

**A Discrete Hartley Transform based on  
Simpson's rule**

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# A Discrete Hartley Transform based on Simpson's rule

by

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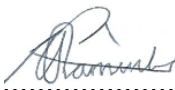
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# Preface

The work described in this dissertation was carried out in the School of Mathematical Sciences, University of KwaZulu-Natal, Durban, from February 2015 to December 2015, under the supervision of Dr P. Singh and co-supervised by Dr V. Singh. This study represents original work by the author and has not otherwise been submitted in any form for any degree or diploma to any other tertiary institution. Where use has been made of the work of others, it is duly acknowledged in the text.

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# FACULTY OF SCIENCE AND AGRICULTURE

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# Dedication

This thesis is dedicated to my father Dave Ramsunder, my mother Saira Ramsunder, my brother Mihir Ramsunder and all the family members who have been lost over the years.

# Acknowledgement

I would firstly like to thank God and my immediate family which include my parents Dave and Saira Ramsunder as well as my brother Mihir Ramsunder for making me the person I am today. My utmost gratitude and appreciation is expressed to my supervisor Dr Pravin Singh and co-supervisor Dr Virath Singh. Dr P.Singh and Dr V.Singh have provided me with a vast set of new skills in both life and mathematics. They have ensured that the work presented is of the highest quality and I have always challenged myself to make an impression and meet their high standards. Although the work was demanding, I have not once been afraid or stressed to ask either of my supervisors for assistance.

I would like to also thank the National Research Foundation (NRF) for providing me with funding required to pursue my studies further. A special thank you to Mr Soren Greenwood for the computer support as well as insightful conversations.

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# Notation

$\mathbf{A}, \mathbf{A}_N$	Matrix of order $N$
$C(k), C(k, l)$	Approximation to Hartley coefficients
$\mathbf{f}$	Signal or image matrix
$\mathbf{f}_e$	Even part of signal $\mathbf{f}$
$\mathbf{f}_o$	Odd part of signal $\mathbf{f}$
$\mathbf{f}_R$	Signal $\mathbf{f}$ reversed
$\mathbf{f}_0$	Even indexed components of a signal
$\mathbf{f}_1$	Odd indexed components of a signal
$\mathcal{H}$	Hartley transformation matrix
$\mathbf{H}$	Simpson Hartley transformation matrix
$\mathbf{I}, \mathbf{I}_N$	Identity matrix of order $N$
$\mathbf{P}, \mathbf{P}_N$	Permutation matrix of order $N$
$\otimes$	Convolution
$\star$	Correlation
$\cdot\star$	Element-wise multiplication

# Abstract

The Discrete Hartley Transform and Discrete Fourier Transform are classical transformations designed for efficient computations in the frequency domain. We introduce a relatively new transformation based on the existing Discrete Hartley Transform by applying Simpson's quadrature for  $N = 4m + 2$  quadrature nodes. The majority of our investigation involves exploring the mathematical properties satisfied by our newly derived transformation. We formulate the convolution and cross correlation properties both in the real and frequency domain. An intensive spectral analysis is performed to ascertain the multiplicities of the eigenvalues corresponding to the transformation matrix.

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# Introduction

In chapter 1, we provide a brief literature survey of the Discrete Hartley Transform (DHT) and show how it arises from trapezoidal quadrature.

In chapter 2, we derive the Simpson Discrete Hartley Transform (SDHT) from Simpson's quadrature. We provide an expression for the transformation matrix as well as an expression for the inverse. We investigate the effect of the transform on a real signal.

In chapter 3, we provide detailed proofs for several properties satisfied by the Simpson Discrete Hartley Transform. We prove the convolution property for both the real and frequency domains as well as the cross correlation property. We also prove various other properties such as the frequency shift property and the duality property.

In chapter 4, we provide a detailed spectral analysis of the transformation matrix. We use the duality property to obtain an expression for the minimal polynomial. From the minimal polynomial we obtain four distinct eigenvalues for the transformation matrix. We employ powers of the traces of the matrix and the theory of Gauss sums to obtain the multiplicities of the eigenvalues.

In chapter 5, we derive a two-dimensional Simpson Discrete Hartley Transform.

In chapter 6, we apply this transform to the useful area of image processing namely

watermarking and cleaning sinusoidal noise in the frequency domain.

# Chapter 1

## Discrete Hartley Transform

### 1.1 Introduction

Ralph Hartley was an innovative scientist responsible for mathematical triumphs such as the Hartley transform and Hartley oscillator [9]. The Hartley transform is closely related to the Fourier transform. These are both integral transforms, which however have one main difference being that the former is real valued whilst the latter is generally complex valued [9].

While investigating the relationship of Fourier analysis to images in 1983, Ronald Bracewell, who was in the research field of electronic engineering discovered a new factorization of the discrete Fourier transformation matrix which resulted in a fast algorithm for spectral analysis [4]. This discovery led to the proposal of the Discrete Hartley Transform (DHT) and the first Fast Hartley Transform (FHT) algorithm by Ronald Bracewell in 1983 [5]. The DHT is closely related to the Discrete Fourier transform (DFT) whereby both consist of discrete periodic data. Ronald Bracewell

also proposed that the FHT algorithm is more computationally efficient than the FFT (Fast DFT) when the data is real, however this was disproved by tweaking the FFT for real inputs [7].

The properties of the DHT enables one to design fast algorithms to compute it [14] and thus allows applications in various areas of signal processing such as power engineering and digital filtering [10]. The DHT is also used extensively in biomedical imaging for edge enhancement and medical resonance imaging and in addition can also be useful in image compression and reconstruction.

## 1.2 Derivation of the DHT

We will now show how the DHT arises by applying trapezoidal quadrature. Recall that the coefficients in a Hartley series for a continuous,  $2\pi$ -periodic function  $f(x)$  have the form

$$c(k) = \frac{1}{2\pi} \int_0^{2\pi} f(x) \text{cas}(kx) dx, \quad (1.1)$$

where  $\text{cas}(x) = \cos(x) + \sin(x)$ . Divide the interval  $[0, 2\pi]$  into  $N$  equally spaced subintervals of length  $\omega = \frac{2\pi}{N}$  with nodes  $x_j = \omega j$ ,  $j = 0, 1, \dots, N$ . The approximation  $C(k)$  to  $c(k)$  using the trapezoidal rule gives

$$\begin{aligned} C(k) &= \frac{1}{2N} \left[ f(x_0) \text{cas}(kx_0) + f(x_N) \text{cas}(kx_N) + 2 \sum_{j=1}^{N-1} f(x_j) \text{cas}(kx_j) \right] \\ &= \frac{1}{2N} \left[ 2f(x_0) \text{cas}(kx_0) + 2 \sum_{j=1}^{N-1} f(x_j) \text{cas}(kx_j) \right] \\ &= \frac{1}{N} \sum_{j=0}^{N-1} f(x_j) \text{cas}(kx_j). \end{aligned} \quad (1.2)$$

We used the fact that  $f(x_0) = f(x_N)$  due to the periodicity of  $f$ . We adopt the notation  $f(x_j) = f(j)$  and rewrite equation (1.2) as

$$C(k) = \frac{1}{N} \sum_{j=0}^{N-1} f(j) \text{cas}(kj\omega). \quad (1.3)$$

The derivation above enables us to define the DHT of a  $N$  point real sequence  $f(j)$ ,  $j = 0, 1, \dots, N - 1$  by [6]

$$F(k) = \mathcal{H}\mathbf{f}(k) = \sum_{j=0}^{N-1} \text{cas}(k\omega j) f(j), \quad (1.4)$$

$k = 0, 1, \dots, N - 1$  along with transformation matrix  $\mathcal{H}$  defined by components  $h_{kj} = \text{cas}(k\omega j)$ ,  $j = 0, 1, \dots, N - 1$ ,  $k = 0, 1, \dots, N - 1$ . To be consistent we have chosen to index the components from zero.

In order to recover the signal  $\mathbf{f} = [f(0), f(1), \dots, f(N-1)]^T$  multiply (1.4) by  $\text{cas}(k\omega p)$ ,  $p \in \{0, 1, \dots, N - 1\}$  and sum over  $k$  to obtain

$$\sum_{k=0}^{N-1} \text{cas}(k\omega p) F(k) = \sum_{j=0}^{N-1} \left[ \sum_{k=0}^{N-1} \text{cas}(k\omega j) \text{cas}(k\omega p) \right] f(j). \quad (1.5)$$

We will now prove the following lemma.

**Lemma 1.1.**

$$\sum_{k=0}^{N-1} \text{cas}(k\omega j) \text{cas}(k\omega p) = \begin{cases} N & \text{when } j = p, \\ 0 & \text{when } j \neq p, \end{cases} \quad (1.6)$$

where  $p \in \{0, 1, \dots, N - 1\}$ .

*Proof.* Using the identity

$$\text{cas}(A) \text{cas}(B) = \cos(A - B) + \sin(A + B), \quad (1.7)$$

enables us to expand the left hand side of (1.6) which results in

$$\sum_{k=0}^{N-1} \cos(k\omega(j - p)) + \sum_{k=0}^{N-1} \sin(k\omega(j + p)). \quad (1.8)$$

Let  $m = j - p$  then  $m \in \{-N + 1, -N + 2, \dots, N - 1\}$  and it suffices to show that

$$\sum_{k=0}^{N-1} \cos(k\omega m) = \begin{cases} N & \text{when } m = 0, \\ 0 & \text{when } m \neq 0. \end{cases} \quad (1.9)$$

When  $m = 0$  the result is trivial. Consider the case when  $m \neq 0$ . Let

$$\begin{aligned} S_m &= \sum_{k=0}^{N-1} e^{ik\omega m} = 1 + e^{i\omega m} + e^{2i\omega m} + \dots + e^{i\omega m(N-1)} \\ &= 1 + \sum_{k=1}^{N-1} e^{ik\omega m}. \end{aligned} \quad (1.10)$$

Multiply equation (1.10) by  $e^{i\omega m}$  to get

$$\begin{aligned} S_m(e^{i\omega m}) &= \sum_{k=0}^{N-1} e^{i(k+1)\omega m} \\ &= \sum_{k=1}^N e^{ik\omega m} \\ &= \sum_{k=1}^{N-1} e^{ik\omega m} + e^{i\omega Nm}. \end{aligned} \quad (1.11)$$

Subtracting (1.11) from (1.10) yields

$$S_m = \frac{1 - e^{i\omega Nm}}{1 - e^{i\omega m}}. \quad (1.12)$$

We observe that  $S_m = 0$  since the numerator is zero and the denominator is non zero.

Taking the real part of  $S_m$  proves (1.9). Furthermore it may be shown in a similar manner that

$$\sum_{k=0}^{N-1} \sin(k\omega(j+p)) = 0 \quad (1.13)$$

□

Applying lemma 1.1 to the right hand side of (1.5) yields  $Nf(p)$ . Finally from (1.5),

by replacing the index  $p$  by  $j$ , we get the inverse DHT

$$f(j) = \frac{1}{N} \sum_{k=0}^{N-1} \text{cas}(k\omega j) F(k). \quad (1.14)$$

## Chapter 2

# Simpson Discrete Hartley

## Transform

Applying Simpson quadrature to equation (1.1) yields

$$\begin{aligned} C(k) &= \frac{1}{3N} \left[ f(x_0) \operatorname{cas}(kx_0) + f(x_N) \operatorname{cas}(kx_N) + 2 \sum_{j=1}^{\frac{N}{2}-1} f(x_{2j}) \operatorname{cas}(kx_{2j}) \right. \\ &\quad \left. + 4 \sum_{j=0}^{\frac{N}{2}-1} f(x_{2j+1}) \operatorname{cas}(kx_{2j+1}) \right] \\ &= \frac{1}{N} \left[ \frac{2}{3} \sum_{j=0}^{\frac{N}{2}-1} \operatorname{cas}(2k\omega j) f(2j) + \frac{4}{3} \sum_{j=0}^{\frac{N}{2}-1} \operatorname{cas}(k\omega(2j+1)) f(2j+1) \right]. \end{aligned} \quad (2.1)$$

Hence the Simpson Discrete Hartley Transform (SDHT) is defined component-wise by

$$\mathbf{Hf}(k) = \frac{2}{3} \sum_{j=0}^{\frac{N}{2}-1} \operatorname{cas}(2k\omega j) f(2j) + \frac{4}{3} \sum_{j=0}^{\frac{N}{2}-1} \operatorname{cas}(k\omega(2j+1)) f(2j+1). \quad (2.2)$$

Define

$$F_0(k) = \frac{2}{3} \sum_{j=0}^{\frac{N}{2}-1} \operatorname{cas}(2k\omega j) f(2j) \quad (2.3)$$

and

$$F_1(k) = \frac{4}{3} \sum_{j=0}^{\frac{N}{2}-1} \operatorname{cas}(k\omega(2j+1)) f(2j+1), \quad (2.4)$$

for  $k = 0, 1, 2, \dots, N - 1$ . Using the fact that the cas function is anti-periodic in  $\pi$  it follows from (2.4) that

$$\begin{aligned}
F_1(k + \frac{N}{2}) &= \frac{4}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}((k + \frac{N}{2})\omega(2j + 1))f(2j + 1) \\
&= \frac{4}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(k\omega(2j + 1) + (2j + 1)\pi)f(2j + 1) \\
&= -\frac{4}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(k\omega(2j + 1))f(2j + 1) \\
&= -F_1(k).
\end{aligned} \tag{2.5}$$

Define  $\mathbf{F}_1 = [F_1(0), F_1(1), \dots, F_1(\frac{N}{2} - 1)]^T$  then the vector  $\mathbf{F}_1$  is anti-periodic in  $\frac{N}{2}$ . Similarly it may be shown that the vector  $\mathbf{F}_0 = [F_0(0), F_0(1), \dots, F_0(\frac{N}{2} - 1)]^T$  is periodic in  $\frac{N}{2}$ .

Equation (2.3) leads us to define a  $(\frac{N}{2} \times \frac{N}{2})$  matrix  $\mathbf{A}$  by

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \text{cas}(2\omega) & \text{cas}(4\omega) & \cdots & \text{cas}(N_1\omega) \\ 1 & \text{cas}(4\omega) & \text{cas}(8\omega) & \cdots & \text{cas}(2N_1\omega) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \text{cas}(N_1\omega) & \text{cas}(2N_1\omega) & \cdots & \text{cas}(\frac{N_1}{2}N_1\omega) \end{bmatrix}. \tag{2.6}$$

Hence (2.3) may be written as

$$\mathbf{F}_0 = \frac{2}{3} \mathbf{A} \mathbf{f}_0,$$

where  $\mathbf{f}_0 = [f(0), f(2), \dots, f(N - 2)]^T$ . Equation (2.4) leads us to define a  $(\frac{N}{2} \times \frac{N}{2})$

matrix  $\mathbf{B}$  by

$$\mathbf{B} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ \text{cas}(\omega) & \text{cas}(3\omega) & \text{cas}(5\omega) & \cdots & \text{cas}((N_1 + 1)\omega) \\ \text{cas}(2\omega) & \text{cas}(6\omega) & \text{cas}(10\omega) & \cdots & \text{cas}(2(N_1 + 1)\omega) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \text{cas}(\frac{N_1}{2}\omega) & \text{cas}(\frac{N_1}{2}3\omega) & \text{cas}(\frac{N_1}{2}5\omega) & \cdots & \text{cas}(\frac{N_1}{2}(N_1 + 1)\omega) \end{bmatrix}, \quad (2.7)$$

where  $N_1 = N - 2$ . Hence (2.4) may be written as

$$\mathbf{F}_1 = \frac{4}{3}\mathbf{B}\mathbf{f}_1,$$

where  $\mathbf{f}_1 = [f(1), f(3), \dots, f(N - 1)]^T$ .

Considering (2.2) for  $k = 0, 1, \dots, \frac{N}{2} - 1$  the right hand side can be written in the form

$$\frac{2}{3}\mathbf{A}\mathbf{f}_0 + \frac{4}{3}\mathbf{B}\mathbf{f}_1.$$

Considering (2.2) for  $k = \frac{N}{2}, \frac{N}{2} + 1, \dots, N - 1$  the right hand side can be written in the form

$$\frac{2}{3}\mathbf{A}\mathbf{f}_0 - \frac{4}{3}\mathbf{B}\mathbf{f}_1,$$

where we have used the periodicity and anti-periodicity of  $\mathbf{F}_0$  and  $\mathbf{F}_1$  respectively.

Finally (2.2) in block matrix vector form is represented by

$$\mathbf{H}\mathbf{f} = \frac{2}{3} \begin{bmatrix} \mathbf{A} & 2\mathbf{B} \\ \mathbf{A} & -2\mathbf{B} \end{bmatrix} \begin{bmatrix} \mathbf{f}_0 \\ \mathbf{f}_1 \end{bmatrix}. \quad (2.8)$$

Since  $[\mathbf{f}_0 \ \mathbf{f}_1]^T = \mathbf{P}\mathbf{f}$  where the  $(N \times N)$  permutation matrix  $\mathbf{P}$  is defined by  $P_{j,2j} = 1$  and  $P_{\frac{N}{2}+j,2j+1} = 1$  for  $j = 0, 1, \dots, \frac{N}{2} - 1$ . With this substitution in (2.8) we recover the transformation matrix

$$\mathbf{H} = \frac{2}{3} \begin{bmatrix} \mathbf{A} & 2\mathbf{B} \\ \mathbf{A} & -2\mathbf{B} \end{bmatrix} \mathbf{P}. \quad (2.9)$$

Taking the inner product of the  $j_{th}$  and  $p_{th}$  columns of  $\mathbf{A}$  results in

$$\sum_{i=0}^{\frac{N}{2}-1} \text{cas}(2ij\omega) \text{cas}(2ip\omega) = \begin{cases} \frac{N}{2} & \text{when } j = p, \\ 0 & \text{when } j \neq p, \end{cases} \quad (2.10)$$

where we have used a variation of lemma 1.1. Hence we conclude that  $\sqrt{\frac{2}{N}}\mathbf{A}$  is orthogonal. Since  $\mathbf{A}$  is symmetric it follows that  $\sqrt{\frac{2}{N}}\mathbf{A}$  is involutory. Similarly it may be shown that  $\sqrt{\frac{2}{N}}\mathbf{B}$  is orthogonal.

Using these properties it may be shown by multiplication that the inverse of  $\mathbf{H}$  is given by

$$\mathbf{H}^{-1} = \frac{3}{2N}\mathbf{P}^T \begin{bmatrix} \mathbf{A} & \mathbf{A} \\ \frac{1}{2}\mathbf{B}^T & -\frac{1}{2}\mathbf{B}^T \end{bmatrix}. \quad (2.11)$$

From (2.11) the inverse transform may be represented component-wise by

$$f(2j) = \frac{3}{2N} \sum_{k=0}^{\frac{N}{2}-1} \text{cas}(2k\omega j) \mathbf{H}\mathbf{f}(k), \quad (2.12)$$

$$f(2j+1) = \frac{3}{4N} \sum_{k=0}^{\frac{N}{2}-1} \text{cas}(k\omega(2j+1)) \mathbf{H}\mathbf{f}(k). \quad (2.13)$$

# Chapter 3

## Properties of the SDHT

In this chapter we state and prove various properties of the SDHT.

An even signal  $\mathbf{f}$  is defined as one for which  $f(-j) = f(j)$  whilst an odd signal is defined as one for which  $f(-j) = -f(j)$ .

### 3.1 Transform of an even signal

Given an even signal  $\mathbf{f}$  then  $\mathbf{Hf}(-k) = \mathbf{Hf}(k)$ .

*Proof.* Replace  $k$  by  $-k$  in equation (2.2) to obtain

$$\mathbf{Hf}(-k) = \frac{2}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(-2k\omega j) f(2j) + \frac{4}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(-k\omega(2j+1)) f(2j+1) \quad (3.1)$$

$$= \frac{2}{3} \sum_{j=-\frac{N}{2}+1}^0 \text{cas}(2k\omega j) f(-2j) + \frac{4}{3} \sum_{j=-\frac{N}{2}+1}^0 \text{cas}(k\omega(2j-1)) f(-2j+1) \quad (3.2)$$

$$= \frac{2}{3} \sum_{j=-\frac{N}{2}+1}^0 \text{cas}(2k\omega j) f(-2j) + \frac{4}{3} \sum_{j=-\frac{N}{2}}^{-1} \text{cas}(k\omega(2j+1)) f(-2j-1) \quad (3.3)$$

$$= \frac{2}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(2k\omega j) f(2j) + \frac{4}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(k\omega(2j+1)) f(2j+1) \quad (3.4)$$

$$= \mathbf{Hf}(k),$$

where we replaced  $j$  by  $-j$  from (3.1) to (3.2). From (3.2) to (3.3) we have replaced  $j$  by  $j + 1$  in the second summation. We notice that the arguments of both summations in (3.3) are periodic in  $j$  of periodicity  $\frac{N}{2}$ , hence we may sum over any period of length  $\frac{N}{2}$  [2] resulting in (3.4).  $\square$

## 3.2 Transform of an odd signal

Given an odd signal  $\mathbf{f}$  then  $\mathbf{Hf}(-k) = -\mathbf{Hf}(k)$ .

*Proof.* Replace  $k$  by  $-k$  in equation (2.2) to obtain

$$\mathbf{Hf}(-k) = \frac{2}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(-2k\omega j) f(2j) + \frac{4}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(-k\omega(2j+1)) f(2j+1) \quad (3.5)$$

$$= \frac{2}{3} \sum_{j=-\frac{N}{2}+1}^0 \text{cas}(2k\omega j) f(-2j) + \frac{4}{3} \sum_{j=-\frac{N}{2}+1}^0 \text{cas}(k\omega(2j-1)) f(-2j+1) \quad (3.6)$$

$$= \frac{2}{3} \sum_{j=-\frac{N}{2}+1}^0 \text{cas}(2k\omega j) f(-2j) + \frac{4}{3} \sum_{j=-\frac{N}{2}}^{-1} \text{cas}(k\omega(2j+1)) f(-2j-1) \quad (3.7)$$

$$= -\frac{2}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(2k\omega j) f(2j) - \frac{4}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(k\omega(2j+1)) f(2j+1) \quad (3.8)$$

$$= -\mathbf{Hf}(k).$$

From (3.7) to (3.8) we have used the fact that  $\mathbf{f}$  is an odd signal.  $\square$

## 3.3 Time reversal

By this property it is meant that the reversal of the transformed signal is equal to the transform of the reversed signal, that is  $\mathbf{Hf}(-k) = \mathbf{Hf}_{\mathbf{R}}(k)$  where

$$\mathbf{f}_{\mathbf{R}} = [f(0), f(N-1), \dots, f(1)]^T.$$

*Proof.* Replace  $k$  by  $-k$  in equation (2.2) to get

$$\mathbf{Hf}(-k) = \frac{2}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(-2k\omega j) f(2j) + \frac{4}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(-k\omega(2j+1)) f(2j+1) \quad (3.9)$$

$$= \frac{2}{3} \sum_{j=1}^{\frac{N}{2}} \text{cas}(-2k\omega(-j + \frac{N}{2})) f(2(-j + \frac{N}{2})) \quad (3.10)$$

$$+ \frac{4}{3} \sum_{j=1}^{\frac{N}{2}} \text{cas}(-k\omega(2(-j + \frac{N}{2}) + 1)) f(2(-j + \frac{N}{2}) + 1),$$

where we have replaced  $j$  by  $-j + \frac{N}{2}$ .

