ALGEBRAIZING DEDUCTIVE SYSTEMS

by

CLINT JOHANN VAN ALTEN

Submitted in fulfilment of the academic

requirements for the degree of

Master of Science

in the

Department of Mathematics and Applied Mathematics,

University of Natal

Pietermaritzburg

1995

Preface

The work described in this thesis was carried out under the supervision of Prof James G. Raftery, Department of Mathematics and Applied Mathematics, University of Natal, Pietermaritzburg.

The thesis represents original work by the author and has not been submitted in any form to another University. Where use has been made of the work of others it has been duly acknowledged in the text.

Acknowledgements

I wish to thank both my supervisors Prof Teo Sturm and Prof James Raftery for overseeing this degree: Teo for starting me on this project and for his encouragement; James for all his patience, understanding and thoroughness in seeing it through to completion.

I wish to thank the Mathematics departments of the Pietermaritzburg and Durban campuses of the University of Natal for making their facilities available to me.

I thank the University of Natal and the Foundation for Research Development for financial assistance.

Thanks to my family for all their support and encouragement.

Abstract

Chapter 0. For completeness we present (mostly without proof) the background results of universal algebra that are necessary for the rest of the thesis. We also present here some proofs of results concerning quasivarieties that are less accessible in the literature.

Chapter 1. A *deductive system* S is defined in terms of axioms and inference rules and equivalent characterizations in terms of algebraic closure operators and, in particular, the 'lattice of theories' are given. S can be considered as an absolutely free algebra (of formulas) together with a *consequence relation* $\vdash_S$ satisfying some natural conditions. 'Theories' of S are sets of formulas 'closed' under the axioms and inference rules. A number of examples of deductive systems are given. We generalize the notion of deductive system to *k-deductive system* (k a positive integer) and show how the 2-deductive system $S_{\mathfrak{K}}$ can be obtained from any quasivariety $\mathfrak{K}$. A *matrix model* (*S-matrix*) $\mathcal{A}$ consists of an algebra of the same type as the language of S with an associated subset (*S-filter*) F such that $\vdash_S$ is reflected in a natural way by membership of F. The *Leibniz operator* Ω associates with each theory T a congruence relation ΩT on the algebra of formulas in the following way: Formulas φ and ψ are identified by ΩT if either one can replace the other as a subformula in an arbitrary formula ϑ without affecting the truth of ϑ relative to T. The Leibniz operator leads to the definition of a *reduced matrix*, and we show that the class of reduced matrices form a 'matrix semantics'. Two deductive systems are *equivalent* if there are mutually inverse interpretations of the consequence relations of each in the other. This is characterized by the property that there exists an isomorphism between their corresponding lattices of theories that satisfies an additional natural condition. We show how matrix models can be presented as universal Horn classes in the sense of first-order logic.

Chapter 2. *Protoalgebraic* k-deductive systems are introduced; these may be characterized as k-deductive systems for which the Leibniz operator Ω is monotonic. Protoalgebraicity is also characterized with respect to S-matrices and S-filters in a number of ways. A class of reduced S-matrices is the class of reduced matrices of a protoalgebraic k-deductive system iff it is closed under subdirect products. We present an internal characterization of protoalgebraicity in the spirit of Mal'cev-type conditions. *Congruential* and *weakly congruential* k-deductive systems are stronger than protoalgebraic ones, and their reduced model classes are characterized by the fact that they are closed under both submatrices and products (and also ultraproducts in the congruential case). A number of examples are provided.

Chapter 3. A class $\mathfrak{K}$ of algebras is an *algebraic semantics* for a k-deductive system S if the consequence relation $\vdash_S$ can be interpreted in the (semantical) equational consequence relation $\models_{\mathfrak{K}}$ of $\mathfrak{K}$ in a natural way. $\mathfrak{K}$ is called an *equivalent algebraic semantics* if there is an inverse interpretation of $\models_{\mathfrak{K}}$ in $\vdash_S$. If S has an equivalent algebraic semantics then it is called *algebraizable.* Every algebraizable k-deductive system has a unique quasivariety among its algebraic semantics, called its *equivalent quasivariety semantics.* Conversely, a quasivariety $\mathfrak{K}$ is the equivalent algebraic semantics of a 1-deductive system if certain Mal'cev-type conditions hold in $\mathfrak{K}$. A k-deductive system is algebraizable iff the Leibniz operator is monotonic, continuous and injective; iff it is an isomorphism between the lattice of theories and the lattice of $\mathfrak{K}$-congruences of its algebra of formulas (where $\mathfrak{K}$ can be taken as its equivalent quasivariety semantics); iff it is equivalent (in the sense of Chapter 1) to a 2-deductive system $S_{\mathfrak{K}}$ for some quasivariety $\mathfrak{K}$. The metalogical property of having the Gödel rule (or G-rule) (for algebraizable 1-deductive systems) corresponds to relative T-regularity of the equivalent quasivariety semantics (for some equationally definable constant T) but does not imply relative ideal-determination. Moreover, the equivalent quasivariety semantics has the relative shifting lemma, but need not be relative congruence modular (contrasting with the situation for varieties). A number of examples are provided.

Chapter 4. Here we investigate the metalogical properties called the deduction theorems. The local deduction-detachment theorem (LDDT) is a very general version of the deduction theorem (DDT). Both properties have matrix analogues. An algebraizable k-deductive system with equivalent quasivariety semantics $\mathfrak{K}$ has the LDDT iff $\mathfrak{K}$ has the relative congruence extension property. S has the DDT iff $\mathfrak{K}$ has equationally definable principal relative congruences. The classes of matrix models and reduced matrix models of k-deductive systems with the (L)DDT are characterized. In the case of $S_{\mathfrak{K}}$, where $\mathfrak{K}$ is a quasivariety, these characterizations lead to results about quasivarieties.

Chapter 5. We present as a case study the deductive system **BCK** of Meredith that exemplifies a number of the results from previous chapters. **BCK** is algebraizable with equivalent quasivariety semantics the quasivariety $\mathfrak{BCK}$ of BCK-algebras (in the sense of Iséki). Logic-driven proofs of existing algebraic results for $\mathfrak{BCK}$ are presented, e.g., relative congruence distributivity, possession of locally equationally definable principal relative congruences, hence also the relative congruence extension property. Strongly algebraizable axiomatic extensions of **BCK** have as their equivalent variety semantics just the varieties of BCK-algebras. Several of these are studied.

List of Contents

Introduction

This thesis is concerned mainly with the 'algebraization' of logics (alias deductive systems). Certain logics are well known to be 'algebraizable' in the (somewhat imprecise) sense that a class of algebras exists in which the logic's consequence relation is reflected faithfully. Standard examples are the Classical Propositional Calculus (denoted **CPC**), Intuitionistic Propositional Calculus (denoted **IPC**) and the modal logics **K** and $\boldsymbol{S4}$. It was Tarski who produced the algebraization of the **CPC**. He introduced the 'algebra of propositional formulas' and defined on it the relation $\equiv$, defined by

$$\varphi \equiv \psi \quad \text{iff} \quad \varphi \rightarrow \psi \text{ and } \psi \rightarrow \varphi \text{ are theorems of } \mathbf{CPC},$$

and showed that $\equiv$ forms what is now called a 'congruence relation' on this algebra. The corresponding quotient algebra is a Boolean algebra and the theorems of **CPC** (i.e., tautologies) coincide precisely with the formulas equivalent to some fixed but arbitrarily chosen tautology of **CPC** (e.g., T). A similar construction with the **IPC** creates a Heyting algebra, and from **K** and $\boldsymbol{S4}$ we get a modal algebra and an interior algebra respectively.

The quotient algebra obtained by factoring the algebra of propositional formulas by the relation $\equiv$ has come to be known as the 'Tarski-Lindenbaum algebra' of the logic. The study of a logic to which this construction can be applied can to a large extent be reduced to the study of this algebra, and for this the well-developed apparatus of universal algebra is available. This is the primary motivation of the algebraic logician.

While certain logics have been 'algebraized' in the above way, prior to the publication of Blok and Pigozzi's 'Algebraizable Logics' [BP89a], no precise general notion of 'algebraizability' of a logic was available. There remained significant logics that had not been algebraized in the Tarski-Lindenbaum sense, for example, the modal logics $\boldsymbol{S5}^C$ and $\boldsymbol{S5}^W$ and the logics **E** and **R** of entailment and relevance, respectively. In the cases of $\boldsymbol{S5}^C$ and $\boldsymbol{S5}^W$, the Tarski-Lindenbaum construction fails because $\equiv$ is not a congruence relation on the algebra of formulas, since it is not compatible with the unary operator $\Box$. This is a consequence of the fact that these logics do not have the rule of necessitation (i.e., we cannot deduce $\Box p$ from p). In the cases **E** and **R**, the relation $\equiv$ is a congruence, but the sets of theorems of **E** and **R** do not coincide with a congruence class of $\equiv$.

The question arises whether these logics can be algebraized in another manner or whether they are inherently nonalgebraizable. The same question may be asked of an arbitrary logic. In response to this question, Blok and Pigozzi have developed a general theory of algebraizability of logics. They approach the problem from an abstract point of view. A logic (which we shall call a deductive system) is defined (in Chapter 1) by a set of axioms and rules of inference over a given language. Moreover, we define a 'consequence relation' $\vdash$ between sets of formulas (over the language) and single formulas that encapsulates all possible derivations of the logic. A logic is said to be 'algebraizable' if there exists a class of algebras (which we can take as a quasivariety) whose equational theory is reflected in and reflects the consequence relation of the logic. This definition is made precise in Chapter 3. This criterion is naturally applicable to those logics that have been 'algebraized' previously, and the resultant algebras are the same. Moreover, following [BP89a], we show in Chapter 3 that it is possible to algebraize the logic **R** of relevance and certain modal logics. The logic **E** is not algebraizable in this sense however. In fact **E** is an example of a logic that is 'congruential' but not algebraizable. We also consider a number of pure implicational logics (i.e., logics whose language contains only an $\rightarrow$). Of these, we show that **BCK**, $\mathbf{RMO}_{\rightarrow}$ and $\mathbf{RM}_{\rightarrow}$ are algebraizable, while **BCI**, $\mathbf{E}_{\rightarrow}$ and $\mathbf{R}_{\rightarrow}$ are not. Amongst other examples, we shall show that the $\{\leftrightarrow\}$-fragments of **CPC** and **IPC** are algebraizable.

A number of characterizations of algebraizable logics (due to Blok and Pigozzi) are presented. It becomes evident that quasivarieties (classes of algebras closed under subalgebras, products and ultraproducts) are the natural algebraic counterparts to algebraizable logics. When the quasivariety is a variety, we call the logic 'strongly algebraizable'. We add to Blok and Pigozzi's results by presenting a characterization of such logics in Chapter 3.

In many of the main results presented in this thesis, the 'Leibniz operator' Ω plays an integral role. It applies to 'theories' of a deductive system, i.e., sets of formulas 'closed' under the axioms and inference rules. It associates with each theory T a congruence relation ΩT on the algebra of formulas in the following way: Formulas φ and ψ are identified by ΩT if either one can replace the other as a subformula in an arbitrary formula ϑ without affecting the truth of ϑ

relative to T. (Here the truth or falsity of ϑ is determined by whether or not ϑ is contained in T.) A logic turns out to be algebraizable if and only if the Leibniz relation is monotonic, continuous and injective; if and only if it is an isomorphism between the lattice of theories and the lattice of relative congruences of the formula algebra.

That the conditions presented are all characterizations of algebraizability provides convincing evidence that Blok and Pigozzi's definition of an algebraizable logic is the correct one. The Tarski-Lindenbaum algebra construction is not lost in the characterizations. In the logics mentioned above that are algebraizable by the Tarski-Lindenbaum method, the relation $\equiv$ coincides with the Leibniz relation ΩT, where T is the theory consisting of all theorems.

The correspondence between algebra and logic allows one to apply algebraic results to certain logics and deduce new results. The abstract approach to algebraizing logics investigated here allows one to find universal algebraic counterparts for many metalogical notions. For example, the metalogical property of having a deduction theorem corresponds to the universal algebraic property of a quasivariety having EDPRC (EDPC for varieties). We shall be investigating this connection and various related ones in Chapter 4. Another example is the Gödel rule. We shall see in Chapter 3 how this metalogical property corresponds to relative T-regularity for a 'truth' constant T.

Much of the above theory may be developed under assumptions far weaker than algebraizability. In particular, logics for which the Leibniz operator is monotonic are called 'protoalgebraic'. These deductive systems will be studied in detail in Chapter 2, and characterized in many different ways. Though not algebraizable in the above sense, they are amenable to algebraic treatment if we regard the study of relational structures as a part of algebra. In this broad sense, the title of this thesis should be considered to include their study. They turn out to include all logics having a deduction theorem. The intermediate notions of 'weakly congruential' and 'congruential' deductive systems, which are algebraically motivated, are also considered.

Chapter 5 is a case study in which the aforementioned theory is applied to the relatively 'weak' but still algebraizable logic **BCK**. Except in Chapter 5, the approach adopted follows that of the series of papers of Blok and Pigozzi [BP89a], [BP89b], [BP88], [BP92]. Most of the main results of this thesis are modifications or extensions of their work. The thesis [Pała94] was also consulted.

Chapter 0
Preliminaries

In this chapter we present much of the mathematical background needed for the rest of the thesis. We do assume a familiarity with elementary set theory. Most of the results presented are well-known for which reason we do not give many proofs, but rather give references to where proofs may be found. The proofs that are included are of results that will probably be known to experts, but for which suitable published proofs were either unavailable or in less accessible references. This applies particularly to the section on quasivarieties (0.4).

The set of *natural numbers* is $\omega = \{0, 1, 2, \dots\}$. For a set X, the set of all subsets of X is called the *power set of* X and denoted by $\mathcal{P}(X)$. The set of all *finite* subsets of X is denoted by $\mathcal{P}_\omega(X)$. For sets X, Y, let X^Y be the set of all functions mapping Y to X, i.e., all $f : Y \to X$. For a set X and a natural number n, X^n is therefore the set of all functions from an n-element set to X, i.e., X^n is a set whose elements can be considered n-tuples $\langle x_1, \dots, x_n \rangle$, where $x_1, \dots, x_n \in X$. An *n-ary operation* on a set X is a function from X^n to X. An *n-ary relation* on a set X is a subset of X^n. A 1-ary operation (resp. relation) is called *unary*; a 2-ary operation (resp. relation) is called *binary*. For a set X, we define the binary relation $I_X = \{(x, x)\,;\, x \in X\}$. Let $f : X \to Y$ be a function. We shall often write fx for $f(x)$ when $x \in X$, and we write $f(Z)$ for the image $\{fz\,;\, z \in Z\}$ of Z under f when $Z \subseteq X$. For $Z \subseteq X$, the restriction of f to Z is denoted by $f \mid Z$. If n is a nonzero natural number, then we often write $i \leq n$ for $i \in \{1, 2, \dots, n\}$. Note that the lastmentioned set begins at 1 and not 0.

0.1 LATTICE-THEORETIC PRELIMINARIES

There are two standard ways of defining semilattices and lattices – one puts them on the same algebraic footing as groups or rings, while the other is based on the notion of order.

An ordered pair $\mathbf{L} = \langle L\,;\, \vee \rangle$ (resp. $\langle L\,;\, \wedge \rangle$), such that L is a nonempty set and $\vee$ (resp. $\wedge$) is a binary operation on L (i.e., a function from L^2 to L) called *join* (resp. *meet*), is called a *join-semilattice* (resp. *meet-semilattice*) if it satisfies the following identities for all $a, b, c \in L$:

$a \vee b = b \vee a$, (resp. $a \wedge b = b \wedge a$) [*commutative laws*]

$a \vee (b \vee c) = (a \vee b) \vee c$, (resp. $a \wedge (b \wedge c) = (a \wedge b) \wedge c$) [*associative laws*]

$a \vee a = a$, (resp. $a \wedge a = a$) [*idempotent laws*]

An ordered triple $\mathbf{L} = \langle L\, ;\, \wedge\, ,\, \vee \rangle$ such that L is a nonempty set and $\wedge$ and $\vee$ are binary operations on L, called *join* and *meet*, respectively, is called a *lattice* if $\langle L\, ;\, \wedge \rangle$ is a meet-semilattice and $\langle L\, ;\, \vee \rangle$ is a join-semilattice and the following identities hold for all $a, b \in L$:

$a = a \vee (a \wedge b)$, $a = a \wedge (a \vee b)$ [*absorption laws*]

Let L be a nonempty set and let $\leq$ be a binary relation on L (i.e., $\leq\, \subseteq L^2$). We write $a \leq b$ if $(a, b) \in\, \leq$. Then $\leq$ is called a *partial order* on L and the pair $\langle L\, ;\, \leq \rangle$ is called an *ordered set* or *partially ordered set* if the following conditions hold for all $a, b, c \in L$:

$a \leq a$ [*reflexivity*],

if $a \leq b$ and $b \leq c$ then $a \leq c$ [*transitivity*],

if $a \leq b$ and $b \leq a$ then $a = b$ [*anti-symmetry*].

A partial order $\leq$ on a set L is a *linear order* if for all $a, b \in L$ the following condition is satisfied:

$$a \leq b \quad \text{or} \quad b \leq a.$$

A partially ordered set $\langle L\, ;\, \leq \rangle$ is called a *linearly ordered set* or a *chain* if $\leq$ is a linear order. For $a, b \in L$, we use the expression $a < b$ to mean $a \leq b$ but $a \neq b$. Also, for $a, b \in L$, we say that b *covers* a, or a *is covered by* b, if $a < b$ and whenever $c \in L$ and $a \leq c \leq b$ it follows that $a = c$ or $b = c$. We use the notation $a \prec b$ to denote that a is covered by b. Let $X \subseteq L$ where $\langle L\, ;\, \leq \rangle$ is a partially ordered set. An element $b \in L$ is an *upper* (resp. *lower*) *bound* of X if $a \leq b$ (resp. $b \leq a$) for all $a \in X$. An upper (resp. lower) bound of X is a *least upper bound* (resp. *greatest lower bound*) or *supremum* (resp. *infimum*) of X if for all upper (resp. lower) bounds c of X, we have $b \leq c$ (resp. $c \leq b$). If the least upper bound (resp. greatest lower bound) of X exists, then it is unique and we denote it by $\bigvee^{\mathbf{L}} X$ (resp. $\bigwedge^{\mathbf{L}} X$), where $\mathbf{L} = \langle L\, ;\, \leq \rangle$. (We sometimes drop the superscript if $\mathbf{L}$ is understood.)

A partially ordered set $\mathbf{L} = \langle L\, ;\, \leq \rangle$ is called a *join-semilattice* (resp. *meet-semilattice*) if $\bigvee^{\mathbf{L}} X$ (resp. $\bigwedge^{\mathbf{L}} X$) exists for every finite subset X of L. If $\langle L\, ;\, \vee \rangle$ is a join-semilattice in the sense of the first definition, and we define a relation $\leq$ on L by $a \leq b$ iff $a \vee b = b$, then $\langle L; \leq \rangle$ is a join-semilattice in the sense of the second definition. Similarly, if $\langle L; \wedge \rangle$ is a meet-semilattice in the sense of the first definition, then $\langle L; \leq \rangle$, with $\leq$ defined by $a \leq b$ iff $a \wedge b = a$, is a meet-semilattice in the sense of the second definition. Conversely, if $\langle L; \leq \rangle$ is a join-semilattice in the sense of the second definition, then $\langle L; \vee \rangle$, with $\vee$ defined by $a \vee b = \bigvee^{\mathbf{L}}\{a, b\}$, is a join-semilattice in the second sense. A similar statement is true for meet-semilattices, with $a \wedge b = \bigwedge^{\mathbf{L}}\{a, b\}$. We shall, at times, confuse the algebraic and order-theoretic notions of semilattice systematically, as is standard practice.

A subset $X \subseteq L$ of a partially ordered set $\langle L; \leq \rangle$ is said to be *upwardly directed* (resp. *downwardly directed*) if for every $a, b \in X$, there exists $c \in X$ such that $a, b \leq c$ (resp. $c \leq a, b$).

An element a of a partially ordered set $\langle L; \leq \rangle$ is said to be *least* (resp. *greatest*) if for every $b \in L$, $a \leq b$ (resp. $b \leq a$). If a least element exists in a join-semilattice, it is unique and is often denoted $\perp$ or 0, and if a greatest element exists in a meet-semilattice, it is unique and is often denoted T or 1. A join-semilattice $\langle L; \vee \rangle$ is called a *join-semilattice with* 0 if it has a least element. A meet-semilattice $\langle L; \wedge \rangle$ is called a *meet-semilattice with* 1 if it has a greatest element.

A partially ordered set $\langle L; \leq \rangle$ is called a *lattice* and $\leq$ a *lattice order* if for all finite subsets X of L, the least upper bound and greatest lower bound exist. If $\langle L; \wedge, \vee \rangle$ is a lattice, as in the first definition, then $\langle L; \leq \rangle$ is a lattice as in the second definition, where $\leq$ is defined by $a \leq b$ iff $a \wedge b = a$ iff $a \vee b = b$. Conversely, if $\mathbf{L} = \langle L; \leq \rangle$ is a lattice as defined here, then $\langle L; \wedge, \vee \rangle$ is a lattice in the sense of the first definition, where $\wedge$ and $\vee$ are defined by $a \wedge b = \bigwedge^{\mathbf{L}}\{a, b\}$, $a \vee b = \bigvee^{\mathbf{L}}\{a, b\}$. These two constructions are inverses. Throughout the text we will use both the above definitions of lattices, and we will make free use the notions associated with either definition.

A lattice $\mathbf{L} = \langle L; \leq \rangle$ is called *complete* if $\bigwedge^{\mathbf{L}} X$ and $\bigvee^{\mathbf{L}} X$ exist for every subset X of L. In fact, a partially ordered set $\mathbf{L} = \langle L; \leq \rangle$ is complete if either $\bigwedge^{\mathbf{L}} X$ exists for every $X \subseteq L$ or $\bigvee^{\mathbf{L}} X$ exists for every $X \subseteq L$. A complete lattice $\mathbf{L} = \langle L; \leq \rangle$ always has a greatest and a least element, namely $\bigvee^{\mathbf{L}} L$ and $\bigwedge^{\mathbf{L}} L$. For every set X, $\langle \mathcal{P}(X), \subseteq \rangle$ is a complete lattice. Its corresponding meet and join operations are $\cap$ and $\cup$ and its greatest and least elements are X and $\emptyset$, respectively.

Two lattices $\mathbf{L}_1 = \langle L_1, \wedge_1, \vee_1 \rangle$, $\mathbf{L}_2 = \langle L_2, \wedge_2, \vee_2 \rangle$ are said to be *isomorphic*, written $\mathbf{L}_1 \cong \mathbf{L}_2$, if there exists a bijection f from L_1 to L_2 such that for every $a, b \in L_1$ the following two equations hold:

$$f(a \wedge_1 b) = f(a) \wedge_2 f(b) \quad \text{and} \quad f(a \vee_1 b) = f(a) \vee_2 f(b).$$

Such an f is called an *isomorphism.* If $\langle L_1 ; \leq_1 \rangle$ and $\langle L_2 ; \leq_2 \rangle$ are partially ordered sets and f is a map from L_1 to L_2, then we say that f is *order-preserving* or *monotonic* if $a \leq_1 b$ implies $f(a) \leq_2 f(b)$ for all $a, b \in L_1$. We say that f is *order-reflecting* if $f(a) \leq_2 f(b)$ implies $a \leq_1 b$ for every $a, b \in L_1$.

0.1.1 LEMMA [BS81, Theorem 2.3, p8]

Let $\mathbf{L}_1 = \langle L_1, \wedge_1, \vee_1 \rangle$, $\mathbf{L}_2 = \langle L_2, \wedge_2, \vee_2 \rangle$ be lattices and let $\leq_1$ and $\leq_2$ be their associated lattice orders. Then $\mathbf{L}_1$ and $\mathbf{L}_2$ are isomorphic iff there exists a bijection f from L_1 to L_2 such that f is both order-preserving and order-reflecting. □

A lattice $\langle L, \wedge, \vee \rangle$ is said to be *distributive* if it satisfies each of the following equations for all $a, b, c \in L$:

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \quad \text{and} \quad a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

In fact it is sufficient that either one of the above equations hold. Moreover, the first equation holds iff $a \wedge (b \vee c) \leq (a \wedge b) \vee (a \wedge c)$ since $(a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee c)$ for all $a, b, c \in L$. Similarly, the second equation holds iff $a \vee (b \wedge c) \geq (a \vee b) \wedge (a \vee c)$. A lattice $\langle L, \wedge, \vee \rangle$ is said to be *modular* if it satisfies the following condition for all $a, b, c \in L$:

$$a \leq b \text{ implies } a \vee (b \wedge c) = b \wedge (a \vee c).$$

A distributive lattice is clearly modular.

Let $\mathbf{L} = \langle L ; \leq \rangle$ be a join-semilattice (resp. meet-semilattice). A nonempty subset G of L is called an *ideal* (resp. *filter*) of $\mathbf{L}$ if and only if for any $a, b \in L$, we have

(i) $a, b \in G$ implies $a \vee b \in G$ (resp. $a, b \in G$ implies $a \wedge b \in G$),

(ii) $a \in G$ and $b \leq a$ implies $b \in G$ (resp. $a \in G$ and $a \leq b$ implies $b \in G$).

If in addition, $G \neq L$, we say that G is a *proper ideal* (resp. *proper filter*) of $\mathbf{L}$. The set of all ideals of $\mathbf{L}$ is denoted $Id\,\mathbf{L}$. We denote by $(x] = \{y \in L ; y \leq x\}$ the *ideal of* $\mathbf{L}$ *generated by* x (it is the intersection of all ideals containing x). Similarly, we denote by $[x) = \{y \in L ; x \leq y\}$ the *filter* *of* $\mathbf{L}$ *generated by* x.

Let $\mathbf{L} = \langle L; \leq \rangle$ be a join-semilattice with 0. It is easy to see that the set $Id\,\mathbf{L}$ is closed under arbitrary intersections. If we define $I \wedge J = I \cap J$ and

$$I \vee J = \{x \in L;\, x \leq i \vee j \text{ for some } i \in I \text{ and } j \in J\}$$

for all $I, J \in Id\,\mathbf{L}$, then $\mathbf{Id}\,\mathbf{L} = \langle Id\,\mathbf{L}, \vee, \wedge \rangle$ is a lattice whose associated lattice order is $\subseteq$. Moreover, since $Id\,\mathbf{L}$ is closed under arbitrary intersections, it follows that $\mathbf{Id}\,\mathbf{L}$ is a *complete* lattice.

We say that the condition $D(a,b)$ holds in $\mathbf{L}$ if, for all $c, d \in L$,

$a \vee c \geq b$ and $a \vee d \geq b$ implies there exists $e \in L$ such that $e \leq c$, $e \leq d$ and $a \vee e \geq b$.

0.1.2 LEMMA [BP88, Lemma 4.1] (see also [Grä78, Lemma 1, p99])

Let $\mathbf{L} = \langle L, \vee \rangle$ *be a join-semilattice with* 0. *The lattice* $Id\,\mathbf{L}$ *is distributive if and only if* $D(a,b)$ *holds in* $\mathbf{L}$ *for all* $a, b \in L$.

Proof. ($\Leftarrow$) Let I, J, K be ideals of $\mathbf{L}$. We must show that

$$I \vee (J \wedge K) = (I \vee J) \wedge (I \vee K).$$

Since the inclusion from left to right always holds, we need only show the reverse inclusion. Let $x \in (I \vee J) \wedge (I \vee K)$. Then, by definition, there exist $i, i' \in I$, $j \in J$ and $k \in K$ such that $x \leq i \vee j$ and $x \leq i' \vee k$, hence $x \leq (i \vee i') \vee j$ and $x \leq (i \vee i') \vee k$. Since $D(i \vee i', x)$ holds, there exists an $\ell \in L$ such that $\ell \leq j$ and $\ell \leq k$ and $x \leq (i \vee i') \vee \ell$. Since $\ell \in J \cap K$ and $i \vee i' \in I$, $x \in I \vee (J \wedge K)$ as required.

($\Rightarrow$) Let $a, b \in L$. To show that $D(a,b)$ holds, suppose there exist $c, d \in L$ such that $a \vee c \geq b$ and $a \vee d \geq b$. Then $b \in ((a] \vee (c]) \wedge ((a] \vee (d])$, i.e., $b \in (a] \vee ((c] \wedge (d])$, since $\mathbf{L}$ is distributive. Thus there exist $f \leq a$ and $e \in (c] \cap (d]$ with $b \leq f \vee e$. Finally, $a \vee e \geq f \vee e \geq b$, $e \leq c$ and $e \leq d$. □

A join-semilattice $\mathbf{L} = \langle L; \vee \rangle$ (resp. meet-semilattice $\mathbf{L} = \langle L; \wedge \rangle$) is said to be *generated* by a set $X \subseteq L$ if for every $a \in L$, there exist some $x_1, \ldots, x_n \in X$ such that $a = \bigvee^{\mathbf{L}}\{x_1, \ldots, x_n\}$ (resp. $a = \bigwedge^{\mathbf{L}}\{x_1, \ldots, x_n\}$).

0.1.3 LEMMA [BP88, Lemma 4.2]

Let $\mathbf{L} = \langle L, \vee \rangle$ *be a join-semilattice with* 0 *generated by* $X \subseteq L$. *If* $D(a,b)$ *holds for all* $a, b \in X$, *then* $D(a,b)$ *holds for all* $a, b \in L$.

Proof. First consider the case $a \in X, b \in L$. There exists a positive integer n and $x_i \in X$ for $i \leq n$ such that $b = \bigvee_{i \leq n} x_i$. Proceed by induction on n. For $n = 1$, $b \in X$, so $D(a,b)$ holds by assumption. Assume that $n > 1$, and that $D(a,b')$ holds, where $b' = \bigvee_{i \leq n-1} x_i$. Let $c, d \in L$ such that $b \leq a \vee c$ and $b \leq a \vee d$. Since $b' \leq b$, we have $b' \leq a \vee c$ and $b' \leq a \vee d$, hence there exists $e' \in L$ such that $b' \leq a \vee e'$, $e' \leq c$ and $e' \leq d$. Now, $x_n \leq a \vee c$ and $x_n \leq a \vee d$ since $x_n \leq b$, hence, since $D(a, x_n)$ holds, there exists $e'' \in L$ such that $x_n \leq a \vee e''$, $e'' \leq c$ and $e'' \leq d$. If we set $e = e' \vee e''$, then $e \leq c$, $e \leq d$ and

$$a \vee e = a \vee e' \vee e'' = a \vee e' \vee a \vee e'' \geq x_n \vee b' = b,$$

so $D(a,b)$ holds and the induction is complete.

Next, suppose that a, b are arbitrary elements of L, say $a = \bigvee_{i \leq n} x_i$ where $x_i \in X$ for all $i \leq n$. Proceed by induction again. If $n = 1$, then $a \in X$ and $D(a,b)$ holds by the above result. If $n > 1$, set $a' = \bigvee_{i \leq n-1} x_i$ and assume that $D(a', b)$ holds. Let $c, d \in L$ such that $a \vee c \geq b$, $a \vee d \geq b$. Then $b \leq a' \vee x_n \vee c$ and $b \leq a' \vee x_n \vee d$. Since $D(a', b)$ holds, there exists $e' \in L$ such that $a' \vee e' \geq b$, $e' \leq x_n \vee c$ and $e' \leq x_n \vee d$. But, since $D(x_n, e')$ holds, there exists $e \in L$ such that $e' \leq x_n \vee e$, $e \leq c$ and $e \leq d$, hence

$$b \leq a' \vee e' \leq a' \vee x_n \vee e = a \vee e,$$

and the induction is complete. □

Let $\mathbf{L} = \langle L; \vee \rangle$ be a join-semilattice and $\emptyset \neq L' \subseteq L$. If for any $a, b \in L'$, we have $a \vee b \in L'$ then we call $\langle L'; \vee \mid (L')^2 \rangle$ a *join-subsemilattice* of $\mathbf{L}$, and we abbreviate $\langle L'; \vee \mid (L')^2 \rangle$ by $\langle L'; \vee \rangle$. *Meet-subsemilattices* of meet-semilattices are defined dually. If $\mathbf{L} = \langle L; \wedge, \vee \rangle$ is a lattice and $\emptyset \neq L' \subseteq L$, such that $\langle L'; \vee \rangle$ is a join-subsemilattice of $\langle L; \vee \rangle$ and $\langle L'; \wedge \rangle$ is a meet-subsemilattice of $\langle L; \wedge \rangle$ then $\langle L'; \wedge \mid (L')^2, \vee \mid (L')^2 \rangle$ (abbreviated $\langle L'; \wedge, \vee \rangle$) is called a *sublattice* of $\mathbf{L}$. Similarly, if we denote a (semi)lattice by $\langle L; \leq \rangle$ then a subsemilattice will be denoted by $\langle L'; \leq \rangle$, which really abbreviates $\langle L'; \leq \cap (L')^2 \rangle$.

Let $\langle L; \leq \rangle$ be a complete lattice. An element $a \in L$ is called *compact* if, whenever $X \subseteq L$

and $a \leq \bigvee X$, there exists a finite subset X' of X such that $a \leq \bigvee X'$. The set of all compact elements of L is denoted by L^c. For example, for any set Y, the compact elements of the complete lattice $\langle \mathcal{P}(Y), \subseteq \rangle$ are the finite subsets of Y. A complete lattice $\langle L; \leq \rangle$ is called *algebraic* if every element of L is the join of a set of compact elements of L. Clearly, $\langle \mathcal{P}(Y); \subseteq \rangle$ is an algebraic lattice for every set Y. Algebraic lattices are called 'algebraic' because typically, they are the kinds of lattices studied by algebraists, e.g., the ideal lattices of rings, the subgroup or normal subgroup lattices of groups, the ideal or filter lattices of Boolean algebras, etc. (ordered, in each of these cases, by set inclusion). For all of these lattices, the term 'compact' turns out to have exactly the same meaning as 'finitely generated'. We shall give a standard test for algebraicity of a complete lattice and a well-known characterization of the compact elements of such a lattice. Some more terminology is required first.

Given a complete lattice $\mathbf{L} = \langle L; \leq \rangle$, a subset X of L is called a *closure system* in $\mathbf{L}$ if for every $Y \subseteq X$, $\bigwedge^{\mathbf{L}} Y \in X$. (This forces X to contain the greatest element of L, viz., $\bigwedge^{\mathbf{L}} \emptyset$; in particular, $X \neq \emptyset$ and $\langle X; \leq \rangle$ is a complete lattice in its own right.) If in addition, we have $\bigvee^{\mathbf{L}} Y \in X$ for every nonempty upwardly directed subset Y of X, then X is called an *algebraic closure system* in $\mathbf{L}$. With a closure system X in a complete lattice $\langle L; \leq \rangle$, we associate a mapping $u = u_X : L \to L$ defined by $u(x) = \bigwedge^{\mathbf{L}} \{z \in X ; x \leq z\}$. This map has the following properties:

(i) $\qquad x \leq u(x) = u(u(x))$;

(ii) $\qquad x \leq y$ implies $u(x) \leq u(y)$

for all $x, y \in L$. The range $u(L)$ of this map is just X. If X is an algebraic closure system then we also have

(iii) $\qquad u(x) = \bigvee^{\mathbf{L}} \{u(y) ; y \leq x \text{ and } y \text{ is compact in } \mathbf{L}\} \quad (x \in L)$.

Mappings $u : L \to L$ satisfying (i) and (ii) are called *closure operators* on $\langle L; \leq \rangle$. For such a map u, we always have

(0.1.1) $\qquad \bigvee^{u(\mathbf{L})} u(Y) = u(\bigvee^{\mathbf{L}} Y)$ for any $Y \subseteq L$ (where $u(\mathbf{L}) = \langle u(L); \leq \rangle = \langle X; \leq \rangle$)

([BS81, Theorem 5.2, p18]). A closure operator u on $\langle L; \leq \rangle$ is called an *algebraic closure operator* on $\langle L; \leq \rangle$ if it satisfies (iii) (for all $x \in L$). The elements of L of the form $u(x)$ (for some $x \in L$) are called *closed (with respect to u)*. Every (algebraic) closure operator on $\langle L; \leq \rangle$ has the form u_X

for some (algebraic) closure system X in $\langle L\,;\,\leq\rangle$, viz., for $X = u(L)$, and the correspondence $X \mapsto u_X$ is one-to-one. Also, every (algebraic) closure system X in $\langle L\,;\,\leq\rangle$ is in the range $u(L)$ of a suitable (algebraic) closure operator u on $\langle L\,;\,\leq\rangle$, viz., $u = u_X$, and the correspondence $u \mapsto u_X$ is one-to-one. The aforementioned two correspondences are mutually inverse bijections between the set of (algebraic) closure systems in $\langle L\,;\,\leq\rangle$ and the set of (algebraic) closure operators on $\langle L\,;\,\leq\rangle$. The following results are to be found in most introductory lattice theory texts or, e.g., [Grä79, Theorem 5 and Lemma 5, p25].

0.1.4 PROPOSITION

The following conditions on a lattice $\mathbf{L} = \langle L\,;\,\leq\rangle$ *are equivalent:*

(i) $\mathbf{L}$ *is an algebraic lattice;*

(ii) *there exists a set* S *and an algebraic closure system* X *in the complete lattice* $\langle \mathcal{P}(S),\,\subseteq\rangle$ *of all subsets of* S *(ordered by inclusion) such that* $\mathbf{L}$ *is isomorphic to the lattice* $\langle X\,;\,\subseteq\rangle$. □

0.1.5 PROPOSITION

Let $\mathbf{L} = \langle L\,;\,\leq\rangle$ *be an algebraic lattice and* u *an algebraic closure operator on* $\mathbf{L}$. *Then* $y \in L$ *is a compact element of* $\langle u(L)\,;\,\leq\rangle$ *iff* $y = u(x)$ *for some compact element* x *of* $\langle L\,;\,\leq\rangle$. □

0.1.6 COROLLARY

Let S *be a set and* $X \subseteq \mathcal{P}(S)$. *Then* $\langle X\,;\,\subseteq\rangle$ *is an algebraic lattice if and only if* X *is closed under arbitrary intersections and* $\bigcup Y \in X$ *for any nonempty upwardly directed subset* Y *of* $\langle X\,;\,\subseteq\rangle$. *In this case, the map* $Z \mapsto u(Z) = \bigcap\{A \in X\,;\, A \supseteq Z\}$ $(Z \in \mathcal{P}(S))$ *is the algebraic closure operator on* $\langle \mathcal{P}(S)\,;\,\subseteq\rangle$ *corresponding to* X *and the compact elements of* $\langle X\,;\,\subseteq\rangle$ *are just the elements of the form* $u(Z)$, *where* Z *is any finite subset of* S. □

In Chapter 4, we shall make essential use of the following proposition, which combines several classical results from lattice theory. Proofs of all statements in the proposition may be found in [Grä79, Chapter 0, §6].

0.1.7 PROPOSITION

(i) [Grä79, Theorem 2, p22] *The ideal lattice* $\mathbf{Id}\,\mathbf{L}$ *of a join-semilattice* $\mathbf{L}$ *with* 0 *is an algebraic lattice.*

(ii) [Grä79, Theorem 3, p22] *If* $\langle L; \leq \rangle$ *is an algebraic lattice and* 0 *is the least element of* L *then the set* L^c *of all compact elements of* L *is a join-subsemilattice of* L *(i.e.,* L^c *is closed under finite joins), and* $0 \in L^c$.

(iii) [Grä79, Theorem 3, pp22-23] *If* $\langle L; \leq \rangle$ *is an algebraic lattice then the map* $x \mapsto \{y \in L^c;\, y \leq x\}$ $(x \in L)$ *is a lattice isomorphism from* L *onto the ideal lattice* $\mathbf{Id}\,\mathbf{L}^c$ *of the join-semilattice* L^c *(with* 0*) of all compact elements of* L*. Therefore:*

(iv) [Grä79, Theorem 5, pp25-26] *A lattice is algebraic if and only if it is isomorphic to the ideal lattice of some join-semilattice with* 0. □

Let $\mathbf{L} = \langle L, \wedge \rangle$ be a meet-semilattice with 1. We say that $\mathbf{L}$ is *Brouwerian* if, for all $a, b \in L$, there exists an element $a \rightarrow^{\mathbf{L}} b \in L$ such that

$$a \rightarrow^{\mathbf{L}} b = \max\{c \in L;\, a \wedge c \leq b\}.$$

We then call $a \rightarrow^{\mathbf{L}} b$ the *relative pseudocomplement* of a with respect to b. Let $\mathbf{L} = \langle L, \vee \rangle$ be a join-semilattice with 0. We say that $\mathbf{L}$ is *dually Brouwerian* if, for all $a, b \in L$, there exists an element $a *^{\mathbf{L}} b \in L$ such that

$$a *^{\mathbf{L}} b = \min\{c \in L;\, a \vee c \geq b\}.$$

We then call $a *^{\mathbf{L}} b$ the *dual relative pseudocomplement* of a with respect to b.

0.1.8 LEMMA [BP88, Lemma 7.4]

Let $\mathbf{L} = \langle L, \vee \rangle$ *be a join-semilattice with* 0 *such that* $\mathbf{L}$ *is generated by* $X \subseteq L$*. Then* $\mathbf{L}$ *is dually Brouwerian if and only if* $a *^{\mathbf{L}} b$ *exists for all* $a, b \in X$.

Proof. Suppose $a *^{\mathbf{L}} b$ exists for all $a, b \in X$. Let $a \in X$ and $b \in L$. Then there exist a positive integer n and $x_i \in X$ for $i \leq n$, such that $\bigvee_{i \leq n} x_i = b$. We proceed by induction on n. If $n = 1$, then $a *^{\mathbf{L}} b$ exists by assumption. Suppose $n > 1$ and $a *^{\mathbf{L}} \bigvee_{i \leq n-1} x_i$ exists, say $d = a *^{\mathbf{L}} \bigvee_{i \leq n-1} x_i$. Thus $a \vee d \geq \bigvee_{i \leq n-1} x_i$. Since $a, x_n \in X$, $a *^{\mathbf{L}} x_n$ exists and $a \vee (a *^{\mathbf{L}} x_n) \geq x_n$. Set $e = d \vee (a *^{\mathbf{L}} x_n)$. Then

$$a \vee e = (a \vee d) \vee (a \vee (a *^{\mathbf{L}} x_n)) \geq (\bigvee_{i \leq n-1} x_i) \vee x_n = b.$$

Moreover, if $a \vee f \geq b$, then

$$a \vee f \geq \bigvee_{i \leq n-1} x_i, \quad \text{hence } f \geq d,$$

and $$a \vee f \geq x_n, \quad \text{hence } f \geq a *^{\mathbf{L}} x_n,$$

implying that $a \vee f \geq d \vee (a *^{\mathbf{L}} x_n) = e$. Thus $a *^{\mathbf{L}} b$ exists and the induction is complete.

Now, suppose $a, b \in L$. Then there exist a positive integer n and $x_i \in X$ for $i \leq n$ such that $\bigvee_{i \leq n} x_i = a$. We proceed by induction on n. If $n = 1$, then $a *^{\mathbf{L}} b$ exists by the previous paragraph. Suppose $(\bigvee_{i \leq n-1} x_i) *^{\mathbf{L}} b$ exists, say $d = (\bigvee_{i \leq n-1} x_i) *^{\mathbf{L}} b$. Then $(\bigvee_{i \leq n-1} x_i) \vee d \geq b$. Since $x_n \in X$, the previous paragraph implies that $x_n *^{\mathbf{L}} d$ exists. Set $e = x_n *^{\mathbf{L}} d$. Then $x_n \vee e \geq d$, hence

$$a \vee e = (\bigvee_{i \leq n-1} x_i) \vee x_n \vee e \geq (\bigvee_{i \leq n-1} x_i) \vee d \geq b.$$

Suppose $a \vee f \geq b$. Then $(\bigvee_{i \leq n} x_i) \vee f \geq b$, hence

$$(\bigvee_{i \leq n-1} x_i) \vee x_n \vee f \geq b,$$

so $$x_n \vee f \geq (\bigvee_{i \leq n-1} x_i) *^{\mathbf{L}} b = d,$$

and $$f \geq x_n *^{\mathbf{L}} d = e.$$

Thus $e = a *^{\mathbf{L}} b$ and the proof is complete. □

0.1.9 LEMMA [BP88, Lemma 7.5]

Let $\mathbf{L}$ *be an algebraic lattice and* $\mathbf{L}^c$ *its join-semilattice of compact elements. If* $a, b \in L^c$, *then* $a *^{\mathbf{L}^c} b$ *exists if and only if* $a *^{\mathbf{L}} b$ *exists, and if they do, then* $a *^{\mathbf{L}^c} b = a *^{\mathbf{L}} b$.

Proof. (⇒) Suppose that $a *^{\mathbf{L}^c} b = d \in L^c$. Then $d \in L$ and $a \vee d \geq b$, by definition. Now, suppose $c \in L$ such that $a \vee c \geq b$. We can write $c = \bigvee_{i \in I} c_i$ for some set $\{c_i \,;\, i \in I\}$ with each $c_i \in L^c$, so $a \vee \bigvee_{i \in I} c_i \geq b$. Since b is compact in $\mathbf{L}$, there exists a finite $I_0 \subseteq I$ such that $a \vee \bigvee_{i \in I_0} c_i \geq b$. Then $\bigvee_{i \in I_0} c_i \in L^c$, so

$$d = a *^{\mathbf{L}^c} b \leq \bigvee_{i \in I_0} c_i \leq c,$$

hence $d = \bigwedge\{c \in L \,;\, a \vee c \geq b\} = a *^{\mathbf{L}} b$, and the implication is proved.

(⇐) Suppose $a *^{\mathbf{L}} b = d \in L$. Then $a \vee d \geq b$. Since $\mathbf{L}$ is algebraic, there exist $c_j \in L^c$, $j \in J$, such that $d = \bigvee_{j \in J} c_j$. Thus $a \vee (\bigvee_{j \in J} c_j) \geq b$. Since b is a compact element of $\mathbf{L}$, there exists a finite set $J_0 \subseteq J$ such that $a \vee (\bigvee_{j \in J_0} c_j) \geq b$. By definition of d, we have $d \leq \bigvee_{j \in J_0} c_j$, hence

$\bigvee_{j \in J_0} c_j = d$, hence d is a compact element of $\mathbf{L}$. □

0.2 UNIVERSAL ALGEBRAIC PRELIMINARIES

Recall that for a nonempty set A and a natural number n we have $A^0 = \{\emptyset\}$, and, for $n > 0$, A^n is the set of n-tuples of elements from A. An *n-ary operation on A* is any function $f: A^n \to A$; we say that n is the *arity* (*rank*) of f, written $ar(f) = n$. An n-ary operation, for some n, is called a *finitary* operation. The image of $\langle a_1, \ldots, a_n \rangle$ under an n-ary operation f is denoted by $f(a_1, \ldots, a_n)$. An operation f on A is called a *nullary* operation (or *constant*) if its arity is zero; it is completely determined by the image $f(\emptyset)$ in A of the element $\emptyset$ in A^0, and it is convenient to identify it with the element $f(\emptyset)$. Thus a nullary operation is thought of as an element of A. An operation f on A is called *unary*, *binary* or *ternary* if its arity is 1, 2 or 3, respectively.

A *language* or *type* of algebras is a pair $\boldsymbol{\mathcal{L}} = \langle \mathcal{L}, ar \rangle$, where $\mathcal{L}$ is a set (whose elements are called *function symbols*) and $ar: \mathcal{L} \to \omega$ is a map (called the *arity function*). For each $f \in \mathcal{L}$, $ar(f)$ is called the *arity* (or *rank*) of f, and f is said to be an *$ar(f)$-ary function symbol*. If $\boldsymbol{\mathcal{L}}$ is a type of algebras then an *algebra* $\mathbf{A}$ of *type* $\boldsymbol{\mathcal{L}}$ (also called an *$\boldsymbol{\mathcal{L}}$-algebra*) is an ordered pair $\langle A; L \rangle$ where A is a nonempty set and $L = \{f^{\mathbf{A}}; f \in \mathcal{L}\}$ is a set indexed by $\mathcal{L}$ such that for each $f \in \mathcal{L}$, with $ar(f) = n$ say, $f^{\mathbf{A}}$ is an n-ary operation on A. The set A is called the *universe* (or *underlying set*) of $\mathbf{A} = \langle A; L \rangle$, and the $f^{\mathbf{A}}$'s are called the *fundamental operations of* $\mathbf{A}$. If $\mathcal{L}$ is finite, say $\mathcal{L} = \{f_1, \ldots, f_m\}$, we often write $\langle A; f_1^{\mathbf{A}}, \ldots, f_m^{\mathbf{A}} \rangle$ for $\langle A; L \rangle$ and we say that $\langle A; f_1^{\mathbf{A}}, \ldots, f_m^{\mathbf{A}} \rangle$ has type $\langle ar(f_1), \ldots, ar(f_m) \rangle$. An algebra is *finite* if A is a finite set, and is *trivial* if A has precisely one element. We often simply write $\mathcal{L}$ for $\boldsymbol{\mathcal{L}}$, and we often write f for a fundamental operation $f^{\mathbf{A}}$. Unless otherwise specified, it will be understood that the universe of algebras $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots$ are $A, B, C, \ldots$, respectively.

Let $\mathbf{A}$ and $\mathbf{B}$ be two algebras of the same type. Then $\mathbf{B}$ is a *subalgebra* of $\mathbf{A}$ if $B \subseteq A$ and every fundamental operation of $\mathbf{B}$ is the restriction of the corresponding operation of $\mathbf{A}$, i.e., for each function symbol f, $f^{\mathbf{B}} = f^{\mathbf{A}} \mid B^{ar(f)}$. In this case, we call B a *subuniverse* of $\mathbf{A}$. Given an algebra $\mathbf{A}$ define, for every $X \subseteq A$,

$$\mathrm{Sg}^{\mathbf{A}}(X) = \bigcap \{B;\, X \subseteq B \text{ and } B \text{ is a subuniverse of } \mathbf{A}\}.$$

We read $\mathrm{Sg}^{\mathbf{A}}(X)$ as 'the subuniverse generated by X'. We denote by $\mathrm{Sub}\,\mathbf{A}$ the set of subuniverses of $\mathbf{A}$. For $X \subseteq A$ we say X *generates* $\mathbf{A}$ (or $\mathbf{A}$ *is generated by* X) if $\mathrm{Sg}^{\mathbf{A}}(X) = A$. The algebra $\mathbf{A}$ is *finitely* (resp. *countably*) *generated* if it is generated by a finite (resp. countable) set. We drop the superscript from $\mathrm{Sg}^{\mathbf{A}}(X)$ if $\mathbf{A}$ is understood.

Note that a lattice $\mathbf{L} = \langle L\,;\, \wedge\,,\, \vee\,\rangle$ is an algebra of type $\langle 2,2\rangle$. Moreover, the sublattices of $\mathbf{L}$ are precisely the subalgebras of $\mathbf{L}$.

Let $\mathbf{A} = \langle A\,; L\rangle$ be an algebra of type $\boldsymbol{\mathfrak{L}} = \langle \mathfrak{L}, ar\rangle$. Let $\mathfrak{L}' \subseteq \mathfrak{L}$ and let $L' = \{f^{\mathbf{A}}\,;\, f \in \mathfrak{L}'\}$. The algebra $\mathbf{A}' = \langle A\,;\, L'\rangle$ is called the $\mathfrak{L}'$-*reduct* of $\mathbf{A}$. If $\mathbf{B}$ is a subalgebra of $\mathbf{A}'$ and $\mathbf{A}'$ is the $\mathfrak{L}'$-reduct of $\mathbf{A}$ then $\mathbf{B}$ is called an $\mathfrak{L}'$-*subreduct* of $\mathbf{A}$.

Recall that a binary relation R on X is an *equivalence relation* if the following conditions hold:

$$(x,x) \in R \quad \text{for all } x \in X \qquad [\textit{reflexivity}],$$

$$\text{for all } x,y \in X, \quad (x,y) \in R \ \text{ implies } \ (y,x) \in R \qquad [\textit{symmmetry}],$$

$$\text{for all } x,y,z \in X, \quad (x,y),(y,z) \in R \ \text{ implies } (x,z) \in R \qquad [\textit{transitivity}].$$

The set of all equivalence relations on a set X is denoted $\mathrm{Eq}(X)$. If R is an equivalence relation on a set X, then for each $x \in X$, we define the *equivalence class* of x (*modulo* R) by

$$x/R = \{y \in X\,;\, (x,y) \in R\}$$

and set

$$X/R = \{x/R\,;\, x \in X\}.$$

Let $\mathbf{A}$ be an algebra of type $\boldsymbol{\mathfrak{L}}$ and let Φ be an equivalence relation on A. Then Φ is called a *congruence* on $\mathbf{A}$ if Φ satisfies the following *compatibility property*:

For each n-ary function symbol $f \in \mathfrak{L}$ and elements $a_1,\ldots,a_n,b_1,\ldots,b_n \in A$,

$$\text{if } (a_i,b_i) \in \Phi \text{ for each } i \le n \text{ then } (f^{\mathbf{A}}(a_1,\ldots,a_n), f^{\mathbf{A}}(b_1,\ldots,b_n)) \in \Phi.$$

The classes a/Φ, for $a \in A$, are called the *congruence classes* of $\mathbf{A}$ (modulo Φ).

The compatibility property allows one to introduce an algebraic structure on the set of equivalence classes A/Φ that is inherited from $\mathbf{A}$. For if $a_1,\ldots,a_n \in A$ and f is an n-ary symbol in $\mathfrak{L}$, then the easiest choice of a value in A/Φ for f applied to $\langle a_1/\Phi,\ldots,a_n/\Phi\rangle$ would be $f^{\mathbf{A}}(a_1,\ldots,a_n)/\Phi$. And indeed, this defines a function on A/Φ if and only if the compatibility property holds.

The set of all congruences on $\mathbf{A}$ is denoted by $\mathrm{Con}\,\mathbf{A}$. Let $\Phi \in \mathrm{Con}\,\mathbf{A}$. Then the *quotient algebra of* $\mathbf{A}$ *by* Φ, denoted $\mathbf{A}/\Phi$, is the algebra of type $\mathcal{L}$ whose universe is A/Φ and whose fundamental operations satisfy

$$f^{\mathbf{A}/\Phi}(a_1/\Phi, \ldots, a_n/\Phi) = f^{\mathbf{A}}(a_1, \ldots, a_n)/\Phi$$

where $a_1, \ldots, a_n \in A$ and f is an n-ary function symbol of $\mathcal{L}$. It is easy to prove that $\mathrm{Con}\,\mathbf{A}$ is closed under arbitrary intersections and unions of upwardly directed sets (with respect to $\subseteq$). For $X \subseteq A^2$, let $\Theta^{\mathbf{A}}(X)$ denote the congruence on $\mathbf{A}$ generated by the set X, i.e., the intersection of all congruences on $\mathbf{A}$ containing X. For $X \subseteq A$, let $\Theta^{\mathbf{A}}(X)$ denote the congruence generated by the set X^2. A congruence $\Phi \in \mathrm{Con}\,\mathbf{A}$ is said to be *finitely generated* if $\Phi = \Theta^{\mathbf{A}}(X)$ for some finite subset X of A^2. We abbreviate $\Theta^{\mathbf{A}}(\{(a,b)\})$ by $\Theta^{\mathbf{A}}(a,b)$. By Theorem 0.1.6, we get

0.2.1 THEOREM [BS81, Theorem 5.5]

For each algebra $\mathbf{A}$*, there is an algebraic closure operator* $\Theta^{\mathbf{A}}$ *on* $\langle \mathcal{P}(A^2); \subseteq \rangle$ *such that the closed subsets of* A^2 *are precisely the congruences on* $\mathbf{A}$*. Thus* $\mathbf{Con}\,\mathbf{A} = \langle \mathrm{Con}\,\mathbf{A}; \subseteq \rangle$ *is an algebraic lattice. For an algebra* $\mathbf{A}$*, the compact members of* $\mathrm{Con}\,\mathbf{A}$ *are the finitely generated members of* $\mathrm{Con}\,\mathbf{A}$. □

It is well-known that if $\mathbf{L} = \langle L; \wedge, \vee \rangle$ is a lattice then $\mathbf{Con}\,\mathbf{L}$ is a distributive lattice and that if $\Phi \in \mathrm{Con}\,\mathbf{L}$ and $a, b, c \in L$ with $a \le c \le b$ and $(a,b) \in \Phi$, then $(a,c), (c,b) \in \Phi$.

If R, S are binary relations on a set A then we define

$$R \circ S = \{(a,b) \in A^2 \,;\, \text{for some } c \in A, \text{ we have } (a,c) \in R \text{ and } (c,b) \in S\},$$

$$R \circ_0 R = I_A \ (= \{(a,a) \,;\, a \in A\}),$$

$$R \circ_{n+1} R = (R \circ_n R) \circ R \quad (n \in \omega).$$

A binary reflexive and symmetric relation R on the universe of an algebra $\mathbf{A}$ is called a *tolerance* on $\mathbf{A}$ if it has the compatibility property. In this case $\Theta^{\mathbf{A}}(R)$, the least congruence on $\mathbf{A}$ containing R, is just the *transitive closure* of R, i.e., the least transitive binary relation containing R. That is to say,

$$\Theta^{\mathbf{A}}(R) = \bigcup_{n \in \omega} R \circ_n R = \{(a,b) \in A^2 \,;\, \text{there exists } n \in \omega \text{ and } c_0, \ldots, c_n \in A \text{ such that}$$

$$a = c_0,\ b = c_n \text{ and } (c_i, c_{i+1}) \in R \text{ for all } i = 0, 1, \ldots, n-1\}.$$

Suppose $\mathbf{A}$ and $\mathbf{B}$ are two algebras of the same type. A mapping $h: A \to B$ is called a *homomorphism* from $\mathbf{A}$ to $\mathbf{B}$ if

$$hf^{\mathbf{A}}(a_1, \ldots, a_n) = f^{\mathbf{B}}(ha_1, \ldots, ha_n)$$

for each n-ary $f \in \mathcal{L}$ and all $a_1, \ldots, a_n \in A$. If the map h is surjective or injective then h is called a *surjective* or *injective homomorphism*, respectively. If h is an injective homomorphism from $\mathbf{A}$ to $\mathbf{B}$ then h is called an *embedding*. We say $\mathbf{A}$ can be *embedded* in $\mathbf{B}$ if there exists an embedding from $\mathbf{A}$ to $\mathbf{B}$. If h is a surjective homomorphism from $\mathbf{A}$ to $\mathbf{B}$ then $\mathbf{B}$ is called a *homomorphic image* of $\mathbf{A}$ under h. If h is a surjective and injective homomorphism from $\mathbf{A}$ to $\mathbf{B}$ then h is called an *isomorphism* from $\mathbf{A}$ to $\mathbf{B}$. In this case we say that $\mathbf{A}$ is isomorphic to $\mathbf{B}$, written $\mathbf{A} \cong \mathbf{B}$. If h is a homomorphism (isomorphism) from $\mathbf{A}$ to $\mathbf{B}$ we may simply say "$h: \mathbf{A} \to \mathbf{B}$ is a homomorphism (an isomorphism)." An isomorphism from $\mathbf{A}$ to $\mathbf{A}$ is called an *automorphism* of $\mathbf{A}$.

0.2.2 THEOREM

(i) [BS81, Theorem 6.2] *If $\mathbf{A}$ is an algebra generated by a set $X \subseteq A$ and $h: \mathbf{A} \to \mathbf{B}$ and $g: \mathbf{A} \to \mathbf{B}$ are two homomorphisms which agree on X (i.e., $ha = ga$ for $a \in X$), then $h = g$.*

(ii) [BS81, Theorem 6.3] *Let $h: \mathbf{A} \to \mathbf{B}$ be a homomorphism. Then the image of A under h is a subuniverse of $\mathbf{B}$, and the inverse image under h of a subuniverse C of $\mathbf{B}$, i.e., $\{a \in A;\ ha \in C\}$, is a subuniverse of $\mathbf{A}$.*

(iii) [BS81, Theorem 6.5] *Suppose $h: \mathbf{A} \to \mathbf{B}$ and $g: \mathbf{B} \to \mathbf{C}$ are homomorphisms (isomorphisms). Then the composition $g \circ h$ is a homomorphism (isomorphism) from $\mathbf{A}$ to $\mathbf{C}$.*

(iv) *Suppose $h: \mathbf{A} \to \mathbf{B}$ is a surjective homomorphism and $\Phi \in \operatorname{Con} \mathbf{A}$. Then $h(\Phi) = \{(ha, hb);\ (a, b) \in \Phi\}$ is a tolerance on $\mathbf{B}$, hence $\Theta^{\mathbf{B}}(h(\Phi))$ is the transitive closure of $h(\Phi)$. Also, if $\eta \in \operatorname{Con} \mathbf{B}$, then $h^{-1}(\eta)$ $(= \{(a, b) \in A^2;\ (ha, hb) \in \eta\})$ is a congruence on $\mathbf{A}$.* □

Let $h: \mathbf{A} \to \mathbf{B}$ be a homomorphism. We denote by $h(\mathbf{A})$ the subalgebra of $\mathbf{B}$ with universe $h(A)$. The *kernel* of h, written $\ker h$, is defined by

$$\ker h = \{(a, b) \in A^2;\ ha = hb\}.$$

0.2.3 THEOREM [BS81, Theorem 6.8, p44]

Let $h: \mathbf{A} \to \mathbf{B}$ be a homomorphism. Then $\ker h$ *is a congruence on* $\mathbf{A}$. □

Let $\mathbf{A}$ be an algebra and let $\Phi \in \operatorname{Con} \mathbf{A}$. The *natural map* $j: \mathbf{A} \to \mathbf{A}/\Phi$ is defined by $ja = a/\Phi$. The natural map is a surjective homomorphism, hence it is sometimes referred to as the *natural homomorphism.*

0.2.4 HOMOMORPHISM THEOREM (First Isomorphism Theorem) [BS81, Theorem 6.12, p46]

Suppose $h: \mathbf{A} \to \mathbf{B}$ is a surjective homomophism. Then there is an isomorphism g from $\mathbf{A}/\ker h$ *to* $\mathbf{B}$ *defined by $h = g \circ j$, where j is the natural homomorphism from* $\mathbf{A}$ *to* $\mathbf{A}/\ker h$. □

Suppose $\mathbf{A}$ is an algebra and $\Phi, \Psi \in \operatorname{Con} \mathbf{A}$ with $\Phi \subseteq \Psi$. Let

$$\Psi/\Phi = \{(a/\Phi, b/\Phi) \in (A/\Phi)^2 \,;\, (a,b) \in \Psi\}.$$

It is easy to check that Ψ/Φ is a congruence on $\mathbf{A}/\Phi$.

0.2.5 THEOREM (Second Isomorphism Theorem) [BS81, Theorem 6.15, p41]

If $\mathbf{A}$ *is an algebra and* $\Phi, \Psi \in \operatorname{Con} \mathbf{A}$ *with* $\Phi \subseteq \Psi$, *then the map*

$$h: (A/\Phi)/(\Psi/\Phi) \to A/\Psi$$

defined by

$$h((a/\Phi)/(\Psi/\Phi)) = a/\Psi$$

is an isomorphism from $(\mathbf{A}/\Phi)/(\Psi/\Phi)$ to $\mathbf{A}/\Psi$. □

0.2.6 CORRESPONDENCE THEOREM [BS81, Theorem 6.20, p49]

Let $\mathbf{A}$ *be an algebra and let* $\Phi \in \operatorname{Con} \mathbf{A}$. *Then the map h from* $[\Phi, A^2]$ *to* $\operatorname{Con} \mathbf{A}/\Phi$, *where* $[\Phi, A^2] = \{\Psi \in \operatorname{Con} \mathbf{A} \,;\, \Phi \subseteq \Psi \subseteq A^2\}$, *defined by $h\Psi = \Psi/\Phi$ is a lattice isomorphism from* $[\mathbf{\Phi}, \mathbf{A^2}]$ *to* $\operatorname{Con} \mathbf{A}/\Phi$, *where* $[\mathbf{\Phi}, \mathbf{A^2}]$ *is the sublattice of* $\mathbf{Con\, A}$ with universe $[\Phi, A^2]$. □

Let $\{\mathbf{A}_i \,;\, i \in I\}$ be an indexed family of algebras of type $\mathcal{L}$. Recall that $\Pi_{i \in I} A_i$ is the set of all functions λ from I to $\bigcup_{i \in I} A_i$ such that $\lambda(i) \in A_i$ for each $i \in I$. We often use the notation $\langle \lambda_i \,;\, i \in I \rangle$ for the element λ of $\Pi_{i \in I} A_i$ such that $\lambda(i) = \lambda_i$ for each $i \in I$. The *direct product* of $\{\mathbf{A}_i \,;\, i \in I\}$, denoted $\mathbf{A} = \Pi_{i \in I} \mathbf{A}_i$, is the algebra with universe $\Pi_{i \in I} A_i$ and such that for $f \in \mathcal{L}$ and $a_1, \ldots, a_n \in \Pi_{i \in I} A_i$,

$$(f^{\mathbf{A}}(a_1,\ldots,a_n))(i) = f^{\mathbf{A}_i}(a_1(i),\ldots,a_n(i))$$

for each $i \in I$, i.e., $f^{\mathbf{A}}$ is defined coordinate-wise. For each $j \in I$, we have the *projection maps* $\pi_j : \Pi_{i \in I} A_i \to A_j$ defined by $\pi_j a = a(j)$, which are also surjective homomorphisms $\pi_j : \Pi_{i \in I} \mathbf{A}_i \to \mathbf{A}_j$. If $I = \{1,2,\ldots,n\}$ we also write $\mathbf{A}_1 \times \ldots \times \mathbf{A}_n$ for $\Pi_{i \in I} \mathbf{A}_i$. Notice that the direct product $\Pi\,\emptyset$ of the empty family of algebras of type $\mathcal{L}$ is a trivial algebra of type $\mathcal{L}$.

An algebra $\mathbf{A}$ is a *subdirect product* of an indexed family $\{\mathbf{A}_i\,;\, i \in I\}$ of algebras if $\mathbf{A}$ is a subalgebra of $\Pi_{i \in I} \mathbf{A}_i$ and $\pi_i(\mathbf{A}) = \mathbf{A}_i$ for each $i \in I$. An embedding $h : \mathbf{A} \to \Pi_{i \in I} \mathbf{A}_i$ is *subdirect* if $h(\mathbf{A})$ is a subdirect product of the $\mathbf{A}_i$. An algebra $\mathbf{A}$ is called *subdirectly irreducible* if for every subdirect embedding $h : \mathbf{A} \to \Pi_{i \in I} \mathbf{A}_i$ there is an $i \in I$ such that $\pi_i \circ h : \mathbf{A} \to \mathbf{A}_i$ is an isomorphism. Equivalently, $\mathbf{A}$ is subdirectly irreducible if and only if $\{\Phi \in \operatorname{Con}\mathbf{A}\,;\, \Phi \neq I_A\}$ (recall that $I_A = \{(a,a)\,;\, a \in A\}$) has a least element (with respect to $\subseteq$).

Let $\mathcal{K}$ be a class of algebras of the same type. We introduce the following operators mapping classes of algebras to classes of algebras:

$\mathbf{A} \in \mathrm{I}(\mathcal{K})$ iff $\mathbf{A}$ is isomorphic to some member of $\mathcal{K}$,

$\mathbf{A} \in \mathrm{S}(\mathcal{K})$ iff $\mathbf{A}$ is a subalgebra of some member of $\mathcal{K}$,

$\mathbf{A} \in \mathrm{H}(\mathcal{K})$ iff $\mathbf{A}$ is a homomorphic image of some member of $\mathcal{K}$,

$\mathbf{A} \in \mathrm{P}(\mathcal{K})$ iff $\mathbf{A}$ is a direct product of a family of algebras in $\mathcal{K}$,

$\mathbf{A} \in \mathrm{P_S}(\mathcal{K})$ iff $\mathbf{A}$ is a subdirect product of a family of algebras in $\mathcal{K}$.

If O_1 and O_2 are two operators on classes of algebras, we write $\mathrm{O}_1\mathrm{O}_2$ for the composition of the two operators, and we write $\mathrm{O}_1 \leq \mathrm{O}_2$ to abbreviate $\mathrm{O}_1(\mathcal{K}) \subseteq \mathrm{O}_2(\mathcal{K})$ for all classes of algebras $\mathcal{K}$. Clearly '$\leq$' has the properties of a partial order. A class $\mathcal{K}$ of algebras is *closed under an operator* O if $\mathrm{O}(\mathcal{K}) \subseteq \mathcal{K}$.

A nonempty class $\mathcal{K}$ of algebras of the same type is called a *variety* if it is closed under subalgebras, homomorphic images and direct products. The intersection of a class of varieties of a fixed type is again a variety, hence for every class $\mathcal{K}$ of algebras of the same type there is a smallest variety containing $\mathcal{K}$. If $\mathcal{K}$ is a class of algebras of the same type, let $\mathrm{V}(\mathcal{K})$ or $\mathcal{K}^{\mathrm{V}}$ denote the smallest variety containing $\mathcal{K}$. We say that $\mathcal{K}^{\mathrm{V}}$ is the *variety generated by* $\mathcal{K}$. A variety $\mathcal{V}$ is *finitely generated* if $\mathcal{V} = \mathcal{K}^{\mathrm{V}}$ for some finite set $\mathcal{K}$ of *finite* algebras. A variety $\mathcal{V}'$ that is a subclass of a variety $\mathcal{V}$ is called a *subvariety* of $\mathcal{V}$.

0.2.7 THEOREM (Tarski) [BS81, Theorem 9.5, p61]

For every class $\mathcal{K}$ of algebras of the same type, $\mathcal{K}^{\mathrm{V}} = \mathrm{HSP}\,(\mathcal{K})$. □

0.2.8 BIRKHOFF'S SUBDIRECT DECOMPOSITION THEOREM [BS81, Theorem 8.6, p58]

An algebra **A** *is isomorphic to a subdirect product of subdirectly irreducible algebras which are homomorphic images of* **A**. *In particular, if $\mathcal{K}$ is a variety, then every member of $\mathcal{K}$ is isomorphic to a subdirect product of subdirectly irreducible members of $\mathcal{K}$.* □

Let $\mathcal{L}$ be a type of algebras. Let X be a set of (distinct) objects called variables. (When given a type $\mathcal{L}$ and a set X of variables, we shall always assume that $X \cap \mathcal{L} = \emptyset$.) The set $T(X)$ of *terms of type $\mathcal{L}$ over X* is the smallest set such that

(i) $X \subseteq T(X)$

(ii) If $t_1, \ldots, t_n \in T(X)$ and $f \in \mathcal{L}$ with $ar(f) = n$ then the formal expression $f(t_1, \ldots, t_n)$ is in $T(X)$. (We take this to imply that $c \in T(X)$ for any nullary $c \in \mathcal{L}$.)

For a binary operation, $\cdot$ say, we usually write $t_1 \cdot t_2$ for $\cdot(t_1, t_2)$. For $t \in T(X)$ we often write t as $t(x_1, \ldots, x_n)$ to indicate that the variables occurring in t are among $x_1, \ldots, x_n$. A term t is n-ary (has *arity* n) if the number of variables appearing explicitly in t is less than or equal to n. We abbreviate $T(\{x_1, \ldots, x_m\})$ by $T(x_1, \ldots, x_m)$.

Given a term $t(x_1, \ldots, x_n)$ of type $\mathcal{L}$ over some set X and given an algebra **A** of type $\mathcal{L}$ we define a map $t^{\mathbf{A}}: A^n \to A$ (called the *term function of t on* **A**) as follows:

(i) If t is a variable x_i, then $t^{\mathbf{A}}(a_1, \ldots, a_n) = a_i$ for $a_1, \ldots, a_n \in A$, i.e., $t^{\mathbf{A}}$ is the ith projection map.

(ii) If t is of the form $f(t_1(x_1, \ldots, x_n), \ldots, t_k(x_1, \ldots, x_n))$, where $f \in \mathcal{L}$ such that $ar(f) = k$, then
$$t^{\mathbf{A}}(a_1, \ldots, a_n) = f^{\mathbf{A}}(t_1^{\mathbf{A}}(a_1, \ldots, a_n), \ldots, t_k^{\mathbf{A}}(a_1, \ldots, a_n)).$$

We take this to imply that if t is a constant symbol $c \in \mathcal{L}$ then $t^{\mathbf{A}}$ is $c^{\mathbf{A}}$. More generally, if $t = f \in \mathcal{L}$ then $t^{\mathbf{A}} = f^{\mathbf{A}}$ and we say that $t^{\mathbf{A}}$ is the *term function* on **A** corresponding to the term t. (Often we will drop the superscript **A**.)

0.2.9 THEOREM [BS81, Theorem 10.3 (c), p64]

For an algebra $\mathbf{A}$ *of type* $\mathcal{L}$ *and a subset* X *of* A,

$$\mathrm{Sg}^{\mathbf{A}}(X) = \{t^{\mathbf{A}}(a_1,\ldots,a_n)\,;\ t \text{ is an } n\text{-ary term of type } \mathcal{L},\ n\in\omega \text{ and } a_1,\ldots,a_n \in X\}. \qquad \square$$

Given $\mathcal{L}$ and X, if $T(X) \neq \emptyset$ then the *term algebra* (also called the *absolutely free algebra*) of type $\mathcal{L}$ over X, written $\mathbf{T}(X)$, has as its universe the set $T(X)$, and the fundamental operations satisfy

$$f^{\mathbf{T}(X)}: \langle t_1,\ldots,t_n\rangle \mapsto f(t_1,\ldots,t_n)$$

for $f\in\mathcal{L}$ with $ar(f)=n$ and $t_i\in T(X)$ for $i\le n$.

Note that $\mathbf{T}(X)$ exists iff $X\neq\emptyset$ or there exists a constant symbol in the type $\mathcal{L}$. Note also that $\mathbf{T}(X)$ is indeed generated by X (by Theorem 0.2.9). Term algebras provide us with the simplest examples of algebras with the 'universal mapping property': Let $\mathcal{K}$ be a class of algebras of type $\mathcal{L}$ and let $\mathbf{B}$ be an algebra of type $\mathcal{L}$ which is generated by a set X. If for every $\mathbf{A}\in\mathcal{K}$ and for every map $\alpha: X\to A$ there is a homomorphism $\beta:\mathbf{B}\to\mathbf{A}$ which extends α (i.e., $\beta x=\alpha x$ for $x\in X$), then we say that $\mathbf{B}$ has the *universal mapping property for* $\mathcal{K}$ *over* X, X is called a set of *free generators* of $\mathbf{B}$, and $\mathbf{B}$ is said to be *freely generated* by X.

0.2.10 THEOREM [BS81, Theorem 10.8, p66]

For any type $\mathcal{L}$ *and set* X *of variables, if* $\mathbf{T}(X)$ *exists then it has the universal mapping property for the class of all algebras of type* $\mathcal{L}$ *over* X. $\square$

Let $\mathcal{K}$ be a family of algebras of type $\mathcal{L}$. Given a set X of variables, define the congruence $\theta_{\mathcal{K}}(X)$ on $\mathbf{T}(X)$ by $\theta_{\mathcal{K}}(X)=\bigcap\Phi_{\mathcal{K}}(X)$, where

$$\Phi_{\mathcal{K}}(X)=\{\Psi\in\mathrm{Con}\,\mathbf{T}(X)\,;\ \mathbf{T}(X)/\Psi\in\mathrm{IS}(\mathcal{K})\};$$

and then define $\mathbf{F}_{\mathcal{K}}(\underline{X})$, the $\mathcal{K}$*-free algebra over* $\underline{X}$, by $\mathbf{F}_{\mathcal{K}}(\underline{X})=\mathbf{T}(X)/\theta_{\mathcal{K}}(X)$, where $\underline{X}=X/\theta_{\mathcal{K}}(X)$. For $x,x_1,\ldots,x_n\in X$ we write $\underline{x}$ for $x/\theta_{\mathcal{K}}(X)$ and for $t=t(x_1,\ldots,x_n)\in T(X)$ we write $\underline{t}$ for $t^{\mathbf{F}_{\mathcal{K}}(\underline{X})}(\underline{x}_1,\ldots,\underline{x}_n)$. If X is finite, say $X=\{x_1,\ldots,x_n\}$, we often write $\mathbf{F}_{\mathcal{K}}(\underline{x}_1,\ldots,\underline{x}_n)$ for $\mathbf{F}_{\mathcal{K}}(\underline{X})$. Let $F_{\mathcal{K}}(\underline{X})$ be the universe of $\mathbf{F}_{\mathcal{K}}(\underline{X})$. Note that $\mathbf{F}_{\mathcal{K}}(\underline{X})$ exists iff $\mathbf{T}(X)$ exists. If in addition, $\mathcal{K}$ contains a nontrivial algebra then the map $x\mapsto\underline{x}$ defines a bijection from X onto $\underline{X}$.

0.2.11 THEOREM

Suppose $\mathbf{T}(X)$ *exists.*

(i) [BS81, Theorem 10.10, p67] *Then* $\mathbf{F}_{\mathcal{K}}(\underline{X})$ *has the universal mapping property for* $\mathcal{K}$ *over* $\underline{X}$.

(ii) [BS81, Theorem 10.12, p68] *For* $\mathcal{K} \neq \emptyset$, $\mathbf{F}_{\mathcal{K}}(\underline{X}) \in \mathrm{ISP}(\mathcal{K})$. *Thus, if* $\mathcal{K}$ *is closed under* I, S *and* P, *in particular if* $\mathcal{K}$ *is a variety, then* $\mathbf{F}_{\mathcal{K}}(\underline{X}) \in \mathcal{K}$. □

Let $\mathbf{A}$ be an $\mathcal{L}$-algebra, and X a set of variables. Let $Y \subseteq X$ and let $\overline{a} \in A^Y$. We call $\overline{a}$ an *interpretation of* Y *in* A. If $t = t(x_1, \ldots, x_m)$ is an $\mathcal{L}$-formula and $x_1, \ldots, x_m \in Y \subseteq X$, then for $\overline{a} \in A^Y$ such that $\overline{a}(x_i) = a_i$ for each $i \leq m$, we write $t^{\mathbf{A}}(\overline{a})$ for $t^{\mathbf{A}}(a_1, \ldots, a_m)$. In this case we also call $a_1, \ldots, a_m$ an *interpretation of* $x_1, \ldots, x_m$ *(respectively) in* A.

An *identity* of type $\mathcal{L}$ over X is a formal expression of the form $s \approx t$, where $s, t \in T(X)$. Let $\mathrm{Id}(X)$ be the set of identities of type $\mathcal{L}$ over X. An algebra $\mathbf{A}$ of type $\mathcal{L}$ *satisfies* an identity $s(x_1, \ldots, x_n) \approx t(x_1, \ldots, x_n)$ (or the identity *is true in* $\mathbf{A}$, or *holds* in $\mathbf{A}$), abbreviated $\mathbf{A} \models s(x_1, \ldots, x_n) \approx t(x_1, \ldots, x_n)$, or $\mathbf{A} \models s \approx t$, or $\models_{\mathbf{A}} s \approx t$ if for every interpretation $a_1, \ldots, a_n$ of the variables $x_1, \ldots, x_n$, respectively, in A, we have $s^{\mathbf{A}}(a_1, \ldots, a_n) = t^{\mathbf{A}}(a_1, \ldots, a_n)$. A class $\mathcal{K}$ of algebras satisfies $s \approx t$, written $\mathcal{K} \models s \approx t$ or $\models_{\mathcal{K}} s \approx t$ if $\models_{\mathbf{A}} s \approx t$ for each $\mathbf{A} \in \mathcal{K}$. If Σ is a set of identities, we say $\mathcal{K}$ satisfies Σ, written $\mathcal{K} \models \Sigma$ or $\models_{\mathcal{K}} \Sigma$ if $\models_{\mathcal{K}} s \approx t$ for each $s \approx t \in \Sigma$.

Suppose $\Sigma = \{s_i \approx t_i\, ;\, i \in I\} \subseteq \mathrm{Id}(X)$ and $s \approx t \in \mathrm{Id}(X)$. We write $\Sigma \models_{\mathcal{K}} s \approx t$ provided that for every $\mathbf{A} \in \mathcal{K}$ and every interpretation $\overline{a} \in A^X$ of X in A such that $s_i^{\mathbf{A}}(\overline{a}) = t_i^{\mathbf{A}}(\overline{a})$ for all $i \in I$, we also have $s^{\mathbf{A}}(\overline{a}) = t^{\mathbf{A}}(\overline{a})$. Given $\mathcal{K}$ and X, let $\mathrm{Id}_{\mathcal{K}}(X) = \{s \approx t \in \mathrm{Id}(X)\, ;\, \mathcal{K} \models s \approx t\}$. We use the symbol $\not\models$ for "does not satisfy."

0.2.12 THEOREM [BS81, Theorem 11.4, pp73-74]

Given a class $\mathcal{K}$ *of algebras of type* $\mathcal{L}$ *and terms* $s, t \in T(X)$ *of type* $\mathcal{L}$ *over a set* X *of variables, we have*

$$\mathcal{K} \models s \approx t \quad \textit{iff} \quad \mathbf{F}_{\mathcal{K}}(\underline{X}) \models s \approx t \quad \textit{iff} \quad \underline{s} = \underline{t} \textit{ in } \mathbf{F}_{\mathcal{K}}(\underline{X}) \quad \textit{iff} \quad (s, t) \in \theta_{\mathcal{K}}(X). \qquad \square$$

Let Σ be a set of identities of type $\mathcal{L}$. A class $\mathcal{K}$ of algebras of type $\mathcal{L}$ is said to be *defined* or *axiomatized* by Σ if $\mathcal{K}$ is the class of all algebras of type $\mathcal{L}$ that satisfy all of the identities of Σ.

0.2.13 THEOREM (Birkhoff) [BS81, Theorem 11.9, pp75-76]

A class $\mathcal{K}$ of algebras of the same type is a variety if and only if $\mathcal{K}$ is axiomatized by some set of identities. □

A class $\mathcal{K}$ of algebras of the same type has an *equationally definable constant* C if and only if there is a term $t(x_1, \ldots, x_n)$ of the type of $\mathcal{K}$ such that

$$\mathcal{K} \models t(x_1, \ldots, x_n) \approx t(y_1, \ldots, y_n)$$

and C abbreviates $t(x_1, \ldots, x_n)$.

Examples

Groups. A *group* **G** is an algebra $\langle G; \cdot, {}^{-1}, 1\rangle$ of type $\langle 2, 1, 0\rangle$ in which the following identities hold:

(G1) $x \cdot (y \cdot z) \approx (x \cdot y) \cdot z,$

(G2) $x \cdot 1 \approx 1 \cdot x \approx x,$

(G3) $x \cdot x^{-1} \approx x^{-1} \cdot x \approx 1.$

A group **G** is *abelian* (or *commutative*) if the following identity holds in **G**:

(G4) $x \cdot y \approx y \cdot x.$

A group is called *Boolean* if the following identity holds in **G**:

(G5) $x \cdot x \approx 1.$

It follows from the above definition and Birkhoff's theorem (Theorem 0.2.8), that the class of all groups is a variety (it is defined by the identities (G1) to (G3)). We denote this variety by $\mathcal{G}$. The class of all abelian groups is a subvariety of $\mathcal{G}$ and we denote it by $\mathcal{AG}$. The class of all Boolean groups is a subvariety of $\mathcal{G}$ (in fact, it is a subvariety of $\mathcal{AG}$) and we denote it by $\mathcal{BG}$.

Lattices. A *lattice* is an algebra $\mathbf{L} = \langle L; \wedge, \vee\rangle$, where $\wedge$ and $\vee$ (called *meet* and *join*, respectively) are binary operations, in which the following identities hold:

(L1) $x \vee y \approx y \vee x, \quad x \wedge y \approx y \wedge x$ [*commutative laws*]

(L2) $x \vee (y \vee z) \approx (x \vee y) \vee z, \quad x \wedge (y \wedge z) \approx (x \wedge y) \wedge z$ [*associative laws*]

(L3) $x \vee x \approx x, \quad x \wedge x \approx x$ [*idempotent laws*]

(L4) $x \approx x \vee (x \wedge y), \quad x \approx x \wedge (x \vee y)$ [*absorption laws*]

A lattice is a *distributive lattice* if it satisfies:

(L5) $x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge z), \quad x \vee (y \wedge z) \approx (x \vee y) \wedge (x \vee z)$ [*distributive laws*]

An algebra $\mathbf{L} = \langle L;\ \wedge, \vee, 0, 1\rangle$ of type $\langle 2,2,0,0\rangle$ is a *bounded lattice* if $\langle L;\ \wedge, \vee\rangle$ is a lattice and $\mathbf{L}$ satisfies:

(L6) $x \wedge 0 \approx 0, \quad x \vee 1 \approx 1.$

Boolean Algebras. A *Boolean algebra* is an algebra $\langle B;\ \wedge, \vee, ', 0, 1\rangle$ of type $\langle 2,2,1,0,0\rangle$ in which the following hold:

(B1) $\langle B;\ \wedge, \vee\rangle$ is a distributive lattice

(B2) $x \wedge 0 \approx 0, \quad x \vee 1 \approx 1$

(B3) $x \wedge (x') \approx 0, \quad x \vee (x') \approx 1.$

The following identities are known to be satisfied by every Boolean algebra:

$(x \wedge y)' = x' \vee y' \qquad (x \vee y)' = x' \wedge y'$ [*De Morgan's Laws*].

It is evident from Birkhoff's theorem that the class of all Boolean algebras is a variety; we denote it by $\mathcal{BA}$. We shall sometimes find it more natural to write $\neg$, $\bot$ and T for $'$, 0 and 1, respectively. The term $\rightarrow$ will be frequently used; it is a binary term defined by $x \rightarrow y = (x') \vee y$. It is easy to show that the underlying lattice order on a Boolean algebra $\mathbf{B}$ satisfies the following property for all $a, b \in B$:

$$a \leq b \quad \text{iff} \quad a \rightarrow^{\mathbf{B}} b = 1^{\mathbf{B}}.$$

Brouwerian Semilattices. An algebra $\mathbf{A} = \langle A;\ \wedge, \rightarrow, 1\rangle$ of type $\langle 2,2,0\rangle$ is a *Brouwerian semilattice* if $\langle A;\ \wedge\rangle$ is a semilattice and the following identities hold in $\mathbf{A}$:

(BS1) $x \rightarrow x \approx 1,$

(BS2) $(x \rightarrow y) \wedge x \approx (y \rightarrow x) \wedge y,$

(BS3) $(x \wedge y) \rightarrow z \approx x \rightarrow (y \rightarrow z).$

In this case 1 is the greatest element of $\langle A;\ \wedge\rangle$. Note that the class of Brouwerian semilattices forms a variety by Birkhoff's theorem.

Heyting algebras. An algebra $\langle H;\ \wedge, \vee, \rightarrow, 0, 1\rangle$ of type $\langle 2,2,2,0,0\rangle$ is a *Heyting algebra* if it satisfies:

(H1) $\langle H\,;\ \wedge\,,\ \vee\,\rangle$ is a distributive lattice

(H2) $x \wedge 0 \approx 0, \quad x \vee 1 \approx 1$

(H3) $x \to x \approx 1$

(H4) $(x \to y) \wedge y \approx y, \quad x \wedge (x \to y) \approx x \wedge y$

(H5) $x \to (y \wedge z) \approx (x \to y) \wedge (x \to z), \quad (x \vee y) \to z \approx (x \to z) \wedge (y \to z).$

By Birkhoff's theorem the class of all Heyting algebras is a variety; we denote it $\mathcal{HA}$. We shall often use $\perp$ and T for 0 and 1, respectively. A unary term $\neg$ is sometimes added to the type of Heyting algebras; it is defined by $\neg\, x = x \to 0$. A Boolean algebra, with the term $\to$ defined as above, satisfies each of the defining identities of a Heyting algebra, so we can consider Boolean algebras stronger than Heyting algebras. Note that the $\{\wedge\,, \to, 1\}$-reduct of a Heyting algebra is a Brouwerian semilattice. In fact, it is well known that the $\{\wedge\,, \to, 1\}$-subreducts of Heyting algebras are exactly the Brouwerian semilattices.

Let $\mathbf{A}$ be a Brouwerian semilattice or a Heyting algebra. The $\to^{\mathbf{A}}$ operation of $\mathbf{A}$ satisfies the following property for all $a, b, c \in A$:

$$a \to^{\mathbf{A}} b = \max\{c \in A\,;\, c \wedge^{\mathbf{A}} a \le b\}.$$

In other words, $\langle A\,;\ \wedge\,\rangle$ is Brouwerian and for all $a, b \in A$, $a \to^{\mathbf{A}} b$ is the relative pseudocomplement of a with respect to b.

As in the case of Boolean algebras, the underlying partial order $\le$ of $\mathbf{A}$ satisfies the following, for all $a, b \in A$:

$$a \le b \quad \text{iff} \quad a \to^{\mathbf{A}} b = 1^{\mathbf{A}}.$$

We list some other easily proved properties of $\le$ that we shall need. For all $a, b, c \in A$:

$$a \le b \to^{\mathbf{A}} a,$$

$$a \le b \quad \text{implies} \quad c \to^{\mathbf{A}} a \le c \to^{\mathbf{A}} b,$$

$$a \le b \quad \text{implies} \quad b \to^{\mathbf{A}} c \le a \to^{\mathbf{A}} c.$$

If the underlying partial order of $\mathbf{A}$ is a chain, then, for all $a, b \in A$:

$$\text{If } \ a \le b \ \text{ then } \ a \to^{\mathbf{A}} b = 1^{\mathbf{A}},$$

$$\text{if } \ a > b \ \text{ then } a \to^{\mathbf{A}} b = b.$$

Wajsberg Algebras. An algebra $\mathbf{W} = \langle W\,;\, \to, \neg\,, 1\rangle$ of type $\langle 2, 1, 0\rangle$ is a *Wajsberg algebra* if it satisfies the following identities:

(W1) $1 \to x \approx x$,

(W2) $(x \to y) \to [(y \to z) \to (x \to z)] \approx 1$,

(W3) $(x \to y) \to y \approx (y \to x) \to x$,

(W4) $((\neg x) \to (\neg y)) \to (y \to x) \approx 1$.

The class of all Wajsberg algebras is evidently also a variety; we denote it by $\mathcal{W}$.

Modal Algebras. An algebra $\mathbf{M} = \langle M;\ \wedge, \vee, \to, \neg, \Box, 0, 1\rangle$ of type $\langle 2,2,2,1,1,0,0\rangle$ is a *modal algebra* if it satisfies the following:

(M1) $\langle M;\ \wedge, \vee, \neg, 0, 1\rangle$ is a Boolean algebra,

(M2) $x \to y \approx (\neg x) \vee y$

(M3) $\Box 1 \approx 1$,

(M4) $\Box(x \wedge y) \approx (\Box x) \wedge (\Box y)$.

The class of all modal algebras is a variety; we denote it by $\mathcal{MA}$. A modal algebra $\mathbf{M} = \langle M;\ \wedge, \vee, \to, \neg, \Box, 0, 1\rangle$ is a *monadic algebra* if it satisfies

(M5) $(\Box x) \wedge x \approx \Box x$ (i.e., $\Box x \leq x$),

(M6) $\Diamond x \wedge (\Box \Diamond x) \approx \Diamond x$, where $\Diamond x = \neg \Box \neg x$ (i.e., $\Diamond x \leq \Box \Diamond x$).

A modal algebra $\mathbf{M} = \langle M;\ \wedge, \vee, \to, \neg, \Box, 0, 1\rangle$ is an *interior algebra* if it satisfies (M5) and

(M7) $(\Box x) \wedge (\Box\Box x) = \Box x$ (i.e., $\Box x \leq \Box\Box x$).

0.3 REDUCED PRODUCTS AND ULTRAPRODUCTS

Let $\mathbf{B}$ be a Boolean algebra or a Heyting algebra. A subset G of B is a *filter* of $\mathbf{B}$ if it is a filter of the underlying lattice of $\mathbf{B}$. The set of all filters of $\mathbf{B}$ is denoted $Fi\,\mathbf{B}$. It is easy to see that $Fi\,\mathbf{B}$ is the universe of a lattice (ordered by $\subseteq$), which we denote by $\mathbf{Fi}\,\mathbf{B}$.

0.3.1 THEOREM (cf. [BS81, Theorem 3.5, p128])

Let $\mathbf{B} = \langle B;\ \wedge, \vee, ', 0, 1\rangle$ (resp. $\langle B;\ \wedge, \vee, \to, 0, 1\rangle$) be a Boolean algebra (resp. Heyting algebra) and let G be a filter of $\mathbf{B}$. Define a binary relation λ_G on B by:

$$(a, b) \in \lambda_G \text{ iff } a \vee (b'), (a') \vee b \in G \text{ [i.e., iff } (a \vee (b')) \wedge ((a') \vee b) \in G]$$

$$\text{(resp. iff } a \to b, b \to a \in G \text{ [i.e., iff } (a \to b) \wedge (b \to a) \in G]).$$

Then λ_G *is a congruence relation on* **B**. *Conversely, if* $\Phi \in \operatorname{Con}\mathbf{B}$, *then* $1/\Phi$ *is a filter of* **B**.

$$\mathbf{Fi\,B} \to \mathbf{Con\,B} \quad ; \quad \mathbf{Con\,B} \to \mathbf{Fi\,B}$$

$$G \mapsto \lambda_G \quad ; \quad \Phi \mapsto 1/\Phi$$

are mutually inverse lattice isomorphisms, so $\langle \mathbf{Fi\,B}, \subseteq \rangle \cong \langle \mathbf{Con\,B}, \subseteq \rangle$. *Therefore* $\langle \mathbf{Fi\,B}, \subseteq \rangle$ *is an algebraic distributive lattice.* □

Let $\mathbf{B} = \langle B; \wedge, \vee, ', 0, 1\rangle$ be a Boolean algebra and G a subset of B. By an *ultrafilter* of **B**, we mean a proper filter G of **B** that is maximal, with respect to $\subseteq$, among all proper filters of **B** (i.e., if F is a proper filter of **B** and if $G \subseteq F$ then $G = F$).

0.3.2 THEOREM [BS81, Theorem 3.12 and Corollary 3.13, pp132-133]

Let G *be a filter on a Boolean algebra* $\mathbf{B} = \langle B; \wedge, \vee, ', 0, 1\rangle$. *Then the following conditions are equivalent:*

(i) G *is an ultrafilter of* **B**;

(ii) *for every* $a \in B$, *exactly one of* a *and* a' *is an element of* G;

(iii) $0 \notin G$ *and for every* $a, b \in B$ *we have* $a \vee b \in G$ *iff* $a \in G$ *or* $b \in G$. □

If $\mathbf{B} = \langle B; \wedge, \vee, ', 0, 1\rangle$ is a Boolean algebra and $a \in B$ then $[a) = \{b \in B; b \geq a\}$ (where $\leq$ is the lattice order of $\langle B; \wedge, \vee\rangle$) is clearly a filter of **B** – called the *principal filter generated by* a – it is clearly the least filter of **B** containing a. An element $a \in B$ is an *atom* of **B** if $0 \prec a$. If a is an atom of **B**, it is easy to see that $[a)$ is an ultrafilter of **B**. Such an ultrafilter is called a *principal* or *fixed* ultrafilter. A *free ultrafilter* is one that is not fixed. A routine application of Zorn's Lemma produces the following result.

0.3.3 THEOREM

Let $\mathbf{B} = \langle B; \wedge, \vee, ', 0, 1\rangle$ *be a Boolean algebra. If* G *is a proper filter of* **B** *then there exists an ultrafilter* U *of* **B** *such that* $G \subseteq U$. □

If I is a set, then the power set $\mathcal{P}(I)$, partially ordered by $\subseteq$, gives rise to a Boolean algebra $\boldsymbol{\mathcal{P}(I)} = \langle \mathcal{P}(I); \cap, \cup, -, \emptyset, I\rangle$, where $-X = I - X = \{a \in I; a \notin X\}$ for all $X \subseteq I$. By a *filter* (resp. *ultrafilter*) *over a set* I, we mean a filter (ultrafilter) on the Boolean algebra $\mathfrak{P}(I)$. If I is an infinite set, then $X \subseteq I$ is called a *cofinite* subset of I if $I - X$ is finite. If I is an infinite set then the set of all cofinite subsets of I is a filter over I; it is called the *Fréchet filter* over I and is not an ultrafilter. By the previous theorem, the Fréchet filter over an infinite set I is contained in a (necessarily free) ultrafilter. Conversely, we have the following well-known fact.

0.3.4 COROLLARY

Every free ultrafilter over I contains the Fréchet filter. □

0.3.5 COROLLARY

For every infinite set I, there exists a free (i.e., nonprincipal) ultrafilter over I. □

Let $\{\mathbf{A}_i\,;\, i \in I\}$ be a family of algebras of the same type and let $a, b \in \Pi_{i \in I} \mathbf{A}_i$. Let G be a filter over the set I. Define a binary relation θ_G on $\Pi_{i \in I} \mathbf{A}_i$ by

$$(a,b) \in \theta_G \quad \text{iff} \quad \{i \in I\,;\, a(i) = b(i)\} \in G.$$

Then θ_G is a congruence relation on $\Pi_{i \in I} \mathbf{A}_i$. The quotient algebra $(\Pi_{i \in I} \mathbf{A}_i)/\theta_G$ is usually called a *reduced product* in the literature. We shall call it a *filtered product* in order to avoid a clash of usage of the term 'reduced' later. We denote the filtered product by $\Pi_{i \in I} \mathbf{A}_i/\theta_G$. If G is actually an ultrafilter over I, we call $\Pi_{i \in I} \mathbf{A}_i/\theta_G$ an *ultraproduct* of the family $\{\mathbf{A}_i\,;\, i \in I\}$. In particular, if $\mathbf{A}_i = \mathbf{A}$ for all $i \in I$, we call $\Pi_{i \in I} \mathbf{A}_i/\theta_G$ an *ultrapower* of $\mathbf{A}$.

0.3.6 LEMMA [BS81, Lemma 6.5, p146]

If $\{\mathbf{A}_i\,;\, i \in I\}$ is a finite set of finite algebras, say $\{\mathbf{B}_1, \ldots, \mathbf{B}_n\}$ (I may be infinite) and U is an ultrafilter over I then the ultraproduct $\Pi_{i \in I} \mathbf{A}_i/\theta_G$ is isomorphic to one of the algebras $\mathbf{B}_1, \ldots, \mathbf{B}_n$, namely to that $\mathbf{B}_j$ such that $\{i \in I\,;\, \mathbf{A}_i = \mathbf{B}_j\} \in U$. □

Let $\mathcal{K}$ be a class of algebras of the same type. We introduce the following operators mapping classes of algebras to classes of algebras:

$\mathbf{A} \in \mathrm{P_F}(\mathcal{K})$ iff $\mathbf{A}$ is a filtered product of a family of algebras in $\mathcal{K}$,

$\mathbf{A} \in \mathrm{P_U}(\mathcal{K})$ iff $\mathbf{A}$ is an ultraproduct of a family of algebras in $\mathcal{K}$.

0.4 QUASIVARIETIES

A class $\mathcal{K}$ of algebras of the same type is called a *quasivariety* if it is closed under I, S, P and P_U. The intersection of a class of quasivarieties is again a quasivariety, hence for every class $\mathcal{K}$ of algebras of the same type there is a smallest quasivariety containing $\mathcal{K}$. If $\mathcal{K}$ is a class of algebras of the same type, let $Q(\mathcal{K})$ or $\mathcal{K}^Q$ denote the smallest quasivariety containing $\mathcal{K}$. We say that $\mathcal{K}^Q$ is the *quasivariety generated by* $\mathcal{K}$. A quasivariety $\mathcal{K}'$ that is a subclass of a quasivariety $\mathcal{K}$ is called a *subquasivariety* of $\mathcal{K}$.

A *quasi-identity* of type $\mathcal{L}$ over X is an identity or a formal expression of the form

$$s_1 \approx t_1 \,\&\ldots\&\, s_n \approx t_n \Rightarrow s \approx t \quad \text{(abbreviated } (\underset{i\leq n}{\&}\, s_i \approx t_i) \Rightarrow s \approx t)$$

where $s_1,\ldots,s_n,t_1,\ldots,t_n,s,t \in T(X)$. We extend the relation $\models_{\mathcal{K}}$ defined for identities: An algebra $\mathbf{A}$ of type $\mathcal{L}$ *satisfies* a quasi-identity of type $\mathcal{L}$

$$s_1(x_1,\ldots,x_n) \approx t_1(x_1,\ldots,x_n) \,\&\ldots\&\, s_n(x_1,\ldots,x_n) \approx t_n(x_1,\ldots,x_n) \Rightarrow s(x_1,\ldots,x_n) \approx t(x_1,\ldots,x_n)$$

(or the quasi-identity *is true in* $\mathbf{A}$, or *holds in* $\mathbf{A}$), abbreviated $\mathbf{A} \models s_1 \approx t_1 \,\&\ldots\&\, s_n \approx t_n \Rightarrow s \approx t$, or $\models_{\mathbf{A}} s_1 \approx t_1 \,\&\ldots\&\, s_n \approx t_n \Rightarrow s \approx t$ if

$$\{s_1 \approx t_1,\ldots,s_n \approx t_n\} \models_{\mathbf{A}} s \approx t.$$

A class $\mathcal{K}$ of algebras satisfies $s_1 \approx t_1 \,\&\ldots\&\, s_n \approx t_n \Rightarrow s \approx t$, written

$$\mathcal{K} \models s_1 \approx t_1 \,\&\ldots\&\, s_n \approx t_n \Rightarrow s \approx t,$$

or $\models_{\mathcal{K}} s_1 \approx t_1 \,\&\ldots\&\, s_n \approx t_n \Rightarrow s \approx t$, if $\models_{\mathbf{A}} s_1 \approx t_1 \,\&\ldots\&\, s_n \approx t_n \Rightarrow s \approx t$ for each $\mathbf{A} \in \mathcal{K}$. If Σ is a set of quasi-identities, we say $\mathcal{K}$ satisfies Σ, written $\models_{\mathcal{K}} \Sigma$ or $\mathcal{K} \models \Sigma$ if $\mathcal{K} \models s_1 \approx t_1 \,\&\ldots\&\, s_n \approx t_n \Rightarrow s \approx t$ for each $(s_1 \approx t_1 \,\&\ldots\&\, s_n \approx t_n \Rightarrow s \approx t) \in \Sigma$. We use the symbol $\not\models$ for "does not satisfy." Let Σ be a set of quasi-identities of type $\mathcal{L}$. A class $\mathcal{K}$ of algebras of type $\mathcal{L}$ is said to be *defined* or *axiomatized* by Σ if $\mathcal{K}$ is the class of all algebras of type $\mathcal{L}$ that satisfies each of the quasi-identities of Σ.

0.4.1 THEOREM (Mal'cev) [BS81, Theorem 2.25, p219]

Let $\mathcal{K}$ be a class of algebras of a fixed type. The following conditions are equivalent:

(i) $\mathcal{K}$ *is axiomatized by a set of quasi-identities,*

(ii) $\mathcal{K}$ *is a quasivariety,*

(iii) $\mathcal{K}$ *is closed under* I, S *and* P_F,

(iv) $\mathcal{K} = ISP_F(\mathcal{K}')$ *for some class* $\mathcal{K}'$,

(v) $\mathcal{K} = ISPP_U(\mathcal{K}')$ *for some class* $\mathcal{K}'$. □

0.4.2 THEOREM (Mal'cev) [Mal73, Corollary 5, p216]

Let $\mathcal{K}$ *be a quasivariety of type* $\boldsymbol{\mathcal{L}} = \langle \mathcal{L}, ar\rangle$ *and let* $\mathcal{L}' \subseteq \mathcal{L}$. *Let* $\mathcal{K}'$ *be the class of all* $\mathcal{L}'$*-reducts of algebras in* $\mathcal{K}$. *Then* $(\mathcal{K}')^Q = S(\mathcal{K}')$, *i.e., the quasivariety generated by* $\mathcal{K}'$ *is the class of all* $\mathcal{L}'$*-subreducts of* $\mathcal{K}$. □

Let $\mathcal{K}$ be a class of algebras of the same type and let **A** be an algebra with the same type as $\mathcal{K}$. A congruence Φ on **A** is called a $\mathcal{K}$-*congruence* (or a *relative congruence*) on **A** if the quotient algebra $\mathbf{A}/\Phi \in \mathcal{K}$. We denote the set of all $\mathcal{K}$-congruences on **A** by $\mathrm{Con}_{\mathcal{K}}\mathbf{A}$.

The following proposition and the subsequent lemma will be needed, of which we have not been able to find proofs in the published literature (although the former result is certainly known to quasivariety specialists).

0.4.3 PROPOSITION

For a quasivariety $\mathcal{K}$ *and an algebra* **A** *with the same type as* $\mathcal{K}$, $\langle \mathrm{Con}_{\mathcal{K}}\mathbf{A}, \subseteq \rangle$ *is an algebraic lattice.*

Proof. To prove that $\mathrm{Con}_{\mathcal{K}}\mathbf{A}$ is algebraic it suffices, by Theorem 0.1.6, to show that $\mathrm{Con}_{\mathcal{K}}\mathbf{A}$ is closed under arbitrary intersections and unions of upwardly directed subsets. Let $\{\Phi_i;\, i \in I\}$ be a set of $\mathcal{K}$-congruences on **A**. Note that $\bigcap\{\Phi_i;\, i \in I\} \in \mathrm{Con}\,\mathbf{A}$ by Theorem 0.2.1. Define a map from $A/\bigcap\{\Phi_i;\, i \in I\}$ to $\Pi_{i \in I} A/\Phi_i$ by

$$a/\bigcap\{\Phi_i;\, i \in I\} \mapsto \langle a/\Phi_i;\, i \in I\rangle.$$

The map is easily seen to be well-defined, injective and a homomorphism from $\mathbf{A}/\bigcap\{\Phi_i;\, i \in I\}$ to $\Pi_{i \in I}\mathbf{A}/\Phi_i$. The image of $\mathbf{A}/\bigcap\{\Phi_i;\, i \in I\}$ under this map is therefore a subalgebra of $\Pi_{i \in I}\mathbf{A}/\Phi_i$ by Theorem 0.2.2 (ii). Since each $\mathbf{A}/\Phi_i \in \mathcal{K}$ by assumption, we have that $\Pi_{i \in I}\mathbf{A}/\Phi_i \in P(\mathcal{K})$. It follows that $\mathbf{A}/\bigcap\{\Phi_i;\, i \in I\} \in ISP(\mathcal{K}) \subseteq \mathcal{K}$, by Theorem 0.4.1, i.e., $\bigcap\{\Phi_i;\, i \in I\} \in \mathrm{Con}_{\mathcal{K}}\mathbf{A}$.

Let $\{\Phi_i;\, i \in I\}$ be an upwardly directed family of $\mathcal{K}$-congruences on **A**. By Theorem 0.2.1,

$\bigcup\{\Phi_i\,;\, i\in I\}\in \operatorname{Con}\mathbf{A}$. Suppose that $\mathbf{B}=\mathbf{A}/\bigcup\{\Phi_i\,;\, i\in I\}\notin \mathcal{K}$. Then, by Theorem 0.4.1, there exists a quasi-identity $s_1\approx t_1\ \&\ \ldots\ \&\ s_n\approx t_n \Rightarrow s\approx t$ satisfied by $\mathcal{K}$ such that

$$\mathbf{A}/\bigcup\{\Phi_i\,;\, i\in I\}\not\models s_1\approx t_1\ \&\ \ldots\ \&\ s_n\approx t_n \Rightarrow s\approx t,$$

i.e., there exists an interpretation $\overline{a}$ of the variables in $s_1,t_1,\ldots,s_n,t_n,s,t$ in $A/\bigcup\{\Phi_i\,;\, i\in I\}$ such that $s_i^{\mathbf{B}}(\overline{a})=t_i^{\mathbf{B}}(\overline{a})$ for each $i\le n$ but $s^{\mathbf{B}}(\overline{a})\ne t^{\mathbf{B}}(\overline{a})$. Now, for each $\ell\le n$, let $s_\ell^{\mathbf{B}}(\overline{a})=b_\ell/\bigcup\{\Phi_i\,;\, i\in I\}$ and $t_\ell^{\mathbf{B}}(\overline{a})=c_\ell/\bigcup\{\Phi_i\,;\, i\in I\}$ for some $b_\ell,c_\ell\in A$. Moreover, let $s^{\mathbf{B}}(\overline{a})=b/\bigcup\{\Phi_i\,;\, i\in I\}$ and $t^{\mathbf{B}}(\overline{a})=c/\bigcup\{\Phi_i\,;\, i\in I\}$ where $b,c\in A$. Then $s_\ell^{\mathbf{B}}(\overline{a})=t_\ell^{\mathbf{B}}(\overline{a})$ implies that $(b_\ell,c_\ell)\in\bigcup\{\Phi_i\,;\, i\in I\}$, i.e., $(b_\ell,c_\ell)\in\Phi_{j_\ell}$ for some $j_\ell\in I$. Since $\{\Phi_i\,;\, i\in I\}$ is directed, there exists $k\in I$ such that $\Phi_{j_\ell}\subseteq\Phi_k$ for each $\ell\le n$. Thus $(b_\ell,c_\ell)\in\Phi_k$, hence $b_\ell/\Phi_k=c_\ell/\Phi_k$ for each $\ell\le n$, and $\mathbf{A}/\Phi_k\in\mathcal{K}$ by assumption, hence $b/\Phi_k=c/\Phi_k$, i.e., $(b,c)\in\Phi_k$. But this implies that $(b,c)\in\bigcup\{\Phi_i\,;\, i\in I\}$, hence that $s^{\mathbf{B}}(\overline{a})=t^{\mathbf{B}}(\overline{a})$. This contradiction implies that $\mathbf{A}/\bigcup\{\Phi_i\,;\, i\in I\}\in\mathcal{K}$. □

We denote the lattice $\langle \operatorname{Con}_{\mathcal{K}}\mathbf{A},\subseteq\rangle$ by $\mathbf{Con}_{\mathcal{K}}\mathbf{A}$. For a quasivariety $\mathcal{K}$, an algebra $\mathbf{A}$ with the same type as $\mathcal{K}$ and $X\subseteq A^2$, let $\Theta_{\mathcal{K}}^{\mathbf{A}}(X)$ denote the $\mathcal{K}$-congruence on $\mathbf{A}$ generated by the set X, i.e., the intersection of all $\mathcal{K}$-congruences on $\mathbf{A}$ containing X. For $X\subseteq A$, let $\Theta_{\mathcal{K}}^{\mathbf{A}}(X)$ denote the $\mathcal{K}$-congruence on $\mathbf{A}$ generated by the set X^2. A $\mathcal{K}$-congruence Φ is said to be *finitely generated* if $\Phi=\Theta_{\mathcal{K}}^{\mathbf{A}}(X)$ for some finite $X\subseteq A^2$. The compact members of $\operatorname{Con}_{\mathcal{K}}\mathbf{A}$ are the finitely generated members of $\operatorname{Con}_{\mathcal{K}}\mathbf{A}$, by Theorem 0.1.6. We abbreviate $\Theta_{\mathcal{K}}^{\mathbf{A}}(\{(a,b)\})$ by $\Theta_{\mathcal{K}}^{\mathbf{A}}(a,b)$.

0.4.4 LEMMA

Let $\mathcal{K}$ be a class of algebras of a fixed type which is closed under I, S *and* P. *Let X be a set of variables and let $\zeta_i,\eta_i\in T(X)$ for $i\in I$ and $\varphi,\psi\in T(X)$. The following conditions are equivalent:*

(i) $\{\zeta_i\approx\eta_i\,;\, i\in I\}\models_{\mathcal{K}}\varphi\approx\psi$,

(ii) *for each $\mathbf{A}\in\mathcal{K}$ and each interpretation $\overline{a}$ of X in A,*

$$\Theta_{\mathcal{K}}^{\mathbf{A}}(\{(\zeta_i^{\mathbf{A}}(\overline{a}),\eta_i^{\mathbf{A}}(\overline{a}))\,;\, i\in I\})\supseteq\Theta_{\mathcal{K}}^{\mathbf{A}}(\varphi^{\mathbf{A}}(\overline{a}),\psi^{\mathbf{A}}(\overline{a})),$$

(iii) $\Theta_{\mathcal{K}}^{\mathbf{F}}(\{(\underline{\zeta_i},\underline{\eta_i})\,;\, i\in I\})\supseteq\Theta_{\mathcal{K}}^{\mathbf{F}}(\underline{\varphi},\underline{\psi})$, *where* $\mathbf{F}=\mathbf{F}_{\mathcal{K}}(\underline{X})$.

Proof. (i)⇒(ii) Let $\mathbf{A}\in\mathcal{K}$ and let $\overline{a}$ be an interpretation of X in A. Take any $\Phi\in\operatorname{Con}_{\mathcal{K}}\mathbf{A}$ such that $\{(\zeta_i^{\mathbf{A}}(\overline{a}),\eta_i^{\mathbf{A}}(\overline{a}))\,;\, i\in I\}\subseteq\Phi$. Let $\overline{a/\Phi}$ denote the interpretation of X in A/Φ defined by $\overline{a/\Phi}(x)=\overline{a}(x)/\Phi$ for all $x\in X$. Now, in $\mathbf{A}/\Phi$, $\zeta_i^{\mathbf{A}/\Phi}(\overline{a/\Phi})=\eta_i^{\mathbf{A}/\Phi}(\overline{a/\Phi})$ for each $i\in I$ and

$\mathbf{A}/\Phi \in \mathcal{K}$, so, by (i),

$$\varphi^{\mathbf{A}/\Phi}(\overline{a/\Phi}) = \psi^{\mathbf{A}/\Phi}(\overline{a/\Phi}),$$

i.e.,

$$(\varphi^{\mathbf{A}}(\overline{a}))/\Phi = (\psi^{\mathbf{A}}(\overline{a}))/\Phi,$$

i.e.,

$$(\varphi^{\mathbf{A}}(\overline{a}), \psi^{\mathbf{A}}(\overline{a})) \in \Phi,$$

i.e.,

$$\Theta^{\mathbf{A}}_{\mathcal{K}}(\varphi^{\mathbf{A}}(\overline{a}), \psi^{\mathbf{A}}(\overline{a})) \subseteq \Phi.$$

Thus

$$\Theta^{\mathbf{A}}_{\mathcal{K}}(\varphi^{\mathbf{A}}(\overline{a}), \psi^{\mathbf{A}}(\overline{a})) \subseteq \Theta^{\mathbf{A}}_{\mathcal{K}}(\{(\zeta_i^{\mathbf{A}}(\overline{a}), \eta_i^{\mathbf{A}}(\overline{a}))\,;\, i \in I\}).$$

(ii)⇒(iii) Follows trivially from the fact that by Theorem 0.2.11 (ii), $\mathbf{F} \in \mathrm{ISP}(\mathcal{K}) \subseteq \mathcal{K}$.

(iii)⇒(i) We can assume without loss of generality that $\mathcal{K}$ contains at least one nontrivial algebra. Take $\mathbf{A} \in \mathcal{K}$ and let $\overline{a}$ be an interpretation of X in A with $\zeta_i^{\mathbf{A}}(\overline{a}) = \eta_i^{\mathbf{A}}(\overline{a})$ for all $i \in I$. We can consider the interpretation $\overline{a}$ as an interpretation of $\underline{X}$ in A also. (See the remark preceding Theorem 0.2.11.) By the universal mapping property (Theorem 0.2.11 (i)), $\overline{a}$ extends to a unique homomorphism $\pi : \mathbf{F} \to \mathbf{A}$. Then, by the Homomorphism Theorem (Theorem 0.2.4),

$$\mathbf{F}/\ker\pi \cong \pi(\mathbf{F}) \in \mathrm{S}(\mathbf{A}) \subseteq \mathcal{K},$$

hence $\ker\pi \in \mathrm{Con}_{\mathcal{K}}\mathbf{A}$. Now, $\pi(\underline{\zeta}_i) = \zeta_i^{\mathbf{A}}(\overline{a}) = \eta_i^{\mathbf{A}}(\overline{a}) = \pi(\underline{\eta}_i)$ for all $i \in I$, so

$$(\underline{\zeta}_i, \underline{\eta}_i) \in \ker\pi \quad \text{for all } i \in I,$$

hence

$$\Theta^{\mathbf{F}}_{\mathcal{K}}(\{(\underline{\zeta}_i, \underline{\eta}_i)\,;\, i \in I\}) \subseteq \ker\pi.$$

By (iii), $\Theta^{\mathbf{F}}_{\mathcal{K}}(\underline{\varphi}, \underline{\psi}) \subseteq \ker\pi$, so $(\underline{\varphi}, \underline{\psi}) \in \ker\pi$, i.e., $\pi(\underline{\varphi}) = \pi(\underline{\psi})$, i.e., $\varphi^{\mathbf{A}}(\overline{a}) = \psi^{\mathbf{A}}(\overline{a})$. This shows that $\{\zeta_i \approx \eta_i\,;\, i \in I\} \models_{\mathbf{A}} \varphi \approx \psi$, hence (i) holds. □

0.4.5 COROLLARY

Let $\mathcal{K}$ be a quasivariety, X a set of variables and $\zeta_1, \ldots, \zeta_m, \eta_1, \ldots, \eta_m, \varphi, \psi \in T(X)$. The following conditions are equivalent:

(i) $\mathcal{K} \models \zeta_1 \approx \eta_1 \,\&\ldots\&\, \zeta_m \approx \eta_m \Rightarrow \varphi \approx \psi$,

(ii) *for each $\mathbf{A} \in \mathcal{K}$ and each interpretation $\overline{a}$ of X in A,*

$$\Theta^{\mathbf{A}}_{\mathcal{K}}(\{(\zeta_i^{\mathbf{A}}(\overline{a}), \eta_i^{\mathbf{A}}(\overline{a}))\,;\, i \le m\}) \supseteq \Theta^{\mathbf{A}}_{\mathcal{K}}(\varphi^{\mathbf{A}}(\overline{a}), \psi^{\mathbf{A}}(\overline{a})),$$

(iii) $\Theta^{\mathbf{F}}_{\mathcal{K}}(\{(\underline{\zeta}_i, \underline{\eta}_i)\,;\, i \le m\}) \supseteq \Theta^{\mathbf{F}}_{\mathcal{K}}(\underline{\varphi}, \underline{\psi})$.

Proof. Noting that a quasivariety is, by definition, closed under I, S and P, and that (i) holds if and only if $\{\zeta_i \approx \eta_i\,;\, i \le m\} \models_{\mathcal{K}} \varphi \approx \psi$, one can see that the corollary follows immediately from the previous theorem. □

The following result will be used in Chapter 3.

0.4.6 THEOREM [DMS87, Theorem 3.4]

Let $\mathcal{V}$ be a variety and let ζ_i, η_i, $i \leq m$, and φ, ψ be n-ary terms of the type of $\mathcal{V}$ in the variables $\overline{x} = x_1, \dots, x_n$. Then the following are equivalent:

(i) $\mathcal{V} \models (\zeta_1 \approx \eta_1 \,\&\, \dots \,\&\, \zeta_m \approx \eta_m) \Rightarrow \varphi \approx \psi$;

(ii) *for some $\ell \in \omega$ there exist $(n+2)$-ary terms $t_1, \dots, t_\ell$ and pairs $(u_j, v_j) \in \{(\zeta_i, \eta_i) \,;\, i \leq m\}$ for $j \leq \ell$ such that $\mathcal{V}$ satisfies the identities*

$$\varphi(\overline{x}) \approx t_1(u_1(\overline{x}), v_1(\overline{x}), \overline{x})$$
$$t_1(v_1(\overline{x}), u_1(\overline{x}), \overline{x}) \approx t_2(u_2(\overline{x}), v_2(\overline{x}), \overline{x})$$
$$\vdots$$
$$t_\ell(v_\ell(\overline{x}), u_\ell(\overline{x}), \overline{x}) \approx \psi(\overline{x});$$

(iii) *for some $\ell \in \omega$ there exist $(n+1)$-ary terms $t_1, \dots, t_\ell$ and pairs $(u_j, v_j) \in \{(\zeta_i, \eta_j), (\eta_i, \zeta_i) \,;\, i \leq m\}$ for $j \leq \ell$ such that $\mathcal{V}$ satisfies the identities*

$$\varphi(\overline{x}) \approx t_1(u_1(\overline{x}), \overline{x})$$
$$t_1(v_1(\overline{x}), \overline{x}) \approx t_2(v_2(\overline{x}), \overline{x})$$
$$\vdots$$
$$t_\ell(v_\ell(\overline{x}), \overline{x}) \approx \psi(\overline{x}).$$
□

A quasivariety $\mathcal{K}$ is called *relatively* T-*regular* if there exists an (equationally definable) constant term T of the type of $\mathcal{K}$ such that, for all $\mathbf{A} \in \mathcal{K}$ and $\Phi, \Psi \in \mathrm{Con}_{\mathcal{K}}\mathbf{A}$,

$$\mathrm{T}^{\mathbf{A}}/\Phi = \mathrm{T}^{\mathbf{A}}/\Psi \text{ implies } \Phi = \Psi.$$

If in addition, $\mathcal{K}$ is a variety, then $\mathrm{Con}_{\mathcal{K}}\mathbf{A} = \mathrm{Con}\,\mathbf{A}$ for all $\mathbf{A} \in \mathcal{K}$, so $\mathcal{K}$ is called T-*regular.*

The following notion of an 'ideal' is due to A. Ursini (see [GU84]).

Let $\mathcal{K}$ be a quasivariety of type $\mathcal{L}$ with constant term T. A term $t(x_1, \dots, x_\ell, y_1, \dots, y_m)$ of type $\mathcal{L}$ over $\{x_1, \dots, x_\ell, y_1, \dots, y_m\}$ is called an *ideal term in* $y_1, \dots, y_m$ *(for* $\mathcal{K}$ *and* T) if

$$\mathcal{K} \models t(x_1, \dots, x_\ell, \mathrm{T}, \dots, \mathrm{T}) \approx \mathrm{T}.$$

Let $\mathbf{A}$ be an algebra of type $\mathcal{L}$. A subset I of A is called an *ideal of* $\mathbf{A}$ (with respect to $\mathcal{K}$ and T) if for every ideal term $t(x_1,\dots,x_\ell,y_1,\dots,y_m)$ in $y_1,\dots,y_m$, we have

$$t^{\mathbf{A}}(a_1,\dots,a_\ell,i_1,\dots,i_m)\in I \text{ for all } a_1,\dots,a_\ell\in A \text{ and } i_1,\dots,i_m\in I.$$

The set of all ideals of **A** is denoted by $Id\,\mathbf{A}$. Since the constant term T is an ideal term (in any variable), it follows trivially that $\mathrm{T}^{\mathbf{A}}\in I$ for every ideal I of **A**. We also have the following easily verified fact:

For any reflexive compatible binary relation Φ on **A**, $\mathrm{T}^{\mathbf{A}}/\Phi$ is an ideal of **A**.

This induces a map $f:\mathrm{Con}_{\mathcal{K}}\mathbf{A}\to Id\,\mathbf{A}$ defined by $f(\Phi)=\mathrm{T}^{\mathbf{A}}/\Phi$. It is evident that $\mathcal{K}$ is relatively T-regular if and only if f is injective for all $\mathbf{A}\in\mathcal{K}$.

The varieties of groups and Boolean algebras are 1-regular, for example. All natural examples of (relatively) T-regular (quasi) varieties that have been identified as such in the literature have a stronger property, viz., their (relative) congruences are 'ideal determined' in the following sense: A quasivariety $\mathcal{K}$ with a constant term T is called *relatively ideal determined (with respect to* T*)* if the map $f:\mathrm{Con}_{\mathcal{K}}\mathbf{A}\to Id\,\mathbf{A}$ defined by $f(\Phi)=\mathrm{T}^{\mathbf{A}}/\Phi$ is a bijection for each $\mathbf{A}\in\mathcal{K}$. If in addition, $\mathcal{K}$ is a variety we call $\mathcal{K}$ *ideal determined (with respect to* T*)*. We shall give an example in Chapter 3 of a relatively T-regular quasivariety that is not relatively ideal determined. We believe that the next result is not in the published literature, although its specialization to varieties is well-known.

0.4.7 THEOREM

Assume the type of the quasivariety $\mathcal{K}$ admits a constant term T. *The following conditions are equivalent:*

(i) $\mathcal{K}$ *is relatively* T*-regular,*

(ii) there exist binary terms $d_1(x,y),\dots,d_n(x,y)$ (of the type of $\mathcal{K}$ over $\{x,y\}$) such that $\mathcal{K}$ satisfies

$$d_i(x,x)\approx \mathrm{T} \textit{ for all } i\le n, \textit{ and}$$

$$\Big(\mathop{\&}_{i\le n} d_i(x,y)\approx \mathrm{T}\Big)\Rightarrow x\approx y.$$

Proof. (i)⇒(ii) Let $\mathbf{F}=\mathbf{F}_{\mathcal{K}}(\underline{x},\underline{y})$. Let

$$U=\{(\mathrm{T}^{\mathbf{F}},\underline{b})\,;\,(\mathrm{T}^{\mathbf{F}},\underline{b})\in\Theta^{\mathbf{F}}_{\mathcal{K}}(\underline{x},\underline{y})\}.$$

(Here, every element of F has the form $\underline{b}=b^{\mathbf{F}}(\underline{x},\underline{y})$ for a term $b(x,y)$ of the type of $\mathcal{K}$, so the definition of U makes sense.) Trivially, $\Theta^{\mathbf{F}}_{\mathcal{K}}(U) \subseteq \Theta^{\mathbf{F}}_{\mathcal{K}}(\underline{x}, \underline{y})$. By definition of U,

$$\mathrm{T}^{\mathbf{F}}/\Theta^{\mathbf{F}}_{\mathcal{K}}(\underline{x}, \underline{y}) \subseteq \mathrm{T}^{\mathbf{F}}/\Theta^{\mathbf{F}}_{\mathcal{K}}(U),$$

so

$$\mathrm{T}^{\mathbf{F}}/\Theta^{\mathbf{F}}_{\mathcal{K}}(\underline{x}, \underline{y}) = \mathrm{T}^{\mathbf{F}}/\Theta^{\mathbf{F}}_{\mathcal{K}}(U).$$

By relative T-regularity, $\Theta^{\mathbf{F}}_{\mathcal{K}}(\underline{x}, \underline{y}) = \Theta^{\mathbf{F}}_{\mathcal{K}}(U)$, hence $\Theta^{\mathbf{F}}_{\mathcal{K}}(U)$ is compact in $\mathrm{Con}_{\mathcal{K}}\mathbf{F}$, by Corollary 0.1.6. Since $\Theta^{\mathbf{F}}_{\mathcal{K}}(U) = \bigvee^{\mathbf{Con}_{\mathcal{K}}\mathbf{F}}\{\Theta^{\mathbf{F}}_{\mathcal{K}}(\mathrm{T}^{\mathbf{F}}, \underline{b}) \,;\, (\mathrm{T}^{\mathbf{F}}, \underline{b}) \in U\}$, it follows that for some finite subset, say $\{(\mathrm{T}^{\mathbf{F}}, \underline{d}_1), \ldots, (\mathrm{T}^{\mathbf{F}}, \underline{d}_n)\}$ of U, we have

$$\begin{aligned}\Theta^{\mathbf{F}}_{\mathcal{K}}(U) &\subseteq \bigvee^{\mathbf{Con}_{\mathcal{K}}\mathbf{F}}\{\Theta^{\mathbf{F}}_{\mathcal{K}}(\mathrm{T}^{\mathbf{F}}, \underline{d}_i) \,;\, i \leq n\} \\ &= \Theta^{\mathbf{F}}_{\mathcal{K}}(\{(\mathrm{T}^{\mathbf{F}}, \underline{d}_i) \,;\, i \leq n\}).\end{aligned}$$

The converse inclusion is also true, trivially. We have shown that

$$\Theta^{\mathbf{F}}_{\mathcal{K}}(\underline{x}, \underline{y}) = \Theta^{\mathbf{F}}_{\mathcal{K}}(\{(\mathrm{T}^{\mathbf{F}}, \underline{d}_i) \,;\, i \leq n\}),$$

hence, by Lemma 0.4.4, $\mathcal{K}$ satisfies the quasi-identity in (ii). It remains to show that $\mathcal{K}$ satisfies $d_i(x, x) \approx \mathrm{T}$, $i \leq n$. Take $j \in \{1, \ldots, n\}$. In Corollary 0.4.5, replace m by 1 and $\zeta_i, \eta_i, \varphi, \psi$ by $x, y, d_j(x, y), \mathrm{T}$, respectively. By the above,

$$(\mathrm{T}^{\mathbf{F}}, \underline{d}_j) \in \Theta^{\mathbf{F}}_{\mathcal{K}}(\underline{x}, \underline{y}),$$

so

$$\Theta^{\mathbf{F}}_{\mathcal{K}}(\underline{x}, \underline{y}) \supseteq \Theta^{\mathbf{F}}_{\mathcal{K}}(\underline{d}_j, \mathrm{T}^{\mathbf{F}}).$$

Lemma 0.4.4 then implies that $\mathcal{K}$ satisfies $x \approx y \Rightarrow d_j(x, y) \approx \mathrm{T}$, from which it follows that $\mathcal{K}$ satisfies $d_j(x, x) \approx \mathrm{T}$.

(ii)⇒(i) Assume that $d_1, \ldots, d_n$ are binary terms as in (ii). Let $\mathbf{A} \in \mathcal{K}$ and suppose that $\Phi, \Psi \in \mathrm{Con}_{\mathcal{K}}\mathbf{A}$ such that $\mathrm{T}^{\mathbf{A}}/\Phi = \mathrm{T}^{\mathbf{A}}/\Psi$. Take $(a, b) \in \Phi$. By reflexivity, $(b, b) \in \Phi$, hence, for each $i \leq n$

$$(d_i^{\mathbf{A}}(a, b), d_i^{\mathbf{A}}(b, b)) \in \Phi.$$

Now, $d_i^{\mathbf{A}}(b, b) = \mathrm{T}^{\mathbf{A}}$, by assumption, so $(d_i^{\mathbf{A}}(a, b), \mathrm{T}^{\mathbf{A}}) \in \Phi$,

i.e.,

$$d_i^{\mathbf{A}}(a, b) \in \mathrm{T}^{\mathbf{A}}/\Phi,$$

so

$$d_i^{\mathbf{A}}(a, b) \in \mathrm{T}^{\mathbf{A}}/\Psi.$$

Thus

$$d_i^{\mathbf{A}}(a, b)/\Psi = \mathrm{T}^{\mathbf{A}}/\Psi,$$

which is equivalent to

$$d_i^{\mathbf{A}/\Psi}(a/\Psi, b/\Psi) = \mathrm{T}^{\mathbf{A}/\Psi}.$$

Since Ψ is a relative congruence, $\mathbf{A}/\Psi \in \mathcal{K}$, so (ii) implies that $a/\Psi = b/\Psi$, i.e., $(a, b) \in \Psi$. This proves the inclusion $\Phi \subseteq \Psi$, and the reverse inclusion follows by symmetry, hence $\mathcal{K}$ is relatively T-regular. □

Let $\mathcal{K}$ be a class of algebras of type $\mathbf{\mathcal{L}} = \langle \mathcal{L}, ar\rangle$ and $\mathcal{K}'$ a class of algebras of type $\mathbf{\mathcal{L}}' = \langle \mathcal{L}', ar'\rangle$. Let X be an infinite set of variables. Suppose:

(i) There is an injection $\alpha : \mathcal{K}' \to \mathcal{K}$ such that for each $\mathbf{A} \in \mathcal{K}'$, $\mathbf{A}$ and $\alpha(\mathbf{A})$ have the same universe.

(ii) For every operation symbol $f \in \mathbf{\mathcal{L}}$ with $ar(f) = n > 0$ there is a term f' of type $\mathbf{\mathcal{L}}'$ over X with $ar'(f') = n$ such that for every $\mathbf{A} \in \mathcal{K}'$, and all $a_1, \ldots, a_n \in A$,

$$f^{\alpha(\mathbf{A})}(a_1, \ldots, a_n) = (f')^{\mathbf{A}}(a_1, \ldots, a_n).$$

(iii) For every constant symbol c of $\mathbf{\mathcal{L}}$, there is a term c' of type $\mathbf{\mathcal{L}}'$ over X such that for every $\mathbf{A} \in \mathcal{K}'$ and all $a_1, \ldots, a_{ar'(c')} \in A$,

$$(c')^{\mathbf{A}}(a_1, \ldots, a_{ar'(c')}) = c^{\alpha(\mathbf{A})}.$$

Then we say that $\mathcal{K}$ is *interpretable* in $\mathcal{K}'$. In this case, for any term t of type $\mathbf{\mathcal{L}}$ over X with $ar(t) = n > 0$, there is clearly a term t' of type $\mathbf{\mathcal{L}}'$ over X of arity $ar'(t') = n$ such that for every $\mathbf{A} \in \mathcal{K}'$ and all $a_1, \ldots, a_n \in A$,

$$t^{\alpha(\mathbf{A})}(a_1, \ldots, a_n) = (t')^{\mathbf{A}}(a_1, \ldots, a_n).$$

We call $(\alpha, {}')$ an *interpretation* of $\mathcal{K}$ in $\mathcal{K}'$ where $'$ denotes the map $t \mapsto t'$ whose domain is the set $T(X)$ of all terms of type $\mathbf{\mathcal{L}}$ over X. If in addition, $\mathcal{K}'$ is interpretable in $\mathcal{K}$ by means of an interpretation $(\beta, {}^*)$ such that

(i) α and β are mutually inverse functions,

(ii) for every term t of type $\mathbf{\mathcal{L}}$ over X,

$$\mathcal{K} \models t(x_1, \ldots, x_{ar(t)}) \approx (t')^*(x_1, \ldots, x_{ar'((t')^*)}),$$

(iii) For every term s of type $\mathbf{\mathcal{L}}'$ over X,

$$\mathcal{K}' \models s(x_1, \ldots, x_{ar'(s)}) \approx (s^*)'(x_1, \ldots, x_{ar((s^*)')})$$

(where $x_1, x_2, \ldots \in X$),

then we say that $\mathcal{K}$ and $\mathcal{K}'$ are *termwise definitionally equivalent.* The standard example used to illustrate this concept is the termwise equivalence of the varieties of Boolean algebras and Boolean rings. (See [BS81, Chapter IV §2] for the details of this example.) Algebras $\mathbf{A}$ and $\mathbf{A}'$ are said to be *termwise definitionally equivalent* if the classes $\{\mathbf{A}\}$ and $\{\mathbf{A}'\}$ are.

