousness with which the participants engaged with the tasks through which data were generated.

Finally, as the researcher was also a teacher at the school where the study was carried out, issues of power dynamics and biases may have impacted the research results. To mediate this, the study was conducted with learners from the after-school programme rather than during the normal teaching time. The after-school programme was an optional programme that provided a space for learners to discuss their learning challenges with the teacher or with peers. As the tasks done at the after-school programme did not contribute to the learners' summative marks and were not used for grading purposes, learners could be expected to experience less pressure to impress the teacher than in the classroom context. The researcher/teacher, also, was free of any motivation to press learners for correct answers or to complete the curriculum; their role was rather to engage with learners about any challenges they encountered in the learning process.

8.9 Conclusion

This chapter has provided a summary of the study's conclusions. Recommendations were made for teaching and learning and also for further research. While due to the localised nature of this case study the findings cannot be generalised to all schools, is hoped that the findings, conclusions and recommendations will assist mathematics teachers and others interested in enhancing mathematics teaching and learning – not only in South Africa, but around the world.

REFERENCES

Acharya, B. R. (2017). Factors affecting difficulties in learning mathematics by mathematics learners. *International Journal of Elementary Education*, *6*(2), 8-15.

Ahmad, H., Al Yakin, A., & Sarbi, S. (2018). The analysis of student error in solving the problem of spherical trigonometry application. In *Journal of Physics: Conference Series* (Vol. 1114, No. 1, p. 012114). IOP Publishing.

Aizikovitsh-Udi, E., & Cheng, D. (2015). Developing critical thinking skills from dispositions to abilities: mathematics education from early childhood to high school. *Creative education*, *6*(04), 455.

Akintoye, A. (2015). Developing theoretical and conceptual frameworks. In *EDMIC research workshop*. Ile-Ife: Faculty of Environmental Design and Management, Obafemi Awolowo University, Ile-Ife, Nigeria.

Alagic, M. (2003). Technology in the mathematics classroom: Conceptual orientation. *Journal of Computers in mathematics and Science Teaching*, *22*(4), 381-399.

Ali, A. M., & Yusof, H. (2011). Quality in qualitative studies: The case of validity, reliability and generalizability. *Issues in Social and Environmental Accounting*, *5*(1/2), 25-64.

Alreshidi, N. A. K. (2023). Enhancing topic-specific prior knowledge of students impacts their outcomes in mathematics. In *Frontiers in Education* (Vol. 8, p. 1050468). Frontiers.

Amineh, R. J., & Asl, H. D. (2015). Review of constructivism and social constructivism. *Journal of Social Sciences, Literature and Languages*, *1*(1), 9-16.

Aminudin, M., Nusantara, T., Parta, I. N., Rahardjo, S., As'ari, A. R., & Subanji, S. (2019). Engaging problems on trigonometry: why were student hard to think critically? In *Journal of Physics: Conference Series* (Vol. 1188, No. 1, p. 012038). IOP Publishing.

Anderson, R. (2007). Being a Mathematics Learner: Four Faces of Identity. *Mathematics Educator*, *17*(1), 7-14.

Anfara, V. A., & Mertz, N. T. (2015). Setting the stage. *Theoretical frameworks in qualitative research (2)*, 1-20.

Arhin, J., & Hokor, E. (2021). Analysis of High School Students' Errors in Solving Trigonometry Problems. *Journal of Mathematics and Science Teacher*, *1*(1).

Arnawa, I. M., & Nita, S. (2019, October). Errors and misconceptions in learning elementary linear algebra. In *Journal of Physics: Conference Series* (Vol. 1321, No. 2, p. 022095). IOP Publishing.

Arnawa, I. M., & Nita, S. (2020). Improvement students' achievement in elementary linear algebra through APOS theory approach.

Arnon, I., Cottrill, J., Dubinsky, E., Oktaç, A., Roa Fuentes, S., Trigueros, M., ... & Weller, K. (2014). The teaching of mathematics using APOS theory. *APOS theory: A Framework for Research and Curriculum Development in Mathematics Education*, Springer New York, NY 57-91.

Asmida, A., Sugiatno, S., & Hartoyo, A. (2018). Developing the mathematics conceptual understanding and procedural fluency through Didactical Anticipatory Approach equipped with teaching aids. *Journal of Education, Teaching and Learning*, *3*(2), 367-372.

Aspers, P., & Corte, U. (2019). What is qualitative in qualitative research. *Qualitative Sociology*, *42*, 139-160.

Austin, Z., & Sutton, J. (2014). Qualitative research: Getting started. *The Canadian journal of hospital pharmacy*, *67*(6), 436.

Ay, Y. (2017). A review of research on the misconceptions in mathematics education. *Education Research Highlights in Mathematics, Science and Technology*, *12*(1), 21-31.

Aygor, N., & Ozdag, H. (2012). Misconceptions in linear algebra: the case of undergraduate students. *Procedia-Social and Behavioral Sciences*, *46*, 2989-2994.

Bansilal, S., Brijlall, D., & Trigueros, M. (2017). An APOS study on pre-service teachers' understanding of injections and surjections. *The Journal of Mathematical Behavior*, *48*, 22-37.

Bansilal, S., Mkhwanazi, T., & Brijlall, D. (2014). An exploration of the common content knowledge of high school mathematics teachers. *Perspectives in Education*, *32*(1), 34-50.

Barnes, H., & Venter, E. (2008). Mathematics as a social construct: Teaching mathematics in context. *Pythagoras*, *2008*(68), 3-14.

Bassey, B. A., Ubi, I., Anagbogu, E. G., & Owan, V. J. (2022). Permutation of the UTME multiple-choice test items on performance in use of English and mathematics among prospective higher education students. *The Journal of Social Sciences Research*, 2411-9458.

Baxter, P., & Jack, S. (2008). Qualitative case study methodology: Study design and implementation for novice researchers. *The qualitative report, 13(*4), 544-559.

Berndt, A. E. (2020). Sampling methods. *Journal of Human Lactation*, *36*(2), 224-226.

Bertram, C., & Christiansen, I. (2014). Understanding research: An introduction to reading research (1st ed.). Van Schaik Publishers.

Bishara, S. (2016). Creativity in unique problem-solving in mathematics and its influence on motivation for learning. *Cogent Education*, *3*(1), 1202604.

Biyela, K., Sibaya, D., & Sibaya, P. (2016). Comparative analysis of mathematics syllabi for secondary school and teacher training programs. International Journal of Educational Sciences, 13(3), 325-334.

Borji, V., & Font, V. (2019). Exploring students' understanding of integration by parts: a combined use of APOS and OSA. *EURASIA Journal of Mathematics, Science and Technology Education*, *15*(7), em1721.

Bornstein, N. M. (2017). *Music of the triangles: How students come to understand trigonometric identities and transformations* [Unpublished doctoral dissertation]. University of New Hampshire.

Brau, B. (2020). Constructivism. In R. Kimmons & S. Caskurlu (Eds.), *The Students' Guide to Learning Design and Research.* EdTech Books. Available at https://edtechbooks.org/studentguide/constructivism

Bressoud, D. M. (2010). Historical reflections on teaching trigonometry. *The Mathematics Teacher*, *104*(2), 106-112.

Brijlall, D. & Maharaj, A. (2010). *An APOS analysis of students' construction of the concept of continuity of a singled-valued function. Proceeding Gordan's Bay Delta*, *9*, 36-49.

Brijlall, D. & Maharaj, A. (2015) Exploring pre-service teachers' mental constructions when solving problems involving infinite sets. *International Journal of Educational Sciences, 9*(3) pp.273-281.

Brijlall, D., & Maharaj, A. (2009). Using an inductive approach for definition making: Monotonicity and boundedness of sequences. *Pythagoras*, *2009*(70), 68-79.

Brijlall, D., & Maharaj, A. (2014). Exploring support strategies for high school mathematics teachers from underachieving schools. *International Journal of Educational Sciences*, *7*(1), 99-107.

Brijlall, D., & Ndlovu, Z. (2013). High school learners' mental construction during solving optimisation problems in Calculus: A South African case study. *South African Journal of Education*, *33*(2), 1-18.

Brodie, K. (2014). Learning about learner errors in professional learning communities. *Educational studies in mathematics*, *85*(2), 221-239.

Cahyani, N. P. I., Suarsana, I. M., & Mahayukti, G. A. (2021). Improving Student's Mathematical Problem-Solving Skills Through Relating-Experiencing-Applying-Cooperating-Transferring Learning Strategy and Graphic Organizer. In *First International Conference* *on Science, Technology, Engineering, and Industrial Revolution (ICSTEIR 2020)* (pp. 337-344). Atlantis Press.

Cangelosi, R., Madrid, S., Cooper, S., Olson, J., & Hartter, B. (2013). The negative sign and exponential expressions: Unveiling students' persistent errors and misconceptions. *The Journal of Mathematical Behavior*, *32*(1), 69-82.

Chagwiza, C., Mutambara, L. H. N., & Sunzuma, G. (2021). Exploring Zimbabwean A-Level Mathematics Learners' Understanding of the Determinant Concept. European Journal of Mathematics and Science Education, 2(2), 85-100.

Challenger, M. (2009). *From triangles to a concept: a phenomenographic study of A-level students' development of the concept of trigonometry* [Unpublished doctoral dissertation]. University of Warwick.

Chen, S. C., Hsiao, M. S., & She, H. C. (2015). The effects of static versus dynamic 3D representations on 10th grade students' atomic orbital mental model construction: Evidence from eye movement behaviors. *Computers in Human Behavior*, *53*, 169-180.

Chigonga, B. (2016). *Learners' Errors When Solving Trigonometric Equations and Suggested Interventions from Grade 12 Mathematics Teachers.* International Society for Technology in Education, 163–176.

Chikiwa, C. (2015). Teaching trigonometry in a grade 11 multilingual mathematics class: A focus teacher code switching practices. In *Iste international conference on mathematics, science and technology education* (p. 15).

Chikiwa, C., & Schäfer, M. (2018). Promoting critical thinking in multilingual mathematics classes through questioning. *Eurasia Journal of Mathematics, Science and Technology Education*, *14*(8), em1562.

Chilisa, B., & Kawulich, B. (2012). Selecting a research approach: Paradigm, methodology and methods. *Doing social research: A global context*, *5*(1), 51-61.

Chowdhury, M. F. (2014). Interpretivism in aiding our understanding of the contemporary social world. *Open Journal of Philosophy*, *4*(03), 432-438.

Choy, L. T. (2014). The strengths and weaknesses of research methodology: Comparison and complimentary between qualitative and quantitative approaches. *IOSR journal of humanities and social science*, *19*(4), 99-104.

Chuang, S. (2021). The applications of constructivist learning theory and social learning theory on adult continuous development. *Performance Improvement*, *60*(3), 6-14.

Ciesielska, M., Boström, K. W., & Öhlander, M. (2018). Observation methods. In Qualitative methodologies in organization studies*: Volume II: Methods and Possibilities*, 33-52.

Cohen, N. (2018). A case study of East Harlem, New York. *Integrating food into urban planning*, *312*.

Cope, D. G. (2014). Methods & Meanings. Credibility and Trustworthiness of Qualitative Research. *Oncology nursing forum*, *41*(1), 89-91.

Corbin, J., & Strauss, A. (2008). Qualitative research. *Techniques and procedures for developing grounded theory*, *3*.

Creswell J. W. (2007). *Qualitative inquiry and research design: Choosing among five approaches.* Sage.

Creswell, J. W. (2012). *Educational Research: Planning, Conducting and Evaluating Quantitative and Qualitative Research* (4th ed.). Pearson Education *260*(1), 375-382.

Creswell, J. W. (2014). *Research design: qualitative, quantitative, and mixed method approaches.* Sage.

Creswell, J. W., & Clark, V. L. P. (2017). *Designing and conducting mixed methods research*. Sage.

Creswell, J. W., & Creswell, J. D. (2017). *Research design: Qualitative, quantitative, and mixed methods approach*. Sage.

Creswell, J. W., & Miller, D. L. (2000). Determining validity in qualitative inquiry. *Theory into Practice, 39*(3), 124-130.

Creswell, J. W., & Poth, C. N. (2016). *Qualitative Inquiry and Research Design: Choosing Among Five Approaches.* Sage.

Cropley, A. (2015). *Introduction to qualitative research methods*. Germany.

Crossman, A., & Nicki, L. C. (2018). *Understanding purposive sampling*. ThoughtCo.

Crowe, S., Cresswell, K., Robertson, A., Huby, G., Avery, A., & Sheikh, A. (2011). The case study approach. *BMC Medical Research Methodology*, *11*(1), 100-108.

Darling-Hammond, L. (2016). Research on teaching and teacher education and its influences on policy and practice. *Educational Researcher*, *45*(2), 83-91.

Darling-Hammond, L., Flook, L., Cook-Harvey, C., Barron, B., & Osher, D. (2020). Implications for educational practice of the science of learning and development. *Applied Developmental Science*, *24*(2), 97-140.

de Guzman, A. B., & Adamos, J. L. (2020). Like the layers of an onion: curricular noticing as a lens to understand the epistemological features of the Philippine K to 12 secondary mathematics curriculum materials. *Educational Research for Policy and Practice*, *19*(3), 389-409.

De Massis, A., & Kotlar, J. (2014). The case study method in family business research: Guidelines for qualitative scholarship. *Journal of Family Business Strategy, 5*(1), 15- 29.

Demir, K., Sutton-Brown, C., & Czerniak, C. (2012). Constraints to changing pedagogical practices in higher education: An example from Japanese lesson study. *International journal of science education*, *34*(11), 1709-1739.

Demir, O. (2012). Students' concept development and understanding of sine and cosine functions. [Unpublished doctoral dissertation]. Amsterdam University. Retrieved from https://esc.fnwi.uva.nl/thesis/centraal/files/f107257570.pdf.

Dennick, R. (2016). Constructivism: reflections on twenty-five years teaching the constructivist approach in medical education. *International journal of medical education*, *7*, 200.

Denzin, N. K., & Lincoln, Y. S. (2011). *The SAGE Handbook of Qualitative Research*. Sage.

Denzin, N. K., Lincoln, Y. S., Giardina, M. D., & Cannella, G. S. (Eds.). (2023). *The Sage handbook of qualitative research*. Sage.

Department of Basic Education (DBE). (2011). Curriculum and Assessment Policy Statement Grade 10-12 Mathematics. Pretoria: Department of Basic Education.

Department of Basic Education (DBE). (2012). Curriculum and Assessment Policy Statement Grade 10-12 Mathematics. Pretoria: Department of Basic Education.

