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DISSIPATIVE GRAVITATING SYSTEMS

BY

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Dissipative Gravitating Systems

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Abstract

In this thesis we investigate the effect of shear on radiating stars undergoing gravitational collapse. The interior spacetime is described by the most general spherically symmetric line element in the absence of rotation. The energy momentum tensor for the stellar interior is taken to be an anisotropic fluid with heat flux. The thermodynamics of a relativistic fluid is reviewed for the Eckart and causal theories. Since the star is radiating energy to the exterior in the form of a radial heat flux, the atmosphere is described by Vaidya's outgoing solution. We provide the matching conditions required for the continuity of the momentum flux across the boundary, which determines the temporal evolution junction conditions for the metric functions. We provide a general method to obtain shearing solutions of the Einstein field equations describing a radiating, collapsing sphere. A particular exact solution satisfying the boundary condition and field equations is found. The validity of this specific model is investigated by employing a causal heat transport equation which yields the temperature profile within the stellar core. The energy conditions are studied and yield interesting features of this particular model which are absent in the shear-free case.

This thesis is dedicated to

my family and friends for their support

and understanding while I was writing this thesis.

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DECLARATION 1 - PLAGIARISM

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DECLARATION 2 - PUBLICATIONS

DETAILS OF CONTRIBUTION TO PUBLICATIONS that form part and/or include research presented in this thesis (include publications in preparation, submitted, in press and published and give details of contributions of each author to experimental work and writing of each publication.)

Publication 1

D Fleming, M Govender and K S Govinder, *A new family of solutions describing dissipative collapse of a fluid sphere*, Int. J. Mod. Phys. D, 2011. (Submitted)

A new family of solutions to the Einstein field equations describing the collapse of a fluid sphere in the presence of heat flux and shear. These solutions ensure that the collapsing fluid is accelerating and provide a generalisation of the geodesic fluid models studied in earlier treatments.

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Chapter 1

Introduction

Relativistic physics is one of the cornerstones of modern theoretical physics. Astrophysics, a branch of relativistic physics, has been striving to explain physical phenomena associated with a bounded local system. For example, a star is of particular interest in the end state of gravitational collapse. The laws of physics affect large gravitational objects as they approach the end state and its possible outcomes, namely black holes, neutron stars, super nova and quark stars [1]. These objects are of particular interest because this is when general relativity is pushed to the limits of it's explanatory power. This brings us to the observation that we know the star undergoes some form of collapse but how do we model such a scenario? With the progress made in observational astrophysics, it has become necessary to model more realistic scenarios of gravitational collapse.

The first significant solution to describe a star's gravitational field was the exterior Schwarzschild solution [2] which gives the exterior gravitational field of a static star with a spherically symmetric mass distribution. This was later expanded by Schwarzschild to include the interior of a star, treating the star as a static incompressible fluid sphere [3]. Reinforcing Schwarzschild's solution was Birkhoff's theorem which indicates that the spacetime geometry outside a general spherically symmetric matter distribution is the Schwarzschild geometry [4].

From this basic initial solution more phenomena were perceived as significant contributors to gravitational collapse, namely anisotropy, bulk viscosity, shear, the electromagnetic field and energy dissipation through radiative processes [5],[6]. One of these phenomena that Birkhoff's theorem did not account for was the radiating nature of a gravitational body. Consequently the Schwarzschild solution needs to be extended to include radiation. The Vaidya solution [7], describes radially directed null radiation creating an atmosphere around the gravitating body. The Vaidya solution is thus a very natural model for stars in comparison to the Schwarzschild solution, as it is clear that stars do radiate matter. The Vaidya solution has also been extended to include charge and the cosmological constant [8] although these will not be covered in this thesis. A further phenomenon that is significant but has not been pursued to its full extent due to the difficulty in modeling, is the effect of rotation.

With the discovery of the Vaidya solution, it became possible to model radiating stars and to investigate dissipative collapse. The junction conditions required for the smooth matching of the interior spacetime to the Vaidya solution was first provided by Santos in 1985 [9]. An important consequence of the junction conditions is that the pressure on the boundary is nonvanishing. This ensures the continuity of the momentum flux across the boundary. Earlier treatments of radiating stellar models were restricted to shear-free, spherically symmetric interiors. These models were generalized to include shear, electromagnetic field, bulk viscosity and the cosmological constant [10]. The first shearing, radiating stellar model within this framework was provided by Naidu *et al* [11] in which the particles were assumed to undergo geodesic motion. These results were extended by Thirukkanesh and Maharaj [12]. Herrera and Santos [15] proposed a shearing model in which the areal radius was always equal to the proper stellar radius, and referred to these models as Euclidean stars. The first exact solution describing Euclidean stars was presented by Govender *et al* [13]. These models were subsequently generalized by Govinder and Govender [14] who found exact solutions describing Euclidean stars with a barotropic equation of state. It was further shown that these models exhibit the horizon-free property, ie, the rate of collapse balances the rate of energy emission preventing the formation of the horizon.

Another important physical aspect to consider is the thermodynamics of the collapsing star. Historically most researchers utilized Eckart's theory to model heat transfer within the stellar core. There are various problems associated with the theory. First and foremost, the theory is noncausal, ie, it predicts superluminous velocities for the dissipative fluxes. Secondly the Eckart theory yields unstable equilibrium states [16]. When the temperature inside an object changes then that temperature change is not instantaneous across the whole object especially if it is massive (like a star). Therefore a relaxation time needs to be included such that it allows the temperature change to occur gradually across the star. We will thus investigate dissipative processes within the framework of extended irreversible thermodynamics [18],[19],[20]. This framework has been used before by others including the treatment of Maharaj *et al* [17].

It is important to consider existing shearing radiating models. Most solutions found do not include shear but include other physical phenomena occurring in the star [9]. Some of the early attempts to incorporate shear were made by Herrera and Martinez [21] but they principally dealt with bulk viscosity. It took the treatment of Chan [22], to include shear viscosity for the first time although there were some issues with his approach. A more comprehensive and clear shear viscosity analysis was completed by Govender *et al* [24]. Their paper did not produce solutions but put together a model for the problem. The first exact model for this framework, where the fluid is acceleration-free, was presented by Naidu *et al* [11]. This model was subsequently generalised by Thirukkanesh and Maharaj [12]. An interesting class of shearing models are the Euclidean stars first proposed by Herrera and Santos [15]. These models exhibit the special feature that the proper radius equals the areal radius throughout the collapse process. Some further exact solutions involving shear can be seen by Di Prisco *et al* [23].

In this thesis we provide a new class of solutions which describe a shearing interior spacetime with acceleration thus providing a wider spectrum of models than the geodesic radiating models.

A quick overview of what is included in each chapter is included below.

In chapter 2, we describe the most basic aspects of relativistic astrophysics by considering differential geometry and the associated field equations. We illustrate how to mathematically describe the various energy conditions or physical aspects of our spacetime given by the radial pressure, density, heat flux and transverse pressure. We also present Vaidya's exterior radiating solution.

Chapter 3 involves the study of stellar boundaries, commonly known in the astrophysics community as "Junction Conditions." We consider the interior spacetime and the exterior spacetime that match at the surface of the star. By matching the exterior and interior at this boundary we demand that the two spacetimes and the extrinsic curvature are continuous at this surface. This matching process results in new constraints to our system that ultimately helps us in making predictions about the system.

Chapter 4 covers irreversible thermodynamics relevant to relativistic astrophysics. In particular, we study the previous work of Eckart [25] and more modern results that include causality to model the thermodynamical nature of a star.

Chapter 5 considers a specific model of a radiating shearing star drawing on what was discussed in chapters two, three and four. A new exact solution is found which is shearing and accelerating. We perform a physical analysis of this solution and show that it is physically reasonable.

Chapter 2

Fundamentals of Relativistic Astrophysics

2.1 Introduction

In this chapter we provide a brief overview of differential geometry that is required for our investigation. In §2.2 we introduce the notion of a four-dimensional psuedo-Riemannian manifold representing the geometry of spacetime. The element required to describe the curvature of spacetime, namely the Riemann tensor, and associated quantities are introduced here. The Einstein field equations, in its generality, are also presented in this subsection. In §2.3 we consider a general spherically symmetric shearing line element which describes the interior of the stellar model. The kinematical quantities such as the acceleration, shear and expansion are calculated for this line element. We assume a perfect fluid energy momentum tensor with heat flux. We derive the Einstein field equations describing the gravitational behaviour of the interior fluid. The exterior Vaidya solution which is the unique spherically symmetric solution representing null radiation is presented in §2.4. The solution describes the exterior spacetime of a relativistic radiating star.

2.2 Differential Geometry and Field Equations

In general relativity we take spacetime to be a four-dimensional differentiable manifold which is defined by a symmetric nonsingular metric tensor field $\boldsymbol{g}$ [26]. The metric tensor field has a signature $(- + + +)$. Coordinates in the manifold are local and given by $(x^a) = (x^0, x^1, x^2, x^3)$ where x^0 is timelike and the other components are spacelike. For this thesis the speed of light and the coupling constant are taken to be unity.

The invariant distance between two infinitesimally separated points in spacetime is defined by [26]

$$ds^2 = g_{ab}dx^a dx^b \tag{2.2.1}$$

which is the most general form of the line element. The connection coefficients, which represent a connection on the manifold, connect spaces of vectors and tensors at nearby events through parallel transport [27]. They are defined in terms of $\boldsymbol{g}$ and it's derivatives (partial derivatives are denoted by commas) to give the following,

$$\Gamma^a{}_{bc} = \frac{1}{2}g^{ad}(g_{cd,b} + g_{db,c} - g_{bc,d}) \tag{2.2.2}$$

The fundamental theorem of Riemann geometry ensures the existence of a unique connection coefficient which preserves the inner product.

The Riemann tensor, $\boldsymbol{R}$, which is a measure of curvature in spacetime, is created from these connection coefficients [26],

$$R^c{}_{abc} = \Gamma^d{}_{ac,b} - \Gamma^d{}_{ab,c} + \Gamma^e{}_{ac}\Gamma^d{}_{eb} - \Gamma^e{}_{ab}\Gamma^d{}_{ec} \tag{2.2.3}$$

By contracting the Riemann tensor (2.2.3) the symmetric Ricci tensor is obtained,

$$\begin{aligned} R_{ab} &= R^c{}_{acb} \qquad (2.2.4) \\ &= \Gamma^c{}_{ab,c} - \Gamma^c{}_{ac,b} + \Gamma^c{}_{dc}\Gamma^d{}_{ab} - \Gamma^c{}_{db}\Gamma^d{}_{ac} \end{aligned}$$

A further contraction of the Ricci tensor (2.2.4) results in the Ricci scalar. This is a scaling constant that assigns a single real number to each point on the Riemannian manifold. As can be seen by the following equation it is determined by the intrinsic geometry of the manifold [26]

$$R = R^a{}_a \tag{2.2.5}$$

$$= g^{ab} R_{ab} \tag{2.2.6}$$

Using the Ricci tensor (2.2.4) and the Ricci scalar (2.2.5) it is possible to construct the Einstein tensor, $\boldsymbol{G}$,

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab} \tag{2.2.7}$$

As both the Ricci tensor and the metric coefficient are symmetric one can see that $\boldsymbol{G}$, in turn, is symmetric. Another property of the Einstein tensor is that it has zero divergence such that (where semicolons denote covariant differentiation),

$$G^{ab}{}_{;b} = 0 \tag{2.2.8}$$

This property follows from the definition of the Einstein tensor (2.2.7). This property is sometimes called the Bianchi identity, and it is a necessary condition to generate the conservation equations via the Einstein equations.

After defining $\boldsymbol{G}$ we need to look at the energy momentum tensor, $\boldsymbol{T}$, which has been separated into three distinct terms,

$$T_{ab} = T^{(1)}_{ab} + T^{(2)}_{ab} + T^{(3)}_{ab} \tag{2.2.9}$$

We decompose these terms to get the following representations

The first term

$$T^{(1)}_{ab} = (\mu + p) u_{ab} + p g_{ab} \tag{2.2.10}$$

represents the perfect fluid where $\boldsymbol{u}$ is a unit timelike four-velocity vector, μ is the energy density and p is the pressure.

