EXPLORING THE INFLUENCE OF MATHEMATICAL REPRESENTATIONS ON 10TH GRADE LEARNERS' UNDERSTANDING OF TRIGONOMETRY IN A POORLY PERFORMING SCHOOL

by

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Chapter 1

Introduction and background to the study

This Chapter introduces the study, which entails understanding of key terms in the context of this study. Chapter 1 also highlights the aims of the study in its given parameters, research questions that drive the research, the intended contribution that the study will make in the research space and its impacts in today's mathematics classrooms. Lastly, this chapter highlights the anticipated outcomes of the research and its given limitations.

1.1 Introduction

There has been a growing interest in the field of Mathematics research on how representations could influence learners' understanding. Studies by Janvier (1987); Lesh, Post, and Behr (1987); NCTM (2000); Schultz and Waters (2000) point out that with or without the use of technology, representations are a useful pedagogical instrument. The National Council of Teachers of Mathematics (NCTM, 2000) in the United States of America placed representations as one of five broad process standards in the Principal and Standards for Educators. This then endorses the importance of this concept worldwide. Nevertheless, its practice is not common in daily teaching practice of most Mathematics educators and so, learners rarely use representations in their learning. Contrary to this, Lesh et al. (1987) asserted that effective teaching occurs when learners use different representations to solve a problem and they are able to understand new or existing knowledge.

It is quite surprising that representations are not commonly used in the daily practice of teaching mathematics and learners rarely engage with its use in their learning. In essence this makes representations to be one of the few essential skills that still need to be utilized, especially in our public schools where results are poor. In mathematics teaching, educators should embrace the importance of allowing learners to develop their own representations that will allow them to solve extreme problems. In this w ay, encouraging learners to incorporate multiple representations in their thinking enables them to solve mathematical problems and understand underlying concepts (NCTM, 2000). A similar view is shared by Adams, Tung, Warfield, Knaub, Mudavanhu and Yong (2000) that reform in mathematics education can be achieved by devising ways of empowering learners to do mathematics. This is significantly important to learners if mathematics is aimed to be understood well.

I believe that all mathematics educators should attempt to help learners to understand mathematics concepts well. The Mathematics Grade 10-12 (Specific Aim 7) Curriculum Assessment Policy Statement stipulates that teachers should make it a point that learners who are doing mathematics are able to "develop problem-solving and cognitive skills. Teaching should not be limited to "how" but should include the "when" and "why" of problem types" (CAPS, 2011, p.8). The curriculum assessment policy statements stipulate that: "Learning procedures and proofs without a good understanding of why they are important will leave learners ill-equipped to use their knowledge in later life" (DoE, 2011, p.8). This is in line with what Shulman (1986) suggested, that teachers must not only enforce rote learning but also help learners to understand reasoning behind the accepted truth. He stated that "they must also be able to explain why a particular proposition is deemed warranted, why it is worth knowing, and how it relates to other propositions, both within the discipline and without, both in theory and in practice" (p. 9).

What seems to dominate in the discussion around teaching mathematics in schools today is that learners are not performing or the fact that teachers are not teaching learners to the best of their abilities. It is however true that "Mathematics teaching is generally conducted in a verbal way, where the teacher orally engages his or her learners with new or old concepts. These words are often abstract cues which the learners have to decipher, and with the added language problems that learners inherently have, these cues often confuse learners" (Mudaly, 2010, p. 27). Furthermore, Mudaly (2010) argue that "Mathematics, being the king amongst abstract languages, conjures up nightmares in the minds of those that are mathematically challenged" (Mudaly, 2010, p. 27).

What appears to be lacking is ideally the teaching that will incorporate learners to be actively involved with their minds and be hands-on in the tasks provided. In these tasks, learners need to be challenged, at the same time be guided through various stages and steps of the discovery of mathematical ideas and concepts. Mudaly (2013) asserted that this will allow for a greater and more meaningful understanding of the solution to the given problem than to copy solutions from the chalkboard. One way of overcoming the talk and chalk is to engage learners in activities that will enforce the use of representations. In this way, learners will be challenged to be actively involved, communicate mathematical ideas and thinking that are generally difficult to convey in teacher-centered classrooms. It is undoubtable that representation creates a thinking space that eventually helps learners to have proficient knowledge and develops their reasoning or sense making skills that enables them to solve problems.

The NCTM (2002) calls for and presents a common foundation of mathematics to be learned by all students, hence supporting the view of teaching using learning aids that will assist learners to shy away from the norm teaching in mathematics classrooms, where teachers focus on the verifying of mathematical truths that are being investigated to using a multiple representation-based instruction. Friendland and Tabach (2001) believe that the use of multiple representations has the potential of making the learning process to be more effective and meaningful. A similar view is shared by Fennema and Franke (1992) who believe that Mathematics complex subject matter should be translated into representations that can be understood by learners. The NCTM (1998) standards, as emphasised by Carthy (2013) stipulated that “Mathematics instructional programs should emphasize mathematical representations to foster understanding of mathematics so that all learners:

- create and use representations to organize, record, and communicate mathematical ideas;
- develop a repertoire of mathematical representations that can be used purposefully, flexibly and appropriately;
- Use representations to model and interpret physical, social and mathematical phenomena” (p. 10).

Notably, there is a significant effect on the understanding and general performance on learners who are taught using multiple representations contrary to the performance obtained by those who are taught using conventional teaching or traditional teaching of chalk and talk. Akkus and Cakiroglu (2009a) cited Seeger, Voigt, and Waschescio (1998) who highlighted the overall delineation of multiple representation, that “it is any kind of mental state with a specific content; a mental reproduction of a former mental state; a picture, symbol, or sign; a something in place of something else" (p. 420). This implies that when teachers use different representation in teaching mathematics they ought to develop and deepen their content and conceptual knowledge.

Mathematical representations can be well-defined as alternative ways in which mathematical ideas can be presented. The manner in which one perceives mathematical ideas is fundamental in concrete understanding of the learned idea. Ozgun-Koca (1998) precisely defined multiple representations as external mathematical embodiments of ideas and concepts to provide the same information in more than one form (p. 3). In essence, one idea can be interpreted in many different ways, which will yield the same solution. What can be drawn from this, is that for learners to construct deeper understanding of what they learn, they ought to integrate different forms of representations. In this regard one will attain all advantages that each representation holds than working with one representation.

It is important that learners become proficient in using representations in such a way that when they solve a problem they can purposefully select the representation which will present their solution better. This essentially means that learners who are taught using multiple representations are in a better position of grasping the content better than those who are taught using traditional talk and chalk. A similar view is shared by ÇIKLA (2004) who articulated that multiple representations hold different meanings in which ideas can be presented in regard to teaching and learning of mathematics. Notably, in multiple representations, one piece of information can be presented in many different ways.

A similar view is shared by Akkus and Cakiroglu (2009) who asserted that for educators to successfully reach all their learners in terms of instruction, they ought to employ the use of representations to enable their learners to better learn mathematics. Akkus and Cakiroglu (2009b) emphasized that learners should be encouraged to think deeply about mathematical concepts and eventually eliminate rote memorization and over emphasizing on rules and algorithms in mathematics. In my teaching, I always make reference to my University mathematics lecturer, Prof. V. Mudaly, who believes that educators should have many ways of solving one problem in mathematics. Most of his tests or exam questions were seeking to know as to how many ways a certain problem could be solved. In brief, effective mathematics teachers represent concepts and demonstrate how to solve problems in more than one way in order to assist all learners to get the most out of mathematics instruction. This is to say that the more strategies and approaches that learners are exposed to, the deeper their conceptual understanding of the content knowledge becomes.

I feel that unlocking learners' potential of doing mathematics depends mainly on teachers. In this regard, teachers need to teach learners how mathematics solutions can be represented in different ways. Possibly, pedagogical and content knowledge may be lacking in some educators and thus they cannot show learners how to solve problems in different ways. Perhaps in well-resourced schools where there are smart boards, projectors, and so on, mathematical ideas are taught in different ways which enable learners to make sense of the learnt content knowledge.

Similar thoughts are shared by the NCTM (2000, p. 67) when they state that, "when students gain access to mathematical representations and the ideas they represent, they have a set of tools that significantly expand their capacity to think mathematically". This means that they are able to solve most of the mathematics problems that they encounter, as they are capable of applying different approaches of problem solving. Moreover, multiple ways of representing solutions is better than one way, in the sense that learners often get to visualize problems that they encounter in different ways as opposed to just one view.

1.2 Statement of the Problem

In this study the researcher's main focus was to explore how the use of mathematical representations influenced learners' understanding of trigonometric functions in a poor performing school. The study attempted to gain more understanding on the types of representations that are commonly used by learners, the learners' ability to use representations and how fluently they can move from one mathematical representation to another. Furthermore, the study investigated whether the use of mathematical software (GeoGebra) could help learners to understand trigonometric functions better or not.

The use of representations seems to be lacking in daily teaching and learning, especially in semi-urban schools and mostly in rural schools which are under-resourced. As mentioned earlier, teachers do not use varying representations in their teaching. Consequently learners rarely display the use of representations in their practice. This is despite the number of advantages that teaching using representations holds over conventional instruction. Lesh et al. (1987) state that multiple representations-based instruction promotes conceptual understanding, meaningful learning where learners make connections between varieties of representational models. In essence, this eliminates the rote learning of facts. In my mathematics classroom, I sometimes think that learners are lazy to engage with a given problem and assume that all answers will come from the calculator. Some learners in my grade 10 class, struggle to solve simple problems in algebra, for example, solving for x in $x + 6 = 8$. Some will try to compute $x + 6$ in their calculators and end up getting an undefined number. My point is that multiple representations could be applied in many mathematics problems; however, teachers and learners, often ignore its use as they are not aware that other representations do exist.

It is important to help learners in grade 10 who will be introduced to trigonometric functions to maximize their use of mathematical representations in their learning. Learners in grade 10 need a good content background in Trigonometry as it carries 40% in their final examination. Nevertheless, it is a topic that requires one's attention as it is required in all three FET grades of their schooling. If learners in grade 10 are able to master the basics of trigonometry and its complexity, then they stand a good chance of excelling in grades 11 and 12 as well. One of the general aims of the South African Curriculum is to prepare learners for further education and training as well as the world of work. In essence, teaching of trigonometric functions should be for lifelong learning not just to gain a minimal pass mark for promotion to the next grade. It is noteworthy that trigonometry is one of the branches of mathematics which integrates with different content areas of mathematics including the 'theorem of Pythagoras' learnt in grade nine Geometry.

In the past decade, there has been an increase in the number of learners who are taking mathematics in the FET phase in South Africa, since the implementation of the National Curriculum Statement (NCS) in 2006 and Curriculum and Assessment Policy Statement (CAPS) in 2012. It is therefore important that these learners attain the important learning outcomes that are stipulated in the curriculum document. The NCS (2003, p. 5) Critical Outcomes and the CAPS (2001, p. 4) general aims of the South African Curriculum relevant to this study require learners to be able to identify and solve problems and make decisions using critical and creative thinking; and more importantly 'communicate effectively using visual, symbolic and/or language skills in various modes'. This then requires both educators and learners to adapt in new ways of teaching and learning where rote memorization of facts is eliminated but allows a space for critical thinking. In essence, when learners are ideally critical thinkers they tend to conceptualize, apply, analyze, synthesize and evaluate given information. They are eager to know 'why' the learning proofs and procedures are the way they are, so they can attain a good understanding (CAPS, 2011, p. 11).

Learners in peri-urban and rural schools come from different backgrounds in terms of their social and academic backgrounds. Being in a school where resources are limited makes it difficult to demonstrate most representations. It is only through the use of multiple representations that one may recognize and enrich all learners with knowledge irrespective of their individual differences. Introducing learners to learning using mathematical software is a nightmare for some learners who have never used a computer before. The same applies to educators who have been in the teaching field for quite some time, and who rarely use a computer. They lack the necessary skills to manipulate software or even to guide learners to discover for themselves. The ideas of multiple representations are deemed to fail in such schools where teachers are not enthusiastic to learn more about how they can manipulate teaching aids to enrich their learners. It is noteworthy that technology forms an integral part of learning today and that teachers should take advantage of it in order to get learners to learn in the language they understand which is 'technology'.

1.3 Purpose and Rationale for the study

The purpose of this study is to examine how the use of representations influences learners' understanding of Trigonometric functions. The research will contrast the conventional teaching instructions with teaching using representations to enhance learners' understanding. The research aims to engage learners in teaching practice in which representations are in use to cater for individual preferences and differences among learners. Moreover, this study has the following aims:

- to investigate how learners use mathematical representations in understanding trigonometric functions.
- to explore how learners fluently move from one mathematical representation to the other.
- to investigate how learners use mathematical software (GeoGebra) to understand trigonometric functions.

1.4 Critical Questions

In order to discover more about how representations can help learners to develop a better understanding of Trigonometry, the following specific research questions were formulated:

- How do learners engage in using mathematical representations to understand Trigonometric functions?
- What representations do learners commonly use in the classroom?
- To what extent do representations including GeoGebra improve learners understanding?
- Is there a correlation between learners' use of representations and their performance in general? Why?

1.5 Significance of the Study

Exploring how the use of representations influences learners' understanding of trigonometry is important in novice 10^{th} grade mathematics learners who know little or nothing about trigonometry. This study will form a platform from which learners will engage fully with the learning of trigonometry to such an extent that they will not only learn how to solve problems of this content area but also know how to approach other relative problems in the mathematics curriculum. This study then narrows its focus on the use of representations in trigonometry especially in the context where learners are performing poorly in mathematics. Davis (2005) asserted that little attention has been paid to trigonometry and different ways in which it has been represented. This then implies that such interventions are needed to ensure that learners become familiar in using representations to solve problems in trigonometry. In addition, more attention is essential in ensuring that trigonometry is not a source of misconception in novice mathematics learners, but part of mathematics where learners can master content with little or no difficulties.

One may emphasise that through the use of representations, learners will be fully aware of their learning abilities, noting their preferred learning abilities and be exposed to different representations so that more learners benefit from many learnt ways in which problems are solved. Through the use of representations, as a teacher, I will be familiar with the preferred learning styles often used by learners. Understanding their preferred learning styles has important implications for their understanding. Gruwell (2007) cited Ignacio Estrada who once said, "If a child can't learn the way we teach, maybe we should teach the way they learn." (p. vii). Mathematics teachers need to understand their learners preferred representations so they can integrate those representations in their lessons.

In the same way Singh (2014, p. 13) asserted that it is essential that teachers acquaint themselves with both the advantages and disadvantages of each representation so that they may be able to more effectively cater for learners' individual styles of thinking. One may add that it becomes advantageous too, when a teacher knows the types of learners that he/she is dealing with as that enables the teacher to use a preferred type of representation when explaining to an individual learner on a one-to-one discussion. This may also help a teacher in class discussions to effectively plan lessons catering for his/her learners' needs in the choice of representation to be used. I believe that this study will give a good platform for mathematics teachers to know their learners better and understand their academic needs.

The importance of this study extends to ensure that through the use of representations, learners get a chance to improve their performance in mathematics at large by exposing them to multiple ways of problem solving. By introducing this method of learning, it is highly likely that learners will be able to employ their learnt means of grasping content to improve their performance. Bruner (1966, p. 48) stipulated that the power of a representation can be described as "its ability, in the hands of a learner, to connect matters that, on the surface, seem quite separate". This then means that when learners are able to make connections of the content learnt, they stand a good chance to improve understanding. While this is true, Greeno and Hall (1997) disapprove teaching representations in isolation or rather as ends in themselves, instead representation should be taken as instruments that are useful in problem solving and conceptual understanding. In the same way, for learners to benefit from using representations they ought to use representations as tools to enhance their understanding.

The other point of interest that is important in this study is getting to know the types of representations that learners commonly use in solving trigonometric problems. This may assist mathematics teachers who are willing to know how they should plan and teach this topic in their classrooms. Through examining series of representations that learners commonly use to answer trigonometric functions, it will be easy for learners to select their preferred representations that they can employ when solving problems in this content area. I believe that not all representations can work in every topic. This belief is influenced by the work of Lamon (2001) who asserted that teachers can assess whether or not learners understand ideas in mathematics by simply looking or examining the representational models they chose to use. Pritchard and Simpson (1999) believed that deep mathematical understanding is verified when learners are able to move between multiple representations and call upon representations that are most beneficial to solve problems and check their solutions. I believe that this study will be a motivation that will enable learners to purposively select appropriate types of representations when solving mathematical problems.

Lastly, one may assert that existing literature does not take into consideration learners in the South African context, especially in schools where learners are performing poorly in mathematics. This then limits the relevance of current literature by the fact that semi-urban schools are still under-resourced and thus the availability of resources is a limiting factor in terms of the representations that learners may use. This comparison is important because it is mostly proclaimed that mostly leaners are not performing well in mathematics while forgetting the availability of basic resources like textbooks. The emphasis of this study though is to make it a point that learners are exposed to multiple forms of representations so they can have limited learning barriers. It is within the heart of this research to focus on how underprivileged learners make use of representations to enhance their understanding.

In a nutshell, this study bridges the existing gap in the use of representations in different contexts. Moreover, successful completion of this work will provide a baseline study upon which further research and investigations on representations may be built on. With the above mentioned points, I believe that some learners who will successfully use representations to their benefit will yield better results in terms of their performance in trigonometric functions and in mathematics at large.

1.6 Delimitation of the study

Although I believe that ‘exploring how learner’s utilization of representations influences their understanding in mathematics’ has a number of significances, it has a few limitations that are beyond the control of a researcher. On that note one may maintain that results obtained from the study may give educationists an idea as to how learners respond to learning using representations. However, the intended outcomes of this study cannot be generalized for all learners as the study itself is limited. Nevertheless, this study may be useful when one needs to look at the trends of how learners’ perform when they are exposed to learning through the use of multiple representations.

The other point is that the study is based in one school where learners are believed to be performing poorly. This might yield results that cannot be generalized to schools which are performing well in mathematics, and so the results cannot be transferable into other school contexts. This study then neglects the role that teachers have in curriculum delivery in the school concerned. It should be noted that learners’ performance relies on a number of aspects. It sometimes depend on how teachers deliver content. This is supported by Shulman (1986) who maintained that representations are an important part of pedagogical content knowledge. Shulman (1986) stated that “for mathematical knowledge to pass on, the teachers should be able to select the most useful forms of representation that could work, selecting the most powerful analogies, explanations, illustrations, examples, and demonstrations in a word, the ways of representing and formulating the subject that make it comprehensible to others” (p. 9). This then is one of the limiting factors in the study as the focus is narrowed to receivers of information (learners) but not those who are responsible for smooth flow of information (teachers). Again many studies, Loewenberg, Thames, and Phelps (2008), Turnuklu and Yesildere (2007) support the idea that teachers ought to have deep knowledge of mathematical content for them to create and use correct representations.

On another note, since the study is based in a school where resources are limited and most of learners are coming a from poor socio-economic background, it is expected that most of them will not benefit from the study since they do not have appropriate mathematical tools that are essential in the study of trigonometry. Most learners have no scientific calculators and it becomes a problem if they are taught a lesson that involves utilization of 'a' calculator. Trigonometry happens to be one of those key areas where learners need to be introduced to in $y = sin \propto, y = cos \propto$ and $y = tan \propto$ and they need to use a calculator to find some values that cannot be easily calculated through the use of a unit circle or special angles. In a nutshell, lack of resources in school is one of the limiting factors that may compromise the results of the study. This does not mean that those learners that do not have basic instruments will be sidelined; however, it is upon themselves and teachers to make arrangement so they can have access to these instruments for them to benefit from lessons.

The study ought to investigate the effect of parameter 'a' and 'q' using multiple representations, in which GoeGebra is one of the essential tools that will be used in the investigations. The use of GeoGebra requires access to computers and internet which is not catered for in the study. Whereas access to computers can be feasible, the other challenge might lie on participants' who are not computer literate. The use of computer will one of essential skills needed in manipulating parameters in a computer. Despite that, one of the key delimitation is that the researcher is the teacher of the participants. The selected learners might not be able to differentiate between the teacher and the researcher, consequently they might want to impress the researcher.

It is customary that research participants may differ in their willingness to participate in the study. At the beginning of the research, most participants are very keen to participate but they get exhausted as the study unfolds. It is common in most research space to encounter individuals who do not share the same sentiments with what research entails. It may be because they are not fond of the ideas that it brings, perhaps moving them for their comfort zone. Such teachers and learners are thus discouraged to further involve themselves in the research. Such incidents may limit the reliability of the results obtained and in some cases it may cause loopholes where the respondents claim to have understood a certain concept simply because they do not want to admit that they have limited understanding. It is then a question of whether the participants in the research are advised that the research will use pseudonym names instead of their real names. Above all, the main limiting factor in this regard is that some proposed questions may take extensive amount of time to complete and thus disadvantaging participants who need more time when reading and responding to the questions. Interview questions that will be given to leaners will have open-ended and close-ended questions. I hope that learners will remain objective and faithful in their responses with or without supervision.

1.7 Methodology

The methodology section discusses the organization of the entire research within its given parameters. The study distinguished the conventional teaching to teaching using multiple representations, through the use of GeoGebra. The study was underpinned by two theories which are Janvier's representation translation model and Lesh's translation model. The study is positioned within a qualitative interpretive case study design of six 10^{th} grade learners who are doing mathematics from one peri-urban school.

1.8 Outline of the study

Chapter 1 gave the introduction, described research questions and highlighting the reasoning behind it. Chapter 2 gives a reviews of what has already been written in mathematical representations in the past. It highlights the theories relevant to the research questions. Chapter 3 describe the theoretical framework that guides this research. Chapter 4 detail the methodology used in the study. Chapter 5 reports on the responses of participants in this study. Lastly, chapter 6 provides a summary, general conclusions and recommendations based on the findings of the study.

1.9 Chapter Summary

This chapter dealt with the introduction and background of the study. In addition, the purpose of the study in its given parameters, critical questions that probed the research were also presented. Furthermore, this chapter highlighted the intended contribution of the study and its anticipated outcomes. The next chapter presents a review of present literature in the field.

Chapter 2
Literature Review

The focus of this Chapter is to review other research that relates to the study of the use of mathematical representations in Trigonometric functions. The key issues to be discussed in this section are the nature of trigonometry, how learners understand it, multiple representations as a teaching and learning aid in teaching trigonometric functions and its implications in South African contexts. This Chapter will detail the use of mathematical software (GeoGebra) to enhance the understanding of trigonometric functions, and discuss multiple intelligences of mathematics learners. It is important to note that my understanding is that the current research in this field has not yet taken into account the context and background of mostly underprivileged, poorly performing mathematics learners in KwaZulu-Natal schools. It is in the best interest of this study to expose learners from such contexts to the idea of mathematical representations as an important learning tool so they can unlock their potential and improve their performance.

2.1 Learners' understanding of trigonometric functions

Trigonometry is a branch of Mathematics, which is difficult for grade 10 learners, who are learning it for the first time. These learners can either understand better or even misunderstand basics of trigonometry. It is important then that they are exposed to correct instruction and learning aids that will assist them to have a better understanding, otherwise they will develop misconceptions, which may be difficult to address later. Reeder and Bated (2008) and Dinkelman (2013) asserted that learners in grades nine to twelve need to understand functions well if they are to succeed in courses that build on quantitative thinking and relationships. In the same way, Clement (2001) claimed that the concept of function is central to learners' ability to describe relationships of change between variables, explain parameter changes, and interpret and analyze graphs.

I believe that learners who have a deep understanding of the knowledge that should be attained in Grade 10 trigonometric functions are likely to do better in grade eleven and twelve trigonometry and stand a better chance to succeeding in college mathematics. This is important as current studies, (Dinkelman (2013); Harris and Bourne (2017); Kitchen, DePree, Celed, and Brinkerhoff (2017) point to the fact that learners from secondary schools are not performing well in college mathematics, and that implies that they are ill-prepared to deal with abstract college content. Breidenbach, Dubinsky, Hawks, and Nichols (1992) maintained that first year students and others who have taken a fair number of mathematics courses, do not have much of an understanding of trigonometric functions. Perhaps this depends on how they were introduced to this concept that made them lose interest. It is notable that in mathematics, few learners tend to like and succeed in trigonometry and there is a large number of learners who hate and struggle with it. Gur (2009) insisted that trigonometry is an area of mathematics that students believe to be particularly difficult and abstract as compared with the other aspects of mathematics. Most studies, Chigonga (2016), Bohlmann, Prince, and Deacon (2017) support the view that trigonometry is one of the sections in mathematics where learners perform poorly. These studies confirm that learners are unable to relate trigonometric functions to previously learned basics to support them when solving difficult trigonometric problems.

The trends from the literature point out that learners are not performing well in trigonometric functions and in mathematics in general. Brijlall, Bansilal and Moore-Russo (2012) confirmed that learners' performance in mathematics has been a problem for many decades. Byers (2010) echoed that learners' difficulties in learning trigonometry are often connected to the difficulties they experience in learning mathematics in general. These researchers articulate that understanding of how learners think when they are engaging in different mathematical problems is a fundamental step in devising ways that will improve their learning. In my experience of teaching trigonometry in grade 10, most learners are able to understand how to find an unknown angle or a side by simply using pneumonics "SOH CAH TOA". However, when they are exposed to problems where they need to sketch trigonometric functions in a Cartesian plane, they tend to find it difficult to understand and to solve.

This explains why most difficult concepts in mathematics are being neglected by most learners in their tests and exams, and they tend to focus on aspects of the curriculum in which they are confident about. What emerges from this is that little attention has been paid to trigonometry and different ways in which it has been represented (Davis, 2005). Now, devising ways of understanding this concept is fundamental in daily teaching and learning. This is important, as these learners from grade 9 have to experience a bigger transition where they have to learn more about trigonometry as a branch of mathematics and be able to attain a deeper understanding of trigonometric functions. It is usual in mathematics classrooms to find learners complaining that teachers do easy (or give them easily solvable problems) examples when explaining a certain concept but when they are assigned with class activities or homework, they tend to struggle as they find those activities more difficult.

I feel that minding the existing gap should be a challenge for all education stakeholders to make it a point that transitional process between 9^{th} grade to 10^{th} grade will not discourage learners and to make them assume that trigonometry is difficult, hence make them experience difficulties in mathematics achievement. In her research, Byers (2010) found out that there exists so many gaps and omissions in a coherent pathway for learners' learning trigonometric functions between high schools and higher education institutions. Byers (2010) maintained that this is the underpinning reason why students have difficulties in university courses. Moreover, she insisted, "if these discrepancies are not addressed, students will continue to struggle. As a result, many learners may prolong the fulfilment of or ultimately abandon their chosen educational and career goals. Not only would this be tragic for them as students, but this could lead to a delay or loss in skilled workers so critical for industries in the engineering field today" (Byers, 2010, p. 9). It is critical to note that learning trigonometry is not just for passing the foundation high school mathematic but it also lays a foundation, which will help learners in their future profession.

Whereas trigonometry is identified as an area of mathematics in which many learners find too abstract and difficult in mathematics, learners need to bear in mind the importance of this content area. Educators need to enunciate the importance of this content area and its enormous uses of trigonometry in real life situation before they can get into the merits of its content. For instance, the technique of triangulation is used in astronomy to measure the distance to nearby stars, in geography to measure distances between landmarks, and in satellite navigation systems. The sine and cosine functions are fundamental to the theory of periodic functions, such as those that describe sound and light waves. My point is that learners in grade 10 need to understand the importance of learning about trigonometry. Once they see and understand why what they are learning matters, they will be able to give trigonometry the attention it deserves.

In my observation over the years, many learners in grade 9 and 10 understand better when they use visual representations. When they are given a relation, it will make more sense to them when it is graphical than algebraic. Nevertheless, learners need to gain the fundamental knowledge of working with different representations in understanding the posed problem. While this is true, Van Dyke (1994) believes that teaching by the use of graphs makes it easy for learners to understand and remember. In the same way, the teaching of trigonometry should be based on these ideas of teaching where learners are given a chance to make sense of graphs (interpreting) and make deductions and predictions.