0.5 FIRST-ORDER STRUCTURES, THEORIES AND MODELS

First-order structures

A (*first-order*[1]) *language* (*without equality*) is an ordered triple $\mathfrak{L} = \langle \mathcal{O}, \mathcal{R}, ar \rangle$ where $\mathcal{O}$ and $\mathcal{R}$ are disjoint sets and ar is a function from $\mathcal{O} \cup \mathcal{R}$ to ω, called the *arity function* of $\mathfrak{L}$. We call $\mathcal{O}$ the set of *operation* (or *function*) *symbols* and $\mathcal{R}$ the set of *relation* (or *predicate*) *symbols* of $\mathfrak{L}$. An element ρ of $\mathcal{O}$ (resp. $\mathcal{R}$) is called an *n-ary operation* (resp. *relation*) *symbol of* $\mathfrak{L}$ if $ar(\rho) = n$.

In this case an ordered triple $\mathbf{A} = \langle A; O; R \rangle$ is called a (*first-order*) *structure of type* $\mathfrak{L}$ (or just an *$\mathfrak{L}$-structure*) if A is a nonempty set, O is a set $\{o^{\mathbf{A}} \colon o \in \mathcal{O}\}$ indexed by $\mathcal{O}$, R is a set $\{r^{\mathbf{A}} \colon r \in \mathcal{R}\}$ indexed by $\mathcal{R}$ and for each ρ in $\mathcal{O}$ (resp. $\mathcal{R}$), if $n = ar(\rho)$ then $\rho^{\mathbf{A}}$ is an n-ary operation (resp. relation) on A (as defined at the beginning of this chapter); we call $\rho^{\mathbf{A}}$ a *fundamental operation* (resp. *relation*) of $\mathbf{A}$. We delete the superscripts when no confusion can arise. We call A the *universe* of $\mathbf{A}$. If $\mathcal{O}$ and $\mathcal{R}$ are finite, say $\mathcal{O} = \{o_1, o_2, \ldots, o_n\}$ and $\mathcal{R} = \{r_1, r_2, \ldots, r_m\}$, we often write $\langle A; o_1, o_2, \ldots, o_n; r_1, r_2, \ldots, r_m \rangle$ for $\mathbf{A}$. Note that if $\mathcal{R} = \emptyset$, we may identify $\mathbf{A} = \langle A; O; \emptyset \rangle$ with the universal algebra $\langle A; O \rangle$.

Let X be a set (whose elements are called *variables*) such that $X \cap (\mathcal{O} \cup \mathcal{R}) = \emptyset$. We define a *term of type* $\mathfrak{L}$ *over* X to be just a term of the type $\langle \mathcal{O}, ar|\,\mathcal{O} \rangle$ (of universal algebras) over X. An *atomic formula of type* $\mathfrak{L}$ (or *atomic $\mathfrak{L}$-formula*) *over* X is defined to be a (purely formal) expression of the form $r(t_1, \ldots, t_n)$ where $r \in \mathcal{R}$, $ar(r) = n$ and $t_1, \ldots, t_n$ are terms of type $\mathfrak{L}$ over X. Notice that if $\approx\, \in \mathcal{R}$ with $ar(\approx) = 2$ and we write $t_1 \approx t_2$ for $\approx(t_1, t_2)$ then all identities of type $\langle \mathcal{O}, ar|\,\mathcal{O} \rangle$ are atomic formulas of $\mathfrak{L}$.

More generally, we define *formulas of type* $\mathfrak{L}$ (or *$\mathfrak{L}$-formulas*) *over* X recursively by the following rules, which involve the (purely formal) logical symbols $\sim$ ('not'), $\Rightarrow$ and $\forall$ as well as symbols of $\mathfrak{L}$ and variables:

(i) any atomic $\mathfrak{L}$-formula over X is an $\mathfrak{L}$-formula over X;

(ii) if $x \in X$ and Φ, Φ_1, Φ_2 are $\mathfrak{L}$-formulas over X then so are

[1]The word 'elementary' is often used as a synonym for 'first-order' in the literature.

$$\sim(\Phi), \quad (\Phi_1) \Rightarrow (\Phi_2), \quad \forall x\,(\Phi).$$

We introduce the following abbreviations for $\mathfrak{L}$-formulas over X:

$(\Phi_1) \sqcup (\Phi_2)$ abbreviates $(\sim(\Phi_1)) \Rightarrow (\Phi_2)$

$(\Phi_1)\ \&\ (\Phi_2)$ abbreviates $\sim((\sim(\Phi_1)) \sqcup (\sim(\Phi_2)))$

$(\Phi_1) \Leftrightarrow (\Phi_2)$ abbreviates $((\Phi_1) \Rightarrow (\Phi_2))\ \&\ ((\Phi_2) \Rightarrow (\Phi_1))$

$\exists x\,(\Phi)$ abbreviates $\sim(\forall x\,(\sim(\Phi)))$.

(We read '$\sqcup$' and '&' as 'or' and 'and', respectively.)

For the present we shall assume that $\mathfrak{L}$ and the set of variables X are fixed and we shall use the expression 'formula' in place of '$\mathfrak{L}$–formula over X'. Where no confusion is possible, we abbreviate formulas by omitting brackets according to standard natural conventions (which we refrain from stating explicitly here).

A formula Φ_1 is a *subformula* of a formula Φ if there is a consecutive string of symbols in the formula Φ which is precisely the formula Φ_1. An occurrence of a variable x in a formula Φ is called *bound* if it is an occurrence of x in a subformula of Φ which is either of the form $\forall x\,\Psi$ or of the form $\exists x\,\Psi$. Otherwise the occurrence of x in Φ is called *free*. The variable x is called *bound* (resp. *free*) *in* Φ if it has a bound (resp. free) occurrence in Φ. An *$\mathfrak{L}$–sentence over X* (briefly, a *sentence*) is a formula with no free variables. We sometimes denote a formula Φ by $\Phi(x_1,\ldots,x_n)$, which should be understood to indicate that the free variables of Φ are *among* $x_1,\ldots,x_n$. Given terms $t_1,\ldots,t_m$, we write $\Phi[t_1/x_1,\ldots,t_m/x_m]$ for the formula obtained by simultaneously replacing all free occurrences of the variable x_i in the formula Φ by the term t_i, for $i = 1,\ldots,m$. When a formula is given as $\Phi(x_1,\ldots,x_n)$, we sometimes denote $\Phi[t_1/x_1,\ldots,t_n/x_n]$ by $\Phi(t_1,\ldots,t_n)$. (Clearly this does not conflict with our earlier usage.) Given a formula Φ and a variable x, a term t is said to be *free for x in Φ* if for any variable y that occurs in t, no free occurrence of x in Φ lies within a subformula of Φ of the form $\forall y\,\Psi$ (equivalently, if no occurrence of any variable in t becomes a bound occurrence of that variable in $\Phi[t/x]$).

Let $\mathbf{A} = \langle A; O; R\rangle$ be an $\mathfrak{L}$–structure. We denote by $\mathfrak{L}_A$ the first-order language obtained from $\mathfrak{L}$ by adding to $\mathcal{O}$ a nullary operation symbol a for each $a \in A$. We may now consider $\mathfrak{L}_A$–formulas of the form $\Phi[a/x]$ for every $\mathfrak{L}$–formula Φ, every variable $x \in X$ and every $a \in A$.

Observe that a sentence of $\mathfrak{L}_A$ is an atomic formula if and only if it is $r(t_1(a_1,\dots,a_m),\dots,t_n(a_1,\dots,a_m))$ for some $r \in \mathfrak{R}$ with $ar(r) = n$, some terms $t_1,\dots,t_n$ of arity m and some $a_1,\dots,a_m \in A$. Given a sentence Φ of $\mathfrak{L}_A$, we define the notion that '**A** *satisfies* Φ', or that 'Φ *is true in* **A**' or 'Φ *holds in* **A**' (written $\mathbf{A} \models \Phi$) recursively as follows:

(i) When Φ is an atomic formula $r(t_1(a_1,\dots,a_m),\dots,t_n(a_1,\dots,a_m))$, we define $\mathbf{A} \models \Phi$ iff $(t_1^{\mathbf{A}}(a_1,\dots,a_m),\dots,t_n^{\mathbf{A}}(a_1,\dots,a_m)) \in r^{\mathbf{A}}$.

(ii) When Φ is $\sim \Phi_1$, we define $\mathbf{A} \models \Phi$ iff it is not the case that $\mathbf{A} \models \Phi_1$ (which we abbreviate as $\mathbf{A} \not\models \Phi_1$).

(iii) When Φ is $\Phi_1 \Rightarrow \Phi_2$, we define $\mathbf{A} \models \Phi$ iff $\mathbf{A} \not\models \Phi_1$ or $\mathbf{A} \models \Phi_2$.

(iv) When Φ is $\forall x\, \Phi_1$, we define $\mathbf{A} \models \Phi$ iff $\mathbf{A} \models \Phi_1[a/x]$ for all $a \in A$.

Notice that in (iv), the fact that Φ is a sentence means that Φ_1 has no free variable except possibly x, so these definitions make sense.

For an $\mathfrak{L}_A$-formula Φ (not necessarily a sentence) whose free variables (listed in the order that they first occur in Φ, from left to right) are $x_1,\dots,x_n$, we say that Φ is *valid* in **A** (or that **A** *satisfies* Φ or that Φ *holds in* **A**) iff $\mathbf{A} \models \forall x_1 \dots \forall x_n \Phi$. In this case, we also write $\mathbf{A} \models \Phi$. (Thus, a sentence is valid in **A** if and only if it is true in **A**.) For a class of structures $\mathcal{K}$, an $\mathfrak{L}$-formula Φ and a set Σ of $\mathfrak{L}$-formulas, we define:

$$\mathcal{K} \models \Phi \text{ iff } \mathbf{A} \models \Phi \text{ for every } \mathbf{A} \in \mathcal{K}; \qquad \mathbf{A} \models \Sigma \text{ iff } \mathbf{A} \models \Phi \text{ for every } \Phi \in \Sigma;$$

$$\mathcal{K} \models \Sigma \text{ iff } \mathbf{A} \models \Sigma \text{ for every } \mathbf{A} \in \mathcal{K}.$$

If $\mathbf{A} \models \Sigma$, we also say that **A** is a *model* of Σ. We use $\models_{\mathcal{K}} \Phi$ to abbreviate $\emptyset \models_{\mathcal{K}} \Phi$, etc. A class $\mathcal{K}$ of $\mathfrak{L}$-structures is said to be *axiomatized* by a set Σ of $\mathfrak{L}$-formulas if $\mathcal{K}$ is exactly the class of all $\mathfrak{L}$-structures **A** for which $\mathbf{A} \models \Phi$ for every $\Phi \in \Sigma$.

Let P be a property which members of a class $\mathcal{K}$ of $\mathfrak{L}$-structures may or may not possess. We say that P is *first-order definable over* $\mathcal{K}$ provided that there exists a set Σ of $\mathfrak{L}$-formulas (or equivalently $\mathfrak{L}$-sentences) such that for each $\mathbf{A} \in \mathcal{K}$, **A** has property P if and only if $\mathbf{A} \models \Sigma$. A subclass $\mathcal{M}$ of $\mathcal{K}$ is called *first-order definable over* $\mathcal{K}$ if and only if the property of being an element of $\mathcal{M}$ is first-order definable over $\mathcal{K}$.

A *universal Horn sentence* is an $\mathfrak{L}$-formula of the form

$$\forall x_1 \ldots \forall x_n [(r_1(x_1,\ldots,x_n) \mathbin{\&} \ldots \mathbin{\&} r_m(x_1,\ldots,x_n)) \Rightarrow r(x_1,\ldots,x_n)]$$

where $r_1,\ldots,r_m,r$ are atomic $\mathfrak{L}$–formulas whose variables are among $x_1,\ldots,x_n$. (We allow $m = 0$, i.e., $\forall x_1 \ldots \forall x_n (r(x_1,\ldots,x_n))$ is also a universal Horn sentence.) A class $\mathcal{K}$ of $\mathfrak{L}$–structures is called a *universal Horn class* if there exists a set of universal Horn sentences which axiomatizes $\mathcal{K}$.

Let $\mathbf{A} = \langle A; O^{\mathbf{A}}; R^{\mathbf{A}} \rangle$ and $\mathbf{B} = \langle B; O^{\mathbf{B}}; R^{\mathbf{B}} \rangle$ be $\mathfrak{L}$–structures. We say that $\mathbf{A}$ is a *substructure* of $\mathbf{B}$ provided that $\langle A; O^{\mathbf{A}} \rangle$ is a subalgebra of $\langle B; O^{\mathbf{B}} \rangle$ and for each $r \in R$ with $ar(r) = n$, say, $r^{\mathbf{A}} = A^n \cap r^{\mathbf{B}}$. We denote the class of substructures of a class $\mathcal{K}$ of $\mathfrak{L}$–structures by $\mathrm{S}(\mathcal{K})$. (This generalizes our use of S as a class operator for algebras.)

A function $h \colon A \to B$ is called a *homomorphism from* $\mathbf{A}$ *to* $\mathbf{B}$ if h is a homomorphism of algebras $\langle A; O^{\mathbf{A}} \rangle \to \langle B; O^{\mathbf{B}} \rangle$ and for each $r \in R$ with $ar(r) = n$, say, and any $a_1,\ldots,a_n \in A$ such that $(a_1,\ldots,a_n) \in r^{\mathbf{A}}$, we have $(h(a_1),\ldots,h(a_n)) \in r^{\mathbf{B}}$. In this case, the image $h(A)$ of A under h is the universe of a substructure of $\mathbf{B}$, which we denote by $h(\mathbf{B})$. If in addition, h is a bijection and for every $r \in R$ with $ar(r) = n$ and any $a_1,\ldots,a_n \in A$ for which $(h(a_1),\ldots,h(a_n)) \in r^{\mathbf{B}}$, we have $(a_1,\ldots,a_n) \in r^{\mathbf{A}}$, then we call h an *isomorphism from* $\mathbf{A}$ *to* $\mathbf{B}$. In this case we say that $\mathbf{A}$ is *isomorphic to* $\mathbf{B}$ and we write $\mathbf{A} \cong \mathbf{B}$ or $h \colon \mathbf{A} \cong \mathbf{B}$. An isomorphism from $\mathbf{A}$ to a substructure of $\mathbf{B}$ is called an *embedding* of $\mathbf{A}$ into $\mathbf{B}$. We denote by $\mathrm{I}(\mathcal{K})$ the closure of a class $\mathcal{K}$ of $\mathfrak{L}$–structures under isomorphism.

Let I be any set and suppose $\mathbf{A}_i = \langle A_i; O_i^{\mathbf{A}_i}; R_i^{\mathbf{A}_i} \rangle$ is an $\mathfrak{L}$–structure for each $i \in I$. We define the *direct product* $\Pi_{i \in I} \mathbf{A}_i$ of the family $\langle \mathbf{A}_i \colon i \in I \rangle$ to be the $\mathfrak{L}$–structure $\mathbf{A} = \langle A; O^{\mathbf{A}}; R^{\mathbf{A}} \rangle$, where $\langle A; O^{\mathbf{A}} \rangle$ is the direct product over I of the algebras $\langle A_i; O_i^{\mathbf{A}_i} \rangle$ and for each $r \in R$ with $ar(r) = n$, say, and any $a_1,\ldots,a_n \in \Pi_{i \in I} A_i$, we have $(a_1,\ldots,a_n) \in r^{\mathbf{A}}$ iff for every $i \in I$, $(a_1(i),\ldots,a_n(i)) \in r^{\mathbf{A}_i}$. In this case the j–th canonical projection map $\pi_j \colon \Pi_{i \in I} \mathbf{A}_i \to \mathbf{A}_j$ is a surjective homomorphism for each $j \in I$. The class of all direct products of members of a class $\mathcal{K}$ is denoted by $\mathrm{P}(\mathcal{K})$. Observe that $\mathrm{P}(\mathcal{K})$ always contains a structure with universe $\{\emptyset\}$. A substructure $\mathbf{B}$ of the direct product $\Pi_{i \in I} \mathbf{A}_i$ is called a *subdirect product* of the family $\langle \mathbf{A}_i \colon i \in I \rangle$ of $\mathfrak{L}$–structures if $\pi_j(A) = A_j$ for all $j \in I$. An embedding h from an $\mathfrak{L}$–structure $\mathbf{C}$ to $\Pi_{i \in I} \mathbf{A}_i$ is called a *subdirect embedding* if $h(\mathbf{C})$ is a subdirect product of $\langle \mathbf{A}_i \colon i \in I \rangle$. The closure of a class $\mathcal{K}$ of $\mathfrak{L}$–structures under subdirect products is denoted by $\mathrm{P_S}(\mathcal{K})$.

Given a family $\langle \mathbf{A}_i = \langle A_i; O_i^{\mathbf{A}_i}; R_i^{\mathbf{A}_i} \rangle : i \in I \rangle$ of $\mathfrak{L}$–structures and a proper filter G on the Boolean algebra of all subsets of I, recall that the binary relation $\theta_G = \{(a,b) \in \Pi_{i \in I} \mathbf{A}_i : \{i \in I : a(i) = b(i)\} \in G\}$ is a congruence of the algebra reduct of $\Pi_{i \in I} \mathbf{A}_i$. We define the *filtered product* $\Pi_{i \in I} \mathbf{A}_i/G$ to be the $\mathfrak{L}$–structure $\mathbf{A} = \langle A; O^{\mathbf{A}}; R^{\mathbf{A}} \rangle$, where $\langle A; O^{\mathbf{A}} \rangle$ is the filtered product of algebras $\Pi_{i \in I} \langle A_i; O_i^{\mathbf{A}_i} \rangle/G$ (as defined in Section 0.3) and for any $r \in R$ with $ar(r) = n$, say, and any $a_1, \ldots, a_n \in \Pi_{i \in I} A_i$, we have $(a_1/G, \ldots, a_n/G) \in r^{\mathbf{A}}$ iff $\{i \in I : (a_1(i), \ldots, a_n(i)) \in r^{\mathbf{A}_i}\} \in G$. If the filter G is an ultrafilter over I then we call $\mathbf{A}$ an *ultraproduct* of the family $\langle \mathbf{A}_i : i \in I \rangle$. If $\mathbf{C}$ is an $\mathfrak{L}$–structure and $\mathbf{A}_i = \mathbf{C}$ for all $i \in I$ then we call $\mathbf{A}$ a *filtered power* (resp. *ultrapower*) of $\mathbf{C}$ and we write $\mathbf{A} = \mathbf{C}^I/G$. We denote by $\mathrm{P_F}(\mathcal{K})$ (resp. $\mathrm{P_U}(\mathcal{K})$) the closure of a class $\mathcal{K}$ of $\mathfrak{L}$–structures under filtered products (resp. ultraproducts). The following famous theorem accounts to a large extent for the attention paid by mathematicians to ultraproducts.

0.5.1 ŁOŚ' THEOREM [BS81, Theorem 2.9, p210]

Given $\mathfrak{L}$–structures $\mathbf{A}_i$, $i \in I$, an ultrafilter U over I, any (first-order) $\mathfrak{L}$–formula $\Phi(x_1, \ldots, x_n)$ and any $a_1, \ldots, a_n \in \Pi_{i \in I} \mathbf{A}_i$, we have

$$\Pi_{i \in I} \mathbf{A}_i/U \models \Phi(a_1/U, \ldots, a_n/U) \;\; \textit{iff} \;\; \{i \in I : \mathbf{A}_i \models \Phi(a_1(i), \ldots, a_n(i))\} \in U.$$

Thus, if a first-order formula holds in all members of a class $\mathcal{K}$ of $\mathfrak{L}$–structures then it holds in any ultraproduct of members of $\mathcal{K}$. In particular, first-order definable properties are preserved by ultraproducts. □

0.5.2 THEOREM (Mal'cev, see e.g. [BS81, Theorem 2.23, p218])

The following conditions are equivalent for any class $\mathcal{K}$ of $\mathfrak{L}$–structures:

(i) $\mathcal{K}$ *is a universal Horn class.*

(ii) $\mathcal{K}$ *is closed under the class operators* I, S *and* $\mathrm{P_F}$.

(iii) $\mathcal{K}$ *is closed under the class operators* I, S, P *and* $\mathrm{P_U}$.

(iv) $\mathcal{K} = \mathrm{ISP_F}(\mathcal{K}')$ *for some class* $\mathcal{K}'$ *of* $\mathfrak{L}$*–structures.*

(v) $\mathcal{K} = \mathrm{ISPP_U}(\mathcal{K}')$ *for some class* $\mathcal{K}'$ *of* $\mathfrak{L}$*–structures.* □

First-order structures with equality

Notice that the notion $\mathbf{A} \models \Sigma$ of the satisfaction of (sets of) identities by a universal algebra

$\mathbf{A} = \langle A; O\rangle$ becomes a special case of the notion of satisfaction defined above, provided that we define $\mathfrak{R} = \{\approx\}$, $ar(\approx) = 2$ and consider the structure $\mathbf{A}' = \langle A; O; \{\approx^{\mathbf{A}}\}\rangle$ in place of $\mathbf{A}$, where $\approx^{\mathbf{A}} = I_A$ $(= \{(a,a) : a \in A\})$. We cannot extend the notion $\mathcal{K} \models \Sigma$ similarly, however, because we cannot require all structures $\mathbf{A}$ of a given type with a binary relation symbol $\approx$ to satisfy $\approx^{\mathbf{A}} = I_A$. For this reason, we make the following definition.

A (*first-order*) *language with equality* is a first-order language $\mathfrak{L} = \langle \mathcal{O}, \mathfrak{R}, ar\rangle$ (as above) with the additional property that $\approx \in \mathfrak{R}$ with $ar(\approx) = 2$. In this case an $\mathfrak{L}$-structure $\mathbf{A}$ is called an $\mathfrak{L}$-*structure with equality* if $\approx^{\mathbf{A}} = I_A$. Then identities $t \approx s$ are atomic $\mathfrak{L}$-formulas. If $\mathcal{K}$ is a class of $\mathfrak{L}$-structures *with equality* then $\mathcal{K} \models t \approx s$ means that the algebra reducts of all members of $\mathcal{K}$ satisfy $t \approx s$ in the sense of universal algebra, as expected.

In order to avoid ambiguity here, it is clearly necessary to specify at the outset whether $\mathfrak{L}$ is being considered as a language *with* equality or as a language *without* equality. If we specify that $\mathfrak{L}$ is a language with equality then a class $\mathcal{K}$ of $\mathfrak{L}$-structures (with equality) is said to be *axiomatized by* a set Σ of $\mathfrak{L}$-formulas if $\mathcal{K}$ is the class of all $\mathfrak{L}$-structures with equality that satisfy Σ. In this case, the members of $\mathcal{K}$ are called the *models of* Σ *with equality.* By a *universal Horn class with equality* shall mean a class $\mathcal{K}$ of $\mathfrak{L}$-structures with equality which is axiomatized by some set Σ of universal Horn sentences. The previous theorem, which characterized universal Horn classes (without equality) remains true for universal Horn classes with equality.

Let $\mathfrak{L} = \langle \mathcal{O}, \mathfrak{R}, ar\rangle$ be a language with equality. A quasi-identity Ψ (in the sense of universal algebra) of type $\langle \mathcal{O}, ar \,|\, \mathcal{O}\rangle$ is clearly *logically equivalent to* the universal Horn $\mathfrak{L}$-sentence $\forall x_1 \ldots \forall x_n \Psi$, where $x_1, \ldots, x_n$ are the variables occurring in Ψ. By this we mean that for any $\mathfrak{L}$-structure $\mathbf{A} = \langle A; O; R\rangle$ (with equality), we have that $\mathbf{A} \models \forall x_1 \ldots \forall x_n \Psi$ iff $\langle A; O\rangle \models \Psi$. It follows that quasivarieties are just the universal Horn classes *with equality* where the binary relation symbol $\approx$ is the *only* relation symbol of the language. The last theorem therefore specializes to our earlier characterization of quasivarieties (Theorem 0.4.1).

First-order theories

Thusfar, we have considered only 'semantic' aspects of first-order languages, i.e., notions of meaning and truth (or satisfiability) of formulas by structures. We now deal with the 'syntactic' aspects of first-order languages, i.e., notions of formal provability and derivability of formulas.

We continue to assume that $\mathfrak{L}$ (as defined at the beginning of this section) is a first-order language (without equality) and that a fixed set X of variables is given. We now assume X to be denumerable. With $\mathfrak{L}$ and X, we associate a set of formulas which we call *logical axioms* of $\mathfrak{L}$ over X. These are just all formulas of the following form, where Φ, Φ_1, Φ_2 are $\mathfrak{L}$-formulas, x is a variable and t is a term.

(A1) $\Phi_1 \Rightarrow (\Phi_2 \Rightarrow \Phi_1)$

(A2) $(\Phi \Rightarrow (\Phi_1 \Rightarrow \Phi_2)) \Rightarrow ((\Phi \Rightarrow \Phi_1) \Rightarrow (\Phi \Rightarrow \Phi_2))$

(A3) $((\sim \Phi_2) \Rightarrow \ \sim \Phi_1) \Rightarrow (((\sim \Phi_2) \Rightarrow \Phi_1) \Rightarrow \Phi_2)$

(A4) $(\forall x\, \Phi) \Rightarrow (\Phi[t/x])$ provided that t is free for x in Φ

(A5) $(\forall x\, (\Phi_1 \Rightarrow \Phi_2)) \Rightarrow (\Phi_1 \Rightarrow (\forall x\, \Phi_2))$ provided that x has no free occurrence in Φ_1.

We also define two *inference rules* (of $\mathfrak{L}$ over X), viz.

(MP) from Φ_1 and $\Phi_1 \Rightarrow \Phi_2$, infer Φ_2

(Gen) from Φ infer $\forall x\, \Phi$

(where Φ, Φ_1, Φ_2 are any $\mathfrak{L}$-formulas and x any variable). (MP) stands for *Modus Ponens* and (Gen) for *Generalization*. More formally one could define (MP) and (Gen) as relations on the set of all $\mathfrak{L}$-formulas over X, e.g., (MP) is really $\{(\Phi_1, \Phi_1 \Rightarrow \Phi_2, \Phi_2) : \Phi_1, \Phi_2$ are $\mathfrak{L}$-formulas over $X\}$. We say that Φ_2 is *directly derivable from* Φ_1 *and* $\Phi_1 \Rightarrow \Phi_2$ *by* (MP) and that $\forall x\, \Phi$ is *directly derivable from* Φ *by* (Gen).

A (*first-order*) $\mathfrak{L}$*-theory* T (*without equality*) *over* X (briefly, an $\mathfrak{L}$*-theory*) is defined whenever in addition to the above, a set Pr of formulas, called the *proper axioms of* T, is given. In this case, we can define a notion of provability in T, as follows. Let $\Sigma \cup \{\Phi\}$ be a set of formulas. A sequence $\Phi_1, \ldots, \Phi_n$ of formulas is called a *proof* (or *derivation*) *of* Φ *from* Σ *in* T if Φ_n is Φ and for each $i \in \{1, \ldots, n\}$, Φ_i is a logical axiom or Φ_i is a proper axiom of T or $\Phi_i \in \Sigma$ or Φ_i is directly derivable from Φ_j, Φ_k by (MP) for some positive integers $j, k < i$, or Φ_i is directly derivable from Φ_j by (Gen) for some positive integer $j < k$. We write $\Sigma \vdash_T \Phi$ (or just $\Sigma \vdash \Phi$) if there is a proof of Φ from Σ in T. We read $\Sigma \vdash_T \Phi$ as 'Σ *syntactically entails* (or *proves*) Φ *in* T'. We abbreviate $\emptyset \vdash_T \Phi$ by $\vdash_T \Phi$ (or just $\vdash \Phi$) and if this is true then Φ is called a *theorem* of T. It is quite common to identify T with its set of theorems. If $\mathrm{Pr} = \emptyset$, we call T a *(first-order) predicate calculus.* If Pr is a set of universal Horn sentences, we call T a *universal Horn theory.*

For any logical axiom Φ and any $\mathfrak{L}$-structure $\mathbf{A}$, we have $\mathbf{A} \models \Phi$. If $\mathbf{A} \models \Psi$ for all proper axioms Ψ of T then $\mathbf{A}$ is called a *model* of T. This does not conflict with our previous usage: $\mathbf{A}$ is a model of T if and only if $\mathbf{A}$ is a model of the set Pr of proper axioms of T in the sense defined previously.

0.5.3 VALIDITY AND COMPLETENESS THEOREM OF FIRST-ORDER LOGIC.

Let Φ be an $\mathfrak{L}$-formula and T an $\mathfrak{L}$-theory. Then $\vdash_T \Phi$ if and only if for every model $\mathbf{A}$ of T, we have $\mathbf{A} \models \Phi$. □

The Validity (or Soundness) Theorem of first-order logic is the assertion that all theorems of T are valid in any model of T. The Completeness Theorem of first-order logic asserts the converse: If a formula Φ is valid in every model of T then $\vdash_T \Phi$. Proofs of these metatheorems may be found, for example, in [Men87, Chapter 2].

First-order theories with equality

Suppose that $\mathfrak{L}$ is a language with equality. By a *(first-order) $\mathfrak{L}$-theory T with equality over X* (briefly an *$\mathfrak{L}$-theory with equality*) we mean an $\mathfrak{L}$-theory T (as defined above) with the additional property that for any variables x, y and any formula Φ, the following formulas are theorems of T:

(A6) $\quad x \approx x$

(A7) $\quad x \approx y \Rightarrow (\Phi \Rightarrow (\Phi[y/x]))$ provided that y is free for x in Φ.

In this case, by a *model of T with equality* we shall mean an $\mathfrak{L}$-structure with equality that is a model of T. It is well known that the Validity and Completeness Theorems may be refined to suit this restricted notion of model. In other words, *given an $\mathfrak{L}$-theory T with equality and a formula Φ, we have $\vdash_T \Phi$ iff for every model $\mathbf{A}$ of T with equality, $\mathbf{A} \models \Phi$.*

Chapter One
Deductive Systems

A deductive system can intuitively be understood to consist of a set of axioms and a set of rules of inference which together represent some method of reasoning. To say that one is working within a given deductive system is to say that one's conclusions are reached purely by means of the given axioms and rules of inference. It is important to note that in the study of deductive systems, it is not the meanings of the premisses and conclusions of a particular deduction that matter, only the method of deduction. For example, consider the following simple deduction in number theory:

x is even. x is not even or 3 divides x. From this we conclude that 3 divides x.

If we represent 'x is even' by p and '3 divides x' by q, and we use the symbols $\neg$ for 'not', $\vee$ for 'or' and $\vdash$ for the deduction, then we can rewrite the above as

$$p, (\neg p) \vee q \vdash q.$$

It is this statement that is of interest to us; the meanings of p and q are irrelevant. In another context, we might use the same argument but on completely different propositions. The need for variables such as p and q is evident and, to that end, we shall specify a fixed set P, whose elements we shall call propositional variables. We furthermore specify that P must be countably infinite so that we can never exhaust the set by finite constructions. Since there are no other restrictions on P, the same set will suffice for all deductive systems.

The symbols $\neg$ and $\vee$ of the above example are called propositional connectives. It is connectives like these that form the so-called language of a deductive system; they are used in conjunction with the set of propositional variables to construct 'sentences' such as $(\neg p) \vee q$. The manner in which these 'sentences' are constructed must conform to some rules, or, roughly speaking, a grammar. By this we mean that the symbol $\vee$, for example, operates on two variables; to write $\vee p$ would be grammatically incorrect. It is necessary, therefore, to associate with each propositional connective a natural number, called its arity, that specifies the number of arguments it takes. Thus the arity of $\vee$ is two. The word 'formula' is commonly used in place of 'sentence'. The set of all formulas constructed using the propositional variables together with the given propositional connectives forms the context, or universe, of the deductive system within which deductions are made.

A deductive system (as will be formally defined in Section 1.1) consists of an ordered pair $\langle \boldsymbol{\mathcal{L}}, \vdash \rangle$, where the $\boldsymbol{\mathcal{L}}$ is the language of the deductive system; it consists of the set of connectives along with an arity function. The symbol $\vdash$, called the consequence relation of the deductive system, is a relation between sets of formulas and single formulas that consists of all the possible deductions that can be made. For example, suppose that from a set Γ of formulas we can deduce a formula φ, then Γ is related to φ by $\vdash$, which we denote by $\Gamma \vdash \varphi$. Inherent in this relation is a set of axioms and a set of inference rules. Typically, the axioms are not 'axioms' that one might associate with a certain mathematical theory, such as the associativity axiom $x*(y*z) = (x*y)*z$ in the theory of groups, but are fixed formulas of the method of reasoning. The formula $p \rightarrow (q \rightarrow p)$, for example, could be taken as an axiom. An example of an inference rule is $\langle \{p, p \rightarrow q\}, q\rangle$ which reads: If p and $p \rightarrow q$ ('p implies q'), then we infer q. Thus the inference rules define the permissible methods of deduction.

The generality of our definition encompasses most deductive systems, including the classically motivated ones. Notice that there are no restrictions on what the axioms and inference rules should be, so it is not necessary to demand that a deductive system be modelled on some intuitive method of reasoning. (The language, too, need not consist of propositional connectives with which intuitive meanings are associated, such as $\rightarrow$, $\neg$ or $\wedge$, but might consist of $\bigstar$, $\bullet$ or $\triangleright$, with which no intuitive meaning is associated. Moreover, the set of connectives need not be finite. In fact, it can have an arbitrarily large cardinality).

The aim of this chapter is to introduce deductive systems formally and to define the concepts that will be necessary for later investigations. In Section 1.1 we consider various (equivalent) definitions of a deductive system and show how the classical propositional calculus can be described using this format. Section 1.2 provides a characterization of deductive systems that is based on algebraic notions, namely, lattices and closure operators, and various properties are exhibited. In Section 1.3 we briefly introduce the idea of matrix models and matrix semantics; they provide a method of algebraization of deductive systems that will be explored in greater depth in later chapters. Further examples of deductive systems are introduced in Section 1.4, many of which will be referred to later. From Section 1.5 onwards we work in the more general setting of k-deductive systems. This requires extending the notions introduced in Sections 1.1, 1.2 and 1.3. This is done in Sections 1.5 and 1.6, where these notions are studied in more depth. The Leibniz equivalence relation defined in Section 1.7 is an inherent notion associated with deductive systems, and as such it plays an integral part in our investigations. This is first demonstrated in Section 1.8. There we investigate various operations on matrix models, and in particular their connection with the Leibniz operator. The question of equivalence of two deductive systems is the topic of Section 1.9. The final section is a brief look at the possibility of using first-order methods to define a deductive system, as opposed to relying on second-order notions such as $\vdash$ that are used in the preceding sections.

The material in this and the following chapters is, for the most part, taken from certain papers of Blok and Pigozzi, especially [BP89a], [BP89b], [BP88] and [BP92].

1.1 FORMULA ALGEBRAS AND DEDUCTIVE SYSTEMS

Let $P = \{\mathrm{p}_1, \mathrm{p}_2, \mathrm{p}_3, \ldots\}$ be a fixed, countably infinite set. The elements of P are called *propositional variables.* The letters $p, q, r, s, p_i, q_i, r_i, s_i$, etc., will be used as metavariables ranging over the set of propositional variables. Let $\mathcal{L}$ be a set of symbols, say $\mathcal{L} = \{f_i;\, i \in I\}$, and let ar be a function that assigns a natural number to every $f_i \in \mathcal{L}$, i.e., $ar : \mathcal{L} \to \omega$. Denote by $\boldsymbol{\mathcal{L}}$ the ordered pair $\langle \mathcal{L}, ar \rangle$. We call $\boldsymbol{\mathcal{L}}$ a *propositional language* and the elements of $\mathcal{L}$ are called *propositional connectives.* The function ar is called the *arity function* of $\boldsymbol{\mathcal{L}}$ and, for every $f_i \in \mathcal{L}$, we call $ar(f_i)$ the *arity* (or *rank*) of f_i. If $ar(f) = n$ we call f_i *n-ary.* We also use the following names: If f_i has arity 0 then f_i is called a *constant* or a *nullary connective*; if f_i has arity 1 then f_i is called *unary,* and if f_i has arity 2 then f_i is called *binary.*

Classical examples of propositional connectives include $\to$, $\wedge$, $\neg$ and $\perp$, understood to mean 'implies', 'and', 'not' and 'falsum', respectively. Note, however, that in this definition it is not necessary for there to be any intuitive meaning assigned to the connectives; every conceivable sort of propositional connective is allowed. Note also that there can be as many propositional connectives as we like, as there is no restriction on the cardinality of the set I (hence on $\mathcal{L}$). The arity function specifies how many arguments a connective should take. For example, we usually have $ar(\bot) = 0$, $ar(\neg) = 1$ and $ar(\rightarrow) = 2$. Strictly speaking, when considering binary connectives such as $\rightarrow$ and $\vee$ we should use the functional notation $\rightarrow(p,q)$ and $\vee(p,q)$, but we usually write $p \rightarrow q$ and $p \vee q$ for convenience. When using this notation, brackets are used liberally to avoid all possible ambiguity.

The set $Fm_{\mathcal{L}}$ of all *$\mathcal{L}$-formulas* is constructed in the following recursive way:

$p \in Fm_{\mathcal{L}}$ for every $p \in P$,

if $\varphi_1, \ldots, \varphi_n \in Fm_{\mathcal{L}}$ and $f \in \mathcal{L}$ with $ar(f) = n$, then $f(\varphi_1, \ldots, \varphi_n) \in Fm_{\mathcal{L}}$.

Note that when $n = 0$, the second defining condition implies that all constants are elements of $Fm_{\mathcal{L}}$. We usually denote $\mathcal{L}$-formulas by lower-case Greek letters, e.g., φ, ψ, ϑ, while sets of $\mathcal{L}$-formulas will usually be denoted by upper case Greek letters, e.g., Γ, Δ, Π. Note that every $\mathcal{L}$-formula contains occurrences of only a finite set of propositional variables. Suppose, for example, that f, g, h are propositional connectives that have arities 3, 2 and 1, respectively. Then the only propositional variables that occur in the $\mathcal{L}$-formula $\varphi = f(g(p,q), h(p), g(r,p))$ are p, q, r. To emphasise this, we would write $\varphi = \varphi(p,q,r)$, and in the general case we would write $\psi = \psi(p_1, \ldots, p_n)$ if ψ is an $\mathcal{L}$-formula whose propositional variables are among $p_1, \ldots, p_n$. In particular, we do not insist that each of the stated variables occurs in the formula. We often use $\overline{p}$ or $\overline{\varphi}$ to denote a sequence (specified or unspecified) of variables or formulas.

The language $\mathcal{L}$ can be considered to be a *type* of algebras (see Section 0.2), hence the $\mathcal{L}$-formulas are just the terms of type $\mathcal{L}$ over the set P (considered as a set of variables). For each $f \in \mathcal{L}$ with $ar(f) = m$, define $f^{\mathbf{Fm}}(\varphi_1, \ldots, \varphi_m) = f(\varphi_1, \ldots, \varphi_m)$. Then $f^{\mathbf{Fm}}$ is an operation on $Fm_{\mathcal{L}}$. Thus the ordered pair $\langle Fm_{\mathcal{L}}, \mathcal{L}^{\mathbf{Fm}} \rangle$ forms an algebra, where $\mathcal{L}^{\mathbf{Fm}} = \{f^{\mathbf{Fm}};\ f \in \mathcal{L}\}$, which is called the *formula algebra (over $\mathcal{L}$)* and is denoted by $\mathbf{Fm}_{\mathcal{L}}$. Note that since P is a fixed set, all that is needed to define $\mathbf{Fm}_{\mathcal{L}}$ is the propositional language $\mathcal{L}$; *the algebra* $\mathbf{Fm}_{\mathcal{L}}$ *can then be considered the absolutely free $\mathcal{L}$-algebra generated by the set P.* This immediately implies that $\mathbf{Fm}_{\mathcal{L}}$ has the universal mapping property for $\mathcal{L}$ over P (Theorem 0.2.10).

In future we shall usually drop the word 'propositional' when talking about propositional languages, propositional variables and propositional connectives. Furthermore, a (propositional) language $\mathcal{L} = \langle \mathcal{L}, ar \rangle$ will simply be denoted by $\mathcal{L}$, so $\mathcal{L}$ will stand for both the language and the set of connectives (and the set of operations on the formula algebra). No confusion will arise. If the language is fixed and understood, then the prefix $\mathcal{L}$- will be dropped from $\mathcal{L}$-formula, and we shall write Fm for $Fm_{\mathcal{L}}$, and $\mathbf{Fm}$ for $\mathbf{Fm}_{\mathcal{L}}$.

By a *substitution* we mean a homomorphism $\sigma : \mathbf{Fm} \rightarrow \mathbf{Fm}$, i.e., a map from Fm into Fm such that $\sigma\varphi(p_1, \ldots, p_n) = \varphi(\sigma p_1, \ldots, \sigma p_n)$ for all $\varphi(p_1, \ldots, p_n) \in Fm$. We call $\sigma\varphi$ a substitution instance of φ. For example, if $\sigma p = r$ and $\sigma q = s \rightarrow r$, then $\sigma(p \rightarrow (q \rightarrow p)) = (\sigma p \rightarrow (\sigma q \rightarrow \sigma p)) = (r \rightarrow ((s \rightarrow r) \rightarrow r))$. Notice that since the formula algebra $\mathbf{Fm}$ is freely generated by P, every substitution is uniquely determined by its restriction to P. Conversely, every map $\tau : P \rightarrow Fm$ extends to a substitution $\sigma_\tau : \mathbf{Fm} \rightarrow \mathbf{Fm}$ by the universal mapping property. The map σ_τ satisfies $\sigma_\tau(\varphi(p_1, \ldots, p_n)) = \varphi(\tau p_1, \ldots, \tau p_n)$ for any $\varphi(p_1, \ldots, p_n) \in Fm$. Thus one can think of a substitution as a renaming of the variables by other variables or by formulas. We shall briefly consider some properties of substitutions as they play a large role in the thesis.

For any substitution $\sigma : \mathbf{Fm} \rightarrow \mathbf{Fm}$, we have that $\sigma^{-1}(P) \subseteq P$: If $\varphi(p_1, \ldots, p_n) \in \sigma^{-1}(P)$, then $\sigma(\varphi(p_1, \ldots, p_n)) = \varphi(\sigma p_1, \ldots, \sigma p_n) \in P$, so, $\varphi(p_1, \ldots, p_n)$ is p_i for some $i \leq n$, hence $\varphi(p_1, \ldots, p_n) \in P$. It follows that if σ is a surjective substitution, then $P = \sigma(\sigma^{-1}(P)) \subseteq \sigma(P)$.

If $\tau : P \rightarrow Fm$ is "surjective on P", i.e., $\tau(P) \supseteq P$, then $\sigma_\tau : \mathbf{Fm} \rightarrow \mathbf{Fm}$ is a surjective substitution. For suppose $\varphi(p_1, \ldots, p_n) \in Fm$. By assumption, there exist $q_i \in P$, $i \leq n$, such that $p_i = \tau q_i$, hence $\varphi(p_1, \ldots, p_n) = \varphi(\tau q_1, \ldots, \tau q_n) = \sigma_\tau \varphi(q_1, \ldots, q_n) \in \sigma_\tau(Fm)$, hence σ_τ is surjective. If $\tau : P \rightarrow Fm$ is injective, i.e., τ is a one-to-one map, then so is $\sigma_\tau : \mathbf{Fm} \rightarrow \mathbf{Fm}$. To see this, let $\varphi(p_1, \ldots, p_n), \psi(q_1, \ldots, q_m) \in Fm$. Then $\sigma_\tau \varphi(p_1, \ldots, p_n) = \sigma_\tau \psi(q_1, \ldots, q_m)$ iff $\varphi(\tau p_1, \ldots, \tau p_n) = \psi(\tau q_1, \ldots, \tau q_m)$. Since we are working in an absolutely free algebra, this holds if and only if $\varphi = \psi$, $n = m$ and $\tau p_i = \tau q_i$ for all $i \leq n$, hence $p_i = q_i$ for each $i \leq n$, as τ is injective.

The previous paragraph implies the result that if $\tau : P \rightarrow P$ is a permutation (an injective and surjective map), then σ_τ will be an automorphism of $\mathbf{Fm}$ (an injective and surjective homomorphism). The converse also holds, namely, if σ is an automorphism of $\mathbf{Fm}$, then $\sigma \mid P$, the restriction of σ to P, is a permutation of P. To see this, we need only show that $\sigma(P) = P$. So, suppose $p \in P$ and $\sigma p = f(\varphi_1,\ldots,\varphi_m) \notin P$ where $\varphi_1,\ldots,\varphi_m \in Fm$ and $f \in \mathcal{L}$. Then there exist $\psi_1,\ldots,\psi_m \in Fm$ such that $\varphi_i = \sigma\psi_i$ for $i \leq m$, hence $\sigma f(\psi_1,\ldots,\psi_m) = f(\sigma\psi_1,\ldots,\sigma\psi_m) = f(\varphi_1,\ldots,\varphi_m)$. Since σ is injective, this implies that $p = f(\psi_1,\ldots,\psi_m)$, contradicting the assumption that $p \in P$. Thus $\sigma(P) \subseteq P$. We know that $P \subseteq \sigma(P)$ since σ is surjective, hence $\sigma(P) = P$.

Let $\mathcal{L}$ be a fixed language. A *(finitary) clause (over $\mathcal{L}$)* is a pair $\langle \Delta, \psi \rangle$ where Δ is a finite set of formulas and ψ is a single formula. For example, the clause $\langle \{p, p \to q\}, q \rangle$ represents *modus ponens*. A formula φ is said to be *directly derivable* from a set Γ of formulas by the clause $\langle \Delta, \psi \rangle$ if there exists a substitution σ such that $\sigma(\Delta) \subseteq \Gamma$ and $\sigma\psi = \varphi$. The formal definition of a deductive system goes as follows:

1.1.1 DEFINITION

Let $\mathcal{L}$ be a fixed language. A *deductive system S (over $\mathcal{L}$)* is determined by a (possibly infinite) set Ir of clauses which we call the *inference rules of S*, and a (possibly infinite) set Ax of formulas, which we call the *axioms of S*. It consists of a pair $S = \langle \mathcal{L}, \vdash_S \rangle$ where $\vdash_S$ is a relation between sets of formulas and single formulas that is defined by the following condition:

For every $\Gamma \subseteq Fm$ and every $\varphi \in Fm$,

> $\Gamma \vdash_S \varphi$ if and only if φ is contained in the smallest set of formulas that includes Γ together with all substitution instances of the axioms of S, and is closed under direct derivability by the inference rules of S.

Deductive systems are sometimes referred to as *logical systems* or simply *logics*. The relation $\vdash_S$ is called the *consequence relation of S*. If $\emptyset \vdash_S \varphi$, then φ is called a *theorem of S*. In this case we just write $\vdash_S \varphi$. (One can read "$\Gamma \vdash_S \varphi$" as "Γ *entails* φ", or "*φ is a consequence of* Γ".) Let $\Delta \subseteq Fm$. A *derivation from Δ* is a nonempty finite sequence of formulas $\psi_1, \psi_2, \ldots, \psi_n$ such that, for each $i \leq n$, one of the following conditions holds:

(i) $\psi_i \in \Delta$ or $\psi_i = \sigma\varphi$ for some $\varphi \in Ax$ and some substitution σ,

(ii) there exists a rule $\langle \Gamma, \varphi \rangle \in Ir$ and a substitution σ such that $\psi_i = \sigma\varphi$ and $\sigma\vartheta \in \{\psi_1,\ldots,\psi_{i-1}\}$ for each $\vartheta \in \Gamma$.

Moreover, we call $\psi_1, \psi_2, \ldots, \psi_n$ a *derivation of φ from Δ* if $\psi_n = \varphi$.

As an example of a deductive system defined in this way, consider the Classical Propositional Calculus, denoted **CPC**. The language is $\mathcal{L} = \{\rightarrow, \wedge, \vee, \neg, \bot, \mathrm{T}\}$, where $\rightarrow, \wedge, \vee$ are binary connectives, $\neg$ is a unary connective and $\bot$ and T are constants. For simplicity we shall write $\neg p$ for $\neg(p)$, $p \rightarrow q$ for $\rightarrow(p,q)$, $p \wedge q$ for $\wedge(p,q)$ and $p \vee q$ for $\vee(p,q)$. Let p, q, r be fixed, but arbitrary propositional variables. The set Ax of axioms of **CPC** consists of (C_1) to (C_{11}) below:

(C_1) $\quad p \rightarrow (q \rightarrow p)$,

(C_2) $\quad (p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$,

(C_3) $\quad ((\neg q) \rightarrow (\neg p)) \rightarrow (p \rightarrow q)$.

Note that the only connectives that occur thusfar in Ax are $\rightarrow$ and $\neg$. In fact, the other connectives are definable in terms of $\rightarrow$ and $\neg$. The remaining axioms serve only to define $\wedge, \vee, \bot$ and T in this way:

(C_4) $\quad (p \wedge q) \rightarrow p$,

(C_5) $\quad (p \wedge q) \rightarrow q$,

(C_6) $\quad (r \rightarrow p) \rightarrow ((r \rightarrow q) \rightarrow (r \rightarrow (p \wedge q)))$,

(C_7) $\quad p \rightarrow (p \vee q)$,

(C_8) $\quad q \rightarrow (p \vee q)$,

(C_9) $\quad (p \rightarrow r) \rightarrow ((q \rightarrow r) \rightarrow ((p \vee q) \rightarrow r))$,

(C_{10}) $\quad \bot \rightarrow p$,

(C_{11}) $\quad p \rightarrow \mathrm{T}$.

The set Ir of inference rules of **CPC** consists only of modus ponens, namely

(MP) $\quad \langle \{p, p \rightarrow q\}, q \rangle$.

The relation $\vdash_S$, associated with a deductive system S, can also be defined recursively in the following way: For every $\Gamma \subseteq Fm$ and every $\varphi \in Fm$,

(i) if $\varphi \in \Gamma$ or φ is a substitution instance of an axiom of S then $\Gamma \vdash_S \varphi$,

(ii) if $\Delta \subseteq Fm$ and φ is directly derivable from Δ by an inference rule of S and for every $\psi \in \Delta$, $\Gamma \vdash_S \psi$, then $\Gamma \vdash_S \varphi$.

From (i) we can deduce immediately that $\vdash_S \varphi$ for every axiom φ of S, and from (ii) we can deduce immediately that $\Delta \vdash_S \psi$ for each inference rule $\langle \Delta, \psi \rangle$ of S. When defining inference rules of S in future, we shall often use $\Delta \vdash_S \psi$ to abbreviate the statement that $\langle \Delta, \psi \rangle$ is an inference rule of S. Similarly, we sometimes write $\vdash_S \varphi$ when defining φ as an axiom of S.

From the above characterization one can easily see that the following conditions are satisfied by the consequence relation $\vdash_S$. For all $\Gamma, \Delta \subseteq Fm$ and $\varphi \in Fm$,

(1.1.1) $\varphi \in \Gamma$ implies $\Gamma \vdash_S \varphi$,

(1.1.2) $\Gamma \vdash_S \varphi$ and $\Gamma \subseteq \Delta$ implies $\Delta \vdash_S \varphi$,

(1.1.3) $\Gamma \vdash_S \varphi$ and $\Delta \vdash_S \psi$ for every $\psi \in \Gamma$ implies $\Delta \vdash_S \varphi$.

It is also true that $\vdash_S$ is *finitary*, by which we mean that $\vdash_S$ satisfies the following property:

(1.1.4) $\Gamma \vdash_S \varphi$ implies there exists some finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash_S \varphi$.

To see that this is true, suppose that $\Gamma \vdash_S \varphi$. We shall consider each of the cases (i) and (ii) above. If $\varphi \in \Gamma$ then $\{\varphi\} \vdash_S \varphi$, and $\{\varphi\}$ is obviously a finite subset of Γ. If φ is a substitution instance of an axiom of S, then $\emptyset \vdash_S \varphi$ and $\emptyset$ is obviously an empty subset of Γ. Next, suppose that there exists a set $\Delta \subseteq Fm$ such that φ is directly derivable from Δ by an inference rule of S and for every $\psi \in \Delta$, we have $\Gamma \vdash_S \psi$ and assume inductively that for each $\psi \in \Delta$ there exists a finite $\Gamma_\psi \subseteq \Gamma$ such that $\Gamma_\psi \vdash_S \psi$. Since φ is directly derivable from Δ, we can assume without loss of generality that Δ is finite, hence the set $\Gamma' = \bigcup \{\Gamma_\psi ; \psi \in \Delta\}$ is a finite subset of Γ. It follows from (1.1.2) that $\Gamma' \vdash_S \psi$ for all $\psi \in \Delta$. Now, since φ is directly derivable from Δ, condition (ii) implies that $\Gamma' \vdash_S \varphi$.

A further property of $\vdash_S$ is that it is *structural*, namely

(1.1.5) $\Gamma \vdash_S \varphi$ implies $\sigma(\Gamma) \vdash_S \sigma\varphi$ for every substitution σ.

To see this, we again proceed inductively, considering each of the cases (i) and (ii). Suppose $\Gamma \vdash_S \varphi$, and σ is a substitution. If $\varphi \in \Gamma$ or φ is a substitution instance of an axiom of S, then (1.1.5) holds, trivially. Next, suppose that there exists a set $\Delta \subseteq Fm$ such that φ is directly derivable from Δ by an inference rule of S and for every $\psi \in \Delta$, we have $\Gamma \vdash_S \psi$ and $\sigma(\Gamma) \vdash_S \sigma\psi$. If φ is directly derivable from Δ by the inference rule $\langle \Pi, \vartheta \rangle$ of S, then there is a substitution τ such that $\tau(\Pi) \subseteq \Delta$ and $\tau\vartheta = \varphi$. The composition $\sigma \circ \tau$ is obviously also a substitution and, moreover, it satisfies $(\sigma \circ \tau)(\Pi) = \sigma(\tau(\Pi)) \subseteq \sigma(\Delta)$ and $(\sigma \circ \tau)\vartheta = \sigma\tau\vartheta = \sigma\varphi$, hence $\sigma\varphi$ is directly derivable from $\sigma(\Delta)$ by $\langle \Pi, \vartheta \rangle$. By (ii), we have that $\sigma(\Gamma) \vdash_S \sigma\varphi$.

It is reasonably intuitive that a deductive system should be finitary. The structurality is an important property in that it says that a rearrangement of the variables does not affect the relation $\vdash_S$. This is certainly to be expected as the set P of variables is an arbitrary set. Los and Suszko [LS58] prove the following converse of the above results: If $\vdash$ is a relation between sets of formulas and single formulas satisfying (1.1.1) to (1.1.5), then $\vdash$ is the consequence relation of some deductive system. Consequently, a deductive system over a set of formulas can be defined by a relation which satisfies the properties (1.1.1) to (1.1.5) This method of defining a deductive system will be utilised in later sections and the relation $\vdash$ will be called a consequence relation. Note that in this case there are no predefined sets of inference rules and axioms of S.

Given a deductive system $S = \langle \mathfrak{L}, \vdash_S \rangle$, there are other deductive systems that one can associate with S, namely extensions, subsystems and fragments. By an *extension of S* we mean a deductive system $S' = \langle \mathfrak{L}', \vdash_{S'} \rangle$, where $\mathfrak{L}' \supseteq \mathfrak{L}$, that satisfies the following condition: For all $\Gamma \subseteq Fm_{\mathfrak{L}}$ and $\varphi \in Fm_{\mathfrak{L}}$, $\Gamma \vdash_S \varphi$ implies $\Gamma \vdash_{S'} \varphi$. In other words, $\vdash_S \subseteq \vdash_{S'}$. If S' has the same language as S and S' is obtained purely by the addition of formulas to the set of axioms of S, then S' is called an *axiomatic extension* of S. If S' is an extension of S, we also call S a *subsystem* of S'.

Suppose $\varphi(p_1, \ldots, p_n)$ is an $\mathfrak{L}$-formula in n variables. We can consider φ as a connective of arity n. Strictly speaking, φ need not be a connective of $\mathfrak{L}$, so we call it an *abbreviation* of $\mathfrak{L}$. For example, in the classical propositional calculus, a common abbreviation is the symbol $\leftrightarrow$; the binary formula $p \leftrightarrow q$ is defined by

$$p \leftrightarrow q = (p \rightarrow q) \wedge (q \rightarrow p).$$

Let $\mathfrak{L}'$ be a set consisting of connectives of $\mathfrak{L}$ and/or abbreviations of $\mathfrak{L}$. Then we can consider $\mathfrak{L}'$ as a propositional language. Let S' be the pair $\langle \mathfrak{L}', \vdash_{S'} \rangle$, where $\vdash_{S'}$ is the restriction of $\vdash_S$ to $\mathfrak{L}'$ in the sense that

$$\Gamma \vdash_{S'} \varphi \text{ if and only if } \Gamma \subseteq Fm_{\mathfrak{L}'}, \varphi \in Fm_{\mathfrak{L}'} \text{ and } \Gamma \vdash_S \varphi.$$

It is easy to see that the properties (1.1.1) to (1.1.5) hold when $\vdash_S$ is replaced by $\vdash_{S'}$ and $Fm_{\mathfrak{L}}$ by $Fm_{\mathfrak{L}'}$, hence S' is a deductive system; it is called the *$\mathfrak{L}'$-fragment of S*. If $\mathfrak{L}'$ has only one element, f say, then we usually write S_f for the $\mathfrak{L}'$-fragment of S. The consequence relation $\vdash_{S'}$ of the $\mathfrak{L}'$-fragment of S consists of all deductions $\Gamma \vdash_S \varphi$ for which the formulas contain only the elements of $\mathcal{L}'$ (and the propositional variables). It is not always clear how to produce from S and $\mathcal{L}'$ an axiomatization for S', nor a finite axiomatization for S' if one exists. In fact, finding a finite axiomatization can be a very difficult problem.

1.2 THE LATTICE OF THEORIES

The properties (1.1.1) to (1.1.3) of a deductive system suggest that some sort of closure operator is inherent in the system (see Section 0.1). For a fixed language $\mathcal{L}$ and a deductive system S, define a map $\mathrm{Cn}_S : \mathcal{P}(Fm) \to \mathcal{P}(Fm)$ by

$$\mathrm{Cn}_S(\Gamma) = \{\varphi \in Fm\,;\, \Gamma \vdash_S \varphi\}.$$

Cn_S is called the *consequence operator* of S and is easily seen to satisfy the following properties, which correspond to the properties (1.1.1) to (1.1.5), respectively,

(1.2.1) $\Gamma \subseteq \mathrm{Cn}_S(\Gamma)$,

(1.2.2) $\Gamma \subseteq \Delta$ implies $\mathrm{Cn}_S(\Gamma) \subseteq \mathrm{Cn}_S(\Delta)$,

(1.2.3) $\mathrm{Cn}_S(\mathrm{Cn}_S(\Gamma)) \subseteq \mathrm{Cn}_S(\Gamma)$,

(1.2.4) $\mathrm{Cn}_S(\Gamma) \subseteq \bigcup \mathrm{Cn}_S(\Gamma')$ where the union is taken over all finite subsets Γ' of Γ,

(1.2.5) $\sigma(\mathrm{Cn}_S(\Gamma)) \subseteq \mathrm{Cn}_S(\sigma(\Gamma))$ for every substitution σ.

By (1.2.1) to (1.2.4), Cn_S is an algebraic closure operator on the complete lattice $\langle \mathcal{P}(Fm), \subseteq \rangle$ (as defined in Section 0.1). Conversely, a function $\mathrm{Cn} : \mathcal{P}(Fm) \to \mathcal{P}(Fm)$ that satisfies (1.2.1) to (1.2.5) gives rise to a consequence relation $\vdash$ defined, for $\Gamma \subseteq Fm$ and $\varphi \in Fm$, by

$$\Gamma \vdash \varphi \text{ if and only if } \varphi \in \mathrm{Cn}(\Gamma).$$

It is easily checked that this is indeed a consequence relation. It is possible, therefore, to define a deductive system as a pair $\langle \mathcal{L}, \mathrm{Cn} \rangle$ where Cn satisfies (1.2.1) to (1.2.5). A set $T \subseteq Fm$ is called a *theory of* S, or an S*-theory* if, for every $\varphi \in Fm$,

$$T \vdash_S \varphi \text{ implies } \varphi \in T.$$

If $\Gamma \subseteq Fm$, then (1.2.3) implies that $\mathrm{Cn}_S(\Gamma)$ is an S-theory. We call $\mathrm{Cn}_S(\Gamma)$ the S*-theory generated by* Γ and, if this is equal to $\mathrm{Cn}_S(\Delta)$ for some finite $\Delta \subseteq Fm$, we say that this S-theory is *finitely generated.* If T is an S-theory, then $\mathrm{Cn}_S(T) = T$. The set of all S-theories, which is denoted by ThS, is therefore the range of the algebraic closure operator Cn_S. It follows that $\mathbf{Th}S = \langle \mathrm{Th}S ; \subseteq \rangle$ is an algebraic lattice in which (arbitrary) infima are just intersections and joins are defined by

$$\bigvee^{\mathbf{Th}S}\{T_i ; i \in I\} = \bigcap\{R \in \mathrm{Th}S ; T_i \subseteq R \text{ for all } i \in I\}$$
$$= \mathrm{Cn}_S(\bigcup\{T_i ; i \in I\})$$

for any family $\{T_i ; i \in I\}$ of S-theories. In particular, the binary join operation of $\mathbf{Th}S$ is defined by:

$$T \vee^{\mathbf{Th}S} U = \bigcap\{R \in \mathrm{Th}S ; T \cup U \subseteq R\} = \mathrm{Cn}_S(T \cup U).$$

The largest element of ThS is the set Fm of all formulas, and the smallest is $\mathrm{Cn}_S(\emptyset)$, the set of all theorems of S. Some properties of $\mathbf{Th}S$ are given in the following lemma, which rephrases some of the above observations in the light of Corollary 0.1.6. A more general result will be proved in detail later (Lemma 1.6.4).

1.2.1 LEMMA [BP89a, Lemma 1.1]

Let S be a deductive system.

(i) ThS *is closed under unions of nonempty upwardly directed sets (i.e., for every nonempty upwardly directed family $\{T_i ; i \in I\}$ of S-theories, $\bigcup\{T_i ; i \in I\} \in \mathrm{Th}S$).*

(ii) The lattice $\mathbf{Th}S$ *is algebraic.*

(iii) The compact elements of $\mathbf{Th}S$ *coincide with the finitely generated S-theories.* □

The consequence operator Cn_S is recovered from the lattice $\mathbf{Th}S$ by

$$\mathrm{Cn}_S(\Gamma) = \bigwedge\{T \in \mathrm{Th}S ; \Gamma \subseteq T\}.$$

It is possible, therefore, to define a deductive system as any one of the pairs $\langle \mathfrak{L}, \vdash_S \rangle$, $\langle \mathfrak{L}, \mathrm{Cn}_S \rangle$ or $\langle \mathfrak{L}, \mathbf{Th}S \rangle$, since they are all equivalent.

Consider the lattice of S-theories again. If T is an S-theory and σ is a substitution, then it is not necessarily true that $\sigma(T)$ will be an S-theory. We shall give an example exhibiting this in the next section. What we can say is that if σ is an automorphism (recall that a substitution is a homomorphism) then $\sigma(T) \in \mathrm{Th}S$. This holds because σ^{-1} is then also an automorphism and thus, by structurality,

$$\sigma^{-1}(\mathrm{Cn}_S(\sigma(T))) \subseteq \mathrm{Cn}_S(\sigma^{-1}(\sigma(T))) \subseteq \mathrm{Cn}_S(T) = T,$$

i.e.,

$$\mathrm{Cn}_S(\sigma(T)) \subseteq \sigma(T).$$

Because this is not true for arbitrary substitutions we define, for each substitution σ, the function $\sigma_S : \mathrm{Th}S \to \mathrm{Th}S$ by $\sigma_S(T) = \mathrm{Cn}_S(\sigma(T))$ for each S-theory T. The next lemma summarises the necessary facts about this function. Again, the lemma is a special case of a later lemma (Lemma 1.5.3, in fact).

1.2.2 LEMMA [BP89a, Lemma 1.2]

Let S be a deductive system.

(i) $\mathrm{Th}S$ *is closed under inverse substitutions (i.e., $\sigma^{-1}(T) \in \mathrm{Th}S$ for every $T \in \mathrm{Th}S$ and every substitution σ).*

(ii) $\sigma_S(\mathrm{Cn}_S(\Gamma)) = \mathrm{Cn}_S(\sigma(\Gamma))$ *for all $\Gamma \subseteq Fm$ and every substitution σ.*

(iii) σ_S *is a join-continuous mapping of* $\mathrm{Th}S$ *into itself (i.e., $\sigma_S\left(\bigvee^{\mathbf{Th}S}_{i \in I} T_i\right) = \bigvee^{\mathbf{Th}S}_{i \in I} \sigma_S(T_i)$ for every family $\{T_i ; i \in I\}$ of S-theories and every substitution σ).* □

1.3 MATRIX SEMANTICS

In this section we begin our investigation into the algebraization of deductive systems. The aim is to translate concepts and results about deductive systems into corresponding ones about algebras. We begin by interpreting the formulas of a deductive system as elements of an algebra. If we interpret the propositional variables as elements of an algebra and the connectives as operations of the same algebra, then a formula (i.e., a grammatically correct formal string of variables and connectives) can be interpreted as the evaluation of a term acting on elements of the algebra, hence as an element of the algebra itself. To be able to do this, it is necessary that the operations of the algebra correspond to the connectives of the deductive system, so only certain types of algebras will do. Suppose $\mathcal{L}$ is a fixed language. We can consider $\mathcal{L}$ to be a *type* of (universal) algebras (see Section 1.1). Thus an *$\mathcal{L}$-algebra* is a structure $\mathbf{A} = \langle A, f^{\mathbf{A}} \rangle_{f \in \mathcal{L}}$, where A is a nonempty set called the *universe* of $\mathbf{A}$, and $f^{\mathbf{A}}$ is an operation on A of rank $ar(f)$ for each connective $f \in \mathcal{L}$. Let $\mathbf{A}$ be an $\mathcal{L}$-algebra, and recall that P is the (denumerable) set of all variables. Let $\bar{a} \in A^P$. We call $\bar{a}$ an *interpretation of P in A*. The set A^P is therefore the set of all interpretations of P in A. Let $Q \subseteq P$ and let $\overline{a} \in A^Q$. We call $\overline{a}$ an *interpretation of Q in A*. If $\varphi = \varphi(p_1, \ldots, p_m)$ is an $\mathcal{L}$-formula and $p_1, \ldots, p_m \in Q \subseteq P$ then, for $\overline{a} \in A^Q$ such that $\overline{a}(p_i) = a_i$ for each $i \leq m$, we write $\varphi^{\mathbf{A}}(\overline{a}) = \varphi^{\mathbf{A}}(a_1, \ldots, a_m)$ to represent the element of A resulting from the evaluation of the formula φ in $\mathbf{A}$ where each p_i is replaced by a_i and each connective f occurring in φ is replaced by the corresponding operation $f^{\mathbf{A}}$ of $\mathbf{A}$. In this case we also call $a_1, \ldots, a_m$ an *interpretation of $p_1, \ldots, p_m$ (respectively) in A*.

Recall that the formula algebra $\mathbf{Fm} = \langle Fm, \mathcal{L} \rangle$ is an absolutely free $\mathcal{L}$-algebra. Since an $\mathcal{L}$-algebra $\mathbf{A}$ has the same type as $\mathbf{Fm}$, we can define the set $\mathrm{Hom}(\mathbf{Fm}, \mathbf{A})$ of all homomorphisms from $\mathbf{Fm}$ into $\mathbf{A}$. Moreover, since $\mathbf{Fm}$ is freely generated by P, we can identify $\mathrm{Hom}(\mathbf{Fm}, \mathbf{A})$ with the set A^P of all maps from P into A. Thus we can identify the set of all interpretations of P in A with the set $\mathrm{Hom}(\mathbf{Fm}, \mathbf{A})$.

In the 'algebraization' of the Classical Propositional Calculus, the correctness of the statement $\Gamma \vdash_{\mathbf{CPC}} \varphi$ can be decided in the Boolean algebra $\mathbf{2} = \langle \{0,1\}\,;\ \wedge^{\mathbf{2}}, \vee^{\mathbf{2}}, \neg^{\mathbf{2}}, 0, 1 \rangle$ in the following way (note that we are using $\neg$ for $'$): First, define the binary operation $\rightarrow^{\mathbf{2}}$ on $\{0,1\}$ by $a \rightarrow^{\mathbf{2}} b = (\neg^{\mathbf{2}} a) \vee^{\mathbf{2}} b$ $(a, b \in \{0,1\})$. Then $\Gamma \vdash_{\mathbf{CPC}} \varphi$ if, for every interpretation $\overline{c}$ of the variables of $\Gamma \cup \{\varphi\}$ in $\{0,1\}$,

$$\psi^{\mathbf{2}}(\overline{c}) = 1 \text{ for all } \psi \in \Gamma \text{ implies } \varphi^{\mathbf{2}}(\overline{c}) = 1.$$

This is the Completeness Theorem of Classical Propositional Calculus. The Validity Theorem of Classical Propositional Calculus asserts that the converse of the above is also true (see Section 1.4).

We can use the Validity Theorem to prove that the set $\mathrm{Th}\,\mathbf{CPC}$, of all theories of $\mathbf{CPC}$, is not closed under substitutions. Let $T = \mathrm{Cn}_{\mathbf{CPC}}(\{p, q \rightarrow r\})$. Then T is a $\mathbf{CPC}$-theory. Let p, q, r be propositional variables such that $p \neq q$, and σ a substitution such that $\sigma p = \sigma q = p$, $\sigma r = r$ and $\sigma s \neq r$ for all other $s \in P$. Then $p = \sigma p \in \sigma(T)$ and $p \rightarrow r = \sigma(q \rightarrow r) \in \sigma(T)$, hence $r \in \mathrm{Cn}_{\mathcal{S}}(\sigma(T))$ by modus ponens. Now, interpret p as 1 and q and r as 0 in $\{0,1\}$, the universe of the two-element Boolean algebra $\mathbf{2}$. Since $0 \rightarrow^{\mathbf{2}} 0 = 1$, the Validity Theorem then implies that $p, q \rightarrow r \nvdash_{\mathbf{CPC}} r$, so $r \notin T$, hence $r \notin \sigma(T)$ by definition of σ. Thus $\sigma(T) \neq \mathrm{Cn}_{\mathcal{S}}(\sigma(T))$.

Unfortunately, not all deductive systems can be 'algebraized' as conveniently as $\mathbf{CPC}$.

An obvious problem that could arise is that there is no definable constant such as '1' with which we could equate interpretations of theorems. Even if there is, would it have validity and completeness behaviour as exemplified by the 2-element Boolean algebra for **CPC**? The following definitions provide a more general approach to overcome these problems. Instead of a single 'truth' constant like 1, we designate a fixed set of 'acceptable truth values' with respect to a fixed algebra.

An $\mathfrak{L}$-*matrix* is a pair $\langle \mathbf{A}, F\rangle$, where $\mathbf{A}$ is an $\mathfrak{L}$-algebra and F is an arbitrary subset of A. The elements of F are called the ***designated elements*** of the $\mathfrak{L}$-matrix $\langle \mathbf{A}, F\rangle$. We denote $\mathfrak{L}$-matrices by the upper-case script letters, $\mathcal{A}$, $\mathfrak{B}$, $\mathfrak{C}$, etc., and we often write $\mathcal{A} = \langle \mathbf{A}, F_{\mathcal{A}}\rangle$ to emphasise the fact that $F_{\mathcal{A}}$ is the set of designated elements of the matrix $\mathcal{A}$. Let $\mathcal{A} = \langle \mathbf{A}, F_{\mathcal{A}}\rangle$ be an $\mathfrak{L}$-matrix. The relation $\models_{\mathcal{A}}$ between sets of formulas and single formulas is defined, for all $\Gamma \subseteq Fm$ and $\varphi \in Fm$, by

$\Gamma \models_{\mathcal{A}} \varphi$ if and only if for every interpretation $\overline{a}$ of the variables of $\Gamma \cup \{\varphi\}$ in A,

$$\psi^{\mathbf{A}}(\overline{a}) \in F_{\mathcal{A}} \text{ for all } \psi \in \Gamma \text{ implies } \varphi^{\mathbf{A}}(\overline{a}) \in F_{\mathcal{A}}.$$

A matrix $\mathcal{A} = \langle \mathbf{A}, F_{\mathcal{A}}\rangle$ is called a *matrix model for S* if, for all $\Gamma \subseteq Fm$ and $\varphi \in Fm$

$$\Gamma \vdash_S \varphi \text{ implies } \Gamma \models_{\mathcal{A}} \varphi.$$

We call $\mathcal{A}$ an *S-matrix* for short in this case.

The set of designated elements can be considered the set of 'acceptable truth values'. One can read the definition of $\Gamma \models_{\mathcal{A}} \varphi$ in the following way: For any interpretation $\overline{a}$ of variables of $\Gamma \cup \{\varphi\}$ in A, if $\psi^{\mathbf{A}}(\overline{a})$ is 'true' for each $\psi \in \Gamma$, then $\varphi^{\mathbf{A}}(\overline{a})$ is 'true' as well. This definition still encompasses the method used on the classical propositional calculus. The $\mathfrak{L}$-matrix $\mathfrak{C} = \langle \mathbf{2}, \{1\}\rangle$ is a matrix model for **CPC**. For suppose that $\Gamma \subseteq Fm$ and $\varphi \in Fm$. By definition, $\Gamma \models_{\mathfrak{C}} \varphi$ if and only if for every interpretation $\overline{c}$ of the variables of $\Gamma \cup \{\varphi\}$ in $\{0, 1\}$, if $\psi^{\mathbf{2}}(\overline{c}) \in \{1\}$ for all $\psi \in \Gamma$, then $\varphi^{\mathbf{2}}(\overline{c}) \in \{1\}$.

Let $\mathcal{A} = \langle \mathbf{A}, F\rangle$ be an $\mathfrak{L}$-matrix. Since $\models_{\mathcal{A}}$ is a relation between sets of formulas and single formulas, it is feasible to ask whether $\models_{\mathcal{A}}$ is the consequence relation of some deductive system. We shall show that $\models_{\mathcal{A}}$ satisfies (1.1.1), (1.1.2), (1.1.3) and (1.1.5), and in Lemma 1.3.1, we show that if $\mathbf{A}$ is a finite algebra then (1.1.4) holds.

It follows easily from the definition that the relation $\models_{\mathcal{A}}$ satisfies (1.1.1), (1.1.2). To see that (1.1.3) holds, suppose that $\Gamma, \Delta \subseteq Fm$ and $\varphi \in Fm$ such that $\Gamma \models_{\mathcal{A}} \varphi$ and $\Delta \models_{\mathcal{A}} \psi$ for every $\psi \in \Gamma$. Let $\overline{a}$ be an interpretation of the variables of $\Delta \cup \{\varphi\}$ in A such that $\vartheta^{\mathbf{A}}(\overline{a}) \in F$ for each $\vartheta \in \Delta$. Then $\psi^{\mathbf{A}}(\overline{a}) \in F$ for each $\psi \in \Gamma$ since $\Delta \models_{\mathcal{A}} \psi$, hence $\varphi^{\mathbf{A}}(\overline{a}) \in F$ since $\Gamma \models_{\mathcal{A}} \varphi$. To see that (1.1.5) holds, suppose $\Gamma \models_{\mathcal{A}} \varphi$ and let σ be a substitution. Recall that we can identify the set of all interpretations of P in A with the set $\mathrm{Hom}(\mathbf{Fm}, \mathbf{A})$. So, let $h : \mathbf{Fm} \to \mathbf{A}$ be a homomorphism such that $h(\sigma\psi) \in F$ for every $\psi \in \Gamma$ (i.e., for every $\sigma\psi \in \sigma\Gamma$). Since $\sigma : \mathbf{Fm} \to \mathbf{Fm}$ is a homomorphism, the composition $h \circ \sigma : \mathbf{Fm} \to \mathbf{A}$ is also a homomorphism, hence $h \circ \sigma$ corresponds to an interpretation of P in A. Furthermore, for all $\psi \in \Gamma$, $(h \circ \sigma)\psi = h(\sigma\psi) \in F$, implying that $(h \circ \sigma)\varphi \in F$ since $\Gamma \models_{\mathcal{A}} \varphi$, hence $h(\sigma\varphi) \in F$. This shows that $\sigma(\Gamma) \models_{\mathcal{A}} \sigma\varphi$.

Although $\models_{\mathcal{A}}$ is not finitary in general, Łos and Suszko prove the following lemma in [LS58]. For the proof of the lemma, we shall digress briefly into the realm of topology. The necessary definitions and results can be found in any standard textbook on topology.

1.3.1 LEMMA

Let $\mathcal{A} = \langle \mathbf{A}, F \rangle$ be an $\mathfrak{L}$-matrix. If $\mathbf{A}$ is a finite $\mathfrak{L}$-algebra (i.e., A is finite), then $\models_{\mathcal{A}}$ is finitary.

Proof. Equip the set A with the discrete topology, i.e., every subset of A is open. Since A is finite, the topological space $T = \langle A, \mathcal{P}(A) \rangle$ is compact and Hausdorff. Recall that P is the set of all variables. Using Tychonoff's Theorem, the product space T^P is also a compact Hausdorff space. Recall, furthermore, that the set A^P is the set of all interpretations of P in A. For each $\varphi(\mathbf{p}_{i_1}, \dots, \mathbf{p}_{i_n}) \in Fm$, define

$$V(\varphi) = \{\overline{a} = \langle a_1, a_2, \dots \rangle \in A^P \,;\, \varphi^{\mathbf{A}}(a_{i_1}, \dots, a_{i_n}) \in F\}^1.$$

We claim that $V(\varphi)$ is a clopen set (i.e., a closed and open set) in T^P: Let $\overline{a} \in V(\varphi)$, i.e., $\varphi^{\mathbf{A}}(a_{i_1}, \dots, a_{i_n}) \in F$. Let π_{i_j} be the projection from A^P onto $A_{i_j} (= A)$ (defined by $\pi_{i_j}(\langle a_1, a_2, \dots \rangle = a_{i_j})$ for each $j \leq n$. Then we have that

$$U = \pi_{i_1}^{-1}(a_{i_1}) \cap \dots \cap \pi_{i_n}^{-1}(a_{i_n})$$

is an open set in T^P since each set $\{a_{i_j}\}$ is open in T. For every $\overline{b} = \langle b_1, b_2, \dots \rangle \in U$ and all $j \leq n$, we have that $b_{i_j} = \pi_{i_j}\overline{b} = a_{i_j}$, hence $\varphi^{\mathbf{A}}(b_{i_1}, \dots, b_{i_n}) = \varphi^{\mathbf{A}}(a_{i_1}, \dots, a_{i_n}) \in F$ and so $\overline{b} \in V(\varphi)$.

[1] It is understood that $\overline{a}(\mathbf{p}_k) = a_k$ for each positive integer k.

Thus $\bar{a} \in U \subseteq V(\varphi)$. Since U is an open neighbourhood of $\bar{a}$ and $\bar{a}$ was an arbitrary element of $V(\varphi)$, we have that $V(\varphi)$ is an open set. We shall denote the complement of a set $X \subseteq A^P$ by X'. Now, for every $\bar{a} \in A^P$, either $\varphi^{\mathbf{A}}(a_{i_1},\ldots,a_{i_n}) \in F$ or $\varphi^{\mathbf{A}}(a_{i_1},\ldots,a_{i_n}) \in A - F$, therefore

$$(V(\varphi))' = \{\bar{a} \in A^P\, ;\, \varphi^{\mathbf{A}}(a_{i_1},\ldots,a_{i_n}) \in A - F\}.$$

Note that the set F is arbitrary, so the same proof used to show that $V(\varphi)$ is open can be used to show that $(V(\varphi))'$ is open. Then, since the complement of $V(\varphi)$ is open, $V(\varphi)$ is a closed set in T^P.

Let $\Gamma \subseteq Fm$ and $\varphi \in Fm$. First, note that

(1.3.1) $$\Gamma \models_{\mathcal{A}} \varphi \text{ if and only if } \bigcap\{V(\psi)\, ;\, \psi \in \Gamma\} \subseteq V(\varphi).$$

For suppose $\bar{a} \in \bigcap\{V(\psi)\, ;\, \psi \in \Gamma\}$, i.e., $\psi^{\mathbf{A}}(\bar{a}) \in F$ for all $\psi \in \Gamma$. If $\Gamma \models_{\mathcal{A}} \varphi$ then $\varphi^{\mathbf{A}}(\bar{a}) \in F$, hence $\bar{a} \in V(\varphi)$ and the implication from left to right holds. Conversely, if the right hand side of (1.3.1) holds, then $\bar{a} \in V(\varphi)$, hence $\Gamma \models_{\mathcal{A}} \varphi$.

Now, suppose that $\Gamma \models_{\mathcal{A}} \varphi$. We shall show that there is a finite $\Delta \subseteq \Gamma$ such that $\Delta \models_{\mathcal{A}} \varphi$. By (1.3.1), $\bigcap\{V(\psi)\, ;\, \psi \in \Gamma\} \subseteq V(\varphi)$,

i.e., $$V(\varphi) \cup \Big(\bigcap\{V(\psi)\, ;\, \psi \in \Gamma\}\Big)' = A^P,$$

equivalently $$V(\varphi) \cup \Big(\bigcup\{(V(\psi))'\, ;\, \psi \in \Gamma\}\Big) = A^P.$$

Since each $(V(\psi))'$ is open, so is $\bigcup\{(V(\psi))'\, ;\, \psi \in \Gamma\}$. Since $V(\varphi)$ is also open and T^P is a compact space, there exists a finite $\Delta \subseteq \Gamma$ such that

$$V(\varphi) \cup \Big(\bigcup\{(V(\psi))'\, ;\, \psi \in \Delta\}\Big) = A^P,$$

hence $$\bigcap\{V(\psi)\, ;\, \psi \in \Delta\} \subseteq V(\varphi),$$

implying, by (1.3.1), that $\Delta \models_{\mathcal{A}} \varphi$. □

1.4 EXAMPLES OF DEDUCTIVE SYSTEMS

In this section we introduce a number of the deductive systems that will be studied in this thesis. Throughout this section we shall assume that p, q, r, s, t are fixed but arbitrary propositional variables. This ensures that in the cases where an explicit axiomatization of a deductive system is given, there are only finitely many axioms. Certain axioms appear in more than one deductive system. As a result, these axioms are sometimes given more than one name.

Classical Propositional Calculus (CPC).

The Classical Propositional Calculus was defined in Section 1.1. We present here some well-known results concerning **CPC**. Recall that $\mathfrak{BA}$ is the variety of all Boolean algebras and that by **2** we mean the Boolean algebra $\langle\{0,1\};\ \wedge^{\mathbf{2}}, \vee^{\mathbf{2}}, \neg^{\mathbf{2}}, 0, 1\rangle$. Recall also that in the language of $\mathfrak{BA}$, we define the binary term $x \to y = (\neg x) \vee y$. Thus the languages of **CPC** and $\mathfrak{BA}$ can be considered the same.

Completeness and Validity Theorems for CPC

For all $\Gamma \subseteq Fm$ *and* $\varphi \in Fm$,

$$\Gamma \vdash_{\mathbf{CPC}} \varphi \quad \textit{iff} \quad \{\psi \approx \mathrm{T}\,;\, \psi \in \Gamma\} \models_{\mathfrak{BA}} \varphi \approx \mathrm{T} \quad \textit{iff} \quad \{\psi \approx \mathrm{T}\,;\, \psi \in \Gamma\} \models_{\mathbf{2}} \varphi \approx \mathrm{T}. \qquad \square$$

The third equivalence of the theorem is a consequence of Birkhoff's Subdirect Decomposition Theorem (Theorem 0.2.8): $\mathfrak{BA} = \mathrm{IP_S}(\mathbf{2})$ $(= \mathrm{HSP}(\mathbf{2})$, since $\mathfrak{BA}$ is a variety) because it is well-known that **2** is (up to isomorphism) the only subdirectly irreducible Boolean algebra. A fundamental result of **CPC** is the deduction theorem. We state it without proof here since we shall give a more general result in Chapter 4. (If one assumes the above Completeness and Validity Theorems, then the Deduction Theorem is an easy corollary.)

Deduction Theorem for CPC

Let $\Gamma \subseteq Fm$ *and* $\varphi, \psi \in Fm$. *Then*

$$\Gamma, \varphi \vdash_{\mathbf{CPC}} \psi \quad \textit{if and only if} \quad \Gamma \vdash_{\mathbf{CPC}} \varphi \to \psi. \qquad \square$$

We shall be using the abbreviation $\leftrightarrow$ (called *equivalence*) frequently; it is a binary formula defined, for $\varphi, \psi \in Fm$, by

$$\leftrightarrow(\varphi, \psi) = \varphi \leftrightarrow \psi = (\varphi \to \psi) \wedge (\psi \to \varphi).$$

Classical Equivalential Calculus ($\mathbf{CPC}_{\leftrightarrow}$).

This is the $\{\leftrightarrow\}$-fragment of **CPC**. In Chapter 3 we shall show that $\mathbf{CPC}_{\leftrightarrow}$ has the finite axiomatization consisting of the axiom

$$(p \leftrightarrow q) \leftrightarrow ((r \leftrightarrow q) \leftrightarrow (p \leftrightarrow r))$$

and the inference rule

$$\langle \{p, p \leftrightarrow q\}, q\rangle.$$

Intuitionistic Propositional Calculus (IPC).

The Intuitionistic Propositional Calculus is the deductive system intended to realise the concept of *intuitionism*, a philosophical approach to mathematics that differs from the classical one. The basic idea behind intuitionism is that the 'truth' of a mathematical statement depends on whether one can explicitly construct a proof of it. The intuitionistic implication connective, $\rightarrow$, can be interpreted as follows: $\varphi \rightarrow \psi$ is 'true' if there exists a method by which a proof of ψ can be deduced from a proof of φ. Intuitionists regard a disjunction $\varphi \vee \psi$ as 'true' if one of the formulas φ, ψ is 'true' and there exists a method by which it is possible to determine which of them is 'true'. An example of a formula that is not considered 'true' by intuitionists is $\varphi \vee (\neg \varphi)$, for there is no general method of finding out, for a given formula φ, whether it is φ or $\neg \varphi$ that is 'true'. It was L.E.J. Brouwer who single-handedly created intuitionism around 1905. A. Heyting formalized and developed many of Brouwer's ideas. We refer the interested reader to [RS63, Chapter IX].

The language of **IPC** is the same as that of **CPC**. The axioms of **IPC** can be taken as (C_1), (C_2), (C_4) - (C_{11}) and

(C_{12}) $\quad (\neg p) \rightarrow (p \rightarrow \bot)$

(C_{13}) $\quad (p \rightarrow \bot) \rightarrow (\neg p)$.

The only inference rule of **IPC** is modus ponens, i.e., $\langle \{p, p \rightarrow q\}, q\rangle$.

Recall that $\mathcal{HA}$ is the variety of all Heyting algebras. Recall also that in the language of $\mathcal{HA}$ we can define the unary operation symbol $\neg$ by $\neg x = x \rightarrow \bot$. Thus **IPC** and $\mathcal{HA}$ can be considered to have the same language. The following well-known result of **IPC** is the analogue of the one for **CPC**. Note, however, that we do not have a two-element algebra for **IPC** that plays a similar role to that of **2** in **CPC**.

Completeness and Validity Theorems for IPC

For all $\Gamma \subseteq Fm$ *and* $\varphi \in Fm$,

$$\Gamma \vdash_{\mathbf{IPC}} \varphi \quad \textit{if and only if} \quad \{\psi \approx \mathrm{T}\,;\, \psi \in \Gamma\} \models_{\mathcal{HA}} \varphi \approx \mathrm{T}. \qquad \square$$

We can deduce from the above result that **IPC** is, indeed, weaker than **CPC**. We claim that $\nvdash_{\mathbf{IPC}} p \vee (\neg p)$. For let $\mathbf{H} = \langle \{\bot, a, \mathrm{T}\}\,;\, \wedge, \vee, \rightarrow, \bot, \mathrm{T}\rangle$ be the three-element linearly ordered Heyting algebra with $\bot < a < \mathrm{T}$ (where $\leq$ is the lattice order) – see Chapter 0 (Section 0.2). Then $\neg a = a \rightarrow \bot = \bot$ and so $a \vee (\neg a) = a \neq \mathrm{T}$, hence $\not\models_{\mathcal{HA}} p \vee (\neg p) \approx \mathrm{T}$. As in the case of **CPC**, the following result will be proved in Chapter 4.

Deduction Theorem for IPC

Let $\Gamma \subseteq Fm$ *and* $\varphi, \psi \in Fm$. *Then*

$$\Gamma, \varphi \vdash_{\mathbf{IPC}} \psi \quad \textit{if and only if} \quad \Gamma \vdash_{\mathbf{IPC}} \varphi \rightarrow \psi. \qquad \square$$

The following proposition presents an explicit description of the **IPC**-matrices; we shall have opportunity to use it in later chapters. Recall from Section 0.2 that the operation $\rightarrow$ on a Heyting algebra $\mathbf{H}$ satisfies the condition

(1.4.1) $$a \rightarrow b = \max\{c \in H\,;\, c \wedge a \leq b\}.$$

Recall, furthermore, that the underlying lattice order on $\mathbf{H}$ can be defined (for all $a, b, c \in H$) by $a \leq b$ if and only if $a \rightarrow b = \mathrm{T}$. Other properties of $\leq$ that we shall need are given in Section 0.2.

1.4.1 PROPOSITION

Let $\mathbf{H} = \langle H\,;\, \wedge, \vee, \rightarrow, \bot, \mathrm{T}\rangle$ *be a Heyting algebra and* $F \subseteq H$. *Then* $\mathcal{A} = \langle \mathbf{H}, F\rangle$ *is an* **IPC**-*matrix if and only if* F *is a filter of the lattice* $\langle H\,;\, \wedge, \vee\rangle$.

Proof. For illustrative purposes and to avoid circularity, we give a proof from first principles. (Shorter proofs, using the Validity and Completeness Theorems for **IPC** or the Deduction Theorem for **IPC**, are of course possible.) Suppose $\mathcal{A} = \langle \mathbf{H}, F\rangle$ is an **IPC**-matrix. By (C_{11}), $\vdash_{\mathbf{IPC}} \mathrm{T} \rightarrow \mathrm{T}$ and $\vdash_{\mathbf{IPC}} (\mathrm{T} \rightarrow \mathrm{T}) \rightarrow \mathrm{T}$, so by (MP), $\vdash_{\mathbf{IPC}} \mathrm{T}$. Consequently, $\mathrm{T}\,(= \mathrm{T}^{\mathbf{H}}) \in F$, so $F \neq \emptyset$. Let $a, b \in H$. If $a \in F$ and $a \leq b$, then $a \rightarrow b = \mathrm{T} \in F$. Since $p, p \rightarrow q \vdash_{\mathbf{IPC}} q$ by (MP), we have $b \in F$. Now, suppose $a, b \in F$. By (C_1) and (C_6),

$$\vdash_{\mathbf{IPC}} p \rightarrow (\mathrm{T} \rightarrow p),$$

$$\vdash_{\mathbf{IPC}} q \rightarrow (\mathrm{T} \rightarrow q),$$

and $$\vdash_{\mathbf{IPC}} (\mathrm{T} \rightarrow p) \rightarrow [(\mathrm{T} \rightarrow q) \rightarrow (\mathrm{T} \rightarrow (p \wedge q))],$$

so, by (MP), $$p, q \vdash_{\mathbf{IPC}} \mathrm{T} \rightarrow (p \wedge q).$$

From $\vdash_{\mathbf{IPC}} \mathrm{T}$ and (MP), we therefore obtain

$$p, q \vdash_{\mathbf{IPC}} p \wedge q.$$

It follows that $a \wedge b \in F$, so F is a filter of the lattice $\langle H ; \wedge , \vee \rangle$.

Now, suppose that F is a filter of the lattice $\langle H ; \wedge , \vee \rangle$. It is not hard to see that for $\mathcal{A}$ to be an **IPC**-matrix we need only prove that $\models_{\mathcal{A}} \varphi$ for each axiom φ of **IPC**, and that $p, p \rightarrow q \models_{\mathcal{A}} q$. (We shall prove a more general result in Section 1.6.) We show that $\varphi^{\mathrm{H}}(\overline{a}) = \mathrm{T}^{\mathrm{H}} (\in F)$ for every interpretation $\overline{a}$ of the variables of φ in H, whenever φ is an axiom of **IPC**. Let a, b, c be an interpretation of p, q, r in H, respectively.

(C_1) Since $a \leq b \rightarrow a$, we have that $a \rightarrow (b \rightarrow a) = \mathrm{T} \in F$.

(C_2) We have $(a \rightarrow (b \rightarrow c)) \wedge (a \rightarrow b) \overset{(H5)}{=} a \rightarrow ((b \rightarrow c) \wedge b) \overset{(H4)}{=} a \rightarrow (b \wedge c) \leq a \rightarrow c$ since $b \wedge c \leq c$. By (1.4.1), therefore, $a \rightarrow (b \rightarrow c) \leq (a \rightarrow b) \rightarrow (a \rightarrow c)$. Thus $(a \rightarrow (b \rightarrow c)) \rightarrow ((a \rightarrow b) \rightarrow (a \rightarrow c)) = \mathrm{T} \in F$.

(C_4) Since $a \wedge b \leq a$, we have that $(a \wedge b) \rightarrow a = \mathrm{T} \in F$.

(C_5) Since $a \wedge b \leq b$, we have that $(a \wedge b) \rightarrow b = \mathrm{T} \in F$.

(C_6) By (H5), $(c \rightarrow a) \wedge (c \rightarrow b) = c \rightarrow (a \wedge b)$, so, by (1.4.1), $c \rightarrow a \leq [(c \rightarrow b) \rightarrow (c \rightarrow (a \wedge b))]$, i.e., $(c \rightarrow a) \rightarrow [(c \rightarrow b) \rightarrow (c \rightarrow (a \wedge b))] = \mathrm{T} \in F$.

(C_7) Since $a \leq a \vee b$, we have that $a \rightarrow (a \vee b) = \mathrm{T} \in F$.

(C_8) Since $b \leq a \vee b$, we have that $b \rightarrow (a \vee b) = \mathrm{T} \in F$.

(C_9) By (H5), $(a \rightarrow c) \wedge (b \rightarrow c) = (a \vee b) \rightarrow c$, so, by (1.4.1), $a \rightarrow c \leq [(b \rightarrow c) \rightarrow ((a \vee b) \rightarrow c)]$, i.e., $(a \rightarrow c) \rightarrow [(b \rightarrow c) \rightarrow ((a \vee b) \rightarrow c)] = \mathrm{T} \in F$.

(C_{10}) Since $\perp \leq a$, we have that $\perp \rightarrow a = \mathrm{T} \in F$.

(C_{11}) Since $a \leq \mathrm{T}$, we have that $a \rightarrow \mathrm{T} = \mathrm{T} \in F$.

(C_{12}) By definition, $\neg a = a \rightarrow \perp$, so $(\neg a) \rightarrow (a \rightarrow \perp) = (a \rightarrow \perp) \rightarrow (a \rightarrow \perp) \overset{(H3)}{=} \mathrm{T} \in F$.

(C_{13}) Using $\neg a = a \rightarrow \perp$ again, $(a \rightarrow \perp) \rightarrow (\neg a) = (a \rightarrow \perp) \rightarrow (a \rightarrow \perp) \overset{(H3)}{=} \mathrm{T} \in F$.

Finally, consider (MP): If $a, a \rightarrow b \in F$, then $a \wedge b \overset{(H4)}{=} a \wedge (a \rightarrow b) \in F$. Since $a \wedge b \leq b$, we deduce that $b \in F$. Thus $p, p \rightarrow q \models_{\mathcal{A}} q$. □

As in the case of **CPC**, we shall be using the abbreviation $\leftrightarrow$; we define, for $\varphi, \psi \in Fm$,

$$\leftrightarrow (\varphi, \psi) = \varphi \leftrightarrow \psi = (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi).$$

Note that

$$\mathcal{HA} \models (\varphi \leftrightarrow \psi \approx \mathrm{T}) \Leftrightarrow (\varphi \rightarrow \psi \approx \mathrm{T} \ \& \ \psi \rightarrow \varphi \approx \mathrm{T}) \Leftrightarrow (\varphi \leq \psi \ \& \ \psi \leq \varphi) \Leftrightarrow (\varphi \approx \psi).$$

We shall need the following theorems of **IPC** in Chapter 2:

(i) $\vdash_{\mathbf{IPC}} (p \leftrightarrow q) \rightarrow [(q \leftrightarrow r) \rightarrow (p \leftrightarrow r)]$,

(ii) $\vdash_{\mathbf{IPC}} (p \leftrightarrow q) \rightarrow [(r \leftrightarrow s) \rightarrow ((p \rightarrow r) \rightarrow (q \rightarrow s))]$,

(iii) $\vdash_{\mathbf{IPC}} (p \leftrightarrow q) \rightarrow [(r \leftrightarrow s) \rightarrow ((p \wedge r) \rightarrow (q \wedge s))]$,

(iv) $\vdash_{\mathbf{IPC}} (p \leftrightarrow q) \rightarrow [(r \leftrightarrow s) \rightarrow ((p \vee r) \rightarrow (q \vee s))]$,

(v) $\vdash_{\mathbf{IPC}} (p \leftrightarrow q) \rightarrow ((\neg p) \leftrightarrow (\neg q))$.

We justify these, assuming the Validity and Completeness Theorems (and hence the Deduction Theorem). By the Deduction Theorem, (i) holds iff $p \leftrightarrow q, q \leftrightarrow r \vdash_{\mathbf{IPC}} p \leftrightarrow r$. By the Validity Theorem, this holds if $p \leftrightarrow q \approx \mathrm{T}, q \leftrightarrow r \approx \mathrm{T} \models_{\mathcal{HA}} p \leftrightarrow r \approx \mathrm{T}$. But, as noted above, $x \leftrightarrow y \approx \mathrm{T}$ is equivalent to $x \approx y$ over $\mathcal{HA}$. Since $p \approx q, q \approx r \models_{\mathcal{HA}} p \approx r$ is true, so is (i). Let $*$ be any of the binary connectives $\rightarrow$, $\wedge$ or $\vee$. By the Deduction Theorem,

$$\vdash_{\mathbf{IPC}} (p \leftrightarrow q) \rightarrow [(r \leftrightarrow s) \rightarrow ((p*r) \rightarrow (q*s))] \quad \text{iff} \quad p \leftrightarrow q, r \leftrightarrow s \vdash_{\mathbf{IPC}} (p*r) \rightarrow (q*s).$$

By the Validity Theorem, this holds if $p \leftrightarrow q \approx \mathrm{T}, r \leftrightarrow s \approx \mathrm{T} \models_{\mathcal{HA}} (p*r) \rightarrow (q*s) \approx \mathrm{T}$ iff $p \approx q, r \approx s \models_{\mathcal{HA}} p*r \approx q*s$. Since the last statement is true when $*$ is $\rightarrow$, $\wedge$ or $\vee$, (ii), (iii) and (iv) hold. Applying the same method to (v), we have that (v) holds iff $p \approx q \models_{\mathcal{HA}} \neg p \approx \neg q$, which is evidently true.

Intuitionistic Equivalential Calculus ($\mathbf{IPC}_{\leftrightarrow}$).

By $\mathbf{IPC}_{\leftrightarrow}$ we mean the $\{\leftrightarrow\}$-fragment of **IPC**. Blok and Pigozzi note in [BP89a] that an axiomatization of the theorems of $\mathbf{IPC}_{\leftrightarrow}$, consisting of one axiom and the following inference rules was produced by Tax, [Tax73]:

$$\langle \{p, p \leftrightarrow q\}, q \rangle,$$

$$\langle \{p\}, q \leftrightarrow (q \leftrightarrow p) \rangle.$$

The deductive system $\mathbf{IPC}^*$ is the $\{\wedge, \vee, \perp, \mathrm{T}\}$-fragment of **IPC**. As an example it is significant in that its language does not contain an $\rightarrow$ connective (or a $\leftrightarrow$ connective).

Modal Logics.

By a *modal logic* we mean a deductive system that has as its language the language of **CPC** together with a unary connective $\Box$, known as a *modality*. In modal logics, the $\Box$'s are interpreted as 'linguistic constructions that qualify assertions about the truth of statements, and express various *modes* of truth' ([Gol94, p1]). The following are examples of modalities (applied to a formula φ):

it is necessarily true that φ,

it is possible that φ is true,

probably φ,

it has always been true that φ,

it will eventually be true that φ,

it ought to be that φ,

it is permissible that φ,

it is known that φ,

it is believed that φ.

Associated with $\Box$ is the unary term $\Diamond\varphi = \neg\Box\neg\varphi$. The interpretation of $\Diamond$ depends on that of $\Box$. For example, if $\Box\varphi$ is understood as 'it is necessarily true that φ', then $\Diamond\varphi$ can be understood to mean 'it is possible that φ'. The concept of a modality was introduced by W.T. Parry [Par39] in connection with the study of the modal calculi of C.I. Lewis [Lew18] and C.H. Langford [LL32]. We list some of the modal logics that we shall be considering.

The modal logic K (named after Kripke) is a deductive system defined over the language $\mathcal{L} = \{\rightarrow, \wedge, \vee, \neg, \bot, \mathrm{T}, \Box\}$, where $\Box$ is unary and all other connectives have arities as in **CPC**. We shall use the convention that both $\neg$ and $\Box$ take precedence over the binary formulas, e.g., $\neg p \rightarrow \Box q = (\neg p) \rightarrow (\Box q)$. The axioms of **K** are

(K_1) All theorems (i.e., tautologies) of **CPC**,

(K_2) $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$.

The inference rules are (MP) and

(Ne) $\langle\{p\}, \Box p\rangle$ [*necessitation*].

We could replace (K_1) (equivalently) by the axioms (C_1) to (C_{11}) of **CPC**, since (MP) is available, which would give us a finite axiomatization of **K**.

The Normal and Quasi-Normal Modal Logics. The *normal modal logics* are those modal logics that are axiomatic extensions of **K**. Let **K′** be the modal logic whose axioms are the set of theorems of **K**, but whose only inference rule is (MP). The modal logics that are axiomatic extensions of **K′** are called the *quasi-normal modal logics.* The modal logic $S5^W$, defined below is quasi-normal.

C.I. Lewis introduced a number of modal logics, of which we shall be concerned with two, viz. $S4$ and $S5$.

The modal logic *S4* is an axiomatic extension of **K** (hence is a normal modal logic). The additional axioms of **S4** are

(K_3) $\Box p \to p$,

(K_4) $\Box p \to \Box\Box p$.

There are various different axiomatic presentations of $S5$, of which the following three are known to have the same theorems (see, e.g., [Por83]).

The modal logic $S5^G$ (*Gödel style* $S5$) is also an axiomatic extension of **K**. The additional axioms of $S5^G$ are (K_3) and

(K_5) $\Diamond p \to \Box \Diamond p$, where $\Diamond p = \neg\Box\neg p$.

The modal logic $S5^C$ (*Carnap style* $S5$), has the same language as **K** and is axiomatized by (K_3) and

(K_6) $\Box\varphi$, where φ is any theorem (i.e., tautology) of **CPC**,

(K_7) $\Box(\Box(p \to q) \to (\Box p \to \Box q))$,

(K_8) $\Box(\Box p \to p)$,

(K_9) $\Box(\Diamond p \to \Box \Diamond p)$, where $\Diamond p = \neg\Box\neg p$.

together with the single inference rule (MP).

The modal logic $S5^W$ (*Wajsberg style* $S5$), over the same language as **K**, is axiomatized by (K_6), (K_8), (K_9) and

(K_{10}) $\Box(\Box(p \to q) \to \Box(\Box p \to \Box q))$,

together with the single inference rule

(K_{11}) $\langle\{p, \Box(p \to q)\}, q\rangle$.

According to [BP89a], $S5^W$ is the closest in spirit of the above three systems to Lewis' original $S5$.

BCK-logic.

This deductive system and several of its extensions are studied in detail in Chapter 5, so we merely present the definition here. The language consists of one binary connective $\to$. The axioms of **BCK** are

(B) $(p \to q) \to ((r \to p) \to (r \to q))$,

(C) $(p \to (q \to r)) \to (q \to (p \to r))$,

(K) $p \to (q \to p)$,

and the single inference rule is (MP).

Entailment and Relevance Logics (E and R).

It is possible to take the view that a necessary condition for the 'correctness' of an inference from φ to ψ is that φ be relevant to ψ. This is not the case in the classical logics **CPC** and **IPC**. The deductive systems *Entailment* and *Relevance*, **E** and **R**, respectively, are designed with the intention of incorporating the ideas of relevance into the logic. A significant example of a formula that does not adhere to the principal of relevance is $\varphi \to (\psi \to \varphi)$. We refer the interested reader to [AB75]. The language of **E** is $\mathcal{L} = \{\wedge, \vee, \to, \neg\}$, where $\wedge, \vee, \to$ are binary and $\neg$ is unary. The axioms of **E** are

(E_1) $p \to p$

(E_2) $(p \to q) \to ((q \to r) \to (p \to r))$

(E_3) $(p \to q) \to ((r \to p) \to (r \to q))$

(E_4) $(p \to (p \to q)) \to (p \to q)$

(E_5) $(p \wedge q) \to p$

(E$_6$) $(p \wedge q) \rightarrow q$

(E$_7$) $((p \rightarrow q) \wedge (p \rightarrow r)) \rightarrow (p \rightarrow (q \wedge r))$

(E$_8$) $p \rightarrow (p \vee q)$

(E$_9$) $p \rightarrow (q \vee p)$

(E$_{10}$) $((p \rightarrow r) \wedge (q \rightarrow r)) \rightarrow ((p \vee q) \rightarrow r)$

(E$_{11}$) $(p \wedge (q \vee r)) \rightarrow ((p \wedge q) \vee r)$

(E$_{12}$) $(p \rightarrow (\neg p)) \rightarrow (\neg p)$

(E$_{13}$) $(p \rightarrow (\neg q)) \rightarrow (q \rightarrow (\neg p))$

(E$_{14}$) $(\neg (\neg p)) \rightarrow p$

(E$_{15}$) $(p \rightarrow q) \rightarrow (((p \rightarrow q) \rightarrow r) \rightarrow r)$

(E$_{16}$) $((\Box p) \wedge (\Box q)) \rightarrow \Box(p \wedge q)$, where $\Box p = (p \rightarrow p) \rightarrow p$.

The inference rules of **E** are (MP) and

(A) $\langle \{p, q\}, p \wedge q \rangle$ [*Adjunction*]

It is no accident that the formula $(p \rightarrow p) \rightarrow p$ is abbreviated $\Box p$. The $\Box$ is intended to be a modality of **E**. In fact, the modal logic $\boldsymbol{S5}^W$ is an extension of **E**. We shall write $\rightarrow_{\mathbf{E}}$ and $\rightarrow_{\mathbf{W}}$ to distinguish between the $\rightarrow$ connectives of **E** and $\boldsymbol{S5}^W$, respectively. Add to the language of **E** the unary connective $\Box$, defined by $\Box\varphi = (\varphi \rightarrow_{\mathbf{E}} \varphi) \rightarrow_{\mathbf{E}} \varphi$, and the binary connective $\rightarrow_{\mathbf{W}}$, defined by $\varphi \rightarrow_{\mathbf{W}} \psi = (\neg \varphi) \vee \psi$. To the language of $\boldsymbol{S5}^W$, add the connective $\rightarrow_{\mathbf{E}}$, defined by $\varphi \rightarrow_{\mathbf{E}} \psi = \Box(\varphi \rightarrow_{\mathbf{W}} \psi)$. Then the extended languages of $\boldsymbol{S5}^W$ and **E** coincide. Moreover, with respect to this formalism, each of the axioms and inference rules of **E** can be proved in $\boldsymbol{S5}^W$ (see [BP89a]). Since $\boldsymbol{S5}^W$ has only one inference rule, corresponding to (MP) for **E**, $\boldsymbol{S5}^W$ is an axiomatic extension of **E** (with the additional connectives).

The deductive system **R** of relevance logic is an axiomatic extension of **E**. It is obtained by adding the axiom

(E$_{17}$) $p \rightarrow ((p \rightarrow p) \rightarrow p)$.

The deductive system **RM** is an axiomatic extension of **R**. It is obtained by adding to **R** the axiom

(M) $p \rightarrow (p \rightarrow p)$ [the *mingle* axiom].

Łukasiewicz Many-Valued Logics.

Philosophical problems arising from the idea that there exist statements which are neither true nor false led Łukasiewicz to develop his many-valued logics. He introduced them by means of certain matrices. Let $\mathbb{Q}\cap[0,1]$ be the set of rational numbers r such that $0\leq r\leq 1$ and let $\mathbf{L}_\omega=\langle \mathbb{Q}\cap[0,1];\to,\neg\rangle$ be the algebra of type $\langle 2,1\rangle$ where

$$a\to b=\min\{1,b+1-a\},\qquad \neg a=1-a.$$

For each integer n with $1\leq n<\omega$, let $\mathbf{L}_n$ be the subalgebra of $\mathbf{L}_\omega$ with universe $\{\frac{r}{n};\, r=0,1,\ldots,n\}$. For each positive $n\leq\omega$, let $\mathcal{L}_n$ be the matrix $\langle \mathbf{L}_n,\{1\}\rangle$. *Łukasiewicz n-valued many-valued logic* is the deductive system $S_n=\langle \mathcal{L},\models_{\mathcal{L}_n}\rangle$, where $\mathcal{L}=\{\to,\neg\}$, $ar(\to)=2$ and $ar(\neg)=1$.

It is shown in [FRT84] that the system S_ω can be axiomatized by

(Ł$_1$) $\quad p\to(q\to p)$

(Ł$_2$) $\quad (p\to q)\to((q\to r)\to(p\to r))$

(Ł$_3$) $\quad ((p\to q)\to q)\to((q\to p)\to p)$

(Ł$_4$) $\quad ((\neg p)\to(\neg q))\to(q\to p)$

together with the inference rule (MP). It was J. Łukasiewicz who conjectured that (Ł$_1$) – (Ł$_4$), (MP) and

(Ł$_5$) $\quad ((p\to q)\to(q\to p))\to(p\to q)$

formed an axiomatization for S_ω. The first proof of the Łukasiewicz conjecture, due to Wajsberg (1935), was never published (see [FRT84]). In 1958, A. Rose and J.B. Rosser [RR58] published a proof of the completeness of the Łukasiewicz axioms. In 1959, C.C. Chang obtained a new proof of the completeness of the Łukasiewicz axioms [Cha58, Cha59].

Three-Valued Paraconsistent Logic (J_3).

Let $\mathcal{L}=\{\to,\neg\}$, where $ar(\to)=2$ and $ar(\neg)=1$, and let $\mathbf{A}=\langle\{0,\frac{1}{2},1\},\mathcal{L}^{\mathbf{A}}\rangle$ be an algebra, where $\mathcal{L}^{\mathbf{A}}=\{\to^{\mathbf{A}},\mathrm{T}^{\mathbf{A}}\}$ and $\to^{\mathbf{A}}$ and $\neg^{\mathbf{A}}$ ($\to$ and $\neg$ for short) are defined by the tables

$\rightarrow$	0	$\frac{1}{2}$	1
0	1	1	1
$\frac{1}{2}$	$\frac{1}{2}$	1	1
1	0	$\frac{1}{2}$	1

$\neg$	
0	1
$\frac{1}{2}$	$\frac{1}{2}$
1	0

Note that $\mathbf{A} = \mathbf{L}_2$. Define the $\mathcal{L}$-matrix $\mathfrak{J}_3 = \langle \mathbf{A}, \{1, \frac{1}{2}\}\rangle$. The *three-valued paraconsistent logic* is defined, as in [BP89b], to be $\mathbf{J_3} = \langle \mathcal{L}, \models_{\mathfrak{J}_3}\rangle$. In other words, if $\Gamma \subseteq Fm$ and $\varphi \in Fm$, then $\Gamma \vdash_{\mathbf{J_3}} \varphi$ if and only if $\Gamma \models_{\mathfrak{J}_3} \varphi$. Since $\mathbf{A}$ is a finite algebra, it follows from Lemma 1.3.1 that $\models_{\mathfrak{J}_3}$ is finitary. By the remarks preceding that lemma, we see that $\models_{\mathfrak{J}_3}$, i.e., $\vdash_{\mathbf{J_3}}$, satisfies properties (1.2.1) to (1.2.5), hence $\mathbf{J_3}$ is, indeed, a deductive system.

Pure Implicational Logics.

In logic and most of mathematics, one is essentially interested in knowing what inferences one can make. Thus it would seem natural to give the $\rightarrow$ connective of certain logics special consideration. By a *pure implicational logic* is meant a logic that has only one binary connective $\rightarrow$. We group together here a number of such logics, including the $\{\rightarrow\}$-fragments of a number the previously defined deductive systems. We shall use the following formulas

(I)	$p \rightarrow p$	[*identity*]
(I′)	$((p \rightarrow q) \rightarrow p) \rightarrow p$	
(B)	$(p \rightarrow q) \rightarrow ((r \rightarrow p) \rightarrow (r \rightarrow q))$	[*transitivity*]
(B′)	$(p \rightarrow q) \rightarrow ((q \rightarrow r) \rightarrow (p \rightarrow r))$	
(C)	$(p \rightarrow (q \rightarrow r)) \rightarrow (q \rightarrow (p \rightarrow r))$	[*commutation*]
(C′)	$(p \rightarrow ((s \rightarrow t) \rightarrow q)) \rightarrow ((s \rightarrow t) \rightarrow (p \rightarrow q))$	[*restricted commutation*]
(E)	$(p \rightarrow (p \rightarrow q)) \rightarrow (p \rightarrow q)$	[*contraction*]
(D)	$(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$	[*distributivity*]
(K)	$p \rightarrow (q \rightarrow p)$	
(M)	$p \rightarrow (p \rightarrow p)$	[*mingle*]

We shall have occasion to use the fact that in the presence of (C) and (MP), (B) and (B′) are interderivable. To see this, let S be a deductive system having (B) and (C) as theorems such that $p, p \to q \vdash_S q$. We have

$$\vdash_S (q \to r) \to ((p \to q) \to (p \to r))$$

by (B), and

$$\vdash_S [(q \to r) \to ((p \to q) \to (p \to r))] \to [(p \to q) \to ((q \to r) \to (p \to r))]$$

by (C). Applying (MP), we get

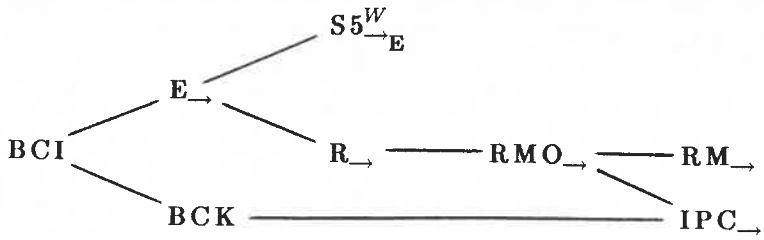
$$\vdash_S (p \to q) \to ((q \to r) \to (p \to r)),$$

i.e., (B$'$) is a theorem of S. That (B) can be derived from (B$'$) and (C) is similarly proved.

In the following table, each of the deductive systems is defined over the language $\mathcal{L} = \{\to\}$, where $\to$ is a binary connective. Those deductive systems whose axioms are stated explicitly all have (MP) as their only inference rule. Reversing historical usage, Blok and Pigozzi use the name (B) for (B$'$) in their papers. However, as the previous paragraph shows, this does not create any problems for the deductive systems defined below as (B) only appears in conjunction with (C). We (temporarily) omit the brackets around the axiom names.

Symbol	Definition	Axioms	Name
$\mathbf{E}_{\to}$		I, B$'$, C$'$, E	*pure entailment*
$\mathbf{R}_{\to}$		I, B, C, E	*relevant implication*
$\mathbf{RMO}_{\to}$		I, B, C, E, M	
$\mathbf{RM}_{\to}$	$\{\to\}$-fragment of **RM**		
BCK		B, C, K	**BCK**-*logic*
BCI		I, B, C	**BCI**-*logic*
$\mathbf{S5}^W_{\to_{\mathrm{E}}}$	$\{\to_{\mathrm{E}}\}$-fragment of $\mathbf{S5}^W$		
$\mathbf{IPC}_{\to}$	$\{\to\}$-fragment of **IPC**	B, C, K, E	*Hilbert logic*
$\mathbf{CPC}_{\to}$	$\{\to\}$-fragment of **CPC**	B, C, K, I$'$	

The theorems of the *pure calculus of entailment*, $\mathbf{E}_{\to}$ (see [AB75, p79]), coincide with those of the $\{\to\}$-fragment of **E**. The theorems of *relevant implication*, $\mathbf{R}_{\to}$, coincide with those of the $\{\to\}$-fragment of **R**. The following diagram taken from [BP89a] displays some of the extension relationships between the above implicative logics. (S_1 is displayed on the left of S_2 iff S_2 is an extension of S_1.)



1.5 k-DEDUCTIVE SYSTEMS

In this section, we generalize the notion of a deductive system to that of a 'k-deductive system', where k is a nonzero natural number. Although we shall not consider examples with $k > 2$, they do exist. In [BP92], Blok and Pigozzi present a k-deductive system for every integer $k \geq 1$, based on the theory of sets. Apart from the greater generality afforded by considering deductive systems of dimension $k > 1$, the main reason for considering them is to create a framework that accomodates deductive systems in the traditional sense as well as quasi-equational theories. The algebraic notions of equivalence and congruence relation rely on 2-tuples, to which 2-deductive systems apply more intuitively than do deductive systems. Furthermore, the algebraic relation $=$ is a binary relation, so the 2-deductive systems allow us to introduce the algebraic notion of equality to deductive systems, as will be seen in Chapter 3.

Let the language $\mathfrak{L}$ be fixed, and let k be a nonzero natural number. A *k-variable* is an ordered k-tuple $\langle p_1, \ldots, p_k\rangle$ where each p_i, $i \leq k$, is a propositional variable. A *k-formula* is an ordered k-tuple $\langle \varphi_1, \ldots, \varphi_k\rangle$ where each φ_i is an $\mathfrak{L}$-formula. We usually denote k-variables by bold lower-case Roman letters, e.g., $\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}$, and k-formulas by bold, lower-case Greek letters, e.g., $\boldsymbol{\varphi}, \boldsymbol{\psi}, \boldsymbol{\vartheta}$ and, unless stated otherwise, we will understand that $\boldsymbol{p} = \langle p_1, \ldots, p_k\rangle$, $\boldsymbol{\varphi} = \langle \varphi_1, \ldots, \varphi_k\rangle$, $\boldsymbol{\psi} = \langle \psi_1, \ldots, \psi_k\rangle$, etc. Sets of k-formulas are usually denoted by upper-case Greek letters (not bold), as for sets of $\mathfrak{L}$-formulas. The set of all k-formulas over the language $\mathfrak{L}$ will be denoted by $Fm^k_{\mathfrak{L}}$, or simply Fm^k if $\mathfrak{L}$ is understood. If $\sigma : \mathbf{Fm} \to \mathbf{Fm}$ is a substitution, $\Gamma \subseteq Fm^k$ and $\boldsymbol{\varphi} \in Fm^k$, then we define

$$\sigma\boldsymbol{\varphi} = \sigma\langle \varphi_1, \ldots, \varphi_k\rangle = \langle \sigma\varphi_1, \ldots, \sigma\varphi_k\rangle,$$

and

$$\sigma(\Gamma) = \{\sigma\boldsymbol{\varphi}\,;\ \boldsymbol{\varphi} \in \Gamma\}.$$

A *k-clause* is a fixed pair $\langle \Gamma, \boldsymbol{\varphi}\rangle$ where Γ is a finite subset of Fm^k and $\boldsymbol{\varphi} \in Fm^k$. A k-formula $\boldsymbol{\psi}$ is said to be *directly derivable* from a set Δ of k-formulas by the k-clause $\langle\Gamma,\varphi\rangle$ if there exists a substitution $\sigma:\mathbf{Fm}\to\mathbf{Fm}$ such that $\sigma(\Gamma)\subseteq\Delta$ and $\sigma\varphi=\psi$. The following definition is a generalization of Definition 1.1.1.

1.5.1 DEFINITION

Let $\mathfrak{L}$ be a fixed language. A *k-deductive system S (over $\mathfrak{L}$)* is determined by a (possibly infinite) set Ir of k-clauses, called the *inference rules of S*, and a (possibly infinite) set Ax of k-formulas, called the *axioms of S*. It consists of a pair $S=\langle\mathfrak{L},\vdash_S\rangle$ where $\vdash_S$ is just a relation between sets of k-formulas and single k-formulas that is defined by the following condition: For every $\Gamma\subseteq Fm^k$ and every $\varphi\in Fm^k$,

> $\Gamma\vdash_S\varphi$ if and only if φ is contained in the smallest set of k-formulas that includes Γ together with all substitution instances of the axioms of S, and is closed under direct derivability by the inference rules of S.

The relation $\vdash_S$ is called the *consequence relation of S*. If S is a k-deductive system, then we sometimes refer to S as a *deductive system of dimension k*. A k-formula φ for which $\emptyset\vdash_S\varphi$ (written $\vdash_S\varphi$ for short) is called a *theorem of S*. Let $\Delta\subseteq Fm^k$. A *derivation from Δ* is a nonempty finite sequence of k-formulas $\psi_1,\psi_2,\dots,\psi_n$ such that, for each $i\le n$, one of the following conditions holds:

(i) $\psi_i\in\Delta$ or $\psi_i=\sigma\varphi$ for some $\varphi\in Ax$ and some substitution σ,

(ii) there exists an inference rule $\langle\Gamma,\varphi\rangle$ of S such that $\psi_i=\sigma\varphi$ and $\sigma\vartheta\in\{\psi_1,\dots,\psi_{i-1}\}$ for each $\vartheta\in\Gamma$.

If, moreover, $\psi_n=\varphi$, we call $\psi_1,\psi_2,\dots,\psi_n$ a *derivation of φ from Δ*.

It is evident that a deductive system as defined in Section 1.1 is a 1-deductive system. In the future we will drop the prefix '1-' when working with (1-) deductive systems. An inference rule $\langle\Gamma,\varphi\rangle$ of S is sometimes written as $\Gamma\vdash_S\varphi$, and an axiom φ of S is sometimes written as $\vdash_S\varphi$. If $\Gamma,\Delta\subseteq Fm^k$ such that $\Gamma\vdash_S\varphi$ for every $\varphi\in\Delta$, then we write $\Gamma\vdash_S\Delta$. For a finite set $\{\psi_1,\dots,\psi_n\}$ of k-formulas we often write $\psi_1,\dots,\psi_n\vdash_S\varphi$ for $\{\psi_1,\dots,\psi_n\}\vdash_S\varphi$. Similarly, if Δ is a finite set, say $\{\varphi_1,\dots,\varphi_m\}$, then we often write $\Gamma\vdash_S\varphi_1,\dots,\varphi_n$ for $\Gamma\vdash_S\{\varphi_1,\dots,\varphi_n\}$.

As for deductive systems, the relation $\vdash_S$ can be defined recursively in the following way: For every $\Gamma \subseteq Fm^k$ and every $\varphi \in Fm^k$,

(i) if $\varphi \in \Gamma$ or φ is a substitution instance of a axiom of S then $\Gamma \vdash_S \varphi$,

(ii) if $\Delta \subseteq Fm^k$ and φ is directly derivable from Δ by an inference rule of S and for every $\psi \in \Delta$, $\Gamma \vdash_S \psi$ then $\Gamma \vdash_S \varphi$.

The definitions of extensions, subsystems and fragments of 1-deductive systems extend in the obvious way to k-deductive systems.

Although the definition of a k-deductive system encompasses the definition of a 1-deductive system, our independent introduction of 1-deductive systems was deliberate. The mention of a (propositional) logic suggests the classical examples of 1-deductive systems (or 'sentential' logics) rather than the examples for $k > 1$. However, such examples do exist and are amenable to treatment similar to that given 1-deductive systems. We present a few examples of 2-deductive systems that will turn out to be essential to our later studies of algebraizable k-deductive systems.

The 2-deductive system $S_{\mathrm{Eq}} = \langle \mathcal{L}, \vdash_{S_{\mathrm{Eq}}} \rangle$ (where S_{Eq} can be thought of as the deductive system associated with equivalence relations) is defined by the following axiom and inference rules:

(1.5.1) $\vdash_{S_{\mathrm{Eq}}} (p, p)$,

(1.5.2) $(p, q) \vdash_{S_{\mathrm{Eq}}} (q, p)$,

(1.5.3) $(p, q), (q, r) \vdash_{S_{\mathrm{Eq}}} (p, r)$,

Note that we use round brackets rather than angled brackets for 2-formulas.

The 2-deductive system $S_{\mathrm{Con}} = \langle \mathcal{L}, \vdash_{S_{\mathrm{Con}}} \rangle$ (where S_{Con} can be thought of as the deductive system associated with congruence relations) is defined by the axiom and inference rules of S_{Eq} together with the following inference rules:

$(1.5.4)_f$ $(p_1, q_1), \ldots, (p_m, q_m) \vdash_{S_{\mathrm{Con}}} (f(p_1, \ldots, p_m), f(q_1, \ldots, q_m))$

for every $f \in \mathcal{L}$ with $ar(f) = m$.

Suppose $\mathfrak{K}$ is a quasivariety axiomatized by a set Id of identities together with a set Qi of quasi-identities that are not identities. We define the 2-deductive system $S_{\mathfrak{K}} = \langle \mathcal{L}, \vdash_{S_{\mathfrak{K}}} \rangle$ as follows: Let $\vdash_{S_{\mathcal{K}}} \supseteq \vdash_{S_{\text{Con}}}$, i.e., $\vdash_{S_{\mathcal{K}}}$ satisfies (1.5.1) to (1.5.3) and $(1.5.4)_f$ for each operation f in the type $\mathcal{L}$ of $\mathcal{K}$, and let

(1.5.5) $\quad \vdash_{S_{\mathcal{K}}} (\varphi, \psi) \quad$ for every $\varphi \approx \psi \in Id$,

(1.5.6) $\quad (\eta_1, \zeta_1), \dots, (\eta_m, \zeta_m) \vdash_{S_{\mathcal{K}}} (\varphi, \psi) \quad$ for every $\left(\underset{i \leq m}{\&} \eta_i \approx \zeta_i \right) \Rightarrow \varphi \approx \psi \in Qi$.

The 2-deductive system $S_{\mathcal{K}}$ depends only on the quasivariety $\mathcal{K}$ and not on its particular axiomatization. In fact, (1.5.5) holds for every identity satisfied by $\mathcal{K}$, and (1.5.6) holds for every quasi-identity satisfied by $\mathcal{K}$. These claims will follow from the fact (mentioned just after Lemma 1.6.9) that the $S_{\mathcal{K}}$-matrices are just the 2-matrices $\langle \mathbf{A}, \Phi \rangle$ such that $\mathbf{A} \in \mathcal{K}$ and $\Phi \in \text{Con}_{\mathcal{K}}\mathbf{A}$. Thus the quasi-equational theory of $\mathcal{K}$ (i.e., the set of all quasi-identities satisfied by $\mathcal{K}$) is captured in our framework of 2-deductive systems.

The consequence relation $\vdash_S$ of a k-deductive system S satisfies the following analogues of properties (1.1.1) to (1.1.5). For all $\Gamma, \Delta \subseteq Fm^k$ and $\varphi \in Fm^k$,

(1.5.7) $\quad \varphi \in \Gamma$ implies $\Gamma \vdash_S \varphi$,

(1.5.8) $\quad \Gamma \vdash_S \varphi$ and $\Gamma \subseteq \Delta$ implies $\Delta \vdash_S \varphi$,

(1.5.9) $\quad \Gamma \vdash_S \varphi$ and $\Delta \vdash_S \psi$ for every $\psi \in \Gamma$ implies $\Delta \vdash_S \varphi$,

(1.5.10) $\quad \Gamma \vdash_S \varphi$ implies there exists some finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash_S \varphi$,

(1.5.11) $\quad \Gamma \vdash_S \varphi$ implies $\sigma(\Gamma) \vdash_S \sigma\varphi$ for every substitution $\sigma : \mathbf{Fm} \to \mathbf{Fm}$.

Note that structurality (1.5.11) is still defined for all substitutions $\sigma : \mathbf{Fm} \to \mathbf{Fm}$ (and not $\sigma : \mathbf{Fm}^k \to \mathbf{Fm}^k$). The following definitions are analogues of those for deductive systems.

Given a k-deductive system S, we define the *consequence operator* of S, $\text{Cn}_S : \mathcal{P}(Fm^k) \to \mathcal{P}(Fm^k)$, by

$$\text{Cn}_S(\Gamma) = \{\varphi \in Fm^k ; \Gamma \vdash_S \varphi\}.$$

This operator satisfies properties (1.2.1) to (1.2.5). We shall write $\text{Cn}_S \varphi$ for $\text{Cn}_S(\{\varphi\})$. A set $T \subseteq Fm^k$ is called a *theory of* S, or an S-*theory*, if

$$T \vdash_S \varphi \quad \text{implies} \quad \varphi \in T, \quad \text{for each } \varphi \in Fm^k.$$

The set of all S-theories is denoted by ThS. Note that, as for deductive systems, Cn_S is an algebraic closure operator on the complete lattice $\langle \mathcal{P}(Fm^k), \subseteq \rangle$. If $\Gamma \subseteq Fm^k$, then (1.2.3) implies that $\text{Cn}_S(\Gamma)$ is an S-theory. We call $\text{Cn}_S(\Gamma)$ the S-*theory generated by* Γ and, if this is equal to $\text{Cn}_S(\Delta)$ for some finite $\Delta \subseteq Fm^k$, we say that this S-theory is *finitely generated*. If T is an S-

theory, then $\mathrm{Cn}_S(T) = T$. The set of all S-theories is therefore the range of the algebraic closure operator Cn_S. As in the case of 1-deductive systems, the set $\mathrm{Th}S$ is the universe of an algebraic lattice $\mathbf{Th}S$ (ordered by set inclusion) in which infima and joins are defined as before. The largest element of $\mathrm{Th}S$ is the set Fm^k of all formulas, and the smallest is $\mathrm{Cn}_S(\emptyset)$, the set of all theorems of S. The following lemma is a generalization of Lemma 1.2.1.

1.5.2 LEMMA

Let $S = \langle \mathfrak{L}, \vdash_S \rangle$ be a k-deductive system.

(i) $\mathrm{Th}S$ *is closed under unions of nonempty upwardly directed sets (i.e., for every nonempty upwardly directed family* $\{T_i; i \in I\}$ *of S-theories,* $\bigcup\{T_i; i \in I\} \in \mathrm{Th}S$*).*

(ii) *The lattice* $\mathbf{Th}S$ *is algebraic.*

(iii) *The compact elements of* $\mathbf{Th}S$ *coincide with the finitely generated S-theories.*

Proof. The result will follow as a corollary to Lemmas 1.6.4 and 1.6.5. □

We showed in Section 1.1 that for an S-theory T of a 1-deductive system S and a substitution σ, $\sigma(T)$ is not necessarily an S-theory. In particular, this holds for k-deductive systems. This leads to the following definition: For every substitution $\sigma : \mathbf{Fm} \to \mathbf{Fm}$, define the function $\sigma_S : \mathrm{Th}S \to \mathrm{Th}S$ by

$$\sigma_S(T) = \mathrm{Cn}_S(\sigma(T)) \text{ for all } T \in \mathrm{Th}S.$$

The following lemma generalizes Lemma 1.2.2 to k-deductive systems.

1.5.3 LEMMA

Let $S = \langle \mathfrak{L}, \vdash_S \rangle$ be a k-deductive system.

(i) $\mathrm{Th}S$ *is closed under inverse substitutions (i.e.,* $\sigma^{-1}(T) \in \mathrm{Th}S$ *for every* $T \in \mathrm{Th}S$ *and every substitution* σ*, where* $\sigma^{-1}(T) = \{\varphi = \langle \varphi_1, \ldots, \varphi_k \rangle \in Fm^k; \langle \sigma\varphi_1, \ldots, \sigma\varphi_k \rangle \in T\}$*).*

(ii) $\sigma_S(\mathrm{Cn}_S(\Gamma)) = \mathrm{Cn}_S(\sigma(\Gamma))$ *for all* $\Gamma \subseteq Fm^k$ *and every substitution* σ.

(iii) σ_S *is a join-continuous mapping of* $\mathrm{Th}S$ *into itself (i.e.,* $\sigma_S\left(\bigvee^{\mathbf{Th}S}_{i \in I} T_i\right) = \bigvee^{\mathbf{Th}S}_{i \in I} \sigma_S(T_i)$ *for every family* $\{T_i; i \in I\}$ *of S-theories and every substitution* σ*).*

Proof. Let $\sigma : \mathbf{Fm} \to \mathbf{Fm}$ be a substitution, $T \in \mathrm{Th}S$ and $\Gamma \subseteq Fm^k$.

(i) By (1.2.5), we have

$$\sigma(\mathrm{Cn}_S(\sigma^{-1}(T))) \subseteq \mathrm{Cn}_S(\sigma(\sigma^{-1}(T))) \subseteq \mathrm{Cn}_S(T) = T,$$

hence $\mathrm{Cn}_S(\sigma^{-1}(T)) \subseteq \sigma^{-1}(T)$, implying $\sigma^{-1}(T) = \mathrm{Cn}_S(\sigma^{-1}(T)) \in \mathrm{Th}S$.

(ii) By the definition of σ_S and (1.2.5) and (1.2.3), we have

$$\sigma_S(\mathrm{Cn}_S(\Gamma)) = \mathrm{Cn}_S(\sigma(\mathrm{Cn}_S(\Gamma))) \subseteq \mathrm{Cn}_S(\mathrm{Cn}_S(\sigma(\Gamma))) \subseteq \mathrm{Cn}_S(\sigma(\Gamma)), \text{ and}$$

$$\mathrm{Cn}_S(\sigma(\Gamma)) \subseteq \mathrm{Cn}_S(\sigma(\mathrm{Cn}_S(\Gamma))) = \sigma_S(\mathrm{Cn}_S(\Gamma)), \text{ hence } \sigma_S(\mathrm{Cn}_S(\Gamma)) = \mathrm{Cn}_S(\sigma(\Gamma)).$$

(iii) Using (ii) and the definition of the join in the lattice $\mathbf{Th}\boldsymbol{S}$, we get

$\sigma_S\Big(\bigvee^S_{i\in I} T_i\Big) = \sigma_S\Big(\mathrm{Cn}_S\Big(\bigcup_{i\in I} T_i\Big)\Big) = \mathrm{Cn}_S\Big(\sigma\Big(\bigcup_{i\in I} T_i\Big)\Big) = \mathrm{Cn}_S\Big(\bigcup_{i\in I}\sigma(T_i)\Big) =$
$\mathrm{Cn}_S\Big(\bigcup_{i\in I}\mathrm{Cn}_S\sigma(T_i)\Big)$ (by (0.1.1)) $= \mathrm{Cn}_S\Big(\bigcup_{i\in I}\sigma_S(T_i)\Big) = \bigvee^S_{i\in I}\sigma_S(T_i)$. □

Note that part (i) above is equivalent to the structurality of Cn_S: Let $\Gamma \subseteq Fm^k$ and let $\sigma: \mathbf{Fm} \to \mathbf{Fm}$ be any substitution. Then $\Gamma \subseteq \sigma^{-1}(\mathrm{Cn}_S(\sigma(\Gamma)))$ since, if $\varphi \in \Gamma$, then $\sigma\varphi \in \sigma(\Gamma)$, which implies $\sigma\varphi \in \mathrm{Cn}_S(\sigma(\Gamma))$ and hence that $\varphi \in \sigma^{-1}(\mathrm{Cn}_S(\sigma(\Gamma)))$. By (i), $\sigma^{-1}(\mathrm{Cn}_S(\sigma(\Gamma)))$ is an S-theory, so $\mathrm{Cn}_S(\Gamma) \subseteq \sigma^{-1}(\mathrm{Cn}_S(\sigma(\Gamma)))$, i.e., $\sigma(\mathrm{Cn}_S(\Gamma)) \subseteq \mathrm{Cn}_S(\sigma(\Gamma))$, proving structurality.