Department of Basic Education (DBE). (2014). *Report on the Annual Assessment of 2014*. Pretoria: Government Printer.

Department of Basic Education (DBE). (2020). Curriculum and Assessment Policy Statement 10-12 Mathematics. Pretoria: Department of Basic Education.

Dickson, A., Emad, K. H., & Joe, A. A. (2018). Theoretical and Conceptual Framework: Mandatory Ingredients of a Quality Research. *International Journal of Scientific Research, 7*(1), 6-9.

Drews, D., Surtees, L., Lawton, F., Dudgeon, J., & Hansen, A. (2020). *Children' s errors in mathematics*. Sage.1-336.

Dubinsky, E. (1991). Constructive aspects of reflective abstraction in advanced mathematics. In *Epistemological foundations of mathematical experience* (pp. 160-202). New York, NY: Springer New York.

Dubinsky, E. (1997). On learning quantification. *Journal of Computers in Mathematics and Science Teaching, 16*(2), 335-362.

Dubinsky, E. (2010). The APOS theory of learning mathematics: Pedagogical applications and results. In *Eighteenth Annual Meeting of the Southern African Association for Research in Mathematics, Science and Technology Education* (pp. 3-18). *Durban, South Africa*.

Dubinsky, E., & McDonald, M. A. (2001). APOS: A constructivist theory of learning in undergraduate mathematics education research. In *The teaching and learning of mathematics at university level(V7): An ICMI study* (pp. 275-282). Springer Netherlands.

Dworkin, S. L. (2012). Sample size policy for qualitative studies using in-depth interviews. *Archives of sexual behavior*, *41*, 1319-1320.

Easwaramoorthy, M., & Zarinpoush, F. (2006). Interviewing for research. *Imagine Canada*, *425*, 1-2.

Ennis, R. H. (2018). Critical thinking across the curriculum: A vision. *Topoi*, *37*, 165-184.

Erath, K., Prediger, S., Quasthoff, U., & Heller, V. (2018). Discourse competence as important part of academic language proficiency in mathematics classrooms: The case of explaining to learn and learning to explain. *Educational Studies in Mathematics*, *99*(2), 161-179.

Ernaningsih, Z., & Wicasari, B. (2017). Analysis of mathematical representation, communication and connection in trigonometry. In *The 2017 International Conference on Research in Education* (pp. 45-57).

Fagerholt, K., Christiansen, M., Hvattum, L. M., Johnsen, T. A., & Vabø, T. J. (2010). A decision support methodology for strategic planning in maritime transportation. *Omega, 38*(6), 465-47

Fahrudin, D., & Pramudya, I. (2019). Profile of students' errors in trigonometry equations. In *Journal of Physics: Conference Series*, 1188 (1), 012044.

Faulkner, F., Breen, C., Prendergast, M., & Carr, M. (2021). Profiling mathematical procedural and problem-solving skills of undergraduate students following a new mathematics curriculum. *International Journal of Mathematical Education in Science and Technology*, 1-30.

Flavell, J. H. (1979). Metacognition and cognitive monitoring: A new area of cognitive–developmental inquiry. *American psychologist*, *34*(10), 906.

Flick, U. (2009). *The sage qualitative research kit.* Sage.

Flyvbjerg, B. (2006). Five misunderstandings about case-study research. *Qualitative inquiry*, *12*(2), 219-245.

Forero, R., Nahidi, S., De Costa, J., Mohsin, M., Fitzgerald, G., Gibson, N., & Aboagye-Sarfo, P. (2018). Application of four-dimension criteria to assess the rigor of qualitative research in emergency medicine. *BMC health services research, 18*(1), 120.

Fraser, S., & Robinson, C. (2004). Paradigms and philosophy. *Doing research with children and young people*, 59-78.

Fuentealba, C., Badillo, E., Sánchez-Matamoros, G., & Cárcamo, A. (2018). The understanding of the derivative concept in higher education. *EURASIA Journal of Mathematics, Science and Technology Education*, *15*(2), em1662.

Fujii, T. (2020). Misconceptions and alternative conceptions in mathematics education. *Encyclopedia of mathematics education*, 625-627.

Gafoor, K. A., & Kurukkan, A. (2015). Why High School Students Feel Mathematics Difficult? An Exploration of Affective Beliefs. *Online Submission*.

Gafoor, K. A., & Sarabi, M. K. (2015). Relating Difficulty in School Mathematics to Nature of Mathematics: Perception of High School Students from Kerala. *Online Submission*.

Gardner, H. E. (2011). *Leading minds: An anatomy of leadership*. Basic Books.

Gerhana, M. T. C., Mardiyana, M., & Pramudya, I. (2017). The effectiveness of project-based learning in trigonometry. *Journal of Physics: Conference Series*, *895,* 1, 012027.

Gernetzky, K. (2012). South Africa bottom of class in math and science. oct 31: *Business Day Live*.

Glaés-Coutts, L., & Nilsson, H. (2021). Who owns the knowledge? Knowledge construction as part of the school improvement process. *Improving Schools*, *24*(1), 62-75.

Govender, R., & Hugo, A. J. (2020). An analysis of the results of literacy assessments conducted in South African primary schools. *South African Journal of Childhood Education*, *10*(1), 1-13.

Grace, S., Higgs, J., & Horsfall, D. (2009). Celebrating writing. (eds) *Writing qualitative research on practice* (pp. 323-330). Brill.

Greene, M. (2020). The Use of a Modified Hess' Cognitive Rigor Matrix to Assess Students' Depth of Knowledge in Key Concepts of Gas Stoichiometry and Chemical Equilibrium [Unpublished doctoral dissertation]. City University, NY.

Gunawan, J. (2015). Ensuring trustworthiness in qualitative research. *Belitung Nursing Journal, 1*(1), 10-11.

Halpern, D. F. (2013). *Thought and knowledge: An introduction to critical thinking*. Psychology Press.

Hamzah, N., Maat, S. M., & Ikhsan, Z. (2021). A systematic review on pupils' misconceptions and errors in trigonometry. *Pegem Journal of Education and Instruction*, *11*(4), 209-218.

Hays, D. G., Wood, C., Dahl, H., & Kirk- Jenkins, A. (2016). Methodological rigor in Journal of Counselling & Development qualitative research articles: A 15- year review. *Journal of Counselling & Development, 94*(2), 172-183.

Healy, M., & Perry, C. (2000). Comprehensive criteria to judge validity and reliability of qualitative research within the realism paradigm. *Qualitative Market Research, 3*(3), 118-126.

Hiller, James. 2016. Epistemological Foundations of Objectivist and Interpretivist Research. In *Music Therapy Research*, 3rd ed. Dallas: Barcelona Publishers, pp. 99–127.

Hitt, F., & Kieran, C. (2009). Constructing knowledge via a peer interaction in a CAS environment with tasks designed from a task–technique–theory perspective. *International Journal of Computers for Mathematical Learning*, *14*, 121-152.

Hughes, P. (2010). Paradigms, methods and knowledge. *Doing early childhood research: International perspectives on theory and practice*, *2*, 35-61.

Hurrell, D. (2021). Conceptual knowledge or procedural knowledge or conceptual knowledge and procedural knowledge: Why the conjunction is important to teachers. *Australian Journal of Teacher Education (Online)*, *46*(2), 57-71.

Imenda, S. (2014). Is there a conceptual difference between theoretical and conceptual frameworks? *Journal of social sciences*, *38*(2), 185-195.

Inglis, C. E. (2015). *A mathematical treatise on vibrations in railway bridges*. Cambridge University Press.

Ishimaru, A. M., Barajas-López, F., & Bang, M. (2015). Centering family knowledge to develop children's empowered mathematics identities. *Journal of Family Diversity in Education*, *1*(4), 1-21.

Jaleel, S. (2016). A Study on the Metacognitive Awareness of Secondary School Students. *Universal Journal of Educational Research*, *4*(1), 165-172.

Jameel, H. T., & Ali, H. H. (2016). Causes of poor performance in mathematics from the perspective of students, teachers and parents. *American Scientific Research Journal for Engineering, Technology, and Sciences*, *15*(1), 122-136.

James, M., Applefield, R., & Mahnaz, M. (2001). *Constructivism in theory and practice: Toward a better understanding.* Watson School of Education, The University of North Carolina at Wilmington. http://doi. org/10.1007/s13398-014-0173-7.2.

Jojo, Z. (2019). Mathematics education system in South Africa. *Education systems around the world*, vol.1, pp.129-140.

Jojo, Z. M. (2015). Comparative study on structural organisation of Mathematics Continuous Professional Development (MCPD) in selected developing and developed countries. *International Journal of Educational Sciences*, *8*(1), 229-240.

Jojo, Z. M. M. (2011). *An APOS exploration of conceptual understanding of the chain rule in calculus by first year engineering students* [Unpublished doctoral dissertation]. University of KwaZulu-Natal.

Jojo, Z. M. M., Maharaj, A., & Brijlall, D. (2013). From human activity to conceptual understanding of a mathematical concept. *Journal of Research in Mathematics Education*, *2*(1), 77-99.

Jojo, Z.M.M., Brijlall, D., & Maharaj, A. (2010). Reflective abstraction and Mathematics Education: The genetic decomposition of the chain rule. *Proceedings of the Eighteenth Annual Meeting of the Southern African Association for Research in Mathematics, Science and Technology Education, 1,* 63-7. SAARMSTE.

Jonassen, D. H. (2000). Toward a design theory of problem solving. *Educational technology research and development*, *48*(4), 63-85.

Josefsson, T. (2016). How good are case studies as scientific products? Halmstad: Halmstad university

Kablan, Z., & Uğur, S. S. (2021). The relationship between routine and non-routine problem solving and learning styles. *Educational Studies*, *47*(3), 328-343.

Kaczmarczyk, L. C., Petrick, E. R., East, J. P., & Herman, G. L. (2010). Identifying student misconceptions of programming. In *Proceedings of the 41st ACM technical symposium on Computer science education* (pp. 107-111).

Kaitera, S., & Harmoinen, S. (2022). Developing Mathematical Problem-Solving Skills in Primary School by Using Visual Representations on Heuristics. *LUMAT: International Journal on Math, Science and Technology Education*, *10*(2), 111-146.

Kamber, D., & Takaci, D. (2018). On problematic aspects in learning trigonometry. *International Journal of Mathematical Education in Science and Technology*, *49*(2), 161-175.

Kapasa, J., Samuels, E. B., & Mulendema, P. J. (2015). Problems Faced by Students and Lecturers in Teaching and Learning of Approximation in Primary Colleges of Education: A Case Study of 4 Primary Colleges on the Copperbelt Province-Zambia. *Journal of Education and Practice*, *6*(17), 137-146.

Kazunga, C, & Bansilal, S. (2018). Misconceptions About Determinants. In: Stewart, S., Andrews-Larson, C., Berman, A., Zandieh, M. (eds) Challenges and Strategies in Teaching Linear Algebra. ICME-13 Monographs. Springer, Cham. https://doi.org/10.1007/978-3-319-66811-6_6

Kepceoglu, I. (2016). Teaching a Concept with GeoGebra: Periodicity of Trigonometric Functions. *Educational Research and Reviews*, *11*(8), 573-581.

Kersaint, G., Thompson, D. R., & Petkova, M. (2013). *Teaching Mathematics to English Language Learners (2nd ed.).* Routledge.

Khudhair, L. K., & Jasim, A. B. M. (2021). Analyzing the content of the mathematics textbook of the fifth grade preparatory according to the depth of knowledge levels. *Journal of Contemporary Issues in Business and Government, 27*(1).

Kilpatrick, J., Swafford, J., & Findell, B. (2002). Adding it up: Helping children learn mathematics. *The National Academies Press. The book is available free on the Web. Accessed*, *2*(4), 04.

Kim, Y., & Baylor, A. L. (2006). A social-cognitive framework for pedagogical agents as learning companions. *Educational technology research and development*, *54*, 569-596.

Kissane, B., & Kemp, M. (2014 A model for the educational role of calculators. In: 19th Asian Technology Conf in Mathematics, 26-30 November 2014, Yogyakarta, Indonesia. https://researchrepository.murdoch.edu.au/id/eprint/24816/

Kivunja, C. (2018). Distinguishing between theory, theoretical framework, and conceptual framework: A systematic review of lessons from the field. *International Journal of Higher Education*, *7*(6), 44-53.

Kivunja, C., & Kuyini, A. B. (2017). Understanding and applying research paradigms in educational contexts. *International Journal of Higher Education, 6*(5), 26-41.

Kiwanuka, H. N., Van Damme, J., Van Den Noortgate, W., Anumendem, D. K., & Namusisi, S. (2015). Factors affecting Mathematics achievement of first-year secondary school students in Central Uganda. *South African Journal of Education*, *35*(3).

Korstjens, I., & Moser, A. (2018). Series: Practical guidance to qualitative research. Part 4: Trustworthiness and publishing. *European Journal of General Practice, 24*(1), 120-124.

Koyunkaya, M. Y. (2016). Mathematics education graduate students' understanding of trigonometric ratios. *International Journal of Mathematical Education in Science and Technology, 47*(7), 1028-1047.

Krusenvik L. (2016). Using case studies as a scientific method: Advantages and disadvantages. [Unpublished thesis]. Halmstad University.

Kuzle, A. (2013). Patterns of Metacognitive Behaviour During Mathematics Problem Solving in a Dynamic Geometry Environment. *International Journal of Mathematics Education, 8*(1), 20-39.

Lai, E. R. (2011). Metacognition: A literature review. *Always learning: Pearson research report*, *24*, 1-40.

Leung, L. (2015). Validity, reliability, and generalizability in qualitative research. *Journal of family medicine and primary care, 4*(3), 324.

Lincoln, Y. & Guba, E. (2005). Techniques for Collecting and Analyzing Data, in A.C. Rusaw (ed.). Transforming the Character of Public Organizations: Techniques for Change Agents, pp. 27-40.

Lincoln, Y. S., & Guba, E. G. (1985). *Naturalistic inquiry*, 75. Sage.

Lodge, J. M., Kennedy, G., Lockyer, L., Arguel, A., & Pachman, M. (2018). Understanding difficulties and resulting confusion in learning: An integrative review. In *Frontiers in Education* (Vol. 3, p. 49). Frontiers Media.

Madonsela, P. S., Ndlovu, Z., & Brijlall, D, D. (2020). An APOS exploration of students' solving three dimensional problems using Trigonometry. *Technology Reports of Kansai University*, 62(4).

Maharaj, A. (2008). Some insights from research literature for teaching and learning mathematics. *South African Journal of Education*, *28*(3), 401-414.

