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NEW SOLUTIONS FOR NONLINEAR PERFECT FLUIDS

Banele Jabula Nzama

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Banele Jabula Nzama

Submitted in fulfilment of the academic requirements for the degree of Master in Science to the School of Mathematics, Statistics and Computer Science, College of Agriculture, Engineering and Science, University of KwaZulu-Natal.

As the candidate's supervisor, I have approved this dissertation for submission.

Signature:	Dr Riven Narain	December 2016
Signature:	Prof. Sunil Maharaj	December 2016

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Banele Jabula Nzama

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To

My Late Father

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Abstract

We investigate the Einstein system that governs the evolution of uncharged shear-free spherically symmetric fluids. First we present the Einstein equations for the static spherically symmetric gravitational fields in isotropic coordinates. Also the nonstatic spherically symmetric gravitational fields are studied. We have demonstrated that the fundamental differential equation governing the behaviour of the model is of the Emden-Fowler type. Such equations also arise in applications in Newtonian physics. The field equations governing the gravitational behaviour of the model are generated. We integrate the system of partial differential equations and apply a transformation that reduces the system to a second order ordinary differential equation. To solve the resulting ordinary differential equation we employ the method of characteristics to find different expressions for the gravitational potentials. We employ the method of characteristics to obtain first integrals for the Emden-Folwer type equation. To apply the method, we make use of the associated multipliers which are obtained via the Euler operator acting on the arbitrary multiplier and differential equation. These multipliers can be obtained under the various forms of the arbitrary function representing the gravitational potential under which the equation becomes integrable. Thus expanding the differential equation with the associated multiplier, we can find first integrals by solving the system of partial differential equations. The study is comprised of various forms of the multipliers associated to first integrals of the equation in question.

Contents

Chapter 1

Introduction

Currently the theory of general relativity provides the best description of the behaviour of the gravitational field. The predictions of general relativity have been shown to be consistent with observational data in relativistic astrophysics and cosmology. In general relativity the curvature of spacetime is described by the Riemann tensor. The matter content is described by the symmetric energy momentum tensor; in this thesis we consider only neutral perfect fluid matter distributions. More general matter distributions, for example including the effects of the electromagnetic field, may be added for other applications. It is for these reasons that general relativity is considered the best theory for describing strong gravitational fields, the early universe and present observations. For a topical review on general relativity and applications see Ashtekar and Petkov (2014).

The Einstein field equations relate the matter content to the curvature. In the presence of an electromagnetic field these equations have to be supplemented with

Maxwell's equations which incorporate charge and current. The field equations satisfy the conservation laws through the Bianchi identity. Determining explicit solutions to the Einstein field equations is necessary for astrophysical and cosmological applications. The field equations may be modified to include higher order curvature terms; an example of a modified gravity theory is given by Chilambwe *et al* (2005).

Spherically symmetric gravitational fields are important to describe the kinematics and dynamics of cosmological models. Such models have been widely studied by a number of researchers. The earliest example of solutions which are isotropic and homogeneous are the Friedmann universes, the Robertson-Walker spacetimes and their generalizations. These spacetimes admit rotational and translational Killing vectors. Examples of spacetimes with a conformal symmetry are given by Castejon-Amendo and Coley (1992), Coley and Tupper (1990), Dyer *et al* (1987) and Maharaj *et al* (1993). Other works admit inheriting Killing vectors with spherical symmetry such as Lortan *et al* (2001) and Tupper *et al* (2003). A comprehensive list of known exact solutions, containing spacetimes with spherical geometry and others, is given by Krasinski (1997) and Stephani *et al* (2003). We find that many of these analyses reduce to studying a differential equation of the Emden-Fowler type. The reader is referred, for example to the investigations of Maharaj *et al* (1996), Srivastava (1987) and Stephani (1983). The particular Emden-Fowler differential equation will be shown to arise in spherical fields and we also generate new families of solutions.

We summarize the contents of our analyses. This dissertation has the following breakdown:

• Chapter 2: We briefly discuss the concepts of general relativity essential for this thesis. We briefly consider the spacetime geometry and the matter distribution that lead to the formulation of the Einstein field equations. We highlight the crucial physical concepts that are essential for the determination of a relativistic gravitational model.

• Chapter 3: The relativistic static gravitational model in isotropic coordinates is considered. We generate the Einstein field equations and the condition of pressure isotropy for the shear-free spacetime in isotropic coordinates. We introduce new variables to transform these equations into equivalent forms that can generate new exact solutions. The condition of pressure isotropy is the master equation in the integration process.

• Chapter 4: The Einstein field equations are generated in terms of isotropic coordinates where the potentials are functions of the radial and timelike coordinates. We consider neutral perfect fluid matter distributions in static spherically symmetric spacetimes. We then transform the resulting field equation into a second order differential equation which is of the Emden-Fowler type.

• Chapter 5: Possible combinations of the form of the arbitrary function, related to the potentials and its associated multiplier are generated via the characteristic approach These forms make the model integrable. We illustrate the mathematical process that has to be followed to find the multipliers with a particular example. The other integrable cases are summarized.

• Chapter 6: The first integrals for the Emden-Fowler equation studied are generated. The process of finding the first integral is shown in detail with a particular example. The various other first integrals that are possible are given in form of a summary.

• Chapter 7: A brief overview of the chapters and the results are provided.

Chapter 2

Differential geometry

2.1 Introduction

A variety of matter distributions arise in physical applications in cosmology and relativistic astrophysics in different scenarios as pointed out by Will (1981). Spherically symmetric matter distributions are best described by Einstein's theory of general relativity for strong gravitational fields. In this chapter we briefly consider the background theory that provides us with the structure to generate a relativistic gravitational model. We give a brief outline of the differential geometry and the matter distribution that lead to the Einstein field equations. For more detailed information on differential manifolds and tensor analysis the reader is referred to Bishop and Goldberg (1968), Misner *et al* (1973) and Wald (1984). The metric tensor field and the metric connection coefficients are introduced in §2.2. Then the Riemann tensor, the Ricci tensor, Ricci scalar and the Einstein tensor are defined. In §2.3, we consider matter fields by introducing the general energy momentum tensor and the special case for a perfect fluid. We also introduce the barotropic equation of state relating the pressure to the energy density. The Einstein field equations are generated by relating the Einstein tensor to the energy momentum tensor. In §2.4 we provide the mathematical tools necessary for solving the resulting nonlinear perfect fluid model under investigation.

2.2 Spacetime geometry

The local neighbourhood of a point in the spacetime manifold possesses the same structure as the open neighbourhood of a point in $\Re^n$. The global structure of the spacetime manifold in general is different from $\Re^n$. A pseudo-Riemannian manifold is a manifold with an indefinite metric tensor field. In general relativity, we assume that the spacetime **M** is a four-dimensional differentiable manifold endowed with a metric tensor field **g**. The symmetric and nonsingular metric tensor field **g** has signature $(- + ++)$. The metric tensor field **g** represents the gravitational potentials. Points in the manifold are labelled by the real coordinates $(x^a) = (x^0, x^1, x^2, x^3)$, where $x^0 = ct$ (where c is the speed of the light in vacuum) is the timelike coordinate, and x^1, x^2, x^3 are spacelike coordinates. In this dissertation we use the convention that the speed of light $c = 1$.

