Perfect Compactifications of Frames

by

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Submitted in fulfilment of the academic

requirements for the degree of

Doctor of Philosophy

in the

School of Mathematics, Statistics and Computer Science

University of KwaZulu-Natal

Durban

November 2018

Dedicated to:

my daughter Akabongwe Mthethwa and the whole family.

Abstract

We study the compactifications of frames. In particular, we study the compactifications of frames which are perfect. That is, those compactifications for which the right adjoint of the compactification mapping preserves disjoint binary joins. The Stone-Cěch compactification of a completely regular frame and the Freudenthal compactification of a rim-compact frame are known to be examples of such compactifications. We study the Freudenthal compactification of a rim-compact frame with an aim of providing more properties and characterizations of this compactification in the context of frames since this is less studied in the literature compared to the Stone-Cěch compactification of frames. One of the main results that we obtain about the Freudenthal compactification of a rim-compact frame is that it is the minimal perfect compactification for this class of frames and the maximal π-compactification.

The notion of a full π-compact basis is known in the context of spaces. We define an analogous concept in the context of frames and show that the Freudenthal compactification of a rim-compact frame arises from such a basis. We also establish the one-to-one correspondence between such bases and the π-compactifications of a rim-compact frame. The fact that the compactifications arriving from such basis are zero-dimensional is also established.

It is well known that a frame has the least compactification if and only if it is regular continuous. Some conditions under which the least compactification of a regular continuous frame is perfect have been studied by Baboolal in [1] and the study is furthered herein. An N-point compactification of a space is any compactification whose remainder consists of N points. The N-star compactifications of frames are known to be the frame analogue of the N-point compactifications for spaces. It has been shown that the least compactification of a regular continuous frame is an example of an N-star compactification. We study the conditions under which a 2-star compactification of a regular continuous frame is perfect and we conjecture that the results can be generalized to any $N > 1$. We prove that, under perfectness, the 2-star compactification of a regular continuous frame is the only N-star compactification. We also show some results related to the connectedness of the remainder of the 1-star compactification.

Some contribution to the theory of compactifications of frames not relating to perfectness has also been made. The concept and the construction of a freely generated frame is well known. We have shown that any compactification of a frame L can be realized as a frame freely generated by L subject to certain relations.

Preface and declaration

The study presented in this thesis was carried out in the School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Durban. This thesis was completed under the supervision of Prof. Dharmanand Baboolal and Dr. Paranjothi Pillay from March 2015 to November 2018.
The research contained in this thesis represents original work by the author and has not been submitted in any form to another University nor has it been published previously. Where use was made of the work of others it has been duly acknowledged in the text.

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As the candidate's supervisors we have approved this thesis for submission.

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Acknowledgements

I wish to express my sincere gratitude to the following people and organisations who made this thesis possible:

- Firstly, I would like to thank God for his guidance and help in giving me the strength and courage to complete this research.
- Prof. Dharmanand Baboolal for his help and Mathematical guidance throughout the period of this research. I would also like to thank Dr. Paranjothi Pillay for carefully reading my work before submission.
- Prof. Aleš Pultr for his lectures and individual consultation times during each of his visits to UKZN.
- The National Research Foundation for financial assistance for the first three years of this research.
- The School of Mathematics, Statistics and Computer Science (UKZN) for providing me with an academic-friendly environment to do my work.

Glossary of symbols, notation and conventions

$\underline{C}$	category
$\mathbf{Hom}(\boldsymbol{C})$	set of all morphisms between the objects of $\underline{C}$
f, h, g	morphisms
id_X	identity morphism on X
Obj(C)	class of objects of a category $\underline{C}$
L, M, N	frames
I, J, K	ideals
e	top element of a lattice
0	bottom element of a lattice
$\bigvee F$	supremum of a subset F of a poset
$\bigwedge F$	infimum of a subset F of a poset
$\vee$	finite join
$\wedge$	finite meet
$\Rightarrow$	implication
$\Leftrightarrow$	double implication
$\square$	end of the proof

ker	the kernel of the frame homomorphism
$\circ$	composition of function
$cl_X(U)$	closure of an open set U in X
$Fr_X(U)$	$cl_X(U) \setminus U$, i.e., the frontier of the open set U in X
$\in$	element of
$X \setminus Y$	$\{x \in X : x \notin Y\}$
$\bigcup$	arbitrary union
$\bigcap$	arbitrary intersection
$\cap$	finite intersection
$\cup$	finite union
$\subseteq$	subset of
$\subsetneq$	proper subset
$\emptyset$	empty set
$\uparrow a$	$\{x \in L : x \geq a\}$
$\downarrow a$	$\{x \in L : x \leq a\}$
$\mapsto$	maps to
a^-	complement of a
a^*	pseudocomplement of a
DA	downset lattice of A

Contents

Chapter 1

Introduction

In this chapter we provide some historical background of compactifications and a synopsis of this thesis.

The theory of frames is an approach to the study of topology where open set lattices are taken as the primitive notion, at least, from a topologist view point. In this philosophical approach, choice principles such as the Axiom of Choice, which are freely used in non-constructive mathematics like in topological spaces, are very much avoided. A more detailed account on the history and evolution of this theory is given by P. T. Johnstone in [15].

Compactifications of topological spaces is a well studied theory and has a long history spanning several generations. It was B. Banaschewski [6] who first conceived the idea of a frame compactification where the interest was to construct compact Hausdorff extension spaces for a given space in terms of frames. The objective of this was to explicate the lattice theoretic essence of some important constructions in topology. One of these important constructions is the Stone-Čech compactification of a topological space. In a later paper B. Banaschewski and C. Mulvey [7] construct the analogue of this result for frames. Also, P. T. Johnstone [16] provides an alternative construction of this result. Subsequently, Banaschewski [6] provided a very comprehensive view of all compactifications of a frame by introducing the very useful notion of a *strong inclusion*. Our interest will be to investigate those compactifications of frames that are referred to as perfect compactifications. These are those compactifications for which the right adjoint of the compactification mapping preserves disjoint binary joins. The motivation for studying this comes from a paper by E.G. Skljarenko [23] where the notion of a perfect compactification of a topological spaces is considered. Coming out of this paper, there are many questions or problems that have a formulation in terms of frames that can be investigated. These questions formed the basis of our research that is presented in this thesis.

Chapter 2 lays down a foundation for the thesis. All the definitions that will be needed in a sequel are discussed. These definitions concern basic theory of frames and are very much well known. Nevertheless, the goal for this chapter is to provide concepts and notations to be used later on. A few relevant well-known results are also furnished, mostly without proofs. The equivalence between spatial frames and a sober topological spaces is outlined and this is one of the core reasons for this chapter.

In Chapter 3 we show that any compactification of a regular frame can be realized as a frame which is freely generated on a suitable meet semilattice subject to certain relations. The way this is done is to recall that each compactification θL of a frame L gives rise to a unique strong inclusion $\lhd$ on L, see Banaschewski [6]. This strong inclusion is then used to define certain relations on DL, the downset lattice of L, and the associated freely generated frame is then shown to be isomorphic with θL. Familiar examples of compactifications which can be regarded as such frames are provided.

It shall be noted that the work in Chapter 3 is an extension of the work presented by the author and his supervisors in [21]. The chapter is ended by showing that a compactification of a rim-compact frame arising from a base satisfying some properties can be realized as a completion of a suitable uniform frame.

Chapter 4 studies those compactifications of frames which are perfect. In this chapter, the reader is reminded of the perfect compactifications of topological spaces as introduced by Skljarenko in [23]. The Stone-Čech compactification of a completely regular topological space and the Freudenthal compactification of a rim-compact topological space are examples of perfect compactifications. The definition of a Freudenthal compactification for the class of rim-compact frames was given by Baboolal in [3] where he also proved the perfectness of this compactification in the frame language. We show that the Freudenthal compactification of a rim-compact frame arises from what we call a *full* π-compact basis. This type of a basis is then studied and the elements of this basis are characterised in terms of perfectness. Several other characterizations of a full π-compact basis are given. The one-to-one correspondence between the full π-compact basis and the π-compactification is furnished, the π-compactification being a zero-dimensional compactification corresponding to a strong inclusion which is defined using the element(s) of a π-compact basis. Hence, there exists a relationship between the ordering of the π-compactifications and the ordering of full π-compact bases.

In Chapter 5 we look at the Freudenthal compactification for rim-compact frames more closely with the aim of providing more properties and information about it. As a result, the minimality of this compactification with respect to perfectness is established for this class of frames. Several other properties satisfied by this compactification are analysed and the uniqueness of this compactification with respect to these properties is detailed. More importantly, this chapter is closed by showing that the category of compact regular frames form a coreflective subcategory of the category of rim-compact regular frames with morphisms being what we call the *F-maps*. The coreflection in this regard is shown to be the Freudenthal compactification.

The least compactification of a regular continuous frame was introduced by Banaschewski [6]. This is the frame counterpart of the well-known Alexandroff one-point compactification of a topological space in the following sense: If X is locally compact then $\mathcal{O}X$ is continuous and a topological space has a smallest compactification if and only if it is locally compact, on the other hand, Banaschewski [6] showed that a frame has the smallest compactification if and only if it is regular continuous. Baboolal [1] studied the conditions under which the least compactification of a regular continuous frame is perfect. In [2], Baboolal developed some theory about the N-star compactification, which is the frame analogue of the well-known N-point compactification for topological spaces. Chapter 6, the last chapter, deals with the conditions under which the N-star compactification of a regular continuous frame is perfect. These conditions are studied under the setting where $N = 2$ and we conjecture that all the results can be generalised to any $N > 1$. In this chapter, we exhibit the connectedness of the remainder of the least compactification of a non-compact regular frame. Amongst other results, we show that a 2-star compactification of a non-compact regular continuous frame L is perfect if and only if it is the only 2-star compactification of L, up to equivalence. Moreover, if a 2-star compactification is perfect, then there exists no other N-star compactification of L for any

$N > 1$. The latter and its converse are also shown to be true for the least compactification of a non-compact regular frame L.

Chapter 2

Preliminaries

In this chapter we provide some basic definitions and results regarding frames which are needed in the subsequent chapters. Results and definitions contained in this chapter are either part of mathematical folklore or can be accessed from the literature. For the details of this introductory material, the reader is referred to the reading of [14] and [22], these can be taken as main references for results in frames.

2.1 Lattices, ideals and filters

This section covers definitions from lattice theory that are going to be used in the sequel.

Definition 2.1.1. A **lattice** is a set A, endowed with a partial order $\leq$, in which every finite subset $S \subseteq A$ has both a greatest lower bound (also called a **meet**, written as $\bigwedge S$) and a least upper bound (also called a **join**, written as $\bigvee S$). A **sublattice** of a lattice A is a subset $S \subseteq A$ that is closed under the same meets and joins. If all finite meets exist, but not all finite joins, then A is called a **meet-semilattice** (**join-semilattice** is a dual notion of the meet-semilattice). If all joins and meets exist, including the infinite ones, then A is called a **complete lattice**. A lattice A is called **distributive** if for any $a, b, c \in A$, the **distributive law** $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ holds true.

We note that a sublattice of a lattice forms a lattice.

Remark 2.1.2. In the definition above, suppose that $S = \{x, y\}$, then we write $x \vee y$ for $\bigvee S$ and $x \wedge y$ for $\bigwedge S$. If $S = \emptyset$, we write 0 for $\bigvee \emptyset$. Vacuously, 0 is the least (or bottom) element of A. Similarly, we write e (or 1) for $\bigwedge \emptyset$, in this case e is the top element of A.

Definition 2.1.3. A **meet-semilattice homomorphism** (**join-semilattice homomorphism**) is a map $f : A \to B$ between two meet-semilattice (join-semilattice homomorphism) A and B preserving the top elements (bottom elements) and whenever $a, b \in A$, we have $f(a \wedge b) = f(a) \wedge f(b)$ $[f(a \vee b) = f(a) \vee f(b)]$.

We sometimes refer to a meet-semilattice or a join-semilattice as just a semilattice. A well-known example of a semilattice is the so called free semilattice generated by a set (or a free semilattice on a set).

Definition 2.1.4. Let X be a set. The **free semilattice** generated by X is the set FX of finite subset of X, the semilattice operation is the set union.

Lemma 2.1.5. *[14, Lemma 4.4 (Ch.I)] Let X be a set and FX be a free semilattice on X. Then any set $S \in FX$ is uniquely expressible as a finite union of singletons. Moreover, any map $f : X \to A$, where A is a semilattice, can be extended uniquely to a semilattice homomorphism $\hat{f} : FX \to A$*

defined by $\hat{f}(S) = \bigvee_A \{f(x) : x \in S\}$. *That is, we have the following situation:*

$$\begin{array}{ccc} X & \xrightarrow{i_X} & FX \\ & \searrow^{f} & \downarrow{\scriptstyle \exists! \hat{f}} \\ & & A \end{array}$$

where i_X *is the inclusion map defined by* $i_X(x) = \{x\}$.

It should be noted that some authors do not require a lattice to have empty joins and empty meets. That is, the existence of the top and the bottom element is not always required. In our setting, we require the existence of these joins and meets (so, we consider the empty set as a finite set) and such lattices are termed bounded as presented below.

Definition 2.1.6. A lattice in which the bottom element 0 and the top element e exist is called a **bounded** lattice.

The well-known notion of ideals and filters will be used throughout, so we provide the definition below.

Definition 2.1.7. Let L be a frame and $J \subseteq L$. We say that J is an **ideal** of L if the following conditions hold true:

1. $0 \in J$
2. For all finite subset $F \subseteq J \Rightarrow \bigvee F \in J$ and
3. $a \leq b \in J \Rightarrow a \in J$.

The smallest ideal that contains a given point is called a **principal ideal**.

Definition 2.1.8. A **filter** of a frame L is a non-empty subset F of L such that:

1. $a, b \in F \Rightarrow a \wedge b \in F$ and

2. $a \geq b \in F \Rightarrow a \in F$.

A filter F is called **proper** if $F \neq L$ (or if $0 \notin F$).

2.2 Some categorical concepts

In this section we provide the definitions of some concepts from category theory which are going to be used in the subsequent chapters. Most of these concepts are well known, we provide their definitions for the sake of completeness. For more information on category theory, the reader is referred to the book by Mac Lane [18].

Definition 2.2.1. A **category** $\underline{C}$ consists of the following three things:

1. A class, **Obj(C)**, of **objects** of $\underline{C}$. The elements of **Obj(C)** will be denoted by capital letters $A, B, C \ldots$

2. A class of **morphisms**, **Hom(*C*)**, whose elements are denoted by lower-case letters $f, g, h \ldots$ Each morphism has a **domain** and a **codomain** which are elements of **Obj(C)**. We write $f : A \rightarrow B$ to mean that f is a morphism with A and B as its domain and codomain, respectively. In this case, we write $dom(f) = A$ and $codom(f) = B$.

3. A **composition law** which assigns to each pair of morphisms (f, g), with $dom(f) = cod(g)$, a composite morphism $g \circ f : dom(g) \rightarrow codom(f)$. Furthermore, the composition is required to satisfy the following axioms:

   (a) The composition is **associative**. That is, $(f \circ g) \circ h = f \circ (g \circ h)$ whenever compositions are defined.

(b) For each $A \in \mathbf{Obj(C)}$, there exists an **identity morphism** $id_A : A \to A$ satisfy $f \circ id_A = f$ and $id_A \circ g = g$, whenever the composition $\circ$ is defined.

The **opposite** of a category $\underline{C}$, denoted by $\underline{C}^{op}$, is a category with the same object as $\underline{C}$ and whenever $f : A \to B$ is a morphism in $\underline{C}$ then $f : B \to A$ is a morphism in $\underline{C}^{op}$.

A **subcategory** $\underline{S}$ of a category $\underline{C}$ is a category whose objects are in $\mathbf{Obj(C)}$ and whose morphisms are in $\mathbf{Hom(\mathit{C})}$ with the same identity morphism and composition of morphism. A subcategory $\underline{S}$ of a category $\underline{C}$ is called **full** if the set of morphisms between a given pair of objects in $\underline{S}$ is the same as the set of morphisms of that pair in $\underline{C}$.

We say that a subcategory $\underline{S}$ is **reflective** in the category $\underline{C}$ if for each $A \in \mathbf{Obj(C)}$, there is a morphism $h_A : A \to hA$ in $\underline{C}$ (here $hA \in \mathbf{Obj(S)}$), such that for every object B in $\mathbf{Obj(S)}$ and a morphism $f : A \to B$ in $\underline{C}$, there is a unique morphism $g : hA \to B$ such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{h_A} & hA \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \exists! g} \\ & & B \end{array}$$

Here, the map h_A is called the **reflection map** and hA is called the **reflection** of A. The dual notion of a reflective subcategory is the notion of a **coreflective** subcategory, i.e., all the arrows in the diagram above are reversed. In this case, hA is called the **coreflection** of A.

We note that we normally abbreviate $f \circ g$ as fg. Another important notion is that of a functor. Roughly, a functor is a function between categories that sends objects to objects and morphisms to morphisms, and respects compositions and identities. One can view a functor as a morphism between categories. We provide its formal definition and that of a natural transformation between functors below.

Definition 2.2.2. Let $\underline{C}$ and $\underline{D}$ be categories. A **functor** is a function $F : \underline{C} \to \underline{D}$ that assigns to each $A \in \mathbf{Obj(C)}$ an object $FA \in \mathbf{Obj(D)}$ and to each morphism $h : A \to B$ in $\underline{C}$ a morphism $Fh : FA \to FB$ in $\underline{D}$ such that:

1. $F(f \circ g) = Ff \circ Fg$, whenever $f \circ g$ is defined.
2. $F(id_A) = id_{FA}$.

A **natural transformation** $\eta : F \to G$ between two functors $F : \underline{C} \to \underline{D}$ and $G : \underline{C} \to \underline{D}$ is a function such that:

1. For each $A \in \mathbf{Obj(C)}$, we have a morphism $\eta_A : FA \to GA$ in $\underline{D}$.
2. For each morphism $h : A \to B$ in $\underline{C}$, the following square diagram commutes:

$$\begin{array}{ccc} FA & \xrightarrow{\eta_A} & GA \\ {\scriptstyle Fh}\downarrow & & \downarrow{\scriptstyle Gh} \\ FB & \xrightarrow{\eta_B} & GB \end{array}$$

Remark 2.2.3. In the definition above, we gave the definition of what is sometimes referred to as a *covariant functor*. A *contravariant functor* is the same except that, instead of $Fh : FA \to FB$ we have $Fh : FB \to FA$, for each morphism $h : A \to B$ in $\underline{C}$.

Another central notion of category theory that we are going to be using a lot is that of adjunctions. We provide with the definition of this concept below.

Definition 2.2.4. Let $\underline{C}$ and $\underline{D}$ be categories. Two functors $F : \underline{C} \to \underline{D}$ and $G : \underline{D} \to \underline{C}$ are **adjoint** if there exist natural transformations $\eta : id_{\underline{C}} \to GF$ and $\varepsilon : id_{\underline{D}} \to FG$ (called the **unit** and **counit**, respectively) such that the following diagrams commute:

$$\begin{array}{ccc} FA & \xrightarrow{\varepsilon_{FA}} & FGFA \\ & \searrow^{id_{FA}} & \downarrow{\scriptstyle F\circ\eta_A} \\ & & FA \end{array} \quad \text{and} \quad \begin{array}{ccc} GB & \xrightarrow{\eta_{GB}} & GFGB \\ & \searrow^{id_{GB}} & \downarrow{\scriptstyle G\circ\varepsilon_B} \\ & & GB \end{array}$$

for any $A \in \mathbf{Obj(C)}$, $B \in \mathbf{Obj(D)}$. We right $F \dashv G$ and we say that G is the **right adjoint** of F. In the case where F and G are contravariant functors, the counit becomes $\varepsilon : FG \to id_{\underline{D}}$ and we say that F is the **left adjoint** of G (in this case, arrows are reversed in the diagrams above). The commutative triangles above are called the **triangular identities**.

2.3 Frames and frame homomorphisms

In what follows, all the lattices are bounded.

Definition 2.3.1. A **frame** L is a complete lattice which satisfies the infinite distributive law:

$$x \wedge \bigvee S = \bigvee \{x \wedge s : s \in S\}$$

for every $x \in L$ and every $S \subseteq L$. The top element and the bottom element of L will be denoted by e and 0, respectively (as seen in Remark 2.1.2). A **frame homomorphism** is a map $h : L \to M$, between the frames L and M, that preserves finite meets, including e, and arbitrary joins, including 0.

The identity map of any frame and the composite of frame homomorphisms are frame homomorphisms. We therefore have a category $\underline{FRM}$ of frames and frame homomorphisms.

Definition 2.3.2. Any frame homomorphism $h : L \to M$ has a **right adjoint** $r : M \to L$ defined by $r(y) = \bigvee\{x \in L : h(x) \leq y\}$.

Remark 2.3.3. The concept of the right adjoint defined above is a special case of the concept of the right adjoint defined in Definition 2.2.4. However, we will not discuss this any further.

Some properties of a right adjoint which we are going to use are captured below. We do not provide the proofs because these properties are well known and easy to verify.

Lemma 2.3.4. *If $h : M \to L$ is a frame homomorphism and $r : L \to M$ is the right adjoint of h, then the following properties hold true;*

1. $r(a \wedge b) = r(a) \wedge r(b)$, *for all* $a, b \in L$.
2. *If* $a \leq b$, *then* $r(a) \leq r(b)$.
3. $a \leq rh(a)$.
4. $h(x) \leq y$ *if and only if* $x \leq r(y)$ *for all* $x \in M$ *and all* $y \in L$.
5. *If* h *is onto then* $r(y) = \bigvee\{x \in L : h(x) = y\}$; *i.e.,* $r(y)$ *is precisely the largest element mapped by* h *to* y.

Definition 2.3.5. Let L be a frame. The **open quotient** induced by an element $a \in L$ is the set $\downarrow a = \{x \in L : x \leq a\}$. The **closed quotient** (or **closed sublocale**) induced by an element $a \in L$ is given by $\uparrow a = \{x \in L : x \geq a\}$.

For more information on sublocales, see [[14], Ch. II] or [[22], Ch. III].

Definition 2.3.6. For any partially ordered set X, a non-empty subset $A \subseteq X$ is called a **downset** if whenever $a \leq x$ and $x \in A$, then $a \in A$. An **upset** is a dual notion of a downset.

Remark 2.3.7. Any downset A can be expressed as $A = \bigcup\{\downarrow a : a \in A\}$. Let A be a bounded meet-semilattice and DA be the set of all downsets of A. It is part of the folklore that the set DA partially ordered by the set inclusion is a frame. Moreover, if L is a frame then the join map $\bigvee : DL \to L$ is a frame homomorphism.

A complete Boolean algebra is an example of a frame, where for each element a there is an element a^- with the property that $a \wedge a^- = 0$ and $a \vee a^- = e$. In general, this is not true in a frame. However, we have the notion of a pseudocomplement defined below.

Definition 2.3.8. Let L be a frame and $a \in L$. The **pseudocomplement** of a is the element defined by $a^* = \bigvee\{x \in L : x \wedge a = 0\}$. We say that an element a is **complemented** if $a \vee a^* = e$.

The following is well known and easy to show and hence the proof is omitted.

Lemma 2.3.9. *Let L be a frame. For any $a, b \in L$, we have:*

1. $a \leq a^{**}$.
2. $a^* \wedge b^* = (a \vee b)^*$.
3. $a \leq b \Rightarrow b^* \leq a^*$.
4. $a^* = a^{***}$.

Special kinds of maps are introduced below and their significance will become transparent in the subsequent chapters.

Definition 2.3.10. A map h is **dense** if whenever $h(x) = 0$, then $x = 0$ and **codense** if whenever $h(x) = e$, then $x = e$.

Properties satisfied by the right adjoint of a dense and onto frame homomorphism due to Banaschewski [5] are recorded below.

Lemma 2.3.11. *If $h : M \to N$ is a dense and onto frame homomorphism, then h and its right adjoint $r : N \to M$ satisfy $h(x^*) = h(x)^*$ and $r(a^*) = r(a)^*$ for all $x \in M$ and $a \in N$.*

Definition 2.3.12. Let L be a frame and $S \subseteq L$. Then S is called a **cover** of L if $\bigvee S = e$. The collection of all covers of L will be denoted by $\mathbf{Cov(L)}$. An element $a \in L$ is called compact if whenever $a \leq \bigvee S$, where $S \subseteq L$, then $a \leq \bigvee F$, for some finite $F \subseteq S$. We say that L is **compact** if whenever $e = \bigvee S$, then $e = \bigvee F$ for some finite $F \subseteq S$, i.e., L is compact if the top element is compact.

We use the idea of a uniform topological space from the covering point of view (see Tukey [25] or Page [24]) to define a uniform frame (see Křìž [17] or Banaschewski and Pultr [8]).

Definition 2.3.13. Let L be a frame, C and D be covers of L. We say that C **refines** D, denoted by $C \leq D$, if for each $c \in C$, there is $d \in D$ such that $c \leq d$. We also denote the cover $\{c \wedge d : c \in C, d \in D\}$ by $C \wedge D$.

For any cover C and $x \in L$, let $Cx = \bigvee\{c \in C : c \wedge x \neq 0\}$. The cover $\{Cx : x \in C\}$ shall be denoted by C^*. We say that C **star-refines** D if $C^* \leq D$ (mostly written as $C \leq^* D$ in the literature).

Let $\mathfrak{U}$ be a non-empty family of covers of L. We say that $(L, \mathfrak{U})$ is a **uniform frame** if the following conditions are satisfied;

1. if $C \in \mathfrak{U}$, $D \in Cov(L)$ and $C \leq D$, then $D \in \mathfrak{U}$,

2. if $C, D \in \mathfrak{U}$, then $C \wedge D \in \mathfrak{U}$,

3. if for any $D \in \mathfrak{U}$, there exists $C \in \mathfrak{U}$, such that $C \leq^* D$, and

4. for each $a \in L$, $a = \bigvee\{x \in L : Ax \leq a$, for some $A \in \mathfrak{U}\}$.

It is also common to say that $\mathfrak{U}$ is a **compatible uniform structure** (a **uniformity**) on L if condition 3 is satisfied. The members of the uniformity $\mathfrak{U}$ are called **uniform covers**.

To have a category of uniform frames, we provide the description of the morphisms in the following definition.

Definition 2.3.14. A function $f : (L, \mathfrak{U}) \to (M, \mathfrak{V})$ between uniform frames is called a **uniform frame map** if the following conditions are satisfied:

1. $f : L \to M$ is a frame map

2. $C \in \mathfrak{U}$ implies that $f(C) = \{f(c) : c \in C\} \in \mathfrak{V}$.

We denote the category of uniform frames and uniform frame maps by $\underline{UNIFRM}$.

Just like in topological spaces, we have the following definition of a uniformity base (or subbase).

Definition 2.3.15. If $\mathfrak{U}$ is a uniformity on L, a subcollection $\mathfrak{B} \subseteq \mathfrak{U}$ is called a **basis for the uniformity** $\mathfrak{U}$ if for each $U \in \mathfrak{U}$, there exists $V \in \mathfrak{B}$ such that $V \leq U$. A **uniformity subbase** is a subcollection of uniform covers

$\mathfrak{D}_{\mathfrak{U}}$ such that the set of all finite meets of the members of $\mathfrak{D}_{\mathfrak{U}}$ forms a basis for the uniformity $\mathfrak{U}$, i.e., for each $U \in \mathfrak{U}$, there exists $V_1, V_2, \ldots, V_n \in \mathfrak{D}$ such that $\bigwedge_{i=1}^{n} V_i \leq U$.

The following is easy to check.

Lemma 2.3.16. *A set $\mathfrak{B} \subseteq Cov(L)$ is a basis for some uniformity $\mathfrak{U}$ on L if and only if it satisfies the following conditions:*

1. *$A, B \in \mathfrak{B}$ implies $C \leq A \wedge B$, for some $C \in \mathfrak{B}$.*

2. *$A \in \mathfrak{B}$ implies $B \leq^* A$, for some $B \in \mathfrak{B}$.*

3. *For each $a \in L$, $a = \bigvee\{x \in L : Ax \leq a, \text{ for some } A \in \mathfrak{B}\}$.*

Definition 2.3.17. Let L be a frame and $B \subseteq L$. We call B a **basis** (or **base**, we use both terms) for L if each $a \in L$ is expressible as a join of some elements in B.

2.4 Topological spaces and frames

Topological spaces give rise to an important class of frames. For any topological space X, the lattice of open sets $\mathcal{O}X$, partially ordered by set inclusion, is a frame. Furthermore, any continuous function $f : X \to Y$, between topological spaces gives rise to a frame homomorphism $\mathcal{O}f : \mathcal{O}Y \to \mathcal{O}X$, taking $U \in \mathcal{O}Y$ to $f^{-1}(U) \in \mathcal{O}X$. The resulting correspondence, from spaces to frames and continuous maps to frame homomorphism, constitute a contravariant functor $\mathcal{O} : \underline{TOP} \to \underline{FRM}$, where $\underline{TOP}$ denotes the category of topological spaces and continuous maps.

For each frame L, let ΣL be the **spectrum** of L. That is, ΣL is the topological space of all frame homomorphisms $\xi : L \to \mathbf{2}$, where $\mathbf{2} = \{0, 1\}$

is the two-element frame. The open sets of the spectrum are given by $\Sigma_a = \{\xi \in \Sigma L : \xi(a) = 1\}$ for each $a \in L$. For each frame homomorphism $h : L \to M$, the associated continuous map $\Sigma h : \Sigma M \to \Sigma L$ takes each $\xi \in \Sigma M$ to ξh. The continuity results from the identity map $(\Sigma h)^{-1}(\Sigma_a) = \Sigma_{h(a)}$. We therefore have a **spectrum functor** $\Sigma : \underline{FRM} \to \underline{TOP}$. We now have the following situation:

$$\underline{TOP} \quad \underset{\Sigma}{\overset{\mathcal{O}}{\rightleftarrows}} \quad \underline{FRM}$$

where the functors $\mathcal{O}$ and Σ are adjoint on the right, and the corresponding adjunctions are defined as follows:

$$\eta_L : L \to \mathcal{O}\Sigma L \quad \text{is the frame homomorphism defined by} \quad \eta_L(a) = \Sigma_a$$

and

$$\varepsilon_X : X \to \Sigma\mathcal{O}X \quad \text{is the continuous map defined by} \quad \varepsilon_X(x) = \tilde{x}$$

where for each $U \in \mathcal{O}X$

$$\tilde{x}(U) = \begin{cases} 1 & \text{if} \quad x \in U \\ 0 & \text{if} \quad x \notin U. \end{cases}$$

By the general theory of contravariant functors which are adjoint on the right, the functors $\mathcal{O}$ and Σ induce a dual equivalence between the full subcategory of $\underline{TOP}$ determined by the topological spaces X for which $\varepsilon_X : X \to \Sigma\mathcal{O}X$ is a homeomorphism (such an X is called **sober**) and the full subcategory of $\underline{\text{FRM}}$ determined by those frames for which $\eta_L : L \to \mathcal{O}\Sigma L$ is an isomorphism (such an L is called **spatial**).