Equation (3.10) simplifies to

$$\begin{aligned} \mathbf{Hf}(-k) &= \frac{2}{3} \sum_{j=1}^{\frac{N}{2}} \text{cas}(2k\omega j - k\omega N) f(N - 2j) \\ &+ \frac{4}{3} \sum_{j=1}^{\frac{N}{2}} \text{cas}(k\omega(2j - N - 1)) f(N - 2j + 1) \end{aligned} \quad (3.11)$$

$$\begin{aligned} &= \frac{2}{3} \sum_{j=1}^{\frac{N}{2}} \text{cas}(2k\omega j - 2k\pi) f(N - 2j) \\ &+ \frac{4}{3} \sum_{j=1}^{\frac{N}{2}} \text{cas}(k\omega(2j - 1) - 2k\pi) f(N - 2j + 1) \end{aligned} \quad (3.12)$$

$$\begin{aligned} &= \frac{2}{3} \sum_{j=1}^{\frac{N}{2}} \text{cas}(2k\omega j) f(-2j) \\ &+ \frac{4}{3} \sum_{j=1}^{\frac{N}{2}} \text{cas}(k\omega(2j - 1)) f(-2j + 1), \end{aligned} \quad (3.13)$$

$$\begin{aligned} &= \frac{2}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(2k\omega j) f(-2j) \\ &+ \frac{4}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(k\omega(2j + 1)) f(-2j - 1) \end{aligned} \quad (3.14)$$

$$= \mathbf{Hf}_{\mathbf{R}}(k).$$

From (3.12) to (3.13) we have used the periodicity of both  $\mathbf{f}$  and the cas function. From

(3.13) to (3.14) we replaced  $j$  by  $j + 1$  in the second summation.  $\square$

In order to proceed we require the following lemma.

**Lemma 3.1.** *The double angle formula of the cas function is given by*

$$\text{cas}(A+B) = \frac{1}{2} [\text{cas}(A) \text{cas}(B) + \text{cas}(A) \text{cas}(-B) + \text{cas}(-A) \text{cas}(B) - \text{cas}(-A) \text{cas}(-B)].$$

*Proof.*

$$\begin{aligned} \text{cas}(A+B) &= \cos(A+B) + \sin(A+B) \\ &= \cos(A) \cos(B) + \cos(A) \sin(B) + \sin(A) \cos(B) - \sin(A) \sin(B) \\ &= \cos(A) \text{cas}(B) + \sin(A) \text{cas}(-B) \\ &= \frac{1}{2} [\text{cas}(A) + \text{cas}(-A)] \text{cas}(B) + \frac{1}{2} [\text{cas}(A) - \text{cas}(-A)] \text{cas}(-B) \\ &= \frac{1}{2} [\text{cas}(A) \text{cas}(B) + \text{cas}(A) \text{cas}(-B) + \text{cas}(-A) \text{cas}(B) \\ &\quad - \text{cas}(-A) \text{cas}(-B)], \end{aligned}$$

where we have used  $\cos(A) = \frac{1}{2} [\text{cas}(A) + \text{cas}(-A)]$  and  $\sin(A) = \frac{1}{2} [\text{cas}(A) - \text{cas}(-A)]$ .

□

Given a sequence  $\mathbf{f}$  define the even part  $\mathbf{f}_e$  by  $f_e(j) = \frac{f(j)+f(-j)}{2}$  and the odd part  $\mathbf{f}_o$  by  $f_o(j) = \frac{f(j)-f(-j)}{2}$ . In addition, define the vector  $\mathbf{v}$  by  $v(j) = \text{cas}(k_0 \omega j)$ ,  $j = 0, 1, \dots, N-1$ , where  $k_0$  is an integer.

### 3.4 Frequency shift

A shift by  $k_0$  of the transformed signal  $\mathbf{f}$  is equivalent to the transformed signal  $\mathbf{v} \star \mathbf{f}_e + \mathbf{v}_R \star \mathbf{f}_o$  where  $\mathbf{v}_R$  is the reversal of  $\mathbf{v}$  and  $\star$  represents element-wise multiplication.

*Proof.* From (2.2) replacing  $k$  by  $k - k_0$  gives

$$\mathbf{Hf}(k - k_0) = \frac{2}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}((k - k_0)\omega 2j) f(2j) + \frac{4}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}((k - k_0)\omega(2j + 1)) f(2j + 1). \quad (3.15)$$

Applying lemma 3.1 to equation (3.15) yields

$$\begin{aligned} & \mathbf{Hf}(k - k_0) \\ &= \frac{1}{2} \left[ \frac{2}{3} \sum_{j=0}^{\frac{N}{2}-1} \left( \text{cas}(2k\omega j) \text{cas}(-2k_0\omega j) + \text{cas}(2k\omega j) \text{cas}(2k_0\omega j) \right. \right. \\ & \quad \left. \left. + \text{cas}(-2k\omega j) \text{cas}(-2k_0\omega j) - \text{cas}(-2k\omega j) \text{cas}(2k_0\omega j) \right) \right] f(2j) \\ & \quad + \frac{1}{2} \left[ \frac{4}{3} \sum_{j=0}^{\frac{N}{2}-1} \left( \text{cas}(k\omega(2j + 1)) \text{cas}(-k_0\omega(2j + 1)) \right. \right. \\ & \quad \left. \left. + \text{cas}(k\omega(2j + 1)) \text{cas}(k_0\omega(2j + 1)) \right. \right. \\ & \quad \left. \left. + \text{cas}(-k\omega(2j + 1)) \text{cas}(-k_0\omega(2j + 1)) \right. \right. \\ & \quad \left. \left. - \text{cas}(-k\omega(2j + 1)) \text{cas}(k_0\omega(2j + 1)) \right) \right] f(2j + 1). \end{aligned} \quad (3.16)$$

Rearranging yields

$$\begin{aligned} & \mathbf{Hf}(k - k_0) \\ &= \frac{1}{2} \left[ \left( \frac{2}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(2k\omega j) \text{cas}(-2k_0\omega j) f(2j) \right. \right. \\ & \quad \left. \left. + \frac{4}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(k\omega(2j + 1)) \text{cas}(-k_0\omega(2j + 1)) f(2j + 1) \right) \right] \end{aligned} \quad (3.17)$$

$$\begin{aligned} & \quad + \left( \frac{2}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(2k\omega j) \text{cas}(2k_0\omega j) f(2j) \right. \\ & \quad \left. + \frac{4}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(k\omega(2j + 1)) \text{cas}(k_0\omega(2j + 1)) f(2j + 1) \right) \end{aligned} \quad (3.18)$$

$$\begin{aligned}
& + \left( \frac{2}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(-2k\omega j) \text{cas}(-2k_0\omega j) f(2j) \right. \\
& + \left. \frac{4}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(-k\omega(2j+1)) \text{cas}(-k_0\omega(2j+1)) f(2j+1) \right) \quad (3.19)
\end{aligned}$$

$$\begin{aligned}
& - \left( \frac{2}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(-2k\omega j) \text{cas}(2k_0\omega j) f(2j) \right. \\
& + \left. \frac{4}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(-k\omega(2j+1)) \text{cas}(k_0\omega(2j+1)) f(2j+1) \right). \quad (3.20)
\end{aligned}$$

Statement (3.19) can be simplified by replacing  $j$  by  $-j$  in the first term and  $j$  by  $-j-1$  in the second term to give

$$\frac{2}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(2k\omega j) \text{cas}(2k_0\omega j) f(-2j) + \frac{4}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(k\omega(2j+1)) \text{cas}(k_0\omega(2j+1)) f(-2j-1). \quad (3.21)$$

Similarly statement (3.20) simplifies to

$$\begin{aligned}
& - \frac{2}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(2k\omega j) \text{cas}(-2k_0\omega j) f(-2j) \\
& - \frac{4}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(k\omega(2j+1)) \text{cas}(-k_0\omega(2j+1)) f(-2j-1). \quad (3.22)
\end{aligned}$$

We can combine (3.18) and (3.21) to yield

$$\begin{aligned}
& \frac{2}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(2k\omega j) \text{cas}(2k_0\omega j) \left[ \frac{f(2j) + f(-2j)}{2} \right] \\
& + \frac{4}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(k\omega(2j+1)) \text{cas}(k_0\omega(2j+1)) \left[ \frac{f(2j+1) + f(-2j-1)}{2} \right]. \quad (3.23)
\end{aligned}$$

In a similar manner we can combine (3.17) and (3.22) to get

$$\begin{aligned}
& \frac{2}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(2k\omega j) \text{cas}(-2k_0\omega j) \left[ \frac{f(2j) - f(-2j)}{2} \right] \\
& + \frac{4}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(k\omega(2j+1)) \text{cas}(-k_0\omega(2j+1)) \left[ \frac{f(2j+1) - f(-2j-1)}{2} \right]. \quad (3.24)
\end{aligned}$$

With this simplification (3.15) reduces to

$$\begin{aligned}\mathbf{Hf}(k - k_0) &= \mathbf{H}[\mathbf{v} \star \mathbf{f}_e] + \mathbf{H}[\mathbf{v}_R \star \mathbf{f}_o] \\ &= \mathbf{H}[\mathbf{v} \star \mathbf{f}_e + \mathbf{v}_R \star \mathbf{f}_o].\end{aligned}\tag{3.25}$$

□

### 3.5 Duality

By this we mean the effect of  $\mathbf{H}^2$  on  $\mathbf{f}$ . From (2.2) we obtain the even components

$$\begin{aligned}\mathbf{H}^2\mathbf{f}(2k) &= \frac{2}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(4k\omega j) \mathbf{Hf}(2j) + \frac{4}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(2k\omega(2j+1)) \mathbf{Hf}(2j+1)\end{aligned}\tag{3.26}$$

$$\begin{aligned}&= \frac{2}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(4k\omega j) \left[ \frac{2}{3} \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(4j\omega p) f(2p) + \frac{4}{3} \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(2j\omega(2p+1)) f(2p+1) \right] \\ &+ \frac{4}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(2k\omega(2j+1)) \left[ \frac{2}{3} \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(2\omega p(2j+1)) f(2p) \right. \\ &\left. + \frac{4}{3} \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(\omega(2j+1)) f(2p+1) \right]\end{aligned}\tag{3.27}$$

$$= \frac{4}{9} \sum_{p=0}^{\frac{N}{2}-1} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(4k\omega j) \text{cas}(4j\omega p) f(2p)\tag{3.28}$$

$$+ \frac{8}{9} \sum_{p=0}^{\frac{N}{2}-1} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(4k\omega j) \text{cas}(2\omega j(2p+1)) f(2p+1)\tag{3.29}$$

$$+ \frac{8}{9} \sum_{p=0}^{\frac{N}{2}-1} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(2k\omega(2j+1)) \text{cas}(2p\omega(2j+1)) f(2p)\tag{3.30}$$

$$+ \frac{16}{9} \sum_{p=0}^{\frac{N}{2}-1} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(2k\omega(2j+1)) \text{cas}(\omega(2p+1)(2j+1)) f(2p+1)\tag{3.31}$$

Statement (3.28) simplifies to

$$\frac{4}{9} \sum_{p=0}^{\frac{N}{2}-1} \sum_{j=0}^{\frac{N}{2}-1} \cos(4\omega j(k-p)) f(2p) + \frac{4}{9} \sum_{p=0}^{\frac{N}{2}-1} \sum_{j=0}^{\frac{N}{2}-1} \sin(4\omega p(k+p)) f(2p).\tag{3.32}$$

It may be proved as in (1.13) that the second term in (3.32) is zero. There is contribution from the first term in (3.32) only when  $p = k$ . Hence (3.32) simplifies to  $\frac{2}{9}Nf(2k)$ .

Similarly (3.29) reduces to

$$\frac{8}{9} \sum_{p=0}^{\frac{N}{2}-1} \sum_{j=0}^{\frac{N}{2}-1} \cos(2\omega j(2k - 2p - 1))f(2p + 1) + \frac{8}{9} \sum_{p=0}^{\frac{N}{2}-1} \sum_{j=0}^{\frac{N}{2}-1} \sin(2\omega j(2k + 2p + 1))f(2p + 1). \quad (3.33)$$

Once again the second summation in (3.33) can be shown to be zero. It must be observed that  $2k - 2p - 1$  is non zero. There is contribution from the first term only when  $2k - 2p - 1 = \frac{N}{2}$ . Hence (3.33) simplifies to  $\frac{4}{9}Nf(2k + \frac{N}{2})$ . Similarly (3.30) simplifies to  $\frac{4}{9}Nf(2k)$  and (3.31) simplifies to  $-\frac{8}{9}Nf(2k + \frac{N}{2})$ .

In summary we combine all four terms to get the duality property for the even components, namely

$$\begin{aligned} \mathbf{H}^2\mathbf{f}(2k) &= \frac{2}{9}Nf(2k) + \frac{4}{9}Nf(2k + \frac{N}{2}) + \frac{4}{9}Nf(2k) - \frac{8}{9}Nf(2k + \frac{N}{2}) \\ &= \frac{2}{3}Nf(2k) - \frac{4}{9}Nf(2k + \frac{N}{2}). \end{aligned} \quad (3.34)$$

The odd components for the duality property is done in a similar manner to yield

$$\begin{aligned} \mathbf{H}^2\mathbf{f}(2k + 1) &= \frac{4}{9} \sum_{p=0}^{\frac{N}{2}-1} \sum_{j=0}^{\frac{N}{2}-1} \cos(2j\omega(2k + 1)) \cos(4j\omega p)f(2p) \end{aligned} \quad (3.35)$$

$$+ \frac{8}{9} \sum_{p=0}^{\frac{N}{2}-1} \sum_{j=0}^{\frac{N}{2}-1} \cos(2\omega j(2k + 1)) \cos(2\omega j(2p + 1))f(2p + 1) \quad (3.36)$$

$$+ \frac{8}{9} \sum_{p=0}^{\frac{N}{2}-1} \sum_{j=0}^{\frac{N}{2}-1} \cos(\omega(2j + 1)(2k + 1)) \cos(2\omega p(2j + 1))f(2p) \quad (3.37)$$

$$\begin{aligned}
& + \frac{16}{9} \sum_{p=0}^{\frac{N}{2}-1} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(\omega(2j+1)(2k+1)) \text{cas}(\omega(2j+1)(2p+1)) \\
& \times f(2p+1). \tag{3.38}
\end{aligned}$$

Statements (3.35)-(3.38) evaluate to  $\frac{2}{9}Nf(2k+1+\frac{N}{2})$ ,  $\frac{4}{9}Nf(2k+1)$ ,  $-\frac{4}{9}Nf(2k+1+\frac{N}{2})$  and  $\frac{8}{9}Nf(2k+1)$  respectively.

In summary we combine all four terms to get the duality property for the odd components, namely

$$\begin{aligned}
& \mathbf{H}^2 \mathbf{f}(2k+1) \\
& = \frac{2}{9}Nf(2k+1+\frac{N}{2}) + \frac{4}{9}Nf(2k+1) - \frac{4}{9}Nf(2k+1+\frac{N}{2}) + \frac{8}{9}Nf(2k+1) \\
& = \frac{4}{3}Nf(2k+1) - \frac{2}{9}Nf(2k+1+\frac{N}{2}). \tag{3.39}
\end{aligned}$$

Let  $\mathbf{f}$  and  $\mathbf{g}$  be vectors of period  $N$ . The circular convolution  $\mathbf{z} = \mathbf{f} \otimes \mathbf{g}$  is defined component-wise by

$$z(j) = \sum_{m=0}^{N-1} f(m)g(j-m), \tag{3.40}$$

for  $j = 0, 1, \dots, N-1$ .

### 3.6 Convolution in the real domain

We derive an expression for the transform of the convolution  $\mathbf{f} \otimes \mathbf{g}$ .

$$\mathbf{H}\mathbf{z}(k) = \frac{2}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(2k\omega j)z(2j) + \frac{4}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(k\omega(2j+1))z(2j+1), \tag{3.41}$$

where

$$z(2j) = \sum_{m=0}^{\frac{N}{2}-1} f(2m)g(2j-2m) + \sum_{m=0}^{\frac{N}{2}-1} f(2m+1)g(2j-2m-1), \quad (3.42)$$

$$z(2j+1) = \sum_{m=0}^{\frac{N}{2}-1} f(2m)g(2j+1-2m) + \sum_{m=0}^{\frac{N}{2}-1} f(2m+1)g((2j+1)-(2m+1)). \quad (3.43)$$

Upon substitution of (3.42) and (3.43) into (3.41), we get

**H $z(k)$**

$$\begin{aligned} &= \frac{2}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(2k\omega j)z(2j) + \frac{4}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(k\omega(2j+1))z(2j+1) \\ &= \frac{2}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(2k\omega j) \left[ \sum_{m=0}^{\frac{N}{2}-1} f(2m)g(2j-2m) + \sum_{m=0}^{\frac{N}{2}-1} f(2m+1)g(2j-2m-1) \right] \\ &+ \frac{4}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(k\omega(2j+1)) \left[ \sum_{m=0}^{\frac{N}{2}-1} f(2m)g(2j+1-2m) \right. \\ &\left. + \sum_{m=0}^{\frac{N}{2}-1} f(2m+1)g((2j+1)-(2m+1)) \right] \end{aligned} \quad (3.44)$$

$$= \frac{2}{3} \left[ \sum_{j=0}^{\frac{N}{2}-1} \sum_{m=0}^{\frac{N}{2}-1} \text{cas}(2k\omega j) f(2m)g(2(j-m)) \right] \quad (3.45)$$

$$+ \sum_{j=0}^{\frac{N}{2}-1} \sum_{m=0}^{\frac{N}{2}-1} \text{cas}(2k\omega j) f(2m+1)g(2(j-m)-1) \quad (3.46)$$

$$+ \frac{4}{3} \left[ \sum_{j=0}^{\frac{N}{2}-1} \sum_{m=0}^{\frac{N}{2}-1} \text{cas}(k\omega(2j+1)) f(2m)g(2(j-m)+1) \right] \quad (3.47)$$

$$+ \sum_{j=0}^{\frac{N}{2}-1} \sum_{m=0}^{\frac{N}{2}-1} \text{cas}(k\omega(2j+1)) f(2m+1)g(2(j-m)) \quad (3.48)$$

Statement (3.45) simplifies to

$$\begin{aligned}
& \frac{2}{3} \sum_{l=-m}^{\frac{N}{2}-1-m} \sum_{m=0}^{\frac{N}{2}-1} \text{cas}(2k\omega(l+m))f(2m)g(2l) \\
&= \frac{1}{3} \sum_{l=0}^{\frac{N}{2}-1} \sum_{m=0}^{\frac{N}{2}-1} \left[ \text{cas}(2k\omega l) \text{cas}(2k\omega m) + \text{cas}(2k\omega l) \text{cas}(-2k\omega m) + \text{cas}(-2k\omega l) \text{cas}(2k\omega m) \right. \\
&\quad \left. - \text{cas}(-2k\omega l) \text{cas}(-2k\omega m) \right] f(2m)g(2l). \tag{3.49}
\end{aligned}$$

The reindexing of the argument of  $g$  allows the double summation to be written as products of single summations giving

$$\frac{1}{3} \left[ \sum_{m=0}^{\frac{N}{2}-1} \text{cas}(2k\omega m) f(2m) \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(2k\omega l) g(2l) \right. \tag{3.50}$$

$$+ \sum_{m=0}^{\frac{N}{2}-1} \text{cas}(-2k\omega m) f(2m) \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(2k\omega l) g(2l) \tag{3.51}$$

$$+ \sum_{m=0}^{\frac{N}{2}-1} \text{cas}(2k\omega m) f(2m) \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(-2k\omega l) g(2l) \tag{3.52}$$

$$\left. - \sum_{m=0}^{\frac{N}{2}-1} \text{cas}(-2k\omega m) f(2m) \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(-2k\omega l) g(2l) \right]. \tag{3.53}$$

Statement (3.46) simplifies to

$$\begin{aligned}
& \frac{2}{3} \sum_{l=-m-1}^{\frac{N}{2}-1-m-1} \sum_{m=0}^{\frac{N}{2}-1} \text{cas}(2k\omega(l+m+1))f(2m+1)g(2l+1) \\
&= \sum_{l=0}^{\frac{N}{2}-1} \sum_{m=0}^{\frac{N}{2}-1} \text{cas}(k\omega(2l+1) + k\omega(2m+1))f(2m+1)g(2l+1) \\
&= \frac{1}{3} \sum_{l=0}^{\frac{N}{2}-1} \sum_{m=0}^{\frac{N}{2}-1} \left[ \text{cas}(k\omega(2m+1)) \text{cas}(k\omega(2l+1)) + \text{cas}(k\omega(2m+1)) \text{cas}(-k\omega(2l+1)) \right. \\
&\quad \left. + \text{cas}(-k\omega(2m+1)) \text{cas}(k\omega(2l+1)) - \text{cas}(-k\omega(2m+1)) \text{cas}(k\omega(2l+1)) \right] \\
&\quad \times f(2m+1)g(2l+1) \tag{3.54}
\end{aligned}$$