The line element is given by

$$ds^2 = g_{ab} dx^a dx^b, \tag{2.1}$$

which measures the infinitesimal interval between neighbouring points on a curve. In the line element (2.1), **g** represents the metric tensor field. We use the line element

(2.1) to generate the metric connection coefficients

$$\Gamma^a{}_{bc} = \frac{1}{2} g^{ad}(g_{cd,b} + g_{bd,c} - g_{bc,d}), \tag{2.2}$$

where commas denote partial differentiation. We use the definition of the connection coefficients in equation (2.2) to generate the Riemann curvature tensor **R** which is given by

$$R^a{}_{bcd} = \Gamma^a{}_{bd,c} - \Gamma^a{}_{bc,d} + \Gamma^a{}_{ec}\Gamma^e{}_{bd} - \Gamma^a{}_{ed}\Gamma^e{}_{bc}, \tag{2.3}$$

which is nonvanishing in general since the covariant derivative is not commutative. We contract the Riemann curvature (2.3) to get the Ricci tensor as follows

$$\begin{aligned} R_{ab} &= R^c{}_{acb}, \\ &= \Gamma^c{}_{ab,c} - \Gamma^c{}_{ac,b} + \Gamma^c{}_{dc}\Gamma^d{}_{ab} - \Gamma^c{}_{db}\Gamma^d{}_{ac}. \end{aligned} \tag{2.4}$$

A second contraction yields the Ricci scalar R which has the form

$$\begin{aligned} R &= R^a{}_a \\ &= g^{ab} R_{ab}. \end{aligned} \tag{2.5}$$

We use the Ricci tensor (2.4) and the Ricci scalar (2.5) to form the Einstein tensor **G** which is given by

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}. \tag{2.6}$$

Note that the divergence of the Einstein tensor is zero, *i.e.*

$$G^{ab}{}_{;b} = 0. \tag{2.7}$$

This is sometimes called the Bianchi identity and generates the conservation laws via the field equations.

2.3 Matter fields

The matter content is decribed by the energy momentum tensor **T**. The energy momentum tensor is given by

$$T^{ab} = (\rho + p) u^a u^b + p g^{ab} + q^a u^b + q^b u^a + \pi^{ab}, \tag{2.8}$$

where ρ is the energy density, p is the isotropic pressure, **q** is the heat flux vector ($q_a u^a = 0$), π^{ab} is the anisotropic stress tensor ($\pi_{ab} u^a = 0 = \pi^a{}_a$) and **u** is a timelike four-velocity ($u^a u_a = -1$). The terms for the heat flux and the anisotropic stress vanish in perfect fluids ($q^a = 0, \pi^{ab} = 0$). Then the energy momentum tensor for a perfect fluid has the form

$$T^{ab} = (\rho + p) u^a u^b + p g^{ab}. \tag{2.9}$$

For many applications it is required that the matter distribution satisfies the barotropic equation of state

$$p = p(\rho). \tag{2.10}$$

A particular case is the equation of state

$$p = a\rho + b, \tag{2.11}$$

where a and b are constants. The above form is often assumed in cosmology and is called the linear equation of state. The parameter a (with $b = 0$) has different values which describe familiar matter distributions: dust ($a = 0$), radiation ($a = \frac{1}{3}$) and stiff matter ($a = 1$). When $a \neq 0$ and $b \neq 0$ then the equation of state (2.11) includes matter distributions for quark, strange and exotic configurations (Komathiraj and Maharaj 2007, Mak and Harko 2004, Sharma and Maharaj 2007). Another case is the polytropic equation of state which has the form

$$p = k\rho^{1+\frac{1}{n}}, \tag{2.12}$$

where k and n are constants. This equation of state is assumed in describing gravitational systems in relativistic astrophysics (Shapiro and Teukolsky 1983).

The Einstein field equations follow by relating (2.6) to (2.8) so that

$$\begin{aligned} G^{ab} &= R^{ab} - \frac{1}{2}Rg^{ab} \\ &= T^{ab}, \end{aligned} \tag{2.13}$$

where the coupling constant is set to unity. The field equations (2.13) govern the interaction between the curvature of the spacetime and the matter distribution. From (2.13) and (2.7) we have the result

$$T^{ab}{}_{;b} = 0, \tag{2.14}$$

which is the conservation law for matter. In general the field equations (2.13) are a highly nonlinear system of differential equations which are difficult to integrate without making simplifying assumptions. For detailed information on general relativity and the formulation of the Einstein field equations the reader is referred to de Felice and Clark (1990), Narlikar (2002) and Stephani (2004). Exact solutions to the field equations which are applicable in many physically relevant relativistic models are listed in Krasinski (1997) and Stephani *et al* (2003).

2.4 Differential equations, multipliers and first integrals

To describe a physical system it is often necessary to integrate a differential equation and generate a first integral. We briefly explain the idea of a first integral and its importance in the physical sciences via an example. Consider the free particle equation

$$y'' = 0. \tag{2.15}$$

If we integrate this equation once then we get

$$y' = I_1,$$

where I_1, is a constant. We say that

$$I_1 = y', \tag{2.16}$$

is a first integral of equation (2.15). In physics we talk of a constant of the motion or a conservation law. The main requirement for a nontrivial function I, to be a first integral of an ordinary differential equation is that

$$I' = 0,$$

when the equation is taken into account. We can usually find an infinite number of first integrals for a given equation (especially because any function of a first integral is itself a first integral) but there are only a finite number of independent first integrals. In the case of equation (2.15) we can multiply by x and integrate to obtain another first integral given by

$$I_2 = y - xy',$$

which is independent of equation (2.16). Since both I_1 and I_2 are independent, we can write down the general solution to equation (2.15) by eliminating y' from each expression. This leads to

$$y = I_2 + xI_1,$$

which we recognise as the general solution of equation (2.15). It is clear that, in simple cases, we can calculate the first integrals by inspection.

Intrinsic to a Lie algebraic treatment of differential equations is the universal space $\mathcal{A}$. A locally analytic function $f(x, u, u_1, \ldots, u_k)$ of a finite number of variables $\{u, u_1, \ldots, u_k\}$ denotes the collections of all first, second,$\cdots$, k^{th}-order partial derivatives, that is

$$u_i^\alpha = D_i(u^\alpha), \qquad u_{ij}^\alpha = D_jD_i(u^\alpha), \dots \tag{2.17}$$

respectively, with the total differentiation operator with respect to x^i given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \dots \qquad i = 1, \dots, m \tag{2.18}$$

The space $\mathcal{A}$ is the vector of all differential functions of all finite orders and forms an algebra. A total derivative converts any differential function of order k to a differential function of order $k+1$. Hence, the space $\mathcal{A}$ is closed under total derivations D_i. The Euler operator is defined by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1}(-1)^s D_{i_1}\dots D_{i_s}\frac{\partial}{\partial u_{i_1\dots i_s}}, \qquad \alpha = 1, \quad \dots, m. \tag{2.19}$$

Consider an rth-order system of partial differential equations of n independent variables $x = (x^1, x^2, \dots, x^n)$ and m dependent variables $u = (u^1, u^2, \dots, u^m)$ so that

$$E(x, u, u_{(1)}, \dots, u_{(r)}) = 0, \qquad \mu = 1, \dots, \tilde{m}. \tag{2.20}$$

It can be shown that every multiplier $Q(x, u, u_{(1)} \dots)$ can be admitted such that

$$\frac{\delta}{\delta_u} Q(x, u, u_{(1)} \dots)\ E(x, u, u_{(1)}, \dots, u_{(r)}) = 0, \tag{2.21}$$

where $\frac{\delta}{\delta_u}$ is the Euler operator, holds identically for the equation $E(x, u, u_{(1)}, \dots, u_{(r)})$. A current $\phi = (\phi^1, \dots, \phi^n)$ is conserved if it satisfies

$$D_i\phi^i = 0, \tag{2.22}$$

along the solutions of (2.17). It can be shown that every admitted conservation law arises from multipliers $Q_u(x, u, u_1 \dots)$ such that,

$$Q_uE^u = D_i\phi^i \tag{2.23}$$

holds identically for some current ϕ^i. The conserved current may then be obtained by the homotopy operator.