Maharaj, A. (2014). An APOS analysis of natural science students' understanding of integration. REDIMAT. *Journal of Research in Mathematics Education, 3*(1), 54-73.

Makonye, J. P., & Fakude, J. (2016). A study of errors and misconceptions in the learning of addition and subtraction of directed numbers in grade 8. *SAGE Open*, *6*(4), 2158244016671375.

Mampane, R., & Bouwer, C. (2011). The influence of township schools on the resilience of their learners. *South African Journal of Education*, *31*(1).

Martínez-Planell, R., & Delgado, A. C. (2016). The unit circle approach to the construction of the sine and cosine functions and their inverses: An application of APOS theory. *The Journal of Mathematical Behavior*, *43*, 111-133.

Maryati, I., & Priatna, N. (2018). Analysis of statistical misconception in terms of statistical reasoning. In *Journal of Physics: Conference Series* (Vol. 1013, No. 1, p. 012206). IOP Publishing.

Mavhunga, F., Chigonga, B., Kibirige, I. & Ndlovu, E. (2015). *The State of Mathematics, Science and Technology Education at Five Schools in Limpopo Province.* International Conference Proceedings of the Institute for Science and Technology Education (ISTE). Mopani Camp in Kruger National Park. Limpopo, South Africa.

May, V., & Courtney, S. (2016). Developing meaning in trigonometry. *Illinois Mathematics Teacher*, *63*(1), 25-33.

Mbewe, T. L. (2013). *Misconceptions and errors in Algebra at grade 11 levael: The case of two selected secondary schools in Petauke District: University of Zambia.* [Unpublished doctoral dissertation].

Mbugua, Z.K., Kibet, K., Muthaa, G.M. and Nkonke, G.R. (2012) Factors Contributing to Students Poor Performance in Mathematics at Kenya Certificate of Secondary Education in Kenya: A Case of Baringo County, Kenya. *American International Journal of Contemporary Research, 2*, 87-91.

McGhie, V., Venter, A., & Dos Reis, K. (2020). Business Education Learners in the Further Education and Training Phase: Towards the Development of a South African Readiness Model to Strengthen Learners' Academic Performance. *Journal of Student Affairs in Africa*, *8*(2), 17-29.

McLeod, S. (2016). *Psychology Perspectives | Simply Psychology*. [online] Simplypsychology.org. Available at: http://www.simplypsychology.org/perspective.html: accessed in January 2022

Mensah, F. S. (2017). Ghanaian Senior High School students' error in learning of trigonometry. *International Journal of Environmental and Science Education*, *12*(0), 8.

Mevarech, Z. R., & Kramarski, B. (1997). IMPROVE: A multidimensional method for teaching mathematics in heterogeneous classrooms. *American educational research journal*, *34*(2), 365-394.

Mishra, L. (2020). Conception and misconception in teaching arithmetic at primary level. *Journal of Critical Reviews*, *7*(5), 936-939.

Mji, A., & Makgato, M. (2006). Factors associated with high school learners' poor performance: a spotlight on mathematics and physical science. *South African Journal of Education*, *26*(2), 253-266.

Mlachila, M.M.; Moeletsi, T. (2019). Struggling to Make the Grade: A Review of The Causes and Consequences of the Weak Outcomes of South Africa's Education System; International Monetary Fund: Johannesburg, South Africa, 2019

Mogashoa, T., & Mboweni, L. (2017). Challenges and factors contributing to learner absenteeism in selected primary schools in Bohlabela district of the Mpumalanga Province. *Journal of Educational Studies*, *16*(1), 21-33.

Mohyuddin, R. G., & Khalil, U. (2016). Misconceptions of Students in Learning Mathematics at Primary Level. *Bulletin of Education and Research*, *38*(1), 133-162.

Moreeng, B. B., & Du Toit, E. (2013). The powerful learning environment and history learners in the Free State Province. *Yesterday and today*, (9), 00-00.

Morgan, H. (2022). Conducting a qualitative document analysis. *The Qualitative Report*, *27*(1), 64-77.

Mudrikah, A. (2016). Problem-Based Learning Associated by Action-Process-Object-Schema (APOS) Theory to Enhance Students' High Order Mathematical Thinking Ability. *International Journal of Research in Education and Science*, *2*(1), 125-135.

Mullis, I. V., Martin, M. O., & Foy, P. (2011). The impact of reading ability on TIMSS mathematics and science achievement at the fourth grade: An analysis by item reading demands. *TIMSS and PIRLS*, 67-108.

Mullis, I. V., Martin, M. O., Foy, P., Kelly, D. L., & Fishbein, B. (2019). TIMSS 2019 international results in mathematics and science. *Retrieved from Boston College, TIMSS & PIRLS International Study Center website: https://timssandpirls. bc. edu/timss2019/international results*.

Mulwa, E. C. (2014). The Role of the Language of Mathematics in Students' Understanding of Number Concepts in Eldoret Municipality, Kenya. *International Journal of Humanities and Social Science*, *4*(3), 264-274.

Mulwa, E. C. (2015). Difficulties Encountered by Students in the Learning and Usage of Mathematical Terminology: A Critical Literature Review. *Journal of Education and Practice*, *6*(13), 27-37.

Munna, A. S., & Kalam, M. A. (2021). Teaching and learning process to enhance teaching effectiveness: a literature review. *International Journal of Humanities and Innovation (IJHI)*, *4*(1), 1-4.

Mupa, P., & Chinooneka, T. I. (2015). Factors Contributing to Ineffective Teaching and Learning in Primary Schools: Why Are Schools in Decadence? *Journal of Education and Practice*, *6*(19), 125-132.

Musonda, A. (2017). Algebraic competences and emotional intelligence of first year Bachelor of Science in Mathematics and Science Education students at the Copperbelt University in Zambia. *Tuning Journal for Higher Education*, *5*(1), 171-195.

Mutambara, L. H. N., & Bansilal, S. (2022). Misconceptions and Resulting Errors Displayed by in Service Teachers in the Learning of Linear Independence. *International Electronic Journal of Mathematics Education*, *17*(4).

Mutodi, P., Mosimege, M., & Maphutha, K. (2023). Learners' Misconceptions and Errors when Solving Inequalities. *Mathematics Education Journal*, *7*(2), 136-158.

Mvududu, N., & Thiel-Burgess, J. (2012). Constructivism in practice: The case for English language learners. *International Journal of Education*, *4*(3), 108-118.

Nabie, M. J., Akayuure, P., Ibrahim-Bariham, U. A., & Sofo, S. (2018). Trigonometric Concepts: Pre-Service Teachers' Perceptions and Knowledge. *Journal on Mathematics Education*, *9*(1), 169-182.

Naidoo, J., & Govender, R. (2014). Exploring the use of a dynamic online software programme in the teaching and learning of trigonometric graphs. *Pythagoras*, *35*(2), 1-13.

Naidoo, K., & Naidoo, R. (2016). From 2D to MD: Implications for teaching secondary school undergraduate students in a pre-service mathematics course at a university of technology. *International Journal of Engineering Technology Science and Research, 3*(9), 46- 50.

Narayan, R., Rodriguez, C., Araujo, J., Shaqlaih, A., & Moss, G. (2013). Constructivism: Constructivist learning theory. In B. J. Irby, G. Brown, R. Lara-Alecio & S. Jackson (Eds), *The handbook of educational theories* (pp. 169–183). Information Age Publishing.

Nath, K., & Veneesha, V. (2009). Language issues in teaching and learning of mathematics. ERIC online submission.

National Council of Teachers of Mathematics. (2012). *Research Pre-Session*. National Council of Teachers of Mathematics. NCTM.

National Research Council. (2000). *How People Learn: Brain, Mind, Experience, and School: Expanded Edition.* The National Academies Press.

National Research Council. (2012). *Education for life and work: Developing transferable knowledge and skills in the 21st century*. National Academies Press.

Ndlovu, A. Z. (2012). *Grade 12 Learners' Construction of knowledge of Optimisation Problems.* [Unpublished master's dissertation]. University of KwaZulu Natal.

Ndlovu, M., & Mji, A. (2012). Alignment between South African mathematics assessment standards and the TIMSS assessment frameworks. *Pythagoras*, *33*(3), 1-9.

Ndlovu, Z. (2014). *Exploring Pre-service Teachers Mental Constructions of Matrix Algebra Concepts: A South African case study.* [Unpublished doctoral thesis]. University of KwaZulu Natal.

Ndlovu, Z., & Brijlall, D. (2015). Pre-service teachers' mental constructions of concepts in matrix algebra. *African Journal of Research in Mathematics, Science and Technology Education*, *19*(2), 156-171.

Ndlovu, Z., & Brijlall, D. (2016). Pre-service mathematics teachers' mental constructions of the determinant concept. *International Journal of Educational Sciences, 14*(1-2), 145-156.

Ndlovu, Z., Amin, N., & Samuel, M. A. (2017). Examining pre-service teachers' subject matter knowledge of school mathematics concepts. *Journal of Education,* 70, 46-72.

Ndlovu, Z., (2017). Grade 12 Mathematics learners' narratives of the effectiveness of collaborative in mathematics classrooms. *International Journal of Sciences and Research,73,* 50-64.

Ngcobo, A. Z., Madonsela, S. P., & Brijlall, D. (2019). The teaching and learning of trigonometry. *The Independent Journal of Teaching and Learning*, *14*(2), 72-91.

Ngobeni, N. R., Chibambo, M. I., & Divala, J. J. (2023). Curriculum transformations in South Africa: some discomforting truths on interminable poverty and inequalities in schools and society. In *Frontiers in Education* (Vol. 8, p. 1132167). Frontiers.

Ngu, B. H., & Phan, H. P. (2020). Learning to solve trigonometry problems that involve algebraic transformation skills via learning by analogy and learning by comparison. *Frontiers in Psychology*, *11*, 558773.

Njoroge, G. (2022). Demystifying Mathematics: handling learning difficulties in mathematics among low achievers in Kenyan schools. *Journal of Language, Technology & Entrepreneurship in Africa*, *13*(1), 75-96.

Noble, H., & Smith, J. (2015). Issues of validity and reliability in qualitative research. *Evidence-based nursing*, *18*(2), 34-35.

Nurmeidina, R., & Rafidiyah, D. (2019). Analysis of Students' Difficulties in Solving Trigonometry Problems. In *PROFUNEDU 2019: Proceedings of the 4th Progressive and Fun Education International Conference, Profunedu 2019, 6-8 August 2019, Makassar, Indonesia* (p. 9). European Alliance for Innovation.

Nyumba, T., Wilson, K., Derrick, C. J., & Mukherjee, N. (2018). The use of focus group discussion methodology: Insights from two decades of application in conservation. *Methods in Ecology and evolution*, *9*(1), 20-32.

OECD. (2014). *PISA 2012 Results: Creative Problem Solving. Students' skills in tackling real-life problems.* (Volume V). OECD Publishing. https://doi.org/10.1787/9789264208070-en

Ojose, B. (2015). Students' misconceptions in mathematics: Analysis of remedies and what research says. *Ohio Journal of School Mathematics, 72,* 30-34.

Olivier, A. (1989). Handling pupils' misconceptions. *Pythagoras, 21,* 9–19.

Organisation for Economic Co-Operation and Development (OECD) (2016). *PISA 2015 Results in Focus*. Retrieved from http://www.oecd.org/pisa/pisa-2015-results-in-focus.pdf.

Orhun, N. (2010). The gap between real numbers and trigonometric relations. *Quaderni di Ricerca in Didattica*, *20*, 175-184.

Orhun, N. (2013). The effects of learning styles on high school students' achievement on a mathematics course. *Educational research and reviews*, *8*(14), 1158-1165.

Orhun, N. (2015) Students' Mistakes and Misconceptions on Teaching of Trigonometry. *Journal of Curriculum Studies*, *32*(6), 797-820.

Ortiz, E. (2016). The problem-solving process in a mathematics classroom. *Transformations*, *1*(1), 4-13.

Osanloo, A., & Grant, C. (2016). Understanding, selecting, and integrating a theoretical framework in dissertation research: Creating the blueprint for your "house". *Administrative Issues Journal: Connecting Education, Practice, and Research*, *4*(2), 7.

Özreçberoğlu, N., & Çağanağa, Ç. K. (2018). Making it count: Strategies for improving problem-solving skills in mathematics for students and teachers' classroom management. *Eurasia Journal of Mathematics, Science and Technology Education*, *14*(4), 1253-1261.

Palinkas, L. A., Horwitz, S. M., Green, C. A., Wisdom, J. P., Duan, N., & Hoagwood, K. (2015). Purposeful sampling for qualitative data collection and analysis in mixed method implementation research. *Administration and policy in mental health and mental health services research*, *42*, 533-544.

Patton, M. Q. (1990). *Qualitative evaluation and research methods*. SAGE Publications, inc.

Perkins, D. N. (1993). Person-plus: A distributed view of thinking and learning. *Distributed cognitions: Psychological and educational considerations*, 88-110.

Permata, L. D., Kusmayadi, T. A., & Fitriana, L. (2018). Mathematical problem-solving skills analysis about word problems of linear program using IDEAL problem solver. In *Journal of Physics: Conference Series* (Vol. 1108, No. 1, p. 012025). IOP Publishing.

Permatasari, D. (2016). The role of productive struggle to enhance learning mathematics with understanding. In *Proceedings of 3rd International Conference on Research, Implementation and Education of Mathematics and Science* (pp. 95-100).

Phonapichat, P., Wongwanich, S., & Sujiva, S. (2014). An analysis of elementary school students' difficulties in mathematical problem solving. *Procedia-social and behavioral sciences*, *116*, 3169-3174.

Piaget, J. (1964). Cognitive development in children: Piaget development and learning. *Journal, of Research in Science Teaching*, *2*, 176-186.

Piaget, J. (1967). On the development of memory and identity.inc

Piaget, J. (1985). *The equilibration of cognitive structures: The central problem of intellectual development*. University of Chicago Press.

Piaget, J. (1995). *Sociological studies*. Psychology Press.

Piaget, J., & García, R. (1989). *Psicogénesis e historia de la ciencia*. Siglo Veintiuno Editores, S.A.

Pitsoe, V. J., & Maila, W. M. (2012). Towards constructivist teacher professional development. *Journal of Social Sciences*, *8*(3), 318-324.

Plomp, T. (2013). Educational design research: An introduction. 11-50.inc

Pournara, C., Adler, J., Pillay, V., & Hodgen, J. (2015). Can improving teachers' knowledge of mathematics lead to gains in learners' attainment in Mathematics? *South African Journal of Education*, *35*(3), 1-10.