The reindexing of the argument of  $g$  once again allows the double summation to be written as products of single summations giving

$$\frac{1}{3} \left[ \sum_{m=0}^{\frac{N}{2}-1} \text{cas}(k\omega(2m+1))f(2m+1) \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(k\omega(2l+1))g(2l+1) \right] \quad (3.55)$$

$$+ \sum_{m=0}^{\frac{N}{2}-1} \text{cas}(k\omega(2m+1))f(2m+1) \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(-k\omega(2l+1))g(2l+1) \quad (3.56)$$

$$+ \sum_{m=0}^{\frac{N}{2}-1} \text{cas}(-k\omega(2m+1))f(2m+1) \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(k\omega(2l+1))g(2l+1) \quad (3.57)$$

$$- \sum_{m=0}^{\frac{N}{2}-1} \text{cas}(-k\omega(2m+1))f(2m+1) \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(-k\omega(2l+1))g(2l+1) \Big]. \quad (3.58)$$

It should be noted that (3.47) and (3.48) can be simplified in a similar manner to obtain

$$\frac{2}{3} \left[ \sum_{m=0}^{\frac{N}{2}-1} \text{cas}(2k\omega m)f(2m) \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(k\omega(2l+1))g(2l+1) \right] \quad (3.59)$$

$$+ \sum_{m=0}^{\frac{N}{2}-1} \text{cas}(2k\omega m)f(2m) \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(-k\omega(2l+1))g(2l+1) \quad (3.60)$$

$$+ \sum_{m=0}^{\frac{N}{2}-1} \text{cas}(-2k\omega m)f(2m) \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(k\omega(2l+1))g(2l+1) \quad (3.61)$$

$$- \sum_{m=0}^{\frac{N}{2}-1} \text{cas}(-2k\omega m)f(2m) \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(-k\omega(2l+1))g(2l+1) \Big] \quad (3.62)$$

and

$$\frac{2}{3} \left[ \sum_{m=0}^{\frac{N}{2}-1} \text{cas}(k\omega(2m+1))f(2m+1) \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(2k\omega l)g(2l) \right] \quad (3.63)$$

$$+ \sum_{m=0}^{\frac{N}{2}-1} \text{cas}(k\omega(2m+1))f(2m+1) \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(-2k\omega l)g(2l) \quad (3.64)$$

$$+ \sum_{m=0}^{\frac{N}{2}-1} \text{cas}(-k\omega(2m+1))f(2m+1) \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(2k\omega l)g(2l) \quad (3.65)$$

$$- \sum_{m=0}^{\frac{N}{2}-1} \text{cas}(-k\omega(2m+1))f(2m+1) \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(-2k\omega l)g(2l) \Big]. \quad (3.66)$$

For any signal  $\mathbf{x}$  we make the following observation

$$\mathbf{H}\mathbf{x}(k) = \frac{2}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(2k\omega j)x(2j) + \frac{4}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(k\omega(2j+1))x(2j+1), \quad (3.67)$$

$$\mathbf{H}\mathbf{x}(k + \frac{N}{2}) = \frac{2}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(2k\omega j)x(2j) - \frac{4}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(k\omega(2j+1))x(2j+1). \quad (3.68)$$

By adding (3.67) to (3.68) we obtain the result

$$\sum_{j=0}^{\frac{N}{2}-1} \text{cas}(2k\omega j)x(2j) = \frac{3}{4}\mathbf{H}\mathbf{x}(k) + \frac{3}{4}\mathbf{H}\mathbf{x}(k + \frac{N}{2}). \quad (3.69)$$

Similarly it may be shown that

$$\sum_{j=0}^{\frac{N}{2}-1} \text{cas}(k\omega(2j+1))x(2j+1) = \frac{3}{8}\mathbf{H}\mathbf{x}(k) - \frac{3}{8}\mathbf{H}\mathbf{x}(k + \frac{N}{2}), \quad (3.70)$$

$$\sum_{j=0}^{\frac{N}{2}-1} \text{cas}(-2k\omega j)x(2j) = \frac{3}{4}\mathbf{H}\mathbf{x}(-k) + \frac{3}{4}\mathbf{H}\mathbf{x}(-k - \frac{N}{2}), \quad (3.71)$$

$$\sum_{j=0}^{\frac{N}{2}-1} \text{cas}(-k\omega(2j+1))x(2j+1) = \frac{3}{8}\mathbf{H}\mathbf{x}(-k) - \frac{3}{8}\mathbf{H}\mathbf{x}(-k - \frac{N}{2}). \quad (3.72)$$

By using the results in (3.69)-(3.72) in (3.50)-(3.53), (3.55)-(3.58), (3.59)-(3.62) and (3.63)-(3.66) we arrive at

$$\begin{aligned} & \mathbf{H}\mathbf{z}(k) \\ &= \frac{39}{64} \left[ \mathbf{H}\mathbf{f}(k)\mathbf{H}\mathbf{g}(k) + \mathbf{H}\mathbf{f}(k)\mathbf{H}\mathbf{g}(-k) + \mathbf{H}\mathbf{f}(-k)\mathbf{H}\mathbf{g}(k) - \mathbf{H}\mathbf{f}(-k)\mathbf{H}\mathbf{g}(-k) \right] \\ & - \frac{9}{64} \left[ \mathbf{H}\mathbf{f}(k + \frac{N}{2})\mathbf{H}\mathbf{g}(k + \frac{N}{2}) + \mathbf{H}\mathbf{f}(k + \frac{N}{2})\mathbf{H}\mathbf{g}(-k - \frac{N}{2}) + \mathbf{H}\mathbf{f}(-k - \frac{N}{2})\mathbf{H}\mathbf{g}(k + \frac{N}{2}) \right. \\ & \left. - \mathbf{H}\mathbf{f}(-k - \frac{N}{2})\mathbf{H}\mathbf{g}(-k - \frac{N}{2}) \right] + \frac{9}{64} \left[ \mathbf{H}\mathbf{f}(k)\mathbf{H}\mathbf{g}(k + \frac{N}{2}) + \mathbf{H}\mathbf{f}(k)\mathbf{H}\mathbf{g}(-k - \frac{N}{2}) \right. \\ & \left. + \mathbf{H}\mathbf{f}(-k)\mathbf{H}\mathbf{g}(k + \frac{N}{2}) - \mathbf{H}\mathbf{f}(-k)\mathbf{H}\mathbf{g}(-k - \frac{N}{2}) + \mathbf{H}\mathbf{f}(k + \frac{N}{2})\mathbf{H}\mathbf{g}(k) \right. \\ & \left. + \mathbf{H}\mathbf{f}(k + \frac{N}{2})\mathbf{H}\mathbf{g}(-k) + \mathbf{H}\mathbf{f}(-k - \frac{N}{2})\mathbf{H}\mathbf{g}(k) - \mathbf{H}\mathbf{f}(-k - \frac{N}{2})\mathbf{H}\mathbf{g}(-k) \right]. \quad (3.73) \end{aligned}$$

Define vectors  $\mathbf{S}$  and  $\mathbf{T}$  as follows

$$\mathbf{S} = \mathbf{Hf} \star \mathbf{Hg} + \mathbf{Hf} \star \mathbf{Hg}_R + \mathbf{Hf}_R \star \mathbf{Hg} - \mathbf{Hf}_R \star \mathbf{Hg}_R, \quad (3.74)$$

$$\mathbf{T} = \mathbf{Hf} \star \mathbf{Hg}_{\frac{N}{2}} + \mathbf{Hf} \star \mathbf{Hg}_{\frac{N}{2}R} + \mathbf{Hf}_R \star \mathbf{Hg}_{\frac{N}{2}} - \mathbf{Hf}_R \star \mathbf{Hg}_{\frac{N}{2}R}, \quad (3.75)$$

where  $\mathbf{Hg}_R$  represents the reversal of  $\mathbf{Hg}$ ,  $\mathbf{Hg}_{\frac{N}{2}}$  represents  $\mathbf{Hg}$  shifted by  $\frac{N}{2}$  and  $\mathbf{Hg}_{\frac{N}{2}R}$  represents the reversal of  $\mathbf{Hg}_{\frac{N}{2}}$ . A similar explanation applies to  $\mathbf{f}$ . Finally we can express equation (3.73) in the form

$$\mathbf{H}(\mathbf{f} \otimes \mathbf{g}) = \frac{39}{64}\mathbf{S} - \frac{9}{64}\mathbf{S}_{(N/2)} + \frac{9}{64}(\mathbf{T} + \mathbf{T}_{(N/2)}). \quad (3.76)$$

We now prove the following lemma.

**Lemma 3.2.**

$$\begin{aligned} \cos(C) \cos(A - B) &= \frac{\cos(A)}{2} [\cos(C - B) + \cos(C + B)] \\ &\quad + \frac{\cos(-A)}{2} [\cos(-C - B) - \cos(-C + B)]. \end{aligned}$$

*Proof.*

$$\begin{aligned} \cos(C) \cos(A - B) &= \cos(C) [\sin(A - B) + \cos(A - B)] \\ &= \cos(C) [\sin(A)(\sin(B) + \cos(B)) + \cos(A)(\cos(B) - \sin(B))] \\ &= \cos(C) [\sin(A) \cos(B) + \cos(A) \cos(-B)] \\ &= \frac{1}{2} \cos(C) \cos(B) [\cos(A) - \cos(-A)] \\ &\quad + \frac{1}{2} \cos(C) \cos(-B) [\cos(A) + \cos(-A)] \\ &= \frac{1}{2} \cos(C) \cos(B) \cos(A) - \frac{1}{2} \cos(C) \cos(B) \cos(-A) \\ &\quad + \frac{1}{2} \cos(C) \cos(-B) \cos(A) + \frac{1}{2} \cos(C) \cos(-B) \cos(-A) \\ &= \frac{\cos(A)}{2} [\cos(C) \cos(B) + \cos(C) \cos(-B)] \\ &\quad + \frac{\cos(-A)}{2} [\cos(C) \cos(-B) - \cos(C) \cos(B)]. \end{aligned} \quad (3.77)$$

In order to simplify the terms in (3.77) we use the following identities.

$$\begin{aligned} \text{cas}(C + B) &= \frac{1}{2} \left[ \text{cas}(C) \text{cas}(B) + \text{cas}(C) \text{cas}(-B) + \text{cas}(-C) \text{cas}(B) \right. \\ &\quad \left. - \text{cas}(-C) \text{cas}(-B) \right] \end{aligned} \quad (3.78)$$

$$\begin{aligned} \text{cas}(C - B) &= \frac{1}{2} \left[ \text{cas}(C) \text{cas}(-B) + \text{cas}(C) \text{cas}(B) + \text{cas}(-C) \text{cas}(-B) \right. \\ &\quad \left. - \text{cas}(-C) \text{cas}(B) \right] \end{aligned} \quad (3.79)$$

$$\begin{aligned} \text{cas}(-C + B) &= \frac{1}{2} \left[ \text{cas}(-C) \text{cas}(B) + \text{cas}(-C) \text{cas}(-B) + \text{cas}(C) \text{cas}(B) \right. \\ &\quad \left. - \text{cas}(C) \text{cas}(-B) \right] \end{aligned} \quad (3.80)$$

$$\begin{aligned} \text{cas}(-C - B) &= \frac{1}{2} \left[ \text{cas}(-C) \text{cas}(-B) + \text{cas}(-C) \text{cas}(B) + \text{cas}(C) \text{cas}(-B) \right. \\ &\quad \left. - \text{cas}(C) \text{cas}(B) \right]. \end{aligned} \quad (3.81)$$

Adding (3.78) and (3.79) gives

$$\text{cas}(C) \text{cas}(B) + \text{cas}(C) \text{cas}(-B) = \text{cas}(C + B) + \text{cas}(C - B). \quad (3.82)$$

Subtracting (3.80) from (3.81) gives

$$\text{cas}(C) \text{cas}(-B) - \text{cas}(C) \text{cas}(B) = \text{cas}(-C - B) - \text{cas}(-C + B). \quad (3.83)$$

With this substitution in (3.77) we complete the proof.  $\square$

### 3.7 Convolution in the frequency domain

We derive an expression for  $\mathbf{Hf} \otimes \mathbf{Hg}$ . This is also known as the product theorem.

$$\begin{aligned} (\mathbf{Hf} \otimes \mathbf{Hg})(j) &= \sum_{m=0}^{N-1} \mathbf{Hf}(m) \mathbf{Hg}(j - m) \\ &= \sum_{m=0}^{\frac{N}{2}-1} \mathbf{Hf}(2m) \mathbf{Hg}(j - 2m) + \sum_{m=0}^{\frac{N}{2}-1} \mathbf{Hf}(2m + 1) \mathbf{Hg}(j - 2m - 1). \end{aligned} \quad (3.84)$$

We now simplify the first summation in (3.84). This reads

$$\begin{aligned}
& \sum_{m=0}^{\frac{N}{2}-1} \mathbf{Hf}(2m) \mathbf{Hg}(j-2m) \\
&= \sum_{m=0}^{\frac{N}{2}-1} \left( \left[ \frac{2}{3} \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(4m\omega p) f(2p) + \frac{4}{3} \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(2m\omega(2p+1)) f(2p+1) \right] \times \right. \\
& \left. \left[ \frac{2}{3} \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(2l\omega(j-2m)) g(2l) + \frac{4}{3} \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(\omega(j-2m)(2l+1)) g(2l+1) \right] \right) \\
&= \frac{4}{9} \sum_{m=0}^{\frac{N}{2}-1} \sum_{p=0}^{\frac{N}{2}-1} \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(4m\omega p) \text{cas}(2l\omega(j-2m)) f(2p) g(2l) + \tag{3.85}
\end{aligned}$$

$$\frac{8}{9} \sum_{m=0}^{\frac{N}{2}-1} \sum_{p=0}^{\frac{N}{2}-1} \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(4m\omega p) \text{cas}(\omega(j-2m)(2l+1)) f(2p) g(2l+1) + \tag{3.86}$$

$$\frac{8}{9} \sum_{m=0}^{\frac{N}{2}-1} \sum_{p=0}^{\frac{N}{2}-1} \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(2m\omega(2p+1)) \text{cas}(2l\omega(j-2m)) f(2p+1) g(2l) + \tag{3.87}$$

$$\frac{16}{9} \sum_{m=0}^{\frac{N}{2}-1} \sum_{p=0}^{\frac{N}{2}-1} \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(2m\omega(2p+1)) \text{cas}(\omega(j-2m)(2l+1)) f(2p+1) g(2l+1). \tag{3.88}$$

Using lemma 3.2, (3.85) simplifies to

$$\begin{aligned}
& \frac{4}{9} \sum_{m=0}^{\frac{N}{2}-1} \sum_{p=0}^{\frac{N}{2}-1} \sum_{l=0}^{\frac{N}{2}-1} \left( \frac{\text{cas}(2j\omega l)}{2} [\text{cas}(4m\omega p + 4m\omega l) + \text{cas}(4m\omega p - 4m\omega l)] \right. \\
& \left. + \frac{\text{cas}(-2j\omega l)}{2} [\text{cas}(-4m\omega p - 4m\omega l) - \text{cas}(-4m\omega p + 4m\omega l)] \right) f(2p) g(2l) \tag{3.89}
\end{aligned}$$

$$= \frac{2}{9} \sum_{m=0}^{\frac{N}{2}-1} \sum_{p=0}^{\frac{N}{2}-1} \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(2j\omega l) \text{cas}(4m\omega(p+l)) f(2p) g(2l) \tag{3.90}$$

$$+ \frac{2}{9} \sum_{m=0}^{\frac{N}{2}-1} \sum_{p=0}^{\frac{N}{2}-1} \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(2j\omega l) \text{cas}(4m\omega(p-l)) f(2p) g(2l) \tag{3.91}$$

$$+ \frac{2}{9} \sum_{m=0}^{\frac{N}{2}-1} \sum_{p=0}^{\frac{N}{2}-1} \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(-2j\omega l) \text{cas}(-4m\omega(p+l)) f(2p) g(2l) \tag{3.92}$$

$$- \frac{2}{9} \sum_{m=0}^{\frac{N}{2}-1} \sum_{p=0}^{\frac{N}{2}-1} \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(-2j\omega l) \text{cas}(-4m\omega(l-p)) f(2p) g(2l). \tag{3.93}$$

The summations in (3.90) may be further simplified as follows.

$$\frac{2}{9} \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(2j\omega l) \sum_{p=0}^{\frac{N}{2}-1} \sum_{m=0}^{\frac{N}{2}-1} \text{cas}(4m\omega(p+l)) f(2p)g(2l) \quad (3.94)$$

$$= \frac{2}{9} \sum_{p=0}^{\frac{N}{2}-1} \text{cas}((N-2p)\omega j) f(2p)g(N-2p) \sum_{m=0}^{\frac{N}{2}-1} \text{cas}(4\pi m) \quad (3.95)$$

$$= \frac{N}{9} \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(-2p\omega j) f(2p)g(-2p). \quad (3.96)$$

We have used the fact that the only contribution to the summation in (3.94) occurs for  $p+l = \frac{N}{2}$ , the periodicity  $N$  of  $\mathbf{g}$  and the periodicity  $2\pi$  of the cas function. The summations in (3.91) may be further simplified as follows.

$$\frac{2}{9} \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(2j\omega l) \sum_{p=0}^{\frac{N}{2}-1} \sum_{m=0}^{\frac{N}{2}-1} \text{cas}(4m\omega(p-l)) f(2p)g(2l) \quad (3.97)$$

$$= \frac{2}{9} \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(2j\omega p) f(2p)g(2p) \sum_{m=0}^{\frac{N}{2}-1} 1 \quad (3.98)$$

$$= \frac{N}{9} \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(2j\omega p) f(2p)g(2p). \quad (3.99)$$

We have used the fact that the only contribution to the summation in (3.97) occurs for  $p=l$ . The summations in (3.92) may be further simplified as follows.

$$\frac{2}{9} \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(-2j\omega l) \sum_{p=0}^{\frac{N}{2}-1} \sum_{m=0}^{\frac{N}{2}-1} \text{cas}(-4m\omega(p+l)) f(2p)g(2l) \quad (3.100)$$

$$= \frac{2}{9} \sum_{p=0}^{\frac{N}{2}-1} \text{cas}((2p-N)\omega j) f(2p)g(N-2p) \sum_{m=0}^{\frac{N}{2}-1} \text{cas}(-4\pi m) \quad (3.101)$$

$$= \frac{N}{9} \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(2p\omega j) f(2p)g(-2p). \quad (3.102)$$

We have used the fact that the only contribution occurs for  $p+l = \frac{N}{2}$ . The summations in (3.93) may be further simplified as follows.

$$\frac{2}{9} \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(-2j\omega l) \sum_{p=0}^{\frac{N}{2}-1} \sum_{m=0}^{\frac{N}{2}-1} \text{cas}(-4m\omega(l-p)) f(2p)g(2l) \quad (3.103)$$

$$= \frac{2}{9} \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(-2j\omega p) f(2p)g(2p) \sum_{m=0}^{\frac{N}{2}-1} 1 \quad (3.104)$$

$$= \frac{N}{9} \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(-2j\omega p) f(2p)g(2p). \quad (3.105)$$

We have used the fact that the only contribution to the summation occurs for  $p = l$ .

Using lemma 3.2, (3.86) simplifies to

$$\begin{aligned} & \frac{8}{9} \sum_{m=0}^{\frac{N}{2}-1} \sum_{p=0}^{\frac{N}{2}-1} \sum_{l=0}^{\frac{N}{2}-1} \left( \frac{\text{cas}(\omega j(2l+1))}{2} [\text{cas}(4m\omega p + 2\omega m(2l+1)) \right. \\ & \left. + \text{cas}(4m\omega p - 2\omega m(2l+1))] + \frac{\text{cas}(-\omega j(2l+1))}{2} \times \right. \\ & \left. [\text{cas}(-4m\omega p - 2\omega m(2l+1)) - \text{cas}(-4m\omega p + 2\omega m(2l+1))] \right) f(2p)g(2l+1) \quad (3.106) \end{aligned}$$

$$= \frac{4}{9} \sum_{m=0}^{\frac{N}{2}-1} \sum_{p=0}^{\frac{N}{2}-1} \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(\omega j(2l+1)) \text{cas}(4m\omega p + 2\omega m(2l+1)) f(2p)g(2l+1) \quad (3.107)$$

$$+ \frac{4}{9} \sum_{m=0}^{\frac{N}{2}-1} \sum_{p=0}^{\frac{N}{2}-1} \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(\omega j(2l+1)) \text{cas}(4m\omega p - 2\omega m(2l+1)) f(2p)g(2l+1) \quad (3.108)$$

$$\begin{aligned} & + \frac{4}{9} \sum_{m=0}^{\frac{N}{2}-1} \sum_{p=0}^{\frac{N}{2}-1} \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(-\omega j(2l+1)) \text{cas}(-4m\omega p - 2\omega m(2l+1)) f(2p) \\ & \times g(2l+1) \quad (3.109) \end{aligned}$$

$$\begin{aligned} & - \frac{4}{9} \sum_{m=0}^{\frac{N}{2}-1} \sum_{p=0}^{\frac{N}{2}-1} \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(-\omega j(2l+1)) \text{cas}(-4m\omega p + 2\omega m(2l+1)) f(2p) \\ & \times g(2l+1). \quad (3.110) \end{aligned}$$

The summations in (3.107) may be further simplified as follows.

$$\frac{4}{9} \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(\omega j(2l+1)) \sum_{p=0}^{\frac{N}{2}-1} \sum_{m=0}^{\frac{N}{2}-1} \text{cas}(2\omega m(2p+2l+1)) f(2p)g(2l+1) \quad (3.111)$$

$$= \frac{4}{9} \sum_{p=0}^{\frac{N}{2}-1} \text{cas}\left(\left(\frac{N}{2}-2p\right)\omega j\right) f(2p)g\left(\frac{N}{2}-2p\right) \sum_{m=0}^{\frac{N}{2}-1} \text{cas}(2\pi m) \quad (3.112)$$

$$= \frac{2N}{9} \sum_{p=0}^{\frac{N}{2}-1} (-1)^j \text{cas}(-2p\omega j) f(2p)g\left(\frac{N}{2}-2p\right). \quad (3.113)$$

We have used the fact that the only contribution to the summation occurs for  $2p+2l+1 \in \{\frac{N}{2}, \frac{3N}{2}\}$ . Now  $\frac{3N}{2}$  is equivalent to  $\frac{N}{2}$  due to periodicity  $N$ . Therefore it suffices to consider only  $2p+2l+1 = \frac{N}{2}$ . Hence  $\text{cas}\left(\left(\frac{N}{2}-2p\right)\omega j\right) = (-1)^j \text{cas}(-2p\omega j)$ . The

summations in (3.108) may be further simplified as follows.

$$\frac{4}{9} \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(\omega j(2l+1)) \sum_{p=0}^{\frac{N}{2}-1} \sum_{m=0}^{\frac{N}{2}-1} \text{cas}(2\omega m(2p-2l-1)) f(2p)g(2l+1) \quad (3.114)$$

$$= \frac{4}{9} \sum_{p=0}^{\frac{N}{2}-1} \text{cas}\left(\left(2p+\frac{N}{2}\right)\omega j\right) f(2p)g\left(2p+\frac{N}{2}\right) \sum_{m=0}^{\frac{N}{2}-1} \text{cas}(2\pi m) \quad (3.115)$$

$$= \frac{2N}{9} \sum_{p=0}^{\frac{N}{2}-1} (-1)^j \text{cas}(2p\omega j) f(2p)g\left(2p+\frac{N}{2}\right). \quad (3.116)$$

We used the fact that the only contribution to the summation occurs for  $2p-2l-1 = \frac{N}{2}$  and the fact that  $\text{cas}\left(\left(2p+\frac{N}{2}\right)\omega j\right) = (-1)^j \text{cas}(2p\omega j)$ . The summations in statement

(3.109) may be further simplified as follows.

$$\frac{4}{9} \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(-\omega j(2l+1)) \sum_{p=0}^{\frac{N}{2}-1} \sum_{m=0}^{\frac{N}{2}-1} \text{cas}(-2m\omega(2p+2l+1)) f(2p)g(2l+1) \quad (3.117)$$

$$= \frac{4}{9} \sum_{p=0}^{\frac{N}{2}-1} \text{cas}\left(-\left(\frac{N}{2}-2p\right)\omega j\right) f(2p)g\left(\frac{N}{2}-2p\right) \sum_{m=0}^{\frac{N}{2}-1} \text{cas}(-2\pi m) \quad (3.118)$$

$$= \frac{2N}{9} \sum_{p=0}^{\frac{N}{2}-1} (-1)^j \text{cas}(2p\omega j) f(2p)g\left(\frac{N}{2}-2p\right). \quad (3.119)$$

We used the fact that the only contribution in the summation occurs for  $2p+2l+1 \in \{\frac{N}{2}, \frac{3N}{2}\}$  which is equivalent to  $2p+2l+1 = \frac{N}{2}$ . The summations in statement (3.110)

may be further simplified as follows.

$$\frac{4}{9} \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(-\omega j(2l+1)) \sum_{p=0}^{\frac{N}{2}-1} \sum_{m=0}^{\frac{N}{2}-1} \text{cas}(2\omega m(-2p+2l+1)) f(2p)g(2l+1) \quad (3.120)$$

$$= \sum_{l=0}^{\frac{N}{2}-1} \text{cas}((\frac{N}{2}+2p)\omega j) f(2p)g(2p+\frac{N}{2}) \sum_{m=0}^{\frac{N}{2}-1} \text{cas}(2\pi m) \quad (3.121)$$

$$= \frac{2N}{9} \sum_{p=0}^{\frac{N}{2}-1} (-1)^j \text{cas}(-2p\omega j) f(2p)g(2p+\frac{N}{2}). \quad (3.122)$$

We used the fact that the only contribution in the summation occurs for  $-2p+2l+1 = \frac{N}{2}$ .