Another way in which first integrals can be found is through the use of symmetry analysis. In general, we must take into account the notion of symmetries. Suppose that equation (2.20) has a symmetry

$$G = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}.$$

A function $f(x,y,y')$ also possesses the symmetry G if

$$\begin{aligned} G^{[1]}f &= 0 \\ \xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} + (\eta' - y'\xi')\frac{\partial f}{\partial y'} &= 0. \end{aligned}$$

This has an associated Lagrange's system

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dy'}{\eta' - y'\xi'}.$$

The two characteristics are, say u and v so that

$$f(x,y,y') = g(u,v).$$

Now f is a first integral of the differential equation $E = 0$ if

$$\begin{aligned} \left.\frac{df}{dx}\right|_{E=0} &= 0 \\ \Longrightarrow\ u'\frac{\partial g}{\partial u} + v'\frac{\partial g}{\partial v} &= 0, \end{aligned}$$

which has the associated Lagrange's system given by

$$\frac{du}{u'} = \frac{dv}{v'}.$$

The system has the one characteristic w, and so $f(x,y,y') = h(w)$, where h is an arbitrary function of its argument. Usually h is taken to be the identity, but this is not necessary. In general a scalar ordinary differential equation of the n^{th} order

$$E(x, y, y', \ldots, y^{(n)}) = 0,$$

has a first integral

$$I = f\left(x, y, y', \ldots, y^{(n-1)}\right),$$

associated with the symmetry

$$G = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y},$$

if

$$G^{[n-1]}f = 0,$$

and

$$\left.\frac{df}{dx}\right|_{E=0}.$$

2.4.1 The multiplier approach in this study

The method of obtaining first integrals with an associated multiplier used in our study is demonstrated below. Suppose the given multiplier is $Q(x, u, u_{(1)} \ldots)$ found via the equation (2.21) and the generalised first integral is $I(x, u, u_{(1)}, \ldots, u_{(r)})$, we can obtain the first integral by solving the system of partial differential equations via the formula

$$Q(x, u, u_{(1)} \ldots)\ E(x, u, u_{(1)}, \ldots, u_{(r)}) = D_i\ I(x, u, u_{(1)}, \ldots, u_{(r)}). \tag{2.24}$$

We demonstrate the method in the given example. Consider a differential equation given by

$$2(x-1)yy'' + 2(1-2x)yy' + (1-4x+3x^2)y'^2 = 0, \tag{2.25}$$

where our multiplier is $\mu = \frac{e^{2x}}{y'^2}$. By applying equation (2.25) to (2.24), we obtain

$$\frac{e^{2x}}{y'^2}\left[2(x-1)yy'' + 2(1-2x)yy' + (1-4x+3x^2)y'^2\right] = \frac{\partial I}{\partial x} + y'\frac{\partial I}{\partial y} + y''\frac{\partial I}{\partial y'}.$$

By equating coefficients of y'',

$$\frac{\partial I}{\partial y'} = \frac{e^{2x}}{y'^2}\left[2(x-1)y\right]. \tag{2.26}$$

Integrating equation (2.26) results in,

$$I = -\frac{2}{y'}e^{2x}y(x-1) + \psi(x,y).$$

By differentiating and equating corresponding coefficients we have

$$\begin{aligned} \frac{\partial I}{\partial y} &= -\frac{2}{y'}e^{2x}(x-1) + \psi_y(x,y), \\ \frac{\partial I}{\partial x} &= -\frac{4}{y'}e^{2x}y(x-1) - \frac{2}{y'}ye^{2x} + \psi_x(x,y), \end{aligned}$$

which results in

$$\begin{aligned} 2e^{2x}(1-2x)\frac{y}{y'} + e^{2x}(1-4x+3x^2) &= \frac{\partial I}{\partial x} + y'\frac{\partial I}{\partial y}, \\ 2e^{2x}(1-2x)y + e^{2x}y'(1-4x+3x^2) &= -4e^{2x}y(x-1) + 2ye^{2x} \\ &\quad +\psi_x + y'\left[-2e^{2x}(x-1) + y'\psi_y\right]. \end{aligned}$$

Separate by powers of y' we then obtain

$$\begin{aligned} y'^2 &: \quad \psi_y = 0 \quad \Longrightarrow \quad \psi(x), \\ y' &: \quad e^{2x}(1-4x+3x^2) + 2e^{2x}(x-1) = \psi_x, \\ y'^0 &: \quad 2e^{2x}(1-2x)y = -4e^{2x}y(x-1) - 2ye^{2x}. \end{aligned}$$

Thus $\psi(x,y)$ is given by

$$\psi = \int e^{2x}(3x^2 - 2x - 1)dx.$$

Integrating this by parts yields,

$$I_1 = \frac{1}{2}e^{2x}(3x^2 - 2x - 1) - \frac{1}{2}\int e^{2x}(6x - 2)dx.$$

Then integrating the second half of I_1 by parts, we get

$$I_2 = \frac{1}{4}e^{2x}(6x - 2) - \int \frac{6}{4}e^{2x}dx = \frac{1}{4}e^{2x}(6x - 2) - \frac{3}{4}e^{2x} + C.$$

And thus, I_1 results in

$$I_1 = \frac{1}{2}e^{2x}(3x^2 - 2x - 1) - \frac{1}{4}e^{2x}(6x - 2) + \frac{3}{4}e^{2x} + C,$$

where C is the integration constant.

Chapter 3

Isotropic coordinates

3.1 Introduction

In this chapter we consider the static relativistic gravitational model in isotropic coordinates. Particular solutions in isotropic coordinates have been found which are useful in astrophysical applications (Stephani *et al* 2003). In §3.2, we consider the spacetime geometry of the shear-free spacetime in isotropic coordinates. We generate the nonzero components of the connection coefficients, the Ricci tensor, the Ricci scalar and the Einstein tensor. We consider the energy momentum for the perfect fluid matter distribution in §3.3. The components of the energy momentum tensor are related to the components of the Einstein tensor to generate the Einstein field equations. We deduce the condition of pressure isotropy from the Einstein field equations. The condition of pressure isotropy is a second order differential equation with variable coefficients. This is a nonlinear differential equation in general. We analyse two sets of transformation that enable us to express the condition of pressure isotropy in equivalent form. The transformations lead to a simpler ordinary differential equation for isotropic pressures.

3.2 Spacetime geometry

In this section we consider the isotropic line element which has the following form

$$ds^2 = -A^2(r)dt^2 + B^2(r)[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi)], \tag{3.1}$$

where $A(r)$ and $B(r)$ are arbitrary functions. This line element is accelerating but is shear-free. The line element is used to describe relativistic compact objects such as neutron stars in astrophysics.

The line element (3.1) is important for the determination of the connection coefficients $\Gamma^a{}_{bc}$. We use equation (2.2) and the above isotropic line element (3.1) to determine the nonvanishing connection coefficients:

$$\Gamma^0{}_{10} = \frac{A'}{A},$$

$$\Gamma^1{}_{00} = \frac{AA'}{B^2},$$

$$\Gamma^1{}_{11} = \frac{B'}{B},$$

$$\Gamma^1{}_{22} = -r^2\left(\frac{B'}{B} + \frac{1}{r}\right),$$

$$\Gamma^1{}_{33} = -r^2\sin^2\theta\left(\frac{B'}{B} + \frac{1}{r}\right),$$

$$\Gamma^2{}_{12} = \frac{B'}{B} + \frac{1}{r},$$

$$\Gamma^2{}_{33} = -\sin\theta\cos\theta,$$

$$\Gamma^3{}_{13} = \frac{B'}{B} + \frac{1}{r},$$

$$\Gamma^3{}_{23} = \cot\theta.$$

The primes denote differentiation with respect to the radial coordinate r. By using the above connection coefficients we generate the Ricci tensor components for the line element (3.1). We substitute the above connection coefficients in equation (2.4) which is the general form for the Ricci tensor in order to obtain the following nonvanishing components

$$R_{00} = \frac{A}{B^2}\left[A'' + A'\left(\frac{B'}{B} + \frac{2}{r}\right)\right], \tag{3.2a}$$

$$R_{11} = -\left(\frac{A''}{A} - \frac{A'}{A}\frac{B'}{B}\right) - 2\left[\frac{B''}{B} - \frac{B'}{B}\left(\frac{B'}{B} - \frac{1}{r}\right)\right], \tag{3.2b}$$

$$R_{22} = -r^2\left(\frac{A'}{A}\frac{B'}{B} + \frac{1}{r}\frac{A'}{A}\right) - r^2\left(\frac{B''}{B} + \frac{3}{r}\frac{B'}{B}\right), \tag{3.2c}$$

$$R_{33} = \sin^2\theta R_{22}, \tag{3.2d}$$

with $R_{ab} = 0$ for $a \neq b$.