The second term

$$T^{(2)}_{ab} = q_a u_b + q_b u_a + (p_r - p_\perp) \chi_a \chi_b \tag{2.2.11}$$

represents the dynamically anisotropic stress energy tensor where q_a is the heat flow vector, p_r is the radial pressure and $p_\perp$ is the transverse pressure. Note that the vector χ_a is orthogonal to u^a. The heat flow vector and stress tensor satisfy the conditions

$$q^a u_b = 0 \tag{2.2.12}$$

$$\Pi^{ab} u_b = 0 \tag{2.2.13}$$

where $\Pi_{ab} = (p_r - p_\perp)\chi_a\chi_b$. These quantities are defined relative to the fluid four-velocity [8] where the heat flow vector q_a is given by

$$q^a = (0, q, 0, 0)\,, \tag{2.2.14}$$

The third item

$$T^{(3)}_{ab} = \epsilon \xi_a \xi_b \tag{2.2.15}$$

is the energy momentum tensor for null radiation where ϵ is the radiation energy density and ξ_a is the null four-vector.

The Einstein field equations can thus be written as

$$G_{ab} = T_{ab} \tag{2.2.16}$$

where we use geometrized units. The Einstein field equations (2.2.16) couple spacetime geometry (2.2.7) to the matter content (2.2.9).

2.3 Spherically Symmetric, Shearing Spacetimes

The most general line element for a spherically symmetric spacetime in the absence of rotation and in comoving coordinates can be written as,

$$ds^2 = -A^2dt^2 + B^2dr^2 + Y^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.3.1}$$

where A, B and Y are all functions of the coordinates t and r. The fluid four-velocity $\boldsymbol{u}$ is timelike and is given by

$$u^a = \frac{1}{A}\delta^a_0 \tag{2.3.2}$$

The kinematical quantities are given by σ (the magnitude of the shear), $\dot{u}^a$ (the four-acceleration vector), Θ (the expansion scalar) and ν_{ab} (the vorticity tensor).

The expansion scalar and the fluid four-acceleration are given by the equations

$$\Theta = u^a{}_{;a}, \qquad\qquad a_a = u_{a;b}u^b,$$

and the shear tensor by

$$\sigma_{ab} = u_{(a;b)} + a_{(a}u_{b)} - \frac{1}{3}\Theta(g_{ab} + u_a u_b).$$

These kinematical quantities contribute to our knowledge about the evolution of the system in question. Later we will draw from them and comment on the physical nature of the system. The kinematical quantities for the line element (2.3.1) are

$$\dot{u}^a = \left(0, \frac{A'}{A}, 0, 0\right) \tag{2.3.3a}$$

$$\Theta = \frac{1}{A}\left(\frac{\dot{B}}{B} + 2\frac{\dot{Y}}{Y}\right) \tag{2.3.3b}$$

$$\sigma^1_1 = \sigma^2_2 = -\frac{1}{2}\sigma^3_3 = \frac{1}{3A}\left(\frac{\dot{Y}}{Y} - \frac{\dot{B}}{B}\right) \tag{2.3.3c}$$

$$\nu_{ab} = 0 \tag{2.3.3d}$$

where dots and primes denote differentiation with respect to t and r respectively.

Using equation (2.2.3) we find the nonzero Ricci tensor components to be,

$$
\begin{aligned}
R_{00} &= -\frac{\ddot{B}}{B} + \frac{\dot{A}}{A}\frac{\dot{B}}{B} + 2\frac{\dot{A}}{A}\frac{\dot{Y}}{Y} - 2\frac{\ddot{Y}}{Y} \\
&\quad + \frac{A^2}{B^2}\left(\frac{A''}{A} - \frac{A'}{A}\frac{B'}{B} + 2\frac{A'}{A}\frac{Y'}{Y}\right) \\
R_{01} &= 2\left(\frac{\dot{B}}{B}\frac{Y'}{Y} + \frac{A'}{A}\frac{Y'}{Y} - \frac{\dot{Y}'}{Y}\right) \\
R_{11} &= -\frac{A''}{A} + \frac{A'}{A}\frac{B'}{B} + 2\frac{B'}{B}\frac{Y'}{Y} - 2\frac{Y''}{Y} \\
&\quad + \frac{B^2}{A^2}\left(\frac{\ddot{B}}{B} - \frac{\dot{A}}{A}\frac{\dot{B}}{B} + 2\frac{\dot{B}}{B}\frac{\dot{Y}}{Y}\right) \\
R_{22} &= \frac{Y\dot{Y}}{A^2}\left(\frac{\dot{B}}{B} - \frac{\dot{A}}{A} + \frac{\dot{Y}}{Y} + \frac{\ddot{Y}}{\dot{Y}}\right) \\
&\quad + \frac{YY'}{B^2}\left(\frac{B'}{B} - \frac{A'}{A} - \frac{Y'}{Y} - \frac{Y''}{Y'}\right) + 1 \\
R_{33} &= \sin^2\theta R_{22}
\end{aligned}
$$

for the line element (2.3.1). From the above components we find the Ricci scalar (2.2.5) to be,

$$
\begin{aligned}
R &= \frac{2}{A^2}\left(\frac{\ddot{B}}{B} - \frac{\dot{A}}{A}\frac{\dot{B}}{B} + 2\frac{\dot{B}}{B}\frac{\dot{Y}}{Y} - 2\frac{\dot{A}}{A}\frac{\dot{Y}}{T} + \frac{\dot{Y}^2}{Y^2} + 2\frac{\ddot{Y}}{Y}\right) \\
&\quad - \frac{2}{B^2}\left(\frac{A''}{A} - \frac{A'}{A}\frac{B'}{B} - 2\frac{B'}{B}\frac{Y'}{Y} + 2\frac{A'}{A}\frac{Y'}{Y} + \frac{Y'^2}{Y} + 2\frac{Y''}{Y}\right) + \frac{2}{Y^2} \qquad (2.3.4)
\end{aligned}
$$

for the line element (2.3.1).

By making use of the above components, (2.2.7) and (2.3.4) we obtain the Einstein tensor components, as follows,

$$G_{00} = 2\frac{\dot{B}}{B}\frac{\dot{Y}}{Y} + \frac{\dot{Y}^2}{Y^2} - \frac{A^2}{B^2}\left(-2\frac{B'}{B}\frac{Y'}{Y} + \frac{Y'^2}{Y^2} + 2\frac{Y''}{Y}\right) + \frac{A^2}{Y^2} \tag{2.3.5a}$$

$$G_{01} = 2\left(\frac{\dot{B}}{B}\frac{Y'}{Y} + \frac{A'}{A}\frac{Y'}{Y} - \frac{\dot{Y}'}{Y}\right) \tag{2.3.5b}$$

$$G_{11} = \frac{B^2}{A^2}\left(2\frac{\dot{A}}{A}\frac{\dot{Y}}{Y} - \frac{\dot{Y}^2}{Y^2} - 2\frac{\ddot{Y}}{Y}\right) + 2\frac{A'}{A}\frac{Y'}{Y} + \frac{Y'^2}{Y^2} - \frac{B^2}{Y^2} \tag{2.3.5c}$$

$$\begin{aligned} G_{22} = & -\frac{Y^2}{A^2}\left(\frac{\ddot{B}}{B} - \frac{\dot{A}}{A}\frac{\dot{B}}{B} + \frac{\dot{B}}{B}\frac{\dot{Y}}{Y} - \frac{\dot{A}}{A}\frac{\dot{Y}}{Y} + \frac{\ddot{Y}}{Y}\right) \\ & + \frac{Y^2}{B^2}\left(\frac{Arr}{A} - \frac{A'}{A}\frac{B'}{B} + \frac{A'}{A}\frac{Y'}{Y} - \frac{B'}{B}\frac{Y'}{Y} + \frac{Y''}{Y}\right) \end{aligned} \tag{2.3.5d}$$

$$G_{33} = \sin^2\theta G_{22} \tag{2.3.5e}$$

which are in a simple form that is consistent with spherical symmetry. The Einstein field equations for the interior of an anisotropic shearing matter distribution may be derived. The following illustrates the physical aspects of the star that we glean from the Einstein field equations,

$$\mu = \frac{1}{A^2}\left(2\frac{\dot{B}}{B}+\frac{\dot{Y}}{Y}\right)\frac{\dot{Y}}{Y} - \frac{1}{B^2}\left[2\frac{Y''}{Y}+\left(\frac{Y'}{Y}\right)^2 - 2\frac{B'}{B}\frac{Y'}{Y} - \left(\frac{B}{Y}\right)^2\right] \quad (2.3.6a)$$

$$p_r = -\frac{1}{A^2}\left[2\frac{\ddot{Y}}{Y} - \left(2\frac{\dot{A}}{A}-\frac{\dot{Y}}{Y}\right)\frac{\dot{Y}}{Y}\right] + \frac{1}{B^2}\left(2\frac{A'}{A}+\frac{Y'}{Y}\right)\frac{Y'}{Y} - \frac{1}{Y^2} \quad (2.3.6b)$$

$$p_\perp = -\frac{1}{A^2}\left[\frac{\ddot{B}}{B}+\frac{\ddot{Y}}{Y}-\frac{\dot{A}}{A}\left(\frac{\dot{B}}{B}+\frac{\dot{Y}}{Y}\right)+\frac{\dot{B}}{B}\frac{\dot{Y}}{Y}\right]$$

$$+\frac{1}{B^2}\left[\frac{A''}{A}+\frac{Y''}{Y}-\frac{A'}{A}\frac{B'}{B}+\left(\frac{A'}{A}-\frac{B'}{B}\right)\frac{Y'}{Y}\right] \quad (2.3.6c)$$

$$q = \frac{2}{AB^2}\left(\frac{\dot{Y}'}{Y}-\frac{\dot{B}}{B}\frac{Y'}{Y}-\frac{\dot{Y}}{Y}\frac{A'}{A}\right) \quad (2.3.6d)$$

where μ is the energy density within the star, and q is the measure of heat flow.

To illustrate how these equations can be reduced to a simpler form we simply let the shear vanish by letting

$$\frac{\dot{Y}}{Y} = \frac{\dot{B}}{B} \quad (2.3.7)$$

This results in the line element changing from (2.3.1) to the following:

$$ds^2 = -A^2dt^2 + B^2\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \quad (2.3.8)$$

This line element is simultaneously comoving and isotropic. The Einstein field equations, for this line element, now reduce to,

$$\mu = \frac{3}{A^2}\frac{\dot{B}^2}{B^2} - \frac{1}{B^2}\left(2\frac{B''}{B} - \frac{B'^2}{B^2} + \frac{4}{r}\frac{B'}{B}\right) \tag{2.3.9a}$$

$$\begin{aligned} p_r &= \frac{1}{A^2}\left(-2\frac{B\ddot{B}}{B} - \frac{\dot{B}3^2}{B^2} + 2\frac{\dot{A}}{A}\frac{\dot{B}}{B}\right) \\ &\quad + \frac{1}{B^2}\left(\frac{B'^2}{B^2} + 2\frac{A'}{A}\frac{B'}{B} + \frac{2}{r}\frac{A'}{A} + \frac{2}{r}\frac{B'}{B}\right) \end{aligned} \tag{2.3.9b}$$

$$\begin{aligned} p_\perp &= -2\frac{1}{A^2}\frac{\ddot{B}}{B} + 2\frac{\dot{A}}{A^3}\frac{\dot{B}}{B} - \frac{1}{A^2}\frac{\dot{B}^2}{B^2} + \frac{1}{r}\frac{A'}{A}\frac{1}{B^2} \\ &\quad + \frac{1}{r}\frac{B'}{B^3} + \frac{A''}{A}\frac{1}{B^2} - \frac{B'^2}{B^4} + \frac{B''}{B^3} \end{aligned} \tag{2.3.9c}$$

$$q = -\frac{2}{AB^2}\left(-\frac{\dot{B}'}{B} + \frac{B'\dot{B}}{B^2} + \frac{A'}{A}\frac{\dot{B}}{B}\right) \tag{2.3.9d}$$

Thus we see how the field equations define the gravitational interactions of a stellar body in the simple shear-free case.