We conclude this section with a lemma that we shall need subsequently. For $\vartheta(p,\overline{r}) \in Fm^k$ and $\varphi \in Fm$, define $\vartheta[\varphi/p] = \vartheta(\varphi,\overline{r})$.

1.5.4 LEMMA

Assume $\sigma: \mathbf{Fm} \to \mathbf{Fm}$ is a surjective substitution. For every $\vartheta \in Fm^k$ and every variable p occurring in ϑ, there exists a $\vartheta' \in Fm^k$ and a variable q such that $\sigma(\vartheta'[\varphi/q]) = \vartheta[\sigma\varphi/p]$ for every $\varphi \in Fm$.

Proof. Using the fact that $\sigma^{-1}(P) \subseteq P$ for every substitution σ (proved in Section 1.1) and the assumption that σ is surjective, we have that for each variable r there exists a variable r' such that $\sigma r' = r$. Let ϑ' be obtained from $\vartheta = \vartheta(p,\overline{r})$ ($\overline{r}$ is a list of variables) by simultaneously replacing each variable r different from p by r', and p by any variable q different from all the r''s. Then $\vartheta' = \vartheta(q,\overline{r'})$, hence

$$\sigma(\vartheta'[\varphi/q]) = \sigma(\vartheta(\varphi,\overline{r'})) = \vartheta(\sigma\varphi,\overline{r}) = \vartheta[\sigma\varphi/p].$$ □

1.6 MATRIX SEMANTICS FOR k-DEDUCTIVE SYSTEMS

Let $\mathcal{L}$ be a fixed language and k a nonzero natural number. A *k-matrix* is a pair $\langle \mathbf{A}, F\rangle$ where $\mathbf{A}$

is an $\mathcal{L}$-algebra and $F \subseteq A^k$. An $\mathcal{L}$-matrix is, therefore, simply a 1-matrix. The elements of F are called *designated elements.* If $\varphi = \langle \varphi_1, \dots, \varphi_k \rangle \in Fm^k$ and the variables of the φ_i's are among $p_1, \dots, p_n$, we write

$$\varphi = \varphi(p_1, \dots, p_n) = \langle \varphi_1(p_1, \dots, p_n), \dots, \varphi_k(p_1, \dots, p_n) \rangle,$$

and if $\mathbf{A}$ is an $\mathcal{L}$-algebra and $\overline{a} = \langle a_1, \dots, a_n \rangle \in A^n$ is an interpretation of $p_1, \dots, p_n$ in A, write

$$\varphi^{\mathbf{A}}(\overline{a}) = \varphi^{\mathbf{A}}(a_1, \dots, a_n) = \langle {\varphi_1}^{\mathbf{A}}(\overline{a}), \dots, {\varphi_k}^{\mathbf{A}}(\overline{a}) \rangle.$$

If $\mathcal{A} = \langle \mathbf{A}, F \rangle$ is a k-matrix, then we define the relation $\models_{\mathcal{A}}$ as follows. For all $\Gamma \subseteq Fm^k$ and $\varphi \in Fm^k$,

$$\Gamma \models_{\mathcal{A}} \varphi \quad \text{iff} \quad \text{for every interpretation } \overline{a} \text{ of the variables of } \Gamma \cup \{\varphi\} \text{ in } A,$$

$$\psi^{\mathbf{A}}(\overline{a}) \in F \text{ for every } \psi \in \Gamma \text{ implies } \varphi^{\mathbf{A}}(\overline{a}) \in F.$$

If $\Gamma, \Delta \subseteq Fm^k$, then we define

$$\Gamma \models_{\mathcal{A}} \Delta \text{ if and only if } \Gamma \models_{\mathcal{A}} \varphi \text{ for every } \varphi \in \Delta.$$

If M is a class of k-matrices, then the relation $\models_M$ is defined, for all $\Gamma \subseteq Fm^k$ and $\varphi \in Fm^k$, by

$$\Gamma \models_M \varphi \quad \text{iff} \quad \Gamma \models_{\mathcal{A}} \varphi \text{ for every } k\text{-matrix } \mathcal{A} \in M.$$

Again, if $\Delta \subseteq Fm^k$, then we define

$$\Gamma \models_M \Delta \text{ if and only if } \Gamma \models_M \varphi \text{ for every } \varphi \in \Delta.$$

If $\Gamma, \Delta \subseteq Fm^k$ and M is a class of k-matrices, then, by

$$\Gamma \mathrel{=\!\mid\!\models_M} \Delta$$

we mean that $\Gamma \models_M \Delta$ and $\Delta \models_M \Gamma$.

The reader may notice that k-matrices may be considered as first-order structures (see Section 0.5) over a language which consists of $\mathcal{L}$ together with one k-ary relation symbol. This connection will be explored in more detail in the final section of this chapter; it guides several of the definitions that will follow.

1.6.1 DEFINITION

Let S be a k-deductive system. A k-matrix $\mathcal{A} = \langle \mathbf{A}, F \rangle$ is called a *matrix model* for S, or an *S-matrix*, if, for all $\Gamma \subseteq Fm^k$ and $\varphi \in Fm^k$,

$$\Gamma \vdash_S \varphi \quad \text{implies} \quad \Gamma \models_{\mathcal{A}} \varphi.$$

We denote the class of all matrix models of S by Mod$S$. Matrix models of S whose $\mathcal{L}$-algebras are the formula algebra **Fm** are called *formula matrix models.* A class M of matrix models of S is called a *matrix semantics* for S provided that, for all $\Gamma \subseteq Fm^k$ and $\varphi \in Fm^k$,

$$\Gamma \vdash_S \varphi \quad \text{if and only if} \quad \Gamma \models_M \varphi.$$

As was the case for 1-deductive systems, it is easy to see that the relation $\models_M$ satisfies properties (1.5.7), (1.5.8), (1.5.9) and (1.5.11) (with $\vdash_S$ replaced by $\models_M$). For a k-matrix $\mathcal{A}$ to be an S-matrix it is sufficient that $\models_{\mathcal{A}} \varphi$ for every axiom φ of S, and that $\psi_1, \ldots, \psi_n \models_{\mathcal{A}} \varphi$ for every inference rule $\langle \{\psi_1, \ldots, \psi_n\}, \varphi \rangle$ of S. To see this, suppose that $\Gamma \vdash_S \varphi$. If $\varphi \in \Gamma$, then $\Gamma \models_{\mathcal{A}} \varphi$ by the definition of $\models_{\mathcal{A}}$. If φ is a substitution instance of an axiom of S, then $\models_{\mathcal{A}} \varphi$, by assumption, hence $\Gamma \models_{\mathcal{A}} \varphi$. Suppose $\Delta \subseteq Fm^k$ and φ is directly derivable from Δ by the inference rule $\langle \Pi, \vartheta \rangle$ of S and for every $\psi \in \Delta$, $\Gamma \vdash_S \psi$. Assume inductively that $\Gamma \models_{\mathcal{A}} \psi$ for each $\psi \in \Delta$. There exists a substitution σ such that $\sigma(\Pi) \subseteq \Delta$ and $\sigma\vartheta = \varphi$, and, by assumption, $\Pi \models_{\mathcal{A}} \vartheta$. Since $\models_{\mathcal{A}}$ is structural, $\sigma(\Pi) \models_{\mathcal{A}} \sigma\vartheta$, hence, by (1.5.8), $\Delta \models_{\mathcal{A}} \varphi$. Since $\Gamma \models_{\mathcal{A}} \psi$ for each $\psi \in \Delta$, $\Gamma \models_{\mathcal{A}} \varphi$, by (1.5.9).

If $\mathcal{A}$ is a k-matrix and φ is an axiom of S, we often say that $\mathcal{A}$ is *closed under* φ if $\models_{\mathcal{A}} \varphi$. Similarly, if $\langle \Gamma, \varphi \rangle$ is an inference rule of S then we often say that $\mathcal{A}$ is *closed under* $\langle \Gamma, \varphi \rangle$ if $\Gamma \models_{\mathcal{A}} \varphi$. Thus, by the previous paragraph, $\mathcal{A}$ is an S-matrix iff $\mathcal{A}$ is closed under each of the axioms and inference rules of S.

An example of a matrix semantics for **CPC** is the class $\{\langle \mathbf{B}, \{1^{\mathbf{B}}\}\rangle ; \mathbf{B}$ a Boolean algebra$\}$. This follows from the Completeness and Validity theorems of **CPC**. This is not the only example, however. The set $\{\langle \mathbf{2}, \{1\}\rangle\}$, where $\mathbf{2}$ is the 2-element Boolean algebra is also a matrix semantics for **CPC**. That $\langle \mathbf{2}, \{1\}\rangle$ is an S-matrix was shown, in Section 1.3, to be a consequence of the Validity Theorem of Classical Propositional Calculus. The converse follows from the Completeness Theorem of Propositional Calculus. The following theorem can be considered as a joint validity and completeness theorem for k-deductive systems with respect to matrix models of S.

1.6.2 THEOREM [BP92, Theorem 4.2]

Let S be a k-deductive system and M the class of all matrix models of S, i.e., $M = \mathrm{Mod}S$, or let

M be the class of all formula matrix models of S. Then M is a matrix semantics for S.

Proof. Suppose that $\Gamma \subseteq Fm^k$ and $\varphi \in Fm^k$ such that $\Gamma \models_M \varphi$. Let $\mathcal{A} = \langle \mathbf{Fm}, \mathrm{Cn}_S(\Gamma) \rangle$. Recall from the second paragraph of Section 1.3 that an interpretation of P in Fm can be identified with a homomorphism from $\mathbf{Fm}$ into $\mathbf{Fm}$, i.e., a substitution. Let $\Delta \subseteq Fm^k$ and $\psi \in Fm^k$ such that $\Delta \vdash_S \psi$. Let $\sigma : \mathbf{Fm} \to \mathbf{Fm}$ be an interpretation of P in Fm such that $\sigma(\Delta) \subseteq \mathrm{Cn}_S(\Gamma)$. Then $\Gamma \vdash_S \sigma(\Delta)$. Now $\sigma(\Delta) \vdash_S \sigma(\psi)$ by structurality hence, by (1.5.9), $\Gamma \vdash_S \sigma(\psi)$. Thus $\sigma(\psi) \subseteq \mathrm{Cn}_S(\Gamma)$ and so $\Delta \models_{\mathcal{A}} \psi$. Thus $\mathcal{A}$ is a formula matrix model of S so $\mathcal{A} \in M$. By assumption $\Gamma \models_{\mathcal{A}} \varphi$. If we interpret $\Gamma \cup \{\varphi\}$ simply as $\Gamma \cup \{\varphi\}$ in Fm, then $\Gamma \subseteq \mathrm{Cn}_S(\Gamma)$ implies $\varphi \in \mathrm{Cn}_S(\Gamma)$, i.e., $\Gamma \vdash_S \varphi$. Conversely, by the definition of an S-matrix, if $\Gamma \vdash_S \varphi$ then $\Gamma \models_M \varphi$. □

1.6.3 DEFINITION

Let $\mathcal{A} = \langle \mathbf{A}, F_{\mathcal{A}} \rangle$ be a k-matrix. A set $F \subseteq A^k$ is called an *S-filter of $\mathcal{A}$* if $\langle \mathbf{A}, F \rangle$ is an S-matrix and $F \supseteq F_{\mathcal{A}}$. The set of all S-filters of $\mathcal{A}$ is denoted by $\mathrm{Fi}^S\mathcal{A}$. If $\{G_i\,;\, i \in I\}$ is a set of S-filters of $\mathcal{A}$, then $\bigcap \{G_i\,;\, i \in I\}$ is an S-filter of $\mathcal{A}$ as well. (If $\Gamma \vdash_S \varphi$ and $\overline{a}$ is an interpretation of the variables of $\Gamma \cup \{\varphi\}$ in A such that $\psi^{\mathbf{A}}(\overline{a}) \in \bigcap \{G_i\,;\, i \in I\}$ for every $\psi \in \Gamma$, then $\psi^{\mathbf{A}}(\overline{a}) \in G_i$ for every $\psi \in \Gamma$ and every $i \in I$. This implies that $\varphi^{\mathbf{A}}(\overline{a}) \in G_i$ for every $i \in I$, hence $\varphi^{\mathbf{A}}(\overline{a}) \in \bigcap \{G_i\,;\, i \in I\}$). It follows that $\mathrm{Fi}^S\mathcal{A}$ is a closure system on the complete lattice of all subsets of A^k (ordered by set inclusion). This allows us to define, for all $X \subseteq A^k$,

$$\mathrm{Fg}^S_{\mathcal{A}} X = \bigcap \{F \in \mathrm{Fi}^S\mathcal{A}\,;\, F \supseteq X\}.$$

We call $\mathrm{Fg}^S_{\mathcal{A}} X$ the *S-filter of $\mathcal{A}$ generated by X*. If X is finite, then $\mathrm{Fg}^S_{\mathcal{A}} X$ is said to be *finitely generated*; if X consists of a single element, $\boldsymbol{a}$ say, then $\mathrm{Fg}^S_{\mathcal{A}}\{\boldsymbol{a}\}$ is called a *principal filter* of $\mathcal{A}$ and we write $\mathrm{Fg}^S_{\mathcal{A}} \boldsymbol{a}$ instead of $\mathrm{Fg}^S_{\mathcal{A}}\{\boldsymbol{a}\}$. Observe that $\mathrm{Fg}^S_{\mathcal{A}}$ is a closure operator on the complete lattice of all subsets of A^k corresponding to $\mathrm{Fi}^S\mathcal{A}$ (in the sense of Section 0.1). In particular, $\mathrm{Fi}^S\mathcal{A}$ is the range of $\mathrm{Fg}^S_{\mathcal{A}}$ and is itself the universe of the complete lattice $\mathbf{Fi}^S\boldsymbol{\mathcal{A}} = \langle \mathrm{Fi}^S\mathcal{A}, \cap, \vee \rangle$, whose partial order is set inclusion. (Arbitrary) infima coincide with intersections in this lattice, while the join in $\mathrm{Fi}^S\mathcal{A}$ of a family $\{F_i\,;\, i \in I\}$ of S-filters of $\mathcal{A}$ is $\mathrm{Fg}^S_{\mathcal{A}}(\bigcup_{i \in I} F_i)$.

1.6.4 LEMMA [BP89b, Proposition 2.2.2]

Let S be a k-deductive system and $\mathcal{A}$ a k-matrix.

(i) $\mathrm{Fi}^S\mathcal{A}$ *is closed under unions of nonempty upwardly directed sets (i.e., for every nonempty upwardly directed family* $\{F_i;\, i \in I\}$ *of S-filters of* $\mathcal{A}$, $\bigcup_{i \in I} F_i \in \mathrm{Fi}^S\mathcal{A}$*).*

(ii) $\mathbf{Fi}^S\boldsymbol{\mathcal{A}}$ *is algebraic.*

(iii) *The compact elements of* $\mathbf{Fi}^S\boldsymbol{\mathcal{A}}$ *coincide with the finitely generated S-filters of* $\mathcal{A}$.

Proof. We shall show that the union of any nonempty upwardly directed family of closed sets (i.e., S-filters of $\mathcal{A}$) is a closed set (S-filter of $\mathcal{A}$). This immediately proves (i). Furthermore, this will prove that $\mathrm{Fi}^S\mathcal{A}$ is an algebraic closure system (i.e., that $\mathrm{Fg}^S_{\mathcal{A}}$ is an algebraic closure operator), from which the statements (ii) and (iii) will follow (by Corollary 0.1.6).

Let $\{F_i;\, i \in I\}$ be an upwardly directed subset of $\mathrm{Fi}^S\mathcal{A}$ and suppose that $\Gamma \subseteq Fm^k$ and $\varphi \in Fm^k$ such that $\Gamma \vdash_S \varphi$. Then there exists a finite set $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash_S \varphi$. Let $\overline{a}$ be an interpretation of the variables of the k-formulas of $\Gamma' \cup \{\varphi\}$ in A such that

$$\psi^{\mathbf{A}}(\overline{a}) \in \bigcup_{i \in I} F_i \quad \text{for each } \psi \in \Gamma'.$$

Since there are only finitely many ψ's in Γ', there exists a finite $J \subseteq I$ such that

$$\psi^{\mathbf{A}}(\overline{a}) \in \bigcup_{i \in J} F_i \quad \text{for each } \psi \in \Gamma'.$$

Since $\{F_i;\, i \in I\}$ is upwardly directed, there exists $\ell \in I$ such that $F_i \subseteq F_\ell$ for all $i \in J$, hence $\bigcup_{i \in J} F_i \subseteq F_\ell$. Thus $\psi^{\mathbf{A}}(\overline{a}) \in F_\ell$ for each $\psi \in \Gamma'$, implying that $\varphi^{\mathbf{A}}(\overline{a}) \in F_\ell$ since F_ℓ is an S-filter of $\mathcal{A}$. So, $\varphi^{\mathbf{A}}(\overline{a}) \in \bigcup_{i \in I} F_i$, which shows that $\Gamma' \models_{\mathcal{A}'} \varphi$, where $\mathcal{A}' = \langle \mathbf{A}, \bigcup_{i \in I} F_i \rangle$. It then follows that $\Gamma \models_{\mathcal{A}'} \varphi$, hence $\bigcup_{i \in I} F_i$ is an S-filter of $\mathcal{A}$. □

The following useful proposition clarifies the connection between S-theories and S-filters and between the consequence relation $\vdash_S$ and S-filters. (It is used tacitly in the literature without explicit statement or proof). Let S be a k-deductive system and let $\emptyset \neq Q \subseteq P$. By $Fm(Q)$ we mean the set of all (1-)formulas of Fm in which only the variables of Q occur. Clearly $Fm(Q)$ is the universe of the absolutely free $\mathcal{L}$-algebra over Q, which we denote by $\mathbf{Fm}(Q)$. We use $(Fm(Q))^k$ to denote the set of k-tuples $\langle \varphi_1, \ldots, \varphi_k \rangle$, where $\varphi_i \in Fm(Q)$ for each $i \leq k$.

1.6.5 PROPOSITION

Let S be a k-deductive system. Let $\emptyset \neq Q \subseteq P$.

(i) *Let $T \subseteq (Fm(Q))^k$. Then $\mathcal{A} = \langle \mathbf{Fm}(Q), T\rangle$ is a matrix model of S if and only if $T = \mathrm{Cn}_S(T) \cap (Fm(Q))^k$. For a matrix model $\mathcal{A} = \langle \mathbf{Fm}(Q), T\rangle$, we have that $\mathrm{Fi}^S\mathcal{A} = \{U \cap (Fm(Q))^k ; U \in \mathrm{Th}S$ and $U \supseteq T\}$. In particular, $\mathcal{A} = \langle \mathbf{Fm}, T\rangle$ is a formula matrix model if and only if T is an S-theory. The S-filters of $\mathcal{A}$ are just the S-theories containing T.*

(ii) *Let $\mathcal{A} = \langle \mathbf{Fm}(Q), T \cap (Fm(Q))^k\rangle$ be an S-matrix, where $T \in \mathrm{Th}S$, and let $\Gamma \subseteq (Fm(Q))^k$. Then*

$$\mathrm{Fg}^S_{\mathcal{A}}\Gamma = (T \vee \mathrm{Cn}_S(\Gamma)) \cap (Fm(Q))^k.$$

In particular, if $\mathcal{A} = \langle \mathbf{Fm}, T_{\mathcal{A}}\rangle$ is a formula matrix model of S then, for $\Gamma \subseteq Fm^k$, $\mathrm{Fg}^S_{\mathcal{A}}\Gamma = T_{\mathcal{A}} \vee \mathrm{Cn}_S(\Gamma)$.

(iii) *Let $\mathcal{A} = \langle \mathbf{Fm}(Q), \mathrm{Cn}_S(\Gamma) \cap (Fm(Q))^k\rangle$ be an S-matrix, where $\Gamma \subseteq (Fm(Q))^k$, and let $\varphi, \psi \in (Fm(Q))^k$. Then*

$$\Gamma, \varphi \vdash_S \psi \quad \textit{if and only if} \quad \psi \in \mathrm{Fg}^S_{\mathcal{A}}\varphi.$$

In particular, the above equivalence holds for a formula matrix model $\mathcal{A} = \langle \mathbf{Fm}, \mathrm{Cn}_S(\Gamma)\rangle$, where $\Gamma \subseteq Fm^k$ and $\varphi, \psi \in Fm^k$.

Proof. (i) Suppose that $\mathcal{A} = \langle \mathbf{Fm}(Q), T\rangle$ is a matrix model of S. The inclusion from left to right is trivial. Conversely, let $\varphi \in \mathrm{Cn}_S(T) \cap (Fm(Q))^k$. Then $T \vdash_S \varphi$, hence $T \models_{\mathcal{A}} \varphi$, by assumption. Fix any $r \in Q$. Define a map $j : P \to Fm(Q)$ by $jq = q$ for all $q \in Q$ and $jp = r$ for all $p \in P - Q$. We can consider j as an interpretation of P, the set of all variables, in $Fm(Q)$. For all $\psi \in T$ we have

$$j\psi(\overline{p}) = \psi(j\overline{p}) = \psi(\overline{p}) \in T,$$

hence $\varphi(\overline{p}) = j\varphi(\overline{p}) \in T$.

Conversely, suppose $T = \mathrm{Cn}_S(T) \cap (Fm(Q))^k$. Let $\Gamma \subseteq Fm^k$ and $\varphi \in Fm^k$ such that $\Gamma \vdash_S \varphi$, and let $h \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{Fm}(Q))$ extend an interpretation of the variables of $\Gamma \cup \{\varphi\}$ in $Fm(Q)$ such that

$$h\psi \in T \text{ for all } \psi \in \Gamma, \text{ i.e.,}$$

$$h\psi \in \mathrm{Cn}_S(T) \cap (Fm(Q))^k \text{ for all } \psi \in \Gamma, \text{ so}$$

$$T \vdash_S h\psi \text{ for all } \psi \in \Gamma. \tag{1.6.1}$$

We can consider h as a substitution, hence $\Gamma \vdash_S \varphi$ implies $h(\Gamma) \vdash_S h\varphi$, by structurality. Since

$T \vdash_S h\psi$ for all $h\psi \in h(\Gamma)$, (1.5.9) implies that $T \vdash_S h\varphi$, i.e., $h\varphi \in \mathrm{Cn}_S(T)$. Since $h\varphi \in (Fm(Q))^k$ as well, we have that $h\varphi \in T$. Thus $\Gamma \models_{\mathcal{A}} \varphi$ and $\mathcal{A}$ is a matrix model for S. The other statements follow from the fact that when $Q = P$, $\mathbf{Fm}(Q) = \mathbf{Fm}$ and membership of $(Fm(Q))^k$ becomes redundant.

(ii) Let $\mathcal{A} = \langle \mathbf{Fm}(Q), T \cap (Fm(Q))^k \rangle$ be a matrix model of S where $T \in \mathrm{Th}S$. By Definition 1.6.3,

$$
\begin{aligned}
\mathrm{Fg}^S_{\mathcal{A}}\Gamma &= \bigcap \{F \in \mathrm{Fi}^S\mathcal{A} \,;\, F \supseteq \Gamma\} \\
&= \bigcap \{U \cap (Fm(Q))^k \,;\, U \in \mathrm{Th}S,\ U \supseteq T \text{ and } U \supseteq \Gamma\} \\
&\quad \text{[by (i) and Def. 1.6.3]} \\
&= (Fm(Q))^k \cap \Big(\bigcap \{U \,;\, U \in \mathrm{Th}S,\ U \supseteq T \text{ and } U \supseteq \Gamma\}\Big) \\
&= (Fm(Q))^k \cap \Big(\bigcap \{U \in \mathrm{Th}S \,;\, U \supseteq (T \cup \Gamma)\}\Big) \\
&= (Fm(Q))^k \cap (\mathrm{Cn}_S(T \cup \Gamma)) \\
&= (Fm(Q))^k \cap (T \vee \mathrm{Cn}_S(\Gamma)).
\end{aligned}
$$

As in (i), the second statement follows from the fact that when $Q = P$, $\mathbf{Fm}(Q) = \mathbf{Fm}$ and membership of $(Fm(Q))^k$ becomes redundant.

(iii)
$$
\begin{aligned}
\psi \in \mathrm{Fg}^S_{\mathcal{A}}\varphi \quad &\text{iff} \quad \psi \in (\mathrm{Cn}_S(\Gamma) \vee \mathrm{Cn}_S\varphi) \cap (Fm(Q))^k \qquad \text{[by (ii)]} \\
&\text{iff} \quad \psi \in \mathrm{Cn}_S(\Gamma \cup \{\varphi\}) \\
&\text{iff} \quad \Gamma, \varphi \vdash_S \psi.
\end{aligned}
$$
□

It is worth noting that in (ii) of the above lemma, if the formula matrix model of S is taken to be $\mathcal{A} = \langle \mathbf{Fm}, \mathrm{Cn}_S(\emptyset) \rangle$, then $\mathrm{Fg}^S_{\mathcal{A}}\Gamma = \mathrm{Cn}_S(\Gamma)$ for all $\Gamma \subseteq Fm^k$, hence $\mathrm{Th}S = \mathrm{Fi}^S\mathcal{A}$. Recall Lemma 1.5.2; the lattice $\mathbf{Th}S$ of S-theories used there is thus the lattice $\mathbf{Fi}^S\mathcal{A}$, where $\mathcal{A} = \langle \mathbf{Fm}, \mathrm{Cn}_S(\emptyset) \rangle$. Lemma 1.5.2 is therefore a special case of Lemma 1.6.4. We will make frequent use of the results of Lemma 1.6.5 in future proofs without referencing them explicitly.

1.6.6 LEMMA [BP89b, Lemma 2.2.3]

Let S be a k-deductive system, $\mathcal{A} = \langle \mathbf{A}, F_{\mathcal{A}} \rangle$ a k-matrix, $X \subseteq A^k$ and $\boldsymbol{a} \in A^k$. Then $\boldsymbol{a} \in \mathrm{Fg}^S_{\mathcal{A}}X$ if and only if $\boldsymbol{a} \in \mathrm{Fg}^S_{\mathcal{A}}X'$ for some finite $X' \subseteq X$.

Proof. We have that

$$a \in \mathrm{Fg}^S_{\mathcal{A}} X \quad \text{if and only if} \quad \mathrm{Fg}^S_{\mathcal{A}} a \subseteq \mathrm{Fg}^S_{\mathcal{A}} X = \bigvee\{\mathrm{Fg}^S_{\mathcal{A}} b\, ;\, b \in X\}.$$

But $\mathrm{Fg}^S_{\mathcal{A}} a$ is compact in $\mathbf{Fi}^S \mathcal{A}$ by Lemma 1.6.4 (i), so, by Lemma 1.6.4 (ii), there exists a finite $X' \subseteq X$ such that

$$\mathrm{Fg}^S_{\mathcal{A}} a \subseteq \bigvee\{\mathrm{Fg}^S_{\mathcal{A}} b\, ;\, b \in X'\} = \mathrm{Fg}^S_{\mathcal{A}} X'. \qquad \square$$

The following lemma presents an explicit characterization of the elements of an S-filter generated by a given set.

1.6.7 LEMMA

Let S be a k-deductive system, $\mathcal{A} = \langle \mathbf{A}, F\rangle$ a k-matrix and $X \subseteq A^k$. Let Y be the set of all $b \in A^k$ that satisfy the following condition: There exists a positive integer n and $b^1, b^2, \ldots, b^n \in A^k$ with $b^n = b$ such that for each $i \leq n$, either

(1) $b^i \in X \cup F$, or

(2) there exists a finite subset $\Gamma \cup \{\varphi\}$ of Fm^k and an interpretation $\overline{c^i} = c^i_1, \ldots, c^i_{m(i)}$ of the variables of $\Gamma \cup \{\varphi\}$ in A such that

(i) $\Gamma \vdash_S \varphi$,

(ii) $\{\psi^{\mathbf{A}}(\overline{c^i})\, ;\, \psi \in \Gamma\} \subseteq \{b^j\, ;\, j < i\}$,

(iii) $\varphi^{\mathbf{A}}(\overline{c^i}) = b^i$.

Then $Y = \mathrm{Fg}^S_{\mathcal{A}} X$.

Proof. By definition, $X \cup F \subseteq Y$. If $\Sigma \vdash_S \mu$ then, since S is finitary, we have $\Sigma' \vdash_S \mu$ for some finite $\Sigma' \subseteq \Sigma$, say $\Sigma' = \{\psi_1, \ldots, \psi_s\}$. Let $p_1, \ldots, p_r$ be the variables occurring in $\Sigma' \cup \{\mu\}$ and let $d_1, \ldots, d_r$ be an interpretation of these variables (respectively) in A such that $\psi^{\mathbf{A}}_j(\overline{d}) \in Y$ for all $j \leq s$. For each $j \leq s$, there exist $b^1_j, \ldots, b^{n(j)}_j \in A^k$, as in the definition of Y, with $b^{n(j)}_j = \psi^{\mathbf{A}}_j(\overline{d})$. The sequence

$$b^1_1, \ldots, b^{n(1)}_1, \ldots, b^1_s, \ldots, b^{n(s)}_s, \mu^{\mathbf{A}}(\overline{d})$$

establishes that $\mu^{\mathbf{A}}(\overline{d}) \in Y$. Thus $Y \in \mathrm{Fi}^S \mathcal{A}$, so $\mathrm{Fg}^S_{\mathcal{A}} X \subseteq Y$.

Conversely, let G be an S-filter of $\mathcal{A}$ with $X \subseteq G$. Let $b \in Y$ and let $b^1, \ldots, b^n$ be as in the definition of Y, with $b^n = b$. We claim that $b^i \in G$ for each $i \leq n$. This is true if $b^i \in X \cup F$. Otherwise there exist $\Gamma, \varphi, \overline{c^i}$ as in the definition of Y satisfying (i), (ii) and (iii). If $i = 1$, then

$\vdash_S \varphi$ and $b^i = \varphi^{\mathbf{A}}(\overline{c^i}) \in G$, since $G \in \mathrm{Fi}^S\mathcal{A}$. Otherwise, by (ii) and an induction hypothesis, $\psi^{\mathbf{A}}(\overline{c^i}) \in G$, since $G \in \mathrm{Fi}^S\mathcal{A}$. So the claim is true, and in particular, $b = b^n \in G$. Thus $Y \subseteq G$. This shows that $Y \subseteq \mathrm{Fg}^S_{\mathcal{A}}X$, and so $Y = \mathrm{Fg}^S_{\mathcal{A}}X$. □

1.6.8 COROLLARY

Let S be a k-deductive system, $\mathcal{A} = \langle \mathbf{A}, F\rangle$ a k-matrix and $a, b \in A^k$, such that $b \in \mathrm{Fg}^S_{\mathcal{A}}\, a$. Then there exists a finitely generated subalgebra $\mathbf{A}'$ of $\mathbf{A}$, with $a, b \in (A')^k$, such that $b \in \mathrm{Fg}^S_{\mathcal{A}'}\, a$, where $\mathcal{A}'$ is the submatrix $\langle \mathbf{A}', F \cap (A')^k\rangle$ of $\mathcal{A}$.

Proof. Set $X = \{a\}$ in the lemma. By the lemma, since $b \in \mathrm{Fg}^S_{\mathcal{A}}\, a$, there exist $b^1, \ldots, b^n \in A^k$ with $b^n = b$ as described in the definition of Y. To each $b^i \notin \{a\} \cup F$, there correspond Γ, φ and $c^i_1, \ldots, c^i_{m(i)} \in A$ satisfying (i), (ii) and (iii). Set

$$Z = \{a_1, \ldots, a_k, b_1, \ldots, b_k\} \cup (\bigcup \{\{b^i_1, \ldots, b^i_k\}\,;\, b^i \in F\}) \cup (\bigcup \{\{c^i_1, \ldots, c^i_{m(i)}\}\,;\, b^i \notin \{a\} \cup F\}).$$

Let $\mathcal{A}' = \mathrm{Sg}^{\mathbf{A}}(Z)$. Then A' is the universe of a finitely generated subalgebra $\mathbf{A}'$ of $\mathbf{A}$ and $\mathcal{A}' = \langle \mathbf{A}', F \cap (A')^k\rangle$ is a submatrix of $\mathcal{A}$. For any ψ in any of the $\Gamma \cup \{\varphi\}$ involved in the above account of the fact that $b \in \mathrm{Fg}^S_{\mathcal{A}}\, a$, we have $\psi^{\mathbf{A}'}(\overline{c^i}) = \psi^{\mathbf{A}}(\overline{c^i}) \in A'$, by definition of $\mathcal{A}'$. It therefore follows directly from the lemma that $b \in \mathrm{Fg}^S_{\mathcal{A}'}\, a$. □

Recall the 2-deductive systems S_{Eq}, S_{Con} and $S_{\mathfrak{K}}$ from Section 1.5. Let $\mathbf{A}$ be an $\mathfrak{L}$-algebra and $F \subseteq A^2$, and suppose that $\mathcal{A} = \langle \mathbf{A}, F\rangle$ is an S_{Eq}-matrix. Take any $a, b, c \in A$. We can consider a, b, c as an interpretation of the variables p, q, r in A, respectively. From (1.5.1) we get that $(a, a) \in F$, hence F is a reflexive relation on A. From (1.5.2) we get that if $(a, b) \in F$ then $(b, a) \in F$, hence F is a symmetric relation on A, and similarly (1.5.3) tells us that F is transitive, therefore F is an equivalence relation on A.

Conversely, if R is any equivalence relation on A, then $\mathcal{A} = \langle \mathbf{A}, R\rangle$ is an S_{Eq}-matrix. To see this, suppose that $\varphi, \psi, \vartheta \in Fm$ and that $\overline{a}$ is an interpretation of the variables of φ, ψ, ϑ in A such that $(\varphi^{\mathbf{A}}(\overline{a}), \psi^{\mathbf{A}}(\overline{a})), (\psi^{\mathbf{A}}(\overline{a}), \vartheta^{\mathbf{A}}(\overline{a})) \in R$. Since Φ is an equivalence relation on A, $(\varphi^{\mathbf{A}}(\overline{a}), \vartheta^{\mathbf{A}}(\overline{a})) \in \Phi$ as well, thus $(\varphi, \psi), (\psi, \vartheta) \models_{\mathcal{A}} (\varphi, \vartheta)$. Similarly, $\models_{\mathcal{A}} (\varphi, \varphi)$ and $(\varphi, \psi) \models_{\mathcal{A}} (\psi, \varphi)$. By the note following Definition 1.6.1, $\mathcal{A}$ is an S_{Eq}-matrix.

If $\mathcal{A} = \langle \mathbf{A}, F\rangle$ is an S_{Con}-matrix then, since $\vdash_{S_{\mathrm{Eq}}} \subseteq \vdash_{S_{\mathrm{Con}}}$, F is an equivalence relation on A. Now, if $a_1,\dots,a_m,b_1,\dots,b_m \in A$ then we can consider $a_1,\dots,a_m,b_1,\dots,b_m$ as an interpretation of the variables $p_1,\dots,p_m,q_1,\dots,q_m$ in A. If $(a_1,b_1),\dots,(a_m,b_m) \in F$ then, by $(1.5.4)_f$, $(f^{\mathbf{A}}(a_1,\dots,a_m), f^{\mathbf{A}}(b_1,\dots,b_m)) \in F$ for each $f \in \mathfrak{L}$ with $ar(f)=m$, hence F is a congruence relation on $\mathbf{A}$. Conversely, it is easy to show, as was done for S_{Eq}, that all 2-matrices of the form $\langle \mathbf{A}, \Phi\rangle$, where $\Phi \in \mathrm{Con}\,\mathbf{A}$, are S_{Con}-matrices.

We claim that the $S_{\mathfrak{K}}$-matrices are precisely the 2-matrices $\langle \mathbf{A}, \Phi\rangle$, where Φ is a $\mathfrak{K}$-congruence of $\mathbf{A}$ (i.e., $\Phi \in \mathrm{Con}\,\mathbf{A}$ and $\mathbf{A}/\Phi \in \mathfrak{K}$). To show this we shall need the following

1.6.9 LEMMA

Let $\mathfrak{K}$ be a quasivariety. For every 2-matrix $\mathcal{A} = \langle \mathbf{A}, F\rangle$ with $F \in \mathrm{Con}\,\mathbf{A}$ and every quasi-identity $\left(\&_{i\le m}\eta_i \approx \zeta_i\right) \Rightarrow \varphi \approx \psi$ of the language of $\mathfrak{K}$,

$\mathbf{A}/F \models \left(\underset{i\le m}{\&}\eta_i \approx \zeta_i\right) \Rightarrow \varphi \approx \psi$ *if and only if* $(\eta_1,\zeta_1),\dots,(\eta_m,\zeta_m) \models_{\mathcal{A}} (\varphi,\psi)$.

Proof. Let $\mathcal{A} = \langle \mathbf{A}, F\rangle$ be any 2-matrix. Let $(\eta_1,\zeta_1),\dots,(\eta_m,\zeta_m),(\varphi,\psi) \in Fm^2$. Then

$(\eta_1,\zeta_1),\dots,(\eta_m,\zeta_m) \models_{\mathcal{A}} (\varphi,\psi)$

iff for every interpretation $\overline{a}$ of the variables of the η_i's, ζ_i's, φ and ψ in A,

$$(\eta_i^{\mathbf{A}}(\overline{a}), \zeta_i^{\mathbf{A}}(\overline{a})) \in F \text{ for all } i \le m \text{ implies } (\varphi^{\mathbf{A}}(\overline{a}), \psi^{\mathbf{A}}(\overline{a})) \in F$$

iff for every interpretation $\overline{a}$ of the variables of the η_i's, ζ_i's, φ and ψ in A

$$\eta_i^{\mathbf{A}}(\overline{a})/F = \zeta_i^{\mathbf{A}}(\overline{a})/F \text{ for all } i \le m \text{ implies } \varphi^{\mathbf{A}}(\overline{a})/F = \psi^{\mathbf{A}}(\overline{a})/F,$$

iff for every interpretation $\overline{a}/F$ of the variables of the η_i's, ζ_i's, φ, ψ in A/F

$$\eta_i^{\mathbf{A}/F}(\overline{a}/F) = \zeta_i^{\mathbf{A}/F}(\overline{a}/F) \text{ for all } i \le m \text{ implies } \varphi^{\mathbf{A}/F}(\overline{a}/F) = \psi^{\mathbf{A}/F}(\overline{a}/F)$$

iff $\mathbf{A}/F \models \left(\underset{i\le m}{\&}\eta_i \approx \zeta_i\right) \Rightarrow \varphi \approx \psi$. □

Let $\mathfrak{K}$ be a quasivariety axiomatized by a set Qi of quasi-identities and let $\mathcal{A} = \langle \mathbf{A}, \Phi\rangle$ be an $S_{\mathfrak{K}}$-matrix. Since $\vdash_{S_{\mathfrak{K}}} \supseteq \vdash_{S_{\mathrm{Con}}}$, $\Phi \in \mathrm{Con}\,\mathbf{A}$, hence the factor algebra $\mathbf{A}/\Phi$ exists. Suppose $\left(\&_{i\le m}\eta_i \approx \zeta_i\right) \Rightarrow \varphi \approx \psi \in Qi$. Then, by definition of $S_{\mathfrak{K}}$, $(\eta_1,\zeta_1),\dots,(\eta_m,\zeta_m) \vdash_{S_{\mathfrak{K}}} (\varphi,\psi)$, hence $(\eta_1,\zeta_1),\dots,(\eta_m,\zeta_m) \models_{\mathcal{A}} (\varphi,\psi)$. Now, $\mathbf{A}/\Phi \models \left(\&_{i\le m}\eta_i \approx \zeta_i\right) \Rightarrow \varphi \approx \psi$, by Lemma 1.6.9. Therefore $\mathbf{A}/\Phi \in \mathfrak{K}$.

Conversely, suppose that $\mathcal{A} = \langle \mathbf{A}, \Phi\rangle$ is a 2-matrix with $\Phi \in \mathrm{Con}\,\mathbf{A}$ for which $\mathbf{A}/\Phi \in \mathfrak{K}$. Let $\Gamma \subseteq Fm^2$ and $(\varphi,\psi) \in Fm^2$ such that $\Gamma \vdash_{S_{\mathfrak{K}}} (\varphi,\psi)$. Then there exist a finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash_{S_{\mathfrak{K}}} (\varphi, \psi)$. It follows easily by induction on the length of a derivation of (φ, ψ) from Γ' in $S_{\mathfrak{K}}$ that

$$\{\eta \approx \zeta \,;\, (\eta, \zeta) \in \Gamma'\} \models_{\mathfrak{K}} \varphi \approx \psi.$$

Since $\mathbf{A}/\Phi \in \mathfrak{K}$,

$$\mathbf{A}/\Phi \models \Big(\mathop{\&}_{(\eta,\zeta) \in \Gamma'} \eta \approx \zeta \Big) \Rightarrow \varphi \approx \psi,$$

so Lemma 1.6.9 implies that $\Gamma' \models_{\mathcal{A}} (\varphi, \psi)$, which trivially implies that $\Gamma \models_{\mathcal{A}} (\varphi, \psi)$. Thus $\mathcal{A}$ is an $S_{\mathfrak{K}}$-matrix. These observations and the fact that $\operatorname{Mod} S_{\mathfrak{K}}$ is a matrix semantics for $S_{\mathfrak{K}}$ show that the definition of the 2-deductive system $S_{\mathfrak{K}}$ does not depend on the particular quasi-equational axiomatization of $\mathfrak{K}$ that is given.

1.7 THE LEIBNIZ EQUIVALENCE RELATION

In the study of modern mathematical logic, the question of how to define 'equality' between two formulas arises. This question has its roots in the philosophical question of what constitutes equality. It was Leibniz who proposed that two objects should be considered equal if and only if they have identical properties. In the context of mathematical objects, the 'properties' referred to here must include all possible definable properties. If, however, we restrict ourselves to properties definable in first-order terms, i.e., within a first-order language (or a propositional language), then it is possible for two objects to have the same properties yet be distinct. (For example, there is a countable model of the first-order theory of the real numbers. Since $\mathbb{R}$ is uncountable, it is not equal to this model, nevertheless, in the first-order language $\{+, -, \cdot, 0, 1, \leq\}$, $\mathbb{R}$ and the countable model have the same properties.) Similarly, in the context of deductive systems, we want to know what can be said about two (not necessarily equal) objects that have the same properties as describable in the given language. This is the idea behind the following definition. (Unless otherwise stated, $\mathfrak{L}$ is a fixed language and k is a nonzero natural number.)

1.7.1 DEFINITION

Let $\mathbf{A}$ be any $\mathfrak{L}$-algebra. For each $F \subseteq A^k$, the *Leibniz equivalence relation* on $\mathbf{A}$, denoted $\Omega_{\mathbf{A}} F$, is defined by the following condition:

(1.7.1) $\quad (a, b) \in \Omega_{\mathbf{A}} F$ iff for all $\varphi(p, \overline{q}) \in Fm^k$ and all $\overline{c} \in A^m$,

$$\varphi^{\mathbf{A}}(a,\bar{c}) \in F \ \text{ iff } \ \varphi^{\mathbf{A}}(b,\bar{c}) \in F.$$

It is evident that $\Omega_{\mathbf{A}}F$ is an equivalence relation on A. If Φ is a congruence relation on an algebra $\mathbf{A}$ and $\boldsymbol{a} = \langle a_1,\ldots,a_k\rangle, \boldsymbol{b} = \langle b_1,\ldots,b_k\rangle \in A^k$, then we write $(\boldsymbol{a},\boldsymbol{b}) \in \Phi^{[k]}$ if $(a_i,b_i) \in \Phi$ for each $i \leq k$. Note that $\Phi^{[k]}$ is, in fact, a congruence on $\mathbf{A}^k$.

1.7.2 DEFINITION

Let $\mathbf{A}$ be an $\mathfrak{L}$-algebra and $F \subseteq A^k$. A congruence relation Φ on $\mathbf{A}$ is said to be *compatible* with F if, whenever $\boldsymbol{a},\boldsymbol{b} \in A^k$, $(\boldsymbol{a},\boldsymbol{b}) \in \Phi^{[k]}$ and $\boldsymbol{a} \in F$, we have $\boldsymbol{b} \in F$ as well. In other words, if $\boldsymbol{a} \in F$, then $\boldsymbol{a}/\Phi^{[k]}$, the congruence class of $\Phi^{[k]}$ containing $\boldsymbol{a}$, is a subset of F as well.

1.7.3 THEOREM [BP89a, Theorem 1.5]

For any $\mathfrak{L}$-algebra $\mathbf{A}$ and any $F \subseteq A^k$, $\Omega_{\mathbf{A}}F$ is the largest congruence on $\mathbf{A}$ that is compatible with F.

Proof. To see that $\Omega_{\mathbf{A}}F$ is a congruence relation on $\mathbf{A}$, suppose that $(a_i,b_i) \in \Omega_{\mathbf{A}}F$ for each $i \leq n$ and that f is an n-ary operation on A. Take any $\varphi(p,\bar{q}) \in Fm^k$ and $\bar{c} \in A^m$. Define

$$\vartheta(p_1,\ldots,p_n,\bar{q}) = \varphi(f(p_1,\ldots,p_n),\bar{q}),$$

(where the variables p_i are distinct from those in $\bar{q}$). By repeated application of (1.7.1) we get that, for all $\bar{c} \in A^k$,

$$\vartheta^{\mathbf{A}}(a_1,\ldots,a_n,\bar{c}) \in F \quad \text{iff} \quad \vartheta^{\mathbf{A}}(b_1,\ldots,b_n,\bar{c}) \in F,$$

i.e.,

$$\varphi^{\mathbf{A}}(f^{\mathbf{A}}(a_1,\ldots,a_n),\bar{c}) \in F \quad \text{iff} \quad \varphi^{\mathbf{A}}(f^{\mathbf{A}}(b_1,\ldots,b_n),\bar{c}) \in F,$$

hence $(f^{\mathbf{A}}(a_1,\ldots,a_n), f^{\mathbf{A}}(b_1,\ldots,b_n)) \in \Omega_{\mathbf{A}}F$, so $\Omega_{\mathbf{A}}F$ is a congruence relation.

Next suppose that $\boldsymbol{a} = \langle a_1,\ldots,a_k\rangle$, $\boldsymbol{b} = \langle b_1,\ldots,b_k\rangle \in A^k$ such that $\boldsymbol{a} \in F$ and $(\boldsymbol{a},\boldsymbol{b}) \in (\Omega_{\mathbf{A}}F)^{[k]}$. Take any k-variable $\boldsymbol{p}$; since $(a_i,b_i) \in \Omega_{\mathbf{A}}F$ for each $i \leq k$, we have $\boldsymbol{p}^{\mathbf{A}}(a_1,\ldots,a_k) \in F$ if and only if $\boldsymbol{p}^{\mathbf{A}}(b_1,\ldots,b_k) \in F$, which is equivalent to saying $\boldsymbol{a} \in F$ if and only if $\boldsymbol{b} \in F$. By assumption, $\boldsymbol{a} \in F$, hence $\boldsymbol{b} \in F$ also, implying $\Omega_{\mathbf{A}}F$ is compatible with F.

Now suppose that Φ is a congruence on $\mathbf{A}$ that is compatible with F. Take $(a,b) \in \Phi$; for every $\varphi(p,\bar{q}) \in Fm^k$ and $\bar{c} \in A^m$ we have, since Φ is a congruence and each φ_i is a term of the absolutely free algebra over $\mathfrak{L}$, that

$$(\varphi^{\mathbf{A}}(a,\bar{c}), \varphi^{\mathbf{A}}(b,\bar{c})) \in \Phi^{[k]}.$$

By the compatibility of Φ with F we get

$$\varphi^{\mathbf{A}}(a,\bar{c}) \in F \quad \text{iff} \quad \varphi^{\mathbf{A}}(b,\bar{c}) \in F,$$

hence $(a,b) \in \Omega_{\mathbf{A}}F$, implying that $\Phi \subseteq \Omega_{\mathbf{A}}F$. □

According to Definition 1.7.1, the set F can be chosen arbitrarily. However, the sets that are of real interest to us are those that are S-filters of some S-matrix. For this reason we shall restrict our attention to S-filters.

1.7.4 DEFINITION

Let S be a k-deductive system and $\mathcal{A} = \langle \mathbf{A}, F_{\mathcal{A}} \rangle$ an S-matrix. We define the *Leibniz operator* of $\mathcal{A}$, denoted $\Omega_{\mathcal{A}}$, as a map $\Omega_{\mathcal{A}} : \mathrm{Fi}^S\mathcal{A} \rightarrow \mathrm{Con}\,\mathbf{A}$, where $\Omega_{\mathcal{A}}F = \Omega_{\mathbf{A}}F$ for every $F \in \mathrm{Fi}^S\mathcal{A}$. In the case where $\mathcal{A}$ is a formula matrix model, i.e., $\mathcal{A} = \langle \mathbf{Fm}, T \rangle$ for some $T \in \mathrm{Th}S$, we simply write ΩU for $\Omega_{\mathcal{A}}U$, where $U \in \mathrm{Th}S$.

The Leibniz operator is intended to represent some form of equality between elements of a matrix model $\mathcal{A}$, hence we should expect it to satisfy some properties similar to those of $=$. We have already noted that $\Omega_{\mathbf{A}}F$, hence also $\Omega_{\mathcal{A}}F$, is an equivalence relation on A, and Theorem 1.7.3 shows that it is, in fact, a congruence relation on $\mathbf{A}$. But what does it mean for $\Omega_{\mathcal{A}}F$ to be compatible with the S-filter F of $\mathcal{A}$? Recall from Section 1.3 that the elements of F can be thought of as 'acceptable truth values' in some sense. So Definition 1.7.1 expresses the idea that we identify (by $\Omega_{\mathcal{A}}F$) those elements a and b for which the 'truth' of each k-formula $\varphi(p,\bar{q})$ is the same when p is interpreted as either a or b. If $\boldsymbol{a} \in F$, i.e., $\boldsymbol{a}$ is an 'acceptable truth value', and each a_i can be replaced by some b_i and without affecting the 'truth' of the interpretation of $\varphi(p,\bar{q})$, then we should expect that $\boldsymbol{b}$ is also an 'acceptable truth value'. This is precisely the statement that $\Omega_{\mathcal{A}}F$ is compatible with F. That $\Omega_{\mathcal{A}}F$ is the largest congruence with this property means that it is not possible to identify any more elements and still preserve the above property.

Consider the 2-deductive system S_{Eq}. Let $\mathcal{A} = \langle \mathbf{A}, R \rangle$ be an S_{Eq}-matrix. Then, by previous results, R is an equivalence relation on A. Since $\Omega_{\mathcal{A}}R$ is the largest congruence on $\mathbf{A}$

compatible with R, we have

$$\Omega_{\mathcal{A}} R = \bigvee\{\Phi \in \operatorname{Con} \mathbf{A} ; \Phi \text{ is compatible with } R\}.$$

We claim that a congruence Φ is compatible with R if and only if $\Phi \subseteq R$. To see this, note that Φ is compatible with R if and only if, for all $a_1, a_2, b_1, b_2 \in A$,

$$(a_1, a_2) \in R \text{ and } (a_1, b_1), (a_2, b_2) \in \Phi \text{ imply } (b_1, b_2) \in R.$$

Suppose the latter condition holds, and let $(a, b) \in \Phi$. We have $(a, a) \in R$ and $(a, a), (a, b) \in \Phi$ since R and Φ are reflexive, hence $(a, b) \in R$, implying that $\Phi \subseteq R$. Conversely, suppose $\Phi \subseteq R$. Let $(a_1, a_2) \in R$ and $(a_1, b_1), (a_2, b_2) \in \Phi$. Then $(a_1, b_1), (a_2, b_2) \in R$ and $(b_1, a_1), (a_1, a_2), (a_2, b_2) \in R$ since R is symmetric, hence $(b_1, b_2) \in R$ since R is transitive. We thus have

$$\Omega_{\mathcal{A}} R = \bigvee\{\Phi \in \operatorname{Con} \mathbf{A} ; \Phi \subseteq R\}.$$

Next, consider S_{Con}. Let $\mathcal{A} = \langle \mathbf{A}, \Phi \rangle$ be an S_{Con}-matrix, then Φ is a congruence on $\mathbf{A}$. As before,

$$\Omega_{\mathcal{A}} \Phi = \bigvee\{\Pi \in \operatorname{Con} \mathbf{A} ; \Pi \text{ is compatible with } \Phi\}.$$

Since Φ is obviously an equivalence relation, the earlier claim still holds, so

$$\Omega_{\mathcal{A}} \Phi = \bigvee\{\Pi \in \operatorname{Con} \mathbf{A} ; \Pi \subseteq \Phi\},$$

but since Φ is compatible with Φ, trivially, we get that $\Omega_{\mathcal{A}} \Phi = \Phi$.

Finally, consider $S_{\mathcal{K}}$ ($\mathcal{K}$ a quasivariety). Recall that $\mathcal{A} = \langle \mathbf{A}, \Phi \rangle$ is an $S_{\mathcal{K}}$-matrix if and only if $\Phi \in \operatorname{Con}_{\mathcal{K}} \mathbf{A}$. Thus, if $\mathcal{A}$ is an $S_{\mathcal{K}}$-matrix, then Φ is a congruence on $\mathbf{A}$, hence $\Omega_{\mathcal{A}} \Phi = \Phi$ as well.

1.8 MATRIX HOMOMORPHISMS AND REDUCED MATRICES

A matrix is basically an algebra with an associated subset. Thus we should be able to extend various definitions regarding algebras to matrices. In this section we consider submatrices, matrix homomorphisms and matrix isomorphisms, that are obviously intended to correspond to subalgebras, homomorphisms and isomorphisms. We also introduce the idea of a reduction between matrices, which is a stronger version of a matrix homomorphism. Unless otherwise stated we shall assume that $\mathcal{L}$ is a fixed language and S is a fixed k-deductive system for some nonzero natural number k.

1.8.1 DEFINITION

Let $\mathcal{A} = \langle \mathbf{A}, F_{\mathcal{A}} \rangle$, $\mathfrak{B} = \langle \mathbf{B}, F_{\mathfrak{B}} \rangle$ be any k-matrices. We say that $\mathfrak{B}$ is a *submatrix* of $\mathcal{A}$ if $\mathbf{B}$ is a subalgebra of $\mathbf{A}$ and $(B^k) \cap F_{\mathcal{A}} = F_{\mathfrak{B}}$. By a *matrix homomorphism* between $\mathcal{A}$ and $\mathfrak{B}$ we mean a homomorphism $h: \mathbf{A} \to \mathbf{B}$ such that $(h(a_1), \ldots, h(a_k)) \in F_{\mathfrak{B}}$ whenever $(a_1, \ldots, a_k) \in F_{\mathcal{A}}$. (Henceforth we shall abbreviate $\{(h(a_1), \ldots, h(a_k));\, (a_1, \ldots, a_k) \in F_{\mathcal{A}}\}$ with $h(F_{\mathcal{A}})$.) We write $h: \mathcal{A} \to \mathfrak{B}$ in this case. We call h *surjective* or *injective* if h is a surjection or injection from $\mathbf{A}$ to $\mathbf{B}$. If h is a surjective and injective matrix homomorphism for which $h(F_{\mathcal{A}}) = F_{\mathfrak{B}}$ then h is called a *matrix isomorphism*, and $\mathcal{A}$ and $\mathfrak{B}$ are said to be *isomorphic*. A matrix homomorphism $h: \mathcal{A} \to \mathfrak{B}$ between $\mathcal{A}$ and $\mathfrak{B}$ is said to be *reductive* from $\mathcal{A}$ to $\mathfrak{B}$ if h is surjective and $F_{\mathcal{A}} = h^{-1}(F_{\mathfrak{B}})$. $\mathfrak{B}$ is then called a *reduction* of $\mathcal{A}$ and $\mathcal{A}$ is called an *expansion* of $\mathfrak{B}$.

The following proposition is an adapted version of [BP92, Proposition 5.1].

1.8.2 PROPOSITION

Let $\mathcal{A} = \langle \mathbf{A}, F_{\mathcal{A}} \rangle$ and $\mathfrak{B} = \langle \mathbf{B}, F_{\mathfrak{B}} \rangle$ be k-matrices and S a k-deductive system.

(i) If $\mathcal{A}$ is an S-matrix and $\mathfrak{B}$ is a submatrix of $\mathcal{A}$, then $\mathfrak{B}$ is also an S-matrix.

Let $\mathfrak{B}$ be a reduction of $\mathcal{A}$, and let $h: \mathcal{A} \to \mathfrak{B}$ be the corresponding reductive homomorphism.

(ii) For each $\varphi(p_1, \ldots, p_m) \in Fm^k$ and all $a_1, \ldots, a_m \in A$,

$$\varphi^{\mathbf{A}}(a_1, \ldots, a_m) \in F_{\mathcal{A}} \quad \textit{iff} \quad \varphi^{\mathbf{B}}(ha_1, \ldots, ha_m) \in F_{\mathfrak{B}}.$$

(iii) $\mathcal{A}$ is an S-matrix if and only if $\mathfrak{B}$ is an S-matrix.

Proof. (i) Let $\Gamma \subseteq Fm^k$ and $\varphi \in Fm^k$ such that $\Gamma \vdash_S \varphi$. Let $\bar{b}$ be an interpretation of the variables of $\Gamma \cup \{\varphi\}$ in B such that $\psi^{\mathbf{B}}(\bar{b}) \in F_{\mathfrak{B}}$ for each $\psi \in \Gamma$. Since $\mathbf{B}$ is a subalgebra of $\mathbf{A}$, $\psi^{\mathbf{B}}(\bar{b}) = \psi^{\mathbf{A}}(\bar{b})$. By definition of a submatrix, $F_{\mathfrak{B}} = F_{\mathcal{A}} \cap B^k$, so $\psi^{\mathbf{B}}(\bar{b}) = \psi^{\mathbf{A}}(\bar{b}) \in F_{\mathcal{A}}$. Since $\mathcal{A}$ is an S-matrix, $\Gamma \models_{\mathcal{A}} \varphi$, hence $\varphi^{\mathbf{B}}(\bar{b}) = \varphi^{\mathbf{A}}(\bar{b}) \in F_{\mathcal{A}}$. Thus $\varphi^{\mathbf{B}}(\bar{b}) \in F_{\mathcal{A}} \cap B^k = F_{\mathfrak{B}}$, so $\Gamma \models_{\mathfrak{B}} \varphi$ and $\mathfrak{B}$ is an S-matrix.

Let $\mathfrak{B}$ be a reduction of $\mathcal{A}$, and let $h: \mathcal{A} \to \mathfrak{B}$ be the corresponding reductive homomorphism.

(ii) If $\varphi^{\mathbf{A}}(a_1, \ldots, a_m) \in F_{\mathcal{A}}$, then

$$\varphi^{\mathbf{B}}(ha_1, \ldots, ha_m) = h\varphi^{\mathbf{A}}(a_1, \ldots, a_m) \in h(F_{\mathcal{A}}) = F_{\mathfrak{B}}.$$

Conversely, if $\varphi^{\mathbf{B}}(ha_1,\ldots,ha_m) \in F_{\mathfrak{B}}$, then

$$\varphi^{\mathbf{A}}(a_1,\ldots,a_m) \in h^{-1}(\varphi^{\mathbf{B}}(ha_1,\ldots,ha_m)) \subseteq h^{-1}(F_{\mathfrak{B}}) = F_{\mathcal{A}}.$$

(iii) Suppose $\mathcal{A}$ is an S-matrix. Let $\Gamma \subseteq Fm^k$ and $\varphi \in Fm^k$ such that $\Gamma \vdash_S \varphi$, and let $\bar{b}$ be an interpretation of the variables of the k-formulas of $\Gamma \cup \{\varphi\}$ in B such that $\psi^{\mathbf{B}}(\bar{b}) \in F_{\mathfrak{B}}$ for each $\psi \in \Gamma$. Since h is surjective, there is an interpretation $\bar{a}$ of the variables occurring in the k-formulas of $\Gamma \cup \{\varphi\}$ in A such that $h\bar{a} = \bar{b}$. By part (ii), as $\psi^{\mathbf{B}}(h\bar{a}) = \psi^{\mathbf{B}}(\bar{b}) \in F_{\mathfrak{B}}$, we have $\psi^{\mathbf{A}}(\bar{a}) \in F_{\mathcal{A}}$ for all $\psi \in \Gamma$. Since $\mathcal{A}$ is an S-matrix, $\varphi^{\mathbf{A}}(\bar{a}) \in F_{\mathcal{A}}$ as well, so $\varphi^{\mathbf{B}}(h\bar{a}) = h\varphi^{\mathbf{A}}(\bar{a}) \in h(F_{\mathcal{A}})$ or, equivalently, $\varphi^{\mathbf{B}}(\bar{b}) \in F_{\mathfrak{B}}$, so $\Gamma \models_{\mathfrak{B}} \varphi$.

Conversely, suppose $\mathfrak{B}$ is an S-matrix. Again, let $\Gamma \subseteq Fm^k$ and $\varphi \in Fm^k$ such that $\Gamma \vdash_S \varphi$, and let $\bar{a}$ be an interpretation of the variables of the k-formulas of $\Gamma \cup \{\varphi\}$ in A such that $\psi^{\mathbf{A}}(\bar{a}) \in F_{\mathcal{A}}$ for each $\psi \in \Gamma$. Then $\psi^{\mathbf{B}}(h\bar{a}) = h\psi^{\mathbf{A}}(\bar{a}) \in h(F_{\mathcal{A}})$, i.e., $\psi^{\mathbf{B}}(h\bar{a}) \in F_{\mathfrak{B}}$ for all $\psi \in \Gamma$. Since $h\bar{a}$ is an interpretation of the variables of the k-formulas of $\Gamma \cup \{\varphi\}$ in B and $\mathfrak{B}$ is an S-matrix, $\varphi^{\mathbf{B}}(h\bar{a}) \in F_{\mathfrak{B}}$, hence, by (i), $\varphi^{\mathbf{A}}(\bar{a}) \in F_{\mathcal{A}}$, so $\Gamma \models_{\mathcal{A}} \varphi$. □

1.8.3 COROLLARY [BP92, Proposition 4.1]

If $\mathcal{A} = \langle \mathbf{A}, F_{\mathcal{A}} \rangle$ and $\mathfrak{B} = \langle \mathbf{B}, F_{\mathfrak{B}} \rangle$ are k-matrices and $h: \mathcal{A} \to \mathfrak{B}$ is a surjective matrix homomorphism then $h^{-1}(G)$ is an S-filter of $\mathcal{A}$ for each $G \in \mathrm{Fi}^S\mathfrak{B}$.

Proof. We show that $\langle \mathbf{A}, h^{-1}(G) \rangle$ is an S-matrix. By definition, $G \supseteq F_{\mathfrak{B}}$, hence $h^{-1}(G) \supseteq h^{-1}(F_{\mathfrak{B}}) \supseteq F_{\mathcal{A}}$. Now, $h: \langle \mathbf{A}, h^{-1}(G) \rangle \to \langle \mathbf{B}, G \rangle$ is surjective, hence $h(h^{-1}(G)) = G$, and therefore h is reductive from $\langle \mathbf{A}, h^{-1}(G) \rangle$ to $\langle \mathbf{B}, G \rangle$. Since $\langle \mathbf{B}, G \rangle$ is an S-matrix, the previous theorem implies that $\langle \mathbf{A}, h^{-1}(G) \rangle$ is an S-matrix as well. □

Note that Lemma 1.5.3 (i) may be considered a corollary of this result. Given a matrix homomorphism $h: \mathcal{A} \to \mathfrak{B}$ between k-matrices $\mathcal{A}$ and $\mathfrak{B}$ and an S-filter F of $\mathcal{A}$, the set $h(F)$ is not necessarily an S-filter of $\mathfrak{B}$. For example, suppose that h is a homomorphism between the S-matrices $\langle \mathbf{Fm}, T \rangle$ and $\langle \mathbf{Fm}, U \rangle$, where T and U are S-theories. Since $h: \mathbf{Fm} \to \mathbf{Fm}$, h is a substitution. It was shown in Section 1.3, that $h(T)$ need not be an S-theory, hence $\langle \mathbf{Fm}, h(T) \rangle$ need not be an S-matrix. If we want h to induce a map between the lattices of S-filters of $\mathcal{A}$ and $\mathfrak{B}$, it is necessary to define the map $h_S: \mathrm{Fi}^S\mathcal{A} \to \mathrm{Fi}^S\mathfrak{B}$ in the following way: For every $F \in \mathrm{Fi}^S\mathcal{A}$, define

$$h_S(F) = \mathrm{Fg}^S_{\mathfrak{B}}\, h(F).$$

1.8.4 LEMMA [BP92, Lemma 7.4]

Let $\mathcal{A} = \langle \mathbf{A}, F_{\mathcal{A}} \rangle$ *and* $\mathfrak{B} = \langle \mathbf{B}, F_{\mathfrak{B}} \rangle$ *be k-matrices and* $h : \mathcal{A} \to \mathfrak{B}$ *a surjective matrix homomorphism. If* $\ker h$ *is compatible with* $F \in \mathrm{Fi}^S \mathcal{A}$ *(where* $\ker h = \{(a, b) \in A^2;\ ha = hb\} \in \mathrm{Con}\,\mathbf{A}$*), then* $h_S(F) = h(F)$.

Proof. We need to show that $h(F) \in \mathrm{Fi}^S \mathfrak{B}$. Suppose $\Gamma \vdash_S \varphi$, and $\bar{b}$ is an interpretation of the variables in the k-formulas of $\Gamma \cup \{\varphi\}$ in B such that $\psi^{\mathbf{B}}(\bar{b}) \in h(F)$ for each $\psi \in \Gamma$. Choose an interpretation $\bar{a}$ of the variables in the k-formulas of $\Gamma \cup \{\varphi\}$ in A such that $h\bar{a} = \bar{b}$. Since $h(\psi^{\mathbf{A}}(\bar{a})) = \psi^{\mathbf{B}}(\bar{b}) \in h(F)$ for all $i \leq k$, we have $h(\psi^{\mathbf{A}}(\bar{a})) = h\boldsymbol{x}$ for some $\boldsymbol{x} \in F$, i.e., $(\psi^{\mathbf{A}}(\bar{a}), \boldsymbol{x}) \in \ker h$. Since $\ker h$ is compatible with F, $\psi^{\mathbf{A}}(\bar{a}) \in F$. This holds for all $\psi \in \Gamma$, hence $\varphi^{\mathbf{A}}(\bar{a}) \in F$, implying $\varphi^{\mathbf{B}}(\bar{b}) \in h(F)$. □

To end this section we consider the Leibniz operator once more. We first present a lemma that we shall need in Section 2.2.

1.8.5 LEMMA [BP92, Lemma 5.4]

Let S be a k-deductive system. Let $\mathcal{A} = \langle \mathbf{A}, F_{\mathcal{A}} \rangle$ *and* $\mathfrak{B} = \langle \mathbf{B}, F_{\mathfrak{B}} \rangle$ *be k-matrices and* $h : \mathcal{A} \to \mathfrak{B}$ *a surjective matrix homomorphism. Then for every* $F \in \mathrm{Fi}^S \mathfrak{B}$, $\Omega_{\mathcal{A}} h^{-1}(F) = h^{-1}(\Omega_{\mathfrak{B}} F)$.

Proof. First note that $h^{-1}(F) \in \mathrm{Fi}^S \mathcal{A}$ by Corollary 1.8.3. Since $\Omega_{\mathfrak{B}} F$ is a congruence on $\mathbf{B}$, it follows from Theorem 0.2.2 (iv) that $h^{-1}(\Omega_{\mathfrak{B}} F)$ is a congruence on $\mathbf{A}$. To see that it is compatible with $h^{-1}(F)$, suppose $\boldsymbol{a} \in h^{-1}(F)$ and $(\boldsymbol{a}, \boldsymbol{b}) \in (h^{-1}(\Omega_{\mathfrak{B}} F))^{[k]}$. Then $h\boldsymbol{a} \in F$ and $(h\boldsymbol{a}, h\boldsymbol{b}) \in (\Omega_{\mathfrak{B}} F)^{[k]}$, hence $h\boldsymbol{b} \in F$ since $\Omega_{\mathfrak{B}} F$ is compatible with F, so $\boldsymbol{b} \in h^{-1}(\Omega_{\mathfrak{B}} F)$. Thus $h^{-1}(\Omega_{\mathfrak{B}} F) \subseteq \Omega_{\mathcal{A}} h^{-1}(F)$ by Theorem 1.7.3. The opposite inclusion is equivalent to $h(\Omega_{\mathcal{A}} h^{-1}(F)) \subseteq \Omega_{\mathfrak{B}} F$, which will follow by Theorem 1.7.3 if we can show that $\Theta^{\mathbf{B}}(h(\Omega_{\mathcal{A}} h^{-1}(F)))$ is compatible with F. Since h is surjective, it follows from Theorem 0.2.2 (iv) that $\Theta^{\mathbf{B}}(h(\Omega_{\mathcal{A}} h^{-1}(F)))$ is the transitive closure of $h(\Omega_{\mathcal{A}} h^{-1}(F))$. Thus it suffices to show that $h(\Omega_{\mathcal{A}} h^{-1}(F))$ is compatible with F. For in that case, suppose $(\boldsymbol{a}, \boldsymbol{b}) \in (\Theta^{\mathbf{B}}(h(\Omega_{\mathcal{A}} h^{-1}(F))))^{[k]}$ and $\boldsymbol{a} \in F$, where $\boldsymbol{a} = \langle a_1, \ldots, a_k \rangle$ and $\boldsymbol{b} = \langle b_1, \ldots, b_k \rangle$. For each $i \leq k$, $(a_i, b_i) \in$

$\Theta^{\mathbf{B}}(h(\Omega_{\mathcal{A}}h^{-1}(F)))$, so there exists a positive integer $n(i)$ and $c_i^1,\dots,c_i^{n(i)} \in B$ such that $a_i = c_i^1$, $b_i = c_i^{n(i)}$ and $(c_i^j, c_i^{j+1}) \in h(\Omega_{\mathcal{A}}h^{-1}(F))$ for $j = 1,\dots,n(i)-1$. Since $h(\Omega_{\mathcal{A}}h^{-1}(F))$ is a reflexive relation, we may assume that there is a positive integer n with $n(i) = n$ for $i = 1,\dots,k$. Let $\boldsymbol{c}^j = \langle c_1^j,\dots,c_k^j\rangle \in B^k$. Then $(\boldsymbol{c}^j, \boldsymbol{c}^{j+1}) \in (h(\Omega_{\mathcal{A}}h^{-1}(F)))^{[k]}$ for each $j \in \{1,\dots,n-1\}$. Since $\boldsymbol{a} = \boldsymbol{c}^1 \in F$, it follows that $\boldsymbol{c}^2 \in F$ and, repeating this argument, that $\boldsymbol{b} = \boldsymbol{c}^n \in F$, as required.

Let $\boldsymbol{b}, \boldsymbol{b}' \in B^k$ such that $(\boldsymbol{b}, \boldsymbol{b}') \in (h(\Omega_{\mathcal{A}}h^{-1}(F)))^{[k]}$ and $\boldsymbol{b} \in F$. There exist $\boldsymbol{a}, \boldsymbol{a}' \in A^k$ such that $(\boldsymbol{a}, \boldsymbol{a}') \in (\Omega_{\mathcal{A}}h^{-1}(F))^{[k]}$ and $h\boldsymbol{a} = \boldsymbol{b}$, $h\boldsymbol{a}' = \boldsymbol{b}'$. Then $\boldsymbol{a} \in h^{-1}(F)$ implies $\boldsymbol{a}' \in h^{-1}(F)$ (since $\Omega_{\mathcal{A}}h^{-1}(F)$ is compatible with $h^{-1}(F)$), so $\boldsymbol{b}' = h\boldsymbol{a}' \in F$. □

The following corollary, which is a special case of Lemma 1.8.5, is proved from first principles in [BP89a].

1.8.6 COROLLARY [BP89a, Lemma 4.4 (ii)]

Let S be a k-deductive system. For every $T \in \mathrm{Th}S$ and every surjective substitution σ, $\sigma^{-1}(\Omega T) = \Omega\sigma^{-1}(T)$, hence $\Omega(\mathbf{Th}\boldsymbol{S})$ is closed under inverse substitution.

Proof. Let $\mathfrak{B} = \langle \mathbf{Fm}, \emptyset\rangle$ and $\mathcal{A} = \langle \mathbf{Fm}, \sigma^{-1}(F_{\mathfrak{B}})\rangle$. If we regard σ as a homomorphism from $\mathbf{Fm}$ to $\mathbf{Fm}$, then $\sigma : \mathcal{A} \to \mathfrak{B}$ is a matrix homomorphism and, by assumption, σ is surjective. The result follows from Lemma 1.8.5 and Proposition 1.6.5. □

We shall use the following convention: If $\langle \mathbf{A}, F_{\mathcal{A}}\rangle$ is a k-matrix and $\Phi \in \mathrm{Con}\,\mathbf{A}$ then for any $F \subseteq A^k$ we denote by F/Φ the set $\{\langle a_1/\Phi,\dots,a_k/\Phi\rangle ; \boldsymbol{a} = \langle a_1,\dots,a_k\rangle \in F\}$.

1.8.7 LEMMA [BP88, Lemma 3.3]

Let $\mathcal{A} = \langle \mathbf{A}, F_{\mathcal{A}}\rangle$ be a k-matrix and $\mathfrak{B} = \langle \mathbf{B}, F_{\mathfrak{B}}\rangle$ a submatrix of $\mathcal{A}$. Let Φ be a congruence on $\mathbf{A}$ that is compatible with $F_{\mathcal{A}}$ (e.g., $\Phi = \Omega_{\mathcal{A}}F_{\mathcal{A}}$). Then $\langle \mathbf{B}/(\Phi \cap B^2), F_{\mathfrak{B}}/(\Phi \cap B^2)\rangle$ is isomorphic to a submatrix of the k-matrix $\langle \mathbf{A}/\Phi, F_{\mathcal{A}}/\Phi\rangle$.

Proof. Observe first that $\Phi \cap B^2 \in \mathrm{Con}\,\mathbf{B}$, so the claim makes sense. Let $\mathfrak{B}' = \langle \mathbf{B}/\Phi, F_{\mathcal{A}}/\Phi \cap (B/\Phi)^k\rangle$, where $\mathbf{B}/\Phi$ is the subalgebra of $\mathbf{A}/\Phi$ with universe $B/\Phi = \{b/\Phi ; b \in B\}$. (This is a subalgebra of $\mathbf{A}/\Phi$ by Theorem 0.2.2 (ii) – consider the composition of the inclusion map $\mathbf{B} \to \mathbf{A}$ and the canonical map $\mathbf{A} \to \mathbf{A}/\Phi$.) By its definition, $\mathfrak{B}'$

is a submatrix of $\langle \mathbf{A}/\Phi, F_{\mathcal{A}}/\Phi\rangle$. We shall show that $\langle \mathbf{B}/(\Phi \cap B^2), F_{\mathfrak{B}}/(\Phi \cap B^2)\rangle$ is isomorphic to $\mathfrak{B}'$. The map $h: B/(\Phi \cap B^2) \to B/\Phi$ defined by $h(b/(\Phi \cap B^2)) = b/\Phi$ defines an isomorphism of algebras $\mathbf{B}/(\Phi \cap B^2) \to \mathbf{B}/\Phi$. Since we are given that $F_{\mathfrak{B}} = F_{\mathcal{A}} \cap B^k$, we have that $h(F_{\mathfrak{B}}/(\Phi \cap B^2)) = F_{\mathfrak{B}}/\Phi = F_{\mathcal{A}}/\Phi \cap B^k/\Phi = F_{\mathcal{A}}/\Phi \cap (B/\Phi)^k$. This shows that h is a matrix isomorphism. □

It is common practice to identify $\langle \mathbf{B}/(\Phi \cap B^2), F_{\mathfrak{B}}/(\Phi \cap B^2)\rangle$ with $\mathfrak{B}'$ and to call it a submatrix of $\langle \mathbf{A}/\Phi, F_{\mathcal{A}}/\Phi\rangle$.

In Section 1.7 we noted that the Leibniz relation identifies those elements of a matrix model that are equivalent with respect to a given S-filter. This leads to the question of whether there are S-matrices for which the Leibniz relation identifies no two distinct elements, i.e., the S-matrix cannot be reduced any further. Thus we make the following definition.

1.8.8 DEFINITION

An S-matrix $\mathcal{A} = \langle \mathbf{A}, F\rangle$ is said to be *reduced* if $\Omega_{\mathcal{A}} F = I_A$. The class of all reduced S-matrices is denoted $\mathrm{Mod}^* S$.

If $\mathcal{A} = \langle \mathbf{A}, F\rangle \in \mathrm{Mod}S$, is it possible to construct a reduced matrix from $\mathcal{A}$? In Theorem 1.7.3, we noted that the relation $\Omega_{\mathcal{A}} F$ is a congruence relation on the underlying algebra $\mathbf{A}$ that is compatible with F. Congruence relations are significant in that they allow one to construct a factor algebra from the original algebra. For any S-matrix $\mathcal{A} = \langle \mathbf{A}, F\rangle$, define the matrix

$$\mathcal{A}^* = \langle \mathbf{A}/\Omega_{\mathcal{A}} F, F/\Omega_{\mathcal{A}} F\rangle,$$

where $F/\Omega_{\mathcal{A}} F = \{\langle a_1/\Omega_{\mathcal{A}} F, \ldots, a_k/\Omega_{\mathcal{A}} F\rangle ; \langle a_1, \ldots, a_k\rangle \in F\}$. The canonical homomorphism $h: \mathcal{A} \to \mathcal{A}^*$, defined by $ha = a/\Omega_{\mathcal{A}} F$ for each $a \in A$, is obviously surjective and satisfies $F = h^{-1}(F/\Omega_{\mathcal{A}} F)$, hence h is reductive from $\mathcal{A}$ to $\mathcal{A}^*$. By Proposition 1.8.2, therefore, $\mathcal{A}^*$ is an S-matrix. Furthermore, $\mathcal{A}^*$ is a reduced S-matrix: If $(a/\Omega_{\mathcal{A}} F, b/\Omega_{\mathcal{A}} F) \in \Omega_{\mathcal{A}^*}(F/\Omega_{\mathcal{A}} F)$ then it is not hard to infer from Definition 1.7.1 that $(a, b) \in \Omega_{\mathcal{A}} F$, i.e., $a/\Omega_{\mathcal{A}} F = b/\Omega_{\mathcal{A}} F$, hence $\Omega_{\mathcal{A}^*}(F/\Omega_{\mathcal{A}} F) = I_{A^*}$, where $A^* = A/\Omega_{\mathcal{A}} F$.

The following important theorem shows that in the study of matrix models, it is sufficient to consider only the reduced matrix models.

1.8.9 THEOREM [BP92, Theorem 5.6]

Mod^*S *is a matrix semantics for* S.

Proof. We must show that, for all $\Gamma \subseteq Fm^k$ and $\varphi \in Fm^k$, $\Gamma \vdash_S \varphi$ if and only if $\Gamma \models_{\mathrm{Mod}^*S} \varphi$. The implication from left to right holds by definition. Conversely, suppose $\Gamma \models_{\mathfrak{B}} \varphi$ for every $\mathfrak{B} \in \mathrm{Mod}^*S$. Let $\mathcal{A} = \langle \mathbf{A}, F \rangle \in \mathrm{Mod}S$. Then $\mathcal{A}^* = \langle \mathbf{A}/\Omega_{\mathcal{A}}F, F/\Omega_{\mathcal{A}}F \rangle \in \mathrm{Mod}^*S$, so $\Gamma \models_{\mathcal{A}^*} \varphi$ by assumption. Since $\mathcal{A}$ was chosen arbitrarily, we have that $\Gamma \models_{\mathrm{Mod}S} \varphi$. Recalling from Theorem 1.6.2 that $\mathrm{Mod}S$ is a matrix semantics for S, we get that $\Gamma \vdash_S \varphi$, hence $\Gamma \models_{\mathcal{A}} \varphi$. □

Let $\mathcal{A} = \langle \mathbf{A}, \Phi \rangle$ be an S_{Con}-matrix. Then $\mathcal{A}$ is reduced if and only if $\Omega_{\mathcal{A}}\Phi = I_A$, but $\Omega_{\mathcal{A}}\Phi = \Phi$, hence $\mathcal{A}$ is reduced if and only if $\Phi = I_A$. So

$$\mathrm{Mod}^*S_{\mathrm{Con}} = \{\langle \mathbf{A}, I_A \rangle\, ; \mathbf{A} \text{ is an } \mathfrak{L}\text{-algebra}\}.$$

Since the I_A's are the usual interpretation of the equality symbol, $\mathrm{Mod}^*S_{\mathrm{Con}}$ is essentially a class of algebras and need not be thought of as a class of matrices (yet it captures the full deductive force of S_{Con}, in view of the previous theorem).

Let $\mathcal{A} = \langle \mathbf{A}, \Phi \rangle$ be an $S_{\mathfrak{K}}$-matrix. Again, $\Omega_{\mathcal{A}}\Phi = \Phi$, so the reduced $S_{\mathfrak{K}}$-matrices are of the form $\langle \mathbf{A}, I_A \rangle$. In fact,

$$\mathrm{Mod}^*S_{\mathfrak{K}} = \{\langle \mathbf{A}, I_A \rangle\, ; \mathbf{A} \in \mathfrak{K}\}.$$

This follows from the fact that if $\langle \mathbf{A}, I_A \rangle$ is an $S_{\mathfrak{K}}$-matrix, then I_A must be a $\mathfrak{K}$-congruence, hence $\mathbf{A} \cong \mathbf{A}/I_A \in \mathfrak{K}$. Again, the class $\mathrm{Mod}^*S_{\mathfrak{K}}$ is essentially a class of algebras, namely the quasivariety $\mathfrak{K}$. Thus the consequence relation of the 2-deductive system $S_{\mathfrak{K}}$ is determined by $\mathfrak{K}$, so the definition of $S_{\mathfrak{K}}$ is the 'correct' one.

1.9 EQUIVALENCE OF DEDUCTIVE SYSTEMS

Two deductive systems of the same dimension are identical if and only if they have the same language and consequence relation. But how does one compare two deductive systems of different dimension? Obviously they cannot be identical, but we are able to define a notion of equivalence.

Loosely, we say that they are equivalent if it is possible to translate formulas from one deductive system into the other in such a way that the relation of consequence is preserved, and vice versa. Furthermore, the two translations must be mutually inverse in some sense. In particular, for every deduction $\Gamma \vdash \varphi$ of the one deductive system, there is an equivalent deduction in the other. In this section, we investigate this notion of equivalence.

Let S_1 be a k-deductive system and S_2 an ℓ-deductive system, where $1 \leq \ell, k \in \omega$. Assume that S_1 and S_2 have the same language $\mathcal{L}$. By a (k,ℓ)-*translation* we mean a finite set τ of ℓ-formulas in k variables, i.e., $\tau = \{\tau^1, \ldots, \tau^n\}$ for some integer $n \geq 1$ and

$$\tau^i(p_1, \ldots, p_k) = \langle \tau_1^i(p_1, \ldots, p_k), \ldots, \tau_\ell^i(p_1, \ldots, p_k)\rangle \quad \text{for each } i \leq n.$$

If $\varphi = \langle \varphi_1, \ldots, \varphi_k\rangle$ is a k-formula, then we define $\tau(\varphi)$ to be the set of ℓ-formulas,

$$\tau(\varphi) = \{\langle \tau_1^i(\varphi_1, \ldots, \varphi_k), \ldots, \tau_\ell^i(\varphi_1, \ldots, \varphi_k)\rangle \,;\, i \leq n\}.$$

If Γ is a set of k-formulas then by $\tau(\Gamma)$ we mean $\bigcup\{\tau(\psi) \,;\, \psi \in \Gamma\}$.

By an *interpretation* of S_1 in S_2 we mean a (k,ℓ)-translation τ that satisfies, for all $\Gamma \subseteq Fm^k$ and $\varphi \in Fm^k$,

(1.9.1) $$\Gamma \vdash_{S_1} \varphi \quad \text{iff} \quad \tau(\Gamma) \vdash_{S_2} \tau(\varphi).$$

We shall need the following abbreviation: For a k-deductive system S and $\Gamma, \Delta \subseteq Fm^k$, we write $\Gamma \dashv\vdash_S \Delta$ to mean '$\Gamma \vdash_S \Delta$ and $\Delta \vdash_S \Gamma$'.

1.9.1 DEFINITION

We say that S_1 and S_2 are *equivalent* if there is an interpretation τ of S_1 in S_2 and an interpretation ρ of S_2 in S_1 such that τ and ρ are *inverse* in the following sense:

(1.9.2) $$\varphi \dashv\vdash_{S_1} \rho(\tau(\varphi)) \quad \text{for all } \varphi \in Fm^k, \text{ and}$$

(1.9.3) $$\varphi \dashv\vdash_{S_2} \tau(\rho(\varphi)) \quad \text{for all } \varphi \in Fm^\ell.$$

The following proposition gives ostensibly weaker conditions that will be sufficient for S_1 and S_2 to be equivalent.

1.9.2 PROPOSITION [BP89b, Proposition 4.1]

Let S_1 be a k-deductive system and S_2 an ℓ-deductive system, where $1 \leq \ell, k < \omega$. Then S_1 and

S_2 are equivalent if there exists a (k,ℓ)-translation τ and an (ℓ,k)-translation ρ such that τ is an interpretation of S_1 in S_2 and (1.9.3) is satisfied.

Proof. We need to show that ρ is an interpretation of S_2 in S_1 and also that (1.9.2) holds. Let $\Gamma \subseteq Fm^\ell$ and $\varphi \in Fm^\ell$. By (1.9.3), $\varphi \dashv\vdash_{S_2} \tau(\rho(\varphi))$ and $\Gamma \dashv\vdash_{S_2} \tau(\rho(\Gamma))$ (since $\psi \dashv\vdash_{S_2} \tau(\rho(\psi))$ for each $\psi \in \Gamma$), therefore

$$\begin{aligned}\Gamma \vdash_{S_2} \varphi \quad &\text{iff} \quad \tau(\rho(\Gamma)) \vdash_{S_2} \tau(\rho(\varphi)) \\ &\text{iff} \quad \rho(\Gamma) \vdash_{S_1} \rho(\varphi) \qquad \text{[since } \tau \text{ is an interpretation],}\end{aligned}$$

implying that ρ is an interpretation of S_2 in S_1. To see that (1.9.2) holds, note that, for all $\varphi \in Fm^k$,

$$\varphi \dashv\vdash_{S_1} \rho(\tau(\varphi)) \quad \text{iff} \quad \tau(\varphi) \dashv\vdash_{S_2} \tau(\rho(\tau(\varphi))),$$

and the right hand side holds by (1.9.3) since $\tau(\varphi) \subseteq Fm^\ell$. □

We shall show that the notion of equivalence is an equivalence relation on the class of all deductive systems. Only transitivity of the relation not obvious. Let S_1 be a k-deductive system, S_2 an ℓ-deductive system and S_3 an m-deductive system. Suppose S_1 and S_2 are equivalent and τ is an interpretation of S_1 in S_2 and ρ is an interpretation of S_2 in S_1 such that τ and ρ are inverse. Suppose S_2 and S_3 are equivalent and η is an interpretation of S_2 in S_3 and ζ is an interpretation of S_3 in S_2 such that η and ζ are inverse. We shall use Proposition 1.9.2 to show that S_1 and S_3 are equivalent with $\eta\tau$ an interpretation of S_1 in S_3 and $\rho\zeta$ an interpretation of S_3 in S_1 such that $\eta\tau$ and $\rho\zeta$ are inverse. Let $\Gamma \subseteq Fm^k$ and $\varphi \in Fm^k$. Then

$$\begin{aligned}\Gamma \vdash_{S_1} \varphi \quad &\text{iff} \quad \tau(\Gamma) \vdash_{S_2} \tau(\varphi) \\ &\text{iff} \quad \eta(\tau(\Gamma)) \vdash_{S_3} \eta(\tau(\varphi)).\end{aligned}$$

Thus $\eta\tau$ is an interpretation of S_1 in S_3. Now suppose $\psi \in Fm^m$. Then

$$\begin{aligned}\psi \dashv\vdash_{S_3} \eta(\tau(\rho(\zeta(\psi)))) \quad &\text{iff} \quad \zeta(\psi) \dashv\vdash_{S_2} \zeta(\eta(\tau(\rho(\zeta(\psi))))) \\ &\text{iff} \quad \zeta(\psi) \dashv\vdash_{S_2} \tau(\rho(\zeta(\psi))) \\ &\qquad [\zeta(\eta(\tau(\rho(\zeta(\psi))))) \dashv\vdash_{S_2} \tau(\rho(\zeta(\psi))) \text{ since } \eta \text{ and } \zeta \text{ are inverse}] \\ &\text{iff} \quad \zeta(\psi) \dashv\vdash_{S_2} \zeta(\psi) \qquad \text{[since } \tau \text{ and } \rho \text{ are inverse],}\end{aligned}$$

which is evidently true. Thus $\psi \dashv\vdash_{S_3} \eta(\tau(\rho(\zeta(\psi))))$ so S_1 and S_3 are equivalent.

Although the definition of equivalence is given for deductive systems of possibly different dimensions, it is interesting to note that there exist *distinct* deductive systems of the same dimension that are nevertheless equivalent. The following example of two 1-deductive systems that are equivalent and not identical is taken from [DDT, p15]. (Note that two deductive systems are identical if and only if they have the same language and consequence relation, but that a single deductive system may be presentable by means of various different sets of axioms and inference rules). We shall consider the three-valued paraconsistent logic $\mathbf{J_3}$ and Łukasiewicz 3-valued many-valued logic S_2 (see Section 1.4). Recall that the language for both these deductive systems is $\mathcal{L} = \{\rightarrow, \neg\}$, where $ar(\rightarrow) = 2$ and $ar(\neg) = 1$. Recall, moreover, that $\mathbf{J_3} = \langle \mathcal{L}, \models_{\mathfrak{J}_3} \rangle$ and $S_2 = \langle \mathcal{L}, \models_{\mathcal{L}_3} \rangle$, where $\mathfrak{J}_3 = \langle \mathbf{A}, \{1, \frac{1}{2}\} \rangle$, $\mathcal{L}_3 = \langle \mathbf{A}, \{1\} \rangle$ and $\mathbf{A} = \langle \{0, \frac{1}{2}, 1\}, \mathcal{L}^{\mathbf{A}} \rangle$ is an $\mathcal{L}$-algebra, whose operations $\rightarrow^{\mathbf{A}}$ and $\neg^{\mathbf{A}}$ ($\rightarrow$ and $\neg$ for short) are defined by the tables

$\rightarrow$	0	$\frac{1}{2}$	1
0	1	1	1
$\frac{1}{2}$	$\frac{1}{2}$	1	1
1	0	$\frac{1}{2}$	1

	$\neg$
0	1
$\frac{1}{2}$	$\frac{1}{2}$
1	0

We will need the abbreviations $\Diamond p = (\neg p) \rightarrow p$ and $\Box p = \neg(\Diamond(\neg p))$. Then $\Diamond$ and $\Box$ take the following values

	$\Diamond$
0	0
$\frac{1}{2}$	1
1	1

	$\Box$
0	0
$\frac{1}{2}$	0
1	1

Let $\varphi(p) = ((\neg p) \rightarrow p) \rightarrow p$. Then we get the following table:

a	$\varphi^{\mathbf{A}}(a)$
0	1
$\frac{1}{2}$	$\frac{1}{2}$
1	1

Since $\varphi^{\mathbf{A}}(a) \in \{\frac{1}{2}, 1\}$ for all $a \in A$, we have $\vdash_{\mathbf{J_3}} \varphi$. However, since $\varphi^{\mathbf{A}}(a) \notin \{1\}$ for $a = \frac{1}{2} \in A$, we have $\nvdash_{S_2} \varphi$. So $\mathbf{J_3}$ and S_2 are different deductive systems. Now define $(1,1)$-translations $\boldsymbol{\tau}$ and $\boldsymbol{\rho}$ by $\boldsymbol{\tau}(p) = \Diamond p$ and $\boldsymbol{\rho}(p) = \Box p$. We have the following (where $\psi(\overline{p}) \in Fm$):

$\psi^{\mathbf{A}}(\overline{a})$	$\Diamond\,\psi^{\mathbf{A}}(\overline{a})$
0	0
$\frac{1}{2}$	1
1	1

therefore $\Diamond\,\psi^{\mathbf{A}}(\overline{a}) \in \{1\}$ if and only if $\psi^{\mathbf{A}}(\overline{a}) \in \{\frac{1}{2}, 1\}$. Since $\mathfrak{J}_3 = \langle \mathbf{A}, \{1, \frac{1}{2}\}\rangle$ and $\mathfrak{L}_3 = \langle \mathbf{A}, \{1\}\rangle$, we have, for all $\Gamma \subseteq Fm$ and $\varphi \in Fm$,

$$\Gamma \models_{\mathfrak{J}_3} \varphi \quad \text{iff} \quad \{\Diamond\,\psi\,;\,\psi \in \Gamma\} \models_{\mathfrak{L}_3} \Diamond\,\varphi, \text{ i.e.,}$$

$$\Gamma \vdash_{\mathbf{J_3}} \varphi \quad \text{iff} \quad \tau(\Gamma) \vdash_{S_2} \tau(\varphi),$$

hence τ is an interpretation of $\mathbf{J_3}$ in S_2. Finally we have, for $\psi(\overline{p}) \in Fm$,

$\psi^{\mathbf{A}}(\overline{a})$	$\Box\,\psi^{\mathbf{A}}(\overline{a})$	$\Diamond\,\Box\,\psi^{\mathbf{A}}(\overline{a})$
0	0	0
$\frac{1}{2}$	0	0
1	1	1

$\psi^{\mathbf{A}}(\overline{a})$	$\Diamond\,\psi^{\mathbf{A}}(\overline{a})$	$\Box\,\Diamond\,\psi^{\mathbf{A}}(\overline{a})$
0	0	0
$\frac{1}{2}$	1	1
1	1	1

so $\psi^{\mathbf{A}}(\overline{a}) \in \{1\}$ if and only if $\Diamond\,\Box\,\psi^{\mathbf{A}}(\overline{a}) \in \{1\}$, and $\psi^{\mathbf{A}}(\overline{a}) \in \{\frac{1}{2}, 1\}$ if and only if $\Box\,\Diamond\,\psi^{\mathbf{A}}(\overline{a}) \in \{\frac{1}{2}, 1\}$, i.e.,

$$\psi \mathrel{=\!\!|}\models_{\mathfrak{L}_3} \Diamond\,\Box\,\psi,$$

and

$$\psi \mathrel{=\!\!|}\models_{\mathfrak{J}_3} \Box\,\Diamond\,\psi.$$

By Proposition 1.9.2, $\mathbf{J_3}$ and S_2 are equivalent.

Since deductive systems can be defined in terms of their lattices of theories, it is to be expected that the equivalence of deductive systems can as well. This is indeed true, which is shown in Theorem 1.9.4. The result of that theorem seems very natural, and it is used later in connection with the algebraization process.

Let S_1 be a k-deductive system and S_2 an ℓ-deductive system, where $1 \le \ell, k \in \omega$, and suppose τ is a (k, ℓ)-translation. For each $T \in \mathrm{Th}S_1$, set

$$\hat{\tau}(T) = \mathrm{Cn}_{S_2}\tau(T),$$

and for each substitution σ and $T \in \mathrm{Th}S_i$, $i = 1, 2$, set

$$\sigma_{S_i}(T) = \mathrm{Cn}_{S_i}\sigma(T).$$

A map $f:\mathrm{Th}S_1 \to \mathrm{Th}S_2$ is said to *commute with substitutions* if $f(\sigma_{S_1}(T)) = \sigma_{S_2} f(T)$ for all $T \in \mathrm{Th}S_1$ and all substitutions σ. In other words, f commutes with substitutions if and only if $f(\mathrm{Cn}_{S_1}(\sigma(T))) = \mathrm{Cn}_{S_2}(\sigma(f(T)))$. Recall that a substitution can be understood to be a renaming of variables (by formulas). In this context, this definition can be read to say: If the variables of an S_1-theory T are renamed and a new S_1-theory generated thereby, then its image under f is the same as that obtained by taking the image of T under f first, then renaming the variables and generating a new S_2-theory.

1.9.3 LEMMA [BP89b, Lemma 4.3]

Let S_1 be a k-deductive system and S_2 an ℓ-deductive system, where $1 \le \ell, k \in \omega$, and suppose that S_1 and S_2 are equivalent under the interpretations τ from S_1 to S_2 and ρ from S_2 to S_1. Then

(i) $\widehat{\tau}: \mathbf{Th}S_1 \to \mathbf{Th}S_2$ *is a lattice isomorphism, and*

(ii) $\widehat{\tau}$ *commutes with substitutions.*

Proof. (i) If $T, U \in \mathrm{Th}S_1$ such that $T \subseteq U$, then obviously $\tau(T) \subseteq \tau(U)$, and $\mathrm{Cn}_{S_2}(\tau(T)) \subseteq \mathrm{Cn}_{S_2}(\tau(U))$, implying that $\widehat{\tau}$ is order-preserving. Similarly, $\widehat{\rho}$ is also order-preserving. Since $\varphi \dashv\vdash_{S_1} \rho(\tau(\varphi))$ for all $\varphi \in Fm^k$, we have

$$\begin{aligned}
\rho(\mathrm{Cn}_{S_2}(\tau(T))) \vdash_{S_1} \varphi \quad &\text{iff} \quad \rho(\mathrm{Cn}_{S_2}(\tau(T))) \vdash_{S_1} \rho(\tau(\varphi)) \\
&\text{iff} \quad \mathrm{Cn}_{S_2}(\tau(T)) \vdash_{S_2} \tau(\varphi) \\
&\text{iff} \quad \tau(T) \vdash_{S_2} \tau(\varphi) \qquad \text{[by (1.5.9)]} \\
&\text{iff} \quad T \vdash_{S_1} \varphi,
\end{aligned}$$

hence $\varphi \in \mathrm{Cn}_{S_1}(\rho(\mathrm{Cn}_{S_2}\tau(T))) = \widehat{\rho}(\widehat{\tau}(T))$ if and only if $\varphi \in T$. This means that $\widehat{\rho}(\widehat{\tau}(T)) = T$, hence $\widehat{\rho} \circ \widehat{\tau}$ is the identity on $\mathrm{Th}S_1$. Similarly, $\widehat{\tau} \circ \widehat{\rho}$ is the identity on $\mathrm{Th}S_2$, therefore $\widehat{\tau}$ and $\widehat{\rho}$ are mutually inverse bijections, and the result follows from Lemma 0.1.1.

(ii) Let σ be a substitution. Recall that by structurality, for $\Gamma \subseteq Fm^\ell$,

$$\sigma(\mathrm{Cn}_{S_2}(\Gamma)) \subseteq \mathrm{Cn}_{S_2}(\sigma(\Gamma)).$$

Using this fact, we get that

$\mathrm{Cn}_{S_2}(\sigma(\mathrm{Cn}_{S_2}(\Gamma))) \subseteq \mathrm{Cn}_{S_2}(\mathrm{Cn}_{S_2}(\sigma(\Gamma))) = \mathrm{Cn}_{S_2}(\sigma(\Gamma)) \subseteq \mathrm{Cn}_{S_2}(\sigma(\mathrm{Cn}_{S_2}(\Gamma)))$, i.e,

$$\mathrm{Cn}_{S_2}(\sigma(\mathrm{Cn}_{S_2}(\Gamma))) = \mathrm{Cn}_{S_2}(\sigma(\Gamma)). \tag{1.9.4}$$

Since $\Gamma \vdash_{S_1} \varphi$ implies $\tau(\Gamma) \vdash_{S_2} \tau(\varphi)$ for all $\Gamma \subseteq Fm^k$ and $\varphi \in Fm^k$, we have

$$\tau(\mathrm{Cn}_{S_1}(\Gamma)) \subseteq \mathrm{Cn}_{S_2}\tau(\Gamma).$$

Using the same argument as above, we similarly get

$$\mathrm{Cn}_{S_2}(\tau(\mathrm{Cn}_{S_1}(\Gamma))) = \mathrm{Cn}_{S_2}\tau(\Gamma). \tag{1.9.5}$$

Now, for $T \in \mathrm{Th}S_1$,

$$\begin{aligned}
\widehat{\tau}(\sigma_{S_1}(T)) &= \widehat{\tau}(\mathrm{Cn}_{S_1}(\sigma(T))) \\
&= \mathrm{Cn}_{S_2}(\tau(\mathrm{Cn}_{S_1}(\sigma(T)))) \\
&= \mathrm{Cn}_{S_2}(\tau(\sigma(T))) && \text{[by (1.9.5)]} \\
&= \mathrm{Cn}_{S_2}(\sigma(\tau(T))) \\
&= \mathrm{Cn}_{S_2}(\sigma(\mathrm{Cn}_{S_2}(\tau(T)))) && \text{[by (1.9.4)]} \\
&= \sigma_{S_2}(\widehat{\tau}(T)).
\end{aligned}$$

□

The next theorem shows that properties (i) and (ii) actually characterize the equivalence of arbitrary deductive systems. The theorem stated here is a slightly adapted version of [BP89b, Lemma 4.4].

1.9.4 THEOREM

Let S_1 and S_2 be deductive systems. The following are equivalent:

(i) S_1 and S_2 are equivalent,

(ii) there exists an isomorphism from $\mathbf{Th}S_1$ onto $\mathbf{Th}S_2$ that commutes with substitutions,

(iii) there exists an isomorphism from $\mathbf{Th}S_1$ onto $\mathbf{Th}S_2$ that commutes with surjective substitutions.

Proof. (i)⇒(ii) This result is proved in the preceding lemma.

(ii)⇒(iii) Trivial.

(iii)⇒(i) Assume that $f : \mathbf{Th}S_1 \to \mathbf{Th}S_2$ is an isomorphism that commutes with surjective substitutions. Let the dimensions of S_1 and S_2 be k and ℓ respectively. Set $T = \mathrm{Cn}_{S_1}(\langle p_1, \ldots, p_k \rangle)$ and set $T' = f(T)$. Since T is a compact element of $\mathrm{Th}S_1$ and f is an isomorphism, T' is a compact element of $\mathrm{Th}S_2$, hence T' is finitely generated, by Lemma 1.5.2. Let $T' = \mathrm{Cn}_{S_2}(\{\chi_i ; i \le n\})$, where χ_i is an ℓ-formula for each $i \le n$. Let the variables occurring in the χ_i's be among $p_1, \ldots, p_k, r_1, \ldots, r_m$, i.e.,

$$\chi_i = \chi_i(p_1, \ldots, p_k, r_1, \ldots, r_m) \text{ for each } i \le n.$$

Define a surjective substitution $\sigma: \mathbf{Fm} \to \mathbf{Fm}$ as follows: Set $\sigma p_i = p_i$ for each $i \le k$, and $\sigma r_i = p_1$ for $i \le m$. Since σ is thusfar defined only on a finite number of variables, it is possible to define $\sigma: P \to Fm$ such that $\sigma(P) \supseteq P$. (Recall that P is countably infinite.) By the remarks about substitutions in Section 1.1, we can extend σ to a surjective substitution, which we also call σ, from **Fm** onto **Fm**. Now,

(1.9.6) $$\sigma\chi_i = \chi_i(p_1, \ldots, p_k, p_1, \ldots, p_1) \text{ for each } i \le n.$$

Using the assumption (iii), we get that

$\sigma_{S_1}(T) = \sigma_{S_1}(\mathrm{Cn}_{S_1}\langle p_1, \ldots, p_k\rangle) = \mathrm{Cn}_{S_1}\sigma\langle p_1, \ldots, p_k\rangle = \mathrm{Cn}_{S_1}\langle p_1, \ldots, p_k\rangle = T$, therefore $T' = f(T)$ $= f(\sigma_{S_1}(T)) = \sigma_{S_2}(f(T)) = \sigma_{S_2}(T') = \sigma_{S_2}(\mathrm{Cn}_{S_2}(\{\chi_i\,;\, i \le n\})) = \mathrm{Cn}_{S_2}(\sigma(\mathrm{Cn}_{S_2}(\{\chi_i\,;\, i \le n\})))$ $= \mathrm{Cn}_{S_2}(\{\sigma\chi_i\,;\, i \le n\})$ (as argued in the justification of (1.9.4) in the proof of Lemma 1.9.3).

From (1.9.6) we get

(1.9.7) $$T' = \mathrm{Cn}_{S_2}(\{\chi_i(p_1, \ldots, p_k, p_1, \ldots, p_1)\,;\, i \le n\}).$$

This effectively removes the unwanted variables $r_1, \ldots, r_m$ which allows us to define a (k, ℓ)-translation τ by

$$\tau(\langle p_1, \ldots, p_k\rangle) = \{\chi_i(p_1, \ldots, p_k, p_1, \ldots, p_1)\,;\, i \le n\}.$$

Then (1.9.7) says that $T' = \mathrm{Cn}_{S_2}(\tau(\langle p_1, \ldots, p_k\rangle))$. We shall show that τ is an interpretation of S_1 in S_2. To that end, let $\varphi = \langle \varphi_1, \ldots, \varphi_k\rangle$ be any k-formula and define a surjective substitution σ as follows: Set $\sigma p_i = \varphi_i$ for all $i \le k$, i.e., $\sigma\langle p_1, \ldots, p_k\rangle = \varphi$. Since σ is thusfar defined only for a finite number of variables, it is possible to define $\sigma: P \to Fm$ such that $\sigma(P) \supseteq P$. As before we can extend σ to a surjective homomorphism, also called σ, from **Fm** onto **Fm**. Then, using (iii), $f(\mathrm{Cn}_{S_1}\varphi) = f(\mathrm{Cn}_{S_1}\sigma\langle p_1, \ldots, p_k\rangle) = f(\sigma_{S_1}(\mathrm{Cn}_{S_1}\langle p_1, \ldots, p_k\rangle)) = \sigma_{S_2}(f(\mathrm{Cn}_{S_1}\langle p_1, \ldots, p_k\rangle)) =$ $\sigma_{S_2}(f(T)) = \sigma_{S_2}(T') = \sigma_{S_2}(\mathrm{Cn}_{S_2}\tau(\langle p_1, \ldots, p_k\rangle)) = \mathrm{Cn}_{S_2}(\sigma(\tau(\langle p_1, \ldots, p_k\rangle))) =$ $\mathrm{Cn}_{S_2}(\tau(\sigma\langle p_1, \ldots, p_k\rangle)) = \mathrm{Cn}_{S_2}(\tau(\varphi))$, i.e., $f(\mathrm{Cn}_{S_1}(\varphi)) = \mathrm{Cn}_{S_2}(\tau(\varphi))$.

Let $\Gamma \subseteq Fm^k$. Since f is a lattice isomorphism,

$$
\begin{aligned}
f(\mathrm{Cn}_{S_1}(\Gamma)) &= f\Big(\bigvee_{\varphi \in \Gamma}^{\mathbf{Th}S_1} \mathrm{Cn}_{S_1}\varphi\Big) \\
&= \bigvee_{\varphi \in \Gamma}^{\mathbf{Th}S_2} f(\mathrm{Cn}_{S_1}\varphi) \\
&= \bigvee_{\varphi \in \Gamma}^{\mathbf{Th}S_2} \mathrm{Cn}_{S_2}(\tau(\varphi)) \qquad \text{[by the above]}
\end{aligned}
$$

$$= \mathrm{Cn}_{S_2}\Big(\bigcup_{\varphi \in \Gamma} \tau(\varphi)\Big)$$

$$= \mathrm{Cn}_{S_2}(\tau(\Gamma)),$$

Now, if $\Gamma \subseteq Fm^k$ and $\varphi \in Fm^k$, then

$$\begin{aligned} \Gamma \vdash_{S_1} \varphi \quad & \text{iff} \quad \mathrm{Cn}_{S_1}\varphi \subseteq \mathrm{Cn}_{S_1}(\Gamma) \\ & \text{iff} \quad f(\mathrm{Cn}_{S_1}\varphi) \subseteq f(\mathrm{Cn}_{S_1}(\Gamma)) \\ & \text{iff} \quad \mathrm{Cn}_{S_2}(\tau(\varphi)) \subseteq \mathrm{Cn}_{S_2}(\tau(\Gamma)) \\ & \text{iff} \quad \tau(\Gamma) \vdash_{S_2} \tau(\varphi), \end{aligned}$$

hence τ is an interpretation of S_1 in S_2.

Since f is an isomorphism, $f^{-1}: \mathbf{Th}\boldsymbol{S_2} \to \mathbf{Th}\boldsymbol{S_1}$ is also an isomorphism. Furthermore, f^{-1} commutes with substitutions: If σ is a substitution, then

$$\begin{aligned} f^{-1}(\sigma_{S_2}(T)) = \sigma_{S_1}(f^{-1}(T)) \quad & \text{iff} \quad \sigma_{S_2}(T) = f(\sigma_{S_1}(f^{-1}(T))) \\ & \text{iff} \quad \sigma_{S_2}(T) = \sigma_{S_2}(f \circ f^{-1}(T)), \end{aligned}$$

which is certainly true. This allows us to define an interpretation ρ of S_2 in S_1 in the same way that τ was defined. In particular we have, for $\varphi \in Fm^{\ell}$, that $f^{-1}(\mathrm{Cn}_{S_2}\varphi) = \mathrm{Cn}_{S_1}(\rho(\varphi))$, therefore

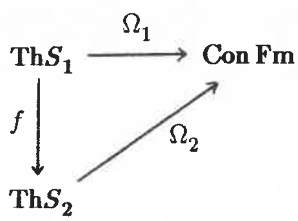
$$\mathrm{Cn}_{S_2}\varphi = f(\mathrm{Cn}_{S_1}(\rho(\varphi))) = \mathrm{Cn}_{S_2}(\tau(\rho(\varphi))),$$

implying that $\varphi \dashv\vdash_{S_2} \tau(\rho(\varphi))$. τ and ρ therefore satisfy the conditions of Proposition 1.9.2, so S_1 and S_2 are equivalent. □

The following theorem looks at the interaction between the Leibniz operator and isomorphisms on the lattices of theories that satisfy condition (ii) of the previous theorem. The corollary that follows will be applied in Chapter 3.

1.9.5 THEOREM [BP89b, Theorem 4.5]

Let S_1 be a k-deductive system and S_2 an ℓ-deductive system, where $1 \leq k, \ell \in \omega$, and suppose that $f: \mathbf{Th}\boldsymbol{S_1} \to \mathbf{Th}\boldsymbol{S_2}$ is an isomorphism that commutes with substitutions. For $i = 1, 2$, let $\mathcal{A}_i = \langle \mathbf{Fm}, \mathrm{Cn}_{S_i}(\emptyset) \rangle$, and set $\Omega_i = \Omega_{\mathcal{A}_i}$, for convenience. Then the following diagram commutes, i.e., for each $T \in \mathrm{Th}S_1$, $\Omega_1(T) = \Omega_2(f(T))$.



Proof. Let $T \in \mathrm{Th}S_1$ and $\Phi \in \mathrm{Con}\,\mathbf{Fm}$. The result will follow if we can show that Φ is compatible with T exactly when Φ is compatible with $f(T)$. Then, since $\Omega_1(T) \in \mathrm{Con}\,\mathbf{Fm}$ is compatible with T, we have that $\Omega_1(T)$ is compatible with $f(T)$, and hence $\Omega_1(T) \subseteq \Omega_2(f(T))$. The converse inclusion follows similarly.

Assume Φ is compatible with T. By Theorem 1.9.4, S_1 and S_2 are equivalent. As in the proof of Theorem 1.9.4, let $\boldsymbol{\tau}$ be an interpretation of S_1 in S_2 and $\boldsymbol{\rho}$ an interpretation of S_2 in S_1 such that $\boldsymbol{\tau}$ and $\boldsymbol{\rho}$ are inverse to eachother as in the sense of Definition 1.9.1, $f = \widehat{\boldsymbol{\tau}}$ and $f^{-1} = \widehat{\boldsymbol{\rho}}$. Then

$$\boldsymbol{\varphi} \dashv\vdash_{S_1} \boldsymbol{\rho}(\boldsymbol{\tau}(\boldsymbol{\varphi})) \quad \text{and} \quad \boldsymbol{\psi} \dashv\vdash_{S_2} \boldsymbol{\tau}(\boldsymbol{\rho}(\boldsymbol{\psi}))$$

for all $\boldsymbol{\varphi} \in Fm^k$ and $\boldsymbol{\psi} \in Fm^\ell$. Let $\boldsymbol{\varphi} = \langle \varphi_1, \ldots, \varphi_\ell \rangle$, $\boldsymbol{\psi} = \langle \psi_1, \ldots, \psi_\ell \rangle \in Fm^\ell$ such that $\boldsymbol{\varphi} \in f(T)$ and $(\boldsymbol{\varphi}, \boldsymbol{\psi}) \in \Phi^{[\ell]}$. To prove that Φ is compatible with $f(T)$ we need to show that $\boldsymbol{\psi} \in f(T)$. Note that $T = f^{-1}(f(T)) = \widehat{\boldsymbol{\rho}}(f(T)) \supseteq \boldsymbol{\rho}(f(T))$. Note also that $\boldsymbol{\varphi} \in f(T)$ implies $f(T) \vdash_{S_2} \boldsymbol{\varphi}$, which implies $\boldsymbol{\rho}(f(T)) \vdash_{S_1} \boldsymbol{\rho}(\boldsymbol{\varphi})$, therefore $T \vdash_{S_1} \boldsymbol{\rho}(\boldsymbol{\varphi})$, i.e., $\boldsymbol{\rho}(\boldsymbol{\varphi}) \subseteq T$. Now suppose $\boldsymbol{\rho} = \{\boldsymbol{\rho}^i \,;\, i \leq n\}$, where $\boldsymbol{\rho}^i \langle p_1, \ldots, p_\ell \rangle = \langle \rho^i_1 \langle p_1, \ldots, p_\ell \rangle, \ldots, \rho^i_k \langle p_1, \ldots, p_\ell \rangle \rangle$. Then

$$\boldsymbol{\rho}^i(\boldsymbol{\varphi}) = \langle \rho^i_1(\varphi_1, \ldots, \varphi_\ell), \ldots, \rho^i_k(\varphi_1, \ldots, \varphi_\ell) \rangle.$$

Since $(\boldsymbol{\varphi}, \boldsymbol{\psi}) \in \Phi^{[\ell]}$, $(\varphi_i, \psi_i) \in \Phi$ for $i \leq \ell$, therefore $(\rho^i_j \langle \varphi_1, \ldots, \varphi_\ell \rangle, \rho^i_j \langle \psi_1, \ldots, \psi_\ell \rangle) \in \Phi$ for $i \leq n$ and $j \leq k$, hence $(\boldsymbol{\rho}^i \langle \varphi_1, \ldots, \varphi_\ell \rangle, \boldsymbol{\rho}^i \langle \psi_1, \ldots, \psi_\ell \rangle) \in \Phi^{[\ell]}$ for $i \leq n$. But we have already shown that $\boldsymbol{\rho}^i \langle \varphi_1, \ldots, \varphi_\ell \rangle \in T$ for $i \leq n$, and since Φ is compatible with T by assumption, $\boldsymbol{\rho}^i \langle \psi_1, \ldots, \psi_\ell \rangle \in T$ as well. So $\boldsymbol{\rho}(\boldsymbol{\psi}) = \{\boldsymbol{\rho}^i(\boldsymbol{\psi}) \,;\, i \leq n\} \subseteq T$, hence $\boldsymbol{\tau}(\boldsymbol{\rho}(\boldsymbol{\psi})) \subseteq \boldsymbol{\tau}(T) \subseteq \widehat{\boldsymbol{\tau}}(T) = f(T)$. Since $\boldsymbol{\psi} \dashv\vdash_{S_2} \boldsymbol{\tau}(\boldsymbol{\rho}(\boldsymbol{\psi}))$, we have $\boldsymbol{\psi} \in f(T)$, so it follows that Φ is compatible with $f(T)$. The reverse implication is similarly proved. □

1.9.6 COROLLARY [BP89b, Corollary 4.6]

If $\mathcal{K}_1$ and $\mathcal{K}_2$ are quasivarieties (over the same language) such that $S_{\mathcal{K}_1}$ and $S_{\mathcal{K}_2}$ are equivalent then $S_{\mathcal{K}_1} = S_{\mathcal{K}_2}$ and $\mathcal{K}_1 = \mathcal{K}_2$.

Proof. As in Theorem 1.9.5, let $\mathcal{A}_i = \langle \mathbf{Fm}, \mathrm{Cn}_{S_{\mathcal{K}_i}}(\emptyset)\rangle$ for $i = 1,2$, and set $\Omega_i = \Omega_{\mathcal{A}_i}$. Recall that the theories of $S_{\mathcal{K}_1}$ and $S_{\mathcal{K}_2}$ are the $\mathcal{K}_1$- and $\mathcal{K}_2$-congruences of $\mathbf{Fm}$ respectively. Recall, furthermore, that for each $\mathcal{K}_1$-congruence Φ of $\mathbf{Fm}$, $\Omega_1\Phi = \Phi$, hence $\Omega_1 : \mathrm{Th}S_{\mathcal{K}_1} \to \mathrm{Con}\,\mathbf{Fm}$ is the identity map. Similarly $\Omega_2 : \mathrm{Th}S_{\mathcal{K}_1} \to \mathrm{Con}\,\mathbf{Fm}$ is also the identity map.

Let $f : \mathbf{Th}\boldsymbol{S}_{\mathcal{K}_1} \to \mathbf{Th}\boldsymbol{S}_{\mathcal{K}_2}$ be an isomorphism that commutes with substitutions. Such a function exists by Theorem 1.9.4, since $S_{\mathcal{K}_1}$ and $S_{\mathcal{K}_2}$ are equivalent. If $\Phi \in \mathrm{Th}S_{\mathcal{K}_1}$, then, by Theorem 1.9.5, $\Omega_2 f(\Phi) = \Omega_1\Phi$. This, in turn, gives us that $f(\Phi) = \Omega_2 f(\Phi) = \Omega_1\Phi = \Phi$, implying that f is the identity map and therefore that $\mathrm{Th}S_{\mathcal{K}_1} = \mathrm{Th}S_{\mathcal{K}_2}$, i.e., $S_{\mathcal{K}_1} = S_{\mathcal{K}_2}$.

Assume that $\mathcal{K}_1 \neq \mathcal{K}_2$. Take $\mathbf{A} \in \mathcal{K}_1 - \mathcal{K}_2$. Since $\mathcal{K}_2$ is a quasivariety, there is a quasi-identity

$$\Big(\mathop{\&}_{i \leq r} \alpha_i \approx \beta_i\Big) \Rightarrow \alpha \approx \beta$$

($r \in \omega$) in variables $x_1, \dots, x_n$ satisfied by all the algebras of $\mathcal{K}_2$ and violated by $\mathbf{A}$. We can therefore choose $a_1, \dots, a_n \in A$ such that $\alpha_i^{\mathbf{A}}(a_1, \dots, a_n) = \beta_i^{\mathbf{A}}(a_1, \dots, a_n)$ for all $i \leq r$ but $\alpha^{\mathbf{A}}(a_1, \dots, a_n) \neq \beta^{\mathbf{A}}(a_1, \dots, a_n)$. Then the subalgebra $\mathbf{A}'$ of $\mathbf{A}$ generated by $\{a_1, \dots, a_n\}$ is in $\mathrm{S}(\mathcal{K}_1) = \mathcal{K}_1$, $\mathbf{A}' \notin \mathcal{K}_2$ and $\mathbf{A}'$ is finitely generated. Let $h : \mathbf{Fm} \to \mathbf{A}$ be a surjective homomorphism and set $\Phi = \ker h$. By the Correspondence Theorem of universal algebra, $\mathbf{Fm}/\Phi \cong \mathbf{A} \in \mathcal{K}_1$, hence Φ is a $\mathcal{K}_1$-congruence. Φ is not a $\mathcal{K}_2$-congruence, however, since $\mathbf{A} \notin \mathcal{K}_2$. What we have, therefore, is that $\Phi \in \mathrm{Th}S_{\mathcal{K}_1}$ but $\Phi \notin \mathrm{Th}S_{\mathcal{K}_2}$, so $\mathrm{Th}S_{\mathcal{K}_1} \neq \mathrm{Th}S_{\mathcal{K}_2}$ or, $S_{\mathcal{K}_1} \neq S_{\mathcal{K}_2}$. By this contradiction, $\mathcal{K}_1 = \mathcal{K}_2$. □

1.10 *k*-DEDUCTIVE SYSTEMS AS FIRST-ORDER THEORIES

This brief section is devoted to a study of an alternative characterization of the consequence relation of a deductive system. Let S be a k-deductive system over a language $\mathcal{L}$. The consequence relation $\vdash_S$, as defined in Section 1.5, is not a first-order notion as it is a relation between *sets* of k-formulas and single k-formulas. Moreover, the alternative characterizations in terms of closure operators or algebraic lattices are not first-order notions either. It is possible, however, to characterize deductive systems in first-order terms, as was first observed by Bloom, in [Bl75], for 1-deductive systems. This result is noteworthy in that it means that, in the metalogical study of k-deductive systems, all the results of first-order mathematics are available. We refer the reader to Section 0.5 for the necessary definitions.

If we add to the language $\mathfrak{L}$ a single k-ary predicate symbol D, then we get a first-order language, $\mathfrak{L}_D$, (without equality) whose extra-logical symbols are D and the connectives of $\mathfrak{L}$, now thought of as operation symbols of the appropriate rank. The $\mathfrak{L}_D$-structures are of the form $\langle A, \mathfrak{L}^A, D^A\rangle$, where A is a set of elements, $\mathfrak{L}^A$ is a set of operations on A corresponding to the language $\mathfrak{L}$, and D^A is a k-ary relation on A. In other words, $\mathbf{A} = \langle A, \mathfrak{L}^A\rangle$ is an $\mathfrak{L}$-algebra and $D^A \subseteq A^k$, i.e., $\langle\langle A, \mathfrak{L}^A\rangle, D^A\rangle = \langle \mathbf{A}, D^A\rangle$ is a k-matrix. Thus the $\mathfrak{L}_D$-structures may be identified with the k-matrices and for consistency, we shall write $\mathfrak{L}^A$ and D^A as $\mathfrak{L}^{\mathbf{A}}$ and $D^{\mathbf{A}}$ henceforth. If we consider k-matrices $\mathcal{A}$ and $\mathfrak{B}$ as $\mathfrak{L}_D$-stuctures, then Definition 1.8.1 implies that $\mathfrak{B}$ is a submatrix of $\mathcal{A}$ iff $\mathfrak{B}$ is a substructure of $\mathcal{A}$. Moreover $h: \mathcal{A} \to \mathfrak{B}$ is a matrix homomorphism (resp. isomorphism) from $\mathcal{A}$ to $\mathfrak{B}$ iff it is a homomorphism (resp. isomorphism) between the structures $\mathcal{A}$ and $\mathfrak{B}$. Note that for any k-formula $\boldsymbol{\varphi} = \langle \varphi_1, \dots, \varphi_k\rangle \in Fm^k$ with variables $p_1, \dots, p_n$, $D(\boldsymbol{\varphi}) = D(\varphi_1, \dots, \varphi_k)$ is valid in the $\mathfrak{L}_D$-structure $\langle A, \mathfrak{L}^{\mathbf{A}}, D^{\mathbf{A}}\rangle$ if and only if for every interpretation $\overline{a}$ of the variables $p_1, \dots, p_n$ in A we have $\boldsymbol{\varphi}^{\mathbf{A}}(\overline{a}) \in D^{\mathbf{A}}$ (where $\boldsymbol{\varphi}^{\mathbf{A}} \in \mathfrak{L}^{\mathbf{A}}$ is the operation on A corresponding to $\boldsymbol{\varphi}$).

With each axiom $\boldsymbol{\varphi}$ of S, we associate the universal Horn sentence $\forall\overline{p}(D\boldsymbol{\varphi})$, where $\overline{p}$ is a list of all the variables occurring in $\boldsymbol{\varphi}$. With each inference rule, $\langle\{\boldsymbol{\psi}_1, \dots, \boldsymbol{\psi}_n\}, \boldsymbol{\varphi}\rangle$ of S we associate the universal Horn sentence

$$\forall\overline{p}(D\boldsymbol{\psi}_1 \,\&\, \dots \,\&\, D\boldsymbol{\psi}_n \Rightarrow D\boldsymbol{\varphi}),$$

where $\overline{p}$ is a list of all the variables occurring in the $\boldsymbol{\psi}$'s and in $\boldsymbol{\varphi}$. Let $T(S)$ be the first-order (elementary) universal Horn theory over $\mathfrak{L}_D$ whose proper axioms are all these sentences. Recall that $\vdash_{T(S)}$ denotes the consequence relation relative to $T(S)$ in the usual first-order sense. As noted above, the models for $T(S)$ can be considered as k-matrices. We claim that the models of $T(S)$ are precisely the matrix models of S. For suppose $\mathcal{A} = \langle \mathbf{A}, F\rangle$ is a k-matrix that is a model of $T(S)$. Then, by the Validity and Completeness Theorems of First-Order Logic, $\vdash_{T(S)} \forall\overline{p}(D\boldsymbol{\varphi})$ if and only if for every interpretation $\overline{a}$ of the variables $\overline{p}$ in A, we have $\boldsymbol{\varphi}^{\mathbf{A}}(\overline{a}) \in F$, which holds if and only if $\models_{\mathcal{A}} \boldsymbol{\varphi}$, by definition of $\models_{\mathcal{A}}$. Similarly, $\vdash_{T(S)} \forall\overline{p}(D\boldsymbol{\psi}_1 \,\&\, \dots \,\&\, D\boldsymbol{\psi}_n \Rightarrow D\boldsymbol{\varphi})$ if and only if for every interpretation $\overline{a}$ of the variables $\overline{p}$ in A such that $\boldsymbol{\psi}_i^{\mathbf{A}}(\overline{a}) \in F$ for all $i \leq n$, we have $\varphi^{\mathbf{A}}(\overline{a}) \in F$, which holds if and only if $\psi_1, \ldots, \psi_n \models_{\mathcal{A}} \varphi$. So, if φ is an axiom of S, then $\forall\overline{p}(D\varphi)$ is a proper axiom of $T(S)$, hence $\models_{\mathcal{A}} \varphi$. If $\langle\{\psi_1, \ldots, \psi_n\}, \varphi\rangle$ is an inference rule of S then $\forall\overline{p}(D\psi_1 \,\&\ldots\&\, D\psi_n \Rightarrow D\varphi)$ is a proper axiom of $T(S)$, hence $\psi_1, \ldots, \psi_n \models_{\mathcal{A}} \varphi$. By the note following Definition 1.6.1, this shows that $\mathcal{A}$ is an S-matrix. Conversely, suppose $\mathcal{A} = \langle \mathbf{A}, F\rangle$ is an S-matrix. For every universal Horn sentence $\forall\overline{p}(D\varphi)$ that is a proper axiom of $T(S)$, φ is a axiom of S, hence $\models_{\mathcal{A}} \varphi$. For every universal Horn sentence $\forall\overline{p}(D\psi_1 \,\&\ldots\&\, D\psi_n \Rightarrow D\varphi)$ that is a proper axiom of $T(S)$, $\langle\{\psi_1, \ldots, \psi_n\}, \varphi\rangle$ is a inference rule of S, hence $\psi_1, \ldots, \psi_n \models_{\mathcal{A}} \varphi$. Thus $\mathcal{A}$ satisfies every axiom of $T(S)$, implying that it is a model for $T(S)$, which proves the claim. This leads to the following result, which is stated in [BP89a].

1.10.1 PROPOSITION

For all $\psi_1, \ldots, \psi_n, \varphi \in Fm^k$, *we have*

$$\psi_1, \ldots, \psi_n \vdash_S \varphi \quad \textit{if and only if} \quad \vdash_{T(S)} \forall\overline{p}(D\psi_1 \,\&\ldots\&\, D\psi_n \Rightarrow D\varphi).$$

Proof. By the Completeness and Validity theorems of first-order theories (Theorem 0.5.3), the right hand side holds if and only if $\forall\overline{p}(D\psi_1 \,\&\ldots\&\, D\psi_n \Rightarrow D\varphi)$ is valid in every model of $T(S)$. By the preceding claim, this is true if and only if $\psi_1, \ldots, \psi_n \models_{\mathcal{A}} \varphi$ for every $\mathcal{A} \in \mathrm{Mod}S$. Recall from Theorem 1.6.2 that the class $\mathrm{Mod}S$ forms a matrix semantics for S, hence this holds if and only if $\psi_1, \ldots, \psi_n \vdash_S \varphi$. □

Conversely, with every universal Horn theory T of the first-order language $\mathcal{L}_D$, we can associate a k-deductive system $S(T)$. Suppose that T has a set Γ of universal Horn sentences as its set of proper axioms. For every universal Horn sentence $\forall\overline{p}(D\varphi)$ in Γ, choose φ to be an axiom of $S(T)$, and for every universal Horn sentence $\forall\overline{p}(D\psi_1 \,\&\ldots\&\, D\psi_n \Rightarrow D\varphi)$ in Γ, choose $\langle\{\psi_1, \ldots, \psi_n\}, \varphi\rangle$ to be an inference rule of $S(T)$. We specify that $S(T)$ has no other axioms or inference rules. It is easy to see that $S(T(S)) = S$ and $T(S(T)) = T$. Thus there is a one-to-one correspondence between k-deductive systems over $\mathcal{L}$ and universal Horn theories over $\mathcal{L}_D$, where D is a k-ary predicate symbol.

Chapter Two
Protoalgebraic Deductive Systems

The general aim of this thesis is to investigate those deductive systems to which universal algebraic methods can be applied. The class of 'protoalgebraic' k-deductive systems introduced in this chapter forms a very wide class of such k-deductive systems. It includes all the classical deductive systems as well as all those introduced in Section 1.4, bar one. The notion of a protoalgebraic k-deductive system is formally defined in terms of the Leibniz operator Ω. We shall see, however, that it is also definable in terms of matrix models and filters, namely by the 'filter correspondence property'. It is this property that evidences the universal algebraic nature of these deductive systems, for it is an extension of the Correspondence theorem of universal algebra. Section 2.1 looks at these notions, with Theorem 2.1.3 presenting a compilation of various equivalent characterizations of protoalgebraicity. In that section we also present a deductive system that is not protoalgebraic. Although k-deductive systems exist that are not protoalgebraic, it does seem that protoalgebraic k-deductive systems form the widest class of k-deductive systems with a reasonable model theory. This claim is explored in Section 2.2. Section 2.3 presents an internal characterization of protoalgebraic k-deductive systems, in other words, a characterization depending only on the existence of certain 'equivalence k-formulas' within a given k-deductive system and not on the Leibniz operator or on matrix models. This idea is taken further in Section 2.4 where we tie up the link between the equivalence k-formulas and the Leibniz operator. In Section 2.5, we define some classes of k-deductive systems that satisfy stronger conditions than protoalgebraicity, namely 'congruential' and 'weakly congruential' k-deductive systems. Lastly, we consider some examples in Section 2.6 which distinguish between the notions 'protoalgebraic', 'weakly congruential' and 'congruential'.

2.1 PROTOALGEBRAIC k-DEDUCTIVE SYSTEMS

2.1.1 DEFINITION

A k-deductive system S is called *protoalgebraic* if, for all S-theories T,

$$(\varphi,\psi)\in(\Omega_{\mathbf{Fm}}T)^{[k]} \text{ implies } T,\varphi\vdash_S\psi \text{ and } T,\psi\vdash_S\varphi.$$

Recall that the intuitive idea behind the Leibniz relation is that it identifies those k-formulas that are 'equivalent' with respect to some fixed S-theory. If we extend that intuition to this definition, we can read it as: If φ and ψ are k-formulas that are 'equivalent' with respect to a fixed S-theory T, then φ and ψ are interderivable relative to T.

S_{Eq} is a natural example of a protoalgebraic 2-deductive system since it is based on an algebraic notion, namely that of an equivalence relation. To see this, recall from Section 1.6 that the S_{Eq}-theories are exactly the equivalence relations on **Fm**, and for each equivalence relation R, $\Omega_{\mathbf{Fm}}R=\bigvee\{\Phi\in\mathrm{Con}\,\mathbf{Fm}\,;\Phi\subseteq R\}\subseteq R$. If $((\varphi_1,\varphi_2),(\psi_1,\psi_2))\in(\Omega_{\mathbf{Fm}}R)^{[2]}$, then $(\varphi_1,\psi_1),(\varphi_2,\psi_2)\in\Omega_{\mathbf{Fm}}R$, hence $(\varphi_1,\psi_1),(\varphi_2,\psi_2)\in R$. By the symmetry rule of S_{Eq}, $(\psi_1,\varphi_1),(\varphi_1,\varphi_2),(\varphi_2,\psi_2)\in\mathrm{Cn}_{S_{\mathrm{Eq}}}(R\cup\{(\varphi_1,\varphi_2)\})$, hence $(\psi_1,\psi_2)\in\mathrm{Cn}_{S_{\mathrm{Eq}}}(R\cup\{(\varphi_1,\varphi_2)\})$ by the transitivity rule of S_{Eq}. Thus

$$R,(\varphi_1,\varphi_2)\vdash_{S_{\mathrm{Eq}}}(\psi_1,\psi_2),$$

and similarly $\quad R,(\psi_1,\psi_2)\vdash_{S_{\mathrm{Eq}}}(\varphi_1,\varphi_2),\quad$ proving protoalgebraicity.