We use the Ricci tensor components (3.2) and equation (2.5), which is the definition of the Ricci scalar, to compute the value

$$R = -\frac{2}{B^2}\left[\frac{A''}{A} + \frac{A'}{A}\left(\frac{B'}{B} + \frac{2}{r}\right)\right] - \frac{2}{B^2}\left[2\frac{B''}{B} - \frac{B'}{B}\left(\frac{B'}{B} - \frac{4}{r}\right)\right], \tag{3.3}$$

in terms of the potentials A and B.

In equation (2.6) we defined the Einstein tensor. For isotropic coordinates we use the Ricci tensor components (3.2) and the Ricci scalar (3.3) to generate the nonvanishing components of the Einstein tensor. These are given by the following equations

$$G_{00} = -\left(\frac{A}{B}\right)^2\left[2\frac{B''}{B} - \frac{B'}{B}\left(\frac{B'}{B} - \frac{4}{r}\right)\right], \tag{3.4a}$$

$$G_{11} = 2\frac{A'}{A}\left(\frac{B'}{B} + \frac{1}{r}\right) + \frac{B'}{B}\left(\frac{B'}{B} + \frac{2}{r}\right), \tag{3.4b}$$

$$G_{22} = r^2\left(\frac{A''}{A} + \frac{1}{r}\frac{A'}{A}\right) + r^2\left[\frac{B''}{B} - \frac{B'}{B}\left(\frac{B'}{B} - \frac{1}{r}\right)\right], \tag{3.4c}$$

$$G_{33} = \sin^2\theta G_{22}, \tag{3.4d}$$

with $G_{ab} = 0$ for $a \neq b$.

3.3 Einstein field equations

Since the fluid four velocity is comoving we have $u^a = \frac{1}{A}\delta^a_0$ for the metric (3.1). The nonvanishing energy momentum tensor components are given by

$$T_{00} = \rho A^2, \tag{3.5a}$$

$$T_{11} = pB^2, \tag{3.5b}$$

$$T_{22} = pB^2r^2, \tag{3.5c}$$

$$T_{33} = \sin^2\theta T_{22}, \tag{3.5d}$$

with $T_{ab} = 0$ for $a \neq b$.

We use the Einstein tensor components (3.4) in conjunction with the energy momentum tensor components (3.5) in isotropic coordinates to generate the Einstein field equations. We thus obtain the following field equations

$$\rho = -\frac{1}{B^2}\left[2\frac{B''}{B} - \frac{B'}{B}\left(\frac{B'}{B} - \frac{4}{r}\right)\right], \tag{3.6a}$$

$$p = 2\frac{A'}{A}\left(\frac{B'}{B^3} + \frac{1}{r}\frac{1}{B^2}\right) + \frac{B'}{B^3}\left(\frac{B'}{B} + \frac{2}{r}\right), \tag{3.6b}$$

$$p = \frac{1}{B^2}\left(\frac{A''}{A} + \frac{1}{r}\frac{A'}{A}\right) + \frac{1}{B^2}\left[\frac{B''}{B} - \frac{B'}{B}\left(\frac{B'}{B} - \frac{1}{r}\right)\right], \tag{3.6c}$$

for isotropic coordinates.

Equating equation (3.6b) to (3.6c) gives the following equation

$$\frac{A''}{A}\frac{1}{B^2} + \frac{B''}{B^3} - \frac{1}{r}\frac{A'}{A}\frac{1}{B^2} - 2\frac{B^2}{B^4} - 2\frac{B'}{B^3}\left(\frac{A'}{A} + \frac{1}{r}\right) = 0, \tag{3.7}$$

which is the condition of pressure isotropy. The above equation (3.7) is the master equation for the gravitating model in isotropic coordinates. The condition of pressure isotropy has equivalent forms. A second equivalent version of (3.7) is the more compact form

$$\frac{A''}{A} + \frac{B''}{B} = \left(\frac{A'}{A} + \frac{B'}{B}\right)\left(2\frac{B'}{B} + \frac{1}{r}\right), \tag{3.8}$$

in terms of the potentials A and B. A third form is found using the transformation $x = r^2$. Then a more compact form is

$$\left(\frac{A}{B}\right)_{xx} = 2A\left(\frac{1}{B}\right)_{xx}. \tag{3.9}$$

The forms (3.8) and (3.9) are particularly suited to conformally flat metrics. They have been used by Herrera *et al* (2004) and Maharaj and Govender (2005) to study radiating relativistic stars undergoing gravitational collapse with vanishing Weyl stresses.

Chapter 4

Expanding models

4.1 Introduction

In this chapter we consider the Einstein field equations in terms of isotropic coordinates which are comoving in a shear-free geometry. The field equations are then expressed in an equivalent form which may be easier to integrate. In §4.2 we analyse the spacetime geometry for spherically symmetric gravitational fields which depend on time, by specifying the line element in isotropic form. The components of the connection coefficients, the Ricci tensor, the Ricci scalar and the Einstein tensor are explicitly generated in this section. In §4.3 we compute the Einstein field equations by relating the components of the energy momentum tensor for the perfect fluid to the components of the Einstein tensor. The condition of pressure isotropy is also found in this section. It is possible to write the Einstein field equations in a different form by introducing new variables. In this section we consider particular transformations that are relevant to the relativistic gravitational model. These transformations were first studied by Buchdahl (1959), Durgapal and Bannerji (1983), Srivastava (1987), Fodor (2000), and Tewari and Pant (2010). The condition of pressure isotropy is also written in new variables using the relevant transformations. Particular exact solutions are found in chapter 6 in terms of elementary functions for the condition of pressure isotropy. These functions are new solutions to the field equations.

4.2 Spacetime geometry

We consider spacetime which is nonstatic and spherically symmetric and define local coordinates $(x^a) = (t, r, \theta, \phi)$. In contrast to the case studied in chapter 3, the spacetime manifold is expanding and the gravitational potentials are dependent on time. Then the shear-free line element in comoving coordinates can be written as

$$ds^2 = -e^{2\nu(t,r)}dt^2 + e^{2\lambda(t,r)}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)] \tag{4.1}$$

where $\nu(r,t)$ and $\lambda(r,t)$ are arbitrary functions representing the gravitational potentials. This line element was first studied systematically by Kustaanheimo and Qvist (1948) and is mostly used to represent cosmological models. It may also be used to describe physical processes in relativistic compact objects such as superdense stars and neutron stars in astrophysics (Komathiraj and Maharaj 2007, Sharma *et al* 2001, Thirukkanesh and Maharaj 2008).

The line element is very important for the calculation of the connection coefficients $\Gamma^a{}_{bc}$ which are defined in equation (2.2). The nonvanishing connection coefficients for the metric (4.1) are

$$
\begin{aligned}
\Gamma^0{}_{00} &= \nu_r, \\
\Gamma^0{}_{10} &= \nu_r, \\
\Gamma^0{}_{11} &= \lambda_t \exp(2\lambda - 2v), \\
\Gamma^0{}_{33} &= r^2 \lambda_t \sin^2\theta \exp(2\lambda - 2\nu), \\
\Gamma^1{}_{00} &= \nu_r \exp(2\nu - 2\lambda), \\
\Gamma^1{}_{10} &= \lambda_t, \\
\Gamma^1{}_{11} &= \lambda_r, \\
\Gamma^1{}_{22} &= -r(1 + r\lambda_r), \\
\Gamma^1{}_{33} &= -r \sin^2\theta (1 + r\lambda_r),
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^2{}_{12} &= \frac{1 + r\lambda_r}{r}, \\
\Gamma^2{}_{21} &= \lambda_t, \\
\Gamma^2{}_{33} &= -\sin\theta\cos\theta, \\
\Gamma^3{}_{13} &= \frac{1 + r\lambda_r}{r}, \\
\Gamma^3{}_{30} &= \lambda_t.
\end{aligned}
$$

In the above equations, subscripts denote partial differentiation with respect to the timelike and radial coordinates t and r respectively. By using the above connection coefficients we can generate the nonvanishing Ricci tensor components for the line element (4.1). Substituting the above connection coefficients in the definition of the