2.4 Exterior Spacetime: Vaidya Solution

Previously the exterior solutions for gravitational collapse, based on the Schwarzschild model for exterior spacetime, suffered from neglecting the radiative heat flow from the star [28]. It was with the advent of the Vaidya solution that a more realistic model of radiative gravitational collapse, where the collapsing core radiated energy to the exterior spacetime, came to pass [7].

Vaidya's solution is given by the following metric,

$$ds^2 = -\left(1 - \frac{2m(v)}{\mathsf{r}}\right)dv^2 - 2dvd\mathsf{r} + \mathsf{r}^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{2.4.1}$$

where v represents the retarded time coordinate. In this result $m(v)$ can be interpreted as the Newtonian mass of the body generating the gravity as observed at infinity, and essentially represents gravitational energy contained within the radius of the star. In order to separate the interior and exterior spacetimes it should be noted that we use the coordinate v and r to represent the temporal and radial coordinates for the exterior spacetime. Using (2.2.3) we find the only nonzero Ricci tensor component to be

$$R_{00} = -\frac{2}{\mathsf{r}^2}\frac{dm}{dv} \tag{2.4.2}$$

The Ricci scalar for the Vaidya line element (2.4.1) vanishes giving $R = 0$.

Thus we find for the Einstein tensor (2.2.7) in this case to be,

$$G_{ab} = -\frac{2}{\mathsf{r}^2}\frac{dm}{dv}\delta^0_a\delta^0_b \tag{2.4.3}$$

and the energy momentum tensor for null radiation is given by

$$T^{(3)}_{ab} = \epsilon\xi_a\xi_b \tag{2.4.4}$$

For the Vaidya solution the null four-vector is given by $\xi_a = (1, 0, 0, 0)$. Thus using (2.4.3) and (2.4.4) we have for the energy density of the null radiation

$$\epsilon = -\frac{2}{\mathsf{r}^2}\frac{dm}{dv} \tag{2.4.5}$$

Since the star is radiating energy to the exterior spacetime we have that $\frac{dm}{dv} \leq 0$.

Chapter 3

Chapter 3: Junction Conditions

3.1 Introduction

The boundary of a radiating star divides spacetime into two distinct regions, the interior spacetime described by a general shearing line element in the absence of rotation and the exterior spacetime represented by Vaidya's solution. Since the star is radiating energy to the exterior spacetime, there is associated with this radiation a momentum flux across the stellar boundary. In order for the flux to be continuous across the boundary, we require that the interior spacetime match smoothly to the exterior Vaidya spacetime. The junction conditions required for the smooth matching of the two spacetimes was first provided by Santos [9]. It was shown that a spherically symmetric, shear-free line element could be matched to Vaidya's outgoing solution. If we further assume that the stellar interior is dissipating energy in the form of a radial heat flux, then the junction conditions imply that the pressure on the surface of the star is nonvanishing. The Santos junction conditions have subsequently been generalised to include the electromagnetic field, shear, bulk viscosity, cosmological constant as well as matching to the generalised Vaidya solution [29]. In section §3.2 we introduce the general conditions required for the smooth matching of the interior spacetime to the exterior spacetime across a timelike hypersurface. In section §3.3 we match the interior spacetime to the timelike hypersurface which defines the boundary of the star. The exterior spacetime is matched to the hypersurface in §3.4. In §3.5 we provide the complete set of junction conditions required for the smooth matching of the interior spacetime to the exterior Vaidya solution.

3.2 Line Elements and Matching Conditions

The interior and exterior spacetimes are both described by distinct, smooth four-dimensional manifolds, $\mathcal{M}^-$ for the interior and $\mathcal{M}^+$ for the exterior, the boundary being Σ which is a timelike hypersurface. The Darmois conditions were used for the derivation of the junction conditions as they were shown to be the most convenient and reliable [30].

We assume that Σ has an intrinsic metric $g_{\alpha\beta}$ where the intrinsic co-ordinates are given by ξ^a where $\alpha = 1, 2, 3$. On the surface Σ we have

$$ds^2_\Sigma = g_{\alpha\beta} d\xi^\alpha d\xi^\beta \tag{3.2.1}$$

The line elements for the interior and exterior spacetimes ($\mathcal{M}^\pm$), respectively, assume the form

$$ds^2{}_\pm = g_{ab} d\mathcal{X}^a_\pm d\mathcal{X}^b_\pm \tag{3.2.2}$$

The coordinates in ($\mathcal{M}^\pm$) are $\chi^a_\pm$, where $a = 0, 1, 2, 3$. When approaching the boundary we demand that

$$(ds^2_\pm)_\Sigma = ds^2_\Sigma \tag{3.2.3}$$

where $(\ )_\Sigma$ represents the value of ( ) at the boundary. It can be seen then that the coordinates of ($\mathcal{M}^\pm$) are given by $\mathcal{X}^a_\pm = \mathcal{X}^a_\pm(\xi^\alpha)$.

The continuity of Σ results in the first junction condition. The second junction condition arises from the requirement that there is continuity of the extrinsic curvature of Σ across the boundary, such that,

$$K^{+}_{\alpha\beta} = K^{-}_{\alpha\beta} \tag{3.2.4}$$

where

$$K^{\pm}_{\alpha\beta} \equiv -n^{\pm}_{a}\frac{\partial^2 \mathcal{X}^{a}_{\pm}}{\partial\xi^{\alpha}\partial\xi^{\beta}} - n^{\pm}_{a}\Gamma^{a}{}_{cd}\frac{\partial\mathcal{X}^{c}_{\pm}}{\partial\xi^{\alpha}}\frac{\partial\mathcal{X}^{d}_{\pm}}{\partial\xi^{\beta}} \tag{3.2.5}$$

The quantities $n^{\pm}_{a}(\mathcal{X}^{b}_{\pm})$ are the components of the vector normal to the boundary Σ. These junction conditions are not necessarily the only junction conditions that have been used. In various papers different junction conditions have been used at the boundary. For a more detailed analysis of these different junction conditions, used in the literature, see [30]. For a deeper discussion of junction conditions at the boundary in general relativity see [31]. The junction conditions described here are the same as those used in [32] and [33].

3.3 Matching the interior to the boundary

The intrinsic metric of Σ is given by

$$ds^2_{\Sigma} = -d\eta^2 + \mathcal{Y}^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{3.3.1}$$

with coordinates $\xi^{\alpha} = (\eta, \theta, \phi)$ and $\mathcal{Y} = \mathcal{Y}(\eta)$. It must be noted that the time coordinate η is defined **only** on the boundary Σ. Using comoving coordinates, we take $\mathcal{M}^{-}$, the interior spacetime, to be described by the most general spherically symmetric line element as defined by equation (2.3.1), repeated below,

$$ds^2 = -A^2dt^2 + B^2dr^2 + Y^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{3.3.2}$$

The boundary of the interior matter distribution is given by

$$f(r,t) = r - r_{\Sigma} = 0 \tag{3.3.3}$$

where the radius at the boundary (r_Σ) is constant. The vector with components $\dfrac{\partial f}{\partial \mathcal{X}^a_-}$ is orthogonal to Σ. Hence the unit vector normal to Σ, n^-_a, must be of the form,

$$n^-_a = [0, B(r_\Sigma, t), 0, 0] \tag{3.3.4}$$

The first junction condition (3.2.3) for metrics (3.3.1) and (3.3.2) will result in the restrictions

$$A(r_\Sigma, t)\dot{t} \;=\; 1 \tag{3.3.5}$$

$$Y(r_\Sigma, t) \;=\; \mathcal{Y}(\eta) \tag{3.3.6}$$

where dots represent differentiation with respect to η. The extrinsic curvature $K^-_{\alpha\beta}$ is calculated from (2.2.2),(3.3.2),(3.2.5) and (3.3.4). The nonzero components of $K^-_{\alpha\beta}$ are,

$$K^-_{\eta\eta} \;=\; \left(-\frac{1}{B}\frac{A'}{A}\right)_\Sigma \tag{3.3.7a}$$

$$K^-_{\theta\theta} \;=\; \left(\frac{YY'}{B}\right)_\Sigma \tag{3.3.7b}$$

$$K^-_{\phi\phi} \;=\; \sin^2\theta\, K^-_{\theta\theta} \tag{3.3.7c}$$

which is valid on the surface Σ.

3.4 Matching the exterior to the boundary

We describe the exterior spacetime manifold, $\mathcal{M}^+$, of the radiating star using the Vaidya line element equation 2.4.1 as outlined in chapter 2:

$$ds^2 = -\left(1 - \frac{2m(v)}{\mathsf{r}}\right)dv^2 - 2dvd\mathsf{r} + \mathsf{r}^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{3.4.1}$$

Similar to the interior spacetime case, the equation at the boundary Σ of $\mathcal{M}^+$ is

$$f(\mathsf{r}, v) = \mathsf{r} - \mathsf{r}_\Sigma(v) = 0 \tag{3.4.2}$$

Note that $\mathsf{r}_\Sigma(v)$ is a function of v. From the above we find the orthogonal vector to Σ is given by

$$\frac{\partial f}{\partial \mathcal{X}_+^a} = \left(-\frac{d\mathsf{r}_\Sigma}{dv}, 1, 0, 0\right) \tag{3.4.3}$$

The unit normal to Σ can then be interpreted as

$$n_a^+ = \left(1 - \frac{2m}{\mathsf{r}_\Sigma} + 2\frac{d\mathsf{r}_\Sigma}{dv}\right)^{-\frac{1}{2}}\left(-\frac{d\mathsf{r}_\Sigma}{dv}, 1, 0, 0\right) \tag{3.4.4}$$

For $\mathcal{M}^+$, the first junction condition (3.2.3), for the line elements (3.3.1 and 3.4.1) generates equations similar to that seen in the interior case,

$$\mathsf{r}_\Sigma(v) = \mathcal{Y}(\eta) \tag{3.4.5a}$$

$$\left(1 - \frac{2m}{\mathsf{r}} + 2\frac{d\mathsf{r}}{dv}\right)_\Sigma = \left(\frac{1}{\dot{v}^2}\right)_\Sigma \tag{3.4.5b}$$

Using (3.4.5b) we can reformulate equation (3.4.4) to read,

$$n_a^+ = (-\dot{\mathsf{r}}, \dot{v}, 0, 0) \tag{3.4.6}$$

In order to find K^+ for the exterior spacetime according to equation (3.2.5) we must perform some long and tedious calculations to find the nonvanishing components of the extrinsic curvature tensor, as given below

$$K^{+}_{\eta\eta} = \left[\frac{\ddot{v}}{\dot{v}} - \dot{v}\frac{m}{\mathsf{r}^2}\right]_{\Sigma} \tag{3.4.7a}$$

$$K^{+}_{\theta\theta} = \left[\dot{v}\left(1 - \frac{2m}{\mathsf{r}}\right)\mathsf{r} + \mathsf{r}\dot{\mathsf{r}}\right]_{\Sigma} \tag{3.4.7b}$$

$$K^{+}_{\phi\phi} = \sin^2\theta\, K^{+}_{\theta\theta} \tag{3.4.7c}$$

As can be seen by the subscript Σ, these forms are valid at the boundary.