Using lemma 3.2, (3.87) simplifies to

$$\begin{aligned} & \frac{8}{9} \sum_{m=0}^{\frac{N}{2}-1} \sum_{p=0}^{\frac{N}{2}-1} \sum_{l=0}^{\frac{N}{2}-1} \left( \frac{\text{cas}(2j\omega l)}{2} [\text{cas}(2m\omega(2p+1) + 4l\omega m) \right. \\ & \left. + \text{cas}(2m\omega(2p+1) - 4l\omega m)] + \frac{\text{cas}(-2j\omega l)}{2} [\text{cas}(-2m\omega(2p+1) - 4l\omega m) \right. \\ & \left. - \text{cas}(-2m\omega(2p+1) + 4l\omega m)] \right) f(2p+1)g(2l) \quad (3.123) \end{aligned}$$

$$= \frac{4}{9} \sum_{m=0}^{\frac{N}{2}-1} \sum_{p=0}^{\frac{N}{2}-1} \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(2j\omega l) \text{cas}(2m\omega(2p+1) + 4l\omega m) f(2p+1)g(2l) \quad (3.124)$$

$$+ \frac{4}{9} \sum_{m=0}^{\frac{N}{2}-1} \sum_{p=0}^{\frac{N}{2}-1} \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(2j\omega l) \text{cas}(2m\omega(2p+1) - 4l\omega m) f(2p+1)g(2l) \quad (3.125)$$

$$+ \frac{4}{9} \sum_{m=0}^{\frac{N}{2}-1} \sum_{p=0}^{\frac{N}{2}-1} \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(-2j\omega l) \text{cas}(-2m\omega(2p+1) - 4l\omega m) f(2p+1)g(2l) \quad (3.126)$$

$$- \frac{4}{9} \sum_{m=0}^{\frac{N}{2}-1} \sum_{p=0}^{\frac{N}{2}-1} \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(-2j\omega l) \text{cas}(-2m\omega(2p+1) + 4l\omega m) f(2p+1)g(2l). \quad (3.127)$$

The summations in (3.124) may be further simplified as follows.

$$\frac{4}{9} \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(2j\omega l) \sum_{p=0}^{\frac{N}{2}-1} \sum_{m=0}^{\frac{N}{2}-1} \text{cas}(2m\omega(2p+2l+1)) f(2p+1)g(2l) \quad (3.128)$$

$$= \frac{4}{9} \sum_{p=0}^{\frac{N}{2}-1} \text{cas}((\frac{N}{2}-2p-1)\omega j) f(2p+1)g(\frac{N}{2}-2p-1) \sum_{m=0}^{\frac{N}{2}-1} \text{cas}(2\pi m) \quad (3.129)$$

$$= \frac{2N}{9} \sum_{p=0}^{\frac{N}{2}-1} (-1)^j \text{cas}(-\omega j(2p+1)) f(2p+1) g(\frac{N}{2} - 2p - 1). \quad (3.130)$$

We have used the fact that the only contribution in the summation essentially occurs when  $2p+2l+1 = \frac{N}{2}$ . The summations in (3.125) may be further simplified as follows.

$$\frac{4}{9} \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(2j\omega l) \sum_{p=0}^{\frac{N}{2}-1} \sum_{m=0}^{\frac{N}{2}-1} \text{cas}(2m\omega(2p-2l+1)) f(2p+1) g(2l) \quad (3.131)$$

$$= \frac{4}{9} \sum_{p=0}^{\frac{N}{2}-1} \text{cas}((\frac{N}{2} + 2p + 1)\omega j) f(2p+1) g(\frac{N}{2} + 2p + 1) \sum_{m=0}^{\frac{N}{2}-1} \text{cas}(2\pi m) \quad (3.132)$$

$$= \frac{2N}{9} \sum_{p=0}^{\frac{N}{2}-1} (-1)^j \text{cas}(j\omega(2p+1)) f(2p+1) g(\frac{N}{2} + 2p + 1). \quad (3.133)$$

We have used the fact that the only contribution in the summation occurs when  $2p - 2l + 1 = \frac{N}{2}$  and periodicity  $2\pi$  of the cas function. The summations in (3.126) may be further simplified as follows.

$$\frac{4}{9} \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(-2j\omega l) \sum_{p=0}^{\frac{N}{2}-1} \sum_{m=0}^{\frac{N}{2}-1} \text{cas}(-2m\omega(2p+2l+1)) f(2p+1) g(2l) \quad (3.134)$$

$$= \frac{4}{9} \sum_{p=0}^{\frac{N}{2}-1} \text{cas}((\frac{N}{2} + 2p + 1)\omega j) f(2p+1) g(\frac{N}{2} - 2p - 1) \sum_{m=0}^{\frac{N}{2}-1} \text{cas}(-2\pi m) \quad (3.135)$$

$$= \frac{2N}{9} \sum_{p=0}^{\frac{N}{2}-1} (-1)^j \text{cas}(j\omega(2p+1)) f(2p+1) g(\frac{N}{2} - 2p - 1). \quad (3.136)$$

We have used the fact that the only contribution in the summation essentially occurs when  $2p+2l+1 = \frac{N}{2}$  and periodicity  $2\pi$  of the cas function. The summations in (3.127) may be further simplified as follows.

$$\frac{4}{9} \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(-2j\omega l) \sum_{p=0}^{\frac{N}{2}-1} \sum_{m=0}^{\frac{N}{2}-1} \text{cas}(2m\omega(-2p+2l-1)) f(2p+1) g(2l) \quad (3.137)$$

$$= \frac{4}{9} \sum_{p=0}^{\frac{N}{2}-1} \text{cas}((\frac{N}{2} - 2p - 1)\omega j) f(2p+1) g(\frac{N}{2} + 2p + 1) \sum_{m=0}^{\frac{N}{2}-1} \text{cas}(2\pi m) \quad (3.138)$$

$$= \frac{2N}{9} \sum_{p=0}^{\frac{N}{2}-1} (-1)^j \text{cas}(-j\omega(2p+1)) f(2p+1) g(\frac{N}{2} + 2p + 1). \quad (3.139)$$

We have used the fact that the only contribution in the summation occurs when  $-2p + 2l - 1 = \frac{N}{2}$ .

Using lemma 3.2, (3.88) simplifies to

$$\begin{aligned} & \frac{16}{9} \sum_{m=0}^{\frac{N}{2}-1} \sum_{p=0}^{\frac{N}{2}-1} \sum_{l=0}^{\frac{N}{2}-1} \left( \frac{\text{cas}(\omega j(2l+1))}{2} [\text{cas}(2m\omega(2p+1) + 2m\omega(2l+1)) \right. \\ & \quad \left. + \text{cas}(2m\omega(2p+1) - 2m\omega(2l+1))] + \frac{\text{cas}(-\omega j(2l+1))}{2} \times \right. \\ & \quad \left. [\text{cas}(-2m\omega(2p+1) - 2m\omega(2l+1)) - \text{cas}(-2m\omega(2p+1) + 2m\omega(2l+1))] \right) \\ & \quad \times f(2p+1)g(2l+1) \end{aligned} \tag{3.140}$$

$$\begin{aligned} & = \frac{8}{9} \sum_{m=0}^{\frac{N}{2}-1} \sum_{p=0}^{\frac{N}{2}-1} \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(\omega j(2l+1)) \text{cas}(2m\omega(2p+1) + 2m\omega(2l+1)) f(2p+1) \\ & \quad \times g(2l+1) \end{aligned} \tag{3.141}$$

$$\begin{aligned} & + \frac{8}{9} \sum_{m=0}^{\frac{N}{2}-1} \sum_{p=0}^{\frac{N}{2}-1} \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(\omega j(2l+1)) \text{cas}(2m\omega(2p+1) - 2m\omega(2l+1)) f(2p+1) \\ & \quad \times g(2l+1) \end{aligned} \tag{3.142}$$

$$\begin{aligned} & + \frac{8}{9} \sum_{m=0}^{\frac{N}{2}-1} \sum_{p=0}^{\frac{N}{2}-1} \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(-\omega j(2l+1)) \text{cas}(-2m\omega(2p+1) - 2m\omega(2l+1)) f(2p+1) \\ & \quad \times g(2l+1) \end{aligned} \tag{3.143}$$

$$\begin{aligned} & - \frac{8}{9} \sum_{m=0}^{\frac{N}{2}-1} \sum_{p=0}^{\frac{N}{2}-1} \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(-\omega j(2l+1)) \text{cas}(-2m\omega(2p+1) + 2m\omega(2l+1)) f(2p+1) \\ & \quad \times g(2l+1). \end{aligned} \tag{3.144}$$

The summations in statement (3.141) may be further simplified as follows.

$$\frac{8}{9} \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(\omega j(2l+1)) \sum_{p=0}^{\frac{N}{2}-1} \sum_{m=0}^{\frac{N}{2}-1} \text{cas}(4m\omega(p+l+1)) f(2p+1)g(2l+1) \quad (3.145)$$

$$= \frac{8}{9} \sum_{p=0}^{\frac{N}{2}-1} \text{cas}((N-2p-1)\omega j) f(2p+1)g(-2p-1) \sum_{m=0}^{\frac{N}{2}-1} \text{cas}(4\pi m) \quad (3.146)$$

$$= \frac{4N}{9} \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(-\omega j(2p+1)) f(2p+1)g(-2p-1). \quad (3.147)$$

We used the fact that the only contribution to the summation occurs for  $p+l+1 = \frac{N}{2}$ .

The summations in statement (3.142) may be further simplified as follows.

$$\frac{8}{9} \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(\omega j(2l+1)) \sum_{p=0}^{\frac{N}{2}-1} \sum_{m=0}^{\frac{N}{2}-1} \text{cas}(4m\omega(p-l)) f(2p+1)g(2l+1) \quad (3.148)$$

$$= \frac{8}{9} \sum_{p=0}^{\frac{N}{2}-1} \text{cas}((2p+1)\omega j) f(2p+1)g(2l+1) \sum_{m=0}^{\frac{N}{2}-1} 1 \quad (3.149)$$

$$= \frac{4N}{9} \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(\omega j(2p+1)) f(2p+1)g(2p+1). \quad (3.150)$$

We used the fact that the only contribution to the summation occurs for  $p = l$ . The

summations in statement (3.143) may be further simplified as follows.

$$\frac{8}{9} \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(-\omega j(2l+1)) \sum_{p=0}^{\frac{N}{2}-1} \sum_{m=0}^{\frac{N}{2}-1} \text{cas}(-4m\omega(p+l+1)) f(2p+1)g(2l+1) \quad (3.151)$$

$$= \frac{8}{9} \sum_{p=0}^{\frac{N}{2}-1} \text{cas}((N+2p+1)\omega j) f(2p+1)g(-2p-1) \sum_{m=0}^{\frac{N}{2}-1} \text{cas}(-4\pi m) \quad (3.152)$$

$$= \frac{4N}{9} \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(\omega j(2p+1)) f(2p+1)g(-2p-1). \quad (3.153)$$

We used the fact that the only contribution to the summation occurs for  $p+l+1 = \frac{N}{2}$ .

The summations in statement (3.144) may be further simplified as follows.

$$\frac{8}{9} \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(-\omega j(2l+1)) \sum_{p=0}^{\frac{N}{2}-1} \sum_{m=0}^{\frac{N}{2}-1} \text{cas}(4m\omega(l-p)) f(2p+1)g(2l+1) \quad (3.154)$$

$$= \frac{8}{9} \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(-\omega j(2p+1)) f(2p+1)g(2p+1) \sum_{m=0}^{\frac{N}{2}-1} 1 \quad (3.155)$$

$$= \frac{4N}{9} \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(-\omega j(2p+1)) f(2p+1)g(2p+1). \quad (3.156)$$

We used the fact that the only contribution to the summation occurs for  $p = l$ .

Combining (3.96), (3.99), (3.102) and (3.105) statement (3.85) simplifies to

$$\begin{aligned} & \frac{N}{9} \left[ \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(2p\omega j) f(2p)g(2p) + \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(2p\omega j) f(2p)g(-2p) \right. \\ & \left. + \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(-2p\omega j) f(2p)g(-2p) - \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(-2p\omega j) f(2p)g(2p) \right]. \quad (3.157) \end{aligned}$$

Combining (3.113), (3.116), (3.119) and (3.122), statement (3.86) simplifies to

$$\begin{aligned} & \frac{2N}{9} \left[ \sum_{p=0}^{\frac{N}{2}-1} (-1)^j \text{cas}(-2p\omega j) f(2p)g\left(\frac{N}{2} - 2p\right) + \sum_{p=0}^{\frac{N}{2}-1} (-1)^j \text{cas}(2p\omega j) f(2p)g\left(2p + \frac{N}{2}\right) \right. \\ & \left. + \sum_{p=0}^{\frac{N}{2}-1} (-1)^j \text{cas}(2p\omega j) f(2p)g\left(\frac{N}{2} - 2p\right) - \sum_{p=0}^{\frac{N}{2}-1} (-1)^j \text{cas}(-2p\omega j) f(2p)g\left(2p + \frac{N}{2}\right) \right]. \quad (3.158) \end{aligned}$$

Combining (3.130), (3.133), (3.136) and (3.139), statement (3.87) simplifies to

$$\begin{aligned} & \frac{2N}{9} \left[ \sum_{p=0}^{\frac{N}{2}-1} (-1)^j \text{cas}(-\omega j(2p+1)) f(2p+1)g\left(\frac{N}{2} - 2p - 1\right) \right. \\ & + \sum_{p=0}^{\frac{N}{2}-1} (-1)^j \text{cas}(j\omega(2p+1)) f(2p+1)g\left(\frac{N}{2} + 2p + 1\right) \\ & + \sum_{p=0}^{\frac{N}{2}-1} (-1)^j \text{cas}(j\omega(2p+1)) f(2p+1)g\left(\frac{N}{2} - 2p - 1\right) \\ & \left. - \sum_{p=0}^{\frac{N}{2}-1} (-1)^j \text{cas}(-j\omega(2p+1)) f(2p+1)g\left(\frac{N}{2} + 2p + 1\right) \right]. \quad (3.159) \end{aligned}$$

Combining (3.147), (3.150), (3.153) and (3.156), statement (3.88) simplifies to

$$\begin{aligned}
& \frac{4N}{9} \left[ \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(-\omega j(2p+1))f(2p+1)g(-2p-1) \right. \\
& \quad + \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(\omega j(2p+1))f(2p+1)g(2p+1) \\
& \quad + \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(\omega j(2p+1))f(2p+1)g(-2p-1) \\
& \quad \left. - \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(-\omega j(2p+1))f(2p+1)g(2p+1) \right]. \tag{3.160}
\end{aligned}$$

By a similar analysis the second summation in (3.84) simplifies to the following four terms:

$$\begin{aligned}
& \frac{4}{9} \sum_{m=0}^{\frac{N}{2}-1} \sum_{p=0}^{\frac{N}{2}-1} \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(2\omega p(2m+1)) \text{cas}(2\omega l(j-2m-1))f(2p)g(2l) \\
& = \frac{N}{9} \left[ \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(2p\omega j)f(2p)g(2p) + \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(2p\omega j)f(2p)g(-2p) \right. \\
& \quad \left. + \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(-2p\omega j)f(2p)g(-2p) - \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(-2p\omega j)f(2p)g(2p) \right]. \tag{3.161}
\end{aligned}$$

$$\begin{aligned}
& \frac{8}{9} \sum_{m=0}^{\frac{N}{2}-1} \sum_{p=0}^{\frac{N}{2}-1} \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(2p\omega(2m+1)) \text{cas}(\omega(2l+1)(j-2m-1))f(2p)g(2l+1) \\
& = -\frac{2N}{9} \left[ \sum_{p=0}^{\frac{N}{2}-1} (-1)^j \text{cas}(-2p\omega j)f(2p)g\left(\frac{N}{2}-2p\right) + \sum_{p=0}^{\frac{N}{2}-1} (-1)^j \text{cas}(2p\omega j)f(2p)g\left(2p+\frac{N}{2}\right) \right. \\
& \quad \left. + \sum_{p=0}^{\frac{N}{2}-1} (-1)^j \text{cas}(2p\omega j)f(2p)g\left(\frac{N}{2}-2p\right) - \sum_{p=0}^{\frac{N}{2}-1} (-1)^j \text{cas}(-2p\omega j)f(2p)g\left(2p+\frac{N}{2}\right) \right]. \tag{3.162}
\end{aligned}$$

$$\begin{aligned}
& \frac{8}{9} \sum_{m=0}^{\frac{N}{2}-1} \sum_{p=0}^{\frac{N}{2}-1} \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(\omega(2p+1)(2m+1)) \text{cas}(2\omega l(j-2m-1)) f(2p+1)g(2l) \\
&= -\frac{2N}{9} \left[ \sum_{p=0}^{\frac{N}{2}-1} (-1)^j \text{cas}(-\omega j(2p+1)) f(2p+1)g\left(\frac{N}{2}-2p-1\right) \right. \\
&+ \sum_{p=0}^{\frac{N}{2}-1} (-1)^j \text{cas}(j\omega(2p+1)) f(2p+1)g\left(\frac{N}{2}+2p+1\right) \\
&+ \sum_{p=0}^{\frac{N}{2}-1} (-1)^j \text{cas}(j\omega(2p+1)) f(2p+1)g\left(\frac{N}{2}-2p-1\right) \\
&\left. - \sum_{p=0}^{\frac{N}{2}-1} (-1)^j \text{cas}(-j\omega(2p+1)) f(2p+1)g\left(\frac{N}{2}+2p+1\right) \right]. \tag{3.163}
\end{aligned}$$

$$\begin{aligned}
& \frac{16}{9} \sum_{m=0}^{\frac{N}{2}-1} \sum_{p=0}^{\frac{N}{2}-1} \sum_{l=0}^{\frac{N}{2}-1} \text{cas}(\omega(2p+1)(2m+1)) \text{cas}(\omega(j-2m-1)(2l+1)) f(2p+1)g(2l+1) \\
&= \frac{4N}{9} \left[ \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(-\omega j(2p+1)) f(2p+1)g(-2p-1) \right. \\
&+ \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(\omega j(2p+1)) f(2p+1)g(2p+1) \\
&+ \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(\omega j(2p+1)) f(2p+1)g(-2p-1) \\
&\left. - \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(-\omega j(2p+1)) f(2p+1)g(2p+1) \right]. \tag{3.164}
\end{aligned}$$

An interesting observation is that (3.158) is the negative of (3.162) as well as (3.159) is the negative of (3.163) whilst (3.157) and (3.160) are duplicates of (3.161) and (3.164) respectively. Hence equation (3.84) simplifies to

$$(\mathbf{Hf} \otimes \mathbf{Hg})(j) = \frac{2N}{9} \left[ \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(2p\omega j) f(2p)g(2p) + \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(2p\omega j) f(2p)g(-2p) \right]$$

$$\begin{aligned}
& + \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(-2p\omega j) f(2p)g(-2p) - \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(-2p\omega j) f(2p)g(2p) \Big] \\
& + \frac{8N}{9} \left[ \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(\omega j(2p+1)) f(2p+1)g(2p+1) \right. \\
& + \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(\omega j(2p+1)) f(2p+1)g(-2p-1) \\
& + \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(-\omega j(2p+1)) f(2p+1)g(-2p-1) \\
& \left. - \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(-\omega j(2p+1)) f(2p+1)g(2p+1) \right]. \tag{3.165}
\end{aligned}$$