Ricci tensor equation (2.4) we obtain the following components

$$
\begin{aligned}
R_{00} &= \frac{1}{re^{2\lambda}}[r\nu_r\lambda_r e^{2\nu} + r\nu_r^2 e^{2\nu} + r\nu_{rr}e^{2\nu} - 3r\lambda_{tt}e^{2\lambda} - 3r\lambda_t^2 e^{2\lambda} \\
&\quad +3r\lambda_t e^{2\lambda} + 2\nu_r e^{2\nu}], \qquad (4.2a)
\end{aligned}
$$

$$
R_{10} = -2\lambda_{tr} + 2\nu_r\lambda_t,
$$

$$
\begin{aligned}
R_{11} &= -\frac{1}{re^{2\nu}}[2\lambda_r e^{2\nu} + 2r\lambda_{rr}e^{2\nu} - 3r\lambda_t^2 e^{2\lambda} - r\lambda_t\nu_t e^{2\lambda} - r\lambda_{tt}e^{2\lambda} \\
&\quad +r\nu_{rr}e^{2\nu} - 2r\nu_r\lambda_r e^{2\nu} + r\nu_r^2 e^{2\nu}], \qquad (4.2b)
\end{aligned}
$$

$$
\begin{aligned}
R_{22} &= -\frac{r}{e^{2\nu}}[3\lambda_r e^{2\nu} + r\lambda_{rr}e^{2\nu} + r\lambda_r^2 e^{2\nu} - 3r\lambda_t^2 e^{2\lambda} + r\lambda_t\nu_t e^{2\nu} - r\lambda_{tt}e^{2\lambda} \\
&\quad +\nu_r e^{2\nu} + r\nu_r\lambda_r e^{2\nu}], \qquad (4.2c)
\end{aligned}
$$

$$
R_{33} = \sin^2\theta\, R_{22}, \qquad (4.2d)
$$

with $R_{ab} = 0$ for $a \neq b$ for the other combinations. We then compute the Ricci scalar which is obtained from the nonvanishing Ricci tensor components. The Ricci scalar has the form

$$
\begin{aligned}
R &= \frac{2}{re^{(2(\lambda+\nu))}}[-4\lambda_r e^{2\nu} - 2r\lambda_{rr}e^{2\nu} + 6\lambda_t^2 e^{2\lambda} - 3r\lambda_t\nu_t e^{2\lambda} + 3r\lambda_{tt}e^{2\lambda} \qquad (4.3) \\
&\quad -r\nu_{rr}e^{2\nu} - r\nu_r\lambda_r e^{2\nu} - r\nu_r^2 e^{2\nu} - r\lambda_r^2 e^{2\nu} - 2\nu_r e^{2\nu}]
\end{aligned}
$$

for a shear-free spherically symmetric metric.

We defined the Einstein tensor (2.6) in tensor form in terms of the Ricci tensor and Ricci scalar. The relevant components of the Einstein tensor are given by

$$G_{00} = \frac{1}{re^{2\lambda}} \left[3r\lambda_t^2 e^{2\lambda} - 4\lambda_r e^{2\nu} - 2r\lambda_{rr} e^{2\nu} - r\lambda_r^2 e^{2\nu}\right], \tag{4.4a}$$

$$G_{10} = -2\lambda_{tr} + 2\nu_r \lambda_t,$$

$$\begin{aligned} G_{11} = &- \frac{1}{re^{2\nu}}[-2\lambda_r e^{2\nu} + 3r\lambda_t^2 e^{2\nu} - 2r\lambda_t \nu_t e^{2\lambda} \\ &+ 2r\lambda_t^2 e^{2\lambda} - 2r\nu_r \lambda_r e^{2\nu} - r\lambda_r^2 e^{2\nu} - 2\nu_r e^{2\nu}], \end{aligned} \tag{4.4b}$$

$$\begin{aligned} G_{22} = &- \frac{r}{e^{2\nu}}[-\lambda_r e^{2\nu} - r\lambda_{rr} e^{2\nu} + 3r\lambda_t^2 e^{2\nu} - 2r\lambda_t \nu_t e^{2\lambda} \\ &+ 2r\lambda_{tt} e^{2\lambda} - \nu_r e^{2\nu} - r\nu_{rr} e^{2\nu} - r\nu_r^2 e^{2\nu}], \end{aligned} \tag{4.4c}$$

$$G_{33} = -\sin^2\theta G_{22}, \tag{4.4d}$$

with $G_{ab} = 0$ for $a \neq b$ for the other combinations.

4.3 Einstein field equations

Since the fluid four-velocity is comoving we have $u^a = e^{-\nu}\delta_0^a$ for the metric (4.1). The nonvanishing components of the energy momentum tensor are given by

$$T_{00} = \rho e^{2\nu}, \tag{4.5a}$$

$$T_{11} = p e^{2\lambda}, \tag{4.5b}$$

$$T_{22} = p r^2, \tag{4.5c}$$

$$T_{33} = \sin^2\theta T_{22}, \tag{4.5d}$$

with $T_{ab} = 0$ for $a \neq b$. The Einstein field equations for a perfect uncharged fluid can be written as a system

$$\rho = \frac{3\lambda_t^2}{e^{2\nu}} - \frac{1}{e^{2\lambda}}\left[2\lambda_{rr} + \lambda_r^2 + \frac{4\lambda_r}{r}\right], \tag{4.6a}$$

$$p = \frac{1}{e^{2\nu}}\left[-3\lambda^2 - 2\lambda_{tt} + 2\nu_t\lambda_t\right] + \frac{1}{e^{2\lambda}}\left[\lambda_r^2 + 2\nu_r\lambda_r + \frac{2\nu_r}{r} + \frac{2\lambda_r}{r}\right], \tag{4.6b}$$

$$p = \frac{1}{e^{2\nu}}\left[-3\lambda^2 - 2\lambda_{tt} + 2\nu_t\lambda_t\right] + \frac{1}{e^{2\lambda}}\left[\nu_{rr} + \nu_r\lambda_r^2 + \frac{\nu_r}{r} + \frac{\lambda_r}{r} + \lambda_{rr}\right], \tag{4.6c}$$

$$0 = \nu_r\lambda_t - \lambda_{tr}. \tag{4.6d}$$

In the above ρ is the energy density and p is the isotropic pressure which are measured relative to the four-velocity vector $u^a = (e^{-\nu}, 0, 0, 0)$. Subscripts refer to partial derivatives with respect to that variable. The Einstein system (4.6) is a coupled system of equations in the variables ρ, p, ν and λ. The system of partial differential equations (4.6) can be simplified to produce a single underlying nonlinear second order equation.

The system (4.6) may be extended to include the presence of the electromagnetic in which case we need to study the Einstein-Maxwell equations

Note that (4.6d) can be written as

$$\nu_r = (\ln \lambda_t)_r.$$

Then (4.6b) and (4.6c) imply that

$$\left[e^{\lambda}\left(\lambda_{rr} - \lambda_r^2 - \frac{\lambda_r}{r}\right)\right]_t = 0,$$

which after integration leads to

$$-\tilde{F} = e^{\lambda}\left[\lambda_{rr} - \lambda_r^2 - \frac{\lambda_r}{r}\right],$$

and the potential ν has been eliminated. Then from further simplifications of (4.6), the Einstein field equations can be written in a more compact equivalent form as

$$\rho = 3e^{2h} - e^{-2\lambda}\left[2\lambda_{rr} + \lambda_r^2 + \frac{4\lambda_r}{r}\right], \tag{4.7a}$$

$$p = \frac{1}{\lambda_t e^{3\lambda}}\left[e^{\lambda}\left(\lambda_r^2 + \frac{2\lambda_r}{r}\right) - e^{3\lambda+2h(t)}\right], \tag{4.7b}$$

$$e^{\nu} = \lambda_t e^{-h}, \tag{4.7c}$$

$$-\tilde{F} = e^{\lambda}\left[\lambda_{rr} - \lambda_r^2 - \frac{\lambda_r}{r}\right], \tag{4.7d}$$

where $h(t)$ and $\tilde{F}(r)$ are functions of integration.