3.5 Summary

The first junction condition for the interior and exterior spacetimes from §3.3 and §3.4 can be written as,

$$A(r_{\Sigma}, t)\dot{t} = 1 \tag{3.5.1}$$

$$Y(r_{\Sigma}, t) = \mathcal{Y}(\eta) \tag{3.5.2}$$

$$\mathsf{r}_{\Sigma}(v) = \mathcal{Y}(\eta) \tag{3.5.3}$$

$$\left(1 - \frac{2m}{\mathsf{r}} + 2\frac{d\mathsf{r}}{dv}\right)_{\Sigma} = \left(\frac{1}{\dot{v}^2}\right)_{\Sigma} \tag{3.5.4}$$

As η was defined as only an intermediate variable we can eliminate it from the above equations. We thus find that both the sufficient and necessary conditions on the spacetime for the first junction condition to be valid are

$$A(r_{\Sigma}, t)dt = \left(1 - \frac{2m}{\mathsf{r}_{\Sigma}} + 2\frac{d\mathsf{r}_{\Sigma}}{dv}\right)^{\frac{1}{2}} dv \tag{3.5.5}$$

$$Y(r_{\Sigma}, t) = \mathsf{r}_{\Sigma}(v) \tag{3.5.6}$$

By equating the appropriate extrinsic curvature components in (3.4.7) and (3.3.7) we generate the second set of junction conditions (3.2.4) These are given by

$$\left(-\frac{1}{B}\frac{A'}{A}\right)_\Sigma = \left[\frac{\ddot{v}}{\dot{v}} - \dot{v}\frac{m}{\mathsf{r}^2}\dot{v}\right]_\Sigma \tag{3.5.7}$$

$$\left(\frac{YY'}{B}\right)_\Sigma = \left[\dot{v}(1 - \frac{2m}{\mathsf{r}} + \mathsf{r}\dot{\mathsf{r}}\right]_\Sigma \tag{3.5.8}$$

After eliminating r, $\dot{\mathsf{r}}$ and $\dot{v}$ we can find the expression of the mass function in terms of the metric function. After a long calculation we find that,

$$m(v) = \left[\frac{Y}{2}\left(1 + \frac{\dot{Y}^2}{A^2} - \frac{Y'^2}{B^2}\right)\right]_\Sigma \tag{3.5.9}$$

We can take $m(v)$ as representing the total gravitational mass within the surface Σ. We can also use (3.3.6), (3.3.5) and (3.4.5a) to write

$$\dot{\mathsf{r}}_\Sigma = \left(\frac{\dot{Y}}{A}\right)_\Sigma \tag{3.5.10}$$

Using the above for $\dot{\mathsf{r}}_\Sigma$ and substituting (3.5.9) in (3.5.8) we obtain

$$\dot{v}_\Sigma = \left(\frac{\dot{Y}}{A} + \frac{Y'}{B}\right)^{-1}_\Sigma \tag{3.5.11}$$

We now differentiate the above with respect to η and make use of (3.3.5) and (3.3.6) to write

$$\dot{v}_\Sigma = \left[-\frac{1}{A}\left(\frac{\dot{Y}}{A} + \frac{Y'}{B}\right)^{-2}\left(\frac{\dot{Y}'}{B} - \frac{\dot{B}Y'}{B^2} - \frac{\dot{A}\dot{Y}}{A^2} + \frac{\ddot{Y}}{A}\right)\right]_\Sigma \tag{3.5.12}$$

From here we substitute (3.3.6), (3.4.5a), (3.5.9), (3.5.11) and (3.5.12) into (3.5.7) which produces,

$$\begin{aligned}\left(-\frac{1}{B}\frac{A'}{A}\right)_\Sigma = &\left[\left(-\frac{\dot{Y}'}{B} + \frac{\dot{B}Y'}{B^2} + \frac{\dot{A}\dot{Y}}{A^2} - \frac{\ddot{Y}}{A} - \frac{\dot{Y}^2}{2AY} + \frac{A}{2Y}\left(\frac{Y'^2}{B^2} - 1\right)\right)\times\right.\\ &\left.\times\left(\frac{\dot{Y}}{A} + \frac{Y'}{B}\right)^{-1}\right]_\Sigma\end{aligned} \tag{3.5.13}$$

When we multiply the above by $\left(\frac{\dot{Y}}{A}+\frac{Y'}{B}\right)$ and simplify, we come to the following result (the same as Santos [9])

$$(p_r)_\Sigma = (qB)_\Sigma \tag{3.5.14}$$

It is clear from (3.5.14) that the radial pressure, at the boundary of the dissipating star, can only be zero when q becomes zero. In this case there is no radial heat flow and the exterior spacetime consequently is not the Vaidya spacetime but is described by the vacuum Schwarzschild spacetime.

Chapter 4

Thermodynamics

4.1 Introduction

A common approach in astrophysics is to look at processes as being perfect fluid approximations in which a fluid in equilibrium generates no entropy and no "frictional" type heating occurs. This is due to the dynamics being reversible and without dissipation. While this is adequate for some processes, it is not suitable for others, such as dissipative phenomena. Dissipative effects require us to develop a relativistic theory of dissipative fluids and treat the fluid as behaving irreversibly. This form of irreversible thermodynamics has been covered extensively in [18]. This paper does cover the relativistic thermodynamical processes but it focuses on the nonrelativistic case. For a full relativistic description the reader is referred to [19],[20],[34]. As has been mentioned in [16] the most comprehensive approach to irreversible thermodynamics is via nonequilibrium kinetic theory. It however, was also pointed out that this method is very complicated and a standard phenomenological approach is more than adequate and much simpler to communicate. Reference can still be made to relativistic kinetic theory to support results.

4.2 Irreversible Thermodynamics

Irreversible thermodynamics was first extended to the relativistic sphere of physics by Eckart in 1940 [25]. This theory and it's later modified version by Landau and Lifschitz in 1959 [35], had the same major problem as in the Newtonian theory. Dissipative perturbations are assumed to propagate instantaneously throughout the fluid (essentially achieving super-luminal speeds).

This propagation assumption, while adequate for small scale nonrelativistic objects, does not work well with relativistic theory and the massive objects it deals with. If we choose to use this assumption in relativity theory then we find the equilibrium states that are predicted are, in fact, unstable.

Mueller developed an extended non-relativistic version of irreversible thermodynamics that included causality [36]. This was, later, extended independently by Israel and Stewart [20], to a relativistic form. The extended theory was known due to two influential aspects. It takes into account causality in thermodynamics by treating dissipative perturbations as being non-instantaneous. A second order term within the thermodynamics arises because the entropy includes a second order term in the dissipative variables. This theory uses transient variables because it looks beyond the semi-stationary form of classical theory to incorporate transient phenomena on the scale of the mean free path.

For this chapter we follow the notation of Maartens [16].

4.3 Entropy in Irreversible Thermodynamics

Both standard and extended irreversible thermodynamical theories impose conservation of particle number and the energy momentum tensor. Thus we have

$$n^{\alpha}{}_{;\alpha} = 0, \;\; T^{\alpha\beta}{}_{;\beta} = 0 \tag{4.3.1}$$

Thus $\dot{n}+3Hn=0$, where H is the generalised Newtonian expansion rate [16] and n is the number density.

In irreversible thermodynamics entropy is no longer conserved, and is expected to increase with time as is suggested by the second law of thermodynamics. Entropy production is given by divergence of the entropy flux vector. Thus the covariant form of the second law of thermodynamics is

$$S^{\alpha}{}_{;\alpha} \geq 0 \tag{4.3.2}$$

In the perfect fluid case entropy is typically represented by [16]

$$S^{\alpha} = Sn^{\alpha} \tag{4.3.3}$$

With the inclusion of a dissipation term we find entropy to be

$$S^{\alpha} = Snu^{\alpha} + \frac{R^{\alpha}}{T} \tag{4.3.4}$$

where entropy and temperature are still related via the Gibb's equation, given below

$$TdS = d\left(\frac{\rho}{n}\right) + pd\left(\frac{1}{n}\right) \tag{4.3.5}$$

The dissipative contributions of S^{α} and R^{α} are both assumed to be algebraic functions of n^{α} and $T^{\alpha\beta}$ (ie, $R^{\alpha} = R^{\alpha}(n^{\beta}, T^{\mu\nu})$) that vanish in equilibrium:

$$R^{\alpha} = 0$$

This essentially goes back to the idea that non-equilibrium states are assumed to be adequately specified by n^{α} and $T^{\alpha\beta}$ which are the hydrodynamical tensors.

One can see in [20] how the above corresponds to truncation in the non-equilibrium distribution function of relativistic kinetic theory.

4.4 Irreversible Thermodynamics - From Eckart Onwards

Eckart theory makes a very simple assumption about the structure of R^α, in that it is linear in the dissipative quantities. This produces only one possible vector that can then be created from the dissipative quantities

$$f(n,\alpha)\Pi u^\alpha + g(\rho,n)q^\alpha \tag{4.4.1}$$

It is expected that the entropy density $(u_\alpha S^\alpha)$ will be maximum at equilibrium,

$$\left[\frac{\partial}{\partial \Pi}(-u_\alpha S^\alpha)\right]_{equil} = 0. \tag{4.4.2}$$

This in turn implies that $f = 0$. We also expect then in a comoving instantaneous orthonormal frame (IOF) that $\frac{q_\alpha}{T} = (0, \frac{\vec{q}}{T})$ which is the entropy flux due to heat flow. Thus we now expect $g = 1$ and (4.3.4) becomes,

$$S^\alpha = Snu^\alpha + \frac{q^\alpha}{T} \tag{4.4.3}$$

Using the Gibb's equation (4.3.5), the conservation equations (4.3.1) and the altered conservation equations of [16], the divergence of (4.4.3) becomes

$$TS^\alpha{}_{;\alpha} = -\left[3H\Pi + (D_\alpha \ln T + \dot{u}_\alpha)q^\alpha + \sigma_{\alpha\beta}\pi^{\alpha\beta}\right] \tag{4.4.4}$$

where $D_\alpha \ln T$ is the covariant spatial derivative of $\ln T$.

The equilibrium conditions from relativistic kinetic theory lead to the vanishing of each factor multiplying the dissipative terms on the right leading to $S^\alpha{}_{;\alpha} = 0$.

For (4.4.4) to satisfy (4.3.2), we impose a linear relationship between the thermodynamical "fluxes" Π, q_α, $\pi_{\alpha\beta}$ and the corresponding thermodynamical "forces" H,

$\dot{u} + D_\alpha \ln T$ and $\sigma_{\alpha\beta}$:

$$\Pi = -3\zeta H \tag{4.4.5a}$$

$$q_\alpha = -\lambda(D_\alpha T + T\dot{u}_\alpha) \tag{4.4.5b}$$

$$\pi_{\alpha\beta} = -2\eta\sigma_{\alpha\beta} \tag{4.4.5c}$$

It is of interest to note that, should pressure be isotropic, then the left hand side of (4.4.5c) vanishes, implying that σ must be zero. This implies that the shear viscosity must be zero. Without shear viscosity there is no shear. In some previous papers they have claimed to include anisotropy without shear which clearly makes no sense [37].

The equations of (4.4.5) are the constitutive equations for dissipative quantities in Eckart's theory of relativistic irreversible thermodynamics. It can also be seen that they are the relativistic generalisations of the corresponding Newtonian laws as shown below,

$$\Pi = -3\zeta\overrightarrow{\nabla}.\overrightarrow{v} \quad (Stokes) \tag{4.4.6}$$

$$q_\alpha = -\lambda\overrightarrow{\nabla}T \quad (Fourier) \tag{4.4.7}$$

$$\pi_{ij} = -2\eta\sigma_{ij} \quad (Newton) \tag{4.4.8}$$

One should note that if one tries to use an IOF in (4.4.5) then an acceleration term arises, that can be seen in (4.4.5b), due to the inertia of heat energy (as noted by Eckart). Essentially this tells us that a heat flux is predicted to arise from the accelerated matter even in the absence of a temperature gradient. As with the Newtonian laws we expect the thermodynamical coefficients to be given by

- $\zeta(\rho, n)$ which is the bulk viscosity.

- $\lambda(\rho, n)$ which is the thermal conductivity.

- $\eta(\rho, n)$ which is the shear viscosity.

Using (4.4.5) we find (4.4.4) becomes,

$$S^{\alpha}{}_{;\alpha} = \frac{\Pi^2}{\zeta T} + \frac{q_\alpha q^\alpha}{\lambda T^2} + \frac{\pi_{\alpha\beta}\pi^{\alpha\beta}}{2\eta T} \tag{4.4.9}$$

which will always be greater than or equal to zero provided that $\zeta \geq 0$, $\lambda \geq 0$ and $\eta \geq 0$.