The dual category $\underline{FRM}^{op}$ of $\underline{FRM}$ can be viewed as an extension of the category $\underline{TOP}$. This category is denoted by $\underline{LOC}$ and its objects are called **locales**.

2.5 Separation axioms in frames

Here we provide some frame notions corresponding to some notions in topological spaces. The following definitions in frames arise naturally from spaces in the sense that if a property of a topological space X can be formulated in $\mathcal{O}X$, it then becomes meaningful in $\underline{FRM}$.

Definition 2.5.1. For any $a, b \in L$, we say a is **rather below** b, written as $a \prec b$, if $a \wedge c = 0$ and $c \vee b = e$, for some $c \in L$.

Some properties of the rather below relation are presented in the lemma below. These properties are well known, therefore we omit the proofs.

Lemma 2.5.2. *Let L be a frame. Let $a, b, x, y \in L$. The following properties hold true:*

1. *If $a \prec b$, then $a \leq b$.*
2. *If $x \leq a \prec b \leq y$, then $x \prec y$.*
3. *If $x \prec b$ and $y \prec b$, then $x \vee y \prec b$.*
4. *If $x \prec a$ and $x \prec b$, then $x \prec a \wedge b$.*
5. *$a \prec b$ if and only if $a^* \vee b = e$, where a^* is the pseudocomplement of a.*
6. *If $a \prec b$, then $b^* \prec a^*$.*
7. *If $f : L \to M$ is a frame homomorphism and $a \prec b$ in L, then $f(a) \prec f(b)$*

in L.

Definition 2.5.3. A frame L is called **regular** if $a = \bigvee\{x \in L : x \prec a\}$, for all $a \in L$.

Regular frames form a full subcategory of the category of frames and frame homomorphisms and this category will be denoted by <u>$REGFRM$</u>.

Definition 2.5.4. A frame L is called **completely regular** if for each $a \in L$, $a = \bigvee\{x \in L : x \prec\prec a\}$ where $x \prec\prec a$ means that there exists a doubly indexed sequence of elements in L, $(x_{i,k})_{i=0,1,2,\ldots;\, k=0,1,2,\ldots,2^i}$ such that, for all i and k:

1. $x = x_{i,0}$ and $x_{i,2^i} = a$

2. $x_{i,k} \prec x_{i,k+1}$ and $x_{i,k} = x_{i+1,2k}$.

If $x \prec\prec a$, we say that x is **completely below** a.

Remark 2.5.5. If $a \prec\prec b$ in L, then $a \prec\prec c \prec\prec b$ for some $c \in L$. Trivially, $\prec\prec$ is preserved by frame homomorphisms since this is the case for the relation $\prec$.

The following notion in frames can be viewed as the analogue of a locally compact topological space.

Definition 2.5.6. A complete lattice L is called **continuous** if for $a \in L$, $a = \bigvee\limits_{x \ll a} x$, where $x \ll a$ means for any $S \subseteq L$ with $a \leq \bigvee S$, there exists a finite $F \subseteq S$ such that $x \leq \bigvee F$. If $x \ll a$ we say that x is **well below** a.

For a locally compact topological space X, it is true that for any $U, V \in \mathcal{O}X$, we have that $U \ll V$ if and only if $U \subseteq K \subseteq V$ for some compact subset K of X. Thus, if X is locally compact, then $\mathcal{O}X$ is continuous (see Section 5.1 in [22], chapter VII).

Remark 2.5.7. It is true that if $a \ll b$ in a continuous lattice L, then $a \ll c \ll b$, for some $c \in L$ (see subsection 5.1.1 in [22], chapter VII).

2.6 Compactifications

The concept of compactifications in spaces has an analogous concept in frames. Since we deal with the compactifications of frames extensively in all the chapters that follows, we present the definition below.

Definition 2.6.1. A **compactification** of a frame L is a compact regular frame M together with a dense onto frame homomorphism $h : M \to L$.

The set $Idl(L)$ of all lattice ideals of a frame L, partially ordered by set inclusion, is a compact frame (see Proposition 4.1.1 in [22], chapter VII). For any frame homomorphism $h : L \to M$, let $\mathcal{J}h : Idl(L) \to Idl(M)$ be the frame homomorphism assigning to each ideal $J \in Idl(L)$ the ideal generated by $h(J)$ in M (i.e., the smallest ideal containing $h(J)$ in M, which is $\downarrow h(J)$). Then $\mathcal{J} : \underline{FRM} \to \underline{FRM}$ is a functor. There is a natural transformation $\eta : \mathcal{J} \to \mathrm{id}_{\underline{FRM}}$ with components given by the join map $\bigvee : Idl(L) \to L$, taking each $J \in Idl(L)$ to its join in L.

The definition of the Stone-Čech compactification for frames arises naturally from the definition of this compactification for topological spaces as explained below.

Example 2.6.2. A T_0 topological space X is completely regular if and only if the frame $\mathcal{O}X$ of its open sets is completely regular. Also, X is compact if and only if $\mathcal{O}X$ is compact. Let L be a frame. The compact completely regular coreflection of L is given by the join map $\bigvee : \mathfrak{J}L \to L$ where $\mathfrak{J}L$ is the subframe of $Idl(L)$ consisting of **completely regular ideals** $J \subseteq L$, i.e., those ideals J such that, for each $a \in J$, there exist $b \in J$ with $a \prec\prec b$ (see Proposition 4.2.1 in [22], chapter VII). The map $\bigvee : \mathfrak{J}L \to L$ is onto if and only if L is completely regular. Therefore, for L completely regular, the compact completely regular universal coreflection $\bigvee : \mathfrak{J}L \to L$ is the frame counterpart of the Stone-Čech compactification βX of a Tychonoff space X. We shall therefore refer to this coreflection as βL and call it the **Stone-Čech compactification for** L.

2.7 Congruences and quotients

Congruences on frames are introduced here and they are used to define the notion of a remainder of a frame in a compactification later on.

Definition 2.7.1. Let L be a frame. A **subframe** of L is a subset of L which is closed under finite meets and arbitrary joins. A **congruence** on a frame L is an equivalence relation which is also a subframe of the frame $L \times L$.

Definition 2.7.2. The **kernel** of any frame homomorphism $h : L \to M$ is the set

$$\ker h = \{(x, y) \in L \times L : h(x) = h(y)\}.$$

Remark 2.7.3. $\ker h$ is an example of a congruence. In fact, any congruence is equal to the kernel of some frame homomorphism.

Definition 2.7.4. The **congruence lattice** of a frame L, is the set

$$\mathcal{C}L = \{\Theta : \Theta \quad \text{is a congruence on} \quad L\}.$$

Set inclusion is a partial order on $\mathcal{C}L$. In fact, $\mathcal{C}L$ is a frame (see Theorem 5.14 in [12]) with the top element $\nabla = L \times L$ and the bottom element $\Delta = \{(x, x) : x \in L\}$. Let I be any index set and $\Theta_i, i \in I$, be congruences on a frame L. Certainly $\bigcap_{i \in F \subseteq I, F \text{ finite}} \Theta_i \in \mathcal{C}L$. In $\mathcal{C}L$, we define joins as follows:

$$\bigvee_{i\in I}\Theta_i = \bigcap\{\Psi \in \mathcal{C}L : \Theta_i \subseteq \Psi, \quad \text{for all} \quad i \in L\}.$$

We then have the infinite distributive law $\Theta \cap \bigvee_{i\in I}\Theta_i = \bigvee_{i\in I}(\Theta \cap \Theta_i)$, for all $\Theta \in \mathcal{C}L$.

Example 2.7.5. Apart from the kernel of a frame homomorphism, we have other special examples of congruences. Let L be a frame and $a \in L$. The sets

$$\nabla_a = \{(x, y) \in L \times L : x \vee a = y \vee a\}$$

and

$$\Delta_a = \{(x, y) \in L \times L : x \wedge a = y \wedge a\}$$

are congruences on L. These type of congruences will play a role when defining the remainder of a non-compact regular continuous frame in any of its compactifications.

Definition 2.7.6. Let Θ be a congruence on a frame L. The **quotient frame** induced by Θ is the frame L/Θ consisting of all the equivalence classes of Θ. The **natural frame homomorphism** is the onto frame homomorphism $\mu : L \to L/\Theta$ defined by $\mu(x) = [x]$, where $[x]$ is the equivalence class containing x.

The following proposition is well known (the so called **Kernel Factorisation Lemma**) and the proof is provided since some construction in the next chapter heavily relies on this result.

Proposition 2.7.7. *Let $h : L \to M$ be a frame homomorphism such that $\Theta \subseteq \ker h$, where Θ is a congruence on L. Then there exists a unique frame homomorphism $\hat{h} : L/\Theta \to M$ such that $h = \hat{h}\mu$, where $\mu : L \to L/\Theta$ is the natural quotient frame homomorphism.*

Proof. Define $\hat{h} : L/\Theta \to M$ by $\hat{h}(\mu(x)) = h(x)$, for all $x \in L$. Then $\hat{h}$ is well-defined. For, if $\mu(x) = \mu(y)$, then $[x] = [y]$. This implies that $(x, y) \in \Theta \subseteq \ker h$. However, $(x, y) \in \ker h$ implies that $h(x) = h(y)$.

We now prove that $\hat{h}$ is a frame homomorphism. It is clear that $\hat{h}(\mu(0)) = h(0) = 0$ and $\hat{h}(\mu(e)) = h(e) = e$. Let $\mu(x), \mu(y) \in L/\Theta$. Then,

$$\hat{h}(\mu(x) \wedge \mu(y)) = \hat{h}(\mu(x \wedge y)) = h(x \wedge y) = h(x) \wedge h(y) = \hat{h}(\mu(x)) \wedge \hat{h}(\mu(y)).$$

Also,

$$\hat{h}(\bigvee_{i\in I} \mu(x_i)) = \hat{h}(\mu(\bigvee_{i\in I} x_i)) = h(\bigvee_{i\in I} x_i) = \bigvee_{i\in I} h(x_i) = \bigvee_{i\in I} \hat{h}(\mu(x_i)).$$

For uniqueness of $\hat{h}$, suppose there exists another frame homomorphism $f : L/\Theta \to M$ such that $h = f\mu$. Then, for all $x \in L$, we have $h(x) = f(\mu(x))$, that is, $\hat{h}(\mu(x)) = f(\mu(x))$, so $\hat{h} = f$. □

A direct consequence of the above result is given below (the so called **Image Factorisation Lemma**).

Corollary 2.7.8. *Let $h : L \to M$ be an onto frame homomorphism. Then $L/\ker h \cong M$.*

Chapter 3

Compactifications from Generators and Relations

The work in this chapter has been published by the author and his supervisors in [21]. The main purpose of this chapter is to show that any compactification of a frame L can be realized as a frame freely generated by L subject to certain relations. The concept and construction of a freely generated frame can be found in the book of Johnstone [14], where the constructed frame is given as a frame of C-ideals on a suitable meet-semilattice A. This frame of C-ideals is shown to be a sublocale of the free frame DA of all downsets of A. As examples, we show that the Stone-Čech compactification of a completely regular frame, the Freudenthal compactification of a rim-compact frame (due to Baboolal [3]), the least compactification of a non-compact regular continuous frame due to Banaschewski [6] (which is the frame counterpart of the Alexandroff one point compactification) and its generalisation due to Baboolal [2], namely the N-star compactifications, can be realized as frames that are freely generated. Not included in [21] is the proof of the fact that a compactification of rim-compact frames can be realized as a completion of a uniform frame on L. We end this chapter providing a detailed proof of this fact.

3.1 Freely generated frames and the relation R

We begin with the following definition, which may be obtained from Johnstone [14].

Definition 3.1.1. Let A be any meet-semilattice. By a **coverage** on A, we mean a function C assigning to each $a \in A$ a set $C(a)$ of subsets of $\downarrow a$. The elements of $C(a)$ are called the **coverings** of a, and they satisfy the **meet-stability property**:

$$\text{If } S \in C(a), \text{ then } \{s \wedge b : s \in S\} \in C(b), \text{ for all } b \leq a.$$

The pair (A, C) is called a **site**. A semilattice homomorphism $f : A \to B$ (A a meet-semilattice, and B a frame) is said to **transform covers to joins** if $f(a) = \bigvee_B \{f(s) : s \in S\} = \bigvee_{s \in S} f(s)$, for each $a \in A$ and each $S \in C(a)$. We say that a frame GA is **freely generated** by a site (A, C) if there exists a meet-semilattice homomorphism $f : A \to GA$ which transforms covers to joins and has the property that if $g : A \to L$ is another meet-semilattice homomorphism to a frame L transforming covers to joins, then there exists a unique frame homomorphism $h : GA \to L$ which makes the following diagram commute:

$$\begin{array}{ccc} A & \xrightarrow{f} & GA \\ & \searrow^{g} & \downarrow{\scriptstyle \exists! h} \\ & & L \end{array}$$

In our treatment of freely generated frames we do not require the coverings $S \in C(a)$ to satisfy the meet stability property. This is because the freely generated frame is constructed in terms of the congruence generated by the relations on the frame of downsets DA of A.

Definition 3.1.2. Let A be a meet-semilattice and DA be the frame of downsets of A. For each $a \in A$, let $C(a)$ be a collection of subsets of $\downarrow a$. We define a relation R on DA by $(\bigcup_{s \in S} \downarrow s, \downarrow a) \in R$, for each $a \in A$, and for each $S \in C(a)$.

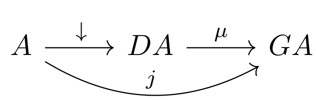
We point out that R is not a congruence on DA. However, we have the smallest congruence containing R on DA, namely

$$\Theta_R = \bigcap \{\Theta \in \mathcal{C}DA : R \subseteq \Theta\},$$

where $\mathcal{C}DA$ is the collection of all congruences on DA. We now have the following result which is of course well known, but we present the details here since this gives the description of generators and relations we use in the sequel.

Proposition 3.1.3. *Let A be a meet-semilattice. The frame $GA = DA/\Theta_R$ is freely generated subject to the relations R on DA.*

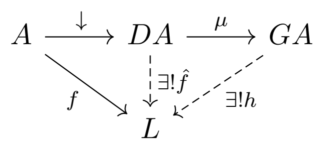
Proof. We note that the map $\downarrow : A \to DA$, defined by, $a \mapsto \downarrow a$, is a meet-semilattice homomorphism since $\downarrow e = A$, $\downarrow (a \wedge b) = \downarrow a \cap \downarrow b$, for all $a, b \in A$ and $\downarrow$ certainly preserves the order. Let $j = \mu \downarrow$, where $j : DA \to DA/\Theta_R$ is the natural frame homomorphism. Clearly j is a meet-semilattice homomorphism. We have the following situation:



We need to show that j transforms covers to joins and is universal among such maps. Let $a \in A$ and $S \in C(a)$. Then

$$\bigvee_{s\in S} j(s) = \bigvee_{s\in S} \mu{\downarrow}s = \mu(\bigvee_{s\in S}{\downarrow}s) = \mu(\bigcup_{s\in S}{\downarrow}s) = \mu({\downarrow}a) = j(a).$$

Now suppose $f : A \to L$ is another meet-semilattice homomorphism, where L is a frame such that $f(a) = \bigvee_{s\in S} f(s)$, for every $a \in A$ and every $S \in C(a)$. We prove that the outer triangle in the following diagram commutes:



Since DA is the free frame on the meet-semilattice A, there exists a unique frame homomorphism $\hat{f}$ such that $\hat{f}\downarrow = f$.

We now show that the outer triangle commutes. Since $f(a) = \bigvee_{s\in S} f(s)$, for every $a \in A$ and every $S \in C(a)$, then

$$\hat{f}({\downarrow}a) = f(a) = \bigvee_{s\in S} f(s) = \bigvee_{s\in S} \hat{f}({\downarrow}s) = \hat{f}(\bigcup_{s\in S}{\downarrow}s).$$

Therefore, $(\bigcup_{s\in S}\downarrow s, \downarrow a) \in \ker \hat{f}$. Thus $R \subseteq \ker \hat{f}$, and so $\Theta_R \subseteq \ker \hat{f}$. By Proposition 2.7.7, there exists a unique frame homomorphism $h : GA \to L$ such that $\hat{f} = h\mu$. Thus $hj = h\mu\downarrow = \hat{f}\downarrow = f$, i.e., the bigger triangle commutes. $\square$

3.2 Strong inclusions, generators and relations

We recall the concept of a strong inclusion on a frame in the following definition (see Banaschewski [6]).

Definition 3.2.1. A **strong inclusion** on a frame L is a binary relation $\lhd$ on L such that

1. if $x \leq a \lhd b \leq y$, then $x \lhd y$,

2. $\lhd$ is a sublattice of $L \times L$,

3. $a \lhd b$ implies $a \prec b$,

4. $a \lhd b$ implies $a \lhd c \lhd b$, for some $c \in L$ (interpolation property),

5. $a \lhd b$ implies $b^* \lhd a^*$ and

6. $a = \bigvee \{x \in L : x \lhd a\}$, for all $a \in L$.

In the proposition below, we define a relation which satisfies every strong inclusion property but the interpolation one and eventually get an extension of a regular frame from it.

Proposition 3.2.2. *Let B be a basis for a regular frame L. Suppose B satisfies the following conditions:*

(b1) $a, b \in B$ implies $a \wedge b \in B$ and $a \vee b \in B$, and

(b2) $a \in B$ implies $a^* \in B$.

Define a relation $\lhd$ on L by: $a \lhd b$ if and only if there exists $c \in B$ such that $a \prec c \prec b$. Then $\lhd$ satisfies the following properties:

1. *if $x \leq a \lhd b \leq y$, then $x \lhd y$,*

2. *$\lhd$ is a sublattice of $L \times L$,*

3. *$a \lhd b$ implies $a \prec b$,*

4. *$a \lhd b$ implies $b^* \lhd a^*$ and*

5. *$a = \bigvee \{x \in L : x \lhd a\}$, for all $a \in L$.*

Proof. 1. Suppose $x \leq a \lhd b \leq y$, then $a \prec c \prec b$, for some $c \in B$. We now have $x \leq a \prec c \prec b \leq y$, so $x \prec c \prec y$ i.e., $x \lhd y$.

2. Since $a \in B$ implies $a^* \in B$, then $a \wedge a^* \in B$ i.e., $0 \in B$. Therefore $0^* = e \in B$. Also note that $0 \wedge e = 0$ and $e \vee 0 = e$, this implies that $0 \prec 0$. Therefore $0 \prec 0 \prec 0$, i.e., $0 \lhd 0$. Similarly, $e \lhd e$. Moreover, $x \lhd a$ and $x \lhd b$ implies $x \prec c_1 \prec a$, and $x \prec c_2 \prec b$, for some $c_1, c_2 \in B$. Therefore, $x \wedge x \prec c_1 \wedge c_2 \prec a \wedge b$. That is, $x \prec c_1 \wedge c_2 \prec a \wedge b$, hence $x \lhd a \wedge b$. Also $x, y \lhd a$ implies $x \vee y \lhd a$. Thus $\lhd$ is a sublattice of $L \times L$.

3. The fact that $a \lhd b$ implies $a \prec b$ comes from the definition of $\lhd$.

4. Suppose $a \lhd b$, then $a \prec c \prec b$, for some $c \in B$. Therefore, $b^* \prec c^* \prec a^*$. Since $c^* \in B$, then $b^* \lhd a^*$.

5. Let $a \in L$. Since L is regular, $a = \bigvee\limits_{y \prec a} y$. However, B is a basis for L, so $y = \bigvee\limits_{b \in B' \subseteq B} b$, clearly $b \leq y$. Use the regularity of L on b to write $b = \bigvee\limits_{x \prec b} x$. We now have

$$a = \bigvee_{y \prec a} \bigvee_{b \in B' \subseteq B} \bigvee_{x \prec b} x.$$

   We have $x \prec b \leq y \prec a$, so $x \prec b \prec a$, i.e., $x \lhd a$. Hence $a = \bigvee\{x \in L : x \lhd a\}$.

$\square$

Remark 3.2.3. Let L be a regular frame. For $a \in L$, let $C(a) = \{S_a\}$, where $S_a = \{x \in L : x \lhd a\}$ and $\lhd$ is defined as in Proposition 3.2.2. We note that $x \lhd a$ implies $x \prec a$, and the latter implies $x \leq a$. It follows that $S_a \subseteq \downarrow a$, for each $a \in L$. We define a relation R on DL by firstly imposing $(\bigcup\limits_{s \in S_a} \downarrow s, \downarrow a) \in R$, for each $a \in L, a \neq e$. For $a = e$, we have $e = \bigvee\{x \in L : x \lhd e\}$. However $x \lhd e$ is true for all $x \in L$. Therefore

$S_e = \{x \in L : x \triangleleft e\} = L = \downarrow e$. We have

$$(\bigcup_{s \in S_e} \downarrow s, \downarrow e) = (\bigcup_{s \in \downarrow e} \downarrow s, \downarrow e) = (\downarrow e, \downarrow e) \in R.$$

Furthermore, if $x \triangleleft y$, then $x \prec y$, i.e., $x^* \vee y = e$. Define $S_{xy} = \{x^*, y\}$, whenever $x \triangleleft y$. So, we also want

$$(\bigcup_{s \in S_{xy}} \downarrow s, \downarrow e) = (\downarrow x^* \cup \downarrow y, \downarrow e) \in R.$$

As a consequence of this, we get the following result.

Proposition 3.2.4. *Let L be a regular frame, $C(a) = \{S_a\}$, where $S_a = \{x \in L : x \triangleleft a\}$, $(\bigcup_{s \in S_a} \downarrow s, \downarrow a) \in R$, for each $e \neq a \in L$, and*

$$(\downarrow x^* \cup \downarrow y, \downarrow e) \in R$$

whenever $x \triangleleft y$. Then $j : L \to GL$ transforms covers to joins and is universal among such maps.

Proof. We note that

$$j(a) = \mu(\downarrow a) = \mu(\bigcup_{s \in S_a} \downarrow s) = \bigvee_{s \in S_a} \mu \downarrow s = \bigvee_{s \in S_a} j(s).$$

Furthermore, if $x \triangleleft y$, then

$$\begin{aligned} j(e) &= \mu \downarrow e \\ &= \mu(\downarrow (x^* \vee y)) \qquad \text{(because } e = x^* \vee y\text{)} \\ &= \mu(\downarrow x^* \cup \downarrow y) \qquad \text{(since } (\downarrow x^* \cup \downarrow y, \downarrow e) \in R\text{)} \\ &= \mu(\bigcup_{s \in S_{xy}} \downarrow s) \\ &= \bigvee_{s \in S_{xy}} \mu(\downarrow s) \\ &= \bigvee_{s \in S_{xy}} j(s). \end{aligned}$$

By the proof of Proposition 3.1.3, j is universal amongst all the maps that transform covers to joins. That is, if $f : L \to M$ is a frame homomorphism which transforms covers to joins, then there is a unique map $h : GL \to M$, such that $hj = f$. $\square$

In the following proposition, we show that GL is always regular.

Proposition 3.2.5. *The frame $GL = DL/\Theta_R$ is regular for any regular frame L, having a basis B satisfying the conditions (b1) and (b2) provided in Proposition 3.2.2.*

Proof. We recall that we have the following situation:

$$L \xrightarrow{\ \downarrow\ } DL \xrightarrow{\ \mu\ } GL, \qquad j = \mu \circ \downarrow : L \to GL$$

Since μ is onto, for each $Y \in GL$, we have $Y = \mu(A)$ for some $A \in DL$. It follows that $\mu(A) = \mu(\bigcup_{a\in A} \downarrow a) = \bigvee_{a\in A} \mu(\downarrow a)$. For each $a \in A$, write $a = \bigvee\{x \in L : x \triangleleft a\}$ and consider the fact that $(\bigcup_{s\in S_a} \downarrow s, \downarrow a) \in R$, where $S_a = \{x \in L : x \triangleleft a\}$. But $\bigcup_{s\in S_a} \downarrow s = \bigcup_{x\triangleleft a} \downarrow x$, so $(\bigcup_{x\triangleleft a} \downarrow x, \downarrow a) \in R$. This implies

$$\mu(\downarrow a) = \mu(\bigcup_{x\triangleleft a} \downarrow x) = \bigvee_{x\triangleleft a} \mu(\downarrow x).$$

We now have

$$\mu(A) = \bigvee_{a\in A} \mu(\downarrow a) = \bigvee_{a\in A} \bigvee_{x\triangleleft a} \mu(\downarrow x).$$

We need to show that $\mu(\downarrow x) \prec \mu(A)$. If $x \triangleleft a$, then $(\downarrow x^* \cup \downarrow a, \downarrow e) \in R$. Therefore $\mu(\downarrow e) = \mu(\downarrow x^* \cup \downarrow a)$, that is, $\mu(\downarrow e) = \mu(\downarrow x^*) \vee \mu(\downarrow a)$. Consequently $\mu(\downarrow x) \prec \mu(\downarrow a)$ for each $x \triangleleft a$, showing regularity on GL. $\square$

We now apply Proposition 3.2.4 to the identity map and the fact that GL is regular to get an extension of a frame L.

Proposition 3.2.6. *Let L be a regular frame having a basis B satisfying the conditions (b1) and (b2) provided in Proposition 3.2.2. Then there exists a dense, onto frame homomorphism $\phi : GL \to L$.*

Proof. The identity map $id : L \to L$ transforms covers to joins. Indeed, note that $a = \bigvee\{x \in L : x \triangleleft a\}$, for all $a \in L$. Let $x \in S_a$. So, $x \triangleleft a$ and therefore $a = \bigvee_{x \in S_a} x$. We have

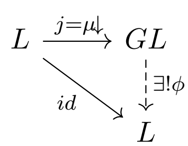
$$id(a) = id(\bigvee_{x\in S_a} x) = \bigvee_{x\in S_a} x = \bigvee_{x\in S_a} id(x).$$

Also, if $x \triangleleft y$, then $e = x^* \vee y$, hence

$$id(e) = id(x^* \vee y) = x^* \vee y = \bigvee_{s\in S_{xy}} s = \bigvee_{s\in S_{xy}} id(s).$$

Thus $id : L \to L$ transforms covers to joins.

By Proposition 3.2.4, there exists a unique frame homomorphism $\phi : GL \to L$, which makes the following diagram commute:



Since $\phi j = id$, ϕ must be onto.

Now, let $X \in GL$. Then $X = \mu(A)$, for some $A \in DL$. We have

$$X = \mu(A) = \mu(\bigcup_{a\in A} \downarrow a) = \bigvee_{a\in A} \mu(\downarrow a) = \bigvee_{a\in A} j(a),$$

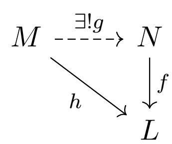
so that $\phi(X) = \phi(\bigvee_{a\in A} j(a)) = \bigvee_{a\in A} \phi(j(a)) = \bigvee_{a\in A} id(a) = \bigvee_{a\in A} a$.

If $\phi(X) = 0$, then $\bigvee_{a\in A} a = 0$, that is $a = 0$, for all $a \in A$. Hence $A = \downarrow 0$. Therefore $X = \mu(A) = \mu(\downarrow 0) = j(0) = O_{GL}$ and consequently ϕ is dense. □

So, if L is regular then GL is regular and $\phi : GL \to L$ is a dense and onto frame homomorphism. We can therefore view (GL, ϕ) as an extension of a regular frame L.

3.3 Compactifictions as freely generated frames

Let KL be the set of all (up to isomorphism) compactifications of L, partially ordered by $(M, h) \leq (N, f)$ if and only if there exists a frame homomorphism $g : M \to N$ making the following diagram commute:



We say that the compactifications (M, h) and (N, f) are **isomorphic** and write $(M, h) \cong (N, f)$ (or simply $M \cong N$, if the maps are understood) if $(M, h) \leq (N, f)$ and $(N, f) \leq (M, h)$. It is known that if $g : M \to N$ exists, then g is unique and one-to-one (and therefore dense) (see Banaschewski [5]).

Let SL be the set of all strong inclusions on L, which are relations on L, and so can be partially ordered by set inclusion. Banaschewski [6] shows that $KL \cong SL$, by defining maps $KL \to SL$ and $SL \to KL$ which are order preserving and inverses of each other. Without proving anything, we state how Banaschewski [6] defines these maps.

The map $KL \to SL$ is defined as follows: for any compactification (M, h) of L, let $r : L \to M$ be the right adjoint of h. Define a relation $\triangleleft_r$ on L by $x \triangleleft_r y$ if and only if $r(x) \prec r(y)$. Then $\triangleleft_r$ becomes a strong inclusion on L. So (M, h) is mapped to $\triangleleft_r$.

The map $SL \to KL$ is defined as follows: Let $\triangleleft$ be a strong inclusion on L. Call an ideal $J \subseteq L$ *strongly regular* with respect to with $\triangleleft$, if for each $x \in J$, there exists $y \in J$ such that $x \lhd y$. Let θL be the set of all strongly regular ideals of L with respect to $\lhd$. Then $\bigvee : \theta L \to L$ is dense and onto, its right adjoint $k : L \to \theta L$ is defined by $k(a) = \{x \in L : x \lhd a\}$. The set θL is a regular subframe of $Idl(L)$, the frame of ideals of L. Therefore $(\theta L, \bigvee)$ is a compactification of L. So $\lhd$ is mapped to $(\theta L, \bigvee)$. Due to this correspondence, we note that $(M, h) \cong (N, f)$ if and only if the corresponding strong inclusion coincides.

We show that any compactification $(\theta L, \bigvee)$ of a frame L can be realized as a freely generated frame subject to certain relations. To do this, we need a few more results. The following result is part of the folklore (see eg. Banaschewski [5]).