Above all, devising a way of understanding trigonometric functions is fundamental in daily teaching and learning. The literature points out that the use of multiple representations is an important goal for schooling, which is to foster the development of learners' minds by engaging them in sophisticated and substantial deep learning opportunities of understanding the curriculum taught (Blumenfeld, Soloway, Marx, Krajcik, Guzdial, Palincsar, 1991). In the same way, Bruner (1966. p. 48) stipulated that the power of a representation can be described as its ability, in the hands of a learner, to connect matters that, on the surface, seem quite separate. If learners are to succeed in mathematics, particularly in trigonometric functions they have to be taught differently by employing the use of representations in their learning.

2.2 Multiple Representations in Mathematics

The National Council of Teachers of Mathematics (NCTM, 2000) defined representations as different ways in which mathematical ideas can be represented, that include: concrete materials, pictures, tables, graphs, number and letter symbols, spreadsheet displays, and many other ways. The NCTM (2000) articulates that the ways in which these ideas are represented are important in how learners understand the ideas presented. When learners have accessed these ideas they are able to make connections in what they are learning and moreover develop critical thinking skills that enable them to interpret and shape physical, social and mathematical phenomena to suit their individual educational needs.

It is important to note that for the last two decades mathematics education community has elevated representations as valuable teaching and learning tools for communicating both information and understanding (NCTM, 2000). This idea of putting emphasis on the use of representations is observed on the five broad goals of what mathematics instruction should enable learners across the globe to do. The recommendation articulates that learners at all levels of education should ‘‘create and use representations to organize, record, and communicate mathematical ideas; select, apply, and translate among mathematical representations to solve problems and use representations to model and interpret physical, social, and mathematical phenomena’’ (NCTM 2000, p. 64).

This then implies that mathematics learners should maximize their use of representations in their learning at all times so they can become better at what they do in terms of knowledge grasping. The argument by Schultz and Waters (2000) is noteworthy, that although representation eliminates rote learning to more meaningful learning, they should not be taught as ends in themselves but to be considered as useful tools for constructing understanding of mathematical ideas.

The NCTM (2000) stresses that learners should be encouraged to use and create multiple representations in their learning in order to develop and deepen their understanding of mathematical concepts. Moreover, switching among different representations will make them become better at what they do while making a platform for concrete understanding. Cleaves (2008) alluded that making use of multiple representations in mathematics classrooms can serve as instructional tools that allow learners to look at problems from a different perspective other than a narrow view of a problem. They should always bear in mind as to how many ways they can solve each problem they encounter. That will enable them to have a better understanding relatively to solving a problem in one way. Concisely, the use of multiple representations provides better lenses of viewing a given problem and maximize chances of getting the better understanding hence the correct solution to a given problem. CAPS (2011. p. 8) emphasizes that “learning procedures and proofs without a good understanding of why they are important will leave learners ill-equipped to use their knowledge in later life”.
Ainsworth (2006) stated that multiple representations “support the construction of deeper understanding when learners integrate information from the use of multiple representations to achieve insight that would be difficult to achieve with only a single representation” (p.189). This implies that the combination of different representations enables a learner to use different

varying computational skills and processes that will lead to one solution. This then helps a learner to benefit from the advantages that are carried by different representations. It is noteworthy that the use of multiple representations then provides sufficient, fair knowledge that an individual representation cannot present. In this regard, there seems to be consensus that representations are a key role in their problem-solving. Exposing grade 10 learners in the use of different multiple representations is believed to enhance learners understanding and thus improve their results (Ainsworth, 2006). In the context where learners are performing poorly in mathematics, they need to understand each representation in isolation first, thereafter be introduced to different types of representations. This will then enable them to know how and how to select appropriate representations in different problems.

2.3 Teaching using representations

While representations are important constructs for the teaching and learning of mathematics, it is important to note that teachers are curriculum drivers and so they need to be more knowledgeable in terms of curriculum delivery and the utilization of multiple representations in the classroom. Cai (2005) articulated that teachers have a great influence on the methods that learners use when solving problems in mathematics. This then implies that learners' abilities to solve problems, or the methods they use to problem-solve, is directly influenced by teachers and their teaching. This remark signifies the power that teachers hold in making it a point that learners are exposed to different representations. Cai (2005) alluded to the fact that different representations that teachers normally make use of during their teaching affect the knowledge, and accordingly, the success of learners in the classroom. A similar view is shared by Bal (2014) who asserted that for teachers to expand learners understanding, they must make use of multiple representations in their teaching of concepts, as an alternative to using just one type of representation, thus ensuring improvement in learners' understanding of mathematics concepts. This will make sure that there is a fair transmission between representations.

Gulkilik and Arikan (2012) pointed out that while pre-service maths teachers feel that mathematical ideas would be ideally constructed with the aid of bridging between different representations, they encountered challenges in relating one representation to the other. This confirms Ainsworth, Bibby, and Woods (1998) theory that although translation within and among representation to the other is ideally the goal of mathematics, it is not an easy matter for most learners. Perhaps this is due to many factors, which may amongst others, include that learners are glued to their preferred representations than be open to other representations. In addition, the problem may lie with educators who do not pave clearer connections among different representations during their teaching. This then causes learners to be ill prepared to work systematically with representations to and from the problem and its solution. It is even more important that multiple representations are used in schools that are performing poorly in mathematics to improve learners understanding of mathematics concepts. This also probe teachers to provide learners alternative ways to demonstrate their understanding of the mathematics problems.

Some learners encounter difficulties in using representations. Ainsworth (2006) articulates the functions that each representation serves, often depend upon learners' knowledge and goals, and not the system designer's intent. Ainsworth (2006) provides an example that a learner may be familiar with tables and extend his or her knowledge to graphs (extension), while another learner may already be familiar with both but not have considered their relationship (relation). The FET Mathematic CAPS (2011, p.12) emphasize that learners should be able to "work with relationships between variables in terms of numerical, graphical, verbal and symbolic representations of functions and move flexibly between these representations (tables, graphs, words, and formulae)". This then challenges educators to make sure that they close the existing gaps, which may hinder learners from solving problems in mathematics. In essence, the use of multiple representations in mathematics classrooms is a solid teaching and learning aid in accelerating the correct understanding of the content.

Tripathi (2008) criticized the idea of using only one representation. She maintained that it only highlights one aspect of the mathematical concept, thus "to restrict oneself to any one mathematical representation is to approach a concept blindfolded. A holistic picture of the concept begins to emerge only when one removes the blindfold and looks at the idea from different perspectives" (Tripathi, 2008. p.438). On this note, learners' representations and their ability to move from one representation to another is a strong indication that learners explicitly understand the concept. This then invites educators to think about how they can integrate these ideas into their daily teaching so that learners can become fluent in shifting between representations so they will have different point of views when solving problems. I believe that the use of multiple representations in mathematics classrooms will ideally give learners different lenses to look at the problem while also providing varying computational processes as they draw alternative inferences about the given domain.

The new reformed curriculum invites all interested stakeholders to use new ideas that aim to better the understanding of mathematics for our learners. Thompson and Chappell (2007) claim that when learners are introduced to learning mathematics using multiple representations, they tend to learn mathematics as an integrated whole rather than as a set of disconnected concepts. It is at the heart of the mathematics curriculum that traditional ways that promoted rote learning should be eliminated in today's teaching of mathematics. The two related critical outcomes in the curriculum assessment policy statement require learners' "to be able to communicate effectively using visual, symbolic and or language skills in various modes; demonstrate an understanding of the world as a set of related systems by recognizing that problem-solving contexts do not exist in isolation" D.o.E (2011, p.4).

Whereas these outcomes are more important in promoting the use of multiple representations, Hirsch, Weinhold, and Nichols (1991) as cited in Weber (2005) felt that "in many instances, trigonometry instructions emphasize on procedural, paper-and-pencil skills at the expense of deeper understanding" (p. 307). As a result, learners lack a basic understanding of trigonometric functions in grades 10 to 12. They believe that such instruction deviates from the goals stipulated in the NCTM (2000) standards and that teachers must shy away from such teaching which encourages "memorization of isolated facts and procedures and proficiency with paper-and-pencil tests [and move towards] programs that emphasise conceptual understanding, multiple representations and connections, mathematical modelling, and problem-solving" (p. 98).

In the context of teaching trigonometric functions, there are multiple ways in which learners can be taught and that excludes the memorization of facts, which seems to prevail in most mathematics lessons. Teachers need to be creative in designing investigations and activities that learners will engage with into discovering for themselves the important deductions that are traditionally taught using memorization. In the overview of content knowledge in grades 10 to 12 CAPS, it is said that learners need to "generate as many graphs as necessary, initially by means of point-by-point plotting, supported by available technology, to make and test conjectures and hence generalise the effect of the parameter which results in a vertical shift and that which results in a vertical stretch and /or a reflection" (D.o.E, 2011, p. 12).

Dewey (1985) articulated that learners learn better by doing so as they self-discover important facts. He believed that mathematical ideas are not merely based on abstractions but they are also connected in some essential way with their objects or representations. This then challenges learners to be active participants in their construction of knowledge. When learners are active in constructing their knowledge they tend to understand many ways of solving a problem. Consequently, Skemp (1987) believed that learners should be able to explain why the procedures they apply are mathematically correct and justify why mathematical concepts have the properties that they do. This application of knowledge seems to be lacking in understanding trigonometric functions. It is by multiple representations that mathematics learners could be fluent in making mathematical deductions to support their understanding.

2.4 Teaching using GeoGebra

Mathematics has become a subject where learners are eager to use technology to solve mathematical problems. It is sad though, that the South African classroom is still dominated by traditional teaching of 'talk and chalk'. The teaching of mathematics remains teacher-centered as opposed to learner-centered where learners discover mathematical facts on their own. In essence, the NCTM (2000) noted the impact of technology in teaching and learning mathematics, by recognizing that classroom technology has a great positive influence. The same view is shared by Zengin, Furkan, and Kutluca (2012) that integrating the educational technology into mathematics lessons does not only improve academic achievements but also improves how learners visualize mathematical problems; thus their attention grows as they visualize concepts which are predominantly difficult.

GeoGebra is one of the interesting Dynamic Mathematical Software (DMS) that is free and easy to use. GeoGebra is mainly used in mathematics teaching and learning in secondary schools and higher education institutions. Hohenwarter and Lavicza (2007) asserted that GeoGebra combines the simplicity of use of the dynamic geometry software with certain features of a computer algebra system and hence, allows for making connections between the mathematical disciplines of geometry, algebra, and calculus. GeoGebra is one of the innovative teaching and learning resources, which is very interactive software, and learners can gain more by investigating interesting features with the guidance of a teacher. It is an educational goal that technology should be embedded within teaching and learning. It eases the understanding of complex problems that could be impractical or inefficient to clarify using traditional ways of teaching.

The integration of GeoGebra like all mathematical technology innovations requires new teaching methods that are flexible enough to allow learners to use computers. Learners today are very socially interactive and their computer skills are far better than their teachers. They have access to computers and cell phones that allow internet access. Likewise, Bennett, Maton, & Kervin (2008) maintain that "these young people are said to have been immersed in technology all their lives, imbuing them with sophisticated technical skills and learning preferences for which traditional education is unprepared" (p. 1). If learners could be given guidance towards working with GeoGebra, they will gain more knowledge than what they are limited to in mathematics classrooms. Dewey (1985) pragmatically maintained that each child is active, inquisitive and wants to explore. Whereas, the majority of learners are from poor socio background, where they have no access to computers, in nowadays almost every household have two or more smartphones which could be used to access the internet and thereof explore the amazing features of GeoGebra. In some cases, learners could request the access to use computers at school and enhance their understanding of some abstract mathematics concepts. Teachers should be the driving force and promote the need of using mathematical software other than the use of textbooks or relying on teachers' notes on the chalkboard.

Weber (2005) noted that in mathematics education research, there is consensus about the goals of mathematics courses that is to learn with understanding rather than memorizing procedures. Weber (2005) added that several researchers articulate that current teaching practices in trigonometry classrooms do not seem geared toward developing learners' understanding of trigonometric functions. What seems to be lacking, is the use of interactive software like GeoGebra to allow learners to manipulate software so they can attain concrete understanding.

It is important to note that most countries are making use of technology in teaching mathematics; however, in mathematics classrooms in South Africa, particularly in my context, teachers either do not know how to use technology or there are no teaching resources that support the use of classroom technology. NCTM (2000) states that if teachers can integrate different learning aids in their everyday practice, they can provide inventive opportunities for their learners and nurture the acquisition of mathematical knowledge and skills.

There seems to be a common positive impact of utilizing mathematical learning representations to enhance learners' understanding. Inquiry based classrooms urge that teachers make use of representations, in particular visual representations that are depicted through images, illustrations, graphs, symbols, words to ensure that their learners gain maximum knowledge. A similar view is shared by Mudaly and Rampersad (2010) that "The use of visual representations such as diagrams, tables, models, graphs and pictures in the teaching and learning of mathematics has been at the forefront of recent research. The visual and symbolic nature of functional relationships is portrayed by using multiple representational modes such as tables, formulae, and graphs" (p.37).

Making such connections require teachers to use appropriate classroom technology and integrate it with the use of multiple representations so that learners can benefit from mathematics classrooms. GeoGebra is not a remedy alone, it is in the best interest of a teacher to use it efficiently so that learners' understanding can be enriched and thus their performance be improved. Ertsen (2014) said that a poor workman blames his tools, and so mathematics teachers ought to engage themselves in professional development programs that will enable them to make use of technology efficiently.

2.5 Multiple Intelligences of mathematics learners

The goal of schooling system is to prepare learners to be best at what they do, to refine their knowledge and skills to meet their area of expertise. Different disciplines and subjects in the schooling system are to help learners to realise and refine their capabilities and skills. South African schools offer more than 40 subjects and mathematics is among these subjects. These subjects are offered based on geographic area, especially where each subject skills and knowledge are essential. In each discipline, there are a variety of ways in which knowledge can be taught to learners, and those ‘ways’ are not always used in a classroom setting.

Teachers as deliveres of curriculum have their own pedagogical content knowledge and unique ways in which they transmit their knowledge. If one is to glance at classrooms of different teachers, one will note that in some classrooms learners will be seated quietly while the teacher delivers the content (‘talk & chalk’). In other classrooms one may observe learners who are sitting in groups perhaps walking around, helping each other while the teacher moves around the class facilitating. These are a few examples of daily activities in ‘normal’ teaching and learning. The above-mentioned classrooms rarely promote individual learning abilities as they are normally dictated by a teacher and not necessarily beneficial to all learners. Learning that promotes individual learning abilities as articulated by Gardner (1987) suggests that human beings possess a range of capacities and potentials (Multiple intelligences) and these can be put to many productive uses. Gardner (1999) suggested that multiple intelligences should be mobilized at schools and at various institutions of society.

Prior to the theory of multiple intelligences, intelligence was measured based on standardized IQ tests. According to the analogy of the IQ tests, the intelligence of people was arranged into three categories. Firstly, those who are gifted, those who are in the middle and those who are intelligently challenged. This is a limited way of viewing the capabilities of individuals in the society; a more holistic view that includes eight other types of intelligences as proposed by Gardner (1987). These are linguistic intelligence (word smart), logical-mathematical intelligence (number/reasoning smart), spatial intelligence (picture smart), bodily-kinesthetic intelligence (body smart), musical intelligence (music smart), interpersonal intelligence (people smart), intrapersonal intelligence (self-smart), and naturalist intelligence (nature smart). This then suggests that learners in a class have different capabilities that need to be acknowledged. This study sought to acknowledge that learners are different, they have different capabilities. Relevant to this study, in mathematics classrooms different educational needs of learners needs to be acknowledge and embraced. Learners who learn better when problems are presented visually should be catered for. In essence, when teachers design instruction materials they should ensure that it caters for different capabilities as depicted in Figure 2.1.

To embrace such differences when teaching trigonometric functions, one needs to make use of different teaching and learning styles, and more importantly assessing knowledge in different ways to accommodate each intelligence. Gardner (2000) criticized the education system which makes an assumption that all learner can uniformly perceive the same knowledge in the same way. Turgut (2013) cited Albert Einstein who criticized our education system, maintaining that "Everybody is a genius. But if you judge a fish by its ability to climb a tree, it will live its whole life believing that it is stupid" (p. 68). Figure 2.1 shows a typical instructional method applied in most mathematics classrooms

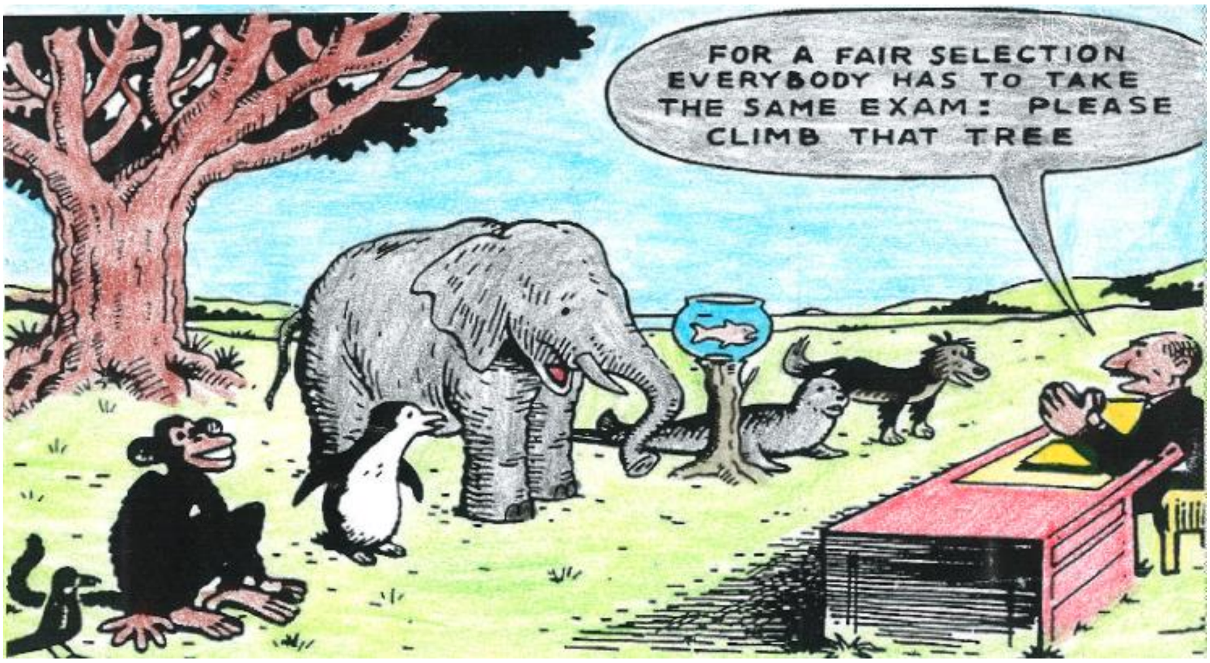


Figure 2. 1: Typical instructional method applied in most mathematics classrooms

Adapted from https://blogs.lt.vt.edu/benb89/2012/10/01/reaction-to-chaper-4-of-instructional-strategies-for-middle-and-secondary-social-studies/

Those in favor of multiple intelligences have noted the worth of using it in schools, and a sense of loss for those who do not conform to its ideology. Armstrong (2000, p.15) maintained that "The good news is that the theory of multiple intelligences has grabbed the attention of many educators, and hundreds of schools are currently using its philosophy to redesign the way it educates children. The bad news is that there are thousands of schools still out there that teach in the same old dull way, through dry lectures, and boring worksheets and textbooks. Armstrong believed that the challenge to eradicate this method of teaching of memorisation of facts is "to get this information out to many more teachers, school administrators, and others who work with children so that each child has the opportunity to learn in ways harmonious with their unique minds" (p. 15).

This is to say that we need more educators who will advocate for change in the way trigonometric functions are taught and mathematics in general. In the case of trigonometric functions, presenting a sine function, one may present it algebraically as: $(x) = \sin x$, in the table form in a given domain, graphical as a wave, and so on. In this way, each individual learner may use his or her intelligence or capabilities to best describe or present a solution to a posed question. It is essential that teachers then know how to assess each individual task in different ways as learners will have different representations to one posed question. This also suggests that teachers should think through their tasks as to how best they will accommodate and benefit each intelligence in task items, to accommodate a range of thinking in mathematics. While this is a great method, it is not always possible in daily teaching given number of constrains and challenges that teachers face every day. I believe that more that the more teachers accommodate different intelligences in their tasks and teaching, more learners will benefit from mathematics lessons.

2.6 Chapter summary

Chapter two discussed how mathematical representation has been used in mathematics and in other disciplines to improve learners' understanding and performance in general. It has been noted that trigonometry is one of the sources of difficulties in mathematics and that the reason is that it has been poorly understood by teachers and consequently their learners. From the discussion, it has arisen that using dynamic interactive software like GeoGebra can enhance learning and allow learners to be active participants in constructing their knowledge. The implications point out that in some South African schools where resources are limited, it will be a challenge for teachers and learners to have and use these useful learning aids. In the case of acknowledging individual capabilities, teachers need to be workshopped about the new interventions that will help them to plan their assessment efficiently and accommodate each range of multiple intelligences in the mathematics classrooms. The next chapter deals with the theoretical framework that informs this study.

Chapter 3
Theoretical Framework

3.1 Introduction

Whereas, the focus of Chapter One and Two were to outline the background and the literature that inform this study respectively, the aim of this Chapter is to detail the theoretical framework that underpins this study. Sinclair (2007) stated that "a theoretical framework can be thought of as a map or travel plan. When planning a journey in unfamiliar country, people seek as much knowledge as possible about the best way to travel, using previous experience and the accounts of others who have been on similar trips" (p. 1). The theoretical perspectives that will provide lenses to understand, describe and examine representations that learners' use in their learning will incorporate, Janvier's model of multiple representations and Lesh's translation model.

3.2 Janvier's Model of Multiple Representations

There has been a growing interest on how learners use multiple representation in their learning of mathematics. In 1987, Janvier made an enormous contribution to our current use and understanding of multiple representations. To him, a representation may well be viewed as a unifying of written symbols, real objects and mental images. Janvier's translation is based on external representations in which he included tables, graphs, formulation, verbal descriptions and objects. In his model, the translation generally happens when moving from one vertex to another.

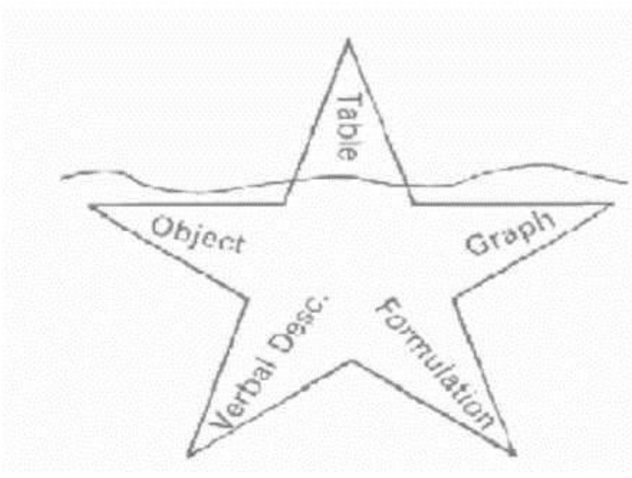


Figure 3.1: Janvier (1987) visual resemblance between a representation and a star.

In his view, learners who have mastered mathematical concepts easily move from one representation to another. Janvier (1987) argues that the ability of making direct and indirect translations between and within representations increases the possibility of learner's mathematical understanding.

In the same way, Larvor (2016) believes that translation from a different field enhances our knowledge of the mathematical subject. He insisted that mathematicians ideally move from and within various notations to gain deeper meaning of what they learn. In his view, Larvor (2016) maintained that "moving from one notation to another for the same mathematical content is indeed a good strategy to discover new relations" (p. 48). I believe that a learner's ability to know and use different types of representations may help them to better remember what they have learnt. Having many ways of arriving at a solution allows one to evaluate each representation, noting its strength and weakness. Similar thoughts are shared by Duval (1999) who asserted that learners who are successful in mathematics are able work with different representations and move fluently from one representation to another.

In this study, learners were asked to make use of different representations to show their understanding of mathematics concepts. Since the interest was in understanding trigonometric functions, learners were to use the table of values to get corresponding y-values for every input x-value (input-output). These values were used to plot point-by-point on the Cartesian plane, joining the points accurately; learners could see different graphs of sine, cosine and tangent functions. Because trigonometric graphs possess different properties, learners were asked to give verbal descriptions based on what they observed and to note effect of each parameter in each function. The aim was to allow a reasonable understanding of each trigonometric function. Janvier's theory asserts that, for meaningful learning to take place, learners must be able to fluently use and move from one representation to another. This implies that for concrete understanding to take place, the learner should be able to flexibly use the translation process; this is by choosing the appropriate representation mode. To think and communicate mathematical ideas relies on learner's ability to translate from one form of representation to another. Table 3.1 shows how different modes of representations could be used to translate from one representation form to another.

Table 3.1: Janvier's representation translation model

To From	Situations, Verbal Description	Tables	Graphs	Formulae
Situations, Verbal Description		Measuring	Sketching	Modelling
Tables	Reading		Plotting	Fitting
Graphs	Interpretation	Reading off		Curve fitting
Formulae	Parameter recognition	Computing	Sketching	

Adapted from Janvier (1987, p. 28)

Janvier (1987) stated that translation is a process that involves the transfer from one representation form to other forms of representations. Janvier argue that translation involves two modes of representation, which are, the source being the initial representation and the target being the final representation. This can be seen when a learner is able to do translation from 'table to formula' and interchangeable from 'formula to table' the learner has gained deeper understanding of mathematical concepts. In table 3.1, Janvier (1987) recognizes some translations as direct and some as indirect. The NCTM believe that, when learners have a complete translation ability the more sound the understanding of mathematical ideas. The flexibly movement from one representation to the next could conquers understanding of trigonometric functions. Furthermore, learners in grade 10 will be exposed use different types of representation to understand the concept of Trigonometric functions.

3.3 Lesh's Translation Model

The second framework that informs this study is Lesh's translation model, which provides great lenses on how learners utilize representations and how they able to make translations among and within representations around a mathematical concept. Lesh's translation model within representations, suggests that mathematical knowledge can be viewed in five interrelated and interconnected modes. These are; spoken symbols, manipulative aids, real world situation, written symbols and pictorial representations (Lesh, Post, and Behr 1987, p.34)

Lesh, Post, and Behr (1987) emphasized the necessity of translation within and across the five representation modes for deep internalization and concrete understanding. Lesh's model suggests that learners should use different modes interchangeably; hence, there is no more important mode than the other. Lesh, Post, and Behr (1987, p.34) simplified these representation modes as:

- Spoken symbols, which could be everyday language that learners use even at home.
- Manipulatives, these are objects that learners are able to feel, helpful effective tools that assist in enhancing the development of mathematics concepts.
- Real world situations, these are typical examples that are referred to so that learners can relate unfamiliar mathematical knowledge with their real lives.
- Written symbols, these are in which specialized sentences and phrases take place.
- Moreover, pictures, these are keyways to explain what words cannot explain accurately.

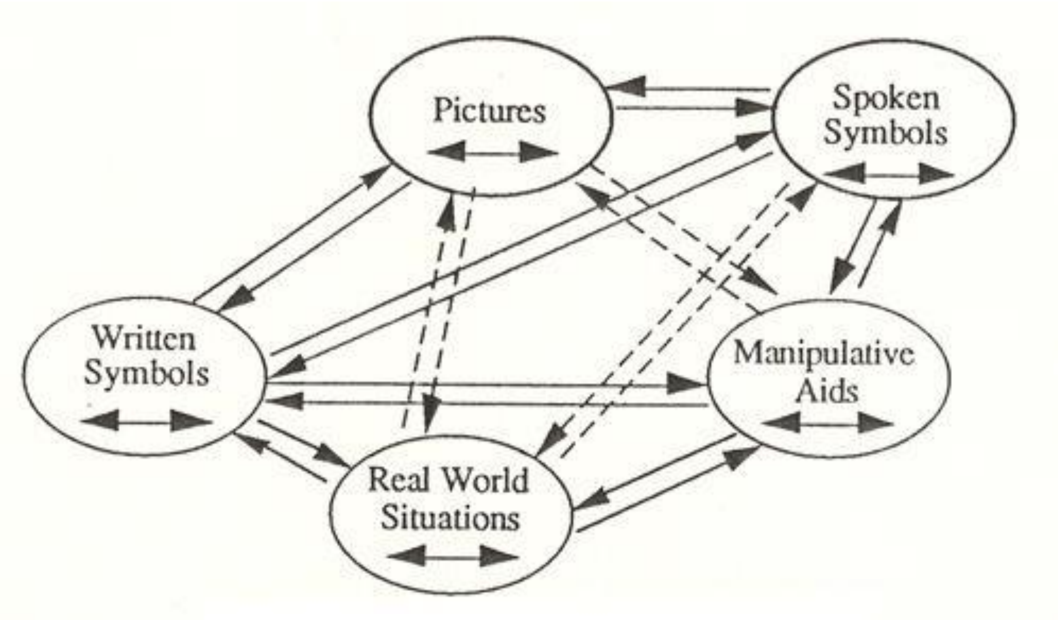


Figure 3.2: Lesh's translation between modes of representation. (Adapted from Lesh, Post, & Behr (1987))

This is an important implication in successfully translating from one representation mode to another. Lesh et al. (1987) suggested that when learners are solving mathematical problems, they should ideally recognize that each problem is embedded in a multiplicity of different representational systems, flexibly move within given representational system, and accurately translate the idea from one representational system to another.