Using the Gibbs equation (4.3.5), number conservation and energy momentum conservation again lead us to find a time evolution equation for the entropy

$$Tn\dot{S} = -3H\Pi - q^{\alpha}{}_{;\alpha} - \dot{u}_\alpha q^\alpha - \sigma_{\alpha\beta}\pi^{\alpha\beta} \tag{4.4.10}$$

A substantial amount of papers have used Eckart's version of irreversible thermodynamics. The problem with it, as mentioned earlier, is it's non-causality which can be seen if one turns the thermodynamic force off suddenly. When this occurs the temperature throughout the fluid changes instantaneously violating relativistic causality.

4.5 Moving Towards a Causal Theory of Irreversible Thermodynamics

We have seen in the previous section that Eckart's assumption with respect to R^α is just too simple. It leaves out the causal nature of temperature propagation thus leading to incorrect predications of stability states.

A more general form of R^α, that is at most second order in the dissipative fluxes [16], has been inputted below

$$S^\mu = Snu^\mu - \left(\gamma_0\Pi^2 + \gamma_1 q_\nu q^\nu + \gamma_2 \pi_{\nu\kappa}\pi^{\nu\kappa}\right)\frac{u^\mu}{T} + \frac{\alpha_0 \Pi q^\mu}{T} + \frac{\alpha_1 \pi^{\mu\nu} q_\nu}{T} \tag{4.5.1}$$

where, $\gamma_A(\rho, n) \geq 0$ are the the thermodynamic coefficients for all scalar, vector and tensor dissipative contributions to the entropy density, and $\alpha_A(\rho, n)$ are the thermodynamic viscous/heat capacity coefficients.

We can now use (4.5.1) to find the effective entropy density (measured by comoving observers)

$$-u_\mu S^\mu = S_n - \frac{1}{2T}\left(\gamma_0 \Pi^2 + \gamma_1 q_\mu q^\mu + \gamma_2 \pi_{\mu\nu}\pi^{\mu\nu}\right) \tag{4.5.2}$$

which has been modified to be independent of the viscous/heat capacity coupling coefficients, ie,

$$\alpha_0 = \alpha_1 = 0 \tag{4.5.3}$$

Similar to our approach with respect to the Eckart formalism we want to look at the divergence of (4.5.1) with (4.5.3) which follows from the Gibbs equation (4.3.5) and the conservation equations. We obtain

$$\begin{aligned} TS^\alpha{}_{;\alpha} \quad = \quad & -\Pi\left[3H + \gamma_0\dot{\Pi} + \frac{1}{2}T\left(\frac{\gamma_1}{T}u^\alpha\right)_{;\alpha}\Pi\right] \\ & -q^\alpha\left[D_\alpha \ln T + \dot{u}_\alpha + \gamma_1 \dot{q}_\alpha + \frac{1}{2}\left(\frac{\gamma_1}{T}u^\mu\right)_{;\mu} q_\alpha\right] \\ & -\pi^{\alpha\mu}\left[\sigma_{\alpha\mu} + \gamma_2\dot{\pi}_{\alpha\mu} + \frac{1}{2}T\left(\frac{\gamma_2}{T}u^\nu\right)_{;\nu}\pi_{\alpha\mu}\right] \end{aligned} \tag{4.5.4}$$

where the above three terms represent the evolution of the bulk viscosity, heat flux and shear respectively.

In order to obey the second law of thermodynamics (4.3.2) we impose a linear relationship between the thermodynamic fluxes and dissipative forces. This leads to the following transport equations,

$$\tau_0\dot{\Pi} + \Pi \quad = \quad -3\zeta H - \left[\zeta T\left(\frac{\tau_0}{\frac{1}{2}\zeta T}u^\alpha\right)_{;\alpha}\Pi\right] \tag{4.5.5a}$$

$$\tau_1 h^\beta{}_\alpha \dot{q}_\beta + q_\alpha \quad = \quad -\lambda(D_\alpha T + T\dot{u}_\alpha) - \left[\frac{1}{2}\lambda T^2\left(\frac{\tau_1}{\lambda T}u^\beta\right)_{;\beta} q_\alpha\right] \tag{4.5.5b}$$

$$\tau_2 h^\mu{}_\alpha h^\nu{}_\beta \dot{\pi}_{\mu\nu} + \pi_{\alpha\beta} \quad = \quad -2\eta\sigma_{\alpha\beta} - \left[\eta T\left(\frac{\tau_2}{2\eta T}u^\nu\right)_{;\nu}\pi_{\alpha\beta}\right] \tag{4.5.5c}$$

where the relaxational times $\tau_0(\rho, n)$, $\tau_1(\rho, n)$ and $\tau_2(\rho, n)$ become

$$
\begin{aligned}
\tau_0 &= \zeta\gamma_0 \\
\tau_1 &= \lambda T\gamma_1 \\
\tau_2 &= 2\eta\gamma_2
\end{aligned}
$$

As was mentioned earlier as long as ζ, λ and η are ≥ 0 then the entropy production rate will be greater than or equal to zero.

In addition to the viscous/heat coupling, relativistic kinetic theory predicts that in general there will be a coupling between the heat flux and the anisotropic pressure to the vorticity, which, unlike the shear, does not vanish at equilibrium. We see these couplings giving rise to the following additions to the right hand side of (4.5.5b) and (4.5.5c) respectively,

$$+\lambda T\chi_1\omega_{\alpha\beta}q^{\beta}$$

and

$$+2\eta\chi_2\pi^{\mu}_{<\alpha}\omega_{\beta>\mu}$$

where χ_1 and χ_2 are the thermodynamic coupling coefficients.

We can omit the terms in brackets from equations (4.5.5a), (4.5.5b) and (4.5.5c). The new equations have been referred to in the published literature as the truncated Israel-Stewart equations [20]. There are only certain conditions that can then be considered reasonable for the truncated equations

$$\tau_0\dot{\Pi} + \Pi = -3\zeta H \tag{4.5.6a}$$

$$\tau_1 h^{\beta}{}_{\alpha}\dot{q}_{\beta} + q_{\alpha} = -\lambda(D_{\alpha}T + T\dot{u}_{\alpha}) \tag{4.5.6b}$$

$$\tau_2 h^{\mu}{}_{\alpha}h^{\nu}{}_{\beta}\dot{\pi}_{\mu\nu} + \pi_{\alpha\beta} = -2\eta\sigma_{\alpha\beta} \tag{4.5.6c}$$

If we compare the above equations, representing the Israel-Stewart transport equations with the Eckart transport equations, we find that the Israel-Stewart equations are differential evolution equations whereas Eckart are merely algebraic equations. The new evolution terms with the relaxation time coefficients τ_A have now introduced the much needed causality to the irreversible thermodynamic theory. If we were to set $\tau_0 = \tau_1 = \tau_2 = 0$ we would then recover the Eckart transport equations [11].

Based on our line element (2.3.1) introduced in chapter 2 we expect (4.5.6b) to become,

$$\tau_1(qB)_{,t} + A(qB) = -\lambda\frac{(AT)'}{B} \tag{4.5.7}$$

Note that(4.5.7) has a large impact on the causal temperature profile as it is linked to the heat transport equation. Equation (4.5.7) governs the behaviour of the temperature and we can quite clearly see that by setting $\tau_1 = 0$ that we will obtain the familiar Fourier heat transport equation,

$$A(qB) = -\lambda\frac{(AT)'}{B} \tag{4.5.8}$$

which can predict reasonable temperatures provided that the fluid is close to a quasi-stationary equilibrium. In our models that we will be analyzing we assume that the energy is carried away from the star by massless particles that are thermally generated with energies of the order of kT. Thus we expect the thermal conductivity to take the form,

$$\lambda = \varphi T^3 \tau_c \tag{4.5.9}$$

where φ (≥ 0) is some constant, and τ_c is the mean collision time between the massless and massive particles. If we model the system like this then we expect the power law behaviour to look like,

$$\tau_c = \left(\frac{\alpha}{\varphi}\right) T^{-\omega} \tag{4.5.10}$$

where α (≥ 0) and ω (≥ 0) are some positive constants. We can test to see if the power law found is reasonable by setting ω to $\frac{2}{3}$ in which case we recover the case of thermally generated neutrinos in neutron stars. We can also see in (4.5.10) that an increase in temperature will result in decreasing collision time, which is in line with what we would expect from any decent prediction mechanism. There is a special case when $\omega = 0$ that results in a constant collision time but this will only give a reasonable model for a very limited range of temperatures.

If we consider the treatment in [39], then we can adopt a similar assumption, in that the velocity of the dissipative propagation is comparable to the adiabatic speed of sound. This is satisfied provided that relaxation time is proportional to the collision time as shown below,

$$\tau = \left(\frac{\beta\varphi}{\alpha}\right)\tau_c \tag{4.5.11}$$

where τ (≥ 0) is a constant. The case where $\tau = 0$ would apply for the noncausal temperature predictions. One can think of τ as being the 'causality index' which measures the strength of the relaxations effects.

We can now use our definition of λ from (4.5.9) and τ from (4.5.11) in (4.5.7) so that we find,

$$\beta(qB)_{,t}T^{-\omega} + A(qB) = -\alpha\frac{T^{3-\omega}(AT)'}{B} \tag{4.5.12}$$

where $(qB)_{,t}$ is differentiation with respect to t. By setting $\beta = 0$, we can then find all the noncausal solutions of (4.5.12), [38]

$$\begin{aligned} (AT)^{4-\omega} &= \frac{\omega-4}{\alpha}\int A^{4-\omega}qB^2dr + F(t) \ \ ,\omega \neq 4 \\ \ln(AT) &= -\frac{1}{\alpha}\int qB^2dr + F(t) \ \ ,\omega = 4 \end{aligned} \tag{4.5.13}$$

where $F(t)$ is some arbitary function of integration. The quantity $F(t)$ is fixed by the expression for the temperature of the star (which is found from the luminosity) at its surface.

If we want to consider the relaxational effects due to shear then we must use (4.5.5c) as a definition of the relaxation time for the shear stress. Using line element (2.3.1) equation (4.5.5c) is reduced to,

$$\tau_2 = \frac{-P}{\dot{P} + \frac{8}{15} r_0 \sigma T^4} \tag{4.5.14}$$

where the coefficient of shear viscosity for a radiative fluid

$$\eta = \frac{4}{15} r_0 T^4 \tau_1 \tag{4.5.15}$$

has been used. In the above we have set $P = \frac{1}{3}(p_\perp - p_r)$ and r_0 is the radiation constant for photons. Also τ_2 is assumed to be proportional to τ_c.

Chapter 5

A Radiating Model

5.1 Introduction

It is the purpose of this chapter to seek solutions to the Einstein field equations describing a spherically symmetric matter distribution with shear and nonvanishing acceleration. We show that the boundary condition for a general spherically symmetric interior matter distribution, matched to Vaidya's outgoing solution, leads to a Riccati equation. We solve this equation for particular combinations of the metric functions. We further investigate the thermal behaviour of a particular model and relate our results to the acceleration-free models developed in the literature.

5.2 Shearing Spacetimes

The interior spacetime of the collapsing sphere is described by the general spherically symmetric, shearing metric in comoving coordinates (2.3.1) which is, repeated below

$$ds^2 = -A^2 dt^2 + B^2 dr^2 + Y^2 (d\theta^2 + \sin^2\theta d\phi^2)\,, \tag{5.2.1}$$

where the metric functions $A = A(t, r)$, $B = B(t, r)$ and $Y = Y(t, r)$ are yet to be determined.