S_{Con} is also a protoalgebraic 2-deductive system. Recall that the S_{Con}-theories are precisely the congruence relations on **Fm**. In particular, each S_{Con}-theory is an equivalence relation and $\Omega_{\mathbf{Fm}}\Phi=\Phi$. Since $\vdash_{S_{\mathrm{Eq}}}\subseteq\vdash_{S_{\mathrm{Con}}}$, it follows immediately from the preceding paragraph that whenever $((\varphi_1,\varphi_2),(\psi_1,\psi_2))\in(\Omega_{\mathbf{Fm}}\Phi)^{[2]}$

$$\Phi,(\varphi_1,\varphi_2)\vdash_{S_{\mathrm{Con}}}(\psi_1,\psi_2)$$

and $\quad \Phi,(\psi_1,\psi_2)\vdash_{S_{\mathrm{Con}}}(\varphi_1,\varphi_2),$

hence S_{Con} is protoalgebraic.

Recall from Section 1.8 that if $\mathcal{A}$ and $\mathfrak{B}$ are k-matrices and $h:\mathcal{A}\to\mathfrak{B}$ is a matrix homomorphism then the map $h_S:\mathrm{Fi}^S\mathcal{A}\to\mathrm{Fi}^S\mathfrak{B}$ is defined for every $F\in\mathrm{Fi}^S\mathcal{A}$ by $h_S(F)=\mathrm{Fg}^S_{\mathfrak{B}}h(F)$. Recall also that $h:\mathcal{A}\to\mathfrak{B}$ is a reductive matrix homomorphism if h is a surjective homomorphism and $h^{-1}(F_{\mathfrak{B}})=F_{\mathcal{A}}$.

2.1.2 DEFINITION

A k-deductive system S has the *filter correspondence property* if, for every reductive matrix homomorphism $h: \mathcal{A} \to \mathfrak{B}$ between S-matrices $\mathcal{A} = \langle \mathbf{A}, F_{\mathcal{A}} \rangle$ and $\mathfrak{B} = \langle \mathbf{B}, F_{\mathfrak{B}} \rangle$ and every $F \in \mathrm{Fi}^S \mathcal{A}$,

$$h^{-1}(h_S(F)) = F.$$

The following theorem collects together a number of equivalent characterizations that appear in [BP89a], [BP89b], [BP88] or [BP92]. In fact, each of the first three properties given have been taken as the definition for protoalgebraicity in at least one of these works. The property (iv) appears in [BP92] and is called the *compatibility property* there. Property (viii) does not appear explicitly in any paper and is included here by the author. We have corrected the original versions of (vi) and (vii) which were given without the requirement that h be reductive in [BP89b].

2.1.3 THEOREM

Let S be a k-deductive system. The following are equivalent:

(i) *S is protoalgebraic;*

(ii) *for all $T, U \in \mathrm{Th}S$, if $T \subseteq U$, then $\Omega T \subseteq \Omega U$, i.e. the operator Ω is monotonic on $\mathrm{Th}S$;*

(iii) *for all S-matrices $\mathcal{A}$ and all $E, F \in \mathrm{Fi}^S \mathcal{A}$, if $E \subseteq F$ then $\Omega_{\mathcal{A}} E \subseteq \Omega_{\mathcal{A}} F$, i.e., the operator $\Omega_{\mathcal{A}}$ is monotonic on $\mathrm{Fi}^S \mathcal{A}$;*

(iv) *for every S-matrix $\mathcal{A} = \langle \mathbf{A}, F_{\mathcal{A}} \rangle$, if $\Phi \in \mathrm{Con}\,\mathbf{A}$ is compatible with $F_{\mathcal{A}}$, then Φ is compatible with every S-filter of $\mathcal{A}$;*

(v) *S has the filter correspondence property;*

(vi) *if $\mathcal{A} = \langle \mathbf{A}, F_{\mathcal{A}} \rangle$, $\mathfrak{B} = \langle \mathbf{B}, F_{\mathfrak{B}} \rangle$ are S-matrices and $h: \mathcal{A} \to \mathfrak{B}$ is a reductive matrix homomorphism, and if $Y \subseteq B^k, X \subseteq A^k$ such that $h(X) = Y$, then*

$$h^{-1}(\mathrm{Fg}^S_{\mathfrak{B}} Y) = \mathrm{Fg}^S_{\mathcal{A}} X;$$

(vii) *if $\mathcal{A} = \langle \mathbf{A}, F_{\mathcal{A}} \rangle$, $\mathfrak{B} = \langle \mathbf{B}, F_{\mathfrak{B}} \rangle$ are S-matrices and $h: \mathcal{A} \to \mathfrak{B}$ is a reductive matrix homomorphism, and if $Y \subseteq B^k, X \subseteq A^k$ such that Y is finite and $h(X) = Y$, then*

$$h^{-1}(\mathrm{Fg}^S_{\mathfrak{B}} Y) = \mathrm{Fg}^S_{\mathcal{A}} X;$$

(viii) *for every S-matrix $\mathcal{A} = \langle \mathbf{A}, F_{\mathcal{A}} \rangle$, if $(\boldsymbol{a}, \boldsymbol{b}) \in (\Omega_{\mathcal{A}} F_{\mathcal{A}})^{[k]}$, then $\boldsymbol{a} \in \mathrm{Fg}^S_{\mathcal{A}} \boldsymbol{b}$ and $\boldsymbol{b} \in \mathrm{Fg}^S_{\mathcal{A}} \boldsymbol{a}$.*

Proof. (i)⇒(ii) Let $T, U \in \mathrm{Th}S$ such that $T \subseteq U$. To show that $\Omega T \subseteq \Omega U$ we shall show that ΩT is compatible with U. Then the result will follow by Theorem 1.7.3. If $\boldsymbol{\varphi} \in U$ and $(\boldsymbol{\varphi}, \boldsymbol{\psi}) \in (\Omega T)^{[k]}$ then, by assumption, $T, \boldsymbol{\varphi} \vdash_S \boldsymbol{\psi}$, i.e., $\boldsymbol{\psi} \in \mathrm{Cn}_S(T \cup \{\boldsymbol{\varphi}\})$. Since $T \subseteq U$ and $\boldsymbol{\varphi} \in U$, $\mathrm{Cn}_S(T \cup \{\boldsymbol{\varphi}\}) \subseteq U$, hence $\boldsymbol{\psi} \in U$. So ΩT is compatible with U.

(ii)⇒(iii) Let $\mathcal{A} = \langle \mathbf{A}, F_{\mathcal{A}} \rangle$ be an S-matrix and let $E, F \in \mathrm{Fi}^S \mathcal{A}$ such that $E \subseteq F$. Suppose that $\Omega_{\mathcal{A}} E \nsubseteq \Omega_{\mathcal{A}} F$. Let $a, b \in A$ such that $(a,b) \in \Omega_{\mathcal{A}} E$ and $(a,b) \notin \Omega_{\mathcal{A}} F$. By Definition 1.7.1, there exists a k-formula $\boldsymbol{\alpha}(p, \overline{q})$ such that, for some $\overline{c} \in A^m$,

$$\boldsymbol{\alpha}^{\mathbf{A}}(a, \overline{c}) \in F \quad \text{and} \quad \boldsymbol{\alpha}^{\mathbf{A}}(b, \overline{c}) \notin F$$

or

$$\boldsymbol{\alpha}^{\mathbf{A}}(a, \overline{c}) \notin F \quad \text{and} \quad \boldsymbol{\alpha}^{\mathbf{A}}(b, \overline{c}) \in F.$$

Without loss of generality we can assume that the first statement holds. Let us assume also, for the present, that the algebra $\mathbf{A}$ is countably generated, i.e., that $\mathbf{A} = \mathrm{Sg}^{\mathbf{A}}(X)$ for some $X \subseteq A$ with $|X| \leq \omega$. Let r be a variable different from p and from all co-ordinates of $\overline{q}$. Since $P - \{p, r, \overline{q}\}$ is denumerable and X is countable, there is a surjection $h: P \to X$ such that $hp = a$, $hr = b$, $h\overline{r} = \overline{c}$. Since $\mathbf{Fm}$ is the absolutely free $\mathcal{L}$-algebra over P, the universal mapping property (Theorem 0.2.10) says that h extends to a homomorphism $\mathbf{Fm} \to \mathbf{A}$ which is surjective (since X generates $\mathbf{A}$). We shall also call this homomorphism h.

Let $\mathcal{B} = \langle \mathbf{Fm}, h^{-1}(F_{\mathcal{A}}) \rangle$. Then $h: \mathcal{B} \to \mathcal{A}$ is a surjective (in fact reductive) matrix homomorphism so, by Corollary 1.8.3, $\mathcal{B}$ is an S-matrix and $h^{-1}(E)$, $h^{-1}(F)$ are S-theories. We have $h^{-1}(E) \subseteq h^{-1}(F)$, so $\Omega h^{-1}(E) \subseteq \Omega h^{-1}(F)$, by assumption (ii). We also have $h^{-1}(\Omega_{\mathcal{A}} E) = \Omega h^{-1}(E)$ by Lemma 1.8.5. We define

$$h^{-1}((\Omega_{\mathcal{A}} E)^{[k]}) = \{(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in (Fm^k)^2 ; (h\boldsymbol{\alpha}, h\boldsymbol{\beta}) \in (\Omega_{\mathcal{A}} E)^{[k]}\}.$$

We claim that $h^{-1}((\Omega_{\mathcal{A}} E)^{[k]}) \subseteq (\Omega h^{-1}(E))^{[k]}$. Indeed, if $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in h^{-1}((\Omega_{\mathcal{A}} E)^{[k]})$, then for each $i \leq k$, $(\alpha_i, \beta_i) \in h^{-1}(\Omega_{\mathcal{A}} E)$, hence $(\alpha_i, \beta_i) \in \Omega h^{-1}(E)$, implying that $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in (\Omega h^{-1}(E))^{[k]}$, as required.

Since $(hp, hr) = (a, b)$ and $(a,b) \in \Omega_{\mathcal{A}} E$, we have that $(p, r) \in h^{-1}(\Omega_{\mathcal{A}} E)$. Also, $(h\boldsymbol{\alpha}(p, \overline{q}), h\boldsymbol{\alpha}(r, \overline{q})) = (\boldsymbol{\alpha}^{\mathbf{A}}(a, \overline{c}), \boldsymbol{\alpha}^{\mathbf{A}}(b, \overline{c})) \in (\Omega_{\mathcal{A}} E)^{[k]}$ since $\Omega_{\mathcal{A}} E$ is a congruence on $\mathbf{A}$ and $(a,b) \in \Omega_{\mathcal{A}} E$. Thus $(\boldsymbol{\alpha}(p, \overline{q}), \boldsymbol{\alpha}(r, \overline{q})) \in h^{-1}((\Omega_{\mathcal{A}} E)^{[k]})$, implying $(\boldsymbol{\alpha}(p, \overline{q}), \boldsymbol{\alpha}(r, \overline{q})) \in (\Omega h^{-1}(E))^{[k]}$ by the above result. Thus $(\boldsymbol{\alpha}(p, \overline{q}), \boldsymbol{\alpha}(r, \overline{q})) \in (\Omega h^{-1}(F))^{[k]}$ since $\Omega h^{-1}(E) \subseteq \Omega h^{-1}(F)$. Now, $\boldsymbol{\alpha}(p, \overline{q}) \in h^{-1}(F)$ since $h\boldsymbol{\alpha}(p, \overline{q}) = \boldsymbol{\alpha}^{\mathbf{A}}(a, \overline{c}) \in F$. By the compatibility of $\Omega h^{-1}(F)$ with $h^{-1}(F)$, we deduce that $\boldsymbol{\alpha}(r, \overline{q}) \in h^{-1}(F)$, hence $\boldsymbol{\alpha}^{\mathbf{A}}(b, \overline{c}) = h\boldsymbol{\alpha}(r, \overline{q}) \in F$, which is a contradiction. So $\Omega_{\mathcal{A}} E \subseteq \Omega_{\mathcal{A}} F$ provided that $\mathbf{A}$ is countably generated.

Now, suppose $\mathbf{A}$ is not countably generated. Let $\mathbf{A}'$ be the subalgebra of $\mathbf{A}$ generated by $\{a,b,\bar{c}\}$ and let $\mathcal{A}' = \langle \mathbf{A}', F_{\mathcal{A}} \cap (A')^k \rangle$. Then $\mathcal{A}'$ is a submatrix of $\mathcal{A}$, hence $\mathcal{A}'$ is an S-matrix, by Proposition 1.8.2 (i). It also follows from Proposition 1.8.2 (i) that $E \cap (A')^k$ and $F \cap (A')^k$ are S-filters of $\mathcal{A}'$. Of course, $E \cap (A')^k \subseteq F \cap (A')^k$. Now for any $\boldsymbol{\eta}(s,\bar{t}) \in Fm^k$ and $\bar{d} \in (A')^k$, $\boldsymbol{\eta}^{\mathbf{A}'}(a,\bar{d}) = \boldsymbol{\eta}^{\mathbf{A}}(a,\bar{d})$ and this lies in $E \cap (A')^k$ if and only if it lies in E. The same applies to $\boldsymbol{\eta}^{\mathbf{A}'}(b,\bar{d})$, so we can deduce from $(a,b) \in \Omega_{\mathcal{A}} E$ and Definition 1.7.1 that $(a,b) \in \Omega_{\mathcal{A}'}(E \cap (A')^k)$. Since $\boldsymbol{\alpha}^{\mathbf{A}'}(a,\bar{c}) = \boldsymbol{\alpha}^{\mathbf{A}}(a,\bar{c}) \in F \cap (A')^k$ but $\boldsymbol{\alpha}^{\mathbf{A}'}(b,\bar{c}) = \boldsymbol{\alpha}^{\mathbf{A}}(b,\bar{c}) \notin F \cap (A')^k$, Definition 1.7.1 tells us that $(a,b) \notin \Omega_{\mathcal{A}'}(F \cap (A')^k)$. Thus $\Omega_{\mathcal{A}'}(E \cap (A')^k) \not\subseteq \Omega_{\mathcal{A}'}(F \cap (A')^k)$. But since $\mathbf{A}'$ is countably (in fact, finitely) generated, this contradicts the result just proved. This concludes the proof of (iii).

(iii)⇒(iv) Let $\mathcal{A} = \langle \mathbf{A}, F_{\mathcal{A}} \rangle$ be an S-matrix and $\Phi \in \operatorname{Con} \mathbf{A}$ such that Φ is compatible with $F_{\mathcal{A}}$. Then $\Phi \subseteq \Omega_{\mathcal{A}} F_{\mathcal{A}}$ by Theorem 1.7.3. Let $G \in \mathrm{Fi}^S \mathcal{A}$. Then $F_{\mathcal{A}} \subseteq G$ by definition, so $\Omega_{\mathcal{A}} F_{\mathcal{A}} \subseteq \Omega_{\mathcal{A}} G$ by the assumption (iii), therefore $\Phi \subseteq \Omega_{\mathcal{A}} F_{\mathcal{A}} \subseteq \Omega_{\mathcal{A}} G$. If $\boldsymbol{a} \in G$ and $(\boldsymbol{a},\boldsymbol{b}) \in \Phi^{[k]}$, then $(\boldsymbol{a},\boldsymbol{b}) \in (\Omega_{\mathcal{A}} G)^{[k]}$, hence $\boldsymbol{b} \in G$, implying that Φ is compatible with G.

(iv)⇒(v) Let $\mathcal{A} = \langle \mathbf{A}, F_{\mathcal{A}} \rangle$, $\mathcal{B} = \langle \mathbf{B}, F_{\mathcal{B}} \rangle$ be S-matrices, let $F \in \mathrm{Fi}^S \mathcal{A}$ and let $h : \mathcal{A} \to \mathcal{B}$ be a reductive matrix homomorphism. Since $h(F_{\mathcal{A}}) = F_{\mathcal{B}}$, it is easily seen that $\ker h$ is compatible with $F_{\mathcal{A}}$, therefore assumption (iv) says that $\ker h$ is compatible with F. Since h is surjective, by assumption, we can deduce, by Lemma 1.8.4, that $h_S(F) = h(F)$. So,

$$F \subseteq h^{-1}(h(F)) = h^{-1}(h_S(F)).$$

To see that $h^{-1}(h(F)) \subseteq F$, suppose $\boldsymbol{a} \in h^{-1}(h(F))$, i.e., $h\boldsymbol{a} = h\boldsymbol{b}$ for some $\boldsymbol{b} \in F$. Since F is compatible with $\ker h$ and $(\boldsymbol{a},\boldsymbol{b}) \in (\ker h)^{[k]}$, we have $\boldsymbol{a} \in F$. Thus $h^{-1}(h(F)) \subseteq F$, implying $F = h^{-1}(h(F)) = h^{-1}(h_S(F))$.

(v)⇒(vi) Let $\mathcal{A} = \langle \mathbf{A}, F_{\mathcal{A}} \rangle$, $\mathcal{B} = \langle \mathbf{B}, F_{\mathcal{B}} \rangle$ be S-matrices and let $h : \mathcal{A} \to \mathcal{B}$ be a reductive matrix homomorphism. Let $Y \subseteq B^k, X \subseteq A^k$ such that $h(X) = Y$. Then the filter correspondence property implies

$$\mathrm{Fg}^S_{\mathcal{A}} X = h^{-1}(h_S(\mathrm{Fg}^S_{\mathcal{A}} X)) \supseteq h^{-1}(\mathrm{Fg}^S_{\mathcal{B}} h(X)) = h^{-1}(\mathrm{Fg}^S_{\mathcal{B}} Y).$$

Conversely, suppose $h(X) \subseteq G \in \mathrm{Fi}^S \mathcal{B}$. Then $X \subseteq h^{-1}(G) \in \mathrm{Fi}^S \mathcal{A}$, by Corollary 1.8.3, so $\mathrm{Fg}^S_{\mathcal{A}} X$

$\subseteq h^{-1}(G)$, i.e., $h(\mathrm{Fg}^S_{\mathcal{A}}X) \subseteq G$. By definition of $\mathrm{Fg}^S_{\mathfrak{B}}h(X)$, it follows that $h(\mathrm{Fg}^S_{\mathcal{A}}X) \subseteq \mathrm{Fg}^S_{\mathfrak{B}}h(X)$, hence $\mathrm{Fg}^S_{\mathcal{A}}X \subseteq h^{-1}(\mathrm{Fg}^S_{\mathfrak{B}}Y)$, implying $\mathrm{Fg}^S_{\mathcal{A}}X = h^{-1}(\mathrm{Fg}^S_{\mathfrak{B}}Y)$.

(vi)⇒(vii) Obvious.

(vii)⇒(viii) Let $\mathcal{A} = \langle \mathbf{A}, F_{\mathcal{A}} \rangle$ be an S-matrix and let $(\boldsymbol{a}, \boldsymbol{b}) \in (\Omega_{\mathcal{A}} F_{\mathcal{A}})^{[k]}$. In Section 1.8 we defined the matrix $\mathcal{A}^* = \mathcal{A}/\Omega_{\mathcal{A}}F_{\mathcal{A}} = \langle \mathbf{A}/\Omega_{\mathcal{A}}F_{\mathcal{A}}, F_{\mathcal{A}}/\Omega_{\mathcal{A}}F_{\mathcal{A}} \rangle$ and showed that the canonical homomorphism $h: \mathcal{A} \to \mathcal{A}^*$ is reductive. We claim that for any $G \in \mathrm{Fi}^S\mathcal{A}$, $h^{-1}(h_S(G)) \subseteq G$: Let $\boldsymbol{x} \in h^{-1}(h_S(G))$. Then $h\boldsymbol{x} \in h_S(G) = \mathrm{Fg}^S_{\mathcal{A}^*}h(G)$. By Lemma 1.6.6, there exists $X \subseteq G$ such that $h(X)$ is finite and $h\boldsymbol{x} \in \mathrm{Fg}^S_{\mathcal{A}^*}h(X)$, hence

$$\boldsymbol{x} \in h^{-1}(\mathrm{Fg}^S_{\mathcal{A}^*}h(X)) = \mathrm{Fg}^S_{\mathcal{A}}X \qquad \text{[by assumption (vii)]}$$
$$\subseteq G \qquad [X \subseteq G,\ G \in \mathrm{Fi}^S\mathcal{A}],$$

which proves the claim. Since $(\boldsymbol{a}, \boldsymbol{b}) \in (\Omega_{\mathcal{A}}F_{\mathcal{A}})^{[k]}$, we have $\boldsymbol{a}/(\Omega_{\mathcal{A}}F_{\mathcal{A}})^{[k]} = \boldsymbol{b}/(\Omega_{\mathcal{A}}F_{\mathcal{A}})^{[k]}$, i.e., $h\boldsymbol{a} = h\boldsymbol{b}$, hence

$$\boldsymbol{a} \in h^{-1}(h(\boldsymbol{b})) \subseteq h^{-1}(h(\mathrm{Fg}^S_{\mathcal{A}}\boldsymbol{b}))$$
$$\subseteq h^{-1}(h_S(\mathrm{Fg}^S_{\mathcal{A}}\boldsymbol{b}))$$
$$\subseteq \mathrm{Fg}^S_{\mathcal{A}}\boldsymbol{b} \qquad \text{[by the above claim]},$$

and finally $\boldsymbol{a} \in \mathrm{Fg}^S_{\mathcal{A}}\boldsymbol{b}$. By symmetry we get that $\boldsymbol{b} \in \mathrm{Fg}^S_{\mathcal{A}}\boldsymbol{a}$.

(viii)⇒(i): Let T be an S-theory. Then $\mathcal{A} = \langle \mathbf{Fm}, T \rangle$ is an S-matrix by Proposition 1.6.5 (i). Suppose that $\boldsymbol{\varphi}, \boldsymbol{\psi} \in Fm^k$ such that $(\boldsymbol{\varphi}, \boldsymbol{\psi}) \in (\Omega_{\mathcal{A}}T)^{[k]}$. Then, by assumption (viii), we have

$$\boldsymbol{\psi} \in \mathrm{Fg}^S_{\mathcal{A}}\boldsymbol{\varphi} \quad \text{and} \quad \boldsymbol{\varphi} \in \mathrm{Fg}^S_{\mathcal{A}}\boldsymbol{\psi},$$

i.e., $\boldsymbol{\psi} \in \mathrm{Cn}_S(T \cup \{\boldsymbol{\varphi}\})$ and $\boldsymbol{\varphi} \in \mathrm{Cn}_S(T \cup \{\boldsymbol{\psi}\})$, [by Proposition 1.6.5 (iii)]

hence $T, \boldsymbol{\varphi} \vdash_S \boldsymbol{\psi}$ and $T, \boldsymbol{\psi} \vdash_S \boldsymbol{\varphi}$. □

2.1.4 COROLLARY

Let S be a protoalgebraic k-deductive system. Let $\mathcal{A} = \langle \mathbf{A}, F_{\mathcal{A}} \rangle$, $\mathfrak{B} = \langle \mathbf{B}, F_{\mathfrak{B}} \rangle$ be S-matrices and let $h: \mathcal{A} \to \mathfrak{B}$ be a reductive matrix homomorphism. Then, for $\boldsymbol{a} \in A^k$, $\boldsymbol{b} \in B^k$ such that $h\boldsymbol{a} = \boldsymbol{b}$,

$$h^{-1}(\mathrm{Fg}^S_{\mathfrak{B}}\boldsymbol{b}) = \mathrm{Fg}^S_{\mathcal{A}}\boldsymbol{a} \quad \text{and} \quad \mathrm{Fg}^S_{\mathfrak{B}}\boldsymbol{b} = h(\mathrm{Fg}^S_{\mathcal{A}}\boldsymbol{a}).$$

Proof. This is a special case of property (vii). □

2.1.5 COROLLARY [BP89a, Lemma 4.4 (ii)]

Let S be a protoalgebraic k-deductive system. Let $\mathcal{A} = \langle \mathbf{A}, F \rangle$ be an S-matrix. For every system F_i, $i \in I$, of S-filters of $\mathcal{A}$, $\Omega_{\mathcal{A}}\left(\bigcap_{i \in I} F_i\right) = \bigcap_{i \in I} \Omega_{\mathcal{A}} F_i$, hence $\Omega_{\mathcal{A}}(\mathrm{Fi}^S \mathcal{A})$ is closed under arbitrary intersections. In particular, $\Omega(\mathrm{Th}S)$ is closed under arbitrary intersections.

Proof. Since S is protoalgebraic, Theorem 2.1.3 states that $\Omega_{\mathcal{A}} : \mathrm{Fi}^S \mathcal{A} \to \mathrm{Con}\,\mathbf{A}$ is monotonic, hence $\Omega_{\mathcal{A}}\left(\bigcap_{i \in I} F_i\right) \subseteq \bigcap_{i \in I} \Omega_{\mathcal{A}} F_i$. For the reverse inclusion, first note that $\bigcap_{i \in I} \Omega_{\mathcal{A}} F_i$ is a congruence on $\mathbf{A}$. It is easy to see that $\bigcap_{i \in I} \Omega_{\mathcal{A}} F_i$ is compatible with $\bigcap_{i \in I} F_i$, hence $\bigcap_{i \in I} \Omega_{\mathcal{A}} F_i \subseteq \Omega_{\mathcal{A}}\left(\bigcap_{i \in I} F_i\right)$, and the result follows. (The last assertion follows from Lemma 1.6.5 (i)). □

A very useful consequence of a k-deductive system being protoalgebraic is the following: Let $\mathcal{A} = \langle \mathbf{A}, F_{\mathcal{A}} \rangle$ and $\mathfrak{B} = \langle \mathbf{B}, F_{\mathfrak{B}} \rangle$ be S-matrices and $h : \mathcal{A} \to \mathfrak{B}$ a reductive matrix homomorphism. Consider the induced map $h_S : \mathrm{Fi}^S \mathcal{A} \to \mathrm{Fi}^S \mathfrak{B}$ defined by $h_S(F) = \mathrm{Fg}^S_{\mathfrak{B}} h(F)$ for each S-filter F of $\mathcal{A}$. We claim that h_S is a lattice isomorphism from $\mathbf{Fi}^S \mathcal{A}$ onto $\mathbf{Fi}_S \mathfrak{B}$. To see that h_S is injective, let $F, G \in \mathrm{Fi}^S \mathcal{A}$ such that $h_S(F) = h_S(G)$. By the filter correspondence property, we have

$$F = h^{-1}(h_S(F)) = h^{-1}(h_S(G)) = G.$$

To see that h_S is surjective, let $H \in \mathrm{Fi}^S \mathfrak{B}$. Since h is surjective, Corollary 1.8.3 states that $h^{-1}(H)$ is an S-filter of $\mathcal{A}$. It follows that $h_S(h^{-1}(H)) = \mathrm{Fg}^S_{\mathfrak{B}} h(h^{-1}(H)) = \mathrm{Fg}^S_{\mathfrak{B}} H = H$. To see that h_S is order-reflecting, suppose that $F, G \in \mathrm{Fi}^S \mathcal{A}$ such that $h_S(F) \subseteq h_S(G)$. Then $h^{-1}(h_S(F)) \subseteq h^{-1}(h_S(G))$. By the filter correspondence property,

$$F = h^{-1}(h_S(F)) \subseteq h^{-1}(h_S(G)) = G,$$

i.e., $F \subseteq G$. Since h_S is obviously order-preserving, it follows by Lemma 0.1.1 that h_S is a lattice isomorphism.

This property is especially significant if we take $\mathfrak{B}$ to be the reduced S-matrix $\mathcal{A}^* = \langle \mathbf{A}/\Omega_{\mathcal{A}} F_{\mathcal{A}}, F_{\mathcal{A}}/\Omega_{\mathcal{A}} F_{\mathcal{A}} \rangle$ and h the canonical homomorphism, i.e., $h\boldsymbol{a} = \boldsymbol{a}/\Omega_{\mathcal{A}} F_{\mathcal{A}}$ for each $\boldsymbol{a} \in A^k$. Note that h is reductive. By the above remarks, therefore, h_S is an isomorphism between $\mathbf{Fi}^S \mathcal{A}$ and $\mathbf{Fi}^S \mathcal{A}^*$. This allows us to restrict our attention to reduced S-matrices when S is a protoalgebraic k-deductive system, i.e., to the class $\mathrm{Mod}^* S$. Recall that the S_{Con}-matrices are precisely the matrices $\langle \mathbf{A}, \Phi \rangle$, where $\Phi \in \mathrm{Con}\,\mathbf{A}$, and that $\Omega_{\mathcal{A}} \Phi = \Phi$ for all $\Phi \in \mathrm{Con}\,\mathbf{A}$. Thus, if

$\mathcal{A} = \langle \mathbf{A}, \Phi \rangle$ is an S_{Con}-matrix, then $\mathcal{A}^* = \langle \mathbf{A}/\Phi, I_{A/\Phi} \rangle$. Let $h : \mathcal{A} \to \mathcal{A}^*$ be the canonical homomorphism. Then $h_{S_{\text{Con}}} : \mathbf{Fi}^S \mathcal{A} \to \mathbf{Fi}^S \mathcal{A}^*$ is an isomorphism. But this is precisely the map $h_{S_{\text{Con}}} : [\Phi, A^2] \to \text{Con}\, \mathbf{A}/\Phi$ defined by

$$h_{S_{\text{Con}}} \Psi = \text{Fg}^S_{\mathcal{A}^*} \Psi/\Phi = \{(a/\Phi, b/\Phi) \in (A/\Phi)^2 ; (a, b) \in \Psi\} = \Psi/\Phi.$$

This is the Correspondence Theorem of universal algebra (Theorem 0.2.6).

Recall from Section 1.8 that the class of reduced S_{Con}-matrices consists of all matrices of the form $\langle \mathbf{A}, I_A \rangle$, where $\mathbf{A}$ is any $\mathcal{L}$-algebra, and the class of reduced $S_{\mathcal{K}}$-matrices, where $\mathcal{K}$ is a quasivariety, consists of all matrices of the form $\langle \mathbf{A}, I_A \rangle$, where $\mathbf{A} \in \mathcal{K}$. In both these cases the matrix models are essentially reduced to algebras, reinforcing the connection between protoalgebraic k-deductive systems and algebra.

Although most classical deductive systems are protoalgebraic, it is not a completely general property of deductive systems. The following theorem gives an example of a deductive system that is not protoalgebraic. Recall from Section 1.4 that $\mathbf{IPC}^*$ is the $\{\wedge, \vee, \perp, \mathrm{T}\}$-reduct of $\mathbf{IPC}$.

2.1.6 THEOREM

$\mathbf{IPC}^*$ *is not protoalgebraic.*

Proof. This theorem will be proved by contradicting condition (iii) of Theorem 2.1.3. Let $\mathbf{H} = \langle \{\mathrm{T}, a, b, \perp\}; \wedge, \vee, \to, \perp, \mathrm{T} \rangle$ be the 4-element chain Heyting algebra. In other words, $\langle \{\mathrm{T}, a, b, \perp\}; \wedge, \vee \rangle$ is a chain and, say, $\perp < b < a < \mathrm{T}$, as in Figure 2.1. The unary operation $\neg$ is defined by $\neg \mathrm{T} = \neg a = \neg b = \perp$ and $\neg \perp = \mathrm{T}$. The binary operation $\to$ is defined by the condition:

$$x \to y = \max\{z \in \{\mathrm{T}, a, b, \perp\} ; z \wedge x \leq y\},$$

where $\leq$ is the lattice order. (See the discussion of Heyting algebras in Section 0.2.) Let the $\{\wedge, \vee, \perp, \mathrm{T}\}$-reduct of $\mathbf{H}$ be denoted by $\mathbf{A}$.

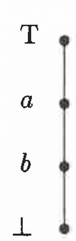


Fig. 2.1: **H**

Set $F_1 = \{\mathrm{T}\}$ and $F_2 = \{a, \mathrm{T}\}$. Since F_1 and F_2 are lattice filters of $\mathbf{H}$, it follows from Proposition 1.4.1 that $\langle \mathbf{H}, F_1 \rangle$ and $\langle \mathbf{H}, F_2 \rangle$ are matrix models of **IPC**. Since **IPC*** is a fragment of **IPC**, we have, for all $\Gamma \subseteq Fm_{\mathcal{L}'}$ and $\varphi \in Fm_{\mathcal{L}'}$ (where $\mathcal{L}'$ is the language of **IPC***), that

$$\Gamma \vdash_{\mathbf{IPC}^*} \varphi \quad \text{if and only if} \quad \Gamma \vdash_{\mathbf{IPC}} \varphi.$$

Thus, for each **IPC**-matrix $\mathfrak{B}$, if $\Gamma \vdash_{\mathbf{IPC}^*} \varphi$, then $\Gamma \vdash_{\mathbf{IPC}} \varphi$, hence $\Gamma \models_{\mathfrak{B}} \varphi$. In particular, if $\Gamma \vdash_{\mathbf{IPC}^*} \varphi$, then $\Gamma \models_{\langle \mathbf{H}, F_1 \rangle} \varphi$ and $\Gamma \models_{\langle \mathbf{H}, F_2 \rangle} \varphi$. Since the connective $\rightarrow$ does not occur in Γ or φ, this implies that $\Gamma \models_{\langle \mathbf{A}, F_1 \rangle} \varphi$ and $\Gamma \models_{\langle \mathbf{A}, F_2 \rangle} \varphi$, i.e., $\langle \mathbf{A}, F_1 \rangle$ and $\langle \mathbf{A}, F_2 \rangle$ are **IPC***-matrices.

If we set $\mathcal{A} = \langle \mathbf{A}, F_1 \rangle$, then F_1 and F_2 are **IPC***-filters of $\mathcal{A}$ and $F_1 \subseteq F_2$. We shall show, however, that $\Omega_{\mathcal{A}} F_1 \not\subseteq \Omega_{\mathcal{A}} F_2$. Note first that if $x \in \{b, a, \mathrm{T}\}$ and $(\perp, x) \in \Phi$ for some $\Phi \in \mathrm{Con}\,\mathbf{A}$, then $(\neg \perp, \neg x) \in \Phi$, i.e., $(\mathrm{T}, \perp) \in \Phi$. Consider $\Omega_{\mathcal{A}} F_1$: If $(x, \mathrm{T}) \in \Omega_{\mathcal{A}} F_1$, then $x \in F_1$ since $\Omega_{\mathcal{A}} F_1$ is compatible with F_1, i.e., $x = \mathrm{T}$, therefore $\Omega_{\mathcal{A}} F_1$ does not identify T with any different element. Also, $\Omega_{\mathcal{A}} F_1$ does not identify $\perp$ with any different element either otherwise $(\mathrm{T}, \perp) \in \Omega_{\mathcal{A}} F_1$, which would imply that $\perp \in F_1$, by compatibility. Thus the largest congruence compatible with F_1 is, therefore, $\Theta^{\mathbf{A}}(a, b) = I_A \cup \{(a, b), (b, a)\} = \Omega_{\mathcal{A}} F_1$.

Consider $\Omega_{\mathcal{A}} F_2$: If $(x, a) \in \Omega_{\mathcal{A}} F_2$, then $x = a$ or $x = \mathrm{T}$, and if $(x, \mathrm{T}) \in \Omega_{\mathcal{A}} F_2$, then $x = a$ or $x = \mathrm{T}$ again. Since $\Omega_{\mathcal{A}} F_2$ cannot identify $\perp$ with anything else (otherwise $(\mathrm{T}, \perp) \in \Omega_{\mathcal{A}} F_2$, implying that $\perp \in F_2$, by compatibility), we have that $\Omega_{\mathcal{A}} F_2 = \Theta^{\mathbf{A}}(a, \mathrm{T}) = I_A \cup \{(a, \mathrm{T}), (\mathrm{T}, a)\}$. Obviously, $\Omega_{\mathcal{A}} F_1 \not\subseteq \Omega_{\mathcal{A}} F_2$. □

2.2 MODEL THEORY FOR PROTOALGEBRAIC *k*-DEDUCTIVE SYSTEMS

We have shown in Theorem 1.8.9 that the class Mod^*S of reduced matrices of a k-deductive system S forms a matrix semantics for S, in other words, $\vdash_S = \models_{\mathrm{Mod}^*S}$. If S is protoalgebraic we showed, moreover, that when considering S-filters, it is sufficient to work with the class Mod^*S of reduced matrices rather than $\mathrm{Mod}S$ inasmuch as $\mathbf{Fi}^S\mathcal{A} \cong \mathbf{Fi}^S\mathcal{A}^*$ for each S-matrix $\mathcal{A}$. A natural question to ask is whether the class Mod^*S has any closure properties that characterize it as the reduced model class of a protoalgebraic k-deductive system. Theorem 2.2.3 offers a positive answer, namely that Mod^*S is closed under subdirect products if and only if S is protoalgebraic. The necessary concepts are defined in Definitions 2.2.1 and 2.2.2. The theorem suggests that protoalgebraic k-deductive systems are the widest class of k-deductive systems having a reasonable model theory.

2.2.1 DEFINITION

A class H of reduced k-matrices of the form Mod^*S for some k-deductive system S is called a *reduced universal Horn k-class*. A reduced universal Horn k-class is called a *k-protoquasivariety* if it is the class of all reduced matrix models of a protoalgebraic k-deductive system.

The terminology is motivated by the fact that Mod$S$ may be regarded as a universal Horn class (see Section 1.10). In Chapter 3 we shall establish a connection between certain deductive systems with properties stronger than protoalgebraicity, and quasivarieties; this will explain our choice of the term 'protoquasivariety'.

The following matrix definitions are analogous to those for universal algebra. In each case, it is the filters that require an appropriate definition.

2.2.2 DEFINITION

Let $\mathcal{A}_i = \langle \mathbf{A}_i, F_i \rangle$ be a k-matrix for each $i \in I$, where I is a set. We may regard each $\mathcal{A}_i$ as an $\mathfrak{L}_D$-structure $\langle \mathbf{A}_i, \mathfrak{L}^{\mathbf{A}_i}, F_i \rangle$ (in the sense of Section 1.10) where $\mathfrak{L}_D$ is the first-order language (without equality) $\langle \mathfrak{L}, \{D\}, ar \rangle$ which extends the language $\mathfrak{L}$ by the addition of one relation symbol D with $ar(D) = k$: Since we have seen that the notions of submatrix, homomorphism and isomorphism of matrices coincide respectively with those of $\mathfrak{L}_D$-substructure, $\mathfrak{L}_D$-homomorphism and $\mathfrak{L}_D$-isomorphism, we shall use $\mathrm{S}(\mathfrak{K})$, $\mathrm{H}(\mathfrak{K})$ and $\mathrm{I}(\mathfrak{K})$ to denote, respectively, the closure of a class $\mathfrak{K}$ of k-matrices under submatrices, (matrix-) homomorphic and isomorphic images. For each $f \in \mathfrak{L}$, $f^{\mathbf{A}_i}$ is, as before, just the interpretation in the algebra $\mathbf{A}_i$ of f, while $D^{\mathbf{A}_i} = F_i$. From Section 0.5, we also obtain the following notions of direct and subdirect products of k-matrices: The *direct product* of $\mathcal{A}_i$, $i \in I$, is the k-matrix

$$\prod_{i \in I} \mathcal{A}_i = \left\langle \prod_{i \in I} \mathbf{A}_i, \prod_{i \in I} F_i \right\rangle,$$

(where, for simplicity, we identify $\Pi_{i \in I} F_i$ with $\{\bar{a} = (a_1, \ldots, a_k) \in (\Pi_{i \in I} A_i)^k ; (a_1(i), \ldots, a_k(i)) \in F_i$ for each $i \in I\}$). The index set I may be empty, in which case $\Pi_{i \in I} \mathbf{A}_i$ is the trivial one-element algebra. A submatrix $\mathcal{B}$ of $\Pi_{i \in I} \mathcal{A}_i$ is a *subdirect product* of the system $\{\mathcal{A}_i ; i \in I\}$ if the projection $\pi_i : B \to A_i$ is surjective for every $i \in I$. (This forces π_i to be a surjective matrix homomorphism as a consequence of the definition of submatrix.)

2.2.3 THEOREM [BP92, Theorem 9.3]

(i) *Every k-protoquasivariety is closed under subdirect products and in particular under direct products.*

(ii) *Conversely, every reduced universal Horn k-class that is closed under subdirect products is a k-protoquasivariety.*

Proof. (i) Let $\mathfrak{B} = \langle \mathbf{B}, F_{\mathfrak{B}} \rangle$. Assume Q is a k-protoquasivariety and $\mathfrak{B}$ is a subdirect product of $\mathcal{A}_i = \langle \mathbf{A}_i, F_{\mathcal{A}_i} \rangle, i \in I$, where $\mathcal{A}_i \in \mathrm{Q}$ for each $i \in I$. For each $i \in I$, let $\pi_i : B \to A_i$ be the projection map, and set $F_i = \pi_i^{-1}(F_{\mathcal{A}_i})$. By Corollary 1.8.3, $F_i \in \mathrm{Fi}^S\mathfrak{B}$. We have that

$$\bigcap_{i \in I} F_i = \bigcap_{i \in I} \pi_i^{-1}(F_{\mathcal{A}_i}) = \Big(\prod_{i \in I} F_{\mathcal{A}_i} \Big) \cap B = F_{\mathfrak{B}}. \tag{2.2.1}$$

Since $\mathcal{A}_i$ is reduced

$$\Omega_{\mathfrak{B}} F_i = \Omega_{\mathfrak{B}}(\pi_i^{-1}(F_{\mathcal{A}_i})) \stackrel{*}{=} \pi_i^{-1}(\Omega_{\mathcal{A}_i} F_{\mathcal{A}_i}) = \pi_i^{-1}(I_{A_i}).$$

The equality denoted $\stackrel{*}{=}$ follows from Lemma 1.8.5, since π_i is surjective. Now, writing $\boldsymbol{a}_i$ for the i-th co-ordinate of $\bar{\boldsymbol{a}}$ (i.e., $\boldsymbol{a}_i = \pi_i \bar{\boldsymbol{a}}$) and the same for $\bar{\boldsymbol{b}}$,

$$\begin{aligned} (\bar{\boldsymbol{a}}, \bar{\boldsymbol{b}}) \in \textstyle\bigcap_{i \in I} \pi_i^{-1}(I_{A_i}) \quad & \text{iff} \quad \pi_i \bar{\boldsymbol{a}} = \pi_i \bar{\boldsymbol{b}} \ \text{ for all } i \in I \\ & \text{iff} \quad \boldsymbol{a}_i = \boldsymbol{b}_i \ \text{ for all } i \in I \\ & \text{iff} \quad \bar{\boldsymbol{a}} = \bar{\boldsymbol{b}}, \end{aligned}$$

hence $\bigcap_{i \in I} \Omega_{\mathfrak{B}} F_i = \bigcap_{i \in I} \pi_i^{-1}(I_{A_i}) = I_B$. By (2.2.1) and the fact that $\Omega_{\mathfrak{B}}$ is monotonic (Theorem 2.1.3),

$$\Omega_{\mathfrak{B}} F_{\mathfrak{B}} = \Omega_{\mathfrak{B}}\Big(\bigcap_{i \in I} F_i \Big) \subseteq \Omega_{\mathfrak{B}} F_i \ \text{ for each } i \in I,$$

hence $\Omega_{\mathfrak{B}} F_{\mathfrak{B}} \subseteq \bigcap_{i \in I} \Omega_{\mathfrak{B}} F_i = I_B$, so $\Omega_{\mathfrak{B}} F_{\mathfrak{B}} = I_B$. Thus $\mathfrak{B}$ is a reduced matrix.

(ii) Assume H is a universal Horn k-class that is closed under subdirect products. Let S be the k-deductive system for which $\mathrm{H} = \mathrm{Mod}^*S$. Let $\mathcal{A} = \langle \mathbf{A}, F_{\mathcal{A}} \rangle$ be an S-matrix, and let $F, G \in \mathrm{Fi}^S\mathcal{A}$ such that $F \subseteq G$. We shall show that $\Omega_{\mathcal{A}} F \subseteq \Omega_{\mathcal{A}} G$ and then invoke Theorem 2.1.3. Let $f : \mathbf{A} \to \mathbf{A}/\Omega_{\mathcal{A}} F$ and $g : \mathbf{A} \to \mathbf{A}/\Omega_{\mathcal{A}} G$ be the canonical homomorphisms and let h be the product of f and g, i.e., $h : \mathbf{A} \to \mathbf{A}/\Omega_{\mathcal{A}} F \times \mathbf{A}/\Omega_{\mathcal{A}} G$ is defined by $h(a) = \langle f(a), g(a) \rangle$. Set $\Phi = \Omega_{\mathcal{A}} F \cap \Omega_{\mathcal{A}} G$. For all $a, b \in A$, we have

$$ha = hb \quad \text{iff} \quad \langle a/\Omega_{\mathcal{A}}F, a/\Omega_{\mathcal{A}}G\rangle = \langle b/\Omega_{\mathcal{A}}F, b/\Omega_{\mathcal{A}}G\rangle$$
$$\text{iff} \quad (a,b) \in \Omega_{\mathcal{A}}F \text{ and } (a,b) \in \Omega_{\mathcal{A}}G$$
$$\text{iff} \quad (a,b) \in \Phi,$$

so Φ is the kernel of h. By the Homomorphism Theorem of universal algebra (Theorem 0.2.4), $\mathbf{A}/\Phi$ is isomorphic to a subalgebra of $\mathbf{A}/\Omega_{\mathcal{A}}F \times \mathbf{A}/\Omega_{\mathcal{A}}G$. Furthermore, we have that

$$h^{-1}(F/\Omega_{\mathcal{A}}F \times G/\Omega_{\mathcal{A}}G) = f^{-1}(F/\Omega_{\mathcal{A}}F) \cap g^{-1}(G/\Omega_{\mathcal{A}}G) = F \cap G = F,$$

hence $\langle \mathbf{A}/\Phi, F/\Phi\rangle$ is isomorphic to a subdirect product of $\langle \mathbf{A}/\Omega_{\mathcal{A}}F, F/\Omega_{\mathcal{A}}F\rangle$ and $\langle \mathbf{A}/\Omega_{\mathcal{A}}G, G/\Omega_{\mathcal{A}}G\rangle$. The latter two matrices are reduced models of S, i.e., they are members of H, therefore $\langle \mathbf{A}/\Phi, F/\Phi\rangle$ must also be a member of H, as H is closed under subdirect products. Set $\mathcal{A}' = \langle \mathbf{A}/\Phi, F/\Phi\rangle$. Then $\Omega_{\mathcal{A}'}(F/\Phi) = I_{A/\Phi}$, i.e., $\Omega_{\mathcal{A}'}(F/(\Omega_{\mathcal{A}}F \cap \Omega_{\mathcal{A}}G)) = I_{A/\Phi}$. From the previous sentence and the Correspondence Theorem, it follows that $\Omega_{\mathcal{A}}F \cap \Omega_{\mathcal{A}}G$ is the largest congruence on $\mathbf{A}$ that contains Φ and is compatible with F. Since it is easy to see that Φ is compatible with F, the largest congruence on $\mathbf{A}$ compatible with F must contain Φ and therefore coincides with the largest congruence on $\mathbf{A}$ containing Φ that is compatible with F, i.e., $\Omega_{\mathcal{A}}F = \Omega_{\mathcal{A}}F \cap \Omega_{\mathcal{A}}G$, so $\Omega_{\mathcal{A}}F \subseteq \Omega_{\mathcal{A}}G$. □

2.3 AN INTERNAL CHARACTERIZATION OF PROTOALGEBRAIC *k*-DEDUCTIVE SYSTEMS

Theorem 2.1.3 provided a number of characterizations of protoalgebraicity in terms of the Leibniz operator and matrix models, while Theorem 2.2.3 provided a characterization in terms of the class of reduced matrices. In each of these cases it was necessary to draw on concepts dependent on but external to the deductive system in question. In this section we show that it is possible to characterize protoalgebraicity from within the deductive system itself, namely, by employing only the language $\mathcal{L}$ and the consequence relation $\vdash_S$ of the deductive system. There is a parallel here with the concept of Mal'cev conditions for classes of algebras, again reinforcing the algebraic character of these deductive systems.

For a k-deductive system S, let $\tilde{z} = \langle z_1, \ldots, z_{k-1}\rangle$ be a fixed $(k-1)$-tuple, where $z_1, \ldots, z_{k-1}$ are propositional variables. For $\varphi \in Fm$ and $i < k$, define $\tilde{z}[\varphi/i] = \langle z_1, \ldots, z_{i-1}, \varphi, z_i, \ldots, z_{k-1}\rangle$. Also, define $\tilde{z}[\varphi/k] = \langle z_1, \ldots, z_{k-1}, \varphi\rangle$. Note that if $k = 1$, then $\tilde{z}$

is the empty string and $\tilde{z}[\varphi/1]$ is φ. By $\{p,q,\tilde{z}\}$ we mean the set $\{p,q,z_1,\ldots,z_{k-1}\}$.

2.3.1 DEFINITION

Let S be a k-deductive system and I an index set. Let $p,q,z_1,\ldots,z_{k-1}$ be distinct variables. A system $\boldsymbol{\Delta}_i(p,q,\tilde{z})$, $i \in I$, of k-formulas, is called a *system of equivalence k-formulas with parameters* $\tilde{z}$ *for* S if

(2.3.1) $\quad \vdash_S \boldsymbol{\Delta}_i(p,p,\tilde{z})$ for all $i \in I$.

(2.3.2) $\quad \tilde{z}[p/j], \{\boldsymbol{\Delta}_i(p,q,\tilde{z}); i \in I\} \vdash_S \tilde{z}[q/j]$ for all $j \leq k$.

Recall that by $\boldsymbol{\Delta}_i(p,q,\tilde{z})$ we mean the k-tuple $\langle \Delta_{i1}(p,q,\tilde{z}),\ldots,\Delta_{ik}(p,q,\tilde{z})\rangle$. A system $\boldsymbol{\Delta}_i(p,q)$, $i \in I$, of k-formulas, is called a *system of equivalence k-formulas without parameters for* S if

$(2.3.1)'$ $\quad \vdash_S \boldsymbol{\Delta}_i(p,p)$ for all $i \in I$.

$(2.3.2)'$ $\quad \tilde{z}[p/j], \{\boldsymbol{\Delta}_i(p,q); i \in I\} \vdash_S \tilde{z}[q/j]$ for all $j \leq k$.

The system $\boldsymbol{\Delta}_i(p,q,\tilde{z})$, $i \in I$, (or $\boldsymbol{\Delta}_i(p,q)$, $i \in I$) is called *finite* if I is finite.

Note that if $k=1$, i.e., $\tilde{z}$ is the empty string, then the system $\Delta_i(p,q)$, $i \in I$, of (1-) formulas is a system of equivalence formulas with parameters $\tilde{z}$ for S if and only it is a system of equivalence formulas without parameters for S. Note also that since any k-deductive system S is finitary, if there exists a system of equivalence k-formulas with parameters $\tilde{z}$ (or without parameters) for S then there exists a *finite* system of equivalence k-formulas with parameters $\tilde{z}$ (without parameters) for S. Finally, note that if (2.3.1) and (2.3.2) (or $(2.3.1)'$ and $(2.3.2)'$) hold in a deductive system S, then they also hold in every extension of S.

The following theorem is an adapted version of Theorem 13.2 of [BP92]. The original theorem does not include the parameters $\tilde{z}$, without which the theorem is false. This error was noted and corrected by Pałasińska, [Pała94]. We exhibit a counterexample, also due to Pałasińska, [Pała94], proving that the original theorem is indeed false. As noted in the previous paragraph, however, when considering 1-deductive systems, the parameters $\tilde{z}$ are redundant, hence Theorem 13.2 of [BP92] is true for $k=1$. We include this result as a corollary to the following theorem.

2.3.2 THEOREM [Pała94, Theorem 3.10]

A k-deductive system is protoalgebraic if and only if it has a finite system of equivalence k-

formulas with parameters $\tilde{z}$.

Proof. ($\Leftarrow$) Assume S has a finite system of equivalence k-formulas with parameters $\tilde{z}$, $\Delta_1(p,q,\tilde{z}),\ldots,\Delta_n(p,q,\tilde{z})$ say. To prove this implication we will show that S satisfies condition (iii) of Theorem 2.1.3. Let $\mathcal{A}=\langle \mathbf{A},F_{\mathcal{A}}\rangle$ be an S-matrix, and let $F,G\in \mathrm{Fi}^S\mathcal{A}$ such that $F\subseteq G$. We shall show that $\Omega_{\mathcal{A}}F$ is compatible with G. Suppose $\boldsymbol{a}=\langle a_1,\ldots,a_k\rangle$, $\boldsymbol{b}=\langle b_1,\ldots,b_k\rangle\in A^k$ such that $\boldsymbol{a}\in G$ and $(\boldsymbol{a},\boldsymbol{b})\in(\Omega_{\mathcal{A}}F)^{[k]}$. Then, since $\Omega_{\mathcal{A}}F\in \mathrm{Con}\,\mathbf{A}$,

$$(\Delta_i^{\mathbf{A}}(a_j,a_j,\bar{c}),\ \Delta_i^{\mathbf{A}}(a_j,b_j,\bar{c}))\in(\Omega_{\mathcal{A}}F)^{[k]}\quad \text{for all } i\le n;\ j\le k \text{ and } \bar{c}\in A^{k-1}.$$

By (2.3.1), since $\mathcal{A}$ is an S-matrix and F is an S-filter of $\mathcal{A}$, we can deduce that

$$\Delta_i^{\mathbf{A}}(a_j,a_j,\bar{c})\in F\quad \text{for } i\le n;\ j\le k \text{ and } \bar{c}\in A^{k-1},$$

hence

$$\Delta_i^{\mathbf{A}}(a_j,b_j,\bar{c})\in F\quad \text{for } i\le n;\ j\le k \text{ and } \bar{c}\in A^{k-1},$$

since $\Omega_{\mathcal{A}}F$ is compatible with F. Thus, since $F\subseteq G$,

$$\Delta_i^{\mathbf{A}}(a_j,b_j,\bar{c})\in G\quad \text{for } i\le n;\ j\le k \text{ and } \bar{c}\in A^{k-1}.$$

Thus, using the S-matrix $\langle \mathbf{A},G\rangle$ and (2.3.2), we have that for every $\ell\le k$, if

$$\langle a_1,\ldots,a_\ell,b_{\ell+1},\ldots,b_k\rangle\in G,$$

then

$$\langle a_1,\ldots,a_{\ell-1},b_\ell,b_{\ell+1},\ldots,b_k\rangle\in G.$$

Since $\boldsymbol{a}=\langle a_1,\ldots,a_k\rangle\in G$, it follows that $\boldsymbol{b}=\langle b_1,\ldots,b_k\rangle\in G$, therefore $\Omega_{\mathcal{A}}F$ is compatible with G, so $\Omega_{\mathcal{A}}F\subseteq\Omega_{\mathcal{A}}G$, and S is protoalgebraic.

($\Rightarrow$) Assume that S is protoalgebraic, i.e., each condition of Theorem 2.1.3 holds. Let p,q be variables such that $p,q,z_1,\ldots,z_{k-1}$ are distinct. Set

$$T=\{\boldsymbol{\vartheta}(p,q,\tilde{z})\in(Fm(p,q,\tilde{z}))^k;\ \vdash_S\boldsymbol{\vartheta}(p,p,\tilde{z})\}.$$

We claim that $\mathcal{A}=\langle \mathbf{Fm}(p,q,\tilde{z}),T\rangle$ is an S-matrix. By Proposition 1.6.5 (i), we need to show that $T=\mathrm{Cn}_S(T)\cap(Fm(p,q,\tilde{z}))^k$. That $T\subseteq\mathrm{Cn}_S(T)\cap(Fm(p,q,\tilde{z}))^k$ is obvious. Suppose $\boldsymbol{\varphi}\in\mathrm{Cn}_S(T)\cap(Fm(p,q,\tilde{z}))^k$. Then $\boldsymbol{\varphi}=\boldsymbol{\varphi}(p,q,\tilde{z})\in(Fm(p,q,\tilde{z}))^k$ and $T\vdash_S\boldsymbol{\varphi}$. Let σ be the substitution defined by $\sigma q=p$ and $\sigma x=x$ for all $x\in P-\{q\}$. By structurality we have that $\sigma(T)\vdash_S\sigma\boldsymbol{\varphi}$, i.e.,

$$\{\boldsymbol{\vartheta}(p,p,\tilde{z});\ \boldsymbol{\vartheta}(p,q,\tilde{z})\in T\}\vdash_S\boldsymbol{\varphi}(p,p,\tilde{z}).$$

Since each of the premisses is an S-theorem by definition, we get that $\vdash_S\boldsymbol{\varphi}(p,p,\tilde{z})$, i.e., $\boldsymbol{\varphi}(p,q,\tilde{z})\in T$. This proves that $\mathrm{Cn}_S(T)\cap(Fm(p,q,\tilde{z}))^k\subseteq T$, and hence that $T=\mathrm{Cn}_S(T)\cap(Fm(p,q,\tilde{z}))^k$.

We now claim that $(p,q)\in\Omega_{\mathcal{A}}T$. We shall prove this using Definition 1.7.1. Let $\varphi(r,\overline{s})\in Fm^k$, where $\overline{s}=\langle s_1,\ldots,s_m\rangle$ is a list of variables, and suppose that for some $\psi_i(p,q,\widetilde{z})\in Fm(p,q,\widetilde{z})$, $i\leq m$, we have $\varphi(p,\psi_1(p,q,\widetilde{z}),\ldots,\psi_m(p,q,\widetilde{z}))\in T$. We need to show that $\varphi(q,\psi_1(p,q,\widetilde{z}),\ldots,\psi_m(p,q,\widetilde{z}))\in T$. By assumption,

$$\vdash_S \varphi(p,\psi_1(p,p,\widetilde{z}),\ldots,\psi_m(p,p,\widetilde{z})). \tag{2.3.3}$$

Set

$$\vartheta(p,q,\widetilde{z})=\varphi(q,\psi_1(p,q,\widetilde{z}),\ldots,\psi_m(p,q,\widetilde{z})).$$

It follows from (2.3.3) that $\vdash_S \vartheta(p,p,\widetilde{z})$, i.e., $\vartheta(p,q,\widetilde{z})\in T$, i.e., $\varphi(q,\psi_1(p,q,\widetilde{z}),\ldots,\psi_m(p,q,\widetilde{z}))$ $\in T$, as required.

Let $i\leq k$. Since S is protoalgebraic, $\Omega_{\mathcal{A}}T\subseteq\Omega_{\mathcal{A}}\mathrm{Fg}^S_{\mathcal{A}}(T\cup\{\widetilde{z}\,[p/i]\})$, hence $(p,q)\in$ $\Omega_{\mathcal{A}}\mathrm{Fg}^S_{\mathcal{A}}(T\cup\{\widetilde{z}\,[p/i]\})$. Thus, since $\widetilde{z}\,[p/i]\in T\cup\{\widetilde{z}\,[p/i]\}$, $\widetilde{z}\,[q/i]\in\mathrm{Fg}^S_{\mathcal{A}}(T\cup\{\widetilde{z}\,[p/i]\})$, i.e.,

$$T,\widetilde{z}\,[p/i]\vdash_S\widetilde{z}\,[q/i].$$

Since S is finitary, there exists a finite subset $\{\Delta_1(p,q,\widetilde{z}),\ldots,\Delta_n(p,q,\widetilde{z})\}$ of T such that

$$\Delta_1(p,q,\widetilde{z}),\ldots,\Delta_n(p,q,\widetilde{z}),\widetilde{z}\,[p/i]\vdash_S\widetilde{z}\,[q/i].$$

Thus $\Delta_1(p,q,\widetilde{z}),\ldots,\Delta_n(p,q,\widetilde{z})$ satisfy (2.3.2). By the definition of T, we have that $\vdash_S\Delta_i(p,p,\widetilde{z})$, hence $\Delta_1(p,q,\widetilde{z}),\ldots,\Delta_n(p,q,\widetilde{z})$ satisfy (2.3.1) as well, so $\Delta_1(p,q,\widetilde{z}),\ldots,$ $\Delta_n(p,q,\widetilde{z})$ form a system of equivalence k-formulas with parameters $\widetilde{z}$ for S. □

2.3.3 COROLLARY

A 1-deductive system S is protoalgebraic if and only if there exists a finite system of equivalence formulas without parameters for S. □

The following example shows that in certain protoalgebraic k-deductive systems ($k>1$) it is essential for a system of congruence k-formulas to have parameters $\widetilde{z}$. Let $\mathcal{L}=\{\cdot\}$ where $\cdot$ is a binary connective. Let S be the 2-deductive system defined by the following axiom and inference rules

$$\vdash_S(p,p), \tag{2.3.4}$$

$$(p,r),(p\cdot r,q\cdot r)\vdash_S(q,r), \tag{2.3.5}$$

$$(r,p),(p\cdot r,q\cdot r)\vdash_S(r,q). \tag{2.3.6}$$

We first claim that $(p\cdot r,q\cdot r)$ forms a system of equivalence 2-formulas (with parameter r) for S, i.e., that

$$\vdash_S (p \cdot r, p \cdot r),$$

$$(p, r), (p \cdot r, q \cdot r) \vdash_S (q, r),$$

and

$$(r, p), (p \cdot r, q \cdot r) \vdash_S (r, q).$$

Each of these conditions holds by (2.3.4), (2.3.5) and (2.3.6), respectively, hence S is a protoalgebraic 2-deductive system. Next we claim that S does not have a system of equivalence 2-formulas without parameters.

So, let $\Delta(p, q) = \{\Delta_i(p, q) \,;\, i \in I\}$ be a set of 2-formulas in the variables p and q. Set

$$\widehat{\Delta}(p, q) = (\mathrm{Cn}_S \Delta(p, q)) \cap (Fm(p, q))^2.$$

By Proposition 1.6.5, $\langle \mathbf{Fm}(p, q), \widehat{\Delta}(p, q) \rangle$ is an S-matrix. We claim that

(2.3.7) $$\mathrm{Cn}_S(\Delta(p, q) \cup \{(p, r)\}) = \widehat{\Delta}(p, q) \cup \{(p, r)\} \cup \{(\varphi, \varphi) \,;\, \varphi \in Fm\}$$

In view of (2.3.4), it is clear that the inclusion from right to left holds. Let X denote the right hand side. Evidently, $\Delta(p, q) \cup \{(p, r)\} \subseteq X$. Thus we need only show that X is an S-theory, i.e., that $\langle \mathbf{Fm}, X \rangle$ is an S-matrix. Clearly, $\langle \mathbf{Fm}, X \rangle$ is closed under (2.3.4). For the inference rule (2.3.5), suppose (φ, ψ), $(\varphi \cdot \psi, \vartheta \cdot \psi) \in X$ for some $\varphi, \psi, \vartheta \in Fm$. We need to show that $(\vartheta, \psi) \in X$ as well. If $\varphi = \vartheta$, then $(\vartheta, \psi) = (\varphi, \psi) \in X$, by assumption. So assume $\varphi \neq \vartheta$. Note that $(\varphi \cdot \psi, \vartheta \cdot \psi) \neq (p, r)$ since p, r are variables and $\varphi \cdot \psi, \vartheta \cdot \psi$ are composite formulas. Thus, by (2.3.7), we may assume that $(\varphi \cdot \psi, \vartheta \cdot \psi) \in \widehat{\Delta}(p, q)$. In particular, the only variables occurring in φ, ψ, ϑ are p and q. Thus, if (φ, ψ) is contained in the third component of X, then it is also contained in $\widehat{\Delta}(p, q)$. Since r does not occur in φ, ψ, $(\varphi, \psi) \neq (p, r)$. By (2.3.7), therefore, $(\varphi, \psi) \in \widehat{\Delta}(p, q)$. It follows that $(\vartheta, \psi) \in \widehat{\Delta}(p, q)$ and hence $(\vartheta, \psi) \in X$ as well, so $\langle \mathbf{Fm}, X \rangle$ is closed under (2.3.5).

The proof for (2.3.6) is similar: Suppose (ψ, φ), $(\varphi \cdot \psi, \vartheta \cdot \psi) \in X$ for some $\varphi, \psi, \vartheta \in Fm$. If $\varphi = \vartheta$, then $(\psi, \vartheta) = (\psi, \varphi) \in X$, by assumption. So assume $\varphi \neq \vartheta$. Then, as before, $(\varphi \cdot \psi, \vartheta \cdot \psi) \in \widehat{\Delta}(p, q)$ and the only variables occurring in φ, ψ, ϑ are p and q. In particular, $\varphi \neq r$ and therefore $(\psi, \varphi) \neq (p, r)$. So $(\psi, \varphi) \in \widehat{\Delta}(p, q)$ or $\varphi = \psi$. In either case, $(\psi, \varphi) \in \widehat{\Delta}(p, q)$. It follows that $(\psi, \vartheta) \in \widehat{\Delta}(p, q) \subseteq X$, thus proving that X is an S-theory and that (2.3.7) holds.

Now, $(q, r) \notin X$, hence $(q, r) \notin \mathrm{Cn}_S(\Delta(p, q) \cup \{(p, r)\})$. This shows that $\Delta(p, q)$ is not a system of equivalence 2-formulas for S. Since $\Delta(p,q)$ was an arbitrary set of 2-formulas in p and q, it follows that S does not have a system of equivalence 2-formulas without parameters.

Theorem 2.3.2 and Corollary 2.3.3 allow one to determine whether a k-deductive system is protoalgebraic by finding a system of equivalence k-formulas with parameters $\tilde{z}$ (a system of equivalence formulas, in the 1-deductive case). For example, suppose S is a 1-deductive system that has a binary connective (or abbreviation), $\rightarrow$ say, in its language. Set $\Delta(p,q) = p \rightarrow q$. Then (2.3.1) and (2.3.2) become, respectively,

$$\vdash_S p \rightarrow p \quad \text{and} \quad p, p \rightarrow q \vdash_S q.$$

If both of these conditions hold, then $\Delta(p,q)$ will constitute a system of equivalence (1-) formulas for S, and S will be protoalgebraic. Since these conditions are certainly satisfied by **CPC**, **IPC**, **BCK** and **BCI** to name but a few, we immediately have that these deductive systems (and their $\{\rightarrow\}$-fragments) are protoalgebraic. The choice of equivalence k-formulas is not necessarily unique. Another example of a system of equivalence formulas for **CPC** is $\Delta(p,q) = p \leftrightarrow q$, which immediately implies that $\mathbf{CPC}_{\leftrightarrow}$ is protoalgebraic.

2.4 REPRESENTATIONS OF EQUALITY

The Leibniz equivalence relation of Section 1.7 is a relation inherent to any deductive system. In this section we shall show that the Leibniz equivalence relation (or operator) associated with a protoalgebraic k-deductive system is representable in terms of so-called 'congruence k-formulas with parameters $\tilde{z}$' (Corollary 2.4.6), hence the title of this section. In doing so, we present another characterization of protoalgebraic k-deductive systems in terms of congruence k-formulas with parameters $\tilde{z}$. The results of this section originated in [BP92], but the error in Theorem 13.2 of that paper has affected the truth of their original statements. These repercussions have been explored by Pałasińska and the corrected results have been taken from [Pała94].

We first introduce the following notion, which extends Definition 2.3.1: Let S be a k-deductive system and I an index set. For the rest of this section, let $\tilde{z} = \langle z_1, \ldots, z_{k-1} \rangle$ be as defined in the beginning of Section 2.3 and let p, q be variables such that $p, q, z_1, \ldots, z_{k-1}$ are distinct. A system $\Delta_i(p, q, w_1, \ldots, w_{m_i}, \tilde{z})$, $i \in I$, of k-formulas in the variables $p, q, z_1, \ldots,$ z_{k-1}

and possible additional variables $w_1,\ldots,w_{m_i}$ is called a *generalized system of equivalence k-formulas with parameters* $\tilde{z}$ *for* S if

(2.4.1) $\vdash_S \Delta_i(p,p,w_1,\ldots,w_{m_i},\tilde{z})$ for all $i \in I$.

(2.4.2) $\tilde{z}[p/j],\ \{\Delta_i(p,q,w_1,\ldots,w_{m_i},\tilde{z})\,;\ i \in I\} \vdash_S \tilde{z}[q/j]$ for all $j \leq k$.

2.4.1 DEFINITION

Let S be a k-deductive system and I an index set. Suppose that for each $i \in I$, $\Delta_i(p,q,w_1,\ldots,w_{m_i},\tilde{z})$ is a k-formula in the variables $p,q,z_1,\ldots,\ z_{k-1}$ and possible additional variables $w_1,\ldots,w_{m_i}$. Let

$$\Delta = \{\Delta_i(p,q,w_1,\ldots,w_{m_i},\tilde{z})\,;\ i \in I\}.$$

We call Δ a *generalized system of congruence k-formulas for* S *with parameters* $\tilde{z}$ iff Δ is a generalized system of equivalence k-formulas with parameters $\tilde{z}$ for S and

(2.4.3) $$\{\Delta_i(p,q,w_1,\ldots,w_{m_i},\tilde{z})\,;\ i \in I\} \vdash_S \Delta_\ell(\varphi(p,\overline{v}),\varphi(q,\overline{v}),w_1,\ldots,w_{m_i},\tilde{z})$$

for each $\varphi(p,\overline{v}) \in Fm$ and all $\ell \in I$.

Now, suppose that for each $i \in I$, $\Delta_i(p,q,\tilde{z})$ is a k-formula in the variables $p,q,z_1,\ldots,z_{k-1}$. We call $\Delta = \{\Delta_i(p,q,\tilde{z})\,;\ i \in I\}$ a *system of congruence k-formulas with parameters* $\tilde{z}$ *for* S iff Δ is a system of equivalence k-formulas with parameters $\tilde{z}$ for S and

(2.4.3)′ $$\{\Delta_i(p,q,\tilde{z})\,;\ i \in I\} \vdash_S \Delta_\ell(\varphi(p,\overline{v}),\varphi(q,\overline{v}),\tilde{z})$$

for each $\varphi(p,\overline{v}) \in Fm$ and all $\ell \in I$.

Suppose that for each $i \in I$, $\Delta_i(p,q)$ is a k-formula in the variables p,q. We call $\Delta = \{\Delta_i(p,q)\,;\ i \in I\}$ a *system of congruence k-formulas without parameters for* S iff Δ is a system of equivalence k-formulas without parameters for S and

(2.4.3)″ $$\{\Delta_i(p,q)\,;\ i \in I\} \vdash_S \Delta_\ell(\varphi(p,\overline{v}),\varphi(q,\overline{v}))$$

for each $\varphi(p,\overline{v}) \in Fm$ and all $\ell \in I$.

The system $\Delta_i(p,q,w_1,\ldots,w_{m_i},\tilde{z})$, $i \in I$, (or $\Delta_i(p,q,\tilde{z})$, $i \in I$, or $\Delta_i(p,q)$, $i \in I$) is called *finite* if I is finite.

Note that if $k = 1$, i.e., $\tilde{z}$ is the empty string, then the system $\Delta_i(p,q)$, $i \in I$, is a system (resp. generalized system) of congruence formulas with parameters $\tilde{z}$ for S if and only if it is a system (resp. generalized system) of congruence formulas without parameters for S. Note also that if S satisfies (2.4.3) (or (2.4.3)$'$ or (2.4.3)$''$) and if S' is an extension of S that has the same language as S, then (2.4.3) (or (2.4.3)$'$ or (2.4.3)$''$) still holds when S is replaced by S'.

2.4.2 THEOREM (cf. [Pała94, Theorem 3.22])

Let S be a k-deductive system:

(i) Suppose the system $\Delta_i(p,q,w_1,\ldots,w_{m_i},\tilde{z})$, $i \in I$, is a generalized system of congruence k-formulas with parameters $\tilde{z}$ for S. Then for each S-matrix $\mathcal{A} = \langle \mathbf{A}, F\rangle$,

(2.4.4) $\quad (a,b) \in \Omega_{\mathcal{A}}F$ *iff* $\Delta_i^{\mathbf{A}}(a,b,\overline{f},\overline{d}) \in F$ *for each* $i \in I$, $\overline{f} \in A^{m_i}$ *and* $\overline{d} \in A^{k-1}$.

(ii) Suppose the system $\Delta_i(p,q,\tilde{z})$, $i \in I$, is a system of congruence k-formulas with parameters $\tilde{z}$ for S. Then for each S-matrix $\mathcal{A} = \langle \mathbf{A}, F\rangle$,

(2.4.4)$'$ $\quad (a,b) \in \Omega_{\mathcal{A}}F$ *iff* $\Delta_i^{\mathbf{A}}(a,b,\overline{d}) \in F$ *for each* $i \in I$ *and* $\overline{d} \in A^{k-1}$.

(iii) Let $\Delta_i(p,q)$, $i \in I$, be a system of k-formulas. The following are equivalent:

(1) The system $\Delta_i(p,q)$, $i \in I$, is a system of congruence k-formulas without parameters for S,

(2) for each S-matrix $\mathcal{A} = \langle \mathbf{A}, F\rangle$,

(2.4.4)$''$ $\quad (a,b) \in \Omega_{\mathcal{A}}F$ *iff* $\Delta_i^{\mathbf{A}}(a,b) \in F$ *for each* $i \in I$.

Proof. (i) Let $\mathcal{A} = \langle \mathbf{A}, F\rangle$ be an S-matrix. Let Φ be the set of all pairs $(a,b) \in A^2$ such that the condition on the right hand side of (2.4.4) holds. To prove the result, we shall show that $\Omega_{\mathcal{A}}F = \Phi$. First we show that Φ is a congruence relation on $\mathbf{A}$. It follows immediately from (2.4.1) that Φ is reflexive. Let $(a,b) \in \Phi$. To show that Φ is symmetric, we need to show that $\Delta_i^{\mathbf{A}}(b,a,\overline{f},\overline{d}) \in F$ for each $i \in I$, $\overline{f} \in A^{m_i}$ and $\overline{d} \in A^{k-1}$.

Claim. *Let $i \in I$, $j \leq k$, $\overline{f} \in A^{m_i}$ and $\overline{d} \in A^{k-1}$. If*

$$\langle \Delta_{i1}^{\mathbf{A}}(a,a,\overline{f},\overline{d}),\ldots,\Delta_{ij}^{\mathbf{A}}(a,a,\overline{f},\overline{d}),\Delta_{ij+1}^{\mathbf{A}}(b,a,\overline{f},\overline{d}),\ldots,\Delta_{ik}^{\mathbf{A}}(b,a,\overline{f},\overline{d})\rangle \in F,$$

then $$\langle \Delta_{i1}^{\mathbf{A}}(a,a,\overline{f},\overline{d}),\ldots,\Delta_{ij-1}^{\mathbf{A}}(a,a,\overline{f},\overline{d}),\Delta_{ij}^{\mathbf{A}}(b,a,\overline{f},\overline{d}),\ldots,\Delta_{ik}^{\mathbf{A}}(b,a,\overline{f},\overline{d})\rangle \in F.$$

Proof. Since $(a,b) \in \Phi$, we have $\Delta_i^{\mathbf{A}}(a,b,\overline{f},\overline{d}) \in F$. Denote $y_1,\ldots,y_{m_i}$ by $\overline{y}$ and $v_1,\ldots,v_{k-1}$ by $\overline{v}$, where the y_i, v_j are variables and $p,q,u,\overline{y},\overline{v}$ are distinct. Set $\varphi(p,u,\overline{y},\overline{v}) = \Delta_{ij}(p,u,\overline{y},\overline{v})$. Let $\ell \in I$, $\overline{g} \in A^{m_\ell}$ and $\overline{e} \in A^{k-1}$ and, using $a,b,a,\overline{f},\overline{d},\overline{g},\overline{e}$ as an interpretation of $p,q,u,\overline{y},\overline{v}$, $w_1,\ldots,w_{m_\ell},\tilde{z}$ in A, we get, from (2.4.3) and the fact that $\langle \mathbf{A}, F\rangle$ is an S-matrix that

$$\Delta_\ell^{\mathbf{A}}(\varphi^{\mathbf{A}}(a,a,\overline{f},\overline{d}),\varphi^{\mathbf{A}}(b,a,\overline{f},\overline{d}),\overline{g},\overline{e}) \in F,$$

i.e., $$\Delta_\ell^{\mathbf{A}}(\Delta_{ij}^{\mathbf{A}}(a,a,\overline{f},\overline{d}),\Delta_{ij}^{\mathbf{A}}(b,a,\overline{f},\overline{d}),\overline{g},\overline{e}) \in F.$$

Now, set $\overline{e} = \langle \Delta_{i1}^{\mathbf{A}}(a,a,\overline{f},\overline{d}),\ldots,\Delta_{ij-1}^{\mathbf{A}}(a,a,\overline{f},\overline{d}),\Delta_{ij+1}^{\mathbf{A}}(b,a,\overline{f},\overline{d}),\ldots,\Delta_{ik}^{\mathbf{A}}(b,a,\overline{f},\overline{d})\rangle$. Taking

$\Delta^{\mathbf{A}}_{ij}(a,a,\overline{f},\overline{d})$, $\Delta^{\mathbf{A}}_{ij}(b,a,\overline{f},\overline{d})$, $\overline{g},\overline{e}$ as an interpretation of $p,q,\overline{w},\widetilde{z}$ in (2.4.2) and using the fact that $\langle \mathbf{A}, F\rangle$ is an S-matrix, we obtain from (2.4.2) exactly the statement of the Claim. Since $\langle \Delta^{\mathbf{A}}_{i1}(a,a,\overline{f},\overline{d}),\ldots,\Delta^{\mathbf{A}}_{ik}(a,a,\overline{f},\overline{d})\rangle \in F$ (by (2.4.1)), it follows from repeated application of the Claim that $\langle \Delta^{\mathbf{A}}_{i1}(b,a,\overline{f},\overline{d}),\ldots,\Delta^{\mathbf{A}}_{ik}(b,a,\overline{f},\overline{d})\rangle \in F$, hence Φ is symmetric.

Next we show that Φ is transitive. Let $(a,b),(b,c)\in\Phi$. To show that Φ is transitive, we need to show that $\mathbf{\Delta}^{\mathbf{A}}_i(a,c,\overline{f},\overline{d})\in F$ for all $i\in I$, $\overline{f}\in A^{m_i}$ and $\overline{d}\in A^{k-1}$.

Claim. *Let $i\in I$, $j\le k$, $\overline{f}\in A^{m_i}$ and $\overline{d}\in A^{k-1}$. If*

$$\langle \Delta^{\mathbf{A}}_{i1}(a,b,\overline{f},\overline{d}),\ldots,\Delta^{\mathbf{A}}_{ij}(a,b,\overline{f},\overline{d}),\Delta^{\mathbf{A}}_{ij+1}(a,c,\overline{f},\overline{d}),\ldots,\Delta^{\mathbf{A}}_{ik}(a,c,\overline{f},\overline{d})\rangle \in F,$$

then $$\langle \Delta^{\mathbf{A}}_{i1}(a,b,\overline{f},\overline{d}),\ldots,\Delta^{\mathbf{A}}_{ij-1}(a,b,\overline{f},\overline{d}),\Delta^{\mathbf{A}}_{ij}(a,c,\overline{f},\overline{d}),\ldots,\Delta^{\mathbf{A}}_{ik}(a,c,\overline{f},\overline{d})\rangle \in F.$$

Proof. Since $(a,b),(b,c)\in\Phi$, we have $\mathbf{\Delta}^{\mathbf{A}}_i(a,b,\overline{f},\overline{d}),\mathbf{\Delta}^{\mathbf{A}}_i(b,c,\overline{f},\overline{d})\in F$. Denote $y_1,\ldots,y_{m_i}$ by $\overline{y}$ and $v_1,\ldots,v_{k-1}$ by $\overline{v}$, where the y_i, v_j are variables and $p,q,u,\overline{y},\overline{v}$ are distinct. Set $\varphi(p,u,\overline{y},\overline{v}) = \Delta_{ij}(u,p,\overline{y},\overline{v})$. For any $\ell\in I$, $\overline{g}\in A^{m_\ell}$ and $\overline{e}\in A^{k-1}$, using $b,c,a,\overline{f},\overline{d},\overline{g},\overline{e}$ as an interpretation of $p,q,u,\overline{y},\overline{v},w_1,\ldots,w_{m_\ell},\widetilde{z}$, we get, from (2.4.3) and the fact that $\langle\mathbf{A},F\rangle$ is an S-matrix that

$$\mathbf{\Delta}^{\mathbf{A}}_{\ell}(\varphi^{\mathbf{A}}(b,a,\overline{f},\overline{d}),\varphi^{\mathbf{A}}(c,a,\overline{f},\overline{d}),\overline{g},\overline{e})\in F,$$

i.e., $$\mathbf{\Delta}^{\mathbf{A}}_{\ell}(\Delta^{\mathbf{A}}_{ij}(a,b,\overline{f},\overline{d}),\Delta^{\mathbf{A}}_{ij}(a,c,\overline{f},\overline{d}),\overline{g},\overline{e})\in F.$$

Now, set $\overline{e}=\langle \Delta^{\mathbf{A}}_{i1}(a,b,\overline{f},\overline{d}),\ldots,\Delta^{\mathbf{A}}_{ij-1}(a,b,\overline{f},\overline{d}),\Delta^{\mathbf{A}}_{ij+1}(a,c,\overline{f},\overline{d}),\ldots,\Delta^{\mathbf{A}}_{ik}(a,c,\overline{f},\overline{d})\rangle$. Taking $\Delta^{\mathbf{A}}_{ij}(a,b,\overline{f},\overline{d})$, $\Delta^{\mathbf{A}}_{ij}(a,c,\overline{f},\overline{d})$, $\overline{g},\overline{e}$ as an interpretation of $p,q,\overline{w},\widetilde{z}$ in (2.4.2) and using the fact that $\langle\mathbf{A},F\rangle$ is an S-matrix, we obtain from (2.4.2) exactly the statement of the Claim. Since $\langle \Delta^{\mathbf{A}}_{i1}(a,b,\overline{f},\overline{d}),\ldots,\Delta^{\mathbf{A}}_{ik}(a,b,\overline{f},\overline{d})\rangle\in F$, it follows by repeated application of the Claim that $\langle \Delta^{\mathbf{A}}_{i1}(a,c,\overline{f},\overline{d}),\ldots,\Delta^{\mathbf{A}}_{ik}(a,c,\overline{f},\overline{d})\rangle\in F$, hence Φ is transitive.

To see that Φ has the substitution property, suppose that $(a_1,b_1),\ldots,(a_m,b_m)\in\Phi$ and $f\in\mathfrak{L}$ with $ar(f)=m$. Let $i\in I$, $\overline{f}\in A^{m_i}$ and $\overline{d}\in A^{k-1}$. By the definition of Φ,

$$\mathbf{\Delta}^{\mathbf{A}}_i(a_j,b_j,\overline{f},\overline{d})\in F \quad\text{for each } j\le m.$$

By (2.4.3), $$\mathbf{\Delta}^{\mathbf{A}}_i(f^{\mathbf{A}}(a_1,a_2,\ldots,a_m),f^{\mathbf{A}}(b_1,a_2,\ldots,a_m),\overline{f},\overline{d})\in F,$$

and by (2.4.3) again, $\mathbf{\Delta}^{\mathbf{A}}_i(f^{\mathbf{A}}(b_1,a_2,a_3,\ldots,a_m),f^{\mathbf{A}}(b_1,b_2,a_3,\ldots,a_m),\overline{f},\overline{d})\in F$.

By transitivity of Φ we have that

$$\mathbf{\Delta}^{\mathbf{A}}_i(f^{\mathbf{A}}(a_1,a_2,a_3,\ldots,a_m),f^{\mathbf{A}}(b_1,b_2,a_3,\ldots,a_m),\overline{f},\overline{d})\in F.$$

Continuing in this way, we obtain the desired result, namely

$$\mathbf{\Delta}^{\mathbf{A}}_i(f^{\mathbf{A}}(a_1,\ldots,a_m),f^{\mathbf{A}}(b_1,\ldots,b_m),\overline{f},\overline{d})\in F.$$

Thus Φ is a congruence.

The compatibility of Φ with F is proved as follows: Suppose that $(\boldsymbol{a}, \boldsymbol{b}) \in \Phi^{[k]}$ and $\boldsymbol{a} \in F$. Then $(a_j, b_j) \in \Phi$ for each $j \leq k$.

Claim. *Let $i \in I$ and $j \leq k$. If*

$$\langle a_1, \ldots, a_j, b_{j+1}, \ldots, b_k \rangle \in F,$$

then
$$\langle a_1, \ldots, a_{j-1}, b_j, \ldots, b_k \rangle \in F.$$

Proof. Since $(a_j, b_j) \in \Phi$, it follows that

$$\Delta_i^{\mathbf{A}}(a_j, b_j, \overline{f}, \overline{e}) \in F \quad \text{for each } \overline{f} \in A^{m_i} \text{ and } \overline{e} \in A^{k-1}.$$

Set $\overline{e} = \langle a_1, \ldots, a_{j-1}, b_{j+1}, \ldots, b_k \rangle$. By the hypothesis of the Claim and (2.4.2), the result follows.

Since $\boldsymbol{a} \in F$, it follows by repeated application of the Claim that $\boldsymbol{b} \in F$, hence Φ is compatible with F.

This implies that $\Phi \subseteq \Omega_{\mathcal{A}} F$. For the reverse inclusion, suppose that $(a, b) \in \Omega_{\mathcal{A}} F$. Let $i \in I$, $\overline{f} \in A^{m_i}$ and $\overline{d} \in A^{k-1}$. Since $\Omega_{\mathcal{A}} F$ is a congruence relation, we immediately have that

$$(\Delta_i^{\mathbf{A}}(a, a, \overline{f}, \overline{d}), \Delta_i^{\mathbf{A}}(a, b, \overline{f}, \overline{d})) \in (\Omega_{\mathcal{A}} F)^{[k]}.$$

But this implies that $\Delta_i^{\mathbf{A}}(a, b, \overline{f}, \overline{d}) \in F$ by the compatibility of $\Omega_{\mathcal{A}} F$ with F and (2.3.1). Thus $(a, b) \in \Phi$, and so $\Omega_{\mathcal{A}} F \subseteq \Phi$.

The second statement of the theorem can be deduced from the above proof in the following way: Set $\Delta_i'(p, q, w_1, \ldots, w_{m_i}, \tilde{z}) = \Delta_i(p, q, \tilde{z})$ for each $i \in I$. Then $\Delta_i'(p, q, w_1, \ldots, w_{m_i}, \tilde{z})$, $i \in I$, forms a generalized system of congruence k-formulas with parameters $\tilde{z}$ for S iff $\Delta_i(p, q, \tilde{z})$, $i \in I$, forms a system of congruence k-formulas with parameters $\tilde{z}$, and (2.4.4) holds iff (2.4.4)$'$ holds. By the above proof, therefore, (2.4.4)$'$ holds.

In the third statement of the theorem, (1)$\Rightarrow$(2) can be deduced from (ii) in the following way: Set $\Delta_i'(p, q, \tilde{z}) = \Delta_i(p, q)$ for each $i \in I$. Then $\Delta_i'(p, q, \tilde{z})$, $i \in I$, forms a system of congruence k-formulas with parameters $\tilde{z}$ for S iff $\Delta_i(p, q)$, $i \in I$, forms a system of congruence k-formulas without parameters, and (2.4.4)$'$ holds iff (2.4.4)$''$ holds. Thus (2.4.4)$''$ is true.

(2)$\Rightarrow$(1) We need to show that $\Delta_i(p, q)$, $i \in I$, forms a system of equivalence k-formulas without parameters for S, i.e., that (2.3.1)$'$ and (2.3.2)$'$ hold, and also, by Definition 2.4.1, that (2.4.3)$''$

holds. That $(2.3.1)'$ holds is trivial since $\Omega_{\mathcal{A}}F$ is reflexive. To see that $(2.3.2)'$ holds, let $\mathcal{A} = \langle \mathbf{Fm}, T\rangle$, where

$$T = \mathrm{Cn}_S(\{\widetilde{z}\,[p/j]\} \cup \{\Delta_i(p,q)\,;\, i \in I\}).$$

Since $\Delta_i(p,q) \in T$ for each $i \in I$, we have that $(p,q) \in \Omega_{\mathcal{A}}T$. This implies, by the definition of $\Omega_{\mathcal{A}}T$, that $\widetilde{z}\,[p/j] \in T$ if and only if $\widetilde{z}\,[q/j] \in T$. Consequently $\widetilde{z}\,[q/j] \in T$, i.e.,

$$\widetilde{z}\,[p/j], \{\Delta_i(p,q)\,;\, i \in I\} \vdash_S \widetilde{z}\,[q/j].$$

To see that (2.4.3) holds, recall from Theorem 1.6.2 that Mod$S$ is a matrix semantics for S. Suppose that $\varphi(p,\overline{v}) \in Fm$. We claim that for each $\ell \in I$,

$$\{\Delta_i(p,q)\,;\, i \in I\} \models_{\mathrm{Mod}S} \Delta_\ell(\varphi(p,\overline{v}), \varphi(q,\overline{v})),$$

which will imply (2.4.3). So, let $\mathcal{A} = \langle \mathbf{A}, F\rangle \in \mathrm{Mod}S$ and let a, b be interpretations of the variables p, q in A, respectively, such that $\Delta_i^{\mathbf{A}}(a,b) \in F$ for all $i \in I$. Then, by assumption, $(a,b) \in \Omega_{\mathcal{A}}F$. Since $\Omega_{\mathcal{A}}F$ is a congruence relation on $\mathbf{A}$, it follows that $(\varphi^{\mathbf{A}}(a,\overline{c}), \varphi^{\mathbf{A}}(b,\overline{c})) \in \Omega_{\mathcal{A}}F$. By assumption again, we get that $\Delta_i^{\mathbf{A}}(\varphi^{\mathbf{A}}(a,\overline{c}), \varphi^{\mathbf{A}}(b,\overline{c})) \in F$ for all $i \in I$, hence the claim holds since $\mathcal{A}$ was arbitrary, and it implies $(2.4.3)''$. □

The condition $(2.4.3)''$ of Definition 2.4.1 is not easily checked directly for a given deductive system. In the following proposition, we present an equivalent formulation of $(2.4.3)''$ that we shall use in later sections when proving that a given system of equivalence formulas is a system of congruence formulas.

2.4.3 PROPOSITION

Let $\Delta_i(p,q)$, $i \in I$, be a system of equivalence k-formulas without parameters for S. The following are equivalent:

(i) $$\{\Delta_i(p,q)\,;\, i \in I\} \vdash_S \Delta_\ell(\varphi(p,\overline{v}), \varphi(q,\overline{v}))$$
for each $\varphi(p,\overline{v}) \in Fm$ and all $\ell \in I$,

(ii) $$\{\Delta_j(p_i,q_i)\,;\, i \le m;\, j \in I\} \vdash_S \Delta_\ell(f(p_1,\ldots,p_m), f(q_1,\ldots,q_m))$$
for each $f \in \mathfrak{L}$ with $ar(f) = m$ and all $\ell \in I$.

In particular, a system $\Delta_i(p,q)$, $i \in I$, of equivalence k-formulas without parameters is a system of congruence k-formulas without parameters if and only if (ii) holds.

Proof. (i)$\Rightarrow$(ii) Let $f \in \mathfrak{L}$ with $ar(f) = m$. We may clearly assume without loss of generality that the variables $p_1,\dots,p_m,q_1,\dots,q_m$ are all distinct (in view of structurality). Let

$$F = \mathrm{Cn}_S(\{\Delta(p_i,q_i)\,;\, i \le m\})$$

and let $\mathcal{A} = \langle \mathbf{Fm}, F\rangle$. Since $\mathcal{A}$ is an S-matrix, it follows from (i) (which says that $\Delta(p,q)$ is a system of congruence k-formulas without parameters for S) and Theorem 2.4.2 that

$$\Omega_{\mathcal{A}}F = \{(\varphi,\psi) \in Fm^2\,;\, \{\Delta(p_i,q_i)\,;\, i \le m\} \vdash_S \Delta_\ell(\varphi,\psi) \text{ for all } \ell \in I\}. \tag{2.4.5}$$

By (i) and the fact that $p_1,\dots,p_m,q_1,\dots,q_m$ are all distinct, we have, for any $\ell \in I$, that

$$\Delta(p_1,q_1) \vdash_S \Delta_\ell(f(p_1,p_2,\dots,p_m), f(q_1,p_2,\dots,p_m)).$$

By (2.4.5), therefore,

$$(f(p_1,p_2,\dots,p_m), f(q_1,p_2,\dots,p_m)) \in \Omega_{\mathcal{A}}F.$$

Similarly,
$$(f(q_1,p_2,p_3,\dots,p_m), f(q_1,q_2,p_3,\dots,p_m)) \in \Omega_{\mathcal{A}}F,$$

$$\vdots$$

$$(f(q_1,\dots,q_{m-1},p_m), f(q_1,\dots,q_{m-1},q_m)) \in \Omega_{\mathcal{A}}F$$

and since $\Omega_{\mathcal{A}}F$ is a congruence (in particular, it is transitive),

$$(f(p_1,\dots,p_m), f(q_1,\dots,q_m)) \in \Omega_{\mathcal{A}}F.$$

By (2.4.5), we have

$$\{\Delta(p_i,q_i)\,;\, i \le m\} \vdash_S \Delta_\ell(f(p_1,\dots,p_m), f(q_1,\dots,q_m))$$

for each $\ell \in I$, as required.

(ii) $\Rightarrow$ (i) We proceed by induction on the complexity of $\varphi(p,\bar{r})$. Suppose $\varphi(p,\bar{r})$ is a variable. Then the result reduces either to

$$\Delta(p,q) \vdash_S \Delta_\ell(p,q) \quad (\text{all } \ell \in I),$$

or to
$$\Delta(p,q) \vdash_S \Delta_\ell(r_j,r_j) \quad (\text{all } \ell \in I)$$

for some variable r_j. If $\varphi(p,\bar{r})$ is a constant, say $\varphi(p,\bar{r}) = \mathrm{T}$, then the result reduces to

$$\Delta(p,q) \vdash_S \Delta_\ell(\mathrm{T},\mathrm{T}) \quad (\text{all } \ell \in I).$$

The first case is trivially true and both the last two cases are true by (2.3.1). Now suppose, inductively, that $\varphi(p,\bar{r}) = f(\psi_1(p,\bar{r}),\dots,\psi_m(p,\bar{r}))$, where $f \in \mathcal{L}$ with $ar(f) = m > 0$, and

$$\Delta(p,q) \vdash_S \Delta_\ell(\psi_i(p,\bar{r}), \psi_i(q,\bar{r}))$$

for each $i \le m$ and each $\ell \in I$. We may clearly assume without loss of generality that the variables $p,q,\bar{r}$ are all distinct (in view of structurality). By (ii), and structurality,

$$\{\Delta(\psi_i(p,\bar{r}),\psi_i(q,\bar{r}))\,;\, i \le m\} \vdash_S \Delta_\ell(f(\psi_1(p,\bar{r}),\dots,\psi_m(p,\bar{r})), f(\psi_1(q,\bar{r}),\dots,\psi_m(q,\bar{r}))),$$

i.e., $$\{\Delta(\psi_i(p,\overline{r}),\psi_i(q,\overline{r}))\,;\, i\le m\}\vdash_S \Delta_\ell(\varphi(p,\overline{r}),\varphi(q,\overline{r}))$$

for all $\ell\in I$. Consequently,

$$\Delta(p,q)\vdash_S \Delta_\ell(\varphi(p,\overline{r}),\varphi(q,\overline{r}))\quad \text{for all } \ell\in I.$$

The remaining statements follow immediately from Definition 2.4.1. □

2.4.4 LEMMA [Pała94, Lemma 3.21]

Let S be a protoalgebraic k-deductive system and $\mathcal{A}=\langle \mathbf{A},F\rangle$ an S-matrix. Let $\Delta_i(p,q,\widetilde{z})$, $i\in I$, be a system of equivalence k-formulas with parameters $\widetilde{z}$ for S (which exist by Theorem 2.3.2). Define

$$\widehat{\Delta}=\{\Delta_i(\varphi(p,\overline{v}),\varphi(q,\overline{v}),\widetilde{z})\,;\, i\in I \text{ and } \varphi(p,\overline{v})\in Fm, \text{ where } \overline{v} \text{ is a finite sequence of variables}\}.$$

Then $\widehat{\Delta}$ is a generalized system of congruence k-formulas with parameters $\widetilde{z}$ for S. In particular, $\langle a,b\rangle\in\Omega_{\mathcal{A}}F$ if and only if $\Delta_i^{\mathbf{A}}(\varphi^{\mathbf{A}}(a,\overline{c}),\varphi^{\mathbf{A}}(b,\overline{c}),\overline{d})\in F$ for each $i\in I$, all $\varphi(p,\overline{q})\in Fm$ and $\overline{c}\in A^m$, where $\overline{c}$ has the same length as $\overline{v}$, and $\overline{d}\in A^{k-1}$.

Proof. By definition, $\widehat{\Delta}$ satisfies (2.4.3). Since $\vdash_S\Delta_i(p,p,\widetilde{z})$, for each $i\in I$, by (2.3.1) and structurality, we have that $\widehat{\Delta}$ satisfies (2.4.1). Since $\Delta_i(p,q,\widetilde{z})$, $i\in I$, satisfies (2.3.2) and $\{\Delta_i(p,q,\widetilde{z})\,;\, i\in I\}\subseteq\widehat{\Delta}$, it follows that $\widehat{\Delta}$ satisfies (2.4.2). Thus $\widehat{\Delta}$ is a generalized system of congruence k-formulas with parameters $\widetilde{z}$ for S. The second statement follows by Theorem 2.4.2 (i) and the definition of $\widehat{\Delta}$. □

Knowing that protoalgebraicity is definable purely in terms of the Leibniz operator, one would expect that a converse of Lemma 2.4.4 holds, i.e., if one can define $\Omega_{\mathcal{A}}F$ in terms of a fixed set of k-formulas in the sort of way exemplified by Lemma 2.4.4, then the associated k-deductive system is protoalgebraic. The following theorem proves that this is indeed the case, hence we have another characterization of protoalgebraic k-deductive systems.

2.4.5 THEOREM

A k-deductive system S is protoalgebraic if and only if it has a generalized system of congruence k-formulas with parameters $\widetilde{z}$ for S.

Proof. ($\Rightarrow$) Assume that S is protoalgebraic. By Theorem 2.3.2 there exists a finite system of equivalence k-formulas with parameters $\widetilde{z}$ for S, say $\Delta_1(p,q,\widetilde{z}),\ldots,\Delta_n(p,q,\widetilde{z})$. By Lemma

2.4.4, there exists a generalized system of congruence k-formulas with parameters $\tilde{z}$ for S.

($\Leftarrow$) Let $\Delta = \{\Delta_i(p,q,w_1,\ldots,w_{m_i},\tilde{z}\,);\, i \in I\}$ be a generalized system of congruence k-formulas with parameters $\tilde{z}$ for S, and let $\mathcal{A} = \langle \mathbf{A}, F_{\mathcal{A}}\rangle$ be an S-matrix. We show that $\Omega_{\mathcal{A}}$ is monotonic, which will imply that S is protoalgebraic by Theorem 2.1.3 (iii). Let $F, G \in \mathrm{Fi}^S\mathcal{A}$ such that $F \subseteq G$ and let $(a,b) \in \Omega_{\mathcal{A}}F$. Then

$$\Delta_i^{\mathbf{A}}(a,b,\overline{f},\overline{d}) \in F \text{ for all } i \in I,\ \overline{f} \in A^{m_i} \text{ and } \overline{d} \in A^{k-1},$$

by Theorem 2.4.2 (i). Since $F \subseteq G$, we therefore have that

$$\Delta_i^{\mathbf{A}}(a,b,\overline{f},\overline{d}) \in G \text{ for all } i \in I,\ \overline{f} \in A^{m_i} \text{ and } \overline{d} \in A^{k-1},$$

hence, by Theorem 2.4.2 (i) again, $(a,b) \in \Omega_{\mathcal{A}}G$. Thus $\Omega_{\mathcal{A}}F \subseteq \Omega_{\mathcal{A}}G$. □

2.5 WEAKLY CONGRUENTIAL AND CONGRUENTIAL k-DEDUCTIVE SYSTEMS

In Theorem 2.4.5, we provided a characterization of protoalgebraic k-deductive systems in terms of generalized congruence k-formulas with parameters $\tilde{z}$. The form of the k-formulas at issue is very general, and it is natural to consider more refined systems of k-formulas. These can be used to define deductive systems which are stronger than protoalgebraic deductive systems, namely 'congruential' and 'weakly congruential' ones. For both these types of k-deductive system we present a model-theoretic characterization of the classes of reduced matrix models, as was done in Section 2.2 for protoalgebraic k-deductive systems. We also present a characterization of weakly congruential k-deductive systems in terms of the Leibniz operator and a similar partial result for congruential k-deductive systems.

2.5.1 DEFINITION

A k-deductive system is called *weakly congruential* if it has a system of congruence k-formulas without parameters, and it is called *congruential* if it has a *finite* system of congruence k-formulas without parameters.

It is evident from the definition that a congruential k-deductive system is weakly congruential, and from Theorems 2.4.2 and 2.4.5 that a weakly congruential k-deductive system is protoalgebraic.

By Theorem 2.4.2, a k-deductive system S is weakly congruential if and only if there exists a system $\Delta_i(p,q)$, $i \in I$, of k-formulas such that for each S-matrix $\mathcal{A} = \langle \mathbf{A}, F\rangle$,

$$(a,b) \in \Omega_{\mathcal{A}} F \quad \text{if and only if} \quad \Delta_i^{\mathbf{A}}(a,b) \in F \ \text{ for each } i \in I.$$

S is congruential if and only if there exists a finite system $\Delta_i(p,q)$, $i \in I$, that has the above property. It is evident from Definition 2.5.1 that every extension S' of a (weakly) congruential k-deductive system S that has the same same language as S is also (weakly) congruential. If the language of S' contains additional connectives and for each additional connective f, with $ar(f) = m$ say,

$$\{\Delta_j(p_i, q_i)\,;\, i \leq m\,;\, j \in I\} \vdash_S \Delta_\ell(f(p_1,\dots,p_m), f(q_1,\dots,q_m)) \ \text{ for each } \ell \in I,$$

then S' is weakly congruential.

Recall from Section 1.10 the first-order universal Horn theory $T(S)$ (without equality) over the language $\mathcal{L}_D$. Let I be a set and $\mathcal{A}_i = \langle \mathbf{A}_i, F_{\mathcal{A}_i}\rangle \in \mathrm{Mod}S$ for each $i \in I$. Let $\mathfrak{F} \subseteq \mathfrak{P}(I)$ be a filter of the Boolean algebra of all subsets of I. By regarding each $\mathcal{A}_i$ as an $\mathcal{L}_D$-structure (where $\mathcal{L}_D$ is $\mathcal{L}$ augmented by a single k-ary relation symbol D, and $D^{\mathbf{A}_i} = F_{\mathcal{A}_i}$) we get from Section 0.5 the following notions of filtered products and ultraproducts of k-matrices:

We identify $(\Pi_{i \in I} A_i)^k$ with $\Pi_{i \in I} A_i^k$ under the natural map

$$\langle\langle a_{i1}\,;\, i \in I\rangle, \dots, \langle a_{ik}\,;\, i \in I\rangle\rangle \mapsto \langle\langle a_{i1},\dots,a_{ik}\rangle\,;\, i \in I\rangle.$$

We define

$$F^{\mathfrak{F}}_{\Pi\mathcal{A}_i} = \{\overline{a} = \langle\langle a_{i1}\,;\, i \in I\rangle, \dots, \langle a_{ik}\,;\, i \in I\rangle\rangle \in (\Pi_{i \in I} A_i)^k\,;\, \{i \in I\,;\, \langle a_{i1},\dots,a_{ik}\rangle \in F_{\mathcal{A}_i}\} \in \mathfrak{F}\}.$$

Let $\overline{a} = \langle a_i\,;\, i \in I\rangle$ and $\overline{b} = \langle b_i\,;\, i \in I\rangle$. Recall that

$$\theta(\mathfrak{F}) = \{(\overline{a},\overline{b}) \in (\prod_{i \in I} A_i)^2\,;\, \{i\,;\, a_i = b_i\} \in \mathfrak{F}\},$$

$$\prod_{i \in I} \mathbf{A}_i/\mathfrak{F} = (\prod_{i \in I} \mathbf{A}_i)/\theta(\mathfrak{F}).$$

If we define

$$F_{\Pi\mathcal{A}_i}/\mathfrak{F} = F^{\mathfrak{F}}_{\Pi\mathcal{A}_i}/\theta(\mathfrak{F}) \quad (= \{\overline{a}/\theta(\mathfrak{F})\,;\, \overline{a} \in F^{\mathfrak{F}}_{\Pi\mathcal{A}_i}\}),$$

then

$$\prod_{i \in I} \mathcal{A}_i/\mathfrak{F} = \langle \prod_{i \in I} \mathbf{A}_i/\mathfrak{F}, F_{\Pi\mathcal{A}_i}/\mathfrak{F}\rangle$$

is the filtered product of the structures $\mathcal{A}_i$, $i \in I$. We shall also call $\Pi_{i \in I} \mathcal{A}_i/\mathfrak{F}$ the *matrix filtered product* of $\{\mathcal{A}_i\,;\, i \in I\}$ by $\mathfrak{F}$. If $\mathfrak{F}$ is an ultrafilter of $\mathfrak{P}(I)$, then $\prod_{i \in I} \mathcal{A}_i/\mathfrak{F}$ is called the *matrix ultraproduct* of $\{\mathcal{A}_i\,;\, i \in I\}$ by $\mathfrak{F}$. For a class H of k-matrices, define

$$\mathrm{P_F}(H) \text{ (resp. } \mathrm{P_U}(H)) = \{ \prod_{i \in I} \mathcal{A}_i/\mathcal{F}\,;\ \mathcal{A}_i \in H \text{ for all } i \in I \text{ and } \mathcal{F} \text{ is a filter (resp. ultrafilter) of } \mathcal{P}(I)\}.$$

Now let $T(S)^{\approx}$ denote the first-order theory *with equality* over the language $\mathcal{L}_D$, augmented by the binary predicate symbol $\approx$, whose proper axioms are just those of $T(S)$. It is easy to see that the structures (resp. models) with equality for $T(S)^{\approx}$ are just the triples $\langle \mathbf{A}, F, I_A\rangle$, where $\langle \mathbf{A}, F\rangle$ is a k-matrix (resp. S-matrix) and I_A is the identity relation on A. We shall confuse $\langle \mathbf{A}, F, I_A\rangle$ and $\langle \mathbf{A}, F\rangle$ systematically. Notice that submatrices, products, etc. of matrices $\langle \mathbf{A}, F\rangle$ may also be regarded, in the obvious way, as substructures, products, etc. of the structures $\langle \mathbf{A}, F, I_A\rangle$ with respect to the language of $T(S)^{\approx}$, so no harm will come of this convention.

2.5.2 LEMMA [BP92, Corollary 13.6]

Let S be a congruential k-deductive system and let $\Delta_1(p,q), \ldots, \Delta_n(p,q)$ be a finite system of congruence k-formulas without parameters for S.

(i) *An S-matrix $\mathcal{A}$ is reduced if and only if, as a model of $T(S)^{\approx}$, it satisfies the universal Horn sentence*

$$\forall x\, \forall y\, (D(\Delta_1(x,y))\, \&\ldots \&\, D(\Delta_n(x,y)) \Rightarrow x \approx y).$$

(ii) *The class* Mod$^*S$ *is closed under the formation of submatrices and filtered products.*

Proof. (i) By the Validity and Completeness Theorems of First-Order Theories with equality (the equality-analogue of Theorem 0.5.3),

$$\vdash_{T(S)} \forall x\, \forall y\, (D(\Delta_1(x,y))\, \&\ldots \&\, D(\Delta_n(x,y)) \Rightarrow x \approx y)$$

iff for every model $\mathcal{A} = \langle \mathbf{A}, F\rangle$ of $T(S)^{\approx}$ (i.e., every S-matrix) and every $a, b \in A$,

$$\Delta_i^{\mathbf{A}}(a,b) \in F \text{ for all } i \leq n \text{ implies } a = b.$$

Since an S-matrix $\mathcal{A} = \langle \mathbf{A}, F\rangle$ is reduced iff $\Omega_{\mathcal{A}} F = I_A$ (i.e., $(a,b) \in \Omega_{\mathcal{A}} F$ iff $a = b$), the above argument, together with (2.4.4)$''$ yield statement (i). Statement (ii) follows immediately from (i), the fact that $T(S)^{\approx}$ is axiomatized by universal Horn sentences and the equality-analogue of Theorem 0.5.2. □

We now present some model-theoretic results concerning congruential and weakly congruential k-deductive systems.

2.5.3 DEFINITION

A reduced universal Horn k-class is called a *k-(weakly) congruential quasivariety* if it is the class of all reduced matrices of some (weakly) congruential k-deductive system.

2.5.5 THEOREM [BP92, Theorem 13.12]

Let H be a reduced universal Horn k-class.

(i) H *is a k-protoquasivariety if and only if it closed under* $\mathrm{P_S}$.

(ii) H *is a k-weakly congruential quasivariety if and only if it is closed under* S *and* P.

(iii) H *is a k-congruential quasivariety if and only if it is closed under* S, P *and* $\mathrm{P_U}$.

Proof. (i) It follows trivially from the definitions that H is closed under (matrix-) isomorphic images. The result follows from Theorem 2.2.3.

(ii) Let H be a k-weakly congruential quasivariety, i.e., $H = \mathrm{Mod}^*S$, where S is a weakly congruential k-deductive system. Then H is a k-protoquasivariety, so it is closed under $\mathrm{P_S}$, hence it is also closed under P. Let $\Delta_i(p,q)$, $i \in I$, be a system of congruence k-formulas without parameters for S. Let $\mathcal{A} = \langle \mathbf{A}, F\rangle \in H$. By assumption, $\mathcal{A}$ is reduced, so $\Omega_{\mathcal{A}}F = I_A$. Let $\mathfrak{B} = \langle \mathbf{B}, G\rangle$ be a submatrix of $\mathcal{A}$. For all $a, b \in B$, we have, by Theorem 2.4.2 (iii) that

$$(a,b) \in \Omega_{\mathfrak{B}}G \quad \text{if and only if} \quad \Delta_i^{\mathbf{B}}(a,b) \in G \ \text{ for all } i \in I.$$

Since $\mathbf{B}$ is a subalgebra of $\mathbf{A}$, we have $\Delta_i^{\mathbf{B}}(a,b) = \Delta_i^{\mathbf{A}}(a,b)$ and, since $G = F \cap B^k$, $\Delta_i^{\mathbf{B}}(a,b) \in G$ if and only if $\Delta_i^{\mathbf{A}}(a,b) \in F$. Hence $(a,b) \in \Omega_{\mathfrak{B}}G$ if and only if $(a,b) \in \Omega_{\mathcal{A}}F$ if and only if $a = b$ (since $\Omega_{\mathcal{A}}F = I_A$). Thus $\Omega_{\mathfrak{B}}G = I_B$, and $\mathfrak{B}$ is reduced. This proves that H is closed under S.

Now, assume that $\mathrm{S}(H) \subseteq H$ and $\mathrm{P}(H) \subseteq H$. Then H is closed under $\mathrm{P_S}$ and is therefore a k-protoquasivariety. Thus we can assume $H = \mathrm{Mod}^*S$, where S is protoalgebraic, so there exists a generalized system $\Delta_i(p, q, w_1, \ldots, w_{m_i}, \tilde{z}\,)$, $i \in I$, of congruence k-formulas with parameters $\tilde{z}$ for S (Theorem 2.4.5). Let $\{\Delta'_j(p,q)\,;\, j \in J\}$ be the set of all k-formulas of the form $\Delta_i(p,q,\varphi_1(p,q),\ldots,\varphi_{m_i}(p,q),\tau_1(p,q),\ldots,\tau_{k-1}(p,q))$, where $\varphi_1(p,q),\ldots,\varphi_{m_i}(p,q)$, $\tau_1(p,q),\ldots,$ $\tau_{k-1}(p,q)$ are formulas in the variables p, q. To show that H is a k-weakly congruential quasivariety, we shall prove that $\Delta'_j(p,q)$, $j \in J$, is a system of congruence k-formulas without parameters for S. Let $\mathcal{A} = \langle \mathbf{A}, F\rangle$ be an S-matrix and $a, b \in A$. Let $\mathbf{B}$ be the subalgebra of $\mathbf{A}$

generated by $\{a,b\}$ and $\mathfrak{B}$ the submatrix $\langle \mathbf{B}, F \cap B^k\rangle$ of $\mathcal{A}$. Each element of B is of the form $\tau^{\mathbf{A}}(a,b)$ for some formula $\tau(p,q)$ (by Theorem 0.2.9).

By Lemma 1.8.7, $\mathfrak{C} = \langle \mathbf{B}/(\Omega_{\mathcal{A}}F \cap B^2), (F \cap B^k)/(\Omega_{\mathcal{A}}F \cap B^2)\rangle$ is isomorphic to a submatrix of $\mathcal{A}^* = \langle \mathbf{A}/\Omega_{\mathcal{A}}F, F/\Omega_{\mathcal{A}}F\rangle$. Since $\mathcal{A}^*$ is reduced and $\mathrm{IS}(H) \subseteq H$, $\mathfrak{C} \in H$, so $\mathfrak{C}$ is reduced. Thus, by the Correspondence Theorem (Theorem 0.2.6), $\Omega_{\mathcal{A}}F \cap B^2$ is the largest congruence on $\mathbf{B}$ that is compatible with $F \cap B^k$, i.e.,

$$\Omega_{\mathfrak{B}}(F \cap B^k) = \Omega_{\mathcal{A}}F \cap B^2.$$

Using the fact that $\Delta_i(p,q,w_1,\ldots,w_{m_i},\tilde{z})$, $i \in I$, is a generalized system of congruence k-formulas with parameters $\tilde{z}$ for S we get

$$\begin{aligned}
(a,b) \in \Omega_{\mathcal{A}}F \quad &\text{iff} \quad (a,b) \in (\Omega_{\mathcal{A}}F) \cap B^2 \qquad\qquad [\text{since } a,b \in B]\\
&\text{iff} \quad (a,b) \in \Omega_{\mathfrak{B}}(F \cap B^k)\\
&\text{iff} \quad \Delta_i^{\mathbf{B}}(a,b,\overline{f},\overline{c}) \in F \cap B^k \text{ for all } i \in I,\ \overline{f} \in B^{m_i} \text{ and } \overline{c} \in B^{k-1} \ [\text{Thm } 2.4.2\,(\text{i})]\\
&\text{iff} \quad \Delta_i^{\mathbf{B}}(a,b,\varphi_1^{\mathbf{B}}(a,b),\ldots,\varphi_{m_i}^{\mathbf{B}}(a,b),\tau_1^{\mathbf{B}}(a,b),\ldots,\tau_{k-1}^{\mathbf{B}}(a,b)) \in F \cap B^k\\
&\qquad \text{for all } i \in I \text{ and all formulas } \varphi_1(p,q),\ldots,\varphi_{m_i}(p,q),\tau_1(p,q),\ldots,\tau_{k-1}(p,q).\\
&\text{iff} \quad \Delta_i^{\mathbf{A}}(a,b,\varphi_1^{\mathbf{A}}(a,b),\ldots,\varphi_{m_i}^{\mathbf{A}}(a,b),\tau_1^{\mathbf{A}}(a,b),\ldots,\tau_{k-1}^{\mathbf{A}}(a,b)) \in F\\
&\qquad \text{for all } i \in I \text{ and all formulas } \varphi_1(p,q),\ldots,\varphi_{m_i}(p,q),\tau_1(p,q),\ldots,\tau_{k-1}(p,q).\\
&\text{iff} \quad \Delta_j'^{\mathbf{A}}(a,b) \in F \quad \text{for all } j \in J.
\end{aligned}$$

Since a,b are arbitrary elements of A, we can deduce from Theorem 2.4.2 (iii) that $\Delta'_j(p,q)$, $j \in J$, is a system of congruence k-formulas without parameters for S and therefore that H is a k-weakly congruential quasivariety.

(iii) Let $H = \mathrm{Mod}^*S$ and assume H is a k-congruential quasivariety, i.e., we may assume that S is congruential. Since H is also a k-weakly congruential quasivariety, (ii) implies that H is closed under S and P, and in Lemma 2.5.2, we showed that H is closed under $\mathrm{P_U}$. Conversely, suppose that $\mathrm{S}(H) \subseteq H$, $\mathrm{P}(H) \subseteq H$ and $\mathrm{P_U}(H) \subseteq H$. By (ii), we know that there exists a system $\Delta_i(p,q)$, $i \in I$, of congruence k-formulas without parameters for S. We shall show that some finite subset of $\{\Delta_i(p,q);\ i \in I\}$ must be a system of congruence formulas without parameters for S by assuming otherwise and obtaining a contradiction. By definition, for all $\mathcal{A} = \langle \mathbf{A}, F\rangle \in \mathrm{Mod}S$,

$$\Omega_{\mathcal{A}}F = \{(a,b) \in A^2;\ \Delta_i^{\mathbf{A}}(a,b) \in F \ \text{ for all } i \in I\}.$$

Assume that for every finite subset J of I, $\Delta_j(p,q)$, $j \in J$, does not form a system of congruence k-formulas without parameters for S. Then, for each $J \in \mathcal{P}_\omega(I)$ (recall that $\mathcal{P}_\omega(I)$ is the set of all finite subsets of I), there exists an S-matrix $\mathcal{A}_J = \langle \mathbf{A}_J, F_J \rangle$ such that $\Omega_{\mathcal{A}_J} F_J \subsetneq \Phi_J$, where

$$\Phi_J = \{(a,b) \in A_J^2 ; \Delta_j^{\mathbf{A}_J}(a,b) \in F_J \text{ for all } j \in J\}.$$

We may assume without loss of generality that each $\mathcal{A}_J$ is reduced (for otherwise we could take $\mathcal{A}_J^* = \langle \mathbf{A}_J/\Omega_{\mathcal{A}_J} F_J, F_J/\Omega_{\mathcal{A}_J} F_J \rangle$ for $\mathcal{A}_J$), hence that $\mathcal{A}_J \in H$ for each $J \in \mathcal{P}_\omega(I)$. It follows that for every $J \in \mathcal{P}_\omega(I)$, $\Phi_J \neq I_{A_J}$.

For every $J \in \mathcal{P}_\omega(I)$, let $\widehat{J} = \{K \in \mathcal{P}_\omega(I) ; J \subseteq K\}$. Since $\{M \in \mathcal{P}(\mathcal{P}_\omega(I)) ; M \supseteq \widehat{J}$ for some $J \in \mathcal{P}_\omega(I)\}$ is clearly a filter of $\mathcal{P}(\mathcal{P}_\omega(I))$, it follows from Theorem 0.3.3 that there exists an ultrafilter $\mathcal{U}$ on the Boolean algebra of all subsets of $\mathcal{P}_\omega(I)$ that includes $\widehat{J}$ for all $J \in \mathcal{P}_\omega(I)$. Let

$$\mathcal{B} = \prod_{J \in \mathcal{P}_\omega(I)} \mathcal{A}_J/\mathcal{U}.$$

By assumption, $\mathcal{B} \in H$. For each $J \in \mathcal{P}_\omega(I)$, choose $(a_J, b_J) \in \Phi_J$ such that $a_J \neq b_J$. Let $\overline{a} = \langle a_J ; J \in \mathcal{P}_\omega(I) \rangle/\mathcal{U}$ and $\overline{b} = \langle b_J ; J \in \mathcal{P}_\omega(I) \rangle/\mathcal{U}$. Clearly $\overline{a} \neq \overline{b}$. Let $i \in I$. For each $J \in \mathcal{P}_\omega(I)$ such that $i \in J$, we have $\Delta_i^{\mathbf{A}_J}(a_J, b_J) \in F_J$ since $(a_J, b_J) \in \Phi_J$. Consequently,

$$\{J \in \mathcal{P}_\omega(I) ; \Delta_i^{\mathbf{A}_J}(a_J, b_J) \in F_J\} \supseteq \widehat{\{i\}} \in \mathcal{U},$$

and thus $\Delta_i^{\mathbf{B}}(\overline{a}, \overline{b}) \in F_{\prod \mathcal{A}_J}/\mathcal{U}$ $(= F_{\mathcal{B}})$ for all $i \in I$. So $(\overline{a}, \overline{b}) \in \Omega_{\mathcal{B}} F_{\mathcal{B}}$, by Theorem 2.4.2 (iii), since $\Delta_i(p,q)$, $i \in I$, is a system of congruence k-formulas without parameters for S. But this implies that $\overline{a} = \overline{b}$ because $\mathcal{B} \in H$ (hence $\mathcal{B}$ is reduced). From this contradiction, we deduce that some finite subset of $\{\Delta_i(p,q) ; i \in I\}$ must be a system of congruence k-formulas without parameters for S, so H is a k-congruential quasivariety. □

Finally, we present a characterization of weakly congruential k-deductive systems in terms of the Leibniz operator, which appears in [BP92, Theorem 13.13]. The same theorem of [BP92] presented a similar characterization for congruential k-deductive systems whose proof was shown in [Pała94] to be invalid. The proof of one implication is correct (this is our Theorem 2.5.6 (ii) below), but whether the other implication is true is an open question.

Recall that a set $\{F_i ; i \in I\}$ of S-filters on a matrix model $\mathcal{A}$ of some k-deductive system S is said to be *directed* if for all $i, j \in I$ there exists a $k \in I$ such that $F_i \subseteq F_k$ and $F_j \subseteq F_k$. If $\{F_i ; i \in I\}$ is directed then, by Lemma 1.6.4 (i), $\bigcup_{i \in I} F_i$ is an S-filter of $\mathcal{A}$. The Leibniz operator $\Omega_{\mathcal{A}}$ is said to be *continuous* if $\Omega_{\mathcal{A}}\left(\bigcup_{i\in I}F_i\right) = \bigcup_{i\in I}\Omega_{\mathcal{A}}F_i$ for any directed set $\{F_i;\, i\in I\}$ of S-filters of $\mathcal{A}$.

2.5.6 THEOREM

(i) *A k-deductive system S is weakly congruential if and only if the Leibniz operator $\Omega_{\mathcal{A}}$ is monotonic for all S-matrices $\mathcal{A}$ and*

$$\Omega_{\mathfrak{B}}(F\cap B^k) = (\Omega_{\mathcal{A}}F)\cap B^2$$

for every S-matrix $\mathcal{A} = \langle \mathbf{A}, F\rangle$ and every submatrix $\mathfrak{B} = \langle \mathbf{B}, F\cap B^k\rangle$ of $\mathcal{A}$.

(ii) *If a k-deductive system S is congruential then the Leibniz operator $\Omega_{\mathcal{A}}:\mathrm{Fi}^S\mathcal{A}\to \mathrm{Con}\,\mathbf{A}$ is both monotonic and continuous for every S-matrix $\mathcal{A} = \langle \mathbf{A}, F_{\mathcal{A}}\rangle$.*

Proof. (i) ($\Leftarrow$) Since $\Omega_{\mathcal{A}}$ is monotonic for all $\mathcal{A}$, S is protoalgebraic (Theorem 2.1.3) and, by Theorem 2.4.5, there exists a generalized system of congruence k-formulas with parameters $\tilde{z}$ for S, say $\Delta_i(p,q,w_1,\ldots,w_{m_i},\tilde{z}\,)$, $i\in I$. Define $\{\Delta'_j(p,q);\, j\in J\}$ to be the set of all k-formulas of the form $\Delta_i(p,q,\varphi_1(p,q),\ldots,\varphi_{m_i}(p,q),\tau_1(p,q),\ldots,\tau_{k-1}(p,q))$, where $i\in I$ and $\varphi_1,\ldots,\varphi_{m_i}$, $\tau_1,\ldots,\tau_{k-1}$ are formulas that contain only the variables p and q. It follows exactly as in the proof of Theorem 2.5.5 (ii) that $\{\Delta'_j(p,q);\, j\in J\}$ is a system of congruence k-formulas without parameters for S.

($\Rightarrow$) Monotonicity of $\Omega_{\mathcal{A}}$ (for all $\mathcal{A}$) follows from Theorem 2.1.3 since weakly congruential implies protoalgebraic. Let $\Delta_i(p,q), i\in I$, be a system of congruence k-formulas without parameters for S, let $\mathcal{A} = \langle \mathbf{A}, F\rangle$ be an S-matrix, and let $\mathfrak{B} = \langle \mathbf{B}, F\cap B^k\rangle$ be a submatrix of $\mathcal{A}$. For all $a,b\in A$,

$$\begin{aligned}(a,b)\in\Omega_{\mathfrak{B}}(F\cap B^k) \quad &\text{iff}\quad (a,b)\in B^2 \text{ and } \Delta_i^{\mathbf{A}}(a,b)\in F\cap B^k \ \text{ for all } i\in I\\ &\text{iff}\quad (a,b)\in B^2 \text{ and } \Delta_i^{\mathbf{A}}(a,b)\in F \ \text{ for all } i\in I\\ &\text{iff}\quad (a,b)\in(\Omega_{\mathcal{A}}F)\cap B^2,\end{aligned}$$

hence $\Omega_{\mathfrak{B}}(F\cap B^k) = (\Omega_{\mathcal{A}}F)\cap B^2$.

(ii) Suppose that S is congruential and let $\Delta_1(p,q),\ldots,\Delta_n(p,q)$ be a finite system of congruence k-formulas without parameters for S. Monotonicity of $\Omega_{\mathcal{A}}$ (for all $\mathcal{A}$) follows since congruential implies protoalgebraic. Let $\mathcal{A} = \langle \mathbf{A}, F_{\mathcal{A}}\rangle$ be an S-matrix. Let $\{F_j;\, j\in J\}$ be a directed set of S-filters of $\mathcal{A}$. Set $G = \bigcup_{j\in J}F_j$. Since $\Omega_{\mathcal{A}}$ is monotonic, $\bigcup_{j\in J}\Omega_{\mathcal{A}}F_j\subseteq\Omega_{\mathcal{A}}G$. For the reverse inclusion, suppose $(a,b) \in \Omega_{\mathcal{A}} G$. Then $\Delta_i^{\mathbf{A}}(a,b) \in G$ for each $i \leq n$, hence there exist $j_i \in J$, for $i \leq n$, such that $\Delta_i^{\mathbf{A}}(a,b) \in F_{j_i}$. Since $\{F_j\,;\, j \in J\}$ is directed, there is an $\ell \in J$ such that $F_{j_i} \subseteq F_\ell$ for each $i \leq n$, hence $(a,b) \in \Omega_{\mathcal{A}} F_\ell \subseteq \bigcup_{j \in J} \Omega_{\mathcal{A}} F_j$. □

2.6 EXAMPLES

We present here an example from [BP92, p28] of a protoalgebraic 1-deductive system that is not weakly congruential. We show that the deductive system's class of reduced matrices is not closed under submatrices. Thus a k-protoquasivariety need not be closed under submatrices and in particular, by Theorem 2.5.5, its associated k-deductive system need not be weakly congruential.

Let $\mathcal{L} = \{\rightarrow\}$, where $\rightarrow$ is a binary connective. Let S be the (1-)deductive system over $\mathcal{L}$ defined by the axiom $p \rightarrow p$ and the inference rule $\langle\{p, p \rightarrow q\}, q\rangle$. Let $\mathbf{A} = \langle\{0,a,b,1\}\,;\, \rightarrow\rangle$ be an algebra of type $\langle 2\rangle$, where $\rightarrow$ is defined by the following table:

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	0	1	0	1
b	0	0	1	1
1	0	a	b	1

By definition, $\Delta(p,q) = p \rightarrow q$ forms a system of equivalence (1-)formulas for S, so it follows from Corollary 2.3.3 that S is protoalgebraic. Set $F = \{1, a\}$. It is easy to see that $\mathcal{A} = \langle \mathbf{A}, F\rangle$ is an S-matrix. We claim that $\mathcal{A}$ is reduced. Any congruence on $\mathbf{A}$ other than I_A which is compatible with F would have to contain $\Theta^{\mathbf{A}}(0,b)$ or $\Theta^{\mathbf{A}}(a,1)$. We have $(0,1) = (b \rightarrow a, 0 \rightarrow a) \in \Theta^{\mathbf{A}}(0,b)$, so $\Theta^{\mathbf{A}}(0,1) \subseteq \Theta^{\mathbf{A}}(0,b)$. Also, $(0,1) = (b \rightarrow a, b \rightarrow 1) \in \Theta^{\mathbf{A}}(a,1)$, so $\Theta^{\mathbf{A}}(0,1) \subseteq \Theta^{\mathbf{A}}(a,1)$. Since $1 \in F$ and $0 \notin F$, neither $\Theta^{\mathbf{A}}(0,b)$ nor $\Theta^{\mathbf{A}}(a,1)$ is compatible with F, hence $\Omega_{\mathcal{A}} F = I_A$, and $\mathcal{A}$ is reduced. Let $\mathbf{B}$ be the subalgebra of $\mathbf{A}$ with universe $\{0,a,1\}$. Then $\mathcal{B} = \langle \mathbf{B}, F\rangle$ is an S-matrix and a submatrix of $\mathcal{A}$. We have $\Theta^{\mathbf{B}}(a,1) = I_A \cup \{(a,1),(1,a)\}$, hence $\Omega_{\mathcal{B}} F = \Theta^{\mathbf{B}}(a,1)$, since $\Theta^{\mathbf{B}}(a,1)$ is the largest congruence of $\mathbf{B}$ compatible with F. Thus $\mathcal{B}$ is not reduced. This shows that $\mathrm{Mod}^* S$ is not closed under submatrices. By Theorem 2.5.5, S is not weakly congruential.

Next, we present a variation of an example from [BP92, p30] of a weakly congruential 1-deductive system that is not congruential. Let $\mathcal{L} = \{\wedge, \vee, \rightarrow, \neg, \Box, \bot, \top\}$ be the language of modal logics. Define $\Box^0 p = p$, and $\Box^i p = \Box\Box^{i-1} p$ for each positive integer i. Let S be the modal logic over the language $\mathcal{L}$ whose only inference rule is (MP) and whose axioms are:

(i) $\vdash_S \Box^i\varphi$ for all $i \in \omega$ whenever $\vdash_{\mathbf{IPC}}\varphi$,

(ii) $\vdash_S (\Box^i(p \rightarrow q)) \rightarrow ((\Box^i p) \rightarrow (\Box^i q))$ for all $i \in \omega$,

(iii) $\vdash_S \Box^{i+1}(p \leftrightarrow q) \rightarrow \Box^i((\Box p) \leftrightarrow (\Box q))$ for all $i \in \omega$,

where $p \leftrightarrow q$ abbreviates $(p \rightarrow q) \wedge (q \rightarrow p)$.

2.6.1 THEOREM

The deductive system S, defined above, is weakly congruential, with a system of congruence formulas without parameters for S given by $\{\Box^i(p \leftrightarrow q)\,;\, i \in \omega\}$.

Proof. Let $\mathcal{A} = \langle \mathbf{A}, F\rangle$ be an S-matrix. Let $\Phi = \{(a,b) \in A^2\,;\, \Box^i(a \leftrightarrow b) \in F$ for all $i \in \omega\}$. We shall show that Φ is a congruence relation on $\mathbf{A}$ that is compatible with F. Evidently Φ is reflexive and symmetric. Suppose $(a,b),(b,c) \in \Phi$. In Section 1.4, we showed that the following is a theorem of **IPC**:

$$(p \leftrightarrow q) \rightarrow [(q \leftrightarrow r) \rightarrow (p \leftrightarrow r)],$$

hence, by (i) $\vdash_S \Box^i((p \leftrightarrow q) \rightarrow [(q \leftrightarrow r) \rightarrow (p \leftrightarrow r)])$ for each $i \in \omega$.

By (ii) and (MP), $\vdash_S \Box^i(p \leftrightarrow q) \rightarrow [\Box^i(q \leftrightarrow r) \rightarrow \Box^i(p \leftrightarrow r)]$ for each $i \in \omega$,

hence, by (MP), $\Box^i(p \leftrightarrow q), \Box^i(q \leftrightarrow r) \vdash_S \Box^i(p \leftrightarrow r)$ for each $i \in \omega$.

Thus $\Box^i(a \leftrightarrow b), \Box^i(b \leftrightarrow c) \in F$ implies that $\Box^i(a \leftrightarrow c) \in F$ for all $i \in \omega$, i.e., $(a,c) \in \Phi$, so Φ is transitive.

Suppose $(a,b),(c,d) \in \Phi$. Using the following theorem of **IPC**, which was proved in Section 1.4:

$$(p \leftrightarrow q) \rightarrow [(r \leftrightarrow s) \rightarrow ((p \rightarrow r) \leftrightarrow (q \rightarrow s))],$$

it follows from (i), (ii) and (MP) that

$$\Box^i(p \leftrightarrow q), \Box^i(r \leftrightarrow s) \vdash_S \Box^i((p \rightarrow r) \leftrightarrow (q \rightarrow s)) \quad \text{for each } i \in \omega.$$

Thus $\Box^i(a \leftrightarrow b), \Box^i(c \leftrightarrow d) \in F$ implies that $\Box^i((a \rightarrow c) \leftrightarrow (b \rightarrow d)) \in F$ for each $i \in \omega$, i.e., $(a \rightarrow c, b \rightarrow d) \in \Phi$. It follows similarly that $(a \wedge c, b \wedge d) \in \Phi$ and $(a \vee c, b \vee d) \in \Phi$ from the following theorems of **IPC**, which were proved in Section 1.4:

$$(p \leftrightarrow q) \rightarrow [(r \leftrightarrow s) \rightarrow ((p \wedge r) \leftrightarrow (q \wedge s))],$$

$$(p \leftrightarrow q) \rightarrow [(r \leftrightarrow s) \rightarrow ((p \vee r) \leftrightarrow (q \vee s))].$$

Suppose $(a, b) \in \Phi$. Using the following theorem of **IPC**, also proved in Section 1.4:

$$(p \leftrightarrow q) \rightarrow ((\neg p) \leftrightarrow (\neg q)),$$

it follows from (i), (ii) and (MP) that

$$\Box^i(p \leftrightarrow q) \vdash_S \Box^i((\neg p) \leftrightarrow (\neg q)) \quad \text{for all } i \in \omega.$$

Thus $\Box^i(a \leftrightarrow b) \in F$ implies $\Box^i((\neg a) \leftrightarrow (\neg b)) \in F$ for each $i \in \omega$, i.e., $(\neg a, \neg b) \in \Phi$. From (iii) and (MP), we get

$$\Box^{i+1}(p \leftrightarrow q) \vdash_S \Box^i((\Box p) \leftrightarrow (\Box q)) \quad \text{for all } i \in \omega.$$

Thus $\Box^{i+1}(a \leftrightarrow b) \in F$ implies $\Box^i((\Box a) \leftrightarrow (\Box b)) \in F$ for each $i \in \omega$, i.e., $(\Box a, \Box b) \in \Phi$.