Further simplification is possible by eliminating the exponential factor e^{λ} in the condition of pressure isotropy (4.7d). We introduce the new variables

$$x = r^2, \tag{4.8a}$$

$$y = e^{-\lambda}. \tag{4.8b}$$

Then (4.7d) becomes

$$y_{xx}(t,x) = \frac{\tilde{F}(r)}{4r^2} y^2.$$

We then let

$$K(r) = \frac{\tilde{F}(r)}{4r^2}, \tag{4.9}$$

so that we can express (4.7d) in the more compact form

$$y_{xx}(t,x) = K(x) y^2(t,x). \tag{4.10}$$

This equation will be investigated for its integrability conditions in the next chapter. Note that a solution to the differential equation (4.10) generates an exact solution to the Einstein system (4.7) by construction.

Chapter 5

Multiplier formulations for the model

5.1 Introduction

We consider the integrability of the second order differential equation of the form

$$y_{xx} = K(x)y^n. \tag{5.1}$$

In the study of solutions of Einstein's field equations, the equation (5.1) occurs in different physical situations. For example when $n = 2$, this differential equation arises in the description of shear-free spherically symmetric perfect fluid solutions. Other applications of this equation are considered in the treatment of Krasinski (1997). In the previous chapter, y is a function of t and x. It is possible that y is a function of x only. We will consider only the static case with $n = 2$ in our study under which the equation becomes

$$y''(x) = K(x)y^2(x), \tag{5.2}$$

where the primes represent differentiation with respect to x only and $K(x)$ is arbitrary. The solutions of equation (5.2) are important in relativistic astrophysics. They are applicable to highly dense objects, superdense stars, neutron stars, quark stars and strange stars. Leach and Maharaj (1992) indicated that (5.2) is applicable to the Newtonian systems involving plasmas, spherical gas clouds and particle motion in an axially symmetric magnetic field.

5.2 Integration process

To integrate (5.2) we apply the method outlined in §2.4. This involves finding the multiplier μ so that (5.2) can be written in the form (2.24). This can be achieved for particular forms of the function K. We will present the various cases for which this is possible in the next section. To illustrate the method let us write (2.24), in our case, in the form

$$\mu[y'' - Ky^2] = D_x I, \tag{5.3}$$

where I is the first integral. Then solving (2.21) gives specific forms of K and μ. This is a complicated procedure in practice. However this is possible in particular cases. We will illustrate the process by an example . If we take

$$K = \frac{F_1''}{F_1 y}, \tag{5.4a}$$

$$\mu = F_1, \tag{5.4b}$$

then (5.3) becomes

$$F_1''y - F_1''y = I_x + y'I_y + y''I_y'. \tag{5.5}$$

Hence according to the theory in §2.4, equation (5.5) is integrable.

5.3 Multipliers for the model

As pointed out earlier it is necessary to find the multipliers to proceed with the integration of (5.2). We now list the various multipliers for the equation (5.2) through the formula in (2.23) using the method demonstrated in the example given in §5.2.

5.3.1 Case 1

The function K given by

$$K(x) = \frac{F_1''(x)}{F_1 y}, \tag{5.6}$$

has an associated multiplier

$$\mu(x, y, y') = F_1. \tag{5.7}$$

5.3.2 Case 2

In a more general form of the function K given by

$$K(x) = \left[\int \frac{F_2'''}{\left(\exp\left(\frac{F_2}{yC_1}\right)\right)^2} dx + C_2 \right] \int \exp\left(\frac{2F_2'}{yC_1}\right) dx, \tag{5.8}$$

we have the associated multiplier

$$\mu(x, y, y') = C_1 y' + F_2'. \tag{5.9}$$

5.3.3 Case 3

It is possible that K is an explicit function in y. Then the form

$$K(x) = \frac{1}{2C_2^2 y}, \tag{5.10}$$

has an associated multiplier

$$\mu(x, y, y') = C_1 y' + \frac{a_2}{2\exp\left(\frac{x}{C_2}\right)}\left[a_1^2\left(\exp\left(\frac{x}{C_2}\right)\right)^2 + 1\right], \tag{5.11}$$

where the constants $a_1 = \exp\left(\frac{C_3}{C_2}\right)$ and $a_2 = \left(-\frac{C_4}{a_1^2}\right)^{\frac{1}{2}}$.

Two simple cases are contained in (5.11) depending on the values of a_1 and a_2.

Case 3.1: When $a_2 = 0$, we obtain a simplified form for the multiplier

$$\mu(x, y, y') = C_1 y'. \tag{5.12}$$

Case 3.2 : When $a_2 = 1$ and $a_1 = 0$, we obtain another simplified form for the multiplier

$$\mu(x, y, y') = C_1 y' + \frac{1}{2}\exp\left(-\frac{x}{C_2}\right). \tag{5.13}$$

5.3.4 Case 4

It is possible that the quantity K is a constant. When $K(x) \in \mathbb{R}$, it has an associated multiplier of the form

$$\mu(x, y, y') = C_1 y + C_2 \exp\left[\sqrt{2Kyx}\right] + C_3 \exp\left[-\sqrt{2Kyx}\right]. \tag{5.14}$$

For particular values of the constants C_1 ,C_2 and C_3 we obtain simpler forms of the multiplier μ. The various cases are considered below.

Case 4.1: When $C_1 = 1, C_2 = C_3 = 0$, we obtain a simplified form for the multiplier

$$\mu(x, y, y') = y. \tag{5.15}$$

Case 4.2: When $C_2 = 1, C_1 = C_3 = 0$, we obtain the simplified form for the multiplier

$$\mu(x, y, y') = \exp\left[\sqrt{2Kyx}\right]. \tag{5.16}$$

Case 4.3: When $C_3 = 1, C_1 = C_2 = 0$, we obtain the simplified form for the multiplier

$$\mu(x, y, y') = \exp\left[-\sqrt{2Kyx}\right]. \tag{5.17}$$

Case 4.4: When $C_1 = C_2 = 1, C_3 = 0$, we obtain the simplified form for the multiplier

$$\mu(x, y, y') = y + \exp\left[\sqrt{2Kyx}\right]. \tag{5.18}$$

Case 4.5: When $C_1 = C_3 = 1, C_2 = 0$, we obtain the simplified form for the multiplier

$$\mu(x, y, y') = y + \exp\left[-\sqrt{2Kyx}\right]. \tag{5.19}$$

Case 4.6: When $C_2 = C_3 = 1, C_1 = 0$, we obtain the simplified form for the multiplier

$$\mu(x, y, y') = \exp\left[\sqrt{2Kyx}\right] + \exp\left[-\sqrt{2Kyx}\right]. \tag{5.20}$$

5.3.5 Case 5

It is possible that K vanishes. Then the choice

$$K(x) = 0, \tag{5.21}$$

it has an associated multiplier

$$\mu(x, y, y') = C_1 x + F_2(y'). \tag{5.22}$$

A simple subcase arises in (5.22).

Case 5.1: When $C_1 = 0$, we obtain a simplified form for the multiplier

$$\mu(x, y, y') = F_2(y'). \tag{5.23}$$

Chapter 6

First integrals for model

6.1 Introduction

The equations in astrophysical problems involve second order differential equations. First integrals are the result of integrating once, to reduce the second order equations to a first order differential equation. First integrals were introduced in physical problems which dealt with laws of motion and later on used in many other fields of applied mathematics. Any quantities which are not changing with time are called “first integrals". There are many ways in which first integrals can be found. If we are able to reduce our equation from second order to a first order differential equation, it becomes much simpler to solve. Many astrophysical problems can be quite complicated to solve directly. By the use of first integrals, we are able to obtain exact solutions to our systems. These solutions may allow us to make physical conclusions regarding the behaviour of our systems. We use the multipliers generated in chapter 5 for equation (5.2) to find first integrals for all the cases that are listed in §5.3.

6.2 Formulation of the first integrals

There are different methods to finding first integrals; here we are use the characteristic approach. This technique produces a general form for the multiplier, allowing us to extract other simplified forms of $K(x)$ in (5.2). The functions F, y and y' and their derivatives are functions of x unless otherwise specified.