The matter content for the interior is described by the energy momentum tensor (2.2.9),

$$T_{ab} = (\mu + p_\perp)u_a u_b + p_\perp g_{ab} + (p_r - p_\perp)\chi_a \chi_b + q_a u_b + q_b u_a \tag{5.2.2}$$

where μ, p_r, $p_\perp$ and q^a are defined in §2.2. The fluid four-velocity $\boldsymbol{u}$ and the heat flow vector q^a are,

$$u^a = \frac{1}{A}\delta^a_0 \tag{5.2.3}$$

$$q^a = (0, q, 0, 0) \tag{5.2.4}$$

and $q^a u_a = 0$ ensures radial heat dissipation. We further have

$$\chi^a \chi_a = 1, \qquad \qquad \chi^a u_a = 0. \tag{5.2.5}$$

For the comoving line element (5.2.1) the kinematical quantities take the following forms

$$a_1 = \frac{A'}{A} \tag{5.2.6}$$

$$\Theta = \frac{1}{A}\left(\frac{\dot{B}}{B} + 2\frac{\dot{Y}}{Y}\right) \tag{5.2.7}$$

$$\sigma = \frac{1}{A}\left(\frac{\dot{A}}{A} - \frac{\dot{Y}}{Y}\right) \tag{5.2.8}$$

where $a_1 = \dot{\mu}^1$. We then find the thermodynamical quantities for the interior of the star

$$\mu = \frac{1}{A^2}\left(2\frac{\dot{B}}{B}+\frac{\dot{Y}}{Y}\right)\frac{\dot{Y}}{Y} - \frac{1}{B^2}\left[2\frac{Y''}{Y}+\left(\frac{Y'}{Y}\right)^2 - 2\frac{B'}{B}\frac{Y'}{Y} - \left(\frac{B}{Y}\right)^2\right] \tag{5.2.9}$$

$$p_r = -\frac{1}{A^2}\left[2\frac{\ddot{Y}}{Y} - \left(2\frac{\dot{A}}{A}-\frac{\dot{Y}}{Y}\right)\frac{\dot{Y}}{Y}\right] + \frac{1}{B^2}\left(2\frac{A'}{A}+\frac{Y'}{Y}\right)\frac{Y'}{Y} - \frac{1}{Y^2} \tag{5.2.10}$$

$$\begin{aligned} p_\perp = & -\frac{1}{A^2}\left[\frac{\ddot{B}}{B}+\frac{\ddot{Y}}{Y}-\frac{\dot{A}}{A}\left(\frac{\dot{B}}{B}+\frac{\dot{Y}}{Y}\right)+\frac{\dot{B}}{B}\frac{\dot{Y}}{Y}\right] \\ & +\frac{1}{B^2}\left[\frac{A''}{A}+\frac{Y''}{Y}-\frac{A'}{A}\frac{B'}{B}+\left(\frac{A'}{A}-\frac{B'}{B}\right)\frac{Y'}{Y}\right] \end{aligned} \tag{5.2.11}$$

$$q = \frac{2}{AB^2}\left(\frac{\dot{Y}'}{Y} - \frac{\dot{B}}{B}\frac{Y'}{Y} - \frac{\dot{Y}}{Y}\frac{A'}{A}\right) \tag{5.2.12}$$

This is an underdetermined system of four coupled partial differential equations in the seven unknowns $A, B, Y, \mu, p_r, p_\perp$ and q.

5.3 Exterior Spacetime and Junction Conditions

The exterior spacetime is taken to be Vaidya's outgoing solution

$$ds^2 = -\left(1-\frac{2m(v)}{\mathsf{r}}\right)dv^2 - 2dvd\mathsf{r} + \mathsf{r}^2\left(\mathsf{d}\theta^2 + \sin^2\theta\mathsf{d}\phi^2\right) \tag{5.3.1}$$

where $m(v)$ is not constant. The necessary conditions for the smooth matching of the interior spacetime (5.2.1) to the exterior spacetime (5.3.1) have been discussed in chapter 3. We present the main results that are necessary for modeling a radiating star. The continuity of the intrinsic and extrinsic curvature components of the interior and exterior spacetimes across a timelike boundary are, given by

$$m(v)_\Sigma = \left\{\frac{Y}{2}\left[\left(\frac{\dot{Y}}{A}\right)^2 - \left(\frac{Y'}{B}\right)^2 + 1\right]\right\}_\Sigma \tag{5.3.2}$$

$$(p_r)_\Sigma = (qB)_\Sigma \tag{5.3.3}$$

Relation (5.3.3) determines the temporal evolution of the collapsing star.

5.4 Temporal Evolution

The junction condition 5.3.3 yields

$$\begin{aligned} \dot{B} &= \left(\frac{Y}{2AY'}\right)\left[2\frac{\ddot{Y}}{Y}+\left(\frac{\dot{Y}}{Y}\right)^2-2\frac{\dot{A}}{A}\frac{\dot{Y}}{Y}+\frac{A^2}{Y^2}\right]B^2 \\ &\quad +\left[\frac{\dot{Y}'}{Y'}-\frac{A'}{A}\frac{\dot{Y}}{Y'}\right]B-\frac{A}{2}\left[\frac{Y'}{Y}+2\frac{A'}{A}\right] \end{aligned} \tag{5.4.1}$$

which is of the form

$$\dot{B}=\mathcal{C}_0(t)B^2+\mathcal{C}_1(t)B+\mathcal{C}_2(t) \tag{5.4.2}$$

We now attempt to integrate (5.4.1) for some special cases.

5.4.1 Case I

If we set

$$\frac{2\ddot{Y}}{Y}+\frac{\dot{Y}^2}{Y^2}-\frac{2\dot{A}}{A}\frac{\dot{Y}}{Y}+\frac{A^2}{Y^2}=0 \tag{5.4.3}$$

then (5.4.1) reduces to a linear equation. We note that (5.4.3) can be recast into the following form

$$\dot{A}-\left[\frac{\ddot{Y}}{\dot{Y}}+\frac{\dot{Y}}{2Y}\right]A=A^3\left(\frac{1}{2Y\dot{Y}}\right)$$

where is a Bernoulli equation in the variable A. This equation can be integrated in general to yield

$$A^2=\frac{Y\dot{Y}^2}{f(r)-Y} \tag{5.4.4}$$

where $f(r)$ is a function of integration. With the result (5.4.4), we find that (5.4.1) becomes

$$\dot{B}-\left[\frac{\dot{Y}'}{Y'}-\frac{A'}{A}\frac{\dot{Y}}{Y'}\right]B+\frac{A}{2}\left[\frac{Y'}{Y}+2\frac{A'}{A}\right]=0 \tag{5.4.5}$$

which is linear in B. We are in a position to integrate (5.4.5) which then allows us to obtain a particular solution of the junction condition (5.4.1) given by

$$A = \sqrt{\frac{Y\dot{Y}^2}{f(r) - Y}} \tag{5.4.6}$$

$$B = \frac{g(r) - \int \left(A' + \frac{AY'}{2Y}\right) \exp\left[\int \left(\frac{A'\dot{Y}}{AY'} - \frac{\dot{Y}'}{Y'}\right) dt\right] dt}{\exp\left[\int \left(\frac{A'\dot{Y}}{AY'} - \frac{\dot{Y}'}{Y'}\right) dt\right] dt} \tag{5.4.7}$$

$$Y = Y(t, r) \tag{5.4.8}$$

where $g(r)$ is the second function of integration. We believe that (5.4.6)-(5.4.8) is a new solution to the boundary condition (5.4.1) which allows for nonzero acceleration of the fluid particles. We note that once the arbitrary function Y is specified, we can then obtain explicit forms for $A(r, t)$ and $B(r, t)$ which completely specifies the gravitational behaviour of our model.

5.4.2 Case II

If we set

$$\frac{Y'}{Y} + 2\frac{A'}{A} = 0 \tag{5.4.9}$$

then equation (5.4.1) becomes a Bernoulli equation. Integrating equation (5.4.9) we get

$$Y = \frac{C_1(t)}{A^2} \tag{5.4.10}$$

where $C_1(t)$ is a function of integration. Substituting (5.4.10) into (5.4.1) we obtain

$$\begin{aligned} &\dot{B} - \left[\frac{3}{2}\frac{\dot{C}_1}{C_1} - 4\frac{\dot{A}}{A} + \frac{\dot{A}'}{A'}\right] B \\ &= \left[\frac{7}{2}\frac{\dot{A}\dot{C}_1}{AA'C_1} - 5\frac{\dot{A}^2}{A^2A'} - \frac{\ddot{C}_1}{2C_1A'} + \frac{\ddot{A}}{AA'} - \frac{\dot{C}_1{}^2}{4C_1^2A'} - \frac{A^6}{4C_1^2A'}\right] B^2 \end{aligned} \tag{5.4.11}$$

which is a Bernoulli equation in the variable B.

Even though the coefficients of (5.4.11) contain the functions A, C_1 and their derivatives; it can be integrated in general and we get

$$B = \frac{A' C_1^{3/2}}{A^4[\int I dt + h(r)]} \tag{5.4.12}$$

where $h(r)$ is a function of integration and for convenience we have set

$$I = -\frac{7}{2}\frac{C_1^{1/2}\dot{A}\dot{C}_1}{A^5} + 5\frac{\dot{A}^2 C_1^{3/2}}{A^6} + \frac{\ddot{C}_1 C_1^{1/2}}{2A^4} - \frac{\ddot{A}C_1^{3/2}}{A^5} + \frac{\dot{C_1}^2}{4C_1^{1/2}A^4} + \frac{A^2}{4C_1^{1/2}}$$

Therefore the functions

$$A = A(t,r) \tag{5.4.13}$$

$$B = \frac{A' C_1^{3/2}}{A^4[\int I dt + h(r)]} \tag{5.4.14}$$

$$Y = \frac{C_1}{A^2} \tag{5.4.15}$$

satisfy the junction condition (5.4.1). Note that the gravitational potential $A(t,r)$ is an arbitrary function in this class of solution. Once A is specified, together with the integration function C_1, the explicit form for I can be determined.

5.4.3 Case III

If we set

$$\frac{\dot{Y}'}{Y'} - \frac{A'}{A}\frac{\dot{Y}}{Y'} = 0 \tag{5.4.16}$$

then equation (5.4.1) has the form of an inhomogeneous Riccati equation. Integrating equation (5.4.16) we get

$$A = \alpha(t)\dot{Y}$$

where $\alpha(t)$ is an integration function. In this case (5.4.1) becomes

$$\dot{B} = \left[\frac{\dot{Y}(1+\alpha^2)}{2YY'\alpha} - \frac{\dot{\alpha}}{\alpha^2 Y'}\right]B^2 - \left[\dot{Y}'\alpha + \frac{\dot{Y}Y'\alpha}{2Y}\right] \tag{5.4.17}$$

This is an inhomogenous Riccati equation which is difficult to analyse in general.