We wish to express equation (3.165) in terms of the transform and this requires some manipulation to yield

$$\begin{aligned}
& (\mathbf{Hf} \otimes \mathbf{Hg})(j) \\
& = \frac{N}{3} \left[ \frac{2}{3} \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(2p\omega j) f(2p)g(2p) + \frac{4}{3} \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(\omega j(2p+1)) f(2p+1)g(2p+1) \right. \\
& + \frac{2}{3} \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(2p\omega j) f(2p)g(-2p) + \frac{4}{3} \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(\omega j(2p+1)) f(2p+1)g(-2p-1) \\
& + \frac{2}{3} \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(2p\omega j) f(-2p)g(2p) + \frac{4}{3} \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(\omega j(2p+1)) f(-2p-1)g(2p+1) \\
& \left. - \frac{2}{3} \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(2p\omega j) f(-2p)g(-2p) + \frac{4}{3} \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(\omega j(2p+1)) f(-2p-1)g(-2p-1) \right] \\
& + \frac{4N}{9} \left[ \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(\omega j(2p+1)) f(2p+1)g(2p+1) \right. \\
& + \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(\omega j(2p+1)) f(2p+1)g(-2p-1) \\
& \left. + \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(\omega j(2p+1)) f(-2p-1)g(2p+1) \right]
\end{aligned}$$

$$\begin{aligned}
& - \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(\omega j(2p+1))f(-2p-1)g(-2p-1) \Big] \\
& = \frac{N}{3} \left[ \mathbf{H}(\mathbf{f} \star \mathbf{g})(j) + \mathbf{H}(\mathbf{f} \star \mathbf{g}_R)(j) + \mathbf{H}(\mathbf{f}_R \star \mathbf{g})(j) - \mathbf{H}(\mathbf{f}_R \star \mathbf{g}_R)(j) \right] \\
& + \frac{4N}{9} \left[ \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(\omega j(2p+1))f(2p+1)g(2p+1) \right. \\
& + \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(\omega j(2p+1))f(2p+1)g(-2p-1) \\
& + \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(\omega j(2p+1))f(-2p-1)g(2p+1) \\
& \left. - \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(\omega j(2p+1))f(-2p-1)g(-2p-1) \right]. \tag{3.166}
\end{aligned}$$

It should be noted that the reason we have arrived at a reversal of signals is simply due to the fact that we have replaced  $p$  by  $-p$  in order to introduce the transform. The remaining summations in (3.166) may be written in terms of the transform by employing (3.69) and (3.70) to obtain

$$\sum_{p=0}^{\frac{N}{2}-1} \text{cas}(\omega j(2p+1))f(2p+1)g(2p+1) = \frac{3}{8} \left[ \mathbf{H}(\mathbf{f} \star \mathbf{g})(j) - \mathbf{H}(\mathbf{f} \star \mathbf{g})(j + \frac{N}{2}) \right]. \tag{3.167}$$

$$\sum_{p=0}^{\frac{N}{2}-1} \text{cas}(\omega j(2p+1))f(2p+1)g(-2p-1) = \frac{3}{8} \left[ \mathbf{H}(\mathbf{f} \star \mathbf{g}_R)(j) - \mathbf{H}(\mathbf{f} \star \mathbf{g}_R)(j + \frac{N}{2}) \right]. \tag{3.168}$$

$$\sum_{p=0}^{\frac{N}{2}-1} \text{cas}(\omega j(2p+1))f(-2p-1)g(2p+1) = \frac{3}{8} \left[ \mathbf{H}(\mathbf{f}_R \star \mathbf{g})(j) - \mathbf{H}(\mathbf{f}_R \star \mathbf{g})(j + \frac{N}{2}) \right]. \tag{3.169}$$

$$\sum_{p=0}^{\frac{N}{2}-1} \text{cas}(\omega j(2p+1))f(-2p-1)g(-2p-1) = \frac{3}{8} \left[ \mathbf{H}(\mathbf{f}_R \star \mathbf{g}_R)(j) - \mathbf{H}(\mathbf{f}_R \star \mathbf{g}_R)(j + \frac{N}{2}) \right]. \tag{3.170}$$

We now define a vector  $\mathbf{S}$  by

$$\mathbf{S} = \mathbf{f} \star \mathbf{g} + \mathbf{f} \star \mathbf{g}_R + \mathbf{f}_R \star \mathbf{g} - \mathbf{f}_R \star \mathbf{g}_R \tag{3.171}$$

thus allowing us to write the final form of equation (3.84) as

$$(\mathbf{H}\mathbf{f} \otimes \mathbf{H}\mathbf{g}) = \frac{N}{2} \left( \mathbf{H}\mathbf{S} - \frac{1}{3} \mathbf{S}_{(N/2)} \right). \quad (3.172)$$

The circular cross correlation  $\mathbf{r} = \mathbf{f} \star \mathbf{g}$  is defined component-wise by

$$r(j) = \sum_{m=0}^{N-1} f(m)g(j+m). \quad (3.173)$$

The cross correlation property bares striking similarities to that of the convolution property. As such we will not consider the detailed derivation for this property, rather we will use an alternative approach to arrive at the final result.

Component-wise the effect of the transpose on a signal can be obtained from (2.9) and is given by [15]

$$\begin{aligned} \mathbf{H}^T \mathbf{f}(2k) &= \frac{2}{3} \sum_{j=0}^{N-1} \text{cas}(2k\omega j) f(j) \\ &= \frac{2}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(4k\omega j) f(2j) + \frac{2}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(2k\omega(2j+1)) f(2j+1) \end{aligned} \quad (3.174)$$

and

$$\begin{aligned} \mathbf{H}^T \mathbf{f}(2k+1) &= \frac{4}{3} \sum_{j=0}^{N-1} \text{cas}((2k+1)\omega j) f(j) \\ &= \frac{4}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}((2k+1)\omega 2j) f(2j) + \frac{4}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}((2k+1)\omega(2j+1)) f(2j+1). \end{aligned} \quad (3.175)$$

We see that the index  $2k + \frac{N}{2}$  is odd, hence from (3.175) it follows that

$$\mathbf{H}^T \mathbf{f}(2k + \frac{N}{2}) = \frac{4}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(4k\omega j) f(2j) - \frac{4}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(2k\omega(2j+1)) f(2j+1). \quad (3.176)$$

We see that the index  $2k + \frac{N}{2} + 1$  is even, hence from (3.174) it follows that

$$\mathbf{H}^T \mathbf{f}(2k+1 + \frac{N}{2}) = \frac{2}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}((2k+1)\omega 2j) f(2j) - \frac{2}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}((2k+1)\omega(2j+1)) f(2j+1). \quad (3.177)$$

From (2.2) and (3.174) it follows that

$$\mathbf{Hf}(2k) = \mathbf{H}^T \mathbf{f}(2k) + \frac{2}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(2k\omega(2j+1))f(2j+1). \quad (3.178)$$

From (3.174) and (3.176) it can be shown that

$$\frac{2}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(2k\omega(2j+1))f(2j+1) = \frac{1}{2} \mathbf{H}^T \mathbf{f}(2k) - \frac{1}{4} \mathbf{H}^T \mathbf{f}(2k + \frac{N}{2}). \quad (3.179)$$

Hence (3.178) can be written in the form

$$\mathbf{Hf}(2k) = \frac{3}{2} \mathbf{H}^T \mathbf{f}(2k) - \frac{1}{4} \mathbf{H}^T \mathbf{f}(2k + \frac{N}{2}). \quad (3.180)$$

It may be shown similarly from (2.2), (3.175) and (3.177) that

$$\mathbf{Hf}(2k + \frac{N}{2}) = \frac{3}{4} \mathbf{H}^T \mathbf{f}(2k + \frac{N}{2}) - \frac{1}{2} \mathbf{H}^T \mathbf{f}(2k). \quad (3.181)$$

From (3.180) and (3.181) we may recover the effect of the transpose on a signal to get

$$\mathbf{H}^T \mathbf{f}(2k) = \frac{3}{4} \mathbf{Hf}(2k) + \frac{1}{4} \mathbf{Hf}(2k + \frac{N}{2}) \quad (3.182)$$

$$\mathbf{H}^T \mathbf{f}(2k + \frac{N}{2}) = \frac{1}{2} \mathbf{Hf}(2k) + \frac{3}{2} \mathbf{Hf}(2k + \frac{N}{2}). \quad (3.183)$$

### 3.8 Cross correlation

It is shown that the even component of the circular cross correlation is given by [15]

$$\begin{aligned} \mathbf{Hr}(2k) &= \frac{9}{8} \left[ \mathbf{H}^T \mathbf{f}(2k) \mathbf{H}^T \mathbf{g}(2k) - \mathbf{H}^T \mathbf{f}(2k) \mathbf{H}^T \mathbf{g}(-2k) \right. \\ &\quad \left. + \mathbf{H}^T \mathbf{f}(-2k) \mathbf{H}^T \mathbf{g}(2k) + \mathbf{H}^T \mathbf{f}(-2k) \mathbf{H}^T \mathbf{g}(-2k) \right] \\ &- \frac{3}{32} \left[ \mathbf{H}^T \mathbf{f}(2k + \frac{N}{2}) \mathbf{H}^T \mathbf{g}(2k + \frac{N}{2}) - \mathbf{H}^T \mathbf{f}(2k + \frac{N}{2}) \mathbf{H}^T \mathbf{g}(-2k - \frac{N}{2}) \right. \\ &\quad \left. + \mathbf{H}^T \mathbf{f}(-2k - \frac{N}{2}) \mathbf{H}^T \mathbf{g}(2k + \frac{N}{2}) + \mathbf{H}^T \mathbf{f}(-2k - \frac{N}{2}) \mathbf{H}^T \mathbf{g}(-2k - \frac{N}{2}) \right]. \quad (3.184) \end{aligned}$$

Using (3.182) and (3.183) in (3.184) we may write the correlation purely in terms of the transform to get

$$\begin{aligned}
\mathbf{Hr}(2k) &= \frac{9}{8} \left[ \left( \frac{3}{4} \mathbf{Hf}(2k) + \frac{1}{4} \mathbf{Hf}(2k + \frac{N}{2}) \right) \left( \frac{3}{4} \mathbf{Hg}(2k) + \frac{1}{4} \mathbf{Hg}(2k + \frac{N}{2}) \right) \right. \\
&\quad - \left( \frac{3}{4} \mathbf{Hf}(2k) + \frac{1}{4} \mathbf{Hf}(2k + \frac{N}{2}) \right) \left( \frac{3}{4} \mathbf{Hg}(-2k) + \frac{1}{4} \mathbf{Hg}(-2k - \frac{N}{2}) \right) \\
&\quad + \left( \frac{3}{4} \mathbf{Hf}(-2k) + \frac{1}{4} \mathbf{Hf}(-2k - \frac{N}{2}) \right) \left( \frac{3}{4} \mathbf{Hg}(2k) + \frac{1}{4} \mathbf{Hg}(2k + \frac{N}{2}) \right) \\
&\quad \left. + \left( \frac{3}{4} \mathbf{Hf}(-2k) + \frac{1}{4} \mathbf{Hf}(-2k - \frac{N}{2}) \right) \left( \frac{3}{4} \mathbf{Hg}(-2k) + \frac{1}{4} \mathbf{Hg}(-2k - \frac{N}{2}) \right) \right] \\
&\quad - \frac{3}{32} \left[ \left( \frac{1}{2} \mathbf{Hf}(2k) + \frac{3}{2} \mathbf{Hf}(2k + \frac{N}{2}) \right) \left( \frac{1}{2} \mathbf{Hg}(2k) + \frac{3}{2} \mathbf{Hg}(2k + \frac{N}{2}) \right) \right. \\
&\quad - \left( \frac{1}{2} \mathbf{Hf}(2k) + \frac{3}{2} \mathbf{Hf}(2k + \frac{N}{2}) \right) \left( \frac{1}{2} \mathbf{Hg}(-2k) + \frac{3}{2} \mathbf{Hg}(-2k - \frac{N}{2}) \right) \\
&\quad + \left( \frac{1}{2} \mathbf{Hf}(-2k) + \frac{3}{2} \mathbf{Hf}(-2k - \frac{N}{2}) \right) \left( \frac{1}{2} \mathbf{Hg}(2k) + \frac{3}{2} \mathbf{Hg}(2k + \frac{N}{2}) \right) \\
&\quad \left. + \left( \frac{1}{2} \mathbf{Hf}(-2k) + \frac{3}{2} \mathbf{Hf}(-2k - \frac{N}{2}) \right) \left( \frac{1}{2} \mathbf{Hg}(-2k) + \frac{3}{2} \mathbf{Hg}(-2k - \frac{N}{2}) \right) \right] \quad (3.185)
\end{aligned}$$

$$\begin{aligned}
&= \frac{39}{64} \left[ \mathbf{Hf}(2k) \mathbf{Hg}(2k) - \mathbf{Hf}(2k) \mathbf{Hg}(-2k) + \mathbf{Hf}(-2k) \mathbf{Hg}(2k) + \mathbf{Hf}(-2k) \mathbf{Hg}(-2k) \right] \\
&\quad - \frac{9}{64} \left[ \mathbf{Hf}(2k + \frac{N}{2}) \mathbf{Hg}(2k + \frac{N}{2}) - \mathbf{Hf}(2k + \frac{N}{2}) \mathbf{Hg}(-2k - \frac{N}{2}) \right. \\
&\quad \left. + \mathbf{Hf}(-2k - \frac{N}{2}) \mathbf{Hg}(2k + \frac{N}{2}) + \mathbf{Hf}(-2k - \frac{N}{2}) \mathbf{Hg}(-2k - \frac{N}{2}) \right] \\
&\quad + \frac{9}{64} \left[ \mathbf{Hf}(2k) \mathbf{Hg}(2k + \frac{N}{2}) - \mathbf{Hf}(2k) \mathbf{Hg}(-2k - \frac{N}{2}) \right. \\
&\quad \left. + \mathbf{Hf}(-2k) \mathbf{Hg}(2k + \frac{N}{2}) + \mathbf{Hf}(-2k) \mathbf{Hg}(-2k - \frac{N}{2}) + \mathbf{Hf}(2k + \frac{N}{2}) \mathbf{Hg}(2k) \right. \\
&\quad \left. - \mathbf{Hf}(2k + \frac{N}{2}) \mathbf{Hg}(-2k) + \mathbf{Hf}(-2k - \frac{N}{2}) \mathbf{Hg}(2k) + \mathbf{Hf}(-2k - \frac{N}{2}) \mathbf{Hg}(-2k) \right]. \quad (3.186)
\end{aligned}$$

It may be proved that (3.186) is also valid for the odd components. In order to write the correlation in a more compact form we define two vectors

$$\mathbf{R} = \mathbf{Hf} \cdot \star \mathbf{Hg} - \mathbf{Hf} \cdot \star \mathbf{Hg}_R + \mathbf{Hf}_R \cdot \star \mathbf{Hg} + \mathbf{Hf}_R \cdot \star \mathbf{Hg}_R \quad (3.187)$$

$$\mathbf{M} = \mathbf{Hf} \cdot \star \mathbf{Hg}_{\frac{N}{2}} - \mathbf{Hf} \cdot \star \mathbf{Hg}_{\frac{N}{2}R} + \mathbf{Hf}_R \cdot \star \mathbf{Hg}_{\frac{N}{2}} + \mathbf{Hf}_R \cdot \star \mathbf{Hg}_{\frac{N}{2}R}. \quad (3.188)$$

Hence the correlation can be written as

$$\mathbf{Hr} = \frac{39}{64}\mathbf{R} - \frac{9}{64}\mathbf{R}_{(N/2)} + \frac{9}{64}(\mathbf{M} + \mathbf{M}_{(N/2)}). \quad (3.189)$$

# Chapter 4

## Spectrum of $\mathbf{H}$

From (3.34) and (3.39) we deduce that the matrix  $\mathbf{H}^2$  is tridiagonal. The diagonal of  $\mathbf{H}^2$  has the form  $[\frac{2}{3}N, \frac{4}{3}N, \frac{2}{3}N, \frac{4}{3}N, \dots, \frac{2}{3}N, \frac{4}{3}N]$  of length  $N$ , the super diagonal has the form  $[-\frac{4}{9}N, -\frac{2}{9}N, -\frac{4}{9}N, -\frac{2}{9}N, \dots, -\frac{2}{9}N, -\frac{4}{9}N]$  of length  $\frac{N}{2}$  located at column  $\frac{N}{2}$  in the  $0_{th}$  row and the sub diagonal has the form  $[-\frac{2}{9}N, -\frac{4}{9}N, -\frac{2}{9}N, -\frac{4}{9}N, \dots, -\frac{4}{9}N, -\frac{2}{9}N]$  of length  $\frac{N}{2}$  located at row  $\frac{N}{2}$  in the  $0_{th}$  column.

Let  $\mathbf{X} = [X(0), X(1), X(2), \dots, X(N-1)]$  be an eigenvector of  $\mathbf{H}^2$  which corresponds to the eigenvalue  $\lambda$ , then we can write

$$\mathbf{H}^2 \mathbf{X} = \lambda \mathbf{X}. \quad (4.1)$$

Now let  $p$  be the index such that  $|X(2k)| \leq |X(2p)|$  and  $m$  be the index such that  $|X(2k+1)| \leq |X(2m+1)|$  for  $k = 0, 1, \dots, \frac{N}{2} - 1$ . Using the duality property (3.34), (3.39) results in

$$\mathbf{H}^2 \mathbf{X}(2p) = \frac{2}{3}NX(2p) - \frac{4}{9}NX(2p + \frac{N}{2}) \quad (4.2)$$

$$\mathbf{H}^2 \mathbf{X}(2p + \frac{N}{2}) = \frac{4}{3}NX(2p + \frac{N}{2}) - \frac{2}{9}NX(2p). \quad (4.3)$$

Since  $\mathbf{X}$  is an eigenvector of  $\mathbf{H}^2$  it must be true that

$$\frac{2}{3}NX(2p) - \frac{4}{9}NX(2p + \frac{N}{2}) = \lambda X(2p) \quad (4.4)$$

$$\frac{4}{3}NX(2p + \frac{N}{2}) - \frac{2}{9}NX(2p) = \lambda X(2p + \frac{N}{2}). \quad (4.5)$$

Hence system (4.4)-(4.5) can be written as

$$(\frac{2}{3}N - \lambda)X(2p) - \frac{4}{9}NX(2p + \frac{N}{2}) = 0 \quad (4.6)$$

$$-\frac{2}{9}NX(2p) + (\frac{4}{3}N - \lambda)X(2p + \frac{N}{2}) = 0. \quad (4.7)$$

Now if  $X(2p) = X(2m + 1) = 0$  then  $\mathbf{X}$  cannot be an eigenvector. If  $X(2p) \neq 0$  then  $X(2p + \frac{N}{2}) = 0$  will contradict (4.7). Hence the linear system (4.6)-(4.7) has a non trivial solution.

Using the duality property (3.34), (3.39) results in

$$\mathbf{H}^2\mathbf{X}(2m + 1) = \frac{4}{3}NX(2m + 1) - \frac{2}{9}NX(2m + 1 + \frac{N}{2}) \quad (4.8)$$

$$\mathbf{H}^2\mathbf{X}(2m + 1 + \frac{N}{2}) = -\frac{4}{9}NX(2m + 1) + \frac{2}{3}NX(2m + 1 + \frac{N}{2}). \quad (4.9)$$

Since  $\mathbf{X}$  is an eigenvector of  $\mathbf{H}^2$  it must be true that

$$\frac{4}{3}NX(2m + 1) - \frac{2}{9}NX(2m + 1 + \frac{N}{2}) = \lambda X(2m + 1) \quad (4.10)$$

$$-\frac{4}{9}NX(2m + 1) + \frac{2}{3}NX(2m + 1 + \frac{N}{2}) = \lambda X(2m + 1 + \frac{N}{2}). \quad (4.11)$$

Hence system (4.10)-(4.11) can be written as

$$(\frac{4}{3}N - \lambda)X(2m + 1) - \frac{2}{9}NX(2m + 1 + \frac{N}{2}) = 0 \quad (4.12)$$

$$-\frac{4}{9}NX(2m + 1) + (\frac{2}{3}N - \lambda)X(2m + 1 + \frac{N}{2}) = 0. \quad (4.13)$$

If  $X(2m + 1) \neq 0$  then  $X(2m + 1 + \frac{N}{2}) = 0$  will contradict (4.13). Hence the linear

system (4.12)-(4.13) has a non trivial solution. Linear systems (4.6)-(4.7) and (4.12)-(4.13) imply that

$$\begin{vmatrix} \frac{2}{3}N - \lambda & -\frac{4}{9}N \\ -\frac{2}{9}N & \frac{4}{3}N - \lambda \end{vmatrix} = 0. \quad (4.14)$$

This results in

$$\left(\frac{2}{3}N - \lambda\right)\left(\frac{4}{3}N - \lambda\right) - \frac{8}{81}N^2 = 0, \quad (4.15)$$

which can easily be solved to yield the eigenvalues of  $\mathbf{H}^2$ , namely  $r_1^2 = N + \frac{N}{9}\sqrt{17}$  and  $r_2^2 = N - \frac{N}{9}\sqrt{17}$  where

$$r_1 = \frac{\sqrt{9 + \sqrt{17}}\sqrt{N}}{3} \quad (4.16)$$

$$r_2 = \frac{\sqrt{9 - \sqrt{17}}\sqrt{N}}{3}. \quad (4.17)$$

Hence we may deduce that the spectrum  $\sigma(\mathbf{H}) \subseteq \{\pm r_1, \pm r_2\}$ .

## 4.1 Minimal polynomial

The minimal polynomial  $P_m$  of an  $n \times n$  matrix  $\mathbf{H}$  over a field  $\mathbb{F}$  is the monic polynomial over  $\mathbb{F}$  of least degree such that  $P_m(\mathbf{H}) = \mathbf{0}$ . Any other polynomial  $Q$  with  $Q(\mathbf{H}) = \mathbf{0}$  is a polynomial multiple of  $P_m$ .