Thus Φ is a congruence on **A**. To see that Φ is compatible with F, suppose $a \in F$ and $(a, b) \in \Phi$. Then $a \leftrightarrow b \in F$ and $\vdash_{\mathbf{IPC}} (p \leftrightarrow q) \rightarrow (p \rightarrow q)$, hence $a \rightarrow b \in F$, so (MP) implies that $b \in F$. Consequently, $\Phi \subseteq \Omega_{\mathcal{A}} F$ by Theorem 1.7.3. Conversely, suppose $(a, b) \in \Omega_{\mathcal{A}} F$. Set $\varphi(p, q) = \Box^i(p \leftrightarrow q)$ for some $i \in \omega$. Interpreting p as a and q as b in A, Definition 1.7.1 states

$$\varphi^{\mathbf{A}}(a, b) \in F \quad \text{iff} \quad \varphi^{\mathbf{A}}(b, b) \in F,$$

i.e.,

$$\Box^i(a \leftrightarrow b) \in F \quad \text{iff} \quad \Box^i(b \leftrightarrow b) \in F.$$

Since $\vdash_{\mathbf{IPC}} p \leftrightarrow p$, we have $\vdash_S \Box^i(p \leftrightarrow p)$, hence $\Box^i(b \leftrightarrow b) \in F$. Thus $\Box^i(a \leftrightarrow b) \in F$, hence $(a, b) \in \Phi$, implying that $\Omega_{\mathcal{A}} F = \Phi$. This proves that $\{\Box^i(p \leftrightarrow q) ; i \in \omega\}$ forms a system of congruence formulas without parameters for S, hence that S is weakly congruential. $\Box$

2.6.2 THEOREM

The deductive system S, defined above, is not congruential.

Proof. We first prove the following Claim:

Claim: Let **A** be an $\mathcal{L}$-algebra whose $\{\wedge, \vee, \rightarrow, \bot, \mathrm{T}\}$-reduct is a linearly ordered Heyting algebra and whose $\Box$ operation has the following properties: $\Box \mathrm{T} = \mathrm{T}$, $\Box \bot = \bot$ and if $a \leq b$, then $\Box a \leq \Box b$ for all $a, b \in A$ (hence $\Box^i a \leq \Box^i b$ for all $i \in \omega$). Let $F \subseteq A$ be a filter of the underlying lattice of **A**. Then $\mathcal{A} = \langle \mathbf{A}, F \rangle$ is an S-matrix.

Proof. Since the $\{\wedge, \vee, \rightarrow, \bot, \mathrm{T}\}$-reduct of **A** is a linearly ordered Heyting algebra, we have that for all $a, b \in A$, $a \to b = \mathrm{T}$ if $a \leq b$, $a \to b = b$ if $a > b$, $\neg a = \bot$ if $a \neq \bot$ and $\neg \bot = \mathrm{T}$ (see Section 0.2). To prove that $\mathcal{A}$ is an S-matrix, it suffices to show that $\models_{\mathcal{A}} \varphi$ for each of the axioms φ of S and that $p, p \to q \models_{\mathcal{A}} q$. Whenever $\vdash_{\mathbf{IPC}} \varphi$, $i \in \omega$ and $\bar{a}$ is an interpretation of the variables of φ in A, we have $\varphi^{\mathbf{A}}(\bar{a}) = \mathrm{T} \in F$ and so $\Box^i \varphi^{\mathbf{A}}(\bar{a}) = \Box^i \mathrm{T} = \mathrm{T} \in F$, hence $\models_{\mathcal{A}} \varphi$ for each of the axioms φ stated in (i). Let a, b be an interpretation of the variables p, q in A. Note that if $a \leq b$, then $a \leftrightarrow b = (a \to b) \wedge (b \to a) = \mathrm{T} \wedge a = a$.

(ii) Assume first that $a \leq b$. Thus $\Box^i(a \to b) = \Box^i \mathrm{T} = \mathrm{T}$ and $\Box^i a \to \Box^i b = \mathrm{T}$, hence $\Box^i(a \to b) \to (\Box^i a \to \Box^i b) = \mathrm{T} \in F$. Now suppose that $a > b$. Then $\Box^i(a \to b) = \Box^i b$ and $\Box^i a \to \Box^i b = \Box^i b$, hence $\Box^i(a \to b) \to (\Box^i a \to \Box^i b) = \mathrm{T} \in F$.

(iii) Assume, without loss of generality that $a \leq b$, hence $\Box a \leq \Box b$. Thus $a \leftrightarrow b = a$ and $(\Box a) \leftrightarrow (\Box b) = \Box a$. So, $(\Box^{i+1}(a \leftrightarrow b)) \to \Box^i((\Box a) \leftrightarrow (\Box b)) = \Box^{i+1} a \to \Box^i(\Box a) = \Box^{i+1} a \to \Box^{i+1} a = \mathrm{T} \in F$.

Lastly, consider (MP). Suppose $a \in F$ and $a \to b \in F$. If $a \leq b$ then $b \in F$ since F is a lattice filter, and if $a > b$ then $b = a \to b \in F$. This proves the Claim.

For each integer n, where $n \geq 2$, let $C_n = \{0, 1, 2, \ldots, n\}$. Let $\mathbf{C}_n$ be the $\mathcal{L}$-algebra whose $\{\wedge, \vee, \to, \bot, \mathrm{T}\}$-reduct is the $(n+1)$-element linearly ordered Heyting algebra with universe C_n (hence $\mathrm{T}^{\mathbf{C}_n} = n$ and $\bot^{\mathbf{C}_n} = 0$) and whose $\Box$ operation is defined as follows: $\Box n = n$, $\Box 0 = 0$ and $\Box a = a - 1$ for $0 < a < n$. Let $F_n = \{1, 2, \ldots, n\}$. Note that if $a \leq b$, then $\Box^i a \leq \Box^i b$ for all $i \in \omega$. It follows from the previous claim that $\mathfrak{C}_n = \langle \mathbf{C}_n, F_n \rangle$ is an S-matrix for each n. Each $\mathfrak{C}_n$ is also reduced, as is shown in the next paragraph.

The only congruences on $\mathbf{C}_n$ that might be compatible with F_n are I_{C_n} and congruences of the form $\Theta^{\mathbf{C}_n}(X)$, where $X \subseteq C_n - \{0\}$. So, let $\Phi = \Theta^{\mathbf{C}_n}(X)$, where $X \subseteq C_n - \{0\}$ and let $a, b \in C_n - \{0\}$ such that $a < b$ and $(a, b) \in \Phi$. Then $(0, \Box^a b) = (\Box^a a, \Box^a b) \in \Phi$, where $\Box^a b \neq 0$, hence Φ is not compatible with F_n. Thus $\Omega_{\mathfrak{C}_n} F_n = I_{C_n}$, and $\mathfrak{C}_n$ is reduced.

Let $\mathcal{U}$ be a free ultrafilter over $I = \{2, 3, 4, \ldots\}$. Let $\mathfrak{C} = \langle \mathbf{C}, F \rangle = \langle \Pi_{n \in I} \mathbf{C}_n / \mathcal{U}, \Pi_{n \in I} F_n / \mathcal{U} \rangle$ be the matrix ultraproduct of $\{\mathfrak{C}_n ; n \in I\}$. Note that $\langle 1, 1, 1, \ldots \rangle / \mathcal{U} \in F$.

Since each of the defining identities of a Heyting algebra is a first-order formula, it follows from Los' Theorem (Theorem 0.5.1) that the $\{\wedge, \vee, \to, \bot, \mathrm{T}\}$-reduct of $\mathbf{C}$ is a Heyting algebra.

Also, by Łos' Theorem, it is linearly ordered by the relation $a \leq b$ iff $a \rightarrow b = \mathrm{T}$, since each $\mathbf{C}_n$ is, and we can describe the property of being linearly ordered by the first-order sentence

$$\forall x\, \forall y\, ((x \rightarrow y = \mathrm{T}) \sqcup (y \rightarrow x = \mathrm{T})).$$

Similarly the $\Box$ operation of $\mathbf{C}$ satisfies $\Box\mathrm{T} = \mathrm{T}$, $\Box\bot = \bot$ and if $a \leq b$, then $\Box^i a \leq \Box^i b$ for all $a, b \in C$ and $i \in \omega$.

Claim: The S-matrix $\mathfrak{C}$ is not reduced.

Proof. For each $n \in \omega$, define $n^* \in \Pi_{n \in I} C_n$ and $\overline{n} \in C$ by

$$n^*(i) = n \ \text{ if } n \leq i\,;\ n^*(i) = i \text{ otherwise},$$

$$\overline{n} = n^*/\mathcal{U}.$$

For each $n \in \omega$, $\overline{n} < \overline{n+1}$ because $\{i \in I\,;\, n^*(i) < (n+1)^*(i)\} = \{i \in I\,;\, i \geq n+1\} \in \mathcal{U}$ (since $\mathcal{U}$ contains the cofinite subsets of I, by Corollary 0.3.4). We claim that there does not exist an $x \in C$ such that $\overline{n} < x < \overline{n+1}$. For suppose otherwise, say $\overline{n} < y/\mathcal{U} < \overline{n+1}$ ($y \in \Pi_{i \in I} C_i$). Then

$$J = \{i \in I\,;\, n^*(i) < y(i) < (n+1)^*(i)\} \in \mathcal{U}.$$

So, for each $i \in J$, if $i \geq n+1$ then $n < y(i) < n+1$. This is impossible, so $J \cap \{i \in I\,;\, i \geq n+1\} = \emptyset$. But $\{i \in I\,;\, i \geq n+1\}$ is cofinite, hence is in $\mathcal{U}$, implying that $\emptyset \in \mathcal{U}$! Thus, in $\mathbf{C}$, $\bot^{\mathbf{C}} = \overline{0}$ and (recall that $\prec$ is the covering symbol)

(2.6.1) $$\overline{0} \prec \overline{1} \prec \overline{2} \prec \ldots,$$

i.e., for each $n \in \omega$, $\overline{n}$ is covered by $\overline{n+1}$. Let $N = \{\overline{n}\,;\, n \in \omega\}$ and $G = C - N$. Note that $G \subseteq F$ since $\overline{1} \in F$ and C is a chain. We show that G is an 'open filter' of $\mathbf{C}$, i.e., a filter of the lattice $\langle C\,;\, \leq \rangle$ such that $\Box g \in G$ whenever $g \in G$.

We first show that $G \neq \emptyset$, by showing that $e/\mathcal{U} \in G$, where $e = \langle 1, 2, 3, \ldots \rangle \in \Pi_{i \in I} C_i$. Suppose on the contrary, that $e/\mathcal{U} = \overline{m}$, where $m \in \omega$. This means that if $X = \{i \in I\,;\, e(i) = m^*(i)\}$ then $X \in \mathcal{U}$. But $X = \{i \in I\,;\, i - 1 = m\} \subseteq \{m+1\}$, so $I - X$ is cofinite, hence $I - X \in \mathcal{U}$. But then $\emptyset = X \cap (I - X) \in \mathcal{U}$! So $e \in G$, hence $G \neq \emptyset$.

Since $\langle C\,;\, \leq \rangle$ is a chain, it follows from (2.6.1) that G is a lattice filter of $\langle C\,;\, \leq \rangle$. Since $\mathrm{T}^{\mathbf{C}}$ is the greatest element of $\langle C\,;\, \leq \rangle$ (by Łos' Theorem), we therefore have $\mathrm{T}^{\mathbf{C}} \in G$. Suppose $a \in \Pi_{n \in I} C_n$ such that $a/\mathcal{U} \in G$ but $\Box(a/\mathcal{U}) \notin G$, say $\Box(a/\mathcal{U}) = \overline{n}$ for some $n \in \omega$. If $n = 0$ then $\Box(a/\mathcal{U}) = \overline{0}$ so for some $U \in \mathcal{U}$, we have $(\Box a)(i) = 0$ for all $i \in U$, hence $a(i) = 0$ or $a(i) = 1$ for all $i \in U$. Let $U_0 = \{i \in U\,;\, a(i) = 0\}$, $U_1 = \{i \in U\,;\, a(i) = 1\}$. We claim that $U_0 \in \mathcal{U}$ or $U_1 \in \mathcal{U}$.

Otherwise $I-U_0, I-U_1 \in \mathfrak{U}$, so $I-U = I-(U_0 \cup U_1) = (I-U_0) \cap (I-U_1) \in \mathfrak{U}$, implying that $\emptyset = U \cap (I-U) \in \mathfrak{U}$, which is a contradiction. If $U_0 \in \mathfrak{U}$ then $a/\mathfrak{U} = \overline{0} \notin G$, and if $U_1 \in \mathfrak{U}$ then $a/\mathfrak{U} = \overline{1} \notin G$, hence $n \neq 0$. Now, for every $n \in \omega$,

$$\mathfrak{C}_n \models \forall x \, \forall y \, (\Box x \approx \Box y \not\approx \bot \Rightarrow x \approx y),$$

hence $$\mathfrak{C} \models \forall x \, \forall y \, (\Box x \approx \Box y \not\approx \bot \Rightarrow x \approx y) \qquad \text{[by Łos' Theorem].}$$

We have $\Box(a/\mathfrak{U}) = \overline{n} = \Box\overline{(n+1)} \neq \overline{0} = \bot$ (in **C**), so $a/\mathfrak{U} = \overline{(n+1)} \notin G$, a contradiction. This proves that $\Box(a/\mathfrak{U}) \in G$.

Now, define

$$\Phi = \{(a,b) \in C^2 \,;\, a \leftrightarrow b \in G\}.$$

Since **C** has a Heyting algebra as a reduct, Φ is a congruence of this Heyting algebra by Theorem 0.3.1. Moreover, by Łos' Theorem, we have, for any $a, b \in C$,

(2.6.2) $$a \leftrightarrow b = \mathrm{T}^{\mathbf{C}} \text{ if } a = b \text{ and } a \leftrightarrow b = a \wedge b \text{ otherwise.}$$

From (2.6.2) and the fact that G is an open filter of **C**, it follows easily that Φ is compatible with $\Box$ also, i.e., that $\Phi \in \mathrm{Con}\,\mathbf{C}$. We claim that Φ is compatible with F. Since each F_n is a filter of $\langle C_n ; \leq \rangle$, it follows easily from Łos' Theorem that F is a filter of $\langle C ; \leq \rangle$, and since $\overline{0} \notin F$ and $\overline{1} \in F$, we have $F = \{c \in C \,;\, 1 \leq c\}$. So we have to show that for any $(a,b) \in \Phi$, if $a \neq \overline{0}$ then $b \neq \overline{0}$. Suppose to the contrary, that $(a, \overline{0}) \in \Phi$ and that $a \neq \overline{0}$. Then

$$\overline{0} = \overline{0} \wedge a = \overline{0} \leftrightarrow a \in G,$$

a contradiction. This shows that Φ is compatible with F, so $\Phi \subseteq \Omega_{\mathfrak{C}} F$. We need to know that $\Phi \neq I_C$. We showed that $\mathrm{T}^{\mathbf{C}}, e/\mathfrak{U} \in G$ where $e = \langle 1,2,3,\ldots \rangle$. Note that $e/\mathfrak{U} \neq \mathrm{T}^{\mathbf{C}}$ since $\{i \in I \,;\, e(i) = i\} = \emptyset \notin \mathfrak{U}$. Since $e/\mathfrak{U} \wedge \mathrm{T}^{\mathbf{C}} = e/\mathfrak{U} \in G$, we have $(e/\mathfrak{U}, \mathrm{T}^{\mathbf{C}}) \in \Phi$, so $\Phi \neq I_C$. Thus $\Omega_{\mathfrak{C}} F \neq I_C$, i.e., $\mathfrak{C}$ is not reduced, hence $\mathrm{Mod}^* S$ is not closed under $\mathrm{P_U}$. By Theorem 2.5.5 (iii), S is not congruential. $\Box$

Lastly we provide some examples of congruential deductive systems. In the next chapter we introduce the notion of an 'algebraizable deductive system' and we shall show that an 'algebraizable deductive system' is also congruential. As will be proved in Section 3.3, the examples given here are not algebraizable.

2.6.3 THEOREM

A pure implicational logic satisfying (I), (B), (B′) *and* (MP) *is congruential with a finite system of congruence formulas without parameters given by* $\Delta_1(p,q) = p \rightarrow q$, $\Delta_2(p,q) = q \rightarrow p$.

Proof. Let S be a pure implicational logic satisfying (I), (B), (B′) and (MP), and let $\mathcal{A} = \langle \mathbf{A}, F\rangle$ be an S-matrix. Set

$$\Phi = \{(a,b) \in A^2;\ a \rightarrow b \in F \ \text{ and } \ b \rightarrow a \in F\}$$

(we write $\rightarrow$ for $\rightarrow^{\mathbf{A}}$). We shall use Theorem 1.7.3 to show that $\Omega_{\mathcal{A}} F = \Phi$. First we show that Φ is a congruence on $\mathbf{A}$. By (I), $\vdash_S p \rightarrow p$, hence $a \rightarrow a \in F$ for all $a \in A$, so Φ is reflexive. It follows trivially that Φ is symmetric. To see that Φ is transitive, consider the following:

$$\vdash_S (q \rightarrow p) \rightarrow ((r \rightarrow q) \rightarrow (r \rightarrow p)) \qquad \text{[by (B)]},$$

so
$$q \rightarrow p \vdash_S (r \rightarrow q) \rightarrow (r \rightarrow p) \qquad \text{[by (MP)]},$$

hence
$$q \rightarrow p, r \rightarrow q \vdash_S r \rightarrow p \qquad \text{[by (MP)]}.$$

It follows similarly, using (B′) rather than (B), that $p \rightarrow q, q \rightarrow r \vdash_S p \rightarrow r$, so

$$p \rightarrow q, q \rightarrow p, q \rightarrow r, r \rightarrow q \vdash_S p \rightarrow r, r \rightarrow p.$$

Let $(a,b),(b,c) \in \Phi$, i.e., $a \rightarrow b, b \rightarrow a, b \rightarrow c, c \rightarrow b \in F$. Interpreting p as a, q as b and r as c in A, we can deduce from the above that $a \rightarrow c, c \rightarrow a \in F$, i.e., $(a,c) \in \Phi$.

Next, we need that if $(a,b),(c,d) \in \Phi$, then $(a \rightarrow c, b \rightarrow d) \in \Phi$. Now,

$$\vdash_S (q \rightarrow p) \rightarrow ((p \rightarrow r) \rightarrow (q \rightarrow r)) \qquad \text{[by (B′)]}$$

and
$$\vdash_S (p \rightarrow q) \rightarrow ((q \rightarrow r) \rightarrow (p \rightarrow r)) \qquad \text{[by (B′)]},$$

hence, by (MP),

(2.6.3) $$p \rightarrow q, q \rightarrow p \vdash_S (p \rightarrow r) \rightarrow (q \rightarrow r), (q \rightarrow r) \rightarrow (p \rightarrow r).$$

Now,
$$\vdash_S (r \rightarrow s) \rightarrow ((q \rightarrow r) \rightarrow (q \rightarrow s)) \qquad \text{[by (B)]},$$

and
$$\vdash_S (s \rightarrow r) \rightarrow ((q \rightarrow s) \rightarrow (q \rightarrow r)). \qquad \text{[by (B)]},$$

therefore, by (MP),

(2.6.4) $$r \rightarrow s, s \rightarrow r \vdash_S (q \rightarrow r) \rightarrow (q \rightarrow s), (q \rightarrow s) \rightarrow (q \rightarrow r).$$

Interpreting p as a, q as b, r as c and s as d in A, (2.6.3) and (2.6.4) imply that $(a \rightarrow c) \rightarrow (b \rightarrow c)$, $(b \rightarrow c) \rightarrow (a \rightarrow c), (b \rightarrow c) \rightarrow (b \rightarrow d), (b \rightarrow d) \rightarrow (b \rightarrow c) \in F$, hence $(a \rightarrow c, b \rightarrow c), (b \rightarrow c, b \rightarrow d) \in \Phi$. By the transitivity of Φ, therefore, $(a \rightarrow c, b \rightarrow d) \in \Phi$. Thus Φ is a congruence on $\mathbf{A}$. To see that Φ is compatible with F, suppose $a \in F$ and $(a,b) \in \Phi$. Then $a \rightarrow b \in F$, hence it follows from (MP) that

$b \in F$. Thus, by Theorem 1.7.3, $\Phi \subseteq \Omega_{\mathcal{A}} F$.

Now, suppose $(a,b) \in \Omega_{\mathcal{A}} F$. Set $\varphi(p,q) = p \to q$. By Definition 1.7.1, $\varphi^{\mathbf{A}}(a,b) \in F$ if and only if $\varphi^{\mathbf{A}}(a,a) \in F$. Now, $\varphi^{\mathbf{A}}(a,a) = a \to a \in F$, by (I), hence $\varphi^{\mathbf{A}}(a,b) = a \to b \in F$. Similarly, $b \to a \in F$, hence $(a,b) \in \Phi$ and so $\Omega_{\mathcal{A}} F \subseteq \Phi$. Thus $\Omega_{\mathcal{A}} F = \Phi$, i.e.,

$$(a,b) \in \Omega_{\mathcal{A}} F \text{ if and only if } \Delta_1^{\mathbf{A}}(a,b), \Delta_2^{\mathbf{A}}(a,b) \in F,$$

so $\Delta_1(p,q)$, $\Delta_2(p,q)$ form a finite system of congruence formulas without parameters for S. □

2.6.4 COROLLARY

The pure implicational logics **BCI**, $\mathbf{R}_{\to}$, $\mathbf{E}_{\to}$ *and* $S5^{W}_{\to_{\mathbf{E}}}$ *are congruential deductive systems with congruence formulas* $\Delta_1(p,q) = p \to q$, $\Delta_2(p,q) = q \to p$ *(replace* $\to$ *by* $\to_{\mathbf{E}}$ *in* $S5^{W}_{\to_{\mathbf{E}}}$*).*

Proof. In Section 1.4, it was shown that (B′) is derivable by (MP) from (B) and (C), hence the previous theorem implies that **BCI** and $\mathbf{R}_{\to}$ are congruential. To see that $\mathbf{E}_{\to}$ is congruential, we need only show that it satisfies (B). By (C′) (replacing p by $r \to p$, s by p, t by q and q by $r \to q$),

$$\vdash_{\mathbf{E}_{\to}} [(r \to p) \to ((p \to q) \to (r \to q))] \to [(p \to q) \to ((r \to p) \to (r \to q))],$$

hence (B′) and (MP) imply $\vdash_{\mathbf{E}_{\to}} (p \to q) \to ((r \to p) \to (r \to q))$, i.e., $\mathbf{E}_{\to}$ satisfies (B). Since $S5^{W}_{\to_{\mathbf{E}}}$ is an axiomatic extension of $\mathbf{E}_{\to}$ (see Section 1.4), it is congruential and its congruence formulas are given by $\Delta_1(p,q) = p \to_{\mathbf{E}} q$, $\Delta_2(p,q) = q \to_{\mathbf{E}} p$. By the definition of $\to_{\mathbf{E}}$, this system is equivalent to $\Delta_1(p,q) = \Box(p \to q)$, $\Delta_2(p,q) = \Box(q \to p)$. □

Note that we cannot deduce that the $\{\to\}$-fragment of $\mathbf{S5}^{W}$ is congruential from the fact that $\mathbf{E}_{\to}$ is congruential because in extending **E** to $\mathbf{S5}^{W}$, the $\to$ connective of **E** (which we denoted $\to_{\mathbf{E}}$) is added to the language of $\mathbf{S5}^{W}$, being defined by $\varphi \to_{\mathbf{E}} \psi = \Box(\varphi \to_{\mathbf{W}} \psi)$, where $\to_{\mathbf{W}}$ is the $\to$ connective of $\mathbf{S5}^{W}$. The $\{\to_{\mathbf{E}}\}$-reduct of the enriched deductive system $\mathbf{S5}^{W}$ is congruential with congruence formulas $\Delta_1(p,q) = \Box(p \to q)$, $\Delta_2(p,q) = \Box(q \to p)$.

2.6.5 COROLLARY

The deductive systems **E** *and* $\mathbf{S5}^{W}$ *are congruential.*

Proof. We know that $\mathbf{E}_{\to}$ is congruential with congruence formulas $\Delta(p,q) = \{\Delta_1(p,q), \Delta_2(p,q)\}$, where $\Delta_1(p,q) = p \to q$ and $\Delta_2(p,q) = q \to p$. By Definition 2.4.1 and

Proposition 2.4.3, all that is needed to show that $\mathbf{E}$ is congruential is that

$$\Delta(p,q),\Delta(r,s)\vdash_{\mathbf{E}}\Delta(f(p,r),f(q,s)),$$

where f is either $\wedge$ or $\vee$, and

$$\Delta(p,q)\vdash_{\mathbf{E}}\Delta(\neg p,\neg q).$$

Consider $\wedge$: Since $\vdash_{\mathbf{E}}((p\wedge r)\to p)\to[(p\to q)\to((p\wedge r)\to q)]$, by ($E_2$), and $\vdash_{\mathbf{E}}(p\wedge r)\to p$, by ($E_5$), (MP) implies that $\vdash_{\mathbf{E}}(p\to q)\to((p\wedge r)\to q)$. Thus, by (MP) again,

$$p\to q\vdash_{\mathbf{E}}(p\wedge r)\to q.$$

Similarly $\vdash_{\mathbf{E}}((p\wedge r)\to r)\to[(r\to s)\to((p\wedge r)\to s)]$, by ($E_2$) and $\vdash_{\mathbf{E}}(p\wedge r)\to r$, by ($E_6$), so $\vdash_{\mathbf{E}}(r\to s)\to((p\wedge r)\to s)$ by (MP). Thus, by (MP) again,

$$r\to s\vdash_{\mathbf{E}}(p\wedge r)\to s.$$

Applying the inference rule (A), we get

$$p\to q,r\to s\vdash_{\mathbf{E}}((p\wedge r)\to q)\wedge((p\wedge r)\to s).$$

By (E_7),

$$\vdash_{\mathbf{E}}[((p\wedge r)\to q)\wedge((p\wedge r)\to s)]\to[(p\wedge r)\to(q\wedge s)],$$

hence, by (MP),

$$p\to q,r\to s\vdash_{\mathbf{E}}(p\wedge r)\to(q\wedge s).$$

Similarly, $q\to p,s\to r\vdash_{\mathbf{E}}(q\wedge s)\to(p\wedge r)$, hence $\Delta(p,q),\Delta(r,s)\vdash_{\mathbf{E}}\Delta(p\wedge r,q\wedge s)$.

Consider $\vee$: By (E_2), $\vdash_{\mathbf{E}}(p\to q)\to[(q\to(q\vee s))\to(p\to(q\vee s))]$, hence, by (MP), $p\to q\vdash_{\mathbf{E}}$ $(q\to(q\vee s))\to(p\to(q\vee s))$. By (MP) and ($E_8$), we get

$$p\to q\vdash_{\mathbf{E}}p\to(q\vee s).$$

Similarly, by (E_2), $\vdash_{\mathbf{E}}(r\to s)\to[(s\to(q\vee s))\to(r\to(q\vee s))]$, hence, by (MP), $r\to s\vdash_{\mathbf{E}}$ $(s\to(q\vee s))\to(r\to(q\vee s))$. By (MP) and ($E_9$), we get

$$r\to s\vdash_{\mathbf{E}}r\to(q\vee s).$$

Applying the inference rule (A), we get

$$p\to q,r\to s\vdash_{\mathbf{E}}(p\to(q\vee s))\wedge(r\to(q\vee s)).$$

Now, by (E_{10}),

$$\vdash_{\mathbf{E}}[(p\to(q\vee s))\wedge(r\to(q\vee s))]\to[(p\vee r)\to(q\vee s)],$$

hence we can deduce, using (MP), that

$$p\to q,r\to s\vdash_{\mathbf{E}}(p\vee r)\to(q\vee s).$$

Similarly, $q \to p, s \to r \vdash_{\mathbf{E}} (q \vee s) \to (p \vee r)$, hence $\Delta(p,q), \Delta(r,s) \vdash_{\mathbf{E}} \Delta(p \vee r, q \vee s)$.

Consider $\neg$: First note that, by (E_{13}), $\vdash_{\mathbf{E}} [(\neg q) \to (\neg q)] \to [q \to (\neg\neg q)]$. Using (MP) and ($E_1$), we can deduce that $\vdash_{\mathbf{E}} q \to (\neg\neg q)$. Now, by ($E_3$),

$$\vdash_{\mathbf{E}} [q \to (\neg\neg q)] \to [(p \to q) \to (p \to (\neg\neg q))],$$

hence, by (MP), $\vdash_{\mathbf{E}} (p \to q) \to (p \to (\neg\neg q))$. By ($E_3$),

$$\vdash_{\mathbf{E}} [(p \to (\neg\neg q)) \to ((\neg q) \to (\neg p))] \to [((p \to q) \to (p \to (\neg\neg q))) \to ((p \to q) \to ((\neg q) \to (\neg p)))].$$

Since $\vdash_{\mathbf{E}} (p \to (\neg\neg q)) \to ((\neg q) \to (\neg p))$, by ($E_{13}$), (MP), applied twice, gives us

$$\vdash_{\mathbf{E}} (p \to q) \to ((\neg q) \to (\neg p)).$$

Finally, (MP) gives us $p \to q \vdash_{\mathbf{E}} (\neg q) \to (\neg p)$. Similarly, $q \to p \vdash_{\mathbf{E}} (\neg p) \to (\neg q)$, hence, $\Delta(p,q) \vdash_{\mathbf{E}} \Delta(\neg p, \neg q)$.

Thus **E** is congruential. It was noted in Section 1.4 that $\mathbf{S5}^{\boldsymbol{W}}$ is an axiomatic extension of **E**, hence it too is congruential. It follows from Section 1.4 that a system of congruence formulas for $\mathbf{S5}^{\boldsymbol{W}}$ is given by $\Delta_1(p,q) = \Box(p \to q)$, $\Delta_2(p,q) = \Box(q \to p)$. $\Box$

Chapter 3
Algebraizable Deductive Systems

The notion of an 'equivalent algebraic semantics' for a deductive system is defined by Blok and Pigozzi in [BP89a]. The aim of their definition is to associate a class of algebras with a given deductive system in such a way that the consequence relation of the deductive system reflects and is reflected in the equational theory of the class of algebras. This allows one to use universal algebraic results to produce results about the deductive system in question. In fact, the nature of the 'equivalence' between a deductive system and its corresponding class of algebras allows one to produce results about the class of algebras from the deductive system as well (as we shall exhibit in Chapter 5). Blok and Pigozzi applied the term 'algebraizable deductive system' to those deductive systems for which an equivalent algebraic semantics exists. In this chapter we investigate 'algebraizable deductive systems'. For this reason, much of the material contained here comes from the Memoir [BP89a]. Blok and Pigozzi's investigation into the subject continued in [BP89b], [BP88] and [BP92] and in this chapter we collect a number of the characterizations of algebraizable deductive systems that appear in these papers. In fact, we combine all these characterizations in one theorem, namely Theorem 3.1.11. Following that theorem, we present results that focus more explicitly on the equivalent quasivariety semantics (Definition 3.1.4) and the 'defining equations' (Definition 3.1.2) and 'equivalence formulas' (Definition 3.1.4) of such a deductive system. We also consider fragments and extensions of deductive systems.

Section 3.2 is concerned with a special class of algebraizable 1-deductive system, namely those that satisfy the so-called 'Gödel rule' (G-rule, for short). This class of 1-deductive systems contains a number of well-known deductive systems (as will be pointed out in Section 3.3). Of interest to us is the fact that the logical property of satisfying the G-rule corresponds to an algebraic property of the equivalent quasivariety semantics. In particular, the equivalent quasivariety semantics of an algebraizable 1-deductive system that has the G-rule is relatively T-regular (as defined in Section 0.4) with respect to a constant term T. Conversely, every relatively T-regular quasivariety is the equivalent quasivariety semantics of an algebraizable 1-deductive system that has the G-rule.

Finally, Section 3.3 provides a number of examples of deductive systems that exemplify some of the results of the first two sections.

3.1 ALGEBRAIZABLE $\mathcal{k}$-DEDUCTIVE SYSTEMS

Unless stated otherwise, we shall assume that a language $\mathcal{L}$ is fixed and that k is a fixed, nonzero natural number. Let $\mathcal{K}$ be a quasivariety of type $\mathcal{L}$, and recall from Section 0.2 that $\mathrm{Id}(P)$ is the set of all identities $\varphi \approx \psi$, where φ and ψ are $\mathcal{L}$-terms over P (i.e., $\mathcal{L}$-formulas). We have the following:

3.1.1 LEMMA

Let $\mathcal{K}$ be a class of $\mathcal{L}$-algebras. Recall that $\mathcal{K}^{\mathrm{Q}}$ is the quasivariety generated by $\mathcal{K}$.

(i) $\models_{\mathcal{K}}$ *is structural, (i.e., for all $\Gamma \subseteq \mathrm{Id}(P)$, $\varphi \approx \psi \in \mathrm{Id}(P)$ and each substitution σ, if $\Gamma \models_{\mathcal{K}} \varphi \approx \psi$ then $\sigma(\Gamma) \models_{\mathcal{K}} \sigma\varphi \approx \sigma\psi$).* [BP89a, Lemma 2.1]

(ii) $\models_{\mathcal{K}}$ *is finitary if and only if $\models_{\mathcal{K}}$ coincides with $\models_{\mathcal{K}^{\mathrm{Q}}}$. (We say that $\models_{\mathcal{K}}$ is finitary if for all $\Gamma \subseteq \mathrm{Id}(P)$ and $\varphi \approx \psi \in \mathrm{Id}(P)$, $\Gamma \models_{\mathcal{K}} \varphi \approx \psi$ implies that there exists a finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \models_{\mathcal{K}} \varphi \approx \psi$.)*

Proof. (i) Let $\mathbf{A} \in \mathcal{K}$. For a formula $\vartheta(\overline{p}) = \vartheta(p_1, p_2, \ldots)$ (i.e., an $\mathcal{L}$-term over P), a substitution σ and an interpretation $\overline{a} = a_1, a_2, \ldots$ (with $\overline{a}(p_i) = a_i$ for each i) of the variables of ϑ in A, we have $(\sigma\vartheta(\overline{p}))^{\mathbf{A}}(\overline{a}) = (\vartheta(\overline{\sigma p}))^{\mathbf{A}}(\overline{a}) = \vartheta^{\mathbf{A}}(\overline{(\sigma p)^{\mathbf{A}}(\overline{a})})$. Now, suppose $\Gamma \models_{\mathcal{K}} \varphi \approx \psi$ and that $\overline{a}$ is an interpretation of the variables $\overline{p}$ of $\Gamma \cup \{\varphi, \psi\}$ in A. If $(\sigma\zeta)^{\mathbf{A}}(\overline{a}) = (\sigma\eta)^{\mathbf{A}}(\overline{a})$ for all $\zeta \approx \eta \in \Gamma$, then $\zeta^{\mathbf{A}}(\overline{(\sigma p)^{\mathbf{A}}(\overline{a})}) = \eta^{\mathbf{A}}(\overline{(\sigma p)^{\mathbf{A}}(\overline{a})})$ for all $\zeta \approx \eta \in \Gamma$, hence $\varphi^{\mathbf{A}}(\overline{(\sigma p)^{\mathbf{A}}(\overline{a})}) = \psi^{\mathbf{A}}(\overline{(\sigma p)^{\mathbf{A}}(\overline{a})})$ since $\mathbf{A} \in \mathcal{K}$ and the map $p_i \mapsto (\sigma p_i)^{\mathbf{A}}(\overline{a})$ is an interpretation of P in A. This means that $(\sigma\varphi)^{\mathbf{A}}(\overline{a}) = (\sigma\psi)^{\mathbf{A}}(\overline{a})$, hence $\{\sigma\zeta \approx \sigma\eta\,;\, \zeta \approx \eta \in \Gamma\} \models_{\mathcal{K}} \sigma\varphi \approx \sigma\psi$, implying that $\models_{\mathcal{K}}$ is structural.

(ii) Suppose that $\models_{\mathcal{K}}$ is finitary. Since $\mathcal{K} \subseteq \mathcal{K}^{\mathrm{Q}}$, we have $\models_{\mathcal{K}^{\mathrm{Q}}} \subseteq \models_{\mathcal{K}}$. For the reverse inclusion, suppose that $\Gamma \models_{\mathcal{K}} \varphi \approx \psi$. Then there exists a finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \models_{\mathcal{K}} \varphi \approx \psi$. If $\Gamma' = \{\zeta_1 \approx \eta_1, \ldots, \zeta_n \approx \eta_n\}$ then $\Gamma' \models_{\mathcal{K}} \varphi \approx \psi$ is equivalent to (see Section 0.4)

$$\models_{\mathcal{K}} (\zeta_1 \approx \eta_1 \ \& \ \ldots \ \& \ \zeta_n \approx \eta_n) \Rightarrow \varphi \approx \psi.$$

This is a quasi-identity, hence it is also satisfied by $\mathcal{K}^{Q}$, from which we deduce that $\Gamma' \models_{\mathcal{K}^Q} \varphi \approx \psi$, which implies that $\Gamma \models_{\mathcal{K}^Q} \varphi \approx \psi$, hence $\models_{\mathcal{K}} \subseteq \models_{\mathcal{K}^Q}$.

Conversely, suppose $\models_{\mathcal{K}}$ coincides with $\models_{\mathcal{K}^Q}$ and that $\Gamma \models_{\mathcal{K}} \varphi \approx \psi$. Then $\Gamma \models_{\mathcal{K}^Q} \varphi \approx \psi$. By the Lemma 0.4.4, we have that for each $\mathbf{A} \in \mathcal{K}$ and each interpretation $\overline{a}$ of the variables of $\Gamma \cup \{\varphi, \psi\}$ in A, $\Theta^{\mathbf{A}}_{\mathcal{K}}(\{(\zeta^{\mathbf{A}}(\overline{a}), \eta^{\mathbf{A}}(\overline{a}))\,;\, \zeta \approx \eta \in \Gamma\}) \supseteq \Theta^{\mathbf{A}}_{\mathcal{K}}(\varphi^{\mathbf{A}}(\overline{a}), \psi^{\mathbf{A}}(\overline{a}))$. Since the lattice $\mathbf{Con}_{\mathcal{K}}\mathbf{A}$ is algebraic (Proposition 0.4.3) and $\Theta^{\mathbf{A}}_{\mathcal{K}}(\varphi^{\mathbf{A}}(\overline{a}), \psi^{\mathbf{A}}(\overline{a}))$ is compact in $\mathbf{Con}_{\mathcal{K}}\mathbf{A}$, there exists a finite $\Gamma' \subseteq \Gamma$ such that

$$\Theta^{\mathbf{A}}_{\mathcal{K}}(\{(\zeta^{\mathbf{A}}(\overline{a}), \eta^{\mathbf{A}}(\overline{a}))\,;\, \zeta \approx \eta \in \Gamma'\}) \supseteq \Theta^{\mathbf{A}}_{\mathcal{K}}(\varphi^{\mathbf{A}}(\overline{a}), \psi^{\mathbf{A}}(\overline{a})).$$

By the Lemma 0.4.4, this implies that $\Gamma' \models_{\mathcal{K}^Q} \varphi \approx \psi$, so $\models_{\mathcal{K}}$ is finitary. □

The above lemma shows that for a quasivariety $\mathcal{K}$, the relation $\models_{\mathcal{K}}$ is both structural and finitary. We also have that the relation $\models_{\mathcal{K}}$ satisfies the following properties, for all $\Gamma, \Delta \subseteq \mathrm{Id}\,(P)$ and $\varphi \approx \psi \in \mathrm{Id}\,(P)$,

if $\varphi \approx \psi \in \Gamma$ then $\Gamma \models_{\mathcal{K}} \varphi \approx \psi$,

if $\Gamma \models_{\mathcal{K}} \varphi \approx \psi$ and $\Gamma \subseteq \Delta$, then $\Delta \models_{\mathcal{K}} \varphi \approx \psi$,

if $\Gamma \models_{\mathcal{K}} \varphi \approx \psi$ and $\Delta \models_{\mathcal{K}} \zeta \approx \eta$ for all $\zeta \approx \eta \in \Gamma$, then $\Delta \models_{\mathcal{K}} \varphi \approx \psi$.

Thus the relation $\models_{\mathcal{K}}$ satisfies properties similar to those that define deductive systems (the difference being that $\models_{\mathcal{K}}$ is a relation between sets of identities and single identities). The 2-deductive system $S_{\mathcal{K}}$ is also an attempt to present the quasi-equational theory of a quasivariety $\mathcal{K}$ as a deductive system in our sense. The relation $\models_{\mathcal{K}}$ is reproduced without any loss of information as $\vdash_{S_{\mathcal{K}}}$ in this construction as all that is done is to replace the formal expression $\varphi \approx \psi$ by the formal expression (φ, ψ). Later we show that with certain quasivarieties it is possible to associate a 1-deductive system. Nevertheless, the strong connections between quasivarieties in general and k-deductive systems should not be overlooked. In Theorem 3.1.11, we show that a k-deductive system can be considered algebraizable (in a soon to be defined sense) if and only if it is equivalent to the 2-deductive system associated with a quasivariety.

More precisely, we have the following connection between $\models_{\mathcal{K}}$ and $\vdash_{S_{\mathcal{K}}}$: For all $\Gamma \subseteq \mathrm{Id}\,(P)$ and $\varphi \approx \psi \in \mathrm{Id}\,(P)$, (3.1.1) $\Gamma \models_{\mathfrak{K}} \varphi \approx \psi$ if and only if $\{(\zeta, \eta) ; \zeta \approx \eta \in \Gamma\} \vdash_{S_{\mathfrak{K}}} (\varphi, \psi)$.

To see this, recall from Theorem 1.8.9 that $\mathrm{Mod}^* S_{\mathfrak{K}}$ is a matrix semantics for $S_{\mathfrak{K}}$, hence the right hand side holds if and only if

$$\{(\zeta, \eta) ; \zeta \approx \eta \in \Gamma\} \models_{\mathrm{Mod}^* S_{\mathfrak{K}}} (\varphi, \psi)$$

if and only if
$$\{(\zeta, \eta) ; \zeta \approx \eta \in \Gamma\} \models_{\mathcal{A}} (\varphi, \psi)$$

for every $\mathcal{A} = \langle \mathbf{A}, I_A \rangle \in \mathrm{Mod}^* S_{\mathfrak{K}}$. In turn, this holds if and only if, for every $\mathbf{A} \in \mathfrak{K}$ and every interpretation $\bar{a}$ of the variables of Γ and (φ, ψ) in A, $\varphi^{\mathbf{A}}(\bar{a}) = \psi^{\mathbf{A}}(\bar{a})$ whenever $\zeta^{\mathbf{A}}(\bar{a}) = \eta^{\mathbf{A}}(\bar{a})$ for all $\zeta \approx \eta \in \Gamma$. This is precisely what the left hand side of (3.1.1) says.

3.1.2 DEFINITION

Let S be a k-deductive system. A class $\mathfrak{K}$ of algebras is called an *algebraic semantics* for S if there exists a natural number $m \geq 1$ and 1-formulas $\delta_1(\boldsymbol{p}), \ldots, \delta_m(\boldsymbol{p})$, and $\varepsilon_1(\boldsymbol{p}), \ldots, \varepsilon_m(\boldsymbol{p})$, where $\boldsymbol{p} = \langle p_1, \ldots, p_k \rangle$ is a k-variable, such that for all $\Gamma \subseteq Fm^k$ and $\varphi \in Fm^k$,

(3.1.2) $\Gamma \vdash_S \varphi$ if and only if $\{\delta_i(\boldsymbol{\psi}) \approx \varepsilon_i(\boldsymbol{\psi}) ; \boldsymbol{\psi} \in \Gamma, i \leq m\} \models_{\mathfrak{K}} \{\delta_i(\boldsymbol{\varphi}) \approx \varepsilon_i(\boldsymbol{\varphi}) ; i \leq m\}$.

Then $\delta_1(\boldsymbol{p}) \approx \varepsilon_1(\boldsymbol{p}), \ldots, \delta_m(\boldsymbol{p}) \approx \varepsilon_m(\boldsymbol{p})$ are called *defining equations for S and $\mathfrak{K}$*.

We shall be using (3.1.2) fairly often, so it would be a good idea to introduce an abbreviation for it. If $\delta_1(\boldsymbol{p}), \ldots, \delta_m(\boldsymbol{p})$ is any system of 1-formulas in the k-variable $\boldsymbol{p} = \langle p_1, \ldots, p_k \rangle$, then, for any $\boldsymbol{\varphi} \in Fm^k$, we abbreviate the system $\delta_1(\boldsymbol{\varphi}), \ldots, \delta_m(\boldsymbol{\varphi})$ by $\boldsymbol{\delta}(\boldsymbol{\varphi})$. We similarly abbreviate the system $\varepsilon_1(\boldsymbol{\varphi}), \ldots, \varepsilon_m(\boldsymbol{\varphi})$ by $\boldsymbol{\varepsilon}(\boldsymbol{\varphi})$. A system $\delta_1(\boldsymbol{p}) \approx \varepsilon_1(\boldsymbol{p}), \ldots, \delta_m(\boldsymbol{p}) \approx \varepsilon_m(\boldsymbol{p})$ of defining equations is written as $\boldsymbol{\delta}(\boldsymbol{p}) \approx \boldsymbol{\varepsilon}(\boldsymbol{p})$. We then abbreviate (3.1.2) as

(3.1.3) $\Gamma \vdash_S \boldsymbol{\varphi}$ if and only if $\{\boldsymbol{\delta}(\boldsymbol{\psi}) \approx \boldsymbol{\varepsilon}(\boldsymbol{\psi}) ; \boldsymbol{\psi} \in \Gamma\} \models_{\mathfrak{K}} \boldsymbol{\delta}(\boldsymbol{\varphi}) \approx \boldsymbol{\varepsilon}(\boldsymbol{\varphi})$.

Note that since $\vdash_S$ is finitary, we can always take Γ to be finite in (3.1.3). Then the right-hand side of (3.1.3) holds if and only if $\mathfrak{K}$ satisfies

$$\Big(\mathop{\&}_{\boldsymbol{\psi} \in \Gamma} \boldsymbol{\delta}(\boldsymbol{\psi}) \approx \boldsymbol{\varepsilon}(\boldsymbol{\psi}) \Big) \Rightarrow \delta_i(\boldsymbol{\varphi}) \approx \varepsilon_i(\boldsymbol{\varphi}),$$

for each $i \leq m$. This is a system of quasi-identities, hence they are also satisfied by $\mathfrak{K}^{\mathrm{Q}}$. So (3.1.3) holds when $\mathfrak{K}$ is replaced by $\mathfrak{K}^{\mathrm{Q}}$, which leads to the following:

3.1.3 COROLLARY

If $\mathfrak{K}$ is an algebraic semantics for a k-deductive system S, then so is $\mathfrak{K}^{\mathrm{Q}}$. □

An algebraic semantics that is a quasivariety (resp. variety) is called a *quasivariety* (resp. *variety*) *semantics*. As a result of the above corollary we get that a k-deductive system has an algebraic semantics if and only if it has a quasivariety semantics.

Consider the Classical Propositional Calculus (**CPC**) introduced in Section 1.1. The Validity and Completeness Theorems of **CPC** combine to give the following:

$$(3.1.4) \qquad \Gamma \vdash_{\mathbf{CPC}} \varphi \text{ if and only if } \{\psi \approx \mathrm{T}\,;\, \psi \in \Gamma\} \models_{\mathbf{2}} \varphi \approx \mathrm{T}$$
$$\text{if and only if } \{\psi \approx \mathrm{T}\,;\, \psi \in \Gamma\} \models_{\mathcal{BA}} \varphi \approx \mathrm{T}.$$

If we set $\delta(p) = p$ and $\varepsilon(p) = \mathrm{T}$, then (3.1.4) corresponds to (3.1.3). Thus both the singleton $\{\mathbf{2}\}$ and the class $\mathcal{BA}$ form algebraic semantics for **CPC**. Since $\mathcal{BA}$ is a variety, hence also a quasivariety, we have that $\mathcal{BA}$ is a variety semantics for **CPC**. Recall that $\mathcal{BA} = \{\mathbf{2}\}^{\mathrm{Q}}$, illustrating the previous corollary.

3.1.4 DEFINITION

Let S be a k-deductive system and $\mathcal{K}$ an algebraic semantics for S with defining equations $\delta_1(\boldsymbol{p}) \approx \varepsilon_1(\boldsymbol{p}), \ldots, \delta_m(\boldsymbol{p}) \approx \varepsilon_m(\boldsymbol{p})$. The class $\mathcal{K}$ is said to be an *equivalent algebraic semantics* for S if there exists a natural number $n \geq 1$ and a finite system $\boldsymbol{\Delta}_1(p,q), \ldots, \boldsymbol{\Delta}_n(p,q)$ of k-formulas in two variables such that, for all $\varphi \approx \psi \in \mathrm{Id}\,(P)$,

$$(3.1.5) \qquad \varphi \approx \psi \mathrel{=\!||\!\!=_{\mathcal{K}}} \{\delta_i(\boldsymbol{\Delta}_j(\varphi,\psi)) \approx \varepsilon_i(\boldsymbol{\Delta}_j(\varphi,\psi))\,;\, i \leq m\,;\, j \leq n\}.$$

Then $\boldsymbol{\Delta}_1(p,q), \ldots, \boldsymbol{\Delta}_n(p,q)$ are called *equivalence k-formulas for S and $\mathcal{K}$*.

Note that the $\boldsymbol{\Delta}_i$'s are k-formulas in two variables and not in two k-variables. As the terminology suggests, a system of equivalence k-formulas for S, as defined here, will turn out to be a system of equivalence (in fact, congruence) k-formulas for S, in the sense of Section 2.3, having additional properties also: see Theorem 3.1.11. Again, (3.1.5) could do with some abbreviating. We write $\boldsymbol{\Delta}(p,q)$ for the system $\boldsymbol{\Delta}_1(p,q), \ldots, \boldsymbol{\Delta}_n(p,q)$ of k-formulas. Let $\delta(\boldsymbol{\Delta}(\varphi,\psi)) \approx \varepsilon(\boldsymbol{\Delta}(\varphi,\psi))$ stand for the set $\{\delta_i(\boldsymbol{\Delta}_j(\varphi,\psi)) \approx \varepsilon_i(\boldsymbol{\Delta}_j(\varphi,\psi))\,;\, i \leq m\,;\, j \leq n\}$. Then (3.1.5) is abbreviated as

$$(3.1.6) \qquad \varphi \approx \psi \mathrel{=\!||\!\!=_{\mathcal{K}}} \delta(\boldsymbol{\Delta}(\varphi,\psi)) \approx \varepsilon(\boldsymbol{\Delta}(\varphi,\psi)).$$

It is evident that (3.1.6) is equivalent to a set of quasi-identities just as (3.1.3) is. (The set of identities on the right is finite.) So (3.1.6) must also hold for $\mathcal{K}^{\mathrm{Q}}$. This leads to the following:

3.1.5 COROLLARY [BP89a, Corollary 2.11]

If $\mathcal{K}$ is an equivalent algebraic semantics for a k-deductive system S, then so is $\mathcal{K}^{Q}$. □

An equivalent algebraic semantics that is a quasivariety is called an *equivalent quasivariety semantics.* If an equivalent quasivariety semantics is a variety, then it is called an *equivalent variety semantics.* The above corollary implies that a k-deductive system has an equivalent algebraic semantics if and only if it has an equivalent quasivariety semantics.

In the case of **CPC** and $\mathcal{BA}$, a system of equivalence (1-)formulas is given by $\Delta(p,q) = p \leftrightarrow q$. If we take $\mathcal{BA}$ as the algebraic semantics, then (3.1.6) becomes

$$\varphi \approx \psi \mathrel{=\mid\models_{\mathcal{BA}}} \varphi \leftrightarrow \psi \approx \mathrm{T}. \tag{3.1.7}$$

To see that this indeed holds, recall that in the language of $\mathcal{BA}$, $\varphi \leftrightarrow \psi$ is defined as $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$, which is defined as $((\neg\varphi) \vee \psi) \wedge ((\neg\psi) \vee \varphi)$. The implication from left to right in (3.1.7) is obvious, since $(\neg\varphi) \vee \varphi \approx \mathrm{T}$. Now, suppose $((\neg\varphi) \vee \psi) \wedge ((\neg\psi) \vee \varphi) \approx \mathrm{T}$. This implies that $(\neg\varphi) \vee \psi \approx \mathrm{T}$ and $(\neg\psi) \vee \varphi \approx \mathrm{T}$. Using identities of $\mathcal{BA}$ (see Section 0.2), we get that $\mathcal{BA}$ satisfies the following:

$$\begin{aligned}
\varphi &\approx \varphi \wedge \mathrm{T} \approx \varphi \wedge ((\neg\varphi) \vee \psi) \approx (\varphi \wedge (\neg\varphi)) \vee (\varphi \wedge \psi) \\
&\approx \bot \vee (\varphi \wedge \psi) \approx (\neg \mathrm{T}) \vee (\varphi \wedge \psi) \approx (\neg((\neg\psi) \vee \varphi)) \vee (\varphi \wedge \psi) \\
&\approx (\psi \wedge (\neg\varphi)) \vee (\varphi \wedge \psi) \approx \psi \wedge ((\neg\varphi) \vee \varphi) \approx \psi \wedge \mathrm{T} \\
&\approx \psi,
\end{aligned}$$

hence (3.1.7) holds. By Definition 3.1.4, $\mathcal{BA}$ forms an equivalent variety semantics for **CPC**.

3.1.6 LEMMA [BP89a, Corollary 2.9]

If $\mathcal{K}$ is an equivalent algebraic semantics for S with defining equations $\delta_1(\boldsymbol{p}) \approx \varepsilon_1(\boldsymbol{p}),\ldots,$ $\delta_m(\boldsymbol{p}) \approx \varepsilon_m(\boldsymbol{p})$ and equivalence k-formulas $\boldsymbol{\Delta}_1(p,q),\ldots,\boldsymbol{\Delta}_n(p,q)$, then, for every $\Gamma \subseteq \mathrm{Id}(P)$, $\varphi \approx \psi \in \mathrm{Id}(P)$ and $\boldsymbol{\vartheta} \in Fm^k$,

(i) $\Gamma \models_{\mathcal{K}} \varphi \approx \psi$ *iff* $\{\boldsymbol{\Delta}(\zeta,\eta)\,;\, \zeta \approx \eta \in \Gamma\} \vdash_S \boldsymbol{\Delta}(\varphi,\psi)$,

(ii) $\boldsymbol{\vartheta} \dashv\vdash_S \{\boldsymbol{\Delta}(\delta_i(\boldsymbol{\vartheta}),\varepsilon_i(\boldsymbol{\vartheta}))\,;\, i \leq m\}$.

Conversely, if there exists a system $\delta_1(\boldsymbol{p}),\ldots,\delta_m(\boldsymbol{p})$, $\varepsilon_1(\boldsymbol{p}),\ldots,\varepsilon_m(\boldsymbol{p})$ of 1-formulas in one k-variable and a system $\boldsymbol{\Delta}_1(p,q),\ldots,\boldsymbol{\Delta}_n(p,q)$ of k-formulas in 2 variables that satisfy (i) and (ii), *then* $\mathcal{K}$ *is an equivalent algebraic semantics for* S *with equivalence k-formulas* $\Delta_1(p,q),\dots,\Delta_n(p,q)$ *and defining equations* $\delta_1(\boldsymbol{p}) \approx \varepsilon_1(\boldsymbol{p}),\dots,\delta_m(\boldsymbol{p}) \approx \varepsilon_m(\boldsymbol{p})$.

Proof. ($\Rightarrow$)

(i) $\{\Delta(\zeta,\eta)\,;\,\zeta \approx \eta \in \Gamma\} \vdash_S \Delta(\varphi,\psi)$

iff $\{\delta(\Delta(\zeta,\eta)) \approx \varepsilon(\Delta(\zeta,\eta))\,;\,\zeta \approx \eta \in \Gamma\} \models_{\mathcal{K}} \delta(\Delta(\varphi,\psi)) \approx \varepsilon(\Delta(\varphi,\psi))$ [by (3.1.3)]

iff $\Gamma \models_{\mathcal{K}} \varphi \approx \psi$ [by (3.1.6)].

(ii) $\boldsymbol{\vartheta} \dashv\vdash_S \{\Delta(\delta_i(\boldsymbol{\vartheta}),\varepsilon_i(\boldsymbol{\vartheta}))\,;\,i \le m\}$

iff $\delta(\boldsymbol{\vartheta}) \approx \varepsilon(\boldsymbol{\vartheta}) =\mid\models_{\mathcal{K}} \{\delta(\Delta(\delta_i(\boldsymbol{\vartheta}),\varepsilon_i(\boldsymbol{\vartheta}))) \approx \varepsilon(\Delta(\delta_i(\boldsymbol{\vartheta}),\varepsilon_i(\boldsymbol{\vartheta})))\,;\,i \le m\}$ [by (3.1.3)]

iff $\delta(\boldsymbol{\vartheta}) \approx \varepsilon(\boldsymbol{\vartheta}) =\mid\models_{\mathcal{K}} \{\delta_i(\boldsymbol{\vartheta}) \approx \varepsilon_i(\boldsymbol{\vartheta})\,;\,i \le m\}$ [by (3.1.6)].

The last condition is certainly true, hence (ii) holds.

For the converse, suppose $\Gamma \subseteq Fm^k$ and $\boldsymbol{\vartheta} \in Fm^k$. Then $\Gamma \vdash_S \boldsymbol{\vartheta}$ if and only if

$\{\Delta(\delta_i(\boldsymbol{\eta}),\varepsilon_i(\boldsymbol{\eta}))\,;\,\boldsymbol{\eta} \in \Gamma\,;\,i \le m\} \vdash_S \{\Delta(\delta_i(\boldsymbol{\vartheta}),\varepsilon_i(\boldsymbol{\vartheta}))\,;\,i \le m\}$ [by (ii)]

iff $\{\delta(\boldsymbol{\eta}) \approx \varepsilon(\boldsymbol{\eta})\,;\,\boldsymbol{\eta} \in \Gamma\} \models_{\mathcal{K}} \delta(\boldsymbol{\vartheta}) \approx \varepsilon(\boldsymbol{\vartheta})$ [by (i)],

hence (3.1.3) holds. One can see that (3.1.6) holds from the following:

$\varphi \approx \psi =\mid\models_{\mathcal{K}} \delta(\Delta(\varphi,\psi)) \approx \varepsilon(\Delta(\varphi,\psi))$

iff $\Delta(\varphi,\psi) \dashv\vdash_S \{\Delta(\delta_i(\Delta(\varphi,\psi)),\varepsilon_i(\Delta(\varphi,\psi)))\,;\,i \le m\}$ [by (i)]

iff $\Delta(\varphi,\psi) \dashv\vdash_S \Delta(\varphi,\psi)$. [by (ii)] □

The idea of an algebraic semantics is to interpret $\vdash_S$ in $\models_{\mathcal{K}}$, i.e., to translate the statement $\Gamma \vdash_S \boldsymbol{\varphi}$ ($\Gamma \cup \{\boldsymbol{\varphi}\} \subseteq Fm^k$) into a corresponding algebraic one, as done in (3.1.3). This means that it is possible to determine whether or not $\Gamma \vdash_S \boldsymbol{\varphi}$ purely by considering a corresponding relationship in the equational theory of $\mathcal{K}$. The idea of an equivalent algebraic semantics extends this idea to allow one to interpret $\models_{\mathcal{K}}$ in $\vdash_S$ as well, i.e., to translate the statement $\Gamma \models_{\mathcal{K}} \zeta \approx \eta$ ($\Gamma \cup \{\zeta \approx \eta\} \subseteq \mathrm{Id}(P)$) into a logical deduction, as done in Lemma 3.1.6 (i). Similarly, this allows one to decide whether $\Gamma \models_{\mathcal{K}} \zeta \approx \eta$ by considering a corresponding statement about S. Most importantly, these two interpretations are inverse to one another in the following sense: Suppose $\boldsymbol{\varphi}$ is a k-formula. Applying the defining equations, we get the identities $\delta_i(\boldsymbol{\varphi}) \approx \varepsilon_i(\boldsymbol{\varphi})$, $i \le m$. Now, applying the equivalence k-formulas to these identities, we get the set of k-formulas $\{\Delta(\delta_i(\boldsymbol{\varphi}),\varepsilon_i(\boldsymbol{\varphi}))\,;\,i \le m\}$. Lemma 3.1.6 (ii) is precisely the statement that this set and $\boldsymbol{\varphi}$ are inter-derivable. On the other hand, suppose $\zeta \approx \eta$ is an identity. Applying the equivalence k-formulas we get the k-formulas $\Delta(\zeta,\eta)$. Then, applying the defining equations, we get the identities $\delta_i(\Delta(\zeta,\eta)) \approx \varepsilon_i(\Delta(\zeta,\eta))$, $i \leq m$. The statement (3.1.6) of Definition 3.1.4 says precisely that these identities are equivalent to $\zeta \approx \eta$ over $\mathcal{K}$.

(In the above discussion we have used the expressions 'interpretation' and 'equivalence' in a natural but loose way. These words were given precise definition in Section 1.9. In Theorem 3.1.11, we shall see that our intuitive usage is consistent with the definitions of Section 1.9.)

Note that conditions (i) and (ii) of Lemma 3.1.6 are not used in the definitions of algebraic and equivalent algebraic semantics. In fact, Lemma 3.1.6 shows that they are consequences of statements (3.1.3) and (3.1.6) (and vice-versa). In the case of **CPC** and $\mathcal{BA}$, Lemma 3.1.6 (i) and (ii) correspond to the following: For every set Γ of identities, every identity $\varphi \approx \psi$ and every formula φ,

$$\Gamma \models_{\mathcal{BA}} \varphi \approx \psi \quad \text{iff} \quad \{\zeta \leftrightarrow \eta\,;\, \zeta \approx \eta \in \Gamma\} \vdash_{\mathbf{CPC}} \varphi \leftrightarrow \psi,$$

$$\varphi \dashv\vdash_{\mathbf{CPC}} \varphi \leftrightarrow \top.$$

The idea that a k-deductive system can be equivalent to a class of algebras with respect to two interpretations which are mutually inverse in the above sense has led Blok and Pigozzi to propose the following definition of algebraizability of deductive systems in [BP89a] for 1-deductive systems, and more generally in [BP94].

3.1.7 DEFINITION

A k-deductive system is called *algebraizable* if there exists an equivalent algebraic semantics for it. A k-deductive system is called *strongly algebraizable* if there exists an equivalent variety semantics for it.

Note that since the class $\mathcal{BA}$ of all Boolean algebras is a variety, our earlier remarks show that **CPC** is strongly algebraizable.

We immediately have the following corollary regarding fragments of deductive systems.

3.1.8 COROLLARY [BP89a, Corollary 2.12]

Let $S = \langle \mathcal{L}, \vdash_S \rangle$ be a k-deductive system. If S is algebraizable, then so is any $\mathcal{L}'$-fragment of S, where $\mathcal{L}'$ contains all the connectives that occur in a system of equivalence k-formulas and defining equations for an equivalent quasivariety semantics $\mathfrak{K}$ of S. Moreover, if $\mathfrak{K}'$ is the class of all $\mathcal{L}'$-reducts of members of $\mathfrak{K}$, then $\mathrm{S}\,\mathfrak{K}'$ is the equivalent quasivariety semantics for the $\mathcal{L}'$-fragment of S and the equivalence k-formulas and defining equations of S and $\mathfrak{K}$ continue to function as such for the $\mathcal{L}'$-fragment and $\mathrm{S}\,\mathfrak{K}'$.

Proof. It is clear that the conditions (i) and (ii) of Lemma 3.1.6 continue to hold when S is replaced by its $\mathcal{L}'$-fragment and $\mathfrak{K}$ is replaced by $\mathfrak{K}'$, hence the $\mathcal{L}'$-fragment is algebraizable with equivalent algebraic semantics $\mathfrak{K}'$ and the same defining equations and equivalence k-formulas as S. By Theorem 0.4.2, $(\mathfrak{K}')^{\mathrm{Q}} = \mathrm{S}\,\mathfrak{K}'$, which completes the proof. □

The following theorem presents necessary and sufficient conditions for a quasivariety to be the equivalent quasivariety semantics of an algebraizable 1-deductive system.

3.1.9 THEOREM [BKP, Theorem 2.5]

Let $\mathfrak{K}$ be a quasivariety. Then $\mathfrak{K}$ is the equivalent quasivariety semantics of a 1-deductive system if and only if there exist binary formulas $\Delta_1(p,q),\dots,\Delta_n(p,q)$ and unary formulas $\delta_1(p),\dots,\delta_m(p)$, $\varepsilon_1(p),\dots,\varepsilon_m(p)$ such that $\mathfrak{K}$ satisfies

(i) $\quad \delta_j(\Delta_i(p,p)) \approx \varepsilon_j(\Delta_i(p,p))$ *for all* $i \le n$ *and* $j \le m$,

(ii) $\quad \Big(\mathop{\&}_{j \le m} \mathop{\&}_{i \le n} \delta_j(\Delta_i(p,q)) \approx \varepsilon_j(\Delta_i(p,q)) \Big) \Rightarrow p \approx q.$

Proof. ($\Rightarrow$) If $\mathfrak{K}$ is the equivalent quasivariety semantics of a 1-deductive system S, say, then (3.1.5) holds, and this implies both (i) and (ii).

($\Leftarrow$) Suppose the described Δ_i's, δ_j's and ε_j's exist. Define a 1-deductive system $S = \langle \mathcal{L}, \vdash_S \rangle$ as follows: Let $\mathcal{L}$ be the type of $\mathfrak{K}$. For $\Gamma \subseteq Fm_{\mathcal{L}}$ and $\varphi \in Fm_{\mathcal{L}}$, let

$$\Gamma \vdash_S \varphi \text{ if and only if } \{\delta(\psi) \approx \varepsilon(\psi)\,;\, \psi \in \Gamma\} \models_{\mathfrak{K}} \delta(\varphi) \approx \varepsilon(\varphi).$$

By Lemma 3.1.1, $\vdash_S$ is both structural and finitary. Since the conditions (1.1.1), (1.1.2) and (1.1.3) hold trivially, S is a deductive system. To show that $\delta_1(p) \approx \varepsilon_1(p),\dots,\delta_m(p) \approx \varepsilon_m(p)$ forms a system of defining equations for S and $\mathfrak{K}$, we need to show that (3.1.3) holds. But this is an immediate consequence of the definition of $\vdash_S$. To show that $\Delta_1(p,q),\dots,\Delta_n(p,q)$ forms a system of equivalence formulas for S and $\mathfrak{K}$ we must establish (3.1.5). The implication from right to left follows immediately from (ii), and the implication from left to right follows from (i). By Definition 3.1.7, S is algebraizable with equivalent quasivariety semantics $\mathfrak{K}$, defining equations $\delta_1(p) \approx \varepsilon_1(p), \ldots, \delta_m(p) \approx \varepsilon_m(p)$ and equivalence formulas $\Delta_1(p,q), \ldots, \Delta_n(p,q)$. □

The following lemma extends [BP89a, Lemma 2.13] to k-deductive systems.

3.1.10 LEMMA [BP89a, Lemma 2.13]

Let S be an algebraizable k-deductive system with equivalent algebraic semantics $\mathfrak{K}$ and equivalence formulas $\Delta_1(p,q), \ldots, \Delta_n(p,q)$. Then, for all $\varphi, \psi, \vartheta \in Fm$, we have

(i) $\vdash_S \Delta(\varphi, \varphi)$,

(ii) $\Delta(\varphi, \psi) \vdash_S \Delta(\psi, \varphi)$,

(iii) $\Delta(\varphi, \psi), \Delta(\psi, \vartheta) \vdash_S \Delta(\varphi, \vartheta)$,

(iv) *If $\vartheta = \vartheta(p, \overline{r})$, where p does occur in ϑ and $\overline{r}$ is a list of all variables in ϑ other than p, then* $\Delta(\varphi, \psi) \vdash_S \Delta(\vartheta(\varphi, \overline{\eta}), \vartheta(\psi, \overline{\eta}))$.

Proof. Parts (i), (ii) and (iii) are all similarly proved, so we shall only prove (iii) and (iv). By Lemma 3.1.6 (i), (iii) holds if and only if $\varphi \approx \psi, \psi \approx \vartheta \models_{\mathfrak{K}} \varphi \approx \vartheta$, and this is obviously true. Now, (iv) holds if and only if $\varphi \approx \psi \models_{\mathfrak{K}} \vartheta(\varphi, \overline{\eta}) \approx \vartheta(\psi, \overline{\eta})$, which is clearly also true. □

The following theorem combines results from the sources [BP89a], [BP89b] and [BP92]. The conditions (i), (ii), (v) and (vii) are from [BP89a], (iii) and (iv) appear in [BP92] and (vi) and (viii) appear in [BP89b]. The results from [BP89a] have been suitably generalized to k-deductive systems.

3.1.11 THEOREM

Let S be a k-deductive system. The following are equivalent:

(i) *S is algebraizable.*

(ii) *There exists a system $\Delta_1(p,q), \ldots, \Delta_n(p,q)$ of k-formulas in two variables and a system $\delta_1(\boldsymbol{p}), \ldots, \delta_m(\boldsymbol{p}), \varepsilon_1(\boldsymbol{p}), \ldots, \varepsilon_m(\boldsymbol{p})$ of 1-formulas in k variables $p_1, \ldots, p_k$ such that for every $\varphi, \psi \in Fm$,*

(1) $\vdash_S \Delta(\varphi, \varphi)$,

(2) $\Delta(\varphi,\psi)\vdash_S \Delta(\psi,\varphi)$,

(3) $\Delta(\varphi,\psi),\Delta(\psi,\vartheta)\vdash_S \Delta(\varphi,\vartheta)$,

for every $f\in\mathfrak{L}$ *with* $ar(f)=\ell$, *and* $\varphi_1,\ldots,\varphi_\ell,\psi_1,\ldots,\psi_\ell\in Fm$,

(4) $\Delta(\varphi_1,\psi_1),\ldots,\Delta(\varphi_\ell,\psi_\ell)\vdash_S \Delta(f(\varphi_1,\ldots,\varphi_\ell),f(\psi_1,\ldots,\psi_\ell))$

and, for every $\boldsymbol{\vartheta}\in Fm^k$,

(5) $\boldsymbol{\vartheta}\dashv\vdash_S\{\Delta(\delta_i(\boldsymbol{\vartheta}),\varepsilon_i(\boldsymbol{\vartheta}))\,;\,i\leq m\}$.

(iii) *S has a finite system $\Delta_1(p,q),\ldots,\Delta_n(p,q)$ of congruence k-formulas without parameters and a system $\delta_1(\boldsymbol{p}),\ldots,\delta_m(\boldsymbol{p}),\varepsilon_1(\boldsymbol{p}),\ldots,\varepsilon_m(\boldsymbol{p})$ of 1-formulas in k variables such that*

$$\boldsymbol{\vartheta}\dashv\vdash_S\{\Delta(\delta_i(\boldsymbol{\vartheta}),\varepsilon_i(\boldsymbol{\vartheta}))\,;\,i\leq m\}\quad \text{for all } \boldsymbol{\vartheta}\in Fm^k.$$

(iv) *For every S-matrix $\mathcal{A}=\langle\mathbf{A},F_{\mathcal{A}}\rangle$, the Leibniz operator $\Omega_{\mathcal{A}}:\mathrm{Fi}^S\mathcal{A}\to\mathrm{Con}\,\mathbf{A}$ is monotonic, continuous and injective.*

(v) *$\Omega:\mathbf{Th}S\to\mathbf{Con\,Fm}$ is monotonic, continuous and injective.*

(vi) *There exists a quasivariety $\mathfrak{K}$ such that $\Omega(\mathrm{Th}S)=\mathrm{Th}S_{\mathfrak{K}}$ and $\Omega:\mathbf{Th}S\to\mathbf{Th}S_{\mathfrak{K}}$ is an isomorphism.*

(vii) *There exists a quasivariety $\mathfrak{K}$ and an isomorphism $f:\mathbf{Th}S\to\mathbf{Th}S_{\mathfrak{K}}$ that commutes with substitutions.*

(viii) *There exists a quasivariety $\mathfrak{K}$ such that S is equivalent to $S_{\mathfrak{K}}$ (in the sense of Definition 1.9.1).*

Proof. (i)⇒(ii) Suppose S is algebraizable with equivalent quasivariety semantics $\mathfrak{K}$, defining equations $\delta_1(\boldsymbol{p})\approx\varepsilon_1(\boldsymbol{p}),\ldots,\delta_m(\boldsymbol{p})\approx\varepsilon_m(\boldsymbol{p})$ and equivalence k-formulas $\Delta_1(p,q),\ldots,\Delta_n(p,q)$. We claim that these systems satisfy (1) to (5). The first three are proved in Lemma 3.1.10 and condition (4) follows easily from Lemma 3.1.10 (iv) and (3). Condition (5) is proved in Lemma 3.1.6, hence this implication is true.

(ii)⇒(iii) Let $\Delta_1(p,q),\ldots,\Delta_n(p,q)$ and $\delta_1(\boldsymbol{p}),\ldots,\delta_m(\boldsymbol{p}),\varepsilon_1(\boldsymbol{p}),\ldots,\varepsilon_m(\boldsymbol{p})$ be as in (ii). All we need to show is that $\Delta_1(p,q),\ldots,\Delta_n(p,q)$ forms a system of congruence k-formulas without parameters for S. We first show that $\Delta_1(p,q),\ldots,\Delta_n(p,q)$ is a system of equivalence k-formulas without parameters as defined in Definition 2.3.1. Condition (1) with p replacing φ is the necessary (2.3.1)′. To see that (2.3.2)′ holds, let $p,q,z_1,\ldots,z_{k-1}$ be distinct variables. Let $\ell\leq k$ and let $\tilde{z}$, $\tilde{z}\,[p/\ell]$ and $\tilde{z}\,[q/\ell]$ be as defined in Section 2.3. Observe that an (inductive) generalization of (4), replacing $f \in \mathcal{L}$ by any $\eta \in Fm$, is clearly true. Note that each δ_i is a formula, so it follows from this generalization of (4) that

$$(3.1.8) \qquad \Delta(z_1, z_1), \ldots, \Delta(z_{\ell-1}, z_{\ell-1}), \Delta(p, q), \Delta(z_\ell, z_\ell), \ldots, \Delta(z_{k-1}, z_{k-1}) \vdash_S \{\Delta(\delta_i(\tilde{z}\,[p/\ell]), \delta_i(\tilde{z}\,[q/\ell]))\,;\, i \leq m\},$$

and, similarly,

$$(3.1.9) \qquad \Delta(z_1, z_1), \ldots, \Delta(z_{\ell-1}, z_{\ell-1}), \Delta(p, q), \Delta(z_\ell, z_\ell) \ldots, \Delta(z_{k-1}, z_{k-1}) \vdash_S \{\Delta(\varepsilon_i(\tilde{z}\,[p/\ell]), \varepsilon_i(\tilde{z}\,[q/\ell]))\,;\, i \leq m\},$$

Using (5) and (2), respectively, we have that

$$(3.1.10) \qquad \tilde{z}\,[p/\ell] \dashv\vdash_S \{\Delta(\delta_i(\tilde{z}\,[p/\ell]), \varepsilon_i(\tilde{z}\,[p/\ell]))\,;\, i \leq m\} \dashv\vdash_S \{\Delta(\varepsilon_i(\tilde{z}\,[p/\ell]), \delta_i(\tilde{z}\,[p/\ell]))\,;\, i \leq m\}.$$

Collecting all these results together, we get

$$\tilde{z}\,[p/\ell], \Delta(p, q) \vdash_S \tilde{z}\,[p/\ell], \Delta(z_1, z_1), \ldots, \Delta(z_{\ell-1}, z_{\ell-1}), \Delta(p, q), \Delta(z_\ell, z_\ell) \ldots, \Delta(z_{k-1}, z_{k-1})$$

by (1), hence, by (3.1.8),

$$\tilde{z}\,[p/\ell], \Delta(p, q) \vdash_S \tilde{z}\,[p/\ell], \{\Delta(\delta_i(\tilde{z}\,[p/\ell]), \delta_i(\tilde{z}\,[q/\ell]))\,;\, i \leq m\}.$$

By (3.1.10),

$$\tilde{z}\,[p/\ell], \Delta(p, q) \vdash_S \{\Delta(\varepsilon_i(\tilde{z}\,[p/\ell]), \delta_i(\tilde{z}\,[p/\ell]))\,;\, i \leq m\}, \{\Delta(\delta_i(\tilde{z}\,[p/\ell]), \delta_i(\tilde{z}\,[q/\ell]))\,;\, i \leq m\},$$

hence, by (3) and (2),

$$\tilde{z}\,[p/\ell], \Delta(p, q) \vdash_S \{\Delta(\varepsilon_i(\tilde{z}\,[p/\ell]), \delta_i(\tilde{z}\,[q/\ell]))\,;\, i \leq m\}$$

and

$$\tilde{z}\,[p/\ell], \Delta(p, q) \vdash_S \{\Delta(\delta_i(\tilde{z}\,[q/\ell]), \varepsilon_i(\tilde{z}\,[p/\ell]))\,;\, i \leq m\}.$$

By (3.1.9),

$$\tilde{z}\,[p/\ell], \Delta(p, q) \vdash_S \{\Delta(\varepsilon_i(\tilde{z}\,[p/\ell]), \varepsilon_i(\tilde{z}\,[q/\ell]))\,;\, i \leq m\},$$

hence, by (3) and (5)

$$\tilde{z}\,[p/\ell], \Delta(p, q) \vdash_S \{\Delta(\delta_i(\tilde{z}\,[q/\ell]), \varepsilon_i(\tilde{z}\,[q/\ell]))\,;\, i \leq m\}$$

and

$$\tilde{z}\,[p/\ell], \Delta(p, q) \vdash_S \tilde{z}\,[q/\ell],$$

and so (2.3.2)$'$ holds and $\Delta_1(p, q), \ldots, \Delta_n(p, q)$ form a system of equivalence k-formulas without parameters for S. The result now follows from Definition 2.3.1, Definition 2.4.1, (4) and Proposition 2.4.3.

(iii)$\Rightarrow$(iv) By Definition 2.5.1, S is a congruential k-deductive system, hence, by Theorem 2.5.6, $\Omega_{\mathcal{A}}$ is monotonic and continuous for each S-matrix $\mathcal{A} = \langle \mathbf{A}, F_{\mathcal{A}} \rangle$. To see that $\Omega_{\mathcal{A}}$ is injective, let

$F, G \in \mathrm{Fi}^S \mathcal{A}$ such that $\Omega_{\mathcal{A}} F = \Omega_{\mathcal{A}} G$. Let $\boldsymbol{a} \in A^k$ and let $\boldsymbol{p}$ be any k-variable. Then $\boldsymbol{a}$ can be considered an interpretation of $\boldsymbol{p}$ in A. By assumption, $\boldsymbol{p} \dashv\vdash_S \{\Delta(\delta_i(\boldsymbol{p}), \varepsilon_i(\boldsymbol{p}))\,;\, i \leq m\}$, hence

$$\begin{aligned}
\boldsymbol{a} \in F \quad & \text{iff} \quad \Delta_\ell^{\mathbf{A}}(\delta_i^{\mathbf{A}}(\boldsymbol{a}), \varepsilon_i^{\mathbf{A}}(\boldsymbol{a})) \in F \quad \text{for all } \ell \leq n\,; i \leq m \\
& \text{iff} \quad (\delta_i^{\mathbf{A}}(\boldsymbol{a}), \varepsilon_i^{\mathbf{A}}(\boldsymbol{a})) \in \Omega_{\mathcal{A}} F \quad \text{for all } i \leq m \qquad \text{[by Theorem 2.4.2]} \\
& \text{iff} \quad (\delta_i^{\mathbf{A}}(\boldsymbol{a}), \varepsilon_i^{\mathbf{A}}(\boldsymbol{a})) \in \Omega_{\mathcal{A}} G \quad \text{for all } i \leq m \\
& \text{iff} \quad \Delta_\ell^{\mathbf{A}}(\delta_i^{\mathbf{A}}(\boldsymbol{a}), \varepsilon_i^{\mathbf{A}}(\boldsymbol{a})) \in G \quad \text{for all } \ell \leq n\,; i \leq m \qquad \text{[by Theorem 2.4.2]} \\
& \text{iff} \quad \boldsymbol{a} \in G,
\end{aligned}$$

hence $F = G$, implying that $\Omega_{\mathcal{A}}$ is injective.

(iv)⇒(v) Take $\mathcal{A} = \langle \mathbf{Fm}, \mathrm{Cn}_S(\emptyset) \rangle$. Then $\Omega_{\mathcal{A}} = \Omega$, so the result follows trivially.

(v)⇒(vi) Set $K = \{\mathbf{Fm}/\theta\,;\, \theta \in \Omega(\mathrm{Th}S)\}$, and let $\mathcal{K}$ be the quasivariety generated by K. Recall that the $S_{\mathcal{K}}$-theories are exactly the $\mathcal{K}$-congruences of $\mathbf{Fm}$ (see Section 1.7), so we need to show that Ω is an isomorphism between $\mathbf{Th}S$ and $\mathbf{Con}_{\mathcal{K}}\mathbf{Fm}$. For each $\theta \in \Omega(\mathrm{Th}S)$, θ is obviously a $\mathcal{K}$-congruence of $\mathbf{Fm}$ (since $\mathbf{Fm}/\theta \in \mathcal{K}$), hence $\Omega(\mathrm{Th}S) \subseteq \mathrm{Con}_{\mathcal{K}}\mathbf{Fm}$. For the reverse inclusion, we need the following

Claim: Let $\theta \in \Omega(\mathrm{Th}S)$ and let $h: \mathbf{Fm} \to \mathbf{Fm}/\theta$ be any homomorphism with the property that for each $p \in P$, $hp \in Fm/\theta$ is the image under h of an infinite number of variables. Then the kernel of h is of the form $\sigma^{-1}(\theta)$ for some surjective substitution $\sigma: \mathbf{Fm} \to \mathbf{Fm}$.

Proof. We shall construct a surjective substitution σ such that $\sigma p \in hp$ for each variable p. Recall from Section 1.1 that $P = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \ldots\}$. Let $Q = \{q_1, q_2, q_3, \ldots\}$ where $q_j = \mathbf{p}_j$ for all j. Set $\sigma \mathbf{p}_1 = q_i$, where i is the least integer such that $q_i \in h\mathbf{p}_1$. Then $\sigma \mathbf{p}_1 \in h\mathbf{p}_1$ as required. Now reorder the set Q by simultaneously renaming q_i as q_1, q_1 as q_2, q_2 as $q_3, \ldots,$ q_{i-1} as q_i. Set $\sigma \mathbf{p}_2 = q_i$, where i is the least integer such that $i > 1$ and $q_i \in h\mathbf{p}_2$. Then $\sigma \mathbf{p}_2 \in h\mathbf{p}_2$ as required. Now reorder Q by simultaneously renaming q_i as q_2, q_2 as $q_3, \ldots,$ q_{i-1} as q_i. This process can be continued for all $i < \omega$ because at every step there are infinitely many variables q in $\{q_{i+1}, q_{i+2}, \ldots\}$ for which $q \in h\mathbf{p}_i$, by assumption. If we complete this induction, then we will have constructed a bijective map $\sigma: P \to Q$, viz., $\sigma: \mathbf{p}_i \mapsto q_i$, that extends to a surjective substitution (since $Q = P$) which we will also call σ, such that $\sigma p \in hp$ for each $p \in P$. Note that $hp = (\sigma p)/\theta$ for each $p \in P$.

Let $f:\mathbf{Fm}\to\mathbf{Fm}/\theta$ be the canonical homomorphism. Then $(f\circ\sigma)p=f(\sigma p)=(\sigma p)/\theta=hp$ for each $p\in P$. Since $f\circ\sigma$ and h agree on the generators of $\mathbf{Fm}$, we have that $f\circ\sigma=h$. Now, $(\varphi,\psi)\in\ker h$ iff $h\varphi=h\psi$ iff $f(\sigma\varphi)=f(\sigma\psi)$ iff $(\sigma\varphi,\sigma\psi)\in\theta$ iff $(\varphi,\psi)\in\sigma^{-1}(\theta)$, hence $\ker h=\sigma^{-1}(\theta)$, which proves the claim.

Let $\Phi\in\mathrm{Con}_{\mathcal{K}}\mathbf{Fm}$, and assume first that Φ is finitely generated, say $\Phi=\Theta^{\mathbf{Fm}}_{\mathcal{K}}(\Gamma)$ for some finite $\Gamma\subseteq Fm^2$. Suppose that $(\varphi,\psi)\notin\Phi$. By considering the $S_{\mathcal{K}}$-matrix $\langle\mathbf{Fm},\Phi\rangle$, we get $\Gamma\nvdash_{S_{\mathcal{K}}}(\varphi,\psi)$. By (3.1.1), we have that this is equivalent to $\{\eta\approx\zeta\,;\,(\eta,\zeta)\in\Gamma\}\not\models_{\mathcal{K}}\varphi\approx\psi$. Since Γ is finite, this can also be written as

$$\not\models_{\mathcal{K}}\Big(\mathop{\&}_{(\eta,\zeta)\in\Gamma}\eta\approx\zeta\Big)\Rightarrow\varphi\approx\psi. \tag{3.1.11}$$

Since $\mathcal{K}$ is generated by K, there must exist $\mathbf{Fm}/\theta\in K$ such that the quasi-identity in (3.1.11) fails in $\mathbf{Fm}/\theta$. Thus there exists an interpretation $h_{(\varphi,\psi)}:\mathbf{Fm}\to\mathbf{Fm}/\theta$ of the variables of P in Fm/θ such that

$$h_{(\varphi,\psi)}\eta=h_{(\varphi,\psi)}\zeta\quad\text{for each }(\eta,\zeta)\in\Gamma,\quad\text{but}\quad h_{(\varphi,\psi)}\varphi\neq h_{(\varphi,\psi)}\psi. \tag{3.1.12}$$

Let $X\subseteq P$ be the set of all variables occurring in any of the $(\zeta,\eta)\in\Gamma$ or in (φ,ψ). Since Γ is finite, X is finite as well, hence the set of all variables not in X is infinite. Without affecting property (3.1.12) of $h_{(\varphi,\psi)}$, we could modify $h_{(\varphi,\psi)}$ by mapping those variables not in X anywhere we want to. Since $P-X$ is countably infinite and the set $\{h_{(\varphi,\psi)}p\in Fm/\theta\,;\,p\in P\}$ is countable, we can define a map $\tau:(P-X)\to Fm/\theta$ in such a way that $h_{(\varphi,\psi)}p$ is the image under τ of infinitely many variables of $P-X$ for each $p\in P$. If we now define $h'_{(\varphi,\psi)}:\mathbf{Fm}\to\mathbf{Fm}/\theta$ by specifying that $h'_{(\varphi,\psi)}p=h_{(\varphi,\psi)}p$ for all $p\in X$ and $h'_{(\varphi,\psi)}p=\tau p$ for all $p\in P-X$, then $h'_{(\varphi,\psi)}$ still satisfies (3.1.12) and it also satisfies the hypothesis of the Claim.

Now $(\varphi,\psi)\notin\ker h'_{(\varphi,\psi)}$, but $\Gamma\subseteq\ker h'_{(\varphi,\psi)}$, hence $\Phi=\Theta^{\mathbf{Fm}}_{\mathcal{K}}(\Gamma)\subseteq\ker h'_{(\varphi,\psi)}$ since $\ker h'_{(\varphi,\psi)}$ is a $\mathcal{K}$-congruence (by the Homomorphism Theorem, $\mathbf{Fm}/\ker h'_{(\varphi,\psi)}\cong h'_{(\varphi,\psi)}(\mathbf{Fm})\in\mathrm{S}(\mathbf{Fm}/\theta)\subseteq\mathcal{K}$). Since (φ,ψ) was an arbitrary element satisfying $(\varphi,\psi)\notin\Phi$, we may construct an $h'_{(\varphi,\psi)}$ with the aforementioned properties for any such (φ,ψ) and we have that

$$\Phi\subseteq\bigcap\{\ker h'_{(\varphi,\psi)}\,;\,(\varphi,\psi)\notin\Phi\}.$$

The reverse inclusion also holds, because if $(\varphi,\psi)\notin\Phi$ then $(\varphi,\psi)\notin\ker h'_{(\varphi,\psi)}$ (by construction of $h'_{(\varphi,\psi)}$), therefore we have

$$\Phi=\bigcap\{\ker h'_{(\varphi,\psi)}\,;\,(\varphi,\psi)\notin\Phi\},$$

where each $\ker h'_{(\varphi,\psi)} \in \mathrm{Th}S_{\mathfrak{K}}$. By the claim, each $\ker h'_{(\varphi,\psi)}$ is of the form $\sigma^{-1}(\theta)$ for some surjective substitution σ and some $\theta \in \Omega(\mathrm{Th}S)$. By Corollary 1.8.6, $\ker h'_{(\varphi,\psi)} = \sigma^{-1}(\theta) \in \Omega(\mathrm{Th}S)$, hence Φ is an intersection of elements of $\Omega(\mathrm{Th}S)$, so Corollary 2.1.5 gives us that $\Phi \in \Omega(\mathrm{Th}S)$.

Now, suppose that Φ is an arbitrary $\mathfrak{K}$-congruence. Set

$$\mathcal{C} = \{\Theta^{\mathbf{Fm}}_{\mathfrak{K}}(\Gamma)\,;\, \Gamma \subseteq \Phi \text{ and } \Gamma \text{ is finite}\}.$$

Then $\Phi = \bigcup \mathcal{C}$ since $\mathbf{Th}S_{\mathfrak{K}}$ is an algebraic lattice under set inclusion, by Lemma 1.5.2. By the above, $\mathcal{C} \subseteq \Omega(\mathrm{Th}S)$, say $\mathcal{C} = \{\Omega T_i\,;\, i \in I\}$, where $T_i \in \mathrm{Th}S$ for each $i \in I$. We claim that $\{T_i\,;\, i \in I\}$ is upwardly directed. For suppose $T_i, T_j \in \{T_i\,;\, i \in I\}$. Then there exist $\Gamma_i, \Gamma_j \subseteq \Phi$ such that Γ_i, Γ_j are finite and $\Omega T_i = \Theta^{\mathbf{Fm}}_{\mathfrak{K}}(\Gamma_i)$ and $\Omega T_j = \Theta^{\mathbf{Fm}}_{\mathfrak{K}}(\Gamma_j)$. Set $\Omega T_\ell = \Theta^{\mathbf{Fm}}_{\mathfrak{K}}(\Gamma_i \cup \Gamma_j) \in \mathcal{C}$. The assumption that Ω is monotonic implies, by Corollary 2.1.5, that

$$\begin{aligned}\Omega(T_i \cap T_\ell) &= \Omega T_i \cap \Omega T_\ell \\ &= \Theta^{\mathbf{Fm}}_{\mathfrak{K}}(\Gamma_i) \cap \Theta^{\mathbf{Fm}}_{\mathfrak{K}}(\Gamma_i \cup \Gamma_j) \\ &= \Theta^{\mathbf{Fm}}_{\mathfrak{K}}(\Gamma_i) \\ &= \Omega T_i.\end{aligned}$$

The assumption that Ω is injective implies, therefore, that $T_i \cap T_\ell = T_i$, i.e., that $T_i \subseteq T_\ell$. It follows similarly that $T_j \subseteq T_\ell$, hence the set $\{T_i\,;\, i \in I\}$ is directed. Now, we can use the assumption that Ω is continuous and deduce that

$$\Phi = \bigcup \mathcal{C} = \bigcup_{i \in I} \Omega T_i = \Omega(\bigcup_{i \in I} T_i),$$

hence $\Phi \in \Omega(\mathrm{Th}S)$. This proves that $\mathrm{Con}_{\mathfrak{K}}\mathbf{Fm} \subseteq \Omega(\mathrm{Th}S)$ and hence that $\Omega(\mathrm{Th}S) = \mathrm{Con}_{\mathfrak{K}}\mathbf{Fm}$.

Finally, in view of Lemma 0.1.1, we need to show that Ω is order reflecting (we have assumed that Ω is order preserving and injective). So, suppose $T, U \in \mathrm{Th}S$ such that $\Omega T \subseteq \Omega U$. Since Ω is monotonic, Corollary 2.1.5 implies that $\Omega(T \cap U) = \Omega T \cap \Omega U = \Omega T$. Consequently, since Ω is injective, we have that $T \cap U = T$, i.e., $T \subseteq U$.

(vi)$\Rightarrow$(vii) Assume that $\Omega : \mathbf{Th}S \to \mathbf{Th}S_{\mathfrak{K}}$ is an isomorphism. To prove this implication, we shall show that Ω commutes with *surjective* substitutions and employ Theorem 1.9.4. Let σ be a surjective substitution and $T \in \mathrm{Th}S$. We first show that $\sigma_{S_{\mathfrak{K}}}(\Omega T) \subseteq \Omega\sigma_S(T)$. Take $(\varphi, \psi) \in \sigma(\Omega T)$ with, say $\sigma\varphi' = \varphi$ and $\sigma\psi' = \psi$, where $(\varphi', \psi') \in \Omega T$. Assume that $(\varphi, \psi) \notin \Omega\sigma_S(T)$.

By Definition 1.7.1, there exists a $\zeta(p,\overline{r}) \in Fm^k$ and formulas $\overline{\eta}$ such that, say, $\zeta(\varphi,\overline{\eta}) \in \sigma_S(T)$ and $\zeta(\psi,\overline{\eta}) \notin \sigma_S(T)$. Let $\overline{u}$ be a list of the variables occurring in $\overline{\eta}$ and let p' be a variable distinct from all of these. Define $\vartheta(p',\overline{u}) = \zeta(p',\overline{\eta})$ (so that $\vartheta[\tau/p'] = \zeta(\tau,\overline{\eta})$ for any $\tau \in Fm$). Since σ is surjective, we can apply Lemma 1.5.4, hence there exists a $\vartheta'(q,\overline{s}) \in Fm^k$ such that

$$\sigma(\vartheta'[\varphi'/q]) = \vartheta[\sigma\varphi'/p'] = \zeta(\sigma\varphi',\overline{\eta}) = \zeta(\varphi,\overline{\eta}),$$

and

$$\sigma(\vartheta'[\psi'/q]) = \vartheta[\sigma\psi'/p'] = \zeta(\sigma\psi',\overline{\eta}) = \zeta(\psi,\overline{\eta}).$$

So $\vartheta'[\varphi'/q] \in \sigma^{-1}(\zeta(\varphi,\overline{\eta}))$, hence $\vartheta'[\varphi'/q] \in \sigma^{-1}(\sigma_S(T))$. Similarly, $\vartheta'[\psi'/q] \notin \sigma^{-1}(\sigma_S(T))$. By Definition 1.7.1, again, this means that $(\varphi',\psi') \notin \Omega\sigma^{-1}(\sigma_S(T))$. Now, Ω is an order-preserving map by hypothesis, and $T \subseteq \sigma^{-1}(\sigma_S(T))$, hence we have that $(\varphi',\psi') \notin \Omega T$. This contradiction implies that $\sigma(\Omega T) \subseteq \Omega\sigma_S(T)$. Now,

$$\sigma_{S_{\mathcal{K}}}(\Omega T) = \mathrm{Cn}_{S_{\mathcal{K}}}\sigma(\Omega T) = \Theta^{\mathbf{Fm}}_{\mathcal{K}}\sigma(\Omega T) \subseteq \Theta^{\mathbf{Fm}}_{\mathcal{K}}(\Omega\sigma_S(T)) = \Omega\sigma_S(T),$$

with the last equality following from the hypothesis that $\Omega\sigma_S(T)$ is a $\mathcal{K}$-congruence of $\mathbf{Fm}$.

For the reverse inclusion, note that since $\sigma_{S_{\mathcal{K}}}(\Omega T)$ is a $\mathcal{K}$-congruence and Ω is a surjective map, there must exist a $U \in \mathrm{Th}S$ such that $\sigma_{S_{\mathcal{K}}}(\Omega T) = \Omega U$. Then

$$\Omega T \subseteq \sigma^{-1}(\sigma_{S_{\mathcal{K}}}(\Omega T)) = \sigma^{-1}(\Omega U) = \Omega\sigma^{-1}(U),$$

by Corollary 1.8.6. Thus $T \subseteq \sigma^{-1}(U)$ since Ω is an isomorphism, so $\sigma(T) \subseteq \sigma(\sigma^{-1}(U)) = U$ since σ is surjective. Finally, $\sigma_S(T) = \mathrm{Cn}_S\sigma(T) \subseteq \mathrm{Cn}_S U \subseteq U$ since U is an S-theory, hence $\Omega\sigma_S(T) \subseteq \Omega U = \sigma_{S_{\mathcal{K}}}(\Omega T)$.

(vii)⇒(viii) This is proved in Theorem 1.9.4.

(viii)⇒(i) Let $\boldsymbol{\tau} = \{\boldsymbol{\tau}^1,\ldots,\boldsymbol{\tau}^m\}$ be an interpretation of S in $S_{\mathcal{K}}$ and $\boldsymbol{\rho} = \{\boldsymbol{\rho}^1,\ldots,\boldsymbol{\rho}^n\}$ an interpretation of $S_{\mathcal{K}}$ in S satisfying (1.9.2) and (1.9.3). Since $S_{\mathcal{K}}$ is a 2-deductive system, each $\boldsymbol{\tau}^i$ is a 2-formula in one k-variable, i.e., $\boldsymbol{\tau}^i = (\tau^i_1(p_1,\ldots,p_k), \tau^i_2(p_1,\ldots,p_k))$. We claim that $\tau^1_1 \approx \tau^1_2,\ldots,\tau^m_1 \approx \tau^m_2$ form a system of defining equations for S and $\mathcal{K}$. Let $\Gamma \subseteq Fm^k$ and $\boldsymbol{\varphi} \in Fm^k$. Then

$$\Gamma \vdash_S \boldsymbol{\varphi} \quad \text{iff} \quad \boldsymbol{\tau}(\Gamma) \vdash_{S_{\mathcal{K}}} \boldsymbol{\tau}(\boldsymbol{\varphi}) \qquad \text{[by (1.9.1)]}$$

$$\text{iff} \quad \{\tau^i_1(\boldsymbol{\psi}) \approx \tau^i_2(\boldsymbol{\psi})\,;\, i \le m, \boldsymbol{\psi} \in \Gamma\} \models_{\mathcal{K}} \tau^j_1(\boldsymbol{\varphi}) \approx \tau^j_2(\boldsymbol{\varphi}) \quad \text{for each } j \le m,$$

by (3.1.1). This shows that $\mathcal{K}$ is an algebraic semantics for S.

Next, we claim that $\boldsymbol{\rho}^1, \ldots, \boldsymbol{\rho}^n$ form a system of equivalence k-formulas for S and $\mathcal{K}$. Note that $\boldsymbol{\rho}^i = \langle \rho_1^i(p_1, p_2), \ldots, \rho_k^i(p_1, p_2) \rangle$, for each $i \leq n$, i.e., each $\boldsymbol{\rho}^i$ is a k-formula in two variables. Let $(\varphi, \psi) \in Fm^2$. Then

$$(\varphi, \psi) \dashv\vdash_{S_{\mathcal{K}}} \tau(\{\boldsymbol{\rho}^1(\varphi, \psi), \ldots, \boldsymbol{\rho}^n(\varphi, \psi)\}) \qquad \text{[by (1.9.3)]}$$

so

$$(\varphi, \psi) \dashv\vdash_{S_{\mathcal{K}}} \{(\tau_1^i(\rho_1^j(\varphi, \psi), \ldots, \rho_k^j(\varphi, \psi)), \tau_2^i(\rho_1^j(\varphi, \psi), \ldots, \rho_k^j(\varphi, \psi))) \, ; i \leq m \, ; \, j \leq n\}$$

i.e.,

$$\varphi \approx \psi \mathrel{=\!\!||\!\!=_{\mathcal{K}}} \{\tau_1^i(\rho_1^j(\varphi, \psi), \ldots, \rho_k^j(\varphi, \psi)) \approx \tau_2^i(\rho_1^j(\varphi, \psi), \ldots, \rho_k^j(\varphi, \psi)) \, ; i \leq m \, ; \, j \leq n\},$$

by (3.1.1). Thus $\mathcal{K}$ is an equivalent algebraic semantics for S and $\mathcal{K}$, so S is algebraizable. □

It is evident from condition (iii) that an algebraizable k-deductive system is congruential, hence also weakly congruential and protoalgebraic.

It is possible for an algebraizable k-deductive system to have more than one system of defining equations or equivalence k-formulas. In the following theorem, however, we shall see that different sets of equivalence k-formulas must be inter-derivable and that different sets of defining equations must be equivalent over $\mathcal{K}$. We shall see, moreover, that the equivalent quasivariety semantics of an algebraizable k-deductive system must be unique.

3.1.12 THEOREM [BP89a, Theorem 2.15]

Let S be an algebraizable k-deductive system. Let $\mathcal{K}$ and $\mathcal{K}'$ be two equivalent algebraic (resp. quasivariety) semantics for S. Then $\mathcal{K}^{\mathrm{Q}} = \mathcal{K}'^{\mathrm{Q}}$ (resp. $\mathcal{K} = \mathcal{K}'$). Let $\boldsymbol{\Delta}_1(p, q), \ldots, \boldsymbol{\Delta}_n(p, q)$ and $\delta_1(\boldsymbol{p}) \approx \varepsilon_1(\boldsymbol{p}), \ldots, \delta_m(\boldsymbol{p}) \approx \varepsilon_m(\boldsymbol{p})$ be equivalence k-formulas and defining equations for S and $\mathcal{K}$, respectively, and let $\boldsymbol{\Delta}'_1(p, q), \ldots, \boldsymbol{\Delta}'_{n'}(p, q)$ and $\delta'_1(\boldsymbol{p}) \approx \varepsilon'_1(\boldsymbol{p}), \ldots, \delta'_{m'}(\boldsymbol{p}) \approx \varepsilon'_{m'}(\boldsymbol{p})$ be equivalence k-formulas and defining equations for S and $\mathcal{K}'$, respectively. Then

$$\boldsymbol{\Delta}'_1(p, q), \ldots, \boldsymbol{\Delta}'_{n'}(p, q) \dashv\vdash_S \boldsymbol{\Delta}_1(p, q), \ldots, \boldsymbol{\Delta}_n(p, q),$$

and

$$\{\delta'_i(\boldsymbol{p}) \approx \varepsilon'_i(\boldsymbol{p}) \, ; i \leq m'\} \mathrel{=\!\!||\!\!=_{\mathcal{K}}} \{\delta_i(\boldsymbol{p}) \approx \varepsilon_i(\boldsymbol{p}) \, ; i \leq m\}.$$

Proof. We first prove that the two systems of equivalence k-formulas are inter-derivable. If we consider that each $\Delta'_{ij}(p, q)$ is merely an $\mathcal{L}$-formula, then, by Lemma 3.1.10 (iv),

(3.1.13) $$\boldsymbol{\Delta}(p, q) \vdash_S \{\boldsymbol{\Delta}(\Delta'_{ij}(p, p), \Delta'_{ij}(p, q)) \, ; i \leq n', \; j \leq k\}.$$

In Theorem 3.1.11, we showed that the equivalence k-formulas of condition (i) satisfy (ii) and thereby also (iii), i.e., $\boldsymbol{\Delta}'_1(p, q), \ldots, \boldsymbol{\Delta}'_{n'}(p, q)$ is a system of equivalence k-formulas as in Definition 2.3.1. Thus we get that (2.3.1)′ holds, i.e., $\vdash_S \boldsymbol{\Delta}'(p, p)$, i.e.,

$$(3.1.14) \qquad \vdash_S \{\langle \Delta'_{i1}(p,p), \ldots, \Delta'_{ik}(p,p)\rangle\,;\, i \le n'\}.$$

Since $(2.3.2)'$ also holds, we can use k applications of it to get, for each $i \le n'$,

$$(3.1.15) \quad \langle \Delta'_{i1}(p,p), \ldots, \Delta'_{ik}(p,p)\rangle, \{\Delta(\Delta'_{ij}(p,p), \Delta'_{ij}(p,q))\,;\, j \le k\} \vdash_S \langle \Delta'_{i1}(p,q), \ldots, \Delta'_{ik}(p,q)\rangle.$$

So, from (3.1.13), (3.1.14) and (3.1.15) we get

$$\Delta(p,q) \vdash_S \{\Delta(\Delta'_{ij}(p,p), \Delta'_{ij}(p,q))\,;\, i \le n'\,;\, j \le k\}, \{\langle \Delta'_{i1}(p,p), \ldots, \Delta'_{ik}(p,p)\rangle\,;\, i \le n'\},$$

so $\quad \Delta(p,q) \vdash_S \{\langle \Delta'_{i1}(p,q), \ldots, \Delta'_{ik}(p,q)\rangle\,;\, i \le n'\}$,

hence $\Delta(p,q) \vdash_S \Delta'(p,q)$, and, symmetrically, $\Delta'(p,q) \vdash_S \Delta(p,q)$.

With this result we can now show that $\mathcal{K}^{\mathbf{Q}} = \mathcal{K}'^{\mathbf{Q}}$. Let $\left(\&_{i \le \ell} \zeta_i \approx \eta_i\right) \Rightarrow \varphi \approx \psi$ be a quasi-identity over the language of S. We have that

$\mathcal{K} \models \left(\&_{i \le \ell} \zeta_i \approx \eta_i\right) \Rightarrow \varphi \approx \psi$

iff $\quad \{\zeta_i \approx \eta_i\,;\, i \le \ell\} \models_{\mathcal{K}} \varphi \approx \psi$

iff $\quad \{\Delta(\zeta_i, \eta_i)\,;\, i \le \ell\} \vdash_S \Delta(\varphi, \psi)$ [by Lemma 3.1.6 (i)]

iff $\quad \{\Delta'(\zeta_i, \eta_i)\,;\, i \le \ell\} \vdash_S \Delta'(\varphi, \psi)$ [by the above]

iff $\quad \{\zeta_i \approx \eta_i\,;\, i \le \ell\} \models_{\mathcal{K}'} \varphi \approx \psi$ [by Lemma 3.1.6 (i)]

iff $\quad \mathcal{K}' \models \left(\&_{i \le \ell} \zeta_i \approx \eta_i\right) \Rightarrow \varphi \approx \psi$

Thus $\mathcal{K}$ and $\mathcal{K}'$ satisfy the same quasi-identities, i.e., $\mathcal{K}^{\mathbf{Q}} = \mathcal{K}'^{\mathbf{Q}}$. To complete the proof, we have

$\{\delta'_i(\boldsymbol{p}) \approx \varepsilon'_i(\boldsymbol{p})\,;\, i \le m'\} =\!\mid\models_{\mathcal{K}} \{\delta_i(\boldsymbol{p}) \approx \varepsilon_i(\boldsymbol{p})\,;\, i \le m\}$

iff $\quad \{\Delta(\delta'_i(\boldsymbol{p}), \varepsilon'_i(\boldsymbol{p}))\,;\, i \le m'\} \dashv\vdash_S \{\Delta(\delta_i(\boldsymbol{p}), \varepsilon_i(\boldsymbol{p}))\,;\, i \le m\}$

iff $\quad \{\Delta'(\delta'_i(\boldsymbol{p}), \varepsilon'_i(\boldsymbol{p}))\,;\, i \le m'\} \dashv\vdash_S \{\Delta(\delta_i(\boldsymbol{p}), \varepsilon_i(\boldsymbol{p}))\,;\, i \le m\}$

iff $\quad \boldsymbol{p} \dashv\vdash_S \boldsymbol{p}$ [by Lemma 3.1.6],

which is clearly true. □

With the knowledge that the equivalent quasivariety semantics of an algebraizable k-deductive system is unique, we now seek a description of it. In the following corollary we give one in terms of the Leibniz operator of the matrix $\langle \mathbf{Fm}, \mathrm{Cn}_S(\emptyset)\rangle$. Furthermore, we show that the equivalent quasivariety semantics satisfies each of the conditions (vi), (vii) and (viii) of Theorem 3.1.11.

3.1.13 COROLLARY

Let S be an algebraizable k-deductive system. Let $\mathcal{K}$ be the quasivariety generated by $K =$

$\{\mathbf{Fm}/\theta\,;\,\theta\in\Omega(\mathrm{Th}S)\}$ *(as in the proof of the implication (v)⇒(vi) of Theorem 3.1.11), i.e.,* $\mathcal{K}=\mathrm{ISPP}_{\mathrm{U}}(K)$. *Then* $\mathcal{K}$ *is the equivalent quasivariety semantics for* S *and*

(i) $\Omega:\mathbf{Th}S\to\mathbf{Th}S_{\mathcal{K}}$ *is an isomorphism which commutes with substitutions,*

(ii) S *is equivalent to* $S_{\mathcal{K}}$,

Proof. Note that in the proof of Theorem 3.1.11, the quasivariety $\mathcal{K}$ defined here is proved to satisfy condition (vi), and the same $\mathcal{K}$ is used for the proofs of (vi) ⇒ (vii) and (vii) ⇒ (viii) with Ω for f in (vii). In the proof of (viii) ⇒ (i) it is shown that a quasivariety satisfying (viii) is an equivalent algebraic (hence quasivariety) semantics for S. Consequently, we must have that $\mathcal{K}$ is the unique quasivariety semantics for S, which completes the proof. □

By Theorem 3.1.11 (vi), the Leibniz operator is an isomorphism from $\mathbf{Th}S$ onto the lattice of $\mathcal{K}$-congruences of $\mathbf{Fm}$, where, by the above corollary, we can take $\mathcal{K}$ to be the equivalent quasivariety semantics of S. As is shown in the following corollary, the same holds for the Leibniz operator associated with each matrix model.

3.1.14 COROLLARY [BP89a, Theorem 5.1]

Let S *be a* k*-deductive system and* $\mathcal{K}$ *a quasivariety.*

(i) *The following are equivalent:*

(i′) S *is algebraizable with equivalent quasivariety semantics* $\mathcal{K}$,

(i″) *If* $\mathbf{A}$ *is an* $\mathcal{L}$*-algebra,* $F_{\mathcal{A}}=\bigcap\{F\subseteq A^k\,;\,\langle\mathbf{A},F\rangle$ *is an* S*-matrix}* *(i.e.,* $F_{\mathcal{A}}$ *is the least subset of* A^k *such that* $\langle\mathbf{A},F_{\mathcal{A}}\rangle$ *is an* S*-matrix) and* $\mathcal{A}=\langle\mathbf{A},F_{\mathcal{A}}\rangle$ *then* $\Omega_{\mathcal{A}}:\mathbf{Fi}^S\mathcal{A}\to\mathbf{Con}_{\mathcal{K}}\mathbf{A}$ *is an isomorphism.*

(ii) *Assume* S *is algebraizable with equivalent quasivariety semantics* $\mathcal{K}$, *and let* $\delta_1(\boldsymbol{p})\approx\varepsilon_1(\boldsymbol{p}),\ldots,\delta_m(\boldsymbol{p})\approx\varepsilon_m(\boldsymbol{p})$ *be a system of defining equations for* S *and* $\mathcal{K}$. *Let* $\mathbf{A}$ *be an* $\mathcal{L}$*-algebra, let* $F_{\mathcal{A}}$ *be the least subset of* A^k *for which* $\langle\mathbf{A},F_{\mathcal{A}}\rangle$ *is an* S*-matrix, and let* $\mathcal{A}=\langle\mathbf{A},F_{\mathcal{A}}\rangle$. *For each* $\Phi\in\mathrm{Con}_{\mathcal{K}}\mathbf{A}$, *define*

$$H_{\mathcal{A}}(\Phi)=\{\boldsymbol{a}\in A^k\,;\,(\delta_i^{\mathbf{A}}(\boldsymbol{a}),\varepsilon_i^{\mathbf{A}}(\boldsymbol{a}))\in\Phi\ \text{ for } i\leq m\}.$$

Then $H_{\mathcal{A}}$ *is the inverse of* $\Omega_{\mathcal{A}}$.

Proof. The implication from (i″) to (i′) is proved by taking $\mathcal{A}=\langle\mathbf{Fm},\mathrm{Cn}_S(\emptyset)\rangle$ and employing Theorem 3.1.11 and the previous corollary. Assume now that (i′) holds, and that

$\Delta_1(p,q),\ldots,\Delta_n(p,q)$ are equivalence k-formulas for S and $\mathfrak{K}$. Let $\mathcal{A} = \langle \mathbf{A}, F_{\mathcal{A}} \rangle$ be as described in (i″). By Theorem 3.1.11 (iv) and Lemma 0.1.1, we need only show that $\Omega_{\mathcal{A}}$ maps $\mathrm{Fi}^S\mathcal{A}$ onto the whole of $\mathrm{Con}_{\mathfrak{K}}\mathbf{A}$ and that $\Omega_{\mathcal{A}}$ is order reflecting. First we shall show that $\Omega_{\mathcal{A}}F$ is a $\mathfrak{K}$-congruence for every $F \in \mathrm{Fi}^S\mathcal{A}$. By Theorem 3.1.11 (iii), $\Delta_1(p,q),\ldots,\Delta_n(p,q)$ form a finite system of congruence k-formulas without parameters for S, hence, by Theorem 2.4.2,

$$\Omega_{\mathcal{A}}F = \{(a,b) \in A^2 ;\ \Delta_i^{\mathbf{A}}(a,b) \in F \ \text{ for all } i \le n\}.$$

Suppose that
$$\mathfrak{K} \models \Big(\mathop{\&}_{i \le \ell} \zeta_i \approx \eta_i \Big) \Rightarrow \varphi \approx \psi,$$
or, equivalently,
$$\{\zeta_i \approx \eta_i ;\ i \le \ell\} \models_{\mathfrak{K}} \varphi \approx \psi.$$

Let $\bar{a}$ be an interpretation of the variables of the $\zeta_i, \eta_i, \varphi, \psi$ in A such that $(\zeta_i^{\mathbf{A}}(\bar{a}), \eta_i^{\mathbf{A}}(\bar{a})) \in \Omega_{\mathcal{A}}F$ for all $i \le \ell$. Then, by the above, we have that

$$\Delta_j^{\mathbf{A}}(\zeta_i^{\mathbf{A}}(\bar{a}), \eta_i^{\mathbf{A}}(\bar{a})) \in F \text{ for all } j \le n \text{ and all } i \le \ell.$$

Since $\mathfrak{K}$ is an equivalent algebraic semantics for S, Lemma 3.1.6 (i) implies that

$$\{\Delta_j(\zeta_i, \eta_i) ;\ j \le n ;\ i \le \ell\} \vdash_S \{\Delta_j(\varphi, \psi) ;\ j \le n\}.$$

Thus, since F is an S-filter of $\mathcal{A}$, we get $\Delta_j^{\mathbf{A}}(\varphi^{\mathbf{A}}(\bar{a}), \psi^{\mathbf{A}}(\bar{a})) \in F$ for all $j \le n$. Consequently, $(\varphi^{\mathbf{A}}(\bar{a}), \psi^{\mathbf{A}}(\bar{a})) \in \Omega_{\mathcal{A}}F$, so $\mathbf{A}/\Omega_{\mathcal{A}}F \in \mathfrak{K}$, implying that $\Omega_{\mathcal{A}}F$ is a $\mathfrak{K}$-congruence.

Next, we show that $\Omega_{\mathcal{A}}$ is surjective. Let $\delta_1(\boldsymbol{p}) \approx \varepsilon_1(\boldsymbol{p}),\ldots,\delta_m(\boldsymbol{p}) \approx \varepsilon_m(\boldsymbol{p})$ be a system of defining equations for S and $\mathfrak{K}$. Let $\Phi \in \mathrm{Con}_{\mathfrak{K}}\mathbf{A}$, and let $H_{\mathcal{A}}\Phi$ be the subset of A^k defined in part (ii) of this theorem. We show that $H_{\mathcal{A}}\Phi$ is an S-filter of $\mathcal{A}$. Suppose that $\Gamma \vdash_S \varphi$ and assume without loss of generality that Γ is finite. Let $\bar{a}$ be an interpretation of the variables of $\Gamma \cup \{\varphi\}$ in A such that $\boldsymbol{\psi}^{\mathbf{A}}(\bar{a}) \in H_{\mathcal{A}}\Phi$ for all $\boldsymbol{\psi} \in \Gamma$. Then $(\delta_i^{\mathbf{A}}(\boldsymbol{\psi}^{\mathbf{A}}(\bar{a})), \varepsilon_i^{\mathbf{A}}(\boldsymbol{\psi}^{\mathbf{A}}(\bar{a}))) \in \Phi$ for all $\boldsymbol{\psi} \in \Gamma$ and $i \le m$. Now, $\Gamma \vdash_S \boldsymbol{\varphi}$

iff
$$\{\delta(\boldsymbol{\psi}) \approx \varepsilon(\boldsymbol{\psi}) ;\ \boldsymbol{\psi} \in \Gamma\} \models_{\mathfrak{K}} \delta(\boldsymbol{\varphi}) \approx \varepsilon(\boldsymbol{\varphi})$$
iff
$$\{(\delta(\boldsymbol{\psi}), \varepsilon(\boldsymbol{\psi})) ;\ \boldsymbol{\psi} \in \Gamma\} \vdash_{S_{\mathfrak{K}}} (\delta(\boldsymbol{\varphi}), \varepsilon(\boldsymbol{\varphi})).$$

Since $\Phi \in \mathrm{Con}_{\mathfrak{K}}\mathbf{A}$, $\langle \mathbf{A}, \Phi \rangle$ is an $S_{\mathfrak{K}}$-matrix. Thus, since $(\delta_i^{\mathbf{A}}(\boldsymbol{\psi}^{\mathbf{A}}(\bar{a})), \varepsilon_i^{\mathbf{A}}(\boldsymbol{\psi}^{\mathbf{A}}(\bar{a}))) \in \Phi$ for all $\boldsymbol{\psi} \in \Gamma$ and $i \le m$, we have that $(\delta_i^{\mathbf{A}}(\boldsymbol{\varphi}^{\mathbf{A}}(\bar{a})), \varepsilon_i^{\mathbf{A}}(\boldsymbol{\varphi}^{\mathbf{A}}(\bar{a}))) \in \Phi$ for all $i \le m$. Thus $\boldsymbol{\varphi}^{\mathbf{A}}(\bar{a}) \in H_{\mathcal{A}}\Phi$, and so $\langle \mathbf{A}, H_{\mathcal{A}}\Phi \rangle$ is an S-matrix. Thus, $F_{\mathcal{A}} \subseteq H_{\mathcal{A}}\Phi$ and so $H_{\mathcal{A}}\Phi \in \mathrm{Fi}^S\mathcal{A}$. We claim that $\Omega_{\mathcal{A}}H_{\mathcal{A}}\Phi = \Phi$. By the above remarks, for all $a,b \in A$, $(a,b) \in \Omega_{\mathcal{A}}H_{\mathcal{A}}\Phi$ if and only if $(\delta_i^{\mathbf{A}}(\Delta_j^{\mathbf{A}}(a,b)), \varepsilon_i^{\mathbf{A}}(\Delta_j^{\mathbf{A}}(a,b))) \in \Phi$ for all $i \le m$ and $j \le n$. Now, using (3.1.1), (3.1.5) becomes

$$(\varphi, \psi) \dashv\vdash_{S_{\mathfrak{K}}} \{(\delta_i(\Delta_j(\varphi,\psi)), \varepsilon_i(\Delta_j(\varphi,\psi))) ;\ i \le m ;\ j \le n\}.$$

Thus it follows that $(\delta_i^{\mathbf{A}}(\Delta_j^{\mathbf{A}}(a,b)), \varepsilon_i^{\mathbf{A}}(\Delta_j^{\mathbf{A}}(a,b))) \in \Phi$ for each $i \leq m$ and $j \leq n$ if and only if $(a,b) \in \Phi$, which proves the claim that $\Omega_{\mathcal{A}} H_{\mathcal{A}} \Phi = \Phi$, and hence that $\Omega_{\mathcal{A}}$ is surjective.

Now, suppose that F, G are S-filters of $\mathcal{A}$ such that $\Omega_{\mathcal{A}} F \subseteq \Omega_{\mathcal{A}} G$. Since S is protoalgebraic, we can apply Corollary 2.1.5 and get that

$$\Omega_{\mathcal{A}}(F \cap G) = \Omega_{\mathcal{A}} F \cap \Omega_{\mathcal{A}} G = \Omega_{\mathcal{A}} F.$$

By Theorem 3.1.11 (iv), we have that $\Omega_{\mathcal{A}}$ is injective, so $F \cap G = F$, i.e., $F \subseteq G$, hence $\Omega_{\mathcal{A}}$ is order reflecting. This proves part (i). Furthermore, this helps in proving part (ii), for now all that is needed is to show that $H_{\mathcal{A}} \Omega_{\mathcal{A}} F = F$ for every $F \in \mathrm{Fi}^S \mathcal{A}$. If $\boldsymbol{a} \in A^k$, then $\boldsymbol{a} \in H_{\mathcal{A}} \Omega_{\mathcal{A}} F$ if and only if $(\boldsymbol{\delta}^{\mathbf{A}}(\boldsymbol{a}), \boldsymbol{\varepsilon}^{\mathbf{A}}(\boldsymbol{a})) \in (\Omega_{\mathcal{A}} F)^{[k]}$. This holds if and only if $\Delta_j^{\mathbf{A}}(\delta_i(\boldsymbol{a}), \varepsilon_i(\boldsymbol{a})) \in F$ for all $i \leq m$ and $j \leq n$. But since F is an S-filter of $\mathcal{A}$ this holds if and only if $\boldsymbol{a} \in F$, by Lemma 3.1.6 (ii). This proves part (ii). □

As a sort of converse to the previous corollary, the following corollary shows that if the class of reduced matrix models of an algebraizable k-deductive system is known, then it is possible to give a description of the equivalent quasivariety semantics in terms of the reduced matrix models.

3.1.15 COROLLARY [BP89a, Corollary 5.3]

Let S be an algebraizable k-deductive system. Then the equivalent quasivariety semantics for S, $\mathfrak{K}$ say, is the class of all algebra reducts of Mod^*S, *i.e.,*

(3.1.16) $$\mathfrak{K} = \{\mathbf{A} ; \langle \mathbf{A}, F \rangle \in \mathrm{Mod}^* S \text{ for some } F \subseteq A^k\}.$$

Moreover, for each $\mathbf{A} \in \mathfrak{K}$, there exists a unique $F \subseteq A^k$ such that $\langle \mathbf{A}, F \rangle \in \mathrm{Mod}^* S$.

Proof. Let $\mathcal{A} = \langle \mathbf{A}, F \rangle \in \mathrm{Mod}^* S$. Then F is an S-filter of $\mathcal{A}' = \langle \mathbf{A}, F_{\mathcal{A}} \rangle$ where $F_{\mathcal{A}}$ is the least subset of A^k for which $\langle \mathbf{A}, F_{\mathcal{A}} \rangle$ is an S-matrix. Clearly $\Omega_{\mathcal{A}} F = \Omega_{\mathcal{A}'} F$. By Corollary 3.1.14 (i), $\Omega_{\mathcal{A}} F$ is a $\mathfrak{K}$-congruence of $\mathbf{A}$, so $\mathbf{A}/\Omega_{\mathcal{A}} F \in \mathfrak{K}$. But since $\mathcal{A}$ is reduced, $\Omega_{\mathcal{A}} F = I_A$, hence $\mathbf{A} \cong \mathbf{A}/\Omega_{\mathcal{A}} F \in \mathfrak{K}$. Conversely, let $\mathbf{A} \in \mathfrak{K}$, let $F_{\mathcal{A}}$ be the least subset of A^k for which $\langle \mathbf{A}, F_{\mathcal{A}} \rangle$ is an S-matrix and let $\mathcal{A} = \langle \mathbf{A}, F_{\mathcal{A}} \rangle$. Then I_A is the least $\mathfrak{K}$-congruence of $\mathbf{A}$ and $F_{\mathcal{A}}$ is the least S-filter of $\mathcal{A}$, hence, by Corollary 3.1.14 (i), $F_{\mathcal{A}}$ is unique in $\mathrm{Fi}^S \mathcal{A}$ such that $\Omega_{\mathcal{A}} F_{\mathcal{A}} = I_A$. Since $F_{\mathcal{A}}$ is contained in any $G \subseteq A^k$ with $\langle \mathbf{A}, G \rangle \in \mathrm{Mod} S$, $F_{\mathcal{A}}$ is unique among subsets G of A^k with $\Omega_{\mathcal{A}} G = I_A$. Thus $\langle \mathbf{A}, F_{\mathcal{A}} \rangle \in \mathrm{Mod}^* S$ is the required S-matrix. □

Returning to Theorem 3.1.11, we now show that the formulas defined in (ii) and (iii) there can be taken as defining equations and equivalence k-formulas.

3.1.16 COROLLARY

Let S be a k-deductive system and $\mathcal{K}$ the class defined in (3.1.16). Let $\Delta_1(p,q),\dots,\Delta_n(p,q)$ be a system of k-formulas in 2 variables and $\delta_1(\boldsymbol{p}),\dots,\delta_m(\boldsymbol{p})$, $\varepsilon_1(\boldsymbol{p}),\dots,\varepsilon_m(\boldsymbol{p})$ a system of 1-formulas in one k-variable. The following are equivalent:

(i) S is algebraizable with equivalent quasivariety semantics $\mathcal{K}$, equivalence k-formulas $\Delta_1(p,q),\dots,\Delta_n(p,q)$ and defining equations $\delta_1(\boldsymbol{p})\approx\varepsilon_1(\boldsymbol{p}),\dots,\delta_m(\boldsymbol{p})\approx\varepsilon_m(\boldsymbol{p})$.

(ii) The systems $\Delta_1(p,q),\dots,\Delta_n(p,q)$ and $\delta_1(\boldsymbol{p}),\dots,\delta_m(\boldsymbol{p})$, $\varepsilon_1(\boldsymbol{p}),\dots,\varepsilon_m(\boldsymbol{p})$ satisfy the conditions of 3.1.11 (ii).

(iii) The systems $\Delta_1(p,q),\dots,\Delta_n(p,q)$ and $\delta_1(\boldsymbol{p}),\dots,\delta_m(\boldsymbol{p})$, $\varepsilon_1(\boldsymbol{p}),\dots,\varepsilon_m(\boldsymbol{p})$ satisfy the conditions of 3.1.11 (iii).

Proof. That (i)⇒(ii) and (ii)⇒(iii) is evident from the proof of the respective implications in Theorem 3.1.11. Assume that (iii) holds. We shall use Lemma 3.1.6 to prove the result, namely, that $\Delta_1(p,q),\dots,\Delta_n(p,q)$ forms a system of equivalence k-formulas and that $\delta_1(\boldsymbol{p})\approx\varepsilon_1(\boldsymbol{p}),\dots,\delta_m(\boldsymbol{p})\approx\varepsilon_m(\boldsymbol{p})$ forms a system of defining equations for S and $\mathcal{K}$. Lemma 3.1.6 (ii) holds by hypothesis. For Lemma 3.1.6 (i), let $\Gamma\subseteq \mathrm{Id}(P)$ and $\varphi\approx\psi\in\mathrm{Id}(P)$. Then

$$\{\Delta(\zeta,\eta)\,;\,\zeta\approx\eta\in\Gamma\}\vdash_S\Delta(\varphi,\psi)\quad\text{iff}\quad\{\Delta(\zeta,\eta)\,;\,\zeta\approx\eta\in\Gamma\}\models_{\mathrm{Mod}^*S}\Delta(\varphi,\psi),$$

since Mod^*S is a matrix semantics for S, by Theorem 1.8.9. Let $\mathcal{A}=\langle\mathbf{A},F\rangle\in\mathrm{Mod}^*S$. Then

$\{\Delta(\zeta,\eta)\,;\,\zeta\approx\eta\in\Gamma\}\models_{\mathcal{A}}\Delta(\varphi,\psi)$

iff for any interpretation $\overline{a}$ of the variables of Γ and $\varphi\approx\psi$ in A,

$\Delta^{\mathbf{A}}(\zeta^{\mathbf{A}}(\overline{a}),\eta^{\mathbf{A}}(\overline{a}))\in F$ for all $\zeta\approx\eta\in\Gamma$ implies $\Delta^{\mathbf{A}}(\varphi^{\mathbf{A}}(\overline{a}),\psi^{\mathbf{A}}(\overline{a}))\in F$

iff for any interpretation $\overline{a}$ of the variables of Γ and $\varphi\approx\psi$ in A,

$(\zeta^{\mathbf{A}}(\overline{a}),\eta^{\mathbf{A}}(\overline{a}))\in\Omega_{\mathcal{A}}F$ for all $\zeta\approx\eta\in\Gamma$ implies $(\varphi^{\mathbf{A}}(\overline{a}),\psi^{\mathbf{A}}(\overline{a}))\in\Omega_{\mathcal{A}}F$ [by (2.4.4)″]

iff for any interpretation $\overline{a}$ of the variables of Γ and $\varphi\approx\psi$ in A,

$\zeta^{\mathbf{A}}(\overline{a})=\eta^{\mathbf{A}}(\overline{a})$ for all $\zeta\approx\eta\in\Gamma$ implies $\varphi^{\mathbf{A}}(\overline{a})=\psi^{\mathbf{A}}(\overline{a}))$ [since $\Omega_{\mathcal{A}}F=I_A$]

iff $\Gamma\models_{\mathbf{A}}\varphi\approx\psi$.

By the definition of $\mathcal{K}$, we have the required result, namely

$$\{\Delta(\zeta,\eta)\,;\,\zeta\approx\eta\in\Gamma\}\vdash_S\Delta(\varphi,\psi)\quad\text{iff}\quad\Gamma\models_{\mathcal{K}}\varphi\approx\psi.\qquad\square$$

3.1.17 COROLLARY [BP89a, Corollary 4.9]

Let S be an algebraizable k-deductive system, let $\mathcal{K}$ be its equivalent quasivariety semantics and let $\boldsymbol{\delta}(\boldsymbol{p}) \approx \boldsymbol{\varepsilon}(\boldsymbol{p})$ and $\Delta(p,q)$ be defining equations and equivalence k-formulas for S and $\mathcal{K}$. Then an extension S' of S with the same language as S is algebraizable and $\boldsymbol{\delta}(\boldsymbol{p}) \approx \boldsymbol{\varepsilon}(\boldsymbol{p})$ and $\Delta(p,q)$ are defining equations and equivalence k-formulas for S' and its equivalent quasivariety semantics. An arbitrary extension $S'' = \langle \mathcal{L}'', \vdash_{S''} \rangle$ of S is algebraizable if for every $f \in \mathcal{L}'' - \mathcal{L}$ with $ar(f) = \ell$, and all $\varphi_1, \ldots, \varphi_\ell, \psi_1, \ldots, \psi_\ell \in Fm$,

$$\Delta(\varphi_1, \psi_1), \ldots, \Delta(\varphi_\ell, \psi_\ell) \vdash_{S''} \Delta(f(\varphi_1, \ldots, \varphi_\ell), f(\psi_1, \ldots, \psi_\ell)).$$

In this case, $\boldsymbol{\delta}(\boldsymbol{p}) \approx \boldsymbol{\varepsilon}(\boldsymbol{p})$ and $\Delta(p,q)$ are defining equations and equivalence k-formulas for S'' and its equivalent quasivariety semantics.

Proof. Conditions (1) to (5) of Theorem 3.1.11 (ii) hold for S and they continue to hold regardless of any additions to the axioms or the inference rules of S. Thus S' is algebraizable and by Corollary 3.1.16 the same defining equations and equivalence k-formulas continue to function as such for S'. If S'' satisfies the conditions of the present corollary, then it satisfies the conditions of Theorem 3.1.11 (iii), hence it is algebraizable and, by Corollary 3.1.16, the same defining equations and equivalence k-formulas continue to function as such for S''. □

Note that if S' is an extension of S obtained purely by the addition of axioms and inference rules, then the equivalent quasivariety semantics $\mathcal{K}'$ for S' will satisfy all the identities and quasi-identities of the equivalent quasivariety semantics $\mathcal{K}$ for S (and possibly some more), hence $\mathcal{K}'$ will be a subquasivariety of $\mathcal{K}$.

The following theorem is perhaps the most useful in determining the equivalent quasivariety semantics of an algebraizable k-deductive system (if the axioms and inference rules are known). For here we give an explicit axiomatization based on the defining equations and equivalence k-formulas.

3.1.18 THEOREM [BP89a, Theorem 2.17]

Let S be a k-deductive system defined by a set Ax of axioms and a set Ir of inference rules of S. Assume that S is algebraizable with equivalence k-formulas $\Delta_1(p,q), \ldots, \Delta_n(p,q)$ and defining *equations $\delta_1(p) \approx \varepsilon_1(p), \ldots, \delta_m(p) \approx \varepsilon_m(p)$. Then the unique equivalent quasivariety semantics for S is axiomatized by the identities*

(i) $\delta_i(\varphi) \approx \varepsilon_i(\varphi)$ *for all* $\varphi \in Ax$ *and* $i \leq m$,

(ii) $\delta_i(\Delta_j(p,p)) \approx \varepsilon_i(\Delta_j(p,p))$ *for all* $i \leq m$ *and* $j \leq n$,

together with the following quasi-identities

(iii) $\left(\mathop{\&}_{i \leq m} \mathop{\&}_{j \leq \ell} \delta_i(\psi_j) \approx \varepsilon_i(\psi_j)\right) \Rightarrow \delta_h(\varphi) \approx \varepsilon_h(\varphi)$ *for each* $\langle\{\psi_1, \ldots, \psi_\ell\}, \varphi\rangle \in Ir$ *and* $h \leq m$,

(iv) $\left(\mathop{\&}_{i \leq m} \mathop{\&}_{j \leq n} \delta_i(\Delta_j(p,q)) \approx \varepsilon_i(\Delta_j(p,q))\right) \Rightarrow p \approx q$.

Proof. Let $\mathcal{K}'$ be the quasivariety defined by (i)-(iv). We shall show that $\mathcal{K}'$ is the equivalent quasivariety semantics for S. The quasi-identity (iv) implies that

$$\delta(\Delta(p,q)) \approx \varepsilon(\Delta(p,q)) \models_{\mathcal{K}'} p \approx q,$$

and the identity (ii) implies that

$$p \approx q \models_{\mathcal{K}'} \delta(\Delta(p,q)) \approx \varepsilon(\Delta(p,q)),$$

hence (3.1.6) holds. Let $\Gamma \subseteq Fm^k$ and set

$$X = \{\varphi \in Fm^k ; \{\delta(\psi) \approx \varepsilon(\psi) ; \psi \in \Gamma\} \models_{\mathcal{K}'} \delta(\varphi) \approx \varepsilon(\varphi)\}.$$

We shall show that (3.1.3) holds by showing that $\Gamma \vdash_S \varphi$ if and only if $\varphi \in X$. From (i) it follows that $Ax \subseteq X$, and from (iii) it follows that X is closed under the inference rules of S, hence, X is an S-theory. If $\Gamma \vdash_S \varphi$ then $X \vdash_S \varphi$ (since $\Gamma \subseteq X$), so $\varphi \in X$. Conversely, suppose that $\varphi \in X$. Let $\mathcal{K}$ be the equivalent quasivariety semantics of S. Then $\mathcal{K}$ satisfies (i)-(iv) (by (3.1.3) and (3.1.6)), so $\mathcal{K} \subseteq \mathcal{K}'$. Since $\varphi \in X$,

$$\{\delta(\psi) \approx \varepsilon(\psi) ; \psi \in \Gamma\} \models_{\mathcal{K}'} \delta(\varphi) \approx \varepsilon(\varphi),$$

hence

$$\{\delta(\psi) \approx \varepsilon(\psi) ; \psi \in \Gamma\} \models_{\mathcal{K}} \delta(\varphi) \approx \varepsilon(\varphi).$$

By (3.1.6), $\Gamma \vdash_S \varphi$. □

Since the notion of equivalence is an equivalence relation on the class of deductive systems, we obtain the following result from the remarks following Proposition 1.9.2 and Theorem 3.1.11 (viii).