Lemma 3.3.1. *Let $h : M \to N$ be a frame homomorphism. Let M be regular, then h is one-to-one if and only if h is codense.*

Lemma 3.3.2. *Let $h : M \to N$ be a frame homomorphism, where M is regular and N is compact regular. If h is dense and onto, then h is one-to-one, and hence is an isomorphism.*

Proof. Since M is regular, by Lemma 3.3.1, it is enough to show that h is codense. Write $x = \bigvee_{a \prec x} a$ and suppose that $h(x) = e$. Realise that $e = h(x) = h(\bigvee_{a \prec x} a) = \bigvee_{a \prec x} h(a) = \bigvee_{i=1}^{n} h(a_i)$, where $a_i \prec x$, by compactness of N. Therefore $\bigvee_{i=1}^{n} h(a_i) = h(\bigvee_{i=1}^{n} a_i)$, where $\bigvee_{i=1}^{n} a_i \prec x$. Let $y = \bigvee_{i=1}^{n} a_i$. Hence $e = h(y)$ with $y \prec x$. It follows that $0 = e^* = (h(y))^* = h(y^*)$, since h is dense and onto. Again, by the density of h, we have $y^* = 0$. Since $y \prec x$, then $y^* \vee x = e$. Thus $0 \vee x = e$, and consequently $x = e$. So h is codense. □

We are now ready to prove the major result.

Proposition 3.3.3. *Let $(\theta L, \bigvee)$ be a compactification of L. Then $\theta L \cong GL$, where GL is a freely generated frame subject to certain relations.*

Proof. Let $\triangleleft^L$ be a strong inclusion associated with the compactification $(\theta L, \bigvee)$ and $k(a) = \{x \in L : x \triangleleft^L a\}$ be the right adjoint of $\bigvee$. We note that $x \triangleleft^L a$ implies $x \prec a$, and this implies $x \leq a$. Thus $C(a) = \{S_a : a \in L\}$ is a collection of subsets of $\downarrow a$, where $S_a = \{x : x \triangleleft^L a\}$. One has a relation R on DL by imposing $(\bigcup_{s \in S_a} \downarrow s, \downarrow a) \in R$, for each $a \in L, a \neq e$. For $a = e$, we define $C(e) = \{S_{xy} : x \triangleleft^L y\}$ where $S_{xy} = \{x^*, y\}$. We also declare that $(\downarrow x^* \cup \downarrow y, \downarrow e) \in R$. Let Θ_R be the smallest congruence on DL containing R and $GL = DL/\Theta_R$

It is clear that $k(e) = L$. The map k preserves the order, for if $a \leq b$, and $x \in k(a)$, then $x \leq x \triangleleft^L a \leq b$ implies $x \triangleleft^L b$. That is $k(a) \subseteq k(b)$. Also, $x \in k(a \wedge b)$ if and only if $x \triangleleft^L a \wedge b$, and this is the case if and only if $x \triangleleft^L a$ and $x \triangleleft^L b$, the latter holds if and only if $x \in k(a) \cap k(b)$, i.e., $k(a \wedge b) = k(a) \cap k(b)$. Therefore k preserves meets, so k is a meet-semilattice homomorphism.

We now show that k transforms covers to joins. For $a \neq e$, we show that $k(a) = \bigvee_{x \in S_a} k(x)$, that is, to show that $k(a) = \bigvee_{x \triangleleft^L a} k(x)$. Let $y \in k(a)$. This implies $y \triangleleft^L a$, therefore $y \triangleleft^L x \triangleleft^L a$, for some $x \in L$. So, $y \triangleleft^L x$ for some $x \triangleleft^L a$, that is, $y \in k(x)$ for some $x \triangleleft^L a$. Thus $y \in \bigvee_{x \triangleleft^L a} k(x)$. On the other hand, if $y \in \bigvee_{x \triangleleft^L a} k(x)$, then $y \in k(x)$ for some $x \triangleleft^L a$, using the general fact about ideals that $\bigcup_{i \in I} J_i = \bigcup\{\bigcup_{i \in F} J_i : F \subseteq I,\ F \text{ finite}\}$ and $x_i \triangleleft^L a$ for $i = 1, 2, \ldots, n$ implies $x_1 \vee x_2 \cdots \vee x_n \triangleleft^L a$. Therefore $y \triangleleft^L x \triangleleft^L a$, implies that $y \triangleleft^L a$, that is, $y \in k(a)$.

Moreover, suppose $x \triangleleft^L y$, and consider $S_{xy} = \{x^*, y\}$. We show that $k(e) = \bigvee_{s \in S_{xy}} k(s)$, that is, we show that $k(e) = k(x^*) \vee k(y)$. Since $\triangleleft^L$ interpolates, we can find $a, b \in L$ such that $x \triangleleft^L a \triangleleft^L b \triangleleft^L y$, hence $y^* \triangleleft^L$

$b^* \triangleleft^L a^* \triangleleft^L x^*$. But $a \triangleleft^L b$ implies $a \prec b$, therefore $a^* \vee b = e$. Since $a^* \vee b \in k(x^*) \vee k(y)$, then $e \in k(x^*) \vee k(y)$. That is, $k(x^*) \vee k(y) = k(e)$. Since $j : L \to GL$ is universal amongst the maps which transforms covers to joins, it follows by Proposition 3.1.3 that there exists a unique frame homomorphism which makes the following diagram commute:

$$\begin{array}{ccc} L & \xrightarrow{j} & GL \\ & \searrow^{k} & \downarrow \exists!\phi \\ & & \theta L \end{array}$$

The frame homomorphism $\phi : GL \to \theta L$ is onto. To see this, let $J \in \theta L$. We first show that $J = \bigcup_{a \in J} k(a)$. If $x \in J$, then $x \triangleleft^L a$, for some $a \in J$, by strong regularity of J. Thus $x \in k(a)$ for some $a \in J$, that is, $x \in \bigcup_{a \in J} k(a)$. On the other hand, if $x \in \bigcup_{a \in J} k(a)$, then $x \in k(a)$, for some $a \in J$. That is, $x \triangleleft^L a$, which implies that $x \prec a$, and so $x \leq a$, for some $a \in J$. Since J is an ideal, then $x \in J$. We need to find $Y \in GL$ such that $\phi(Y) = J$. Since J is an ideal, then J is a downset. Therefore $J = \bigcup_{a \in J} \downarrow a$, and so $\mu(J) = \mu(\bigcup_{a \in J} \downarrow a) = \bigvee_{a \in J} \mu(\downarrow a) = \bigvee_{a \in J} j(a)$, where $\mu : DL \to DL/\Theta_R = DL$ is the natural map. Let $Y = \bigvee_{a \in J} j(a)$. Then $\phi(Y) = \phi(\bigvee_{a \in J} j(a)) = \bigvee_{a \in J} \phi(j(a)) = \bigcup_{a \in J} k(a) = J$.

We now show that $\phi : GL \to \theta L$ is dense. If $X \in GL$, then $X = \mu(A) = \bigvee_{a \in A} j(a)$. Hence $\phi(X) = \bigvee_{a \in A} \phi(j(a)) = \bigvee_{a \in A} k(a)$. If $\phi(X) = \{0\}$, then $\bigvee_{a \in A} k(a) = \{0\}$, that is $k(a) = \{0\}$, for all $a \in A$. Therefore $a = \bigvee\{x \in L : x \triangleleft^L a\} = \bigvee k(a) = \bigvee\{0\} = 0$, for all $a \in A$. So $X = \mu(\{0\}) = \mu(\downarrow 0)$. Therefore ϕ is dense.

To show that $\phi : GL \to \theta L$ is one-to-one, we note that GL is regular by Proposition 3.2.5. Also, $(\theta L, \bigvee)$ is a compactification, so θL is compact and regular. Since ϕ is dense and onto, then by Lemma 3.3.2, $\phi : GL \to \theta L$ is one-to-one. So that $\theta L \cong GL$, as desired. $\square$

As an application of the above result, we give some well-known examples of compactifications which are isomorphic to frames that are freely generated.

Definition 3.3.4. A **rim-compact** frame L is a regular frame such that each $a \in L$ is a join of elements u, where $\uparrow(u \vee u^*)$ is compact.

Remark 3.3.5. The above definition arises naturally from the topological spaces counterpart. The details shall be discussed in the next section. We defined the concept above for the sake of providing the following example.

Example 3.3.6. Let L be rim-compact and B be a basis satisfying:

$(b1)$ $a, b \in B$ implies $a \wedge b \in B$ and $a \vee b \in B$, and

$(b2)$ $a \in B$ implies $a^* \in B$.

$(b3)$ $\uparrow(a \vee a^*)$ is compact for all $a \in B$

Baboolal [3] shows that, for a rim-compact frame L, the relation in Proposition 3.2.2 defined by: $a \lhd b$ if and only if there exists $c \in B$ such that $a \prec c \prec b$, is a strong inclusion. Let $(\gamma_B L, \bigvee)$ be the compactification of L corresponding to $\lhd$. So, $\gamma_B L$ is the set of all ideals which are strongly regular with respect to $\lhd$.

Take $B_L = \{u \in L : \uparrow(u \vee u^*) \text{ is compact}\}$. Baboolal [3] shows B_L is a basis such that $a \vee b, a \wedge b \in B_L$, and $a^* \in B_L$, for every $a, b \in B_L$. The compactification $(\gamma L, \bigvee)$ corresponding to the strong inclusion $\lhd_L$ defined by: $a \lhd_L b$ if and only if there exists $c \in B_L$ such that $a \prec c \prec b$ is called the *Freudenthal compactification* (this compactification will be studied in detail in Chapter 5) of L and B_L is called the *Freudenthal base*. Using the $\lhd_L$ instead of $\lhd^L$, the proof of Proposition 3.3.3 shows that GL isomorphic to the Freudenthal compactification.

Example 3.3.7. Let L be a completely regular frame. Banaschewski [6] showed that $\prec\!\prec$ is a strong inclusion on L which corresponds to the Stone-Čech compactification βL determined by the coreflection $\bigvee : \mathfrak{J}L \to L$ provided in Example 2.6.2. Hence, replacing the strong inclusion $\triangleleft^L$ by the strong inclusion $\prec\!\prec$ and following the proof of Proposition 3.3.3, we get that GL is isomorphic to βL.

Example 3.3.8. Banaschewski [6] proved that if a frame is continuous and regular, then there is a smallest strong inclusion $\blacktriangleleft$ on L defined by: $a \blacktriangleleft b$ if and only if $a \prec b$ and either $\uparrow a^*$ or $\uparrow b$ is compact. The compactification corresponding to $\blacktriangleleft$ is the least compactification ξL of L and this is the analogue of the Alexandroff one-point compactification for a locally compact non-compact Hausdorff space (more on this in Chapter 6).

Baboolal [2] introduced the concept of an $N-$*star compactification* which generalises the concept of the Alexandroff one-point compactification to a compactification with N points, where $N > 1$. For any positive integer N, an $N-$*star* of any frame L is defined to be a collection of N mutually disjoint elements of L, say $\{u_1, u_2, \ldots, u_N\}$, such that $\uparrow(u_1 \vee u_2 \vee \cdots \vee u_N)$ is compact, while for each i, $1 \leq i \leq N$, $\uparrow(u_1 \vee u_2 \vee \cdots \vee u_{i-1} \vee u_{i+1} \cdots \vee u_N)$ is not compact. In the case where $N = 1$, the latter is interpreted to mean L is not compact.

Let L be a regular continuous frame and let $\{u_1, u_2, \ldots, u_N\}$ be an $N-$star of L. Let $N_i = \{x \in L : \uparrow(u_1 \vee u_2 \vee \cdots \vee u_{i-1} \vee x \vee u_{i+1} \cdots \vee u_N)\text{ is compact}\}$. Define a relation $\blacktriangleleft_N$ by: $a \blacktriangleleft_N b$ if and only if $a \prec b$ and for each i, either $a^* \in N_i$ or $b \in N_i$. Baboolal [2] proved that the relation $\blacktriangleleft_N$ is a strong inclusion. Since any $N-$star, α_N, of L can be associated with a strong inclusion $\blacktriangleleft_N$, we can then associate α_N with a compactification

$\bigvee : \alpha_N L \to L$ arising from $\blacktriangleleft_N$. Call the compactification $\bigvee : \alpha_N L \to L$ an $N-$*star compactification* of L, and any compactification isomorphic with this one shall be called such. Thus if we replace $\triangleleft^L$ by $\blacktriangleleft_N$ in the proof of Proposition 3.3.3, then θL will be precisely $\alpha_N L$, the $N-$star compactification, so $GL \cong \alpha_N L$. In the special case where $N = 1$, $\blacktriangleleft = \blacktriangleleft_N$ and therefore $GL \cong \xi L$.

3.4 Compactifications of rim-compact frames as completions of uniform frames

In this section, we show that any compactification of a rim-compact frame L having a π-compact base B can be realized as the completion of some suitable uniform frame. The way this is done is as follows; we use B to define a strong inclusion $\triangleleft_B$ on L. This strong inclusion gives rise to a unique compactification $\gamma_B L$. A compatible uniform structure $\mathfrak{U}$ on L will be defined using the strong inclusion $\triangleleft_B$. The completion $(\tilde{L}, \tilde{\mathfrak{U}})$ of the uniform frame $(L, \mathfrak{U})$ is then shown to be isomorphic with the compactification $\gamma_B L$ of L.

3.4.1 Rim-compact frames and π-compact basis

In Definition 3.4.3, a rim-compact frame was defined. This concept naturally arises from topological spaces. We start by providing the following definition which can be obtained from the paper of Skljarenko [23].

Definition 3.4.1. A Hausdorff topological space X is called **rim-compact** (or **peripherally (bi)compact**) if there exists (in this space) a basis consisting of open sets U with compact frontiers, i.e.,

$$Fr_X(U) = cl_X(U) \setminus U = cl_X(U) \cap cl_X(X \setminus U)$$

is compact for all open sets U in the base.

The motivation of the definition of a rim-compact frame given in Definition 3.4.3 follows from the remark below.

Remark 3.4.2. Let X be a topological space. For any $V \in \mathcal{O}X$, the frame map $\uparrow V \to \mathcal{O}(X \setminus V)$ defined by $U \mapsto U \cap (X \setminus V)$, for all $V \subseteq U$, is an isomorphism. Therefore $\uparrow V \cong \mathcal{O}(X \setminus V)$ as frames.

Let $U \in \mathcal{O}X$ and U^* be the pseudocomplement of U in $\mathcal{O}X$. Therefore U^* is the largest open set in $\mathcal{O}X$ whose intersection with U is empty, that is $U^* = X \setminus cl_X(U)$. Therefore $\uparrow(U \cup U^*)$ is compact if and only if $\mathcal{O}(X \setminus (U \cup U^*))$. However, $X \setminus (U \cup U^*) = (X \setminus U) \cap (X \setminus U^*) = (X \setminus U) \cap cl_X(U) = Fr_X(U)$. Thus, $\uparrow(U \cup U^*)$ is compact if and only if $\mathcal{O}(Fr_X(U))$ is compact. This implies that the topological space X is rim-compact if and only if the frame $\mathcal{O}X$ is also rim-compact.

The dual equivalence induced by the functors Σ and $\mathcal{O}$ on the category of spatial frames and the category of sober topological spaces (i.e., $C = cl_X(\{x\})$ for a unique $x \in X$, whenever C is an irreducible closed subset of X) helps us to deduce the following: since every rim-compact space X is sober and $\mathcal{O}X$ is rim-compact for such topological spaces, the functor $\mathcal{O}$ embeds the category of rim-compact spaces into the category of rim-compact frames. We can therefore realize the category of rim-compact frames as the generalisation of the category of rim-compact spaces. For convenience we state the definition of a rim-compact frame again below.

Definition 3.4.3. A **rim-compact** frame L is a regular frame such that each $a \in L$ is a join of elements u, where $\uparrow(u \vee u^*)$ is compact.

Let $\mathfrak{B}$ be a basis for a topological space X such that each member of $\mathfrak{B}$ has a compact frontier. Construct a basis $\mathfrak{B}_X$ by taking the totality of those open sets that may be obtained from the members of $\mathfrak{B}$ by means of a finite number of the following operations:

- finite intersections and unions
- complementation of closures.

This basis $\mathfrak{B}_X$ also consists of sets with compact frontiers. Moreover, if $U_1, U_2, \ldots, U_n \in \mathfrak{B}_X$ then $X \setminus cl_X(U_i)$, $\bigcup_{i=1}^{n} U_i$, $\bigcap_{i=1}^{n} U_i, U_i \setminus cl_X(U_j) \in \mathfrak{B}_X$. To formalise this idea, we have the following definition (see Skljarenko [23] or Dickman and McCoy [10]) .

Definition 3.4.4. A basis $\mathfrak{B}_X$ of open sets for a topological space X is called a π**-compact** basis if the following conditions are satisfied:

1. if $A, B \in \mathfrak{B}_X$ then $A \cap B, A \cup B \in \mathfrak{B}_X$;
2. if $A \in \mathfrak{B}_X$, then $X \setminus cl_X(A) \in \mathfrak{B}_X$;
3. if $A \in \mathfrak{B}_X$, then $Fr_X(A)$ is compact.

The frame counterpart of the definition of a π-compact basis is therefore given as follows.

Definition 3.4.5. Let L be a rim-compact frame. A π**-compact** base B for L is a basis for L such that

1. $a, b \in B$ implies $a \wedge b$, $a \vee b \in B$,
2. $a \in B$ implies $a^* \in B$,
3. $a \in B$ implies $\uparrow(a \vee a^*)$ is compact.

3.4.2 Compactification of a rim-compact frame as a completion of a suitable uniform frame

Any rim-compact frame has a uniform structure arising from a π-compact basis. To prove this we need the following lemma, its proof can be found in Baboolal [3]. However, since the techniques employed in the proof of this lemma are used throughout (and for the sake of completeness) we provide the proof below.

Lemma 3.4.6. *Let L be a rim-compact frame and B be a π-compact basis for L. If $w \in L$ and $u \in B$ with $w \vee u = e$, then there exists $v \in B$ such that $v \prec u$ and $w \vee v = e$.*

Proof. Take $w \in L$ and $u \in B$ with $w \vee u = e$. Since L regular and B is a basis for L, then $w = \bigvee\{x : x \prec w, x \in B\}$. We then have $u \vee \bigvee\{x : x \prec w, x \in B\} = e$ and hence $u \vee u^* \vee \bigvee\{x : x \prec w, x \in B\} = e$. By the compactness of $\uparrow(u \vee u^*)$, it follows that $u \vee u^* \vee \bigvee_{i=1}^{n}\{x_i : x_i \prec w, x_i \in B\} = e$. Let $z = \bigvee_{i=1}^{n}\{x_i : x_i \prec w, x_i \in B\}$. Therefore $z \in B, z \prec w$ and $u \vee u^* \vee z = e$. Put $v = u \wedge z^*$. We note that $v \in B$ and $w \vee v = w \vee (u \wedge z^*) = (w \vee u) \wedge (w \vee z^*) = e \wedge e$. Furthermore, $v \prec u$. Indeed, $v \wedge (u^* \vee z) = (v \wedge u^*) \vee (v \wedge z) = (u \wedge z^* \wedge u^*) \wedge (v \wedge z \wedge z) = 0 \wedge 0 = 0$ and $u \vee (u^* \vee z) = e$. □

In defining a uniformity on L, we use the following remark for selecting the binary covers. We also use this remark for other purposes at a later stage of this thesis.

Remark 3.4.7. Let L be rim-compact and B be a π-compact basis for L. We recall that we have a strong inclusion $\triangleleft_B$ defined by: $a \triangleleft_B b$ if and only if there exists $c \in B$ such that $a \prec c \prec b$. This implies that $a^* \vee c = e$. By the above lemma, there exists $v \in B$, such that $v \prec c$ and $a^* \vee v = e$. Therefore $a \prec v \prec c \prec b$. In a similar way, we can get $w \in B$ such that $a \prec v \prec w \prec c \prec b$. This means that $a \triangleleft_B w \triangleleft_B b$, where $w \in B$. So, the strong inclusion $\triangleleft_B$ does not only interpolate, but the element witnessing the interpolation can be chosen to belong to the basis B.

Frith [12] introduced the method of binary covers to define a uniformity on a completely regular frame. Also, he showed that any totally bounded uniformity gives rise to a strong inclusion and conversely. The proposition below (where we show that any rim-compact frame possesses a uniformity arising from a π-compact base) is a special case of his work.

Proposition 3.4.8. *Let L be a rim-compact frame with a π-compact basis B. Then L has a compatible uniform structure.*

Proof. Let $a \triangleleft_B b$ where $a, b \in B$. Let $C_a^b = \{a^*, b\}$. Clearly, C_a^b is a cover of L. Since $a \triangleleft_B b$, we can find $c, d \in B$ such that $a \triangleleft_B c \triangleleft_B d \triangleleft_B b$. We show that $C = C_a^c \wedge C_c^d \wedge C_d^b \leq^* C_a^b$. We note $a \triangleleft_B c \triangleleft_B d \triangleleft_B b$ implies $a \leq c \leq d \leq b$ and hence $b^* \leq d^* \leq c^* \leq a^*$. We use this in the following calculations to enumerate the elements of C:

$$\begin{aligned} C &= C_a^c \wedge C_c^d \wedge C_d^b \\ &= \{a^*, c\} \wedge \{c^*, d\} \wedge \{d^*, b\} \\ &= \{a^* \wedge c^* \wedge d^*, a^* \wedge c^* \wedge b, a^* \wedge d \wedge b, c \wedge d \wedge b, 0\} \\ &= \{d^*, c^* \wedge b, a^* \wedge d, c, 0\} \end{aligned}$$

To show that $C \leq^* C_a^b$, we need to show that $C^* = \{Cx : x \in C\} \leq C_a^b$. We list the elements of C^* by doing some calculations.

Since $d^* \wedge (c^* \wedge b) = d^* \wedge b \neq 0, d^* \wedge (a^* \wedge d) = 0$ and $d^* \wedge c = 0$, then $Cd^* = (c^* \wedge b) \vee d^*$.

We have seen that $d^* \wedge (c^* \wedge b) \neq 0$. Also, $(c^* \wedge b) \wedge (a^* \wedge d) = c^* \wedge d \neq 0$ and $(c^* \wedge b) \wedge c = 0$. Therefore $C(c^* \wedge b) = (c^* \wedge b) \vee [d^* \vee (a^* \wedge d)]$.

Clearly, $(a^* \wedge d) \wedge d^* = 0$, and we know that $(a^* \wedge d) \wedge (c^* \wedge b) \neq 0$. Also $(a^* \wedge d) \wedge c = a^* \wedge c \neq 0$, and so $C(a^* \wedge d) = (a^* \wedge d) \vee (c^* \wedge b) \vee c$.

We have seen that $d^* \wedge c = 0$ and $(a^* \wedge d) \wedge c \neq 0$, and it is clear that $(c^* \wedge b) \wedge c = 0$. Thus $Cc = c \vee (a^* \wedge d)$. So, we have that:

• $Cd^* = (c^* \wedge b) \vee d^*$

• $C(c^* \wedge b) = (c^* \wedge b) \vee [d^* \vee (a^* \wedge d)]$

• $C(a^* \wedge d) = (a^* \wedge d) \vee (c^* \wedge b) \vee c$

• $Cc = c \vee (a^* \wedge d)$

It is now transparent that for each $u \in C^* = \{Cd^*, C(c^* \wedge b), C(a^* \wedge d), Cc\}$, either $u \leq a^*$ or $u \leq b$. For example, since $c^* \leq a^*$, then $c^* \wedge b \leq a^*$. Combining this with the fact that $d^* \leq a^*$ we get $Cd^* = (c^* \wedge b) \vee d^* \leq a^*$. Using similar arguments, we can also show that $C(c^* \wedge b) \leq a^*, C(a^* \wedge d) \leq b$ and $Cc \leq b$. Therefore $C \leq^* C_a^b$. Define $\mathfrak{U}_B = \{C_a^b : a \lhd_B b, a, b \in B\}$ and let $\mathfrak{U}$ be the family of all covers having $\mathfrak{U}_B$ as a subbase. We only need to check that $\mathfrak{U}$ is a compatible uniform structure on L.

Since B is a base and $\lhd_B$ is a strong inclusion on L, then $a = \bigvee\{x \in B : x \lhd_B a\}$ for each $a \in L$. We note that $x \lhd_B a$ implies $(C_x^a)x = \bigvee\{y \in C_x^a : y \wedge x \neq 0\} = a$. We now have $a = \bigvee\{x : (C_x^a)x = a\}$, hence $a = \bigvee\{x : (C_x^a)x \leq a\}$. □

Remark 3.4.9. For a rim-compact frame L with a π-compact basis B, we have found a uniform frame $(L, \mathfrak{U})$. Let $\tilde{L}$ be the frame freely generated by

L subject to the following relations:

1. $j(0) = 0$,

2. $j(a) = \bigvee_{x \lhd_B a} j(x)$, for each $a \in L$, and

3. $\bigvee_{x \in U} j(x) = e$, for each $U \in \mathfrak{U}$,

That is, we have the following diagram:

$$L \xrightarrow{\downarrow} DL \xrightarrow{\mu} \tilde{L} = DL/\Theta_R. \qquad (j = \mu \circ \downarrow : L \to \tilde{L})$$

Here R is as in Remark 3.2.3. We note the following:

1. $j(0) = 0 \Rightarrow \mu \downarrow 0 = 0 \Rightarrow \mu(\emptyset) = 0 \Rightarrow (\emptyset, \downarrow 0) \in R$.

2. $j(a) = \bigvee_{x \lhd_B a} j(x) \Rightarrow \mu \downarrow a = \bigvee_{x \lhd_B a} \mu \downarrow x \Rightarrow \mu \downarrow a = \mu(\bigcup_{x \lhd_B a} \downarrow x) \Rightarrow (\bigcup_{x \lhd_B a} \downarrow x, \downarrow a) \in R$, for each $a \in L$.

3. $\bigvee_{x \in U} j(x) = e \Rightarrow \bigvee_{x \in U} \mu \downarrow x = e \Rightarrow \mu(\bigcup_{x \in U} \downarrow x) = \mu(L) \Rightarrow (\bigcup_{x \in U} \downarrow x, L) \in R$, for each $U \in \mathfrak{U}$.

Furthermore, the identity map $id : L \to L$ is a meet-semilattice map (being a frame map) that satisfies the relations specified from 1 to 3 above and transform covers to joins. That $j : L \to \tilde{L}$ is a meet-semilattice homomorphism that is universal among the maps that transforms covers to joins is the content of Propositions 3.2.4 and 3.1.3. Therefore there exists a unique frame homomorphism $\rho : \tilde{L} \to L$ such that $\rho j = id$.

We now show that the frame $\tilde{L}$ possesses a uniform structure on it. This uniform structure is inherited from the one we defined on L in Proposition 3.4.8.

Proposition 3.4.10. *Let $(L, \mathfrak{U})$ be the uniform frame obtained in the proof in Proposition 3.4.8 and $\mathfrak{B}_j = \{j(U) : U \in \mathfrak{U}\}$, where $j(U) = \{j(u) : u \in U\}$. Then $\mathfrak{B}_j$ is a basis for a uniformity on $\tilde{L}$. That is, $(\tilde{L}, \tilde{\mathfrak{U}})$ is a uniform frame, where $\tilde{\mathfrak{U}}$ is the set of all covers of $\tilde{L}$ refined by the members of $\mathfrak{B}_j$.*

Proof. First note that $\bigvee_{x \in U} j(x) = e$, i.e., $j(U)$ is a cover for each $U \in \mathfrak{U}$. We make use of Lemma 2.3.16 to show that $\tilde{\mathfrak{U}}$ is a uniformity on $\tilde{L}$.

1. For each $U, V \in \mathfrak{U}$, we have

$$\begin{aligned} j(U \wedge V) &= \{j(u \wedge v) : u \in U,\, v \in V\} \\ &= \{j(u) \wedge j(v) : u \in U,\, v \in V\} \quad (j \text{ preserves finite meets}) \\ &= j(U) \wedge j(V). \end{aligned}$$

   Since $U \wedge V \in \mathfrak{U}$, then $j(U \wedge V) \in \mathfrak{B}_j$.

2. Let $U \in \mathfrak{U}$. We can find $V \in \mathfrak{U}$ such that $V \leq^* U$. We show that $j(V) \leq^* j(U)$. To see this, take $x \in V$. Then $Vx \leq u$, for some $u \in U$. We claim that $j(V)j(x) = \bigvee\{j(v) \in j(V) : j(v) \wedge j(x) \neq 0\} \leq j(u)$. We note that $j(v) \wedge j(x) \neq 0 \Rightarrow j(v \wedge x) \neq 0 \Rightarrow v \wedge x \neq 0 \Rightarrow v \leq Vx \leq u \Rightarrow j(v) \leq j(u) \Rightarrow j(V)j(x) \leq j(u)$. Hence $j(V) \leq^* j(U)$.

3. To show the compatibility of $\tilde{\mathfrak{U}}$, note from the remark provided above that $j(a) = \bigvee_{x \triangleleft_B a} j(x)$, for each $a \in L$. Now, from the proof of Proposition 3.4.8, $x \triangleleft_B a \Rightarrow a = \bigvee\{x : (C_x^a)x \leq a\}$. However, since $C_x^a \in \mathfrak{U}$, then $(C_x^a)x \leq a \Rightarrow j(C_x^a)j(x) \leq j(a)$. That is,

$j(a) = \bigvee\{j(x) : j(C_x^a)j(x) \leq j(a)\}$. If $z \in \tilde{L} = DL/\Theta_R$, then

$$\begin{aligned} z &= \bigvee\{j(a) : a \in K \subseteq L\}, \text{ (because } j(a)\text{'s generate } \tilde{L}) \\ &= \bigvee_{a \in K} \quad \bigvee_{(C_x^a)j(x) \leq j(a)} j(x) \\ &= \bigvee_{(C_x^a)j(x) \leq z} j(x) \\ &= \bigvee\{j(x) : (C_x^a)j(x) \leq z\}. \end{aligned}$$

Therefore $\tilde{\mathfrak{U}} = \{T \in Cov(\tilde{L}) : j(U) \leq T \text{ for some } U \in \mathfrak{U}\}$ is a compatible uniformity on $\tilde{L}$. □

Remark 3.4.11. We recall that a metric space (X, ρ) is said to be *complete* if every Cauchy sequence converges. More generally, a uniform space $(X, \mathfrak{U})$ is said to be *complete* if every Cauchy filter converges. These definitions are point based and hence cannot be utilised in frames. A correct notion of completion in frames (or locales), which is accepted as a natural one was introduced by Křìž in [17]. This definition arises quite naturally if one realizes that a uniform space $(X, \mathfrak{U})$ is complete if and only if whenever X is densely embedded into a uniform space $(Y, \mathfrak{V})$, then X is isomorphic to Y.