In order to derive an expression for the minimal polynomial of  $\mathbf{H}$  we require the effect of  $\mathbf{H}^4$  on a signal  $\mathbf{f}$ . Hence

$$\begin{aligned} \mathbf{H}^4\mathbf{f}(2k) &= \mathbf{H}^2[\mathbf{H}^2\mathbf{f}](2k) \\ &= \mathbf{H}^2\mathbf{g}(2k), \end{aligned} \quad (4.18)$$

where  $\mathbf{g} = \mathbf{H}^2 \mathbf{f}$ . By recursively using the duality property (3.34) and (3.39) we have

$$\begin{aligned}
\mathbf{H}^2 \mathbf{g}(2k) &= \frac{2}{3} N g(2k) - \frac{4}{9} N g(2k + \frac{N}{2}) \\
&= \frac{2}{3} N \mathbf{H}^2 \mathbf{f}(2k) - \frac{4}{9} N \mathbf{H}^2 \mathbf{f}(2k + \frac{N}{2}) \\
&= \frac{2}{3} N \left[ \frac{2}{3} N f(2k) - \frac{4}{9} N f(2k + \frac{N}{2}) \right] - \frac{4}{9} N \left[ \frac{4}{3} N f(2k + \frac{N}{2}) - \frac{2}{9} N f(2k) \right] \\
&= \frac{4}{9} N^2 f(2k) - \frac{8}{27} N^2 f(2k + \frac{N}{2}) - \frac{16}{27} N^2 f(2k + \frac{N}{2}) + \frac{8}{81} N^2 f(2k) \\
&= \frac{44}{81} N^2 f(2k) - \frac{8}{9} N^2 f(2k + \frac{N}{2}) \\
&= \mathbf{H}^4 \mathbf{f}(2k).
\end{aligned} \tag{4.19}$$

We also note that

$$\begin{aligned}
-(r_1^2 + r_2^2) \mathbf{H}^2 \mathbf{f}(2k) &= -2N \left( \frac{2}{3} N f(2k) - \frac{4}{9} N f(2k + \frac{N}{2}) \right) \\
&= -\frac{4}{3} N^2 f(2k) + \frac{8}{9} N^2 f(2k + \frac{N}{2})
\end{aligned} \tag{4.20}$$

and

$$r_1^2 r_2^2 \mathbf{I} \mathbf{f}(2k) = \frac{64}{81} N^2 f(2k). \tag{4.21}$$

Summing equations (4.19), (4.20) and (4.21) we observe the following

$$\begin{aligned}
&\left[ \mathbf{H}^4 - (r_1^2 + r_2^2) \mathbf{H}^2 + r_1^2 r_2^2 \mathbf{I} \right] \mathbf{f}(2k) \\
&= \frac{44}{81} N^2 f(2k) - \frac{8}{9} N^2 f(2k + \frac{N}{2}) - \frac{4}{3} N^2 f(2k) + \frac{8}{9} N^2 f(2k + \frac{N}{2}) + \frac{64}{81} N^2 f(2k) \\
&= \frac{108}{81} N^2 f(2k) - \frac{4}{3} N^2 f(2k) \\
&= 0.
\end{aligned} \tag{4.22}$$

In a similar manner it may be shown that

$$\left[ \mathbf{H}^4 - (r_1^2 + r_2^2) \mathbf{H}^2 + r_1^2 r_2^2 \mathbf{I} \right] \mathbf{f}(2k + 1) = 0.$$

Hence we may conclude that

$$\mathbf{H}^4 - (r_1^2 + r_2^2)\mathbf{H}^2 + r_1^2 r_2^2 \mathbf{I} = \mathbf{0}. \quad (4.23)$$

It follows from (4.23) that the minimal polynomial has degree four and is given by

$$\begin{aligned} P_4(\lambda) &= \lambda^4 - (r_1^2 + r_2^2)\lambda^2 + r_1^2 r_2^2 \\ &= (\lambda^2 - r_1^2)(\lambda^2 - r_2^2) \\ &= (\lambda - r_1)(\lambda + r_1)(\lambda - r_2)(\lambda + r_2). \end{aligned} \quad (4.24)$$

In our case we can see that  $\mathbf{H}$  is diagonalizable since the minimal polynomial has linear elementary divisors [13], furthermore  $\sigma(\mathbf{H}) = \{\pm r_1, \pm r_2\}$ .

We now derive an expression for the effect of  $\mathbf{H}^3$  on a signal  $\mathbf{f}$ . Hence

$$\begin{aligned} \mathbf{H}^3 \mathbf{f}(2k) &= \mathbf{H}^2 [\mathbf{H} \mathbf{f}](2k) \\ &= \mathbf{H}^2 \mathbf{g}(2k), \end{aligned} \quad (4.25)$$

where  $\mathbf{g} = \mathbf{H} \mathbf{f}$ . Using the duality property again we arrive at

$$\begin{aligned} &\mathbf{H}^2 \mathbf{g}(2k) \\ &= \frac{2}{3} N g(2k) - \frac{4}{9} N g(2k + \frac{N}{2}) \\ &= \frac{2}{3} N \mathbf{H} \mathbf{f}(2k) - \frac{4}{9} N \mathbf{H} \mathbf{f}(2k + \frac{N}{2}) \\ &= \frac{2}{3} N \left[ \frac{2}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(4k\omega j) f(2j) + \frac{4}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(2k\omega(2j+1)) f(2j+1) \right] \\ &\quad - \frac{4}{9} N \left[ \frac{2}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(2(2k + \frac{N}{2})\omega j) f(2j) + \frac{4}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(\omega(2k + \frac{N}{2})(2j+1)) f(2j+1) \right]. \end{aligned} \quad (4.26)$$

Using the fact that the cas function is periodic in  $2\pi$  and anti-periodic in  $\pi$  (4.26)

simplifies to

$$\begin{aligned}
\mathbf{H}^3\mathbf{f}(2k) &= \frac{4}{9}N \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(4k\omega j)f(2j) + \frac{8}{9}N \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(2k\omega(2j+1))f(2j+1) \\
&\quad - \frac{8}{27}N \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(4k\omega j)f(2j) + \frac{16}{27}N \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(2k\omega(2j+1))f(2j+1) \\
&= \frac{4}{27}N \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(4k\omega j)f(2j) + \frac{40}{27}N \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(2k\omega(2j+1))f(2j+1). \quad (4.27)
\end{aligned}$$

It can be shown in a similar manner that

$$\mathbf{H}^3\mathbf{f}(2k+1) = \frac{20}{27}N \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(2\omega j(2k+1))f(2j) + \frac{56}{27}N \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(\omega(2j+1)(2k+1))f(2j+1). \quad (4.28)$$

## 4.2 Traces of powers of $\mathbf{H}$

For future use we require expressions for the traces of  $\mathbf{H}$ ,  $\mathbf{H}^2$  and  $\mathbf{H}^3$ . For this we require the following summary

$$\mathbf{H}\mathbf{f}(2k) = \frac{2}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(4k\omega j)f(2j) + \frac{4}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(\omega(2j+1)(2k+1))f(2j+1) \quad (4.29)$$

$$\begin{aligned}
\mathbf{H}\mathbf{f}(2k+1) &= \frac{2}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(2\omega j(2k+1))f(2j) + \frac{4}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(\omega(2j+1)(2k+1))f(2j+1) \quad (4.30)
\end{aligned}$$

$$\mathbf{H}^2\mathbf{f}(2k) = \frac{2}{3}Nf(2k) - \frac{4}{9}Nf(2k + \frac{N}{2}) \quad (4.31)$$

$$\mathbf{H}^2\mathbf{f}(2k+1) = \frac{4}{3}Nf(2k+1) - \frac{2}{9}Nf(2k+1 + \frac{N}{2}) \quad (4.32)$$

$$\mathbf{H}^3\mathbf{f}(2k) = \frac{4}{27}N \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(4k\omega j)f(2j) + \frac{40}{27}N \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(2k\omega(2j+1))f(2j+1) \quad (4.33)$$

$$\begin{aligned} \mathbf{H}^3 \mathbf{f}(2k+1) &= \frac{20}{27} N \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(2\omega j(2k+1)) f(2j) + \frac{56}{27} N \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(\omega(2j+1)(2k+1)) \\ &\times f(2j+1). \end{aligned} \quad (4.34)$$

By expressing the right hand side of (4.29) and (4.30) in matrix vector form we observe that the diagonal entries of  $\mathbf{H}$  consist of alternating terms of the form  $\frac{2}{3} \text{cas}((2p)^2\omega)$  and  $\frac{4}{3} \text{cas}((2p+1)^2\omega)$  hence

$$\text{trace}(\mathbf{H}) = \frac{2}{3} \sum_{p=0}^{\frac{N}{2}-1} \text{cas}((2p)^2\omega) + \frac{4}{3} \sum_{p=0}^{\frac{N}{2}-1} \text{cas}((2p+1)^2\omega). \quad (4.35)$$

The diagonal of  $\mathbf{H}^2$  has already been expressed on page 43 and can be expressed as

$$\begin{aligned} \text{trace}(\mathbf{H}^2) &= \frac{N}{2} \left( \frac{2N}{3} + \frac{4N}{3} \right) \\ &= N^2. \end{aligned} \quad (4.36)$$

The trace of  $\mathbf{H}^3$  is deduced (from (4.33),(4.34)) in a similar manner to the trace of  $\mathbf{H}$  to give

$$\text{trace}(\mathbf{H}^3) = \frac{4}{27} N \sum_{p=0}^{\frac{N}{2}-1} \text{cas}((2p)^2\omega) + \frac{56}{27} N \sum_{p=0}^{\frac{N}{2}-1} \text{cas}((2p+1)^2\omega). \quad (4.37)$$

In order to further simplify the trace of  $\mathbf{H}$  we consider the first term of (4.35) which simplifies as follows

$$\begin{aligned} &\sum_{p=0}^{\frac{N}{2}-1} \text{cas}(4p^2\omega) \\ &= 1 + \sum_{p=1}^{\frac{N}{2}-1} \text{cas}(4p^2\omega) \\ &= 1 + \sum_{p=1}^{\frac{N}{4}-\frac{1}{2}} \text{cas}(4p^2\omega) + \sum_{p=\frac{N}{4}+\frac{1}{2}}^{\frac{N}{2}-1} \text{cas}(4p^2\omega) \end{aligned}$$

$$\begin{aligned}
&= 1 + \sum_{p=1}^{\frac{N}{4}-\frac{1}{2}} \text{cas}(4p^2\omega) + \sum_{p=1}^{\frac{N}{4}-\frac{1}{2}} \text{cas}\left(4\left(p + \frac{N}{4} - \frac{1}{2}\right)^2\omega\right) \\
&= 1 + \sum_{p=1}^{\frac{N}{4}-\frac{1}{2}} \text{cas}(4p^2\omega) + \sum_{p=1}^{\frac{N}{4}-\frac{1}{2}} \text{cas}\left(\left(2p + \frac{N}{2} - 1\right)^2\omega\right). \tag{4.38}
\end{aligned}$$

The argument of the summation of the sum in the third term of (4.38) simplifies as follows

$$\begin{aligned}
&\text{cas}\left(\left(2p - 1 + \frac{N}{2}\right)^2\omega\right) \\
&= \text{cas}\left(\left(2p - 1\right)^2\omega + (2p - 1)N\omega + \frac{N^2}{4}\omega\right) \\
&= \text{cas}\left(\left(2p - 1\right)^2\omega + (2p - 1)2\pi + \frac{N}{2}\pi\right) \\
&= \text{cas}\left(\left(2p - 1\right)^2\omega + \frac{N}{2}\pi\right) \\
&= \text{cas}\left(\left(2p - 1\right)^2\omega + (2m + 1)\pi\right) \\
&= -\text{cas}\left(\left(2p - 1\right)^2\omega\right). \tag{4.39}
\end{aligned}$$

Using (4.39) in (4.38) we obtain

$$\sum_{p=0}^{\frac{N}{2}-1} \text{cas}(4p^2\omega) = 1 + \sum_{p=1}^{\frac{N}{4}-\frac{1}{2}} \text{cas}(4p^2\omega) - \sum_{p=1}^{\frac{N}{4}-\frac{1}{2}} \text{cas}\left(\left(2p - 1\right)^2\omega\right). \tag{4.40}$$

Now let us investigate the second term of (4.35) which simplifies as follows

$$\begin{aligned}
&\sum_{p=0}^{\frac{N}{2}-1} \text{cas}\left(\left(2p + 1\right)^2\omega\right) \\
&= \sum_{p=0}^{\frac{N}{4}-\frac{1}{2}} \text{cas}\left(\left(2p + 1\right)^2\omega\right) + \sum_{p=\frac{N}{4}+\frac{1}{2}}^{\frac{N}{2}-1} \text{cas}\left(\left(2p + 1\right)^2\omega\right) \\
&= \sum_{p=0}^{\frac{N}{4}-\frac{1}{2}} \text{cas}\left(\left(2p + 1\right)^2\omega\right) + \sum_{p=1}^{\frac{N}{4}-\frac{1}{2}} \text{cas}\left(\left(2\left(p + \frac{N}{4} - \frac{1}{2}\right) + 1\right)^2\omega\right) \\
&= \sum_{p=1}^{\frac{N}{4}+\frac{1}{2}} \text{cas}\left(\left(2p - 1\right)^2\omega\right) + \sum_{p=1}^{\frac{N}{4}-\frac{1}{2}} \text{cas}\left(\left(2\left(p + \frac{N}{4} - \frac{1}{2}\right) + 1\right)^2\omega\right)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{p=1}^{\frac{N}{4}-\frac{1}{2}} \text{cas}((2p-1)^2\omega) + \text{cas}\left(\left(2\left(\frac{N}{4} + \frac{1}{2}\right) - 1\right)^2\omega\right) + \sum_{p=1}^{\frac{N}{4}-\frac{1}{2}} \text{cas}\left(\left(2\left(p + \frac{N}{4} - \frac{1}{2}\right) + 1\right)^2\omega\right) \\
&= \sum_{p=1}^{\frac{N}{4}-\frac{1}{2}} \text{cas}((2p-1)^2\omega) - 1 + \sum_{p=1}^{\frac{N}{4}-\frac{1}{2}} \text{cas}\left(\left(2\left(p + \frac{N}{4} - \frac{1}{2}\right) + 1\right)^2\omega\right). \tag{4.41}
\end{aligned}$$

The argument of the summation of the sum in the third term of (4.41) simplifies as follows

$$\begin{aligned}
&\text{cas}\left(\left(2\left(p + \frac{N}{4} - \frac{1}{2}\right) + 1\right)^2\omega\right) \\
&= \text{cas}\left(4p^2\omega + 2pN\omega + \frac{N^2}{4}\omega\right) \\
&= \text{cas}\left(4p^2\omega + 2p2\pi + \frac{N}{2}\pi\right) \\
&= \text{cas}\left(4p^2\omega + \frac{N}{2}\pi\right) \\
&= \text{cas}\left(4p^2\omega + (2m+1)\pi\right) \\
&= -\text{cas}(4p^2\omega). \tag{4.42}
\end{aligned}$$

Using (4.42) in (4.41) we obtain

$$\sum_{p=0}^{\frac{N}{2}-1} \text{cas}((2p+1)^2\omega) = -1 - \sum_{p=1}^{\frac{N}{4}-\frac{1}{2}} \text{cas}(4p^2\omega) + \sum_{p=1}^{\frac{N}{4}-\frac{1}{2}} \text{cas}((2p-1)^2\omega). \tag{4.43}$$

From (4.40) and (4.43) we deduce that  $\sum_{p=0}^{\frac{N}{2}-1} \text{cas}(4p^2\omega) = -\sum_{p=0}^{\frac{N}{2}-1} \text{cas}((2p+1)^2\omega)$ .

With this substitution in (4.35) we obtain

$$\text{trace}(\mathbf{H}) = -\frac{2}{3} \sum_{p=0}^{\frac{N}{2}-1} \text{cas}(4p^2\omega). \tag{4.44}$$

In order to explicitly evaluate the sum in (4.44) we employ the result of generalized quadratic Gauss sums [3] stated in the theorem below.

**Theorem 4.1.** *The reciprocity formula for generalized quadratic Gauss sums can be written as*

$$\sum_{p=0}^{|c|-1} e^{\frac{i\pi(ap^2+bp)}{c}} = \left|\frac{c}{a}\right|^{\frac{1}{2}} e^{\frac{i\pi(|ac|-b^2)}{4ac}} \sum_{p=0}^{|a|-1} e^{\frac{-i\pi(cp^2+bp)}{a}} \tag{4.45}$$

where  $a$ ,  $b$  and  $c$  are integers with  $ac \neq 0$  and  $ac + b$  being even.

Choosing parameters  $c = \frac{N}{2}$ ,  $a = 4$  and  $b = 0$  in theorem 4.1 gives

$$\begin{aligned}
\sum_{p=0}^{\frac{N}{2}-1} e^{\pi i \left( \frac{4p^2}{N} \right)} &= \sum_{p=0}^{\frac{N}{2}-1} e^{i4p^2\omega} \\
&= \sqrt{\frac{N}{8}} e^{\frac{\pi}{4}i} \sum_{p=0}^3 e^{-\pi i \left( \frac{N}{2} \frac{p^2}{4} \right)} \\
&= \sqrt{\frac{N}{8}} e^{\frac{\pi}{4}i} \sum_{p=0}^3 e^{-\pi i \frac{N}{8} p^2} \\
&= \frac{1}{2} \sqrt{\frac{N}{2}} e^{\frac{\pi}{4}i} [1 + e^{-i\frac{\pi}{8}N} + e^{-i\frac{4\pi}{8}N} + e^{-i\frac{9\pi}{8}N}] \\
&= \frac{1}{2} \sqrt{\frac{N}{2}} e^{\frac{\pi}{4}i} [1 + e^{-i\frac{\pi}{8}N} + e^{-i\frac{\pi}{8}N} + e^{-i\frac{\pi}{2}N}] \\
&= \frac{1}{2} \sqrt{\frac{N}{2}} e^{\frac{\pi}{4}i} [1 + 2e^{-i\frac{\pi}{8}N} - 1] \\
&= \sqrt{\frac{N}{2}} e^{i\left(\frac{\pi}{4} - \frac{\pi}{8}N\right)}. \tag{4.46}
\end{aligned}$$

Thus if we take the real and imaginary parts respectively of equation (4.46) we arrive at

$$\sum_{p=0}^{\frac{N}{2}-1} \cos(4p^2\omega) = \sqrt{\frac{N}{2}} \cos\left(\frac{\pi}{4} - \frac{\pi}{8}N\right) \tag{4.47}$$

$$\sum_{p=0}^{\frac{N}{2}-1} \sin(4p^2\omega) = \sqrt{\frac{N}{2}} \sin\left(\frac{\pi}{4} - \frac{\pi}{8}N\right). \tag{4.48}$$

We then add equations (4.47) and (4.48) to get

$$\sum_{p=0}^{\frac{N}{2}-1} \text{cas}(4p^2\omega) = \sqrt{\frac{N}{2}} \text{cas}\left(\frac{\pi}{4} - \frac{\pi}{8}N\right). \tag{4.49}$$

Hence

$$\text{trace}(\mathbf{H}) = -\frac{2}{3} \sqrt{\frac{N}{2}} \text{cas}\left(\frac{\pi}{4} - \frac{\pi}{8}N\right). \tag{4.50}$$

A similar calculation yields

$$\text{trace}(\mathbf{H}^3) = -\frac{52}{27} N \sqrt{\frac{N}{2}} \text{cas}\left(\frac{\pi}{4} - \frac{\pi}{8}N\right). \tag{4.51}$$

### 4.3 Multiplicity of eigenvalues

Let  $m_{10}, m_{11}, m_{20}, m_{21}$  represent the multiplicities of the eigenvalues  $r_1, -r_1, r_2, -r_2$  respectively. Using the fact that the trace of a matrix is the sum of all its eigenvalues we form the following system of linear equations in order to ascertain the multiplicities of the eigenvalues of  $\mathbf{H}$ .

$$m_{10} + m_{11} + m_{20} + m_{21} = N \quad (4.52)$$

$$r_1 m_{10} - r_1 m_{11} + r_2 m_{20} - r_2 m_{21} = \text{trace}(\mathbf{H}) \quad (4.53)$$

$$r_1^2 m_{10} + r_1^2 m_{11} + r_2^2 m_{20} + r_2^2 m_{21} = \text{trace}(\mathbf{H}^2) \quad (4.54)$$

$$r_1^3 m_{10} - r_1^3 m_{11} + r_2^3 m_{20} - r_2^3 m_{21} = \text{trace}(\mathbf{H}^3). \quad (4.55)$$

In order to solve the system (4.52)-(4.55) we let

$$A = m_{10} + m_{11} \quad (4.56)$$

$$B = m_{20} + m_{21} \quad (4.57)$$

$$C = m_{10} - m_{11} \quad (4.58)$$

$$D = m_{20} - m_{21}. \quad (4.59)$$

Hence (4.52) and (4.54) can be written as

$$A + B = N \quad (4.60)$$

$$Ar_1^2 + Br_2^2 = N^2. \quad (4.61)$$

The simplified system (4.60)-(4.61) can be solved easily by Cramer's rule. This leads us to the solution

$$A = \frac{Nr_2^2 - N^2}{r_2^2 - r_1^2} \quad (4.62)$$

$$B = \frac{N^2 - Nr_1^2}{r_2^2 - r_1^2}. \quad (4.63)$$

In a similar manner we simplify equations (4.53) and (4.55) which leads us to

$$r_1C + r_2D = -\frac{2}{3}\sqrt{\frac{N}{2}} \operatorname{cas}\left(\frac{\pi}{4} - \frac{\pi}{8}N\right) \quad (4.64)$$

$$r_1^3C + r_2^3D = -\frac{52}{27}N\sqrt{\frac{N}{2}} \operatorname{cas}\left(\frac{\pi}{4} - \frac{\pi}{8}N\right). \quad (4.65)$$

System (4.64)-(4.65) yields the solution

$$C = \frac{\sqrt{\frac{N}{2}} \operatorname{cas}\left(\frac{\pi}{4} - \frac{\pi}{8}N\right)\left(\frac{52}{27}Nr_2 - \frac{2}{3}r_2^3\right)}{r_1r_2^3 - r_2r_1^3} \quad (4.66)$$

$$D = \frac{\sqrt{\frac{N}{2}} \operatorname{cas}\left(\frac{\pi}{4} - \frac{\pi}{8}N\right)\left(\frac{2}{3}r_1^3 - \frac{52}{27}Nr_1\right)}{r_1r_2^3 - r_2r_1^3}. \quad (4.67)$$

Since  $r_2^2 - r_1^2 = -\frac{2}{9}\sqrt{17}N$ , this substitution in (4.62) yields

$$\begin{aligned} A &= \frac{N(r_2^2 - N)}{r_2^2 - r_1^2} \\ &= \frac{N^2\left(\frac{9-\sqrt{17}}{9} - 1\right)}{-\frac{2}{9}\sqrt{17}N} \\ &= \frac{N}{2}. \end{aligned} \quad (4.68)$$