This is an important principle in mathematics grades R to12 which aims to produce learners that are able to employ more active and critical learning ways of arriving at solutions to posed problems. Department of Basic Education (DoE (2011)) affirms that an active and critical approach to learning that does not foster rote and uncritical learning should be put in action in mathematics. In the same way Mudaly and Rampersad (2010) urge educators to emphasise the visual relationships between the symbolic and graphical representations in their teaching. They stated that "The use of visualisation skills in the teaching and learning of graphical functional relationships is essential for conceptual understanding. While there is still a need to teach according to rules and algebraic procedures, concepts in functional relationships cannot be conceived without graphical visualisation" (p. 47).

This study is embedded in Lesh's Translation model because of its convenience on representing a problem using different ways. Representation mode, shown in Figure 3.2, are to ensure that learners gain substantial understanding of the learnt knowledge. Learners are highly likely to gain deeper understanding when they are able to freely move from one representation mode to the other, when solving problems. Lesh's Translation model assures that when learners chose suitable representation notation, describing the problem in their own words, giving examples of similar problems or even giving examples of their real life situation/ context, they give assurance that they understand mathematics.

Whereas Lesh's translation and representation model suggests that if a learner understands mathematical idea/s, the learner would be able to make translation between and within different modes of representations. This model suggest that learners should deepen their understanding of mathematical ideas, in the sense that before translation occurs the learner should be able to interpret the problem that is presented so that the meaning could be translated within different modes of representations. In this study, learners are to represent trigonometric functions in different ways, using table of values, sketch graphs using points from the table of values, interpret the graphs verbally and give meaningful description on the parameters affected. In enhancing understanding this study sought to trigger learners to recognize that mathematics ideas are embedded in a variety of qualitatively different representational systems, and they can flexibly manipulate and translate ideas from one representational systems to another.

On the other view, Janvier (1987) refer the psychological processes that a learner undergoes through when moving from one representation to the next. Janvier believes that profound meaning making and its translation, takes place when a mathematic concept is understood by being translated from word problem to symbolic or conversely. Janvier (1987) further argue that although translation involve two modes of representation, he articulates that instructional methods employed by teachers should include different types of representations that will enable learners to flexible move from one representation to the next.

3.4 Chapter summary

This chapter detailed the theoretical framework that informs this study. The two theoretical perspectives of examining representations that were discussed are Janvier's model of multiple representations and Lesh's translation model. The fundamentals of these theories is that they address multiple ways in which a problem can be represented, where the learners apply different methods to solve problems. Lesh suggested that knowing different strategies is not sufficient, rather making translations between and within different modes of representations to present a solution can guarantee concrete understanding. In a nutshell, translating an idea from one representation to another makes learners to employ critical learning ways. The next chapter deals with the research methodology, which entails the organization of the entire research project.

Chapter 4

Research Methodology

4.1 Introduction

Whereas preceding Chapters orientated the research, the aim of this chapter is to detail the research methodology that has been used in this study to obtain the data. In this Chapter, the organization of the entire research will be explicitly elaborated in such a way that the research goals can be feasible to achieve. Briefly, this can be viewed as a plan of procedures to answer the research questions. Mason (2002) defined a research design as the logic through which a researcher addresses the research questions. Cohen and Manion (2000) elaborated that research design can be viewed as a: "systematic way of gathering data from a given population so as to understand a phenomenon and to generalize facts obtained from a larger population" (p. 44).

In essence, this chapter discusses the methodology and research design employed in this study. Furthermore, this chapter will shed light on the geographical area of the study, population, sample of the population, data collection instrument and its validity, administration of the data collection instrument and lastly details the framework of the data analysis.

Briefly, the purpose of this study is to examine how the use of mathematical representations influences learners' understanding of Trigonometric functions. The research contrasted the conventional teaching instructions with teaching using representations to enhance learners' understanding. The research aimed at engaging learners in teaching practice in which representations are in use to cater for individual preferences and differences among learners. The results of this study have the potential to inform discussions around the teaching and learning of trigonometry in secondary schools. It will further give learners hands-on experience with GeoGebra to investigate trigonometric functions, hence reflect on their experience.

This research intends to answer the following research questions:

1. How do learners engage in using mathematical representations to understand trigonometric functions?
2. What representations do learners commonly use in the classroom?
3. To what extent do representations improve learners understanding?
4. Is there a correlation between learners' use of representations and their performance in general? Why?

4.2 Qualitative Approach

This research was a qualitative study. The approach used in this study was systematic and aimed at gaining in-depth comprehension of investigations from learners and has important implications for how learners understand trigonometric functions. In order for the research to capture learners' understanding, the study included different representations of functions so one can possibly understand how learners make use of each representation. This then takes different forms, either algebraically, table wise, graphical, symbolical and or a verbal description. This was a feasible approach since the research was conducted at a school where it became easier to gather all required data.

It was then essential to give a description and gain an understanding of the social phenomena in question as it is. Denzin and Lincoln (2000) stressed that "qualitative researchers study things in their natural setting, attempting to make sense of, or interpret phenomena in terms of the meaning people bring to them" (p.3). This essentially means that researchers are concerned with developing a detailed explanation of how things happen in the way they do. I think that is an important observation as the participants' perspectives is gained whilst they are in their natural setting. This study then adopts a qualitative approach because it depends on human understanding and perception of the phenomenon.

Yin (2013) views qualitative research as an empirical inquiry that aims to investigate the phenomenon in its real life situation. A similar view is shared by Creswell (2013) who asserted that for the meaningful data on a qualitative research, "researchers need to gather information by actually talking directly to people and seeing them behave and act within their context" (p.185). In essence, the researcher does not simply send the data collection instrument to the participants to complete but goes the extra mile to reach the research site and have direct contact with the participants for more understanding of their natural setting. This is important as the research takes place in a real-life context which is contrary to an experimental context where there is test- retest setting.

Furthermore, Schumacher and McMillan (2006) stated that this approach is an inquiry where the researcher generates data through interacting with selected participants in their natural settings. Similar view is shared by Creswell (2013) who defined a qualitative study as: " a type of educational research in which the researcher relies on the view of participants, asks broad, general questions, collects data consisting largely of words (or texts) from participants, describes and analyzes these words for themes, and conducts the inquiry in a subjective, biased manner" (p. 39).

For this approach to be more fruitful to understand each individual case, the data collected was more qualitative than quantitative, which is an important approach to obtain in-depth data. Particularly, this study follows qualitative research procedures which are important for interpreting group shared patterns of behaviour, beliefs, and language that develop over time. This study did not employ the use of quantitative study as a research approach because the interest was to study learners' understanding. This was going to be impossible to understanding as Miller (2017) asserted that "the quantitative nature does not rely on any assumption about what motivates human beings, rather than relying on an overly materialistic view of human nature" (p. 83). Furthermore, Miller (2017) argues that "Quantitative research exists to give us an eagle's eye view of social phenomena. It is designed to give us a general picture of some phenomenon in which we are interested, provided that there has been sufficient instances of the phenomenon at hand. As with the most powerful tools, though it can go wrong" (p.80).

This qualitative study adopts an inductive approach, meaning that all conclusions that will be made will be based on the data collected. The aim is to produce new knowledge rather than verifying existing facts. Researchers using inductive approach move from specific observations to broader generalizations and theories. On the other hand, researchers are concerned with proving theories that have already been proposed. According to Creswell (2007), the inductive approach is based on perspectives that arise from the parameters of collected data. This is contrary to a deductive approach where ideas are predetermined before the data is collected.

Trochim (2006) maintained that "anything that is qualitative can be assigned meaningful numerical values. These values can then be manipulated to help us achieve greater insight into the meaning of the data and to help us examine specific hypotheses" (p.1). Trochim (2006) believes that qualitative and quantitative data are inseparable and therefore cannot be isolated.

4.3 Research Paradigm

Kelly, Lesh, and Baek (2014) defined a paradigm as a belief system that guides the way in which research is conducted. Guba and Lincoln (1994) viewed a paradigm as a world view, a total framework of beliefs, values, and methods in which a study takes place or occurs. In the same way, Byers (2010) views a paradigm as a worldview that defines what is acceptable to research and how it should be researched, mainly for researchers who hold the same worldview. Furthermore, Guba and Lincoln (1994) asserted that “a paradigm is a set of basic beliefs (or metaphysics) that deals with ultimate or first principles. It represents a worldview that defines, for its holder, the nature of the ‘world’, the individual's place in it, and the range of possible relationships between that world and its parts, as, for example, cosmologies and theologies do” (p. 108). What seems to be the common understanding of these researchers is that they view a paradigm as a belief system that informs how a study should be done systematically.

Guba and Lincoln (1994) proposed four fundamental paradigms for qualitative research, which are positivism, post-positivism, critical theory, and constructivism. According to Terre Blanche and Durrheim (1999), under each research paradigm there is a distinct set of concepts that is embedded. Their view is that a paradigm will be an all-encompassing system of interrelated practice that define the nature of inquiry along these three dimensions which are ontology, epistemology, and methodology. A paradigm informs how a research should be conducted. In support of this view, Bertram and Christiansen (2014) stated that a research paradigm informs a researcher about what is acceptable to research and how it should be researched. Bertram and Christiansen (2014) stated that a research paradigm determines for a researcher the kinds of questions be asked, what can be observed, how data should be collected and how findings should be interpreted (p. 22).

Theoretically, interpretive paradigm allows researchers to view the world through the insights and understandings of those who are participating in the research. Interpretivism then uses gathered data from their participants’ experiences to make deductions. Researchers under this paradigm seek to understand participants understanding and experiences they hold. They want to get better insight and more in-depth information to describe the patterns in the subject of their interest. This is why they engage in the situation from the participants’ point of view. Because of the proposed research questions, aimed to understand the nature of how learners best understand trigonometric functions when exposed to different teaching and learning aids, the expected outcomes are leaning more on the interpretive paradigm. Bertram and Christiansen (2014) alluded to the fact that interpretivism targets an understanding of social behaviour, and how people make meaning of their experiences. In this background, exposing learners to different learning models have a potential to help them understand better and to alter their ways of doing mathematics to the use of multiple representations.

Phothongsunan (2015) stated that the distinguishing factor of interpretive paradigm from other paradigms is that it does not regard the social world as 'out there' but believe that meaning making is constructed by human beings. Phothongsunan (2015) stated that "Participants are taken as people who make meaning of their social world, interpretive researchers seek to investigate how humans perceive and make sense of this world. From the interpretive researcher, there can be no truly objective position. He or she becomes part of the research as a meaning-maker interacting with the other meaning-makers. The research becomes the construction of meanings between the participants, one of whom is the researcher himself or herself" (p. 1). This study relies mostly on the learners' feedback on their experiences of using GeoGebra and other types of representations in relation to traditional rote learning. Consequently, the results obtained may be used to understand the difficulties and strengths that learners in grade 10 possess when using multiple representations.

4.4 Case Study

Since this study is a qualitative study which is within the interpretive paradigm, the research approach that was more relevant to this study was the 'case study'. Starman (2013) stated that qualitative research is characterized by an interpretative paradigm, which emphasizes subjective experiences and the meanings they have for an individual. This research was designed as a case study, where the case to be understood was the 10^{th}-grade mathematics learners understanding of trigonometric functions. This is on the basis of Lincoln and Denzin (1994) that a 'case' is a "phenomenon of some sort occurring in a bounded context" (p. 440). What the study seeks to understand is whether or not, the teaching using multiple representations which include innovative teaching and learning aids, yield positive results in learners' better understanding Trigonometric functions.

In this research, case studies were used to gain more in-depth data by focusing on a small group, to obtain detailed information in order to understand how each individual learner engages in the use of representations and how this impacts individual learning of trigonometric functions. Because the study aimed at understanding shared patterns in learners' understanding, case studies then provided clearer lenses to explore in-depth information, based on extensive data collection (Creswell, 2007). Hartley (2004) claimed that a case study involves a comprehensive investigation which is from the data collected over a long period of time, hence of phenomena in their given local context. This is to provide an analysis of the context and processes which provide light in the theoretical issues being studied.

Because this study aims to understand and explore learners' understanding of trigonometric functions, the research method that was more appropriate was a case study. In essence, qualitative research aims at understanding, and therefore, the case study is a method of qualitative research that allows understanding of the social phenomenon. Stake (2000) asserted that a case study is a common framework for conducting qualitative research. Stake (1995) defined a case study as the "study of the particular" (p.253). A similar definition is shared by Navarro Sada and Maldonado (2007) who asserted that "it is a single instance of a bounded system such as a child, a class, school…" (p. 103). The goals of a case study, according to Yin (2009) are to understand complex social phenomena, and real–life events such as organizational and decision-making processes. In the same way, Louis Cohen, Manion, and Morrison (2013) maintained that a case study provides a unique example of real people in real situations.

Yin (2009) claimed that case study is an empirical inquiry that aims to "investigate a contemporary phenomenon in-depth and within its real-life context, especially when the boundaries between phenomenon and context are not clearly evident" (p.18). In this way, case studies are useful in understanding and observing any natural phenomenon which exists in a given set of data. This means that a researcher is able to closely examine the data within a specific context. Zainal (2007) added that it enables researchers to examine data at the micro level. Using the case study will then help in understanding each individual case, the reasons that influence learners' understanding that cannot be detected through surface observation.

Yin (2013) described a case study as a research method that aims to explore, describe and explain natural phenomenon, in the effort to answer the "who, what, where, how and why" research questions (p. 5-6). In essence, this study addresses the 'how, what and why' questions, the ‘who and where’ questions are not in the research questions because answers to those questions are known, such that the ‘who’, are the six grade ten mathematics learners, and the ‘where' question, is the location which is the high school in the Pinetown district. The research was conducted to answer the remainder of the questions as mentioned in chapter one.

In the same way, Baxter and Jack (2008) suggested that a case study design should be considered when:

- “the focus of the study is to answer “how” and “why” questions;
- you cannot manipulate the behaviour of those involved in the study;
- you want to cover contextual conditions because you believe they are relevant to the phenomenon under study;
- Or the boundaries are not clear between the phenomenon and context.” (p. 545).

The fundamental part of this choice of a case study is that the research questions require an extensive and in-depth description of the social phenomenon in question (Yin, 2013). Furthermore, what makes the case study to be the most relevant method in this study is that the researcher needs to understand how learners understand trigonometry and the reasons underpinning learners’ understanding. The questions of interest required a description of what has been happening in participants learning experience and how is it related to their learning using multiple representations.

When Creswell (2002) distinguished case study from ethnography, he stated that “case study researchers may focus on a program, event, or activity involving individuals rather than a group per se. In addition, when case study writers research a group, they may be more interested in describing the activities of the group instead of identifying shared patterns of behaviour exhibited by the group” (p. 24). In essence, the aim is to understand and not to draw patterns for statistical use. The nature of the research questions requires an explanation as to why and how learners have been understanding trigonometric functions and how engaging in the study influences their understanding and use of multiple representations.

This study employed multiple case design. In his view, Yin (2013) asserted that multiple case study enables the researcher to explore differences within and between cases. In the same way, Baxter and Jack (2008) cited Yin (2003) who asserted that “the goal is to replicate findings across cases. Because comparisons will be drawn, it is imperative that the cases are chosen carefully so that the researcher can predict similar results across cases, or predict contrasting results based on a theory" (p.548).

Yin (2013) points out Campbell's (1975) argument that the use of multiple case studies over a single case study helps to have more data for comparative purposes, and that "it is worth having double the amount of work on a single case study" (p.180). Baxter and Jack (2008) insisted that having multiple case study "ensures that the issue is not explored through one lens, but rather a variety of lenses which allows for multiple facets of the phenomenon to be revealed and understood" (p.544). It is noteworthy that using multiple case studies helps the researcher to analyze the data obtained from different cases within its context and through and across different contexts. Moreover, the researcher is able to draw similarities and differences between the cases.

This study reports on six cases of how learners engaged with the trigonometric investigation of parameters, with a special note on what and how they reflected on their experience. In this way, learners gave insights of how they perceived different investigations/ tasks moving from one functional representation to another. Likewise, Baxter and Jack (2008) maintained that a multiple case study, examines several cases to understand the similarities and differences between the cases. This study categorizes participants in three categories, high achievers, medium achievers and low achievers. This is to ensure that results can be categorized and generalized among the three groups of learners.

4.4.1 Anticipated strength and weaknesses of case study

Cohen, Manion, and Morrison (2011) stipulated the general advantages of using a case study research design, is that "the results obtained are more easily understood by a wide audience (including non-academics) as they frequently written in every day, non-professional language" (p.293). However, they asserted that the disadvantage of case studies is that they are not easily open to cross-checking, hence they may be selective, biased, personal and subjective (p. 293). In the same way, Zainal (2007) stated some criticism along the use of case study method, that "It has always been criticized for its lack of rigour and the tendency for a researcher to have a biased interpretation of the data. Grounds for establishing reliability and generality are also subjected to skepticism when a small sampling is deployed. Often time, case study research is dismissed as useful only as an exploratory tool" (p. 5).

4.4.2 Strength of multiple case study research

Rule and John (2011) outline some of the following strengths of selecting multiple cases research design:

- "A range of cases can be chosen to represent the class of cases better.
- Multiple cases studies allow comparison across cases.
- They allow for some breadth as well as depth focus.
- They can accommodate methodological replication (use of the same methods, techniques, and instruments of data collection and analysis).
- Multiple cases are amenable to study within a common theoretical framework" (p. 21).

4.4.3 Limitations of multiple case study research

Rule and John (2011) outline some of the following limitations of selecting multiple cases research design:

- "Researcher might be tempted to look for similarities and disregard differences.
- The specific context of each case might be skimmed over in the quest for generalities.
- It is difficult to replicate exactly the same methodological regime in different cases.
- A multiple case study design still cannot generate findings that represent all cases of the population" (p. 22).

Briefly, this study made use of multiple case study design, as this method allowed to understand how different capable learners comprehend and understand trigonometric functions in light of different teaching and learning methods presented to them. For this study, the advantages are that case study method seemed to supersede its disadvantages. Thus, it was deemed appropriate in the effort of understanding learners' behavior in the given social phenomenon. Case studies were then used to gain an in-depth understanding of learners' understanding of trigonometric functions. The aim here was to gain a holistic understanding of the phenomenon in question.

4.5 Research site

This study was conducted in a school in Pinetown District, which is a public school in KwaZulu-Natal, situated between a semi-suburban area and a squatter camp. This large school caters for approximately 1200 learners mainly from historically disadvantaged communities. Currently, the enrolment of learners who are doing mathematics in school are 120 in the 10^{th} grade, 113 in the 11^{th} grade and 65 in the 12^{th} grade. In this school, Mathematics continues to be one of the subjects in which learners perform poorly (Table 4.1 and Figure 4.1), thus such intervention that seeks to improve learners' understanding of mathematical concepts is vital. I chose this school mainly because I am a mathematics educator in this school and results are a cause for concern. I felt that introducing learners to multiple ways of solving mathematics problems that this study entails, will help interested stakeholders in understanding learners' capabilities and their sources of difficulties.

Table 4.1: Average performance of grade 12 mathematics learners over the years

Year	**Pass Rate (%)**
2014	51
2015	40
2016	33
2017	19

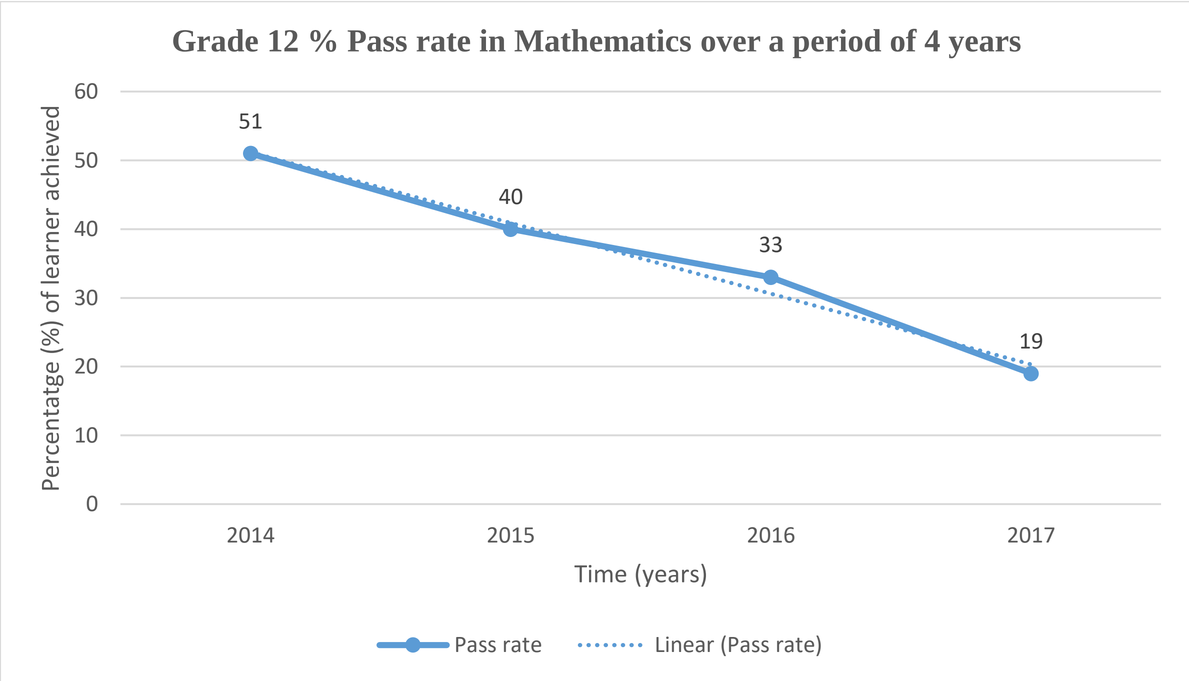


Figure 4.1: Line graph below shows the learners' performance in Mathematics over years

The line graph in Figure 4.1 shows a drastic drop in number of learners who pass mathematics in grade 12 over a period of 4 years, the graph depicts that mathematics is indeed a challenge in this school. The records of the school reveal that learners have been performing very well in the past decades. These results calls for different ways to engage learners in the teaching and learning of mathematics. Engaging learners in multiple ways of learning mathematics would help learners and teachers to be more accommodative in this transition phase. This intervention is an underlining part of this study which consequently aimed to improve learners' performance. In this regard, I feel that this research has the potential to correct educators and learners from routine to more diverse practice of teaching and learning trigonometric functions and mathematics in general.

4.6 Gaining access from the gatekeepers

Gaining access to the school was through a formal letter to the principal (Appendix A3). The letter sought permission to conduct the research at school, particularly with six grade 10 learners who are doing mathematics. It was explained to the principal that the study is purely for academic purposes and there will be no financial gain involved. Because the study involved learners who are still minors, it became a necessity to gain permission from parents/ guardian of all participating learners (Appendix A4).

The consent form was to give parents a description of the research and detailing what the research study expects from the participants. Having noted that the study involved minors, Appendix A4 and appendix A5 were to explain to learners what was expected from them. In a nutshell, for learners to participate, it was essential that there is communication between parents, learners and the researcher. This was in terms of a declaration by parents/guardians, learner assent to participate and signing the declaration by the learner who was participating in the research project.

In this study, consent forms were issued to participants prior to the collection of data. This allowed learners to consent to their anonymous participation. In their consent form, it was explicitly stated that their real names will not be used but pseudonyms will be used (Louis Cohen, Manion, and Morrison (2007). Bertram and Christiansen (2014) explained the importance of informing the parents or guardians if a child is below 18 years old. They noted that because children are special populations, they require special care with regard to obtaining consent. In this sense, consent must be obtained from parents/ legal guardians.

4.7 Sampling

Bertram and Christiansen (2014) defined sampling as a "process of making a decision about which people, settings, events or behaviours to include in the study" (p. 59). Essentially, it is the selection of individuals to be studied. Because this study is qualitative, the selection of participants was purposeful and used criterion-based sampling. This was to ensure that all participants that were sampled had characteristics relevant to the research questions. Whereas, in quantitative research, the research pursues to infer from a sample to a population, so that empirical generalization can be made to many; the goal of qualitative research is to select purposively and study individuals in a sample to gain in-depth understanding. Moreover, this study employs non-probability sampling technique. This essentially means that participants were chosen by the researcher with the full knowledge of their value to the study. Hence, the researcher purposefully, concisely and deliberately selects who may participate in the research.

The research was limited to one school and to Mathematics learners only who are in grade 10. From the grade 10 population, only six participants were selected. Six (6) case studies were purposively selected from a class of 43 learners. Learners were selected based on their academic history in mathematics, availability, and willingness to participate in the study. This is why this research holds some elements of convenience sampling. In this regard, the learners who are representatives of the whole class population were chosen because they are from the class that I am currently teaching and I have taught them earlier in the past years. This essentially means that they were close at hand as opposed to a group which might be selected randomly. Lincoln and Guba (1985) advocate that "sampling is done with some purpose in mind" (p.199). In this study, as stated elsewhere that it aims in obtaining in-depth understanding of the "world as seen through the eyes of the people being studied" (Bryman and Bell, 2015. p. 406). This sample size was deemed appropriate for this qualitative study.

The sampling method that was used in this study was purposive and convenient because the results were to be meaningful and relevant to the cases that were studied (Bertram and Christiansen (2014)). Palinkas, Green, Wisdom, Duan and Hoagwood (2015) stated that “purposeful sampling is widely used in qualitative research for the identification and selection of information-rich cases related to the phenomenon of interest”. In essence, purposive sampling has an important benefit to narrow the research focus to its targeted population. The selection of the participants who were sampled in this study was mainly determined by the researcher. This was to make sure that all the participants meet the requirements and needs for this study. Bertram and Christiansen (2014) suggested that if the researcher wants to gain a deeper understanding of the population and draw conclusions about the population of interest, the researcher ought to choose the sample that will be the representatives of the population.

Coyne (1997) stated that “In the qualitative research, sample selection has a profound effect on the ultimate quality of the research”. This suggests that the sampled participants selected in the study should be a true reflection of the population for fair generalization to be made. Schutt (2006) believed that purposive sampling embraces the selection of participants for specific purposes including being in a unique position in the population they represent, meeting the requirements of the research questions and willing to give evidence on the topic investigated. In this study, the researcher made specific choices as to who is going to be in the sample.

In this study, I used Lincon and Guba’s (1985) approach to detail how trustworthiness was ensured. Lincoln and Guba (1985) suggested that for a qualitative research to be trustworthy it should encompass credibility, transferability, dependability, and confirmability. In the same way, Golafshani (2003) argued that “although reliability and validity are treated separately in quantitative studies, these terms are not viewed separately in qualitative research. Instead, terminology that encompasses both, such as credibility, transferability, and trustworthiness is used” (p.32).

The choice of this class was motivated by the fact that I have taught most of these learners before and thus I know them more than any other grade 10 group. These learners experience different difficulties in mathematics and thus they were selected in relation to their academic performance in mathematics; two high performing learners; two average learners and two low performing learners.