If we take α as a constant and Y to be the separable function

$$Y(t,r) = K(r)C(t) \tag{5.4.18}$$

where $K(r)$ and $C(t)$ are arbitrary functions of r and t respectively. Then (5.4.17) can be written as

$$\dot{B} = \left[\frac{(1+\alpha^2)}{2\alpha K'}\frac{\dot{C}}{C^2}\right] - \frac{3}{2}\alpha K'\dot{C} \tag{5.4.19}$$

5.5 A Particular Radiating Model

In order to study the physical viability of the class of solutions describing dissipative collapse we take a closer look at solutions (5.4.6)–(5.4.8). To this end we choose

$$Y(r,t) = a + cr - bt \tag{5.5.1}$$

which subsequently allows us to obtain

$$A(r,t) = 2\sqrt{\frac{b(a+cr-bt)}{t}} \tag{5.5.2}$$

$$B(r,t) = \frac{a\sqrt{b(a+cr-bt)} - 2ct\sqrt{\frac{b(a+cr-bt)}{t}}}{a+cr-bt} \tag{5.5.3}$$

where we have set $f(r) = a + cr$. The Einstein field equations (5.2.9)–(5.2.12) yield

$$\begin{aligned}
\mu &= \Bigg(4bc^3(cr-bt)^2 + a^3\left(-4bc + \sqrt{a+cr-bt}\sqrt{\frac{b(a+cr-bt)}{t}}\right) \\
&\quad -2a^2\left(-2bc^2(c-r) - 2b^2ct + c^2\sqrt{a+cr-bt}\sqrt{\frac{b(a+cr-bt)}{t}}\right) \\
&\quad -2a\Big(-4bc^4r + 4b^2c^3t + c^3r\sqrt{a+cr-bt}\sqrt{\frac{b(a+cr-bt)}{t}} \\
&\quad -3bc^2t\sqrt{a+cr-bt}\sqrt{\frac{b(a+cr-bt}{t}}\Big)\Bigg) \\
&\quad \Bigg(\sqrt{\frac{b(a+cr-bt}{t}}(-a-cr+bt) \\
&\quad \left(-a\sqrt{a+cr-bt} + 2ct\sqrt{\frac{b(a+cr-bt}{t}}\right)^3\Bigg) && (5.5.4) \\
p_r &= qB = \frac{2c^2}{\left(a\sqrt{a+cr-bt} - 2ct\sqrt{\frac{b(a+cr-bt)}{t}}\right)^2} && (5.5.5)
\end{aligned}$$

The thermodynamical quantities at the centre of star become

$$\begin{aligned}
\mu_0 &= 59\Bigg(4b^3c^3t^2 + a^3\left(-4bc + \sqrt{a-bt}\sqrt{\frac{b(a-bt)}{t}}\right) \\
&\quad -2a^2\left(-2bc^3 - 2b^2ct + c^2\sqrt{a-bt}\sqrt{\frac{b(a-bt)}{t}}\right) \\
&\quad -2a\Big(4b^2c^3t - 3bc^2t\sqrt{a-bt}\sqrt{\frac{b(a-bt}{t}}\Big)\Bigg) \\
&\quad /\left(\sqrt{\frac{b(a-bt}{t}}(-a+bt)\left(-a\sqrt{a-bt} + 2ct\sqrt{\frac{b(a-bt}{t}}\right)^3\right) && (5.5.6) \\
p_{r0} &= Q_0 = \frac{2c^2}{\left(a\sqrt{a-bt} - 2ct\sqrt{\frac{b(a-bt)}{t}}\right)^2} && (5.5.7)
\end{aligned}$$

The kinematical quantities yield

$$\begin{aligned}
\sigma &= \frac{a+cr+2(a+cr)t-2bt^2}{2t\sqrt{\frac{bY}{t}}(-Y)} && (5.5.8) \\
\Theta &= \frac{-2ac\sqrt{Y} - 2c(cr-4bt)\sqrt{Y} - 3at\sqrt{\frac{bY}{t}}}{2Y(3/2)(a\sqrt{Y} - 2ct\sqrt{\frac{bY}{t}})} && (5.5.9) \\
a_1 &= \frac{c}{2(a+cr-bt)} && (5.5.10)
\end{aligned}$$

5.6 Energy conditions and stability

In order to test the physical viability of our solution it is imperative to impose the energy conditions on the interior matter distribution. We firstly require that the energy density, radial pressure and tangential pressure all be positive within the stellar core, ie.,

$$\rho \geq 0, \qquad p_r \geq 0, \qquad p_\perp \geq 0$$

We further require that the energy density and pressure decrease outwards from the centre towards the stellar surface, ie.,

$$\rho' < 0, \qquad p_r' < 0$$

We expect this as the density of the star at the centre is the highest and gradually drops off towards the surface. From Figure 5.2 we note that the radial pressure also drops off as one moves from the centre of the star to the stellar surface. Since $p_r = qB$ for all time, the heat flux is highest at the centre of the core and falls off radially outwards leading to cooling of the surface layers. An interesting feature of this model is that the tangential pressure becomes negative for this particular epoch of the star's evolution. This suggest that the tension on each concentric shell is such that it tends to cause the shell to expand while the sphere is collapsing under gravity. We can think of the negative tangential pressure as contributing to the outward radial pressure thus slowing down the collapse process. Figures 5.4 and 5.5 show $(\rho + p_r)^2 - 4q^2 > 0$ and $\rho - p_r - 2p_\perp + [(\rho + P_r)^2 - 4q^2]^{1/2} > 0$ always. These requirements ensure that all the energy conditions (weak, strong and dominant) are satisfied at all interior points of our stellar model.

5.7 Thermal behaviour

The behaviour of the temperature during dissipative collapse has been extensively studied in shear-free models and shearing models with geodesic flows. Up to this point, the only nonzero accelerating models with shear are the so-called Euclidean stars which were first investigated by Herrera *et al* [15] and expansion free models [23]. The first exact model of a Euclidean star was presented by Govender *et al* [13]. This model has been generalised by Govinder and Govender [14], in which the thermal behaviour and stability of Euclidean stars were investigated . The thermal behaviour of these models were well studied within the context of extended irreversible thermodynamics. Just as in the cases of shear-free models and shearing, geodesic models it was shown that relaxational effects due to heat dissipation lead to higher interior temperatures of the Euclidean stellar core. To this end we investigate the evolution of the temperature profiles in our class of accelerating models with nonzero shear. We employ a causal transport equation for the heat flux in order to display relaxational effects during the collapse process. The truncated causal heat transport equation for the line element (5.2.1) is given by

$$\tau(qB)^{\dot{}} + A(qB) = -\kappa\frac{(AT)'}{B} \tag{5.7.1}$$

as seen in chapter 4, where τ is the relaxation time and κ is the thermal conductivity. Switching 'off' τ leads to the noncausal Fourier heat transport equation

$$A(qB) = -\kappa\frac{(AT)'}{B} \tag{5.7.2}$$

which suffers various pathologies in terms of causality and stability of the equilibrium states. Adopting the thermodynamic coefficients for radiative transfer as outlined in [40], we set

$$\kappa = \gamma T^3\tau_{\rm c}, \qquad\qquad \tau_{\rm c} = \left(\frac{\alpha}{\gamma}\right)T^{-\sigma} \tag{5.7.3}$$

where $\alpha \geq 0$, $\gamma \geq 0$ and $\sigma \geq 0$ are constants. We further assume that the velocity of thermal dissipative signals is comparable to the adiabatic sound speed. This is ensured if the relaxation time is proportional to the collision time:

$$\tau = \left(\frac{\beta\gamma}{\alpha}\right)\tau_{\rm c} \tag{5.7.4}$$

where β (≥ 0) is a constant. Using the above definitions for τ and κ, equation (5.7.1) takes the form

$$\beta(q\dot{B})T^{-\sigma} + A(qB) = -\alpha\frac{T^{3-\sigma}(AT)'}{B}\,. \tag{5.7.5}$$

We are in a position to integrate equation (5.7.5) for the special case of constant collision time corresponding to $\sigma = 0$:

$$\begin{aligned}
&T(r,t) = \Bigg(\frac{1}{X_2^2}\Bigg(X_1^2\Bigg(\\
&\frac{c^2X_1^3\Big(a^2c + 2ac^2d + c^3d^2 - b^2ct^2 - at\sqrt{X_1}\sqrt{\frac{bX_1}{t}}\Big)X_3}{d^2\sqrt{\frac{bX_1}{t}}\Big(a\sqrt{X_1} - 2ct\sqrt{\frac{bX_1}{t}}^6\Big)}\\
&+\frac{1}{b^2X_1^2\psi}4t^2\Bigg(\frac{4b^2cX_1(a\sqrt{X_1} + 2ct\sqrt{\frac{bX_1}{t}})}{3t^2(a^2-4bc^2t)} - \Big(2b^2c\beta\Big(-2c^2dX_4 -\\
&b(a^2-4bc^2t)(a\sqrt{X_1}\log X_1 + 2ct\sqrt{\frac{bX_1}{t}}\log\frac{bX_1}{t})\Big)\Big)/(t^2\sqrt{\frac{bX_1}{t}}(a^2-4bc^2t)^2)\Bigg)\Bigg) -\\
&\frac{4t^2}{(b^2\psi)}\Bigg(\frac{4b^2c(X_2)(a\sqrt{X_2} + 2ct\sqrt{\frac{bX_2}{t}})}{3t^2(a^2-4bc^2t)} -\\
&\Big(2b^2c\beta\Big(-2c^2r\Big(4abc\sqrt{X_2} + a^2\sqrt{\frac{bX_2}{t}} + 4bc^2t\sqrt{\frac{bX_2}{t}}\Big) -\\
&b(a^2-4bc^2t)\Big(a\sqrt{X_2}\log X_2 + 2ct\sqrt{\frac{bX_2}{t}}\log\frac{bX_2}{t}\Big)\Big)\Big)/\Big(t^2\sqrt{\frac{bX_2}{t}}(a^2-4bc^2t)^2\Big)\Bigg)\Bigg)\Bigg)^{1/4}
\end{aligned}$$

where we have defined

$$\begin{aligned}
X_1 &= a + cd - bt\\
X_2 &= a + cr - bt\\
X_3 &= \Big(-ab\sqrt{X_1} + ac\sqrt{\frac{bX_1}{t}} + c\sqrt{\frac{bX_1}{t}}(cd+bt)\Big)\\
X_4 &= \Big(4abc\sqrt{X_1} + a^2\sqrt{\frac{bX_1}{t}} + 4bc^2t\sqrt{\frac{bX_1}{t}}\Big)
\end{aligned}$$

We note, from Figure 5.6, that the causal temperature is everywhere higher than its noncausal counterpart for each interior point of the stellar core. This agrees with earlier results obtained in the shearing case with zero acceleration as well as the models of Euclidean stars studied in [11],[13].

5.8 Discussion

We have modelled a spherically symmetric radiating star undergoing dissipative collapse in the presence of a radially driven heat flux to the exterior spacetime. The exterior spacetime is described by Vaidya's outgoing solution. The matching of the interior spacetime to the exterior spacetime across a timelike hypersurface generates a temporal evolution equation for the metric functions which is a Riccati equation. We integrate this equation under various assumptions to generate several classes of radiating solutions with shear. We further studied the physical viability of one these classes of solutions by imposing the energy conditions. Some of the physical properties of the system are displayed in the Figures 5.1-5.3 and 5.7-5.10. An interesting feature of this particular model is that the tangential pressure becomes negative (Figure 5.3) for a particular period of the collapse. We point out that such a scenario would lead to the slowing down of the collapse as opposed to the case of positive tangential pressure where the collapsing sphere is squeezed into a smaller volume. We also showed that the causal temperature is everywhere higher within the stellar core than the Eckart temperature (Figure 5.6). Up to this point, the shearing radiating solutions that exist in the literature are all acceleration-free [11] apart from the Euclidean star models and expansion-free models [23]. We have investigated the evolution of the temperature for a shearing collapsing fluid with nonzero acceleration. We have found that in the presence of acceleration and shear, the causal temperature within the stellar core is higher than the Eckart temperature. This reinforces the belief that relaxational effects cannot be ignored during dissipative gravitational collapse, even when the stellar fluid is close to hydrostatic equilibrium.