Similarly  $B = \frac{N}{2}$ . We now attempt to simplify  $C$  from (4.66) as follows.

$$\begin{aligned}
C &= \frac{r_2 \sqrt{\frac{N}{2}} \operatorname{cas}\left(\frac{\pi}{4} - \frac{\pi}{8}N\right) \left(\frac{52}{27}N - \frac{2}{3}r_2^2\right)}{r_1 r_2 (r_2^2 - r_1^2)} \\
&= \frac{\sqrt{\frac{N}{2}} \operatorname{cas}\left(\frac{\pi}{4} - \frac{\pi}{8}N\right) \left(\frac{52}{27}N - \frac{2}{27}N(9 - \sqrt{17})\right)}{\frac{1}{3}\sqrt{9 + \sqrt{17}}\sqrt{N}(-\frac{2}{9}\sqrt{17}N)} \\
&= \frac{\sqrt{\frac{N}{2}} \operatorname{cas}\left(\frac{\pi}{4} - \frac{\pi}{8}N\right) \left(\frac{34}{27}N + \frac{2\sqrt{17}}{27}N\right)}{-\frac{2}{27}\sqrt{9 + \sqrt{17}}\sqrt{N}N\sqrt{17}} \\
&= \frac{\frac{2}{27}N\sqrt{N}\frac{1}{\sqrt{2}} \operatorname{cas}\left(\frac{\pi}{4} - \frac{\pi}{8}N\right) (17 + \sqrt{17})}{-\frac{2}{27}\sqrt{9 + \sqrt{17}}\sqrt{N}N\sqrt{17}} \\
&= \frac{\frac{1}{\sqrt{2}} \operatorname{cas}\left(\frac{\pi}{4} - \frac{\pi}{8}N\right) (17 + \sqrt{17})}{-\sqrt{17}(9 + \sqrt{17})} \\
&= \frac{-\operatorname{cas}\left(\frac{\pi}{4} - \frac{\pi}{8}N\right) (17 + \sqrt{17})}{\sqrt{34}(9 + \sqrt{17})} \\
&= -\operatorname{cas}\left(\frac{\pi}{4} - \frac{\pi}{8}N\right). \tag{4.69}
\end{aligned}$$

The result in (4.69) is due to the fact that  $S = \frac{17 + \sqrt{17}}{\sqrt{34(9 + \sqrt{17})}} = 1$  as shown below:

$$\begin{aligned}
S^2 &= \frac{(17)^2 + 34\sqrt{17} + 17}{34(9 + \sqrt{17})} \\
&= \frac{306 + 34\sqrt{17}}{306 + 34\sqrt{17}} \\
&= 1.
\end{aligned}$$

Similarly  $D = \operatorname{cas}\left(\frac{\pi}{4} - \frac{\pi}{8}N\right)$ . From equations (4.56)-(4.59) we obtain

$$A + C = 2m_{10} \tag{4.70}$$

$$C - A = -2m_{11} \tag{4.71}$$

$$B + D = 2m_{20} \tag{4.72}$$

$$D - B = -2m_{21}, \tag{4.73}$$

from which the multiplicities can easily be recovered and expressed as

$$m_{10} = \frac{1}{2} \left[ \frac{N}{2} - \text{cas} \left( \frac{-m\pi}{2} \right) \right] \quad (4.74)$$

$$m_{11} = \frac{1}{2} \left[ \frac{N}{2} + \text{cas} \left( \frac{-m\pi}{2} \right) \right] \quad (4.75)$$

$$m_{20} = \frac{1}{2} \left[ \frac{N}{2} + \text{cas} \left( \frac{-m\pi}{2} \right) \right] \quad (4.76)$$

$$m_{21} = \frac{1}{2} \left[ \frac{N}{2} - \text{cas} \left( \frac{-m\pi}{2} \right) \right], \quad (4.77)$$

by using the fact that  $\text{cas} \left( \frac{\pi}{4} - \frac{\pi}{8} N \right) = \text{cas} \left( \frac{-m\pi}{2} \right)$ .

# Chapter 5

## 2-Dimensional SDHT

The focus of this chapter is going to be the two-dimensional (2-D) SDHT. The digital world has evolved drastically over the past decade with the ability to communicate vast amounts of data in the form of audio or visual signals. This leads us to the fact that digital images are of vital importance to various aspects of life including but not limited to medical imaging, astronomy, image processing, geophysics and magnetic resonance imaging [11]. It may be apparent that images could be contaminated by sinusoidal noise. This can be easily cleaned up in the frequency domain using specific types of filters whose purpose is to eliminate the contaminating noise. As a two-dimensional (2-D) Discrete Hartley Transform is obtained by an application of the trapezoidal rule in 2-D, here we similarly derive a 2-D Simpson Discrete Hartley Transform using Simpson's rule. A 2-D transform is necessary for application to image processing as images are represented digitally in the form of a 2-D array on the gray scale or a collection of 2-D arrays on the colour scale.

## 5.1 Derivation of the 2-D SDHT

In order to proceed we adopt the notation  $\mathbf{H}_N$  where  $N$  denotes the order of the SDHT matrix  $\mathbf{H}$  defined in equation (2.9). This implies that  $\mathbf{H}_N$  is given component-wise by

$$\mathbf{H}_N \mathbf{f}(k) = \frac{2}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(2k\omega_N j) f(2j) + \frac{4}{3} \sum_{j=0}^{\frac{N}{2}-1} \text{cas}(k\omega_N(2j+1)) f(2j+1), \quad (5.1)$$

where  $\omega_N = \frac{2\pi}{N}$  and  $k = 0, 1, \dots, N-1$ . The transformation matrix can be represented by

$$\mathbf{H}_N = \frac{2}{3} \begin{pmatrix} \mathbf{A}_N & 2\mathbf{B}_N \\ \mathbf{A}_N & -2\mathbf{B}_N \end{pmatrix} \mathbf{P}_N, \quad (5.2)$$

where the  $(\frac{N}{2} \times \frac{N}{2})$  matrices  $\mathbf{A}_N$  and  $\mathbf{B}_N$  are defined respectively by

$$\mathbf{A}_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ \text{cas}(\omega_N) & \text{cas}(3\omega_N) & \text{cas}(5\omega_N) & \cdots & \text{cas}((N_1+1)\omega_N) \\ \text{cas}(2\omega_N) & \text{cas}(6\omega_N) & \text{cas}(10\omega_N) & \cdots & \text{cas}(2(N_1+1)\omega_N) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \text{cas}(\frac{N_1}{2}\omega_N) & \text{cas}(\frac{N_1}{2}3\omega_N) & \text{cas}(\frac{N_1}{2}5\omega_N) & \cdots & \text{cas}(\frac{N_1}{2}(N_1+1)\omega_N) \end{bmatrix}, \quad (5.3)$$

$$\mathbf{B}_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ \text{cas}(\omega_N) & \text{cas}(3\omega_N) & \text{cas}(5\omega_N) & \cdots & \text{cas}(N_1\omega_N) \\ \text{cas}(2\omega_N) & \text{cas}(6\omega_N) & \text{cas}(10\omega_N) & \cdots & \text{cas}(2N_1\omega_N) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \text{cas}(N_1\omega_N) & \text{cas}(3N_1\omega_N) & \text{cas}(5N_1\omega_N) & \cdots & \text{cas}(\frac{N_1}{2}(N_1+1)\omega_N) \end{bmatrix} \quad (5.4)$$

and the  $N \times N$  permutation matrix  $\mathbf{P}_N$  defined by components  $P_{N_{(j,2j)}} = 1$  and

$P_{N_{(\frac{N}{2}+j,2j+1)}} = 1$  for  $j = 0, 1, \dots, \frac{N}{2} - 1$  and  $N_1 = N - 2$ .

The two-dimensional Discrete Hartley Transform in a separable form is defined as [16]

$$f(x, y) = \sum_{l=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} C(k, l) \operatorname{cas}\left(\frac{2\pi}{N}xk\right) \operatorname{cas}\left(\frac{2\pi}{M}yl\right), \quad (5.5)$$

with  $f(x, y)$  representing a periodic function of  $2\pi$  in the variables  $x$  and  $y$ . The Hartley coefficients are represented by

$$c(k, l) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} f(x, y) \operatorname{cas}\left(\frac{2\pi}{N}xk\right) \operatorname{cas}\left(\frac{2\pi}{M}yl\right) dx dy. \quad (5.6)$$

In order to proceed with the derivation we need to approximate the double integral (5.6). To do this we employ Simpson's quadrature with the aid of a grid in the  $x$  direction defined by the nodes  $x_n = \frac{2\pi n}{N}$ ,  $n = 0, 1, \dots, N$  with the step length defined by  $\omega_N = \frac{2\pi}{N}$ . With this discretization equation (5.6) can be written as

$$\begin{aligned} & \frac{1}{4\pi^2} \int_0^{2\pi} \frac{\omega_N}{3} \left[ f(x_0, y) \operatorname{cas}\left(\frac{2\pi}{N}x_0k\right) + f(x_N, y) \operatorname{cas}\left(\frac{2\pi}{N}x_Nk\right) \right. \\ & \left. + 2 \sum_{n=1}^{\frac{N}{2}-1} f(x_{2n}, y) \operatorname{cas}\left(\frac{2\pi}{N}x_{2n}k\right) + 4 \sum_{n=0}^{\frac{N}{2}-1} f(x_{2n+1}, y) \operatorname{cas}\left(\frac{2\pi}{N}x_{2n+1}k\right) \right] \operatorname{cas}\left(\frac{2\pi}{M}yl\right) dy \\ & = \frac{\omega_N}{12\pi^2} \left[ \int_0^{2\pi} f(x_0, y) \operatorname{cas}\left(\frac{2\pi}{N}x_0k\right) \operatorname{cas}\left(\frac{2\pi}{M}yl\right) dy + \int_0^{2\pi} f(x_N, y) \operatorname{cas}\left(\frac{2\pi}{N}x_Nk\right) \operatorname{cas}\left(\frac{2\pi}{M}yl\right) dy \right. \\ & \left. + 2 \sum_{n=1}^{\frac{N}{2}-1} \int_0^{2\pi} f(x_{2n}, y) \operatorname{cas}\left(\frac{2\pi}{N}x_{2n}k\right) \operatorname{cas}\left(\frac{2\pi}{M}yl\right) dy \right. \\ & \left. + 4 \sum_{n=0}^{\frac{N}{2}-1} \int_0^{2\pi} f(x_{2n+1}, y) \operatorname{cas}\left(\frac{2\pi}{N}x_{2n+1}k\right) \operatorname{cas}\left(\frac{2\pi}{M}yl\right) dy \right]. \quad (5.7) \end{aligned}$$

Using the fact that  $f(x_0, y) = f(x_N, y)$ , (5.7) simplifies to

$$\begin{aligned} & \frac{\omega_N}{12\pi^2} \left[ 2 \sum_{n=0}^{\frac{N}{2}-1} \int_0^{2\pi} f(x_{2n}, y) \operatorname{cas}\left(\frac{2\pi}{N}x_{2n}k\right) \operatorname{cas}\left(\frac{2\pi}{M}yl\right) dy \right. \\ & \left. + 4 \sum_{n=0}^{\frac{N}{2}-1} \int_0^{2\pi} f(x_{2n+1}, y) \operatorname{cas}\left(\frac{2\pi}{N}x_{2n+1}k\right) \operatorname{cas}\left(\frac{2\pi}{M}yl\right) dy \right] \end{aligned}$$

$$\begin{aligned}
&= \frac{\omega_N}{6\pi^2} \left[ \sum_{n=0}^{\frac{N}{2}-1} \int_0^{2\pi} f(x_{2n}, y) \operatorname{cas}\left(\frac{2\pi}{N}x_{2n}k\right) \operatorname{cas}\left(\frac{2\pi}{M}yl\right) dy \right. \\
&\quad \left. + 2 \sum_{n=0}^{\frac{N}{2}-1} \int_0^{2\pi} f(x_{2n+1}, y) \operatorname{cas}\left(\frac{2\pi}{N}x_{2n+1}k\right) \operatorname{cas}\left(\frac{2\pi}{M}yl\right) dy \right]. \tag{5.8}
\end{aligned}$$

We now apply Simpson quadrature in the  $y$  direction with nodes  $y_m = \frac{2\pi m}{M}$  where  $m = 0, 1, \dots, M$ . We now define  $G(x_{2n}, y) = f(x_{2n}, y) \operatorname{cas}\left(\frac{2\pi}{N}x_{2n}k\right) \operatorname{cas}\left(\frac{2\pi}{M}yl\right)$  and obtain the following approximation

$$\begin{aligned}
\int_0^{2\pi} G(x_{2n}, y) dy &\approx \frac{\omega_M}{3} \left[ G(x_{2n}, y_0) + G(x_{2n}, y_M) \right. \\
&\quad \left. + 2 \sum_{m=1}^{\frac{M}{2}-1} G(x_{2n}, y_{2m}) + 4 \sum_{m=0}^{\frac{M}{2}-1} G(x_{2n}, y_{2m+1}) \right] \\
&= \frac{\omega_M}{3} \left[ 2 \sum_{m=0}^{\frac{M}{2}-1} G(x_{2n}, y_{2m}) + 4 \sum_{m=0}^{\frac{M}{2}-1} G(x_{2n}, y_{2m+1}) \right] \\
&= \frac{2\omega_M}{3} \sum_{m=0}^{\frac{M}{2}-1} G(x_{2n}, y_{2m}) + \frac{4\omega_M}{3} \sum_{m=0}^{\frac{M}{2}-1} G(x_{2n}, y_{2m+1}), \tag{5.9}
\end{aligned}$$

where  $\omega_M = \frac{2\pi}{M}$  is the step length in the  $y$  direction and  $m = 0, 1, \dots, M$ . We now substitute (5.9) into the first part of (5.8) to get

$$\begin{aligned}
&\frac{\omega_N}{6\pi^2} \sum_{n=0}^{\frac{N}{2}-1} \int_0^{2\pi} G(x_{2n}, y) dy \\
&= \frac{\omega_N}{6\pi^2} \sum_{n=0}^{\frac{N}{2}-1} \left[ \frac{2\omega_M}{3} \sum_{m=0}^{\frac{M}{2}-1} G(x_{2n}, y_{2m}) + \frac{4\omega_M}{3} \sum_{m=0}^{\frac{M}{2}-1} G(x_{2n}, y_{2m+1}) \right] \\
&= \frac{\omega_N \omega_M}{9\pi^2} \sum_{n=0}^{\frac{N}{2}-1} \sum_{m=0}^{\frac{M}{2}-1} G(x_{2n}, y_{2m}) + \frac{2\omega_N \omega_M}{9\pi^2} \sum_{n=0}^{\frac{N}{2}-1} \sum_{m=0}^{\frac{M}{2}-1} G(x_{2n}, y_{2m+1}) \\
&= \frac{\omega_N \omega_M}{9\pi^2} \sum_{n=0}^{\frac{N}{2}-1} \sum_{m=0}^{\frac{M}{2}-1} f(x_{2n}, y_{2m}) \operatorname{cas}\left(\frac{2\pi}{N}x_{2n}k\right) \operatorname{cas}\left(\frac{2\pi}{M}y_{2m}l\right) \\
&\quad + \frac{2\omega_N \omega_M}{9\pi^2} \sum_{n=0}^{\frac{N}{2}-1} \sum_{m=0}^{\frac{M}{2}-1} f(x_{2n}, y_{2m+1}) \operatorname{cas}\left(\frac{2\pi}{N}x_{2n}k\right) \operatorname{cas}\left(\frac{2\pi}{M}y_{2m+1}l\right). \tag{5.10}
\end{aligned}$$

Adopting the notation  $f(x_n, y_m) = f(n, m)$ , (5.10) can be written in the simplified

form

$$\begin{aligned} & \frac{4}{9NM} \sum_{n=0}^{\frac{N}{2}-1} \sum_{m=0}^{\frac{M}{2}-1} f(2n, 2m) \operatorname{cas}(2n\omega_N k) \operatorname{cas}(2\omega_M ml) \\ & + \frac{8}{9NM} \sum_{n=0}^{\frac{N}{2}-1} \sum_{m=0}^{\frac{M}{2}-1} f(2n, 2m+1) \operatorname{cas}(2n\omega_N k) \operatorname{cas}(\omega_M(2m+1)l). \end{aligned} \quad (5.11)$$

Similarly the second part of (5.8) can be shown to simplify to

$$\begin{aligned} & \frac{8}{9NM} \sum_{n=0}^{\frac{N}{2}-1} \sum_{m=0}^{\frac{M}{2}-1} f(2n+1, 2m) \operatorname{cas}(\omega_N(2n+1)k) \operatorname{cas}(2m\omega_M l) \\ & + \frac{16}{9NM} \sum_{n=0}^{\frac{N}{2}-1} \sum_{m=0}^{\frac{M}{2}-1} f(2n+1, 2m+1) \operatorname{cas}(\omega_N(2n+1)k) \operatorname{cas}(\omega_M(2m+1)l). \end{aligned} \quad (5.12)$$

We now define the following terms

$$T_1(k, l) = \frac{4}{9} \sum_{n=0}^{\frac{N}{2}-1} \sum_{m=0}^{\frac{M}{2}-1} f(2n, 2m) \operatorname{cas}(2n\omega_N k) \operatorname{cas}(2\omega_M ml) \quad (5.13)$$

$$T_2(k, l) = \frac{8}{9} \sum_{n=0}^{\frac{N}{2}-1} \sum_{m=0}^{\frac{M}{2}-1} f(2n, 2m+1) \operatorname{cas}(2n\omega_N k) \operatorname{cas}(\omega_M(2m+1)l) \quad (5.14)$$

$$T_3(k, l) = \frac{8}{9} \sum_{n=0}^{\frac{N}{2}-1} \sum_{m=0}^{\frac{M}{2}-1} f(2n+1, 2m) \operatorname{cas}(\omega_N(2n+1)k) \operatorname{cas}(2\omega_M ml) \quad (5.15)$$

$$T_4(k, l) = \frac{16}{9} \sum_{n=0}^{\frac{N}{2}-1} \sum_{m=0}^{\frac{M}{2}-1} f(2n+1, 2m+1) \operatorname{cas}(\omega_N(2n+1)k) \operatorname{cas}(\omega_M(2m+1)l). \quad (5.16)$$

We now let

$$F(k, l) = T_1(k, l) + T_2(k, l) + T_3(k, l) + T_4(k, l). \quad (5.17)$$

Thus from (5.8) the approximation  $C(k, l)$  to  $c(k, l)$  is given by

$$C(k, l) = \frac{F(k, l)}{NM}, \quad (5.18)$$

where  $F(k, l)$  is defined as the Simpson Discrete Hartley Transform in 2-D. Summing

(5.13) and (5.15) we get

$$\begin{aligned}
& T_1(k, l) + T_3(k, l) \\
&= \frac{2}{3} \sum_{m=0}^{\frac{M}{2}-1} \left[ \frac{2}{3} \sum_{n=0}^{\frac{N}{2}-1} f(2n, 2m) \operatorname{cas}(2n\omega_N k) \operatorname{cas}(2\omega_M m l) \right. \\
&+ \left. \frac{4}{3} \sum_{n=0}^{\frac{N}{2}-1} f(2n+1, 2m) \operatorname{cas}(\omega_N(2n+1)k) \operatorname{cas}(2\omega_M m l) \right] \\
&= \frac{2}{3} \sum_{m=0}^{\frac{M}{2}-1} \left[ \frac{2}{3} \sum_{n=0}^{\frac{N}{2}-1} f(2n, 2m) \operatorname{cas}(2n\omega_N k) \right. \\
&+ \left. \frac{4}{3} \sum_{n=0}^{\frac{N}{2}-1} f(2n+1, 2m) \operatorname{cas}(\omega_N(2n+1)k) \right] \operatorname{cas}(2\omega_M m l) \\
&= \frac{2}{3} \sum_{m=0}^{\frac{M}{2}-1} \hat{F}(k, 2m) \operatorname{cas}(2\omega_M m l), \tag{5.19}
\end{aligned}$$

where

$$\hat{F}(k, 2m) = \frac{2}{3} \sum_{n=0}^{\frac{N}{2}-1} f(2n, 2m) \operatorname{cas}(2n\omega_N k) + \frac{4}{3} \sum_{n=0}^{\frac{N}{2}-1} f(2n+1, 2m) \operatorname{cas}(\omega_N(2n+1)k). \tag{5.20}$$

In a similar manner by summing (5.14) and (5.16) we get

$$\begin{aligned}
& T_2(k, l) + T_4(k, l) \\
&= \frac{4}{3} \sum_{m=0}^{\frac{M}{2}-1} \left[ \frac{2}{3} \sum_{n=0}^{\frac{N}{2}-1} f(2n, 2m+1) \operatorname{cas}(2n\omega_N k) \operatorname{cas}(\omega_M(2m+1)l) \right. \\
&+ \left. \frac{4}{3} \sum_{n=0}^{\frac{N}{2}-1} f(2n+1, 2m+1) \operatorname{cas}(\omega_N(2n+1)k) \operatorname{cas}(\omega_M(2m+1)l) \right] \\
&= \frac{4}{3} \sum_{m=0}^{\frac{M}{2}-1} \left[ \frac{2}{3} \sum_{n=0}^{\frac{N}{2}-1} f(2n, 2m+1) \operatorname{cas}(2n\omega_N k) \right. \\
&+ \left. \frac{4}{3} \sum_{n=0}^{\frac{N}{2}-1} f(2n+1, 2m+1) \operatorname{cas}(\omega_N(2n+1)k) \right] \operatorname{cas}(\omega_M(2m+1)l) \\
&= \frac{4}{3} \sum_{m=0}^{\frac{M}{2}-1} \hat{F}(k, 2m+1) \operatorname{cas}(\omega_M(2m+1)l), \tag{5.21}
\end{aligned}$$

where

$$\begin{aligned}\hat{F}(k, 2m+1) &= \frac{2}{3} \sum_{n=0}^{\frac{N}{2}-1} f(2n, 2m+1) \text{cas}(2n\omega_N k) \\ &+ \frac{4}{3} \sum_{n=0}^{\frac{N}{2}-1} f(2n+1, 2m+1) \text{cas}(\omega_N(2n+1)k).\end{aligned}\quad (5.22)$$

Using (5.19) and (5.21) allows us to write

$$F(k, l) = \frac{2}{3} \sum_{m=0}^{\frac{M}{2}-1} \hat{F}(k, 2m) \text{cas}(2\omega_M ml) + \frac{4}{3} \sum_{m=0}^{\frac{M}{2}-1} \hat{F}(k, 2m+1) \text{cas}(\omega_M(2m+1)l). \quad (5.23)$$

It can be seen that equations (5.20) and (5.22) are equivalent to the 1-D SDHT on the columns of the  $N \times M$  matrix  $\mathbf{f}$  which consists of elements  $f(n, m)$ . This can be represented as  $\hat{\mathbf{F}} = \mathbf{H}_N \mathbf{f}$ . An interesting observation is that (5.23) represents the 1-D SDHT on the rows of  $\hat{\mathbf{F}}$ . Let us consider transforming the rows of  $\hat{\mathbf{F}}$ . In order to achieve this, it would be appropriate to transform the columns of  $\hat{\mathbf{F}}^T$ . Performing the transformation yields

$$\begin{aligned}\mathbf{F}^T &= \mathbf{H}_M \hat{\mathbf{F}}^T \\ &= \mathbf{H}_M \mathbf{f}^T \mathbf{H}_N^T.\end{aligned}\quad (5.24)$$

From (5.24) we obtain

$$\mathbf{F} = \mathbf{H}_N \mathbf{f} \mathbf{H}_M^T. \quad (5.25)$$

Equation (5.25) is the representation of the 2-D SDHT on matrix  $\mathbf{f}$ . Equation (5.17) essentially represents the Simpson's quadrature for a periodic function in two variables. This is characterized by the mesh  $(x_n, y_m)$  for  $n = 0, 1, \dots, N-1$ ,  $m = 0, 1, \dots, M-1$ . This is illustrated for an image  $\mathbf{f}$  of size  $6 \times 10$  in figure 5.1 with the appropriate weights.