Patten (2017) stated that, a researcher plans who to observe; whether the entire population or just a ‘sample’ of the population. Patten (2017) suggested that the researcher should avoid being biased in the sense that the selected sample includes all types of learners that are likely to be affected. Essentially, this means that all subgroups should be sampled in the research for the overall results to be aligned with reality. Nevertheless, I believe that working with the sampled group will yield rich information. Purposeful sampling allows a researcher to select information-rich cases. In this way the research then uncovers deep insights and in-depth understanding. Palinkas et al. (2015) believe that, “The logic and power of purposeful sampling lie in selecting information-rich cases for in-depth study. Information-rich cases are those from which one can learn a great deal about issues of central importance to the purpose of the inquiry” (p. 264).

As stated already that in this study, all of these learners were from the same class. Recruited learners were asked to seek their parents' permission first before they could partake in the research. They were also advised about the time that they will need to devote to successful completion of the investigation. The criteria that was used were their grade 8 and 9 final mathematics marks.

It should be noted that initially six learners were purposefully and conveniently selected to participate in the study, however, two did not get consent from their parents and therefore were replaced by other learners from the same class who were recruited subsequently. These two participants were selected on the same criteria and their willingness to participate. The important criteria used was based on their differences in their learning in mathematics, which was mostly determined by their academic performance in mathematics.

4.8 Trustworthiness

Guba, 1981; Schwandt, Lincoln, & Guba (2007) stated that qualitative researchers consider that dependability, credibility, transferability and confirmability as trustworthiness criteria ensure the rigour of qualitative findings. In this study, trustworthiness was ensured by scrutinizing the data collection tool. In essence, it was adopted from Mathematics 220, a module at the University of KwaZulu-Natal. The author of the Trigonometric function investigation is Dr. Linda van Laren (UKZN) and the initial editor of the document is Mr

Thokozani Mkhwanazi (UKZN). This investigation was used by preservice teachers to investigate parameters a, b, c and d, in different trigonometric functions, for instance in the sine function ($y = a\sin(bx + c) + d$. The researcher then edited investigation accordingly to suit the expectation of grade 10 in the current CAPS document. Suitable for this study, the focus was on the effect of parameter 'a' and 'd' of different trigonometric functions. Prior to data collection, the research instruments which include a software, GeoGebra was tested by senior lectures for its functionality and the content of the investigation was checked by the school Head of Department (HOD) if it was CAPS aligned or not. Furthermore validity was ensured by the review of the research instrument by the researcher's supervisor.

To increase the trustworthiness of the study, triangulation method of data collection was used, thereby where items in the research instrument that deals with the same concept were asked in different ways to ensure that participants give the best of their ability, and not to impress the researcher. In essence one question was asked in different ways to ensure that there is a correlation between individual cases. Furthermore, trustworthiness in this qualitative study will be established by using Lincoln & Guba (1985, p. 235) model of trustworthiness of research. The four criteria for trustworthiness; credibility, transferability, dependability and confirmability.

4.9 Reliability/ Credibility

Reliability refers to the consistency in measurements, the extent in which measurements can be repeated to obtain the same results. It is actually a measure of accuracy in the research instrument at obtaining results of what it is trying to obtain from the research. Cohen, Manion, and Morrison (2011) believed that "For the research to be reliable it must demonstrate that if it were to be carried out on a similar group of respondents in a similar context, them similar results would be found" (p. 199). Cohen, Manion, and Morrison (2011) maintained that "premises of naturalistic studies include the uniqueness and idiosyncrasy of situations, such that the study cannot be replicated- that is their strength other than their weakness" (p. 202).

This study makes use of multiple data collection instruments. This is to ensure that there is a correlation in the data collected through the use of the task-based activities and the data collected through interviews. I believe that the research instruments used in this study captured learners' understanding of trigonometric functions. However, the research instruments can only capture some basic understanding and not all that an individual learner knows. It should be highlighted that measuring understanding is complex in its nature because it is within the individual and who holds information decides when and how it can be voiced out. In essence, reliability is relative, mainly because the interpretation of data may vary from researchers to researchers. This implies that they lack objectivity and are dispassionate, thus this violates the credibility of the research.

Credibility refers to consistency in measurement that is the need to ensure that results obtained from each case do correlate with itself. In essence, data collected from the interview should correlate with one of the task based activities. In this way, one can argue that results obtained from each case study are repeatable and replicable. This essentially means that reliability in this study was concurred by the use of 'triangulation'. Yin (2011) asserted that using different methods of generating data to test the results have a positive impact to strengthen the validity of the study. The nature of questions in the task based activity (investigations) and interview were designed to get consistency in the responses that learners were to give. In a nutshell, one question was asked in different ways to ensure that there is a correlation between individual cases.

It is noteworthy that the research participants are mainly not first language English speakers, and they struggle with expressing themselves in the way they want to in English. Where English is used as the medium of instruction, there are some challenging factors that one may expect from some participants. It then became difficult for such learners to understand the language that is used in task-based activities, interviews and following steps in a software (GeoGebra). To overcome this, the researcher assisted with a direct translation in elaborating unfamiliar terms in their language of preference (isiZulu). To enforce the validity of data collected, learners were advised to work on their own so they can produce reliable results which are not influenced by their peers or any person around the participants.

4.10 Ethical considerations

Bertram and Christiansen (2014) defined ethics as a measure of right or wrong behaviour. They highlighted that ethical consideration is an important factor that researchers need to consider in a study that involves human beings and animals (p. 65). Vital elements that this research considered under this aspect was the autonomy of participants, non-maleficence of the study to the participants and the beneficence that participants may expect for their participation in the study. While there was no iconic benefit, the benefit of participating in this study may be that it is expected that learners in grade 10 were to get the insight of how they can use different learning aids to improve their understanding of mathematics.

Whereas the selection of participants was through purposive sampling, recruited learners were asked to participate voluntarily. Learners were advised that the participation in the research is not compulsory and solely based on their willingness to participate and their commitment. While the decision to participate in this study was voluntary, learners were advised that their withdrawal to participating in the study for whatever reason will yield no negative consequences. Furthermore, learners were to get consent from their parents as they are still minors and thus they are a 'vulnerable'. It became a necessity to gain permission from parents or guardians of all participating learners. Bertram and Christiansen (2014) stated that "special population, such as children, require particular care with regard to obtaining consent. Researchers need to get the consent of a legal guardian when working with children" (p. 66).

Non-maleficence in the study was ensured in the learner's consent form. This means that the research is expected to bring no harm to the research participants or any other people who may be indirectly involved. Bertram and Christiansen (2014) maintained that "the researcher needs to think about whether their study will do any physical, emotional, social or other harm to any person" (p. 66). Learners were assured that the findings of this research will not be used for any purpose other than the Master's dissertation. In this regard, no harm was caused to the participating learners. The report of this study did not include learners' personal names; this was to protect their identities. Furthermore, any leading information that might trace learners who were participating in the study was not depicted in the public domain. Instead, to ensure anonymity of both school and learners, pseudonyms were used to protect learners and school identity.

Furthermore, this study ensured that learners' participation was anonymous. In their consent form, it was explicitly stated that their real names will not be used but pseudonyms will to be used instead (Cohen, et al. 2007). To guarantee that participants responded candidly, the school name (see Appendix A5) and their names were not used in research and thus all participants were given pseudonyms in the written report of the study. This was accompanied by a signed declaration of informed consent form (to parents in case of participants under the age of 18) which stipulated how their authenticity was to be protected (see Appendix A4). In brief, before the parent or legal guardian agree to participation of his or her child, they were to understand and agree that:

- The child may not directly benefit from taking part in this research.
- The child is free to withdraw from the project at any time and is free to decline to answer particular questions.
- While the information gained in this study will be published as explained, the child will not be identified, and individual information will remain confidential.
- Whether the child participates or not, or withdraws after participating, will have no effect on any treatment or service that is being provided to him/her.
- Whether the child participates or not, or withdraws after participating, will have no effect on his/her progress in his/her course of study, or results gained.
- The child may ask that the interviews/task based activities and observation be stopped at any time, and he/she may withdraw at any time from the session or the research without disadvantage.

4.11 Developing the research instrument

The data of this study was mainly based on task based activities, semi-structured interviews and learning journal extracts (See Appendix A6). In essence, learners were given an investigation task where they were to study the effect of parameters '*a'* and 'q' for three trigonometric functions (sine, cosine, and tangent) as stipulated in the 10^{th} grade CAPS (p.24) document. In this way, learners' sketched graphs manually and used GeoGebra to interpret trigonometric graphs. After every investigation, learners were to reflect on their learning experiences in their learning journals. Participants were then asked to further detail their understanding by responding to series of semi-structured interviews either individually or in a group setting.

Because this is a qualitative study, most of the collected data required a direct link between the researcher and the participants. Mainly the researcher interviewed the participants on a one-to-one basis and indirectly interacted with individuals in a group setting. From learners' investigations, the researcher marked the task and rated each learner. In the process of marking the task, the researcher analyzed the task for each learner, indicating misconceptions, learner's strength and their areas of concern.

The computer software was used to capture and depict the information in the form of a graph and tables that provided a summary of the data (more details in chapter 5). One of the data analysis tools highlights how learners engaged with different reflective tools in their reflections after each investigation. This tool focused on their ability to highlight important points to remember, their levels of understanding of aspects of trigonometric functions, their mistakes, and the lesson learned from those mistakes.

4.12 Data generation

The sequence on how data was generated in this study is as follow:
The primary step was to introduce the selected participants to the idea of multiple representations of a function, detailing the use of different representation to present one function, either in a table, input-process-output, or verbal description.

I then used a computer software (GeoGebra) to represent the three trigonometric functions, demonstrating how to change the parameters of a function using a computer. In this way, trigonometric functions will be represented visually. They could see the function being reflected along the *x- axis*, if the value of '*a'* was positive (+) or negative (-). This fundamental reflection was an important observation for learners in grade 10 to relate with linear functions which they learned in grade 9. Learners were able to recall that adding the value of '*q*' in a function (i.e. $f(x) = \sin x + q$) move the graph q units up, as opposed to subtracting value of 'q' in a function (i.e. $f(x) = \sin x - q$) which move the graph '*q*' units down. This graphical representation of a function allowed learners who are visual to understand this concept better. This is to ensure that learners who are not familiar with the use of computers cannot be disadvantaged. Learners who were not familiar with using the internet to find information (research) were given a tutorial on how to begin using a computer to access the internet. I noted that some learners were excited with this learning experience as they normally use their cell phones to access the internet but not for academic purposes.

On the second day, having demonstrated the basic concepts in sketching trigonometric functions, learners were investigating the effect of parameter 'a' , where 'a' is any real number. They were asked to manually draw functions in a Cartesian plane ($y = \sin x$; $y = 2 \sin x$; $y = -\sin x$ and $y = -2 \sin x$) and then compare the effect of 'a' in ($y = a \sin x$). In this task, learners were expected to discover on their own, the transformation that is affected by the parameter 'a' in the graphs they have sketched. The second part had to do with using a link https://www.geogebra.org/RVTxQ6Vm#material/znb4GNk7 to access GeoGebra and interact with it to manipulate the value of 'a' and respond to questions which speak to the understanding of amplitude, period and general understanding of a function. The same was done for $y = a \cos x$ and $y = a \tan x$. The aim of this task to ensure that learners have a clear understanding of the effect of the parameter 'a'.

After the first investigation, learners were to reflect on their learning experience where they were investigating the effect of parameter 'a' in different trigonometric functions. Some of what learners were expected to reflect on was their current level of understanding. The second part of the reflection was to reflect on their mistakes that facilitated their learning. Here they were expected to identify their sources of misconception and to explicitly describe their learning journey. The last task was to describe the representations that helped them to understand the effect of parameter 'a' in the different trigonometric functions.

The last part of the reflective task was to test learners' understanding. Here learners were given a summative assessment. The summative assessment was to give learners an insight of whether they are understanding the knowledge or not. This, however, was not a determination whether the learner will pass or not, but a tool which a learner can use to determine the areas of concern and strength. For the researcher, it was used to check if the outcome of using all the learning and teaching tools had an impact in improving learners' performance More importantly, it was used to check the representations that learners use mostly and perhaps indicate the benefit of each representation used. One of the research questions seeks to determine the extent to which learners are able to relate and interpret daily life experiences to understand the trigonometric function. The aim was to enforce the relevance of mathematics in the learners' lives.

On the third session, learners were asked to investigate the effect of parameter 'q' or parameter 'd' where 'q' or 'd' is any real number. They were asked to manually draw functions in a Cartesian plane ($y = \sin x, y = \sin x + 1, y = \sin x - 1$ *and* $= a \sin x - \frac{1}{2}$) and then compare the effect of 'q' in ($y = a \sin x + q$). In this task, learners were expected to discover on their own, the transformation that is affected by the parameter 'q' in the graphs they have sketched. The second part had to do with using a link: https://www.geogebra.org/RVTxQ6Vm#material/znb4GNk7 to access GeoGebra and interact with it to manipulate the value of 'q' and respond to questions which speak to the understanding of amplitude, period and general understanding of a function. The same was done for $y = a \cos x + q$ and $y = a \tan x + q$. The aim of this task was to ensure that learners have a clear understanding of the effect of the parameter 'q' on parental functions.

4.13 Content focus

The fundamental aims of the investigation were to ensure that learners have a basic understanding of trigonometric parental functions. In this investigation, learners were to explore the standard equations of the sine, cosine and tangent functions. This was done through the comparison of the sine and cosine graphs as they are very similar, and then the tangent function. It was expected that they will discover that parameter 'a' and 'q' in any function can be adjusted to alter the shape and affects the general behaviour of the parent function. Noting that the tangent function has a different shape to the sine and cosine functions, learners were to discover that its parental graph/ function has a period of 180^{o}. Using a calculator, learners were able to generate a table of values which represent parental functions of the three trigonometric ratios and plotted these graphs from the table.

4.13.1 The effect of 'a' on the basic trigonometric functions

In the first investigation, learners were supposed to investigate the changes to the parent graphs brought about by changes to the value 'a' in the formulae: $y = a \sin x$ and $y = a \cos x$. In this regard, they had to describe in words that adjusting the parameter 'a' in the function $y = a \sin x$) influences a range of a function. An important observation was that while adjusting 'a' changes the amplitude of the function, however, the period of the function does not change. In a way, the obvious observation is that the graph 'stretches' by a factor of *'a'*. Furthermore, learners were to note that if the value of *'a'* is negative, importantly the graph would stretch in the same manner as it would if 'a' was positive, in addition to reflecting across the x-axis. In the last part of the first investigation, the important observation was that sine and the cosine functions are periodical, while on the tangent parental function ($y = a \tan x$), has no amplitude as the y-values are approaching infinity.

4.13.2 The effect of '*a*' – summary

- The value of 'a' affects the amplitude of the graph; the height of the peaks and the depth of the troughs.
- For $a > 1$, there is a vertical stretch and the amplitude increases.
- For $0 < a < 1$, the amplitude decreases.
- For $a < 0$, there is a reflection about the x-axis.
- For $-1 < a < 0$, there is a reflection about the x-axis and the amplitude decreases.
- For $a < -1$, there is a reflection about the x-axis and the amplitude increases.
- Note that amplitude is always positive.

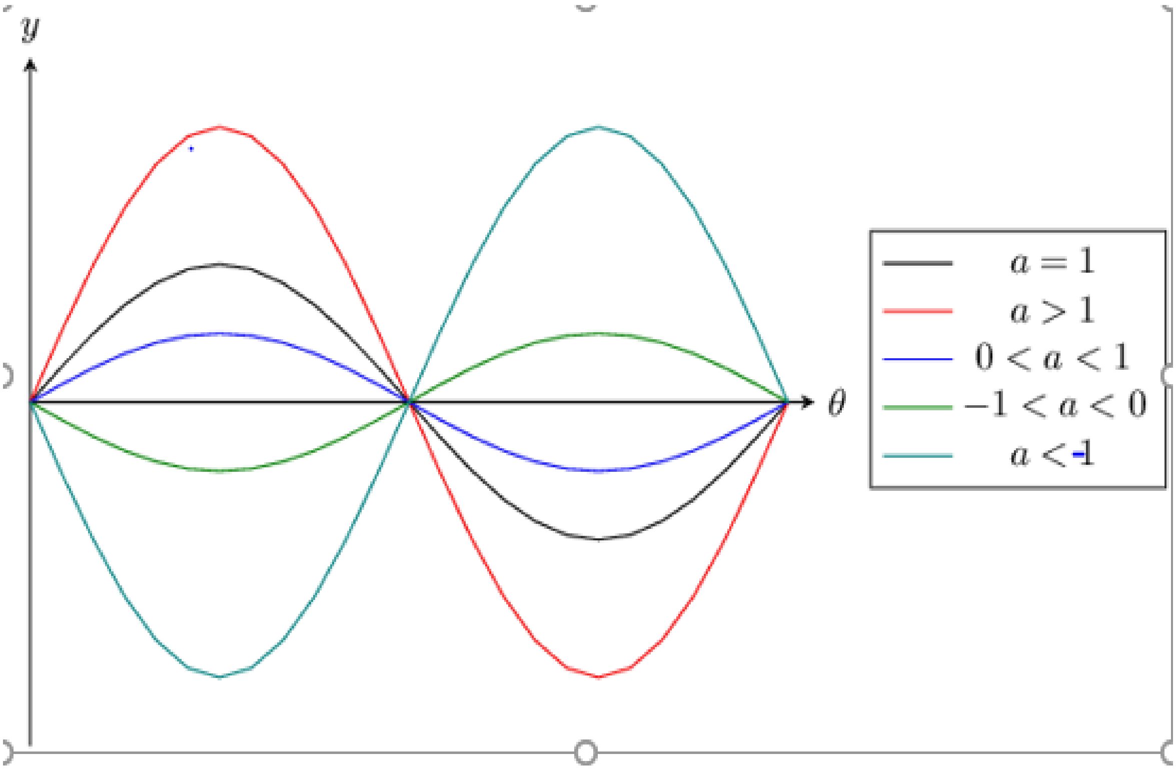


Figure 4.2: The effect of 'a' the graph of $y = a \sin x$

4.13.3 The Effect of ‘q’ on the basic trigonometric functions

The effect of ‘q’ is called a vertical shift or vertical translation since the whole parental graph shifts up or down by ‘q’ units. Briefly, the learners were to understand that, when ‘q’ is more than 0 (positive), the graph is shifted vertically upwards by ‘q’ units. On the other hand, when ‘q’ is less than 0 (negative), the graph is shifted vertically downwards by ‘q’ units.

4.13.4 The effect of ‘q’- summary

- The effect of ‘q’ is called a vertical shift because the whole sine graph shifts up or down by ‘q’ units.
- For $q > 0$, the graph is shifted vertically upwards by ‘q’ units.
- For $q < 0$, the graph is shifted vertically downwards by ‘q’ units.

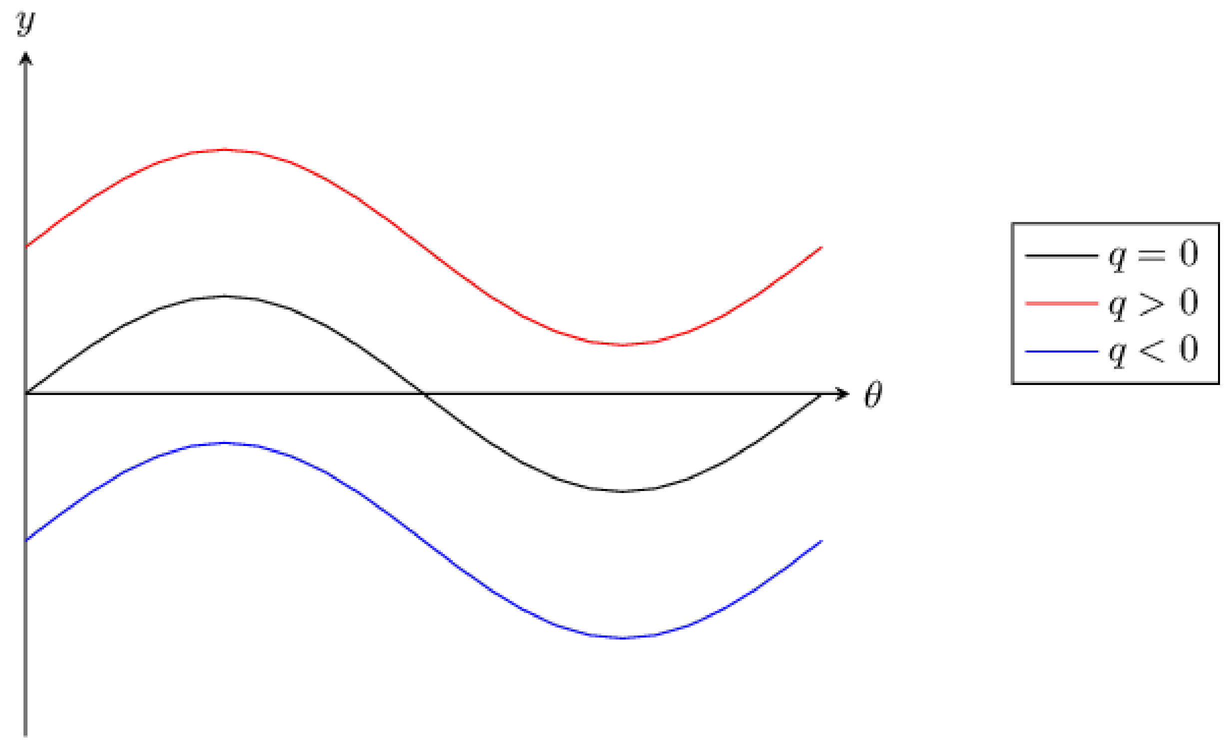


Figure 4. 3: The effect of ‘q’ the graph of $y = \cos x + q$

4.14 Moving from one representation to the next

The level of understanding that was expected from learners was that if they are given an equation ($y = -\frac{1}{2}\cos x + 4$), they had to give a verbal description to say that ‘the cosine function’s amplitude will be halved and the function will be reflected over the x-axis, and the function will be translated four units up. When learners are able to use different representations from equations to generating a table of values, sketching functions using coordinates obtained from the table method to verbal description of the same function. This confirms that a learner has undergone different levels of understanding which is sufficient for productive learning using multiple representations. It is noteworthy that if learners are able to look at a graph and determine the amplitude and period of a function by inspection and then write the equation that satisfies the graph, they would have understood the concept of trigonometric functions.

4.15 Chapter summary

This chapter dealt with the organization of the research, detailed how the research was conducted. This was achieved through presentation of an explicit guiding plan of procedures that used to answer the research questions. In a nutshell, the chapter entailed different aspects that needs to be taken into consideration. Firstly, this study is within the interpretive paradigm and the qualitative approach was deemed more appropriate for this study. Case study was the most efficient method of data collection which aimed to get rich information from the participants. Issues of trustworthiness, reliability, ethical considerations were discussed in detail. Moreover, the basic design of the study and how the research instruments were designed is also detailed in this chapter. The next chapter contains the analysis, presentation and interpretation of the findings resulting from this study.

Chapter 5
Data Analysis

5.1 Introduction

This Chapter reports on the responses of the participants in this study. I share participants' insights, their understanding, experiences and areas of concern. In essence, this Chapter contains the analysis, presentation and interpretation of the findings resulting from this study. To protect the identity of participants, real names were not used.

Table 5.1: Participants pseudonyms

Participants	**Fictitious names**
One	Lindokuhle
Two	Phumlani
Three	Skhona
Four	Nomfundo
Five	Nonto
Six	Yandisa

5.2 Investigating the effect of parameter *'a'*

5.2.1 Investigating the effect of parameter *'a'* in $f(x) = a \sin x$

In the first phase all six participants were expected to investigate the effect of 'a' in $y = a \sin x$. They had to explore the effect of *'a'* in the graph of $y = a \sin x$. In this investigation, participants were to note that adjusting the parameter 'a' changes the amplitude of the function. This investigation reinforced understanding of evaluating trigonometric functions, whereby participants were using calculators to find the value of sin 30^{o}, sin 60^{o}, and so on. In this regard, participants were instructed to complete the table by using their calculator to read off the required ratios and round off their answers to one decimal place.

5.2.2 Tabular representation of $y = a \sin x$, $y = a \cos x$ and $y = a \tan x$

The first part of this investigation, was to investigate the effect of parameter 'a' in $f(x) = a \sin x$, $f(x) = a \cos x$ and $f(x) = a \tan x$, where 'a' is any real number. Here learners were asked to complete the table below by using a calculator to read off the required ratios and also to round off the ratios to one decimal place. Table 5.2 shows the expected solutions for corresponding y-values.

Table 5. 2: Expected responses to Question 1A

	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$y = sin\, x$	0	0.5	0.9	1	0.9	0.5	0	-0.5	-0.9	-1	-0.9	-0.5	0
$y = 2\, sin\, x$	0	1	1.7	2	1.7	1	0	-1	-1.7	-2	-1.7	-1	0
$y = -\, sin\, x$	0	-0.5	-0.9	-1	-0.9	-0.5	0	0.5	0.9	1	0.9	0.5	0
$y = -2\, sin\, x$	0	-1	-1.7	-2	-1.7	-1	0	1	1.7	2	1.7	1	0

In this task, 4 participants managed to come up with correct corresponding y-values (reading off the required ratios), although these values were rounded off to different decimal places. In some cases, learners rounded off to one or two decimal places. Table 5.3 show the mistake that Phumlani made when he was rounding off ratios to one decimal place. He was not consistent, he rounded off some ratios to one decimal place *(sin 30°= 0,5)* and some to two decimal places *(sin 60° = 0,87)* instead of *(sin 60° = 0,9).*

Table 5. 3: Phumlani's answers for Question 1A

	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$y = sin\, x$	0	0,5	0,87	1	0,87	0,5	0	-0,5	-0,87	-1	-0,87	-0,5	0
$y = 2\, sin\, x$	0	1	1,73	2	1,73	1	0	-1	-1,73	-2	-1,73	-1	0
$y = -\, sin\, x$	0	-0,5	-0,87	-1	-0,87	-0,5	0	0,5	0,87	1	0,87	0,5	0
$y = -2\, sin\, x$	0	-1	-1,73	-2	-1,73	-1	0	1	1,73	2	1,73	1	0

The reason why there is an emphasis placed on getting correct values, is because this guarantees that the shape of the graph will be correct. Table 5.4 to Table 5.7 show how Lindokuhle, Skhona, Nomfundo and Nonto completed the table of values and how they were more accurate in computing different trigonometric ratios and rounding off the values to one decimal place respectively.

Table 5. 4: Lindokuhle's answers for Question 1A

A. Complete the following table. Use your calculator to read off the required ratios rounded off to one decimal place.

	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$y = \sin x$	0	0,5	0,9	1	0,9	0,5	0	-0,5	-0,9	-1	-0,9	-0,5	0
$y = 2\sin x$	0	1	1,7	2	1,7	1	0	-1	-1,7	-2	-1,7	-1	0
$y = -\sin x$	0	-0,5	-0,9	-1	-0,9	-0,5	0	0,5	0,9	1	0,9	0,5	0
$y = -2\sin x$	0	-1	-1,7	-2	-1,7	-1	0	1	1,7	2	1,7	1	0

Skhona presentation of the table of values shows more uncertainty, this is deduced from the untidy work where he roughly scratched where he rounded off to decimals. Table 5.5 show his tabular representation of the sine function.