3.1.19 THEOREM

Let S_1 and S_2 be equivalent deductive systems. Suppose that S_1 is a k-deductive system and S_2 is an ℓ-deductive system and let τ be an interpretation of S_1 in S_2 and ρ an interpretation of S_2 in

S_1 such that (1.9.2) and (1.9.3) hold. Let S_1 be algebraizable with equivalent quasivariety semantics $\mathcal{K}$, defining equations $\boldsymbol{\delta}(\boldsymbol{p}) \approx \boldsymbol{\varepsilon}(\boldsymbol{p})$ and equivalence k-formulas $\boldsymbol{\Delta}(p,q)$. Then S_2 is algebraizable with equivalent quasivariety semantics $\mathcal{K}$, defining equations $\boldsymbol{\delta}(\boldsymbol{\rho}(\boldsymbol{p})) \approx \boldsymbol{\varepsilon}(\boldsymbol{\rho}(\boldsymbol{p}))$ and equivalence ℓ-formulas $\boldsymbol{\tau}(\boldsymbol{\Delta}(p,q))$. □

It is possible to characterize strong algebraizability in terms of the relation $\vdash_S$. (The result is new.)

3.1.20 THEOREM

Let S be an algebraizable k-deductive system with equivalent quasivariety semantics $\mathcal{K}$, defining equations $\delta_1(\boldsymbol{p}) \approx \varepsilon_1(\boldsymbol{p}), \ldots, \delta_m(\boldsymbol{p}) \approx \varepsilon_m(\boldsymbol{p})$ and equivalence k-formulas $\boldsymbol{\Delta}_1(p,q), \ldots, \boldsymbol{\Delta}_n(p,q)$. Then S is strongly algebraizable if and only if the following conditions hold:

(i) For every inference rule $\langle \Gamma, \boldsymbol{\varphi} \rangle$ of S, there exist $\ell \in \omega$ and terms $t_1(p_1,p_2,\overline{r}), \ldots, t_\ell(p_1,p_2,\overline{r})$, where $\overline{r}$ is a list of the variables occurring in Γ and $\boldsymbol{\varphi}$ and p_1,p_2 are variables such that $p_1,p_2,\overline{r}$ are distinct, and there exist pairs $(u_i,v_i) \in \{(\delta_j(\boldsymbol{\psi}), \varepsilon_j(\boldsymbol{\psi}))\,;\, \boldsymbol{\psi} \in \Gamma\,;\, j \le m\}$ such that for each $h \le m$,

$$\vdash_S \boldsymbol{\Delta}(\delta_h(\boldsymbol{\varphi}), t_1(u_1,v_1,\overline{r}))$$
$$\vdash_S \boldsymbol{\Delta}(t_1(v_1,u_1,\overline{r}), t_2(u_2,v_2,\overline{r}))$$
$$\vdots$$
$$\vdash_S \boldsymbol{\Delta}(t_{\ell-1}(v_{\ell-1},u_{\ell-1},\overline{r}), t_\ell(u_\ell,v_\ell,\overline{r}))$$
$$\vdash_S \boldsymbol{\Delta}(t_\ell(v_\ell,u_\ell,\overline{r}), \varepsilon_h(\boldsymbol{\varphi})).$$

(ii) There exist terms $s_1(p_1,p_2,p_3,p_4), \ldots, s_r(p_1,p_2,p_3,p_4)$ in four distinct variables, and pairs $(a_i,b_i) \in \{(\delta_j(\boldsymbol{\Delta}_h(p,q)), \varepsilon_j(\boldsymbol{\Delta}_h(p,q)))\,;\, j \le m\,;\, h \le n\}$ such that

$$\vdash_S \boldsymbol{\Delta}(p, s_1(a_1,b_1,p,q))$$
$$\vdash_S \boldsymbol{\Delta}(s_1(b_1,a_1,p,q), s_2(a_2,b_2,p,q))$$
$$\vdots$$
$$\vdash_S \boldsymbol{\Delta}(s_{r-1}(b_{r-1},a_{r-1},p,q), s_r(a_r,b_r,p,q))$$
$$\vdash_S \boldsymbol{\Delta}(s_r(a_r,b_r,p,q), q).$$

Proof. By Lemma 3.1.6 (i), (i) is equivalent to the following: for each $h \le m$,

$$\models_{\mathcal{K}} \delta_h(\boldsymbol{\varphi}) \approx t_1(u_1,v_1,\overline{r})$$

$$\models_{\mathcal{K}} t_1(v_1, u_1, \bar{r}) \approx t_2(u_2, v_2, \bar{r})$$

$$\vdots$$

$$\models_{\mathcal{K}} t_{\ell-1}(v_{\ell-1}, u_{\ell-1}, \bar{r}) \approx t_\ell(u_\ell, v_\ell, \bar{r})$$

$$\models_{\mathcal{K}} t_\ell(v_\ell, u_\ell, \bar{r}) \approx \varepsilon_h(\varphi).$$

Similarly, (ii) is equivalent to the following identities

$$\models_{\mathcal{K}} p \approx s_1(a_1, b_1, p, q)$$

$$\models_{\mathcal{K}} s_1(b_1, a_1, p, q) \approx s_2(a_2, b_2, p, q)$$

$$\vdots$$

$$\models_{\mathcal{K}} s_{r-1}(b_{r-1}, a_{r-1}, p, q) \approx s_r(a_r, b_r, p, q)$$

$$\models_{\mathcal{K}} s_r(b_r, a_r, p, q) \approx q.$$

Suppose that S is strongly algebraizable. Since the equivalent quasivariety semantics of S is unique (Theorem 3.1.12), we may infer that $\mathcal{K}$ is a variety. By Theorem 3.1.18, $\mathcal{K}$ satisfies the quasi-identities stated in condition (iii) and (iv) of that theorem. By Theorem 0.4.6, a variety satisfies a quasi-identity of the form 3.1.18 (iii) if and only if for suitable terms t_i it satisfies the identities equivalent to (i) listed above, and it satisfies 3.1.18 (iv) if and only if it satisfies the identities equivalent to (ii) listed above. Conversely, if S satisfies (i) and (ii), then the above identities hold in $\mathcal{K}$, and it is easy to see that these sets of identities imply the quasi-identities 3.1.18 (iii) and 3.1.18 (iv). Since $\mathcal{K}$ is axiomatized by these quasi-identities and certain identities, it can be axiomatized purely by identities, i.e., it is a variety. □

Clearly another version of the above characterization could be given using the condition in Theorem 0.4.6 (iii) rather than 0.4.6 (ii).

3.2 THE GÖDEL RULE

In this section we shall consider algebraizable 1-deductive systems which satisfy the so-called 'Gödel-rule' (G-rule, for short). This condition is satisfied by many classical algebraizable 1-deductive systems. In the **CPC** and **IPC** cases the G-rule states that $p, q \vdash_{\mathbf{CPC}} p \leftrightarrow q$ (resp. $p, q \vdash_{\mathbf{IPC}} p \leftrightarrow q$). An intuitive interpretation for this inference is: 'If p and q are both true, then p is equivalent to q', or, 'there exists exactly one object corresponding to 'the true' (modulo equivalence). The G-rule was defined by Suszko, [Sus71, p40] and adapted, in [BP89a, p41] to the definition of the G-rule, in terms of equivalence formulas, used here (Definition 3.2.1). Although the (syntactic) definition of the G-Rule may be extended to algebraizable k-deductive systems for $k \geq 2$, we shall not study this 'higher dimensional' analogue. This is because, for example, in the 2-deductive systems of greatest interest to us, the G-Rule proves enormously restrictive. The reader may verify, in particular, that for a quasivariety $\mathfrak{K}$, the 2-deductive system $S_{\mathfrak{K}}$ has the G-rule if and only if $\mathfrak{K}$ is a trivial quasivariety.

Not all algebraizable 1-deductive systems have the G-rule. An example of an algebraizable 1-deductive system that does not have the G-rule is the deductive system **R** of relevance. In Theorem 3.2.3, we present a number of characterizations of the G-rule. Thereafter, in Theorem 3.2.4, we present a number of equivalent conditions that the equivalent quasivariety semantics of an algebraizable 1-deductive system with the G-rule must satisfy. Moreover, we show that such an equivalent quasivariety semantics must be relatively T-regular (see Section 0.4) for a suitable constant term T. The converse of this is not true, however, but we do present a partial converse. Using a logic-driven proof, we show that for quasivarieties, relative T-regularity does not imply relative ideal-determination. Lastly, we present some results on quasivarieties that contrast with their counterparts for varieties. In particular, a relatively T-regular quasivariety satisfies the 'relative shifting lemma' but need not be 'relatively congruence modular'.

3.2.1 DEFINITION

Let S be an algebraizable 1-deductive system with a system of equivalence formulas $\Delta_1(p,q),\ldots,\Delta_n(p,q)$. We say that S has the *Gödel rule* (the G-rule, for short) if

(G-rule) $$\varphi, \psi \vdash_S \Delta_i(\varphi, \psi),$$

for all $\varphi, \psi \in Fm$ and $i \leq n$.

Note that the definition does not depend on the choice of equivalence formulas, in view of Theorem 3.1.12. Note also that an algebraizable extension S' of a 1-deductive system S with the Gödel rule also has the Gödel rule since $\vdash_S \subseteq \vdash_{S'}$. The Classical Propositional Calculus does have the G-rule. Taking $\Delta(p,q) = p \leftrightarrow q$ as our equivalence formula, the G-rule for **CPC** states that for all $\varphi, \psi \in Fm$,

$$\varphi, \psi \vdash_{\mathbf{CPC}} \varphi \leftrightarrow \psi.$$

We can see that this holds by using (3.1.3), which says that

$$\varphi, \psi \vdash_{\mathbf{CPC}} \varphi \leftrightarrow \psi \quad \text{if and only if} \quad \{\varphi \approx \mathrm{T}, \psi \approx \mathrm{T}\} \models_{\mathfrak{BA}} \varphi \leftrightarrow \psi \approx \mathrm{T},$$

which is the same as $\models_{\mathfrak{BA}} \mathrm{T} \leftrightarrow \mathrm{T} \approx \mathrm{T}$, which is certainly true. It follows similarly that **IPC** also has the G-rule. We give more examples of deductive systems with the G-rule in Section 3.3 and Chapter 5.

The following result is a corollary to Theorem 3.1.11 and Corollary 3.1.16

3.2.2 COROLLARY [BP89a, Corollary 4.8]

A sufficient condition for a deductive system S to be algebraizable is that there exists a system $\Delta_1(p,q), \ldots, \Delta_n(p,q)$ of binary formulas satisfying conditions (1) to (4) of Theorem 3.1.11 (ii) (for 1-deductive systems) and, for all $\varphi, \psi \in Fm$,

(6) $\quad \varphi, \Delta_1(\varphi,\psi), \ldots, \Delta_n(\varphi,\psi) \vdash_S \psi,$

(7) $\quad \varphi, \psi \vdash_S \Delta_i(\varphi,\psi)$ *for all* $i \leq n$.

In this event, $\Delta_1(p,q), \ldots, \Delta_n(p,q)$ form a system of equivalence formulas for S and $p \approx \Delta_i(p,p)$, $i \leq n$, form a system of defining equations for S.

Proof. Let $\vartheta \in Fm$. By (7), $\vartheta, \Delta_i(\vartheta,\vartheta) \vdash_S \Delta_j(\vartheta, \Delta_i(\vartheta,\vartheta))$ for all $i, j \leq n$. By (1), $\vdash_S \Delta_i(\vartheta,\vartheta)$ for all $i \leq n$, hence $\vartheta \vdash_S \{\Delta_j(\vartheta, \Delta_i(\vartheta,\vartheta))\,;\, i,j \leq n\}$. Also, $\vdash_S \Delta_i(\vartheta,\vartheta)$ implies $\{\Delta_j(\vartheta, \Delta_i(\vartheta,\vartheta))\,;\, i,j \leq n\} \vdash_S \vartheta$ by (6) and (2). So, by Theorem 3.1.11 (ii), S is algebraizable, and by Corollary 3.1.16, $\Delta_1(p,q), \ldots, \Delta_n(p,q)$ and $p \approx \Delta_1(p,p), \ldots, p \approx \Delta_n(p,p)$ are equivalence formulas and defining equations, respectively, for S and its equivalent quasivariety semantics. □

The converse of this corollary is not true, however, for there do exist algebraizable deductive systems that fail to have the G-rule. In particular, the logic **R** of relevance is an example of this. This is proved in Section 3.3. The (algebraizable) deductive systems that have the G-rule can be characterized in terms of their defining equations, S-matrices or S-theories, as is shown in the following theorem, which extends [BP89a, Corollary 5.4].

3.2.3 THEOREM

Let S be an algebraizable deductive system with equivalent quasivariety semantics $\mathfrak{K}$, and let

$\Delta_1(p,q),\ldots,\Delta_n(p,q)$ be a system of equivalence formulas for S and $\mathcal{K}$. The following are equivalent

(i) *S has the G-rule.*

(ii) *Every reduced S-matrix has exactly one designated element.*

(iii) *There exists an equationally definable constant T in the language of $\mathcal{K}$ such that $p \approx \mathrm{T}$ forms a system of defining equations for S and $\mathcal{K}$.*

(iv) *There exists an equationally definable constant T in the language of $\mathcal{K}$ such that for every S-matrix $\mathcal{A} = \langle \mathbf{A}, F\rangle$, $F = \mathrm{T}^{\mathbf{A}}/\Omega_{\mathcal{A}}F$.*

(v) *For every S-matrix $\mathcal{A} = \langle \mathbf{A}, F\rangle$, F is an equivalence class of $\Omega_{\mathcal{A}}F$.*

(vi) *For every S-theory U, U is an equivalence class of ΩU.*

Proof. (i)⇒(ii) Assume that $\delta_1(p) \approx \varepsilon_1(p),\ldots,\varepsilon_m(p) \approx \delta_m(p)$ is a system of defining equations for S and $\mathcal{K}$. Let $\mathcal{A} = \langle \mathbf{A}, F\rangle$ be a reduced S-matrix. Then $\Omega_{\mathcal{A}}F = I_A$, hence, by Corollary 3.1.14,

$$F = H_{\mathcal{A}}\Omega_{\mathcal{A}}F = \{a \in A\,;\ \delta_i^{\mathbf{A}}(a) = \varepsilon_i^{\mathbf{A}}(a) \text{ for each } i \le m\}.$$

Suppose $a, b \in F$. For any two variables p, q, the G-rule implies that $p, q \vdash_S \Delta_i(p,q)$ for $i \le n$, hence $p, q \models_{\mathcal{A}} \Delta_i(p,q)$ for $i \le n$. If we interpret p as a and q as b, then $a, b \in F$ implies that $\Delta_i^{\mathbf{A}}(a,b) \in F$ for $i \le n$. Thus $\delta_j^{\mathbf{A}}(\Delta_i^{\mathbf{A}}(a,b)) = \varepsilon_j^{\mathbf{A}}(\Delta_i^{\mathbf{A}}(a,b))$ for $i \le n$ and $j \le m$. But $\mathbf{A} \in \mathcal{K}$ by Corollary 3.1.15, so (3.1.6) immediately implies that $a = b$, hence F contains at most one element. From $\vdash_S \Delta_i(p,p)$, we have $\Delta_i^{\mathbf{A}}(a,a) \in F$ for any $a \in A$ and $i \le n$, so F contains exactly one element.

(ii)⇒(iii) Since S is algebraizable, there exists a system of defining equations for S and $\mathcal{K}$. Thus, by Corollary 3.1.16, the system $\Delta_1(p,q),\ldots,\Delta_n(p,q)$ forms a system of congruence formulas without parameters for S in the sense of Chapter 2. In particular, (2.3.1)′ and (2.3.2)′ hold, i.e.,

$$\vdash_S \Delta_i(p,p) \quad \text{for each } i \le n,$$

and

$$p, \Delta_1(p,q),\ldots,\Delta_n(p,q) \vdash_S q.$$

We claim that $p \approx \Delta_i(p,p)$, $i \le n$, form a system of defining equations for S and $\mathcal{K}$. We shall use Corollary 3.1.16 to prove this. Since $\Delta_1(p,q),\ldots,\Delta_n(p,q)$ is a system of congruence formulas without parameters for S, we need only show that

$$\varphi \dashv\vdash_S \{\Delta_i(\varphi, \Delta_j(\varphi,\varphi))\,;\ i \le n\,;\ j \le n\}.$$

Using Theorem 3.1.11 (ii) (2) and the above properties, we have, for each $j \leq n$,

$$\Delta_j(\varphi,\varphi), \{\Delta_i(\varphi, \Delta_j(\varphi,\varphi))\,;\, i \leq n\} \vdash_S \varphi.$$

Since $\vdash_S \Delta_j(\varphi,\varphi)$, we have $\{\Delta_i(\varphi,\Delta_j(\varphi,\varphi))\,;\, i \leq n\} \vdash_S \varphi$, hence

$$\{\Delta_i(\varphi,\Delta_j(\varphi,\varphi))\,;\, i \leq n\,;\, j \leq n\} \vdash_S \varphi.$$

For the converse, since Mod^*S is a matrix semantics for S, we need only show that

$$\varphi \models_{\mathrm{Mod}^*S} \{\Delta_i(\varphi,\Delta_j(\varphi,\varphi))\,;\, i \leq n\,;\, j \leq n\}.$$

By assumption, for each $\mathcal{A} = \langle \mathbf{A}, F\rangle \in \mathrm{Mod}^*S$, F has precisely one element, T^F say. (By Corollary 3.1.15, there exists a *unique* reduced S-matrix that has $\mathbf{A}$ as its underlying algebra, so T^F is well-defined.) Let a, b be an interpretation of the variables p, q in A, respectively. We have $\Delta_i^{\mathbf{A}}(a,a) \in F$ since $\vdash_S \Delta_i(p,p)$, hence $\Delta_i^{\mathbf{A}}(a,a) = \mathrm{T}^F$. Let $\overline{a}$ be an interpretation of the variables of φ in A. Then $\Delta_i^{\mathbf{A}}(\varphi^{\mathbf{A}}(\overline{a}),\varphi^{\mathbf{A}}(\overline{a})) = \mathrm{T}^F$ for each $i \leq n$. Thus, if $\varphi^{\mathbf{A}}(\overline{a}) \in F$, i.e., $\varphi^{\mathbf{A}}(\overline{a}) = \mathrm{T}^F$, then $\Delta_i^{\mathbf{A}}(\varphi^{\mathbf{A}}(\overline{a}), \Delta_j^{\mathbf{A}}(\varphi^{\mathbf{A}}(\overline{a}),\varphi^{\mathbf{A}}(\overline{a}))) = \Delta_i^{\mathbf{A}}(\mathrm{T}^F,\mathrm{T}^F) = \mathrm{T}^F$, i.e., $\Delta_i^{\mathbf{A}}(\varphi^{\mathbf{A}}(\overline{a}), \Delta_j^{\mathbf{A}}(\varphi^{\mathbf{A}}(\overline{a}),\varphi^{\mathbf{A}}(\overline{a}))) \in F$ for each $i, j \leq n$. This proves that

$$\varphi \dashv\vdash_S \{\Delta_i(\varphi,\Delta_j(\varphi,\varphi))\,;\, i \leq n\,;\, j \leq n\},$$

and hence that $p \approx \Delta_i(p,p)$, $i \leq n$, forms a system of defining equations for S and $\mathcal{K}$.

Now consider *any* $\mathbf{A} \in \mathcal{K}$. By Corollary 3.1.15, there exists a unique $F \subseteq A$ with $\langle \mathbf{A}, F\rangle \in \mathrm{Mod}^*S$, i.e., F has precisely one element, T^F say. We have already noted that $\Delta_i^{\mathbf{A}}(a,a) = \mathrm{T}^F$ for each $a \in A$ and $i \leq n$. Thus $\Delta_i(x,x) \approx \Delta_j(y,y)$ is an identity satisfied by $\mathbf{A}$. Since $\mathbf{A}$ was an arbitrary element of $\mathcal{K}$, we have

$$\mathcal{K} \models \Delta_i(x,x) \approx \Delta_j(y,y) \quad \text{for each } i, j \leq n.$$

Choose any $i \leq n$. The term $\Delta_i(x,x)$ therefore acts as a constant in each algebra $\mathbf{A}$ of $\mathcal{K}$. Thus we may denote the term $\Delta_i(x,x)$ by T, whereupon $\mathrm{T}^{\mathbf{A}} = \mathrm{T}^F$ for each $\mathbf{A} \in \mathcal{K}$. This implies that $p \approx \mathrm{T}$ is a system of defining equations for S and $\mathcal{K}$.

(iii)$\Rightarrow$(iv) Let $\langle \mathbf{A}, F\rangle$ be an S-matrix. Then, by Corollary 3.1.14,

$$\begin{aligned} F = H_{\mathcal{A}}\Omega_{\mathcal{A}}F &= \{a \in A\,;\, (\delta^{\mathbf{A}}(a), \varepsilon^{\mathbf{A}}(a)) \in \Omega_{\mathcal{A}}F\} \\ &= \{a \in A\,;\, (a, \mathrm{T}^{\mathbf{A}}) \in \Omega_{\mathcal{A}}F\} \\ &= \mathrm{T}^{\mathbf{A}}/\Omega_{\mathcal{A}}F. \end{aligned}$$

(iv)$\Rightarrow$(v) Trivial.

(v)⇒(vi) Trivial, since $\langle \mathbf{Fm}, U\rangle$ is an S-matrix for every S-theory U.

(vi)⇒(i) Let $\varphi, \psi \in Fm$ and set $\mathcal{A} = \langle \mathbf{Fm}, U\rangle$, where $U = \mathrm{Cn}_S(\{\varphi, \psi\})$. Since U is an equivalence class of $\Omega_{\mathcal{A}}U$, and $\varphi, \psi \in U$, we have $(\varphi, \psi) \in \Omega_{\mathcal{A}}U$. Then, since $\Omega_{\mathcal{A}}U$ is a congruence relation on $\mathbf{Fm}$, $(\Delta_i(\varphi,\varphi), \Delta_i(\varphi,\psi)) \in \Omega_{\mathcal{A}}U$ for each $i \leq n$. Since $\vdash_S \Delta_i(p,p)$, we have $\Delta_i(\varphi,\varphi) \in U$ for each $i \leq n$, so, by the compatibility of $\Omega_{\mathcal{A}}U$ with U, we get $\Delta_i(\varphi,\psi) \in U$, i.e., $\varphi, \psi \vdash_S \Delta_i(\varphi,\psi)$. □

An algebraizable deductive system has, as we know, a unique equivalent quasivariety semantics. The (algebraizable) deductive systems that satisfy the G-rule can be characterized in terms of their equivalent quasivariety semantics. Recall the definitions of (relative) T-regularity, ideals and ideal terms from Section 0.4. The remaining results and examples of this section are not in the published literature.

3.2.4 THEOREM

Let S be an algebraizable deductive system and let $\mathfrak{K}$ be the unique equivalent quasivariety semantics for S. Let $\delta_1(p) \approx \varepsilon_1(p), \ldots, \delta_m(p) \approx \varepsilon_m(p)$ and $\Delta_1(p,q), \ldots, \Delta_n(p,q)$ be defining equations and equivalence formulas for S and $\mathfrak{K}$. The following are equivalent:

(i) *S has the G-rule.*

(ii) $\mathfrak{K} \models \left(\mathop{\&}_{i \leq m} \delta_i(p) \approx \varepsilon_i(p)\right) \& \left(\mathop{\&}_{i \leq m} \delta_i(q) \approx \varepsilon_i(q)\right) \Rightarrow p \approx q$

(iii) *$\mathfrak{K}$ has an equationally definable constant term T such that $\mathfrak{K} \models \mathrm{T} \approx \Delta_i(\mathrm{T},\mathrm{T})$ and $\mathrm{Fi}^S\mathcal{A} \subseteq Id\,\mathbf{A}$ for each S-matrix $\mathcal{A} = \langle \mathbf{A}, F\rangle$, where $\mathbf{A} \in \mathfrak{K}$.*

Moreover, each of the conditions (i) to (iii) implies

(iv) $\mathfrak{K} \models \mathrm{T} \approx \Delta_i(p,p)$ *for all $i \leq n$, and*

$\mathfrak{K} \models \left(\mathop{\&}_{i \leq n} \Delta_i(p,q) \approx \mathrm{T}\right) \Rightarrow p \approx q.$

(v) *$\mathfrak{K}$ is relatively T-regular.*

Proof. (i)⇒(ii) By assumption, $p, q \vdash_S \Delta_i(p,q)$ for all $i \leq n$, hence

$$\mathfrak{K} \models \left(\mathop{\&}_{i \leq m} \delta_i(p) \approx \varepsilon_i(p)\right) \& \left(\mathop{\&}_{i \leq m} \delta_i(q) \approx \varepsilon_i(q)\right) \Rightarrow \delta_i(\Delta_j(p,q)) \approx \varepsilon_i(\Delta_j(p,q))$$

for each $i \leq m$ and $j \leq n$, which we can write as

$$\delta(p) \approx \varepsilon(p), \delta(q) \approx \varepsilon(q) \models_{\mathfrak{K}} \delta(\Delta_i(p,q)) \approx \varepsilon(\Delta_i(p,q)).$$

Since $$\delta(\Delta_i(p,q)) \approx \varepsilon(\Delta_i(p,q)) =\!\mid\models_{\mathcal{K}} p \approx q,$$

the result follows.

(ii)⇒(i) Let $\langle \mathbf{A}, F\rangle$ be a reduced S-matrix, where $\mathbf{A} \in \mathcal{K}$. Then, by Corollary 3.1.14,

$$F = H_{\mathcal{A}}\Omega_{\mathcal{A}}F = H_{\mathcal{A}}I_A = \{a \in A\,;\, \delta_i^{\mathbf{A}}(a) = \varepsilon_i^{\mathbf{A}}(a) \text{ for each } i \leq m\}.$$

If $a, b \in F$, then $\delta_i^{\mathbf{A}}(a) = \varepsilon_i^{\mathbf{A}}(a)$ and $\delta_i^{\mathbf{A}}(b) = \varepsilon_i^{\mathbf{A}}(b)$, implying, by the quasi-identity, that $a = b$, hence F has at most one element. Form $\vdash_S \Delta_i(p,p)$ $(i \leq n)$, we get that F has at least one element. By Theorem 3.2.3, S has the G-rule.

(i)⇒(iii) By (i) and Theorem 3.2.3, there exists an equationally definable constant T in the language of $\mathcal{K}$ such that $p \approx \mathrm{T}$ forms a system of defining equations for S and $\mathcal{K}$. Thus, since $\vdash_S \Delta_i(p,p)$, we have $\mathcal{K} \models \Delta_i(p,p) \approx \mathrm{T}$ for each $i \leq n$. In particular, $\mathcal{K} \models \Delta_i(\mathrm{T},\mathrm{T}) \approx \mathrm{T}$ for all $i \leq n$. Let $\mathcal{A}$ be an S-matrix and F an S-filter of $\mathcal{A}$. Since (i) holds we can use condition (iv) of the Theorem 3.2.3 to deduce that $F = \mathrm{T}^{\mathbf{A}}/\Omega_{\mathcal{A}}F$. Then, since $\mathrm{T}^{\mathbf{A}}/\Omega_{\mathcal{A}}F$ is an ideal (with respect to $\mathcal{K}$ and T) (see Section 0.4), the result follows.

(iii)⇒(i) Since $\mathcal{K} \models \Delta_i(\mathrm{T},\mathrm{T}) \approx \mathrm{T}$ for all $i \leq n$, we have that $\Delta_i(y_1, y_2)$ is an ideal term in y_1, y_2 (with respect to $\mathcal{K}$ and T). If $\varphi, \psi \in Fm$, $\langle \mathbf{A}, F\rangle$ is an S-matrix and $\overline{a}$ is an interpretation of the variables of φ, ψ in A such that $\varphi^{\mathbf{A}}(\overline{a}), \psi^{\mathbf{A}}(\overline{a}) \in F$, then $\Delta_i^{\mathbf{A}}(\varphi^{\mathbf{A}}(\overline{a}), \psi^{\mathbf{A}}(\overline{a})) \in F$ since F is an ideal (by (iii)), hence $\varphi, \psi \vdash_S \Delta_i(\varphi, \psi)$.

[(i)-(iii)]⇒(iv) By Theorem 3.2.3, there exists an equationally definable constant T in the language of $\mathcal{K}$ such that $p \approx \mathrm{T}$ forms a system of defining equations for S and $\mathcal{K}$. Thus, since $\vdash_S \Delta_i(p,p)$, $\mathcal{K} \models \Delta_i(p,p) \approx \mathrm{T}$ for each $i \leq n$. Now, $\mathcal{K} \models \mathrm{T} \approx \mathrm{T}$, hence $\vdash_S \mathrm{T}$ by (3.1.3). By Theorem 3.1.11 (iii), $\Delta_1(p,q), \ldots, \Delta_n(p,q)$ form a system of congruence formulas without parameters for S (in the sense of Chapter 2). In particular, (2.3.2)′ holds, so

$$\mathrm{T}, \Delta_1(\Delta_j(p,q),\mathrm{T}), \ldots, \Delta_n(\Delta_j(p,q),\mathrm{T}) \vdash_S \Delta_j(p,q)$$

for each $j \leq n$. Since $\vdash_S \mathrm{T}$,

$$\{\Delta_i(\Delta_j(p,q),\mathrm{T})\,;\, i,j \leq n\} \vdash_S \{\Delta_j(p,q)\,;\, j \leq n\},$$

hence $\Delta_1(p,q) \approx \mathrm{T}, \ldots, \Delta_n(p,q) \approx \mathrm{T} \models_{\mathcal{K}} p \approx q$, by Lemma 3.1.6 (i).

[(i)-(iii)]⇒(v) This can be deduced immediately from (iv) and Theorem 0.4.7, with $d_i = \Delta_i$ for each $i \leq n$. A direct proof of T-regularity is easily obtained though. Suppose each of the equivalent conditions (i) to (iii) hold. Let $\mathbf{A} \in \mathcal{K}$ and let $F_{\mathcal{A}}$ be the least subset of A such that $\mathcal{A} = \langle \mathbf{A}, F_{\mathcal{A}} \rangle$ is an S-matrix. Let $\theta, \Phi \in \mathrm{Con}_{\mathcal{K}}\mathbf{A}$ such that $\mathrm{T}^{\mathbf{A}}/\theta = \mathrm{T}^{\mathbf{A}}/\Phi$. By Theorem 3.1.14, $\Omega_{\mathcal{A}} : \mathbf{Fi}^S\mathcal{A} \cong \mathbf{Con}_{\mathcal{K}}\mathbf{A}$, so there must exist unique $F, G \in \mathrm{Fi}^S\mathcal{A}$ such that $\Omega_{\mathcal{A}}F = \theta$ and $\Omega_{\mathcal{A}}G = \Phi$, and by (iv) of the Theorem 3.2.3, $F = \mathrm{T}^{\mathbf{A}}/\Omega_{\mathcal{A}}F$ and $G = \mathrm{T}^{\mathbf{A}}/\Omega_{\mathcal{A}}G$. So, $\mathrm{T}^{\mathbf{A}}/\theta = \mathrm{T}^{\mathbf{A}}/\Phi$ implies $\mathrm{T}^{\mathbf{A}}/\Omega_{\mathcal{A}}F = \mathrm{T}^{\mathbf{A}}/\Omega_{\mathcal{A}}G$, which implies $F = G$, hence $\Omega_{\mathcal{A}}F = \Omega_{\mathcal{A}}G$, i.e., $\theta = \Phi$. □

Although it is not true that (v) implies (i) in the previous theorem (which we shall show in Section 3.3), we nevertheless have the following result.

3.2.5 THEOREM

Let $\mathcal{K}$ be a relatively T*-regular quasivariety. Then there exists an algebraizable 1-deductive system with the G-rule that has $\mathcal{K}$ as its equivalent quasivariety semantics.*

Proof. By Theorem 0.4.7 there exist formulas $d_1, \ldots, d_\ell$ such that

(i) $\mathcal{K} \models \mathrm{T} \approx d_i(p,p)$ for all $i \leq \ell$, and

(ii) $\mathcal{K} \models \left(\underset{i \leq \ell}{\&} d_i(p,q) \approx \mathrm{T} \right) \Rightarrow p \approx q.$

Consider Theorem 3.1.9. Setting $\delta(p) = p$, $\varepsilon(p) = \mathrm{T}$ and $\Delta_i(p,q) = d_i(p,q)$ for each $i \leq \ell$, it follows that (i) and (ii) of that theorem coincide with (i) and (ii) here. Thus $\mathcal{K}$ is the equivalent quasivariety semantics of a 1-deductive system S and $p \approx \mathrm{T}$ and $d_1(p,q), \ldots, d_\ell(p,q)$ form systems of defining equations and equivalence formulas for S and $\mathcal{K}$ (see the proof of Theorem 3.1.9). Now, the G-rule holds if and only if

$$p \approx \mathrm{T},\ q \approx \mathrm{T} \models_{\mathcal{K}} d_i(p,q) \approx \mathrm{T}$$

for each $i \leq \ell$. But this is equivalent to $\models_{\mathcal{K}} d_i(\mathrm{T},\mathrm{T}) \approx \mathrm{T}$, which holds by (i), hence S has the G-rule. □

The reason for the failure of the implication (v) to (i) of Theorem 3.2.4 is suggested in the above theorem. For it is possible that there exist more than one 1-deductive system that has $\mathcal{K}$ as its equivalent quasivariety semantics and it is not necessarily true that if one such system has the G-rule then all such systems do. In Section 3.3 we present an example of a quasivariety that is the equivalent quasivariety semantics of two distinct 1-deductive systems such that one has the G-rule and the other does not. In particular, the G-rule is not preserved by equivalence of deductive systems.

Most naturally occurring relatively T-regular quasivarieties enjoy the stronger property of being relatively ideal determined. We shall show that a relatively T-regular quasivariety need not, in general, be relatively ideal determined. It seems that (relative) ideal determination of the equivalent quasivariety semantics of an algebraizable logic fails to reflect any significant properties of the logic (other than those already deducible from relative T-regularity), except in cases that are syntactically rather special.

3.2.6 PROPOSITION

The equivalent quasivariety semantics of an algebraizable deductive system S that has the G-rule need not be relatively ideal determined.

Proof. Let $\mathfrak{L}$ be the language $\langle d, \mathrm{T}\rangle$ with $ar(d) = 2$ and $ar(\mathrm{T}) = 0$. Let $\mathfrak{K}$ be the quasivariety of this type defined by

(i) $$d(x,x) \approx \mathrm{T},$$

(ii) $$d(x,y) \approx \mathrm{T} \Rightarrow x \approx y.$$

By Theorem 0.4.7, $\mathfrak{K}$ is relatively T-regular, so by Theorem 3.2.5, there is an algebraizable 1-deductive system S with the G-rule such that $\mathfrak{K}$ is the equivalent quasivariety semantics of S.

We shall construct an algebra $\mathbf{A} = \langle A, d^{\mathbf{A}}, \mathrm{T}^{\mathbf{A}}\rangle$ that is an element of $\mathfrak{K}$. Set $A = \{0,1,2\}$, and let $\mathrm{T}^{\mathbf{A}} = 0$. Define

$$d(a,b) = 0 \ \text{ if } a = b,$$

and $$d(a,b) = 1 \ \text{ if } a \neq b.$$

The only congruences of $\mathbf{A}$ are I_A, $\Theta^{\mathbf{A}}(0,1) = I_A \cup \{(0,1),(1,0)\}$ and A^2. Let $\Phi = \Theta^{\mathbf{A}}(0,1)$. We show that $\mathbf{A}/\Phi \notin \mathfrak{K}$, hence that Φ is not a relative congruence. It is evident that the congruence classes of $\mathbf{A}/\Phi$ are $\{0,1\}$ and $\{2\}$, and that the following holds:

$$d^{\mathbf{A}/\Phi}(\{0,1\},\{2\}) = d^{\mathbf{A}}(0,2)/\Phi = 1/\Phi = \{0,1\}.$$

Note that $\mathrm{T}^{\mathbf{A}/\Phi} = 0/\Phi = \{0,1\}$, so $d^{\mathbf{A}/\Phi}(\{0,1\},\{2\}) = \mathrm{T}^{\mathbf{A}/\Phi}$, but $\{0,1\} \neq \{2\}$, which contradicts (ii), i.e., $\mathbf{A}/\Phi \not\models d(x,y) \approx \mathrm{T} \Rightarrow x \approx y$, hence $\mathbf{A}/\Phi \notin \mathfrak{K}$. The lattice $\mathbf{Con}_{\mathfrak{K}}\mathbf{A}$ is thus the two-element chain.

Since $\Phi \in \mathrm{Con}\,\mathbf{A}$, it is certainly reflexive and compatible, hence, as noted in Section 0.4, $0/\Phi = \{0,1\}$ is an ideal of $\mathbf{A}$ (with respect to $\mathcal{K}$ and T). Since $\{0\}$ and A are both ideals of $\mathbf{A}$ different from $0/\Phi$, the set $Id\,\mathbf{A}$ has at least three elements and so cannot be mapped bijectively onto $\mathrm{Con}_{\mathcal{K}}\mathbf{A}$, hence $\mathcal{K}$ is not relatively ideal determined. □

3.2.7 COROLLARY

For a quasivariety $\mathcal{K}$, relative T*-regularity does not imply relative ideal-determination.* □

Let $\mathcal{K}$ be a quasivariety (resp. variety) and $\mathbf{A} \in \mathcal{K}$. We say that $\mathbf{A}$ is *relatively congruence modular* (resp. *congruence modular*) if the lattice $\mathrm{Con}_{\mathcal{K}}\,\mathbf{A}$ (resp. $\mathrm{Con}\,\mathbf{A}$) is modular. Every T-regular variety is congruence modular (see [Hag73]), therefore the equivalent variety semantics of a strongly algebraizable deductive system with the G-rule is congruence modular. Relatively congruence modular quasivarieties $\mathcal{K}$ are characterized by the conjunction of two properties, viz. the 'Relative Shifting Lemma' and the 'Extension Principle' [KM92, Theorem 4.1]. The former asserts that there are terms $p_1(x,y,z,w),\ldots,p_n(x,y,z,w)$, $q_1(x,y,z,w),\ldots,q_n(x,y,z,w)$ of the type of $\mathcal{K}$ such that $\mathcal{K}$ satisfies the identities

$$p_i(x,y,y,x) \approx q_i(x,y,y,x) \text{ and } p_i(x,x,y,y) \approx q_i(x,x,y,y) \text{ for } i \leq n,$$

and the quasi-identity

$$p_1(x,y,y,u) \approx q_1(x,y,y,u) \;\&\; \ldots \;\&\; p_n(x,y,y,u) \approx q_n(x,y,y,u) \Rightarrow x \approx u\,.$$

The latter asserts that for any $\mathbf{A} \in \mathcal{K}$ and $\alpha, \beta \in \mathrm{Con}\,\mathbf{A}$, we have $\Theta^{\mathbf{A}}_{\mathcal{K}}(\alpha) \cap \Theta^{\mathbf{A}}_{\mathcal{K}}(\beta) \subseteq \Theta^{\mathbf{A}}_{\mathcal{K}}(\alpha \cap \beta)$ (which implies that the map $\alpha \mapsto \Theta^{\mathbf{A}}_{\mathcal{K}}(\alpha)$ is a lattice homomorphism from $\mathbf{Con}\,\mathbf{A}$ to $\mathbf{Con}_{\mathcal{K}}\,\mathbf{A}$).

3.2.8 PROPOSITION

A relatively T*-regular quasivariety satisfies the Relative Shifting Lemma.*

Proof. Let $\mathcal{K}$ be a relatively T-regular quasivariety and let $d_1,\ldots,d_m$ be binary terms as in the statement of Theorem 0.4.7. Define $p_{ij}(x,y,z,u) = d_i(d_j(x,u), d_j(y,z))$ and $q_{ij}(x,y,z,u) = \mathrm{T}$ for all $i,j \in \{1,\ldots,m\}$. Then $\mathcal{K}$ satisfies $p_{ij}(x,y,y,x) \approx d_i(d_j(x,x), d_j(y,y)) \approx \mathrm{T} \approx q_{ij}(x,y,y,x)$ for all i,j. If $\mathbf{A} \in \mathcal{K}$ and $a,b,d \in A$ with $p^{\mathbf{A}}_{ij}(a,b,b,d) = q^{\mathbf{A}}_{ij}(a,b,b,d)$ for all i,j, then $d^{\mathbf{A}}_i(d^{\mathbf{A}}_j(a,d), d^{\mathbf{A}}_j(b,b)) = \mathrm{T}^{\mathbf{A}}$ for all i,j, i.e., $d^{\mathbf{A}}_i(d^{\mathbf{A}}_j(a,d), \mathrm{T}^{\mathbf{A}}) = \mathrm{T}^{\mathbf{A}}$ for all i,j, so $d^{\mathbf{A}}_j(a,d) = \mathrm{T}^{\mathbf{A}}$ for all j, hence $a = d$. □

Notwithstanding this result, the following example shows that a relatively T-regular quasivariety need not be relatively congruence modular.

3.2.9 EXAMPLE

Let $\mathcal{K}$ be the quasivariety whose type consists of two binary operation symbols d_1, d_2 and a constant symbol T and whose axioms are the identities $d_i(x,x) \approx \mathrm{T}$ $(i = 1,2)$ and the quasi-identity $d_1(x,y) \approx \mathrm{T}$ & $d_2(x,y) \approx \mathrm{T} \Rightarrow x \approx y$. Then $\mathcal{K}$ is relatively T-regular, by Theorem 0.4.7. For a given integer $n \geq 3$, let $A = B \cup I \cup \{0\}$ be a disjoint union, with $B = \{a_0, a_1, \ldots, a_n\}$, $|B| = n+1$, and $I = \{\{i,j\}; i \neq j ; i,j \in \{0,1,\ldots,n\}\}$. Let $\mathbf{A} = \langle A; d_1^{\mathbf{A}}, d_2^{\mathbf{A}}, \mathrm{T}^{\mathbf{A}}\rangle$ be the algebra of type $\langle 2,2,0\rangle$ defined by $\mathrm{T}^{\mathbf{A}} = 0 = d_1^{\mathbf{A}}(c,c) = d_1^{\mathbf{A}}(0,c)$; $d_1^{\mathbf{A}}(c,0) = c$ for all $c \in A$; $d_1^{\mathbf{A}}(a_i, a_j) = \{i,j\}$ for distinct $i,j \in \{0,1,\ldots,n\}$; $d_1^{\mathbf{A}}(a_i, \{j,k\}) = a_i$; $d_1^{\mathbf{A}}(\{j,k\}, a_i) = \{j,k\}$ for $i,j,k \in \{0,1,\ldots,n\}$ with $j \neq k$; $d_1^{\mathbf{A}}(\{i,j\},\{k,\ell\}) = \{i,j\}$ for distinct $\{i,j\},\{k,\ell\} \in I$ and $d_2^{\mathbf{A}}(c,d) = d_1^{\mathbf{A}}(d,c)$ for all $c,d \in A$. It is easily verified that $\mathbf{A} \in \mathcal{K}$. Let π be any partition of B and let $\sim$ be the corresponding equivalence relation on B. Define

$$r = \{(\{i,j\},0); \; a_i \text{ and } a_j \text{ each belong to some nontrivial block of } \pi\}.$$

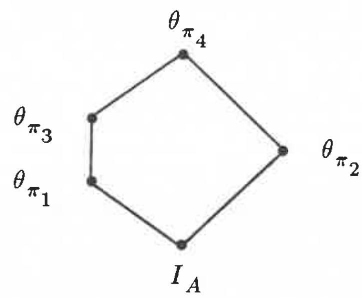
Define

$$r' = \{(\{i,k\},\{j,k\}); \; a_i \text{ and } a_j \text{ each belong to some nontrivial block of } \pi \text{ and } a_k \text{ belongs to a trivial block of } \pi\}.$$

For a binary relation R, define $R^{-1} = \{(y,x); (x,y) \in R\}$. Let s be the transitive closure (in A) of $r \cup r' \cup r^{-1} \cup (r')^{-1}$. Let $\theta_\pi = I_A \cup \sim \cup\, s$. One checks that $\theta_\pi \in \mathrm{Con}\,\mathbf{A}$. Moreover,

$$0/\theta_\pi = \{0\} \cup \{\{i,j\}; \text{ each of } a_i \text{ and } a_j \text{ belongs to some nontrivial block of } \pi\},$$

from which it follows that $\theta_\pi \in \mathrm{Con}_{\mathcal{K}}\mathbf{A}$ if and only if π has at most one nontrivial block. Thus, if $\pi_1, \pi_2, \pi_3, \pi_4$ are the partitions of B whose only nontrivial blocks are, respectively, $\{a_0,a_1\}$, $\{a_2,a_3\}$, $\{a_0,a_1,a_2\}$, $\{a_0,a_1,a_2,a_3\}$, then in $\mathbf{Con}_{\mathcal{K}}\mathbf{A}$, $\theta_{\pi_1} \vee \theta_{\pi_2} = \theta_{\pi_4} = \theta_{\pi_3} \vee \theta_{\pi_2}$, so the following pentagon is a sublattice of $\mathbf{Con}_{\mathcal{K}}\mathbf{A}$:



Thus, in $\mathbf{Con}_{\mathcal{K}}\mathbf{A}$ we have $\theta_{\pi_1} \subseteq \theta_{\pi_3}$ but

$$\theta_{\pi_1} \vee (\theta_{\pi_3} \wedge \theta_{\pi_2}) = \theta_{\pi_1} \vee I_A = \theta_{\pi_1} \neq \theta_{\pi_3} = \theta_{\pi_3} \wedge \theta_{\pi_4} = \theta_{\pi_3} \wedge (\theta_{\pi_1} \vee \theta_{\pi_2}),$$

hence $\mathbf{A}$ is not relatively congruence modular.

3.3 EXAMPLES

We shall investigate the algebraizability of a number of deductive systems here. We refer the reader to Section 1.4 for the necessary definitions.

Classical and Intuitionistic Propositional Calculi.

It was shown early on in Section 3.1 that the Classical Propositional Calculus is strongly algebraizable with equivalence formula $\Delta(p,q) = p \leftrightarrow q$ (abbreviating $(p \rightarrow q) \wedge (q \rightarrow p)$), and defining equation $p \approx \mathrm{T}$. The unique equivalent variety semantics for **CPC** is $\mathcal{BA}$, the variety of Boolean algebras. By Theorem 3.1.18, the equivalent variety semantics for **CPC** is axiomatized by (we shall use T instead of 1 here):

(3.3.1) $\varphi \approx \mathrm{T}$ where φ is any axiom from (C_1) to (C_{11}),

(3.3.2) $p \leftrightarrow p \approx \mathrm{T}$,

(3.3.3) $p \approx \mathrm{T}$ & $p \rightarrow q \approx \mathrm{T} \Rightarrow q \approx \mathrm{T}$,

(3.3.4) $p \leftrightarrow q \approx \mathrm{T} \Rightarrow p \approx q$.

Thus (3.3.1) to (3.3.4) must form an axiomatization of $\mathcal{BA}$. That these identities and quasi-identities are true in all Boolean algebras is evident. That they form an axiomatization of the variety of Boolean Algebras can be directly proved; we shall omit the easy, but lengthy details.

The Completeness and Validity Theorem for **IPC** states that, for $\Gamma \subseteq Fm$ and $\varphi \in Fm$,

$$\Gamma \vdash_{\mathbf{IPC}} \varphi \quad \text{if and only if} \quad \{\psi \approx \mathrm{T}\,;\, \psi \in \Gamma\} \models_{\mathcal{HA}} \varphi \approx \mathrm{T},$$

where $\mathcal{HA}$ is the variety of all Heyting algebras (using T instead of 1). Clearly, this coincides with (3.1.3), with $\delta(p) = p$ and $\varepsilon(p) = \mathrm{T}$. Set $\Delta(p,q) = p \leftrightarrow q$ (abbreviating $(p \to q) \wedge (q \to p)$). For Δ to be an equivalence formula for **IPC**, we need

$$\zeta \approx \eta =\mid\models_{\mathcal{HA}} \zeta \leftrightarrow \eta \approx \mathrm{T},$$

corresponding to (3.1.6). Recall that the natural order on $\mathcal{HA}$, namely, $a \leq b$ if and only if $a \to b = \mathrm{T}^{\mathbf{A}}$ $(a, b \in A, \mathbf{A} \in \mathcal{HA})$ is a lattice order, hence the above statement follows from the anti-symmetry of $\leq$. Consequently, **IPC** is strongly algebraizable with equivalent variety semantics $\mathcal{HA}$, equivalence formula $\Delta(p,q) = p \leftrightarrow q$ and defining equation $p \approx \mathrm{T}$. Evidently, an axiomatization of $\mathcal{HA}$ can be deduced from Theorem 3.1.18.

Since $p, q \vdash_{\mathbf{CPC}} p \leftrightarrow q$ and $p, q \vdash_{\mathbf{IPC}} p \leftrightarrow q$ (which can be deduced from the Completeness Theorems of **CPC** and **IPC**, respectively), both **CPC** and **IPC** have the G-rule. This can also be deduced from Theorem 3.2.3 and the fact that $p \approx \mathrm{T}$ forms a system of defining equations for **CPC** and $\mathcal{BA}$ (resp. **IPC** and $\mathcal{HA}$). The well known fact that the varieties $\mathcal{BA}$ and $\mathcal{HA}$ are T-regular illustrates Theorem 3.2.4.

The defining equations and equivalence formulas used for **CPC** and **IPC** are not unique. An alternative system of defining equations is given by $p \approx p \to p$, and an alternative system of equivalence formulas is $\Delta_1(p,q) = p \to q$, $\Delta_2(p,q) = q \to p$. One can immediately see that the necessary conditions for algebraizability, namely (3.1.3) and (3.1.6), hold when these new systems are used. Thus the $\{\to\}$-fragments of **CPC** and **IPC**, denoted $\mathbf{CPC}_{\to}$ and $\mathbf{IPC}_{\to}$, respectively, are also algebraizable. We shall return to these deductive systems in Chapter 5, where we show, in particular, that they are strongly algebraizable.

Algebraizing Predicate Logic presents certain difficulties, nevertheless, it can be done using the methods of this chapter. Of course, this requires a presentation of predicate logic as a (structural and finitary) deductive system in our sense. This is done in two ways in [BP89a, Appendix C]. The equivalent quasivariety semantics for the two formal presentations that are given there are the varieties of 'cylindric' and 'polyadic' algebras.

Pure Implicative Logics.

3.3.1 THEOREM [BP89a, Theorem 5.10]

A pure implicative logic satisfying (I), (B), (C), (M) *and* (MP) *is algebraizable with equivalence formulas* $\Delta_1(p,q) = p \to q$, $\Delta_2(p,q) = q \to p$ *and defining equations* $p \approx p \to p$.

Proof. Let S be a pure implicative logic satisfying (I), (B), (C), (M) and (MP). To prove that S is algebraizable, we shall prove that condition (iii) of Theorem 3.1.11 holds. As noted in Section 1.4, (B′) is derivable from (B), (C) and (MP), hence (B′) is a theorem of S. Thus, by Theorem 2.6.3, S is congruential with congruence formulas without parameters $\Delta_1(p,q) = p \to q$, $\Delta_2(p,q) = q \to p$. Set $\delta(p) = p$ and $\varepsilon(p) = p \to p$. Let $\varphi \in Fm$. We need to show that

$$\varphi \dashv\vdash_S \Delta_1(\delta(\varphi), \varepsilon(\varphi)), \Delta_2(\delta(\varphi), \varepsilon(\varphi)),$$

i.e.,

$$\varphi \vdash_S \varphi \to (\varphi \to \varphi),$$

$$\varphi \vdash_S (\varphi \to \varphi) \to \varphi,$$

and

$$\varphi \to (\varphi \to \varphi), (\varphi \to \varphi) \to \varphi \vdash_S \varphi.$$

The first follows trivially from the mingle axiom (M). For the second, we have

	$\vdash_S (\varphi \to \varphi) \to (\varphi \to \varphi)$	[by (I)],
implying	$\vdash_S \varphi \to ((\varphi \to \varphi) \to \varphi)$	[by (C) and (MP)],
hence	$\varphi \vdash_S (\varphi \to \varphi) \to \varphi$	[by (MP)].

For the third, we have

	$\vdash_S ((\varphi \to \varphi) \to \varphi) \to ((\varphi \to \varphi) \to \varphi)$	[by (I)],
implying	$\vdash_S (\varphi \to \varphi) \to (((\varphi \to \varphi) \to \varphi) \to \varphi)$	[by (C) and (MP)],
so	$\vdash_S ((\varphi \to \varphi) \to \varphi) \to \varphi$	[by (I) and (MP)],
hence	$\varphi \to (\varphi \to \varphi), (\varphi \to \varphi) \to \varphi \vdash_S \varphi$	[by (MP)].

By Corollary 3.1.16, $\Delta_1(p,q) = p \to q$, $\Delta_2(p,q) = q \to p$ are equivalence formulas and $p \approx p \to p$ is a defining equation for S. □

As a corollary to this theorem, we immediately get that $\mathbf{RMO}_{\to}$ is algebraizable with the given defining equations and equivalence formulas (see Section 1.4). It is shown in [BKP, pp56-57] that $\mathbf{RMO}_{\to}$ is *not* strongly algebraizable. $\mathbf{RM}_{\to}$ is an axiomatic extension of $\mathbf{R}_{\to}$, hence (I), (B) and (C) are theorems of $\mathbf{RM}_{\to}$. Since (M) is, trivially, a theorem of $\mathbf{RM}_{\to}$ (and (MP) is an inference rule of $\mathbf{RM}_{\rightarrow}$) $\mathbf{RM}_{\rightarrow}$ is also algebraizable, by the previous theorem.

3.3.2 THEOREM [BP89a, Theorem 5.9]

None of $\mathbf{E}_{\rightarrow}$, $\mathbf{R}_{\rightarrow}$ *or* **BCI** *is algebraizable.*

Proof. Let $\mathbf{A} = \langle A ; \rightarrow \rangle$ be the algebra with universe $A = \{\mathrm{T}, t, f, \bot\}$ and with $\rightarrow$ defined in the table below.

$\rightarrow$	T	t	f	$\bot$
T	T	$\bot$	$\bot$	$\bot$
t	T	t	f	$\bot$
f	T	$\bot$	t	$\bot$
$\bot$	T	T	T	T

First we show that $\mathbf{R}_{\rightarrow}$ is not algebraizable. Let $\mathcal{A} = \langle \mathbf{A}, \{\mathrm{T}, t\}\rangle$. It is easy to see that $\mathcal{A}$ is an $\mathbf{R}_{\rightarrow}$-matrix. We shall show that $\Omega_{\mathcal{A}}$ is not an injective map and deduce from Theorem 3.1.11 that $\mathbf{R}_{\rightarrow}$ is not algebraizable. Set $F_1 = \{\mathrm{T}, t\}$ and $F_2 = \{\mathrm{T}, t, f\}$. One can easily verify that both F_1 and F_2 are $\mathbf{R}_{\rightarrow}$-filters of $\mathcal{A}$ (trivial for F_1). Now, apart from I_A, a congruence on $\mathbf{A}$ that is compatible with F_1 must contain (t, T) or $(f, \bot)$. But $(\bot, \mathrm{T}) = (\mathrm{T} \rightarrow t, \mathrm{T} \rightarrow \mathrm{T}) \in \Theta^{\mathbf{A}}(t, \mathrm{T})$ and $(\bot, \mathrm{T}) = (f \rightarrow t, \bot \rightarrow t) \in \Theta^{\mathbf{A}}(f, \bot)$, so neither $\Theta^{\mathbf{A}}(t, \mathrm{T})$ nor $\Theta^{\mathbf{A}}(f, \bot)$ is compatible with F_1 (as $\mathrm{T} \in F_1$ but $\bot \notin F_1$). Thus $\Omega_{\mathcal{A}} F_1 = I_A$. Apart from I_A, a congruence on $\mathbf{A}$ that is compatible with F_2 must contain (t, T), (t, f) or (f, T). We have $(\bot, \mathrm{T}) \in \Theta^{\mathbf{A}}(t, \mathrm{T})$, so $\Theta^{\mathbf{A}}(t, \mathrm{T})$ is not compatible with F_2. Now, $(\bot, t) = (f \rightarrow t, f \rightarrow f) \in \Theta^{\mathbf{A}}(t, f)$, hence $\Theta^{\mathbf{A}}(t, f)$ is not compatible with F_2. Also, $(\bot, t) = (\mathrm{T} \rightarrow f, f \rightarrow f) \in \Theta^{\mathbf{A}}(f, \mathrm{T})$, hence $\Theta^{\mathbf{A}}(f, \mathrm{T})$ is not compatible with F_2 either. Thus $\Omega_{\mathcal{A}} F_2 = I_A$, proving that $\Omega_{\mathcal{A}}$ is not injective.

Since $\mathbf{R}_{\rightarrow}$ is an extension of **BCI** (with the same language), it follows from the above and Corollary 3.1.17 that **BCI** is not algebraizable. Similarly, $\mathbf{R}_{\rightarrow}$ is an extension of $\mathbf{E}_{\rightarrow}$ (since **R** is an axiomatic extension of **E**), hence $\mathbf{E}_{\rightarrow}$ is not algebraizable. □

In the literature, a quasivariety $\mathfrak{K}$ of algebras $\langle \mathbf{A}; \rightarrow^{\mathbf{A}}, \mathrm{T}^{\mathbf{A}}\rangle$ called *BCI*-algebras, was introduced in [Isé66] and has been studied extensively. This quasivariety has among its axioms

$$x \rightarrow x \approx \mathrm{T},$$

and $$x \to y \approx \mathrm{T} \ \&\ y \to x \approx \mathrm{T} \Rightarrow x \approx y,$$

so by Theorem 3.1.9, $\mathfrak{K}$ is the equivalent quasivariety semantics of some algebraizable 1-deductive system S. But, by the above, this S is not **BCI**. The term '*BCI*-algebra' should perhaps be regarded as a misnomer for this reason.

Modal logics.

Since **K** is an extension of **CPC**, we wish to apply Corollary 3.1.17 to deduce that **K** is algebraizable. Recall that $p \leftrightarrow q = (p \to q) \wedge (q \to p)$ in **K**. By Corollary 3.1.17, all we need show is that $$p \leftrightarrow q \vdash_{\mathbf{K}} (\Box p) \leftrightarrow (\Box q).$$

By the inference rule (Ne), i.e., $p \vdash_{\mathbf{K}} \Box p$,

$$p \to q \vdash_{\mathbf{K}} \Box(p \to q).$$

By (K_2), i.e., $\vdash_{\mathbf{K}} \Box(p \to q) \to (\Box p \to \Box q)$, and (MP) we therefore get

$$p \to q \vdash_{\mathbf{K}} \Box p \to \Box q.$$

It follows similarly that $q \to p \vdash_{\mathbf{K}} \Box q \to \Box p$. Now, $(p \to q) \wedge (q \to p) \vdash_{\mathbf{K}} p \to q, q \to p$ and $\Box p \to \Box q, \Box q \to \Box p \vdash_{\mathbf{K}} (\Box p \to \Box q) \wedge (\Box q \to \Box p)$ (since **K** is an extension of **CPC**), hence the required result follows. Thus, by Corollary 3.1.17, **K** is algebraizable with the same defining equation and equivalence formula as **CPC**, namely $p \approx \mathrm{T}$ and $\Delta(p,q) = p \leftrightarrow q$, respectively. Moreover, since $\boldsymbol{S5}^G$ and $\boldsymbol{S4}$ are axiomatic extensions of **K**, it follows from Corollary 3.1.17 that $\boldsymbol{S5}^G$ and $\boldsymbol{S4}$ are algebraizable. Theorem 3.1.18 gives the following axiomatization for the equivalent quasivariety semantics for **K**:

(3.3.5) all the axioms of $\mathfrak{BA}$,

(3.3.6) $p \leftrightarrow p \approx \mathrm{T}$,

(3.3.7) $\Box(p \to q) \to (\Box p \to \Box q) \approx \mathrm{T}$,

(3.3.8) $p \approx \mathrm{T} \ \&\ p \to q \approx \mathrm{T} \Rightarrow q \approx \mathrm{T}$,

(3.3.9) $p \approx \mathrm{T} \Rightarrow \Box p \approx \mathrm{T}$,

(3.3.10) $p \leftrightarrow q \approx \mathrm{T} \Rightarrow p \approx q$.

For $\boldsymbol{S5}^G$, the equivalent quasivariety semantics is defined by (3.3.5) to (3.3.10) plus the following

(3.3.11) $\Box p \to p \approx \mathrm{T}$

(3.3.12) $\Diamond p \to \Box \Diamond p \approx \mathrm{T}$ [recall that $\Diamond p = \neg \Box \neg p$]

For $\boldsymbol{S4}$, the equivalent quasivariety semantics is defined by (3.3.5) to (3.3.11) plus

(3.3.13) $\Box p \to \Box\Box p \approx \mathrm{T}.$

3.3.3 THEOREM

The modal logic **K** *is strongly algebraizable with equivalent variety semantics the class* $\mathcal{MA}$ *of all modal algebras. The modal logic* $S5^G$ *is strongly algebraizable with equivalent variety semantics the class of all monadic algebras, i.e., modal algebras that satisfy the following identities*

(i) $\Box p \wedge p \approx \Box p$ *(i.e.,* $\Box p \leq p$*),*

(ii) $\Diamond p \wedge \Box \Diamond p \approx \Diamond p$ *(i.e.,* $\Diamond p \leq \Box \Diamond p$*).*

The modal logic ***S4*** *is strongly algebraizable with equivalent variety semantics the class of all interior algebras, i.e., modal algebras that satisfy (i) and*

(iii) $\Box p \wedge \Box\Box p \approx \Box p$ *(i.e.,* $\Box p \leq \Box\Box p$*).*

Proof. As noted before this theorem, the equivalent quasivariety semantics for **K** is axiomatized by (3.3.5) to (3.3.10). We shall show that (3.3.5) to (3.3.10) axiomatize $\mathcal{MA}$ and, conversely, that $\mathcal{MA}$ satisfies each of (3.3.5) to (3.3.10). Refer to Section 0.2 for an axiomatization of $\mathcal{MA}$. Note first that the quasi-identity (3.3.9) is equivalent to the identity $\Box \mathrm{T} \approx \mathrm{T}$, which is (M3). (M1) and (M2) follow from (3.3.5). Thus we need only show that (M4) is derivable from (3.3.5) to (3.3.10), i.e., we must derive

(3.3.14) $\Box(p \wedge q) \approx \Box p \wedge \Box q.$

Using Boolean identities, we get the following:

$$\begin{aligned}
&\Box(p \wedge q) \leftrightarrow (\Box p \wedge \Box q)\\
&\quad = (\Box(p \wedge q) \to (\Box p \wedge \Box q)) \wedge ((\Box p \wedge \Box q) \to \Box(p \wedge q))\\
&\quad = ((\neg \Box(p \wedge q)) \vee (\Box p \wedge \Box q)) \wedge ((\neg (\Box p \wedge \Box q)) \vee \Box(p \wedge q))\\
&\quad \approx [(\neg \Box(p \wedge q)) \wedge \neg (\Box p \wedge \Box q)] \vee [(\neg \Box(p \wedge q)) \wedge \Box(p \wedge q)] \vee [(\Box p \wedge \Box q) \wedge \neg (\Box p \wedge \Box q)]\\
&\qquad\quad \vee [(\Box p \wedge \Box q) \vee \Box(p \wedge q)] \qquad\qquad [\text{by distributivity}]\\
&\quad \approx [(\neg \Box(p \wedge q)) \wedge \neg (\Box p \wedge \Box q)] \vee [\perp] \vee [\perp] \vee [(\Box p \wedge \Box q) \vee \Box(p \wedge q)]\\
&\quad \approx (\neg [(\Box p \wedge \Box q) \vee \Box(p \wedge q)]) \vee [(\Box p \wedge \Box q) \vee \Box(p \wedge q)] \qquad [\text{by de Morgan law}]\\
&\quad \approx \mathrm{T}.
\end{aligned}$$

Thus, by (3.3.10), we have that $\Box(p \wedge q) \approx \Box p \wedge \Box q$. To see that $\mathcal{MA}$ satisfies (3.3.5) to (3.3.10), all we need show is that (3.3.7) holds. (We are exploiting the aforementioned axiomatization of $\mathcal{BA}$.) So, using Boolean identities again, we have

$$\begin{aligned}
\Box(p \to q) \to (\Box p \to \Box q) &\approx \neg\Box(\neg p \vee q) \vee (\neg\Box p \vee \Box q) \\
&\approx (\neg\Box(\neg p \vee q) \vee \neg\Box p) \vee \Box q && \text{[by associativity of } \vee\text{]} \\
&\approx \neg(\Box(\neg p \vee q) \wedge \Box p) \vee \Box q && \text{[by de Morgan law]} \\
&\approx \neg\Box((\neg p \vee q) \wedge p) \vee \Box q && \text{[by (3.3.14)]} \\
&\approx \neg\Box(q \wedge p) \vee \Box q \\
&\approx \neg(\Box q \wedge \Box p) \vee \Box q && \text{[by (3.3.14)]} \\
&\approx \neg\Box q \vee \neg\Box p \vee \Box q \\
&\approx \mathrm{T} \vee \neg\Box p \\
&\approx \mathrm{T}.
\end{aligned}$$

This proves the result for **K**. Since $\boldsymbol{S5}^G$ is an axiomatic extension of **K**, it follows from Theorem 3.1.17 that $\boldsymbol{S5}^G$ is algebraizable. By Theorem 3.1.18, its equivalent quasivariety semantics is the variety of all modal algebras that satisfy

$$\Box p \to p \approx \mathrm{T} \quad \text{and} \quad \Diamond p \to \Box\Diamond p \approx \mathrm{T}.$$

In particular, the quasivariety is axiomatizable by identities, so it is a variety, hence $\boldsymbol{S5}^G$ is strongly algebraizable. Recall that in $\mathfrak{BA}$, the associated partial order $\leq$ is definable by $a \leq b$ iff $a \to b = \mathrm{T}$. Thus, the above identities are equivalent to (i) and (ii), respectively. ***S4*** is also an axiomatic extension of **K**, hence is also algebraizable. Its equivalent quasivariety semantics is the variety of all modal algebras satisfying the identities

$$\Box p \to p \approx \mathrm{T} \quad \text{and} \quad \Box p \to \Box\Box p \approx \mathrm{T}.$$

As for $\boldsymbol{S5}^G$, ***S4*** is strongly algebraizable. Applying the same comment about the partial order $\leq$ that was used for $\boldsymbol{S5}^G$, we see these identities are equivalent to (i) and (iii), respectively. □

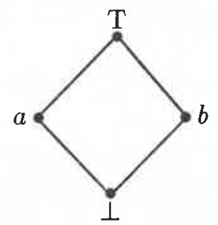
Note that since both **K** and $\boldsymbol{S5}^G$ are algebraizable extensions of **CPC**, it follows by the note after Definition 2.3.1 that both these deductive systems have the G-rule.

We proved in Corollary 2.6.5 that $\boldsymbol{S5}^W$ is congruential. We now show that it is not algebraizable.

3.3.4 THEOREM [BP89a, Theorem 5.5]

The deductive systems $\boldsymbol{S5}^C$ *and* $\boldsymbol{S5}^W$ *are not algebraizable.*

Proof. Define the algebra $\mathbf{A} = \langle A;\ \wedge, \vee, \to, \neg, \Box, \bot, \mathrm{T}\rangle$ to be the modal algebra with universe $A = \{a, b, \mathrm{T}, \bot\}$ such that $\bot < a, b < \mathrm{T}$, $a \not\leq b$, $b \not\leq a$ as in the figure below:



The operations $\wedge$, $\vee$ are interpreted as the lattice operations of $\langle A; \leq \rangle$ and the operations $\rightarrow$, $\neg$ and $\Box$ are explicitly defined by the following tables:

$\rightarrow$	$\bot$	a	b	T
$\bot$	T	T	T	T
a	$\bot$	T	b	T
b	$\bot$	a	T	T
T	$\bot$	a	b	T

$\neg$	
$\bot$	T
a	b
b	a
T	$\bot$

$\Box$	
$\bot$	$\bot$
a	$\bot$
b	$\bot$
T	T

Set $\mathcal{A} = \langle \mathbf{A}, \{\mathrm{T}\}\rangle$. Since $\mathbf{A}$ is a modal algebra and $\Box\mathrm{T} = \mathrm{T}$ and $\mathrm{T} \rightarrow \mathrm{T} = \mathrm{T}$, it follows easily that each axiom of $\boldsymbol{S5}^C$ and $\boldsymbol{S5}^W$ evaluates to T in $\mathbf{A}$ (regardless of the values substituted for variables) and also that $\{\mathrm{T}\}$ is closed under the inference rules (MP) and (K_{11}). Thus $\mathcal{A}$ is an $\boldsymbol{S5}^C$- and an $\boldsymbol{S5}^W$-matrix. Clearly $\{a, b, \bot, \mathrm{T}\}$ is an $\boldsymbol{S5}^C$- and an $\boldsymbol{S5}^W$-filter of $\mathcal{A}$. Set $F = \{a, \mathrm{T}\}$. Again, since each axiom of $\boldsymbol{S5}^C$ and $\boldsymbol{S5}^W$ evaluates to T, and it is easily checked that F is closed under the inference rules (MP) and (K_{11}), F is an $\boldsymbol{S5}^C$- and $\boldsymbol{S5}^W$-filter of $\mathcal{A}$. Similarly $\{b, \mathrm{T}\}$ is also an $\boldsymbol{S5}^C$- and $\boldsymbol{S5}^W$-filter of $\mathcal{A}$. Thus the lattices $\mathbf{Fi}^{\boldsymbol{S5}^C}\mathcal{A}$ and $\mathbf{Fi}^{\boldsymbol{S5}^W}\mathcal{A}$ each have at least four elements. Now it is easy to see that the algebra $\mathbf{A}$ has only two congruences, namely I_A and A^2: Note first that $\Theta^{\mathbf{A}}(\bot, \mathrm{T}) = A^2$ since $\mathbf{A}$ has a lattice reduct (see the remarks following Theorem 0.2.1). We have $(\bot, \mathrm{T}) = (\Box a, \Box\mathrm{T}) \in \Theta^{\mathbf{A}}(a, \mathrm{T})$ and $(\bot, \mathrm{T}) = (\Box b, \Box\mathrm{T}) \in \Theta^{\mathbf{A}}(b, \mathrm{T})$, so $\Theta^{\mathbf{A}}(a, \mathrm{T}) = A^2 = \Theta^{\mathbf{A}}(b, \mathrm{T})$. Since $(b, \mathrm{T}) = (\neg a, \neg\bot) \in \Theta^{\mathbf{A}}(a, \bot)$ and $(a, \mathrm{T}) = (\neg b, \neg\bot) \in \Theta^{\mathbf{A}}(b, \bot)$, it follows that $\Theta^{\mathbf{A}}(a, \bot) = A^2 = \Theta^{\mathbf{A}}(b, \bot)$. Finally, $(b, \mathrm{T}) = (a \rightarrow b, b \rightarrow b) \in \Theta^{\mathbf{A}}(a, b)$, so $\Theta^{\mathbf{A}}(a, b) = A^2$. Thus the Leibniz operators $\Omega_{\mathcal{A}} : \mathbf{Fi}^{\boldsymbol{S5}^C}\mathcal{A} \rightarrow \mathbf{Con}\,\mathbf{A}$ and $\Omega_{\mathcal{A}} : \mathbf{Fi}^{\boldsymbol{S5}^W}\mathcal{A} \rightarrow \mathbf{Con}\,\mathbf{A}$ cannot be injective. By Theorem 3.1.11 (iv), the result follows. $\Box$

From the above theorem we can deduce that $\boldsymbol{S5}^W_{\rightarrow_{\mathbf{E}}}$ (recall from Section 1.4 that $p \rightarrow_{\mathbf{E}} q =$

$\Box(p \to q))$ is not algebraizable. For if it were, then $S5^W$ would be algebraizable as it satisfies the necessary conditions of Corollary 3.1.17 (recall that $S5^W$ is congruential).

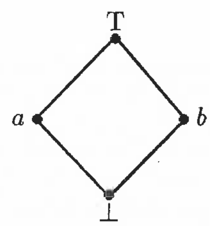
Entailment and Relevance Logics.

We proved in Corollary 2.6.5 that **E** is congruential. We now prove that **E** is not algebraizable.

3.3.5 THEOREM [BP89a, Corollary 5.7]

The deductive system **E** *is not algebraizable.*

Proof. Define the algebra $\mathbf{A} = \langle A\,; \to, \wedge\,, \vee\,, \neg\,\rangle$ as follows: Let $A = \{a, b, \mathrm{T}, \perp\}$ and let $\langle \{a, b, \perp, \mathrm{T}\}\,;\ \wedge\,, \vee\,\rangle$ be a lattice with $\perp\ < a, b < \mathrm{T}$, $a \not\leq b$, $b \not\leq a$, as shown below:



The connectives $\to$ and $\neg$ are defined by the following tables:

$\to$	$\perp$	a	b	T
$\perp$	T	T	T	T
a	$\perp$	T	$\perp$	T
b	$\perp$	$\perp$	T	T
T	$\perp$	$\perp$	$\perp$	T

$\neg$	
$\perp$	T
a	$\perp$
b	$\perp$
T	$\perp$

Set $\mathcal{A} = \langle \mathbf{A}, \{\mathrm{T}\}\rangle$. It can easily be shown that each axiom of **E** evaluates to T and also that $\{\mathrm{T}\}$ is closed under the inference rules (MP) and (A). Thus $\mathcal{A}$ is an **E**-matrix. Clearly A is an **E**-filter of $\mathcal{A}$. Set $F = \{a, \mathrm{T}\}$. It is tediously verified that F is closed under all the axioms of **E** and under the inference rules (MP) and (A), therefore F is an **E**-filter of $\mathcal{A}$. Similarly $\{b, \mathrm{T}\}$ is also an **E**-filter of $\mathcal{A}$. Thus the lattice $\mathbf{Fi}^{\mathbf{E}}\mathcal{A}$ has at least four elements.

We shall show that **A** has only two congruences, namely I_A and A^2. Let $\Phi \in \mathrm{Con}\,\mathbf{A}$. If $(a, b) \in \Phi$, then $(a, \perp) = (a \wedge a, a \wedge b) \in \Phi$ and $(a, \mathrm{T}) = (a \vee a, a \vee b) \in \Phi$, hence $(\perp, \mathrm{T}) \in \Phi$, by transitivity, which implies $\Phi = A^2$. If $(a, \perp) \in \Phi$, then $(\perp, \mathrm{T}) = (\neg a, \neg \perp) \in \Phi$, hence $(a, \mathrm{T}) \in \Phi$, so $\Phi = A^2$. The other cases follow similarly. This implies that the Leibniz operator $\Omega_{\mathcal{A}} : \mathbf{Fi}^{\mathbf{E}}\mathcal{A} \to \mathbf{Con}\,\mathbf{A}$ cannot be injective. By Theorem 3.1.11 (iv), the result follows. □

3.3.6 THEOREM [BP89a, Theorem 5.8]

The deductive systems **R** *and* **RM** *are algebraizable with equivalence formulas* $\Delta_1(p,q) = p \to q$, $\Delta_2(p,q) = q \to p$ *and defining equation* $p \wedge (p \to p) \approx p \to p$.

Proof. By Corollary 2.6.5, **E** is congruential, hence its axiomatic extension **R** is also congruential (with congruence formulas $\Delta_1(p,q)$, $\Delta_2(p,q)$ as above). We shall use condition (iii) of Theorem 3.1.11 to show that **R** is algebraizable. Thus we need only show that

$$p \dashv\vdash_{\mathbf{R}} \{(p \wedge (p \to p)) \to (p \to p),\ (p \to p) \to (p \wedge (p \to p))\}.$$

Consider the following derivation in **R**:

(3.3.15) $\quad \vdash_{\mathbf{R}} p \to ((p \to p) \to p)$ [by (E_{17})]

(3.3.16) $\quad p \vdash_{\mathbf{R}} (p \to p) \to p$ [by (3.3.15) and (MP)]

(3.3.17) $\quad \vdash_{\mathbf{R}} (p \to p) \to (p \to p)$ [by (E_1)]

(3.3.18) $\quad p \vdash_{\mathbf{R}} ((p \to p) \to p) \wedge ((p \to p) \to (p \to p))$ [by (3.3.16), (3.3.17) and (A)]

(3.3.19) $\quad \vdash_{\mathbf{R}} (((p \to p) \to p) \wedge ((p \to p) \to (p \to p))) \to ((p \to p) \to (p \wedge (p \to p)))$ [by (E_7)]

$\quad p \vdash_{\mathbf{R}} (p \to p) \to (p \wedge (p \to p))$ [by (3.3.18), (3.3.19) and (MP)]

Now, from (E_6), we immediately get $\vdash_{\mathbf{R}} (p \wedge (p \to p)) \to (p \to p)$, hence

$$p \vdash_{\mathbf{R}} \{(p \wedge (p \to p)) \to (p \to p),\ (p \to p) \to (p \wedge (p \to p))\}.$$

For the inference in the other direction, note that

$$(p \to p) \to (p \wedge (p \to p)) \vdash_{\mathbf{R}} p \wedge (p \to p),$$

by (E_1) and (MP), and $p \wedge (p \to p) \vdash_{\mathbf{R}} p$ by (E_5), hence

$$\{(p \wedge (p \to p)) \to (p \to p),\ (p \to p) \to (p \wedge (p \to p))\} \vdash_{\mathbf{R}} p.$$

Thus **R** is algebraizable and, by Corollary 3.1.16, $\Delta_1(p,q), \Delta_2(p,q)$ are equivalence formulas and $p \wedge (p \to p) \approx p \to p$ are defining equations for **R** and its equivalent quasivariety semantics. Since **RM** is an axiomatic extension of **R**, it is also algebraizable, with the same defining equations and equivalence formulas. □

In fact, in the paper [FR90] it is shown that **R** is strongly algebraizable and an axiomatization of its equivalent variety semantics is given there. The extension **RM** of **R** is also strongly algebraizable. Its equivalent variety semantics is the variety of Sugihara algebras (cf. [AB75], [Dun70]; also see [BD86]). This variety is defined as follows: Let **Z** be the algebra $\langle Z; \wedge, \vee, \rightarrow, - \rangle$ of type $\langle 2,2,2,1 \rangle$, where Z is the set of integers with the usual ordering, $\wedge, \vee$ are the lattice operations with respect to this order, $-$ is the usual operation of additive inverse and

$$x \rightarrow y = (-x) \vee y \quad \text{if } x \leq y$$

and
$$x \rightarrow y = (-x) \wedge y \quad \text{otherwise.}$$

By a *Sugihara algebra* we will understand any algebra in the variety $\mathfrak{Z}$ generated by **Z** (so $\mathfrak{Z} = \mathrm{HSP}(\mathbf{Z})$).

Let **A** be the algebra defined in Theorem 3.3.2 and let $\mathcal{A} = \langle \mathbf{A}, \{\mathrm{T}, t, f\} \rangle$. As noted in the proof of Theorem 3.3.2, $\mathcal{A}$ is an $\mathbf{R}_{\rightarrow}$-matrix. We have that

$$p, q \not\models_{\mathcal{A}} p \rightarrow q.$$

For if we interpret p as f and q as t in **A**, then both $f, t \in \{\mathrm{T}, t, f\}$, but $p \rightarrow q$ is interpreted as $f \rightarrow t = \perp \notin \{\mathrm{T}, t, f\}$. Thus we get that

$$p, q \nvdash_{\mathbf{R}_{\rightarrow}} p \rightarrow q.$$

Since $\mathbf{R}_{\rightarrow}$ coincides with the $\{\rightarrow\}$-reduct of **R**, it follows that $p, q \nvdash_{\mathbf{R}} p \rightarrow q$. As noted after Definition 3.2.1, a deductive system has the G-rule with respect to a certain system of equivalence formulas if and only if it has the G-rule with respect to every system of equivalence formulas. Thus **R** does not have the G-rule.

In fact **RM** does not have the G-rule either. We shall show that condition (ii) of Theorem 3.2.4 does not hold in $\mathfrak{Z}$. Let **Z** be the Sugihara algebra defined above. By Theorem 3.3.6, $p \wedge (p \rightarrow p) \approx p \rightarrow p$ is a defining equation for **RM**. We shall show that

$$\mathbf{Z} \not\models p \wedge (p \rightarrow p) \approx p \rightarrow p \;\&\; q \wedge (q \rightarrow q) \approx q \rightarrow q \Rightarrow p \approx q.$$

Interpret p and q as 1 and 2, respectively, in Z. Then $1 \wedge (1 \rightarrow 1) = 1 = 1 \rightarrow 1$ and $2 \wedge (2 \rightarrow 2) = 2 = 2 \rightarrow 2$, but $1 \neq 2$. By Theorem 3.2.4 (ii), **RM** does not have the G-rule.

Note that **R** was obtained from **E** by the addition of the axiom $p \rightarrow ((p \rightarrow p) \rightarrow p)$. Recall that $\mathcal{A} = \langle \mathbf{A}, \{a, \mathrm{T}\} \rangle$, as defined in Theorem 3.3.5, is an **E**-matrix. One would expect that $\mathcal{A}$ is *not* an **R**-matrix, otherwise Theorem 3.3.5 would prove that **R** is not algebraizable. This is indeed the case since $a \to ((a \to a) \to a) = \bot$, which is not an element of $\{a, T\}$.

Łukasiewicz Many-Valued Logics.

3.3.7 THEOREM

For each $n \leq \omega$, S_n is algebraizable with equivalence formulas $\Delta_1(p,q) = p \to q$, $\Delta_2(p,q) = q \to p$, defining equation $p \approx p \to p$ and equivalent algebraic semantics $\{\mathbf{L}_n\}$ (hence with equivalent quasivariety semantics $\{\mathbf{L}_n\}^{\mathrm{Q}}$).

Proof. Let $n \leq \omega$ be fixed. To prove that the set $\{\mathbf{L}_n\}$ forms an equivalent algebraic semantics for S_n, we must show that the following two conditions hold, corresponding to (3.1.3) and (3.1.6), respectively (for $\Gamma \cup \{\varphi\} \subseteq Fm$),

(i) $\Gamma \vdash_{S_n} \varphi$ if and only if $\{\psi \approx \psi \to \psi;\ \psi \in \Gamma\} \models_{\mathbf{L}_n} \varphi \approx \varphi \to \varphi$.

(ii) $x \approx y =\!\!|\!\!\models_{\mathbf{L}_n} \{x \to y \approx (x \to y) \to (x \to y), y \to x \approx (y \to x) \to (y \to x)\}$.

For (i), we have, by definition of S_n, that $\Gamma \vdash_{S_n} \varphi$ if and only if $\Gamma \models_{\mathcal{A}} \varphi$, where $\mathcal{A} = \langle \mathbf{L}_n, \{1\} \rangle$. Thus $\Gamma \vdash_{S_n} \varphi$ if and only if for every interpretation $\bar{a}$ of the variables of $\Gamma \cup \{\varphi\}$ in L_n,

(3.3.20) $\quad \psi^{\mathbf{L}_n}(\bar{a}) = 1$ for all $\psi \in \Gamma$ implies $\varphi^{\mathbf{L}_n}(\bar{a}) = 1$.

Now, $\{\psi \approx \psi \to \psi;\ \psi \in \Gamma\} \models_{\mathbf{L}_n} \varphi \approx \varphi \to \varphi$ if and only if for every interpretation $\bar{a}$ of the variables of $\Gamma \cup \{\varphi\}$ in L_n,

(3.3.21) $\quad \psi^{\mathbf{L}_n}(\bar{a}) = \psi^{\mathbf{L}_n}(\bar{a}) \to \psi^{\mathbf{L}_n}(\bar{a})$ for all $\psi \in \Gamma$ implies $\varphi^{\mathbf{L}_n}(\bar{a}) = \varphi^{\mathbf{L}_n}(\bar{a}) \to \varphi^{\mathbf{L}_n}(\bar{a})$.

But since $a \to a = 1$ for each $a \in L_n$, (3.3.21) is evidently equivalent to (3.3.20).

For (ii), let a, b be interpretations of x and y in L_n, respectively. Since $a \to a = 1$ for any $a \in L_n$, in particular $(a \to b) \to (a \to b) = 1$ and $(b \to a) \to (b \to a) = 1$. If $a = b$, then $a \to b = 1$ and $b \to a = 1$, by definition. Conversely, if $a \to b = 1$ and $b \to a = 1$, then

$$1 \leq b + 1 - a \quad \text{and} \quad 1 \leq a + 1 - b,$$

i.e., $a \leq b$ and $b \leq a$, so $a = b$. □

For each $n \leq \omega$, S_n has the G-rule, since, by definition,

$$p, q \vdash_{S_n} p \to q \quad \text{iff} \quad p \approx 1, q \approx 1 \models_{\mathbf{L}_n} p \to q \approx 1 \quad \text{iff} \quad \models_{\mathbf{L}_n} 1 \to 1 \approx 1,$$

and the fact that $\models_{\mathbf{L}_n} 1 \to 1 \approx 1$ follows immediately from the definition of $\to$. Thus, by Theorem 3.2.4, $\{\mathbf{L}_n\}^{\mathrm{Q}}$ is a relatively 1-regular quasivariety.

In Section 1.4, we presented an axiomatization of S_ω. Since S_ω has the G-rule, Theorem 3.2.3 allows us to define a constant term $\mathrm{T} = \Delta_1(x,x) = x \rightarrow x$ in the language of the equivalent quasivariety semantics such that $p \approx \mathrm{T}$ is a defining equation for S_ω. Thus, by Theorem 3.1.18, the equivalent quasivariety semantics for S_ω, namely $\{\mathbf{L}_\omega\}^{\mathrm{Q}}$, can be axiomatized by

(3.3.22) $\quad x \rightarrow (y \rightarrow x) \approx \mathrm{T}$

(3.3.23) $\quad (x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) \approx \mathrm{T}$

(3.3.24) $\quad ((x \rightarrow y) \rightarrow y) \rightarrow ((y \rightarrow x) \rightarrow x) \approx \mathrm{T}$

(3.3.25) $\quad ((\neg x) \rightarrow (\neg y)) \rightarrow (y \rightarrow x) \approx \mathrm{T}$

(3.3.26) $\quad x \approx \mathrm{T} \;\&\; x \rightarrow y \approx \mathrm{T} \Rightarrow y \approx \mathrm{T}$

(3.3.27) $\quad x \rightarrow y \approx \mathrm{T} \;\&\; y \rightarrow x \approx \mathrm{T} \Rightarrow x \approx y.$

We return to this quasivariety in Chapter 5, where we show that it is a variety, hence that S_ω is strongly algebraizable.

Equivalent Deductive Systems.

3.3.8 COROLLARY [BP89b, p21]

The three-valued paraconsistent logic $\mathbf{J_3}$ (refer to the discussion following Proposition 1.9.2) is algebraizable with defining equation $\Diamond p \approx \Diamond(p \rightarrow p)$, equivalence formulas $\Delta_1(p,q) = \Box(p \rightarrow q)$, $\Delta_2(p,q) = \Box(q \rightarrow p)$ and equivalent quasivariety semantics $\{\mathbf{L}_2\}^{\mathrm{Q}}$.

Proof. In Section 1.9 we proved that $\mathbf{J_3}$ and the 3-valued many-valued logic S_2 of Lukasiewicz are equivalent with interpretations $\Diamond$ from $\mathbf{J_3}$ to S_2 and $\Box$ from S_2 to $\mathbf{J_3}$. In Theorem 3.3.7, we proved that S_2 is algebraizable with equivalent quasivariety semantics $\{\mathbf{L}_2\}^{\mathrm{Q}}$, defining equations $p \approx p \rightarrow p$ and equivalence formulas $\Delta_1(p,q) = p \rightarrow q$, $\Delta_2(p,q) = q \rightarrow p$. The result now follows directly from Theorem 3.1.19. □

Note that $\mathbf{J_3}$ does *not* have the G-rule: In view of the comments which follow Definition 3.2.1, it is sufficient to show that $\mathbf{J_3}$ does not have the G-rule with respect to the system $\Delta_1(p,q) = \Box(p \rightarrow q)$, $\Delta_2(p,q) = \Box(q \rightarrow p)$ of equivalence formulas. Now,

$$p, q \vdash_{\mathbf{J_3}} \Box(p \rightarrow q) \quad \text{iff} \quad p, q \models_{\mathfrak{J}_3} \Box(p \rightarrow q),$$

where $\mathfrak{J}_3 = \langle \mathbf{L}_2, \{1, \frac{1}{2}\}\rangle$. Interpreting p as 1 and q as $\frac{1}{2}$ in $\mathbf{L}_2$, we have that $1, \frac{1}{2} \in \{1, \frac{1}{2}\}$, but

$\Box(1 \to \frac{1}{2}) = \Box\frac{1}{2} = 0 \notin \{1, \frac{1}{2}\}$. This shows that the G-rule is not preserved by the relation of equivalence between deductive systems. Moreover, it shows that for a quasivariety $\mathfrak{K}$, there may exist two distinct 1-deductive systems both of whose equivalent quasivariety semantics is $\mathfrak{K}$. Proposition 3.2.6 states that every relatively T-regular quasivariety is the equivalent quasivariety semantics of a 1-deductive system that has the G-rule. As the above example shows, this does not exclude the possibility that there may exist other 1-deductive systems whose equivalent quasivariety semantics is $\mathfrak{K}$ but that do not have the G-rule.

Two Logics with the Same Algebraization.

The deductive systems $\mathbf{J}_3$ and S_2 are examples of distinct deductive systems that have the same equivalent quasivariety semantics. We present here another example that appears in [BP89a]. The implicative logic $\mathbf{RMO}_\to$ is an algebraizable, axiomatic extension of $\mathbf{R}_\to$. We now consider two non-axiomatic extensions of $\mathbf{R}_\to$.

Let $\mathbf{A} = \langle\{\mathrm{T}, t, f, \perp\}, \to\rangle$ be the algebra defined in Theorem 3.3.2. Set

(3.3.28) $$\delta_1(p) = p, \quad \varepsilon_1(p) = p \to p,$$

(3.3.29) $$\delta_2(p) = (p \to (p \to p)) \to p, \quad \varepsilon_2(p) = (p \to (p \to p)) \to (p \to p),$$

Let S_1 and S_2 be the two deductive systems over the language $\mathcal{L} = \{\to\}$ defined as follows for $i = 1, 2$:

(3.3.30) $$\Gamma \vdash_{S_i} \varphi \quad \text{iff} \quad \{\delta_i(\psi) \approx \varepsilon_i(\psi);\ \psi \in \Gamma\} \models_{\mathbf{A}} \delta_i(\varphi) \approx \varepsilon_i(\varphi).$$

It is easy to see that $\models_{\mathbf{A}} = \models_{\mathrm{ISP}(\mathbf{A})}$, but since $\mathbf{A}$ is finite, $\{\mathbf{A}\}^{\mathrm{Q}} = \mathrm{ISPP_U}(\mathbf{A}) = \mathrm{ISP}(\mathbf{A})$ (Lemma 0.3.6). Thus $\models_{\mathbf{A}}$ coincides with $\models_{\{\mathbf{A}\}^{\mathrm{Q}}}$, hence Lemma 3.1.1 (ii) implies that $\models_{\mathbf{A}}$ is finitary, so S_1 and S_2 are indeed deductive systems.

3.3.9 THEOREM [BP89a, Theorem 5.12]

The deductive systems S_1 and S_2 are distinct algebraizable deductive systems with the same equivalent quasivariety semantics $\mathfrak{K} = \{\mathbf{A}\}^{\mathrm{Q}}$. $\mathfrak{K}$ has the same system of equivalence formulas, namely $\Delta(p,q) = \{p \to q, q \to p\}$, with respect to both S_1 and S_2, but different systems of defining equations, namely $\delta_1(p) \approx \varepsilon_1(p)$ and $\delta_2(p) \approx \varepsilon_2(p)$, respectively, where

$$\delta_1(p) = p, \quad \varepsilon_1(p) = p \to p,$$

and $$\delta_2(p) = \varphi \to p, \quad \varepsilon_2(p) = \varphi \to (p \to p),$$

where $\varphi = p \to (p \to p)$.

Proof. We shall use Definitions 3.1.2 and 3.1.4 to show that $\mathcal{K}$ is the equivalent quasivariety semantics for both S_1 and S_2. The equivalence (3.1.3) is precisely the defining condition for $\vdash_{S_i}$, $i = 1, 2$, i.e., (3.3.30), while (3.1.6) is

(3.3.31) $$p \approx q =\!\!||\!\!=_{\mathbf{A}} \delta_i(\Delta(p,q)) \approx \varepsilon_i(\Delta(p,q)).$$

Set $F_1 = \{\mathrm{T}, t\}$ and $F_2 = \{\mathrm{T}, t, f\}$. From the definition of $\to$ we get that $\delta_1^{\mathbf{A}}(\mathrm{T}) = \mathrm{T} = \mathrm{T} \to \mathrm{T} = \varepsilon_1^{\mathbf{A}}(\mathrm{T})$, $\delta_1^{\mathbf{A}}(t) = t = t \to t = \varepsilon_1^{\mathbf{A}}(t)$, $\delta_1^{\mathbf{A}}(f) = f \neq t = f \to f = \varepsilon_1^{\mathbf{A}}(f)$ and $\delta_1^{\mathbf{A}}(\perp) = \perp \neq \mathrm{T} = \perp \to \perp = \varepsilon_1^{\mathbf{A}}(\perp)$. Thus

(3.3.32) $$\delta_1^{\mathbf{A}}(a) = \varepsilon_1^{\mathbf{A}}(a) \quad \text{if and only if} \quad a \in F_1.$$

By a similar computation, it can be shown that

(3.3.33) $$\delta_2^{\mathbf{A}}(a) = \varepsilon_2^{\mathbf{A}}(a) \quad \text{if and only if} \quad a \in F_2.$$

The implication from left to right of (3.3.31) means that $\delta_i^{\mathbf{A}}(\Delta^{\mathbf{A}}(a,a)) = \varepsilon_i^{\mathbf{A}}(\Delta^{\mathbf{A}}(a,a))$ for all $a \in \{\mathrm{T}, t, f, \perp\}$ and $i = 1, 2$. But this follows from (3.3.32), (3.3.33) and the fact that $\Delta_i^{\mathbf{A}}(a,a) = a \to a \in \{\mathrm{T}, t\} \subseteq F_1 \subseteq F_2$ for all $a \in \{\mathrm{T}, t, f, \perp\}$ and $i = 1, 2$. For the implication from right to left, note that if $a \neq b$, then either $a \to b = \perp$ or $b \to a = \perp$, hence $\delta_i^{\mathbf{A}}(\Delta^{\mathbf{A}}(a,b)) \neq \varepsilon_i^{\mathbf{A}}(\Delta^{\mathbf{A}}(a,b))$ for $i = 1, 2$.

This shows that S_1 and S_2 are both algebraizable and that the unique equivalent quasivariety semantics for both of them is $\mathcal{K} = \{\mathbf{A}\}^{\mathbf{Q}}$.

Observe that $f \in F_2$, but $\delta_1^{\mathbf{A}}(f) \to \varepsilon_1^{\mathbf{A}}(f) = f \to (f \to f) = f \to t = \perp \notin F_2$, therefore, by (3.3.32) and (3.3.33), $\delta_2^{\mathbf{A}}(f) = \varepsilon_2^{\mathbf{A}}(f)$ but $\delta_2^{\mathbf{A}}(\delta_1^{\mathbf{A}}(f) \to \varepsilon_1^{\mathbf{A}}(f)) \neq \varepsilon_2^{\mathbf{A}}(\delta_1^{\mathbf{A}}(f) \to \varepsilon_1^{\mathbf{A}}(f))$. Thus

(3.3.34) $$\delta_2(p) \approx \varepsilon_2(p) \not\models_{\mathbf{A}} \delta_2(\delta_1(p) \to \varepsilon_1(p)) \approx \varepsilon_2(\delta_1(p) \to \varepsilon_1(p)).$$

By the definition (3.3.30), we may conclude that $p \nvdash_{S_2} \delta_1(p) \to \varepsilon_1(p)$, but it is true that $p \vdash_{S_1} \delta_1(p) \to \varepsilon_1(p)$ since $p \dashv\vdash_{S_1} \Delta(\delta_1(p), \varepsilon_1(p))$. This shows that S_1 and S_2 are distinct deductive systems. □

Classical Equivalential logic.

Recall that $\mathbf{CPC}_{\leftrightarrow}$ is the $\{\leftrightarrow\}$-fragment of the Classical Propositional Calculus (defined in Section 1.4). It follows from Corollary 3.1.8 that $\mathbf{CPC}_{\leftrightarrow}$ is algebraizable and $\Delta(p,q) = p \leftrightarrow q$ forms a system of equivalence formulas for $\mathbf{CPC}_{\leftrightarrow}$ and its equivalent quasivariety semantics. Thus $\mathbf{CPC}_{\leftrightarrow}$ satisfies the conditions (1) to (4) of Theorem 3.1.11 (ii). Moreover, it satisfies conditions (6) and (7) of Corollary 3.2.2, hence a system of defining equations for $\mathbf{CPC}_{\leftrightarrow}$ and its equivalent quasivariety semantics is $p \approx p \leftrightarrow p$. Since the G-rule is satisfied, Theorem 3.2.3 shows that if we define $\mathrm{T} = \Delta(p, p) = p \leftrightarrow p$ for any variable p then, over the equivalent quasivariety semantics of $\mathbf{CPC}_{\leftrightarrow}$, T is a constant. By Corollary 3.1.8, the equivalent quasivariety semantics for $\mathbf{CPC}_{\leftrightarrow}$ is the class of all $\{\leftrightarrow\}$-subreducts of Boolean algebras. Using the symbol $+$ for $\leftrightarrow$, we see that this is precisely the variety $\mathfrak{BG}$ of Boolean groups (as defined in Section 0.2). We shall present an axiomatization here that was produced by Łukasiewicz, [Łuk70], and prove that it does form an axiomatization of $\mathbf{CPC}_{\leftrightarrow}$.

3.3.10 LEMMA

Let S be the deductive system over the language $\{\leftrightarrow\}$, where $\leftrightarrow$ is binary, that is defined by the single axiom

(i) $\quad (p \leftrightarrow q) \leftrightarrow ((r \leftrightarrow q) \leftrightarrow (p \leftrightarrow r)),$

and the single rule of inference

(ii) $\quad \langle \{p, p \leftrightarrow q\}, q \rangle.$

Then

(3.3.35) $\quad \vdash_S p \leftrightarrow p,$

(3.3.36) $\quad \vdash_S (p \leftrightarrow q) \leftrightarrow (q \leftrightarrow p),$

(3.3.37) $\quad \vdash_S p \leftrightarrow (q \leftrightarrow (q \leftrightarrow p)).$

Proof. (3.3.35) By (i)

$\vdash_S [(p \leftrightarrow q) \leftrightarrow ((r \leftrightarrow q) \leftrightarrow (p \leftrightarrow r))] \leftrightarrow [((p \leftrightarrow q) \leftrightarrow ((r \leftrightarrow q) \leftrightarrow (p \leftrightarrow r))) \leftrightarrow ((p \leftrightarrow q) \leftrightarrow (p \leftrightarrow q))]$, so

(1) $\quad \vdash_S (p \leftrightarrow q) \leftrightarrow (p \leftrightarrow q)$ by (ii), twice.

$\vdash_S ((p \leftrightarrow q) \leftrightarrow (p \leftrightarrow q)) \leftrightarrow [(\varphi \leftrightarrow (p \leftrightarrow q)) \leftrightarrow ((p \leftrightarrow q) \leftrightarrow \varphi)]$ (for any $\varphi \in Fm$) by (i), and

(2) $\quad \vdash_S (\varphi \leftrightarrow (p \leftrightarrow q)) \leftrightarrow ((p \leftrightarrow q) \leftrightarrow \varphi)$ by (1) and (ii), and

$\vdash_S ((p \leftrightarrow q) \leftrightarrow ((r \leftrightarrow q) \leftrightarrow (p \leftrightarrow r))) \leftrightarrow [[((p \leftrightarrow r) \leftrightarrow (r \leftrightarrow q)) \leftrightarrow ((r \leftrightarrow q) \leftrightarrow (p \leftrightarrow r))] \leftrightarrow$

$[(p \leftrightarrow q) \leftrightarrow ((p \leftrightarrow r) \leftrightarrow (r \leftrightarrow q))]]$ by (i), hence

(3) $\quad \vdash_S (p \leftrightarrow q) \leftrightarrow ((p \leftrightarrow r) \leftrightarrow (r \leftrightarrow q))$ by (ii) and (i) and by (ii) and (2) with $\varphi = p \leftrightarrow r$.

$\vdash_S ((p \leftrightarrow q) \leftrightarrow (p \leftrightarrow q)) \leftrightarrow [((p \leftrightarrow q) \leftrightarrow ((p \leftrightarrow r) \leftrightarrow (r \leftrightarrow q))) \leftrightarrow$

$(((p \leftrightarrow r) \leftrightarrow (r \leftrightarrow q)) \leftrightarrow (p \leftrightarrow q))]$ by (i),

(4) $\vdash_S ((p \leftrightarrow r) \leftrightarrow (r \leftrightarrow q)) \leftrightarrow (p \leftrightarrow q)$ by (ii) and (1) and by (ii) and (3), hence

$\vdash_S ((p \leftrightarrow (p \leftrightarrow q)) \leftrightarrow ((p \leftrightarrow q) \leftrightarrow p)) \leftrightarrow (p \leftrightarrow p)$ by (4). Thus

(5) $\vdash_S p \leftrightarrow p$ by (ii) and (2).

(3.3.36) $\vdash_S (q \leftrightarrow q) \leftrightarrow ((p \leftrightarrow q) \leftrightarrow (q \leftrightarrow p))$ by (i), hence

$\vdash_S (p \leftrightarrow q) \leftrightarrow (q \leftrightarrow p)$ by (ii) and (5).

(3.3.37) Suppose that for formulas φ and ψ we have $\vdash_S \varphi \leftrightarrow \psi$. By (3.3.36), we also have $\vdash_S (\varphi \leftrightarrow \psi) \leftrightarrow (\psi \leftrightarrow \varphi)$, hence we can apply (ii) to get $\vdash_S \psi \leftrightarrow \varphi$. We refer to this argument as (6).

$\vdash_S ((q \leftrightarrow p) \leftrightarrow p) \leftrightarrow [(q \leftrightarrow p) \leftrightarrow ((q \leftrightarrow p) \leftrightarrow q)]$ by (i), so

(7) $\vdash_S [(q \leftrightarrow p) \leftrightarrow ((q \leftrightarrow p) \leftrightarrow q)] \leftrightarrow ((q \leftrightarrow p) \leftrightarrow p)$ by (6).

(8) $\vdash_S (p \leftrightarrow (q \leftrightarrow p)) \leftrightarrow ((q \leftrightarrow p) \leftrightarrow p)$ by (3.3.36),

$\vdash_S [(p \leftrightarrow (q \leftrightarrow p)) \leftrightarrow ((q \leftrightarrow p) \leftrightarrow p)] \leftrightarrow$

$[[((q \leftrightarrow p) \leftrightarrow ((q \leftrightarrow p) \leftrightarrow q)) \leftrightarrow ((q \leftrightarrow p) \leftrightarrow p)] \leftrightarrow [(p \leftrightarrow (q \leftrightarrow p)) \leftrightarrow ((q \leftrightarrow p) \leftrightarrow ((q \leftrightarrow p) \leftrightarrow q))]]$

by (i), hence, by (7), (8) and (ii) (twice),

$\vdash_S (p \leftrightarrow (q \leftrightarrow p)) \leftrightarrow ((q \leftrightarrow p) \leftrightarrow ((q \leftrightarrow p) \leftrightarrow q))$.

(9) $\vdash_S ((q \leftrightarrow p) \leftrightarrow ((q \leftrightarrow p) \leftrightarrow q)) \leftrightarrow (p \leftrightarrow (q \leftrightarrow p))$ by (6).

Now $\vdash_S [p \leftrightarrow ((q \leftrightarrow p) \leftrightarrow q)] \leftrightarrow [((q \leftrightarrow p) \leftrightarrow ((q \leftrightarrow p) \leftrightarrow q)) \leftrightarrow (p \leftrightarrow (q \leftrightarrow p))]$ by (i),

so $\vdash_S [((q \leftrightarrow p) \leftrightarrow ((q \leftrightarrow p) \leftrightarrow q)) \leftrightarrow (p \leftrightarrow (q \leftrightarrow p))] \leftrightarrow [p \leftrightarrow ((q \leftrightarrow p) \leftrightarrow q)]$ by (6), hence

(10) $\vdash_S p \leftrightarrow ((q \leftrightarrow p) \leftrightarrow q)$ by (9) and (ii).

(11) $\vdash_S (q \leftrightarrow (q \leftrightarrow p)) \leftrightarrow ((q \leftrightarrow p) \leftrightarrow q)$ by (3.3.36), and

$\vdash_S [(q \leftrightarrow (q \leftrightarrow p)) \leftrightarrow ((q \leftrightarrow p) \leftrightarrow q)] \leftrightarrow [(p \leftrightarrow ((q \leftrightarrow p) \leftrightarrow q)) \leftrightarrow ((q \leftrightarrow (q \leftrightarrow p)) \leftrightarrow p)]$ by (i)

so $\vdash_S (p \leftrightarrow ((q \leftrightarrow p) \leftrightarrow q)) \leftrightarrow ((q \leftrightarrow (q \leftrightarrow p)) \leftrightarrow p)$ by (11) and (ii)

so $\vdash_S (q \leftrightarrow (q \leftrightarrow p)) \leftrightarrow p$ by (10) and (ii)

so $\vdash_S p \leftrightarrow (q \leftrightarrow (q \leftrightarrow p))$ by (6). □

3.3.11 THEOREM [BP89a, Theorem 5.14]

CPC$_{\leftrightarrow}$ *is defined by the single axiom* *(i)* $(p \leftrightarrow q) \leftrightarrow ((r \leftrightarrow q) \leftrightarrow (p \leftrightarrow r)),$

and the single rule of inference

(ii) $\langle \{p, p \leftrightarrow q\}, q \rangle.$

Proof. Let S be the deductive system defined by (i) and (ii). By the lemma, each of (3.3.35), (3.3.36) and (3.3.37) are theorems of S. First, we show that S is algebraizable. Set $\Delta(p,q) = p \leftrightarrow q$. Condition (1) of Theorem 3.1.11 (ii) is (3.3.35), Condition (2) follows from (3.3.36) and the inference rule (ii), and condition (3) follows from axiom (i) and the inference rule (ii). That (4) holds can be seen from the following:

By (i), $\vdash_S (p_1 \leftrightarrow q_1) \leftrightarrow ((p_2 \leftrightarrow q_1) \leftrightarrow (p_1 \leftrightarrow p_2))$ and $\vdash_S (p_2 \leftrightarrow q_2) \leftrightarrow ((q_1 \leftrightarrow q_2) \leftrightarrow (p_2 \leftrightarrow q_1))$ hold, hence, by (ii),

$$p_1 \leftrightarrow q_1 \vdash_S (p_2 \leftrightarrow q_1) \leftrightarrow (p_1 \leftrightarrow p_2)$$

and
$$p_2 \leftrightarrow q_2 \vdash_S (q_1 \leftrightarrow q_2) \leftrightarrow (p_2 \leftrightarrow q_1).$$

Thus
$$p_1 \leftrightarrow q_1, p_2 \leftrightarrow q_2 \vdash_S (p_2 \leftrightarrow q_1) \leftrightarrow (p_1 \leftrightarrow p_2), (q_1 \leftrightarrow q_2) \leftrightarrow (p_2 \leftrightarrow q_1),$$

hence
$$p_1 \leftrightarrow q_1, p_2 \leftrightarrow q_2 \vdash_S (p_1 \leftrightarrow p_2) \leftrightarrow (q_1 \leftrightarrow q_2)$$

by Theorem 3.1.11 (ii) (2) and (3). From (3.3.37) and (ii) it follows that $p, q \vdash_S p \leftrightarrow q$. Thus, by Corollary 3.2.2, S is algebraizable with equivalence formula $\Delta(p,q) = p \leftrightarrow q$ and defining equation $p \approx p \leftrightarrow p$. Let $\mathfrak{K}$ denote the equivalent quasivariety semantics for S and replace $\leftrightarrow$ by $+$ in $\mathfrak{K}$. Since the G-rule holds, Theorem 3.2.3 shows that it is possible to define a constant term 0 in the language of $\mathfrak{K}$ by $0 = p \leftrightarrow p = p + p$ for any variable p. Applying Theorem 3.1.18, we get the following axiom system for $\mathfrak{K}$

(3.3.38) $$(p+q) + ((r+q) + (p+r)) \approx 0$$

(3.3.39) $$p + p \approx 0$$

(3.3.40) $$p \approx 0 \ \& \ p + q \approx 0 \Rightarrow q \approx 0$$

(3.3.41) $$p + q \approx 0 \ \& \ q + p \approx 0 \Rightarrow p \approx q.$$

Next we show that $\mathfrak{K}$ coincides with the class of semigroup reducts of Boolean groups. Clearly each of (3.3.38) to (3.3.41) hold in each Boolean group, i.e., $\mathfrak{BG} \subseteq \mathfrak{K}$. Let $\mathbf{A} = \langle A, + \rangle \in \mathfrak{K}$ and let $a, b, c \in A$. By (3.3.38), we have $(a+a) + ((b+a) + (a+b)) = 0$. Since $a + a = 0$, (3.3.40) implies that $(b+a) + (a+b) = 0$ and, by symmetry, $(a+b) + (b+a) = 0$. Thus $b + a = a + b$, by (3.3.41). So $\mathbf{A}$ is commutative. We shall make use of this fact without explicit mention. Now, (3.3.39) and (3.3.41) immediately imply that $a = b$ if and only if $a + b = 0$, hence, by (3.3.38),

(3.3.42) $$a + b = (c + a) + (b + c) = (a + c) + (b + c)$$

Thus $a + c = b + c$ implies $a = b$, i.e., the right (hence also left) cancellation law holds. As a substitution instance of (3.3.42), we have

(3.3.43) $$a + (b + c) = (a + b) + ((b + c) + b).$$

Using (3.3.42) again, we get $(b + c) + c = ((b + c) + b) + (c + b)$ so, cancelling $b + c$, we get $c = (b + c) + b$. Substituting c for $(b + c) + b$ in (3.3.43), gives the associativity of **A**. Finally, $a + 0 = a + (b + b) = (a + b) + b = a$, hence **A** is a Boolean group and $\mathfrak{K} = \mathfrak{BG}$.

To complete the proof, we note that S and $\mathbf{CPC}_{\leftrightarrow}$ are algebraizable with the same equivalent quasivariety semantics and the same defining equations. Thus, for $\Gamma \subseteq Fm$ and $\varphi \in Fm$, $\Gamma \vdash_S \varphi$ iff $\{\psi \approx 0\,;\, \psi \in \Gamma\} \models_{\mathfrak{BG}} \varphi \approx 0$ iff $\Gamma \vdash_{\mathbf{CPC}_{\leftrightarrow}} \varphi$, hence S and $\mathbf{CPC}_{\leftrightarrow}$ are equal. □

Since $\mathfrak{BG}$ is a variety, $\mathbf{CPC}_{\leftrightarrow}$ is strongly algebraizable.

Intuitionistic Equivalential Logic.

Recall that $\mathbf{IPC}_{\leftrightarrow}$ is the $\{\leftrightarrow\}$-fragment of $\mathbf{IPC}$ (defined in Section 1.4). It follows from Corollary 3.1.8 that $\mathbf{IPC}_{\leftrightarrow}$ is algebraizable and $\Delta(p, q) = p \leftrightarrow q$ forms a system of equivalence formulas for $\mathbf{IPC}_{\leftrightarrow}$ and its equivalent quasivariety semantics. Thus $\mathbf{IPC}_{\leftrightarrow}$ satisfies the conditions (1) to (4) of Theorem 3.1.11 (ii). Moreover, it satisfies conditions (6) and (7) of Corollary 3.2.2, and a system of defining equations for $\mathbf{IPC}_{\leftrightarrow}$ and its equivalent quasivariety semantics is $p \approx p \leftrightarrow p$. Since the G-rule is satisfied, Theorem 3.2.3 shows that if we define a constant term T in the language of $\mathbf{IPC}_{\leftrightarrow}$ by $\mathrm{T} = p \leftrightarrow p$ for any variable p then, over the equivalent quasivariety semantics of $\mathbf{IPC}_{\leftrightarrow}$, T is a constant term. By Corollary 3.1.8, the equivalent quasivariety semantics for $\mathbf{IPC}_{\leftrightarrow}$ is the class of all $\{\leftrightarrow\}$-subreducts of Heyting algebras. It is shown in [BP89a] that this class is precisely the class of *equivalential algebras*, by which we mean the class of all algebras over the language $\{\leftrightarrow\}$, where $ar(\leftrightarrow) = 2$, that satisfy the identities

$$(x \leftrightarrow x) \leftrightarrow y \approx y,$$

$$((x \leftrightarrow y) \leftrightarrow z) \leftrightarrow z \approx (x \leftrightarrow y) \leftrightarrow (y \leftrightarrow z),$$

$$((x \leftrightarrow y) \leftrightarrow ((x \leftrightarrow z) \leftrightarrow z)) \leftrightarrow ((x \leftrightarrow z) \leftrightarrow z) \approx x \leftrightarrow y.$$

Chapter 4
The Deduction Theorems

Certain classical deductive systems, for example **CPC**, **IPC** and $\mathbf{S}_4$, are known to satisfy the metalogical condition of having a deduction theorem. In the cases of **CPC** and **IPC**, this property takes the following form: For all $\Gamma \subseteq Fm$ and $\varphi, \psi \in Fm$,

$$\text{if} \quad \Gamma, \varphi \vdash \psi \quad \text{then} \quad \Gamma \vdash \varphi \to \psi.$$

In the case of $\mathbf{S}_4$ the property is slightly different, with the formula $\Box\varphi \to \psi$ playing the role of $\varphi \to \psi$ here. The converse implication of the deduction theorem is known as the detachment theorem and together the two properties form the deduction-detachment theorem (DDT, for short). In this chapter we shall formalise the notion of a deductive system 'having a deduction-detachment theorem' and generalize it to include k-deductive systems. In fact, initially we consider a much more general form of a deduction-detachment theorem, namely a '*local* deduction-detachment theorem' (LDDT, for short).

We shall only consider protoalgebraic (including algebraizable) k-deductive systems in this chapter. This proves not to be restrictive in the case of the (full) DDT, as we shall show that any k-deductive system with the property is protoalgebraic. The aim of this chapter is to investigate the connection between protoalgebraic k-deductive systems that have a (local) deduction-detachment theorem and their associated classes of matrix models. The 2-deductive systems associated with quasivarieties and, in particular, algebraizable k-deductive systems, are of special interest, for the associated matrix-theoretic results are universal algebraic in nature. In fact, using the 'logical' approach, as we do, certain (known) purely algebraic results can be obtained.

We begin with the more general version of the deduction theorems that we shall consider, namely the LDDT. In Section 4.1 we show that a protoalgebraic k-deductive system has the LDDT if and only if its class of matrix models has 'locally formula definable principal filters', if and only if it has the 'filter extension property'. Corollary 4.1.11 summarises the results of this section, which are then applied to the 2-deductive systems associated with quasivarieties. In Section 4.2, we examine the model classes of protoalgebraic k-deductive systems with the LDDT that are 'filter-distributive'. The main result of the next section, 4.3, states that the property of having the LDDT is preserved by the relation of equivalence of deductive systems, and is applied to the study of algebraizable deductive systems with the LDDT.

The study of protoalgebraic k-deductive systems with the DDT is approached in a similar manner to the LDDT case. In Section 4.4 we show that a protoalgebraic k-deductive system has the DDT if and only if its class of matrix models has 'formula definable principal filters', if and only if the lattice of compact filters of any matrix model is dually Brouwerian. Corollary 4.4.8 summarises the results of this section and, again, these are applied to the 2-deductive systems associated with quasivarieties. The matrix models of protoalgebraic k-deductive systems that have the DDT are shown to be 'filter-distributive' in Section 4.5. In this section we study such classes in more detail. We show in Section 4.6 that the property of having the DDT is also preserved by the relation of equivalence of deductive systems, and we apply this result to the study of algebraizable deductive systems with the DDT. Lastly, a number of examples are considered in Section 4.7.

The results of this chapter are drawn (or adapted) mainly from [BP88] and [BP89b]. Throughout this chapter we assume that k is a positive integer and that $\mathcal{L}$ is a fixed, but arbitrary, language. As before, the set of propositional variables is P.

4.1 LOCAL DEDUCTION-DETACHMENT THEOREMS

4.1.1 DEFINITION

Let S be a k-deductive system, let I be an index set and let $\boldsymbol{p} = \langle p_1, \dots, p_k \rangle$ and $\boldsymbol{q} = \langle q_1, \dots, q_k \rangle$ be k-variables with $p_1, \dots, p_k, q_1, \dots, q_k$ all distinct. For each $i \in I$, let n_i be a positive integer and let $E_i(\boldsymbol{p}, \boldsymbol{q}) = \{\boldsymbol{\eta}^j(\boldsymbol{p}, \boldsymbol{q})\,;\, j \leq n_i\}$ be a set of k-formulas in the $2k$ variables $p_1, \dots, p_k, q_1, \dots, q_k$. The system $\mathcal{E}(\boldsymbol{p}, \boldsymbol{q}) = \{E_i(\boldsymbol{p}, \boldsymbol{q})\,;\, i \in I\}$ is called a *local deduction-detachment system for S* if, for all $\Gamma \subseteq Fm^k$ and $\boldsymbol{\varphi}, \boldsymbol{\psi} \in Fm^k$,

$$\text{(4.1.1)} \qquad \Gamma, \boldsymbol{\varphi} \vdash_S \boldsymbol{\psi} \quad \text{if and only if} \quad \Gamma \vdash_S E_i(\boldsymbol{\varphi}, \boldsymbol{\psi}) \ \text{ for some } i \in I.$$

Here, $\Gamma \vdash_S E_i(\boldsymbol{\varphi}, \boldsymbol{\psi})$ is an abbreviation for $\Gamma \vdash_S \boldsymbol{\eta}^j(\boldsymbol{\varphi}, \boldsymbol{\psi})$ for all $\boldsymbol{\eta}^j \in E_i(\boldsymbol{p}, \boldsymbol{q})$. If there exists a local deduction-detachment system $\mathcal{E}(\boldsymbol{p}, \boldsymbol{q})$ for S, we say that S has the *local deduction-detachment theorem* (*LDDT*, for short) with respect to $\mathcal{E}$.

In the above definition, the implication from left to right in (4.1.1) is known as the *deduction* part, and the implication from right to left as the *detachment* part. Note that $E_i(\varphi,\psi) \vdash_S E_i(\varphi,\psi)$, trivially, hence (4.1.1) implies that $E_i(\varphi,\psi),\varphi \vdash_S \psi$. Also, from $\varphi \vdash_S \varphi$ we obtain $\vdash_S E_i(\varphi,\varphi)$. Examples of 1-deductive systems that have the LDDT are **BCK** (proved in Chapter 5) and the modal logic **K** (proved in Section 4.7). We provide here an example of a 2-deductive system that has the LDDT.

4.1.2 EXAMPLE [BP88, Example 2.3]

Recall from Section 0.2 that $\mathcal{AG}$ is the variety of abelian groups. As in Section 1.5, we define the 2-deductive system $S_{\mathcal{AG}}$ (denoted S for convenience), by the axioms

$$\vdash_S (p,p),$$

$$\vdash_S ((p+q)+r, p+(q+r)),$$

$$\vdash_S (p+0,p),$$

$$\vdash_S (p+(-p),0),$$

$$\vdash_S (p+q,q+p),$$

and the inference rules

$$(p,q) \vdash_S (q,p),$$

$$(p,q),(q,r) \vdash_S (p,r),$$

$$(p_1,q_1),(p_2,q_2) \vdash_S (p_1+p_2, q_1+q_2),$$

$$(p,q) \vdash_S (-p,-q).$$

For each $i \in \mathbb{Z}$ and $\boldsymbol{p} = (p_1,p_2), \boldsymbol{q} = (q_1,q_2) \in Fm^2$ define

$$\boldsymbol{\eta}^i(\boldsymbol{p},\boldsymbol{q}) = (\eta_1^i(\boldsymbol{p},\boldsymbol{q}), \eta_2^i(\boldsymbol{p},\boldsymbol{q})) = (\eta_1^i(p_1,p_2,q_1,q_2), \eta_2^i(p_1,p_2,q_1,q_2)),$$

where $$\eta_1^i(p_1,p_2,q_1,q_2) = i(p_1-p_2),$$

and $$\eta_2^i(p_1,p_2,q_1,q_2) = q_1-q_2.$$

(For each $\varphi \in Fm$, $i\varphi = \varphi+\varphi+\ldots+\varphi$ (i times) if $i \geq 1$, and $i\varphi = -i(-\varphi)$ if $i \leq -1$ and $0\varphi = 0$.) Set $E_i(\boldsymbol{p},\boldsymbol{q}) = \{\boldsymbol{\eta}^i(\boldsymbol{p},\boldsymbol{q})\}$ for each $i \in \mathbb{Z}$. We shall show that S possesses the LDDT with local deduction-detachment set $\mathcal{E}(\boldsymbol{p},\boldsymbol{q}) = \{E_i(\boldsymbol{p},\boldsymbol{q}) ; i \in \mathbb{Z}\}$. Recall from Section 2.1 that $S_{\mathcal{K}}$ is protoalgebraic for any quasivariety $\mathcal{K}$, hence $S = S_{\mathcal{AG}}$ is protoalgebraic. Let $\boldsymbol{\varphi} = (\varphi_1,\varphi_2)$, $\boldsymbol{\psi} = (\psi_1,\psi_2) \in Fm^2$ and $\Gamma \subseteq Fm^2$. Set $\mathcal{A} = \langle \mathbf{Fm}, \mathrm{Cn}_S(\Gamma)\rangle$. Then $\mathcal{A}$ is an S-matrix by Proposition

1.6.5 (i). Recall from Section 1.6 that the S-filters of $\mathcal{A}$ (i.e., S-theories) are exactly the $\mathcal{AG}$-congruences of **Fm**. In particular, this means that $\mathrm{Cn}_S(\Gamma) = \Theta^{\mathbf{Fm}}_{\mathcal{AG}}(\Gamma)$. Set $\Phi = \Theta^{\mathbf{Fm}}_{\mathcal{AG}}(\Gamma)$. Then, as noted in the paragraph preceeding section 1.9, $\Omega_{\mathcal{A}}\Phi = \Phi$, so $\mathcal{A}^* = \langle \mathbf{Fm}/\Phi, I_{Fm/\Phi}\rangle$. Let h be the canonical homomorphism from $\mathcal{A}$ to $\mathcal{A}^*$. Now,

$$\Gamma, \boldsymbol{\varphi} \vdash_S \boldsymbol{\psi} \quad \text{iff} \quad \boldsymbol{\psi} \in \mathrm{Fg}^S_{\mathcal{A}}\boldsymbol{\varphi} \qquad \text{[by Proposition 1.6.5 (iii)]}$$

$$\text{iff} \quad \boldsymbol{\psi} \in h^{-1}(\mathrm{Fg}^S_{\mathcal{A}^*}\boldsymbol{\varphi}/\Phi)$$

[by Corollary 2.1.4, as h is reductive and S is protoalgebraic]

$$\text{iff} \quad \boldsymbol{\psi}/\Phi = h\boldsymbol{\psi} \in \mathrm{Fg}^S_{\mathcal{A}^*}\boldsymbol{\varphi}/\Phi$$

$$\text{iff} \quad (\psi_1/\Phi, \psi_2/\Phi) \in \Theta^{\mathbf{Fm}/\Phi}_{\mathcal{AG}}(\varphi_1/\Phi, \varphi_2/\Phi)$$

$$\text{iff} \quad \psi_1/\Phi - \psi_2/\Phi = i(\varphi_1/\Phi - \varphi_2/\Phi) \quad \text{for some } i \in \mathbb{Z},$$

by a well-known result of abelian groups. Now

$$\psi_1/\Phi - \psi_2/\Phi = i(\varphi_1/\Phi - \varphi_2/\Phi) \quad \text{iff} \quad (i(\varphi_1 - \varphi_2), \psi_1 - \psi_2) \in \Phi$$

$$\text{iff} \quad \Gamma \vdash_S (i(\varphi_1 - \varphi_2), \psi_1 - \psi_2)$$

$$\text{iff} \quad \Gamma \vdash_S \boldsymbol{\eta}^i(\boldsymbol{\varphi}, \boldsymbol{\psi})$$

$$\text{iff} \quad \Gamma \vdash_S E_i(\boldsymbol{\varphi}, \boldsymbol{\psi}).$$

4.1.3 DEFINITION

Let S be a k-deductive system, M a class of S-matrices and $\boldsymbol{p} = \langle p_1, \ldots, p_k\rangle$, $\boldsymbol{q} = \langle q_1, \ldots, q_k\rangle$ k-variables with $p_1, \ldots, p_k, q_1, \ldots, q_k$ all distinct. Let I be an index set and for each $i \in I$, let $E_i(\boldsymbol{p}, \boldsymbol{q})$ be a finite set of k-formulas in the $2k$ variables $p_1, \ldots, p_k, q_1, \ldots, q_k$. We say that M has *locally formula definable principal filters* (*LFDPF*, for short) with *defining system* $\mathcal{E}(\boldsymbol{p}, \boldsymbol{q}) = \{E_i(\boldsymbol{p}, \boldsymbol{q})\,;\, i \in I\}$ if, for all $\mathcal{A} = \langle \mathbf{A}, F_{\mathcal{A}}\rangle \in M$ and $\boldsymbol{a}, \boldsymbol{b} \in A^k$,

$$\boldsymbol{b} \in \mathrm{Fg}^S_{\mathcal{A}}\boldsymbol{a} \quad \text{if and only if} \quad E_i^{\mathbf{A}}(\boldsymbol{a}, \boldsymbol{b}) \subseteq F_{\mathcal{A}} \text{ for some } i \in I. \tag{4.1.2}$$

Note that if $\mathcal{A} = \langle \mathbf{Fm}, T\rangle$ is a formula matrix model and $\boldsymbol{\varphi}, \boldsymbol{\psi} \in Fm^k$, then (4.1.2) states that

$$\boldsymbol{\psi} \in \mathrm{Cn}_S(T \cup \{\boldsymbol{\varphi}\}) \quad \text{if and only if} \quad E_i(\boldsymbol{\varphi}, \boldsymbol{\psi}) \subseteq T \text{ for some } i \in I,$$

i.e.,

$$T, \boldsymbol{\varphi} \vdash_S \boldsymbol{\psi} \quad \text{if and only if} \quad T \vdash_S E_i(\boldsymbol{\varphi}, \boldsymbol{\psi}).$$

4.1.4 THEOREM [BP88, Theorem 2.4]

Let S be a protoalgebraic k-deductive system and $\mathcal{E}(\boldsymbol{p},\boldsymbol{q}) = \{E_i(\boldsymbol{p},\boldsymbol{q})\,;\, i \in I\}$ a system of finite sets of k-formulas in the $2k$ variables $p_1,\ldots,p_k,q_1,\ldots,q_k$. The following are equivalent:

(i) S *has the LDDT with local deduction-detachment system* $\mathcal{E}(\boldsymbol{p},\boldsymbol{q})$,

(ii) Mod$S$ *has LFDPF with defining system* $\mathcal{E}(\boldsymbol{p},\boldsymbol{q})$,

(iii) Mod$^*S$ *has LFDPF with defining system* $\mathcal{E}(\boldsymbol{p},\boldsymbol{q})$.

Proof. (i)⇒(ii) Let $\mathcal{A} = \langle \mathbf{A}, F_{\mathcal{A}}\rangle \in \mathrm{Mod}S$ such that $\mathcal{A}$ is countably generated and let $\boldsymbol{a},\boldsymbol{b} \in A^k$. Then there exists a surjective homomorphism $h\colon \mathbf{Fm} \to \mathbf{A}$ and $\boldsymbol{\varphi},\boldsymbol{\psi} \in Fm^k$ such that $h\boldsymbol{\varphi} = \boldsymbol{a}$ and $h\boldsymbol{\psi} = \boldsymbol{b}$. Set $T = h^{-1}(F_{\mathcal{A}})$. By Corollary 1.8.3, $\mathfrak{B} = \langle \mathbf{Fm}, T\rangle$ is an S-matrix, i.e., T is an S-theory and $h\colon \mathfrak{B} \to \mathcal{A}$ is a reductive matrix homomorphism. Moreover, $\mathrm{Fg}^S_{\mathfrak{B}}\boldsymbol{\varphi} = \mathrm{Cn}_S(T \cup \{\boldsymbol{\varphi}\})$ by Proposition 1.6.5. Since S is protoalgebraic, Corollary 2.1.4 applies, so

$$\begin{aligned}
\boldsymbol{b} \in \mathrm{Fg}^S_{\mathcal{A}}\boldsymbol{a} \quad &\text{iff} \quad h\boldsymbol{\psi} \in \mathrm{Fg}^S_{\mathcal{A}} h\boldsymbol{\varphi} \\
&\text{iff} \quad \boldsymbol{\psi} \in h^{-1}(\mathrm{Fg}^S_{\mathcal{A}} h\boldsymbol{\varphi}) = \mathrm{Fg}^S_{\mathfrak{B}}\boldsymbol{\varphi} && \text{[by Corollary 2.1.4]} \\
&\text{iff} \quad T, \boldsymbol{\varphi} \vdash_S \boldsymbol{\psi} && \text{[by Proposition 1.6.5]} \\
&\text{iff} \quad T \vdash_S E_i(\boldsymbol{\varphi},\boldsymbol{\psi}) \text{ for some } i \in I.
\end{aligned}$$

Now,

$$\begin{aligned}
T \vdash_S E_i(\boldsymbol{\varphi},\boldsymbol{\psi}) \quad &\text{iff} \quad E_i(\boldsymbol{\varphi},\boldsymbol{\psi}) \subseteq T = h^{-1}(F_{\mathcal{A}}) \\
&\text{iff} \quad h(E_i(\boldsymbol{\varphi},\boldsymbol{\psi})) \subseteq F_{\mathcal{A}} \\
&\text{iff} \quad E_i^{\mathbf{A}}(\boldsymbol{a},\boldsymbol{b}) \subseteq F_{\mathcal{A}},
\end{aligned}$$

hence (4.1.2) holds. Next, suppose $\mathcal{A} = \langle \mathbf{A}, F_{\mathcal{A}}\rangle$ is an arbitrary S-matrix and let $\boldsymbol{b} \in \mathrm{Fg}^S_{\mathcal{A}}\boldsymbol{a}$. By Corollary 1.6.8, there is a finitely generated subalgebra $\mathbf{A}'$ of $\mathbf{A}$ with $\boldsymbol{a},\boldsymbol{b} \in (A')^k$ and $\boldsymbol{b} \in \mathrm{Fg}^S_{\mathcal{A}'}\boldsymbol{a}$, where $\mathcal{A}' = \langle \mathbf{A}', (A')^k \cap F_{\mathcal{A}}\rangle$. Since $\mathbf{A}'$ is countably generated, we have that

$$E_i^{\mathbf{A}'}(\boldsymbol{a},\boldsymbol{b}) \subseteq (A')^k \cap F_{\mathcal{A}} \quad \text{for some } i \in I,$$

i.e.,

$$E_i^{\mathbf{A}}(\boldsymbol{a},\boldsymbol{b}) \subseteq F_{\mathcal{A}} \quad \text{for some } i \in I.$$

Conversely, suppose $E_i^{\mathbf{A}}(\boldsymbol{a},\boldsymbol{b}) \subseteq F_{\mathcal{A}}$ for some $i \in I$. We can consider $\boldsymbol{a},\boldsymbol{b}$ as interpretations of $\boldsymbol{p},\boldsymbol{q}$ (respectively) in A. Since $\boldsymbol{p}, E_i(\boldsymbol{p},\boldsymbol{q}) \vdash_S \boldsymbol{q}$, $\boldsymbol{a} \in \mathrm{Fg}^S_{\mathcal{A}}\boldsymbol{a}$ and $E_i^{\mathbf{A}}(\boldsymbol{a},\boldsymbol{b}) \subseteq F_{\mathcal{A}} \subseteq \mathrm{Fg}^S_{\mathcal{A}}\boldsymbol{a}$, it follows that $\boldsymbol{b} \in \mathrm{Fg}^S_{\mathcal{A}}\boldsymbol{a}$ since $\mathrm{Fg}^S_{\mathcal{A}}\boldsymbol{a}$ is an S-filter of $\mathcal{A}$.

(ii)⇒(iii) Trivial.