Prediger, S., Wilhelm, N., Büchter, A., Gürsoy, E., & Benholz, C. (2018). Language proficiency and mathematics achievement. *Journal für Mathematik-Didaktik*, *39*(1), 1-26.

Rafiah, H., & Ekawati, A. (2017). Misconceptions of the Students with High Mathematical Creative Thinking Level in Solving the Geometric Shapes Problems. In *5th SEA-DR (Southeast Asia Development Research) International Conference 2017 (SEADRIC 2017)* (pp. 155-158). Atlantis Press.

Rapley, T., & Rees, G. (2018). Collecting documents as data. *FLICK, Uwe. The sage handbook of qualitative collection.* Sage. 378-391.

Rashidov, A. S. (2022). Using of problem educational technologies in the development of students' creative and logical thinking skills. *Berlin Studies Transnational Journal of Science and Humanities*, *2*(1.5 Pedagogical sciences). 1-13

Ravitch, S. M., & Carl, N. M. (2016). *Qualitative research: Bridging the conceptual, theoretical, and methodological*. Sage.

Rehman, A. A., & Alharthi, K. (2016). An introduction to research paradigms. *International Journal of Educational Investigations*, *3*(8), 51-59.

Reid O'Connor, B., & Norton, S. (2022). Exploring the challenges of learning quadratic equations and reflecting upon curriculum structure and implementation. *Mathematics Education Research Journal*, 1-26.

Rittle-Johnson, B., & Schneider, M. (2015). Developing conceptual and procedural knowledge of mathematics. In R. C. Kadosh & A. Dowker (Eds.), *The Oxford handbook of numerical cognition* (pp. 1118–1134). Oxford University Press.

Roberts, D. A., Yaida, S., & Hanin, B. (2022). *The principles of deep learning theory*. Cambridge University Press.

Robertson, S. A., & Graven, M. (2019). Language as an including or excluding factor in mathematics teaching and learning. *Mathematics Education Research Journal*, *32*(1), 77-101.

Robinson KH (2013) *Innocence, Knowledge and the Construction of Childhood: The Contradictory Nature of Sexuality and Censorship in Children's Contemporary Lives.* Routledge.

Rohid, N. (2019). Students' Mathematical Communication Skills (MCS) in Solving Mathematics Problems: A Case in Indonesian Context. *Anatolian Journal of Education*, *4*(2), 19-30.

Rohimah, S. M., & Prabawanto, S. (2019). Students' difficulty identification in completing the problem of equation and trigonometry identities. I*nternational Journal of Trends in Mathematics Education Research, 2*(1), 34-36.

Sarwadi, H. Roselizawati. H., & Shahrill, M. (2014). Understanding students' mathematical errors and misconceptions: The case of year 11 repeating students. *Mathematics Education Trends and Research*, *2014*, 1-10.

Saxena, S. (2015). *Geometry and Trigonometry.* Britannica Educational Publishing. Available at: https://worldcat.org/title/1042164904.

Seethaler, P. M., Fuchs, L. S., Star, J. R., & Bryant, J. (2011). The cognitive predictors of computational skill with whole versus rational numbers: An exploratory study. *Learning and Individual Differences*, *21*(5), 536-542.

Simons, M. D. (2016). An Ethnomethodological analysis of learners' ways of working in a high-stakes Grade 12 Mathematics National Senior Certificate (NSC) Examination: The case of Trigonometry.

Singh, J., Sajid, M., Yadav, C. S., Singh, S. S., & Saini, M. (2022). A Novel Deep Neural-based Music Recommendation Method considering User and Song Data. In *2022 6th International Conference on Trends in Electronics and Informatics (ICOEI)* (pp. 1-7). IEEE.

Sireci, S. G. (2016). On the validity of useless tests. *Assessment in Education: Principles, Policy & Practice, 23*(2), 226-235.

Sirkiä, T., & Sorva, J. (2012). Exploring programming misconceptions: an analysis of student mistakes in visual program simulation exercises. In *Proceedings of the 12th Koli calling international conference on computing education research* (pp. 19-28).

Siyepu, S. W. (2012). *Analysis of errors in derivatives of trigonometric functions: A case study in an extended curriculum programme* [Unpublished doctoral dissertation]. University of the Western Cape.

Siyepu, S. W. (2015). Analysis of errors in derivatives of trigonometric functions. *International journal of STEM education*, *2*, 1-16.

Starman, A. B. (2013). The case study as a type of qualitative research. *Journal of Contemporary Educational Studies/Sodobna Pedagogika*, *64*(1).

Stenbacka, C. (2001). Qualitative research requires quality concepts of its own. *Management decision*, *39*(7), 551-556.

Stols, G. (2011). Geogebra (Open-Source Software) for High School Teachers. *South Africa (AMESA)*, 32.

Stols, G. (2015). Important to use technology in pre-service teachers'geometry courses? In *INTED2015 Proceedings* (pp. 5165-5165). IATED.

Stone, H. L. (1970). Probability model for estimating three-phase relative permeability. *Journal of petroleum technology*, *22*(02), 214-218.

Stylianides, A. J., & Stylianides, G. J. (2007). Learning mathematics with understanding: A critical consideration of the learning principle in the principles and standards for school mathematics. *The Mathematics Enthusiast*, *4*(1), 103-114.

Sujarwo, M., Sudiyanto, F., & Kurniawan, S. B. (2020). Analysis on Mathematics Learning Misconceptions of the Second-Grade Students of Elementary School in Addition and

Subtraction Integer Topics. In *3rd International Conference on Learning Innovation and Quality Education (ICLIQE 2019)* (pp. 757-764). Atlantis Press.

Suparno, P. (2013). *Misconceptions and Changes in the Concept of Physical Education*. Gramedia Widiasarana Indonesia.

Tall, D. (1999). Reflections on APOS theory in elementary and advanced mathematical thinking. In *Proceedings of the 23rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 148-155).

Tall, D. (2008). The transition to formal thinking in mathematics. *Mathematics Education Research Journal*, *20*(2), 5-24.

Taylor, N. (2021). The dream of Sisyphus: Mathematics education in South Africa. *South African Journal of Childhood Education*, *11*(1), 1-12.

Taylor, P. C., & Medina, M. (2011). Educational research paradigms: From positivism to pluralism. *College Research Journal*, *1*(1), 1-16.

Teegavarapu, S., Summers, J. D., & Mocko, G. M. (2008, January). Case study method for design research: A justification. In *International design engineering technical conferences and computers and information in engineering conference* (Vol. 43284, pp. 495-503).

Telima, A. (2011). Problems of teaching and learning of geometry in secondary schools in River State, Nigeria. *International Journal of Emerging Science, 1*(2). 143-152.

Thanh, N. C., & Thanh, T. (2015). The interconnection between interpretivist paradigm and qualitative methods in education. *American Journal of Educational Science, 1*(2), 24- 27.

Thanheiser, E., & Melhuish, K. (2023). Teaching routines and student-centered mathematics instruction: The essential role of conferring to understand student thinking and reasoning. *The Journal of Mathematical Behavior, 70,* 101032.

Thompson, P. W. (2008). Conceptual analysis of mathematical ideas: Some spadework at the foundations of mathematics education. In *Proceedings of the annual meeting of the*

International Group for the Psychology of Mathematics Education (Vol. 1, pp. 31-49). PME Morelia, Mexico.

Thorogood, N., & Green, J. (2018). Qualitative methods for health research. *Qualitative methods for health research*, 1-440.

Tong, D. H., & Loc, N. P. (2017). students' errors in solving mathematical word problems and their ability in identifying errors in wrong solutions. *European Journal of Education Studies*.

Trigueros, R., Juan, M., & Sandoval, F. (2017). Qualitative and quantitative research instruments: Research tools. *Handbook of research methods and applications in political science*, *2*.inc

Umam, K., & Susandi, A. D. (2022). Critical Thinking Skills: Error Identifications on Students' with APOS Theory. *International Journal of Evaluation and Research in Education*, *11*(1), 182-192.

Usiskin, Z. (2015). What does it mean to understand some mathematics? In *Selected regular lectures from the 12th international congress on mathematical education* (pp. 821-841). Springer International Publishing.

Usman, M. H., & Hussaini, M. M. (2017). Analysis of students' error in learning of trigonometry among senior secondary school students in Zaria Metropolis, Nigeria. *IOSR Journal of Mathematics*, *13*(2), 1-4.

Van de Walle, J.A., Karp, K.S., & Bay-Williams, J.M. (2016) Elementary and middle school mathematics: Teaching developmentally (9^{th} ed.). Pearson [e-text].

van der Walt, M. (2009). Study orientation and knowledge of basic vocabulary in Mathematics in the primary school. *Suid-Afrikaans Tydskrif vir Natuurwetenskap en Tegnologie/South African Journal of Science and Technology*, *28*(4), 378-392.

Van Eeuwijk, P., & Angehrn, Z. (2017). How to conduct a Focus Group Discussion (FGD). Methodological Manual.

Venkat, H., & Adler, J. (2012). Coherence and connections in teachers' mathematical discourses in instruction. *Pythagoras*, *33*(3), 1-8.

Voskoglou, M. G. (2013). An application of the APOS/ACE approach in teaching the irrational numbers. *Journal of Mathematical Sciences and Mathematics Education*, *8*(1), 30-47.

Vygotsky, L. (1998). *Mind in society: The development of higher psychological processes.inc*

Vygotsky, L. S. (1962). *The Development of Scientific Concepts in Childhood.inc.*

Vygotsky, L. S. (1978). Mind and society: The development of higher psychological processes. Harvard University Press

Walida, S., & Hasana, S. N. (2020). The Identification of Students' Misconceptions in Mathematical Induction. *Journal of Education and Learning Mathematics Research (JELMaR)*, *1*(2), 50-57.

Weller, K., Arnon, I., & Dubinsky, E. (2009). Preservice teachers' understanding of the relation between a fraction or integer and its decimal expansion. *Canadian Journal of Science, Mathematics and Technology Education*, *9*(1), 5-28.

Weyer, R. S. (2010). APOS theory as a conceptualisation for understanding mathematics learning. *Summation: Mathematics and computer science scholarship at Ripon*, *3*, 9-15.

Winarso, W. (2017). A case study of misconceptions students in the learning of mathematics; The concept limit function in high school. *Jurnal Riset Pendidikan Matematika*, *4*.

Yin, R. K. (2003). Designing case studies. *Qualitative research methods*, *5*(14), 359-386.

Yin, R. K. (2009). *Case study research: Design and methods* (Vol. 5). sage.

Yin, R. K. (2017). *Case study research and applications: Design and methods.* Sage publications.

Yin, R. K. (2018). *Case study research and applications* (Vol. 6). Thousand Oaks, CA: Sage.

Zainal, Z. (2007). Case study as a research method. *Jurnal Kemanusiaan, 9,* 1-6.

Zawojewski, J. S., Magiera, M., & Lesh, R. (2013). A proposal for a problem-driven mathematics curriculum framework. *The Mathematics Enthusiast*.

Zulkiply, N. (2009). Metacognition and its relationship with students' academic performance. *The International Journal of Learning*, *15*(11), 97-106.

APPENDICES

APPENDIX A: UKZN ETHICAL CLEARANCE LETTER

UNIVERSITY OF KWAZULU-NATAL
INYUVESI YAKWAZULU-NATALI

31 July 2021

Mr Njabulo Happyboy Dube (210500469)
School Of Education
Edgewood Campus

Dear Mr Dube,

Protocol reference number: HSSREC/00003000/2021
Project title: Exploration of Grade 11 learners' mental constructions and difficulties in learning and solving trigonometric equations: a case of one school in Umlazi district
Degree: PhD

Approval Notification – Expedited Application

This letter serves to notify you that your application received on 21 June 2021 in connection with the above, was reviewed by the Humanities and Social Sciences Research Ethics Committee (HSSREC) and the protocol has been granted **FULL APPROVAL**.

Any alteration/s to the approved research protocol i.e. Questionnaire/Interview Schedule, Informed Consent Form, Title of the Project, Location of the Study, Research Approach and Methods must be reviewed and approved through the amendment/modification prior to its implementation. In case you have further queries, please quote the above reference number. PLEASE NOTE: Research data should be securely stored in the discipline/department for a period of 5 years.

This approval is valid until 31 July 2022.
To ensure uninterrupted approval of this study beyond the approval expiry date, a progress report must be submitted to the Research Office on the appropriate form 2 - 3 months before the expiry date. A close-out report to be submitted when study is finished.

All research conducted during the COVID-19 period must adhere to the national and UKZN guidelines.

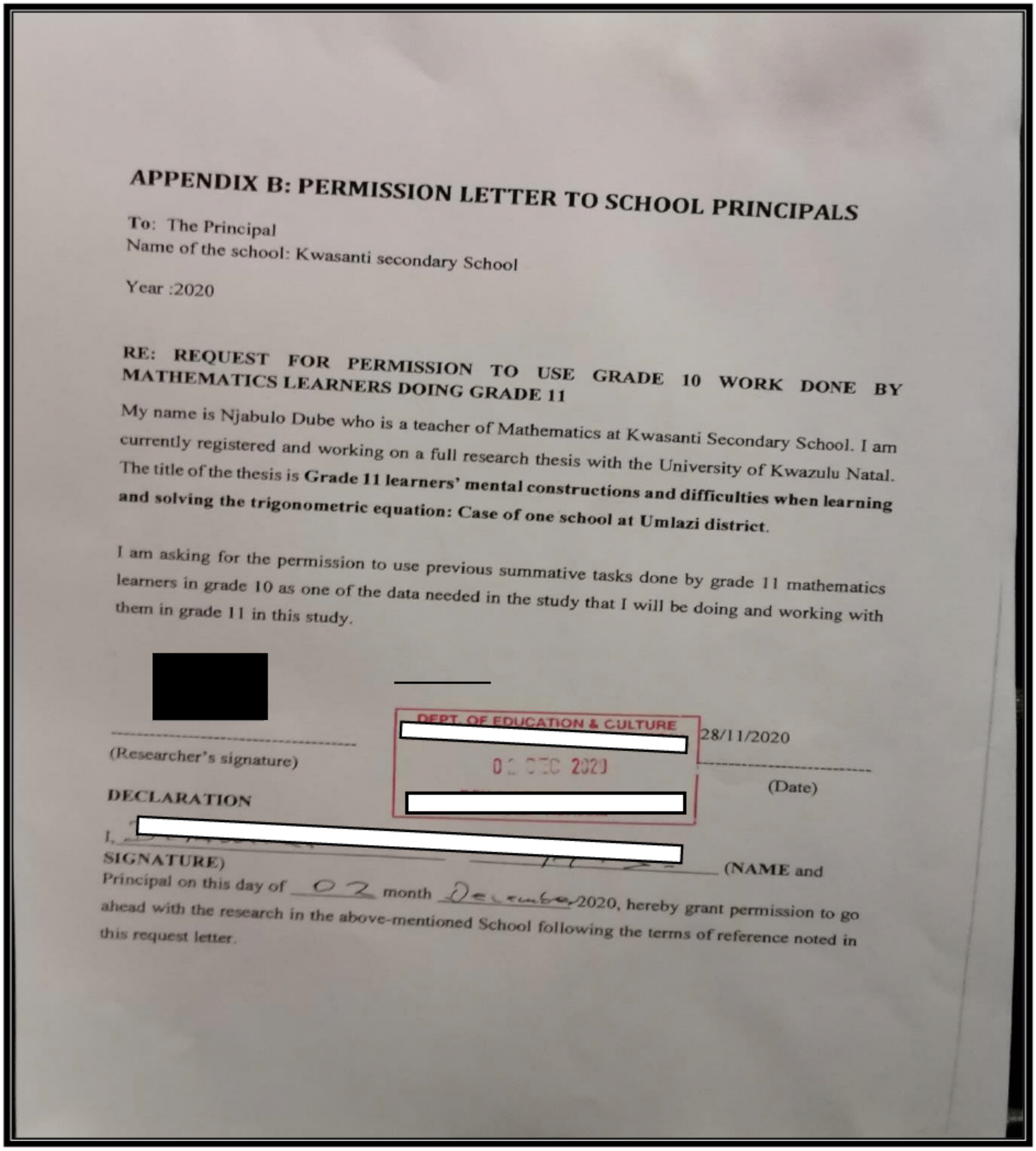
HSSREC is registered with the South African National Research Ethics Council (REC-040414-040).