Table 5. 5: Skhona's answers for Question 1A

A. Complete the following table. Use your calculator to read off the required ratios rounded off to one decimal place.

	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$y = \sin x$	0	0,5	0,9	1	0,9	0,5	0	-0,5	-0,9	-1	-0,9	-0,5	0
$y = 2\sin x$	0	1	1,73	2	1,73	1	0	-1	-1,73	-2	-1,73	-1	0
$y = -\sin x$	0	-0,5	-0,9	-1	-0,9	-0,5	0	0,5	0,87	1	0,9	0,5	0
$y = -2\sin x$	0	-1	-1,73	-2	-1,73	-1	0	1	1,73	2	1,73	1	0

Table 5. 6: Nomfundo's answers for Question 1A

A. Complete the following table. Use your calculator to read off the required ratios rounded off to one decimal place.

	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$y = \sin x$	0	0,5	0,9	1	0,9	0,5	0	-0,5	-9,9	-1	-9,9	-0,5	0
$y = 2\sin x$	0	1	1,7	2	1,7	1	0	-1	-1,7	-2	-1,7	-1	0
$y = -\sin x$	0	-0,5	-0,9	-1	-0,9	-0,5	0	0,5	0,9	1	0,9	0,5	0
$y = -2\sin x$	0	-1	-1,7	-2	-1,7	-1	0	1	1,7	2	1,7	1	0

Table 5. 7: Nonto's answers for Question 1A

A. Complete the following table. Use your calculator to read off the required ratios rounded off to one decimal place.

	0°	30°	60°	90°	120°	150°	180	210	240	270	300	330°	360°
$y = \sin x$	0	0,5	0,9	1	0,9	0,5	0	-0,5	-0,9	-1	-0,9	-0,5	0
$y = 2\sin x$	0	1	1,7	2	1,7	1	0	-1	-1,7	-2	-1,7	-1	0
$y = -\sin x$	0	-0,5	-0,9	-1	-0,9	-0,5	0	0,5	0,9	1	0,9	0,5	0
$y = -2\sin x$	0	-1	-1,7	-2	-1,7	1	0	1	1,7	2	1,7	1	0

Five participants, except Yandisa who did not do the investigation; were able to compute the *y*-values for all the given *x*-values. Lindokuhle presented his tabular representation (Table 5.8) of a cosine with minimal error. He was able to round off all values to the correct 2 decimal places.

Table 5. 8: Lindokuhle's answers to Question 1B

	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$y = cos\, x$	1	0,87	0,5	0	-0,5	-0,81	-1	-0,87	-0,5	0	0,5	0,87	1
$y = 4\, cos\, x$	4	3,46	2	0	-2	-3,46	-4	-3,46	-2	0	2	3,46	4
$y = -\, cos\, x$	-1	-0,87	-0,5	0	0,5	0,81	1	0,87	0,5	0	-0,5	-0,87	-1
$y = -4\, cos\, x$	-4	-3,46	-2	0	2	3,46	4	3,46	2	0	-2	-3,46	-4

Through effective use of a calculator, it was noted that the table method of finding the corresponding *y*-values was completed correctly by all participants. In this case, the tabular representation of a tangent function was completed with minimal or no error by Phumlani. When he was calculating the value of tan 90°, he said that the calculator gave him an 'error' and indicated that in table 5.9.

Table 5. 9: Phumlani's answers to Question 1B

	0°	45°	90°	135°	180	225	270	315	330°	360°
$y = tanx\, x$	0	1	Error	-1	0	1	Error	-1	- 0,58	0
$y = 2\, tanx$	0	2	Error	-2	0	2	Error	-2		0
$y = -\, tan\, x$	0	-1	Error	1	0	-1	Error	1		0
$y = -2\, tan\, x$	0	-2	Error	2	0	-2	Error	2		0

It was noted that the data collection instrument had some errors on the *x*-values which could have confused participants. The value of $x = 330^{o}$ was not within the scale of 45's, and thus participants were asked not to fill in the corresponding y-values. In the same way, the participants were concerned with the behavior of the function as it approached $x = 330^{o}$ as shown in Table 5.9 where Phumlani indicated that the corresponding y-value will be -0,58.

The results emanated from the table of values in the quest of investigating the effect of parameter '*a'* in the tangent function were not compromised by the error in the research instrument. It however gave the researcher the understanding that participants could actually see that there is an error and make means to rectify it. In Table 5.10 Lindokuhle corrected the mistake and consequently presented his solutions in the table.

Table 5.10: Lindokuhle’s answers to Question 1B

1.3 *Investigation 1C: The effect of parameter a in $f(x) = a\tan x$ where a is any real number.*

1. Complete the following table. Use your calculator to read off the required ratios rounded off to one decimal place.

	0°	45°	90°	135°	180	225	270	315	360 330°	~~360°~~
$y = tanx\,x$	0	1	und	-1	0	1	und	-1	0	
$y = 2\ tanx$	0	2	und	-2	0	2	und	-2	0	
$y = -\ tan\,x$	0	-1	und	1	0	-1	und	1	0	
$y = -2\ tan\,x$	0	-2	und	2	0	-2	und	2	0	

In the same way Nomfundo (Table 5.11), stated that the value of tan 90º and tan 270º is undefined. Mathematical reasoning of undefined y-value suggest that as the values of x approaches 90º from the left hand side and from the right hand side, the values of ‘y’ increases endlessly, approaching infinity. Notably ‘undefined’ suggests that the expression does not have meaning and so it is not assigned to an interpretation.

Table 5.11: Nomfundo’s answers to Question 1B

	0°	45°	90°	135°	180	225°	270°	315°	360°	360°
$y = tanx\,x$	0	1	un	-1	0	1	un	-1	0	
$y = 2\ tanx$	0	2	un	-2	0	2	un	-2	0	
$y = -\ tan\,x$	0	-1	un	1	0	-1	un	1	0	
$y = -2\ tan\,x$	0	-2	un	2	0	-2	un	2	0	

After the first part of the investigation, its came to light that that participants are now able to use a calculator to compute the corresponding y-values. This then suggest that plotting the points or coordinates will be much easier as all participants will get the correct shape of different functions.

5.2.3 Graphical representation of $y = a\ \sin x$

After participants have completed filling the table, they were to draw the graphs of the functions $y = sin\,x$; $y = 2\ sin\,x$; $y = -\ sin\,x$; $y = -2\ sin\,x$ on the same system of axes. This was to get an insight of participants’ ability to draw trigonometric graphs. Here, participants had to use points from the table, then plot points in a Cartesian plane to get the correct shape of the graph. Perhaps, some participants had different presentations of the table values to say that they can be presented as ordered pairs (coordinates) before plotting the points in a Cartesian plane. From the results obtained from Table 5.3, Phumlani sketched the graphs of $y = sin\, x$ and $y = 2\ sin\, x$ and not the graphs of $y = -\ sin\, x$ and $y = -2\ sin\, x$.

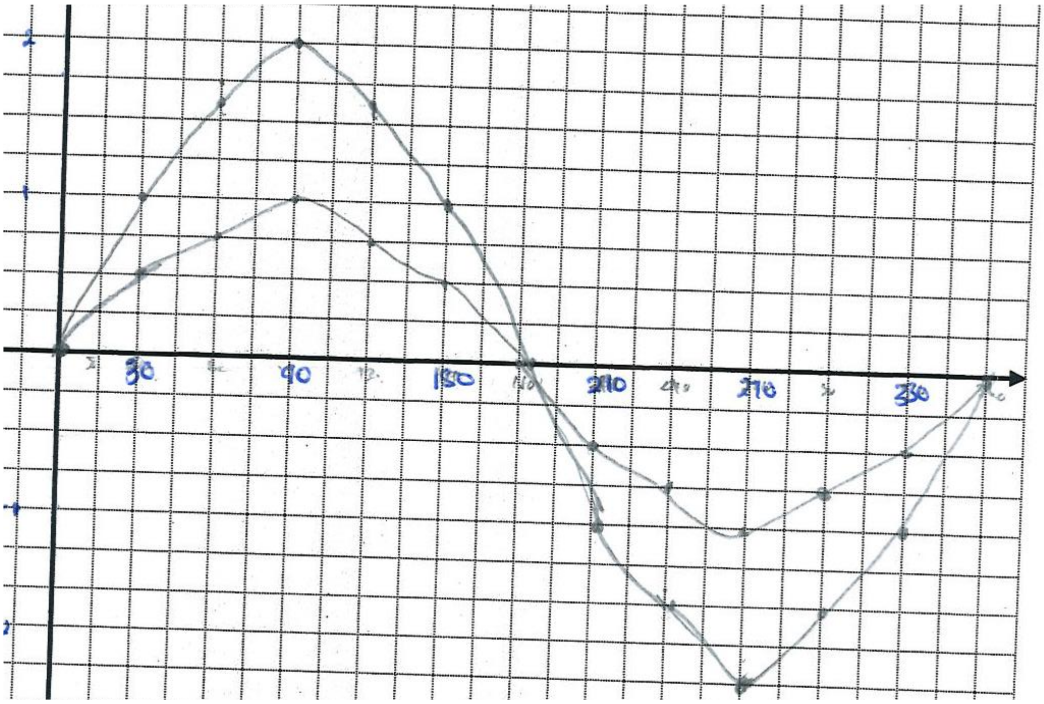


Figure 5.1: Graphs plotted by Phumlani, showing the effect of parameter 'a' in $y = a\ \sin x$

Instructions for drawing the graph emphasised that 'the squared paper must be arranged in landscape format. The scale on the y - axis should be 4 blocks representing 1 unit and the scale on the x -axis should be two blocks representing 30°'. Furthermore, participants were asked to label their graphs clearly. All graphs sketched by participants were correct, meaning that the shape resembled a sine function (wave-like) and points were accurately plotted in the Cartesian plane. What is also important is that all critical points were accurately presented and plotted accordingly.

The graphs in Figure 5.1 were sketched by Phumlani. What can be observed in a positive light is that these graphs have the correct shape, intercepts of each graph are correct, also the minimum and maximum values of each graph are accurately plotted, however the minimum of *y = 2 sin x*, where *x* = 270° was also plotted correctly as four spaces represent one unit, however, Phumlani did not label the graphs. Whilst the learner was able to choose the correct scale for the *x-axis* and *y-axis* he was only able to draw a parental function *y= sin x* and the graph of *y = 2 sin x* on the same system of axes. The two graphs $y = -\ sin\, x$ and $y = -2\ sin\, x$ which are the reflection of the two functions graphs $y = sin\, x$ and $y = 2\ sin\, x$, were not drawn by the learner. What needs to be emphasised is the idea of labelling the graphs, this is if two or more functions are drawn in the same system of axis. Manually drawing graphs is an essential skill tested in grade 10. These graphs include both parameter 'a' and 'q'. In Figure 5.1, Phumlani showed that he is able to sketch the graph of $y = a\ \sin x$ when the values of 'a' are +1 and +2, however he was not sure as to how the graph of $y = -a\ \sin x$ will behave if it is reflected along the *x*-axis. Whereas participants were not told that a negative value of 'a' reflect the graph along the *x*-axis, it was expected that when they interpret graphs they will notice the different functions being reflected along the *x*-axis. This is one of the major problems with most participants who just look at the questions which challenge them to move from their comfort zone to do something that they are not familiar with. If he wanted to, without the fear of drawing something new/ strange he was going to plot the points in a Cartesian plane, then join the points to get the graph.

The graph in Figure 5.2 was sketched by Lindokuhle, where he indicated all the intercepts, minimum and maximum values, accurately plotted the points from the table and used the correct scale for both the *x*-axis and the *y*-axis.

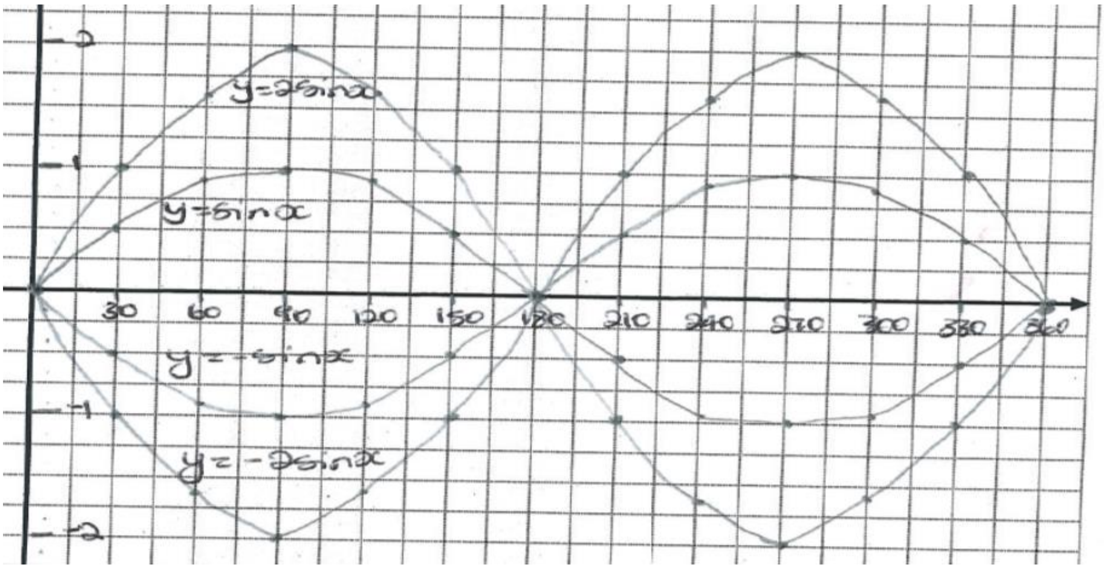


Figure 5. 2: Graphs plotted by Lindokuhle, showing the effect of parameter 'a' in $y = a\ \sin x$

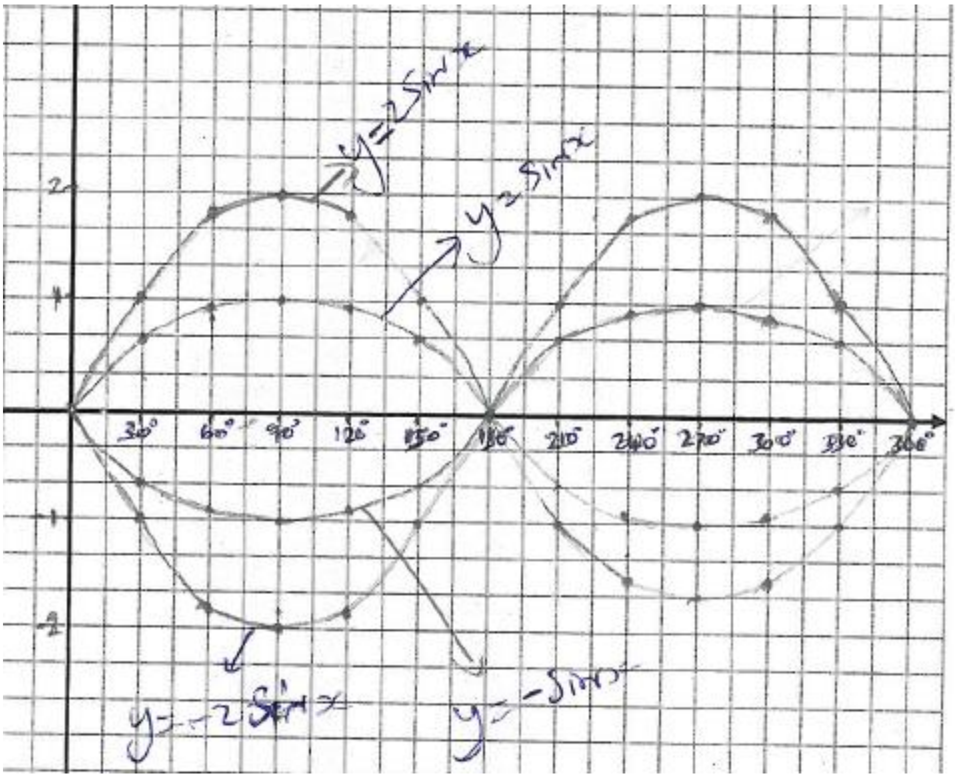


Figure 5. 3: Graphs plotted by Skhona, showing the effect of parameter 'a' in $y = a \ \sin x$

All other 4 participants' were confident when using a calculator to compute the corresponding y-values. This then enabled them to be fluent in plotting the points or coordinates and getting the correct shape/s of different functions.

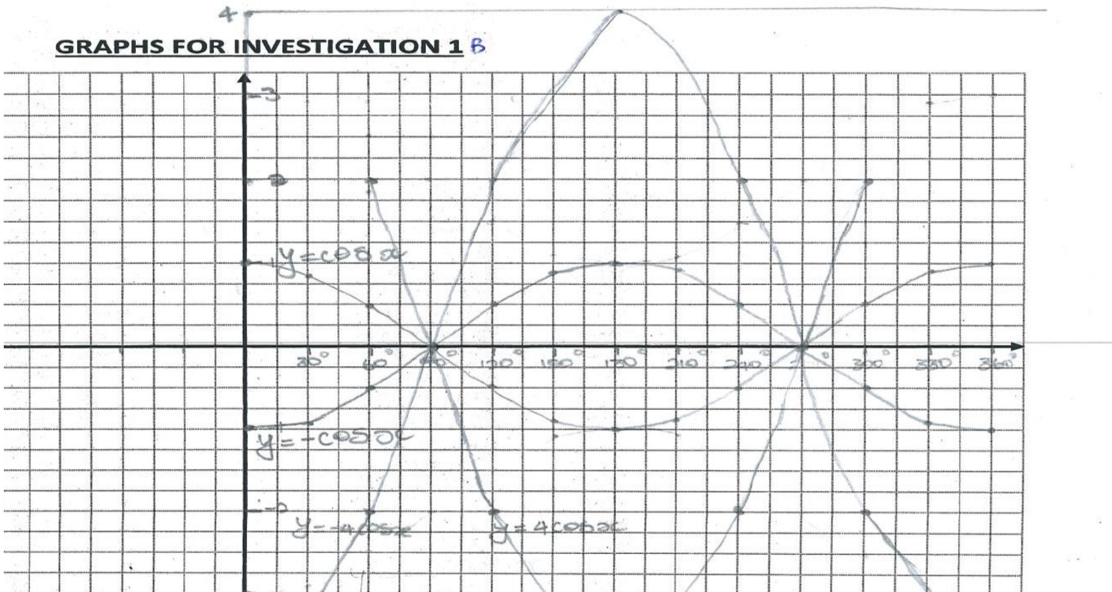


Figure 5. 4: Graphs plotted by Lindokuhle, showing the effects of 'a' in $y = a \ \cos x$

What appears to be a common trend in these graphs drawn by the all participants is that their critical points are correct and shapes of their graphs are generally correct with minor areas of concern. This will help them in scoring marks in their tests or exams. Generally, examiners are looking for the following:

- ✓ correct *x*-intercepts.
- ✓ correct *y*-intercept.
- ✓ asymptotes (in case of tangent function).
- ✓ shape (must pass through critical points).

5.2.4 Identifying the value of '*a*' in the equation and describing its effect

The other point of interest was to understand if participants are able to identify the value of '*a*' in a given function. Participant's responses showed that they know how to identify the value of '*a*' in a given function. I expected that participants would interpret the graphs in order to answer the questions, however in a case where some graphs are missing and the learner was able to list all the values of '*a*' correctly, it will mean that the participants read the values of '*a*' from the equation, for example the value of '*a*' in *y= 1 sin x* will be equal to 1, value of '*a*' in $y = 2\sin x$ will be equal to 2, -1 and -2 for $y = -1\ \sin x$ and $y = -2\ \sin x$ respectively.

Figure 5.5 presents Lindokuhle's answers to the question '*what are the values of* '*a*' *in each of these graphs*?', the graphs in question were: $y = sin\, x$, $y = 2\ sin\, x$, $y = -\ sin\, x$ and $y = -2\ sin\, x$.

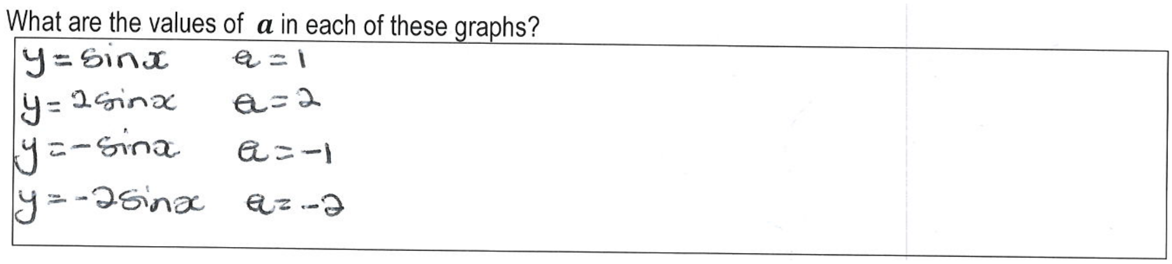


Figure 5. 5: Lindokuhle's answers to Question 1.3 A.

Yandisa presented his solution in words, saying that: 'for *y = sin x*, the value of '*a*' will be 1, for *y = -1 sin x*, the value of *a* will be -1, for *y = 2 sin x*, the value of '*a*' will be 2, and for *y = - 2 sin x*, the value of '*a*' will be -2'.

Nomfundo was able to identify the value of '*a*' in the graphs *y = tan x, y = 2 tan x, y = -1 tan x,* and *y = -2 tan x*. Lindokuhle, Nomfundo and Yandisa noted that the parameter '*a*' in any standard sine, cosine and tangent functions will be the coefficient of the trigonometric function as shown in Figure 5.6.

What are the values of $\boldsymbol{a}$ in each of these graphs?

for the graph of $y = \tan x$, $a = 1$
for the graph of $y = 2\tan x$, $a = 2$
for the graph of $y = -\tan x$, $a = -1$
for the graph of $y = -2\tan x$, $a = -2$.

Figure 5.6: Nomfundo's answers to Question 1.3 C

When participants have drew the graphs of: *y = tan x, y = 2 tan x, y = -tan x* and *y = -2 tan x* they were to compare the three graphs *(y = 2 tan x, y = - tan x and y = -2 tan x)* against the parental function *y = tan x*. This was to ensure that they understand how the value of '*a*' affects the amplitude of the graph. In this regard; increasing the value of '*a*' increases the amplitude This also has important implication on the range of the function. Below are the participants' responses to the question, '*Describe in your own words the effect of 'a' on the graph of* $y = a\, sin\, x$, this was used to capture participants' ability to note the effect of parameter '*a*' in a given function and thereby give a verbal description of the effect on a parental function.

In Skhona's description of the effect of '*a*', he noted that '*a*' *as a parameter, determines the amplitude or the length between the x-axis and the maximum height of the graph*'. Importantly, he noted that the value of '*a*' determines the amplitude of the function. In the same light, Lindokuhle said the following in his description (Figure 5.7):

Describe in your own words the effect of $\boldsymbol{a}$ in the graph of $y = \boldsymbol{a}\, sin\, x$

In the exclusion of the signs, a, determines the height of the wave known as the amplitude. The greater the value of a the higher the amplitude and if it's smaller, the amplitude will also be smaller.

Figure 5.7: Lindokuhle's answers to Question 1.3 A

Lindokuhle mentioned the important thing in Figure 5.7, that irrespective of the value of the parameter ‘a’ sign, whether negative or positive, the amplitude will be positive. This is obviously because the height of the graph (amplitude) can never be negative, the distance is always positive. Mathematically speaking, the amplitude will be determined by the absolute value of ‘a’ ($|a|$).

Nonto detailed her understanding, to say that ‘*the graph will increase in its height or depth if ‘a’ is increased and it will reduce its height and depth if ‘a’ is decreased. ‘A bigger ‘a’ value indicates a greater amplitude*’. This is similar to Yandisa’s description to say that ‘*a*’ *widens up the graph of sine vertically and it also shrinks the graph of sine vertically*’. These two descriptions have the same meaning in a nutshell.

5.2.5 The effect of parameter ‘*a*’

5.2.5.1 The effect of parameter ‘*a*’ *in y = a sin x*

Table 5.12 was used to reinforce understanding of some important characteristics observed with the function *y = sin x* when the parameter was 1, 2, -1 and -2 respectively. To get a brief summary of the characteristics of the function, participants were asked to complete the table in Table 5.12. The summary entails understanding of domain, range, amplitude and period.

Table 5. 12: Expected responses to Question 1.5

	$y = sin\, x$	$y = 2\, sin\, x$	$y = -\, sin\, x$	$y = -2\, sin\, x$
Domain	[0° ; 360°]	[0° ; 360°]	[0° ; 360°]	[0° ; 360°]
Range	[-1 ; 1]	[-2 ; 2]	[-1 ; 1]	[-2 ; 2]
Amplitude	1	2	1	2
Period	360°	360°	360°	360°

Table 5.12 summarises the most important basic concepts that need to be understood for a function. This is because the domain will provide the feasible regions which will restrict the graph on the *x-axis*. Almost all participants managed to come up with correct answers which means that their observation could be described using mathematics symbols and words. A typical example of other cases is Skhona’s responses as shown in Table 5.13:

Table 5. 13: Skhona's answers for Question 1.5

Complete the table below:

	$y = sin\, x$	$y = 2\, sin\, x$	$y = -\, sin\, x$	$y = -2\, sin\, x$
Domain	[0; 360]	[0; 360)	[0; 360)	[0; 360]
Range	[-1; +1]	[-2; 2]	[-1; 1]	[-2; 2]
Amplitude	1	2	1	2
Period	360°	360°	360°	360°

However, one case that should be discussed is Phumlani's responses in the Table 5.14 below.

Table 5. 14: Phumlani's answers for Question 1.5

Complete the table below:

	$y = sin\, x$	$y = 2\, sin\, x$	$y = -\, sin\, x$	$y = -2\, sin\, x$
Domain	[0; 360]	[0; 360]	[0; 360]	[0, 360]
Range	[-1; 1]	[-2, 2]	[-1, 1]	[-2 ; 2]
Amplitude	1	2	1	2
Period	[0; 360]	[0, 360]	[0, 360]	[0, 360]

Skhona (Table 5.13) and Phumlani (Table 5.14) used the same representation of a domain [0° to 360°] where they noted that the domain of all the sine function in question was from 0° to 360°. The use of square brackets has an important mathematical meaning especially in the case of functions, [0°; 360°] as it indicates that the graph starts at x = 0° and goes in cycles up to where x = 360°. Moreover, all these values are 'included' meaning that it is part of this function, they guide us that the function is restricted, and therefore it is bounded between x = 0° and x = 360°.

Another interesting presentation of a domain and range, was used by Nonto to present her solutions (Table 5.15). Here she used $x \in [0^{o}, 360^{o}]$, meaning that the domain of the function $y = sin\ x$ is all the x-values from 0^{o} to 360^{o}

Table 5.15: Nonto's answers for Question 1.5

Complete the table below:

	$y = sin\, x$	$y = sin\, x + 1$	$y = sin\, x - 2$	$y = sin\, x - \frac{1}{2}$
Domain	[0, 360°]	[0; 360°]	[0°, 360°]	[0°, 360°]
Range	[1, -1]	[0, 2]	[-3; -1]	[-1,5 ; 0,5]
Amplitude	1	1	1	1
Period	360°	360°	360°	360°

Yandisa represented his solution in a different manner using interval notation (Table 5.16), specifically the domain and range. This representation is important in understanding why other participants used the square brackets as discussed in Table 5.13 and Table 5.14.

Table 5.16: Yandisa's answers for Question 1.5

Complete the table below:

	$y = \sin x$	$y = 2\sin x$	$y = -\sin x$	$y = -2\sin x$
Domain	$0° \leqslant x \leqslant 360°$	$0° \leqslant x \leqslant 360°$	$0° \leqslant x \leqslant 360°$	$0° \leqslant x \leqslant 360°$
Range	$-1 \leqslant y \leqslant 1$	$-2 \leqslant x \leqslant 2$	$-1 \leqslant x \leqslant 1$	$-2 \leqslant x \leqslant 2$
Amplitude	1	2	1	2
Period	360°	360°	360°	360°

The range of the sine function was correctly presented, for example the range of *y = sin x* is [-1, 1]. This means that the minimum *y*-value is -1 and the maximum y-value will be +1. From the first task, participants were asked to identify the values of '*a*' and describe the effect of '*a*' in different functions. Participants showed that they understood that the value of '*a*' determines the amplitude, and now participants have to incorporate that idea to say that the amplitude will also determine the minimum and the maximum *y*-values i.e. the range of a function. As mentioned above, Phumlani believed that the value of *'a'* determines the amplitude of the graph. He fairly displayed that understanding in the table above.

What seemed to be a challenge in Phumlani's understanding, is the period of a graph and how it can be represented symbolically. He used the domain notation [0°, 360°] to describe the period of a function. Instead of saying that the period of the function *y = sin x* is 360°, he said that the domain is [0°, 360°]. This is an incorrect symbolic representation of a period. What seems to be lacking in his understanding is how the domain differs from a period of a function. His assumption that these terms are interchangeable is incorrect. The period of a graph represent the distance required for the function to complete one full cycle. Phumlani did not note that the graph of *y = sin x* is periodical, meaning that the period will be the length of the repeating pattern on the *x- axis* (written as *x = 360*°). While the minimum and the maximum *x*-values are the domain written as [*min value, max value*]. When '*b'* is 1, the period is 360°. When *b* = 2 the period is 180°. To calculate this, in the parental function $y = a \sin bx + q$, the period will be $\frac{360°}{b}$. However, for grade 10 purposes the period of *y = sin x* and *y = cos x* is generally 360° while the period of the tangent function is 180°.