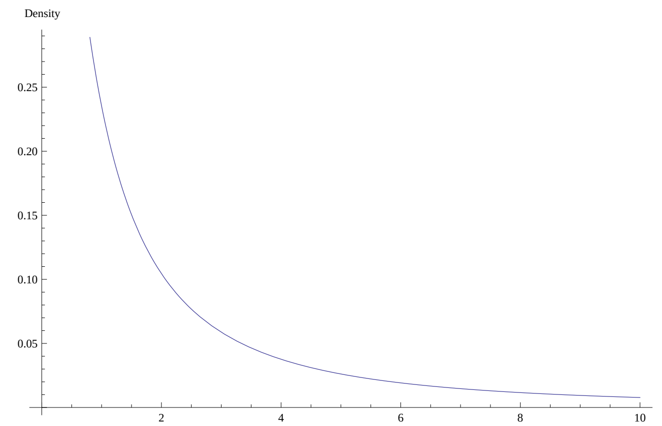

Figure 5.1: Density vs radial coordinate

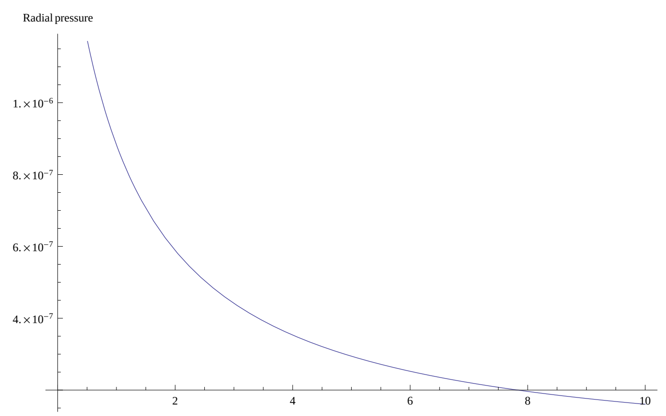

Figure 5.2: Radial pressure vs radial coordinate

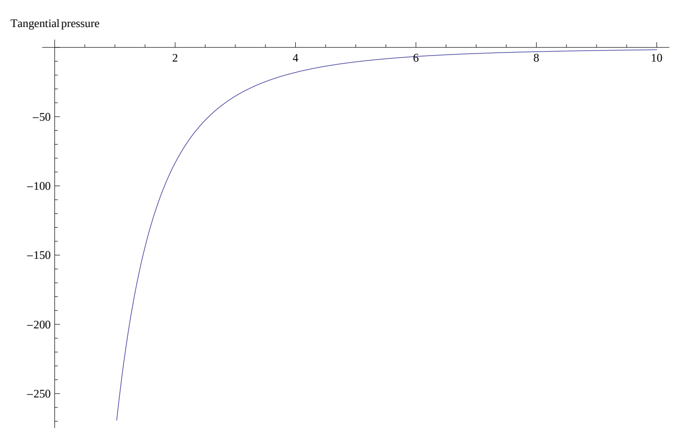

Figure 5.3: Tangential pressure vs radial coordinate

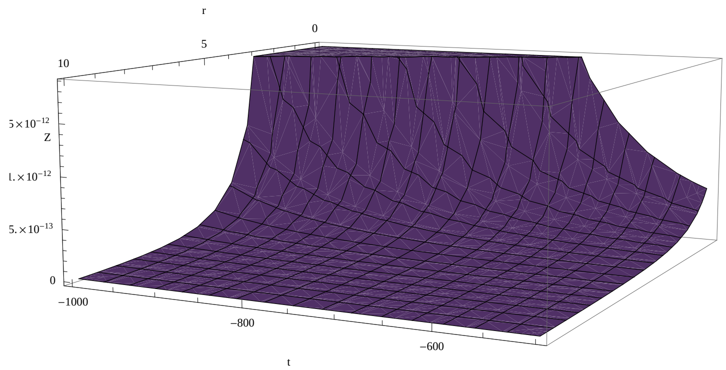

Figure 5.4: $Z = (\rho + p_r)^2 - 4q^2 > 0$ as a function of r and t.

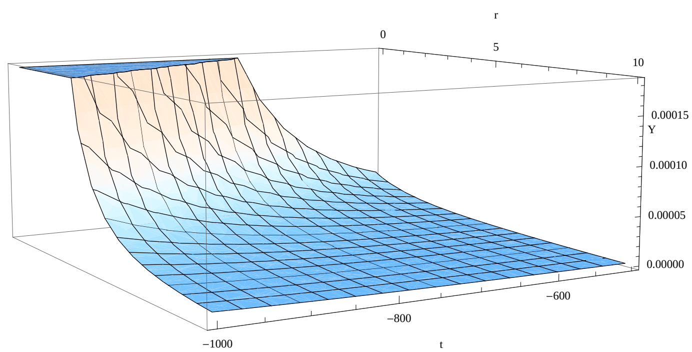

Figure 5.5: $\mathrm{Y} = \rho - p_r - 2P_\perp + [(\rho + p_r)^2 - 4q^2]^{1/2} > 0$ as a function of r and t.

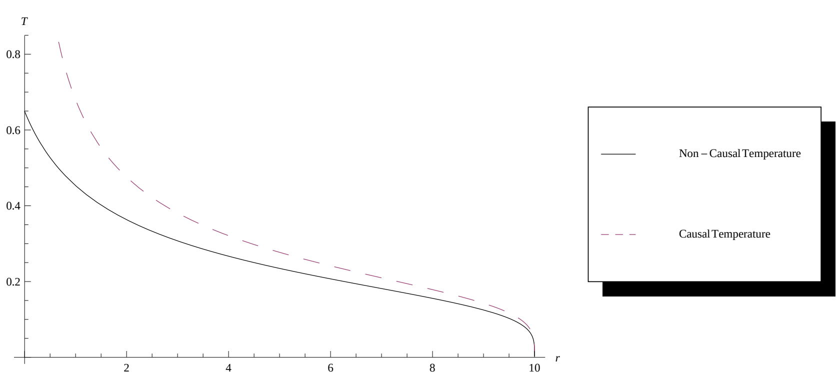

Figure 5.6: Temperature vs radial coordinate

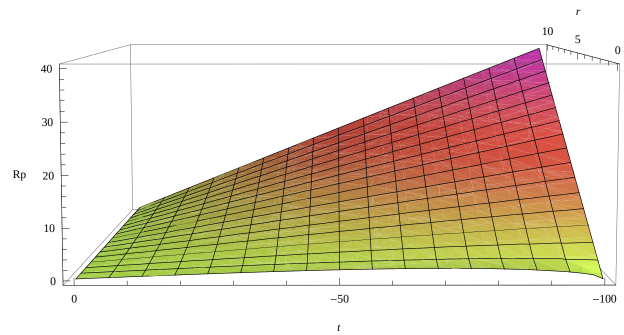

Figure 5.7: Proper radius vs time

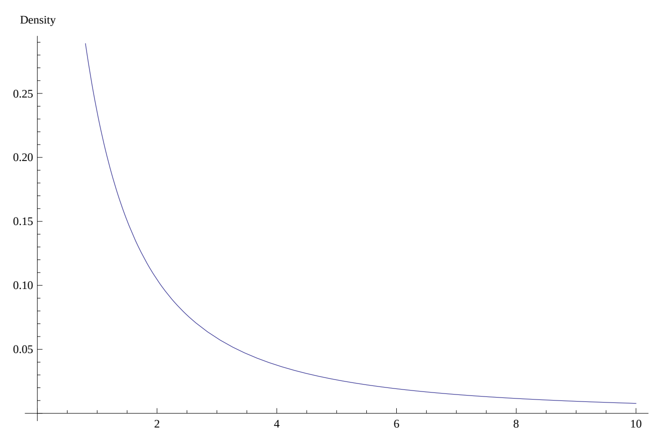

Figure 5.8: Density vs radial coordinate

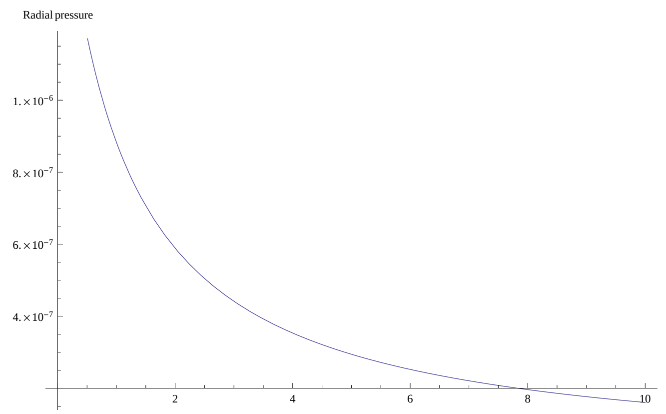

Figure 5.9: Radial pressure vs radial coordinate

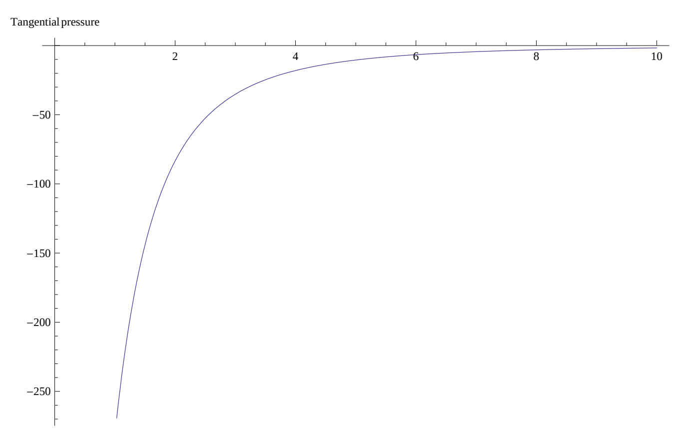

Figure 5.10: Tangential pressure vs radial coordinate

Chapter 6

Conclusion

In our research we modelled spherically symmetric stellar configurations undergoing dissipative gravitational collapse in the form of a radial heat flux. The Einstein field equations for a spherically symmetric, shearing spacetime endowed with an imperfect energy-momentum tensor were derived in detail. Since the star is radiating energy the exterior spacetime is nonempty and is described by Vaidya's outgoing solution. The junction conditions required for the smooth matching of the interior spacetime and exterior spacetime were derived in detail. Our focus was to extend radiating stellar models to include shear and acceleration. There are several shear-free models of dissipative gravitational collapse but the shearing models are quite restricted. These shearing models include Euclidean stars, expansion-free collapse and geodesic collapse. We attempted to generalise the geodesic models to include acceleration. We succeeded in obtaining a wide class of accelerating-shearing models. A detailed analysis of the physics of a particular class of models indicated that these solutions are physically viable for particular epochs of the collapse.

We present here an overview of our results obtained during our investigations:

- We derived in detail the Einstein field equations required for modelling a sphere undergoing dissipative collapse in the presence of shear and the emission of heat.

We presented the Vaidya solution which is needed to model the exterior of the radiating star. The atmosphere is assumed to be composed of null radition, thus representing a radiation zone around the collapsing star.

- The junction conditions first obtained by Santos for shear-free collapse was extended to include shear. We showed in detail that the radial pressure at the boundary of the collapsing star is nonvanishing. We showed that the radial pressure is balanced by the heat flux at the boundary. This condition ensures continuity of the momentum flux across the matching surface.

- The nonvanishing of the radial pressure at the boundary fixes the temporal evolution of our radiating models. We obtained the junction condition that determines the temporal behaviour of the gravitational potentials. The evolution equation (which is a nonlinear, ordinary differential equation) is the most general for such a collapsing model. Unfortunately it cannot be solved exactly in the most general case. By making certain simplifying assumptions we were in a position to obtain a wide variety of radiating, shearing solutions with nonvanishing acceleration. We believe that these solutions have not appeared in the literature and are all new.

- In order to get a handle on the physical viability of these models we investigated the evolution of the temperature profile by employing a causal heat transport equation. We presented the important aspects of thermodynamics necessary to describe fluids leaving hydrostatic equilibrium. We highlighted the importance of relaxational effects when the star is collapsing.

- In order to test the physical viability of our solutions we presented a particular model and investigated its stability and associated thermodynamical properties. We found that this particular model satisfies all the energy conditions except for a particular period in which the tangential pressure becomes negative. We imagine a negative tangential pressure repelling the fluid particles on each spherical shell thus leading to an expansion of these shells. It is speculated that the negative tangential pressure contributes to outward radial pressure. This would mean that as the tangential pressure becomes more negative, this may slow down and possibly halt the collapse.

We now point out several extensions to the results obtained in this thesis. The shearing solutions presented in this work can easily be generalised to include the effect of charge on the collapse process. The interesting result we hope to obtain is the effect of charge on formation or avoidance of the horizon. The role of shear has also been investigated in expansion-free collapse where it was shown that a cavity evolves within the dynamical body. The extension of the Santos junction conditions required for the matching of charged radiating interior to the Vaidya-Reissner-Nordstrom spacetime has already been obtained by Maharaj and Govender [17]. Another important aspect that needs to be investigated is the behaviour of the temperature profile by employing a full causal heat transport equation. The general framework for this investigation has already been provided by Herrera *et al* [21]. Temperature profiles using a full causal approach have not been obtained, even in the shear-free case. We hope that the work presented in this thesis opens up new areas of investigation in particular, the influence of inertia due to heat flow in accelerating models.

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