Yours sincerely,

[redacted]

Professor Dipane Hlalele (Chair)

APPENDIX B: SAMPLE OF LETERS FROM THE PRINCIPAL



APPENDIX B: PERMISSION LETTER TO SCHOOL PRINCIPALS

To: The Principal
Name of the school: Kwasanti secondary School

Year :2020

RE: REQUEST FOR PERMISSION TO USE GRADE 10 WORK DONE BY MATHEMATICS LEARNERS DOING GRADE 11

My name is Njabulo Dube who is a teacher of Mathematics at Kwasanti Secondary School. I am currently registered and working on a full research thesis with the University of Kwazulu Natal. The title of the thesis is **Grade 11 learners' mental constructions and difficulties when learning and solving the trigonometric equation: Case of one school at Umlazi district.**

I am asking for the permission to use previous summative tasks done by grade 11 mathematics learners in grade 10 as one of the data needed in the study that I will be doing and working with them in grade 11 in this study.

[redacted]

(Researcher's signature)

DEPT. OF EDUCATION & CULTURE
02 DEC 2020

28/11/2020
(Date)

DECLARATION

I, ______ (NAME and SIGNATURE) Principal on this day of 02 month December 2020, hereby grant permission to go ahead with the research in the above-mentioned School following the terms of reference noted in this request letter.

APPENDIX B: PERMISSION LETTER TO SCHOOL PRINCIPALS

To: The Principal
Name of the school: Kwasanti High school

Year : 2020

RE: REQUEST FOR PERMISSION TO CONDUCT RESEARCH IN YOUR SCHOOL

My name is Njabulo Dube who is a teacher of Mathematics at [redacted] Secondary School. I am currently registered and working on a full research thesis with the University of Kwazulu Natal. The title of the thesis is: **Grade 11 learners' mental constructions and difficulties when learning and solving the trigonometric equation: Case of one school at Umlazi district.**

Learners are requested to assist in this research project. The study will use interviews, and test. Responses will be treated with confidentiality and pseudonyms will be used instead of the actual names. Participants will be contacted in time for interviews, and they will be randomly selected to participate in this study. Participation will always remain voluntary which means that participant have a choice to withdraw from the study for any reason, anytime if they so wish without any penalties.

[redacted]

DEPT. OF EDUCATION & CULTURE

07 AUG 2020

05/08/2020

(Researcher's signature) (Date)

DECLARATION

I, [redacted] **(NAME and SIGNATURE)**

Principal on this day of 7th month August 2020, hereby grant permission to go ahead with the research in the above-mentioned School following the terms of reference noted in this request letter.

APPENDIX C: KZN DEPARTMENT OF EDUCATION CLEARANCE LETTER FOR CONDUCTING RESEARCH IN SCHOOLS

KWAZULU-NATAL PROVINCE
EDUCATION
REPUBLIC OF SOUTH AFRICA

OFFICE OF THE HEAD OF DEPARTMENT

Private Bag X9137, PIETERMARITZBURG, 3200
Anton Lembede Building, 247 Burger Street, Pietermaritzburg, 3201
Tel: 033 392 1063
Email: Phindile.duma@kzndoe.gov.za

Enquiries: Phindile Duma **Ref.**:2/4/8/1750

Mr NH Dube
PO Box 07
ST WENDOLINS
3609

Dear Mr Dube

PERMISSION TO CONDUCT RESEARCH IN THE KZN DoE INSTITUTIONS

Your application to conduct research entitled: **"GRADE 11 LEARNERS' MENTAL CONSTRUCTIONS AND DIFFICULTIES IN LEARNING AND SOLVING THE TRIGONOMETRIC EQUATION: CASE OF ONE SCHOOL AT UMLAZI DISTRICT"**, in the KwaZulu-Natal Department of Education Institutions has been approved. The conditions of the approval are as follows:

1. The researcher will make all the arrangements concerning the research and interviews.
2. The researcher must ensure that Educator and learning programmes are not interrupted.
3. Interviews are not conducted during the time of writing examinations in schools.
4. Learners, Educators, Schools and Institutions are not identifiable in any way from the results of the research.
5. A copy of this letter is submitted to District Managers, Principals and Heads of Institutions where the Intended research and interviews are to be conducted.
6. The period of investigation is limited to the period from 31 May 2021 to 31 August 2023.
7. Your research and interviews will be limited to the schools you have proposed and approved by the Head of Department. Please note that Principals, Educators, Departmental Officials and Learners are under no obligation to participate or assist you in your investigation.
8. Should you wish to extend the period of your survey at the school(s), please contact Miss Phindile Duma at the contact numbers above.
9. Upon completion of the research, a brief summary of the findings, recommendations or a full report/dissertation/thesis must be submitted to the research office of the Department. Please address it to The Office of the HOD, Private Bag X9137, Pietermaritzburg, 3200.
10. Please note that your research and interviews will be limited to schools and institutions in KwaZulu-Natal Department of Education.

[redacted]

Dr. EV Nzama
Head of Department: Education
Date: 01 June 2021

APPENDIX D: LEANER CONSENT LETTER

Science and Technology Cluster, School of Education,
College of Humanities, University of KwaZulu-Natal,
Edgewood Campus, KwaZulu-Natal

Date

Dear Learner

INFORMED CONSENT LETTER FOR MATHEMATICS LEARNERS

My name is Mr. N. H. Dube I am a Doctor of Philosophy (PhD) student from the Science and Technology Cluster, School of Education, College of Humanities, University of KwaZulu-Natal. I am conducting research titled **'Exploration of Grade 11 learners' mental constructions and difficulties in learning and solving trigonometric equations: a case of one school in Umlazi district'**

In view of the foregoing, I intend to explore learner's mental construction of knowledge and difficulties they face when they are learning and solving trigonometric equations.

The objectives of the research are as follows:

1. To explore Grade 11 learner's mental construction of knowledge when leaning and solving trigonometric equations.

2. To determine the difficulties that hinder the construction of knowledge among grade 11 learners when learning and solving trigonometric equations.

3. To explore the extent, the preliminary genetic decomposition, explain the learners' mental construction of trigonometric equations

Document analysis would involve analyzing learner's responses to classroom tasks done within the lesson and during assessment. Also the observation will also be done in the class during discussion. You are invited to please participate in the study because you are a student who is studying the Mathematics in Grade 11 as the study focusing in the trigonometric equations from grade 11 mathematics. To gather the information, I am interested in requesting you to participate in this project by reflecting critically on a classroom tasks done within the lesson and during assessment. I will also ask you some questions individual interview, each of 25-30 minutes' duration.

This study has been ethically reviewed and approved by the UKZN Humanities and Social Sciences Research Ethics Committee (approval number___).

Please note that:

- Your participation is voluntary. If you do not participate you will not be penalized in any way. No marks will be deducted from your project if you decline to participate.

- Your confidentiality is guaranteed as your inputs will not be attributed to you in person, but reported only as a population member opinion. The individual interviews (1 of each) will last for about 25-30 minutes and may be split depending on your preference.

- Any information given by you cannot be used against you, and the collected data will be used for purposes of this research only.

- Data will be in the form of interview transcripts and document analysis, and will be stored in secure storage and destroyed by shredding after 5 years. Digitally recorded data will be deleted after five years.

- You have a choice to participate, not participate or stop participating in the research. You will not be penalized for taking such an action.

- Your involvement is purely for academic purposes only, and there are no financial benefits involved. However, it is expected that you will gain more knowledge for exam purpose in the trigonometric equations.

Thank you.

Yours faithfully

……………………..

My contact details are as follows:

Email: njabulohappyboy.dube@gmail.com

Cell phone: 0614470809

My supervisor is Dr Zanele Ngcobo. She is a lecturer in the School of Education, College of Humanities, Edgewood Campus, University of KwaZulu-Natal.

My supervisor's contact details are:

Email: NgcoboA2@ukzn.ac.za

Phone number:0724011275

You may also contact the Research Office at:

University of KwaZulu-Natal

Humanities and Social Sciences Research Ethics

Govan Mbeki Centre

Tel +27312604557

Email: HSSREC@ukzn.ac.za

Thank you for reading this document about this research.

DECLARATION OF CONSENT

I ………………………………………………………………………… (Full names of participant) hereby confirm that I have been informed about the study entitled '**Exploration of Grade 11 learners' mental constructions and difficulties in learning and solving trigonometric equations: a case of one school in Umlazi district**

'. I understand the contents of this document and the nature of the research project, and I consent to participating in the research project.

I understand the purpose and procedures of the study (add these again if appropriate).

I have been given an opportunity to answer questions about the study and have had answers to my satisfaction.

I declare that my participation in this study is entirely voluntary and that I may withdraw at any time without negative consequences.

I voluntarily give permission for the interviews to be audio-recorded.

I give permission for my assessment task marks and interview response to be used as a source of data. My identity will not be disclosed and pseudonyms will be used to protect my identity

If I have any further questions/concerns or queries related to the study I understand that I may contact the researcher at. (provide details).

If I have any questions or concerns about my rights as a study participant, or if I am concerned about an aspect of the study or the researcher, then I may contact:

HUMANITIES & SOCIAL SCIENCES RESEARCH ETHICS ADMINISTRATION

RESEARCH OFFICE,

Westville Campus

Govan Mbeki Building

PrivateBagX54001

Durban 4000

KwaZulu-Natal, SOUTH AFRICA Email: HSSREC@ukzn.ac.za

Additional consent, where applicable:

- I am willing to be part of the garden project and interviews. I am also willing to allow recording by the following equipment, and the use of other data:

Digital audio recording interviews	Willing	Not willing
Use of portfolio of evidence		
Use of reflective diary		

……………………………………...

Name of Participant

…………………………………

Signature of Participant **Date:**

APPENDIX D1: LEANER CONSENT LETTER

KwaZulu-Natal, SOUTH AFRICA Email: HSSREC@ukzn.ac.za

Additional consent, where applicable:

- I am willing to be part of the garden project and interviews. I am also willing to allow recording by the following equipment, and the use of other data:

Digital audio recording interviews	Willing	Not willing
Use of portfolio of evidence	✓	
Use of reflective diary	✓	

[redacted]….

Name of Participant

[redacted].

Signature of Participant

6 September 2021

Date:

APPENDIX D2: LEANER CONSENT LETTER

KwaZulu-Natal, SOUTH AFRICA Email: HSSREC@ukzn.ac.za

Additional consent, where applicable:

- I am willing to be part of the garden project and interviews. I am also willing to allow recording by the following equipment, and the use of other data:

Digital audio recording interviews	Willing	Not willing
Use of portfolio of evidence	✓	
Use of reflective diary	✓	

[redacted]

Name of Participant

[redacted]..

Signature of Participant

Date: 6/9/2021

APPENDIX D3: LEANER CONSENT LETTER

KwaZulu-Natal, SOUTH AFRICA Email: HSSREC@ukzn.ac.za

Additional consent, where applicable:

- I am willing to be part of the garden project and interviews. I am also willing to allow recording by the following equipment, and the use of other data:

Digital audio recording interviews	Willing	Not willing
Use of portfolio of evidence	✓	
Use of reflective diary	✓	

[redacted]

Name of Participant

[redacted].......

Signature of Participant

6/9/2021
Date:

APPENDIX D4: LEANER CONSENT LETTER

KwaZulu-Natal, SOUTH AFRICA Email: HSSREC@ukzn.ac.za

Additional consent, where applicable:

- I am willing to be part of the garden project and interviews. I am also willing to allow recording by the following equipment, and the use of other data:

Digital audio recording interviews	Willing	Not willing
Use of portfolio of evidence	✓	
Use of reflective diary	✓	

[redacted]

Name of Participant

[redacted]

Signature of Participant **Date:** 6/9/2021

APPENDIX D5: LEANER CONSENT LETTER

KwaZulu-Natal, SOUTH AFRICA Email: HSSREC@ukzn.ac.za

Additional consent, where applicable:

- I am willing to be part of the garden project and interviews. I am also willing to allow recording by the following equipment, and the use of other data:

Digital audio recording interviews	Willing	Not willing
Use of portfolio of evidence	✓	
Use of reflective diary	✓	

[redacted]

Name of Participant

[redacted]....

Signature of Participant **Date:** 6/9/2021.

APPENDIX D6: LEANER CONSENT LETTER

KwaZulu-Natal, SOUTH AFRICA Email: HSSREC@ukzn.ac.za

Additional consent, where applicable:

- I am willing to be part of the garden project and interviews. I am also willing to allow recording by the following equipment, and the use of other data:

Digital audio recording interviews	Willing	Not willing
Use of portfolio of evidence	✓	
Use of reflective diary	✓	

[redacted]

Name of Participant

[redacted]....