(iii)⇒(i) Let $\Gamma \subseteq Fm^k$, $\varphi, \psi \in Fm^k$ and set $T = \mathrm{Cn}_S(\Gamma)$. Then T is an S-theory and $\mathcal{A} = \langle \mathbf{Fm}, T\rangle \in \mathrm{Mod}S$, hence $\mathcal{A}^* = \mathcal{A}/\Omega_{\mathcal{A}}T = \langle \mathbf{Fm}/\Omega_{\mathcal{A}}T, T/\Omega_{\mathcal{A}}T\rangle \in \mathrm{Mod}^*S$. Set $\mathbf{A}^* = \mathbf{Fm}/\Omega_{\mathcal{A}}T$ and $F_{\mathcal{A}^*} = T/\Omega_{\mathcal{A}}T$. Let $h: \mathcal{A} \to \mathcal{A}^*$ be the canonical homomorphism. Then h is reductive, so Corollary 2.1.4 gives us that $h^{-1}(\mathrm{Fg}^S_{\mathcal{A}^*}h\varphi) = \mathrm{Fg}^S_{\mathcal{A}}\varphi$, so

$$\begin{aligned}
\Gamma, \varphi \vdash_S \psi \quad &\text{iff} \quad \psi \in \mathrm{Fg}^S_{\mathcal{A}}\varphi && \text{[by Proposition 1.6.5]}\\
&\text{iff} \quad \psi \in h^{-1}(\mathrm{Fg}^S_{\mathcal{A}^*}h\varphi)\\
&\text{iff} \quad h\psi \in \mathrm{Fg}^S_{\mathcal{A}^*}h\varphi\\
&\text{iff} \quad E_i^{\mathbf{A}^*}(h\varphi, h\psi) \subseteq F_{\mathcal{A}^*} \quad \text{for some } i \in I,
\end{aligned}$$

since Mod^*S has LFDPF and $\mathcal{A}^* \in \mathrm{Mod}^*S$. Now,

$$\begin{aligned}
E_i^{\mathbf{A}^*}(h\varphi, h\psi) \subseteq F_{\mathcal{A}^*} \quad &\text{iff} \quad h(E_i(\varphi, \psi)) \subseteq F_{\mathcal{A}^*}\\
&\text{iff} \quad E_i(\varphi, \psi) \subseteq h^{-1}(F_{\mathcal{A}^*}) = T\\
&\text{iff} \quad \Gamma \vdash_S E_i(\varphi, \psi).
\end{aligned}$$

□

The equivalence of conditions (iii) and (i) of the previous theorem can be stated in model-theoretic terms, using the terminology of Section 2.2: A k-protoquasivariety, Mod^*S say, has LFDPF with defining system $\mathfrak{E}(\boldsymbol{p}, \boldsymbol{q})$ if and only if its associated protoalgebraic k-deductive system S has the LDDT with local deduction-detachment system $\mathfrak{E}(\boldsymbol{p}, \boldsymbol{q})$. We now wish to find a characterization of k-protoquasivarieties (and the classes $\mathrm{Mod}S$) that have LFDPF.

4.1.5 DEFINITION

Let S be a k-deductive system. An S-matrix $\mathcal{A} = \langle \mathbf{A}, F_{\mathcal{A}}\rangle$ has the *filter-extension property* (*FEP*, for short) if for every submatrix $\mathfrak{B} = \langle \mathbf{B}, F_{\mathfrak{B}}\rangle$ of $\mathcal{A}$ and every S-filter F of $\mathfrak{B}$ there exists an S-filter F' of $\mathcal{A}$ such that $F' \cap (B^k) = F$. If the previous statement holds for all principal S-filters F of $\mathfrak{B}$ then we say that $\mathcal{A}$ has the *principal filter-extension property* (*PFEP*, for short). A class M of S-matrices has the *(principal) filter-extension property* if each of its members does.

4.1.6 LEMMA

Let the S-matrix $\mathcal{A} = \langle \mathbf{A}, F_{\mathcal{A}}\rangle$ have the FEP, let $\mathfrak{B} = \langle \mathbf{B}, F_{\mathfrak{B}}\rangle$ be a submatrix of $\mathcal{A}$ and let $X \subseteq B^k$. Then $\mathrm{Fg}^S_{\mathfrak{B}}X = B^k \cap \mathrm{Fg}^S_{\mathcal{A}}X$.

Proof. It is easy to check that $B^k \cap \mathrm{Fg}^S_{\mathcal{A}}X$ is an S-filter of $\mathfrak{B}$ containing X, so

$\mathrm{Fg}^S_{\mathfrak{B}}X \subseteq B^k \cap \mathrm{Fg}^S_{\mathcal{A}}X$ (regardless of FEP). By FEP, $\mathrm{Fg}^S_{\mathfrak{B}}X = B^k \cap G$ for some $G \in \mathrm{Fi}^S\mathcal{A}$. Now, $X \subseteq G$, so $\mathrm{Fg}^S_{\mathcal{A}}X \subseteq G$, so

$$B^k \cap \mathrm{Fg}^S_{\mathcal{A}}X \subseteq B^k \cap G = \mathrm{Fg}^S_{\mathfrak{B}}X,$$

hence

$$B^k \cap \mathrm{Fg}^S_{\mathcal{A}}X = \mathrm{Fg}^S_{\mathfrak{B}}X.$$

□

4.1.7 THEOREM [BP88, Theorem 3.1]

Let S be a protoalgebraic k-deductive system. The following are equivalent:

(i) S *has the LDDT,*

(ii) Mod$S$ *has the PFEP.*

Proof. (i)⇒(ii) Assume that S has the LDDT. By Theorem 4.1.4, Mod$S$ has LFDPF with defining system $\mathfrak{E}(\boldsymbol{p},\boldsymbol{q}) = \{E_i(\boldsymbol{p},\boldsymbol{q})\,;\, i \in I\}$, say. Let $\mathcal{A} = \langle \mathbf{A}, F_{\mathcal{A}}\rangle$ be an S-matrix, $\mathfrak{B} = \langle \mathbf{B}, F_{\mathfrak{B}}\rangle$ a submatrix of $\mathcal{A}$ and $\boldsymbol{a} \in B^k$. Then $\mathrm{Fg}^S_{\mathfrak{B}}\boldsymbol{a}$ is a principal S-filter of $\mathfrak{B}$; we shall show that $\mathrm{Fg}^S_{\mathfrak{B}}\boldsymbol{a} = \mathrm{Fg}^S_{\mathcal{A}}\boldsymbol{a} \cap B^k$, which will prove (ii). If $\boldsymbol{b} \in B^k$ then, since Mod$S$ has LFDPF, we have that (4.1.2) holds, i.e.,

$$\boldsymbol{b} \in \mathrm{Fg}^S_{\mathcal{A}}\boldsymbol{a} \quad \text{if and only if} \quad E_i^{\mathbf{A}}(\boldsymbol{a},\boldsymbol{b}) \subseteq F_{\mathcal{A}} \text{ for some } i \in I.$$

Both $\boldsymbol{a}$ and $\boldsymbol{b}$ are elements of B^k, hence $E_i^{\mathbf{A}}(\boldsymbol{a},\boldsymbol{b}) = E_i^{\mathbf{B}}(\boldsymbol{a},\boldsymbol{b})$ and $E_i^{\mathbf{A}}(\boldsymbol{a},\boldsymbol{b}) \subseteq B^k$ for each $i \in I$, implying that

$$E_i^{\mathbf{B}}(\boldsymbol{a},\boldsymbol{b}) \subseteq F_{\mathfrak{B}} = F_{\mathcal{A}} \cap B^k \quad \text{if and only if} \quad E_i^{\mathbf{A}}(\boldsymbol{a},\boldsymbol{b}) \subseteq F_{\mathcal{A}}$$

for each $i \in I$. Thus, by (4.1.2) and the fact that $\mathfrak{B} \in$ Mod$S$ (see Proposition 1.8.2 (i)), we have

$$\boldsymbol{b} \in \mathrm{Fg}^S_{\mathfrak{B}}\boldsymbol{a} \quad \text{if and only if} \quad \boldsymbol{b} \in \mathrm{Fg}^S_{\mathcal{A}}\boldsymbol{a}.$$

(ii)⇒(i) Let $\boldsymbol{p} = \langle p_1,\ldots,p_k\rangle$ and $\boldsymbol{q} = \langle q_1,\ldots,q_k\rangle$ be k-variables such that $p_1,\ldots,p_k,q_1,\ldots,q_k$ are all distinct, and recall that $Fm(\boldsymbol{p},\boldsymbol{q})$ denotes the set of all formulas in the $2k$ variables $p_1,\ldots,p_k,q_1,\ldots,q_k$. Recall also that $Fm(\boldsymbol{p},\boldsymbol{q})$ is the universe of the absolutely free $\mathfrak{L}$-algebra generated by $p_1,\ldots,p_k,q_1,\ldots,q_k$, which we denote by $\mathbf{Fm}(\boldsymbol{p},\boldsymbol{q})$. Next, define

$$\mathfrak{E} = \mathfrak{E}(\boldsymbol{p},\boldsymbol{q}) = \{E(\boldsymbol{p},\boldsymbol{q}) \subseteq (Fm(\boldsymbol{p},\boldsymbol{q}))^k\,;\, E(\boldsymbol{p},\boldsymbol{q}) \text{ is finite and } E(\boldsymbol{p},\boldsymbol{q}),\boldsymbol{p} \vdash_S \boldsymbol{q}\}.$$

We claim that S has the LDDT with local deduction detachment system $\mathfrak{E}$. Let $\boldsymbol{\varphi},\boldsymbol{\psi} \in Fm^k$. Consider the substitution σ defined by $\sigma\boldsymbol{p} = \boldsymbol{\varphi}$ and $\sigma\boldsymbol{q} = \boldsymbol{\psi}$ (i.e., $\sigma p_i = \varphi_i$ and $\sigma q_i = \psi_i$ for $i \leq k$) and $\sigma r = r$ for all $r \in P - \{p_1,\ldots,p_k,q_1,\ldots,q_k\}$. Note that σ is well-defined by the choice of $\boldsymbol{p}$ and $\boldsymbol{q}$. Let $E \in \mathfrak{E}$. Since $E(\boldsymbol{p},\boldsymbol{q}),\boldsymbol{p} \vdash_S \boldsymbol{q}$, structurality implies that $\sigma(E(\boldsymbol{p},\boldsymbol{q})),\sigma\boldsymbol{p} \vdash_S \sigma\boldsymbol{q}$, i.e.,

$E(\varphi,\psi), \varphi \vdash_S \psi$. For any $\Gamma \subseteq Fm^k$, therefore, if $\Gamma \vdash_S E(\varphi,\psi)$ then $\Gamma, \varphi \vdash_S \psi$.

Conversely, suppose $\Gamma, \varphi \vdash_S \psi$. Set

$$\mathcal{A} = \langle \mathbf{Fm}(p,q), F_{\mathcal{A}} \rangle, \text{ where } F_{\mathcal{A}} = \sigma^{-1}(\mathrm{Cn}_S(\Gamma)) \cap (Fm(p,q))^k,$$

and set
$$\mathfrak{C} = \langle \mathbf{Fm}, F_{\mathfrak{C}} \rangle, \text{ where } F_{\mathfrak{C}} = \mathrm{Cn}_S(\Gamma).$$

It follows easily from Lemma 1.5.3 (i) and Proposition 1.6.5 (i) that $\mathcal{A}$ and $\mathfrak{C}$ are S-matrices. Let $h : \mathbf{Fm}(p,q) \to \mathbf{Fm}$ be the homomorphism determined by $hp_i = \varphi_i$ and $hq_i = \psi_i$ for $i \leq k$. Then $h : \mathcal{A} \to \mathfrak{C}$ is a matrix homomorphism. Indeed, for any $\eta \in (Fm(p,q))^k$, we have $h\eta = \sigma\eta$, hence $h(F_{\mathcal{A}}) = h(\sigma^{-1}(\mathrm{Cn}_S(\Gamma)) \cap (Fm(p,q))^k) = \sigma(\sigma^{-1}(\mathrm{Cn}_S(\Gamma)) \cap (Fm(p,q))^k) \subseteq \mathrm{Cn}_S(\Gamma)$, but h is not surjective. To remedy this, we set

$$\mathfrak{B} = \langle \mathbf{B}, F_{\mathfrak{B}} \rangle, \text{ where } \mathbf{B} = h(\mathbf{Fm}(p,q)) \text{ and } F_{\mathfrak{B}} = F_{\mathfrak{C}} \cap B^k.$$

Note that both $\varphi, \psi \in B^k$ since $hp = \varphi$ and $hq = \psi$. Moreover, $\mathfrak{B}$ is a submatrix of $\mathfrak{C}$ and $\mathrm{Fg}^S_{\mathfrak{B}}\varphi$ is a principal S-filter of $\mathfrak{B}$, hence the PFEP implies that there exists $F \in \mathrm{Fi}^S\mathfrak{C}$ such that $F \cap B^k = \mathrm{Fg}^S_{\mathfrak{B}}\varphi$. Now $\mathrm{Fg}^S_{\mathfrak{B}}\varphi \subseteq \mathrm{Fg}^S_{\mathfrak{C}}\varphi \cap B^k$ and, conversely, $\varphi \in \mathrm{Fg}^S_{\mathfrak{B}}\varphi$ implies $\varphi \in F$, hence $\mathrm{Fg}^S_{\mathfrak{C}}\varphi \subseteq F$. Thus $\mathrm{Fg}^S_{\mathfrak{C}}\varphi \cap B^k \subseteq F \cap B^k = \mathrm{Fg}^S_{\mathfrak{B}}\varphi$, hence

$$\mathrm{Fg}^S_{\mathfrak{B}}\varphi = \mathrm{Fg}^S_{\mathfrak{C}}\varphi \cap B^k.$$

Since $\Gamma, \varphi \vdash_S \psi$, Proposition 1.6.5 (iii) implies that $\psi \in \mathrm{Fg}^S_{\mathfrak{C}}\varphi$, hence $\psi \in \mathrm{Fg}^S_{\mathfrak{C}}\varphi \cap B^k = \mathrm{Fg}^S_{\mathfrak{B}}\varphi$. Note that $h : \mathcal{A} \to \mathfrak{B}$ is a reductive matrix homomorphism, so we can apply Corollary 2.1.4 (recall that S is protoalgebraic), and get

$$h^{-1}(\mathrm{Fg}^S_{\mathfrak{B}}\varphi) = \mathrm{Fg}^S_{\mathcal{A}}p.$$

Since $hq = \psi \in \mathrm{Fg}^S_{\mathfrak{B}}\varphi$, we get $q \in h^{-1}(\mathrm{Fg}^S_{\mathfrak{B}}\varphi)$, hence $q \in \mathrm{Fg}^S_{\mathcal{A}}p$, i.e., $F_{\mathcal{A}}, p \vdash_S q$, by Proposition 1.6.5 (iii). Recall that $\vdash_S$ is finitary and $F_{\mathcal{A}} \subseteq (Fm(p,q))^k$, so there exists a finite set $E(p,q) \subseteq F_{\mathcal{A}}$ such that $E(p,q), p \vdash_S q$. Thus $E(p,q) \in \mathfrak{E}$ and, moreover, $E(\varphi,\psi) = h(E(p,q)) \subseteq h(F_{\mathcal{A}}) \subseteq F_{\mathfrak{B}} \subseteq F_{\mathfrak{C}} = \mathrm{Cn}_S(\Gamma)$, i.e., $\Gamma \vdash_S E(\varphi,\psi)$. □

Having achieved a connection between the LDDT and the PFEP we now proceed to the FEP in the classes Mod$S$ and Mod$^*S$. The following theorem shows that for the class Mod$S$, having the FEP is a consequence of having the PFEP. Note that this theorem is not restricted to protoalgebraic k-deductive systems.

4.1.8 THEOREM [BP88, Theorem 3.2]

Let S be a k-deductive system. Then Mod$S$ *has the PFEP if and only if it has the FEP.*

Proof. Suppose S has the PFEP. Let $\mathcal{A} = \langle \mathbf{A}, F_{\mathcal{A}} \rangle$ and $\mathfrak{B} = \langle \mathbf{B}, F_{\mathfrak{B}} \rangle$ be S-matrices such that $\mathfrak{B}$ is a submatrix of $\mathcal{A}$. We first consider S-filters of $\mathfrak{B}$ that are finitely generated. We shall show that for any finite $X \subseteq B^k$,

$$\mathrm{Fg}^S_{\mathfrak{B}} X = \mathrm{Fg}^S_{\mathcal{A}} X \cap B^k. \tag{4.1.3}$$

This is achieved by induction on the cardinality of X. If $|X| = 0$, then $X = \emptyset$ and

$$\mathrm{Fg}^S_{\mathfrak{B}} \emptyset = F_{\mathfrak{B}} = F_{\mathcal{A}} \cap B^k = \mathrm{Fg}^S_{\mathcal{A}} \emptyset \cap B^k.$$

Assume that (4.1.3) holds whenever $|X| < n$, for some positive integer n, and suppose that $X = \{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n\} \subseteq B^k$. Let $F = \mathrm{Fg}^S_{\mathfrak{B}}\{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_{n-1}\}$. Then, by the induction hypothesis, there exists an $F' \in \mathrm{Fi}^S \mathcal{A}$ such that $F = F' \cap B^k$. Set $\mathfrak{B}' = \langle \mathbf{B}, F \rangle$ and $\mathcal{A}' = \langle \mathbf{A}, F' \rangle$. Then $\mathfrak{B}'$ is a submatrix of $\mathcal{A}'$ hence, by the PFEP, there exists a $G \in \mathrm{Fi}^S \mathcal{A}'$ such that $\mathrm{Fg}^S_{\mathfrak{B}'} \boldsymbol{b}_n = G \cap B^k$. But since $\mathrm{Fi}^S \mathcal{A}' \subseteq \mathrm{Fi}^S \mathcal{A}$, we get that $G \in \mathrm{Fi}^S \mathcal{A}$ and

$$\mathrm{Fg}^S_{\mathfrak{B}'} \boldsymbol{b}_n = \mathrm{Fg}^S_{\mathfrak{B}}(F \cup \{\boldsymbol{b}_n\}) = \mathrm{Fg}^S_{\mathfrak{B}} X,$$

hence $G \cap B^k = \mathrm{Fg}^S_{\mathfrak{B}} X$ and the induction is complete.

Now suppose that $F \in \mathrm{Fi}^S \mathfrak{B}$ is arbitrary. Recall from Lemma 1.6.4 that filter lattices are algebraic and that F is the join of compact elements of $\mathbf{Fi}^S \mathfrak{B}$, so

$$\begin{aligned} F &= \bigcup \{\mathrm{Fg}^S_{\mathfrak{B}} X \,;\, X \subseteq F \text{ and } X \text{ is finite}\} \\ &= \bigcup \{\mathrm{Fg}^S_{\mathcal{A}} X \cap B^k \,;\, X \subseteq F \text{ and } X \text{ is finite}\} \\ &= \left(\bigcup \{\mathrm{Fg}^S_{\mathcal{A}} X \,;\, X \subseteq F \text{ and } X \text{ is finite}\}\right) \cap B^k \\ &= \mathrm{Fg}^S_{\mathcal{A}} F \cap B^k. \end{aligned}$$

□

In Theorem 4.1.9 we shall show that when establishing the PFEP for a class of matrix models, it is sufficient to consider the reduced matrix models. Recall from Section 1.8 that for an S-matrix $\mathcal{A} = \langle \mathbf{A}, F \rangle$, we have $\mathcal{A}^* = \langle \mathbf{A}/\Omega_{\mathcal{A}} F, F/\Omega_{\mathcal{A}} F \rangle \in \mathrm{Mod}^* S$.

4.1.9 THEOREM [BP88, Theorem 3.4]

Let S be a protoalgebraic k-deductive system and $\mathcal{A}$ an S-matrix. If $\mathcal{A}^*$ has the PFEP, then so does $\mathcal{A}$.

Proof. Let $\mathcal{A} = \langle \mathbf{A}, F_{\mathcal{A}} \rangle$ be an S-matrix and $\mathfrak{B} = \langle \mathbf{B}, F_{\mathfrak{B}} \rangle$ a submatrix of $\mathcal{A}$. Set

$\Phi = \Omega_{\mathcal{A}} F_{\mathcal{A}}$ and let $h : \mathbf{A} \to \mathbf{A}/\Phi$ be the canonical homomorphism. Then h is reductive from $\mathcal{A}$ to $\mathcal{A}^*$. Let $\mathfrak{B}/\Phi = \langle \mathbf{B}/\Phi, F_{\mathcal{A}}/\Phi \cap (B/\Phi)^k \rangle$ (as defined in the proof of Lemma 1.8.7). Clearly, $\mathfrak{B}/\Phi$ is a submatrix of $\mathcal{A}^*$. Let $\boldsymbol{b} \in B^k$ so that $\mathrm{Fg}^S_{\mathfrak{B}} \boldsymbol{b}$ is a principal S-filter of $\mathfrak{B}$. We need to show that $\mathrm{Fg}^S_{\mathfrak{B}} \boldsymbol{b}$ can be extended over $\mathcal{A}$. If we set $g = h \mid \mathfrak{B}$, the restriction of h to $\mathfrak{B}$, then $g : \mathfrak{B} \to \mathfrak{B}/\Phi$ is a surjective homomorphism of matrices and $g^{-1}(F_{\mathcal{A}}/\Phi \cap (B/\Phi)^k) = F_{\mathcal{A}} \cap B^k$, i.e., g is reductive. Since S is protoalgebraic, it follows from Theorem 2.1.3 (vi) that

$$g^{-1}(\mathrm{Fg}^S_{\mathfrak{B}/\Phi}\, g\boldsymbol{b}) = \mathrm{Fg}^S_{\mathfrak{B}} \boldsymbol{b}.$$

Since $\mathcal{A}^*$ has the principal filter extension property there exists an S-filter of $\mathcal{A}^*$, G say, such that $G \cap (B/\Phi)^k = \mathrm{Fg}^S_{\mathfrak{B}/\Phi}\, g\boldsymbol{b}$. Then $h^{-1}(G)$ is an S-filter of $\mathcal{A}$ by Corollary 1.8.3 and

$$\begin{aligned}
\mathrm{Fg}^S_{\mathfrak{B}} \boldsymbol{b} &= g^{-1}(\mathrm{Fg}^S_{\mathfrak{B}/\Phi}\, g\boldsymbol{b}) \\
&= g^{-1}(G \cap (B/\Phi)^k) \\
&= h^{-1}(G \cap (B/\Phi)^k) \cap B^k \\
&= h^{-1}(G) \cap h^{-1}((B/\Phi)^k) \cap B^k \\
&= h^{-1}(G) \cap B^k,
\end{aligned}$$

hence $\mathcal{A}$ has the PFEP. □

The results concerning the matrix model classes Mod$S$ and Mod$^*S$ are summarised in the following corollary. Note that the equivalence of (i) and (iv) can be rephrased in model-theoretic terms: A k-protoquasivariety has the PFEP if and only if it has the FEP.

4.1.10 COROLLARY [BP88, Corollary 3.5]

Let S be a protoalgebraic k-deductive system. The following are equivalent:

(i) Mod$^*S$ *has the PFEP,*

(ii) Mod$S$ *has the PFEP,*

(iii) Mod$S$ *has the FEP,*

(iv) Mod$^*S$ *has the FEP.*

Proof. (i)⇒(ii) is Theorem 4.1.9, (ii)⇒(iii) is Theorem 4.1.8, (iii)⇒(iv) is obvious and so is (iv)⇒(i). □

The theorems that have been proved thusfar can be summed up in a single corollary.

4.1.11 COROLLARY [BP88, Corollary 3.6]

Let S be a protoalgebraic k-deductive system. The following are equivalent:

(i) S *has the LDDT,*

(ii) Mod$S$ *has LFDPF,*

(iii) Mod$S$ *has PFEP,*

(iv) Mod$S$ *has FEP,*

(v) Mod$^*S$ *has LFDPF,*

(vi) Mod$^*S$ *has FEP,*

(vii) Mod$^*S$ *has PFEP.* □

Quasivarieties.

As promised, we now apply the results of this section to the 2-deductive system $S_{\mathcal{K}}$, where $\mathcal{K}$ is a quasivariety.

4.1.12 DEFINITION

A quasivariety $\mathcal{K}$ has *locally equationally definable principal relative congruences* (*LEDPRC*, for short), if there exists a system

$$\mathcal{E}(\boldsymbol{p},\boldsymbol{q}) = \mathcal{E}(p_1,p_2,q_1,q_2) = \{E_i(p_1,p_2,q_1,q_2)\,;\, i \in I\}$$

of finite sets of 4-ary equations, say,

$$E_i(p_1,p_2,q_1,q_2) = \{\zeta^i_j(p_1,p_2,q_1,q_2) \approx \eta^i_j(p_1,p_2,q_1,q_2)\,;\, j \le n_i\},$$

such that for all $\mathbf{A} \in \mathcal{K}$, and all $a,b,c,d \in A$,

$$(c,d) \in \Theta^{\mathbf{A}}_{\mathcal{K}}(a,b) \quad \text{iff} \quad \text{for some } i \in I,\ (\zeta^i_j)^{\mathbf{A}}(a,b,c,d) = (\eta^i_j)^{\mathbf{A}}(a,b,c,d) \ \text{ for all } j \le n_i.$$

Then $\mathcal{E}(\boldsymbol{p},\boldsymbol{q})$ is called a *set of local defining equations for the principal relative congruences* of $\mathcal{K}$ (where $\boldsymbol{p} = \langle p_1,p_2\rangle$, $\boldsymbol{q} = \langle q_1,q_2\rangle$).

Consider the 2-deductive system $S_{\mathcal{K}}$ associated with a quasivariety $\mathcal{K}$, defined in Section 1.5. Recall from Section 1.8 that Mod$^*S_{\mathcal{K}} = \{\langle \mathbf{A}, I_A\rangle\,;\, \mathbf{A} \in \mathcal{K}\}$ and that the lattice $\mathbf{Fi}^{S_{\mathcal{K}}}\mathcal{A}$ of $S_{\mathcal{K}}$-filters of a reduced $S_{\mathcal{K}}$-matrix $\mathcal{A}$ coincides with the lattice $\mathbf{Con}_{\mathcal{K}}\mathbf{A}$ of $\mathcal{K}$-congruences of $\mathbf{A}$. Let I be an index set and let $E_i(\boldsymbol{p},\boldsymbol{q}) = \{(\zeta^i_j(\boldsymbol{p},\boldsymbol{q}), \eta^i_j(\boldsymbol{p},\boldsymbol{q}))\,;\, j \le n_i\}$ be a set of 2-formulas in two 2-variables $\boldsymbol{p} = \langle p_1,p_2\rangle, \boldsymbol{q} = \langle q_1,q_2\rangle$ for each $i \in I$. Set $\mathcal{E}(\boldsymbol{p},\boldsymbol{q}) = \{E_i(\boldsymbol{p},\boldsymbol{q})\,;\, i \in I\}$. Then

$\text{Mod}^*S_{\mathcal{K}}$ has LFDPF with defining system $\mathcal{E}(p,q)$

iff for all $\mathcal{A} = \langle \mathbf{A}, I_A \rangle \in \text{Mod}^*S_{\mathcal{K}}$ and $(a_1,a_2),(b_1,b_2) \in A^2$,

$$(b_1,b_2) \in \text{Fg}_{\mathcal{A}}^{S_{\mathcal{K}}}(a_1,a_2) \quad \text{iff} \quad E_i^{\mathbf{A}}(a_1,a_2,b_1,b_2) \subseteq I_A \ \text{ for some } i \in I,$$

iff for all $\mathbf{A} \in \mathcal{K}$ and $a_1,a_2,b_1,b_2 \in A$ there exists $i \in I$ such that

$$(b_1,b_2) \in \Theta_{\mathcal{K}}^{\mathbf{A}}(a_1,a_2) \quad \text{iff} \quad (\zeta_j^i)^{\mathbf{A}}(a_1,a_2,b_1,b_2) = (\eta_j^i)^{\mathbf{A}}(a_1,a_2,b_1,b_2) \ \text{ for all } j \leq n_i,$$

iff $\mathcal{K}$ has LEDPRC with respect to $\mathcal{E}(p,q)$.

If $\mathcal{K}$ is a variety then every congruence on any $\mathbf{A} \in \mathcal{K}$ is a $\mathcal{K}$-congruence. In this case, if $\mathcal{K}$ has LEDPRC, we drop the adjective "relative" and say that $\mathcal{K}$ has *locally equationally definable principal congruences* (*LEDPC*, for short).

4.1.13 DEFINITION

A quasivariety $\mathcal{K}$ has the *(principal) relative congruence extension property* (*(P)RCEP*, for short), if for all $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ such that $\mathbf{B}$ is a subalgebra of $\mathbf{A}$, and all (principal) $\mathcal{K}$-congruences Φ of $\mathbf{B}$, there exists a $\mathcal{K}$-congruence Φ' of $\mathbf{A}$ such that $\Phi' \cap B^2 = \Phi$. We drop the adjective "relative" and use the abbreviation (P)CEP in the case where $\mathcal{K}$ is a variety.

Consider once again the 2-deductive system $S_{\mathcal{K}}$ associated with a quasivariety $\mathcal{K}$.

$\text{Mod}^*S_{\mathcal{K}}$ has the PFEP

iff for all $\mathcal{A}, \mathcal{B} \in \text{Mod}^*S_{\mathcal{K}}$ such that $\mathcal{B}$ is a submatrix of $\mathcal{A}$, and all principal $S_{\mathcal{K}}$-filters $\text{Fg}_{\mathcal{B}}^{S_{\mathcal{K}}}(b_1,b_2)$ of $\mathcal{B}$, there exists $F \in \text{Fi}^{S_{\mathcal{K}}}\mathcal{A}$ such that $F \cap B^2 = \text{Fg}_{\mathcal{B}}^{S_{\mathcal{K}}}(b_1,b_2)$

iff for all $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ such that $\mathbf{B}$ is a subalgebra of $\mathbf{A}$, and all principal $\mathcal{K}$-congruences $\Theta_{\mathcal{K}}^{\mathbf{B}}(b_1,b_2)$ of $\mathbf{B}$, there exists $F \in \text{Con}_{\mathcal{K}}\mathbf{A}$ such that $F \cap B^2 = \Theta_{\mathcal{K}}^{\mathbf{B}}(b_1,b_2)$

iff $\mathcal{K}$ has the PRCEP.

In particular, a variety $\mathcal{K}$ has the PCEP if and only if $\text{Mod}^*S_{\mathcal{K}}$ has the PFEP. It can similarly be shown that $\text{Mod}^*S_{\mathcal{K}}$ has the FEP if and only if the (quasi) variety $\mathcal{K}$ has the (R)CEP. This leads to the following corollary of Corollary 4.1.11.

4.1.14 COROLLARY [BP88, Corollary 3.7]

Let $\mathcal{K}$ be a quasivariety. The following are equivalent:

(i) $\mathcal{K}$ *has LEDPRC,*

(ii) $\mathcal{K}$ *has the PRCEP,*

(iii) $\mathcal{K}$ *has the RCEP,*

(iv) $S_{\mathcal{K}}$ *has the LDDT.*

Moreover, $\mathcal{K}$ *has LEDPRC with respect to* $\mathcal{E}(p,q)$ *if and only if* $S_{\mathcal{K}}$ *has LDDT with local deduction-detachment system* $\mathcal{E}(p,q)$.

Proof. By the above observations, $\mathcal{K}$ has LEDPRC if and only if $\mathrm{Mod}^*S_{\mathcal{K}}$ has LFDPF which holds if and only if $\mathrm{Mod}^*S_{\mathcal{K}}$ has the PFEP, by Corollary 4.1.11, and this holds if and only if $\mathcal{K}$ has the PRCEP. Thus (i) and (ii) are equivalent. The equivalence of (i) and (iii) and that of (i) and (iv) follow similarly from Corollary 4.1.11. The last statement follows directly from Theorem 4.1.4 and the above observations. □

This generalizes a result of Alan Day [Day70] to the effect that a variety has the CEP if and only if it has the PCEP.

4.2 THE LDDT AND FILTER-DISTRIBUTIVITY

4.2.1 DEFINITION

Let S be a k-deductive system. A class M of S-matrices is called *filter-distributive* if $\mathbf{Fi}^S\mathcal{A}$ is a distributive lattice for all $\mathcal{A} \in M$. If the class $\mathrm{Mod}S$ is filter-distributive then we say that S is *filter-distributive.*

In this section we provide a characterization of protoalgebraic k-deductive systems with the LDDT that are filter-distributive (or, equivalently, of k-protoquasivarieties that have LFDPF and are filter-distributive). We shall use a number of lattice-theoretic definitions and results that were proved in Section 0.1.

Let S be a k-deductive system and $\mathcal{E} = \{E_i\, ;\, i \in I\}$ a family of sets of k-formulas, i.e., $E_i \subseteq Fm^k$ for each $i \in I$. We say that $\mathcal{E}$ is S-*directed* if, for all $i, j \in I$, there exists a $k \in I$ such that $E_i \vdash_S E_k$ and $E_j \vdash_S E_k$.

4.2.2 THEOREM [BP88, Theorem 4.3]

Let S be a protoalgebraic k-deductive system that possesses the LDDT with respect to a local deduction-detachment system $\mathbf{\mathcal{E}}$. Then S is filter-distributive if and only if $\mathbf{\mathcal{E}}$ is S-directed.

Proof. ($\Leftarrow$) Suppose S has the LDDT with respect to $\mathbf{\mathcal{E}} = \mathbf{\mathcal{E}}(\boldsymbol{p},\boldsymbol{q}) = \{E_i(\boldsymbol{p},\boldsymbol{q})\,;\, i \in I\}$, where $\mathbf{\mathcal{E}}$ is S-directed. Let $\mathcal{A} = \langle \mathbf{A}, F_{\mathcal{A}}\rangle$ be an S-matrix. Let $\mathrm{Fc}^S\mathcal{A}$ denote the set of finitely generated S-filters of $\mathcal{A}$. Note that by Lemma 1.6.4 (iii) these are precisely the compact elements of the algebraic lattice $\mathbf{Fi}^S\boldsymbol{\mathcal{A}}$. Let $\mathbf{Fc}^S\boldsymbol{\mathcal{A}} = \langle \mathrm{Fc}^S\mathcal{A}, \vee\rangle$ be the associated join-semilattice with 0, where $F_{\mathcal{A}}$ is the least element. It follows from Proposition 0.1.7 (iii) that $\mathbf{Fi}^S\boldsymbol{\mathcal{A}}$ is isomorphic to the ideal lattice, $\mathbf{Id}\,\mathbf{Fc}^S\boldsymbol{\mathcal{A}}$ of $\mathbf{Fc}^S\boldsymbol{\mathcal{A}}$. The join-semilattice $\mathbf{Fc}^S\boldsymbol{\mathcal{A}}$ is generated by the set $\{\mathrm{Fg}^S_{\mathcal{A}}\boldsymbol{a},\,;\, \boldsymbol{a} \in A^k\}$ so, by Lemmas 0.1.2 and 0.1.3, we need only show that $D(\mathrm{Fg}^S_{\mathcal{A}}\boldsymbol{a}, \mathrm{Fg}^S_{\mathcal{A}}\boldsymbol{b})$ holds in $\mathbf{Fc}^S\boldsymbol{\mathcal{A}}$ for all $\boldsymbol{a},\boldsymbol{b} \in A^k$ in order to conclude that $\mathbf{Fi}^S\boldsymbol{\mathcal{A}}$ is distributive.

So, let $F, G \in \mathrm{Fc}^S\mathcal{A}$ such that $\mathrm{Fg}^S_{\mathcal{A}}\boldsymbol{b} \subseteq \mathrm{Fg}^S_{\mathcal{A}}\boldsymbol{a} \vee F$ and $\mathrm{Fg}^S_{\mathcal{A}}\boldsymbol{b} \subseteq \mathrm{Fg}^S_{\mathcal{A}}\boldsymbol{a} \vee G$. First suppose that $\mathbf{A}$ is countably generated. Since $P - \{p_1,\ldots,p_k,q_1,\ldots,q_k\}$ is denumerable, there is a surjection from this set to a generating subset of $\mathbf{A}$, hence there exists a surjective homomorphism $h\colon \mathbf{Fm} \to \mathbf{A}$ such that $h\boldsymbol{p} = \boldsymbol{a}$, $h\boldsymbol{q} = \boldsymbol{b}$. Let $\mathcal{B} = \langle \mathbf{Fm}, h^{-1}(F_{\mathcal{A}})\rangle$. Then $h\colon \mathcal{B} \to \mathcal{A}$ is a reductive matrix homomorphism. Now,

$$h\boldsymbol{q} = \boldsymbol{b} \in (\mathrm{Fg}^S_{\mathcal{A}}\boldsymbol{a}) \vee F = \mathrm{Fg}^S_{\mathcal{A}}(F \cup \{\boldsymbol{a}\}),$$

so
$$\boldsymbol{q} \in h^{-1}(\mathrm{Fg}^S_{\mathcal{A}}(F \cup \{\boldsymbol{a}\}))$$

and
$$h^{-1}(\mathrm{Fg}^S_{\mathcal{A}}(F \cup \{\boldsymbol{a}\})) = \mathrm{Fg}^S_{\mathcal{B}}(h^{-1}(F) \cup \{\boldsymbol{p}\})$$

(by Theorem 2.1.3 (vi) since, by surjectivity of h, we have $h(h^{-1}(F) \cup \{\boldsymbol{p}\}) = F \cup \{\boldsymbol{a}\}$)

$$= \mathrm{Cn}_S(h^{-1}(F) \cup \{\boldsymbol{p}\}) \vee h^{-1}(F_{\mathcal{A}}) \qquad \text{[by Proposition 1.6.5 (ii)]}$$

$$= \mathrm{Cn}_S(h^{-1}(F) \cup \{\boldsymbol{p}\}) \qquad [\text{as } h^{-1}(F_{\mathcal{A}}) \subseteq h^{-1}(F)].$$

Thus $h^{-1}(F), \boldsymbol{p} \vdash_S \boldsymbol{q}$. Similarly, we get $h^{-1}(G), \boldsymbol{p} \vdash_S \boldsymbol{q}$. By the LDDT there exist $i, j \in I$ such that

$$h^{-1}(F) \vdash_S E_i(\boldsymbol{p},\boldsymbol{q}) \quad \text{and} \quad h^{-1}(G) \vdash_S E_j(\boldsymbol{p},\boldsymbol{q}).$$

Since $\mathbf{\mathcal{E}}$ is S-directed, there exists $k \in I$ such that

$$E_i(\boldsymbol{p},\boldsymbol{q}) \vdash_S E_k(\boldsymbol{p},\boldsymbol{q}) \quad \text{and} \quad E_j(\boldsymbol{p},\boldsymbol{q}) \vdash_S E_k(\boldsymbol{p},\boldsymbol{q}),$$

hence $h^{-1}(F) \supseteq E_k(\boldsymbol{p},\boldsymbol{q})$ and $h^{-1}(G) \supseteq E_k(\boldsymbol{p},\boldsymbol{q})$ since $h^{-1}(F)$ and $h^{-1}(G)$ are S-theories. So

$$E_k^{\mathbf{A}}(\boldsymbol{a},\boldsymbol{b}) = h(E_k(\boldsymbol{p},\boldsymbol{q})) \subseteq F \cap G.$$

Set $H = \mathrm{Fg}^S_{\mathcal{A}} E^{\mathbf{A}}_k(\boldsymbol{a},\boldsymbol{b})$. Then H is a compact S-filter of $\mathcal{A}$ since $E^{\mathbf{A}}_k(\boldsymbol{a},\boldsymbol{b})$ is a finite set and $H \subseteq F$ and $H \subseteq G$. Moreover, $E_k(\boldsymbol{p},\boldsymbol{q}), \boldsymbol{p} \vdash_S \boldsymbol{q}$, by assumption, from which it follows that

$$\boldsymbol{q} \in \mathrm{Fg}^S_{\mathfrak{B}}(E_k(\boldsymbol{p},\boldsymbol{q}) \cup \{\boldsymbol{p}\}),$$

so

$$\begin{aligned} \boldsymbol{b} = h\boldsymbol{q} &\in h(\mathrm{Fg}^S_{\mathfrak{B}}(E_k(\boldsymbol{p},\boldsymbol{q}) \cup \{\boldsymbol{p}\})) \\ &= h(h^{-1}(\mathrm{Fg}^S_{\mathcal{A}}(E^{\mathbf{A}}_k(\boldsymbol{a},\boldsymbol{b}) \cup \{\boldsymbol{a}\})) && \text{[by Theorem 2.1.3 (vi)]} \\ &= \mathrm{Fg}^S_{\mathcal{A}}(E^{\mathbf{A}}_k(\boldsymbol{a},\boldsymbol{b}) \cup \{\boldsymbol{a}\}) && \text{[by surjectivity of } h\text{]} \\ &= \mathrm{Fg}^S_{\mathcal{A}} E^{\mathbf{A}}_k(\boldsymbol{a},\boldsymbol{b}) \vee \mathrm{Fg}^S_{\mathcal{A}} \boldsymbol{a} \\ &= H \vee \mathrm{Fg}^S_{\mathcal{A}} \boldsymbol{a}, \end{aligned}$$

so

$$\mathrm{Fg}^S_{\mathcal{A}} \boldsymbol{b} \subseteq H \vee \mathrm{Fg}^S_{\mathcal{A}} \boldsymbol{a}.$$

This shows that $D(\mathrm{Fg}^S_{\mathcal{A}}\boldsymbol{a}, \mathrm{Fg}^S_{\mathcal{A}}\boldsymbol{b})$ holds in $\mathbf{Fc}^S\boldsymbol{\mathcal{A}}$.

Now, suppose $\mathcal{A}$ is not countably generated. Since $F, G \in \mathrm{Fc}^S\mathcal{A}$, there are finite subsets X, Y of A^k such that $F = \mathrm{Fg}^S_{\mathcal{A}} X$ and $G = \mathrm{Fg}^S_{\mathcal{A}} Y$. Let $\mathbf{A}'$ be the subalgebra of $\mathbf{A}$ generated by the co-ordinates of $X \cup Y \cup \{\boldsymbol{a},\boldsymbol{b}\}$. Let $\mathcal{A}' = \langle \mathbf{A}', (A')^k \cap F_{\mathcal{A}} \rangle$, so that $\mathcal{A}'$ is a submatrix of $\mathcal{A}$. Since S has the LDDT, Mod$S$ has the FEP (by Corollary 4.1.11). Thus, by Lemma 4.1.6,

$$\mathrm{Fg}^S_{\mathcal{A}'}(X \cup \{\boldsymbol{a}\}) = (A')^k \cap \mathrm{Fg}^S_{\mathcal{A}}(X \cup \{\boldsymbol{a}\}),$$

i.e.,

$$\mathrm{Fg}^S_{\mathcal{A}'} X \vee \mathrm{Fg}^S_{\mathcal{A}'} \boldsymbol{a} = (A')^k \cap (\mathrm{Fg}^S_{\mathcal{A}} \boldsymbol{a} \vee F)$$

and similarly

$$\mathrm{Fg}^S_{\mathcal{A}'} Y \vee \mathrm{Fg}^S_{\mathcal{A}'} \boldsymbol{a} = (A')^k \cap (\mathrm{Fg}^S_{\mathcal{A}} \boldsymbol{a} \vee G).$$

Now, $\boldsymbol{b} \in (A')^k \cap (\mathrm{Fg}^S_{\mathcal{A}} \boldsymbol{a} \vee F)$, so

$$\mathrm{Fg}^S_{\mathcal{A}'} \boldsymbol{b} \subseteq \mathrm{Fg}^S_{\mathcal{A}'} X \vee \mathrm{Fg}^S_{\mathcal{A}'} \boldsymbol{a}$$

and similarly

$$\mathrm{Fg}^S_{\mathcal{A}'} \boldsymbol{b} \subseteq \mathrm{Fg}^S_{\mathcal{A}'} Y \vee \mathrm{Fg}^S_{\mathcal{A}'} \boldsymbol{a}.$$

Since $\mathbf{A}'$ is countably (in fact finitely) generated, we may apply the result just established and conclude that there exists $H' \in \mathrm{Fc}^S\mathcal{A}'$ (say $H' = \mathrm{Fg}^S_{\mathcal{A}'} Z$, where Z is a finite subset of $(A')^k$) such that $H' \subseteq \mathrm{Fg}^S_{\mathcal{A}'} X \cap \mathrm{Fg}^S_{\mathcal{A}'} Y$ and $\mathrm{Fg}^S_{\mathcal{A}'} \boldsymbol{b} \subseteq \mathrm{Fg}^S_{\mathcal{A}'} \boldsymbol{a} \vee H'$.

Using the FEP and Lemma 4.1.6 again,

$$\boldsymbol{b} \in \mathrm{Fg}^S_{\mathcal{A}'}(Z \cup \{\boldsymbol{a}\}) = (A')^k \cap \mathrm{Fg}^S_{\mathcal{A}}(Z \cup \{\boldsymbol{a}\}) \subseteq \mathrm{Fg}^S_{\mathcal{A}}(Z \cup \{\boldsymbol{a}\}) = \mathrm{Fg}^S_{\mathcal{A}} \boldsymbol{a} \vee H'',$$

where $H'' = \mathrm{Fg}^S_{\mathcal{A}} Z$ (hence $H'' \in \mathrm{Fc}^S\mathcal{A}$). We therefore have

$$\mathrm{Fg}^S_{\mathcal{A}} \boldsymbol{b} \subseteq \mathrm{Fg}^S_{\mathcal{A}} \boldsymbol{a} \vee H''.$$

Lastly, if T is any S-filter of $\mathcal{A}$ containing X, then $(A')^k \cap T$ is an S-filter of $\mathcal{A}'$ containing X, so $\mathrm{Fg}^S_{\mathcal{A}'} X \subseteq (A')^k \cap T$, hence $Z \subseteq (A')^k \cap T \subseteq T$. Consequently, $H'' = \mathrm{Fg}^S_{\mathcal{A}} Z \subseteq T$ and so

$H'' \subseteq \mathrm{Fg}^S_{\mathcal{A}} X = F$. Similarly, $H'' \subseteq G$, completing the proof that $D(\mathrm{Fg}^S_{\mathcal{A}} \boldsymbol{a}, \mathrm{Fg}^S_{\mathcal{A}} \boldsymbol{b})$ holds in $\mathrm{Fc}^S \mathcal{A}$.

($\Rightarrow$) Suppose $\mathcal{E} = \{E_i(\boldsymbol{p}, \boldsymbol{q})\,;\, i \in I\}$ and choose $E_i = E_i(\boldsymbol{p}, \boldsymbol{q})$, $E_j = E_j(\boldsymbol{p}, \boldsymbol{q}) \in \mathcal{E}$. Let $\mathfrak{B}$ be the formula matrix model $\langle \mathbf{Fm}, \mathrm{Cn}_S(E_i) \cap \mathrm{Cn}_S(E_j) \rangle$ and let $\mathbf{A}$ be the absolutely free $\mathcal{L}$-algebra generated by $\{p_1, \dots, p_k, q_1, \dots, q_k\}$. Set $\mathcal{A} = \langle \mathbf{A}, A^k \cap \mathrm{Cn}_S(E_i) \cap \mathrm{Cn}_S(E_j) \rangle$, so that $\mathcal{A}$ is a submatrix of $\mathfrak{B}$. By the FEP, the Lemma 4.1.6 and Proposition 1.6.5 (ii), we have, for any $X \subseteq A^k$,

$$\mathrm{Fg}^S_{\mathcal{A}} X = A^k \cap \mathrm{Fg}^S_{\mathfrak{B}} X = A^k \cap [(\mathrm{Cn}_S(E_i) \cap \mathrm{Cn}_S(E_j)) \vee \mathrm{Cn}_S(X)]. \tag{4.2.1}$$

Now, $E_i, \boldsymbol{p} \vdash_S \boldsymbol{q}$ and $E_j, \boldsymbol{p} \vdash_S \boldsymbol{q}$, so since $E_i \cup E_j \cup \{\boldsymbol{p}, \boldsymbol{q}\} \subseteq A^k$, we have by Proposition 1.6.5 (iii),

$$\boldsymbol{q} \in \mathrm{Fg}^S_{\mathcal{A}} \boldsymbol{p} \vee \mathrm{Fg}^S_{\mathcal{A}} E_i \quad \text{and} \quad \boldsymbol{q} \in \mathrm{Fg}^S_{\mathcal{A}} \boldsymbol{p} \vee \mathrm{Fg}^S_{\mathcal{A}} E_j,$$

so

$$\begin{aligned}
\boldsymbol{q} &\in (\mathrm{Fg}^S_{\mathcal{A}} \boldsymbol{p} \vee \mathrm{Fg}^S_{\mathcal{A}} E_i) \cap (\mathrm{Fg}^S_{\mathcal{A}} \boldsymbol{p} \vee \mathrm{Fg}^S_{\mathcal{A}} E_j) \\
&= \mathrm{Fg}^S_{\mathcal{A}} \boldsymbol{p} \vee [\mathrm{Fg}^S_{\mathcal{A}} E_i \cap \mathrm{Fg}^S_{\mathcal{A}} E_j] \qquad \text{[by distributivity]} \\
&\doteq [A^k \cap ((\mathrm{Cn}_S(E_i) \cap \mathrm{Cn}_S(E_j)) \vee \mathrm{Cn}_S \boldsymbol{p})] \vee [A^k \cap \mathrm{Cn}_S(E_i) \cap \mathrm{Cn}_S(E_j)] \\
&\text{[by (4.2.1)]} \\
&= A^k \cap [(\mathrm{Cn}_S(E_i) \cap \mathrm{Cn}_S(E_j)) \vee \mathrm{Cn}_S \boldsymbol{p}].
\end{aligned}$$

It follows that $\mathrm{Cn}_S(E_i) \cap \mathrm{Cn}_S(E_j), \boldsymbol{p} \vdash_S \boldsymbol{q}$, hence by the LDDT, there exists $E_k = E_k(\boldsymbol{p}, \boldsymbol{q}) \in \mathcal{E}$ such that

$$\mathrm{Cn}_S(E_i) \cap \mathrm{Cn}_S(E_j) \vdash_S E_k.$$

Certainly, therefore, $\mathrm{Cn}_S(E_i) \vdash_S E_k$ and $\mathrm{Cn}_S(E_j) \vdash_S E_k$, i.e., $E_i \vdash_S E_k$ and $E_j \vdash_S E_k$, so $\mathcal{E}$ is S-directed. □

4.3 EQUIVALENT DEDUCTIVE SYSTEMS AND THE LDDT

Recall that in Section 1.9 we defined a relation of equivalence for arbitrary deductive systems. Here we partly justify that definition by showing that the property of possessing the LDDT is preserved by this relation of equivalence. The main result is obtained in several steps. First we show that if a k-deductive system has the LDDT with a local deduction-detachment system $\mathcal{E}$, then it also has a local deduction-detachment system "for any finite set of premisses". This statement is made precise in Lemma 4.3.1. Unfortunately, for the proof of the main theorem some very technical definitions and a technical lemma are necessary.

Let $p = \langle p_1, \dots, p_k \rangle$ and $q = \langle q_1, \dots, q_k \rangle$ be k-variables and $\{p_1, p_2, p_3, \dots\}$ a countable set of k-variables, where $p_i = \langle p_{i1}, \dots, p_{ik} \rangle$ for each i and $p_{ij} \neq p_{rl}$ unless $(i,j) = (r,\ell)$. Assume also that the p_i's, the p_j's and $p_{r\ell}$'s are all distinct. Let S be a k-deductive system with local deduction-detachment system $\mathcal{E} = \mathcal{E}(p,q) = \{E_i(p,q)\,;\, i \in I\}$. We shall need the following inductively defined sets: Define $I^{(1)} = I$, $E_i^{(1)}(p_1, q) = E_i(p_1, q)$ for each $i \in I$, and $\mathcal{E}^{(1)}(p_1, q) = \{E_i^{(1)}(p_1, q)\,;\, i \in I^{(1)}\}$. Next, we explicitly show the inductive step for $n = 2$ to make the definition more accessible. Let $I^{E_i^{(1)}}$ denote the set of all maps from $E_i^{(1)}(p_1, q)$ into I and set

$$I^{(2)} = \bigcup\{I^{E_i^{(1)}}\,;\, i \in I^{(1)}\}.$$

For each $f \in I^{(2)}$, say $f : E_i^{(1)}(p_1, q) \to I$, set

$$E_f^{(2)}(p_1, p_2, q) = \bigcup\{E_{f(\eta)}(p_2, \eta)\,;\, \eta \in E_i^{(1)}(p_1, q)\},$$

and, finally,

$$\mathcal{E}^{(2)}(p_1, p_2, q) = \{E_f^{(2)}(p_1, p_2, q)\,;\, f \in I^{(2)}\}.$$

Suppose that $n \geq 2$ and that $I^{(n)}$ and $\mathcal{E}^{(n)}(p_1, \dots, p_n, q) = \{E_i^{(n)}(p_1, \dots, p_n, q)\,;\, i \in I^{(n)}\}$ have been defined, where $E_i^{(n)}(p_1, \dots, p_n, q)$ is a finite set of k-formulas for each $i \in I$. For each $i \in I^{(n)}$, let $I^{E_i^{(n)}}$ denote the set of all maps from $E_i^{(n)}(p_1, \dots, p_n, q)$ into I, and set

$$I^{(n+1)} = \bigcup\{I^{E_i^{(n)}}\,;\, i \in I^{(n)}\}.$$

For each $f \in I^{(n+1)}$, say $f : E_i^{(n)}(p_1, \dots, p_n, q) \to I$, set

$$E_f^{(n+1)}(p_1, \dots, p_{n+1}, q) = \bigcup\{E_{f(\eta)}(p_{n+1}, \eta)\,;\, \eta \in E_i^{(n)}(p_1, \dots p_n, q)\},$$

and, finally, set

$$\mathcal{E}^{(n+1)}(p_1, \dots, p_{n+1}, q) = \{E_f^{(n+1)}(p_1, \dots, p_{n+1}, q)\,;\, f \in I^{(n+1)}\}.$$

Note that if the set I contains only one element then for any n, the set of all maps from $E_i^{(n)}(p_1, \dots, p_n, q)$ into I contains precisely one element, hence each $I^{(n)}$ has one element. Then it follows that each $\mathcal{E}^{(n)}$ contains only one element.

4.3.1 LEMMA [BP88, Lemma 5.1]

Let S be a k-deductive system with local deduction-detachment system $\mathcal{E}(p,q)$. For all integers $n \geq 1$, $\delta_1, \dots, \delta_n, \psi \in Fm^k$ and $\Gamma \subseteq Fm^k$, we have

(4.3.1) $\quad \Gamma, \delta_1, \dots, \delta_n \vdash_S \psi$ *if and only if* $\Gamma \vdash_S E_i^{(n)}(\delta_1, \dots, \delta_n, \psi)$ *for some* $i \in I^{(n)}$.

Proof. As one would expect, the proof is done by induction on n. If $n = 1$, the statement of the lemma simplifies to the definition of a local deduction-detachment system. Assume that the statement holds for some $n \geq 1$, and let $\delta_1, \ldots, \delta_{n+1}, \psi \in Fm^k$, $\Gamma \subseteq Fm^k$ and σ a substitution satisfying $\sigma p_i = \delta_i$ for each $i \leq n+1$ and $\sigma q = \psi$. Note that such a σ exists, by choice of the k-variables p_i and q.

($\Leftarrow$) Suppose that

$$\Gamma \vdash_S E_f^{(n+1)}(\delta_1, \ldots, \delta_{n+1}, \psi), \tag{4.3.2}$$

where $f \in I^{(n+1)}$, say $f : E_i^{(n)}(p_1, \ldots, p_n, q) \to I$. Since $E_i^{(n)}(\delta_1, \ldots, \delta_n, \psi) \subseteq Fm^k$ we can apply (4.3.1) and deduce, by the induction assumption, that

$$E_i^{(n)}(\delta_1, \ldots, \delta_n, \psi), \delta_1, \ldots, \delta_n \vdash_S \psi \quad \text{for each } i \in I^{(n)},$$

which implies, since $E_i^{(n)}(\delta_1, \ldots, \delta_n, \psi) = \sigma(E_i^{(n)}(p_1, \ldots, p_n, q))$, that

$$\sigma(E_i^{(n)}(p_1, \ldots, p_n, q)), \delta_1, \ldots, \delta_n \vdash_S \psi \quad \text{for each } i \in I^{(n)}. \tag{4.3.3}$$

For each $\eta \in E_i^{(n)}(p_1, \ldots, p_n, q)$, $E_{f(\eta)}(p, q)$ is an element of the local deduction-detachment system $\mathfrak{E}$, hence

$$E_{f(\eta)}(\delta_{n+1}, \sigma\eta), \delta_{n+1} \vdash_S \sigma\eta.$$

Now,

$$\begin{aligned} E_{f(\eta)}(\delta_{n+1}, \sigma\eta) &= \sigma(E_{f(\eta)}(p_{n+1}, \eta)) \\ &\subseteq \sigma(E_f^{(n+1)}(p_1, \ldots, p_{n+1}, q)) \\ &= E_f^{(n+1)}(\delta_1, \ldots, \delta_{n+1}, \psi), \end{aligned}$$

hence

$$E_f^{(n+1)}(\delta_1, \ldots, \delta_{n+1}, \psi), \delta_{n+1} \vdash_S \sigma\eta \quad \text{for all } \eta \in E_i^{(n)}(p_1, \ldots, p_n, q).$$

We can rephrase this as

$$E_f^{(n+1)}(\delta_1, \ldots, \delta_{n+1}, \psi), \delta_{n+1} \vdash_S \sigma(E_i^{(n)}(p_1, \ldots, p_n, q)).$$

By (4.3.3), we then get that

$$E_f^{(n+1)}(\delta_1, \ldots, \delta_{n+1}, \psi), \delta_1, \ldots, \delta_{n+1} \vdash_S \psi,$$

hence (4.3.2) implies that $\Gamma, \delta_1, \ldots, \delta_{n+1} \vdash_S \psi$, as required.

($\Rightarrow$) Suppose that $\Gamma, \delta_1, \ldots, \delta_{n+1} \vdash_S \psi$, i.e., $\Gamma \cup \{\delta_{n+1}\}, \delta_1, \ldots, \delta_n \vdash_S \psi$. By the induction assumption (replacing Γ by $\Gamma \cup \{\delta_{n+1}\}$) we have

$$\Gamma, \delta_{n+1} \vdash_S E_i^{(n)}(\delta_1, \ldots, \delta_n, \psi) \quad \text{for some } i \in I^{(n)}$$

and, as before, we can write $E_i^{(n)}(\delta_1,\ldots,\delta_n,\psi)=\sigma(E_i^{(n)}(p_1,\ldots,p_n,q))$. For each $\eta\in E_i^{(n)}(p_1,\ldots,p_n,q)$, therefore, $\Gamma,\delta_{n+1}\vdash_S\sigma\eta$. Since $\mathcal{E}$ is a local deduction-detachment system for S, there exists a $j_\eta\in I$ such that

$$\Gamma\vdash_S E_{j_\eta}(\delta_{n+1},\sigma\eta). \tag{4.3.4}$$

It is possible, therefore, to define a function $f:E_i^{(n)}(p_1,\ldots,p_n,q)\to I$ by $f(\eta)=j_\eta$. By definition, $f\in I^{(n+1)}$. Now,

$$\begin{aligned}E_f^{(n+1)}(\delta_1,\ldots,\delta_{n+1},\psi)&=\sigma(E_f^{(n+1)}(p_1,\ldots,p_{n+1},q))\\&=\sigma\Big(\bigcup\{E_{f(\eta)}(p_{n+1},\eta)\,;\,\eta\in E_i^{(n)}(p_1,\ldots,p_n,q)\}\Big)\\&=\bigcup\{\sigma(E_{f(\eta)}(p_{n+1},\eta))\,;\,\eta\in E_i^{(n)}(p_1,\ldots,p_n,q)\}\\&=\bigcup\{E_{f(\eta)}(\delta_{n+1},\sigma\eta)\,;\,\eta\in E_i^{(n)}(p_1,\ldots,p_n,q)\},\end{aligned}$$

hence $\Gamma\vdash_S E_f^{(n+1)}(\delta_1,\ldots,\delta_{n+1},\psi)$, by (4.3.4). □

4.3.2 THEOREM [BP88, Lemma 5.2]

Let S_1 and S_2 be equivalent deductive systems. Then S_1 has the LDDT if and only if S_2 has the LDDT.

Proof. Assume that S_1 is a k-deductive system and S_2 is an ℓ-deductive system. Let $\tau=\{\tau^1,\ldots,\tau^n\}$ be an interpretation of S_1 in S_2 and $\rho=\{\rho^1,\ldots,\rho^m\}$ an interpretation of S_2 in S_1. Recall from Section 1.9 that each τ^i is an ℓ-formula in k variables, and each ρ^i is a k-formula in ℓ variables. Suppose that S_1 has the LDDT with local deduction-detachment system $\mathcal{E}(p,q)$. For all $\Gamma\subseteq Fm^\ell$ and $\varphi,\psi\in Fm^\ell$, we have that

$$\begin{aligned}\Gamma,\varphi\vdash_{S_2}\psi\quad&\text{iff}\quad\rho(\Gamma),\rho(\varphi)\vdash_{S_1}\rho(\psi)\\&\text{iff}\quad\rho(\Gamma),\rho^1(\varphi),\ldots,\rho^m(\varphi)\vdash_{S_1}\rho^j(\psi)\ \text{ for each } j\le m.\end{aligned}$$

By Lemma 4.3.1, for all $j\le m$, the right hand side holds if and only if there exists an $i_j\in I^{(m)}$ such that

$$\rho(\Gamma)\vdash_{S_1}E_{i_j}^{(m)}(\rho^1(\varphi),\ldots,\rho^m(\varphi),\rho^j(\psi)),$$

hence

$$\begin{aligned}\Gamma,\varphi\vdash_{S_2}\psi\quad&\text{iff}\quad\rho(\Gamma)\vdash_{S_1}\bigcup\{E_{i_j}^{(m)}(\rho^1(\varphi),\ldots,\rho^m(\varphi),\rho^j(\psi))\,;\,j\le m\}\\&\text{iff}\quad\tau(\rho(\Gamma))\vdash_{S_2}\tau\Big(\bigcup\{E_{i_j}^{(m)}(\rho^1(\varphi),\ldots,\rho^m(\varphi),\rho^j(\psi))\,;\,j\le m\}\Big).\end{aligned}$$

Define $E'_{\boldsymbol{i}}(p,q)=\tau\Big(\bigcup\{E_{i_j}^{(m)}(\rho^1(p),\ldots,\rho^m(p),\rho^j(q))\,;\,j\le m\}\Big)$, where $\boldsymbol{i}=\langle i_1,\ldots,i_m\rangle\in\big(I^{(m)}\big)^m$, and define $\mathcal{E}'(p,q)=\{E'_{\boldsymbol{i}}(p,q)\,;\,\boldsymbol{i}\in\big(I^{(m)}\big)^m\}$. Since $\Gamma\dashv\vdash_{S_2}\tau(\rho(\Gamma))$, the above equivalences yield

$$\Gamma, \varphi \vdash_{S_2} \psi \quad \text{if and only if} \quad \Gamma \vdash_{S_2} E'_{\boldsymbol{i}}(\varphi, \psi) \text{ for some } \boldsymbol{i} \in \left(I^{(m)}\right)^m,$$

hence $\mathcal{E}'(\boldsymbol{p}, \boldsymbol{q})$ is a local deduction-detachment system for S_2, therefore S_2 has the LDDT. If we assume that S_2 has the LDDT then a symmetrical proof will yield that S_1 has the LDDT. □

In Theorem 3.1.11, we showed that a k-deductive system is algebraizable if and only if it is equivalent to $S_{\mathcal{K}}$, where $\mathcal{K}$ is its equivalent quasivariety semantics. This observation allows us to apply the previous theorem to the study of algebraizable k-deductive systems.

4.3.3 COROLLARY [BP88, Corollary 5.3]

Let S be an algebraizable k-deductive system with equivalent quasivariety semantics $\mathcal{K}$. Then S has the LDDT if and only if $\mathcal{K}$ has the RCEP. In particular, in the case where S is strongly algebraizable, i.e., $\mathcal{K}$ is a variety, S has the LDDT if and only if $\mathcal{K}$ has the CEP.

Proof. By Theorem 3.1.11 (viii), S is equivalent to $S_{\mathcal{K}}$, hence, by the theorem just proved, S has the LDDT if and only if $S_{\mathcal{K}}$ has the LDDT. By Corollary 4.1.14, $S_{\mathcal{K}}$ has the LDDT if and only if $\mathcal{K}$ has the RCEP. □

4.4 DEDUCTION-DETACHMENT THEOREMS

We now proceed to study a stronger form of the LDDT, namely the 'deduction-detachment theorem'. A number of the results of the following sections are special cases of results concerning the LDDT, and the algebraic properties considered here are (stronger) analogues of those considered previously.

4.4.1 DEFINITION

Let S be a k-deductive system and let $p_1, \ldots, p_k, q_1, \ldots, q_k$ be distinct variables with $\boldsymbol{p} = \langle p_1, \ldots, p_k \rangle$ and $\boldsymbol{q} = \langle q_1, \ldots, q_k \rangle$. A finite set $E(\boldsymbol{p}, \boldsymbol{q}) = \{\boldsymbol{\eta}_1(\boldsymbol{p}, \boldsymbol{q}), \ldots, \boldsymbol{\eta}_m(\boldsymbol{p}, \boldsymbol{q})\}$ of k-formulas in the $2k$ variables $p_1, \ldots, p_k, q_1, \ldots, q_k$ is called a *deduction-detachment set for S* if, for all $\Gamma \subseteq Fm^k$ and $\varphi, \psi \in Fm^k$,

$$\Gamma, \varphi \vdash_S \psi \quad \text{if and only if} \quad \Gamma \vdash_S E(\varphi, \psi).$$

(Note that $\Gamma \vdash_S E(\varphi, \psi)$ is an abbreviation of $\Gamma \vdash_S \boldsymbol{\eta}_i(\varphi, \psi)$ for each $i \leq m$.) If there exists a deduction-detachment set for S then we say that S has the *deduction-detachment theorem* (*DDT*, for short).

Note that a deduction-detachment set is a local deduction-detachment system with one element. So if S has the DDT then S certainly has the LDDT as well.

In [BP88] and [BP89b], most of the results about k-deductive systems S that have the DDT make the assumption that S is protoalgebraic. The next theorem, which is not in the published literature, shows that this assumption may be dropped, as it is redundant. This allows us to phrase several subsequent results more strongly than was done in their original sources in the literature.

4.4.2 THEOREM

Let S be a k-deductive system which has the DDT. Then S is protoalgebraic.

Proof. Let $E(\boldsymbol{p},\boldsymbol{q}) = \{\boldsymbol{\eta}_1(\boldsymbol{p},\boldsymbol{q}),\dots,\boldsymbol{\eta}_m(\boldsymbol{p},\boldsymbol{q})\}$ be a deduction-detachment set for S. Let $p, q, z_1,\dots,z_{k-1}$ be variables such that they and $p_1,\dots,p_k,q_1,\dots,q_k$ are all distinct. Recall that for each $\ell \le k$, $\tilde{z}\,[p/\ell]$ means $\langle z_1,\dots,z_{\ell-1},p,z_\ell,\dots,z_{k-1}\rangle$. For each $\ell \le k$ and each $i \le m$, define the k-formula $\Delta_{i\ell}(p,q,\tilde{z}\,) = \boldsymbol{\eta}_i(\tilde{z}\,[p/\ell],\tilde{z}\,[q/\ell])$ and let $\Delta(p,q,\tilde{z}\,)$ denote $\{\Delta_{i\ell}(p,q,\tilde{z}\,)\,;\ \ell \le k\,;\, i \le m\}$. From the fact that for each $\ell \le k$, $\tilde{z}\,[p/\ell] \vdash_S \tilde{z}\,[p/\ell]$, we infer $\vdash_S E(\tilde{z}\,[p/\ell],\tilde{z}\,[p/\ell])$, i.e., $\vdash_S \Delta_{i\ell}(p,p,\tilde{z}\,)$ for all $i \le m$. Thus $\vdash_S \Delta(p,p,\tilde{z}\,)$. From the fact that, for each $\ell \le k$, $E(\tilde{z}\,[p/\ell],\tilde{z}\,[q/\ell]) \vdash_S E(\tilde{z}\,[p/\ell],\tilde{z}\,[q/\ell])$, we obtain

$$\tilde{z}\,[p/\ell], E(\tilde{z}\,[p/\ell],\tilde{z}\,[q/\ell]) \vdash_S \tilde{z}\,[q/\ell],$$

i.e.,

$$\tilde{z}\,[p/\ell], \{\Delta_{i\ell}(p,q,\tilde{z}\,)\,;\ i \le m\} \vdash_S \tilde{z}\,[q/\ell],$$

so

$$\tilde{z}\,[p/\ell], \Delta(p,q,\tilde{z}\,) \vdash_S \tilde{z}\,[q/\ell].$$

Thus $\Delta(p,q,\tilde{z}\,)$ is a finite system of equivalence k-formulas with parameters $\tilde{z}$ for S. By Theorem 2.3.2, S is protoalgebraic. □

4.4.3 DEFINITION

Let S be a k-deductive system and M a class of S-matrices. Let $p_1,\dots,p_k,q_1,\dots,q_k$ be distinct variables with $\boldsymbol{p} = \langle p_1,\dots,p_k\rangle$ and $\boldsymbol{q} = \langle q_1,\dots,q_k\rangle$. We say that M has *formula definable principal filters* (*FDPF*, for short) if there exists a finite set $E(\boldsymbol{p},\boldsymbol{q}) = \{\boldsymbol{\eta}_1(\boldsymbol{p},\boldsymbol{q}),\dots,\boldsymbol{\eta}_m(\boldsymbol{p},\boldsymbol{q})\}$ of k-formulas in the $2k$ variables $p_1, \ldots, p_k, q_1, \ldots, q_k$ such that, for all $\mathcal{A} = \langle \mathbf{A}, F_{\mathcal{A}} \rangle \in M$ and all $\boldsymbol{a}, \boldsymbol{b} \in A^k$,

$$\boldsymbol{b} \in \mathrm{Fg}^S_{\mathcal{A}} \boldsymbol{a} \text{ if and only if } E^{\mathbf{A}}(\boldsymbol{a}, \boldsymbol{b}) \subseteq F_{\mathcal{A}}.$$

Then $E(\boldsymbol{p}, \boldsymbol{q})$ is called a set of *defining formulas* for the principal filters.

Note that if we regard $\mathcal{A}$ as a first-order structure (without equality) (see Section 1.10) then, in terms of the notation of Section 0.5, the right hand side of the equivalence says that $\models_{\mathcal{A}} \boldsymbol{\eta}_i(\boldsymbol{a}, \boldsymbol{b})$ for all $i \leq m$, or simply $\models_{\mathcal{A}} E(\boldsymbol{a}, \boldsymbol{b})$, where $\boldsymbol{\eta}_i(\boldsymbol{a}, \boldsymbol{b})$ is considered as a formula of the language $\mathfrak{L}_A$ got by adding to $\mathfrak{L}$ new constant symbols for all elements of A. If we set $\mathfrak{E}(\boldsymbol{p}, \boldsymbol{q}) = \{E(\boldsymbol{p}, \boldsymbol{q})\}$, then the above definition implies that M has locally formula definable principal filters with defining system $\mathfrak{E}$, as in Definition 4.1.3. The following theorem can therefore be obtained as a corollary to Theorem 4.1.4 and the previous theorem.

4.4.4 THEOREM [BP89b, Theorem 7.1]

Let S be a k-deductive system and $E(\boldsymbol{p}, \boldsymbol{q})$ a finite set of k-formulas in $2k$ variables $p_1, \ldots, p_k$, $q_1, \ldots, q_k$. The following are equivalent:

(i) *S has the DDT with deduction-detachment set $E(\boldsymbol{p}, \boldsymbol{q})$,*

(ii) Mod$S$ *has FDPF with defining formulas $E(\boldsymbol{p}, \boldsymbol{q})$,*

(iii) Mod*S *has FDPF with defining formulas $E(\boldsymbol{p}, \boldsymbol{q})$ and is closed under subdirect products.*

Proof. Let $\mathfrak{E} = \{E(\boldsymbol{p}, \boldsymbol{q})\}$. Note that (i) says that S has the LDDT with local deduction-detachment system $\mathfrak{E}$, and implies that S is protoalgebraic (by the previous theorem). Since (ii) says just that Mod$S$ has LFDPF with defining system $\mathfrak{E}$, the implication (i)⇒(ii) follows from Theorem 4.1.4. Assuming (ii) and considering the formula matrix models $\langle \mathbf{Fm}, \mathrm{Cn}_S(\Gamma) \rangle$ ($\Gamma \subseteq Fm^k$), we obtain from Proposition 1.6.5 that S has the DDT (with deduction-detachment set $E(\boldsymbol{p}, \boldsymbol{q})$) and is therefore protoalgebraic. By Theorem 2.2.3, Mod*S is closed under subdirect products. This clearly suffices in establishing that (ii)⇒(iii). Since (iii) implies that Mod*S has LFDPF with defining system $\mathfrak{E}$, and that S is protoalgebraic (Theorem 2.2.3), the implication (iii)⇒(i) is a special case of Theorem 4.1.4. □

We make the same point here that was made for the LDDT, namely that the equivalence of (iii) and (i) is a matrix model-theoretic result: A reduced universal Horn k-class Mod*S is a k-protoquasivariety having FDPF with defining formulas $E(\boldsymbol{p}, \boldsymbol{q})$ if and only if S has the DDT with deduction-detachment set $E(\boldsymbol{p},\boldsymbol{q})$. We now wish to characterize k-protoquasivarieties (and classes Mod$S$) that have FDPF in terms of the structure of their associated theory (and filter) lattices.

For a k-deductive system S, let TfS denote the set of finitely generated S-theories. Note that by Lemma 1.5.2 (iii) these are precisely the compact elements of the algebraic lattice $\mathbf{Th}S$. Let $\mathbf{Tf}S = \langle \mathrm{Tf}S, \vee \rangle$ be the associated join-semilattice with 0, where $\mathrm{Cn}_S(\emptyset)$ is the least element. Recall also that for an S-matrix $\mathcal{A} = \langle \mathbf{A}, F_{\mathcal{A}} \rangle$, $\mathrm{Fc}^S\mathcal{A}$ denotes the set of all finitely generated S-filters of $\mathcal{A}$ and that $\mathbf{Fc}^S\boldsymbol{\mathcal{A}} = \langle \mathrm{Fc}^S\mathcal{A}, \vee \rangle$ is the corresponding join-semilattice with 0. Recall from Theorem 1.6.4 (iii) that $\mathrm{Fc}^S\mathcal{A}$ is the set of all compact elements of the algebraic lattice $\mathbf{Fi}^S\boldsymbol{\mathcal{A}}$.

4.4.5 THEOREM [BP89b, Theorem 7.6]

Let S be a k-deductive system. The following are equivalent:

(i) *S has the DDT,*

(ii) *S is protoalgebraic and for all $\mathcal{A} \in$ Mod$S$, $\mathbf{Fc}^S\boldsymbol{\mathcal{A}}$ is dually Brouwerian,*

(iii) *S is protoalgebraic and $\mathbf{Tf}S$ is dually Brouwerian.*

Proof. (i)⇒(ii) S is protoalgebraic by Theorem 4.4.2. Let $\mathcal{A} = \langle \mathbf{A}, F_{\mathcal{A}} \rangle \in$ Mod$S$. Note first that as a join-semilattice $\mathbf{Fc}^S\boldsymbol{\mathcal{A}}$ is generated by the set $X = \{\mathrm{Fg}^S_{\mathcal{A}}\boldsymbol{a};\, \boldsymbol{a} \in A^k\} \subseteq \mathrm{Fc}^S\mathcal{A}$, so, by Lemma 0.1.8, all we need to show is that $\mathrm{Fg}^S_{\mathcal{A}}\boldsymbol{a} *^{\mathbf{Fc}^S\boldsymbol{\mathcal{A}}} \mathrm{Fg}^S_{\mathcal{A}}\boldsymbol{b}$ exists in $\mathbf{Fc}^S\boldsymbol{\mathcal{A}}$ for all $\boldsymbol{a}, \boldsymbol{b} \in A^k$. Let $E(\boldsymbol{p},\boldsymbol{q})$ be a deduction-detachment set for S, where $\boldsymbol{p} = \langle p_1, \dots, p_k \rangle$ and $\boldsymbol{q} = \langle q_1, \dots, q_k \rangle$ and the variables $p_1, \dots, p_k, q_1, \dots, q_k$ are all distinct. Note that $E(\boldsymbol{p},\boldsymbol{q})$ is finite by definition, hence $\mathrm{Fg}^S_{\mathcal{A}}(E^{\mathbf{A}}(\boldsymbol{a},\boldsymbol{b})) \in \mathrm{Fc}^S\mathcal{A}$. The result will follow immediately from the following claim.

Claim: $\mathrm{Fg}^S_{\mathcal{A}}\boldsymbol{a} *^{\mathbf{Fc}^S\boldsymbol{\mathcal{A}}} \mathrm{Fg}^S_{\mathcal{A}}\boldsymbol{b} = \mathrm{Fg}^S_{\mathcal{A}}(E^{\mathbf{A}}(\boldsymbol{a},\boldsymbol{b}))$.

Proof. We have $E(\boldsymbol{p},\boldsymbol{q}), \boldsymbol{p} \vdash_S \boldsymbol{q}$. Set $\mathfrak{B} = \langle \mathbf{A}, \mathrm{Fg}^S_{\mathcal{A}}(E^{\mathbf{A}}(\boldsymbol{a},\boldsymbol{b}) \cup \{\boldsymbol{a}\}) \rangle$. Then, since $E(\boldsymbol{p},\boldsymbol{q}), \boldsymbol{p} \models_{\mathfrak{B}} \boldsymbol{q}$, we can interpret $\boldsymbol{p}$ and $\boldsymbol{q}$ as $\boldsymbol{a}$ and $\boldsymbol{b}$ in A, respectively, and deduce that $\boldsymbol{b} \in \mathrm{Fg}^S_{\mathcal{A}}(E^{\mathbf{A}}(\boldsymbol{a},\boldsymbol{b}) \cup \{\boldsymbol{a}\})$, from which it follows that

$$\mathrm{Fg}^S_{\mathcal{A}}\boldsymbol{b} \subseteq \mathrm{Fg}^S_{\mathcal{A}}E^{\mathbf{A}}(\boldsymbol{a},\boldsymbol{b}) \vee \mathrm{Fg}^S_{\mathcal{A}}\boldsymbol{a}.$$

Next, we shall show that $\mathrm{Fg}^S_{\mathcal{A}}E^{\mathbf{A}}(\boldsymbol{a},\boldsymbol{b}) = \min\{F \in \mathbf{Fi}^S\boldsymbol{\mathcal{A}};\, F \vee \mathrm{Fg}^S_{\mathcal{A}}\boldsymbol{a} \supseteq \mathrm{Fg}^S_{\mathcal{A}}\boldsymbol{b}\}$. Note that by Lemma 0.1.9, the use here of the lattice $\mathbf{Fi}^S\boldsymbol{\mathcal{A}}$ rather than the semilattice $\mathbf{Fc}^S\boldsymbol{\mathcal{A}}$ will not compromise the claim. Now, suppose first that $\mathbf{A}$ is countably generated, and let $h: \mathbf{Fm} \to \mathbf{A}$ be a surjective homomorphism such that $h\boldsymbol{p} = \boldsymbol{a}$ and $h\boldsymbol{q} = \boldsymbol{b}$. Set $\mathfrak{C} = \langle \mathbf{Fm}, h^{-1}(F_{\mathcal{A}}) \rangle$. Then $h: \mathfrak{C} \to \mathcal{A}$

is a reductive matrix homomorphism. For $X \subseteq A^k$, we have

$\mathrm{Fg}^S_{\mathcal{A}}\boldsymbol{a} \vee \mathrm{Fg}^S_{\mathcal{A}}X \supseteq \mathrm{Fg}^S_{\mathcal{A}}\boldsymbol{b}$ (in $\mathbf{Fi}^S\mathcal{A}$)

iff $\mathrm{Fg}^S_{\mathcal{A}}(X \cup \{\boldsymbol{a}\}) \supseteq \mathrm{Fg}^S_{\mathcal{A}}\boldsymbol{b}$

iff $h^{-1}(\mathrm{Fg}^S_{\mathcal{A}}(X \cup \{\boldsymbol{a}\})) \supseteq h^{-1}(\mathrm{Fg}^S_{\mathcal{A}}\boldsymbol{b})$ [h is surjective]

iff $\mathrm{Fg}^S_{\mathfrak{C}}(h^{-1}(X) \cup \{\boldsymbol{p}\}) \supseteq \mathrm{Fg}^S_{\mathfrak{C}}\boldsymbol{q}$

[by Theorem 2.1.3 (vi) since $h(h^{-1}(X) \cup \{\boldsymbol{p}\}) = X \cup \{\boldsymbol{a}\}$, by surjectivity, and $h\boldsymbol{q} = \boldsymbol{b}$]

iff $h^{-1}(F_{\mathcal{A}}) \vee \mathrm{Cn}_S(h^{-1}(X) \cup \{\boldsymbol{p}\}) \supseteq h^{-1}(F_{\mathcal{A}}) \vee \mathrm{Cn}_S\boldsymbol{q}$ (in $\mathbf{Th}\mathcal{S}$)

[by Proposition 1.6.5 (ii)]

iff $\boldsymbol{q} \in \mathrm{Cn}_S(h^{-1}(F_{\mathcal{A}}) \cup h^{-1}(X) \cup \{\boldsymbol{p}\})$

iff $h^{-1}(F_{\mathcal{A}}) \cup h^{-1}(X), \boldsymbol{p} \vdash_S \boldsymbol{q}$

iff $h^{-1}(F_{\mathcal{A}}) \cup h^{-1}(X) \vdash_S E(\boldsymbol{p}, \boldsymbol{q})$ [by the DDT]

iff $E(\boldsymbol{p}, \boldsymbol{q}) \subseteq \mathrm{Cn}_S(h^{-1}(F_{\mathcal{A}}) \cup h^{-1}(X)) = h^{-1}(F_{\mathcal{A}}) \vee \mathrm{Cn}_S(h^{-1}(X))$ (in $\mathbf{Th}\mathcal{S}$)

iff $E(\boldsymbol{p}, \boldsymbol{q}) \subseteq \mathrm{Fg}^S_{\mathfrak{C}}h^{-1}(X)$ [by Proposition 1.6.5 (ii)]

iff $E(\boldsymbol{p}, \boldsymbol{q}) \subseteq h^{-1}(\mathrm{Fg}^S_{\mathcal{A}}X)$

[by Theorem 2.1.3 (vi), since $h(h^{-1}(X)) = X$, by surjectivity]

iff $h(E(\boldsymbol{p}, \boldsymbol{q})) \subseteq \mathrm{Fg}^S_{\mathcal{A}}X$

iff $\mathrm{Fg}^S_{\mathcal{A}}X \supseteq E^{\mathbf{A}}(\boldsymbol{a}, \boldsymbol{b})$.

From this we can see that the claim holds for $\mathcal{A}$. Now, let $\mathcal{A}$ be an arbitrary S-matrix and suppose that the claim fails, i.e., there exists a finite set $X \subseteq A^k$ such that

$$\mathrm{Fg}^S_{\mathcal{A}}\boldsymbol{a} \vee \mathrm{Fg}^S_{\mathcal{A}}X \supseteq \mathrm{Fg}^S_{\mathcal{A}}\boldsymbol{b}, \text{ but } \mathrm{Fg}^S_{\mathcal{A}}X \not\supseteq E^{\mathbf{A}}(\boldsymbol{a}, \boldsymbol{b}).$$

Let $\mathbf{A}'$ be the subalgebra of $\mathbf{A}$ generated by the set of all co-ordinates of $X \cup \{\boldsymbol{a}, \boldsymbol{b}\}$ and let $\mathcal{A}' = \langle \mathbf{A}', (A')^k \cap F_{\mathcal{A}} \rangle$, so that $\mathcal{A}'$ is a submatrix of $\mathcal{A}$. By Corollary 4.1.11, $\mathcal{A}$ has the FEP so, by the Lemma 4.1.6, $\mathrm{Fg}^S_{\mathcal{A}'}Y = (A')^k \cap \mathrm{Fg}^S_{\mathcal{A}}Y$ for any $Y \subseteq (A')^k$. We therefore have

$$\mathrm{Fg}^S_{\mathcal{A}'}\boldsymbol{a} \vee \mathrm{Fg}^S_{\mathcal{A}'}X = \mathrm{Fg}^S_{\mathcal{A}'}(\{\boldsymbol{a}\} \cup X) = (A')^k \cap \mathrm{Fg}^S_{\mathcal{A}}(\{\boldsymbol{a}\} \cup X) = (A')^k \cap (\mathrm{Fg}^S_{\mathcal{A}}\boldsymbol{a} \vee \mathrm{Fg}^S_{\mathcal{A}}X) \supseteq$$

$(A')^k \cap \mathrm{Fg}^S_{\mathcal{A}}\boldsymbol{b} = \mathrm{Fg}^S_{\mathcal{A}'}\boldsymbol{b}$ and

$$E^{\mathbf{A}'}(\boldsymbol{a}, \boldsymbol{b}) = E^{\mathbf{A}}(\boldsymbol{a}, \boldsymbol{b}) \not\subseteq (A')^k \cap \mathrm{Fg}^S_{\mathcal{A}}X = \mathrm{Fg}^S_{\mathcal{A}'}X.$$

But $\mathbf{A}'$ is countably (in fact finitely) generated so this contradicts the result just obtained. Our claim is therefore true.

(ii)⇒(iii) is trivial, since $\mathbf{Tf}\mathcal{S} = \mathbf{Fc}^S\mathcal{A}$, where $\mathcal{A} = \langle \mathbf{Fm}, \mathrm{Cn}_S(\emptyset) \rangle$.

(iii)⇒(i) Choose $\boldsymbol{p} = \langle p_1, \ldots, p_k \rangle$ and $\boldsymbol{q} = \langle q_1, \ldots, q_k \rangle$ such that the variables $p_1, \ldots, p_k, q_1, \ldots, q_k$ are all distinct. Since $\mathbf{Tf}\,S$ is dually Brouwerian, $\mathrm{Cn}_S \boldsymbol{p} *^{\mathbf{Tf}\,S} \mathrm{Cn}_S \boldsymbol{q}$ exists, say

$$\mathrm{Cn}_S \boldsymbol{p} *^{\mathbf{Tf}S} \mathrm{Cn}_S \boldsymbol{q} = \mathrm{Cn}_S(\{\boldsymbol{\eta}'_i(\boldsymbol{p}, \boldsymbol{q}, r_1, \ldots, r_n)\,;\, i \le m\})$$

(where the r_i's are distinct from all p_j's and all q_ℓ's). Set $\boldsymbol{\eta}_i(\boldsymbol{p}, \boldsymbol{q}) = \boldsymbol{\eta}'_i(\boldsymbol{p}, \boldsymbol{q}, p_1, \ldots, p_1)$ for each $i \le m$.

Claim: For all $\Gamma \subseteq Fm^k$ and $\boldsymbol{\varphi}, \boldsymbol{\psi} \in Fm^k$,

$$\Gamma, \boldsymbol{\varphi} \vdash_S \boldsymbol{\psi} \quad \text{if and only if} \quad \Gamma \vdash_S \boldsymbol{\eta}_i(\boldsymbol{\varphi}, \boldsymbol{\psi}) \ \text{ for all } i \le m.$$

Proof. Let $h: \mathbf{Fm} \to \mathbf{Fm}$ be any surjective homomorphism such that $h\boldsymbol{p} = \boldsymbol{\varphi}$, $h\boldsymbol{q} = \boldsymbol{\psi}$ and $hr_i = \varphi_1$ for each $i \le n$. Such an h exists, by choice of $\boldsymbol{p}, \boldsymbol{q}, r_1, \ldots, r_n$. Let $\mathcal{A} = \langle \mathbf{Fm}, h^{-1}(\mathrm{Cn}_S(\Gamma)) \rangle$ and $\mathfrak{B} = \langle \mathbf{Fm}, \mathrm{Cn}_S(\Gamma) \rangle$. Then $h: \mathcal{A} \to \mathfrak{B}$ is a reductive matrix homomorphism. Now

$\Gamma, \boldsymbol{\varphi} \vdash_S \boldsymbol{\psi}$	iff	$\boldsymbol{\psi} \in \mathrm{Fg}^S_{\mathfrak{B}} \boldsymbol{\varphi}$	[by Proposition 1.6.5 (iii)]
	iff	$\boldsymbol{q} \in h^{-1}(\mathrm{Fg}^S_{\mathfrak{B}} \boldsymbol{\varphi}) = \mathrm{Fg}^S_{\mathcal{A}} \boldsymbol{p}$	[by Corollary 2.1.4]
	iff	$\mathrm{Cn}_S \boldsymbol{q} \subseteq \mathrm{Cn}_S \boldsymbol{p} \vee h^{-1}(\mathrm{Cn}_S(\Gamma))$ (in $\mathbf{Th}S$)	[by Prop. 1.6.5 (ii)]
	iff	$h^{-1}(\mathrm{Cn}_S(\Gamma)) \supseteq \mathrm{Cn}_S \boldsymbol{p} *^{\mathbf{Tf}S} \mathrm{Cn}_S \boldsymbol{q}$	[by definition]
	iff	$h^{-1}(\mathrm{Cn}_S(\Gamma)) \supseteq \{\boldsymbol{\eta}'_i(\boldsymbol{p}, \boldsymbol{q}, r_1, \ldots, r_n)\,;\, i \le m\}$	
	iff	$\mathrm{Cn}_S(\Gamma) \supseteq \{\boldsymbol{\eta}'_i(\boldsymbol{\varphi}, \boldsymbol{\psi}, \varphi_1, \ldots, \varphi_1)\,;\, i \le m\}$	
	iff	$\Gamma \vdash_S \boldsymbol{\eta}_i(\boldsymbol{\varphi}, \boldsymbol{\psi})$ for all $i \le m$.	

This proves the Claim. If we set $E = E(\boldsymbol{p}, \boldsymbol{q}) = \{\boldsymbol{\eta}_i(\boldsymbol{p}, \boldsymbol{q})\,;\, i \le m\}$, then E forms a deduction-detachment set for S, hence S has the DDT. □

We now collect the above results in one convenient theorem.

4.4.6 THEOREM [BP89b, Theorem 7.7]

Let S be a deductive system. The following are equivalent:

(i) *S has the DDT,*

(ii) Mod$S$ *has FDPF,*

(iii) Mod*S *has FDPF and is closed under subdirect products,*

(iv) *S is protoalgebraic and for all $\mathcal{A} \in$* Mod$S$, $\mathbf{Fc}^S\mathcal{A}$ *is dually Brouwerian,*

(v) *S is protoalgebraic and for all $\mathcal{A} \in \mathrm{Mod}^*S$, $\mathbf{Fc}^S\mathcal{A}$ is dually Brouwerian,*

(vi) *S is protoalgebraic and $\mathbf{Tf}S$ is dually Brouwerian.*

Proof. That (i), (ii), (iii), (iv) and (vi) are equivalent follows from Theorems 4.4.4 and 4.4.5. It is clear that (iv) implies (v), hence we need only show that (v) implies (iv). Let $\mathcal{A} = \langle \mathbf{A}, F_{\mathcal{A}} \rangle \in \mathrm{Mod}S$. Recall from the remarks following Corollary 2.1.5 that $\mathbf{Fi}^S\mathcal{A} \cong \mathbf{Fi}^S\mathcal{A}^*$, with $\mathcal{A}^* = \langle \mathbf{A}/\Omega_{\mathcal{A}}F_{\mathcal{A}}, F_{\mathcal{A}}/\Omega_{\mathcal{A}}F_{\mathcal{A}} \rangle$, since S is protoalgebraic. Since $\mathcal{A}^* \in \mathrm{Mod}^*S$, $\mathbf{Fc}^S\mathcal{A}^*$ is dually Brouwerian by the assumption (v), hence $\mathbf{Fc}^S\mathcal{A}$ is dually Brouwerian as well. □

Quasivarieties.

The conditions (iii) and (v) of Theorem 4.4.6 have algebraic interpretations when the deductive system in question is the 2-deductive system $S_{\mathfrak{K}}$, where $\mathfrak{K}$ is a quasivariety. For, as noted before, the class $\mathrm{Mod}^*S_{\mathfrak{K}}$ is essentially $\mathfrak{K}$ and is closed under subdirect products.

4.4.7 DEFINITION

A quasivariety $\mathfrak{K}$ has *equationally definable principal relative congruences* (*EDPRC*, for short) if there exists a finite set $E(p_1, p_2, q_1, q_2)$ of equations in four variables

$$\eta_{i1}(p_1, p_2, q_1, q_2) \approx \eta_{i2}(p_1, p_2, q_1, q_2),$$

where $i \leq m$, such that, for all $\mathbf{A} \in \mathfrak{K}$ and all $a, b, c, d \in A$

$$(c, d) \in \Theta^{\mathbf{A}}_{\mathfrak{K}}(a, b) \text{ if and only if } \eta^{\mathbf{A}}_{i1}(a, b, c, d) = \eta^{\mathbf{A}}_{i2}(a, b, c, d) \text{ for all } i \leq m.$$

Then $E(p_1, p_2, q_1, q_2)$ is called a *set of defining equations for the principal relative congruences of* $\mathfrak{K}$. We drop the adjective "relative" and use the abbreviation EDPC in the case where $\mathfrak{K}$ is a variety.

In the section on quasivarieties in Section 4.1, it was shown that a quasivariety $\mathfrak{K}$ has LEDPRC with respect to $\mathcal{E}(\boldsymbol{p}, \boldsymbol{q})$ if and only if $\mathrm{Mod}^*S_{\mathfrak{K}}$ has LFDPF with defining system $\mathcal{E}(\boldsymbol{p}, \boldsymbol{q})$. As a corollary to that observation, we have that $\mathfrak{K}$ has EDPRC with defining equations $E(\boldsymbol{p}, \boldsymbol{q})$ if and only if $\mathrm{Mod}^*S_{\mathfrak{K}}$ has FDPF with set of defining formulas $E(\boldsymbol{p}, \boldsymbol{q})$. Thus we have the following corollary to Theorem 4.4.6.

4.4.8 COROLLARY [BP89b, Corollary 7.8]

Let $\mathcal{K}$ be a quasivariety. Then $\mathcal{K}$ has EDPRC if and only if the join-semilattice of compact $\mathcal{K}$-congruences of any algebra $\mathbf{A} \in \mathcal{K}$ is dually Brouwerian.

Proof. By the preceding remarks, $\mathcal{K}$ has EDPRC if and only if $\mathrm{Mod}^*S_{\mathcal{K}}$ has FDPF, and, by Theorem 4.4.6, this is equivalent to the condition that $\mathbf{Fc}^S\mathcal{A}$ be dually Brouwerian for any $\mathcal{A} \in \mathrm{Mod}^*S_{\mathcal{K}}$. If $\mathcal{A} = \langle \mathbf{A}, I_A \rangle \in \mathrm{Mod}^*S_{\mathcal{K}}$, then the compact filters of $\mathcal{A}$ are precisely the compact $\mathcal{K}$-congruences of $\mathbf{A}$, whence the result follows. □

This generalizes Köhler and Pigozzi's result [KP80] that a variety $\mathcal{K}$ has EDPC if and only if every member of $\mathcal{K}$ has a dually Brouwerian join-semilattice of compact congruences.

4.5 DDT AND FILTER-DISTRIBUTIVITY

4.5.1 COROLLARY [BP89b, Corollary 7.9]

Let S be a k-deductive system. If S has the DDT then S is filter-distributive.

Proof. If S has the DDT with deduction-detachment set E say, then S has the LDDT with deduction-detachment system $\mathcal{E} = \{E\}$, and S is protoalgebraic by Theorem 4.4.2. Since $\mathcal{E}$ is, trivially, S-directed, it follows from Theorem 4.2.2 that S is filter-distributive. □

If a k-deductive system has the DDT, then it has the LDDT with a finite local deduction-detachment system. In the next theorem we show that protoalgebraic k-deductive systems that have the DDT are characterized by the properties of having the LDDT with a finite local-deduction-detachment system and filter-distributivity.

4.5.2 COROLLARY

Let S be a k-deductive system. The following are equivalent:

(i) S is protoalgebraic, filter-distributive and has the LDDT with a finite local deduction-detachment system,

(ii) S has the DDT.

Proof. (i)⇒(ii) Suppose S is protoalgebraic, filter-distributive and has the LDDT with finite local deduction-detachment system $\mathcal{E}$. Since S is filter-distributive, Theorem 4.2.2 implies that $\mathbf{\mathcal{E}}$ is S-directed. Thus, from the finiteness of $\mathbf{\mathcal{E}}$, we can deduce that there exists $E \in \mathbf{\mathcal{E}}$ such that $E' \vdash_S E$ for all $E' \in \mathbf{\mathcal{E}}$. For $\Gamma \subseteq Fm^k$ and $\varphi, \psi \in Fm^k$,

$$\Gamma, \varphi \vdash_S \psi \quad \text{if and only if} \quad \Gamma \vdash_S E'(\varphi, \psi) \quad \text{for some } E' \in \mathbf{\mathcal{E}},$$

thus

$$\Gamma, \varphi \vdash_S \psi \quad \text{if and only if} \quad \Gamma \vdash_S E(\varphi, \psi),$$

hence $\{E(\boldsymbol{p}, \boldsymbol{q})\}$ is a local deduction-detachment system for S, so S has the DDT.

(ii)⇒(i) This follows from the previous corollary and Theorem 4.4.2. □

Let S be a k-deductive system over the language $\mathcal{L}$. Recall from Section 1.10 the first-order language $\mathcal{L}_D$ (without equality) whose extra-logical symbols are the primitive connectives of $\mathcal{L}$, now thought of as operation symbols of the appropriate rank, together with a single k-ary predicate symbol D. We showed there that the models for the first-order theory $T(S)$ over the language $\mathcal{L}_D$ are precisely the matrix models of S. The only atomic formulas that occur are formulas of the form $D(\varphi_1(\overline{p}), \ldots, \varphi_k(\overline{p}))$, where $\overline{p}$ abbreviates some $p_1, \ldots, p_m \in P$ and $\varphi_1(\overline{p}), \ldots, \varphi_k(\overline{p}) \in Fm_{\mathcal{L}}$. Let $\phi(\overline{p}) = D(\varphi_1(\overline{p}), \ldots, \varphi_k(\overline{p}))$ be an atomic formula. For an S-matrix $\mathcal{A} = \langle \mathbf{A}, F_{\mathcal{A}} \rangle$, i.e., a first-order structure $\langle A\,; \mathcal{L}^{\mathbf{A}}, D^{\mathbf{A}} \rangle$ with $D^{\mathbf{A}} = F_{\mathcal{A}}$, and for all $\overline{a} \in A^m$, we define

$$\models_{\mathcal{A}} \phi[\overline{a}] \quad \text{iff} \quad \langle \varphi_1^{\mathbf{A}}(\overline{a}), \ldots, \varphi_k^{\mathbf{A}}(\overline{a}) \rangle \in F_{\mathcal{A}}.$$

(In fact, this is a natural extension of the convention established in Section 0.5 for the truth of formulas in the language $\mathcal{L}_A$ got by adding to $\mathcal{L}$ a new constant symbol for each $a \in A$.) As in Section 0.5, this induces in the standard way, a definition of $\models_{\mathcal{A}} \psi[\overline{a}]$ for *any* formula $\psi(\overline{p})$ of $\mathcal{L}_D$. Now, for formulas $\gamma(\overline{p})$ $(\gamma \in \Gamma)$ and $\psi(\overline{p})$ of $\mathcal{L}_D$, define $\Gamma[\boldsymbol{a}, \boldsymbol{b}] \models_{\mathcal{A}} \phi[\boldsymbol{a}, \boldsymbol{b}]$ to mean that $\models_{\mathcal{A}} \phi[\boldsymbol{a}, \boldsymbol{b}]$ whenever $\models_{\mathcal{A}} \gamma[\boldsymbol{a}, \boldsymbol{b}]$ for all $\gamma \in \Gamma$.

4.5.3 DEFINITION

Let S be a k-deductive system and M a class of S-matrices. We say that M has *definable principal filters* (*DPF*, for short) if there exists a first-order formula

$$\alpha(p_1, \ldots, p_k, q_1, \ldots, q_k)$$

(without equality) over the language $\mathcal{L}_D$ whose free variables are among $p_1, \ldots, p_k, q_1, \ldots, q_k$ such that for every $\mathcal{A} = \langle \mathbf{A}, F_{\mathcal{A}} \rangle \in M$ and $\boldsymbol{a}, \boldsymbol{b} \in A^k$ we have

$$\boldsymbol{b} \in \mathrm{Fg}^S_{\mathcal{A}} \boldsymbol{a} \quad \text{if and only if} \quad \models_{\mathcal{A}} \alpha[\boldsymbol{a}, \boldsymbol{b}].$$

4.5.4 LEMMA [BP88, Lemma 4.5]

Let S be a protoalgebraic k-deductive system. Then Mod$S$ *has DPF if and only if* Mod$^*S$ *has DPF.*

Proof. ($\Rightarrow$) Trivial.

($\Leftarrow$) Assume Mod$^*S$ has DPF by means of the first order formula $\alpha(\boldsymbol{p},\boldsymbol{q})$, over the language $\mathcal{L}_D$, where $\boldsymbol{p}=\langle p_1,\dots,p_k\rangle$ and $\boldsymbol{q}=\langle q_1,\dots,q_k\rangle$. Let $\mathcal{A}=\langle \mathbf{A},F_{\mathcal{A}}\rangle$ be an S-matrix, let $\mathcal{A}^*=\langle \mathbf{A}/\Omega_{\mathcal{A}}F_{\mathcal{A}},F_{\mathcal{A}}/\Omega_{\mathcal{A}}F_{\mathcal{A}}\rangle$ be the matrix defined in Section 1.8 and let $h:\mathcal{A}\to\mathcal{A}^*$ be the canonical homomorphism. By Corollary 2.1.4, since h is reductive, we have

$$\boldsymbol{b}\in \mathrm{Fg}^S_{\mathcal{A}}\boldsymbol{a} \quad\text{if and only if}\quad h\boldsymbol{b}\in \mathrm{Fg}^S_{\mathcal{A}^*}h\boldsymbol{a}.$$

Let $\psi(r_1,\dots,r_\ell)$ be a first-order formula over the language $\mathcal{L}_D$. We claim that for every S-matrix $\mathcal{A}=\langle \mathbf{A},F_{\mathcal{A}}\rangle$ and any $c_1,\dots,c_\ell\in A$,

$$\models_{\mathcal{A}}\psi[c_1,\dots,c_\ell] \quad\text{if and only if}\quad \models_{\mathcal{A}^*}\psi[hc_1,\dots,hc_\ell].$$

We prove this by induction on the complexity of formulas. If $\psi(\overline{r})=D(\varphi_1(\overline{r}),\dots,\varphi_k(\overline{r}))$, then

$$\begin{aligned}
\models_{\mathcal{A}}\psi[\overline{c}] \quad &\text{iff}\quad \langle\varphi_1^{\mathbf{A}}(\overline{c}),\dots,\varphi_k^{\mathbf{A}}(\overline{c})\rangle\in F_{\mathcal{A}}\\
&\text{iff}\quad \langle h\varphi_1^{\mathbf{A}}(\overline{c}),\dots,h\varphi_k^{\mathbf{A}}(\overline{c})\rangle=\langle\varphi_1^{\mathbf{A}}(h\overline{c}),\dots,\varphi_k^{\mathbf{A}}(h\overline{c})\rangle\in h(F_{\mathcal{A}})\\
&\qquad\text{[by Theorem 2.1.3 (vi), since } h \text{ is reductive]}\\
&\text{iff}\quad \models_{\mathcal{A}^*}\psi[h\overline{c}] \qquad\qquad \text{[since } h(F_{\mathcal{A}})=F_{\mathcal{A}}/\Omega_{\mathcal{A}}F_{\mathcal{A}}\text{]}.
\end{aligned}$$

If $\psi(\overline{r})$ is $\sim\vartheta(\overline{r})$, then

$$\begin{aligned}
\models_{\mathcal{A}}\psi[\overline{c}] \quad &\text{iff}\quad \not\models_{\mathcal{A}}\vartheta[\overline{c}]\\
&\text{iff}\quad \not\models_{\mathcal{A}^*}\vartheta[h\overline{c}] \qquad\qquad \text{[by the induction hypothesis]}\\
&\text{iff}\quad \models_{\mathcal{A}^*}\psi[h\overline{c}].
\end{aligned}$$

If $\psi(\overline{r})$ is $\vartheta_1(\overline{r})\Rightarrow\vartheta_2(\overline{r})$, then

$$\begin{aligned}
\models_{\mathcal{A}}\psi[\overline{c}] \quad &\text{iff}\quad \not\models_{\mathcal{A}}\vartheta_1[\overline{c}] \text{ or } \models_{\mathcal{A}}\vartheta_2[\overline{c}]\\
&\text{iff}\quad \not\models_{\mathcal{A}^*}\vartheta_1[h\overline{c}] \text{ or } \models_{\mathcal{A}^*}\vartheta_2[h\overline{c}] \qquad \text{[by the induction hypothesis]}\\
&\text{iff}\quad \models_{\mathcal{A}^*}\psi[h\overline{c}].
\end{aligned}$$

Suppose $\psi(\overline{r})$ is $\forall s\,\eta(s,\overline{r})$ (s a variable).

$$\begin{aligned}
\models_{\mathcal{A}}\psi[\overline{c}] \quad &\text{iff}\quad \models_{\mathcal{A}}\forall s\,\eta(s,\overline{c})\\
&\text{iff}\quad \text{for any } d\in A,\ \models_{\mathcal{A}}\eta(d,\overline{c})
\end{aligned}$$

iff for any $d \in A$, $\models_{\mathcal{A}^*} \eta(hd, h\bar{c})$ [by the induction hypothesis]

iff for any $e \in A/\Omega_{\mathcal{A}} F_{\mathcal{A}}$, $\models_{\mathcal{A}^*} \eta(e, h\bar{c})$ [since h is surjective]

iff $\models_{\mathcal{A}^*} \forall s\, \eta(s, h\bar{c})$

iff $\models_{\mathcal{A}^*} \psi[h\bar{c}]$.

This establishes the above claim. Thus

$$\begin{aligned} \boldsymbol{b} \in \mathrm{Fg}^S_{\mathcal{A}} \boldsymbol{a} \quad & \text{iff} \quad h\boldsymbol{b} \in \mathrm{Fg}^S_{\mathcal{A}^*} h\boldsymbol{a} \\ & \text{iff} \quad \models_{\mathcal{A}^*} \alpha[h\boldsymbol{a}, h\boldsymbol{b}] \\ & \text{iff} \quad \models_{\mathcal{A}} \alpha[\boldsymbol{a}, \boldsymbol{b}], \end{aligned}$$

hence Mod$S$ has DPF with respect to the formula $\alpha(\boldsymbol{p}, \boldsymbol{q})$. □

We can now characterize model-theoretically those classes of the form Mod$S$ and Mod*S that have FDPF.

4.5.5 THEOREM [BP88, Theorem 4.6]

Let S be a k-deductive system. The following are equivalent:

(i) S has the DDT,

(ii) Mod$S$ *has FDPF,*

(iii) Mod*S *has FDPF and is closed under subdirect products,*

(iv) Mod*S *has the FEP, DPF and is filter-distributive and is closed under subdirect products,*

(v) S is protoalgebraic and Mod$S$ *has the FEP, DPF and is filter-distributive.*

Proof. The equivalence of conditions (i), (ii) and (iii) is proved in Theorem 4.4.4 and that each of them implies (iv) follows from Corollary 4.1.11, Lemma 4.5.4 and Corollary 4.5.1. If (iv) holds, then it follows from Theorem 2.2.3, Corollary 4.1.11 and Lemma 4.5.4 that S is protoalgebraic and Mod$S$ has the FEP and DPF. That Mod$S$ is filter-distributive follows from the fact that for a protoalgebraic k-deductive system S and an S-matrix $\mathcal{A}$, $\mathbf{Fi}^S\mathcal{A}$ is isomorphic to $\mathbf{Fi}^S\mathcal{A}^*$, hence (v) holds.

(v)⇒(i) Assume S is protoalgebraic and that Mod$S$ has the FEP, is filter-distributive and has DPF by means of the first-order formula $\alpha(\boldsymbol{p}, \boldsymbol{q})$. By Corollary 4.1.11, Mod$S$ has LFDPF, with respect to a defining system $\mathfrak{E} = \{E_i(\boldsymbol{p}, \boldsymbol{q})\,;\, i \in I\}$, say. Thus, for all $\mathcal{A} = \langle \mathbf{A}, F_{\mathcal{A}} \rangle \in$ Mod$S$,

$\boldsymbol{a}, \boldsymbol{b} \in A^k$, we have

$$\boldsymbol{b} \in \mathrm{Fg}^S_{\mathcal{A}}\boldsymbol{a} \quad \text{if and only if} \quad E_i^{\mathbf{A}}(\boldsymbol{a}, \boldsymbol{b}) \subseteq F_{\mathcal{A}} \text{ for some } i \in I.$$

The right hand side can be rephrased as: For some $i \in I$,

$$\eta^{\mathbf{A}}(\boldsymbol{a}, \boldsymbol{b}) \in F_{\mathcal{A}} \text{ for all } \eta = \eta(\boldsymbol{p}, \boldsymbol{q}) \in E_i = E_i(\boldsymbol{p}, \boldsymbol{q}),$$

i.e.,
$$\models_{\mathcal{A}} D(\eta(\boldsymbol{p}, \boldsymbol{q}))[\boldsymbol{a}, \boldsymbol{b}] \quad \text{for all } \eta \in E_i$$

i.e.,
$$\models_{\mathcal{A}} \& \{D(\eta(\boldsymbol{p}, \boldsymbol{q}))\,;\, \eta \in E_i\}[\boldsymbol{a}, \boldsymbol{b}].$$

Thus we have that

$$\models_{\mathcal{A}} \alpha[\boldsymbol{a}, \boldsymbol{b}] \quad \text{if and only if} \quad \models_{\mathcal{A}} \& \{D(\eta(\boldsymbol{p}, \boldsymbol{q}))\,;\, \eta \in E_i\}[\boldsymbol{a}, \boldsymbol{b}] \quad \text{for some } i \in I.$$

For each $i \in I$, denote the first-order formula $\& \{D(\eta(\boldsymbol{p}, \boldsymbol{q}))\,;\, \eta \in E_i\}$ by $\beta_i(\boldsymbol{p}, \boldsymbol{q})$. Then we have that for each $\mathcal{A} \in \mathrm{Mod}S$ and $\boldsymbol{a}, \boldsymbol{b} \in A^k$,

(4.5.1) $$\boldsymbol{b} \in \mathrm{Fg}^S_{\mathcal{A}}\boldsymbol{a} \quad \text{if and only if for some } i \in I, \ \models_{\mathcal{A}} \beta_i[\boldsymbol{a}, \boldsymbol{b}].$$

Claim. There exists a finite $I_0 \subseteq I$ such that (4.5.1) holds with I_0 replacing I.

Proof. Suppose on the contrary that for each $J \in \mathcal{P}_\omega(I)$, there exists an $\mathcal{A}_J = \langle \mathbf{A}_J, F_{\mathcal{A}_J} \rangle \in \mathrm{Mod}S$ and $\boldsymbol{a}_J, \boldsymbol{b}_J \in (A_J)^k$ such that $\boldsymbol{b}_J \in \mathrm{Fg}^S_{\mathcal{A}_J}\boldsymbol{a}_J$ but for all $i \in J$, $\not\models_{\mathcal{A}_J} \beta_i[\boldsymbol{a}_J, \boldsymbol{b}_J]$. For all $J \in \mathcal{P}_\omega(I)$, set $\widehat{J} = \{K \in \mathcal{P}_\omega(I)\,;\, J \subseteq K\}$. It is easy to see that $\{H \subseteq \mathcal{P}_\omega(I)\,;\, H \supseteq \widehat{J}$ for some $J \in \mathcal{P}_\omega(I)\}$ is a proper filter of the Boolean algebra of subsets of $\mathcal{P}_\omega(I)$. By Theorem 0.3.3, this filter extends to an ultrafilter $\mathcal{U}$ of the Boolean algebra. Let $\mathcal{A} = \langle \mathbf{A}, F \rangle$ be the (matrix-) ultraproduct $\Pi_{J \in \mathcal{P}_\omega(I)} \mathcal{A}_J/\mathcal{U}$. Regarding $\mathrm{Mod}S$ as a universal Horn class (see Section 1.10), it follows from Theorem 0.5.2 that $\mathrm{P_U}(\mathrm{Mod}S) \subseteq \mathrm{Mod}S$, so $\mathcal{A} \in \mathrm{Mod}S$.

Set $\boldsymbol{a} = \langle \boldsymbol{a}_J\,;\, J \in \mathcal{P}_\omega(I) \rangle, \boldsymbol{b} = \langle \boldsymbol{b}_J\,;\, J \in \mathcal{P}_\omega(I) \rangle \in (\Pi_{J \in \mathcal{P}_\omega(I)} A_J)^k$. Now, $\mathrm{Mod}S$ has DPF by means of the first-order formula $\alpha(\boldsymbol{p}, \boldsymbol{q})$ and for all $J \in \mathcal{P}_\omega(I)$, $\boldsymbol{b}_J \in \mathrm{Fg}^S_{\mathcal{A}_J}\boldsymbol{a}_J$, hence $\models_{\mathcal{A}_J} \alpha[\boldsymbol{a}_J, \boldsymbol{b}_J]$ for all $J \in \mathcal{P}_\omega(I)$. Thus, by Łoś' Theorem (Theorem 0.5.1), $\models_{\mathcal{A}} \alpha[\boldsymbol{a}/\mathcal{U}, \boldsymbol{b}/\mathcal{U}]$, hence $\boldsymbol{b}/\mathcal{U} \in \mathrm{Fg}^S_{\mathcal{A}}(\boldsymbol{a}/\mathcal{U})$ since $\mathcal{A} \in \mathrm{Mod}S$. So, by (4.5.1), there exists an $i \in I$ such that $\models_{\mathcal{A}} \beta_i[\boldsymbol{a}/\mathcal{U}, \boldsymbol{b}/\mathcal{U}]$, from which it follows that

$$V = \{J \in \mathcal{P}_\omega(I)\,;\, \models_{\mathcal{A}_J} \beta_i[\boldsymbol{a}_J, \boldsymbol{b}_J]\} \in \mathcal{U}.$$

Since $\widehat{\{i\}} \in \mathcal{U}$, we have $V \cap \widehat{\{i\}} \in \mathcal{U}$, so $V \cap \widehat{\{i\}} \neq \emptyset$ since $\mathcal{U}$ is a proper filter. Let $J \in V \cap \widehat{\{i\}}$. Then $\models_{\mathcal{A}_J} \beta_i[\boldsymbol{a}_J, \boldsymbol{b}_J]$ since $J \in V$, but since $J \in \widehat{\{i\}}$, $i \in J$ and so $\not\models_{\mathcal{A}_J} \beta_i[\boldsymbol{a}_J, \boldsymbol{b}_J]$. This contradiction proves the Claim.

Let $I_0 \subseteq I$ such that for any $\mathcal{A} = \langle \mathbf{A}, F_{\mathcal{A}} \rangle \in \mathrm{Mod}S$ and any $\boldsymbol{a}, \boldsymbol{b} \in A^k$, (4.5.1) holds. Set $\mathcal{E}' = \{E_i ; i \in I_0\}$. Then, for all $\mathcal{A} = \langle \mathbf{A}, F_{\mathcal{A}} \rangle \in \mathrm{Mod}S$ and $\boldsymbol{a}, \boldsymbol{b} \in A^k$, we have

$$\begin{aligned} \boldsymbol{b} \in \mathrm{Fg}^S_{\mathcal{A}}\boldsymbol{a} \quad &\text{iff} \quad \models_{\mathcal{A}} \& \{D(\boldsymbol{\eta}(\boldsymbol{p}, \boldsymbol{q})) ; \boldsymbol{\eta} \in E_i\}[\boldsymbol{a}, \boldsymbol{b}] \quad \text{for some } i \in I_0 \\ &\text{iff} \quad \boldsymbol{\eta}(\boldsymbol{a}, \boldsymbol{b}) \in F_{\mathcal{A}} \text{ for all } \boldsymbol{\eta} \in E_i, \text{ for some } i \in I_0 \\ &\text{iff} \quad E_i^{\mathbf{A}}(\boldsymbol{a}, \boldsymbol{b}) \subseteq F_{\mathcal{A}} \text{ for some } i \in I_0. \end{aligned}$$

Thus Mod$S$ has LFDPF with defining system $\mathcal{E}'$, i.e., S has the LDDT with local deduction-detachment system $\mathcal{E}'$ (Corollary 4.1.11). Since S is filter-distributive and $\mathcal{E}'$ is finite, Corollary 4.5.2 implies that S has the DDT. □

Quasivarieties

4.5.6 DEFINITION

Let $\mathcal{K}$ be a quasivariety. We say that $\mathcal{K}$ is *relatively congruence-distributive* (RCD, for short) if, for any $\mathbf{A} \in \mathcal{K}$ the lattice $\mathbf{Con}_{\mathcal{K}}\mathbf{A}$ of $\mathcal{K}$-congruences of $\mathbf{A}$ is distributive. Thus, a variety $\mathcal{K}$ is RCD if and only if it is *congruence distributive* (*CD*, for short).

Let $\mathcal{K}$ be a quasivariety. Recall once again that $\mathrm{Mod}^*S_{\mathcal{K}} = \{\langle \mathbf{A}, I_A \rangle ; \mathbf{A} \in \mathcal{K}\}$. For $\mathcal{A} = \langle \mathbf{A}, I_A \rangle \in \mathrm{Mod}^*S_{\mathcal{K}}$ the lattice of filters $\mathbf{Fi}^{S_{\mathcal{K}}}\mathcal{A}$ is precisely the lattice $\mathbf{Con}_{\mathcal{K}}\mathbf{A}$ of $\mathcal{K}$-congruences of $\mathbf{A}$ (see Sections 1.7 and 1.8). Thus,

$$\mathcal{K} \text{ is RCD} \quad \text{if and only if} \quad \mathrm{Mod}^*S_{\mathcal{K}} \text{ is filter-distributive.}$$

Define $\mathcal{L}_{\approx}$ to be the first-order language with equality whose only non-logical symbols are the connectives of $\mathcal{L}$, interpreted as operation symbols (and the binary predicate symbol $\approx$).

4.5.7 DEFINITION

A quasivariety $\mathcal{K}$ has *definable principal relative congruences* (*DPRC*, for short), if there exists a formula $\phi(p_1, p_2, q_1, q_2)$ in the first-order language with equality $\mathcal{L}_{\approx}$, such that for all $\mathbf{A} \in \mathcal{K}$ and all $a, b, c, d \in A$,

$$(c, d) \in \Theta^{\mathbf{A}}_{\mathcal{K}}(a, b) \quad \text{if and only if} \quad \models_{\mathbf{A}} \phi[a, b, c, d].$$

Since $S_{\mathcal{K}}$ is a 2-deductive system, the predicate symbol D is binary. Thus we can replace all occurrences of $\approx$ in $\phi(p_1,p_2,q_1,q_2)$ by D to obtain a formula $\phi'(p_1,p_2,q_1,q_2)$ over the language $\mathcal{L}_D$. Note that for an atomic formula $(\zeta\approx\eta)(\overline{p})$, where ζ,η are formulas over the language $\mathcal{L}$,

$$\models_{\mathbf{A}} (\zeta\approx\eta)[\overline{a}] \quad \text{for all } \mathbf{A}\in\mathcal{K} \text{ and all } \overline{a}\in A^m$$
$$\text{iff} \quad \models_{\mathcal{K}} (\zeta\approx\eta)(\overline{p})$$
$$\text{iff} \quad \models_{\mathcal{A}} \forall\overline{p}\ D(\zeta,\eta)(\overline{p}) \quad \text{for all } \mathcal{A}=\langle\mathbf{A},I_A\rangle\in\mathrm{Mod}^*S_{\mathcal{K}}.$$

A straightforward inductive argument will show that this result extends to all first-order formulas over the language $\mathcal{L}_{\approx}$. Noting that $\Theta^{\mathbf{A}}_{\mathcal{K}}(a,b)=\mathrm{Fg}^{S_{\mathcal{K}}}_{\mathcal{A}}(a,b)$, one can see that $\mathcal{K}$ has DPRC by means of $\phi(p_1,p_2,q_1,q_2)$ if and only if $\mathrm{Mod}^*S_{\mathcal{K}}$ has DPF by means of the $\mathcal{L}_D$-formula $\phi'(p_1,p_2,q_1,q_2)$.

4.5.8 COROLLARY [BP88, Corollary 4.7 and Corollary 4.8]

Let $\mathcal{K}$ be a (quasi)variety. The following are equivalent:

(i) $\mathcal{K}$ has EDP(R)C,

(ii) $\mathcal{K}$ has the (R)CEP, DP(R)C and is (R)CD,

(iii) $S_{\mathcal{K}}$ has the DDT.

Proof. We have already shown that $\mathcal{K}$ has EDPRC if and only if $\mathrm{Mod}^*S_{\mathcal{K}}$ has FDPF if and only if $S_{\mathcal{K}}$ has the DDT. Thus (i) and (iii) are equivalent. Now $\mathcal{K}$ has the RCEP if and only if $\mathrm{Mod}^*S_{\mathcal{K}}$ has the FEP. We have just shown that $\mathcal{K}$ has DPRC if and only if $\mathrm{Mod}^*S_{\mathcal{K}}$ has DPF, and $\mathcal{K}$ is RCD if and only if $\mathrm{Mod}^*S_{\mathcal{K}}$ is filter-distributive. Thus, the rest of this corollary is a special case of the equivalence of (ii) and (iv) of Theorem 4.5.5. □

4.6 EQUIVALENT DEDUCTIVE SYSTEMS AND THE DDT

In Section 4.3 it was shown that the LDDT is a property that is preserved by the relation of equivalence of deductive systems. Here, we shall see that the same is true for the DDT.

It is shown in the proof of Theorem 4.3.2 that if S_1 and S_2 are equivalent deductive systems and S_1 has the LDDT with local deduction-detachment system $\mathcal{E}(\boldsymbol{p},\boldsymbol{q})=\{E_i(\boldsymbol{p},\boldsymbol{q})\,;\,i\in I\}$ indexed by a set I, then S_2 has the LDDT with local deduction-detachment system $\mathcal{E}'(\boldsymbol{p},\boldsymbol{q})=\{E'_{\boldsymbol{i}}(\boldsymbol{p},\boldsymbol{q})\,;\,\boldsymbol{i}\in\left(I^{(m)}\right)^m\}$ for some integer m. In defining $\mathcal{E}^{(n)}$ for $n\geq 1$, in Section 4.3, we remarked that if the set I contains just one element the so do $I^{(n)}$ and $\mathcal{E}^{(n)}$. It follows then that if S_1 has a local deduction-detachment system containing just one element then so does S_2. The next theorem follows immediately.

4.6.1 THEOREM [BP89b, Theorem 5.2]

Let S_1 and S_2 be equivalent deductive systems. Then S_1 has the DDT if and only if S_2 has the DDT.

4.6.2 COROLLARY [BP89b, Corollary 5.3]

Let S be an algebraizable k-deductive system with equivalent quasivariety semantics $\mathcal{K}$. Then S has the DDT if and only if $\mathcal{K}$ has EDPRC. In particular, if S is a strongly algebraizable k-deductive system with equivalent variety semantics $\mathcal{K}$, then S has the DDT if and only if $\mathcal{K}$ has EDPC.

Proof. By Theorem 3.1.11, S is equivalent to $S_{\mathcal{K}}$, hence, by the theorem just proved, S has the DDT if and only if $S_{\mathcal{K}}$ has the DDT. By Theorem 4.4.4, $S_{\mathcal{K}}$ has the DDT if and only if $\text{Mod}^*S_{\mathcal{K}}$ has FDPF, and, by the observation following Definition 4.4.7, this holds if and only if $\mathcal{K}$ has EDPRC. □

4.7 EXAMPLES

Classical and Intuitionistic Propositional Calculi.

It is well-known that the varieties $\mathcal{BA}$ and $\mathcal{HA}$ of all Boolean algebras and all Heyting algebras (respectively) have EDPC. Since $\mathcal{BA}$ and $\mathcal{HA}$ are the equivalent variety semantics of the algebraizable deductive systems **CPC** and **IPC** (respectively), the well-known facts that **CPC** and **IPC** have the DDT (more precisely,

$$\Gamma, \varphi \vdash_S \psi \quad \text{if and only if} \quad \Gamma \vdash_S \varphi \to \psi$$

for $\Gamma \subseteq Fm$, $\varphi, \psi \in Fm$ and $S = \mathbf{CPC}$ or $S = \mathbf{IPC}$) are special cases of Corollary 4.6.2. In Chapter 5, we shall also see that they are derivable from the fact that certain strongly algebraizable extensions of the logic **BCK** have the DDT. The logic **BCK** itself is an example of an algebraizable deductive system which has the LDDT but does not have the DDT. This will be proved in Chapter 5.

Normal Modal Logics.

Let $\mathcal{L}$ be the language $\{\wedge, \vee, \rightarrow, \neg, \Box, 0, 1\}$ of the variety $\mathcal{MA}$ of modal algebras. Define $\Box^0 x = x$ and $\Box^{n+1} x = \Box(\Box^n x)$ for each $n \in \omega$. Now define $\boxed{0}x = x$ and $\boxed{n+1}x = \boxed{n}x \wedge \Box^{n+1}x$ for each $n \in \omega$. Thus, for $n \in \omega$,

$$\mathcal{MA} \models \boxed{n}x \approx x \wedge \Box x \wedge \Box^2 x \wedge \ldots \wedge \Box^n x.$$

Evidently, if $n \geq m$, then $\mathcal{MA} \models \boxed{n}x \leq \boxed{m}x$.

4.7.1 LEMMA

Let $\mathbf{A}$ *be a modal algebra (see Section 0.2). Then for all* $a, b, c, d \in A$,

$$(c, d) \in \Theta^{\mathbf{A}}(a, b) \quad \textit{iff} \quad c \leftrightarrow d \geq \boxed{n}(a \leftrightarrow b) \quad \textit{for some } n \in \omega.$$

Proof. The following first-order sentence is true in all Boolean algebras since it is equivalent to the conjunction of two quasi-identities (which can be easily checked in the Boolean algebra **2** and **2** generates $\mathcal{BA}$ as a quasivariety):

(4.7.1) $$x \wedge z \approx y \wedge z \Leftrightarrow (x \leftrightarrow y) \geq z.$$

Thus (4.7.1) is also true in every modal algebra. Let $\mathbf{A} \in \mathcal{MA}$ and let $a, b \in A$. Set

(4.7.2) $$\Phi = \{(c, d) \in A^2 ; c \leftrightarrow d \geq \boxed{n}(a \leftrightarrow b) \ \text{ for some } n \in \omega\}.$$

By (4.7.1), $$\Phi = \{(c, d) \in A^2 ; c \wedge \boxed{n}(a \leftrightarrow b) = d \wedge \boxed{n}(a \leftrightarrow b) \ \text{ for some } n \in \omega\}.$$

We claim that Φ is a congruence on $\mathbf{A}$. Let

$$F = \{e \in A ; e \geq \boxed{n}(a \leftrightarrow b) \text{ for some } n \in \omega\}.$$

Since $e \geq \boxed{n}(a \leftrightarrow b)$ and $f \geq \boxed{m}(a \leftrightarrow b)$ imply $e \wedge f \geq (\boxed{n}(a \leftrightarrow b)) \wedge (\boxed{m}(a \leftrightarrow b)) = \boxed{k}(a \leftrightarrow b)$, where $k = \max\{n, m\}$, it follows that F is a filter of the lattice reduct $\langle A ; \wedge, \vee \rangle$ of $\mathbf{A}$. By Theorem 0.3.1, Φ is a congruence of the Boolean reduct $\langle A ; \wedge, \vee, \neg, 0, 1 \rangle$ of $\mathbf{A}$ (recall that $x \leftrightarrow y = ((\neg x) \vee y) \wedge ((\neg y) \vee x)$). Since $\mathcal{MA} \models x \rightarrow y \approx (\neg x) \vee y$, Φ is also compatible with $\rightarrow$. Consider the connective $\Box$: Let $(c, d) \in \Phi$ and suppose $c \wedge \boxed{n}(a \leftrightarrow b) = d \wedge \boxed{n}(a \leftrightarrow b)$ where $n \in \omega$. Then

$$\begin{aligned}
\Box c \wedge \boxed{n+1}(a \leftrightarrow b) &= \Box c \wedge (a \leftrightarrow b) \wedge \Box(a \leftrightarrow b) \wedge \Box^2(a \leftrightarrow b) \wedge \ldots \wedge \Box^{n+1}(a \leftrightarrow b) \\
&= (a \leftrightarrow b) \wedge \Box(c \wedge (a \leftrightarrow b) \wedge \Box(a \leftrightarrow b) \wedge \Box^2(a \leftrightarrow b) \wedge \ldots \wedge \Box^n(a \leftrightarrow b)) \\
&\quad \text{[by (M4) of Section 0.2]} \\
&= (a \leftrightarrow b) \wedge \Box(c \wedge \boxed{n}(a \leftrightarrow b)) \\
&= (a \leftrightarrow b) \wedge \Box(d \wedge \boxed{n}(a \leftrightarrow b))
\end{aligned}$$

$$= (a \leftrightarrow b) \wedge \Box(d \wedge (a \leftrightarrow b) \wedge \Box(a \leftrightarrow b) \wedge \Box^2(a \leftrightarrow b) \wedge \ldots \wedge \Box^n(a \leftrightarrow b))$$

$$= \Box d \wedge (a \leftrightarrow b) \wedge \Box(a \leftrightarrow b) \wedge \Box^2(a \leftrightarrow b) \wedge \ldots \wedge \Box^{n+1}(a \leftrightarrow b) \qquad \text{[by (M4)]}$$

$$= \Box d \wedge \boxed{n+1}(a \leftrightarrow b),$$

hence $(\Box c, \Box d) \in \Phi$. Thus Φ is a congruence on $\mathbf{A}$. Clearly $(a,b) \in \Phi$, so $\Theta^{\mathbf{A}}(a,b) \subseteq \Phi$.

Conversely, let $\Psi \in \operatorname{Con}\mathbf{A}$ such that $(a,b) \in \Psi$. We shall show that $\Phi \subseteq \Psi$. Take $(x,y) \in \Phi$ and suppose that $x \wedge \boxed{n}(a \leftrightarrow b) = y \wedge \boxed{n}(a \leftrightarrow b)$ where $n \in \omega$. Note that $(\boxed{n}(a \leftrightarrow b), 1) \overset{(M3)}{=} (\boxed{n}(a \leftrightarrow b), \boxed{n}1) = (\boxed{n}(a \leftrightarrow b), \boxed{n}(a \leftrightarrow a)) \in \Psi$, hence

$$(x \vee (y \wedge \boxed{n}(a \leftrightarrow b)), x \vee (y \wedge 1)) \in \Psi.$$

Now, $\quad x \vee (y \wedge \boxed{n}(a \leftrightarrow b)) = x \vee (x \wedge \boxed{n}(a \leftrightarrow b)) = x \quad$ and $\quad x \vee (y \wedge 1) = x \vee y,$

hence $(x, x \vee y) \in \Psi$. By symmetry, $(y, x \vee y) \in \Psi$, so $(x,y) \in \Psi$ as required. Thus $\Phi \subseteq \Theta^{\mathbf{A}}(a,b)$, so $\Phi = \Theta^{\mathbf{A}}(a,b)$. □

4.7.2 THEOREM

The modal logic **K** *has the LDDT with local deduction-detachment system* $\{\boxed{n}p \rightarrow q\,;\, n \in \omega\}$.

Proof. Recall that $\mathcal{MA}$ is the variety of modal algebras. By Lemma 4.7.1, for each $\mathbf{A} \in \mathcal{MA}$ and $a, c \in A$, we have $(c,1) \in \Theta^{\mathbf{A}}(a,1)$ iff $c \leftrightarrow 1 \geq \boxed{n}(a \leftrightarrow 1)$ for some $n \in \omega$ iff $c \geq \boxed{n}a$ for some $n \in \omega$ (since $x \leftrightarrow 1 = (x \rightarrow 1) \wedge (1 \rightarrow x) = x$ for all $x \in A$). Now, let $\Gamma \subseteq Fm$ and $\varphi, \psi \in Fm$ such that $\Gamma, \varphi \vdash_{\mathbf{K}} \psi$. Then, by Section 3.3,

$$\{\chi \approx 1\,;\, \chi \in \Gamma\} \cup \{\varphi \approx 1\} \models_{\mathcal{MA}} \psi \approx 1. \tag{4.7.3}$$

We shall show that $\Gamma \vdash_{\mathbf{K}} \boxed{n}\varphi \rightarrow \psi$ for some $n \in \omega$ by showing that $\{\chi \approx 1\,;\, \chi \in \Gamma\} \models_{\mathcal{MA}} \boxed{n}\varphi \rightarrow \psi \approx 1$ for some $n \in \omega$. So, let $\mathbf{A} \in \mathcal{MA}$ and let $\bar{a}$ be an interpretation of the variables of $\Gamma \cup \{\varphi, \psi\}$ in A such that $\chi^{\mathbf{A}}(\bar{a}) = 1$ for each $\chi \in \Gamma$. By (4.7.3) and Lemma 0.4.4, we have that

$$(\psi^{\mathbf{A}}(\bar{a}), 1) \in \Theta^{\mathbf{A}}(\{(\chi^{\mathbf{A}}(\bar{a}), 1)\,;\, \chi \in \Gamma\} \cup \{(\varphi^{\mathbf{A}}(\bar{a}), 1)\}),$$

i.e.,

$$(\psi^{\mathbf{A}}(\bar{a}), 1) \in \Theta^{\mathbf{A}}(\varphi^{\mathbf{A}}(\bar{a}), 1),$$

since $\chi^{\mathbf{A}}(\bar{a}) = 1$ for each $\chi \in \Gamma$. But by the previous lemma, this holds if and only if $\psi^{\mathbf{A}}(\bar{a}) = \psi^{\mathbf{A}}(\bar{a}) \leftrightarrow 1 \geq \boxed{n}(\varphi^{\mathbf{A}}(\bar{a}) \leftrightarrow 1) = \boxed{n}\varphi^{\mathbf{A}}(\bar{a})$ for some $n \in \omega$, i.e., iff $\boxed{n}\varphi^{\mathbf{A}}(\bar{a}) \rightarrow \psi^{\mathbf{A}}(\bar{a}) = 1$ for some $n \in \omega$. Thus $\{\chi \approx 1\,;\, \chi \in \Gamma\} \models_{\mathcal{MA}} \boxed{n}\varphi \rightarrow \psi \approx 1$, i.e., $\Gamma \vdash_{\mathbf{K}} \boxed{n}\varphi \rightarrow \psi$ for some $n \in \omega$.

Conversely, suppose $\Gamma \vdash_{\mathbf{K}} \boxed{n}\varphi \rightarrow \psi$ where $n \in \omega$. By (Ne), we have

$$\varphi \vdash_{\mathbf{K}} \Box\varphi,$$

so $$\varphi \vdash_{\mathbf{K}} \Box\varphi, \Box^2\varphi,$$

$$\vdots$$

so $$\varphi \vdash_{\mathbf{K}} \Box\varphi, \Box^2\varphi, \ldots, \Box^n\varphi.$$

Since **K** is an extension of **CPC**, $p, q \vdash_{\mathbf{K}} p \wedge q$ for $p, q \in P$, hence, by structurality,

$$\varphi \vdash_{\mathbf{K}} \boxed{n}\varphi.$$

Thus $\Gamma, \varphi \vdash_{\mathbf{K}} \boxed{n}\varphi, \boxed{n}\varphi \to \psi$, hence, by (MP), $\Gamma, \varphi \vdash_{\mathbf{K}} \psi$. □

4.7.3 THEOREM [BP89b, Theorem 5.4]

A variety $\mathcal{V}$ of modal algebras has EDPC if and only if it satisfies the identity $\boxed{n}x \approx \boxed{n+1}x$ for some $n \in \omega$.