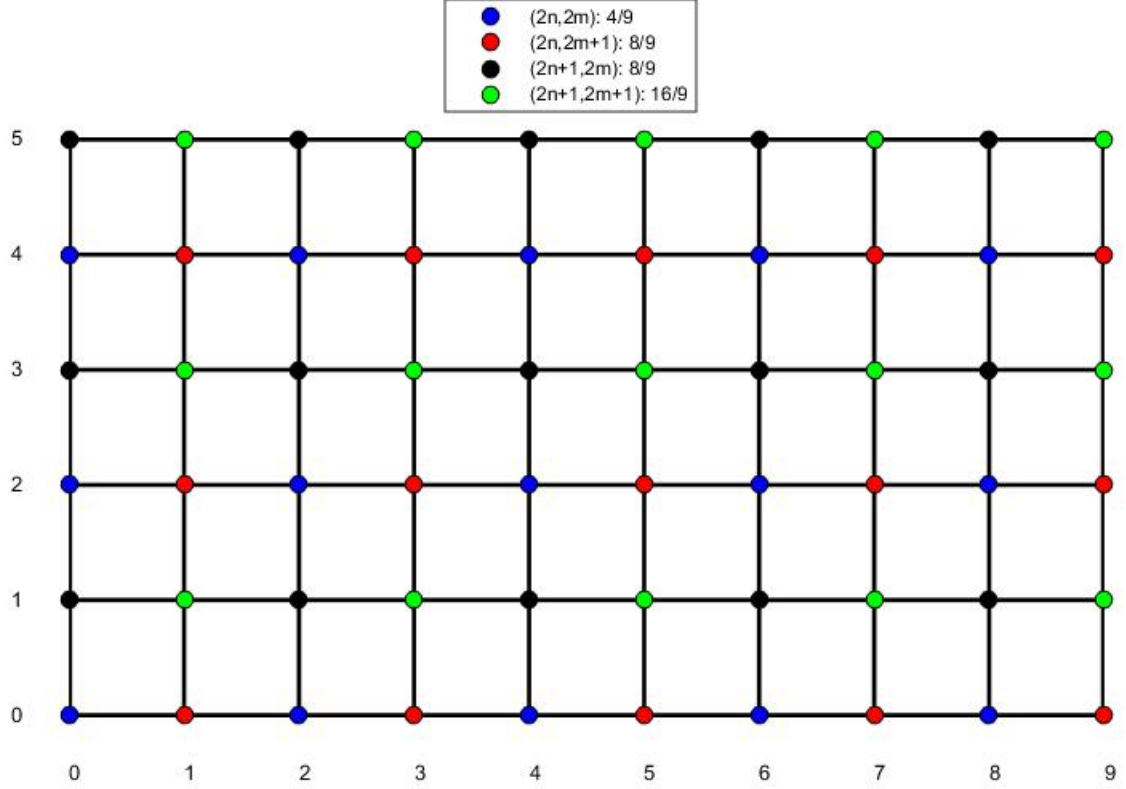


Figure 5.1: A  $6 \times 10$  mesh

Let us now investigate some properties of (5.13)-(5.16).

**Theorem 5.1.** *Let  $\mathbf{T}_1$  be the matrix of size  $\frac{N}{2} \times \frac{M}{2}$  consisting of elements  $T_1(k, l)$ .*

*Then  $\mathbf{T}_1$  is periodic with periodicity of  $\frac{N}{2}$  in the first variable and  $\frac{M}{2}$  in the second variable for  $k = 0, 1, \dots, \frac{N}{2} - 1$  and  $l = 0, 1, \dots, \frac{M}{2} - 1$ .*

*Proof.* From equation (5.13) we have

$$T_1(k, l) = \frac{4}{9} \sum_{n=0}^{\frac{N}{2}-1} \sum_{m=0}^{\frac{M}{2}-1} f(2n, 2m) \text{cas}(2n\omega_N k) \text{cas}(2m\omega_M l).$$

Thus we see that

$$\begin{aligned}
& T_1(k + \frac{N}{2}, l + \frac{M}{2}) \\
&= \frac{4}{9} \sum_{n=0}^{\frac{N}{2}-1} \sum_{m=0}^{\frac{M}{2}-1} f(2n, 2m) \operatorname{cas}(2n\omega_N(k + \frac{N}{2})) \operatorname{cas}(2\omega_M m(l + \frac{M}{2})) \\
&= \frac{4}{9} \sum_{n=0}^{\frac{N}{2}-1} \sum_{m=0}^{\frac{M}{2}-1} f(2n, 2m) \operatorname{cas}(2n\omega_N k + 2\pi n) \operatorname{cas}(2\omega_M ml + 2\pi m) \\
&= \frac{4}{9} \sum_{n=0}^{\frac{N}{2}-1} \sum_{m=0}^{\frac{M}{2}-1} f(2n, 2m) \operatorname{cas}(2n\omega_N k) \operatorname{cas}(2\omega_M ml) \tag{5.26} \\
&= T_1(k, l),
\end{aligned}$$

due to the fact that the cas function is periodic in  $2\pi$ . This completes the proof.  $\square$

**Theorem 5.2.** *Let  $\mathbf{T}_3$  be the matrix of size  $\frac{N}{2} \times \frac{M}{2}$  consisting of elements  $T_3(k, l)$  for  $k = 0, 1, \dots, \frac{N}{2} - 1$  and  $l = 0, 1, \dots, \frac{M}{2} - 1$ . Then  $\mathbf{T}_3$  is anti-periodic in the first variable and periodic in the second variable.*

*Proof.* To proceed we first will show that  $\mathbf{T}_3$  is anti-periodic in the first variable. Recall from equation (5.15) that

$$T_3(k, l) = \frac{8}{9} \sum_{n=0}^{\frac{N}{2}-1} \sum_{m=0}^{\frac{M}{2}-1} f(2n+1, 2m) \operatorname{cas}(\omega_N(2n+1)k) \operatorname{cas}(2\omega_M ml).$$

Thus we see that

$$\begin{aligned}
& T_3(k + \frac{N}{2}, l) \\
&= \frac{8}{9} \sum_{n=0}^{\frac{N}{2}-1} \sum_{m=0}^{\frac{M}{2}-1} f(2n+1, 2m) \operatorname{cas}(\omega_N(2n+1)(k + \frac{N}{2})) \operatorname{cas}(2\omega_M ml) \\
&= \frac{8}{9} \sum_{n=0}^{\frac{N}{2}-1} \sum_{m=0}^{\frac{M}{2}-1} f(2n+1, 2m) (-1)^{2n+1} \operatorname{cas}(\omega_N(2n+1)k) \operatorname{cas}(2\omega_M ml)
\end{aligned}$$

$$\begin{aligned}
&= -\frac{8}{9} \sum_{n=0}^{\frac{N}{2}-1} \sum_{m=0}^{\frac{M}{2}-1} f(2n+1, 2m) \operatorname{cas}(\omega_N(2n+1)k) \operatorname{cas}(2\omega_M ml) \quad (5.27) \\
&= -T_3(k, l),
\end{aligned}$$

due to the fact that the cas function is anti-periodic in  $\pi$ . This completes the first part of the proof. To prove the second part of theorem 5.2 we are required to calculate the following:

$$\begin{aligned}
&T_3(k, l + \frac{M}{2}) \\
&= \frac{8}{9} \sum_{n=0}^{\frac{N}{2}-1} \sum_{m=0}^{\frac{M}{2}-1} f(2n+1, 2m) \operatorname{cas}(\omega_N(2n+1)k) \operatorname{cas}(2\omega_M m(l + \frac{M}{2})) \\
&= \frac{8}{9} \sum_{n=0}^{\frac{N}{2}-1} \sum_{m=0}^{\frac{M}{2}-1} f(2n+1, 2m) \operatorname{cas}(\omega_N(2n+1)k) \operatorname{cas}(2\omega_M ml + 2\pi m) \\
&= \frac{8}{9} \sum_{n=0}^{\frac{N}{2}-1} \sum_{m=0}^{\frac{M}{2}-1} f(2n+1, 2m) \operatorname{cas}(\omega_N(2n+1)k) \operatorname{cas}(2\omega_M ml) \quad (5.28) \\
&= T_3(k, l).
\end{aligned}$$

We have again employed the fact that the cas function is periodic in  $2\pi$ . This completes the proof of theorem 5.2.  $\square$

Let  $\mathbf{T}_4$  be the  $\frac{N}{2} \times \frac{M}{2}$  matrix consisting of elements  $T_4(k, l)$ , then similarly it can be shown that  $T_4(k + \frac{N}{2}, l) = -T_4(k, l)$  and  $T_4(k, l + \frac{M}{2}) = -T_4(k, l)$ ,  $k = 0, 1, \dots, \frac{N}{2} - 1$ ,  $l = 0, 1, \dots, \frac{M}{2} - 1$ . It can be also shown that if we let  $\mathbf{T}_2$  be the  $\frac{N}{2} \times \frac{M}{2}$  matrix consisting of elements  $T_2(k, l)$  then  $T_2(k + \frac{N}{2}, l) = T_2(k, l)$  and  $T_2(k, l + \frac{M}{2}) = -T_2(k, l)$ ,  $k = 0, 1, \dots, \frac{N}{2} - 1$ ,  $l = 0, 1, \dots, \frac{M}{2} - 1$ .

Thus to summarize we see that  $T_1(k, l)$  is periodic in both the first and second variables,  $T_2(k, l)$  is periodic in the first variable and anti-periodic in the second variable,  $T_3(k, l)$  is anti-periodic in the first variable and periodic in the second and finally  $T_4(k, l)$  is

anti-periodic in both the first and second variables. Using the properties listed above we are permitted to write (5.17) in block matrix form as

$$\mathbf{F} = \begin{bmatrix} \mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3 + \mathbf{T}_4 & \mathbf{T}_1 - \mathbf{T}_2 + \mathbf{T}_3 - \mathbf{T}_4 \\ \mathbf{T}_1 + \mathbf{T}_2 - \mathbf{T}_3 - \mathbf{T}_4 & \mathbf{T}_1 - \mathbf{T}_2 - \mathbf{T}_3 + \mathbf{T}_4 \end{bmatrix}. \quad (5.29)$$

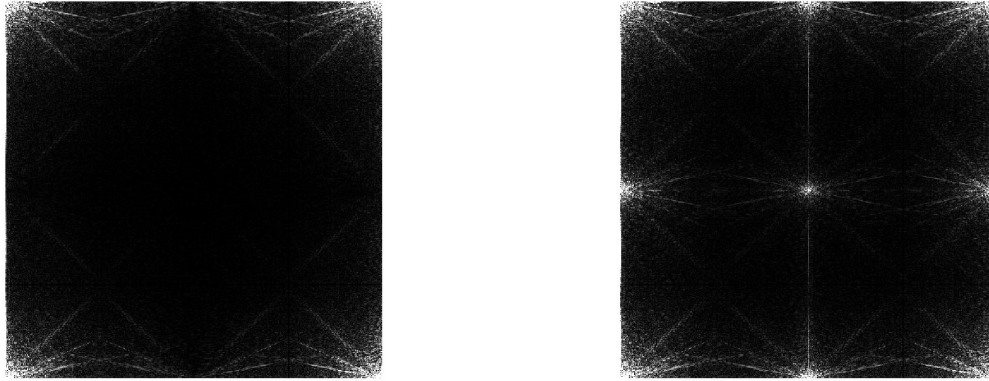
## Chapter 6

# Applications of the 2-D SDHT

In the preceding chapter we have constructed the theory of the 2-D SDHT and we now illustrate it's usefulness to two applications in the frequency domain. Firstly we examine the amplitude spectrum in the frequency domain of the Hartley Transform and SDHT applied to a typical image illustrated in figure 6.1.



Figure 6.1: Cameraman



(a) Frequency domain (Hartley Transform)

(b) Frequency domain (SDHT)

Figure 6.2: Transform of cameraman

We observe four distinct bright regions (corresponding to large amplitudes) in figure 6.2 (a), whilst in figure 6.2 (b) we observe nine such bright regions. The extra regions could be exploited in applications such as watermarking.

## 6.1 Watermarking in the frequency domain

Digital watermarks are of vital importance in this digital age due to the digital medium in which data is transmitted. A digital watermark can be regarded as a covertly embedded distortion in a noise tolerant signal such as audio, image or video data [8]. These watermarks are used to typically verify authenticity or ownership of the signal. Watermarking is the process where the information is hidden within the signal via some sort of algorithmic method [12]. To demonstrate the process of watermarking we will now introduce a watermark into an image. The watermark will be inserted into the frequency domain by using the 2-D SDHT transformation derived in chapter 5. By

implementing equation (5.25) to an image, we successfully insert a watermark into an image of peppers using a distortion algorithm.

In order to do this we first use a function `fftshift` in MATLAB. This essentially moves the low frequency regions to the center and the high frequency into the periphery as illustrated in the figures 6.3 (a) and (b).

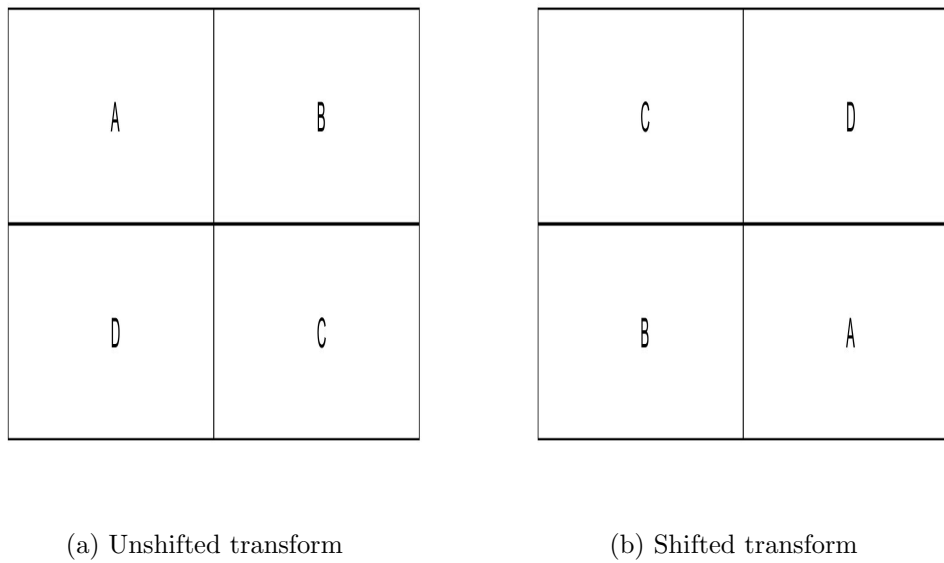
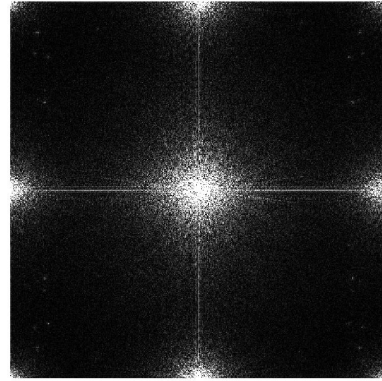


Figure 6.3: `fftshift`

Figure 6.4 (a) is an illustration of the original image while it's corresponding shifted frequency domain is illustrated in figure 6.4 (b).

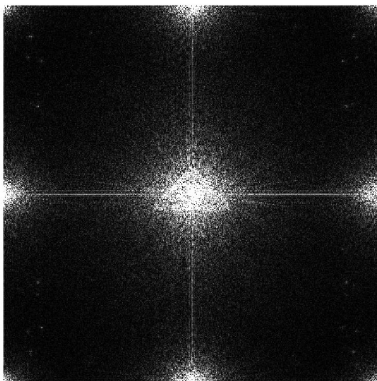


(a) Peppers

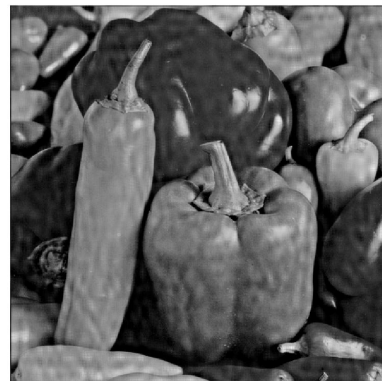


(b) fftshifted transform

Figure 6.4: Peppers and it's transform



(a) Watermark in area of high amplitude.

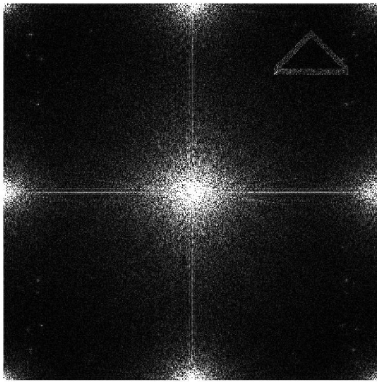


(b) Recovered peppers from figure 6.5 (a)

Figure 6.5: Watermarking in the frequency domain in an area of high amplitude

Figure 6.5 (a) represents a triangle watermark placed in the region of high amplitude and low frequency components. We see that a very small distortion in the central region of the frequency domain results in visible blurriness on the image when transformed back to the real domain. This visible distortion is illustrated in figure 6.5 (b) which is

why it is imperative to place a watermark in an intermediate to high frequency region. Figure 6.6 (a) shown above represents the frequency domain in which we have inserted a watermark in the high frequency region. This results in an unnoticeable difference of the watermarked image when transformed back into the real domain using the inverse transform.



(a) Watermark in area of low amplitude



(b) Recovered peppers from figure 6.6 (a)

Figure 6.6: Watermarking in an intermediate region in the frequency domain

## 6.2 Noise cleaning in the frequency domain

There are various algorithms which are responsible for the removal of unwanted noise in a signal. The cleaning algorithms are also dependent on the type of noise being cleaned up, and the process of removing noise from a signal is known as filtering. This filtering is usually done in the real domain. It is of vital importance that these filter algorithms are implemented in the correct manner as they occur in life altering instruments such as an electrocardiogram machine. The filter for the electrocardiogram machine has to be designed such that only the electric signals from the heart are shown and all other

neurological impulses are suppressed.

The experiment which we are going to conduct comprises the removal of sinusoidal noise from an image. This filtering is best accomplished in the frequency domain.



Figure 6.7: House contaminated with noise

Figure 6.7 is contaminated by unwanted sinusoidal noise. Figure 6.8 illustrates the frequency domain corresponding to figure 6.7. It is observed that the sinusoidal noise appears as sixteen bright crosses.

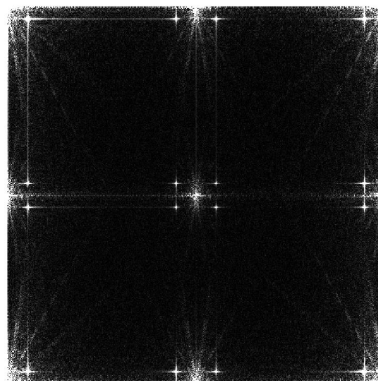


Figure 6.8: Effect of noise in the frequency domain

We remove the noise in the frequency domain and this is achieved by applying a simple block filter algorithm. This is certainly not an optimal filter. This filtration is illustrated in figure 6.9 below.

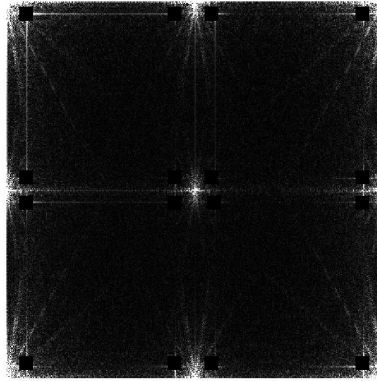


Figure 6.9: Block filter applied in the frequency domain



Figure 6.10: House after applying filter

By inverting the filtered transform we obtain a reasonably cleaned image. Figure 6.10 represents figure 6.7 after the application of a filter. There is clearly visual difference in quality of the image. The lack of perfect clarity in the image is due to the non

optimal filter implemented. We can see that a simple filter algorithm has demonstrated reasonable cleaning ability when transformed into the frequency domain using the 2-D SDHT.

# Conclusion

In this thesis we have introduced a relatively new transform which was derived by application of Simpson's numerical quadrature. We have derived in detail the mathematical properties satisfied by this transform, thus we have placed the SDHT on a firm mathematical foundation. We have not exploited the full potential of the SDHT, but rather restricted it to two applications. Further applicability is left to the field of engineering. We have also not compared the DHT with the SDHT as this was not our intention. A similar investigation of this nature is left as a future task for the dimension  $N \equiv 0 \pmod{4}$ .

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