Signature of Participant

6-09-2021

Date:

APPENDIX D7: LEANER CONSENT LETTER

KwaZulu-Natal, SOUTH AFRICA Email: HSSREC@ukzn.ac.za

Additional consent, where applicable:

- I am willing to be part of the garden project and interviews. I am also willing to allow recording by the following equipment, and the use of other data:

	Willing	Not willing
Digital audio recording interviews		
Use of portfolio of evidence	✓	
Use of reflective diary	✓	

[redacted]

Name of Participant

[redacted]

Signature of Participant

6/9/2021

Date:

APPENDIX E: PARENT CONSENT FORM

Mzali

Ngibhala le ncwadi ukucela imvume yokusebenzisana nomtwana wakho ekwenzeni ucwaningo lwesifundo sezibalo(Mathematics)

Nginguthisha Mr N. Dube ofundisa Kwasanti High school, ngiqhuba izifundo zami enyuvesi yaKwaZulu-Natal (UKZN) kunocwaningo okumele ngilwenze mayelana nale sifundo. Abafundi ngizobabhalisa isivivinyo (Test) ngiphinde ngibabuze imibuzo ngomlomo.

Ngingajabula ukuthola imvume kuwe, ngingajabula uma umzali engasayina lapha ngezansi uma evuma.

☐ Ngiyavuma

☐ Angivumi

--- --

Igama Lomzali Ukusayina

APPENDIX F: GRADE 10 TASK

GRADE 10 MATHEMATICS SHORT TEST

a) $3\sin x = 2$

b) $4\sin x - 3 = 0$

c) $5\cos x + 4 = 0$

d) $x\cos 30° = 2$

e) $4\tan x + 5 = 0$

f) $7\tan x = 10$

g) $\cot x - 2 = 0$

h) $2\,cosec\,x + 3 = 0$

i) $4\sec x - 12 = 0$

j) $5\sin x - 3 = 2\sin x$

APPENDIX G: GRADE 11 TASK

1. Given tan θ = -2.5

1.1 in which quadrant would the function have a solution? Explain

1. Given tan θ = -2.5

1.2 Is the value of 180 < θ or θ < 180.

1. Given tan θ = -2.5

1.3 would you say this function is defined or not? Explain

2 Find the general solution of the following equation.

2.1 $\cos(\theta + 10°) = -0.67$

2 Find the general solution of the following equation.

2.2 $\sqrt{3}\, cos\ \theta - 3\ sin\ \theta = 0$

2 Find the general solution of the following equation.

2.3 $2sin^2\theta - 3\sin\theta = 1$

3. Solve for θ without the use of a calculator if

3.1 $(cos\ \theta - 1)(2sin\ \theta + \sqrt{3}) = 0\ and\ \theta \in [0°; 360°]$

3. Solve for θ without the use of a calculator if

3.2 $sin\ \theta(2cos\ \theta + \sqrt{3}) = 0\ and\ \theta \in [0°; 360°]$

APPENDIX H: INTERVIEW QUESTIONS

Broad themes for interview questions

Name (optional)	
Pseudonym	
Institution	

Interview questions general will develop from learners' response. These are just broad questions.

1. Explain how you solve the question.

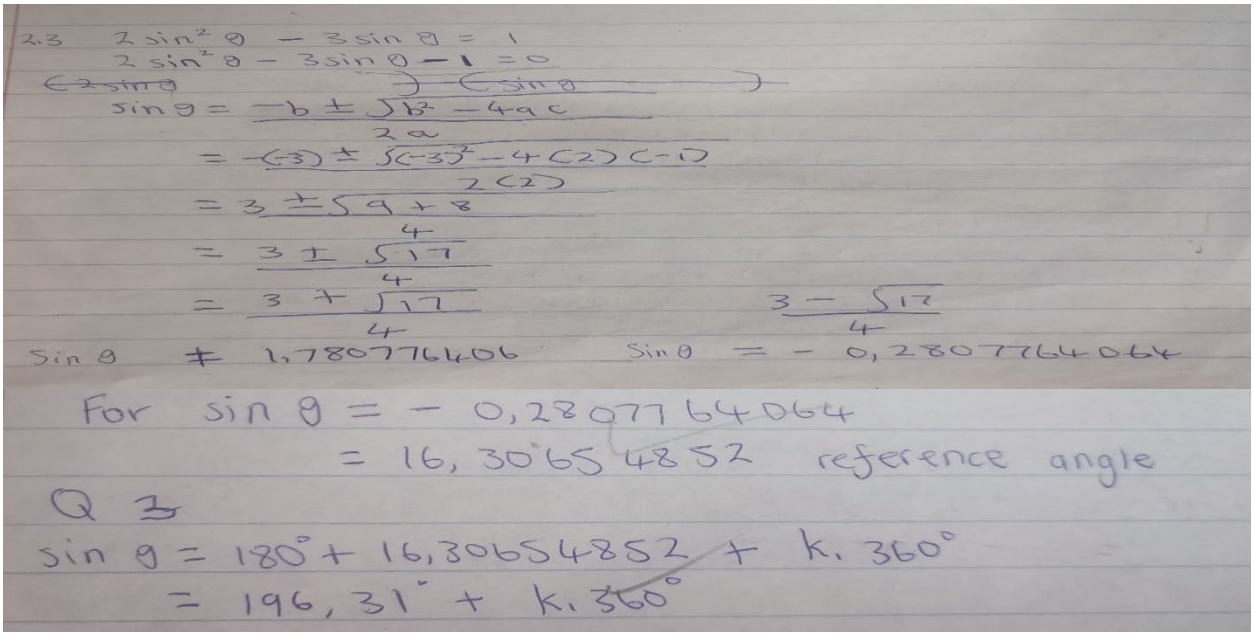
2. Is there any other method you could have used? Explain

3. Will the equation always have a solution?

4. How would you tell if the equation has a solution or not?

APPENDIX I: INTERVIEWS

INTERVIEW WITH MAKWANZA



Researcher: Explain how you solved the problem.

Makwanza: I transposed 1 to the left side, then I was left with a trinomial which is $2sin\theta - 3sin\theta - 1 = 0$. *Ok, I tried to look for factors and I didn't get them. After that I used the quadratic formula, I did a substitution of a=2, b=3, and c=-1. Then I calculate to find the reference angle and I could get a solution to the first one, which was undefined, and the second one got the solution.*

Researcher: Why do you say the other solution was undefined?

Makwanza: When I tried to calculate the reference angle of 1,78 it shows undefined.

Researcher: What does undefined mean?

Makwanza: Mhhh…. [Pause] There is no solution.

Researcher: Why do you think the other coordinates has no solution?

Makwanza: Yoh! [Pause]. Sengiyabona. [Now I see]. When I factorise it, my answer was $\sin\theta = 1.78$. *This is not possible, Sir. No solution.*

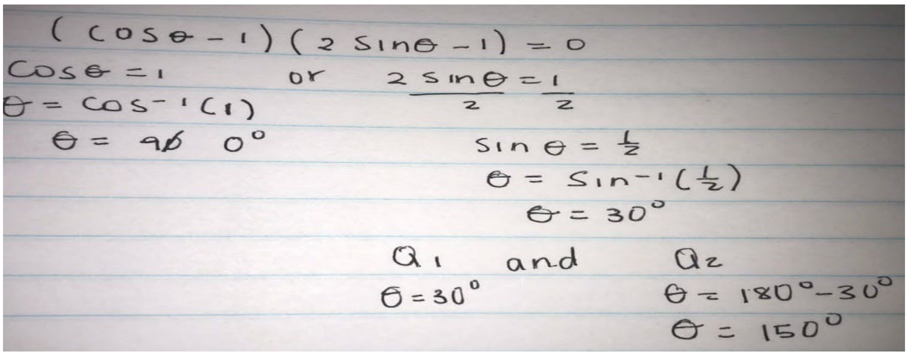
Researcher: Why?

Makwanza: Angithi [Isn't], Sir, the graph of sin turns ku 1 on the y value when x is 90°*, so it cannot go higher than that unless the graph was shifted up vertically? This is normal graph of sin, so the maximum value is 1, yingakho (1,78) enganayo I -x coordinate [That is why there is no x coordinate for 1,78].*

Researcher: Okay, that is good that you can pick up that. Now tell me why you only used the 3rd quadrant to determine the solution?

Makwanza: Eish, sinzima lesisibalo [the sum is difficult], but I see now I should have also included the 4th quadrant, because sin is negative in the 4th quadrant.

INTERVIEW WITH FIHLELA



Researcher: Explain how you solve the question.

Fihlela: What I did here, I took $\cos\theta - 1 = 0$ *and* $2\sin\theta - 1 = 0$ *. In this equation I apply my knowledge of solving the algebraic equation on question 1 when we solve x. Separated by the two equations, first, I solve* $\cos\theta - 1$ *by transposing. Then I had cos θ is equal to 1. The value of θ was equal to 0 degrees. On the other side, I had 2sin θ and transposed 1 to the right and divided by 2 into both sides. I was left with sin θ equal to* $\frac{1}{2}$ *and my theta was 30 degrees.*

Researcher: Is there any other method you could have used? Explain.

Fihlela: No.

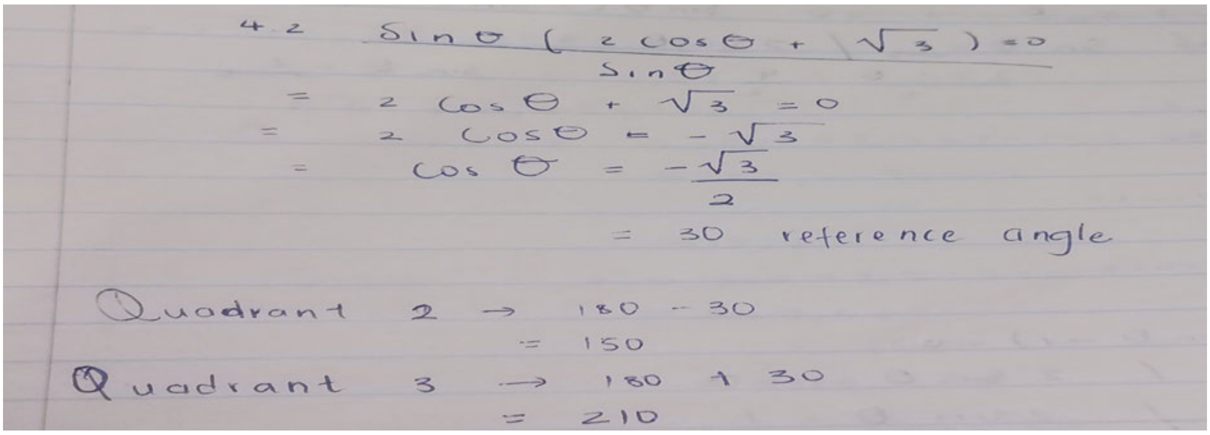
Researcher: Will the equation always have a solution?

Fihlela: Yes.

Researcher: How would you tell if the equation has a solution or not.

Fihlela: If you know about solving algebraic equations in paper 1 of mathematics, it is easy to see how to solve this equation.

INTERVIEW WITH MAKHANYA



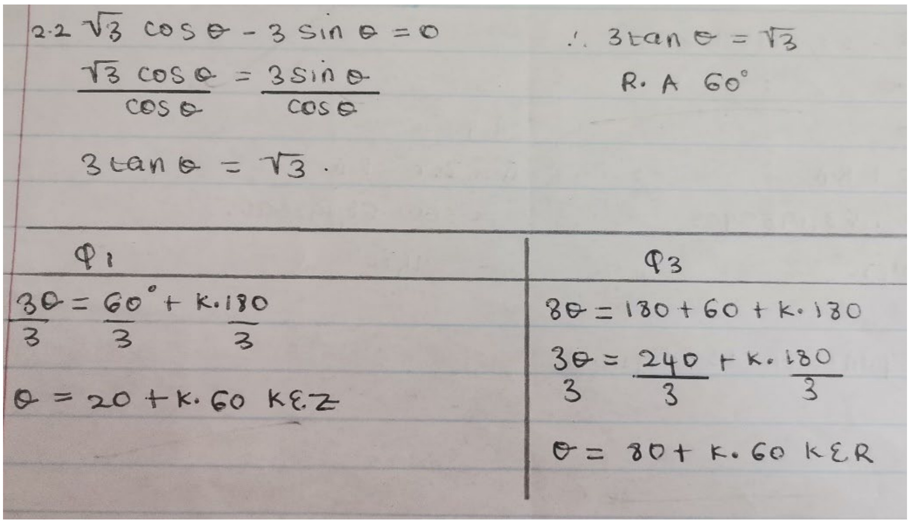
Researcher: *Makhanya you were given* $sin\,\theta(2cos\,\theta + \sqrt{3}) = 0$. *So how did you solve the equation?*

Makhanya: *I answered this question separating* $sin\,\theta = 0$ *and* $2\,cos\,\theta + \sqrt{3} = 0$. *Yabona [you see], Sir, mina [I] what I did is that I transported* $\sqrt{3}$ *to the left and became negative* $\sqrt{3}$ *and I was left with* $2\,cos\,\theta$ *and I divide by 2 both sides. Then I was left with* $cos\,\theta = \frac{-\sqrt{3}}{2}$. *After that, I used the calculator – I do not want to lie sir – and said the shift* $coc^{-1}\left(\frac{\sqrt{3}}{2}\right)$ *and the value of θ= 150. Also,* sin θ=*0 again I used the calculate the answer was* θ= 0.

Researcher: Makhanya, is there any other method you could use to solve it?

Makhanya: Ayi, no sir.

INTERVIEW WITH ZULU



Researcher: How did you find the answer?

Zulu: I divided by 3 on both sides to find the reference angle and I used shift tan of $\frac{\sqrt{3}}{3}$.

Researcher: Can you please do it again, using a calculator?

Zulu: Yes, Sir. [He does it again using a calculator.] Eish, sir: I made a mistake yazi (you know) I divide with $\sqrt{3}, not\ 3,$ *both sides.*

APPENDIX J: EDITOR'S CERTIFICATE

CERTIFICATE OF PROFESSIONAL EDITING

I, Barbara L. Louton, do hereby declare that I am a professional editor with a Bachelor of Arts in Professional Writing and seventeen years of experience as an editor, researcher and writer.

I declare that I was contracted by Njabulo Happyboy Dube (Student number: 210500469), a PhD candidate under the supervision of Prof Zanele Ngcobo in the School of Education at the University of KwaZulu-Natal, to complete a professional edit of his thesis:

'Exploration of Grade 11 learners' mental constructions and difficulties in learning and solving trigonometric equations: a case of one school in Umlazi district'

I declare that I have completed a two-stage professional edit of the document, addressing structural and logical issues, the clarity and flow of language, and correcting grammatical, spelling and formatting errors. Changes were tracked and comments were left for the client, who then make further revisions which were then edited.