Definition 3.4.12. Let $(L, \mathfrak{U})$ and $(M, \mathfrak{V})$ be uniform frames. A uniform frame map $h : (M, \mathfrak{V}) \to (L, \mathfrak{U})$ is said to be a **surjection** if for every $U \in \mathfrak{U}$ there exists $V \in \mathfrak{V}$ such that $h(V) \leq U$. A uniform frame $(L, \mathfrak{U})$ is said to be **complete** if whenever a uniform map $\rho : (M, \mathfrak{V}) \to (L, \mathfrak{U})$ is a dense surjection, then $\rho : M \to L$ is an isomorphism. A complete uniform frame $(M, \mathfrak{V})$ together with a dense surjection $\rho : (M, \mathfrak{V}) \to (L, \mathfrak{U})$ is called a **completion** of $(L, \mathfrak{U})$.

Below, we state the result that the uniform frame $(\tilde{L}, \tilde{\mathfrak{U}})$ provided in the proof of Proposition 3.4.10 is complete, and it is a completion of the uniform frame $(L,\mathfrak{U})$ provided in Proposition 3.4.8. Křìž [17] proved this in a more general setting, where L need not be rim-compact, so we omit the proof.

Proposition 3.4.13. *[17, Section 3] Let $(L,\mathfrak{U})$ be the uniform frame obtained in the proof of Proposition 3.4.8. The uniform frame $(\tilde{L},\tilde{\mathfrak{U}})$ obtained in Proposition 3.4.10 is complete and it is a completion of $(L,\mathfrak{U})$.*

The main aim is to show that the completion $(\tilde{L},\tilde{\mathfrak{U}})$ is isomorphic to the compactification $\gamma_B L$ that corresponds to the strong inclusion $\lhd_B$.

Proposition 3.4.14. *Let L be a rim-compact frame with a π-compact base B. Let $\lhd_B$ be the strong inclusion defined by: $a \lhd_B b$ if and only if there exists $c \in B$ such that $a \prec c \prec b$. Then the compactification $\gamma_B L$ of L associated with $\lhd_B$ is isomorphic with the completion $\tilde{L}$ of L.*

Proof. We have a compactification $\bigvee : \gamma_B L \to L$ and its right adjoint $k : L \to \gamma_B L$ defined by $k(a) = \{x \in L : x \lhd_B a\}$. We have the following:

1. $k(0) = \{x \in L : x \lhd_B 0\} = 0$.

2. $k(a) = \bigvee\limits_{x \lhd_B a} k(x)$ for each $a \in L$, since k transforms covers to joins (see the proof of Proposition 3.3.3).

3. We show that $\bigvee\limits_{a \in U} k(a) = L$, for each $U \in \mathfrak{U}$. Take $U \in \mathfrak{U}$. The cover U is refined by a finite meet of the members of the form C_a^b, where $a \lhd_B b$. Say $C = C_{a_1}^{b_1} \wedge C_{a_2}^{b_2} = \{a_1^* \wedge a_2^*, a_1^* \wedge b_2, a_2^* \wedge b_1, b_1 \wedge b_2\} \leq U$. Since $a_1^* \vee b_1 = e$ and $a_2^* \vee b_2 = e$, then $k(a_1{}^*) \vee k(b_1) = L$ and $k(a_2{}^*) \vee k(b_2) = L$, by the proof of Proposition 3.3.3. Thus $[k(a_1{}^*) \vee k(b_1)] \wedge [k(a_2{}^*) \vee k(b_2)] = L$, that is,

$$[k(a_1{}^*) \wedge k(a_2{}^*)] \vee [k(a_1{}^*) \wedge k(b_2)] \vee [k(b_1) \wedge k(a_2{}^*)] \vee [k(b_1) \wedge \vee k(b_2)] = L.$$

Since k preserve finite meets (see the proof of Proposition 3.3.3 or simply note that k is the right adjoint, so it preserves finite meets), we have $k(a_1^* \wedge a_2^*) \vee k(a_1^* \wedge b_2) \vee k(a_2^* \wedge b_1) \vee k(b_1 \wedge b_2) = L$, i.e., $\bigvee_{c \in C} k(c) = L$. Since $C \leq U$, then for each $c \in C$, we can get $c \leq u$ for some $u \in U$. Therefore $k(c) \subseteq k(u)$, consequently $\bigvee_{c \in C} k(c) \subseteq \bigvee_{a \in U} k(a)$, i.e., $L \subseteq \bigvee_{a \in U} k(a)$ which implies that $\bigvee_{a \in U} k(a) = L$.

So, $k : L \to \gamma_B L$ satisfies conditions (1-3) in Remark 3.4.9. Therefore, there is a unique frame homomorphism $\varrho : \tilde{L} \to \gamma_B L$ which makes the following diagram commute:

$$\begin{array}{ccc} L & \xrightarrow{\;j\;} & \tilde{L} \\ & {\scriptstyle k}\searrow & \Big\downarrow {\scriptstyle \exists! \varrho} \\ & & \gamma_B L \end{array}$$

A proof similar to the one provided in Proposition 3.3.3 shows that the homomorphism ϱ is dense and onto. Since $\tilde{L}$ is regular and $\gamma_B L$ is compact regular, by Lemma 3.3.2 we have that $\varrho : \tilde{L} \to \gamma_B L$ is also one-to-one, making it an isomorphism. Therefore $\gamma_B L \cong \tilde{L}$, as desired. $\square$

Chapter 4

Perfect Compactifications and Full π-Compact Bases

Skljarenko [23] defined a *perfect compactification* of a completely regular space X to be a compactification Y with the property that;

$$Fr_Y(O\langle U\rangle) = cl_Y(Fr_X(U)) \text{ for each open set } U \text{ of } X,$$

where $\langle U\rangle = Y \setminus cl_Y(X \setminus U)$, the largest open set in Y whose intersection with X is U. If the above equation holds for an open set U, we say that the compactification Y *is perfect with respect to* U. Perfect compactifications of topological spaces have been studied extensively. In the literature, several equivalent conditions for perfectness of a compactification of a space are known (eg. see Skljarenko [23], and Dickman and McCoy [10]). Skljarenko proved that the Stone-Čech compactification of a Tychonoff space and the Freudenthal compactification of a rim-compact space are examples of perfect compactifications. It is part of the folklore that the construction of the Freudenthal compactification for spaces is dependant on the Boolean

Ultrafilter Theorem (BUT). Baboolal [3] introduced the concept of perfect compactifications for frames and defined the Freudenthal compactification for the class of rim-compact frames. The construction of the Freudenthal compactification for frames does not depend on any choice principle. This chapter may be viewed as a continuation of the work by Baboolal [3], in the sense that perfectness of compactifications of frames is studied, with more attention to rim-compact frames. We introduce the definition of a full π-compact basis for frames and study the compactifications arising from such bases. Characterizations of these bases in terms of perfectness is given. We also exhibit a one-to-one correspondence between the set of all full π-compact bases and the set of all π-compactifications of a rim-compact frame L. The direct link between the partial order of π-compactifications and containment of their corresponding full π-compact basis is furnished. Of course, most ideas are developed from the work of Skljarenko [23], where the study of perfectness of compactifications for rim-compact spaces is carried out.

4.1 Perfect compactifications of frames

Here we provide a frame analogue of the perfectness of a compactification.

Definition 4.1.1. Let $h : M \to L$ be a compactification of a frame L and $r : L \to M$ be its right adjoint. We say that (M, h) is **perfect with respect to** $u \in L$ if $r(u \vee u^*) = r(u) \vee r(u^*)$. (M, h) is said to be **perfect** if it is perfect with respect to all the elements of L

As a motivation to the definition above, one may observe the following fact.

Lemma 4.1.2. *[3, Remark 3.3] Let $f : X \to Y$ be a compactification of* *a space X. Then $f : X \to Y$ is perfect if and only if $\mathcal{O}f : \mathcal{O}Y \to \mathcal{O}X$ is perfect.*

4.2 π-Compactifications of rim-compact frames

Let B be a π-compact basis for a rim-compact frame L. Let $(\gamma_B L, \bigvee)$ be the compactification associated with the strong inclusion $\lhd_B$ which we defined previously as: For $a, b \in L$, $a \lhd_B b \Leftrightarrow a \prec c \prec b$ for some $c \in B$. Then $(\gamma_B L, \bigvee)$ is perfect with respect to all the members of the basis B. This is a direct consequence of the following result.

Proposition 4.2.1. *[3, Proposition 4.7] Let (M, h) be any compactification such that $(\gamma_B L, \bigvee) \leq (M, h)$. Then (M, h) is perfect with respect to every member of B.*

Skljarenko [23] pointed out that the existence of a one-to-one correspondence between compactifications of a completely regular space X and proximity relations compatible with the topology on this space is attributed to Ju. M. Smirnov. For a rim-compact space X, we have a proximity δ defined by: $A\bar{\delta}B \Leftrightarrow cl_X(A) \subseteq U$, and $cl_X(B) \subseteq X \setminus U$, for some member U of the π-compact basis $\mathfrak{B}_X$ of X. Skljarenko refers to the compactification corresponding to such a proximity relation as a π-*compactifications.* This motivates for the following definition.

Definition 4.2.2. We shall say that a compactification (M, h) of a rim-compact frame L is a π-**compactification** if there is a π-compact basis B of L such that $(\gamma_B L, \bigvee) \cong (M, h)$.

Problem 4.2.3. *[3, Proposition 4.9] Let $(\gamma_B L, \bigvee)$ be the π-compactification with the π-compact basis B. Let $k : L \to \gamma_B L$, defined by*

$k(a) = \{x \in L : x \triangleleft_B a\}$, *be the right adjoint of* π*-compactification* $\bigvee : \gamma_B L \to L$. *Then the set* $k(B) = \{k(b) : b \in B\}$ *is a basis for* $\gamma_B L$.

Lemma 4.2.4. *[3, Lemma 6.1] Let* B *be a* π*-compact basis. Then* $\uparrow k(b \vee b^*) \cong \uparrow b \vee b^*$ *for all* $b \in B$ *where the isomorphism is the restriction* $\bigvee_{\uparrow k(b\vee b^*)}$ *of the join map* $\bigvee : \gamma_B L \to L$ *to the sublocale* $\uparrow k(b \vee b^*)$.

For a topological space X, we say that X is *zero-dimensional* if X is a (non-empty) T_1 space that has a basis consisting of clopen sets. In a natural way, we have the following definition for frames.

Definition 4.2.5. A frame L is said to be **zero-dimensional** if L has a basis B consisting of complemented elements, i.e., $b \vee b^* = e$, for all $b \in B$.

We state the well-known Freudenthal-Morita Theorem which appears in Freudenthal [11], Morita [20] and Skljarenko [23].

Theorem 4.2.6. *Every rim-compact space* X *may be imbedded in a compactum* Y *with a zero-dimensional remainder, i.e.,* $Y \backslash X$ *is zero-dimensional.*

In topological spaces, if $f : X \to Y$ is a compactification of X, the remainder of X in Y is the set $Y \setminus f(X)$. The image, $f(X)$, is normally identified with X and therefore the remainder is simply the annex $Y \setminus f(X)$. In [3], Baboolal argued that if Y is a T_1 space and $f : X \to Y$ is an embedding, then $\mathcal{O}Y/(\ker(Of))^* \cong \mathcal{O}(Y \backslash f(X))$. Motivated by this, we have the following definition of a remainder in the context of frames.

Definition 4.2.7. Let $h : M \to L$ be any compactification of a frame L. The **remainder** of L in this compactification is defined to be the quotient frame M/Θ where $\Theta = (\ker h)^*$, the pseudocomplement of $\ker h$ in the congruence lattice $\mathcal{C}M$.

Baboolal [3] proved the frame counterpart of the Freudenthal-Morita Theorem for frames by proving the following.

Theorem 4.2.8. *[3, Theorem 6.2] Let L be a rim-compact frame. Then the remainder of L in its Freudenthal compactifications zero-dimensional.*

We prove the above theorem in a more general setting in the next section.

Remark 4.2.9. Let B be an arbitrary π-compact basis of a rim-compact frame L. By definition, B is contained in the Freudenthal base $B_L = \{b \in L : \uparrow b \vee b^* \text{ is compact}\}$. Consequently, the Freudenthal compactification γL is the maximal amongst all π-compactifications. So, we have $(\gamma_B L, \bigvee) \leq (\gamma L, \bigvee)$ and hence the theorem above applies for $\gamma_B L$. This exhibits the Freudenthal-Morita theorem for rim-compact frames.

4.3 Full π-compact bases and their associated π-compactifications

We recall the following definition from topological spaces (eg. see Dickman and McCoy [10]).

Definition 4.3.1. If Y is a compactification of a space X, then the remainder $Y \setminus X$ is said to be **zero-dimensionally embedded** in Y provided that each point of $Y \setminus X$ has a neighbourhood base whose frontier in Y misses $Y \setminus X$.

If $Y \setminus X$ is zero-dimensionally embedded in Y, then $Y \setminus X$ is zero-dimensional. This can be deduced by observing the following.

Lemma 4.3.2. *Let X be a subspace of Y. If U is open in Y and $Fr_Y(U) \subseteq X$, then $U \cap (Y \setminus X)$ is clopen in $Y \setminus X$.*

Proof. Suppose $Fr_Y(U) \subseteq X$. We note that $Y = U \cup Fr_Y(U) \cup (Y \setminus cl_Y(U))$. Intersecting with $Y \setminus X$ on both side of the previous equation, gives

$$Y \setminus X = [(Y \setminus X) \cap U] \cup [(Y \setminus X) \cap Fr_Y(U)] \cup [(Y \setminus X) \cap (Y \setminus cl_Y(U))].$$

Since $Fr_Y(U) \subseteq X$, then $(Y \setminus X) \cap Fr_Y(U) = \emptyset$, and hence

$$Y \setminus X = [(Y \setminus X) \cap U] \cup [(Y \setminus X) \cap (Y \setminus cl_Y(U))].$$

Thus, $Y \setminus X$ is a union of two disjoint open sets in $Y \setminus X$. Hence $(Y \setminus X) \cap U$ is clopen in $Y \setminus X$. □

Let Y be a π-compactification of a rim-compact space X with a π-compact base B. In Y, take the system of all open sets U such that $Fr_Y(U) \subseteq X$. We note that $U \cap X$ is an open set in X such that $Fr_Y(U \cap X)$ is compact. Following [23], let $\tilde{B} = \{U \cap X : U \text{ is open in } Y \text{ and } Fr_Y(U) \subseteq X\}$. In [23], it is shown that we always have $B \subseteq \tilde{B}$. The base B is said to be *full* if $B = \tilde{B}$. This will motivate for the frame counterpart definition of a full π-compact base after we observe the following. This is probably well known, we provide the proof for completeness.

Proposition 4.3.3. *Let X be a topological space. If X is dense in Y and U is open in Y, then $Fr_Y(U) \subseteq X$ if and only if $Fr_Y(U) = Fr_X(U \cap X)$.*

Proof. Suppose $Fr_Y(U) \subseteq X$. Take $y \in Fr_Y(U) = cl_Y(U) \setminus U$. Then $y \in X$ and so $y \notin X \cap U$. Therefore, it only remains to show that $y \in cl_X(X \cap U)$. However, $cl_X(X \cap U) = cl_Y(U) \cap X$, so $y \in cl_X(X \cap U)$.

Now, take $y \in Fr_X(U \cap X) = cl_X(X \cap U) \setminus (X \cap U)$. Then $y \in (cl_Y(U) \cap X) \setminus (X \cap U)$, hence $y \in cl_Y(U) \setminus U = Fr_Y(U)$.

Conversely, suppose that $Fr_Y(U) = Fr_X(U \cap X)$. Then $Fr_Y(U) \subseteq X$. □

Definition 4.3.4. *Let B be a π-compact basis of a rim-compact frame L. We set*

$$\tilde{B} = \{\bigvee J : J \in \gamma_B L, \text{ and } \uparrow(J \vee J^*) \cong \uparrow \bigvee(J \vee J^*)\},$$

where the isomorphism is the join map $\bigvee : \gamma_B L \to L$ restricted to the sublocale $\uparrow(J \vee J^*)$.

Lemma 4.3.5. *Let B be a π-compact basis of a rim-compact frame L. Let $\tilde{B} = \{\bigvee J : J \in \gamma_B L, \text{ and } \uparrow(J \vee J^*) \cong \uparrow \bigvee(J \vee J^*)\}$, where the isomorphism is the join map $\bigvee : \gamma_B L \to L$ restricted to the sublocale $\uparrow(J \vee J^*)$. Then $B \subseteq \tilde{B}$.*

Proof. Let $\bigvee : \gamma_B L \to L$ be the π-compactification of a rim-compact frame L and $k : L \to \gamma_B L$ be its right adjoint. Take $b \in B$ and let $J = k(b)$. We note that $\bigvee J = \bigvee k(b) = b$. We show that $b \in \tilde{B}$. However, $b \in \tilde{B} \Leftrightarrow \uparrow k(b) \vee k(b)^* \cong \uparrow \bigvee(k(b) \vee k(b)^*)$, that is, $\uparrow k(b \vee b^*) \cong \uparrow(\bigvee k(b)) \vee (\bigvee k(b^*))$, which is true if and only if $\uparrow k(b \vee b^*) \cong \uparrow b \vee b^*$. The latter follows Lemma 4.2.4. $\square$

This leads to the following definition.

Definition 4.3.6. Let B be a π-compact basis of a rim-compact frame L. We say that B is **full** if $B = \tilde{B}$.

We give a characterization of those ideals $J \in \gamma_B L$ whose join are in the full π-compact base.

Proposition 4.3.7. *Let $\bigvee : \gamma_B L \to L$ be a π-compactification of a rim-compact frame L. For any $J \in \gamma_B L$, $\bigvee J \in \tilde{B}$ if and only if $\uparrow(J \vee J^*) \subseteq k(L)$, where $k : L \to \gamma_B L$ is the right adjoint of $\bigvee$.*

Proof. Let $J \in \gamma_B L$ and suppose that $\bigvee J \in \tilde{B}$. That is, suppose that $\uparrow(J \vee J^*) \cong \uparrow \bigvee(J \vee J^*)$. Take any ideal $I \in \uparrow(J \vee J^*)$. Thus $J \vee J^* \subseteq I$. Let $a = \bigvee I \in L$. Now, $\bigvee(J \vee J^*) \leq \bigvee I = a$. This implies that $a \in \uparrow \bigvee(J \vee J^*)$. Also, $\bigvee(J \vee J^*) \leq a \Rightarrow J \vee J^* \subseteq k(a) \Rightarrow k(a) \in \uparrow (J \vee J^*)$ and $\bigvee k(a) = a = \bigvee I$. Since $\bigvee$ is one-to-one when it is restricted to $\uparrow(J \vee J^*)$, we have $I = k(a)$, so $I \in k(L)$.

Conversely, suppose $\uparrow(J \vee J^*) \subseteq k(L)$. We need to show that $\uparrow(J \vee J^*) \cong \uparrow \bigvee(J \vee J^*)$.

We show that $\bigvee$ is onto. Take $a \in \uparrow \bigvee(J \vee J^*)$. Therefore $\bigvee(J \vee J^*) \leq a \Rightarrow J \vee J^* \subseteq k(a) \Rightarrow k(a) \in \uparrow (J \vee J^*)$. Furthermore, $\bigvee k(a) = a$.

We now show that $\bigvee$ is one-to-one. The frames $\uparrow(J \vee J^*)$ and $\uparrow \bigvee(J \vee J^*)$ are regular sublocales of regular frame, and hence regular. By Lemma 3.3.1, we only need to show that $\bigvee$ is codense. Suppose that $\bigvee K = e$, where $K \in \uparrow (J \vee J^*)$. By assumption, $K \in k(L)$ and so $K = k(a)$ for some $a \in L$. Thus $a = \bigvee k(a) = \bigvee K = e$, that is $a = e$. Hence $K = k(e) = L$, so $\bigvee$ is one-to-one. Hence $\bigvee$ is an isomorphism.

□

Remark 4.3.8. We should remark here that the above proposition holds true for any compactification (M, h) of a regular frame L with a right adjoint $r : L \to M$. That is, for any $x \in M$ we have that $h(x) \in \mathcal{B} = \{h(a) : a \in M, \text{ and } \uparrow(a \vee a^*) \cong \uparrow h(a \vee a^*)\}$, if and only if $\uparrow(x \vee x^*) \subseteq r(L)$. The proof of this is similar to the above proof with $(\gamma_B L, \bigvee)$ replaced with (M, h). The reason why we stick to π-compactifications (and not the general compactifications) will be clear later when we exhibit a one-to one correspondence between the set of all π-compactifications and the set of all full π-compact bases.

From Lemma 4.3.5, we have $B \subseteq \tilde{B}$. We can therefore conclude that $\tilde{B}$ is a base. We prove that $\tilde{B}$ is, in fact, a π-compact basis.

Lemma 4.3.9. *Let B be a π-compact basis of a rim-compact frame L. Then $\tilde{B}$ is a π-compact basis of L.*

Proof. 1. Let $a, b \in \tilde{B}$. We need to show that $a \wedge b, a \vee b \in \tilde{B}$. Now $a = \bigvee J$ and $b = \bigvee K$ for some $J, K \in \gamma_B L$ such that $\uparrow(J \vee J^*) \cong \uparrow\bigvee(J \vee J^*)$ and $\uparrow(K \vee K^*) \cong \uparrow\bigvee(K \vee K^*)$. Now $a \wedge b = (\bigvee J) \wedge (\bigvee K) = \bigvee(J \cap K)$. We need to show that $\uparrow[(J \cap K) \vee (J \cap K)^*] \cong \uparrow\bigvee[(J \cap K) \vee (J \cap K)^*]$. By Proposition 4.3.7, we need to show that $\uparrow[(J \cap K) \vee (J \cap K)^*] \subseteq k(L)$. Now, $\bigvee J, \bigvee K \in \tilde{B}$, and therefore $\uparrow(J \vee J^*) \subseteq k(L)$ and $\uparrow(K \vee K^*) \subseteq k(L)$, by Proposition 4.3.7. Notice that

$$\begin{aligned} (J \cap K) \vee (J \cap K)^* &= [J \vee (J \cap K)^*] \cap [K \vee (J \cap K)^*] \\ &\supseteq (J \vee J^*) \cap (K \vee K^*) \end{aligned}$$

Hence

$$\begin{aligned} \uparrow[(J \cap K) \vee (J \cap K)^* &\subseteq \uparrow[(J \vee J^*) \cap (K \vee K^*)] \\ &= [\uparrow(J \vee J^*)] \vee [\uparrow(K \vee K^*)] \\ &\subseteq k(L) \end{aligned}$$

This shows that $a \wedge b \in \tilde{B}$.

We now show that $a \vee b \in \tilde{B}$. We note that

$$a \vee b = (\bigvee J) \vee (\bigvee K) = \bigvee(J \vee K).$$

We need to show that $\uparrow[(J \vee K) \vee (J \vee K)^*] \cong \uparrow\bigvee[(J \vee K) \vee (J \vee K)^*]$. By Proposition 4.3.7, we need to show that $\uparrow[(J \vee K) \vee (J \vee K)^*] \subseteq k(L)$.

However,

$$\begin{aligned}(J \vee K) \vee (J \vee K)^* &= (J \vee K) \vee (J^* \wedge K^*)\\ &= [(J \vee K) \vee J^*] \wedge [(J \vee K) \vee K^*]\\ &\supseteq (J \vee J^*) \wedge (K \vee K^*)\end{aligned}$$

Hence

$$\begin{aligned}\uparrow[(J \vee K) \vee (J \vee K)^*] &\subseteq \uparrow[(J \vee J^*) \wedge (K \vee K^*)]\\ &= [\uparrow(J \vee J^*)] \vee [\uparrow(K \vee K^*)]\\ &\subseteq k(L)\end{aligned}$$

That is, $a \vee b \in \tilde{B}$.

2. We show that $a \in \tilde{B} \Rightarrow a^* \in \tilde{B}$. Take $a = \bigvee J \in \tilde{B}$. Therefore $\uparrow(J \vee J^*) \subseteq k(L)$. We need to show that $a^* = (\bigvee J)^* = \bigvee J^*$ is in $\tilde{B}$. So, we need to show that $\uparrow(J^* \vee J^{**}) \subseteq k(L)$. The latter holds true since $J \subseteq J^{**}$ and so $J^* \vee J \subseteq J^* \vee J^{**}$. Hence $\uparrow(J^* \vee J^{**}) \subseteq \uparrow(J \vee J^*) \subseteq k(L)$. That is, $a^* \in \tilde{B}$.

3. Let $a \in \tilde{B}$. We need to show that $\uparrow(a \vee a^*)$ is compact. Suppose $a = \bigvee J$, therefore $\uparrow(J \vee J^*) \cong \uparrow\bigvee(J \vee J^*)$. Now, $\uparrow(a \vee a^*) = \uparrow[(\bigvee J) \vee (\bigvee J^*)] = \uparrow\bigvee(J \vee J^*) \cong \uparrow(J \vee J^*)$ and $\uparrow(J \vee J^*)$ is a closed sublocale of a compact frame $\gamma_B L$, and hence compact. Therefore $\uparrow(a \vee a^*)$ is compact.

Therefore, $\tilde{B}$ is a π-compact basis of L. □

Remark 4.3.10. Similarly, for any compactification (M, h) of a regular frame L with a right adjoint $r : L \to M$, if we assume that

$$\mathcal{B} = \{h(a) : a \in M, \text{ and } \uparrow(a \vee a^*) \cong \uparrow h(a \vee a^*)\}$$

is a basis for L, we can then mimic the proof of the above proposition with $(\gamma_B L, \bigvee)$ replaced with (M, h) to show that $\mathcal{B}$ is a π-compact basis of L.

Later, we will show that, not only is $\tilde{B}$ a π-compact basis, but it is also full, i.e., $\tilde{\tilde{B}} = \tilde{B}$. This will be a direct consequence of the fact that, although $\tilde{B}$ might contain B properly, these two bases determine the same strong inclusion, and therefore the π-compactifications corresponding to those strong inclusions are isomorphic. To prove this, we need the following lemmas.

Lemma 4.3.11. *Let B be a π-compact basis of a rim-compact frame L. Let $\bigvee : \gamma_B L \to L$ be a π-compactification of a rim-compact frame L with a right adjoint $k : L \to \gamma_B L$. If $b \in \tilde{B}$, then $k(b \vee b^*) = k(b) \vee k(b^*)$.*

Proof. Let $b \in \tilde{B}$. Then $b = \bigvee J$, $J \in \gamma_B L$ and $\uparrow(J \vee J^*) \cong \uparrow \bigvee (J \vee J^*)$, and therefore $\uparrow(J \vee J^*) \subseteq k(L)$. Hence $J \vee J^* = k(a)$, for some $a \in L$. Taking joins both sides of the previous equation and using the fact that $\bigvee$ is dense and onto, we have $(\bigvee J) \vee (\bigvee J^*) = \bigvee k(a)$, that is, $b \vee b^* = a$.

Now, $\bigvee J = b$ and $\bigvee J^* = b^*$ implies $J \subseteq k(b)$ and $J^* \subseteq k(b^*)$, respectively (see the properties of a right adjoint in Lemma 2.3.4). Thus $k(b \vee b^*) = k(a) = J \vee J^* \subseteq k(b) \vee k(b^*)$. It is trivial that $k(b) \vee k(b^*) \subseteq k(b \vee b^*)$, so we must have that $k(b \vee b^*) = k(b) \vee k(b^*)$. □

We note that the lemma above implies that $\gamma_B L$ is perfect with respect to the elements of $\tilde{B}$.

Lemma 4.3.12. *Let B be a π-compact basis of a rim-compact frame L. Let $a, b \in \tilde{B}$. Let $\lhd_{\tilde{B}}$ be the strong inclusion corresponding to the π-compact basis $\tilde{B}$. If $a \lhd_{\tilde{B}} b$, then $k(a^*) \vee k(b) = L$.*

Proof. Express a as $a = \bigvee J$, $J \in \gamma_B L$ and $\uparrow(J \vee J^*) \cong \uparrow \bigvee (J \vee J^*)$. From the proof of the above lemma, we see that $k(a \vee a^*) = J \vee J^*$. Thus

$\uparrow k(a \vee a^*) = \uparrow (J \vee J^*) \cong \uparrow \bigvee (J \vee J^*) = \uparrow [(\bigvee J) \vee (\bigvee J^*)] = \uparrow (a \vee a^*)$. By the lemma above, $k(a \vee a^*) = k(a) \vee k(a^*)$. So $\uparrow (k(a) \vee k(a^*)) \cong \uparrow (a \vee a^*)$.

We note that $a \lhd_{\tilde{B}} b \Rightarrow a \leq b \Rightarrow k(a) \subseteq k(b) \Rightarrow k(a) \vee k(a^*) \subseteq k(a^*) \vee k(b)$, and so $k(a^*) \vee k(b) \in \uparrow (k(a) \vee k(a^*))$. Furthermore, $\bigvee (k(a^*) \vee k(b)) = (\bigvee k(a^*)) \vee (\bigvee k(b)) = a^* \vee b = e$. Since $\bigvee : \uparrow (k(a) \vee k(a^*)) \to \uparrow (a \vee a^*)$ is an isomorphism, we then have $k(a^*) \vee k(b) = L$. □

We now show that the compactifications $\bigvee : \gamma_B L \to L$ and $\bigvee : \gamma_{\tilde{B}} L \to L$ are isomorphic, for any π-compact basis B of a rim-compact frame L . To show this, we use the fact that there is a one-to-one correspondence between the set of all compactifications of a frame and the set of all strong inclusions on that frame, as discussed on the introduction of Section 3.3.

Proposition 4.3.13. *Let B be a π-compact basis of a rim-compact frame L. Let $\bigvee : \gamma_B L \to L$ and $\bigvee : \gamma_{\tilde{B}} L \to L$ be π-compactifications determined by the π-compact bases B and $\tilde{B}$ respectively. Then $(\gamma_B L, \bigvee) \cong (\gamma_{\tilde{B}} L, \bigvee)$.*

Proof. To show that $(\gamma_B L, \bigvee) \cong (\gamma_{\tilde{B}} L, \bigvee)$, it suffices to show that strong inclusion corresponding to $\bigvee : \gamma_B L \to L$ coincides with the one corresponding to $\bigvee : \gamma_{\tilde{B}} L \to L$. That is, we only need to show that $\lhd_B = \lhd_{\tilde{B}}$. Let $k : L \to \gamma_B L$ be the right adjoint of $\bigvee : \gamma_B L \to L$, defined by $k(a) = \{x \in L : x \lhd_B a\}$. We recall that $a \lhd_B b \Leftrightarrow a \prec c \prec b$, for some $c \in B$. Since $B \subseteq \tilde{B}$, we always have that $\lhd_B \subseteq \lhd_{\tilde{B}}$.