Proof. First suppose that $\mathcal{V}$ satisfies $\boxed{n}x \approx \boxed{n+1}x$ for some $n \in \omega$, i.e.,

$$x \wedge \Box x \wedge \Box^2 x \wedge \ldots \wedge \Box^n x \approx x \wedge \Box x \wedge \Box^2 x \wedge \ldots \wedge \Box^{n+1} x.$$

If we apply $\Box$ to both sides of the identity and use (M4), we get that $\mathcal{V}$ satisfies

$$\Box x \wedge \Box^2 x \wedge \Box^3 x \wedge \ldots \wedge \Box^{n+1} x \approx \Box x \wedge \Box^2 x \wedge \Box^3 x \wedge \ldots \wedge \Box^{n+2} x.$$

Meeting both sides with x, we get that $\boxed{n+1}x \approx \boxed{n+2}x$, hence $\boxed{n}x \approx \boxed{n+2}x$. Continuing in this way, it follows that $\mathcal{V} \models \boxed{n}x \approx \boxed{m}x$ for each $m \geq n$. By the Lemma 4.7.1, for all $\mathbf{A} \in \mathcal{V}$ and $a, b, c, d \in A$, we have that

$$(c, d) \in \Theta^{\mathbf{A}}(a, b) \quad \text{iff} \quad c \leftrightarrow d \geq \boxed{m}(a \leftrightarrow b) \text{ for some } m \in \omega.$$

Now, if $m \geq n$, then $\boxed{n}(a \leftrightarrow b) = \boxed{m}(a \leftrightarrow b)$, so $(c \leftrightarrow d) \geq \boxed{n}(a \leftrightarrow b)$. If $m < n$, then $\boxed{m}(a \leftrightarrow b) \geq \boxed{n}(a \leftrightarrow b)$, hence $c \leftrightarrow d \geq \boxed{n}(a \leftrightarrow b)$. Thus if $(c, d) \in \Theta^{\mathbf{A}}(a, b)$, then $c \leftrightarrow d \geq \boxed{n}(a \leftrightarrow b)$. The converse is also true by the lemma, hence

$$(c, d) \in \Theta^{\mathbf{A}}(a, b) \quad \text{iff} \quad c \leftrightarrow d \geq \boxed{n}(a \leftrightarrow b).$$

Consequently $\mathcal{V}$ has EDPC.

Conversely, suppose $\mathcal{V}$ does not satisfy any of the identities $\boxed{n}x \approx \boxed{n+1}x$, $n \in \omega$. For $n \in \omega$, let $\mathbf{A}_n \in \mathcal{V}$ and $a_n \in A_n$ such that $\boxed{n}a_n \neq \boxed{n+1}a_n$, i.e., $\boxed{n+1}a_n < \boxed{n}a_n$. Suppose $\mathcal{V}$ has EDPC. Then, in particular, $\mathcal{V}$ has DPC, i.e., there exists a formula $\phi(x, y, z, w)$ in the first-order language for modal algebras such that, for all $\mathbf{A} \in \mathcal{V}$ and all $a, b, c, d \in A$, $(c, d) \in \Theta^{\mathbf{A}}(a, b)$ iff $\mathbf{A} \models \phi[a, b, c, d]$. Let $\mathcal{U}$ be a free ultrafilter over the natural numbers and let $\mathbf{A} = \Pi_{i \in \omega} \mathbf{A}_i / \mathcal{U}$, $\bar{a} = \langle a_n ; n \in \omega \rangle$ and $\bar{c} = \langle \boxed{n+1}a_n ; n \in \omega \rangle$. Since for each $n \in \omega$ and $i \geq n$ $(i \in \omega)$, we have

$$\boxed{i+1}\,a_i < \boxed{i}\,a_i \le \boxed{n}\,a_i,$$

so $\boxed{i+1}\,a_i \not\geq \boxed{n}\,a_i$. This says that $\{i \in \omega;\, \bar{c}(i) \not\geq (\boxed{n}\,\bar{a})(i)\} \supseteq \{i \in \omega;\, i \ge n\} \in \mathcal{U}$ (Corollary 0.3.4). Thus, in $\mathbf{A}$, $\bar{c}/\mathcal{U} \not\geq \boxed{n}(\bar{a}/\mathcal{U})$, i.e., $\bar{c}/\mathcal{U} \leftrightarrow 1^{\mathbf{A}} \not\geq \boxed{n}(\bar{a}/\mathcal{U} \leftrightarrow 1^{\mathbf{A}})$. (Note that n was arbitrary.) Since $\mathbf{A} \in \mathcal{V}$ (Theorem 0.4.1), this and Lemma 4.7.1 tell us that $(\bar{c}/\mathcal{U}, 1^{\mathbf{A}}) \notin \Theta^{\mathbf{A}}(\bar{a}/\mathcal{U}, 1^{\mathbf{A}})$. But for each $n \in \omega$,

$$\bar{c}(n) \leftrightarrow 1^{\mathbf{A}_n} = \bar{c}(n) = \boxed{n+1}\,a_n = \boxed{n+1}\,(\bar{a}(n) \leftrightarrow 1^{\mathbf{A}_n}),$$

so $(\bar{c}(n), 1^{\mathbf{A}_n}) \in \Theta^{\mathbf{A}_n}(\bar{a}(n), 1^{\mathbf{A}_n})$, hence $\mathbf{A}_n \models \phi[\bar{a}(n), 1^{\mathbf{A}_n}, \bar{c}(n), 1^{\mathbf{A}_n}]$. But ϕ is a first-order formula, so by Łoś' Theorem, $\mathbf{A} \models \phi[\bar{a}/\mathcal{U}, 1^{\mathbf{A}}, \bar{c}/\mathcal{U}, 1^{\mathbf{A}}]$, i.e., $(\bar{c}/\mathcal{U}, 1^{\mathbf{A}}) \in \Theta^{\mathbf{A}}(\bar{a}/\mathcal{U}, 1^{\mathbf{A}})$. This contradiction shows that $\mathcal{V}$ does not have DPC, and hence does not have EDPC. □

4.7.4 COROLLARY [BP89b, Corollary 5.5]

Let S be a normal modal logic.

(i) S has the DDT iff $\vdash_S \boxed{n}\,p \to \boxed{n+1}\,p$ for some $n \in \omega$.

(ii) Suppose that S has the DDT and let $n \in \omega$ such that $\vdash_S \boxed{n}\,p \to \boxed{n+1}\,p$. Then $\{\boxed{n}\,p \to q\}$ is a deduction-detachment set for S.

Proof. (i) Let $\mathcal{K}$ be the equivalent quasivariety semantics for S. Since S is an axiomatic extension of $\mathbf{K}$, $\mathcal{K}$ is axiomatized by the axioms of $\mathcal{MA}$ and possibly other *identities*, so $\mathcal{K}$ is a variety of modal algebras. Then S has the DDT if and only if $\mathcal{K}$ has EDPC, by Corollary 4.6.2. By the previous theorem, this is the case if and only if $\mathcal{K} \models \boxed{n}\,x \approx \boxed{n+1}\,x$ for some $n \in \omega$, which is equivalent to $\mathcal{K} \models \boxed{n}\,x \to \boxed{n+1}\,x \approx 1$, and hence (by Section 3.3) to $\vdash_S \boxed{n}\,p \to \boxed{n+1}\,p$ for some $n \in \omega$.

(ii) Suppose S has the DDT and let $\mathcal{K}$ be the equivalent variety semantics for S. Then, by (i), we have $\vdash_S \boxed{n}\,p \to \boxed{n+1}\,p$ for some $n \in \omega$. Thus $\mathcal{K} \models \boxed{n}\,p \approx \boxed{n+1}\,p$, and hence, by the previous theorem and its proof, for each $\mathbf{A} \in \mathcal{K}$ and $a, c \in A$, $(c, 1) \in \Theta^{\mathbf{A}}(a, 1)$ iff $c \ge \boxed{n}\,a$. Now, let $\Gamma \subseteq Fm$ and $\varphi, \psi \in Fm$ such that $\Gamma, \varphi \vdash_S \psi$. Then

(4.7.4) $$\{\chi \approx 1;\, \chi \in \Gamma\} \cup \{\varphi \approx 1\} \models_{\mathcal{K}} \psi \approx 1.$$

We shall show that $\Gamma \vdash_S \boxed{n}\,\varphi \to \psi$ by showing that $\{\chi \approx 1;\, \chi \in \Gamma\} \models_{\mathcal{K}} \boxed{n}\,\varphi \to \psi \approx 1$. So, let $\mathbf{A} \in \mathcal{K}$ and let $\bar{a}$ be an interpretation of the variables of $\Gamma \cup \{\varphi, \psi\}$ in A such that $\chi^{\mathbf{A}}(\bar{a}) = 1$ for each $\chi \in \Gamma$. By (4.7.4) and Lemma 0.4.4,

$$(\psi^{\mathbf{A}}(\bar{a}),1) \in \Theta^{\mathbf{A}}(\{(\chi^{\mathbf{A}}(\bar{a}),1)\,;\ \chi \in \Gamma\} \cup \{(\varphi^{\mathbf{A}}(\bar{a}),1)\}),$$

i.e.,
$$(\psi^{\mathbf{A}}(\bar{a}),1) \in \Theta^{\mathbf{A}}(\varphi^{\mathbf{A}}(\bar{a}),1),$$

since $\chi^{\mathbf{A}}(\bar{a}) = 1$ for each $\chi \in \Gamma$. But this holds iff $\psi^{\mathbf{A}}(\bar{a}) \geq \boxed{n}\,\varphi^{\mathbf{A}}(\bar{a})$, i.e., iff $\boxed{n}\,\varphi^{\mathbf{A}}(\bar{a}) \to \psi^{\mathbf{A}}(\bar{a}) = 1$. Thus $\{\chi \approx 1\,;\ \chi \in \Gamma\} \models_{\mathcal{K}} \boxed{n}\,\varphi \to \psi \approx 1$, i.e., $\Gamma \vdash_{\mathcal{S}} \boxed{n}\,\varphi \to \psi$. Conversely, suppose $\Gamma \vdash_{\mathcal{S}} \boxed{n}\,\varphi \to \psi$. As in the proof of Theorem 4.7.2, it follows that $\Gamma, \varphi \vdash_{\mathcal{S}} \psi$. □

In particular, the normal modal logic $\mathcal{S}4$ has the DDT. Let $\mathcal{K}$ be the equivalent variety semantics of $\mathcal{S}4$, i.e., $\mathcal{K}$ is the variety of interior algebras (see Theorem 3.3.3). Since $\vdash_{\mathcal{S}4} \Box p \to \Box\Box p$, we have $\models_{\mathcal{K}} \Box p \to \Box\Box p \approx 1$, hence $\models_{\mathcal{K}} \Box p \leq \Box\Box p$. Since $\vdash_{\mathcal{S}4} \Box\Box p \to \Box p$ by (K_3) and structurality, we have $\models_{\mathcal{K}} \Box\Box p \to \Box p \approx 1$, hence $\models_{\mathcal{K}} \Box\Box p \leq \Box p$. So $\models_{\mathcal{K}} \Box p \approx \Box\Box p$. Thus $\models_{\mathcal{K}} p \wedge \Box p \approx p \wedge \Box p \wedge \Box\Box p$, so $\models_{\mathcal{K}} (p \wedge \Box p) \to (p \wedge \Box p \wedge \Box\Box p) \approx 1$, i.e., $\models_{\mathcal{K}} \boxed{1}\,p \to \boxed{2}\,p \approx 1$. Thus $\vdash_{\mathcal{S}4} \boxed{1}\,p \to \boxed{2}\,p$. By the previous theorem, $\mathcal{S}4$ has the DDT with deduction-detachment set $\{\boxed{1}\,p \to q\}$. Since $\Box p \leq p$ over $\mathcal{K}$, $\boxed{1}\,p \approx \Box p \wedge p \approx \Box p$ over $\mathcal{K}$, so $\{\Box p \to q\}$ is a deduction-detachment set for $\mathcal{S}4$.

Łukasiewicz Many-Valued Logics.

Recall that for $n \leq \omega$, $\mathcal{S}_n$ is the n-valued many-valued Łukasiewicz logic (see Section 1.4). For formulas φ, ψ and $m \in \omega$, we abbreviate by $\varphi^m \to \psi$ the formula $\varphi \to (\varphi \to \ldots (\varphi \to \psi) \ldots)$, where there are precisely m φ's. We shall show in Section 5.3 that $\mathcal{S}_\omega$ has the LDDT but not the DDT. However, we have the following theorem.

4.7.5 THEOREM (cf. [BP89b, Section 5.4.2])

For each positive integer n (i.e., $n < \omega$), $\mathcal{S}_n$ has the DDT with deduction-detachment set $E(p,q) = \{p^n \to q\}$.

Proof. Let $n \geq 1$ be an integer. First, note that $p, p \to q \vdash_{\mathcal{S}_n} q$, i.e., $p, p \to q \models_{\mathcal{L}_n} q$: For if a, b is an interpretation of p, q in $\mathbf{L}_n$ such that $a = 1$ and $a \to b = 1$, then $1 \leq b + 1 - a$, hence $1 = a \leq b$, so $b = 1$. Let $\Gamma \subseteq Fm$ and $\varphi, \psi \in Fm$. It follows from repeated applications of $p, p \to q \models_{\mathcal{L}_n} q$ that if $\Gamma \models_{\mathcal{L}_n} \varphi^n \to \psi$ then $\Gamma, \varphi \models_{\mathcal{L}_n} \psi$, which is the detachment part. Conversely, suppose that $\Gamma, \varphi \models_{\mathcal{L}_n} \psi$. Let $\bar{a}$ be an interpretation of the variables of Γ, φ and ψ in L_n such that $\vartheta^{\mathbf{L}_n}(\bar{a}) = 1$ for each $\vartheta \in \Gamma$. If $\varphi^{\mathbf{L}_n}(\bar{a}) = 1$ then, by assumption, $\psi^{\mathbf{L}_n}(\bar{a}) = 1$ as well, hence

$(\varphi \to \psi)^{\mathbf{L}_n}(\bar{a}) = 1$. Suppose $\varphi^{\mathbf{L}_n}(\bar{a}) \neq 1$. Let $\varphi^{\mathbf{L}_n}(\bar{a}) = \frac{a}{n} \in L_n$, where $0 \leq a < n$, and let $\psi^{\mathbf{L}_n}(\bar{a}) = \frac{b}{n} \in L_n$, where $0 \leq b \leq n$. If $(\frac{a}{n})^i \to \frac{b}{n} = 1$ for some $i < n$, then it follows that $(\frac{a}{n})^n \to \frac{b}{n} = 1$ as well. So, suppose that $(\frac{a}{n})^i \to \frac{b}{n} < 1$ for each $i < n$. Then

$$\tfrac{a}{n} \to \tfrac{b}{n} = \tfrac{b}{n} + 1 - \tfrac{a}{n},$$

$$(\tfrac{a}{n})^2 \to \tfrac{b}{n} = \tfrac{a}{n} \to (\tfrac{a}{n} \to \tfrac{b}{n}) = (\tfrac{a}{n} \to \tfrac{b}{n}) + 1 - \tfrac{a}{n} = (\tfrac{b}{n} + 1 - \tfrac{a}{n}) + 1 - \tfrac{a}{n} = \tfrac{b}{n} + 2(1 - \tfrac{a}{n}),$$

$$(\tfrac{a}{n})^3 \to \tfrac{b}{n} = \tfrac{a}{n} \to ((\tfrac{a}{n})^2 \to \tfrac{b}{n}) = ((\tfrac{a}{n})^2 \to \tfrac{b}{n}) + 1 - \tfrac{a}{n} = (\tfrac{b}{n} + 2(1 - \tfrac{a}{n})) + 1 - \tfrac{a}{n} = \tfrac{b}{n} + 3(1 - \tfrac{a}{n}),$$

$$\vdots$$

$$(\tfrac{a}{n})^{n-1} \to \tfrac{b}{n} = \tfrac{b}{n} + (n-1)(1 - \tfrac{a}{n}),$$

$$(\tfrac{a}{n})^n \to \tfrac{b}{n} = \min\{1, \tfrac{b}{n} + n(1 - \tfrac{a}{n})\}.$$

But $\frac{b}{n} + n(1 - \frac{a}{n}) = \frac{b}{n} + n - a \geq 1$ since $n > a$, hence $(\frac{a}{n})^n \to \frac{b}{n} = 1$. Thus $\Gamma \models_{\mathcal{L}_n} \varphi^n \to \psi$, proving the deduction part. □

Relevance Logic.

The equivalent variety semantics of **R** (see Section 3.3) does not have the congruence extension property (see [BP89b]). By Corollary 4.1.14, therefore, the deductive system **R** does not have the LDDT (and hence not the DDT either). The axiomatic extension **RM** of **R** is strongly algebraizable and its equivalent variety semantics is the variety of Sugihara algebras (see Section 3.3). This variety is locally finite, congruence distributive and has the CEP (see [BD86]). By a result of Fried, Grätzer and Quackenbush [FGQ80, Theorem 6.1], a locally finite, congruence distributive variety with the CEP has EDPC. Thus the variety of Sugihara algebras has EDPC and so **RM** has the DDT, by Corollary 4.6.2. Blok and Pigozzi state the following in [BP89b]:

$$\Gamma, \varphi \vdash_{\mathbf{RM}} \psi \quad \text{iff} \quad \Gamma \vdash_{\mathbf{RM}} (\neg(\varphi \to (\neg\psi)) \vee (\varphi \to \psi)) \wedge ((\neg\varphi) \vee \psi).$$

Equivalential Logic.

In Section 3.3 we showed that the equivalent variety semantics of $\mathbf{CPC}_{\leftrightarrow}$ is the variety $\mathfrak{BG}$ of Boolean groups. It is well-known that this variety is not congruence distributive, hence, by Corollary 4.5.8, $\mathbf{CPC}_{\leftrightarrow}$ does not have the DDT. Of course $\mathbf{CPC}_{\leftrightarrow}$, being algebraizable, is protoalgebraic. This shows that the converse of Theorem 4.4.2 is false.

Chapter 5
BCK: A Case Study

The deductive system **BCK** was introduced by C. A. Meredith but his published work does not contain any very clear introduction to the subject, and prior to the publication of Blok and Pigozzi's Memoir [BP89a], the standard reference to Meredith's logic given by algebraists was A. N. Prior's book "Formal logic": [Pri62, p316]. The name **BCK** is derived from combinator theory, for background on which [CF68] may be consulted. Curry and Feys give an algorithm associating with every 'combinator' an implicational formula; certain natural combinators known as B, C and K translate, under this algorithm, into the formulas (B), (C) and (K) defined in Chapter 1. A discussion of this connection is to be found in [Bun81], for example.

In [Isé66], Kiyoshi Iséki introduced *BCK*-algebras. Dualizing his definition, we may say that a *BCK*-algebra is an algebra $\langle A;\rightarrow,\mathrm{T}\rangle$ of type $\langle 2,0\rangle$ satisfying

(1) $(p\rightarrow q)\rightarrow((q\rightarrow r)\rightarrow(p\rightarrow r))\approx\mathrm{T}$ (2) $p\rightarrow((p\rightarrow q)\rightarrow q)\approx\mathrm{T}$

(3) $p\rightarrow p\approx\mathrm{T}$ (4) $p\rightarrow\mathrm{T}\approx\mathrm{T}$

(5) $p\rightarrow q\approx\mathrm{T}\ \&\ q\rightarrow p\approx\mathrm{T}\ \Rightarrow\ p\approx q.$

The definition has been modified by some authors; it is more standard now to replace (2) and (3) by $\mathrm{T}\rightarrow p\approx p$, to which they are equivalent (in the presence of the other axioms).

Since Iséki was responsible for the name '*BCK*-algebra', it must have been his intention that these algebras would serve as a kind of algebraic semantics for Meredith's logic. His choice of axioms is therefore surprising in that they include none of the natural analogues of (B), (C) or (K). (Axiom (1) corresponds to the (B′) defined in Chapter 1; with Tanaka, he later published a lengthy proof of the analogue of (C) [IT78].) Since the notion of equivalent algebraic (quasivariety) semantics was not available before the mid 1980s and in view of the non-correspondence between Iséki's axioms and the expected ones, it was necessary for Blok and Pigozzi to prove in their Memoir [BP89a] a theorem that is not self-evident (although its statement makes it sound so): **BCK** is algebraizable and the equivalent quasivariety semantics for **BCK** is the quasivariety of all

BCK-algebras. (A self-contained proof will be given here.)

Viewed in isolation, this theorem does not show any connection between **BCK** and any familiar structures of classical algebra. It turns out, however, that BCK-algebras may also be characterized as the residuation subreducts of commutative, partially ordered integral monoids ('pocrims'). This result, which was proved independently in [Pał82], [OK85] and [Fle88] gives BCK-algebras a context that is perhaps more algebraically natural than that afforded by their logical origins.

Research on the deductive system **BCK** diverged somewhat from the vigorous exploration of BCK-algebras that took place in the 1970s and 1980s, mainly in the wake of the survey paper [IT78]. A further survey paper was published by Cornish in 1982 [Cor82], but many important developments have occurred subsequently (particularly Wroński's discovery that BCK-algebras do not form a variety [Wro83].) A survey in Polish appeared around 1987 [Pał]. The paper [BR95] may also serve as an introduction to the subject.

The logic **BCK** makes a good case study for the theory of algebraization: in the first place, it shows that one cannot expect all algebraizable logics to have varieties as equivalent algebraic semantics. In particular, the need to consider relative congruences is evident here. Partly because BCK-algebras do not form a variety and partly because of the 'weakness' of their axioms, some of the desirable algebraic properties of this quasivariety are harder to obtain than is the case, for example with the **CPC** and Boolean algebras. Finally, since the known algebraic properties of BCK-algebras were obtained mostly without reference to Meredith's logic, it is of interest to see how they can be derived from metalogical properties of **BCK** only. The provision of such derivations, which reverses the traditional practice of using algebra to infer facts about logic, is the main purpose of this case study.

5.1 THE DEDUCTIVE SYSTEM BCK

Recall the definition of **BCK** from Section 1.4. The language $\mathcal{L} = \{\rightarrow\}$ consists of one binary connective $\rightarrow$. The axioms are

(B) $(p \to q) \to ((r \to p) \to (r \to q))$,

(C) $(p \to (q \to r)) \to (q \to (p \to r))$,

(K) $p \to (q \to p)$,

and the single inference rule is modus ponens, which we recall is

(MP) $p, p \to q \vdash_{\mathbf{BCK}} q$.

The axiom (B) expresses the condition that implication is transitive. Axiom (C) states that it is permissible to interchange premisses, and axiom (K) states that the strengthening of premisses is permitted. We present some theorems of **BCK**. As noted in Section 1.4, the following formula has been named (B) in some of the papers considered for this thesis:

(B′) $(p \to q) \to ((q \to r) \to (p \to r))$.

It was shown in Section 1.4, that (B′) follows from the axioms (B) and (C), i.e., (B′) is a theorem of **BCK**. The *mingle axiom*, namely

(M) $p \to (p \to p)$,

is also a theorem of **BCK**. This follows from the fact that $p \to (p \to p)$ is a substitution instance of (K), an axiom of **BCK**. The following is also a theorem of **BCK**:

(I) $p \to p$.

To see this, substitute $p \to (p \to p)$ for q in (K), to get

$\vdash_{\mathbf{BCK}} p \to ((p \to (p \to p)) \to p)$.

By (C), $\vdash_{\mathbf{BCK}} (p \to ((p \to (p \to p)) \to p)) \to ((p \to (p \to p)) \to (p \to p))$,

so, by (MP) $\vdash_{\mathbf{BCK}} (p \to (p \to p)) \to (p \to p)$.

The mingle axiom says $\vdash_{\mathbf{BCK}} p \to (p \to p)$,

so (MP) implies $\vdash_{\mathbf{BCK}} p \to p$.

5.1.1 THEOREM [BP89a, Theorem 5.10]

BCK *is algebraizable with equivalence formulas* $\Delta_1(p,q) = p \to q$, $\Delta_2(p,q) = q \to p$ *and defining equation* $p \approx p \to p$.

Proof. This is proved using Theorem 3.3.1. **BCK** satisfies (B), (C) and (MP) by definition, and the mingle axiom (M) is a special case of (K). We have just shown that **BCK** satisfies the axiom (I), hence the result follows. □

We now find the equivalent quasivariety semantics for **BCK**.

5.1.2 DEFINITION

A *BCK-algebra* is an algebra $\mathbf{A} = \langle A\,;\, \rightarrow, \mathrm{T}\rangle$ of type $\langle 2,0\rangle$ satisfying the following three identities and one quasi-identity:

(5.1.1) $\quad (x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) \approx \mathrm{T}$,

(5.1.2) $\quad \mathrm{T} \rightarrow x \approx x$,

(5.1.3) $\quad x \rightarrow \mathrm{T} \approx \mathrm{T}$,

(5.1.4) $\quad (x \rightarrow y \approx \mathrm{T})\ \&\ (y \rightarrow x \approx \mathrm{T}) \Rightarrow x \approx y$.

Let $\mathfrak{BCK}$ be the class of all BCK-algebras. Evidently $\mathfrak{BCK}$ is a quasivariety.

As an example of a BCK-algebra, consider the algebra $\boldsymbol{\omega} = \langle \omega\,;\, \dot{-}\,, 0\rangle$ of type $\langle 2,0\rangle$ where, for $a, b \in \omega$, $\dot{-}$ is defined by $a \dot{-} b = \max\{a - b, 0\}$. Also define $\mathbf{N} = \langle \omega\,; \rightarrow, 0\rangle$ where for $a, b \in \omega$, $a \rightarrow b = b \dot{-} a$. It follows easily that $\mathbf{N}$ is indeed a BCK-algebra.

The smallest non-trivial BCK-algebra is the 2-element BCK-algebra $\mathbf{C}_2 = \langle \{0,1\}\,; \rightarrow, 1\rangle$, where $\rightarrow$ is defined by $x \rightarrow x = 1$ for $x \in \{0,1\}$ and $0 \rightarrow 1 = 1$, $1 \rightarrow 0 = 0$.

We shall investigate some properties of $\mathfrak{BCK}$ that can be derived from the above axioms. From (5.1.3) we can immediately deduce that $\mathrm{T} \rightarrow \mathrm{T} \approx \mathrm{T}$. This gives us the following

$$\begin{aligned} x \rightarrow x &\approx (\mathrm{T} \rightarrow x) \rightarrow (\mathrm{T} \rightarrow x) && \text{[by (5.1.2)]} \\ &\approx \mathrm{T} \rightarrow ((\mathrm{T} \rightarrow x) \rightarrow (\mathrm{T} \rightarrow x)) && \text{[by (5.1.2)]} \\ &\approx (\mathrm{T} \rightarrow \mathrm{T}) \rightarrow ((\mathrm{T} \rightarrow x) \rightarrow (\mathrm{T} \rightarrow x)) && \\ &\approx \mathrm{T} && \text{[by (5.1.1)],} \end{aligned}$$

hence we have the identity

(5.1.5) $\quad x \rightarrow x \approx \mathrm{T}$.

Let $\mathbf{A}$ be an element of $\mathfrak{BCK}$. Define a relation $\leq$ on A by $a \leq b$ if and only if $a \rightarrow b = \mathrm{T}$. Then (5.1.4) immediately implies that if $a \leq b$ and $b \leq a$ then $a = b$. From (5.1.5) we get that $a \leq a$ for all $a \in A$. If $a \leq b$ and $b \leq c$, then $a \rightarrow b = \mathrm{T}$, $b \rightarrow c = \mathrm{T}$, so (5.1.1) implies that

$$\begin{aligned} a \rightarrow c = \mathrm{T} \rightarrow (a \rightarrow c) &= (b \rightarrow c) \rightarrow (a \rightarrow c) && \text{[by (5.1.2)]} \\ &= \mathrm{T} \rightarrow ((b \rightarrow c) \rightarrow (a \rightarrow c)) && \text{[by (5.1.2)]} \end{aligned}$$

$$= (a \to b) \to ((b \to c) \to (a \to c))$$

$$= \mathrm{T} \qquad \text{[by (5.1.1)],}$$

hence $a \leq c$, so $\leq$ is a partial order on A. Furthermore, (5.1.3) implies that T is the greatest element of this order. The relation has the following properties:

(5.1.6) $\quad a \leq b \Rightarrow c \to a \leq c \to b$

(5.1.7) $\quad b \leq c \Rightarrow c \to a \leq b \to a.$

To see (5.1.6), suppose $a \leq b$. Then $a \to b = \mathrm{T}$, so

$$\begin{aligned}(c \to a) \to (c \to b) &= (c \to a) \to (\mathrm{T} \to (c \to b)) && \text{[by (5.1.2)]}\\ &= (c \to a) \to ((a \to b) \to (c \to b)) \\ &= \mathrm{T} && \text{[by (5.1.1)].}\end{aligned}$$

To see (5.1.7), suppose $b \leq c$. Then $b \to c \approx \mathrm{T}$, so

$$\begin{aligned}(c \to a) \to (b \to a) &= \mathrm{T} \to ((c \to a) \to (b \to a)) && \text{[by (5.1.2)]}\\ &= (b \to c) \to ((c \to a) \to (b \to a)) \\ &= \mathrm{T} && \text{[by (5.1.1)].}\end{aligned}$$

The following are all identities satisfied by $\mathfrak{BCK}$

(5.1.8) $\quad x \to (y \to x) \approx \mathrm{T} \quad$ (i.e., $x \leq y \to x$)

(5.1.9) $\quad y \to ((y \to x) \to x) \approx \mathrm{T} \quad$ (i.e., $y \leq (y \to x) \to x$)

(5.1.10) $\quad ((y \to x) \to x) \to x \approx y \to x$

(5.1.11) $\quad z \to (y \to x) \approx y \to (z \to x)$

(5.1.12) $\quad (z \to x) \to ((y \to z) \to (y \to x)) \approx \mathrm{T}.$

In the following derivations, we shall simply write, for example, (1) instead of (5.1.1). Then $\overset{(1)}{\approx}$ will mean 'by (5.1.1)'. We derive (5.1.8) as follows:

$$x \to (y \to x) \overset{(2)}{\approx} \mathrm{T} \to ((\mathrm{T} \to x) \to (y \to x)) \overset{(3)}{\approx} (y \to \mathrm{T}) \to ((\mathrm{T} \to x) \to (y \to x)) \overset{(1)}{\approx} \mathrm{T}.$$

For (9), we have

$$y \to ((y \to x) \to x) \overset{(2)}{\approx} (\mathrm{T} \to y) \to ((y \to x) \to (\mathrm{T} \to x)) \overset{(1)}{\approx} \mathrm{T}.$$

For (10), we start with $y \leq (y \to x) \to x$ from (9). Then $y \to x \geq ((y \to x) \to x) \to x$ by (7). Since $y \to x \leq ((y \to x) \to x) \to x$ is a substitution instance of (9), we get that $((y \to x) \to x) \to x \approx y \to x$ by (4). The standard reference for the identity (11) is a long and complex derivation in [IT78]. We give a simpler derivation. From

$$(y \to (z \to x)) \to (((z \to x) \to x) \to (y \to x)) \overset{(1)}{\approx} \mathrm{T}, \quad \text{and}$$

$$(((z\to x)\to x)\to(y\to x))\to(z\to(y\to x))\overset{(2)}{\approx}\mathrm{T}\to((((z\to x)\to x)\to(y\to x))\to(z\to(y\to x)))$$
$$\overset{(9)}{\approx}(z\to((z\to x)\to x))\to((((z\to x)\to x)\to(y\to x))\to(z\to(y\to x)))\overset{(1)}{\approx}\mathrm{T},$$

we get that $y\to(z\to x)\leq((z\to x)\to x)\to(y\to x)$ and $((z\to x)\to x)\to(y\to x)\leq z\to(y\to x)$, respectively. By the transitivity of $\leq$, we have that $y\to(z\to x)\leq z\to(y\to x)$. By symmetry of the variables, the identity holds.

Finally, to derive (12), all we need do is apply (11) to (1), i.e.,

$$\mathrm{T}\overset{(1)}{\approx}(y\to z)\to((z\to x)\to(y\to x))\overset{(11)}{\approx}(z\to x)\to((y\to z)\to(y\to x)).$$

5.1.3 THEOREM [BP89a, Theorem 5.11]

The class of BCK-algebras is termwise definitionally equivalent to the equivalent quasivariety semantics for **BCK.**

Proof. By Theorem 5.1.1, **BCK** is algebraizable with equivalence formulas $\Delta_1(p,q)=p\to q$, $\Delta_2(p,q)=q\to p$ and defining equation $p\approx p\to p$. Let $\mathcal{K}$ be the equivalent quasivariety semantics for **BCK**. Now, substituting $q\to q$ for q in axiom (K), we get that $\vdash_{\mathbf{BCK}}p\to((q\to q)\to p)$, and then an application of axiom (C) and (MP) to this gives $\vdash_{\mathbf{BCK}}(q\to q)\to(p\to p)$. Thus

$$\vdash_{\mathbf{BCK}}\Delta(p\to p,q\to q).$$

Thus, since $\mathcal{K}$ is the equivalent quasivariety semantics for **BCK**,

$$\models_{\mathcal{K}} x\to x\approx y\to y \qquad \text{[by Lemma 3.1.6 (i)]}.$$

Let $\mathbf{A}=\langle A;\to^{\mathbf{A}}\rangle\in\mathcal{K}$. By the above identity, $a\to^{\mathbf{A}}a=b\to^{\mathbf{A}}b$ for all $a,b\in A$. This makes it possible to define a term T in the language of $\mathcal{K}$ by $\mathrm{T}=x\to x$. For each $\mathbf{A}\in\mathcal{K}$, let $\mathbf{A}'=\langle A;\to^{\mathbf{A}},\mathrm{T}^{\mathbf{A}}\rangle$ be the resultant expansion of $\mathbf{A}$, and let $\mathcal{K}'=\{\mathbf{A}';\mathbf{A}\in\mathcal{K}\}$. We claim that $\mathcal{K}'$ is the class of all *BCK*-algebras. We drop the superscript $\mathbf{A}$ on the operations $\to$ and T if the algebra is understood. Note that an algebra $\langle A;\to,\mathrm{T}\rangle$ of type $\langle 2,0\rangle$ is an element of $\mathcal{K}'$ if and only if $\langle A;\to\rangle\in\mathcal{K}$ and $\mathrm{T}=a\to a$ for all $a\in A$. For such an algebra, the defining equation becomes $x\approx\mathrm{T}$. If we now apply Theorem 3.1.18, we get the following axiom system for $\mathcal{K}'$:

(5.1.13) $\quad (x\to y)\to((z\to x)\to(z\to y))\approx\mathrm{T},$

(5.1.14) $\quad (x\to(y\to z))\to(y\to(x\to z))\approx\mathrm{T},$

(5.1.15) $\quad x\to(y\to x)\approx\mathrm{T},$

(5.1.16) $\quad x\to x\approx\mathrm{T},$

(5.1.17) $(x \to y \approx \mathrm{T})\ \&\ (y \to x \approx \mathrm{T}) \Rightarrow x \approx y,$

(5.1.18) $(x \approx \mathrm{T})\ \&\ (x \to y \approx \mathrm{T}) \Rightarrow y \approx \mathrm{T}.$

It was shown after Definition 5.1.2 that if we assume the axioms of BCK-algebras, then each of the above identities is derivable, by (5.1.12), (5.1.11), (5.1.8) and (5.1.5), respectively. The quasi-identity (5.1.17) is precisely (5.1.4). The quasi-identity (5.1.18) is equivalent to $\mathrm{T} \to y \approx \mathrm{T} \Rightarrow y \approx \mathrm{T}$. But $\mathrm{T} \to y \approx y$ by (5.1.2), so $\mathrm{T} \to y \approx \mathrm{T}$ implies $y \approx \mathrm{T}$.

We now derive the identities and quasi-identities of $\mathfrak{BCK}$ from (5.1.13) to (5.1.18). We need only derive (5.1.1), (5.1.2) and (5.1.3) since (5.1.4) is (5.1.17). For (5.1.1), we have

$$[(y \to z) \to ((x \to y) \to (x \to z))] \to [(x \to y) \to ((y \to z) \to (x \to z))] \overset{(14)}{\approx} \mathrm{T},$$

and

$$(y \to z) \to ((x \to y) \to (x \to z)) \overset{(13)}{\approx} \mathrm{T},$$

hence $(x \to y) \to ((y \to z) \to (x \to z)) \approx \mathrm{T}$, by (5.1.18), so (5.1.1) holds. From (5.1.15) we get that $\mathrm{T} \to (x \to \mathrm{T}) \approx \mathrm{T}$, hence (5.1.18) implies that $x \to \mathrm{T} \approx \mathrm{T}$, so (5.1.3) holds. To derive (5.1.2), we have

$$\mathrm{T} \overset{(14)}{\approx} [\mathrm{T} \to ((\mathrm{T} \to x) \to x)] \to [(\mathrm{T} \to x) \to (\mathrm{T} \to x)] \overset{(16)}{\approx} [\mathrm{T} \to ((\mathrm{T} \to x) \to x)] \to \mathrm{T},$$

and

$$\mathrm{T} \overset{(14)}{\approx} [(\mathrm{T} \to x) \to (\mathrm{T} \to x)] \to [\mathrm{T} \to ((\mathrm{T} \to x) \to x)] \overset{(16)}{\approx} \mathrm{T} \to [\mathrm{T} \to ((\mathrm{T} \to x) \to x)],$$

hence

$$\mathrm{T} \approx \mathrm{T} \to ((\mathrm{T} \to x) \to x),$$

by (17). Thus $(\mathrm{T} \to x) \to x \approx \mathrm{T}$, by (5.1.18). Since we also have $x \to (\mathrm{T} \to x) \approx \mathrm{T}$ by (5.1.15), (5.1.17) implies that $x \approx \mathrm{T} \to x$. □

5.2 PROPERTIES OF BCK AND $\mathfrak{BCK}$

The main drive behind algebraization is to make the machinery of universal algebra available for the study of logics; however, one may reverse this process and derive properties of the equivalent quasivariety semantics of an algebraizable logic from properties of the logic. It is of interest to do this for **BCK** since, historically, the algebraic development detached itself from the logic after Iséki's early papers on the subject. We provide new logic-driven proofs of the significant universal algebraic properties of the quasivariety $\mathfrak{BCK}$.

5.2.1 LEMMA

BCK *has the G-rule (i.e.,* $\varphi, \psi \vdash_S \Delta(\varphi, \psi)$ *for all* $\varphi, \psi \in Fm$*, where* Δ *refers to the equivalence formulas of* **BCK**).

Proof. Recall from Theorem 5.1.1 that **BCK** is algebraizable with equivalence formulas $\Delta_1(p,q) = p \to q, \Delta_2(p,q) = q \to p$. Thus we need to show that $\varphi, \psi \vdash_S \varphi \to \psi$ and $\varphi, \psi \vdash_S \psi \to \varphi$. From the axiom (K), we get that

$$\varphi, \psi \vdash_{\mathbf{BCK}} \varphi, \varphi \to (\psi \to \varphi), \psi, \psi \to (\varphi \to \psi),$$

hence $$\varphi, \psi \vdash_{\mathbf{BCK}} \psi \to \varphi, \varphi \to \psi$$

by two applications of (MP). □

5.2.2 COROLLARY

The quasivariety $\mathfrak{BCK}$ *is relatively* T*-regular. Consequently, every variety of BCK-algebras is* T*-regular.*

Proof. The result follows directly from Theorem 3.2.4. □

5.2.3 DEFINITION

An *ideal* of a *BCK*-algebra $\mathbf{A} = \langle A; \to, \mathrm{T} \rangle$ is a set $F \subseteq A$ such that $\mathrm{T} \in F$ and, whenever $a, a \to b \in F$ we have $b \in F$ as well. The set of all ideals of **A** is denoted by $Id\,\mathbf{A}$. Of course, $\{\mathrm{T}\}, A \in Id\,\mathbf{A}$. We shall show after Corollary 5.2.6 that this notion of an ideal coincides with the notion of an ideal (with respect to $\mathfrak{BCK}$ and T) defined in Section 0.4.

From the point of view of the partial order on A, it would be more natural to use the word 'filter' in place of 'ideal'. We use 'ideal' because it is standard in the theory of *BCK*-algebras and because it avoids confusion with our notion of "S-filter". The connection between ideals and **BCK**-filters is clarified in the next lemma.

5.2.4 LEMMA

The class Mod **BCK**, *of all matrix models of* **BCK**, *is precisely the class* $\{\langle \mathbf{A}, F \rangle; \mathbf{A} \in \mathfrak{BCK}$ *and* $F \in Id\,\mathbf{A}\}$. *Thus, for any* **BCK***-matrix* $\mathcal{A} = \langle \mathbf{A}, F \rangle$, *the* **BCK***-filters of* $\mathcal{A}$ *are just the ideals of* **A** *containing* F.

Proof. Let $\mathbf{A} = \langle \mathbf{A}; \to, \mathrm{T} \rangle \in \mathfrak{BCK}$ and $F \in Id\,\mathbf{A}$. Suppose that $\Gamma \subseteq Fm$, $\varphi \in Fm$ such that $\Gamma \vdash_{\mathbf{BCK}} \varphi$, and let $\bar{a}$ be an interpretation of the variables of $\Gamma \cup \{\varphi\}$ in A such that $\psi^{\mathbf{A}}(\bar{a}) \in F$ for all $\psi \in \Gamma$. To prove that $\varphi^{\mathbf{A}}(\bar{a}) \in F$, we proceed by induction on the complexity of a derivation of φ from Γ. If $\varphi \in \Gamma$, then $\varphi^{\mathbf{A}}(\bar{a}) \in F$, trivially. If φ is a substitution instance of an axiom then $\varphi^{\mathbf{A}}(\bar{a}) = \mathrm{T} \in F$ by Theorems 5.1.1 and 5.1.3. Suppose that φ is directly derived from ψ and $\psi \to \varphi$ by (MP), and assume, inductively, that $\psi^{\mathbf{A}}(\bar{a})$, $\psi^{\mathbf{A}}(\bar{a}) \to \varphi^{\mathbf{A}}(\bar{a}) \in F$. By definition of ideals, we have $\varphi^{\mathbf{A}}(\bar{a}) \in F$. Thus $\Gamma \models_{\langle \mathbf{A}, F \rangle} \varphi$, so $\langle \mathbf{A}, F \rangle$ is a **BCK**-matrix.

Conversely, suppose $\mathcal{A} = \langle \mathbf{A}, F \rangle \in \mathrm{Mod}\,\mathbf{BCK}$. The formula $p \to p$ is an axiom **BCK**, hence, for an interpretation a of p in A, we have $\mathrm{T} = a \to a \in F$. Suppose now that $a, a \to b \in F$. Since $\mathcal{A}$ is a **BCK**-matrix, (MP) implies that $p, p \to q \models_{\mathcal{A}} q$. If we interpret p as a and q as b, then, since $a, a \to b \in F$, we have $b \in F$ as well, hence $F \in Id\,\mathbf{A}$. □

5.2.5 COROLLARY

The ideal lattice $Id\,\mathbf{A}$ of a BCK-algebra $\mathbf{A}$ is algebraic. □

5.2.6 COROLLARY

For any BCK-algebra $\mathbf{A} = \langle A; \to, \mathrm{T} \rangle$, we have $\mathbf{Id}\,\mathbf{A} \cong \mathbf{Con}_{\mathfrak{BCK}}\mathbf{A}$, *the isomorphism being given by*

$$F \mapsto \{(a,b) \in A^2 ;\ a \to b, b \to a \in F\} \quad (F \in Id\,\mathbf{A}).$$

The inverse of the isomorphism is given by

$$\Phi \mapsto \mathrm{T}/\Phi \quad (\Phi \in \mathrm{Con}_{\mathfrak{BCK}}\mathbf{A}).$$

Proof. Set $\mathcal{A} = \langle \mathbf{A}, \{\mathrm{T}\} \rangle \in \mathrm{Mod}\,\mathbf{BCK}$. Then $\mathrm{Fi}^{\mathbf{BCK}}\mathcal{A} = Id\,\mathbf{A}$. Since **BCK** is algebraizable, Corollary 3.1.14 implies that $\Omega_{\mathcal{A}}: \mathbf{Fi}^{\mathbf{BCK}}\mathcal{A} \to \mathbf{Con}_{\mathfrak{BCK}}\mathbf{A}$ is an isomorphism, i.e., $\Omega_{\mathcal{A}}$ is a lattice isomorphism between $\mathbf{Id}\,\mathbf{A}$ and $\mathbf{Con}_{\mathfrak{BCK}}\mathbf{A}$. A system of equivalence formulas for **BCK** is $\Delta_1(p,q) = p \to q$, $\Delta_2(p,q) = q \to p$. Let $F \in Id\,\mathbf{A}$. It was noted in the proof of Corollary 3.1.14 that

$$\begin{aligned}\Omega_{\mathcal{A}} F &= \{(a,b) \in A^2 ;\ \Delta_i^{\mathbf{A}}(a,b) \in F \ \text{ for } i = 1,2\} \\ &= \{(a,b) \in A^2 ;\ a \to b, b \to a \in F\}.\end{aligned}$$

A system of defining equations for **BCK** is $p \approx p \to p$. By Corollary 3.1.14 again, the inverse of $\Omega_{\mathcal{A}}$ is $H_{\mathcal{A}}$, defined, for $\Phi \in \mathrm{Con}_{\mathfrak{BCK}}\mathbf{A}$, by

$$H_{\mathcal{A}}\Phi = \{a \in A\,;\,(a, a \to a) \in \Phi\}$$
$$= \{a \in A\,;\,(a, \mathrm{T}) \in \Phi\}$$
$$= \mathrm{T}/\Phi. \qquad \square$$

We shall need the following abbreviation: For formulas φ and ψ and $i \in \omega$, let $\varphi^i \to \psi$ stand for $\varphi \to (\varphi \to \ldots (\varphi \to \psi) \ldots)$, where there are precisely i occurrences of φ. In particular, $\varphi^0 \to \psi$ abbreviates ψ. The next result is stated without proof in [BP88].

We now show that the notion of an ideal in Definition 5.2.3 coincides with the notion of an ideal in Section 0.4. Let $\mathbf{A} = \langle A\,;\to, \mathrm{T}\rangle \in \mathfrak{BCK}$. Let F be an ideal in the sense of Definition 5.2.3. By Lemma 5.2.4, $\mathcal{A} = \langle \mathbf{A}, F\rangle$ is a **BCK**-matrix. By Corollary 5.2.6, $F = H_{\mathcal{A}}\Omega_{\mathcal{A}}F = \mathrm{T}/\Omega_{\mathcal{A}}F$. Since $\Omega_{\mathcal{A}}F$ is a congruence relation on $\mathbf{A}$, it must be reflexive and compatible. Thus, by the remark following the definition of an ideal in Section 0.4 it follows that $\mathrm{T}/\Omega_{\mathcal{A}}F$ is an ideal of $\mathbf{A}$. Since $F = \mathrm{T}/\Omega_{\mathcal{A}}F$, F is an ideal of $\mathbf{A}$ (for $\mathfrak{BCK}$ and T) in the sense of Section 0.4.

Conversely, let I be an ideal of $\mathbf{A}$ in the sense of Section 0.4. Certainly, $\mathrm{T} \in I$, so $I \neq \emptyset$. Define the term

$$t(x, y, z) = (y \to (z \to x)) \to x.$$

Then $\mathfrak{BCK} \models t(x, \mathrm{T}, \mathrm{T}) \approx (\mathrm{T} \to (\mathrm{T} \to x)) \to x \approx x \to x \approx \mathrm{T}$, hence $t(x, y, z)$ is an ideal term in y, z for $\mathfrak{BCK}$ and T. Thus, if $a, a \to b \in I$ then so is

$$t^{\mathbf{A}}(b, a, a \to b) = (a \to ((a \to b) \to b) \to b \overset{(9)}{=} \mathrm{T} \to b = b.$$

Thus $b \in I$, so I is an ideal in the sense of Definition 5.2.3.

5.2.7 LEMMA

For all $i, j \in \omega$, the following formula is a theorem of **BCK**:

(5.2.1) $\qquad (\varphi^i \to \psi) \to ((\varphi^j \to (\psi \to \zeta)) \to (\varphi^{i+j} \to \zeta)).$

Proof. We shall need the following

Claim 1: If $\vdash_{\mathbf{BCK}} \varphi \to (\psi \to \zeta)$, then $\vdash_{\mathbf{BCK}} (\eta \to \varphi) \to (\psi \to (\eta \to \zeta))$.

Proof. From the axiom (B), we get

$$\vdash_{\mathbf{BCK}} (\varphi \to (\psi \to \zeta)) \to [(\eta \to \varphi) \to (\eta \to (\psi \to \zeta))],$$

hence $\vdash_{\mathbf{BCK}} \varphi \to (\psi \to \zeta)$ and (MP) imply

(5.2.2) $$\vdash_{\mathbf{BCK}}(\eta\to\varphi)\to(\eta\to(\psi\to\zeta)).$$

Now, the axiom (B) gives us

$$\vdash_{\mathbf{BCK}}[(\eta\to(\psi\to\zeta))\to(\psi\to(\eta\to\zeta))]\to[((\eta\to\varphi)\to(\eta\to(\psi\to\zeta)))\to((\eta\to\varphi)\to(\psi\to(\eta\to\zeta)))].$$

By (C),

$$\vdash_{\mathbf{BCK}}(\eta\to(\psi\to\zeta))\to(\psi\to(\eta\to\zeta)),$$

hence, by (MP) this gives us

$$\vdash_{\mathbf{BCK}}((\eta\to\varphi)\to(\eta\to(\psi\to\zeta)))\to((\eta\to\varphi)\to(\psi\to(\eta\to\zeta))).$$

Thus (5.2.2) and (MP) finally give

$$\vdash_{\mathbf{BCK}}(\eta\to\varphi)\to(\psi\to(\eta\to\zeta)),$$

which proves Claim 1.

Claim 2: If $\vdash_{\mathbf{BCK}}\varphi\to(\psi\to\zeta)$, then $\vdash_{\mathbf{BCK}}\varphi\to((\eta\to\psi)\to(\eta\to\zeta))$.

Proof. From the theorem (B′) we get

$$\vdash_{\mathbf{BCK}}(\varphi\to(\psi\to\zeta))\to[((\psi\to\zeta)\to((\eta\to\psi)\to(\eta\to\zeta)))\to(\varphi\to((\eta\to\psi)\to(\eta\to\zeta)))],$$

hence $\vdash_{\mathbf{BCK}}\varphi\to(\psi\to\zeta)$ and (MP) imply

$$\vdash_{\mathbf{BCK}}((\psi\to\zeta)\to((\eta\to\psi)\to(\eta\to\zeta)))\to(\varphi\to((\eta\to\psi)\to(\eta\to\zeta))).$$

Now, since (B) implies that

$$\vdash_{\mathbf{BCK}}(\psi\to\zeta)\to((\eta\to\psi)\to(\eta\to\zeta)),$$

we can use (MP) to deduce that

$$\vdash_{\mathbf{BCK}}\varphi\to((\eta\to\psi)\to(\eta\to\zeta)),$$

which proves Claim 2.

We now proceed by induction on the sum $i+j$. If $i+j=0$, then $i,j=0$, and (5.2.1) reduces to $\psi\to((\psi\to\zeta)\to\zeta)$, which is a theorem of **BCK** as a result of (5.1.9) and Theorems 5.1.1 and 5.1.3. Assume that (5.2.1) is a theorem of **BCK** for some i,j. The formula

$$(\varphi^{i+1}\to\psi)\to((\varphi^{j}\to(\psi\to\zeta))\to(\varphi^{i+1+j}\to\zeta))$$

is an abbreviation for

$$(\varphi\to(\varphi^{i}\to\psi))\to((\varphi^{j}\to(\psi\to\zeta))\to(\varphi\to(\varphi^{i+j}\to\zeta))).$$

But by the induction hypothesis and Claim 1 this formula is a theorem of **BCK**. Next, consider the formula

$$(\varphi^{i}\to\psi)\to((\varphi^{j+1}\to(\psi\to\zeta))\to(\varphi^{i+j+1}\to\zeta)).$$

This is an abbreviation for

$$(\varphi^i \to \psi) \to ((\varphi \to (\varphi^j \to (\psi \to \zeta))) \to (\varphi \to (\varphi^{i+j} \to \zeta))),$$

which is a theorem of **BCK** by Claim 2 and the induction hypothesis. This completes the inductive proof. □

5.2.8 PROPOSITION [BP88]

BCK *has the LDDT with local deduction-detachment system*

$$\mathfrak{S}(p,q) = \{\{p^i \to q\}\,;\, i \in \omega\}.$$

Proof. Let $\Gamma \subseteq Fm$ and $\varphi, \psi \in Fm$. If $\Gamma \vdash_{\mathbf{BCK}} \varphi^i \to \psi$ for some $i \in \omega$, then

$$\Gamma, \varphi \vdash_{\mathbf{BCK}} \varphi, \varphi^i \to \psi$$

so $\quad \Gamma, \varphi \vdash_{\mathbf{BCK}} \varphi, \varphi^{i-1} \to \psi \quad$ [by (MP)]

$\vdots$

so $\quad \Gamma, \varphi \vdash_{\mathbf{BCK}} \varphi, \varphi \to \psi \quad$ [by (MP)]

so $\quad \Gamma, \varphi \vdash_{\mathbf{BCK}} \psi, \quad$ [by (MP)].

This proves the detachment part of the result. Conversely, suppose $\Gamma, \varphi \vdash_{\mathbf{BCK}} \psi$. We need to show that $\Gamma \vdash_{\mathbf{BCK}} \varphi^i \to \psi$ for some $i \in \omega$. This is done using the inductive characterization of $\vdash_{\mathbf{BCK}}$ (see Section 1.1). If $\psi \in \Gamma$ or ψ is a substitution instance of an axiom, then, by (1.1.1) and the axiom (K), $\Gamma \vdash_{\mathbf{BCK}} \psi, \psi \to (\varphi \to \psi)$, hence $\Gamma \vdash_{\mathbf{BCK}} \varphi \to \psi$ by (MP). Suppose ψ is directly derivable by (MP) from χ and $\chi \to \psi$ in a derivation $\eta_1, \ldots, \eta_r = \psi$ of ψ from $\Gamma \cup \{\varphi\}$, so that $\Gamma, \varphi \vdash_{\mathbf{BCK}} \chi$ and $\Gamma, \varphi \vdash_{\mathbf{BCK}} \chi \to \psi$. Since $\chi, \chi \to \psi \in \{\eta_k\,;\, k < r\}$, the induction hypothesis says that for some $i, j \in \omega$, $\Gamma \vdash_{\mathbf{BCK}} \varphi^i \to \chi$ and $\Gamma \vdash_{\mathbf{BCK}} \varphi^j \to (\chi \to \psi)$. From the previous lemma, we have that

$$(\varphi^i \to \chi) \to ((\varphi^j \to (\chi \to \psi)) \to (\varphi^{i+j} \to \psi))$$

is a theorem of **BCK**, hence

$$\Gamma \vdash_{\mathbf{BCK}} \varphi^i \to \chi,\ \varphi^j \to (\chi \to \psi),\ (\varphi^i \to \chi) \to ((\varphi^j \to (\chi \to \psi)) \to (\varphi^{i+j} \to \psi)),$$

so $\Gamma \vdash_{\mathbf{BCK}} \varphi^{i+j} \to \psi$ by (MP) (applied twice). This proves the deduction part of the result. □

5.2.9 COROLLARY

(i) *The quasivariety $\mathfrak{BCK}$ has the RCEP. In particular, every variety of BCK-algebras has the CEP.*

(ii) *The quasivariety* $\mathfrak{BCK}$ *has LEDPRC with a system of local defining equations*

$$\mathcal{E}(p,q,r,s) = \{E_i(p,q,r,s)\,;\, i=1,2\},$$

where
$$E_1(p,q,r,s) = \{(p\rightarrow q)^i \rightarrow [(q\rightarrow p)^j \rightarrow (r\rightarrow s)] \approx \mathrm{T}\,;\, i,j\in\omega\}$$
$$E_2(p,q,r,s) = \{(p\rightarrow q)^k \rightarrow [(q\rightarrow p)^\ell \rightarrow (s\rightarrow r)] \approx \mathrm{T}\,;\, k,\ell\in\omega\}.$$

In particular, every variety of BCK-algebras has LEDPC with a system of local defining equations $\mathcal{E}(p,q,r,s)$.

(iii) *Let* $\mathbf{A} = \langle A\,;\rightarrow,\mathrm{T}\rangle$ *be a BCK-algebra and* $a,b\in A$. *Then the ideal of* $\mathbf{A}$ *generated by* $\{a\}$ *is* $\{e\in A\,;\, a^i\rightarrow e = \mathrm{T}$ *for some* $i\in\omega\}$.

Proof. (i) This follows directly from Proposition 5.2.8 and Corollary 4.3.3.

(ii) By Proposition 5.2.8, **BCK** has the LDDT with local deduction-detachment system $\mathcal{E}(p,q) = \{E_i(p,q)\,;\, i\in\omega\} = \{\{p^i\rightarrow q\}\,;\, i\in\omega\}$. The following sets correspond to those defined in Section 4.3: Set $I^{(1)}=\omega$, $E_i^{(1)}(p_1,q) = \{p_1^i\rightarrow q\}$ and $\mathcal{E}^{(1)}(p_1,q) = \{\{p_1^i\rightarrow q\}\,;\, i\in\omega\}$. Now, set

$$\begin{aligned}\omega^{E_i^{(1)}} &= \{f : E_i^{(1)} \rightarrow \omega\}\\ &= \{\{\langle p_1^i\rightarrow q, j\rangle\}\,;\, j\in\omega\},\\ I^{(2)} &= \bigcup\{\omega^{E_i^{(1)}}\,;\, i\in\omega\}\\ &= \{\{\langle p_1^i\rightarrow q, j\rangle\}\,;\, j\in\omega\,;\, i\in\omega\}.\end{aligned}$$

For $f\in I^{(2)}$, $f = \{\langle p_1^i\rightarrow q, j\rangle\}$, say, set

$$\begin{aligned}E_f^{(2)}(p_1,p_2,q) &= \bigcup\{E_{f(\eta)}(p_2,\eta)\,;\, \eta\in\{p_1^i\rightarrow q\}\}\\ &= E_j(p_2, p_1^i\rightarrow q)\\ &= \{p_2^j\rightarrow(p_1^i\rightarrow q)\}.\end{aligned}$$

Finally, set
$$\begin{aligned}\mathcal{E}^{(2)}(p_1,p_2,q) &= \{\{p_2^j\rightarrow(p_1^i\rightarrow q)\}\,;\, \{\langle p_1^i\rightarrow q, j\rangle\}\in I^{(2)}\}\\ &= \{\{p_2^j\rightarrow(p_1^i\rightarrow q)\}\,;\, i\in\omega\,;\, j\in\omega\}.\end{aligned}$$

Set $S_1 = \mathbf{BCK}$ and $S_2 = S_{\mathfrak{BCK}}$. Let $\boldsymbol{\tau} = \{\tau\}$ be the interpretation of S_1 in S_2 defined by $\tau(\varphi) = (\varphi,\mathrm{T})$ and let $\boldsymbol{\rho} = \{\rho^1,\rho^2\}$ be the interpretation of S_2 in S_1 defined by $\rho^1(\varphi,\psi) = \varphi\rightarrow\psi$, $\rho^2(\varphi,\psi) = \psi\rightarrow\varphi$. By the proof of Theorem 4.3.2, we have that S_2 has the LDDT since S_1 has the LDDT. Let $\boldsymbol{p} = (p_1,p_2)$, $\boldsymbol{q} = (q_1,q_2)$. In the proof of Theorem 4.3.2, we established that a local deduction-detachment system for S_2 is given by

$$\mathcal{E}'(\boldsymbol{p},\boldsymbol{q}) = \{E'_{\boldsymbol{i}}(\boldsymbol{p},\boldsymbol{q})\,;\, \boldsymbol{i}\in\left(I^{(2)}\right)^2\},$$

where $E'_{\boldsymbol{i}}(\boldsymbol{p},\boldsymbol{q}) = \tau\Big(\bigcup\{E_{i_j}^{(2)}(\rho^1(\boldsymbol{p}),\rho^2(\boldsymbol{p}),\rho^j(\boldsymbol{q}))\,;\, j\le 2\}\Big)$, where $\boldsymbol{i} = \langle i_1,i_2\rangle \in \left(I^{(2)}\right)^2$. Now, for $\boldsymbol{i} = \langle i_1,i_2\rangle$, $i_1 = \{\langle p_1^i\rightarrow q, j\rangle\}$, $i_2 = \{\langle p_1^k\rightarrow q,\ell\rangle\}$,

$$E'_{\boldsymbol{i}}(\boldsymbol{p},\boldsymbol{q}) = \tau\Big(\bigcup\{E^{(2)}_{i_j}(\rho^1(\boldsymbol{p}),\rho^2(\boldsymbol{p}),\rho^j(\boldsymbol{q}))\,;\; j\le 2\}\Big)$$
$$= \tau\Big(E^{(2)}_{i_1}(p_1\to p_2, p_2\to p_1, q_1\to q_2)\cup E^{(2)}_{i_2}(p_1\to p_2, p_2\to p_1, q_2\to q_1)\Big)$$
$$= \tau\Big(\{(p_2\to p_1)^j\to((p_1\to p_2)^i\to(q_1\to q_2)), (p_2\to p_1)^\ell\to((p_1\to p_2)^k\to(q_2\to q_1))\}\Big)$$
$$= \{((p_2\to p_1)^j\to((p_1\to p_2)^i\to(q_1\to q_2)),\mathrm{T}),((p_2\to p_1)^\ell\to((p_1\to p_2)^k\to(q_2\to q_1)),\mathrm{T})\}.$$

By Corollary 4.1.14 and its proof, $\mathfrak{BCK}$ has LEDPRC with respect to $\mathcal{E}'(\boldsymbol{p},\boldsymbol{q})$, i.e., if $\mathbf{A}\in\mathfrak{BCK}$ and $a,b,c,d\in A$ then $(c,d)\in\Theta^{\mathbf{A}}_{\mathfrak{BCK}}(a,b)$ if and only if there exists $\boldsymbol{i}=\langle i_1,i_2\rangle\in\left(I^{(2)}\right)^2$, where $i_1=\langle p_1^i\to q,j\rangle$, $i_2=\langle p_1^k\to q,\ell\rangle$ (equivalently, if there exist $i,j,k,\ell\in\omega$) such that

$$(a\to b)^i\to[(b\to a)^j\to(c\to d)]=\mathrm{T}$$

and
$$(a\to b)^k\to[(b\to a)^\ell\to(d\to c)]=\mathrm{T}.$$

(iii) By (ii), $(c,d)\in\Theta^{\mathbf{A}}_{\mathfrak{BCK}}(a,\mathrm{T})$ if and only if there exist $i,j,k,\ell\in\omega$ such that

$$(a\to\mathrm{T})^i\to[(\mathrm{T}\to a)^j\to(c\to d)]=\mathrm{T}$$

and
$$(a\to\mathrm{T})^k\to[(\mathrm{T}\to a)^\ell\to(d\to c)]=\mathrm{T},$$

i.e.,
$$a^j\to(c\to d)=\mathrm{T}\quad\text{and}\quad a^\ell\to(d\to c)=\mathrm{T}.$$

So $\Theta^{\mathbf{A}}_{\mathfrak{BCK}}(a,\mathrm{T})=\{(c,d)\in A^2;\text{ there exist } j,\ell\in\omega \text{ such that } a^j\to(c\to d)=\mathrm{T},\ a^\ell\to(d\to c)=\mathrm{T}\}$. Since $\mathbf{Id}\,\mathbf{A}\cong\mathbf{Con}_{\mathfrak{BCK}}\mathbf{A}$ with the isomorphism given in Corollary 5.2.6, it follows that the ideal of $\mathbf{A}$ generated by $\{a\}$ is equal to $\mathrm{T}/\Theta^{\mathbf{A}}_{\mathfrak{BCK}}(a,\mathrm{T})$, and this is equal to

$$\{e\in A;\,(e,\mathrm{T})\in\Theta^{\mathbf{A}}_{\mathfrak{BCK}}(a,\mathrm{T})\}$$
$$=\{e\in A;\text{ there exist } j,\ell\in\omega\text{ such that } a^j\to(e\to\mathrm{T})=\mathrm{T},\ a^\ell\to(\mathrm{T}\to e)=\mathrm{T}\}$$
$$=\{e\in A;\text{ there exists }\ell\in\omega\text{ such that } a^\ell\to e=\mathrm{T}\}.$$ □

5.2.10 THEOREM [BP89b]

$\mathfrak{BCK}$ does not have equationally definable principal relative congruences (EDPRC).

Proof. We begin by proving a general result for quasivarieties. Let $\mathfrak{K}$ be a quasivariety and $\mathbf{A}\in\mathfrak{K}$; we say that $\mathbf{A}$ is *$\mathfrak{K}$-simple* if the only $\mathfrak{K}$-congruences of $\mathbf{A}$ are I_A and A^2. The subclass of $\mathfrak{K}$ consisting of all $\mathfrak{K}$-simple algebras is denoted $\mathfrak{K}_S$.

Claim: If a quasivariety $\mathfrak{K}$ has EDPRC, then the subclass $\mathfrak{K}_S$ is first-order definable relative to the language of $\mathfrak{K}$.

Proof. Suppose that $\mathfrak{K}$ has EDPRC with defining equations $\eta_{i1}(p_1,p_2,q_1,q_2)\approx\eta_{i2}(p_1,p_2,q_1,q_2)$, $i\le m$. Then we have that $\mathbf{A}\in\mathfrak{K}_S$ if and only if $\mathbf{A}$ satisfies the first-order formula

$$(5.2.3)\qquad \forall x\,\forall y\,\forall z\,\forall w\,(x \not\approx y \Rightarrow \mathop{\&}_{i \leq m} \eta_{i1}(x,y,z,w) \approx \eta_{i2}(x,y,z,w)\,).$$

To see this, recall that for all $a, b, c, d \in A$,

$$(c,d) \in \Theta^{\mathbf{A}}_{\mathcal{K}}(a,b) \text{ if and only if } \eta^{\mathbf{A}}_{i1}(a,b,c,d) = \eta^{\mathbf{A}}_{i2}(a,b,c,d) \text{ for all } i \leq m.$$

For $\mathbf{A} \in \mathcal{K}_S$, we have $\Theta^{\mathbf{A}}_{\mathcal{K}}(a,b) = I_A$ if $a = b$, and $\Theta^{\mathbf{A}}_{\mathcal{K}}(a,b) = A^2$ if $a \neq b$. In other words, if $a \neq b$, then $(c,d) \in \Theta^{\mathbf{A}}_{\mathcal{K}}(a,b)$ for all $c, d \in A$, hence $\mathbf{A}$ satisfies (5.2.3). Conversely, suppose $\mathbf{A}$ satisfies (5.2.3). Let $a, b \in A$ such that $a \neq b$. Then, for all $c, d \in A$,

$$\eta^{\mathbf{A}}_{i1}(a,b,c,d) = \eta^{\mathbf{A}}_{i2}(a,b,c,d) \text{ for all } i \leq m,$$

hence $(c,d) \in \Theta^{\mathbf{A}}_{\mathcal{K}}(a,b)$, so $\Theta^{\mathbf{A}}_{\mathcal{K}}(a,b) = A^2$ and $\mathbf{A}$ is $\mathcal{K}$-simple. This proves the claim.

To prove the theorem, it will suffice to show that $\mathcal{BCK}_S$ is not first-order definable. Recall the BCK-algebra $\mathbf{N} = \langle \omega; \rightarrow, 0\rangle$ from Section 5.1; we shall show that $\mathbf{N}$ is $\mathcal{BCK}$-simple. First we show that $\{0\}$ and ω are its only ideals. Suppose $F \subseteq \omega$ is an ideal of $\mathbf{N}$ and that there exists a nonzero $x \in F$. Then, since $x \rightarrow 2x = 2x \dot{-} x = x \in F$, we also have $2x \in F$. Then $3x \dot{-} 2x = x \in F$, so $3x \in F$. By induction we have that $\ell x \in F$ for all $\ell \in \omega$. Now, for any $y \in \omega$, there exists an $\ell \in \omega$ such that $y \leq \ell x$. (Here and throughout this proof, $\leq$ means the natural order on ω, which is the inverse of the BCK-order of $\mathbf{N}$.) Thus $y \dot{-} \ell x = 0$, implying that $y \in F$ and hence that $F = \omega$. Recalling that $\mathbf{Id}\,\mathbf{N}$ is isomorphic to $\mathbf{Con}_{\mathcal{BCK}}\mathbf{N}$ (Corollary 5.2.6), we get that $\mathbf{Con}_{\mathcal{BCK}}\mathbf{N}$ has only two elements, hence $\mathbf{N}$ is $\mathcal{BCK}$-simple.

Let $\mathcal{U}$ be a free ultrafilter over ω. By Corollary 0.3.4, $\mathcal{U}$ contains all $X \subseteq \omega$ such that $\omega - X$ is finite. Set $\mathbf{A} = \Pi_{i \in \omega}\mathbf{N}/\mathcal{U}$. Then $\mathbf{A} \in P_U(\mathcal{BCK})$, hence $\mathbf{A} \in \mathcal{BCK}$ since every quasivariety is closed under ultraproducts. We show that $\mathbf{A}$ is not $\mathcal{BCK}$-simple. Define the canonical map $i : \omega \rightarrow \Pi_{i \in \omega}\mathbf{N}/\mathcal{U}$ as follows: For all $a \in \omega$, set

$$c_a = (a, a, a, \dots) \in \prod_{i \in \omega} \mathbf{N}$$

and set

$$ia = c_a/\mathcal{U}.$$

Thus,

$$ia = \{\overline{x} \in \prod_{i \in \omega} \mathbf{N};\ \{n \in \omega;\ \overline{x}(n) = a\} \in \mathcal{U}\}.$$

If we set $\overline{p} = (0,1,2,3,\dots) \in \Pi_{i \in \omega}\mathbf{N}$, then $\overline{p}/\mathcal{U} \in \mathbf{A}$, but $\overline{p}/\mathcal{U} \notin i(\omega)$: For suppose $\overline{p}/\mathcal{U} = ia$, where $a \in \omega$. This means that $Z \in \mathcal{U}$, where

$$Z = \{n \in \omega;\ \overline{p}(n) = (ia)(n) = a\}.$$

Obviously, $Z = \{a\}$, so $\omega - Z \in U \subseteq \mathcal{U}$. But $\mathcal{U}$ is a filter of the Boolean algebra of subsets of ω, so

$\emptyset = Z \cap (\omega - Z) \in \mathcal{U}$, contradicting the fact that $\mathcal{U}$ is an ultrafilter (i.e., a maximal *proper* filter) over ω. Thus i is not a surjective map. Next we show that $i(\omega)$ is an ideal of **A**. First, note that $i0 = c_0/\mathcal{U} = 0^{\mathbf{A}}$, hence $0^{\mathbf{A}} \in i(\omega)$. Let $X, Y \in A$ with $Y, X \dot{-} Y$ (i.e., $Y \rightarrow X) \in i(\omega)$, say $ia = Y$ and $ib = X \dot{-} Y$ $(a, b \in \omega)$. We claim that $X \in i(\omega)$: If $\overline{x} \in X$, then, for all $\overline{y} \in Y$, $\overline{x} \dot{-} \overline{y} \in X \dot{-} Y = ib$, hence the set $\{n \in \omega\,; \overline{x}(n) \dot{-} \overline{y}(n) = b\} \in \mathcal{U}$. Since the set $\{n \in \omega\,; \overline{y}(n) = a\} \in \mathcal{U}$, the intersection of these two sets is $\{n \in \omega\,; \overline{x}(n) \dot{-} a = b\}$, which is an element of $\mathcal{U}$, since $\mathcal{U}$ is a filter. Let $V = \{n \in \omega\,; \overline{x}(n) \dot{-} a = b\} \in \mathcal{U}$. There are two cases:

Case 1. Suppose $b = 0$, so that $V = \{n \in \omega\,; \overline{x}(n) \leq a\}$. For $r = 0, 1, \ldots, a$, set $V_r = \{n \in \omega\,; \overline{x}(n) = r\}$, so that

$$V = \bigcup_{0 \leq r \leq a} V_r.$$

Suppose that for all $r \in \{0, 1, \ldots, a\}$, we have $V_r \notin \mathcal{U}$, i.e., $\omega - V_r \in \mathcal{U}$ (see Theorem 0.3.2 (ii)). Then

$$\omega - V = \omega - \bigcup_{0 \leq r \leq a} V_r = \bigcap_{0 \leq r \leq a} (\omega - V_r) \in \mathcal{U}$$

since $\{0, 1, \ldots, a\}$ is finite and $\mathcal{U}$ is a filter, so again, $\emptyset = V \cap (\omega - V) \in \mathcal{U}$, a contradiction. Thus, there exists $r \in \omega$ with $r \leq a$ such that $V_r \in \mathcal{U}$. For this r, we have $X = \overline{x}/\mathcal{U} = ir \in i(\omega)$, as required.

Case 2. Suppose $b \neq 0$. Then $V = \{n \in \omega\,; \overline{x}(n) = a + b\} \in \mathcal{U}$, so $X = \overline{x}/\mathcal{U} = i(a + b) \in i(\omega)$, as required.

This proves that $i(\omega)$ is an ideal of **A**, and, since i is not onto, $i(\omega) \neq A$. Obviously $i(\omega) \neq \{0^{\mathbf{A}}\}$, therefore the lattice **Id A** of all ideals of **A** is not the 2-element chain. Since $\mathbf{Id\,A} \cong \mathbf{Con}_{\mathfrak{BCK}}\mathbf{A}$, we have that $\mathbf{Con}_{\mathfrak{BCK}}\mathbf{A}$ is not the 2-element chain either. Consequently **A** is not $\mathfrak{BCK}$-simple. Note that **A** is an ultrapower of **N**, so **A** must satisfy every first-order formula that **N** satisfies (Theorem 0.5.1). Since **N** is $\mathfrak{BCK}$-simple and **A** is not, it follows that $\mathfrak{BCK}$-simple algebras are not first-order definable over $\mathfrak{BCK}$. By the contrapositive of the Claim, we deduce that $\mathfrak{BCK}$ does not have EDPRC. □

This result, in conjuction with Corollary 4.6.2, immediately gives the following

5.2.11 COROLLARY

BCK *does not have the deduction-detachment theorem (DDT).* □

5.2.12 LEMMA

BCK *is filter-distributive.*

Proof. We know, by Proposition 5.2.8, that **BCK** has the LDDT with deduction-detachment system $\mathcal{S} = \mathcal{S}(p,q) = \{\{p^i \to q\}\,;\, i \in \omega\}$. Also, **BCK** is protoalgebraic (since it is algebraizable). To prove the theorem, we shall show that $\mathcal{S}$ is **BCK**-directed and deduce the result from Theorem 4.2.2. Let $i, j \in \omega$. We need to show that there exists $k \in \omega$ such that

$$p^i \to q \vdash_{\mathbf{BCK}} p^k \to q \quad \text{and} \quad p^j \to q \vdash_{\mathbf{BCK}} p^k \to q.$$

We first show, by induction, that for all $\ell, m \in \omega$, if $\ell \geq m$, then

$$p^m \to q \vdash_{\mathbf{BCK}} p^\ell \to q.$$

For $\ell = m$, the result is trivial. Suppose now that $\ell \geq m$ and $p^m \to q \vdash_{\mathbf{BCK}} p^\ell \to q$. By (K),

$$\vdash_{\mathbf{BCK}} (p^\ell \to q) \to [p \to (p^\ell \to q)],$$

i.e., $\vdash_{\mathbf{BCK}} (p^\ell \to q) \to (p^{\ell+1} \to q)$. By (K), again

$$\vdash_{\mathbf{BCK}} ((p^\ell \to q) \to (p^{\ell+1} \to q)) \to [(p^m \to q) \to ((p^\ell \to q) \to (p^{\ell+1} \to q))],$$

hence, by (MP),

$$\vdash_{\mathbf{BCK}} (p^m \to q) \to ((p^\ell \to q) \to (p^{\ell+1} \to q)).$$

Thus, by (MP),

$$p^m \to q \vdash_{\mathbf{BCK}} (p^\ell \to q) \to (p^{\ell+1} \to q).$$

Since $p^m \to q \vdash_{\mathbf{BCK}} p^\ell \to q$, we finally get, by (MP), that

$$p^m \to q \vdash_{\mathbf{BCK}} p^{\ell+1} \to q,$$

completing the induction. Setting $k = \max\{i, j\}$, we can now deduce that $p^i \to q \vdash_{\mathbf{BCK}} p^k \to q$ and $p^j \to q \vdash_{\mathbf{BCK}} p^k \to q$. Thus $\mathcal{S}$ is **BCK**-directed and the result follows. □

That **BCK** is filter-distributive means that, for all $\mathcal{A} = \langle \mathbf{A}, F_{\mathcal{A}} \rangle \in \mathrm{Mod}\,\mathbf{BCK}$ and $I, J, K \in \mathrm{Fi}^{\mathbf{BCK}}\mathcal{A}$, $I \cap (J \vee K) = (I \cap J) \vee (I \cap K)$. Recalling that $\mathbf{Fi}^{\mathbf{BCK}}\mathcal{A} \cong \mathbf{Con}_{\mathfrak{BCK}}\mathbf{A}$ (whenever $F_{\mathcal{A}}$ is the least subset of A such that $\langle \mathbf{A}, F_{\mathcal{A}} \rangle \in \mathrm{Mod}\,\mathbf{BCK}$), we can take $\mathcal{A} = \langle \mathbf{A}, \{\mathrm{T}^{\mathbf{A}}\} \rangle$ and deduce the following

5.2.13 COROLLARY

$\mathfrak{BCK}$ *is relatively congruence distributive (RCD). Consequently, every variety of BCK-algebras is* *congruence distributive.* □

5.2.14 COROLLARY [Pal81]

Id A *is distributive for every BCK-algebra* **A**. □

5.3 VARIETIES OF *BCK*-ALGEBRAS

A variety of BCK-algebras (i.e., a subvariety of $\mathfrak{BCK}$) may be assumed to be axiomatized by the axioms of $\mathfrak{BCK}$ together with identities of the form $t(\overline{x}) \approx \mathrm{T}$ (in view of (5.1.4)). Such varieties are just the equivalent variety semantics of the strongly algebraizable axiomatic extensions of **BCK**. Here we describe some of the varieties of BCK-algebras that arise naturally from this perspective or that have been studied in the literature.

Commutative *BCK*-algebras.

Consider the deductive system S that is obtained from **BCK** by the addition of the axiom

(A) $\quad ((p \to q) \to q) \to ((q \to p) \to p).$

By Corollary 3.1.17, S is algebraizable with the same defining equations and equivalence formulas as **BCK**. The equivalent quasivariety semantics is the subquasivariety $\mathfrak{T}$ of $\mathfrak{BCK}$ that satisfies the identity

(T) $\quad (x \to y) \to y \approx (y \to x) \to x.$

This follows immediately from Theorem 3.1.18, axiom (A) and (5.1.4). This quasivariety is, in fact, a variety as can be seen from the following: Suppose $\mathbf{A} \in \mathfrak{T}$ and $a, b \in A$ with $a \to b = \mathrm{T}$ and $b \to a = \mathrm{T}$. Then

$$\begin{aligned} a &= \mathrm{T} \to a && \text{[by (5.1.2)]} \\ &= (b \to a) \to a && \\ &= (a \to b) \to b && \text{[by (T)]} \\ &= \mathrm{T} \to b && \\ &= b && \text{[by (5.1.2)],} \end{aligned}$$

hence the quasi-identity (5.1.4) follows from the defining identities of $\mathfrak{T}$. We call $\mathfrak{T}$ the *variety of commutative BCK-algebras.* It was first studied by Tanaka [Tan75].

Let S_{F} be the extension of the deductive system S (defined above) obtained by adding a constant symbol F to the language of **BCK** and the axiom

$$\vdash_{S_{\mathrm{F}}} \mathrm{F} \rightarrow p.$$

Then S_{F} is also algebraizable with the same defining equations and equivalence formulas as S. The equivalent quasivariety semantics is thus termwise definitionally equivalent to the class of all algebras $\mathbf{A} = \langle A\,;\ \rightarrow, \mathrm{T}, \mathrm{F}\rangle$ of type $\langle 2,0,0\rangle$ that satisfy the identities of $\mathcal{T}$ along with the identity

$$\mathrm{F} \rightarrow x \approx \mathrm{T}.$$

This is precisely the class of so-called *bounded commutative BCK-algebras*, denoted $\mathcal{T}_{\mathrm{F}}$, which was first studied in [Tra79]. Every algebra in $\mathcal{T}_{\mathrm{F}}$ has a least element F since the above identity is equivalent to $\mathrm{F} \leq x$. The class $\mathcal{T}_{\mathrm{F}}$ is termwise definitionally equivalent to the variety $\mathcal{W}$ of Wajsberg algebras (see Section 0.2): Add a constant F to the type of $\mathcal{W}$, defined by $\mathrm{F} = \neg\mathrm{T}$, and add a unary operation $\neg$ to the type of $\mathcal{T}_{\mathrm{F}}$, defined by $\neg x = x \rightarrow \mathrm{F}$. Axioms (W1), (W2) and (W3) of Section 0.2 are, respectively, (5.1.2), (5.1.1) and (T), hence they are identities of $\mathcal{T}_{\mathrm{F}}$. To see that (W4) is an identity of $\mathcal{T}_{\mathrm{F}}$, consider the following derivation over $\mathcal{T}_{\mathrm{F}}$:

$$y \rightarrow x \overset{(2)}{\approx} (\mathrm{T} \rightarrow y) \rightarrow (\mathrm{T} \rightarrow x) \approx ((\mathrm{F} \rightarrow y) \rightarrow y) \rightarrow ((\mathrm{F} \rightarrow x) \rightarrow x) \overset{(\mathrm{T})}{\approx} ((y \rightarrow \mathrm{F}) \rightarrow \mathrm{F}) \rightarrow ((x \rightarrow \mathrm{F}) \rightarrow \mathrm{F}),$$

hence

$$\begin{aligned} [(\neg x) \rightarrow (\neg y)] \rightarrow (y \rightarrow x) &\approx [(x \rightarrow \mathrm{F}) \rightarrow (y \rightarrow \mathrm{F})] \rightarrow (y \rightarrow x) && \text{[by definition of } \neg]\\ &\approx [(x \rightarrow \mathrm{F}) \rightarrow (y \rightarrow \mathrm{F})] \rightarrow [((y \rightarrow \mathrm{F}) \rightarrow \mathrm{F}) \rightarrow ((x \rightarrow \mathrm{F}) \rightarrow \mathrm{F})] \\ &\approx \mathrm{T} && \text{[by (5.1.1)]}. \end{aligned}$$

We also need to show that $\mathrm{F} \approx \neg\mathrm{T}$ is an identity of $\mathcal{T}_{\mathrm{F}}$. But this follows easily since $\neg\mathrm{T} \approx \mathrm{T} \rightarrow \mathrm{F} \overset{(2)}{\approx} \mathrm{F}$.

Conversely, we must show that the class $\mathcal{W}$ satisfies the axioms of $\mathcal{BCK}$, (T) and the identities $\mathrm{F} \rightarrow x \approx \mathrm{T}$ and $\neg x \approx x \rightarrow \mathrm{F}$. The identities (W1), (W2) and (W3) are (5.1.2), (5.1.1) and (T), respectively. That (5.1.3) holds in $\mathcal{W}$ is proved in the following way:

$$\begin{aligned} (x \rightarrow \mathrm{T}) \rightarrow \mathrm{T} &\overset{(\mathrm{W3})}{\approx} (\mathrm{T} \rightarrow x) \rightarrow x \overset{(\mathrm{W1})}{\approx} (\mathrm{T} \rightarrow x) \rightarrow (\mathrm{T} \rightarrow x) \overset{(\mathrm{W1})}{\approx} \mathrm{T} \rightarrow ((\mathrm{T} \rightarrow x) \rightarrow (\mathrm{T} \rightarrow x)) \\ &\overset{(\mathrm{W1})}{\approx} (\mathrm{T} \rightarrow \mathrm{T}) \rightarrow ((\mathrm{T} \rightarrow x) \rightarrow (\mathrm{T} \rightarrow x)) \overset{(\mathrm{W2})}{\approx} \mathrm{T}. \end{aligned}$$

Thus

$$x \rightarrow \mathrm{T} \approx x \rightarrow ((x \rightarrow \mathrm{T}) \rightarrow \mathrm{T}) \overset{(\mathrm{W1})}{\approx} (\mathrm{T} \rightarrow x) \rightarrow ((x \rightarrow \mathrm{T}) \rightarrow (\mathrm{T} \rightarrow \mathrm{T})) \overset{(\mathrm{W2})}{\approx} \mathrm{T}.$$

It was shown earlier that (5.1.4) follows from (5.1.1), (5.1.2), (5.1.3) and (T) (i.e., (W3)). Thus

$\mathcal{W}$ is a subvariety of $\mathcal{T}$. In particular, each of the identities of $\mathcal{BCK}$ hold in $\mathcal{W}$. To see that $\neg x \approx x \to \mathrm{F}$ holds in $\mathcal{W}$, consider the following derivation over $\mathcal{W}$:

$$(\neg x) \to \mathrm{F} \approx (\neg x) \to (\neg \mathrm{T}) \overset{(\mathrm{W4})}{\leq} \mathrm{T} \to x \overset{(\mathrm{W1})}{\approx} x, \text{ so}$$

(5.3.1) $$(\neg x) \to \mathrm{F} \leq x.$$

By the $\mathcal{BCK}$ identity $x \leq y \to x$, we have $\neg x \leq (\neg \mathrm{F}) \to (\neg x) \overset{(\mathrm{W4})}{\leq} x \to \mathrm{F}$, so

(5.3.2) $$\neg x \leq x \to \mathrm{F}.$$

A substitution instance of (5.3.2) gives

(5.3.3) $$\neg(\neg x) \leq (\neg x) \to \mathrm{F}.$$

By (5.3.1) and (5.3.3),

(5.3.4) $$\neg(\neg x) \leq x.$$

Now, $x \to \mathrm{F} \leq (\neg(\neg x)) \to \mathrm{F}$ by (5.3.4) and (5.1.7), and $(\neg(\neg x)) \to \mathrm{F} \leq \neg x$ by (5.3.1), so

(5.3.5) $$x \to \mathrm{F} \leq \neg x.$$

By (5.3.2) and (5.3.5) we therefore have that $\neg x \approx x \to \mathrm{F}$.

To see that $\mathrm{F} \to x \approx \mathrm{T}$ holds in $\mathcal{W}$, consider the following derivation over $\mathcal{W}$ (recall that (10) means (5.1.10)):

$$\mathrm{F} \to x \overset{(10)}{\approx} ((\mathrm{F} \to x) \to x) \to x \overset{(\mathrm{W3})}{\approx} ((x \to \mathrm{F}) \to \mathrm{F}) \to x \approx ((\neg x) \to (\neg \mathrm{T})) \to (\mathrm{T} \to x) \overset{(\mathrm{W4})}{\approx} \mathrm{T}.$$

Recall from Section 3.3 that the equivalent quasivariety semantics for the deductive system S_ω is the variety $\mathcal{W}$ of Wajsberg algebras. Thus the deductive systems S_ω and S_F have equivalent quasivariety semantics that are termwise definitionally equivalent. For further algebraic discussion of this semantics, see [BP].

Lastly, we provide the connection between the $\{\to\}$-fragment of S_ω and $\mathcal{BCK}$. The equivalent quasivariety semantics, $\mathcal{K}$ say, of the $\{\to\}$-fragment of S_ω is a subvariety of $\mathcal{T}$. To see this, note first that $\mathcal{K}$ satisfies (3.3.23) and (3.3.27) (of the section on Łukasiewicz logics in Section 3.3), which are (5.1.1) and (5.1.4), respectively. That $\mathcal{K}$ satisfies (T) follows immediately from (3.3.24) and (5.1.4). We shall show that $\mathcal{K}$ also satisfies (5.1.2) and (5.1.3) over $\mathcal{K}$. By (3.3.22),

$$\mathrm{T} \to (x \to \mathrm{T}) \approx \mathrm{T},$$

hence (3.3.26) implies that $x \to \mathrm{T} \approx \mathrm{T}$, i.e., (5.1.3) holds. By (3.3.22) again, we have $x \to (\mathrm{T} \to x) \approx \mathrm{T}$, and, by (T),

$$(\mathrm{T} \to x) \to x \approx (x \to \mathrm{T}) \to \mathrm{T} \approx \mathrm{T} \to \mathrm{T} \approx \mathrm{T},$$

hence $\mathrm{T} \to x \approx x$ by (3.3.27), i.e., (5.1.2) holds.

Since $\mathfrak{K}$ is a subquasivariety of $\mathfrak{BCK}$, it follows that we can dispense with the quasi-identity (3.3.26), and since $\mathfrak{K}$ satisfies (T), we can dispense with the quasi-identity (3.3.27), i.e., $\mathfrak{K}$ is a variety and, in particular, a subvariety of $\mathfrak{T}$. By Corollary 3.1.8, $\mathfrak{K}$ is the class of all $\{\to, \mathrm{T}\}$-subreducts of Wajsberg algebras, and it is known that $\mathfrak{K}$ is a *proper* subvariety of $\mathfrak{T}$. In fact all subdirectly irreducible members of $\mathfrak{K}$ are linearly ordered by $\leq$, [Pał80], [Tra79], while it is easy to construct subdirectly irreducible members of $\mathfrak{T}$ that are not. Bounded commutative *BCK*-algebras are also termwise definitionally equivalent to the so-called MV-algebras of Chang [Cha58], which have a much less elegant definition, [Mun86].

Tarski Algebras.

Consider the $\{\to\}$-fragment of **CPC**, denoted $\mathbf{CPC}_{\to}$. Each of the axioms (B), (C) and (K) is easily seen to be a theorem of $\mathbf{CPC}_{\to}$. Thus $\mathbf{CPC}_{\to}$ is an axiomatic extension of **BCK**, hence Corollary 3.1.17 implies that $\mathbf{CPC}_{\to}$ is algebraizable with the same defining equations and equivalence formulas as **BCK**.

It is well-known that the two-element Boolean algebra **2** is the only subdirectly irreducible Boolean algebra. Equivalently, by Birkhoff's Subdirect Decomposition Theorem (Theorem 0.2.8), $\mathfrak{B}\mathcal{A} = \mathrm{IP_S}(\mathbf{2})$ ($= \mathrm{HSP}(\mathbf{2})$, since $\mathfrak{B}\mathcal{A}$ is a variety). Let $\mathfrak{B}\mathcal{A}_{\to}$ denote the class of all $\{\to\}$-reducts of Boolean algebras (where $x \to y = (x') \vee y$). $\mathrm{S}(\mathfrak{B}\mathcal{A}_{\to})$ is termwise definitionally equivalent to $\mathrm{S}(\mathfrak{B}\mathcal{A}_{\to, \mathrm{T}})$, the class of $\{\to, \mathrm{T}\}$-subreducts of Boolean algebras, since $\mathfrak{B}\mathcal{A}$ satisfies $x \to x \approx \mathrm{T}$. (We are writing T in place of 1.)

It turns out that $\mathrm{S}(\mathfrak{B}\mathcal{A}_{\to, \mathrm{T}})$ is the subquasivariety $\mathfrak{I}$ of $\mathfrak{BCK}$ that satisfies the identity

(I′) $\qquad (x \to y) \to x \approx x.$

(This result is essentially contained in [Kal60].) This class forms a subquasivariety of the variety of commutative *BCK*-algebras and is therefore a variety. To see this we need only show that (T) holds in $\mathfrak{I}$. So,

$$((x \to y) \to y) \to ((y \to x) \to x) \approx (y \to x) \to (((x \to y) \to y) \to x) \qquad \text{[by (5.1.14)]}$$

$$\approx (y \to x) \to (((x \to y) \to y) \to ((x \to y) \to x)) \qquad \text{[by (I$'$)]}$$

$$\approx \mathrm{T} \qquad \text{[by (5.1.13)].}$$

Thus $\mathbf{CPC}_{\to}$ is strongly algebraizable. Its equivalent quasivariety semantics is called the variety of *Tarski algebras* (see [Mon60]) (or *implication algebras*). Observe that by the quasi-identity (5.1.4), (I$'$) is equivalent, over $\mathcal{BCK}$, to the conjunction of $x \to ((x \to y) \to x) \approx \mathrm{T}$ and $((x \to y) \to x) \to x \approx \mathrm{T}$, of which the former conjunct holds in every BCK-algebra. We may deduce that (B), (C), (K) and the formula $((p \to q) \to p) \to p$ (together with (MP)) axiomatize $\mathbf{CPC}_{\to}$.

Recall that $\mathbf{C}_2$ is the two-element BCK-algebra (i.e., the $\{\to, \mathrm{T}\}$-reduct of $\mathbf{2}$). Since the $\{\to, \mathrm{T}\}$-subreducts of subdirect products of $\mathbf{2}$ are just the subdirect products of $\mathbf{C}_2$, it follows that $\mathcal{I} = \mathrm{IP_S}(\mathbf{C}_2) = \mathrm{ISPP_U}(\mathbf{C}_2)$ (by Theorem 0.3.6, since $\mathbf{C}_2$ is finite), the quasivariety generated by $\mathbf{C}_2$. This may be rephrased as an $\{\to\}$-analogue of the (strong) Validity and Completeness Theorems of $\mathbf{CPC}$: For a set $\Gamma \cup \{\varphi\}$ of $\{\to\}$-formulas,

$$\Gamma \vdash_{\mathbf{CPC}_{\to}} \varphi \quad \text{iff} \quad \{\psi \approx \mathrm{T}\,;\, \psi \in \Gamma\} \models_{\mathcal{I}} \varphi \approx \mathrm{T} \quad \text{iff} \quad \{\psi \approx \mathrm{T}\,;\, \psi \in \Gamma\} \models_{\mathbf{C}_2} \varphi \approx \mathrm{T}.$$

Since $\mathcal{I}$ is a variety, we also have $\mathcal{I} = \mathrm{HSP}(\mathbf{C}_2)$. It is easy to see that every nontrivial BCK-algebra $\mathbf{A}$ contains a subalgebra isomorphic to $\mathbf{C}_2$ (for any $a \in A$ such that $a \neq \mathrm{T}$, $\{a, \mathrm{T}\}$ is the universe of such a subalgebra). It follows that $\mathcal{I}$ is the smallest nontrivial subquasivariety of $\mathcal{BCK}$.

Hilbert Algebras.

Consider the $\{\to\}$-fragment of $\mathbf{IPC}$, denoted $\mathbf{IPC}_{\to}$. Just as for $\mathbf{CPC}_{\to}$, it follows that $\mathbf{IPC}_{\to}$ is algebraizable. It is known that the equivalent quasivariety semantics for $\mathbf{IPC}_{\to}$ is precisely the class of all BCK-algebras that satisfy the identity

(E_1) $\qquad y \to (y \to x) \approx y \to x.$

As for $\mathbf{CPC}_{\to}$, one can produce a finite axiomatization of $\mathbf{IPC}_{\to}$, namely the axioms (B), (C), (K) and the formula $(p \to (p \to q)) \to (p \to q)$ (and (MP)). The equivalent quasivariety semantics for $\mathbf{IPC}_{\to}$ is also a variety (we shall prove this in a more general setting in the section on $\mathcal{E}_n$). Thus $\mathbf{IPC}_{\to}$ is also strongly algebraizable. Its equivalent variety semantics, denoted $\mathcal{E}_1$, is called the variety of *Hilbert Algebras* (see [Die66]), or *positive implicative BCK-algebras.* It will be shown later that this variety has EDPC. It is related to $\mathcal{T}$ and $\mathcal{I}$ in the following way: $\mathcal{I} = \mathcal{T} \cap \mathcal{E}_1$. This can be deduced from the following:

5.3.1 PROPOSITION [IT78, Theorem 10]

A BCK-algebra $\mathbf{A} = \langle A ; \rightarrow, \mathrm{T} \rangle$ *satisfies* (T) *and* (E_1) *if and only if it satisfies* (I′).

Proof. Suppose that **A** satisfies (T) and (E_1). Let $a, b \in A$. Then

$$\begin{aligned} ((a \rightarrow b) \rightarrow a) \rightarrow a &= (a \rightarrow (a \rightarrow b)) \rightarrow (a \rightarrow b) && \text{[by (T)]} \\ &= (a \rightarrow b) \rightarrow (a \rightarrow b) && \text{[by } (E_1)\text{]} \\ &= \mathrm{T} && \text{[by (5.1.5)],} \end{aligned}$$

i.e., $(a \rightarrow b) \rightarrow a \leq a$. Since $a \leq (a \rightarrow b) \rightarrow a$ by (5.1.8), (I′) holds in **A**.

The converse follows from the fact that $\mathcal{I}$ is the smallest nontrivial quasivariety of *BCK*-algebras, but we give a direct derivation also. Suppose **A** satisfies (I′). By (5.1.9), $a \leq (a \rightarrow b) \rightarrow b$, hence

$$\begin{aligned} (b \rightarrow a) \rightarrow a &\leq (b \rightarrow a) \rightarrow ((a \rightarrow b) \rightarrow b) && \text{[by (5.1.6)]} \\ &= (a \rightarrow b) \rightarrow ((b \rightarrow a) \rightarrow b) && \text{[by (5.1.11)]} \\ &= (a \rightarrow b) \rightarrow b && \text{[by (I}')\text{].} \end{aligned}$$

Thus $(b \rightarrow a) \rightarrow a \leq (a \rightarrow b) \rightarrow b$. By symmetry, (T) holds in **A**. Note that $b \rightarrow a \leq b \rightarrow (b \rightarrow a)$, by (5.1.8). So we need only show that $b \rightarrow (b \rightarrow a) \leq b \rightarrow a$, i.e., that $(b \rightarrow (b \rightarrow a)) \rightarrow (b \rightarrow a) = \mathrm{T}$, to prove that (E_1) holds in **A**. Now

$$\begin{aligned} (b \rightarrow (b \rightarrow a)) \rightarrow (b \rightarrow a) &= ((b \rightarrow a) \rightarrow b) \rightarrow b && \text{[by (T)]} \\ &= b \rightarrow b && \text{[by (I}')\text{]} \\ &= \mathrm{T}. \end{aligned}$$

□

The Varieties $\mathcal{E}_n$.

Let $n \in \omega$. Recall that $p^n \rightarrow q$ stands for $p \rightarrow (p \rightarrow \ldots (p \rightarrow q) \ldots)$, where there are precisely n occurrences of p. For each $n \in \omega$ such that $n \geq 1$, let S'_n be the deductive system that is the extension of **BCK** by the axiom

(X_n) $\qquad (p^{n+1} \rightarrow q) \rightarrow (p^n \rightarrow q)$.

It follows from Corollary 3.1.17 that S'_n is algebraizable with the same defining equations and equivalence formulas as **BCK**. The equivalent quasivariety semantics for S'_n is thus the class of all *BCK*-algebras satisfying the additional identity

(5.3.6) $(x^{n+1} \to y) \to (x^n \to y) \approx \mathrm{T},$

(where T is defined as $z \to z$ for any z). It is possible to replace this identity by the following

(E_n) $x^{n+1} \to y \approx x^n \to y.$

That (E_n) follows from (5.3.6) can be proved using (5.1.4) and the fact that the identity

$$(x^n \to y) \to (x^{n+1} \to y) \approx (x^n \to y) \to (x \to (x^n \to y)) \overset{(8)}{\approx} \mathrm{T}$$

holds in all BCK-algebras. Conversely, one can deduce (5.3.6) from (E_n) using (5.1.5), which implies that $\mathrm{T} \approx (x^{n+1} \to y) \to (x^n \to y)$. Define $\mathcal{E}_n$ to be the class of all BCK-algebras that satisfy (E_n). $\mathcal{E}_n$ is a variety, as can be seen from the following: By (5.1.9),

$$y \leq (y \to x) \to x,$$

hence by applying (5.1.6) $2n$ times, we get

$$(x \to y)^n \to ((y \to x)^n \to y) \leq (x \to y)^n \to ((y \to x)^n \to ((y \to x) \to x)),$$

i.e., $$(x \to y)^n \to ((y \to x)^n \to y) \leq (x \to y)^n \to ((y \to x)^{n+1} \to x),$$

so, by (E_n), $$(x \to y)^n \to ((y \to x)^n \to y) \leq (x \to y)^n \to ((y \to x)^n \to x).$$

By symmetry and (5.1.11), we have $(x \to y)^n \to ((y \to x)^n \to y) \geq (x \to y)^n \to ((y \to x)^n \to x)$, hence

(5.3.7) $$(x \to y)^n \to ((y \to x)^n \to y) \approx (x \to y)^n \to ((y \to x)^n \to x)$$

since $\leq$ is a partial order. Now, if $x \to y \approx \mathrm{T}$ and $y \to x \approx \mathrm{T}$, then (5.3.7) reduces to

$$\mathrm{T}^n \to (\mathrm{T}^n \to y) \approx \mathrm{T}^n \to (\mathrm{T}^n \to x),$$

hence $y \approx x$. So we can drop the quasi-identity (5.1.4) from the axiomatization of $\mathcal{E}_n$ as it is derivable from (5.3.7), which holds in $\mathcal{E}_n$. Thus $\mathcal{E}_n$ is a variety.

5.3.2 THEOREM

For each positive integer n, S'_n has the DDT with deduction-detachment set

$$E(p,q) = \{p^n \to q\}.$$

Proof. The proof of Theorem 5.2.8 may be used verbatim to show that S'_n has the LDDT with local deduction-detachment system $\mathcal{E}(p,q) = \{\{p^i \to q\}\,;\, i \in \omega\}$. (The only difference is that a derivation of ψ from $\Gamma \cup \{\varphi\}$ may include the axiom (X_n).) Let $\Gamma \subseteq Fm$ and $\varphi, \psi \in Fm$ and suppose that $\Gamma \vdash_{S'_n} \varphi^n \to \psi$. Then

$$\Gamma, \varphi \vdash_{S'_n} \varphi, \varphi^n \to \psi$$

so $$\Gamma, \varphi \vdash_{S'_n} \varphi, \varphi^{n-1} \to \psi \qquad \text{[by (MP)]}$$

$$\vdots$$

so $\Gamma, \varphi \vdash_{S'_n} \varphi, \varphi \to \psi$ [by (MP)]

so $\Gamma, \varphi \vdash_{S'_n} \psi$ [by (MP)].

This proves the detachment part of the result. Suppose now that $\Gamma, \varphi \vdash_{S'_n} \psi$. Then, by the LDDT, there exists some $i \in \omega$ such that $\Gamma \vdash_{S'_n} \varphi^i \to \psi$. We shall show that this implies that $\Gamma \vdash_{S'_n} \varphi^n \to \psi$, which will complete the proof. If $i < n$, then by (K),

$$\vdash_{S'_n} (\varphi^i \to \psi) \to (\varphi \to (\varphi^i \to \psi)),$$

hence by (MP), $\varphi^i \to \psi \vdash_{S'_n} \varphi^{i+1} \to \psi.$

Since $\Gamma \vdash_{S'_n} \varphi^i \to \psi$, we have $\Gamma \vdash_{S'_n} \varphi^{i+1} \to \psi$. By induction it follows that $\Gamma \vdash_{S'_n} \varphi^n \to \psi$.

Now suppose that $i \geq n$. We claim that:

(5.3.8) $$\vdash_{S'_n} (p^{i+1} \to q) \to (p^i \to q).$$

This is proved by induction on i. If $i = n$, then (5.3.8) is precisely (X_n). Suppose that (5.3.8) holds for some $i \geq n$. Then, by (B),

$$\vdash_{S'_n} ((p^{i+1} \to q) \to (p^i \to q)) \to [(p \to (p^{i+1} \to q)) \to (p \to (p^i \to q))],$$

hence, by the assumption (5.3.8) and (MP),

$$\vdash_{S'_n} (p \to (p^{i+1} \to q)) \to (p \to (p^i \to q)).$$

so, by (MP), $p \to (p^{i+1} \to q) \vdash_{S'_n} p \to (p^i \to q),$

i.e., $p^{i+2} \to q \vdash_{S'_n} p^{i+1} \to q$, completing the inductive proof. Now we claim that for any $i \geq n$,

(5.3.9) $$\vdash_{S'_n} (p^i \to q) \to (p^n \to q).$$

Again, this is proved by induction on i. If $i = n$, the result follows from (I). Suppose that (5.3.9) holds for some $i \geq n$. By (B),

$$\vdash_{S'_n} ((p^i \to q) \to (p^n \to q)) \to [((p^{i+1} \to q) \to (p^i \to q)) \to ((p^{i+1} \to q) \to (p^n \to q))].$$

Thus, by the induction assumption, (5.3.8) and two applications of (MP) we get

$$\vdash_{S'_n} (p^{i+1} \to q) \to (p^n \to q),$$

hence the inductive proof is complete. From this it follows by (MP) that $\Gamma \vdash_{S'_n} \varphi^i \to \psi$ implies $\Gamma \vdash_{S'_n} \varphi^n \to \psi$. □

5.3.3 COROLLARY

For each positive integer n, the variety $\mathcal{S}_n$ has EDPC with defining equations

$$(a_1 \to a_2)^n \to (b_1 \to b_2) \approx \mathrm{T}.$$

Proof. That $\mathcal{E}_n$ has EDPC follows from Theorem 5.3.2 and Corollary 4.6.2. The defining equations for $\mathcal{E}_n$ may be obtained in a similar way to that employed in Corollary 5.2.9 for the LEDPRC. □

From this corollary, we immediately have the result that every subvariety of $\mathcal{E}_n$ also has EDPC. In particular, this means that the varieties $\mathcal{E}_1$ and $\mathcal{I}$, of Hilbert and Tarski algebras respectively, have EDPC, and Theorem 5.3.2 generalizes the fact that $\mathbf{IPC}_{\rightarrow}$ and $\mathbf{CPC}_{\rightarrow}$ have the DDT (which also follows from the fact that **IPC** and **CPC** have the DDT with deduction detachment set $\{p \rightarrow q\}$). We have the following converse of the above result

5.3.4 THEOREM [BR, Theorem 4.2]

A variety $\mathcal{V}$ of BCK-algebras has EDPC if and only if $\mathcal{V} \subseteq \mathcal{E}_n$ for some $n \in \omega$.

Proof. Let $\mathcal{V}$ be a variety of BCK-algebras that has EDPC. For an algebra $\mathbf{A} \in \mathcal{V}$ and $a \in A$, let $\langle a \rangle_{\mathbf{A}}$ denote the ideal of $\mathbf{A}$ generated by a. By Corollary 4.5.8, $\mathcal{V}$ has DPC. In view of the isomorphism between the congruence and ideal lattices of any $\mathbf{A} \in \mathcal{V}$ given in Corollary 5.2.6, this implies the existence of a binary first-order formula ϕ in the language of $\mathcal{V}$ such that for all $\mathbf{A} \in \mathcal{V}$ and all $a, b \in A$, $a \in \langle b \rangle_{\mathbf{A}}$ if and only if $\mathbf{A} \models \phi[a,b]$. By Corollary 5.2.9 (iii), this clearly means that

$$\mathbf{A} \models \phi[a,b] \quad \text{iff} \quad \text{there exists } n \in \omega \text{ such that } b^n \rightarrow a = \mathrm{T}.$$

We claim however that there exists a fixed $n \in \omega$ such that

$$\mathbf{A} \models \phi[a,b] \quad \text{iff} \quad b^n \rightarrow a = \mathrm{T}.$$

For if not, then for every $n \in \omega$, there exists $\mathbf{A}_n \in \mathcal{V}$ and $a_n, b_n \in A_n$ such that $a_n \in \langle b_n \rangle_{\mathbf{A}_n}$ but $b_n^n \rightarrow a_n \neq \mathrm{T}$. Let $\mathcal{U}$ be a free ultrafilter over ω, let $\mathbf{A}$ be the ultraproduct $\Pi_{n \in \omega} \mathbf{A}_n / \mathcal{U}$ (so $\mathbf{A} \in \mathcal{V}$) and consider $\bar{a} = \langle a_0, a_1, \ldots \rangle$, $\bar{b} = \langle b_0, b_1, \ldots \rangle \in \Pi_{n \in \omega} A_n$. Since $\mathbf{A}_n \models \phi[a_n, b_n]$ for each $n \in \omega$, we have that $\mathbf{A} \models \phi[\bar{a}/\mathcal{U}, \bar{b}/\mathcal{U}]$ and so $\bar{a}/\mathcal{U} \in \langle \bar{b}/\mathcal{U} \rangle_{\mathbf{A}}$. But for each $n \in \omega$, $\{i \in \omega\,;\, (\bar{b}(i))^n \rightarrow \bar{a}(i) \neq \mathrm{T}\}$ $\supseteq \{i \in \omega\,;\, i \geq n\} \in \mathcal{U}$, so $(\bar{b}/\mathcal{U})^n \rightarrow \bar{a}/\mathcal{U} \neq \mathrm{T}$, contradicting $\bar{a}/\mathcal{U} \in \langle \bar{b}/\mathcal{U} \rangle_{\mathbf{A}}$. This vindicates the above claim. If n is as in the claim, then $\mathcal{V} \subseteq \mathcal{E}_n$. For otherwise, we may choose $\mathbf{B} \in \mathcal{V}$ and $a, b \in B$ such that $b^n \rightarrow a \neq b^{n+1} \rightarrow a$, whence

$$b^n \rightarrow ((b^{n+1} \rightarrow a) \rightarrow a) \overset{(11)}{=} (b^{n+1} \rightarrow a) \rightarrow (b^n \rightarrow a) \neq \mathrm{T}.$$

Thus, if $c = (b^{n+1} \rightarrow a) \rightarrow a$ then $c \notin \langle b \rangle_{\mathbf{B}}$. But this contradicts the fact that $b^{n+1} \rightarrow c \overset{(11)}{=}$

$(b^{n+1} \to a) \to (b^{n+1} \to a) = \mathrm{T}$, completing the proof. □

From our discussion of Wajsberg algebras, we have that the algebra $\mathbf{L}_\omega$ associated with Lukasiewicz's logic S_ω has the property that the $\{\to\}$-reduct of $\mathbf{L}_\omega$ is a BCK-algebra. Thus the subalgebras $\mathbf{L}_n$ of $\mathbf{L}_\omega$ have the same property for each $n \geq 1$. In fact, it is easily checked that the $\{\to\}$-reduct of $\mathbf{L}_n$ is in $\mathcal{E}_n$ for each $n \geq 1$. The proof of the previous theorem may be modified easily to show that a subvariety $\mathcal{V}$ of the variety $\{\mathbf{L}_\omega\}^{\mathrm{Q}}$ has EDPC if and only if $\mathcal{V} \models (E_n)$ for some positive integer n. The algebra $\mathbf{L}_\omega$ does not satisfy (E_n) for any $n \geq 1$, however. For if $n \geq 1$ and $0 \leq m < n$, then

$$\tfrac{n}{n+1} \to \tfrac{m}{n+1} = \min\{1, \tfrac{m}{n+1} + 1 - \tfrac{n}{n+1}\} = \tfrac{m+1}{n+1}.$$

Thus it follows that $(\frac{n}{n+1})^{n+1} \to 0 = \frac{n+1}{n+1} = 1$ but $(\frac{n}{n+1})^{n} \to 0 = \frac{n}{n+1} < 1$, so $\mathbf{L}_\omega \not\models (E_n)$. Thus $\{\mathbf{L}_\omega\}^{\mathrm{Q}}$ does not have EDPC and hence the deductive system S_ω does not have the DDT.

The Variety $\mathcal{J}$.

Lastly, we consider a variety of BCK-algebras that properly contains the join of the varieties $\mathcal{T}$ and $\mathcal{E}_1$. Let $\mathcal{J}$ be the class of algebras of type $\langle 2, 0\rangle$ that is defined by the axioms of $\mathcal{BCK}$ and the identity

(J) $\qquad (((x \to y) \to y) \to x) \to x \approx (((y \to x) \to x) \to y) \to y.$

Let S_J be the extension of **BCK** got by the addition of the axiom

$$[(((p \to q) \to q) \to p) \to p] \to [(((q \to p) \to p) \to q) \to q].$$

By Corollary 3.1.17, S_J is algebraizable with defining equation $p \approx \mathrm{T}$ and equivalence formulas $\Delta_1(p,q) = p \to q$, $\Delta_2(p,q) = q \to p$. The equivalent quasivariety semantics for S_J is easily seen to be the quasivariety of all BCK-algebras that satisfy (J), i.e., $\mathcal{J}$. In fact, S_J is strongly algebraizable, i.e., $\mathcal{J}$ is a variety. For illustrative purposes, we show this by appealing to Theorem 3.1.20.

The only inference rule of S_J is (MP), i.e., $\langle\{p, p \to q\}, q\rangle$. We shall define terms $t_1(x,y,p,q)$, $t_2(x,y,p,q)$ and pairs $(u_1,v_1), (u_2,v_2) \in \{(p,\mathrm{T}), (p \to q, \mathrm{T})\}$. Set

$$t_1(x,y,p,q) = y \to q,$$

$$t_2(x,y,p,q) = x.$$

Let $(u_1,v_1) = (p,\mathrm{T})$ and $(u_2,v_2) = (p \to q, \mathrm{T})$. The required theorems of S_J, as stated in Theorem

3.1.20, are

$$\vdash_{S_J} q \rightarrow (\mathrm{T} \rightarrow q), (\mathrm{T} \rightarrow q) \rightarrow q,$$

$$\vdash_{S_J} (p \rightarrow q) \rightarrow (p \rightarrow q),$$

$$\vdash_{S_J} \mathrm{T} \rightarrow \mathrm{T}.$$

Each of these is easily seen to be a theorem of S_J; in fact they are theorems of **BCK**.

For condition (ii), we shall define terms $s_1(x,y,p,q),\ldots,s_4(x,y,p,q)$ and pairs $(a_i,b_i) \in \{(p \rightarrow q, \mathrm{T}), (q \rightarrow p, \mathrm{T})\}$ for $i \leq 4$. Set

$$s_1(x,y,p,q) = y \rightarrow p,$$

$$s_2(x,y,p,q) = ((y \rightarrow q) \rightarrow p) \rightarrow p,$$

$$s_3(x,y,p,q) = ((x \rightarrow p) \rightarrow q) \rightarrow q,$$

$$s_4(x,y,p,q) = x \rightarrow q.$$

Let $(a_1,b_1) = (a_3,b_3) = (q \rightarrow p, \mathrm{T})$ and $(a_2,b_2) = (a_4,b_4) = (p \rightarrow q, \mathrm{T})$. The required theorems of S_J, as stated in Theorem 3.1.20, are the following:

(i) $\vdash_{S_J} p \rightarrow (\mathrm{T} \rightarrow p),\ (\mathrm{T} \rightarrow p) \rightarrow p$

(ii) $\vdash_{S_J} [(q \rightarrow p) \rightarrow p] \rightarrow [((\mathrm{T} \rightarrow q) \rightarrow p) \rightarrow p],\ [((\mathrm{T} \rightarrow q) \rightarrow p) \rightarrow p] \rightarrow [(q \rightarrow p) \rightarrow p]$

(iii) $\vdash_{S_J} [(((p \rightarrow q) \rightarrow q) \rightarrow p) \rightarrow p] \rightarrow [(((q \rightarrow p) \rightarrow p) \rightarrow q) \rightarrow q]$

and $\vdash_{S_J} [(((q \rightarrow p) \rightarrow p) \rightarrow q) \rightarrow q] \rightarrow [(((p \rightarrow q) \rightarrow q) \rightarrow p) \rightarrow p]$

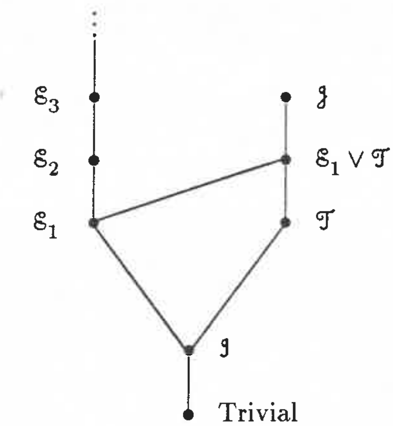
(iv) $\vdash_{S_J} [((\mathrm{T} \rightarrow p) \rightarrow q) \rightarrow q] \rightarrow [(p \rightarrow q) \rightarrow q],\ [(p \rightarrow q) \rightarrow q] \rightarrow [((\mathrm{T} \rightarrow p) \rightarrow q) \rightarrow q]$

(v) $\vdash_{S_J} (\mathrm{T} \rightarrow q) \rightarrow q,\ q \rightarrow (\mathrm{T} \rightarrow q).$

That (i), (ii) (iv) and (v) are theorems of S_J can be seen using the equivalent quasivariety semantics for $\mathfrak{J}$. For (i) holds iff $p \approx \mathrm{T} \rightarrow p$, (ii) holds iff $(q \rightarrow p) \rightarrow p \approx ((\mathrm{T} \rightarrow q) \rightarrow p) \rightarrow p$, (iv) holds iff $((\mathrm{T} \rightarrow p) \rightarrow q) \rightarrow q \approx (p \rightarrow q) \rightarrow q$ and (v) holds iff $\mathrm{T} \rightarrow q \approx q$. Each of these identities is satisfied by $\mathfrak{BCK}$, hence also by $\mathfrak{J}$. That (iii) is a theorem of S_J is immediate from the definition. Thus S_J is strongly algebraizable and $\mathfrak{J}$ is a variety.

The class of subvarieties of the quasivariety $\mathfrak{BCK}$ forms a lattice under the ordering $\subseteq$ [BR95, Theorem 11]. The figure below shows the relationship between the varieties that we have considered here, with respect to this lattice. For a proof of the fact that the join of $\mathfrak{S}_1$ and $\mathfrak{T}$, namely HSP($\mathfrak{S}_1 \cup \mathfrak{T}$), is properly contained in $\mathfrak{J}$, we refer the reader to [Cor81]. If one assumes the relationships in this diagram then the fact that $\mathfrak{J}$ is a variety generalizes the facts that $\mathfrak{S}_1$, $\mathfrak{T}$ and $\mathfrak{J}$

are.



Pocrims.

A deductive system that is algebraizable always has a unique quasivariety as an equivalent algebraic semantics. We have seen that a quasi-equational basis for this quasivariety can be constructed if a set of axioms and inference rules of the deductive system is known. What is not always evident, however, is how (if at all) this quasivariety is related to classical algebra. We have, for example, that every Tarski algebra is a $\{\rightarrow, \mathrm{T}\}$-subreduct of a Boolean algebra, and that every Hilbert algebra is a $\{\rightarrow, \mathrm{T}\}$-subreduct of a Brouwerian semilattice (see [Die66]). In this section we present the class of *pocrims*, which are natural from the point of view of classical algebra, and which are related to the quasivariety $\mathfrak{BCK}$ in a similar way to the above examples.

A *pomonoid* is a structure $\langle A; \cdot, \mathrm{T}; \leq \rangle$, where $\langle A; \cdot, \mathrm{T} \rangle$ is a monoid, i.e., an algebra of type $\langle 2, 0 \rangle$ satisfying the axioms (G1) and (G2) of groups (Section 0.2), and $\leq$ is a partial order on A that is compatible with $\cdot$ in the sense that, whenever $a, b, c \in A$ with $a \leq b$, we have $a \cdot c \leq b \cdot c$ and $c \cdot a \leq c \cdot b$. A pomonoid $\langle A; \cdot, \mathrm{T}; \leq \rangle$ is said to be *commutative* if $\langle A; \cdot, \mathrm{T} \rangle$ is commutative, and *integral* if T is the largest element of A. A commutative pomonoid $\langle A; \cdot, \mathrm{T}; \leq \rangle$ is said to be *residuated* if for any $a, b \in A$, there exists a greatest $c \in A$ such that $c \cdot a \leq b$. We denote this c by $a \rightarrow b$. The fact that such a c always exists and is uniquely determined by a and b means that $\rightarrow$ is a binary operation on A; we call this operation *residuation.* Observe that when $\langle A; \cdot, \mathrm{T}; \leq \rangle$ is integral, we have $a \leq b$ if and only if $a \rightarrow b = \mathrm{T}$ (for all $a, b \in A$). A *pocrim* is a residuated commutative integral pomonoid considered as a structure $\langle A; \cdot, \rightarrow, \mathrm{T}; \leq \rangle$. The class of all pocrims is denoted by $\mathcal{M}b$.

A Boolean algebra $\mathbf{A} = \langle A;\ \wedge, \vee, ', 0, 1\rangle$ can be viewed as a bounded, complemented distributive lattice. Thus $\langle A;\ \wedge, 1;\ \leq\rangle$, where $\leq$ is the lattice order, is a commutative integral pomonoid. If one defines $a \rightarrow b = (a') \vee b$, then $\mathbf{A}$ is residuated with residuation $\rightarrow$, hence the $\langle \wedge, \rightarrow, 1;\ \leq\rangle$-reduct of the (enriched) Boolean algebra $\mathbf{A} = \langle A;\ \wedge, \vee, \rightarrow, ', 0, 1;\ \leq\rangle$ is a pocrim. Similarly, a Heyting algebra $\mathbf{A} = \langle A;\ \wedge, \vee, \rightarrow, 0, 1\rangle$ can be viewed as a bounded distributive lattice with a relative pseudocomplement, i.e., a binary operation $\rightarrow$ that satisfies, for all $a, b, c \in A$,

$$c \leq a \rightarrow b \quad \text{if and only if} \quad c \wedge a \leq b.$$

Thus the $\langle \wedge, \rightarrow, 1;\ \leq\rangle$-reduct of the (enriched) Heyting algebra $\mathbf{A} = \langle A;\ \wedge, \vee, \rightarrow, 0, 1;\ \leq\rangle$, where $\leq$ is the lattice order, is a pocrim.

Since the order $\leq$ on a pocrim is determined by $\rightarrow$ and T, we may as well confuse pocrims systematically with their algebra reducts $\langle A;\cdot, \rightarrow, \mathrm{T}\rangle$. Iséki showed [Isé80] that a pocrim can be defined (equivalently) as an algebra $\langle A;\cdot, \rightarrow, \mathrm{T}\rangle$ of type $\langle 2, 2, 0\rangle$ that is defined by the identities and quasi-identity defining $\mathfrak{BCK}$ and the identity

$$x \rightarrow (y \rightarrow z) \approx (x \cdot y) \rightarrow z.$$

Thus $\mathcal{M}$ is a quasivariety. It is evident from this result that every $\{\rightarrow, \mathrm{T}\}$-subreduct of a pocrim is a BCK-algebra. It was shown independently by Pałasinski [Pał82], Ono and Komori [OK85], and Fleisher [Fle88] that the converse also holds, i.e., every BCK-algebra is a $\{\rightarrow, \mathrm{T}\}$-subreduct of a pocrim. Thus we have the following:

5.3.5 THEOREM [Pał82] [OK85] [Fle88]

The quasivariety $\mathfrak{BCK}$ is precisely the class of $\{\rightarrow, \mathrm{T}\}$-subreducts of $\mathcal{M}$. In other words, an algebra $\mathbf{A} = \langle A;\rightarrow, \mathrm{T}\rangle$ is a BCK-algebra if and only if it is a $\{\rightarrow, \mathrm{T}\}$-subreduct of a pocrim. □

$\{\rightarrow, \wedge\}$-fragments of CPC and IPC.

One can deduce that the $\{\rightarrow, \wedge\}$-fragments of **CPC** and **IPC**, denoted $\mathbf{CPC}_{\{\rightarrow, \wedge\}}$ and $\mathbf{IPC}_{\{\rightarrow, \wedge\}}$, respectively, are algebraizable from Corollary 3.1.17 and the facts that their languages contain the connective $\rightarrow$, which was the only connective used in a system of defining equations and equivalence formulas for **CPC** and **IPC**. By Corollary 3.1.8, the equivalent quasivariety semantics for $\mathbf{CPC}_{\{\rightarrow, \wedge\}}$ is the class of subalgebras of the $\{\rightarrow, \wedge\}$-reducts of Boolean algebras.

This class is precisely the variety of relatively complemented distributive lattices with 0, hence $\mathbf{CPC}_{\{\to,\,\wedge\}}$ is strongly algebraizable. By the same theorem, the equivalent quasivariety semantics for $\mathbf{IPC}_{\{\to,\,\wedge\}}$ is the class of subalgebras of the $\{\to, \wedge\}$-reducts of Heyting algebras, which is precisely the variety of Brouwerian semilattices (as defined in Section 0.2). Thus $\mathbf{IPC}_{\{\to,\,\wedge\}}$ is also strongly algebraizable.

References

[AB75] A.R. Anderson, N.D. Belnap Jr: "Entailment. The Logic of Relevance and Necessity", Princeton University Press, Princeton, 1975.

[BD86] W.J. Blok, W. Dziobiak: On the lattice of quasivarieties of Sugihara algebras, Studia Logica **45** (1986), 275-280.

[BKP] W.J. Blok, P. Köhler, D. Pigozzi: The algebraization of logic, Manuscript.

[BP86] W.J. Blok, D. Pigozzi: Protoalgebraic logics, Studia Logica **45** (1986), 337-369.

[BP88] W.J. Blok, D. Pigozzi: Local deduction theorems in algebraic logic, Colloquia Mathematica Societatis János Bolyai **54** Algebraic Logic, Budapest (Hungary) 1988, 75-109.

[BP89a] W.J. Blok, D. Pigozzi: "Algebraizable logics", Memoirs of the American Mathematical Society: Number 396, Amer. Math. Soc., Providence, 1989.

[BP89b] W.J. Blok, D. Pigozzi: Deduction theorems in algebraic logic, Manuscript.

[BP92] W.J. Blok, D. Pigozzi: Algebraic semantics for universal Horn logic without equality, in "Universal Algebra and Quasigroup Theory," A. Romanowska, J.D.H. Smith (eds.), Heldermann Verlag, Berlin, 1992.

[BP94] W.J. Blok, D Pigozzi: "Abstract algebraic logic", Algebraic Logic and the Methodology of Applying It, Tempus Summer School, July 11-17, 1994, Budapest.

[BP] W.J. Blok, D. Pigozzi: On the structure of varieties with equationally definable principal congruences III, to appear in Algebra Universalis.

[BR95] W.J. Blok, J.G. Raftery: On the quasivariety of BCK-algebras and its subvarieties, Algebra Universalis **33** (1995), 68-90.

[BR] W.J. Blok, J.G. Raftery: Varieties of commutative residuated integral pomonoids and their residuation subreducts, Manuscript.

[Bl75] S.L. Bloom: Some theorems on structural consequence relations, Studia Logica **34** (1975), 1-9

[BS81] S. Burris, H.P. Sankappanavar: "A Course in Universal Algebra," Springer-Verlag, New York, 1981.

[Bun81] M.W. Bunder: Simpler axioms for BCK-algebras and the connection between the axioms and combinators B, C and K, Math. Japonica **26** (1981), 415-418.

[CF68] H.B. Curry, R.Feys: "Combinatorial Logic. Volume I", North Holland, Amsterdam, 2nd printing, 1968.

[Cha58] C.C. Chang: Algebraic analysis of many-valued logics, Trans. Amer. Math. Soc. **88** (1958), 467-490.

[Cha59] C.C. Chang: A new proof of the completeness of the Łukasiewicz axioms, Trans. Amer. Math. Soc. **93** (1959), 74-80.

[Cor81] W.H. Cornish: A large variety of BCK-algebras, Mathematica Japonica **26** (1981), 339-344.

[Cor82] W.H. Cornish: On Iséki's BCK-algebras, in "Lecture Notes in Pure and Applied Mathematics, Vol. 74" (1982), 101-122, Marcel Dekker, New York.

[Day70] A. Day: A note on the congruence extension property, Algebra Universalis **1** (1970), 234-235.

[Die66] A. Diego: "Sur les algèbras de Hilbert", Collection de Logique Mathematique, Series A, No. 21, Gauthier-Villars, Paris, 1966.

[DMS87] B.A. Davey, K.R. Miles, V.J. Schumann: Quasi-identities, Mal'cev conditions and congruence regularity, Acta Sci. Math. **51** (1987), 39-55.

[Dun70] J.M. Dunn: Algebraic completeness results for **R**-mingle and its extensions, J. Symbolic Logic **35** (1970), 1-13.

[FGQ80] E. Fried, G. Grätzer, R. Quackenbush: Uniform congruence schemes, Algebra Universalis **10** (1980), 176-189.

[Fle88] I. Fleischer: Every BCK-algebra is a set of residuables in an integral pomonoid, J. Algebra **119** (1988), 360-365.

[FR90] J.M. Font, G. Rodriguez: Note on algebraic models for relevance logic, Zeitschr. f. math. Logik und Grundlagen d. Math. **36** (1990), 535-540.

[FRT84] J. Font, A.J. Rodriguez, A. Torrens: Wajsberg algebras, Stochastica **8** (1984), 5-31.

[Gai90] H. Gaitan: About quasivarieties of Wajsberg algebras, J. Nonclassical Logic **8** (2) (1991), 79-101.

[Gol94] R. Goldblatt: Modal logics of programs, Lecture Notes for the Workshop on Formal Aspects of Programming, University of Cape Town, 27 June - 8 July 1994.

[Grä78] G. Grätzer, "General Lattice Theory", Birkhäuser Verlag, Basel and Stuttgart, 1978.

[Grä79] G. Grätzer, "Universal Algebra", second edition, Springer-Verlag, New York Inc., 1979.

[GU84] H.P. Gumm, A. Ursini: Ideals in universal algebras, Algebra Universalis **19** (1884), 45-54.

[Hag73] J. Hagemann: On regular and weakly regular congruences, Preprint No. 75, Technische Hoschschule Darmstadt, June 1973.

[Isé66] K. Iséki: An algebra related with a propositional calculus, Proc. Japan Acad. **42** (1966), 26-29.

[Isé80] K. Iséki: On BCK-algebras with condition (S), Math. Japonica **24** (1980), 625-626.

[IT78] K. Iséki, S. Tanaka: An introduction to the theory of BCK-algebras, Math. Japonica **23** (1978), 1-26.

[Kal60] J.A. Kalman: Equational completeness and families of sets closed under subtraction, Nederl. Acad. Wetensch. Proc. Ser. A, **63** (1960), 402-406.

[KM92] K. Kearnes, R. McKenzie: Commutator theory for relatively modular quasivarieties, Trans. Amer. Math. Soc. **331** (1992), 465-502.

[KP80] P. Köhler, D. Pigozzi: Varieties with equationally definable principal congruences, Algebra Universalis **11** (1980), 213-219.

[Lew18] C.I. Lewis: "A Survey of Symbolic Logic", second edition, abridged, Univ. of California Press, New York (Dover), 1960.

[LL32] C.I. Lewis, C.H. Langford: "Symbolic Logic", second edition, The Century Co., New

York and London, New York (Dover), 1959.

[LS58] J. Los, R. Suszko: Remarks on sentential logics, Proc. Kon. Nederl. Akad. van Wetenschappen, Series A **61** (1958), 177-183.

[Łuk70] J. Łukasiewicz: The equivalential calculus, in: "Jan Łukasiewicz. Selected Works", Ed. L. Bordowski, North-Holland Publishing Co., Amsterdam, 1970.

[Mal73] A.I. Mal'cev: "Algebraic Systems", Springer-Verlag, Berlin, Heidelberg, New York, 1973.

[Men87] E. Mendelson: "Introduction to Mathematical Logic", third edition, Wadsworth & Brooks/Cole, Monterey, California, 1987.

[MMT87] R.N. McKenzie, G.F. McNulty, W.F. Taylor: "Algebras, Lattices, Varieties. Volume I", Wadsworth & Brooks/Cole, Monterey, California, 1987.

[Mon60] A. Monteiro: Cours sur les algèbres de Hilbert et de Tarski, Instituto Mat. Univ. del Sur, Bahia Blanca, 1960.

[Mun86] D. Mundici: MV-algebras are categorically equivalent to bounded commutative BCK-algebras, Math. Japonica **31** (1986), 889-894.

[OK85] H. Ono, Y. Komori: Logics without the contraction rule, J. Symbolic Logic **50** (1985), 169-201.

[Pał80] M. Pałasinski: Some remarks on BCK-algebras, Math. Seminar Notes Kobe Univ. **8** (1980), 137-144.

[Pał81] M. Pałasinski: On ideal and congruence lattices of BCK-algebras, Math. Japonica **26** (1981), 543-544.

[Pał82] M. Pałasinski: An embedding theorem for BCK-algebras, Math. Seminar Notes Kobe Univ. **10** (1982), 749-751.

[Pał] M. Pałasinski: Gentzenowskie i inne metody w teorii BCK-algebr. (Gentzen's and some other methods in the theory of BCK-algebras), Manuscript (Polish).

[Pała94] K.M. Pałasinska: Deductive systems and finite axiomatization properties, Ph.D thesis, Iowa State University, Ames, 1994.

[Par39] W.T. Parry: Modalities in the "Survey" system of strict implication, J. Symbolic Logic **4** (1939), 137-154.

[Por83] J. Porte: Axiomatization and independence in $\boldsymbol{S}4$ and $\boldsymbol{S}5$, Reports on Mathematical Logic **16** (1983), 23-33.

[Pri62] A.N. Prior: "Formal Logic," second edition, Oxford University Press, 1962.

[Ras63] H. Rasiowa, R. Sikorski: "The Mathematics of Metamathematics," Państwowe Wydawnictwo Naukowe, Warszawa, 1963.

[RR58] A. Rose, J.B. Rosser: Fragments of many-valued statement calculi, Trans. Amer. Math. Soc. **87** (1958), 1-53.

[Sus71] R. Suszko: Identity connective and modality, Studia Logica **27** (1971), 7-39.

[Tan75] S. Tanaka: On $\wedge$-commutative algebras, Math. Seminar Notes Kobe Univ. **3** (1975), 59-64.

[Tax73] R.E. Tax: On the intuitionistic equivalential calculus, Notre Dame J. Formal Logic **14** (1973), 448-456.

[Tra79] T. Traczyk: On the variety of commutative BCK-algebras, Math. Japonica **24** (1979), 283-292.

[Wro83] A. Wroński: BCK-algebras do not form a variety, Math. Japonica **28** (1983), 211-213.

[Wro85] A. Wroński: An algebraic motivation for BCK-algebras, Math. Japonica **30** (1985), 187-193.

Index

A

B

C

D

E

F

G

H

I

J

K

L

M

N

O

P

Q

R

S

T

U

V

W

List of Symbols and Abbreviations