We indicate how the first integral is generated using the example in §5.2. We have

$$K = \frac{F_1''}{F_1 y}, \tag{6.1a}$$

$$\mu = F_1. \tag{6.1b}$$

Then from the relation given in equation (2.23), we obtain the expression

$$F_1[y'' - F_1''y] = I_x + y'I_y + y''I_{y'}. \tag{6.2}$$

We separate the expression by equating coefficients. Equating coefficients of y'' gives

$$I_y' = F_1. \tag{6.3}$$

Integrating (6.3) gives the first integral

$$I = F_1 y' + \psi(x, y), \tag{6.4}$$

where $\psi(x, y)$ is a result of the integration process. Then differentiating (6.4) partially with respect to x and y we get

$$I_x = y'F_1' + \psi_x(x, y), \tag{6.5a}$$

$$I_y = \psi_y. \tag{6.5b}$$

Substituting (6.5) into (6.2) we get

$$-F_1''y = y'F_1' + \psi_x(x, y) + y'\psi_y(x, y). \tag{6.6}$$

Next we equate powers of y', to obtain

$$F_1' + \psi_y(x, y) = 0. \tag{6.7}$$

Then integrating (6.7) with respect to y, we have

$$\psi(x, y) = -F_1' y + f(x), \tag{6.8}$$

where $f(x)$ is also a result of the integration process. Now differentiating (6.8) partially with respect to x, we get

$$\psi_x(x, y) = -F_1'' y + f'(x), \tag{6.9}$$

and substituting (6.9) into (6.6) we get

$$-F_1'' y = -F_1'' y + f'(x). \tag{6.10}$$

In (6.10) we integrate with respect to x, and get

$$f(x) = C, \tag{6.11}$$

where C is a constant resulting from the integration process. Substituting (6.11) into (6.8) we get

$$\psi(x, y) = -F_1' y + C. \tag{6.12}$$

Hence back substituting (6.12) into (6.4) we get

$$I = F_1' y' - F_1' y + C. \tag{6.13}$$

In this case we can perform a second integration. Rewriting (6.13) we get

$$y' = \frac{I}{F_1} + \frac{F_1'}{F_1} y,$$

which can be further integrated to get

$$\int \frac{I}{F_1} dy + \int \frac{F_1' y}{F_1} dy = \frac{dy}{dx}.$$

This finally results in

$$y = \frac{I}{F_1'} \ln |F_1| + \int \frac{F_1'}{F_1} y dx + C. \tag{6.14}$$

6.3 First integrals

As indicated earlier it is required to find the first integrals in (5.2). We now list the various first integrals for equation (5.2) corresponding to the multipliers given in §5.3 of chapter 5. We have checked the accuracy of our results using the computer software package Maple.

6.3.1 Case 1

The function

$$K(x) = \frac{F_1''(x)}{F_1(x)y}, \tag{6.15}$$

has an associated multiplier

$$\mu(x, y, y') = F_1(x). \tag{6.16}$$

The first integral is

$$I = y' F_1 - y F_1' \tag{6.17}$$

as shown in the previous section a second integration gives

$$y = \frac{I}{F_1'} \ln |F_1| + \int \frac{F_1'}{F_1} y dx + C, \tag{6.18}$$

as shown above. Note that in this case we have been in a position to perform the second integration and (6.18) contains no derivatives in terms of y. This is not possible in general.

6.3.2 Case 2

In this case we look at another possible form of K which is

$$K(x) = \left[\int \frac{F_2'''}{\left(\exp\left(\frac{F_2(x)}{yC_1}\right)\right)^2} dx + C_2 \right] \int \exp\left(\frac{2F_2'(x)}{yC_1}\right) dx, \tag{6.19}$$

and has an associated multiplier of the form

$$\mu(x, y, y') = C_1 y' + F_2'. \tag{6.20}$$

From the relation

$$\mu(x, y, y')[y'' - Ky^2] = I_x + y' I_y + y'' I_{y'},$$

the first integral given by

$$I = \frac{C_1}{2} y'^2 - \frac{C_1}{3} K y^3 + y' F_2' - F_2'' y + \int \left(F_2''' y - K y^2 F_2' \right) dx. \tag{6.21}$$

A simple case is contained in (6.21) when $C_1 = 0$. The associated multiplier takes the form $\mu(x, y, y') = F_2'$ and our first integral is simpler becoming

$$I = y' F_2' - F_2'' y + \int \left(F_2''' y - K y^2 F_2'. \right) dx. \tag{6.22}$$

6.3.3 Case 3

In this case we look at another possible form of K which is

$$K(x) = \frac{1}{2C_2^2 y}, \tag{6.23}$$

and has an associated multiplier of the form

$$\mu(x, y, y') = C_1 y' + \frac{a_2}{2\exp\left(\frac{x}{C_2}\right)} \left[a_1^2 \left(\exp\left(\frac{x}{C_2}\right) \right)^2 + 1 \right], \tag{6.24}$$

where

$$a_1 = \exp\left(\frac{C_3}{C_2}\right), \quad a_2 = \left(-\frac{C_4}{a_1^2}\right)^{\frac{1}{2}}.$$

From the relation

$$\mu(x, y, y')[y'' - Ky^2] = I_x + y'I_y + y''I_{y'},$$

our first integral is given by

$$\begin{aligned} I &= \frac{C_1}{2}y'^2 + \frac{C_1}{3}Ky^3 + \frac{a_2}{2\exp\left(\frac{x}{C_2}\right)}\left[a_1^2\left(\exp\left(\frac{x}{C_2}\right)\right)^2 + 1\right]y' \\ &\quad -\frac{a_2}{C_2}\left[a_1^2\exp\left(\frac{x}{C_2}\right) - \exp\left(-\frac{x}{C_2}\right)\right]y \\ &\quad +a_2\int\frac{1}{2\exp\left(\frac{x}{C_2}\right)}\left[a_1^2\left(\exp\left(\frac{x}{C_2}\right)\right)^2 + 1\right] \\ &\quad +a_2\int\frac{1}{C_2^2}\left[a_1^2\exp\left(\frac{x}{C_2} - C_2\exp\left(-\frac{x}{C_2}\right)\right)\right]. \end{aligned} \tag{6.25}$$

Setting $a_2 = 0$ we get a simpler form of the first integral which is given by

$$I = \frac{C_1}{2}y'^2 + \frac{C_1}{3}Ky^3. \tag{6.26}$$

With the value of $K(x)$ from (6.23) we see that the first integral can be further integrated to get y explicitly

$$\int\left(\frac{2}{C_1}I + \frac{1}{3C_1^2}y^2\right)^{-\frac{1}{2}}dy = dx.$$

Letting $a_3 = \frac{2}{C_1}I$ and $a_4 = \frac{1}{3C_1^2}$ which are constants, we get

$$\int\left(a_3 + a_4y^2\right)^{-\frac{1}{2}}dy = dx,$$

and then through integration we get y explicitly as

$$y = \pm\sqrt{\frac{a_3\exp(2a_4(x + C)) - 1}{2a_4}},$$

which is highly desired. This is one the few cases in which y can be written explicitly as a function of x when solving the Emden-Fowler equation (5.2).

6.3.4 Case 4

In this case we look at another possible form of K which is $K(x) \in \mathbb{R}$, and has an associated multiplier of the form

$$\mu(x,y,y') = C_1 y + C_2 \exp\left[\sqrt{2Kyx}\right] + C_3 \exp\left[-\sqrt{2Kyx}\right]. \tag{6.27}$$

From the relation

$$\mu(x,y,y')[y'' - Ky^2] = I_x + y' I_y + y'' I_{y'}, \tag{6.28}$$

our first integral is given by

$$\begin{aligned} I &= \frac{C_1}{2} y'^2 + \frac{C_1}{3} y^3 + y' C_2 \exp\left[\sqrt{2Kyx}\right] + KC_2 \int \frac{y \exp\left[\sqrt{2Kyx}\right]}{\sqrt{2Kyx}} dy \\ &\quad + y' C_3 \exp\left[-\sqrt{2Kyx}\right] - KC_3 \int \frac{x \exp\left[-\sqrt{2Kyx}\right]}{\sqrt{yx}} dy. \end{aligned} \tag{6.29}$$

Three simple cases are contained in (6.29) depending on the particular values of C_1, C_2 and C_3.