For the reverse inclusion, suppose that $a \lhd_{\tilde{B}} b$. Strong inclusions interpolate, so we can find $c, d \in \tilde{B}$ such that $a \lhd_{\tilde{B}} c \lhd_{\tilde{B}} d \lhd_{\tilde{B}} b$. Since $c, d \in \tilde{B}$, by the lemma above, we have $k(c^*) \vee k(d) = L$. Thus, $e = t \vee s$, for some $t \in k(c^*)$ and $s \in k(d)$. Now, $c = (c \wedge t) \vee (c \wedge s)$. We note that $t \in k(c^*) \Rightarrow t \lhd_B c^* \Rightarrow t \leq c^* \Rightarrow c \wedge t \leq c \wedge c^* = 0$. Therefore $c = c \wedge s \leq s$.

Now $s \in k(d) \Rightarrow s \lhd_B d$. We now have $a \leq c \leq s \lhd_B d \leq b$. So, $a \lhd_B b$. Consequently, $\lhd_{\tilde{B}} \subseteq \lhd_B$. $\square$

We now show that $\tilde{B}$ is full for any given π-compact basis B.

Corollary 4.3.14. *Let B be a π-compact basis of a rim-compact frame L. Then $\tilde{B}$ is a full π-compact basis of L.*

Proof. We need to show that $\tilde{\tilde{B}} = \tilde{B}$. By Proposition 4.3.9, we know that $\tilde{B}$ is a π-compact basis of L. From Proposition 4.3.5, we have $\tilde{B} \subseteq \tilde{\tilde{B}}$. To show the other containment, take $y \in \tilde{\tilde{B}}$. Then $y = \bigvee I$, for some $I \in \gamma_{\tilde{B}} L$, where $\bigvee : \gamma_{\tilde{B}} L \to L$ is the compactification associated with the π-compact basis $\tilde{B}$. Let $\tilde{k} : L \to \gamma_{\tilde{B}} L$ be the right adjoint of the join map $\bigvee : \gamma_{\tilde{B}} L \to L$. To show that $y = \bigvee I \in \tilde{B}$, we need to show that $\uparrow(I \vee I^*) \subseteq k(L)$, where $k : L \to \gamma_B L$ is the right adjoint of join map $\bigvee : \gamma_B L \to L$.

We recall that the right adjoints k and $\tilde{k}$ are defined by $k(a) = \{x \in L : x \lhd_B a\}$ and $\tilde{k}(a) = \{x \in L : x \lhd_{\tilde{B}} a\}$, respectively. Here, the strong inclusion $\lhd_B$ is defined by $x \lhd_B a$ if and only if $x \prec b \prec a$, for some $b \in B$. The strong inclusion $\lhd_{\tilde{B}}$ is defined in a similar way with B replaced by $\tilde{B}$.

Now, since $y = \bigvee I \in \tilde{\tilde{B}}$, then $\uparrow(I \vee I^*) \subseteq \tilde{k}(L)$ by Proposition 4.3.7. However, since $\lhd_B = \lhd_{\tilde{B}}$, we have that $k(L) = \tilde{k}(L)$. Hence, $\uparrow(I \vee I^*) \subseteq \tilde{k}(L)$, as desired. $\square$

For any π-compact basis B of a regular frame L, the pre-images of the join map on the members of the full basis $\tilde{B}$ forms a basis for $\gamma_B L$.

Corollary 4.3.15. *Let B be a π-compact basis of a rim-compact frame L. Then $B_\gamma = \{J \in \gamma_B L : \uparrow(J \vee J^*) \cong \uparrow\bigvee(J \vee J^*)\}$ is a basis for $\gamma_B L$.*

Proof. Let $k : L \to \gamma_B L$ be the right adjoint of join map $\bigvee : \gamma_B L \to L$. Set $k(B) = \{k(b) : b \in B\}$. Then $k(B)$ is a basis for $\gamma_B L$, by Proposition 4.2.3.

In the proof of Proposition 4.3.5, we have seen that $\uparrow(J \vee J^*) \cong \uparrow\bigvee(J \vee J^*)$, for all $J \in k(B)$. Hence $k(B) \subseteq B_\gamma$, and therefore B_γ is a basis for $\gamma_B L$. $\square$

We have seen that if B is a π-compact basis of a rim-compact frame L, then $(\gamma_B L, \bigvee) \cong (\gamma_{\tilde{B}} L, \bigvee)$ and $B_\gamma = \{J \in \gamma_B L : \bigvee J \in \tilde{B}\}$ is a basis for $\gamma_B L$. In general, if (M, h) is a compactification of any rim-compact frame L, and $\mathcal{B} = \{h(a) : a \in M$, and $\uparrow(a \vee a^*) \cong \uparrow h(a \vee a^*)\}$ is a basis for L, then $(M, h) \cong (\gamma_{\mathcal{B}} L, \bigvee)$ if and only if the collection $\mathcal{B}_M = \{a \in M : h(a) \in \mathcal{B}\}$ is a basis for M. Before we provide a proof of this, we need to observe few general well-known facts.

Lemma 4.3.16. *For any frame L, suppose $h : M \to L$ is a dense and onto map with the right adjoint $r : L \to M$. Take $a, b \in M$. If $a \prec b$, then $rh(a) \prec b$.*

Proof. We always have $a^* \leq rh(a^*)$, by Lemma 2.3.4. Since h is dense and onto, we have $rh(a^*) = r(h(a)^*) = (rh(a))^*$ by Lemma 2.3.11. Now $a \prec b \Leftrightarrow e = a^* \vee b \leq rh(a^*) \vee b = (rh(a))^* \vee b$. Therefore $rh(a) \prec b$, by Lemma 2.5.2. $\square$

Lemma 4.3.17. *For any regular frame L, suppose $h : M \to L$ is a compactification of L with the right adjoint $r : L \to M$. Then $r(x) = \bigvee r(y)(y \in L,\, y \lhd x)$, where $\lhd$ is the strong inclusion associated with $h : M \to L$.*

Proof. Take $t \prec r(x)$. Then $rh(t) \prec r(x)$, by Lemma 4.3.16. Therefore $h(t) \lhd x$. So, $t \leq rh(t) \leq \bigvee r(y)(y \in L,\, y \lhd x)$. By regularity of L, we have $r(x) = \bigvee t(t \prec r(x))$. We now have $r(x) = \bigvee t(t \prec r(x)) \leq \bigvee r(y)(y \in L,\, y \lhd x)$. We note that $y \lhd x \Rightarrow y \leq x \Rightarrow r(y) \leq r(x)$. Therefore $\bigvee r(y)(y \in L,\, y \lhd x) \leq r(x)$. Thus $r(x) = \bigvee r(y)(y \in L,\, y \lhd x)$. $\square$

If $h : M \to L$ is any dense onto map with the right adjoint r, we always have $a \leq rh(a)$ and $a^* \leq rh(a^*)$. The equalities hold true for the elements of $\mathcal{B}_M$. We show this in the following lemma.

Lemma 4.3.18. *For any regular frame L, let $h : M \to L$ be a compactification of L with the right adjoint $r : L \to M$. Let*

$$\mathcal{B} = \{h(a) : a \in M, \text{ and } \uparrow(a \vee a^*) \cong \uparrow h(a \vee a^*)\}$$

and $\mathcal{B}_M = \{a \in M : \uparrow(a \vee a^*) \cong \uparrow h(a \vee a^*)\}$ (where the isomorphisms here are via h). If $a \in \mathcal{B}_M$, then $a = rh(a)$ and $a^* = rh(a^*)$.

Proof. Let $a \in \mathcal{B}_M$. We note that $a \vee a^* \leq rh(a) \vee rh(a^*)$ and $h[rh(a) \vee rh(a^*)] = hrh(a) \vee hrh(a^*) = h(a) \vee h(a^*) = h(a \vee a^*)$. Since $h : \uparrow(a \vee a^*) \to \uparrow h(a \vee a^*)$ is one-to-one, we have $rh(a) \vee rh(a^*) = a \vee a^*$. Consequently,

$$\begin{aligned}
rh(a) &= rh(a) \wedge (rh(a) \vee rh(a^*)) \\
&= rh(a) \wedge (a \vee a^*) \\
&= (rh(a) \wedge a) \vee (rh(a) \wedge a^*) \\
&= a \vee (rh(a) \wedge a^*) \\
&\leq a \vee (rh(a) \wedge rh(a^*)) \qquad (\text{since } a^* \leq rh(a^*)) \\
&= a \vee (rh(a \wedge a^*)) \\
&= a \vee rh(0) \\
&= a \vee r(0) \qquad (\text{since } h \text{ is dense}) \\
&= a \vee 0 \\
&= a.
\end{aligned}$$

Therefore $a = rh(a)$. Similarly, $a^* = rh(a^*)$. □

The following result suggests that the only compactifications of a regular frame L arising from a full π-compact basis are the π-compactifications of L associated with some π-compact basis for L. Simply, a regular frame can have a full π-compact basis if and only if L inherited the full π-compact basis from some π-compactification.

Proposition 4.3.19. *Let $h : M \to L$ compactification of a regular frame L with the right adjoint $r : L \to M$. Let $\mathcal{B} = \{h(a) : a \in M,$ and $\uparrow(a \vee a^*) \cong \uparrow h(a \vee a^*)\}$, where the isomorphism is via h. If $\mathcal{B}_M = \{a \in M : h(a) \in \mathcal{B}\}$ is a basis for M, then $(M, h) \cong (\gamma_{\mathcal{B}} L, \bigvee)$.*

Proof. We note that if $\mathcal{B}_M = \{a \in M : h(a) \in \mathcal{B}\}$ is a basis for M, then $\mathcal{B}$ is a basis for L, since h is onto. In fact, $\mathcal{B}$ is a π-compact basis for L. The latter follows by Proposition 4.3.9 and Remark 4.3.10. This implies that L is actually a rim-compact frame, since every $u \in \mathcal{B}$ is such that $\uparrow(u \vee u^*)$ is compact.

It remains to show that the strong inclusion $\lhd$ associated with $h : M \to L$ coincides with the strong inclusion $\lhd_{\mathcal{B}}$ associated with $\bigvee : \gamma_{\mathcal{B}} L \to L$.

Suppose $x \lhd_{\mathcal{B}} y$ in L. Then there exists $a \in M$, such that
$\uparrow(a \vee a^*) \cong \uparrow h(a \vee a^*)$ and $x \prec h(a) \prec y$.

We claim that $rh(a) \prec r(y)$. To show this, note that $(rh(a))^* \vee r(y) = r(h(a)^*) \vee r(y) = r(h(a^*)) \vee r(y)$, by Lemma 2.3.11, since h is dense and onto. Now, $h[r(h(a^*)) \vee r(y)] = hrh(a^*) \vee hr(y) \geq h(a^*) \vee y = (h(a))^* \vee y = e$. We also have $h(a) \prec y \Rightarrow h(a) \leq y \Rightarrow a \leq r(y)$. We always have $a^* \leq rh(a^*)$, so $a \vee a^* \leq rh(a^*) \vee r(y)$, thus $rh(a^*) \vee r(y) \in \uparrow(a \vee a^*)$.

Since $h : \uparrow(a \vee a^*) \to \uparrow h(a \vee a^*)$ is an isomorphism, and hence one-to-one, we therefore have that $r(h(a^*)) \vee r(y) = e$. Hence $rh(a) \prec r(y)$.

Now, $x \prec h(a) \Rightarrow x \leq h(a) \Rightarrow r(x) \leq rh(a) \prec r(y) \Rightarrow r(x) \prec r(y)$. That is, $x \triangleleft y$.

Now, suppose $x \triangleleft y$ in L. The interpolation property of $\triangleleft$ guarantees the existence of $w \in L$ such that $x \triangleleft w \triangleleft y$. That is, $r(x) \prec r(w) \prec r(y)$. Since $\mathcal{B}_M$ is a basis for M, we have that $r(w) = \bigvee\{a \in \mathcal{B}_M : a \leq r(w)\}$. Now $r(x)^* \vee r(w) = e \Rightarrow r(x)^* \vee \bigvee\{a \in \mathcal{B}_M : a \leq r(w)\} = e$. By compactness of M, we have $r(x)^* \vee a_1 \vee a_2 \vee \cdots \vee a_n = e$, where $a_i \in \mathcal{B}_M$ and $a_i \leq r(w)$. Since $\mathcal{B}$ is a π-compact basis, binary joins of elements in $\mathcal{B}$ is also in $\mathcal{B}$. Multiple application of this gives us that $h(a_1) \vee h(a_2) \vee \cdots \vee h(a_n) = h(a_1 \vee a_2 \vee \cdots \vee a_n) \in \mathcal{B}$. Therefore $a_1 \vee a_2 \vee \cdots \vee a_n \in \mathcal{B}_M$. Let $a = a_1 \vee a_2 \vee \cdots \vee a_n$. Hence $r(x)^* \vee a = e$, where $a \in \mathcal{B}_M$, $a \leq r(w)$.

We now have $r(x) \prec a \leq r(w) \prec r(y)$. By Lemma 4.3.18, $a = rh(a)$. Thus $r(x) \prec a = rh(a) \prec r(y) \Rightarrow x \triangleleft h(a) \triangleleft y \Rightarrow x \prec h(a) \prec y \Rightarrow x \triangleleft_{\mathcal{B}} y$, the last implication follows because $h(a) \in \mathcal{B}$. Therefore $(M, h) \cong (\gamma_{\mathcal{B}} L, \bigvee)$. $\square$

We note that in the proof above, the condition that $\mathcal{B}_M$ is a basis for M is necessary to prove that $\triangleleft \subseteq \triangleleft_{\mathcal{B}}$. The converse of this holds, i.e., if $\triangleleft \subseteq \triangleleft_{\mathcal{B}}$, then $\mathcal{B}_M$ is a basis for M. This is noteworthy and thus the proof shall be recorded.

Proposition 4.3.20. *Let (M, h) be a compactification of a regular frame L with a right adjoint $r : L \to M$. Suppose*

$$\mathcal{B} = \{h(a) : a \in M, \text{ and } \uparrow(a \vee a^*) \cong \uparrow h(a \vee a^*)\}$$

is a basis for L. Let $\triangleleft$ be the strong inclusion associated with (M, h) and $\triangleleft_{\mathcal{B}}$ be the strong inclusion corresponding to $(\gamma_{\mathcal{B}} L, \bigvee)$. Then $\triangleleft \subseteq \triangleleft_{\mathcal{B}}$ if and only if $\mathcal{B}_M$ is a basis for M.

Proof. Suppose $\triangleleft \subseteq \triangleleft_{\mathcal{B}}$. Let $t \in M$. Then $t = \bigvee x(x \prec t) = \bigvee y(y \prec x \prec t)$. Now $y \prec x \prec t \Rightarrow y \leq rh(y) \prec x \leq rh(x) \prec t$, by Lemma 4.3.16. Therefore

$rh(y) \prec rh(x)$. Therefore $h(y) \lhd h(x)$, hence $h(y) \lhd_{\mathcal{B}} h(x)$. We can therefore find $a \in \mathcal{B}_M$ such that $h(y) \prec h(a) \prec h(x)$. This implies that $h(y) \leq h(a) \leq h(x)$, which implies that $y \leq rh(y) \leq rh(a) = a \leq rh(x) \prec t$; i.e., $y \leq a \leq t$. Now, $t = \bigvee y \leq \bigvee a \leq t$. Thus $t = \bigvee a$, which makes $\mathcal{B}_M$ a basis for M.

The reverse implication follows from the proof of Proposition 4.3.19. $\square$

We have mentioned earlier in Theorem 4.2.8 that Baboolal [3] showed the Freudenthal-Morita Theorem for frames by showing that the remainder of a rim-compact frame is zero-dimensional in its Freudenthal compactification. This is entirely rooted in a more general observation that we prove after the definition below due to Frith [12].

Definition 4.3.21. Let Θ and Ψ be congruences on a frame L. Set

$$\Theta^{+} = \{(a, b) \in \Theta : a \leq b\}$$

and

$$\Theta \to \Psi = \bigcap_{(a,b) \in \Theta^{+}} \nabla_a \vee \triangle_b \vee \Psi.$$

Remark 4.3.22. Of course, $\Theta \to 0 = \Theta^{*} = \bigcap_{(a,b) \in \Theta^{+}} \nabla_a \vee \triangle_b$, where Θ^{*} is the pseudocomplement of Θ in the congruence lattice $\mathcal{C}L$.

Theorem 4.3.23. *Let L be a regular frame and $h : M \to L$ be a compactification of L with the right adjoint $r : L \to M$. Let*

$$\mathcal{B} = \{h(a) : a \in M, \text{ and } {\uparrow}(a \vee a^{*}) \cong {\uparrow} h(a \vee a^{*})\}.$$

If $\mathcal{B}_M = \{a \in M : h(a) \in \mathcal{B}\}$ is a basis for M, then the remainder of L in (M, h) is zero-dimensional.

Proof. We have to show that the remainder $M/(\ker h)^*$ has a basis of complemented elements. For this, let $\mu : M \to M/(\ker h)^*$ be the natural frame homomorphism. Also let $\mu(\mathcal{B}_M) = \{\mu(b) : b \in \mathcal{B}_M\}$. Since μ is onto and $\mathcal{B}_M$ is a basis for M, then $\mu(\mathcal{B}_M)$ is a basis for $M/(\ker h)^*$.

We now show that each $\mu(a) \in \mu(\mathcal{B}_M)$ is complemented. Hence, we must show that $\mu(a) \vee \mu(a^*) = e$ for all $a \in \mu(\mathcal{B}_M)$, i.e., we need to show that $\mu(a \vee a^*) = \mu(e)$, which is equivalent to showing that $(a \vee a^*, e) \in (\ker h)^*$. From the definition and the remark above, we have

$$(\ker h)^* = \bigcap_{(b,c)\in(\ker h)^+} \nabla_b \vee \triangle_c.$$

Take an arbitrary $(b, c) \in (\ker h)^+$. Therefore $b \leq c$ in M and $h(b) = h(c)$. The task is to show that $(a \vee a^*, e) \in \nabla_b \vee \triangle_c$. That is, we need to prove that $[(a \vee a^*) \vee b] \wedge c = (e \vee b) \wedge c$. Therefore, our problem now reduces to showing that $[(a \vee a^*) \vee b] \wedge c = c$. However, the latter is true if and only if $c \leq (a \vee a^*) \vee b$. Notice that $h :\uparrow (a \vee a^*) \to h(a \vee a^*)$ is an isomorphism since $a \in \mathcal{B}_M$. We now have

$$\begin{aligned} h[(a \vee a^*) \vee b)] &= h(a \vee a^*) \vee h(b) \\ &= h(a \vee a^*) \vee h(b) \qquad \text{(since } h(b) = h(c)) \\ &= h[(a \vee a^*) \vee c)]. \end{aligned}$$

Since $h :\uparrow (a \vee a^*) \to h(a \vee a^*)$ is one-to-one, then $(a \vee a^*) \vee b = (a \vee a^*) \vee c$. This implies that $c \leq (a \vee a^*) \vee b$, as desired.

Thus $(a \vee a^*, e) \in \nabla_b \vee \triangle_c$ for all $(b, c) \in (\ker h)^+$. Therefore $(a \vee a^*, e) \in (\ker h)^*$ and so $\mu(a) \vee \mu(a^*) = e$. Also, trivially $\mu(a) \wedge \mu(a^*) = \mu(a \wedge a^*) = \mu(0) = 0$. Thus $\mu(\mathcal{B}_M)$ is a basis for $M/(\ker h)^*$ that consists of complemented elements. $\square$

Corollary 4.3.24. *Let L be a rim-compact frame with a π-compact basis B. Then the remainder of L in its π-compactification $\gamma_B L$ is zero dimensional. Specifically, the remainder of L in its Freudenthal compactification is zero-dimensional.*

Proof. We have already seen that $B_\gamma = \{J \in \gamma_B L :\uparrow(J \vee J^*) \cong \uparrow \bigvee(J \vee J^*)\}$ is a basis for $\gamma_B L$. By invoking the result above, we get that any π-compactification $\gamma_B L$ (including the Freudenthal compactification) has a zero-dimensional remainder. □

4.4 Some characterizations of a full π-compact basis

The following lemma provides a characterization of the elements belonging to a full π-compact basis with respect to perfectness of the π-compactification.

Lemma 4.4.1. *Let B be a π-compact basis for a regular frame L. Then $u \in \tilde{B}$ if and only if $\uparrow(u \vee u^*)$ is compact and $(\gamma_B L, \bigvee)$ is perfect with respect to u.*

Proof. Let $u \in \tilde{B}$. Therefore $\uparrow(u \vee u^*)$ is compact, since $\tilde{B}$ is a π-compact basis, by Proposition 4.3.9. It follows by Lemma 4.3.11 that $(\gamma_B L, \bigvee)$ is perfect with respect to u.

Conversely, let $u \in L$ be such that $\uparrow(u \vee u^*)$ is compact and suppose that $(\gamma_B L, \bigvee)$ is perfect with respect to u. We show that $u \in \tilde{B}$. We note that $u = \bigvee k(u)$, where $k : L \to \gamma_B L$ is the right adjoint of $\bigvee : \gamma_B L \to L$. Since $k(u) \in \gamma_B L$, we only need to show that $\bigvee :\uparrow(k(u) \vee k(u^*)) \to \uparrow \bigvee(k(u) \vee k(u^*))$ is an isomorphism.

Since $(\gamma_B L, \bigvee)$ is perfect with respect to u, we have that $k(u \vee u^*) = k(u) \vee k(u^*)$. Also, $\bigvee(k(u) \vee k(u^*)) = u \vee u^*$, therefore we need to show that $\bigvee : \uparrow k(u \vee u^*) \to \uparrow (u \vee u^*)$ is an isomorphism. The map
$\bigvee : \uparrow k(u \vee u^*) \to \uparrow (u \vee u^*)$ is onto since $\bigvee : \gamma_B L \to L$ is onto.

To see that $\bigvee : \uparrow k(u \vee u^*) \to \uparrow (u \vee u^*)$ is dense, take $J \in \uparrow k(u \vee u^*)$ and suppose that $\bigvee J = u \vee u^*$. We show that $J = k(u \vee u^*)$. We note that $J \in \uparrow k(u \vee u^*) \Rightarrow k(u \vee u^*) \subseteq J$. Let $x \in J$. Therefore $x \triangleleft_B y$ for some $y \in J$, by strong regularity of J. Since $y \leq \bigvee J = u \vee u^*$, then $x \triangleleft_B u \vee u^*$. Thus $x \in k(u \vee u^*)$, and hence $J = k(u \vee u^*)$. So, $\bigvee : \uparrow k(u \vee u^*) \to \uparrow (u \vee u^*)$ is a dense onto frame homomorphism between regular compact locales. Hence, by Lemma 3.3.2, $\bigvee : \uparrow k(u \vee u^*) \to \uparrow (u \vee u^*)$ is an isomorphism. That is, $u \in \tilde{B}$. □

Baboolal [3] characterized perfectness in terms of strong inclusions as follows:

Proposition 4.4.2. *[3, Proposition 3.7-3.9 and Remark 3.10] Let (M, h) be a compactification of a frame L. Then (M, h) is perfect with respect to an element $y \in L$ if and only if its associated strong inclusion $\triangleleft$ satisfies:*

$$x \leq y, x \triangleleft y \vee y^* \Rightarrow x \triangleleft y, \textit{for all } x, y \in L.$$

We employ the result above together with the following lemma to give yet another characterization of the elements of a full π-compact basis.

Lemma 4.4.3. *Let B be a π-compact basis for a regular frame L. Let $\bigvee : \gamma_B L \to L$ be the π-compactification of a rim-compact frame L with a right adjoint $k : L \to \gamma_B L$. Let $u, v \in L$ be such that $u \wedge v = 0$ and $k(u \vee v) = k(u) \vee k(v)$. Then $x \triangleleft_B (u \vee v)$ implies $(x \wedge u) \triangleleft_B u$ and $(x \wedge v) \triangleleft_B v$.*

Proof. We note that $x \lhd_B u \vee v \Rightarrow x \in k(u \vee v) = k(u) \vee k(v) \Rightarrow x = y \vee z$, for some $y \in k(u), z \in k(v)$. That is, $y \lhd_B u, z \lhd_B v$. Clearly $y \leq x \wedge u$. Also $x \wedge u = (y \wedge u) \vee (z \wedge u) = (y \wedge u) \vee 0 = (y \wedge u) \leq y$. Therefore $y = x \wedge u$. Similarly $z = x \wedge v$. Thus $x \wedge u \lhd_B u$ and $x \wedge v \lhd_B v$. □

We now characterize the elements of the full base $\tilde{B}$ internally using the elements of the associated π-compact basis B.

Proposition 4.4.4. *Let B be a π-compact basis for a regular frame L and $\tilde{B}$ be the corresponding full π-compact basis. Let $u \in L$, be such that $\uparrow(u \vee u^*)$ is compact. Then $u \in \tilde{B}$ if and only if for any $w \in L$, with $w \vee u = e$, there exists $v \in B$ such that $w \vee v = e$, and $v \leq u$.*

Proof. Let $u \in \tilde{B}$. Take $w \in L$ and suppose that $w \vee u = e$.

Therefore $w \vee u \vee u^* = e$. We can express w as $w = \bigvee x(x \lhd_B w)$. Therefore $\bigvee_{x \lhd_B w} (x \vee u \vee u^*) = e$. By compactness of $\uparrow (u \vee u^*)$, we have that $\bigvee_{i=1}^{n} x_i \vee u \vee u^* = e$, where $x_i \lhd_B w$. Let $x = \bigvee_{i=1}^{n} x_i$. We now have $x \vee u \vee u^* = e$ and $x \lhd_B w$. By Remark 3.4.7, we can find $t \in B$ such that $x \lhd_B t \lhd_B w$. Hence $t^* \lhd_B x^* \leq u \vee u^*$. Thus $t^* \lhd_B u \vee u^*$. We also have that $k(u \vee u^*) = k(u) \vee k(u^*)$, by Lemma 4.3.11. Then $t^* \wedge u \lhd_B u$, by the lemma above. Therefore we can find $v \in B$ such that $t^* \wedge u \prec v \prec u$. Now $w \vee v \geq w \vee (t^* \wedge u) = (w \vee t^*) \wedge (w \vee u) = e \wedge e = e$. Thus $w \vee v = e, v \in B$ and $v \leq u$.

Conversely, suppose that u satisfies the conditions of this proposition. We need to show that $u \in \tilde{B}$. By Lemma 4.4.1, it is enough to show that $\gamma_B L$ is perfect with respect to u. We use Proposition 4.4.2. Suppose $x \leq u$ and $x \lhd_B u \vee u^*$. We need to show that $x \lhd_B u$. Now $x \lhd_B u \vee u^* \Rightarrow x \prec u \vee u^* \Rightarrow x^* \vee u \vee u^* = e$. Since $x \leq u$ then $x^* \vee u^* \leq x^*$ and therefore

$x^* \vee u = e$. By assumption, there exists $b \in B$ such that $x^* \vee b = e$ and $b \leq u$. Thus, by Lemma 3.4.6, there exists $v \in B$ such that $v \prec b$ and $x^* \vee v = e$. Now $x \prec v \prec b \leq u \Rightarrow x \triangleleft_B b \leq u \Rightarrow x \triangleleft_B u$. Therefore $(\gamma_B L, \bigvee)$ is perfect with respect to u. Since $\uparrow(u \vee u^*)$ is compact, then $u \in \tilde{B}$, by Lemma 4.4.1. □

The way π-compactifications are ordered is intimately related to the way their corresponding full π-compact basis are ordered. We record the details in the following result.

Proposition 4.4.5. *Let $(\gamma_{B_1} L, \bigvee_1)$ and $(\gamma_{B_2} L, \bigvee_2)$ be π-compactifications of a regular frame L arising from π-compact bases B_1 and B_2, respectively. Then $(\gamma_{B_1} L, \bigvee_1) \leq (\gamma_{B_2} L, \bigvee_1)$ if and only if $\tilde{B}_1 \subseteq \tilde{B}_2$ where $\tilde{B}_i$ is a full π-compact basis of L corresponding to $(\gamma_{B_i} L, \bigvee_i)$.*

Proof. Suppose $(\gamma_{B_1} L, \bigvee) \leq (\gamma_{B_2} L, \bigvee)$ and let $u \in \tilde{B}_1$. Then $\uparrow(u \vee u^*)$ is compact and $(\gamma_{B_2} L, \bigvee)$ is perfect with respect to u, by Proposition 4.2.1. It follows by Lemma 4.4.1 that $u \in \tilde{B}_2$. Thus $\tilde{B}_1 \subseteq \tilde{B}_2$.
Conversely, Suppose that $\tilde{B}_1 \subseteq \tilde{B}_2$. Then $\triangleleft_{\tilde{B}_1} \subseteq \triangleleft_{\tilde{B}_2}$ holds true for the strong inclusions corresponding to $\gamma_{\tilde{B}_1} L$ and $\gamma_{\tilde{B}_2} L$, respectively. By the proof of Proposition 4.3.13, $\triangleleft_{B_i} = \triangleleft_{\tilde{B}_i}$. Therefore $\triangleleft_{B_1} \subseteq \triangleleft_{B_2}$ and hence $(\gamma_{B_1} L, \bigvee_1) \leq (\gamma_{B_2} L, \bigvee_2)$. □

We now establish the one-to-one correspondence between all full π-compact basis and all π-compactifications of L.

Proposition 4.4.6. *There is a one-to-one correspondence between the set of all full π-compact bases and the set of all π-compactifications of a regular frame L.*

Proof. Let **Full**(L) and $\pi(L)$ be the set of all full π-compact bases of L and the set of all π-compactifications of L, respectively. Define a map

$$f : \mathbf{Full}(L) \to \pi(L)$$

by

$$f(B) = (\gamma_B L, \bigvee),$$

for each π-compact basis $B \in \mathbf{Full}(L)$, where $\gamma_B L$ is the π-compactification associated with the strong inclusion $\triangleleft_B$ defined by $a \triangleleft_B b$ if and only if $a \prec c \prec b$, for some $c \in B$.