5.2.5.2 The effect of parameter 'a' in y = *a cos x*

Nomfundo stated that *'the positive value of* '*a*' *widen up the graph and the negative value of* '*a*' *shrink the graph of y = cos x by the factor of* '*a*'. This suggests that the value of '*a*' will have the same effect in the graph of *y = sin x* and the graph of *y = cos x*. Her argument was that '*a*' will widen up the graph of *y = cosx* and also shrinks it by the factor of *'a'*. Moreover, the parameter '*a'* will determine the amplitude, and so the graph will move away from the *x-axis* by '*a*' units, reaching the maximum 'a' units in the *y-axis.* Nomfundo's assumption was that '*if* '*a*' *increases, the amplitude of the graph increases and if* '*a*' *decreases, the amplitude of the graph decreases'.* In Figure 5.8, Lindokuhle added that '*a*' has the ability to flip the graph about the *x*-axis.

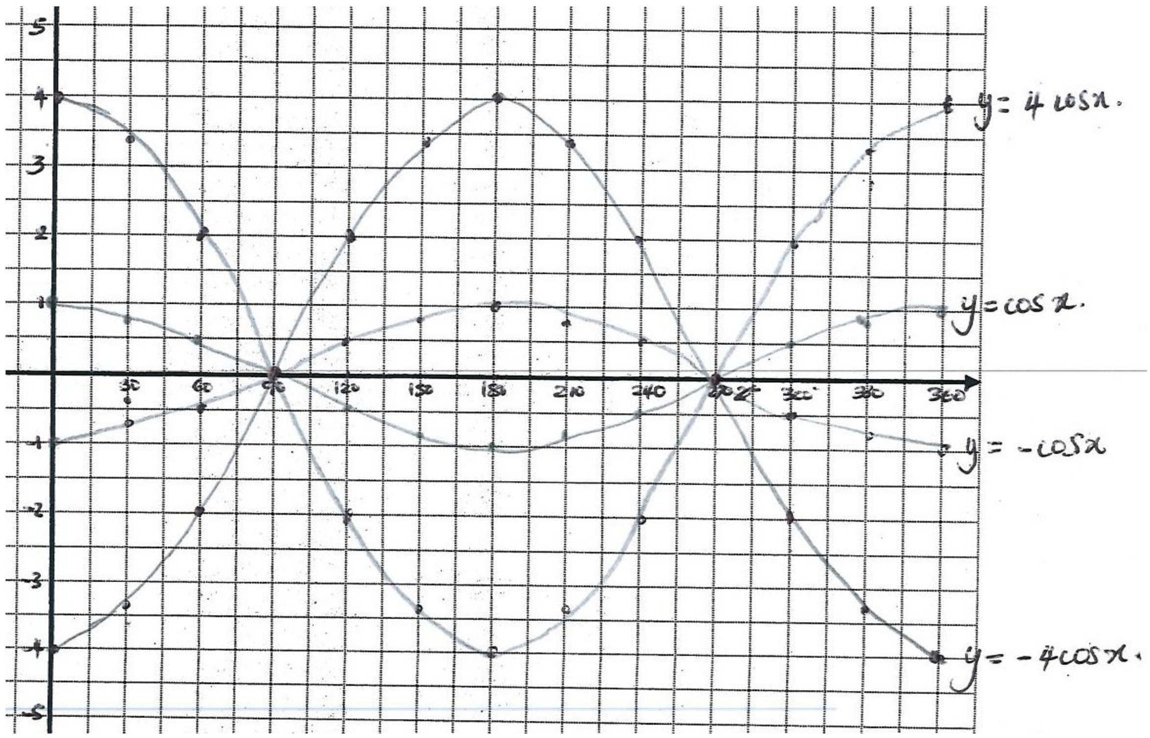


Figure 5. 8: Nomfundo's answers for Question 1.3

Whereas the value of '*a*' has observable effects on the amplitude of a function *y = cos x*, Phumlani believe that it also affects the range of a function. Table 5.17 shows how the range and the amplitude of different *cosine* graphs changed in respects to the change of the parameter '*a*' value.

Table 5. 17: Phumlani's answers to Question 1.5

	$y = \cos x$	$y = 4\cos x$	$y = -\cos x$	$y = -4\cos x$
Domain	0°; 360°	0°, 360°	0°, 360°	0°, 360°
Range	-1; 1	-4; 4	-1, 1	-4, 4
Amplitude	1	4	1	4
Period	360°	360°	360°	360°

From Table 5.17, Phumlani deduced that the parameter '*a*' affects the:

- range of a cosine function
- amplitude of a cosine function

5.2.5.3 The effect of parameter '*a*' in *y* = *a tanx*

The value had the similar effect in the graph of *y* = *tan x* as compared to the graph of sine and cosine function. From Skhona's graphs (Figure 5.9) of the tangent functions, it was clear that the parameter '*a*' caused the graph to change it steepness and the nature of its curve.

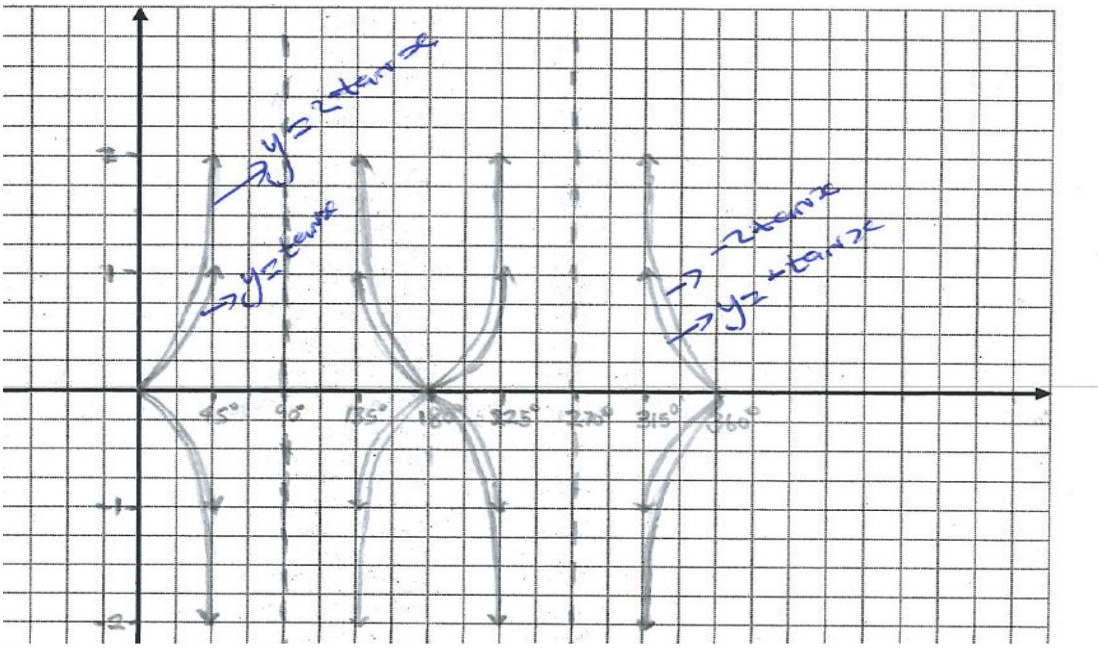


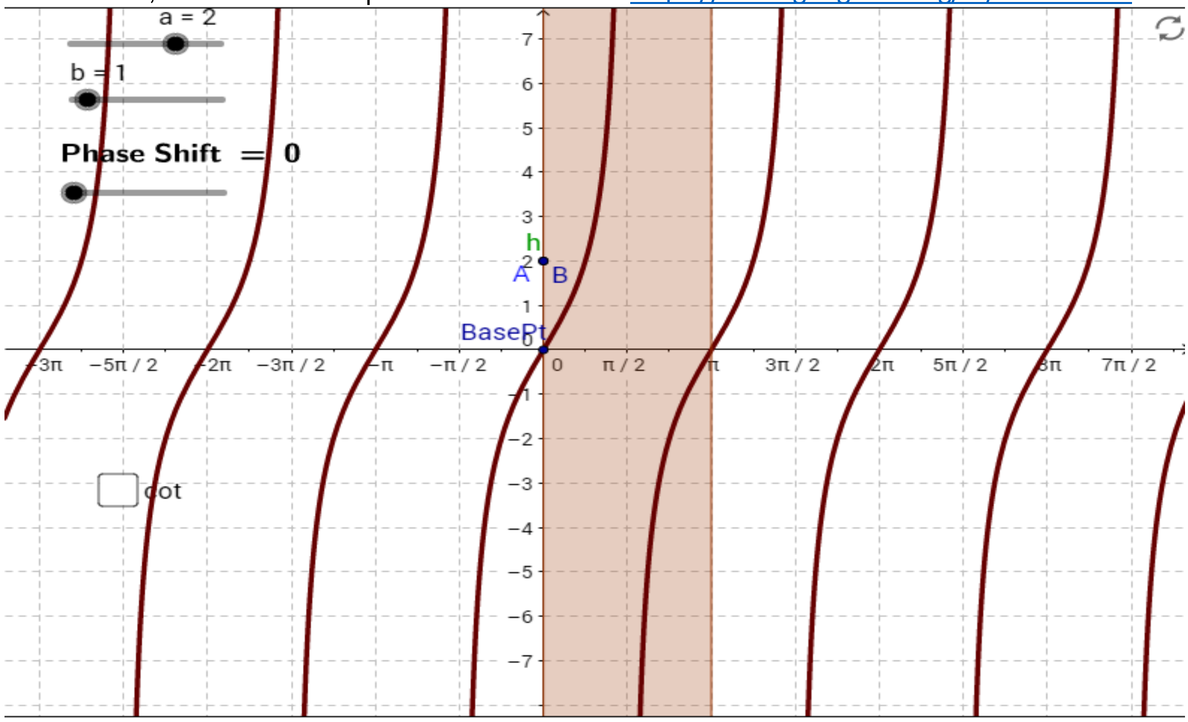
Figure 5. 9: Skhona's answers to Question 3.2

Comparing the graphs of *y* = *tan x*, *y* = - *tan x*, *y* = *2 tan x* and *y* = *-2 tan x,* it can be noted that the effect of '*a*' on shape for *a* > 1 (*y* = 2 tan *x*), branches of *y* = 2 tan *x* are steeper. For 0 < *a* < 1, branches of *y* = tan *x* are less steep and curve more. For *a* < 0, (*y* = -1 tan *x* and *y* = -2 tan *x*), there is a reflection about the *x-axis*. For −1 < *a* < 0, there is a reflection about the *x* -axis and the branches of the graph are less steep. For $a < -1$ (y = -2 tan x), there is a reflection about the x-axis and the branches of the graph are steeper.

Here participants noted that the effect of 'a' was different in the graph of $y = a \tan x$ compared to the graph of $y = a \sin x$ and $y = a \cos x$. Whereas in these two function, the parameter 'a' indicated the amplitude of the graph. In this graph of y = tan x it was representing something else. Phumlani shared that the effect of 'a' stretches the graph away and towards the x-axis as the value of 'a' increases and decreases respectively. He added that 'if 'a' is negative then the graph will be reflected along the x-axis'. The important fact that he noted was that the difference between the effect of parameter 'a' in the graph of y = 2 cos x and y = 2 tan x. He said that: 'the effect of 'a' in the graph of y = 2 cos x is that 'a' determines the amplitude. While on the graph of y = tan x, 'a' makes the graph to stretch away and towards the *x-axis*'. Skhona believed that the value of 'a' have the same effect in the graph of y = 2 cos x and y = 2 tan x, he said that: 'if *'a'* increases, it stretches the graph of the tangent function along the *x-axis* and if *'a'* decreases it also decreases (meaning that it shortens the graph). If the sign of 'a' changes, that is where the graph reflects along *the x-axis'*.

Figure 5.10: shows a screenshot that was taken from Lindokuhle's interaction with the software, Geogebra.

Now, use the link below to access GeoGebra & Interact with the applet for a few minutes changing value of *'a'*, then answer the questions that follows: https://www.geogebra.org/m/d5wBD7U8



7.1 What is the value of a for this parent **tangent function**? What is the **Amplitude** of each graph?

a = 2, since the range of the tangent function can't be defined, the amplitude can't be defined as well.

7.2 What do the parameters a do to the graph of the function $f(x) = \tan x$ under the transformation $y = a^{*}\tan(bx - c) + d$? Explain.

a will cause all the probable y values of the tangent function to be stretched away from the x-axis.

7.3 Is there any relationship between parameter $'a'$ and the **amplitude**? Explain.

a stretches the values of y of the function away from the x-axis.

Figure 5. 10: Lindokuhle's interaction with the software, Geogebra.

It is noteworthy that Lindokuhle (Figure 5.10) believes that the value of '*a*' does not necessarily determine the amplitude but it has an effect in stretching and shrinking the graph of *y* = tan *x* away and towards the *x*-axis. He added that '*a*' *will have the same effect as for y = tan x and y = cos x as it will stretch both graphs away from the x-axis'*. Nomfundo believed that there is no exact relationship between the parameter 'a' and the amplitude. She stated that the graph of *y* = tan *x* has no minimum or maximum value, it rather approaches negative infinity and positive infinity respectively. Nomfundo felt that the value of '*a*' is just a critical point that can help in getting the correct shape of the graph. She insisted that this value does not necessarily determine the amplitude of the graph. This is important in a sense that the graph of *y = tan x* does not have an amplitude and therefore its highest point cannot be the value of '*a*'. Whereas, in the graph of *y = sin x* and *y = cos x* the value of '*a*' determined the amplitude, it is not true for the graph of *y = tan x*.

Whereas the value of '*a*' has observable effects on the amplitude of a function, it also affects the range of a function. Table 5.18 shows the effect of parameter '*a*' in different *tangent* graphs. Table 5.18, shows the table that was completed by Skhona.

Table 5. 18: Skhona's answers to Question 3.5

	$y = \tan x$	$y = 2\tan x$	$y = -\tan x$	$y = -2\tan x$
Domain	0; 360°	0, 360°	0, 360°	0;360°
Range	-∞; ∞	-∞, ∞	-∞ ,∞	-∞, ∞
Amplitude	undifine	undifini	undifine	undifine
Period	180°	180°	180°	180°

Table 5.18 suggests that the range is from negative infinity up to positive infinity. This implies that that there will be no highest value and no defined minimum value in the graph of $y = a \tan x$. This is contrary to the notion that the value of 'a' determines the amplitude and the range, as in the case of $y = a \sin x$ and $y = a \cos x$. Notably, Skhona did not pay special attention to the domain of the tangent function. In fact he did not consider the exclusions where $x = 90^o$ and $x = 270^o$ in the domain of $[0^o; 360^o]$

5.3 Transformation affected by the parameter '*a*'

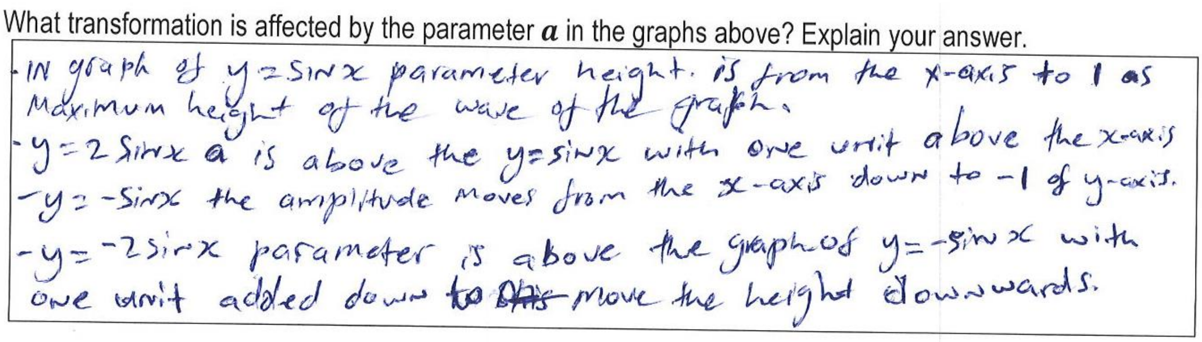
5.3.1 Transformation affected by the parameter '*a*' in the graph of y = *a* sin x

Question 1.6 probed participants to describe the transformation that is affected by changing parameter 'a' in the graphs: $y = sin\, x$; $y = 2\, sin\, x$; $y = -\, sin\, x$ and $y = -2\, sin\, x$. In this question, participants were supposed to give a graphical interpretation in words. By so doing they were demonstrating their understanding of the effect of changing the value of 'a' in the graph of $y = a\, sin\, x$. In conjunction with this, they were expected to describe the transformation that is affected by the parameter 'a' in the graphs. The mathematical transformation that takes place here is called dilation, which is expanding or contracting an object without changing its shape or orientation.

Yandisa stated that ''a' *determines the amplitude, and therefore the graph will stretch vertically by 'a' units. If* 'a' *is negative then the graph is reflected along the x-axis'*. Hence, reflection of a geometric figure creates the mirror image of that figure across the line of reflection. This is contrary to Nonto's description to say that the transformation that will take place is translation, 'Vertical shift or translated about the *x*-axis'. This can be modified to say the graph will be stretched vertically and be reflected along the *x*-axis suppose that '*a*' is negative. What needs to be clarified is that translation happens by moving the graph up or down, which is the effect of '*q*' in $y = asin(x) + q^*$.

Phumlani stated that ''a' *determines the height of the amplitude of the graph'*. What can be said about the description is that it has an element of correct description, while some words like amplitude were used inattentively. Perhaps it would be better to say that, '*the value of 'a'* *determines the amplitude (height of the graph)*. For the transformation affected, Phumlani stated that '*if parameter* '*a*' *is negative the graph will start on the negative side and the amplitude is determined by the parameter* '*a*''. To make this clearer, he meant that the graph of $y = - \sin x$ will start from the point of origin and approach the negative *y*-axis, whereas the graph of $y = \sin x$ starts at the point of origin and approach the positive *y*-axis.

As stated above, what Phumlani described is important in a sense that while he does not use the correct terms that he is expected to use, he however understands that if the function has a negative value of '*a'*, it will start by approaching the negative *y*-axis (for –'*a*') and not the positive (*y*-axis). What needs to be developed from Phumlani's conceptual understanding at this stage is selection and usage of correct words. While verbal description may be challenging in some instances, he seemed to master the use of symbols that are designed to give the same descriptions that he made and other aspects that he neglected in his verbal description of the effect of '*a*'.


What transformation is affected by the parameter a in the graphs above? Explain your answer.

- IN graph of y = Sin x parameter height. is from the x-axis to 1 as Maximum height of the wave of the graph.
- y = 2 Sin x a is above the y = sin x with one unit above the x-axis
- y = - Sin x the amplitude moves from the x-axis down to -1 of y-axis.
- y = -2 sin x parameter is above the graph of y = - sin x with one unit added down ~~to this~~ move the height downwards.

Figure 5. 11: Skhona's responses to Question 1.6

Skhonas's description, shows that he understood that '*a*' determines the height of the graph. This understanding allowed him to conclude that for $y = \sin x$ the amplitude will be one. This then made him to claim that the transformation that took place to produce $y = 2 \sin x$, was that the graph stretched by two units above the *x*-axis.

On the same note, Lindokuhle described the transformation in '*a*' in the sense that multiplying the function by the factor of *'a'* increases the amplitude whereas dividing the function by the factor of '*a*' decreases the amplitude. In the same way he described how the graph will be sketched, noting the change in the amplitude. Figure 5.12 shows Lindokuhle's views about the transformation he observed when manipulating the parameter '*a'* in the sine function.

If the function is multiplied by the parameter a, the amplitude increases and if it is devided by a, then the amplitude decreases. When the ~~graph~~ function is multiplied by the negative parameter then the graph will start from the negative y-axis instead of the positive.

Figure 5. 12: Lindokuhle's responses to Question 1.6.

Nomfundo noted that the value of '*a*' stretches the graph of sine by the factor of '*a*'. This means that if the value of a is equal to two, then the sine graph will be stretched vertically by the factor of two and consequently, the amplitude of that graph will be two. The important fact that Nomfundo mentioned was that dividing the function by '*a*', shrinks the graph vertically by the factor of '*a*'. This concludes that the amplitude will also decrease by the factor of '*a*'. The last point that Nomfundo mentioned was that 'if the parental function $y = sin\ x$ is multiplied by '*a*' negative value of '*a*', then the graph will be flipped oved the *x-axis*. Conversely, if the graph of $y = -\ sin\ x$ is multiplied by the negative value of '*a*', then the graph will be flipped over the *x*-axis again resembling the standard shape of '*a*' sine function. Figure 5.13 is description of the transformation observed when changing the parameter '*a*'.

What transformation is affected by the parameter $\boldsymbol{a}$ in the graphs above? Explain your answer.

The value of a will stretch the graph by a units vertically
⇒ for $a \times \sin x$
The value of a will shrink the graph by a units vertically
⇒ for $a \div \sin x$.
The effect of multiplying the function by (negative) flip the graph over the x-axis.

Figure 5. 13: Nomfundo's responses to Question 1.6

5.3.2 Transformation affected by the parameter '*a*' in the graph of *y* = cos *x*

After careful sketching of the four graphs, Nomfundo observed that the value of '*a*' '*will reflect the graph of y = cos x along the x-axis. The value of* '*a*' *also shrinks and dilates the graph of y = cos x'*. After changing the value of '*a*' in Geogebra, Nomfundo noted that the transformation that took place was that '*the graph of y = cos x moved away from the x-axis by the factor of 4. Since the amplitude was 4, the graph moved away from the x-axis reaching the maximum value of 4 and the minimum value of -4'.* Lindokuhle added that the transformation he observed on the graph of $y = \cos x$ and $y = -\cos x$ was the reflection about the *x*-axis. He said that '*when* '*a*' *is negative the graph will be reflected about the x-axis'.* Skhona used symbols to present his argument. He stated that the transformation that is affected by the change in parameter '*a*' is the reflection about the *x*-axis. This is because when '*a*' is negative, the graph of $y = \cos x$ is reflected about the *x*-axis to produce $y = -\cos x$. Meaning that the coordinates (x, y) changes to $(x, -y)$. He added that as the value of '*a*' increases, the amplitude of the graph also increases.

5.3.3 Transformation affected by the parameter *'a'* in the graph of *y = tan x*

It was noted that most participants agreed that value of '*a*' will stretch the graph of $y = \tan x$. If the value of '*a*' is negative, then the graph will be reflected along the *x*-axis. Lindokuhle gave an example that '*the graph of y = 2 tan x will be the reflection of the graph of y = 2 tan x about the x-axis'.*

Nomfundo (Figure 5.14) believed that the parameter '*a*' has the same function for the tangent, cosine and sine function. Graphically, she presented her solution to say that the difference will be that the tangent function approaches negative and positive *y*-values instead of a distinct value of '*a*'. However, the point that could be read off from the graph is that it is reflected along the *x-axis* when the parameter '*a*' is negative.

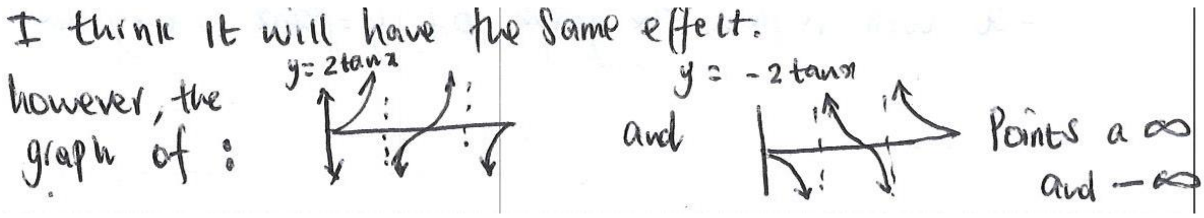


Figure 5. 14: Nomfundo's responses to Question 3.6

5.4 Investigating parameters using GeoGebra

5.4.1 The use of GeoGebra to investigate the effect of '*a*' *in y = a sin x*

Geogebra as a dynamic, interactive software was used for participants to further investigate the effect of different parameters. The focus however was on the parameter '*a*' and parameter '*q*'.

Participants were asked to use the link: http://ggbm.at/fatp4dkx or https://www.geogebra.org/m/RVTxQ6Vm#material/znb4GNk7 to access GeoGebra (Figure 5.15) and interact with the computer software, changing the value of '*a*' and '*q*' to see its effect on the graph. The aim was to allow participants to visualize abstract trigonometric graphs.

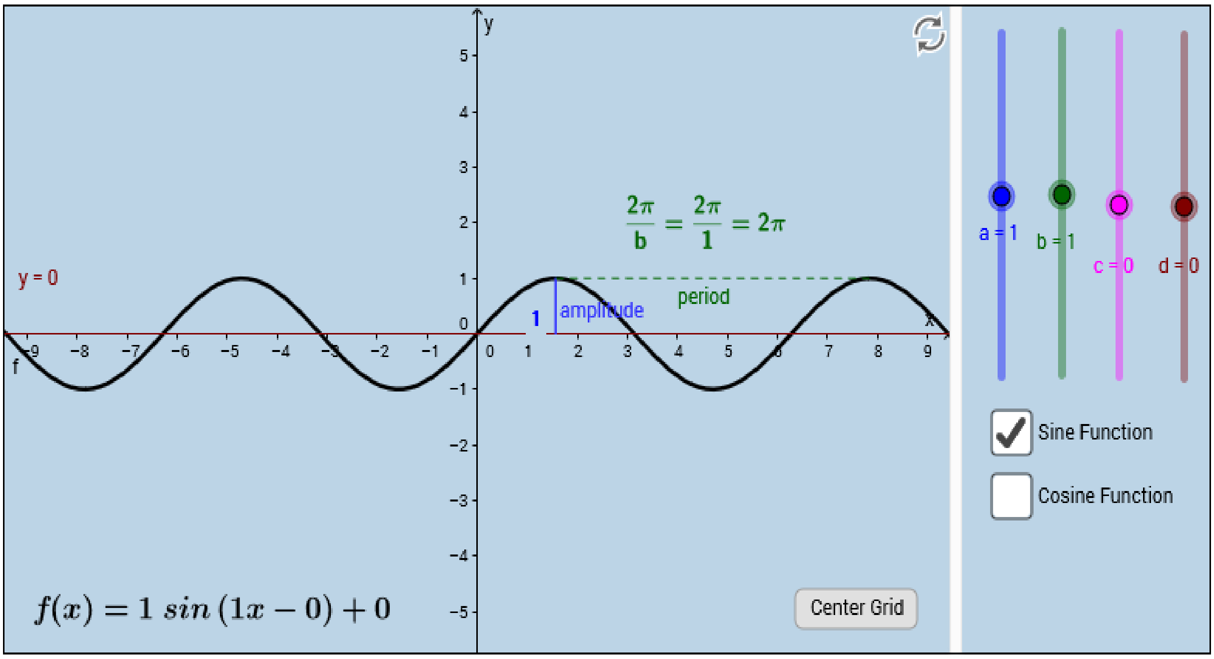


Figure 5. 15: The sine graph used by participants through GeoGebra on the computer.

Whereas, some participants experienced problems in manually sketching the graphs and therefore interpret the graphs they sketched, in this task, participants had the standard parental function of $y = \sin x$ and all they needed was to manipulate the parameter to get a new function immediately drawn for them on the computer. In every class there are participants like Phumlani who was only able to sketch 2 out of 4 graphs, the software was helpful in sketching the graphs. GeoGebra allowed him to look at the graph and change the parameter '*a*' while instantly drawing the new graph with a perimeter of interest. Here the good thing was that the amplitude changed in respect to the parameter '*a*', which allowed him and other participants to make connections.

The first question '*What is the value of 'a' for this parent sine function? What is the Amplitude of the graph?*' seeks to understand if participants were able to look at the function and give their interpretation in words. In the screenshot below I have shown different ways in which participants can find the value of ‘a’ and hence the amplitude of the graph. Figure 5.16 shows different ways in which GeoGebra can be used to understand the effect of ‘a’.