Disclaimer:

Responsibility for the originality and accuracy of the material presented in the edited document lies with the client. I have not verified the originality or accuracy of statements, quotations or citations and references presented in the thesis. Where I have detected inaccuracies I have rectified them or reported them to the client. In addition, the client was free to make further changes to the edited material after the edit was complete.

I can be contacted at:

Cell: 073 766 1139

Email: bellway@gmail.com

Barbara L Louton	[redacted]	11 November 2023
Name	Signed	Date

APPENDIX K: TURNITIN REPORT

Turnitin Originality Report

Processed on: 09-Nov-2023 9:27 PM CAT
ID: 2223088669
Word Count: 41481
Submitted: 1

Njabulo_full_document_4_November.docx By Njabulo Dube

Similarity Index	Similarity by Source	
9%	Internet Sources:	N/A
	Publications:	9%
	Student Papers:	N/A

1% match (Zanele Ndlovu, Deonarain Brijlall. "Pre-service Teachers' Mental Constructions of Concepts in Matrix Algebra", African Journal of Research in Mathematics, Science and Technology Education, 2015)
Zanele Ndlovu, Deonarain Brijlall. "Pre-service Teachers' Mental Constructions of Concepts in Matrix Algebra", African Journal of Research in Mathematics, Science and Technology Education, 2015

1% match (Ilana Arnon, Jim Cottrill, Ed Dubinsky, Asuman Oktaç, Solange Roa Fuentes, Maria Trigueros, Kirk Weller. "APOS Theory", Springer Nature, 2014)
Ilana Arnon, Jim Cottrill, Ed Dubinsky, Asuman Oktaç, Solange Roa Fuentes, Maria Trigueros, Kirk Weller. "APOS Theory", Springer Nature, 2014

< 1% match (Sibawu Witness Siyepu. "Analysis of errors in derivatives of trigonometric functions", International Journal of STEM Education, 2015)
Sibawu Witness Siyepu. "Analysis of errors in derivatives of trigonometric functions", International Journal of STEM Education, 2015

< 1% match (Maharaj, Aneshkumar. "An APOS Analysis of students' understanding of the concept of a limit of a function", Pythagoras, 2011.)
Maharaj, Aneshkumar. "An APOS Analysis of students' understanding of the concept of a limit of a function", Pythagoras, 2011.

< 1% match (Conilius Chagwiza, Lillias Hamufari, Gladys Sunzuma*. "Exploring Zimbabwean A-Level Mathematics Learners' Understanding of the Determinant Concept", European Journal of Mathematics and Science Education, 2021)
Conilius Chagwiza, Lillias Hamufari, Gladys Sunzuma*. "Exploring Zimbabwean A-Level Mathematics Learners' Understanding of the Determinant Concept", European Journal of Mathematics and Science Education, 2021

< 1% match (Jojo, Zingiswa Mybert Monica. "Mental Constructions Formed in the Conceptual Understanding of the Chain Rule", Mediterranean Journal of Social Sciences, 2014.)
Jojo, Zingiswa Mybert Monica. "Mental Constructions Formed in the Conceptual Understanding of the Chain Rule", Mediterranean Journal of Social Sciences, 2014.

< 1% match ("Instructional Design in the Formation of Mental Images and the Genetic Decomposition of a Concept", Journal of Educational and Social Research, 2014.)
"Instructional Design in the Formation of Mental Images and the Genetic Decomposition of a Concept", Journal of Educational and Social Research, 2014.

< 1% match (Julie M. Clark, Francisco Cordero, Jim Cottrill, Bronislaw Czarnocha et al. "Constructing a schema: The case of the chain rule?", The Journal of Mathematical Behavior, 1997)
Julie M. Clark, Francisco Cordero, Jim Cottrill, Bronislaw Czarnocha et al. "Constructing a schema: The case of the chain rule?", The Journal of Mathematical Behavior, 1997

< 1% match ("Encyclopedia of Mathematics Education", Springer Science and Business Media LLC, 2020)
"Encyclopedia of Mathematics Education", Springer Science and Business Media LLC, 2020

< 1% match (Lillias Hamufari Natsai Mutambara, Jane Tendere, Conilius Chagwiza. "Exploring the Conceptual Understanding of the Quadratic Function Concept in Techers' Colleges in Zimbabwe", Eurasia Journal of Mathematics, Science and Technology Education, 2019)
Lillias Hamufari Natsai Mutambara, Jane Tendere, Conilius Chagwiza. "Exploring the Conceptual Understanding of the Quadratic Function Concept in Techers' Colleges in Zimbabwe", Eurasia Journal of Mathematics, Science and Technology Education, 2019

< 1% match (Jogymol Alex, Kuttickattu J. Mammen. "Students' understanding of geometry terminology through the lens of Van Hiele theory", Pythagoras, 2018)
Jogymol Alex, Kuttickattu J. Mammen. "Students' understanding of geometry terminology through the lens of Van Hiele theory", Pythagoras, 2018

< 1% match (Conilius Jaison Chagwiza, Aneshkumar Maharaj, Deonarain Brijlall. "Formulating a Modified Genetic Decomposition for the Concept of 'Limit of a Sequence'", African Journal of Research in Mathematics, Science and Technology Education, 2020)
Conilius Jaison Chagwiza, Aneshkumar Maharaj, Deonarain Brijlall. "Formulating a Modified Genetic Decomposition for the Concept of 'Limit of a Sequence'", African Journal of Research in Mathematics, Science and Technology Education, 2020

< 1% match (Brigitte S. Cypress. "Rigor or Reliability and Validity in Qualitative Research", Dimensions of Critical Care Nursing, 2017)
Brigitte S. Cypress. "Rigor or Reliability and Validity in Qualitative Research", Dimensions of Critical Care Nursing, 2017

< 1% match (Deonarain Brijlall. "chapter 17 Best Practices and Case Studies of Teaching and Learning Mathematics in Higher Education", IGI Global, 2020)
Deonarain Brijlall. "chapter 17 Best Practices and Case Studies of Teaching and Learning Mathematics in Higher Education", IGI Global, 2020

< 1% match (Despina Potari, Olive Chapman. "International Handbook of Mathematics Teacher Education: Volume 1", Brill, 2020)
Despina Potari, Olive Chapman. "International Handbook of Mathematics Teacher Education: Volume 1", Brill, 2020

< 1% match ("The Teaching and Learning of Mathematics at University Level", Springer Nature, 2002)
"The Teaching and Learning of Mathematics at University Level", Springer Nature, 2002

< 1% match (Rafael Martínez-Planell, Angel Cruz Delgado. "The unit circle approach to the construction of the sine and cosine functions and their inverses: An application of APOS theory", The Journal of Mathematical Behavior, 2016)
Rafael Martínez-Planell, Angel Cruz Delgado. "The unit circle approach to the construction of the sine and cosine functions and their inverses: An application of APOS theory", The Journal of Mathematical Behavior, 2016

< 1% match ("Qualitative Methodologies in Organization Studies", Springer Science and Business Media LLC, 2018)
"Qualitative Methodologies in Organization Studies", Springer Science and Business Media LLC, 2018

< 1% match (Ramachandran N., Sivaprakasam P., Thangamani G., Anand G.. "Selecting a suitable Cloud Computing technology deployment model for an academic institute", Campus-Wide Information Systems, 2014)
Ramachandran N., Sivaprakasam P., Thangamani G., Anand G.. "Selecting a suitable Cloud Computing technology deployment model for an academic institute", Campus-Wide Information Systems, 2014

< 1% match (Masoodi Marjan. "Importance of Promoting Metacognitive Awareness at University", Vocational Training: Research And Realities, 2018)
Masoodi Marjan. "Importance of Promoting Metacognitive Awareness at University", Vocational Training: Research And Realities, 2018

< 1% match ("Encyclopedia of Mathematics Education", Springer Science and Business Media LLC, 2014)
"Encyclopedia of Mathematics Education", Springer Science and Business Media LLC, 2014

< 1% match ("Challenges and Strategies in Teaching Linear Algebra", Springer Science and Business Media LLC, 2018)
"Challenges and Strategies in Teaching Linear Algebra", Springer Science and Business Media LLC, 2018

< 1% match (Allan Struthers, Merle Potter. "Differential Equations", Springer Science and Business Media LLC, 2019)
Allan Struthers, Merle Potter. "Differential Equations", Springer Science and Business Media LLC, 2019

< 1% match (Minghui Hou, Alma Jam. "International Education Equity for Doctoral Students: Duoethnographic Reflections from China and Cameroon", International Journal of Doctoral Studies, 2020)
Minghui Hou, Alma Jam. "International Education Equity for Doctoral Students: Duoethnographic Reflections from China and Cameroon", International Journal of Doctoral Studies, 2020

< 1% match (Minghui Hou, Alma Jam. "International Education Equity for Doctoral Students: Duoethnographic Reflections from China and Cameroon", International Journal of Doctoral Studies, 2020)
Minghui Hou, Alma Jam. "International Education Equity for Doctoral Students: Duoethnographic Reflections from China and Cameroon", International Journal of Doctoral Studies, 2020

< 1% match (Süleyman Erkam Sulak. "Investigation of Writing Habits of Primary School Teachers", International Electronic Journal of Elementary Education, 2018)
Süleyman Erkam Sulak. "Investigation of Writing Habits of Primary School Teachers", International Electronic Journal of Elementary Education, 2018

< 1% match ("Creativity and Technology in Mathematics Education", Springer Science and Business Media LLC, 2018)
"Creativity and Technology in Mathematics Education", Springer Science and Business Media LLC, 2018

< 1% match (Budi Halomoan Siregar, Izwita Dewi, Ade Andriani. "Error analysis of mathematics students who are taught by using the book of mathematics learning strategy in solving pedagogical problems based on Polya's four-step approach", Journal of Physics: Conference Series, 2018)
Budi Halomoan Siregar, Izwita Dewi, Ade Andriani. "Error analysis of mathematics students who are taught by using the book of mathematics learning strategy in solving pedagogical problems based on Polya's four-step approach", Journal of Physics: Conference Series, 2018

< 1% match (Naidoo, Jayaluxmi, and Reginald Govender. "Exploring the use of a dynamic online software programme in the teaching and learning of trigonometric graphs", Pythagoras, 2014.)
Naidoo, Jayaluxmi, and Reginald Govender. "Exploring the use of a dynamic online software programme in the teaching and learning of trigonometric graphs", Pythagoras, 2014.

< 1% match (Aisa Oberlin Solomon. "The role of households in solid waste management in East Africa capital cities", Brill, 2011)
Aisa Oberlin Solomon. "The role of households in solid waste management in East Africa capital cities", Brill, 2011

< 1% match (publications)
Bowles, Samuel, Halliday, Simon D.. "Microeconomics", Microeconomics, 2022

< 1% match (Deonarain Brijlall, Aneshkumar Maharaj. "Exploring Pre-service Teachers' Mental Constructions When Solving Problems Involving Infinite Sets", International Journal of Educational Sciences, 2017)
Deonarain Brijlall, Aneshkumar Maharaj. "Exploring Pre-service Teachers' Mental Constructions When Solving Problems Involving Infinite Sets", International Journal of Educational Sciences, 2017

< 1% match (Kaur a/p Swaran Singh, Charanjit, Arshad Abdul Samad, Habsah Hussin, and Tajularipin Sulaiman. "Developing a Portfolio Assessment Model for the Teaching and Learning of English in Malaysian L2 Classroom", English Language Teaching, 2015.)
Kaur a/p Swaran Singh, Charanjit, Arshad Abdul Samad, Habsah Hussin, and Tajularipin Sulaiman. "Developing a Portfolio Assessment Model for the Teaching and Learning of English in Malaysian L2 Classroom", English Language Teaching, 2015.

< 1% match (Sarah Bansilal, Deonarain Brijlall, Maria Trigueros. "An APOS study on pre-service teachers' understanding of injections and surjections", The Journal of Mathematical Behavior, 2017)
Sarah Bansilal, Deonarain Brijlall, Maria Trigueros. "An APOS study on pre-service teachers' understanding of injections and surjections", The Journal of Mathematical Behavior, 2017

< 1% match (Ziphozakhe Theophilus Shasha, Yong Geng, Hua-ping Sun, Walter Musakwa, Lu Sun. "Past, current, and future perspectives on eco-tourism: a bibliometric review between 2001 and 2018", Environmental Science and Pollution Research, 2020)
Ziphozakhe Theophilus Shasha, Yong Geng, Hua-ping Sun, Walter Musakwa, Lu Sun. "Past, current, and future perspectives on eco-tourism: a bibliometric review between 2001 and 2018", Environmental Science and Pollution Research, 2020

< 1% match (Ana Kuzle. "Patterns of Metacognitive Behavior During Mathematics Problem-Solving in a Dynamic Geometry Environment", International Electronic Journal of Mathematics Education, 2013)
Ana Kuzle. "Patterns of Metacognitive Behavior During Mathematics Problem-Solving in a Dynamic Geometry Environment", International Electronic Journal of Mathematics Education, 2013

< 1% match (Kirstin Erath, Susanne Prediger, Uta Quasthoff, Vivien Heller. "Discourse competence as important part of academic language proficiency in mathematics classrooms: the case of explaining to learn and learning to explain", Educational Studies in Mathematics, 2018)
Kirstin Erath, Susanne Prediger, Uta Quasthoff, Vivien Heller. "Discourse competence as important part of academic language proficiency in mathematics classrooms: the case of explaining to learn and learning to explain", Educational Studies in Mathematics, 2018

< 1% match (Sarah Goodrum, Mary K. Evans, Andrew J. Thompson, William Woodward. "Learning from a failure in threat assessment: 11 questions and not enough answers", Behavioral Sciences & the Law, 2019)
Sarah Goodrum, Mary K. Evans, Andrew J. Thompson, William Woodward. "Learning from a failure in threat assessment: 11 questions and not enough answers", Behavioral Sciences & the Law, 2019

< 1% match (Mohammd Nehal Hasnine, Hiroshi Ueda, Mahmoud Mohamed Hussien Ahmed. "Adaptation of AL-TST active learning model in hybrid classroom: Findings from teaching during COVID-19 pandemic in Egypt", Procedia Computer Science, 2022)
Mohammd Nehal Hasnine, Hiroshi Ueda, Mahmoud Mohamed Hussien Ahmed. "Adaptation of AL-TST active learning model in hybrid classroom: Findings from teaching during COVID-19 pandemic in Egypt", Procedia Computer Science, 2022

< 1% match (Sefa Dundar. "Mathematics Teacher-Candidates' Performance in Solving Problems with Different Representation Styles: The Trigonometry Example", EURASIA Journal of Mathematics, Science and Technology Education, 2015)
Sefa Dundar. "Mathematics Teacher-Candidates' Performance in Solving Problems with Different Representation Styles: The Trigonometry Example", EURASIA Journal of Mathematics, Science and Technology Education, 2015

< 1% match (Simon Vurayai. "Rurality and Exclusion in Ordinary Level Mathematics in Zimbabwe: A Document Analysis", International Journal of Learning, Teaching and Educational Research, 2020)
Simon Vurayai. "Rurality and Exclusion in Ordinary Level Mathematics in Zimbabwe: A Document Analysis", International Journal of Learning, Teaching and Educational Research, 2020

< 1% match ("Social Computing and Social Media. Human Behavior", Springer Science and Business Media LLC, 2017)
"Social Computing and Social Media. Human Behavior", Springer Science and Business Media LLC, 2017

< 1% match (Katie N. Dainty. "A Call to Move Beyond the Numbers in DNAR Orders Research", Journal of the American Heart Association, 2020)
Katie N. Dainty. "A Call to Move Beyond the Numbers in DNAR Orders Research", Journal of the American Heart Association, 2020

< 1% match (S M Rohimah, S Prabawanto. "Students' difficulties in solving trigonometric equations and identities", Journal of Physics: Conference Series, 2020)
S M Rohimah, S Prabawanto. "Students' difficulties in solving trigonometric equations and identities", Journal of Physics: Conference Series, 2020

< 1% match (Zidane, Youcef J-T., Agnar Johansen, Anandasivakumar Ekambaram, and Linda C. Hald. "When Stakeholders Shape Successes or Bring Failures – A Case Study of an Algerian Megaproject", Procedia Computer Science, 2015.)
Zidane, Youcef J-T., Agnar Johansen, Anandasivakumar Ekambaram, and Linda C. Hald. "When Stakeholders Shape Successes or Bring Failures – A Case Study of an Algerian Megaproject", Procedia Computer Science, 2015.