Case 4.1: In this case we consider $C_3 = C_2 = 0$ and $C_1 = 1$. The associated multiplier becomes

$$\mu(x,y,y') = y, \tag{6.30}$$

and our first integral is given by

$$I = \frac{1}{2} y'^2 + \frac{1}{3} y^3. \tag{6.31}$$

Case 4.2: In this case we take $C_3 = C_1 = 0$ and $C_2 = 1$. The associated multiplier becomes

$$\mu(x, y, y') = \exp\left[\sqrt{2Kyx}\right], \tag{6.32}$$

and the first integral is given by

$$I = y' \exp\left[\sqrt{2Kyx}\right] + K \int \frac{y \exp\left[\sqrt{2Kyx}\right]}{\sqrt{2Kyx}} dy. \tag{6.33}$$

Case 4.3: In this case we consider $C_1 = C_2 = 0$ and $C_3 = 1$. The associated multiplier becomes

$$\mu(x, y, y') = \exp\left[-\sqrt{2Kyx}\right], \tag{6.34}$$

and our first integral is given by

$$I = K \int \frac{x \exp\left[-\sqrt{2Kyx}\right]}{\sqrt{2Kyx}} dy. \tag{6.35}$$

Case 4.4: In this case we consider $C_3 = C_2 = 1$ and $C_1 = 0$. The associated multiplier becomes

$$\mu(x, y, y') = \exp\left[\sqrt{2Kyx}\right] + \exp\left[-\sqrt{2Kyx}\right], \tag{6.36}$$

and our first integral is given by

$$I = K \int \frac{y \exp\left[\sqrt{2Kyx}\right]}{\sqrt{2Kyx}} dy + y' \exp\left[-\sqrt{2Kyx}\right] - K \int \frac{-\exp\left[-\sqrt{2Kyx}\right]}{\sqrt{2Kyx}} dy. \tag{6.37}$$

Case 4.5: In this case we take $C_3 = C_1 = 1$ and $C_1 = 0$. The associated multiplier becomes

$$\mu(x, y, y') = y + \exp\left[-\sqrt{2Kyx}\right], \tag{6.38}$$

and our first integral is given by

$$I = \frac{1}{2}y'^2 + \frac{1}{3}y^3 + y' \exp\left[-\sqrt{2Kyx}\right] - K \int \frac{x \exp\left[-\sqrt{2Kyx}\right]}{\sqrt{2Kyx}} dy. \tag{6.39}$$

Case 4.6: In this case we consider $C_1 = C_2 = 1$ and $C_3 = 0$, the associated multiplier becomes

$$\mu(x, y, y') = y + \exp\left[\sqrt{2Kyx}\right], \tag{6.40}$$

and our first integral is given by

$$I = \frac{1}{2}y'^2 + \frac{1}{3}y^3 + y' \exp\left[\sqrt{2Kyx}\right] + K \int \frac{y \exp\left[\sqrt{2Kyx}\right]}{\sqrt{yx}} dy. \tag{6.41}$$

6.3.5 Case 5

In this case we investigate another possible form of K which is $K(x) = 0$. We have an associated multiplier of the form

$$\mu(x, y, y') = C_1 x + F_2(y'). \tag{6.42}$$

From the relation

$$\mu(x, y, y')[y'' - Ky^2] = I_x + y' I_y + y'' I_{y'},$$

The first integral given by

$$I = C_1 x y' + y' F_2(y') - C_1 y + C. \tag{6.43}$$

A simpler case is contained in (6.39) when we let $C_1 = 0$.

Case 5.1: With $K(x) = 0$, the associated multiplier of the form $\mu(x, y, y') = F_2(y')$. The first integral given by

$$I = y' F_2(y') + C. \tag{6.44}$$

In this chapter we have found a variety of first integrals for equation (5.2). Each of the integrals generated corresponds to the multipliers in §5.3. It is remarkable that our integration procedure has produced such a variety of first integrals to the Emden-Fowler equation (5.2). There are two cases of particular interest. Firstly, the integral corresponding to Case 3 in §6.3.3 enables us to write the solution for the function y in terms of the variable x explicitly. Secondly we have perfomed a second integration to generate the solution (6.18) in §6.3.1 and no derivatives of y are present. The results for the various cases are tabulated in Table 6.1.

Table 6.1: Summary of results without the special cases*

Case	$K(x)$	Associated multiplier	First integral
1	$\frac{F_1''(x)}{F_1(x)y}$	$F_1(x)$	$y'F_1(x) - yF_1'(x)$
2	F_3	$C_1y' + F_2(x)'$	$\frac{C_1}{2}y'^2(x) - \frac{C_1}{3}Ky^3(x)$
3	$\frac{1}{2C_2^2y(x)}$	F_4	$\frac{C_1}{2}y'^2(x) + \frac{C_1}{3}K(x)y^3(x)$
4	$K \in \mathbb{R}$	$C_1y(x) + C_2e^{\sqrt{2Kxy(x)}}$ $+C_3e^{-\sqrt{2Kxy(x)}}$	$\frac{C_1}{2}y'^2(x) + \frac{C_1}{3}Ky^3(x)$
5	0	$C_1x + F_2(y'(x))$	$C_1xy'(x) + y'(x)F_2(y'(x))$ $-C_1y(x) + C$

In Table 6.1 we have set *

$$F_3 = \left[\left[\int \frac{F_2'''}{\left(\exp\left(\frac{F_2(x)}{y(x)C_1}\right)\right)^2}\right] dx + C_2\right] \int \exp\left(\frac{2F_2'(x)}{yC_1}\right) dx,$$

and

$$F_4 = C_1y'(x) + \frac{a_2}{2\exp\left(\frac{x}{C_2}\right)}\left[a_1^2\left(\exp\left(\frac{x}{C_2}\right)\right)^2 + 1\right].$$

Chapter 7

Conclusion

Our aim in this thesis was to examine the spherically symmetric spacetimes and the Einstein field equations in relativistic astrophysics. Our main objective was to generate new exact solutions of the Einstein field equations with isotropic pressures. Since the Einstein field equations are highly nonlinear in general we used new variables in order to transform the field equations to equivalent forms. We transformed the condition of pressure isotropy by reducing it to less complicated second order differential equations with variable coefficients. We have shown that the fundamental equation governing the gravitational potentials is of the Emden-Fowler type. This equation also arises in Newtonian physics. We obtained several new exact solutions in terms of elementary functions by choosing specific gravitational potentials in order to solve the master equation. We generated a number of algorithms that produce a new solution if a particular model is specified. The new exact solutions are useful in many applications in general relativity, realistic stellar models and Newtonian physics.

We now provide a brief outline of the dissertation by giving the main results achieved in our course of study:

• In chapter 2, we briefly introduced the concepts of differential geometry and the matter distribution that are essential for generating the Einstein field equations. We formulated the Einstein field equations for neutral perfect fluid matter distributions. We also briefly introduced the barotropic equation of state relating the pressure to the energy density. We outlined the physical conditions that are relevant for a realistic relativistic gravitational model.

• In chapter 3, we considered the static gravitational model in isotropic coordinates. We generated the Einstein field equations by using the energy momentum for a perfect fluid. From the Einstein field equations we deduced the condition of pressure isotropy which is a second order differential equation with variable coefficients.

• In chapter 4, we generated the Einstein field equations in terms of isotropic coordinates for neutral perfect fluid matter distribution in nonstatic spherically symmetric spacetimes. We then transformed the resulting field equation into a second order differential equation. We showed that the equation

$$y'' = K(x)y^2, \tag{7.1}$$

is of the Emden-Fowler type.

• in chapter 5, we gave a summary of the possible combinations of the form of the unknown function $K(x)$ and its possible associated multiplier under which the Emden-Fowler model (7.1) was integrable.

• In chapter 6, we made use of the method of multipliers to find first integrals and their explicit solutions where possible. In addition we considered some special cases allowing combinations of constants to vanish. It is interesting to observe that for a particular multiplier it is possible to write the solution for y only in terms of the independent variable x. Our results are summarized in tabular form in Table 6.1.

Chapter 8

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