We show that $f : \mathbf{Full}(L) \to \pi(L)$ is well-defined. Let $B_1, B_2 \in \mathbf{Full}(L)$. Suppose $B_1 = B_2$, then $\triangleleft_{B_1} = \triangleleft_{B_2}$, and therefore $(\gamma_{B_1} L, \bigvee_1) \cong (\gamma_{B_2} L, \bigvee_2)$. So, the π-compactifications associated with B_1 and B_2 are essentially the same.

Now, we show that $f : \mathbf{Full}(L) \to \pi(L)$ is one-to-one. Suppose $f(B_1) = f(B_2)$, where $B_1, B_2 \in \mathbf{Full}(L)$. Then $(\gamma_{B_1} L, \bigvee_1) = (\gamma_{B_2} L, \bigvee_2)$. Since $B_1, B_2 \in \mathbf{Full}(L)$, then $B_1 = \tilde{B_1}$ and $B_2 = \tilde{B_2}$. Hence, $\triangleleft_{B_i} = \triangleleft_{\tilde{B_i}}$, and therefore $(\gamma_{B_i} L, \bigvee) \cong (\gamma_{\tilde{B_i}} L, \bigvee)$, for each i, where $i = 1, 2$. We now have $(\gamma_{\tilde{B_1}} L, \bigvee) \cong (\gamma_{B_1} L, \bigvee_1) = (\gamma_{B_2} L, \bigvee_2) \cong (\gamma_{\tilde{B_2}} L, \bigvee)$, i.e., $(\gamma_{\tilde{B_1}} L, \bigvee) \cong (\gamma_{\tilde{B_2}} L, \bigvee)$. This implies that $(\gamma_{\tilde{B_1}} L, \bigvee_1) \leq (\gamma_{\tilde{B_2}} L, \bigvee_2)$ and $(\gamma_{\tilde{B_2}} L, \bigvee_2) \leq (\gamma_{\tilde{B_1}} L, \bigvee_1)$. By Proposition 4.4.5, we have that $\tilde{B_1} \subseteq \tilde{B_2}$ and $\tilde{B_2} \subseteq \tilde{B_1}$, respectively. That is, $B_1 = B_2$. Thus, $\tilde{B_1} = \tilde{B_2}$. □

We close this chapter by giving an example of a full π-compact basis; the Freudenthal base. The compactification corresponding to this base is studied in the next chapter.

Example 4.4.7. Let $B_L = \{b \in L : \uparrow(b \vee b^*) \text{ is compact}\}$ be the Freudenthal base for a rim-compact frame L. Then B_L is a π-compact basis (see Remark

4.4 in [3]). We always have $B_L \subseteq \tilde{B}_L$, by Lemma 4.3.5. We show that $B_L \supseteq \tilde{B}_L$. If $b \in \tilde{B}_L$, then $\uparrow(b \vee b^*)$ is compact, since $\tilde{B}_L$ is a π-compact basis. That is, $b \in B_L$. Thus $B_L = \tilde{B}_L$.

Chapter 5

The Freudenthal Compactification of Frames

In 1951, the *Freudenthal compactification* was introduced by Freudenthal in [11] where he gave a method of obtaining a compactification X^* of a rim-compact space X such that $X^* \setminus X$ is zero-dimensionally embedded in X^*. Moreover, he showed that if X is rim-compact and second countable, then X^* is maximal with the property that $X^* \setminus X$ is zero-dimensionally embedded in X^*.

In 1952, Morita [20] gave another method of obtaining the Freudenthal compactification of a rim-compact space. The outline of his method is as follows: For a rim-compact Hausdorff space X, let $\mathfrak{M}$ be a set of all coverings with compact boundaries. Then X is a completely regular space and $\mathfrak{M}$ is a uniformity compatible with the topology on X. Let γX be the completion of the uniform space $(X, \mathfrak{M})$. Then γX is a compact Hausdorff space containing X as a dense subset. This unique compactification γX of X is such that the annex $\gamma X \setminus X$ is zero-dimensionally embedded in γX.

Skljarenko [23] developed the Freudenthal compactification of a rim-compact space X via proximities defined using the elements of π-compact basis $\mathfrak{B}$. More specifically, the set $\mathfrak{B}_{\mathfrak{X}}$ consisting of all open sets of X with compact boundaries is the π-compact basis that gives the Freudenthal compactification. The π-compactification μX corresponding to the proximity defined on X using the elements of $\mathfrak{B}_{\mathfrak{X}}$ is such that $\mu X \backslash X$ is zero-dimensionally embedded in μX. The compactification μX is also maximal with this property. He further show that μX is the minimal perfect compactification of X.

This compactification has been studied extensively in the literature (see Dickman [9], Dickman and McCoy [10], etc.).

Following Skljarenko [23], Baboolal [3] defined the Freudenthal compactification for the class of rim-compact frames. He showed that this compactification is perfect and that the remainder of a rim-compact frame in it is zero-dimensional.

In this chapter we look at some properties of the Freudenthal compactification, give some characterizations and prove that it is the minimal perfect compactification for the class of rim-compact frames. We also show that the category of compact regular frames is coreflective in the category of rim-compact frames, the Freudenthal compactification being the coreflection where the maps are what we call the *F-maps* (to be defined later).

5.1 Some characterizations of the Freudenthal compactification of a rim-compact frame

Let us recall the definition of a rim-perfect compactification of a space (see Dickman and McCoy [10]).

Definition 5.1.1. A compactification Y of a space X is called **rim-perfect** provided that whenever U and V are open sets in X with compact frontiers and $cl_X(U) \cap cl_X(V) = \emptyset$, then $cl_Y(U) \cap cl_Y(V) = \emptyset$.

Let γX be the Freudenthal compactification of a rim-compact space X. Morita has characterized γX as follows.

Theorem 5.1.2. *[20, Theorem 1 and Theorem 2] Let X be a rim-compact space. Then γX is the topologically unique Hausdorff compactification with the following properties:*

1. *For any point $x \in \gamma X$ and any neighbourhood U of x, there exists an open set V of γX such that $x \in V \subseteq U$ and $Fr_{\gamma X}(V) \subseteq X$.*
2. *Any two disjoint closed sets A, B of X with compact frontiers have disjoint closures in γX.*

We recall the formal definition of the Freudenthal compactification for frames due to Baboolal [3].

Definition 5.1.3. Let L be a rim-compact frame and take

$$B_L = \{u \in L : \uparrow(u \vee u^*) \text{ is compact}\}.$$

The compactification $(\gamma L, \bigvee)$ corresponding to the strong inclusion $\lhd_L$ defined by: $a \lhd_L b$ if and only if there exists $c \in B_L$ such that $a \prec c \prec b$ is called the **Freudenthal compactification** of L and B_L is called the **Freudenthal base**.

In the Proposition 5.1.2 above, note that (2) implies that γX is rim-perfect. We prove the frame analogue of this characterization below.

Proposition 5.1.4. *Let γL be the Freudenthal compactification of a rim-compact frame L and $k : L \to \gamma L$ be its right adjoint. Then γL is the unique compactification satisfying the following properties:*

1. *γL has $B_\gamma = \{J \in \gamma L : \uparrow(J \vee J^*) \cong \uparrow \bigvee(J \vee J^*)\}$ as a basis.*

2. *If $a \vee b = e$ in L with $\uparrow(a \vee a^*)$ and $\uparrow(b \vee b^*)$ compact, then $k(a) \vee k(b) = L$.*

Proof. 1. Let $B_L = \{b \in L : \uparrow(b \vee b^*) \text{ is compact}\}$ be the Freudenthal base of L. Then $k(B_L) \subseteq B_\gamma$. By Proposition 4.2.3, $k(B_L)$ is a basis for γL. Thus, B_γ is a basis for γL.

2. Suppose that $a \vee b = e$ in L with $\uparrow(a \vee a^*)$ and $\uparrow(b \vee b^*)$ compact and let $x \in L$. Then $a, b \in B_L$. Thus, by Lemma 3.4.6, we can find $s, t \in B_L$ such that $s \prec a, t \prec b$ and $s \vee t = e$. We can do this process again to obtain $s', t' \in B_L$ such that $s' \prec s \prec a, t' \prec t \prec b$ and $s' \vee t' = e$. Therefore $s' \lhd_L a$ and $t' \lhd_L b$. Now, $x = x \wedge e = x \wedge (s' \vee t') = (x \wedge s') \vee (x \wedge t')$ and $x \wedge s' \leq s' \lhd_L a \Rightarrow x \wedge s' \lhd_L a \Rightarrow x \wedge s' \in k(a)$. Similarly, $x \wedge t' \in k(b)$. Hence $x \in k(a) \vee k(b)$ and so $L \subseteq k(a) \vee k(b)$. The other containment is trivial, therefore $k(a) \vee k(b) = L$.

For uniqueness, suppose (M, h) is a compactifiction of L, with the right adjoint $r : L \to M$, satisfying the following properties:

1. M has $\mathcal{B} = \{a \in M : \uparrow(a \vee a^*) \cong \uparrow h(a \vee a^*)\}$ as a basis.

2. If $a \vee b = e$ in L with $\uparrow(a \vee a^*)$ and $\uparrow(b \vee b^*)$ compact, then $r(a) \vee r(b) = e$.

We need to show that $(\gamma L, \bigvee) \cong (M, h)$. Let $\lhd$ be the strong inclusion associated with (M, h). We show that $\lhd = \lhd_L$. Let $x, y \in L$ and suppose that $x \triangleleft y$. Then $x \triangleleft z \triangleleft y$ for some $z \in L$. That is, $r(x) \prec r(z) \prec r(y)$. Therefore $r(x)^* \vee r(z) = e$. Since $\mathcal{B}$ is a basis for M, we can write

$r(z) = \bigvee t(t \in \mathcal{B}, t \leq r(z))$. Thus $r(x)^* \vee \bigvee t(t \in \mathcal{B}, t \leq r(z)) = e$. We have $r(x)^* \vee \bigvee_{i=1}^{n} t_i = e$, by compactness of M, where $t_i \in \mathcal{B}, t_i \leq r(z)$.

We show that $\bigvee_{i=1}^{n} t_i \in \mathcal{B}$. If $t_1, t_2 \in \mathcal{B}$ then $\uparrow(t_i \vee t_i^*) \subseteq r(L)$, $i = 1, 2$ by Proposition 4.3.7 and Remark 4.3.8. Hence, to show that $t_1 \vee t_2 \in \mathcal{B}$, it is enough to show that $\uparrow((t_1 \vee t_2) \vee (t_1 \vee t_2)^*) \subseteq r(L)$. Now

$$(t_1 \vee t_2) \vee (t_1 \vee t_2)^* = [(t_1 \vee t_2) \vee t_1^*] \wedge [(t_1 \vee t_2) \vee t_2^*]$$

and therefore

$$\begin{aligned}
\uparrow((t_1 \vee t_2) \vee (t_1 \vee t_2)^*) &= \uparrow[(t_1 \vee t_2) \vee t_1^*] \vee \uparrow[(t_1 \vee t_2) \vee t_2^*] \\
&\subseteq \uparrow(t_1 \vee t_1^*) \vee \uparrow(t_2 \vee t_2^*) \\
&\subseteq r(L)
\end{aligned}$$

That is, $t_1 \vee t_2 \in \mathcal{B}$. Thus, finite joins of elements in $\mathcal{B}$ are again in $\mathcal{B}$.

Therefore $r(x)^* \vee t = e$, where $t \in \mathcal{B}$ and $t \leq r(z)$. We now have $r(x) \prec t \leq r(z) \prec r(y)$. Therefore $hr(x) \prec h(t) \leq hr(z) \prec hr(y)$, i.e., $x \prec h(t) \prec y$. Since $t \in \mathcal{B}$, then $\uparrow(t \vee t^*) \cong \uparrow(h(t) \vee h(t)^*)$, and therefore $\uparrow(h(t) \vee h(t)^*)$ is compact (being isomorphic to a closed compact sublocale of M). So $h(t) \in B_L$ and therefore $x \triangleleft_L y$.

Now, suppose $x \triangleleft_L y$. Then $x \triangleleft_L z \triangleleft_L y$ for some $z \in B_L$. Hence $x \prec b \prec z \prec c \prec y$ for some $b, c \in B_L$. We note that $b^* \vee c = e$, also $\uparrow(b^* \vee b^{**})$ and $\uparrow(c \vee c^*)$ are compact. By the second property of (M, h), we have $r(b^*) \vee r(c) = e$. Thus $r(b)^* \vee r(c) = e$, since h is dense and onto. That is, $r(b) \prec r(c)$, hence $b \triangleleft c$. We now have $x \leq b \triangleleft c \leq y$, which implies that $x \triangleleft y$. Consequently, $(\gamma L, \bigvee) \cong (M, h)$, as desired. $\square$

Hunsaker and Naimpally [13] characterized the Freudenthal compactification γX of a rim-compact space X using the proximity associated with it as follows.

Theorem 5.1.5. *[13, Theorem 2] Let X be a rim-compact Tychonoff space and let δ be the proximity on X induced by γX. Then $A\bar{\delta}B$ if and only if A and B are contained in two disjoint closed sets with compact frontiers.*

The analogous result for frames does indeed hold true as shown below.

Proposition 5.1.6. *Let γL be the Freudenthal compactification of a rim-compact frame L and $\lhd_L$ be its associated strong inclusion. Then, for any $a, b \in L$, the following holds true; $a \lhd_L b$ if and only if there exists $c, d \in L$ with $\uparrow(d \vee d^*)$ compact such that $a \wedge c = 0, d \leq b$ and $c \vee d = e$.*

Proof. Suppose $a \lhd_L b$. Then $a \prec t \prec b$, for some $t \in B_L$. Therefore $\uparrow(t \vee t^*)$ is compact. Take $c = a^*$ and $d = t$. Hence $\uparrow(d \vee d^*) = \uparrow(t \vee t^*)$ is compact. Then $a \wedge c = a \wedge a^* = 0$ and $d = t \prec b$. The latter implies that $d \leq b$. Also, $c \vee d = a^* \vee t = e$.

Conversely, suppose that there exists $c, d \in L$, such that $a \wedge c = 0$, $d \leq b$ and $c \vee d = e$, where $\uparrow(d \vee d^*)$ is compact. Therefore $d \in B_L$. Now $c \vee d = e$ implies that there exists $v \in B_L$ such that $c \vee v = e$ and $v \prec d$, by Lemma 3.4.6. Now $a \wedge c = 0$ and $v \vee c = e$ implies $a \prec v$. We now have $a \prec v \prec d \leq b$, that is $a \lhd_L d \leq b$. Thus $a \lhd_L b$. □

The following result is due to Baboolal [3].

Proposition 5.1.7. *[3, Proposition 4.10] The Freudenthal compactification γL of a rim-compact frame L is perfect.*

In [23], Skljarenko shows that the maximal π-compactification (i.e., the Freudenthal compactification) of a rim-compact space X coincides with the minimal perfect compactification of X. We prove this in the frame setting.

Proposition 5.1.8. *The Freudenthal compactification γL of a rim-compact frame L is the minimal perfect compactification, up to isomorphism.*

Proof. The Freudenthal compactification γL of a rim-compact frame L is perfect, by Proposition 5.1.7. Let (M, h) be a perfect compactification of L and $\triangleleft$ be its associated strong inclusion. Also let $r : L \to M$ be the right adjoint of $h : M \to L$. We need to show that $(\gamma L, \bigvee) \leq (M, h)$. It is enough to show that $\triangleleft_L \subseteq \triangleleft$, where $\triangleleft_L$ is the strong inclusion associated with γL.

Suppose $a \triangleleft_L b$, where $a, b \in L$. Then $a \prec c \prec b$, for some $c \in B_L$. Then $\uparrow(c \vee c^*)$ is compact. Now $e = c^* \vee b = c^* \vee \bigvee x(x \triangleleft b) = c \vee c^* \vee \bigvee x(x \triangleleft b) = \bigvee(c \vee c^* \vee x)(x \triangleleft b)$. Since $\uparrow(c \vee c^*)$ is compact, we can find $x_1, \ldots, x_n$ with $x_i \triangleleft b$, such that $e = c \vee c^* \vee x_1 \vee \cdots \vee x_n$. Let $z = x_1 \vee \cdots \vee x_n$. Then $z \triangleleft b$ and $e = c \vee c^* \vee z$. We note that $z^* \leq c \vee c^*$. Hence $r(z^*) \leq r(c \vee c^*) = r(c) \vee r(c^*)$, the latter follows from the perfectness of (M, h). We have

$$\begin{aligned}
z \triangleleft b &\Rightarrow r(z) \prec r(b) \\
&\Rightarrow r(z)^* \vee r(b) = e \\
&\Rightarrow r(z^*) \vee r(b) = e \\
&\Rightarrow r(c) \vee r(c^*) \vee r(b) = e \qquad \text{(since } r(z^*) \leq r(c) \vee r(c^*)\text{)} \\
&\Rightarrow r(c^*) \vee r(b) = e \qquad \text{(since } c \leq b\text{)} \\
&\Rightarrow r(c)^* \vee r(b) = e \\
&\Rightarrow r(c) \prec r(b)
\end{aligned}$$

$$\Rightarrow \quad r(a) \le r(c) \prec r(b) \qquad\qquad \text{(since } a \le c\text{)}$$

$$\Rightarrow \quad r(a) \prec r(b)$$

$$\Rightarrow \quad a \lhd b.$$

It is trivial that if (M, h) were to be a minimal perfect compactification of L, then $\lhd \subseteq \lhd_L$, and therefore $(M, h) \cong (\gamma L, \bigvee)$. $\square$

The Freudenthal compactification is the maximal π-compactification in the class of rim-compact frames. We record this below.

Proposition 5.1.9. *The Freudenthal compactification γL of a rim-compact frame L is the maximal π-compactification of L.*

Proof. Let B be a π-compact basis of L. Then $B \subseteq B_L$, where B_L is the Freudenthal base. Consequently, $\lhd_B \subseteq \lhd_{B_L}$, where $\lhd_B$ and $\lhd_{B_L}$ are strong inclusions associated with the π-compactifications $\gamma_B L$ and γL, respectively. Hence $(\gamma_B L, \bigvee) \le (\gamma L, \bigvee)$. $\square$

In the set of all compactifications of a rim-compact frame, the following result provides a description of those compactifications which are bigger than the Freudenthal one, with respect to the partial order of compactifications of frames.

Corollary 5.1.10. *Let $(\gamma L, \bigvee)$ be the Freudenthal compactification of a rim-compact frame L. Let (M, h) be any compactification of L. Then $(\gamma L, \bigvee) \le (M, h)$ if and only if (M, h) is perfect with respect to all the elements $u \in L$ such that $\uparrow(u \vee u^*)$ is compact.*

Proof. If $(\gamma L, \bigvee) \leq (M, h)$, then (M, h) is perfect with respect to all the elements of B_L, by Proposition 4.2.1. That is, (M, h) is perfect with respect to all the elements $u \in L$ such that $\uparrow(u \vee u^*)$ is compact.

Conversely, suppose that (M, h) is perfect with respect to all the elements $u \in L$ such that $\uparrow(u \vee u^*)$ is compact. Let $\triangleleft_L$ and $\triangleleft$ be strong inclusions associated with $(\gamma L, \bigvee)$ and (M, h), respectively. Suppose $a \triangleleft_L b$. An argument similar to the one in Proposition 5.1.8 shows that $a \triangleleft b$ and hence $(\gamma L, \bigvee) \leq (M, h)$. □

5.2 The Freudenthal compactification as a coreflection

In [13], Hunsaker and Naimpally in the context of space deals with the problem of realizing Hausdorff compactifications as epireflections, focusing more on the Freudenthal compactification of rim-compact spaces and the Fan-Gottesman Compactification of a Tychonoff space with a normal base. Here we establish the fact that the category *KRFRM* of compact regular frames is coreflective in the category *PROXFRM* of proximal frames where the morphisms are proximal maps. It will follow that the category *KRFRM* is coreflective in the category *RCFRM* of rim-compact frames, the Freudenthal compactification being the coreflection where the *F-maps* are morphisms. The latter is the frame counterpart of the result by Hunsaker and Naimpally.

5.2.1 The category of compact regular frames is coreflective in the category of proximal frames

We need to recall certain well-known results and observe a few facts from the theory of uniform frames and proximal frames.

Definition 5.2.1. Let $\mathfrak{U}$ be a uniformity on a frame L. For any $x, y \in L$, we define $x \triangleleft_{\mathcal{U}} y$ to mean that there exists a uniform cover $A \in \mathfrak{U}$ such that $Ax \leq y$. If $x \triangleleft_{\mathcal{U}} y$, we say that x is **uniformly below** y. An ideal J will be called **uniformly regular** for each $x \in J$, there exists $y \in J$ such that $x \triangleleft_{\mathcal{U}} y$.

In [8], properties of $\triangleleft_{\mathcal{U}}$ which imply that $\triangleleft_{\mathcal{U}}$ is a strong inclusion on L are provided. However, the details of this were first provided by Frith in [12].

Lemma 5.2.2. *[8] If $(L, \mathfrak{U})$ is a uniform frame, then relation $\triangleleft_{\mathcal{U}}$ defined above is a strong inclusion on L.*

Definition 5.2.3.
1. A frame L is called **proximal** if there is a strong inclusion on it. Let $\triangleleft$ be a strong inclusion on L. We shall refer to the pair $(L, \triangleleft)$ as a **proximal frame**.
2. Let $(L, \triangleleft_1)$ and $(M, \triangleleft_2)$ be proximal frames. A function $f : L \to M$ is called a **proximal map** if it is a frame map and $x \triangleleft_1 y \Rightarrow f(x) \triangleleft_2 f(y)$.
3. Let *PROXFRM* be the category of proximal frames and proximal maps as morphisms.

Frith [12] proved Lemma 5.2.2 by providing a detailed proof of the following.

Lemma 5.2.4. *[12, Proposition 4.18] If $(L, \mathfrak{U})$ is a uniform frame, then $(L, \triangleleft_{\mathcal{U}})$ is a proximal frame.*

The following result is well known and included here for the sake of completeness.

Lemma 5.2.5. *If $\triangleleft$ is a strong inclusion on a compact regular frame L, then $\triangleleft = \prec$.*

Proof. We know that $a \triangleleft b$ implies $a \prec b$, by definition. Now suppose $a \prec b$. Then $a \prec \bigvee x(x \triangleleft b) \Rightarrow a^* \vee \bigvee x(x \triangleleft b) = e$. By compactness of L, we can find $x_1, \ldots, x_n$ such that $a^* \vee x_1 \vee x_2 \vee \cdots \vee x_n = e$ where $x_i \triangleleft b$ for each $i = 1, \ldots, n$. We now have $a^* \vee y = e$, where $y = x_1 \vee x_2 \vee \cdots \vee x_n$ and $y \triangleleft b$. Therefore $a \leq y \triangleleft b \Rightarrow a \triangleleft b$. $\square$

An example of a proximal map is given in the following lemma.

Lemma 5.2.6. *Let $(L, \triangleleft)$ be a proximal frame. Let $\bigvee : \theta L \to L$ be the compactification associated with the strong inclusion $\triangleleft$. Then join map $\bigvee : (\theta L, \prec) \to (L, \triangleleft)$ is a proximal map.*

Proof. Suppose $J \prec I$ in θL. We need to show that $\bigvee J \triangleleft \bigvee I$. That is, we need to show that $k(\bigvee J) \prec k(\bigvee I)$, where $k : L \to \theta L$ is the right adjoint of $\bigvee : \theta L \to L$. However, $J \prec I \Rightarrow k(\bigvee J) \prec I$, by Lemma 4.3.16. We also have $I \subseteq k(\bigvee I)$. Therefore $k(\bigvee J) \prec I \subseteq k(\bigvee I)$, that is, $k(\bigvee J) \prec k(\bigvee I)$. $\square$

The following is also due to Frith [12], the proof is provided for the sake of completeness.

Lemma 5.2.7. *[12, Theorem 4.19] Let $(L, \mathfrak{U})$ and $(M, \mathfrak{V})$ be uniform frames. If a frame map $h : (L, \mathfrak{U}) \to (M, \mathfrak{V})$ is a uniform frame map, then $f : (L, \triangleleft_{\mathcal{U}}) \to (M, \triangleleft_{\mathcal{V}})$ is a proximal map.*

Proof. Suppose $x \lhd_{\mathcal{U}} y$ in L, then there exists a uniform cover $A \in \mathfrak{U}$ such that $Ax \leq y$. Then $h(A) = \{h(a) : a \in A\} \in \mathfrak{V}$.

If $h(a) \wedge h(x) \neq 0$ then $h(a \wedge x) \neq 0$ and therefore $a \wedge x \neq 0$. Thus $a \leq Ax \leq y$ and so $h(a) \leq h(y)$, hence $h(A)h(x) \leq h(y)$, i.e., $h(x) \lhd_{\mathcal{V}} h(y)$ in M. □

Lemma 5.2.8. *[12, Theorem 4.20] Let $(L, \lhd)$ be a proximal frame. Suppose that $a \lhd b$ in L. Set $C_a^b = \{a^*, b\}$. Then C_a^b is a cover. Take $\mathcal{U}$ to be the uniformity having $\mu_{\lhd} = \{C_a^b : a \lhd b\}$ as a sub-basis. Then $\mathcal{U}$ is a compatible uniform structure on L. Moreover, $a \lhd b \Leftrightarrow a \lhd_{\mathcal{U}} b$ (i.e., $\lhd = \lhd_{\mathcal{U}}$).*

On the other hand, Banaschewski and Pultr [8] showed the following.

Lemma 5.2.9. *[8, Proposition 1] If $h : (L, \mathfrak{U}) \to (M, \mathfrak{V})$ is any uniform frame map, then the following diagram commutes:*

$$\begin{array}{ccc} \mathfrak{R}L & \xrightarrow{\mathfrak{R}h} & \mathfrak{R}M \\ {\scriptstyle \rho_L}\downarrow & & \downarrow{\scriptstyle \rho_M} \\ L & \xrightarrow{h} & M \end{array}$$

where ρ_L and ρ_M are join maps and $\mathfrak{R}L$ and $\mathfrak{R}M$ are Samuel compactifications of L and M. That is, $\mathfrak{R}L$ and $\mathfrak{R}M$ are sets of all uniformly regular ideals of L and M, respectively. Here, $\mathfrak{R}h(J)$ is an ideal in $\mathfrak{R}L$ generated by $h(J)$. All these maps are uniform maps.

A direct consequence of the lemmas above is captured in the following lemma.

Lemma 5.2.10. *Let $(L, \lhd_1)$ and $(M, \lhd_2)$ be proximal frames, θL and θM be compactifications corresponding to strong inclusions $\lhd_1$ and $\lhd_2$, respectively. Let $\mathfrak{U}$ and $\mathfrak{V}$ be uniformities associated with $\lhd_1$ and $\lhd_2$, via the method in Lemma 5.2.8. Suppose $h : (L, \lhd_1) \to (M, \lhd_2)$ is a proximal map. Then*

$h : (L, \mathfrak{U}) \to (M, \mathfrak{V})$ is uniform frame map. Also $\mathfrak{R}L = \theta L$ (since $\lhd_1 = \lhd_{\mathcal{U}}$) and similarly $\mathfrak{R}M = \theta M$. Furthermore, we have the following commutative diagram where all the maps are proximal maps:

$$\begin{array}{ccc} (\theta L, \prec) & \xrightarrow{\theta h} & (\theta M, \prec) \\ \rho_L \downarrow & & \downarrow \rho_M \\ (L, \lhd_1) & \xrightarrow{h} & (M, \lhd_2) \end{array}$$

where $\theta h(J)$ is an ideal in θM generated by $h(J)$.

We use the fact that in *REGFRM*, dense maps are monic. We can now finally establish the following result which is one of our main goals.

Proposition 5.2.11. *The category of compact regular frames with proximal maps is coreflective in the category of proximal frames with proximal maps.*

Proof. Let $(M, \lhd)$ be a proximal frame. Suppose $h : (L, \prec) \to (M, \lhd)$ is proximal, where L is compact regular. We need a proximal map (simply a frame map, since we dealing with compact regular frames) $g : L \to \theta M$ that makes the following diagram commute:

$$\begin{array}{ccc} & & (M, \lhd) \\ & \nearrow^{h} & \uparrow \rho_M \\ (L, \prec) & \xrightarrow[g]{} & (\theta L, \prec) \end{array}$$

where ρ_M is the join map associated with the compactification θL. However, we do have the following commutative diagram:

$$\begin{array}{ccc} (L, \prec) & \xrightarrow{h} & (M, \lhd) \\ \uparrow \rho_L & & \rho_M \uparrow \\ (\theta L, \prec) & \xrightarrow{\theta h} & (\theta M, \prec) \end{array}$$

We note that $\rho_L : \theta L \to (L, \prec)$ is an isomorphism by Lemma 3.3.2, since it is dense onto homomorphism between compact regular frames. Now, $\rho_M(\theta h) = h\rho_L \Rightarrow h = \rho_M[(\theta h)(\rho_L)^{-1}]$. We have $h = \rho_M g$ where $g = (\theta h)(\rho_L)^{-1}$. This factorization is unique: for if $h = \rho_M g'$ then $\rho_M g = \rho_M g'$ and therefore $g = g'$. The latter follows since $\rho_M : (\theta M, \prec) \to (M, \triangleleft)$ is dense between regular frames and hence monic i.e., left cancellative. □

5.2.2 The Freudenthal compactification and F-maps

A proximal map between two proximal frames, each possessing the Freudenthal compactification, satisfies the properties provided below.

Proposition 5.2.12. *Let $(L, \triangleleft_L)$ and $(M, \triangleleft_M)$ be rim-compact frames, γL and γM be the Freudenthal compactifications associated with the strong inclusions $\triangleleft_L$ and $\triangleleft_M$, respectively. Then, a frame map $f : (L, \triangleleft_L) \to (M, \triangleleft_M)$ is a proximal map if and only if whenever $a \vee b = e$ in L with $\uparrow(a \vee a^*)$ and $\uparrow(b \vee b^*)$ compact, there exists $c, d \in M$ with $\uparrow(c \vee c^*)$ and $\uparrow(d \vee d^*)$ compact such that $c \leq f(a), d \leq f(b)$, and $c \vee d = e$*

Proof. Suppose $f : (L, \triangleleft_L) \to (M, \triangleleft_M)$ is a proximal map. Let $B_L = \{b \in L : \uparrow(b \vee b^*) \,\text{is compact}\}$ and $B_M = \{b \in M : \uparrow(b \vee b^*) \,\text{is compact}\}$ be the Freudenthal bases for L and M, respectively.