< 1% match ("The Palgrave Handbook of Fulfillment, Wellness, and Personal Growth at Work", Springer Science and Business Media LLC, 2023)
"The Palgrave Handbook of Fulfillment, Wellness, and Personal Growth at Work", Springer Science and Business Media LLC, 2023

< 1% match (Alfredo De Massis, Josip Kotlar. "The case study method in family business research: Guidelines for qualitative scholarship", Journal of Family Business Strategy, 2014)
Alfredo De Massis, Josip Kotlar. "The case study method in family business research: Guidelines for qualitative scholarship", Journal of Family Business Strategy, 2014

< 1% match (Parraguez, M.. "Construction of the vector space concept from the viewpoint of APOS theory", Linear Algebra and Its Applications, 20100401)
Parraguez, M.. "Construction of the vector space concept from the viewpoint of APOS theory", Linear Algebra and Its Applications, 20100401

< 1% match (Cathrine Kazunga, Sarah Bansilal. "Zimbabwean in-service mathematics teachers' understanding of matrix operations", The Journal of Mathematical Behavior, 2017)
Cathrine Kazunga, Sarah Bansilal. "Zimbabwean in-service mathematics teachers' understanding of matrix operations", The Journal of Mathematical Behavior, 2017

< 1% match (Chandra Shekhar Khadka. "Empirical Method in Legal Research", Nepal Law Review, 2019)
Chandra Shekhar Khadka. "Empirical Method in Legal Research", Nepal Law Review, 2019

< 1% match (Nalini Maharajh, Deonarain Brijlall, Nadaraj Govender. "Preservice mathematics students' notions of the concept definition of continuity in calculus through collaborative instructional design worksheets", African Journal of Research in Mathematics, Science and Technology Education, 2013)
Nalini Maharajh, Deonarain Brijlall, Nadaraj Govender. "Preservice mathematics students' notions of the concept definition of continuity in calculus through collaborative instructional design worksheets", African Journal of Research in Mathematics, Science and Technology Education, 2013

< 1% match (Li, Fenfang. "A Study of English Reading Strategies Used by Senior Middle School Students", Asian Social Science, 2010.)
Li, Fenfang. "A Study of English Reading Strategies Used by Senior Middle School Students", Asian Social Science, 2010.

< 1% match (Lillias Mutambara, Sarah Bansilal. "An Exploratory Study on the Understanding of the Vector Subspace Concept", African Journal of Research in Mathematics, Science and Technology Education, 2019)
Lillias Mutambara, Sarah Bansilal. "An Exploratory Study on the Understanding of the Vector Subspace Concept", African Journal of Research in Mathematics, Science and Technology Education, 2019

< 1% match (Ross A. Jackson, Brian L. Heath. "Small Businesses Can Enhance Strategic Insights by Text Mining Organizational Writings", American Journal of Industrial and Business Management, 2023)
Ross A. Jackson, Brian L. Heath. "Small Businesses Can Enhance Strategic Insights by Text Mining Organizational Writings", American Journal of Industrial and Business Management, 2023

< 1% match (Zeynel Kablan, Sevinç Süzer Uğur. "The relationship between routine and non-routine problem solving and learning styles", Educational Studies, 2020)
Zeynel Kablan, Sevinç Süzer Uğur. "The relationship between routine and non-routine problem solving and learning styles", Educational Studies, 2020

< 1% match (Claudio Fuentealba, Edelmira Badillo, Gloria Sánchez-Matamoros, Andrea Cárcamo. "The Understanding of the Derivative Concept in Higher Education", Eurasia Journal of Mathematics, Science and Technology Education, 2018)
Claudio Fuentealba, Edelmira Badillo, Gloria Sánchez-Matamoros, Andrea Cárcamo. "The Understanding of the Derivative Concept in Higher Education", Eurasia Journal of Mathematics, Science and Technology Education, 2018

< 1% match (Martin Wynn, Peter Jones. "New Technology Deployment and Corporate Responsibilities in the Metaverse", Knowledge, 2023)
Martin Wynn, Peter Jones. "New Technology Deployment and Corporate Responsibilities in the Metaverse", Knowledge, 2023

< 1% match (Amanda J. Fairchild, S. Jeanne Horst, Sara J. Finney, Kenneth E. Barron. "Evaluating existing and new validity evidence for the Academic Motivation Scale", Contemporary Educational Psychology, 2005)
Amanda J. Fairchild, S. Jeanne Horst, Sara J. Finney, Kenneth E. Barron. "Evaluating existing and new validity evidence for the Academic Motivation Scale", Contemporary Educational Psychology, 2005

< 1% match (Mahboubeh Taghizadeh, Saba Saadatju. "Engineering students' needs for listening scaffolding strategies and their perceptions of instructors' performance in an academic listening course", Research in Science & Technological Education, 2020)
Mahboubeh Taghizadeh, Saba Saadatju. "Engineering students' needs for listening scaffolding strategies and their perceptions of instructors' performance in an academic listening course", Research in Science & Technological Education, 2020

< 1% match (Mhd Pujiono, Muhammad Kiki Wardana, Rostina Taib, Zuindra ., Wan Anayati. "The Language Attitudes of Indonesian Language Teachers Toward the Acehnese Language", KnE Social Sciences, 2023)
Mhd Pujiono, Muhammad Kiki Wardana, Rostina Taib, Zuindra ., Wan Anayati. "The Language Attitudes of Indonesian Language Teachers Toward the Acehnese Language", KnE Social Sciences, 2023

< 1% match (Vahid Borji, Hassan Alamolhodaei, Farzad Radmehr. "Application of the APOS-ACE Theory to improve Students' Graphical Understanding of Derivative", Eurasia Journal of Mathematics, Science and Technology Education, 2018)
Vahid Borji, Hassan Alamolhodaei, Farzad Radmehr. "Application of the APOS-ACE Theory to improve Students' Graphical Understanding of Derivative", Eurasia Journal of Mathematics, Science and Technology Education, 2018

< 1% match (Kathy Liu Sun, Jennifer L. Ruef. "Examining and conceptualizing the relationship between teacher praise and the co-construction of mathematical competence in classrooms", The Journal of Mathematical Behavior, 2023)
Kathy Liu Sun, Jennifer L. Ruef. "Examining and conceptualizing the relationship between teacher praise and the co-construction of mathematical competence in classrooms", The Journal of Mathematical Behavior, 2023

< 1% match (Kyung Young Lee, Ying Jin, Cheul Rhee, Sung-Byung Yang. "Online consumers' reactions to price decreases: Amazon's Kindle 2 case", Internet Research, 2016)
Kyung Young Lee, Ying Jin, Cheul Rhee, Sung-Byung Yang. "Online consumers' reactions to price decreases: Amazon's Kindle 2 case", Internet Research, 2016

< 1% match (Giedrė Tamoliūnė, Justina Naujokaitienė, Airina Volungevičienė, Elena Trepulė, Rita Misiulienė, Rasa Greenspon. "1. Learning Analytics as a Metacognitive Tool", Walter de Gruyter GmbH, 2021)
Giedrė Tamoliūnė, Justina Naujokaitienė, Airina Volungevičienė, Elena Trepulė, Rita Misiulienė, Rasa Greenspon. "1. Learning Analytics as a Metacognitive Tool", Walter de Gruyter GmbH, 2021

< 1% match (Heather L. Tarr, Aneshkumar Maharaj. "A preliminary genetic decomposition for conceptual understanding of the indefinite integral", The Journal of Mathematical Behavior, 2021)
Heather L. Tarr, Aneshkumar Maharaj. "A preliminary genetic decomposition for conceptual understanding of the indefinite integral", The Journal of Mathematical Behavior, 2021

< 1% match (Khalid Alrashoud, Ryoichi Nakayama. "Chapter 17 Diffusion Policy Assessment of Solar Energy", Springer Science and Business Media LLC, 2019)
Khalid Alrashoud, Ryoichi Nakayama. "Chapter 17 Diffusion Policy Assessment of Solar Energy", Springer Science and Business Media LLC, 2019

< 1% match (Maria C. Marchetti-Mercer. ""If You Uproot a Tree You Leave a Big Hole Behind": Systemic Interconnectedness in Emigration and Family Life", Contemporary Family Therapy, 2016)
Maria C. Marchetti-Mercer. ""If You Uproot a Tree You Leave a Big Hole Behind": Systemic Interconnectedness in Emigration and Family Life", Contemporary Family Therapy, 2016

< 1% match (Sarah Bansilal. "Arriving at the starting point? Exploring a teacher's use of circular reasoning in a mathematics classroom", Education as Change, 2012)
Sarah Bansilal. "Arriving at the starting point? Exploring a teacher's use of circular reasoning in a mathematics classroom", Education as Change, 2012

< 1% match (Emanuele Dolera, Enrico Priola. "A counterexample to $$L^{\infty }$$-gradient type estimates for Ornstein–Uhlenbeck operators", Annali di Matematica Pura ed Applicata (1923 -), 2023)
Emanuele Dolera, Enrico Priola. "A counterexample to $$L^{\infty }$$-gradient type estimates for Ornstein–Uhlenbeck operators", Annali di Matematica Pura ed Applicata (1923 -), 2023

< 1% match (Khemane Thabiso, Padayachee Pragashni, Shaw Corrinne. "Exploring the complexities of swapping the order of integration in double integrals", International Journal of Mathematical Education in Science and Technology, 2023)
Khemane Thabiso, Padayachee Pragashni, Shaw Corrinne. "Exploring the complexities of swapping the order of integration in double integrals", International Journal of Mathematical Education in Science and Technology, 2023

< 1% match (publications)
"DD215 Social Research book 1 isbn9781473029095", Open University

< 1% match (Edgardo M. Santos. "A look into students' conceptual understanding of the definite integral via the APOS model", AIP Publishing, 2019)
Edgardo M. Santos. "A look into students' conceptual understanding of the definite integral via the APOS model", AIP Publishing, 2019

< 1% match (Philip Kwaku Kankam. "Employing Case Study and Survey Designs in Information Research", Journal of Information Technology Research, 2021)
Philip Kwaku Kankam. "Employing Case Study and Survey Designs in Information Research", Journal of Information Technology Research, 2021

< 1% match (Ardhendu Shekhar Singh. "Conducting case study research in non-profit organisations", Qualitative Market Research: An International Journal, 2014)
Ardhendu Shekhar Singh. "Conducting case study research in non-profit organisations", Qualitative Market Research: An International Journal, 2014

< 1% match (Brijlall, Deonarain, and Aneshkumar Maharaj. "Using an inductive approach for definition making: Monotonicity and boundedness of sequences", Pythagoras, 2011.)
Brijlall, Deonarain, and Aneshkumar Maharaj. "Using an inductive approach for definition making: Monotonicity and boundedness of sequences", Pythagoras, 2011.

< 1% match (Shannon E Kelly, David Moher, Tammy J Clifford. "Quality of conduct and reporting in rapid reviews: an exploration of compliance with PRISMA and AMSTAR guidelines", Systematic Reviews, 2016)
Shannon E Kelly, David Moher, Tammy J Clifford. "Quality of conduct and reporting in rapid reviews: an exploration of compliance with PRISMA and AMSTAR guidelines", Systematic Reviews, 2016

< 1% match (Aditya Jain, Daniel Ripa, Luis Torres. "Have companies arisen to the challenge of promoting sustainable work? The role of responsible business practices in the context of evolving employment and working conditions", Safety Science, 2024)
Aditya Jain, Daniel Ripa, Luis Torres. "Have companies arisen to the challenge of promoting sustainable work? The role of responsible business practices in the context of evolving employment and working conditions", Safety Science, 2024

< 1% match (Deonarain Brijlall, Nokwethemba Jubilee Ndlazi. "Analysing engineering students' understanding of integration to propose a genetic decomposition", The Journal of Mathematical Behavior, 2019)
Deonarain Brijlall, Nokwethemba Jubilee Ndlazi. "Analysing engineering students' understanding of integration to propose a genetic decomposition", The Journal of Mathematical Behavior, 2019

< 1% match (Ed Dubinsky. "Isetl: A programming language for learning mathematics", Communications on Pure and Applied Mathematics, 2006)
Ed Dubinsky. "Isetl: A programming language for learning mathematics", Communications on Pure and Applied Mathematics, 2006

< 1% match (publications)
Tamahi KATO (YAMAUCHI). "Women's Empowerment through Small-scale Mixed Porridge Flour Processing in the Morogoro Region of Tanzania", Journal of International Development Studies, 2021