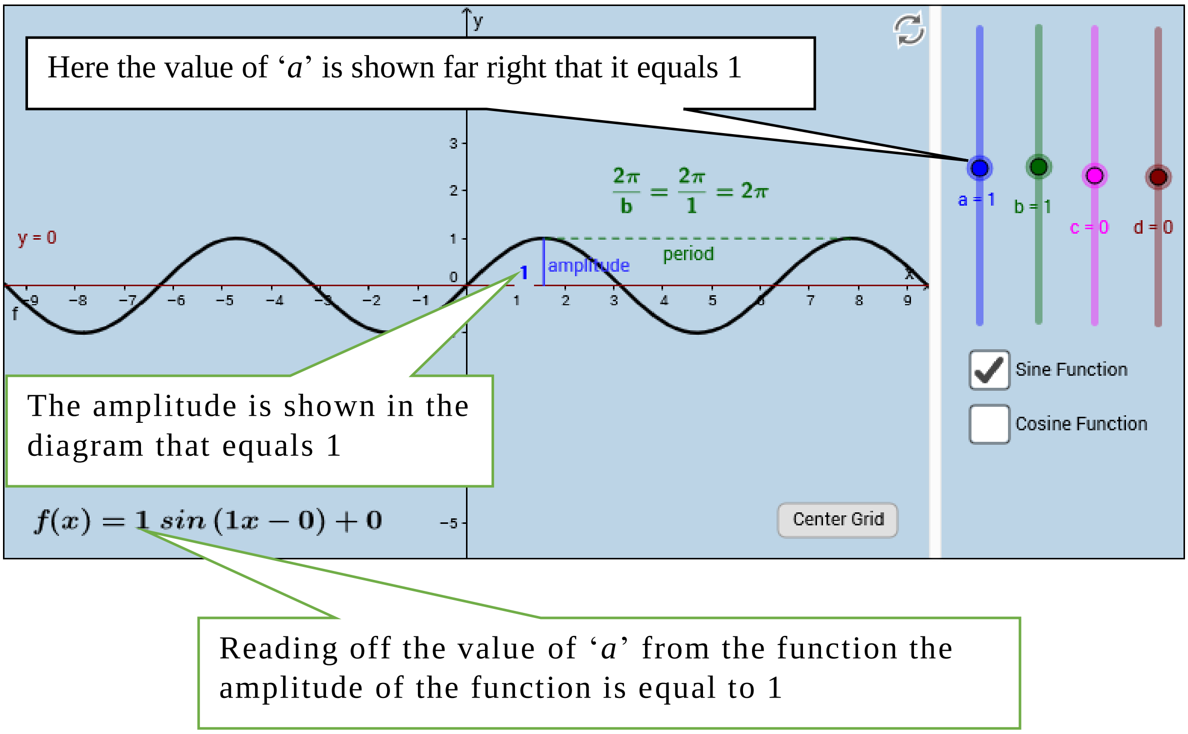
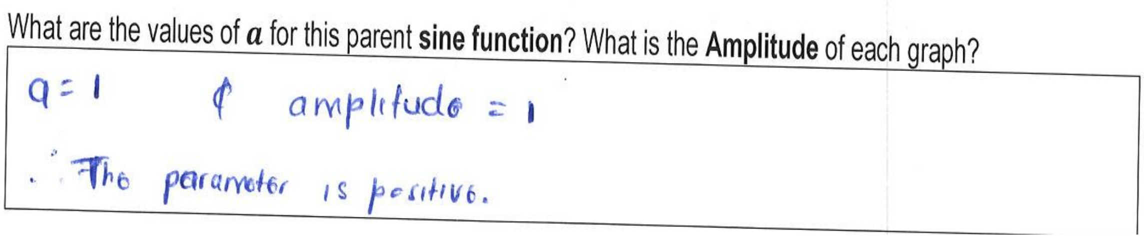


Figure 5. 16: Different ways in which Geogebra can be used to understand the effect of ‘a’

Figure 5.17 was Phumlani’s response which gave satisfaction that he now has an understanding of the parameter ‘a’, its sign (positive or negative) and its relationship with the amplitude.


What are the values of ***a*** for this parent **sine function**? What is the **Amplitude** of each graph?

a = 1 & amplitude = 1

∴ The parameter is positive.

Figure 5. 17: Phumlani’s answers for Question 7.1

After interacting with the applet participants could see that changing the value of ‘a’ affects the amplitude, minimum and maximum turning points. Figure 5.18 shows the change in the value of ‘a’, from $a = 1$, to $a = 4$.

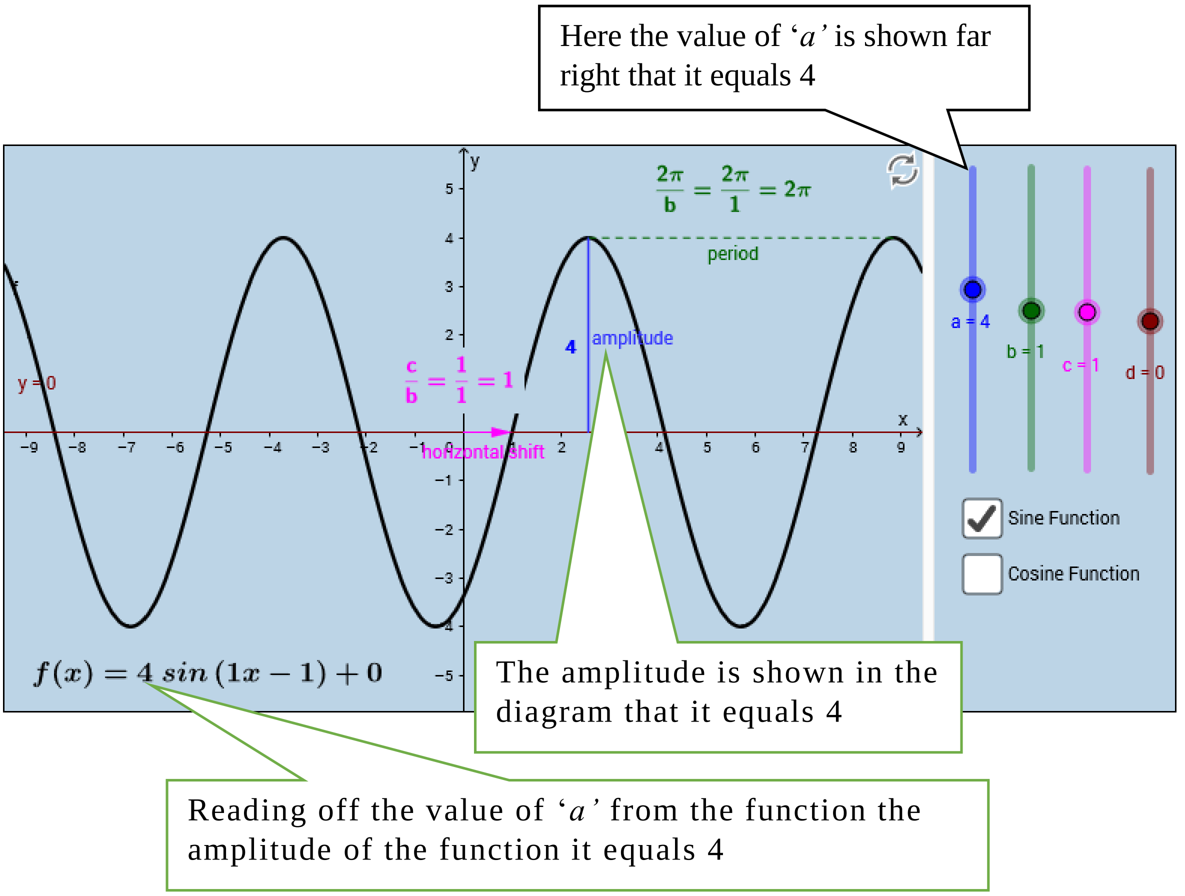


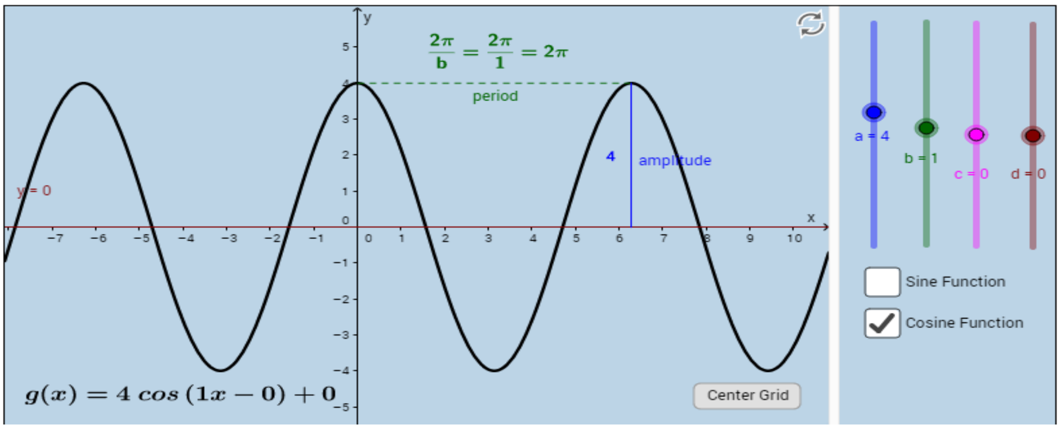
Figure 5. 18: Different ways in which GeoGebra can be used to understand the effect of *'a'*

The important observation that Phumlani noted was that '*a*' determines the amplitude. The same description was shared by Skhona who said that '*the value of* '*a*' *for this parent sine function is positive one, and the amplitude is equal to one...*' Some responses were straight to the point like Nomfundo who said that '*a*' *=1, the amplitude is 1'*, and Lindokuhle who maintained that if *'a' =1, the amplitude is also 1'*. One can see that all these descriptions hold the same view that if the value of '*a*' is 1 the amplitude will also be equal to 1.

5.4.2 The use of GeoGebra to investigate the effect of *'a'* in *y = a* cos *x and y = a sin x.*

Reading off the value of 'a' was approached in different ways, some participants used the graphs that they manually drew, to read off the values of '*a*'. Some participants used the Geogebra software to manipulate the parameter '*a*' and see its effect then identify the parameter '*a*'. Lindokuhle shared that he identified the value of '*a*' through looking at the peak value in the graph which is the maximum, which also determine the amplitude of the graph. Figure 5.19 shows a screenshot that was taken from Phumlani's interaction with the software, GeoGebra.

Now, use the link below to access GeoGebra & Interact with the applet for a few minutes changing the value of '*a*' ,then answer the questions that follow:
https://www.geogebra.org/m/RVTxQ6Vm#material/znb4GNk7 or http://ggbm.at/fatp4dkx



'What is the value of '*a*' for this parent **cosine function**? What is the **Amplitude** of the graph?'

The value of a is 4 and the amplitude of the graph is the distance between x-axis to the turning point of the graph which is 4 units

Figure 5. 19: Phumlani's interaction with Geogebra in understanding the effect of '*a*'

All participants agreed that the value of '*a*' for the graph above is 4. Graphicaly the maximum value is 4 and the minimum value is -4. This concludes that the range of this function is 4. Through the use of Goegebra, it is shown in the graph that the height of the wave or rather the distance from the *x*-axis to the maximum value (turning point) is equal to 4. This also verifies that the amplitude equals 4 indeed. In the diagram above it is also presented in the functional notation that *g(x)= 4 cos(1 x - 0) + 0* and this implies that that the value of '*a*' equals 4, as the standard functional notation is *g(x)= a cos x*. Table 5.19 shows expected answers of the effect of parameter '*a*' in a sine function.

Table 5. 19: The effect of parameter '*a*' in sine function

Function	Value of '*a*'	Effect of parameter '*a*' in a function
$y = sin\,x$	1	For this parent function '*a*' = 1 and the amplitude equals to 1
$y = 2\,sin\,x$	2	The bigger the value of '*a*', the bigger the maximum value will be. The amplitude equals to 2. The graph is stretched away from the *x*-axis (rest position).
$y = -\,sin\,x$	-1	Ideally this will be a reflection of *y = sin x* along the *x*-axis
$y = -2\,sin\,x$	-2	This graph is the reflection of *y = 2 sin x* along the *x*-axis

When participants were asked to explain ‘*what do the parameters* ‘a’ *do to the graph of the function* $f(x) = \sin x$ *under the transformation*: $y = a^{*}sin(bx - c) + d$ ’, they gave meaningful answers. This is because they were unaware that this is similar to the question they have already answered where they were asked to ‘describe *in your own words the effect of* ‘a’ *in the graph of* $y = a \sin x$’. This notion of asking a question in different ways, challenges a learner to think about the problem at a deeper level. Lindokuhle said that ‘*the parameter* ‘a’ *changes the size of the amplitude, if* ‘a’ *gets negative then the graph will be flipped over the y-axis.* It is incorrect to say that the graph will flip over the *y-axis*. What can be said is the graph will be flipped over the *x-axis*. Alternatively, what he needs to consider in his description is that changing the value of ‘a’ to be negative will flip the graph over the *x-axis* or to say that the graph will be reflected along the *x*-axis.

A similar view was shared by Nomfundo who said that *‘the parameter a stretches the graph vertically by a factor of* ‘a’ *for values of* ‘a’ *greater than one. However, the graph will shrink by the factor of* $\frac{1}{a}$, *for values of* ‘a’ *less than one.* Figure 5.20 is the detailed transformation observed by Nomfundo.

* The parameter a Stretch the graph vertically by a factor of a ; a > 1 ; it also shrinks the graph a factor of $\frac{1}{a}$; a < 1
* if a is negative, the the graph flips over the x-axis.

Figure 5. 20: Nomfundo’s answers for Question 7.2

On the same note, Skhona mentioned that the ‘*parameter will determine the amplitude of the graph and moreover it will show the shape of the graph whether it will be negative or positive’*. While the first part about the amplitude is correct, the second point where he articulates that the value of ‘a’ will give us a clue about the shape of the graph is important. Nonto used diagrams to visually express her solution (Figure 5.21). Nonto showed that the value of *‘a’* has an impact in determining the shape of the graph.

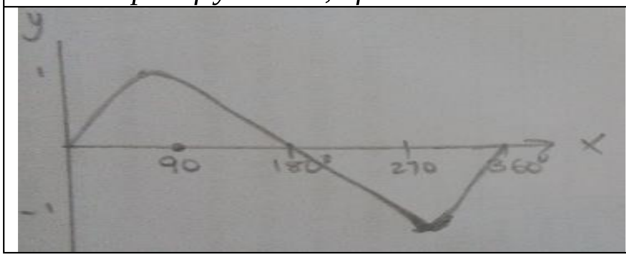
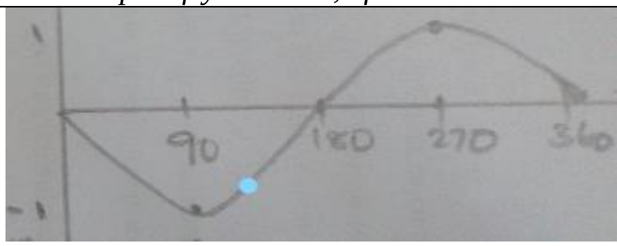
The shape of y=sin x, if a>0	*The shape of y = sin x, if a<0*
	

Figure 5.21: Nonto’s answers for question 7.2

When participants were asked if they have noted '*any relationship between parameter* 'a' *and the amplitude'*, they all confirmed that the parameter 'a' does have an effect on the amplitude. Phumlani shared the same view that the parameter 'a' will ideally determine the amplitude. Whereas, his emphasis is on the amplitude, he made an important observation that the amplitude is always positive. Meaning that the value of 'a' can either be negative or positive, but the '*amplitude will always be positive'*. This emphasizes the point that the amplitude is the height of a wave… and therefore there is no negative height in the real world…while the amplitude can be defined as the maximum displacement of points in a wave.

In essence, the amplitude is a vertical distance from a peak to the equilibrium position. The same was said by Nonto who articulated that '*the change in* 'a' *indicates the change in amplitude, thus when parameter* 'a' *is increased, the maximum height increases'.*

The last question, 'What do you think is the effect of 'a' in the graph of $y = a\ cos\ x$. Use the examples of: $y = 2\ cos\ x$ and $y = -2\ cos\ x$ to explain your conjecture' summed up the understanding of parameter 'a' in a sine function. This also made participants to think about other functions as well. If the effect of a will be the same in the graph of cos and tan. It was interesting to see participants use different methods to give their meaningful answers. Figure 5.33 shows participants' responses, where other participants used GeoGebra to draw the cosine graph and changed the parameter of 'a' to see the effect. Other participants used the function notation to determine the value of 'a' and thus argued that it has the same effect.

In Figure 5.22, participants compared the effect of 'a' in a sine function and cosine function and noted that the parameter 'a' has the same effect in these two functions. The graphical representation that they came up with were sketched using GeoGebra which helped them to compare the effect of 'a' in sine functions. These are the graphs of: $y = 2\ cos\ x$ and $y = -2\ cos\ x$.

5.4.3 Summary of participants comparison of cosine function in respect of parameter ‘a’ in cosine function

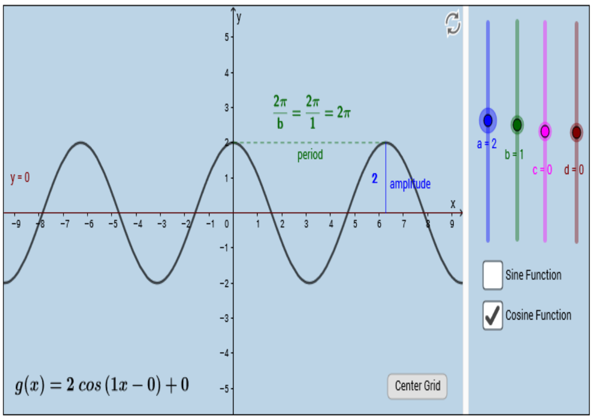
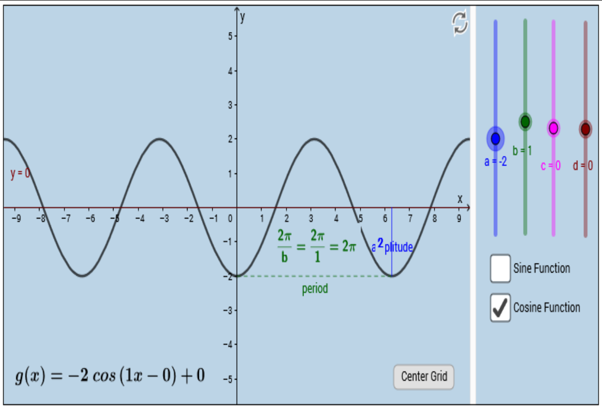
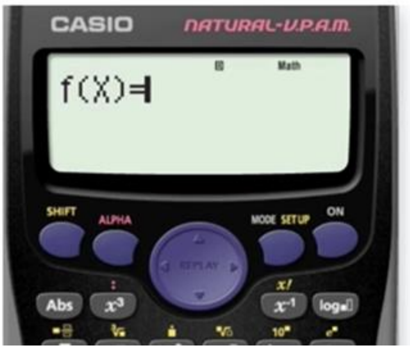
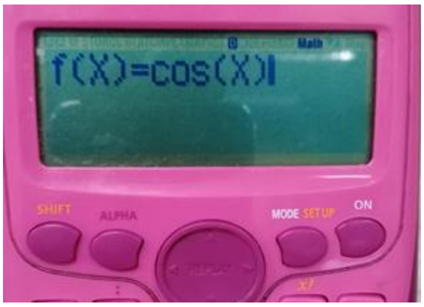
The graph of *y = 2 cos x* Here they noted that parameter a will still represent the amplitude, meaning that if *a* = 2, then the amplitude will be equal to two. This essentially means that the graph will stretch vertically by a factor of 2. Nonto stated that the graph still retains the amplitude of 2. Yandisa concluded that the effect of *a* will still be the same for both sine and cosine functions.	
The graph of $y = -2\cos x$ The same observation was noted that the effect of parameter ‘*a*’ determines the amplitude, however it is always positive. The amplitude is given by the absolute value of a, ($\|a\|$). Furthermore, noting that ‘*a*’ is negative, Lindokuhle added that similar to the graph of sine, the graph of cosine will be reflected about the *x*-axis.	

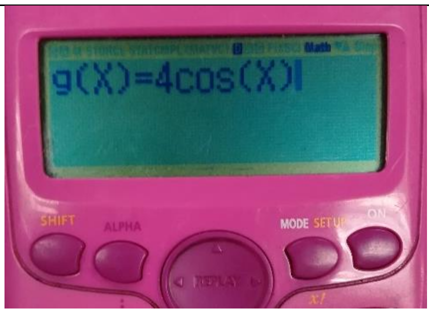
Figure 5. 22: Summary of participants’ comparison of cosine function in respect of parameter ‘a’ in cosine function.

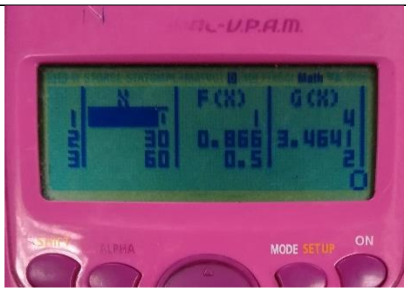
5.5 Investigating the effect of parameter ‘*a*’ in cosine and tangent function

In this investigation all participants were following the steps listed in Figure 5.23. Five of the six participants in this study were able to complete the table of values using their calculators quicker than they were, in the beginning of the study. Here are the steps that Lindokuhle shared to assist other participants in getting the correct table of values (see Table 5.20, Skhona’s table of values (Table 5.20) completed using steps Figure 5.23) using more quick and efficient way. Lindokuhle’s demonstration of using a calculator was chosen because he drew his previous experience in drawing the table of values and then use points to draw linear graphs using a calculator whereas other participants had no prior experience.

5.5.1 Completing a table of values using Table mode in a calculator

Step	Calculator display	Instructions
1		Here participants were instructed to press mode and then choose ‘TABLE’ mode.
2		Participants were instructed to select the Table mode by pressing 3 on their calculators.
3		The next part was about computing the input ‘trigonometric function’ *f(x)* = *cos x*. This was to represent the equation of interest *y* = *cos x*, *y* = *4 cos x*, *y* = - *cos x* and *y* = *-4 cos x*. Then press the equal sign (=) to allow the calculator to move to the next step.
4		The calculator only allowed two functions at a time. For example, the function *f(x)* = *cos x* and *g(x)* = *4 cos x* can be computed at the same time.

		
5		This step required that the learner choose the smallest *x*-value in the function domain. The default setting for tables is: *x* starts at 1, ends at 5, and increases by increments of 1. This value can be changed by pressing the value of interest then the equal sign ‘=’. In this case, participants were given values of *x* from 0^{o} to 360^{o}. The starting point was therefore 0.
6		The ‘End’ was for the highest *x*-value, in the domain of the function. 360^{o} was therefore the endpoint or the last value in this restricted domain.
7		Step was chosen as 30 because the instruction was that ‘the scale on the y - axis should be 4 blocks represent 1 unit and the scale on the x - axis should be two blocks represents 30^{o}’. This implied that since the step is 30, the numbers increased by 30^{o} each time. Participants were then asked to press ‘=’ to generate the table.

<table>
<tr>
<td>8</td>
<td>

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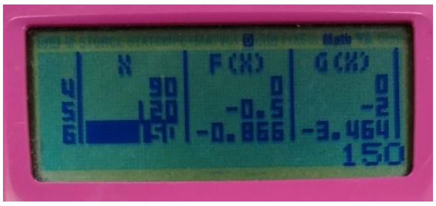

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</td>
<td>It was noted that although the orientation was different from the table… above, but participants were able to get correct y-values [f(x)] for every x-value in order to complete the table.</td>
</tr>
</table>

Figure 5. 23: Completing the table of values using table mode in a calculator

Table 5. 20: Skhona's table of values completed using steps Figure 5.23

1. Complete the following table. Use your calculator to read off the required ratios rounded off to two decimal place.

	0°	30°	60°	90°	120	150	180	210	240	270	300	330°	360°
$y = \cos x$	1	0,87	0,5	0	0,5	-0,87	-1	-0,87	-0,5	0	0,5	0,87	1
$y = 4\cos x$	4	3,46	2	0	-2	-3,46	-4	-3,46	-2	0	2	3,46	4
$y = -\cos x$	-1	-0,87	-0,5	0	0,5	0,87	1	0,87	0,5	0	-0,5	-0,87	-1
$y = -4\cos x$	-4	-3,46	-2	0	2	3,46	4	3,46	2	0	-2	-3,46	-4

5.6 Investigating the effect of parameter *'q'* in *y = sin x + q, y = cos x + q and y = tan x + q*

In the second phase all six participants were expected to investigate the effect of 'q' in *y = sin x, y = cos x and y = tan x respectively.* They had to explore the effect of *'q'* in these three parental functions. In this investigation, participants were to note that adjusting the parameter 'q' shifts the parental function vertically. This vertical translation affects the whole parental graph, when 'q' is less than zero, the graph is shifted vertically downwards by 'q' units. When 'q' is more than zero, the graph is shifted vertically upwards by 'q' units. Similarly to the first investigation, this investigation reinforced the idea evaluating trigonometric functions, whereby participants were using calculators to find the value of sin 30^{o}, sin 60^{o}, and so on. In this regard, participants were instructed to complete the table by using their calculator to read off the required ratios and round off their answers to one decimal place.

5.6.1 Tabular representation of *y = a sin x + q, y = a cos x + q* and *y = tan x + q*

Similar to the investigation of the effect of parameter a, the aim of the second investigation was to explore the effect of parameter 'q' in $y = a\sin x + \mathrm{q}$, $y = a\,cosx + q$ and $y = a\,tanx + q$. Here, participants were asked to complete the table below by using a calculator to read off the required ratios and also to round off the ratios to one decimal place. Table 5.21 shows the expected answers for all corresponding *y*-values.

Table 5. 21: Nomfundos' answers to Question 2A

Investigation 2A: The effect of parameter q in $f(x) = sin\,x + q$ where q is any real number

1. Complete the following table. Use your calculator to read off the required ratios rounded off to one decimal place.

	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$y = sin\,x$	0	0,5	0,9	1	0,9	0,5	0	-0,5	-0,9	-1	-0,9	-0,5	0
$y = sin\,x + 1$	1	1,5	1,9	2	1,9	1,5	1	0,5	0,1	0	0,1	0,5	1
$y = sin\,x - 2$	-2	-1,5	-1,1	-1	-1,1	-1,5	-2	-2,5	-2,9	-3	-2,9	-2,5	-2
$y = sin\,x - \frac{1}{2}$	-0,5	0	0,4	0,5	0,4	0	-0,5	-1	-1,4	-1,5	-1,4	-1	-0,5

In this task, five participants, except Yandisa who did not complete the investigation, were able to get correct corresponding *y*-values. The challenge was on determining the y-values in the graph of *y = - cos x + 2.* Table 5.22 shows a mistake that Nonto and other participants made when determining the *y*-values. It was good to note that Nonto managed to rectify her mistake when she saw that there was no pattern in the *y*- values. This essentially mean that the graph will give us the correct shape if there is a pattern in the table of values.

Table 5. 22: Nonto's answers to Question 2B

. Complete the following table. Use your calculator to read off the required ratios rounded off to one decimal place.

	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300	330°	360°
$y = \cos x$	1	0,9	0,5	0	-0,5	-0,9	-1	-0,9	-0,5	0	0,5	0,9	1
$y = \cos x + 2$	3	2,9	2,5	2	1,5	1,1	1	1,1	1,5	2	2,5	2,9	3
$y = \cos x - 2$	-1	-1,1	-1,5	-2	-2,5	-2,1	-3	-2,9	-2,5	-2	-1,5	-1,1	-1
$y = -\cos x + 2$	~~-3~~	~~-2,9~~	~~-2,5~~	~~+2~~	~~-1,5~~	~~-1,1~~	~~3~~	~~-1,1~~	~~-1,5~~	~~-2~~	~~-2,5~~	~~-2,9~~	~~-3~~
	1	1,1	1,5	2	2,5	2,9	3	2,9	2,5	2	1,5	1,1	1

Skhona's tabular representation (Table 5.23) of different tangent functions i.e. $y = tan\, x$; $y = tan\, x + 2$; $y = -tan\, x$; $y = -tan\, x - 1$, was completed with no error, hence giving us the idea of the expected shape that the graphs of $y = atan\, x + q$.