Take $a \vee b = e$ with $a, b \in B_L$. Then, by Lemma 3.4.6, there exists $s, t \in B_L$ such that $s \prec a, t \prec b$ and $s \vee t = e$. Since $s \leq s^{**}$, we have $s^{**} \vee t = e$. Hence $s^* \prec t \prec b$. Hence $s^* \triangleleft_L b$, and therefore $f(s^*) \triangleleft_M f(b)$, since f is a proximal map.

We use the interpolating property of the strong inclusion $\triangleleft_M$ to get $x, y \in B_M$ such that $f(s^*) \triangleleft_M x \triangleleft_M y \triangleleft_M f(b)$. Then $x^* \vee y = e$ and $x^* \leq f(s^*)^*$. Hence $x^* \wedge f(s^*) = 0$. Now $s \prec a \Rightarrow s^* \vee a = e \Rightarrow f(s^*) \vee f(a) =$

$e \Rightarrow x^* \wedge [f(s^*) \vee f(a)] = x^* \wedge e \Rightarrow [x^* \wedge f(s^*)] \vee [x^* \wedge f(a)] = x^* \Rightarrow x^* \wedge f(a) = x^* \Rightarrow x^* \leq f(a)$ and $x^* \in B_M$ (since B_M is a π-compact basis) so $\uparrow(x^* \vee x^{**})$ is compact. Also $y \leq f(b)$ and $\uparrow(y \vee y^*)$ is compact. Take $c = x^*$ and $d = y$ and we are done.

Conversely, suppose $a \triangleleft_L b$ in L. We need to show that $f(a) \triangleleft_M f(b)$. Take $x, y \in B_L$ such that $a \triangleleft_L x \triangleleft_L y \triangleleft_L b$. Now $x^* \vee y = e$ and $x^*, y \in B_L$. Therefore there exists $c, d \in B_M$ such that $c \leq f(x^*), d \leq f(y)$ and $c \vee d = e$. Now $c \wedge f(x) \leq f(x^*) \wedge f(x) = f(x^* \wedge x) = f(0) = 0$. Thus, $c \wedge f(x) = 0$. Therefore $c \vee d = e \Rightarrow d \wedge f(x) = f(x) \Rightarrow f(x) \leq d$. We now have $f(x) \leq d \leq f(y)$. Also note that $a \prec x \prec y \prec b$, so $f(a) \prec f(x) \leq d \leq f(y) \prec f(b)$, that is, $f(a) \prec d \prec f(b)$ and $d \in B_M$. Thus $f(a) \triangleleft_M f(b)$. □

In a general setting, we give a special name to frame maps satisfying the conditions in the above proposition.

Definition 5.2.13. Let L and M be frames. We call a frame map $f : L \to M$ an F**-map** if whenever $a \vee b = e$ in L with $\uparrow(a \vee a^*)$ and $\uparrow(b \vee b^*)$ compact, there exists $c, d \in M$ with $\uparrow(c \vee c^*)$ and $\uparrow(d \vee d^*)$ compact such that $c \leq f(a), d \leq f(b)$, and $c \vee d = e$.

Remark 5.2.14. In Proposition 5.2.12, we showed that a frame map $f : (L, \triangleleft_L) \to (M, \triangleleft_M)$ is a proximal map if it is an F-map, where L and M are rim-compact frames having Freudenthal compactifications γL and γM, corresponding to strong inclusions $\triangleleft_L$ and $\triangleleft_M$, respectively.

Lemma 5.2.15. *If L and M are compact regular frames, then every frame map $f : L \to M$ is an F-map.*

Proof. Let $a \vee b = e$ in L. Then $\uparrow(a \vee a^*)$ and $\uparrow(b \vee b^*)$ are closed subblocales of a compact frame L, and so they are both compact. In the definition of an F-map above, take $c = f(a)$ and $d = f(b)$. Then $\uparrow(c \vee c^*)$ and $\uparrow(d \vee d^*)$ are compact since they are closed subblocales of a compact regular frame M. Also $c \vee d = f(a) \vee f(b) = f(a \vee b) = f(e) = e$. Therefore $f : L \to M$ is a frame map if and only if $f : L \to M$ is an F-map in $\underline{KRFRM}$. □

A direct proof that the join map associated with the Freudenthal compactification is an F-map is given below. However, this can also be deduced from Lemma 5.2.6 and Proposition 5.2.12.

Lemma 5.2.16. *Let L be a rim-compact frame, γL be the Freudenthal compactification. Then the join map $\bigvee : \gamma L \to L$ is an F-map.*

Proof. Let $I, J \in \gamma L$ be such that $I \vee J = L$, with $\uparrow(I \vee I^*)$ and $\uparrow(J \vee J^*)$ compact. It is always the case that $\uparrow(I \vee I^*)$ and $\uparrow(J \vee J^*)$ are compact since these are closed subblocales of γL. We need to find $c, d \in B_L$ such that $c \leq \bigvee I, d \leq \bigvee J$ and $c \vee d = e$.

Let $k : L \to \gamma L$ be the right adjoint of $\bigvee : \gamma L \to L$. Since $k(B_L)$ is a basis for γL (by Proposition 4.2.3), then $I = \bigvee_{k(a) \subseteq I} k(a)$ and $J = \bigvee_{k(b) \subseteq J} k(b)$, a and b vary over B_L. Therefore $[\bigvee_{k(a) \subseteq I} k(a)] \vee [\bigvee_{k(b) \subseteq J} k(b)] = L$. By compactness of γL, we have $[\bigvee_{i=1}^{n} k(a_i)] \vee [\bigvee_{j=1}^{m} k(b_j)] = L$, where $a_i, b_j \in B_L$, $k(a_i) \subseteq I$ and $k(b_j) \subseteq J$ for each i and each j, respectively. If we take joins on both sides, we get $[\bigvee_{i=1}^{n} a_i] \vee [\bigvee_{j=1}^{m} b_j] = e$. We note that $k(a_i) \subseteq I \Rightarrow a_i \leq \bigvee I$. Similarly $b_j \leq \bigvee J$. Take $c = \bigvee_{i=1}^{n} a_i$ and $d = \bigvee_{j=1}^{m} b_j$, then $c \leq \bigvee I, d \leq \bigvee J$ and $c, d \in B_L$, also $c \vee d = e$. We also have that $\uparrow(c \vee c^*)$ and $\uparrow(d \vee d^*)$ are compact, since $c, d \in B_L$.

□

We conclude this chapter by proving the following main result.

Proposition 5.2.17. *The category of compact regular frames form a coreflective subcategory of the category of rim-compact frames with morphisms being F-maps. The coreflection is given by taking the Freudenthal compactification, for any rim-compact frame.*

Proof. Let L be a rim-compact frame and γL be the Freudenthal compactification with $\triangleleft_L$ the corresponding strong inclusion. Then $\bigvee : \gamma L \to L$ is an F-map by the above Lemma 5.2.16.

We also know that $\bigvee : (\gamma L, \prec) \to (L, \triangleleft_L)$ is a proximal map, by Lemma 5.2.6. Suppose $f : M \to L$ is any F-map where M is compact regular. Since M is compact regular, then it has only one strong inclusion, namely $\prec$. So, the strong inclusion $\triangleleft_M$ associated with the Freudenthal compactification of M is the same as $\prec$.

Hence, $f : (M, \prec) \to (L, \triangleleft_L)$ is proximal by Proposition 5.2.12 and Definition 5.2.13. So, by Proposition 5.2.11 there exists a frame homomorphism $g : M \to \gamma L$ such $\bigvee g = f$. We therefore have the following commutative diagram:

$$\begin{array}{ccc} & & L \\ & \overset{f}{\nearrow} & \uparrow \bigvee \\ M & \overset{g}{\dashrightarrow} & \gamma L \end{array}$$

Since M and γL are compact regular, then $g : M \to \gamma L$ is an F-map by Lemma 5.2.15. Hence the above diagram is commutative where all the maps involved are F-maps.

The uniqueness of g follows from the fact that M and γL are regular and $\bigvee : \gamma L \to L$ is dense and onto and therefore monic. $\square$

Chapter 6

Perfectness of N-star Compactifications

It is well known that a topological space has a smallest compactification if and only if it is locally compact Hausdorff. Also, if X is a locally compact space, then the frame of open sets $\mathcal{O}X$ is locally compact.

Banaschewski [6] showed that a frame has a smallest compactification if and only if it is regular continuous. Therefore, we think of regular continuous frames as the frame analogue of locally compact Hausdorff spaces. Naturally, we also consider the smallest compactification of a frame as the frame analogue of the Alexandroff one-point compactification of a locally compact Hausdorff space.

Some conditions under which the least compactification of a non-compact regular continuous frame is perfect were given by Baboolal in [1]. Notably, one of the characterizations is that the least compactifications of a non-compact regular continuous frame L is perfect if and only if the remainder of L in any of its compactification is compact and connected. This is the case for spaces as well.

Magill [19] introduced the notion of the N-point compactification for topological spaces as any compactification $\zeta(X)$ such that $\zeta(X) \setminus X$ have exactly N points, where N is a positive integer.

Baboolal [2] introduced N-star compactifications for frames as the analogue of N-point compactifications for topological spaces and showed that the least compactification of a non-compact regular continuous frame is the 1-star compactification. The purpose of this chapter is to characterize the perfectness of the N-star compactifications. We focus mainly on the 2-star compactifications and we make a conjecture that the results could be generalised to any $N > 1$.

We also obtain other important results concerning connectedness. For example, we show that the remainder of a non-compact regular continuous frame in its least compactification is connected.

6.1 Connectedness of the remainder of the least compactification of a non-compact regular continuous frame

We recall from Example 3.3.8 that for a non-compact regular continuous frame L, there is the smallest strong inclusion $\blacktriangleleft$ on L defined by:

$$a \blacktriangleleft b \text{ if and only if } a \prec b \text{ and either } \uparrow a^* \text{ or } \uparrow b \text{ is compact}$$

due to Banaschewski [6]. The compactification corresponding to $\blacktriangleleft$ is the least compactification $\bigvee : \xi L \to L$.

This is the analogue of the Alexandroff one-point compactification for a locally compact non-compact Haudorff space.

We state the characterization by Baboolal [1] of those non-compact regular continuous frames for which the least compactification is perfect in the following theorem. This result serves as the basis for this chapter in the sense that the remainder of this chapter aims at extending the content of this theorem to the case where the compactification at hand is an N-star compactification, where $N > 1$.

Theorem 6.1.1. *[1, Theorem 4.2] The following conditions are equivalent for a non-compact regular continuous frame L.*

1. *The least compactification of L is perfect*

2. *Whenever $\uparrow(u \vee v)$ is compact where $u, v \in L$ and $u \wedge v = 0$, then either $k(u) \vee J = L$ or $k(v) \vee J = L$ where $J \subsetneq L$ is the unique element in ξL such that $\downarrow J \xrightarrow{\bigvee} L$ is an isomorphism, and $k : L \to \xi L$ is the right adjoint of $\bigvee$ defined by $k(a) = \{x \in L : x \blacktriangleleft a\}$.*

3. *Whenever $\uparrow(u \vee v)$ is compact where $u, v \in L$ and $u \wedge v = 0$, then either $\uparrow u$ or $\uparrow v$ is compact.*

4. *For every compactification $h : M \to L$, the remainder of L in it is compact and connected*

Theorem 6.1.1 above suggests that connectedness (and compactness) of the remainder of a compactification is an important aspect when studying perfectness of an N-star compactification. We show that the remainder of a non-compact regular continuous frame in the least compactification is connected. We start by making a few of observations from the literature and also prove a few auxiliary results.

We recall the following from Baboolal [1].

Lemma 6.1.2. *[1, Proposition 3.1] If L is a regular continuous frame then $x \ll e$ if and only if $\uparrow x^*$ is compact.*

For a regular continuous frame, we use Lemma 6.1.2 to show that if x is well below y, then x is strongly below y with respect to the least strong inclusion.

Lemma 6.1.3. *Let L be a non-compact regular continuous frame. If $x \ll y$ then $x \blacktriangleleft y$, where $\blacktriangleleft$ is the least strong inclusion.*

Proof. We start by showing that if $x \ll y$ then $x \prec y$. Take $x, y \in L$ such that $x \ll y$. Then by regularity of L, we have $y = \bigvee a(a \prec y)$. Now $x \ll y$ implies that there exists $a_1, \ldots, a_n \in L$ such that $x \leq a_1 \vee \cdots \vee a_n \prec y$. Thus $x \prec y$. Now, $x \ll y \leq e$ implies that $x \ll e$. Therefore by Lemma 6.1.2, we have that $\uparrow x^*$ is compact. So, $x \prec y$ and $\uparrow x^*$ is compact, i.e., $x \blacktriangleleft y$. $\square$

Banaschewski [6] showed that, up to isomorphism, regular continuous frames are exactly the frames $\downarrow a$, where a is a maximal element of some compact regular frame. He shows this by first proving two of the following the results.

Lemma 6.1.4. *[6, Lemma 3] Given a compact regular frame M and $a \in M$, then the frame $\downarrow a$ is continuous.*

Lemma 6.1.5. *[6, Lemma 4] Given a compact regular frame M and $a \in M$, then $M_a = \{a \in M : x \leq a$ or $x \vee a = e\}$ is a regular subframe of M.*

Banaschewski [6] showed that if a frame has the least compactification then it is regular continuous (the converse is also true). A summary of his argument is given below because we use it in describing the remainder of a non-compact regular continuous in any of its compactifications in a sequel:

Let L be any frame which has a smallest compactification $h : M \to L$, say. If L is compact then h is an isomorphism by Lemma 3.3.2. Therefore L will be regular since M is regular. That L is continuous follows from the fact that for a compact regular frame L and for any $x, y \in L$, we have that $x \ll y$ if and only if $x \prec y$.

If L is not compact, then h is not an isomorphism and therefore h is not codense by Lemma 3.3.2. Thus, there exists $a \in M, a < e$ with the property that $h(a) = e$. The frame homomorphism $M_a \to L$ induced by h is again a homomorphism since $h(M_a) = L$, where M_a is defined in Lemma 6.1.5. Hence $M_a = M$ and therefore $x \leq a$ or $x \vee a = e$ for each $x \in M$. The restriction of h to $\downarrow a \subseteq M_a$ induces the isomorphism $h_a :\downarrow a \to L$. From which we conclude that L is continuous, by Lemma 6.1.4.

Remark 6.1.6. Banaschewski [6] observed that the element $a \in M, a < e$ with the property that $h(a) = e$ in the above discussion is maximal. The uniqueness of this element comes from the fact that $h : M \to L$ is the smallest compactification.

Proposition 6.1.7. *[1, Theorem 2.2] Let $h : M \to L$ be any compactification of a non-compact frame L. Then L is regular continuous if and only if $\downarrow a \cong L$ for some $a \in M$.*

Remark 6.1.8. In analysing the proof of the above theorem, we find that for a non-compact regular continuous frame L and any compactification $h : M \to L$ of L, there exists a unique element $a \in M, a < e$ with the property that $h(a) = e$. Furthermore, this a is expressible as $a = \bigvee r(x)(x \ll e)$, where $r : L \to M$ is the right adjoint of $h : M \to L$.

For a non-compact regular continuous frame L, another characterization of the elements of L which are well below the top element is provided below.

Proposition 6.1.9. *For a non-compact regular continuous frame* L, *let* $J = \bigvee k(y)(y \in L, y \ll e)$ *where* $k : L \to \xi L$, *defined by* $k(a) = \{x \in L : x \blacktriangleleft a\}$, *is the right adjoint of the least compactification* $\bigvee : \xi L \to L$. *Then* $x \ll e$ *if and only if* $x \in J$.

Proof. Suppose that $x \in L$ and $x \ll e$. By the interpolation property of the well below relation, we can find $y \in L$ such that $x \ll y \ll e$. From Lemma 6.1.3, we get that $x \blacktriangleleft y$. Hence $x \in k(y)$ and $y \ll e$. Thus, $x \in J$.

Conversely, suppose that $x \in J$. We note that $J = \bigcup k(y)(y \in L, y \ll e)$ because $\bigvee k(y)(y \in L, y \ll e)$ is a directed join. This is so since each $k(y)$ is an ideal of L. Thus, there exists $y \in L$ with $y \ll e$ such that $x \in k(y)$. Hence $x \blacktriangleleft y \ll e$, i.e., $x \ll e$. □

In Definition 4.2.7, for any compactification $h : M \to L$ of L (where L is any regular frame), the remainder of L in this compactification is defined to be the quotient frame M/Θ, where $\Theta = (\ker h)^*$ is the pseudocomplement of $\ker h$ in the congruence lattice $\mathcal{C}M$. We need to get a much clearer description for the remainder of a non-compact regular continuous frame in any of its compactifications.

For any frame M, and any $a \in M$ it is easy to check that $h : M \to \downarrow a$ defined by $x \mapsto x \wedge a$ is a frame homomorphism with $\ker h = \Delta_a$. Likewise, $f : M \to \uparrow a$ defined by $x \mapsto x \vee a$ is a frame homomorphism with $\ker f = \nabla_a$. Therefore, $M/\Delta_a \cong \downarrow a$ and $M/\nabla_a \cong \uparrow a$, by Corollary 2.7.8. We note that $(\Delta_a)^* = \nabla_a$ and $\nabla_a^* = \Delta_a$ in $\mathcal{C}M$. All of these facts can be found in Banaschewski [5].

Proposition 6.1.10. *Let* L *be a non-compact regular continuous frame and* $h : M \to L$ *be any compactification. Then, the remainder of* L *in the compactification* $h : M \to L$ *is isomorphic to* $\uparrow a$, *for some unique* $a \in M$.

Proof. We recall from Proposition 6.1.7 and Remark 6.1.8 that we have a unique element $a_L \in M$, $a_L < e$ with the property that $h(a_L) = e$ and $\downarrow a_L \cong L$ (via h).

We claim that $\ker h = \Delta_{a_L}$. To show this, take $(x, y) \in \ker h$. Therefore $h(x) = h(y)$. Hence $h(x) \wedge h(a_L) = h(y) \wedge h(a_L)$. Thus $h(x \wedge a_L) = h(y \wedge a_L)$. Since $x \wedge a_L, y \wedge a_L \in \downarrow a_L$ and $\downarrow a_L \cong L$ via h, then $x \wedge a_L = y \wedge a_L$. Therefore, $(x, y) \in \Delta_{a_L}$.

Now, take $(x, y) \in \Delta_{a_L}$. Then $x \wedge a_L = y \wedge a_L$, and so $h(x \wedge a_L) = h(y \wedge a_L)$. This implies that $h(x) \wedge h(a_L) = h(y) \wedge h(a_L)$, and therefore $h(x) = h(y)$ since $h(a_L) = e$.

Now, let $\Theta = (\ker h)^*$. The remainder of L in $h : M \to L$ is $M/\Theta = M/(\ker h)^* = M/\Delta_{a_L}^* = M/\nabla_{a_L}$ and M/∇_{a_L} isomorphic to $\uparrow a_L$, by Corollary 2.7.8. $\square$

Motivated by the result above, we define the remainder of a non-compact regular continuous frame L in any compactification $h : M \to L$ to be $\uparrow a_L$.

Let us now recall the definition of a connected element in a frame.

Definition 6.1.11. Let L be a frame. An element $c \in L$ is **connected** if whenever $c = a \vee b$ and $a \wedge b = 0$, then either $a = 0$ or $b = 0$. A frame L is connected if the top element is connected and L is **locally connected** if L has a basis of connected elements.

A simple, yet very useful, observation about connected elements is proved below.

Lemma 6.1.12. *Let $c \in L$ be connected and $\{v_1, v_2, v_3\}$ be a pairwise disjoint set of elements of L such that $v_1 \vee v_2 \vee v_3 = e$. Then $c \leq v_i$ for some $i \in \{1, 2, 3\}$.*

Proof. Since $v_1 \vee v_2 \vee v_3 = e$ then $c = (c \wedge v_1) \vee (c \wedge v_2) \vee (c \vee v_3) = (c \wedge v_1) \vee [c \wedge (v_2 \vee v_3)]$. Now, $v_i \wedge v_j = 0$ for $i \neq j$ implies that $(c \wedge v_1) \wedge [c \wedge (v_2 \vee v_3)] = 0$. Therefore, by connectedness of c, we have that either $c \wedge v_1 = 0$ or $c \wedge (v_2 \vee v_3) = 0$.

If $c \wedge v_1 \neq 0$, then $c \wedge (v_2 \vee v_3) = 0$. Hence, $c = c \wedge v_1$ and therefore $c \leq v_1$.

On the other hand, if $c \wedge (v_2 \vee v_3) \neq 0$ then $c \wedge v_1 = 0$. Therefore $c = c \wedge (v_2 \vee v_3) = (c \wedge v_2) \vee (c \wedge v_3)$. So, again by connectedness of c, either $c \wedge v_2 = 0$ or $c \wedge v_3 = 0$. In either case, we will get that $c \leq v_i$, for some $i = 2, 3$. □

In general, the lemma above holds true for any set $\{v_1, v_2, \ldots, v_n\}$ that consists of pairwise disjoint elements such that $v_1 \vee v_2 \vee \cdots \vee v_n = e$.

We are now ready to prove the connectedness of the remainder of a non-compact regular continuous frame in its least compactification.

Proposition 6.1.13. *Let $\bigvee : \xi L \to L$ be the least compactification of a non-compact regular continuous frame L and let $J \in \xi L$ be the unique element such that $\bigvee : \downarrow J \to L$ is an isomorphism. Then $\uparrow J$ is a connected frame. That is, the remainder of L in $\bigvee : \xi L \to L$ is connected.*

Proof. Suppose $L = I \vee K$ with $I \cap K = J$, where $I, K \in \xi L$. We must show that either $I = J$ or $K = J$. Suppose $I \neq J$, we show that $K = J$. Trivially, $J \subseteq K$.

Now, $J \subsetneq I$. Taking joins on both sides of the equation $L = I \vee K$, we get that $e = i \vee k$, for some $i \in I$ and some $k \in K$.

We show that $i \notin J$. Suppose that $i \in J$, then for any $x \in I$, we would have $x = (x \wedge i) \vee (x \wedge k) \in J \vee (I \cap K) = J \vee J = J$, i.e., $I \subseteq J$, which contradicts $J \subsetneq I$. Therefore $i \notin J$.

We claim that $k \in J$. To prove this, we first note that K is a strongly regular ideal relative to the smallest strong inclusion $\blacktriangleleft$. Therefore $k \in K$ implies that there exists $z \in K$ such that $k \blacktriangleleft z$. Similarly, we can get $l \in I$ such that $i \blacktriangleleft l$. Applying Proposition 6.1.9 and the fact that $i \notin J$, we get that i is not well below e. Therefore, by Lemma 6.1.2, $\uparrow i^*$ is not compact. Since $i \blacktriangleleft l$, we must have that $\uparrow l$ is compact.

Now, if $\uparrow z$ is also compact, then $(\uparrow z) \vee (\uparrow l) = \uparrow (z \wedge l)$ is compact. We note that $z \wedge l \in I \cap K = J$. Therefore, $\uparrow (z \wedge l)^*$ is compact, by Lemma 6.1.2. We therefore have that $[\uparrow (z \wedge l)] \vee [\uparrow (z \wedge l)^*]$ is compact. That is, $\uparrow [(i \wedge l) \wedge (i \wedge l)^*] = \uparrow 0 = L$ is compact, a contradiction. Therefore, $\uparrow z$ is not compact.

We now have that $k \blacktriangleleft z$ and $\uparrow z$ is not compact, from which we conclude that $\uparrow k^*$ is compact. Hence $k \ll e$ by Lemma 6.1.2, therefore $k \in J$ by Proposition 6.1.9.

So, if we take an arbitrary $a \in K$, then we have $a = (a \wedge i) \vee (a \wedge k) \in (I \cap K) \vee J = J \vee J = J$. That is, $K \subseteq J$, and hence $K = J$.

Thus $\uparrow J$ is connected as desired. $\square$

6.2 N-star compactifications of frames and their perfectness

The following was provided in Example 3.3.8 and it is presented formally here.

Definition 6.2.1. Let L be a frame. For any positive integer N, an $N-$***star*** is defined to be a collection of N mutually disjoint elements of L, say $\{u_1, u_2, \ldots, u_N\}$, such that $\uparrow(u_1 \vee u_2 \vee \cdots \vee u_N)$ is compact, while for each

i, $1 \leq i \leq N$, $\uparrow(u_1 \vee u_2 \vee \cdots \vee u_{i-1} \vee u_{i+1} \cdots \vee u_N)$ is not compact. In the case where $N = 1$, the latter is interpreted to mean that L is not compact.

Proposition 6.2.2. *[2, Lemma 3.4] Let L be a regular continuous frame and suppose that $\alpha_N = \{u_1, u_2, \ldots, u_N\}$ is an $N-$star of L. Let*

$$N_i = \{x \in L : \uparrow(u_1 \vee u_2 \vee \cdots \vee u_{i-1} \vee x \vee u_{i+1} \cdots \vee u_N) \textit{ is compact}\}$$

for each $i = 1, \ldots, N$. Define a relation $\blacktriangleleft_N$ by: $a \blacktriangleleft_N b$ if and only if $a \prec b$ and for each i, either $a^* \in N_i$ or $b \in N_i$. Then the relation $\blacktriangleleft_N$ is a strong inclusion.

Remark 6.2.3. Since any $N-$star α_N of L is associated with a strong inclusion $\blacktriangleleft_N$, we can then associate α_N with a compactification $\bigvee : \alpha_N L \to L$ corresponding to $\blacktriangleleft_N$.

Definition 6.2.4. We call the compactification $\bigvee : \alpha_N L \to L$ corresponding to the strong inclusion $\blacktriangleleft_N$ defined in Lemma 6.2.2 an $N-$**star compactification** of L, and any compactification isomorphic with this one shall be called such.

6.2.1 More on the perfectness of the least compactification of a non-compact regular continuous frame

A neat argument that shows that the least compactification of a non-compact regular continuous frame is an example of an N-compactification was provided by Baboolal in [2]. We state the result below.

Proposition 6.2.5. *[2, Remark 3.12 and Proposition 3.13] Let L be a non-compact regular continuous frame. Then the least compactification $\bigvee : \xi L \to L$, of L is an N-star compactification, for $N = 1$. Moreover, L has a unique 1-star compactification, i.e., the least compactification of L.*

Given a non-compact regular continuous frame, if the least compactification is perfect then it is the only $N-$star compactification. We prove this below.

Proposition 6.2.6. *Let L be a non-compact regular continuous frame. Then the least compactification, $\bigvee : \xi L \to L$, of L is perfect if and only L has no $N-$star compactification for any $N > 1$.*

Proof. Assume that $\bigvee : \xi L \to L$ is perfect. Suppose to the contrary that L has an $N-$star compactification for some $N > 1$. Let $\alpha_N = \{u_1, u_2, \ldots, u_N\}$ be an $N-$star of L. Then $\uparrow[u_1 \vee (u_2 \vee \cdots \vee u_N)]$ is compact and
$u_1 \wedge (u_2 \vee \cdots \vee u_N) = 0$.

Thus, by Theorem 6.1.1, either $\uparrow u_1$ or $\uparrow (u_2 \vee \cdots \vee u_N)$ is compact. However, the latter contradicts the fact that $\alpha_N = \{u_1, u_2, \ldots, u_N\}$ is an $N-$star of L. Also, if $\uparrow u_1$ is compact, then so is $\uparrow(u_1 \vee u_2 \vee \cdots \vee u_{N-1})$. We have a contradiction again. Therefore L has no $N-$star compactification for any $N > 1$.

Conversely, suppose that L has no $N-$star compactification for any $N > 1$. We need to show that the least compactification of L is perfect. We use Theorem 6.1.1. Take $u, v \in L$ with $u \wedge v = 0$ and suppose that $\uparrow(u \vee v)$ is compact. We need to show that either $\uparrow u$ or $\uparrow v$ is compact.

If both $\uparrow u$ and $\uparrow v$ are compact, then $(\uparrow u) \vee (\uparrow v) =\uparrow(u \wedge v) =\uparrow 0 = L$ is compact, a contradiction.

If both $\uparrow u$ and $\uparrow v$ are not compact, then $\alpha_2 = \{u, v\}$ would be a 2-star of L. So L has a 2-star compactification, a contradiction. Thus, either $\uparrow u$ or $\uparrow v$ is compact. Therefore $\bigvee : \xi L \to L$ of is perfect by Theorem 6.1.1. $\square$

6.2.2 The 2-star compactification of a non-compact regular continuous frame and its perfectness

For the remainder of this chapter, we shall focus on the 2-star compactification a non-compact regular continuous frame. One of the main results that we obtain is that a 2-star compactification of a non-compact regular continuous frame is perfect if and only if it is the only 2-star compactification. To prove this, we need a some auxiliary results.

The following results is due to Baboolal [1].

Lemma 6.2.7. *[1, Lemma 3.5] Let* $\bigvee : \xi L \to L$ *be the least compactification of a regular continuous frame* L *with the right adjoint* $k : L \to \xi L$ *defined by* $k(a) = \{x \in L : x \blacktriangleleft a\}$*. Let* $J = \bigvee k(y)(y \in L, y \ll e)$*. If* $w \in L$*, then* $\uparrow w$ *is compact if and only if* $k(w) \vee J = L$*.*

Remark 6.2.8. From the proof of Lemma 6.2.7 it can be deduced that $\uparrow w$ compact implies $k(w) \vee J = L$ applies to any strong inclusion of a regular continuous frame L. That is, if $\triangleleft$ is any strong inclusion on L and $\bigvee : \theta L \to L$ is the compactification corresponding to $\triangleleft$ and $k : L \to \theta L$ defined by $k(a) = \{x \in L : x \triangleleft a\}$ is its right adjoint, then $\uparrow w$ compact implies $k(w) \vee J = L$.

One of the important results from Baboolal [1] is that the remainder of a regular continuous frame in any of its 2-star compactifications is disconnected. We state this below.

Proposition 6.2.9. *[1, Proposition 3.7] Let* L *be a regular continuous frame. Suppose* L *has elements* u *and* v *with the property that* $u \wedge v = 0$*,* $\uparrow (u \vee v)$ *is compact, but neither* $\uparrow u$ *nor* $\uparrow v$ *is compact, i.e.,* $\alpha_2 = \{u, v\}$ *is a 2-star of* L*. Let* $N_1 = \{x \in L : \uparrow (x \vee u)$ *is compact*$\}$ *and*

$N_2 = \{x \in L : \uparrow (x \vee v)$ is compact$\}$. Let $\blacktriangleleft_2$ be the strong inclusion given by: $a \blacktriangleleft_2 b$ if and only if $a \prec b$ and for each $i \in \{1, 2\}$, either $a^* \in N_i$ or $b \in N_i$. Then the compactification $\bigvee : \alpha_2 L \to L$ corresponding to $\blacktriangleleft_2$ is such that the remainder of L in it is disconnected.

Remark 6.2.10. In the proof of the result stated above, Baboolal [1] showed that $k_2(u \vee v) = k_2(u) \vee k_2(v)$ where $k_2 : L \to \alpha_2 L$ is the right adjoint of the join map $\bigvee : \alpha_2 L \to L$, and k_2 is defined by $k_2(a) = \{x \in L : x \blacktriangleleft_2 a\}$.

Towards showing that the relation $\blacktriangleleft_2$ is a strong inclusion where $\blacktriangleleft_2$ is defined by $a \blacktriangleleft_2 b$ if and only if $a \prec b$ and for each $i \in \{1, 2\}$, either $a^* \in N_i$ or $b \in N_i$, Baboolal [1] also observed the following:

Lemma 6.2.11. *[1, Lemma 3.2 and Lemma 3.3] Let L be a regular continuous frame. Suppose that $\alpha_2 = \{u, v\}$ is a 2-star of L. Then $N_1 = \{x \in L : \uparrow (x \vee u)$ is compact$\}$ and $N_2 = \{x \in L : \uparrow (x \vee v)$ is compact$\}$ are regular proper filters of L. Moreover, $\uparrow x$ is compact if and only if $x \in N_1 \cap N_2$.*

The following can be viewed as a generalisation of Proposition 6.1.9 to 2-star case.

Proposition 6.2.12. *Let $\alpha_2 = \{u, v\}$ be a 2-star for a non-compact regular continuous frame L. Let $J = \bigvee k_2(y)(y \in L, y \ll e)$, where $k_2 : L \to \alpha_2 L$, defined by $k_2(a) = \{x \in L : x \blacktriangleleft_2 a\}$, is the right adjoint of the 2-star compactification $\bigvee : \alpha_2 L \to L$. For $x \in L$, $x \ll e$ if and only $x \in J$.*

Proof. Suppose $x \ll e$, therefore $\uparrow x^*$ is compact by Lemma 6.1.2. Moreover, $x \ll e$ implies that we can find $y \in L$ such that $x \ll y \ll e$. Now, by regularity of L, $y = \bigvee a(a \prec y)$. Hence $x \le a_1 \vee a_2 \vee \cdots a_n$, with $a_i \prec y$, for each $i = 1, \ldots, n$. So, $x \prec y$.

Now, since $\uparrow x^*$ is compact, then both $\uparrow (x^* \vee u)$ and $\uparrow (x^* \vee v)$ are compact. Therefore $x^* \in N_1 = \{x \in L : \uparrow (x \vee u) \text{ is compact}\}$ and $x^* \in N_2 = \{x \in L : \uparrow (x \vee v) \text{ is compact}\}$.

We now have $x \prec y$, $x^* \in N_1$ and $x^* \in N_2$. So by definition, $x \blacktriangleleft_2 y$. Thus $x \in k_2(y)$, so $x \in J$.

The proof of the converse is similar to the proof of Proposition 6.1.9 with $\blacktriangleleft$ replaced by $\blacktriangleleft_2$ and k replaced by k_2. □

In addition to the property stated in Remark 6.2.10, we show that the elements of a 2-star satisfies the following:

Proposition 6.2.13. *Let $\alpha_2 = \{u, v\}$ ba a 2-star for a non-compact regular continuous frame L. Let $\bigvee : \alpha_2 L \to L$ be the corresponding 2-star compactification and $k_2 : L \to \alpha_2 L$ be the associated right adjoint defined by $k_2(a) = \{x \in L : x \blacktriangleleft_2 a\}$. Then, $k_2(u) \vee J$ and $k_2(v) \vee J$ are connected elements of the frame $\uparrow J$, where $J = \bigvee k_2(y)(y \in L, y \ll e)$.*

Proof. We show that $k_2(u) \vee J$ is connected as an element in $\uparrow J$. Suppose $k_2(u) \vee J = I \vee K$ where $I \cap K = J$. We need to show that $I = J$ or $K = J$. Suppose $I \neq J$, so $J \subsetneq I$.

Now, $\uparrow (u \vee v)$ is compact, therefore $k_2(u \vee v) \vee J = L$, by Remark 6.2.8. We have $k_2(u) \vee k_2(v) \vee J = L$ by Remark 6.2.10. Thus $I \vee K \vee k_2(v) = L$. So there exists elements $i \in I, k \in K, x \in L$ with $x \blacktriangleleft_2 v$, such that $e = i \vee k \vee x$.

Claim 1: We claim that $i \notin J$. To show this, suppose to the contrary that $i \in J$. Take $y \in I$, then $y = (y \wedge i) \vee (y \wedge k) \vee (y \wedge x) \in J \vee (I \cap K) \vee k_2(v) =$

$J \vee k_2(v)$. Also, $y \in I \subseteq I \vee K = k_2(u) \vee J$. Therefore,

$$
\begin{aligned}
y &\in (k_2(u) \vee J) \cap (k_2(v) \vee J) \\
&= [k_2(u) \cap k_2(v)] \vee J \\
&= k_2(u \wedge v) \vee J \\
&= k_2(0) \vee J \\
&= 0 \vee J \\
&= J
\end{aligned}
$$

In the previous calculations, $k_2(0) = 0$ follows because the map $\bigvee : \alpha_2 L \to L$ is dense and onto. Hence $I = J$, a contradiction. Thus $i \notin J$ as claimed.

Claim 2: We claim that $k \in J$. By strong regularity of I and J, we can find $s \in I, t \in K$ such that $i \blacktriangleleft_2 s, k \blacktriangleleft_2 t$, respectively. Let
$N_1 = \{x \in L : \uparrow (x \vee u) \text{ is compact}\}$ and $N_2 = \{x \in L : \uparrow (x \vee v) \text{ is compact}\}$. We also have that $x \blacktriangleleft_2 v$ and $\uparrow v$ is not compact. So, $v \notin N_2$ and therefore $x^* \in N_2$, by definition. Now, $i \vee k \vee x = e$ and so $s \vee t \vee x = e$. Therefore $x^* \leq s \vee t$. Since N_2 is a filter, then $s \vee t \in N_2$.

Now if either $s \in N_1$ or $t \in N_1$, then $s \vee t \in N_1$, since N_1 is a filter. Therefore $s \vee t \in N_1 \cap N_2$, and so $\uparrow (s \vee t)$ is compact, by Lemma 6.2.11. We have $s \vee t \in I \vee K$, so $s \vee t \in k_2(u) \vee J$. That is, we can find $a \in L$ with $a \blacktriangleleft_2 u$ and $j \in J$ such that $s \vee t = a \vee j$. In addition, $\uparrow (a \vee j)$ is compact. Now $a \leq u$, hence $\uparrow (u \vee j) \subseteq \uparrow (a \vee j)$ and so $\uparrow (u \vee j)$ is compact. This implies that $j \in N_1$. Now $j \in J$ implies that $j \ll e$ by Proposition 6.2.12. Therefore $\uparrow j^*$ is compact by Lemma 6.1.2. Hence $j^* \in N_1 \cap N_2$ by Lemma 6.2.11. So, $j^* \in N_1$, and thus $0 = j \wedge j^* \in N_1$. This contradicts the fact that N_1 is a proper filter of L. We therefore conclude that $s \notin N_1$ and $t \notin N_1$.

Now, $i \blacktriangleleft_2 s$ and $s \notin N_1$, so $i^* \in N_1$. We must have that $i^* \notin N_2$, otherwise $i^* \in N_1 \cap N_2$ implies $\uparrow i^*$ is compact, by Lemma 6.2.11. Thus $i \ll e$ by Lemma 6.1.2 and therefore $i \in J$ by Proposition 6.2.12. This contradicts **Claim 1**. Therefore $i^* \notin N_2$, and so $s \in N_2$.

Now if $t \in N_2$, then $s \wedge t \in N_2$. But also $s \wedge t \in I \cap K = J$, therefore $s \wedge t \ll e$ by Proposition 6.2.12. So, $\uparrow (s \wedge t)^*$ is compact by Lemma 6.1.2. Therefore $(s \wedge t)^* \in N_1 \cap N_2$, by Lemma 6.2.11. Thus, $(s \wedge t)^* \in N_2$. Hence $0 = (s \wedge t) \wedge (s \wedge t)^* \in N_2$, a contradiction since N_2 is a proper filter. So $t \notin N_2$.

We now have $k \blacktriangleleft_2 t, t \notin N_1$ and $t \notin N_2$ and therefore $k^* \in N_1 \cap N_2$. That is, $\uparrow k^*$ is compact by Lemma 6.2.11. Thus $k \ll e$ by Lemma 6.1.2. Hence $k \in J$, by Proposition 6.2.12.

The proof that $K = J$ is similar to the case in the third paragraph where we showed that $i \in J$ implies that $I = J$. Thus $k_2(u) \vee J$ is connected as an element of the frame $\uparrow J$. $\square$

If a 2-star compactification of a non-compact regular continuous frame exists and is perfect, then it satisfies some conditions. Some of these conditions are provided in the result below.

Proposition 6.2.14. *Let $\alpha_2 = \{u, v\}$ be a 2-star of a non-compact regular continuous frame L. Suppose that $\bigvee : \alpha_2 L \to L$ is a 2-star compactification corresponding to α_2 with a right adjoint defined by $k_2(a) = \{x \in L : x \blacktriangleleft_2 a\}$. Let $\{u_1, u_2, u_3\}$ be a collection of pairwise disjoint elements of L such that $\uparrow (u_1 \vee u_2 \vee u_3)$ is compact. If $\bigvee : \alpha_2 L \to L$ is perfect then there exists a pair $\{u_i, u_j\} \subseteq \{u_1, u_2, u_3\}$ such that $\uparrow (u_i \vee u_j)$ is compact.*

Proof. Since $\uparrow (u_1 \vee u_2 \vee u_3)$ is compact, then $k_2(u_1 \vee u_2 \vee u_3) \vee J = L$, by Remark 6.2.8, where $J = \bigvee k_2(y)(y \in L, y \ll e)$. By perfectness of

$\bigvee : \alpha_2 L \to L$, we have that $k_2(u_1) \vee k_2(u_2) \vee k_2(u_3) \vee J = L$. The latter can be written as $[k_2(u_1) \vee J] \vee [k_2(u_2) \vee J] \vee [k_2(u_3) \vee J] = L$.

Now $\{k_2(u_1) \vee J, k_2(u_2) \vee J, k_2(u_3) \vee J\}$ is a collection of pairwise disjoint elements in $\uparrow J$. Since for example, $[k_2(u_1) \vee J] \cap [k_2(u_2) \vee J] = J \vee [k_2(u_1) \cap k_2(u_2)] = J \vee k_2(u_1 \wedge u_2) = J \vee k_2(0) = J$. The last equality holds because $k_2(0) = 0$ and this follows since $\bigvee : \alpha_2 L \to L$ is dense.

We note that $k_2(u) \vee J$ and $k_2(v) \vee J$ are connected elements of the frame $\uparrow J$ by Proposition 6.2.13. Also, $k_2(u) \vee J \neq J$ and $k_2(v) \vee J \neq J$. If for example $k_2(u) \vee J = J$ then $k_2(u) \subseteq J$. Since $\uparrow(u \vee v)$ is compact, $k_2(u \vee v) \vee J = L$ by Remark 6.2.8, i.e., $k_2(u) \vee k_2(v) \vee J = L$, by perfectness. Since $k_2(u) \subseteq J$, we have $k_2(v) \vee J = L$. Thus, by Remark 6.2.8, $\uparrow v$ is compact, a contradiction. Hence $k_2(u) \vee J \neq J$ and $k_2(v) \vee J \neq J$.

We now show that there must be a u_i such that $k_2(u_i) \vee J = J$. Suppose there is no such u_i, then $k_2(u_1) \vee J \neq J$, $k_2(u_2) \vee J \neq J$ and $k_2(u_3) \vee J \neq J$. So these are non-zero elements in the frame $\uparrow J$. By Proposition 6.2.13 $k_2(u) \vee J$ is connected in $\uparrow J$, and so Lemma 6.1.12 we have that $k_2(u) \vee J \subseteq k_2(u_i) \vee J$ for some $i \in \{1, 2, 3\}$, say $k_2(u) \vee J \subseteq k_2(u_1) \vee J$. Similarly $k_2(v) \vee J \subseteq k_2(u_j) \vee J$ for some $j \in \{1, 2, 3\}$.

Suppose $u_j = u_1$. Since $\uparrow(u \vee v)$ is compact, then by Remark 6.2.8 we have that $L = [k_2(u \vee v) \vee J] = [k_2(u) \vee J] \vee [k_2(v) \vee J] \subseteq k_2(u_1) \vee J$. Therefore $L = k_2(u_1) \vee J$ and so $\uparrow u_1$ is compact by Remark 6.2.8. Therefore $\uparrow(u_1 \vee u_2)$ is compact.

Suppose $u_j = u_2$. then we have $k_2(u) \vee J \subseteq k_2(u_1) \vee J$ and $k_2(v) \vee J \subseteq k_2(u_2) \vee J$. So, by Remark 6.2.8, $L = [k_2(u_1) \vee J] \vee [k_2(u_2) \vee J] = k_2(u_1) \vee k_2(u_2) \vee J = k_2(u_1 \vee u_2) \vee J$. Hence $k_2(u_1 \vee u_2) \vee J = L$, and hence $\uparrow(u_1 \vee u_2)$ is compact by Remark 6.2.8. Similarly, if $u_j = u_3$, then $\uparrow(u_1 \vee u_3)$ is compact. $\square$

We also prove the following important observation.

Proposition 6.2.15. *Let L be a non-compact regular continuous frame that has a 2-star $\alpha_2 = \{u, v\}$. If the corresponding 2-star compactification $\bigvee : \alpha_2 L \to L$ is perfect, then L has no N-star compactification for $N > 2$.*

Proof. We prove this by contradiction. Let $\bigvee : \alpha_2 L \to L$ be perfect and suppose that $\alpha_N = \{u_1, u_2, \ldots, u_N\}$ is an N-star on L, where $N > 2$. We note that $\uparrow(u_1 \vee u_2 \vee \ldots \vee u_N)$ is compact and $\{u_1, u_2, u_3 \vee u_4 \vee \cdots \vee u_N\}$ is a collection of pairwise disjoint elements of L. So, by Proposition 6.2.14, either $\uparrow(u_1 \vee u_2)$, $\uparrow[u_1 \vee (u_3 \vee u_4 \vee \cdots \vee u_N)]$ or $\uparrow[u_2 \vee (u_3 \vee u_4 \vee \cdots \vee u_N)]$ is compact. However, each of these would contradict the N-star condition. □

Up to equivalence, we can claim that if a 2-star compactification of a non-compact regular continuous frame is perfect then it is the only 2-star compactification. To understand this, we need to state more results from the theory of N-star compactifications developed by Baboolal [2]. We state these results without proof.

Theorem 6.2.16. *[2, Theorem 3.7] Let $N^*(L)$ denote the collection of all the N-stars of L. Define a relation $\sim$ on $N^*(L)$ by:*

$$\alpha_N = \{u_1, u_2, \ldots, u_N\} \sim \beta_N = \{v_1, v_2, \ldots, v_N\}$$

if and only if the elements $u_1, u_2, \ldots, u_N$ and $v_1, v_2, \ldots, v_N$ can be ordered in such a way that

$$\uparrow (v_1 \vee v_2 \vee \cdots \vee v_{i-1} \vee u_i \vee v_{i+1} \vee \cdots v_N)$$

is compact for some $i \in \{1, 2, \ldots, n\}$. Then $\sim$ is an equivalent relation on $N^*(L)$.

Definition 6.2.17. Let $N^*(L)/\sim$ denote the set of all equivalence classes determined by the relation $\sim$ in Theorem 6.2.16 and let $K_N(L)$ denote the collection of all the N-star compactifications of a frame L.

Theorem 6.2.18. *[2, Lemma 3.8 and Theorem 3.10] Let*

$$\alpha_N = \{u_1, u_2, \ldots, u_N\}, \text{ and } \beta_N = \{v_1, v_2, \ldots, v_N\} \in N^*(L).$$

Let

$$N_i^\alpha = \{x \in L : \uparrow(u_1 \vee u_2 \vee \cdots \vee u_{i-1} \vee x \vee u_{i+1} \cdots \vee u_N) \text{ is compact}\}$$

and

$$N_i^\beta = \{x \in L : \uparrow(v_1 \vee v_2 \vee \cdots \vee v_{i-1} \vee x \vee v_{i+1} \cdots \vee v_N) \text{ is compact}\}.$$

Then $\alpha_N \sim \beta_N$ if and only if $N_i^\alpha = N_i^\beta$ for each $i \in \{1, 2, \ldots, n\}$. Moreover, there is a one-to-one correspondence between the elements of $N^*(L)/\sim$ and the elements of $K_N(L)$.

An immediate consequence of the theorem above is stated below.

Corollary 6.2.19. *[2, Corollary 3.11] Let L be a regular continuous frame. Then L has exactly one N-star compactification if and only if L has an N-star and all other N-stars are equivalent to it.*

We use the following from Baboolal [3]

Lemma 6.2.20. *[3, Theorem 3.5] A compactification $h : M \to L$ of a frame L is perfect if and only if $r(u \vee v) = r(u) \vee r(v)$, for any $u, v \in L$ with $u \wedge v = 0$, where $r : L \to M$ is the right adjoint of $h : M \to L$.*

If we assume perfectness of a 2-star compactification of a non-compact regular continuous frame corresponding to a 2-star α_2, then any other 2-star of L is equivalent to α_2. We show this below.

Proposition 6.2.21. *Let L be a non-compact regular continuous frame. Suppose that L has a 2-star compactification $\bigvee : \alpha_2 L \to L$, with the associated 2-star $\alpha_2 = \{u, v\}$. If $\bigvee : \alpha_2 L \to L$ is perfect, then any 2-star of L is equivalent to α_2*

Proof. Let $\beta_2 = \{s, t\}$ be any 2-star of L. We need to show that $\alpha_2 \sim \beta_2$. That is, we need to show that $\uparrow(u \vee t)$ is compact and $\uparrow(s \vee v)$ is compact. Since $s \wedge t = 0$, then $k_2(s \vee t) = k_2(s) \vee k_2(t)$ by Lemma 6.2.20 above, since $\bigvee : \alpha_2 L \to L$ is perfect, where $k_2 : L \to \alpha_2 L$ is the right adjoint of the join map $\bigvee : \alpha_2 L \to L$ defined by $k_2(a) = \{x \in L : x \blacktriangleleft_2 a\}$, where $\blacktriangleleft_2$ is the strong inclusion associated with the 2-star $\alpha_2 = \{u, v\}$. Now, $\uparrow(s \vee t)$ is compact, so $k_2(s \vee t) \vee J = L$ by Remark 6.2.8, where $J = \bigvee k_2(y)(y \in L, y \ll e)$. That is, $k_2(s) \vee k_2(t) \vee J = L$.

We note that $(k_2(s) \vee J) \cap (k_2(t) \vee J) = [k_2(s) \wedge k_2(t)] \vee J = k_2(s \wedge t) \vee J = k_2(0) \vee J = 0 \vee J = J$. Also, $k_2(s) \vee J \neq J$ and $k_2(t) \vee J \neq J$. For, if $k_2(s) \vee J = J$, then $k_2(s) \subseteq J$, so $k_2(t) \vee J = L$. Hence $\uparrow t$ is compact, a contradiction.

So $k_2(s) \vee J \neq J$ and $k_2(t) \vee J \neq J$. Also $k_2(s) \vee k_2(t) \vee J = L$ implies $(k_2(s) \vee J) \vee (k_2(t) \vee J) = L$. By Proposition 6.2.13, $k_2(u) \vee J$ is connected in $\uparrow J$, hence $k_2(u) \vee J \subseteq k_2(s) \vee J$ or $k_2(u) \vee J \subseteq k_2(t) \vee J$, by Lemma 6.1.12. Assume $k_2(u) \vee J \subseteq k_2(s) \vee J$. We cannot also have $k_2(v) \vee J \subseteq k_2(s) \vee J$, otherwise $L = [k_2(u) \vee J] \vee [k_2(v) \vee J] \subseteq k_2(s) \vee J$. Hence $L = k_2(s) \vee J$, from which we get that $\uparrow s$ is compact by Remark 6.2.8, a contradiction. So, $k_2(v) \vee J \subseteq k_2(t) \vee J$ and $k_2(u) \vee J \subseteq k_2(s) \vee J$.

We recall that in the frame $\uparrow J$, we have $(k_2(s) \vee J) \cap (k_2(t) \vee J) = J$ where $k_2(s) \vee J \neq J$ and $k_2(t) \vee J \neq J$. Therefore, by connectedness of $k_2(u) \vee J$ in $\uparrow J$, we cannot have $(k_2(u) \vee J) \cap (k_2(t) \vee J) \neq J$. Otherwise,

$k_2(u) \vee J \subseteq k_2(t) \vee J$ which is impossible. Hence, we have that $(k_2(u) \vee J) \cap (k_2(t) \vee J) = J$. Now $k_2(t) \vee J \subseteq L = (k_2(u) \vee J) \vee (k_2(v) \vee J)$ and therefore

$$\begin{aligned} k_2(t) \vee J &= [(k_2(u) \vee J) \cap (k_2(t) \vee J)] \vee [(k_2(v) \vee J) \cap (k_2(t) \vee J)] \\ &= J \vee [(k_2(v) \vee J) \cap (k_2(t) \vee J)] \\ &= (k_2(v) \vee J) \cap (k_2(t) \vee J). \end{aligned}$$

So, $k_2(t) \vee J \subseteq k_2(v) \vee J$. Similarly $k_2(s) \vee J \subseteq k_2(u) \vee J$. Hence $k_2(v) \vee J = k_2(t) \vee J$ and $k_2(u) \vee J = k_2(s) \vee J$.

We have $(k_2(s) \vee J) \vee (k_2(t) \vee J) = L$. Therefore $(k_2(v) \vee J) \vee (k_2(s) \vee J) = L$. Thus $k_2(v) \vee k_2(s) \vee J = L$. Since we always have $k_2(v) \vee k_2(s) \subseteq k_2(v \vee s)$, then $k_2(v \vee s) \vee J = L$. Therefore $\uparrow (v \vee s)$ is compact by Remark 6.2.8. Similarly $\uparrow (u \vee t)$ is compact. Hence $\alpha_2 \sim \beta_2$. □

A direct consequence of Proposition 6.2.21, Proposition 6.2.15 and Corollary 6.2.19 is stated below.

Corollary 6.2.22. *Let L be a non-compact regular continuous frame. Suppose that L has a 2-star compactification $\bigvee : \alpha_2 L \to L$. If $\bigvee : \alpha_2 L \to L$ is perfect, then it is the only 2-star compactification, up to equivalence. Moreover, there is no N-star compactification for any $N > 2$.*

We now show that the converse of Corollary 6.2.22 holds true.

Proposition 6.2.23. *Let $\alpha_2 = \{u, v\}$ be a 2-star of a non-compact regular continuous frame L. Let $\bigvee : \alpha_2 L \to L$ be the 2-star compactification corresponding to the strong inclusion $\blacktriangleleft_2$ defined by: $a \blacktriangleleft_2 b$ if and only if $a \prec b$ and for each $i \in \{1, 2\}$, either $a^* \in N_i$ or $b \in N_i$, where $N_1 = \{x \in L : \uparrow (x \vee u)$ is compact$\}$ and $N_2 = \{x \in L : \uparrow (x \vee v)$ is compact$\}$.*

Suppose that any 2-star of L is equivalent to α_2 and that L has no N-star for $N > 2$. Then $\bigvee : \alpha_2 L \to L$ is perfect.

Proof. Let $s, t \in L$ be such that $s \wedge t = 0$. To show that $\bigvee : \alpha_2 L \to L$ is perfect, by Lemma 6.2.20, we need to show that $k_2(s \vee t) = k_2(s) \vee k_2(t)$, where $k_2 : L \to \alpha_2 L$ is the right adjoint of the join map $\bigvee : \alpha_2 L \to L$ defined by $k_2(a) = \{x \in L : x \blacktriangleleft_2 a\}$, where $\blacktriangleleft_2$ is the strong inclusion associated with the 2-star $\alpha_2 = \{u, v\}$. It is trivial that $k_2(s) \vee k_2(t) \subseteq k_2(s \vee t)$. To show the reverse inequality, let $x \in k_2(s \vee t)$. Then $x \blacktriangleleft_2 (s \vee t)$, and hence $x \leq (s \vee t)$. Therefore, we have $x = x \wedge (s \vee t) = (x \wedge s) \vee (x \wedge t)$. To show that $x \in k_2(s) \vee k_2(t)$, we need to show that $(x \wedge s) \vee (x \wedge t) \in k_2(s) \vee k_2(t)$. That is, we need to show that $(x \wedge s) \blacktriangleleft_2 s$ and $(x \wedge t) \blacktriangleleft_2 t$. We only show that $(x \wedge s) \blacktriangleleft_2 s$, the arguments to show that $(x \wedge t) \blacktriangleleft_2 t$ will be similar.

Now $x \blacktriangleleft_2 (s \vee t)$ implies that $x \prec (s \vee t)$. So $(x \wedge s) \prec (s \vee t) \wedge s = (s \wedge s) \vee (t \wedge s) = s$. Therefore, $(x \wedge s) \prec s$.

Now either $x^* \in N_1 \cap N_2$ or $x^* \notin N_1 \cap N_2$. We consider these cases separately.

Case 1: Suppose $x^* \in N_1 \cap N_2$. Since $x \wedge s \leq x$, then $x^* \leq (x \wedge s)^*$. Therefore $(x \wedge s)^* \in N_1 \cap N_2$, since each N_i is a filter. Since $(x \wedge s) \prec s$ and $(x \wedge s)^* \in N_1 \cap N_2$, we have $(x \wedge s) \blacktriangleleft_2 s$.

Case 2: Suppose $x^* \notin N_1 \cap N_2$. Here we have three subcases to consider:

1. $x^* \notin N_1$ and $x^* \in N_2$, or
2. $x^* \in N_1$ and $x^* \notin N_2$ or
3. $x^* \notin N_1$ and $x^* \notin N_2$.

We deal with each subcase separately.

1. Suppose $x^* \notin N_1$ and $x^* \in N_2$. We now have $x \blacktriangleleft_2 (s \vee t)$ and $x^* \notin N_1$, therefore $s \vee t \in N_1$. Now $x^* \in N_2$ and $x^* \leq (x \wedge s)^*$, therefore $(x \wedge s)^* \in N_2$, since N_2 is a filter.

If we also have $(x \wedge s)^* \in N_1$, then $(x \wedge s)^* \in N_1 \cap N_2$, i.e., $(x \wedge s) \blacktriangleleft_2 s$.

So, suppose $(x \wedge s)^* \notin N_1$. We claim that $s \in N_1$. To show this we first note that $v \in N_1$ since $\uparrow (u \vee v)$ is compact. Thus $(s \vee t) \wedge v \in N_1$, i.e., $(s \wedge v) \vee (t \wedge v) \in N_1$. Therefore $\uparrow [(s \wedge v) \vee (t \wedge v) \vee u]$ is compact. Let $\alpha = \{s \wedge v, t \wedge v, u\}$. The elements of α are pairwise disjoint but cannot be a 3-star, by the hypothesis. Thus one of the sublocales $\uparrow [(s \wedge v) \vee (t \wedge v)]$, $\uparrow [(s \wedge v) \vee u], \uparrow [(t \wedge v) \vee u]$ must be compact. We note that $(s \wedge v) \vee (t \wedge v) \leq v$, therefore $\uparrow v \subseteq \uparrow [(s \wedge v) \vee (t \wedge v)]$. Hence, if $\uparrow [(s \wedge v) \vee (t \wedge v)]$ is compact then $\uparrow v$ is compact, a contradiction. So $\uparrow [(s \wedge v) \vee (t \wedge v)]$ is not compact. Thus either $\uparrow [(s \wedge v) \vee u]$ or $\uparrow [(t \wedge v) \vee u]$ is compact. Since $(s \wedge v) \vee u \leq (s \vee u)$ and $(t \wedge v) \vee u \leq (t \vee u)$, then $\uparrow (s \vee u) \subseteq \uparrow [(s \wedge v) \vee u]$ and $\uparrow (t \vee u) \subseteq \uparrow [(t \wedge v) \vee u]$. Hence, either $\uparrow (s \vee u)$ or $\uparrow (t \vee u)$ is compact. That is, either $s \in N_1$ or $t \in N_1$.

Suppose $t \in N_1$. Now, $(x \wedge s) \wedge t = 0$. Therefore $t \leq (x \wedge s)^*$, so $(x \wedge s)^* \in N_1$ since N_1 a filter. This contradicts the assumption we are working under that $(x \wedge s)^* \notin N_1$. Therefore $t \notin N_1$ and so $s \in N_1$. Hence $(x \wedge s) \blacktriangleleft_2 s$.

2. Similar arguments as in 1.

3. Suppose $x^* \notin N_1$ and $x^* \notin N_2$. Therefore $x \blacktriangleleft_2 (s \vee t)$ implies that $x \prec (s \vee t)$ and $s \vee t \in N_1 \cap N_2$. Hence $\uparrow (s \vee t)$ is compact by Lemma 6.2.11.

Now, if $\uparrow s$ and $\uparrow t$ are both not compact, then $\beta_2 = \{s, t\}$ is a 2-star. Therefore $\alpha_2 \sim \beta_2$ by the hypothesis. So α_2 and β_2 determine the same compactification. It then follows by Remark 6.2.10 that $k_2(s \vee t) = k_2(s) \vee k_2(t)$. So, in this case, $\bigvee : \alpha_2 L \to L$ is perfect.

Suppose that only $\uparrow s$ is compact, then $s \in N_1 \cap N_2$ by Lemma 6.2.11.

This implies that $(x \wedge s) \blacktriangleleft_2 s$.

Suppose that $\uparrow s$ and $\uparrow t$ are both compact, then $L = \uparrow 0 = \uparrow (s \wedge t) = (\uparrow t) \vee (\uparrow s)$ would be compact, which is a contradiction. Now, $\uparrow s$ is compact $\square$

Remark 6.2.24. We conjecture that all the results that were found for 2-star compactifications can be generalised to N-star compactifications for any $N > 2$. However, the arguments used in the proofs for the 2-star results (eg., see Proposition 6.2.23 above) already suggest that the calculations for $N > 2$ may be more involved and not so easy to handle.

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