Table 5. 23: Skhona's answers to Question 2C

Complete the following table. Use your calculator to read off the required ratios rounded off to one decimal place.

	0°	45°	90°	135°	180°	225°	270°	315°	360°
$y = tanx$	0	1	und	-1	0	1	und	-1	0
$y = tanx + 2$	2	3	und	1	2	3	und	1	2
$y = -\, tan\, x$	0	-1	und	1	0	-1	und	1	0
$y = -tan\, x - 2$	-2	-3	und	-1	-2	-3	und	-1	-2

5.6.2 Graphical representation of *y = a sin x + q*

After participants have completed filling the table, they were to draw the graphs of the functions $y = sin\, x$; $y = sin\, x + 1$; $y = sin\, x - 2$; $y = sin\, x - \frac{1}{2}$ on the same system of axis. Similar to part 1 of this investigation, the aim was to get an insight of the participants' ability to draw trigonometric graphs. Participants used point-by-point to plot points in a Cartesian plane to get the correct shape of the graph. Figure 5.24 – Figure 5.28 shows graphical representation of a sine function with changes in parameter 'q'.

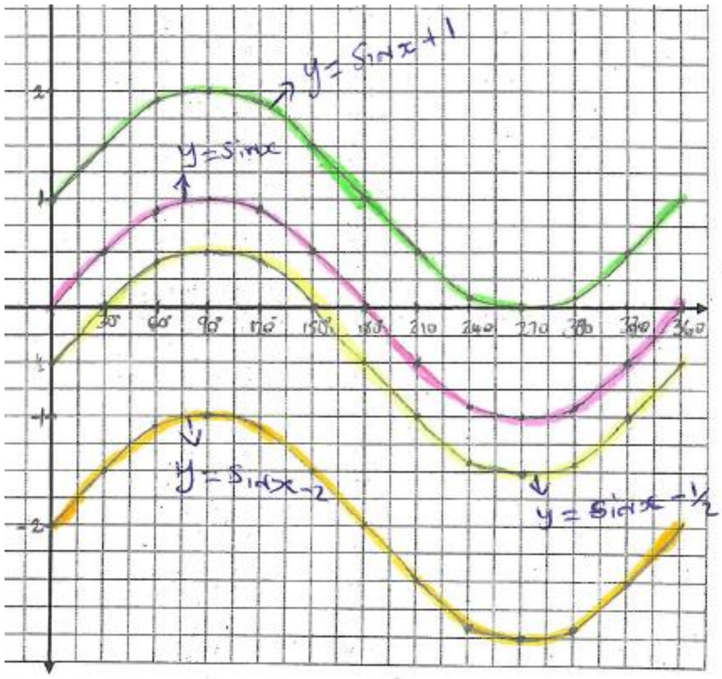


Figure 5. 24: Graphs plotted by Skhona, showing the effect of parameter 'q' in y = *a sin x* + *q*

Whereas Phumlani's graphical representation (Figure 5.25) of the sine function ($y = sin\,x$; $y = sin\,x + 1$; $y = sin\,x -$; $y = sin\,x - \frac{1}{2}$), shows some degree of accuracy, he forgot to label the graphs he drew. Perhaps, this is one are that need to be emphasized to participants that if they have to label graphs, especially when they have to draw more than one graph in one Cartesian plane.

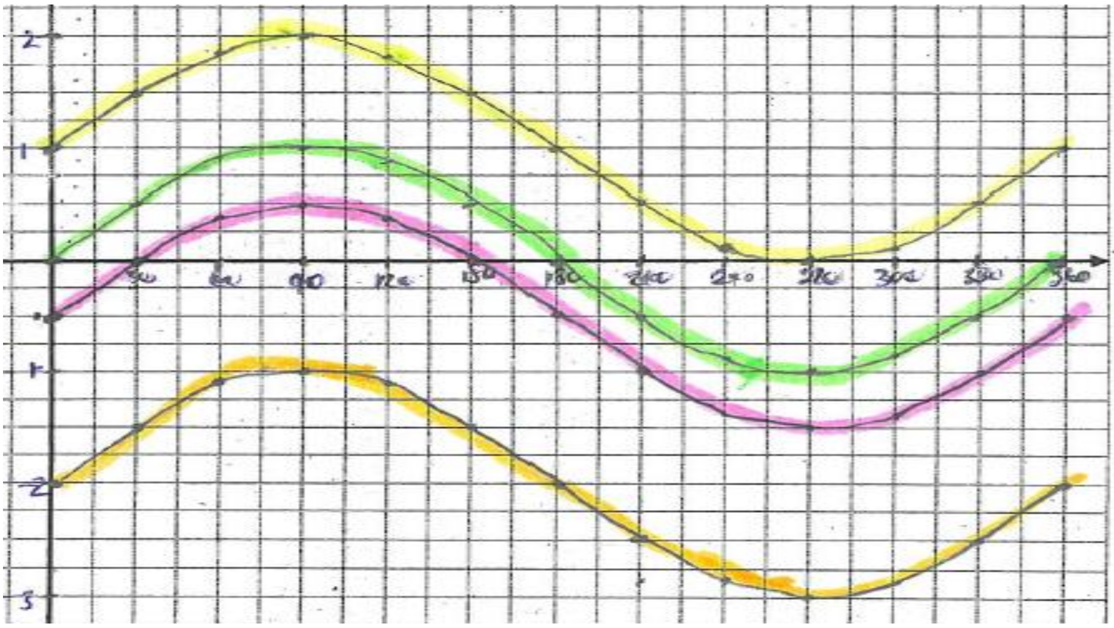


Figure 5.25: Graphs plotted by Phumlani, showing the effect of parameter 'q' in *y*= *a sin x* + *q*.

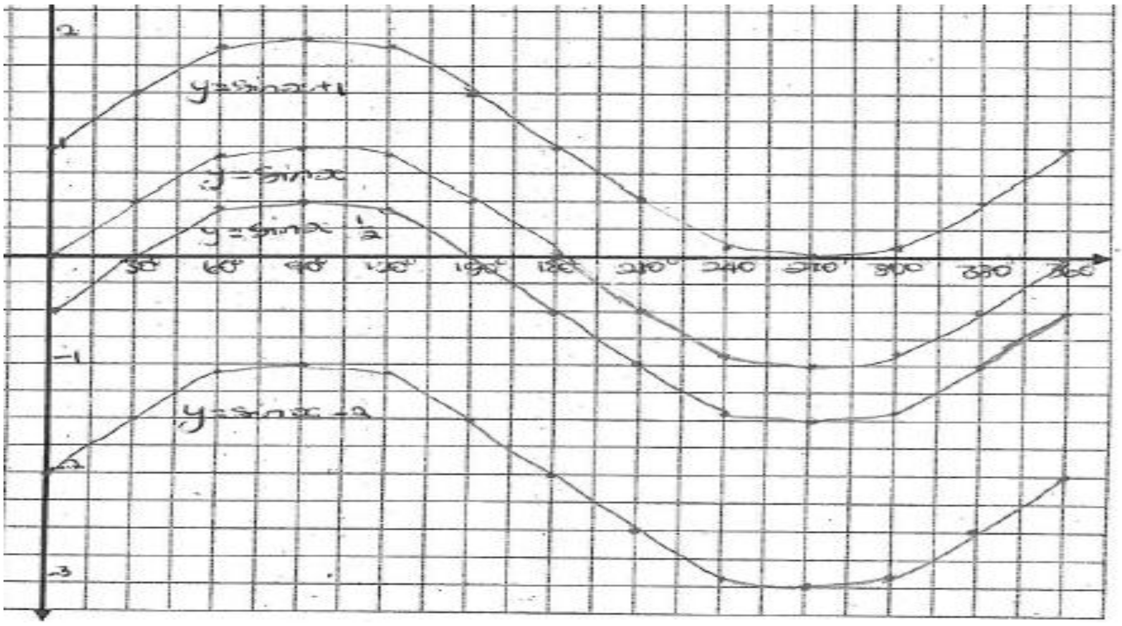


Figure 5.26: Graphs plotted by Lindokuhle, showing the effect of parameter '*q*' in *y = a sin x + q*

What seems to be unique about the graphical representation of the sine function that was drew by Nonto is the (0,5 for two blocks) scale she decided to use for the y-axis. She shows understanding of point by point plotting and increased the degree of accuracy when sketching these graphs.

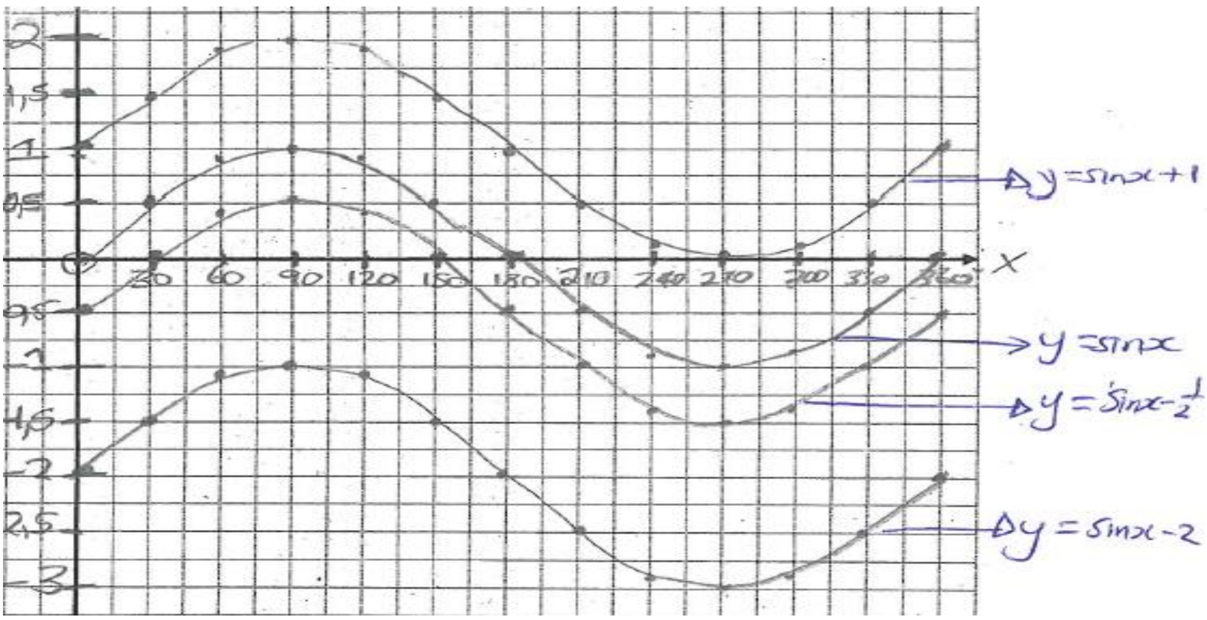


Figure 5. 27: Graphs plotted by Nonto, showing the effect of parameter '*q*' in *y = a sin x + q*

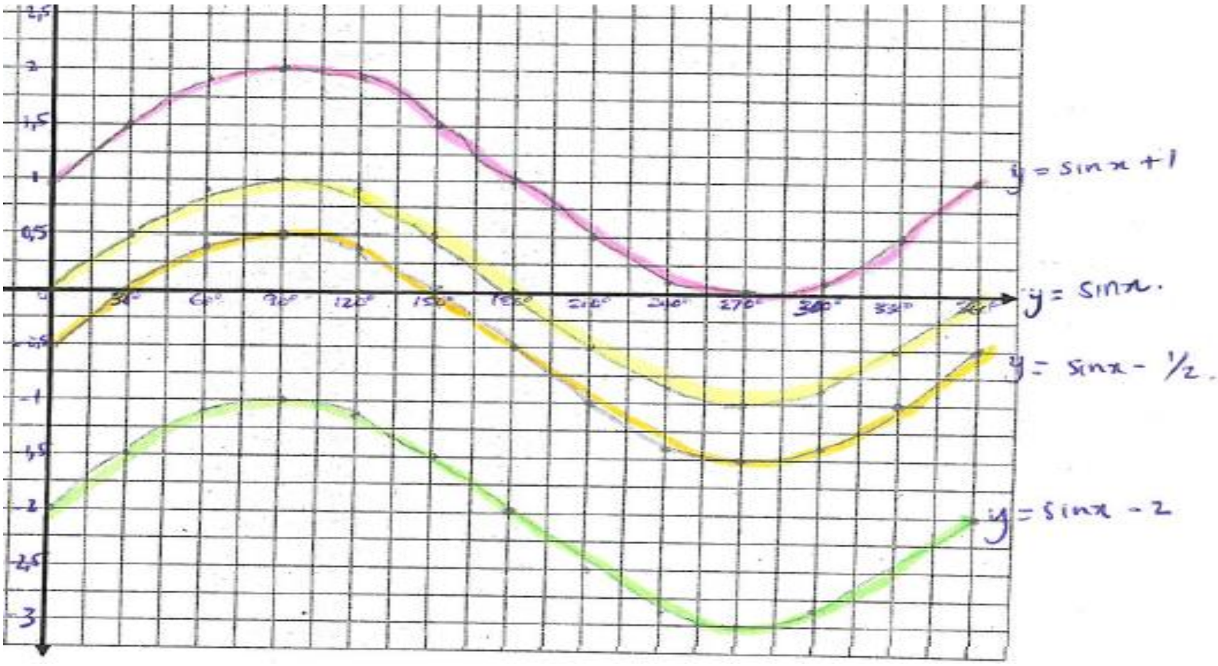


Figure 5. 28: Graphs plotted by Nonto, showing the effect of parameter *'q'* in *y = a sin x + q*

5.6.3 Graphical representation of *y = a cos x + q*

In the second part of the investigations (investigating the effect of 'q') I noted that participants were more confident in computing *y*- values using their calculators. This made them to present accurate graphs as shown in Figure 5.29, where Nomfundo drew the graphs of cosine with different values of parameter 'q'. The graphs in question were; $y = cos\,x$, $y = cos\,x + 2$, $y = cos\,x - 2$ and $y = -\,cos\,x + 2$. From this investigation, participants were able to see that q slides the graph vertically.

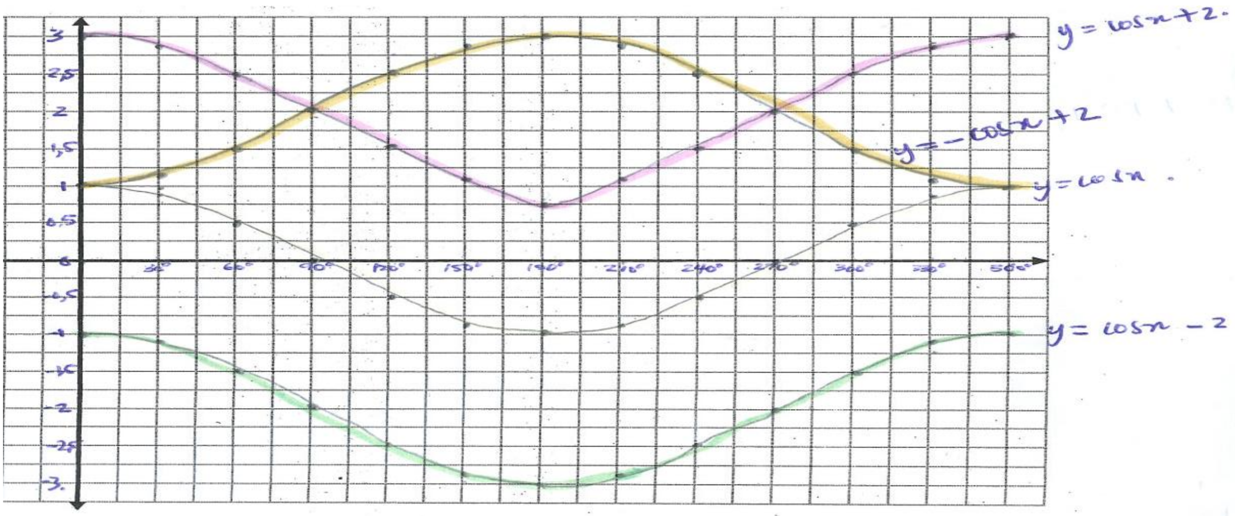
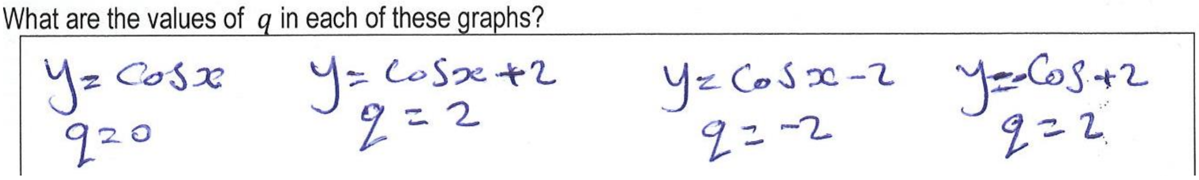


Figure 5. 29: Graphs plotted by Nomfundo, showing the effect of 'q' in $y = a$ cos $x + q$

One of the important observations was that all 6 participants managed to draw the graphs of $y = cos\, x + 2$ and $y = -cos\, x + 2$. These two graphs are reflections of each other. In essence, the understanding of the effect of parameter 'a' and 'q' was asked in this question. It was interesting to note that participants could integrate the knowledge about the effect of 'a' and the effect of 'q' at the same time. This is an important skill that the participants have acquired.

5.6.4 Identifying the value of '*q*' in the equation

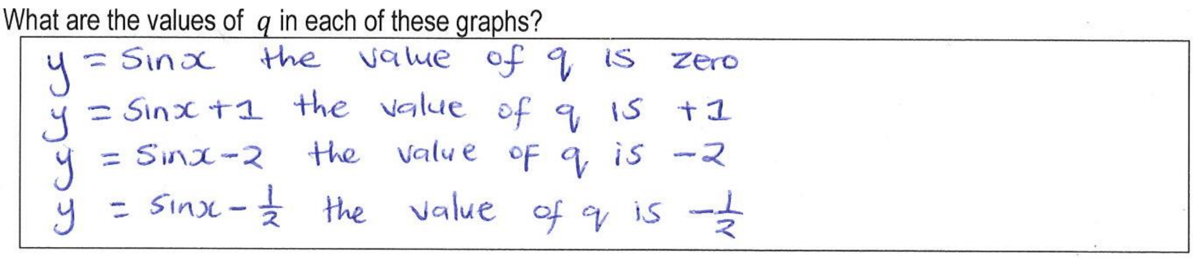
In trying to understand if participants are able to identify the value of 'q' in a given function, participants were given four equations where they were supposed to identify the parameter 'q'. One of the functions in question were, $y = cos\, x$; $y = cos\, x + 2$; $y = cos\, x - 2$ *and* $y = -cos\, x + 2$. Figure 5.30 presents Skhona's answers to the question '*what are the values of 'q' in each of these graphs*?


What are the values of q in each of these graphs?

y = Cosx q = 0	y = Cosx + 2 q = 2	y = Cos x − 2 q = −2	y = −Cos + 2 q = 2

Figure 5. 30: Skhona's answers to Question 2.3 C

Similar understanding is shared by Nonto in Figure 5.31 who expressed her solution in words. The question that she answered was based on identifying the value of 'q' in $y = sin\, x$, $y = sin\, x + 1$; $y = sin\, x - 2$ and $y = sin\, x - \frac{1}{2}$.


What are the values of q in each of these graphs?

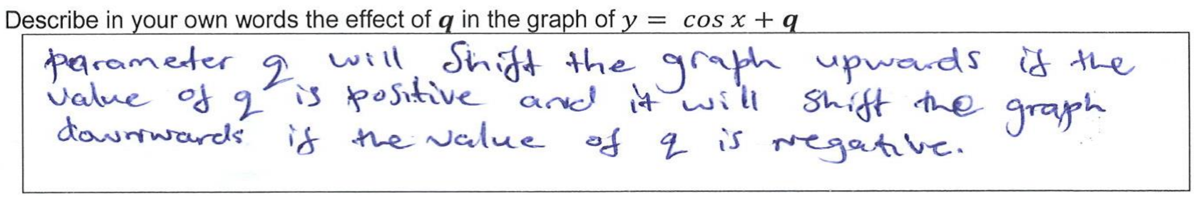
y = Sinx the value of q is zero
y = Sinx + 1 the value of q is +1
y = Sinx − 2 the value of q is −2
y = Sinx − ½ the value of q is −½

Figure 5. 31: Nonto's answers to Question 2.3 B

Importantly, participants were able to compare the standard function $y = a \sin x + q$, with $y = tan\, x$; $y = tan\, x + 2$; $y = -\, tan\, x$ *and* $y = -tan\, x - 2$. This is evident because when there was no value assigned to ‘q’ in a given function, they were able to say that q = 0, whereas in the first investigation when there was no coefficient (value of ‘a’) of a trigonometric function, participants were able to state that it equals 1.

5.6.5 The effect of parameter ‘*q*’

In this phase of the investigation, participants were expected to give a written description of their observations of the effect of ‘q’ in the graphs they drew. The question that was posed to participants was ‘Describe in your own words the effect of ‘q’ in the graph of $y = \ sin\, x + q$. Phumlani said that ‘q’ shifts the graph of y = sin x along the y-axis. On the same note, Lindokuhle added that ‘the parameter ‘q’ shifts the graph vertically. So the graph will shift ‘q’ units up’. Skhona in Figure 5.32 argued that this transformation does not only shifts the graph up but also shift the graph ‘q’ units down.


Describe in your own words the effect of $\boldsymbol{q}$ in the graph of $y = \ cos\, x + \boldsymbol{q}$

parameter q will shift the graph upwards if the value of q is positive and it will shift the graph downwards if the value of q is negative.

Figure 5.32: Skhona’s answers to Question 3B

It was interesting to note that participants used their graphs which they had drawn accurately to answer questions pertaining the domain, range, amplitude and period. Lindokuhle used his graph (Figure 5.33) which he had drawn accurately to complete Table 5.24. In this question they were to show understanding of the domain, range, amplitude and period.

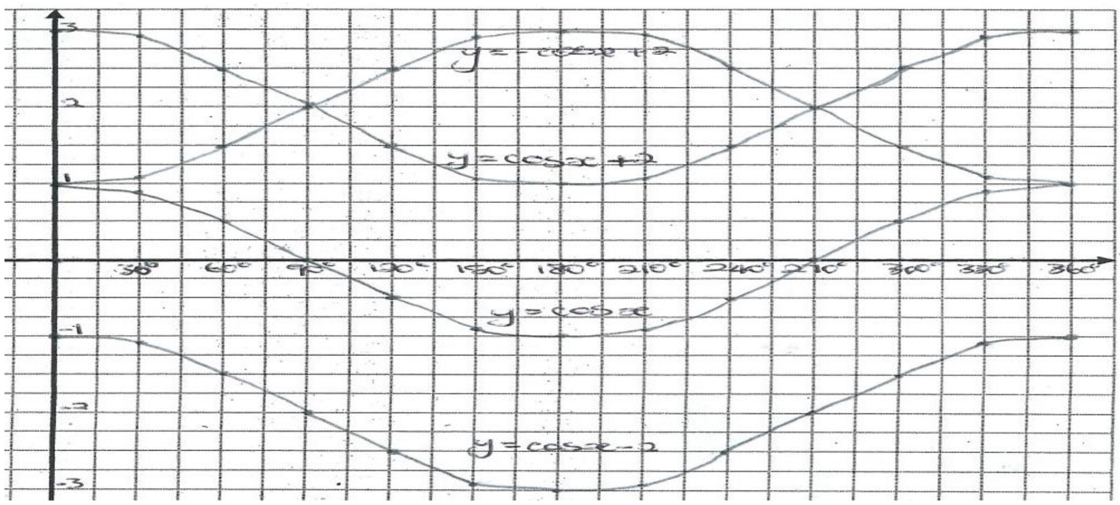


Figure 5.33Lindokuhle’s Graphs, showing the effect of parameter ‘q’ in $y = a$ cos $x + q$

Table 5.24: Lindokuhle's answers to Question 5B.

Complete the table below:

	$y = sin\,x$	$y = sin\,x + 1$	$y = sin\,x - 2$	$y = sin\,x - \frac{1}{2}$
Domain	[0°; 360°]	[0°; 360°]	[0°; 360°]	[0°; 360°]
Range	[-1, 1]	[0, 2]	[-3, -1]	[-1,5; 0,5]
Amplitude	1	1	1	1
Period	360°	360°	360°	360°

It was good to note that most participants were able to move from one functional representation to the next. In this instance, they were able to freely translate between different representations, from written descriptions of functions to graphs (graphical representations) and to tables.

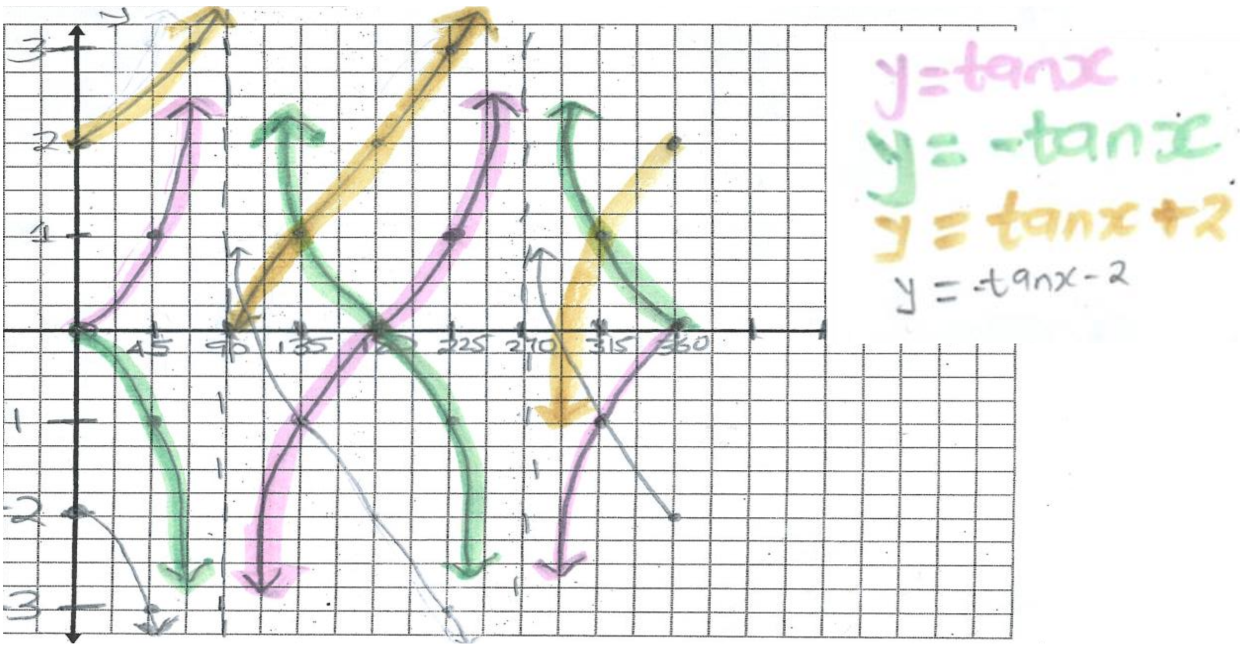


Figure 5.34: Graphs plotted by Nonto, showing the effect of parameter '*q*' in y = *a* tan *x* + *q*

To some participants, it was challenging to interpret the graph of *y* = *tan x* + *q*. Firstly, because in the domain of [0º ; 360º], all real numbers between 0º and 360º are included, but 90º and 270º are excluded. The second part that was challenging was that, there is no amplitude, mainly because the tangent function is not a wave like graph (like sine and cosine function). Figure 5.34 shows Nonto's graphical representation of $y = tan\,x$, $y = -\,tan\,x$, $y = tan\,x + 2$ and $y = tan\,x - 2$

Participants were excited to discover that although the graph of tan was challenging to draw, they interpreted, that *'q'* has the same effect on it. These participants noted that the parameter *‘q’* shifts the whole graph vertically up when it is positive and down when it is negative. Moreover, the *‘q’* value changes the position of the rest position and will change the value of the intercepts but not the asymptotes. Figure 35 is Lindokuhle’s graphical representation of the tangent function under the transformation of *‘q’*.

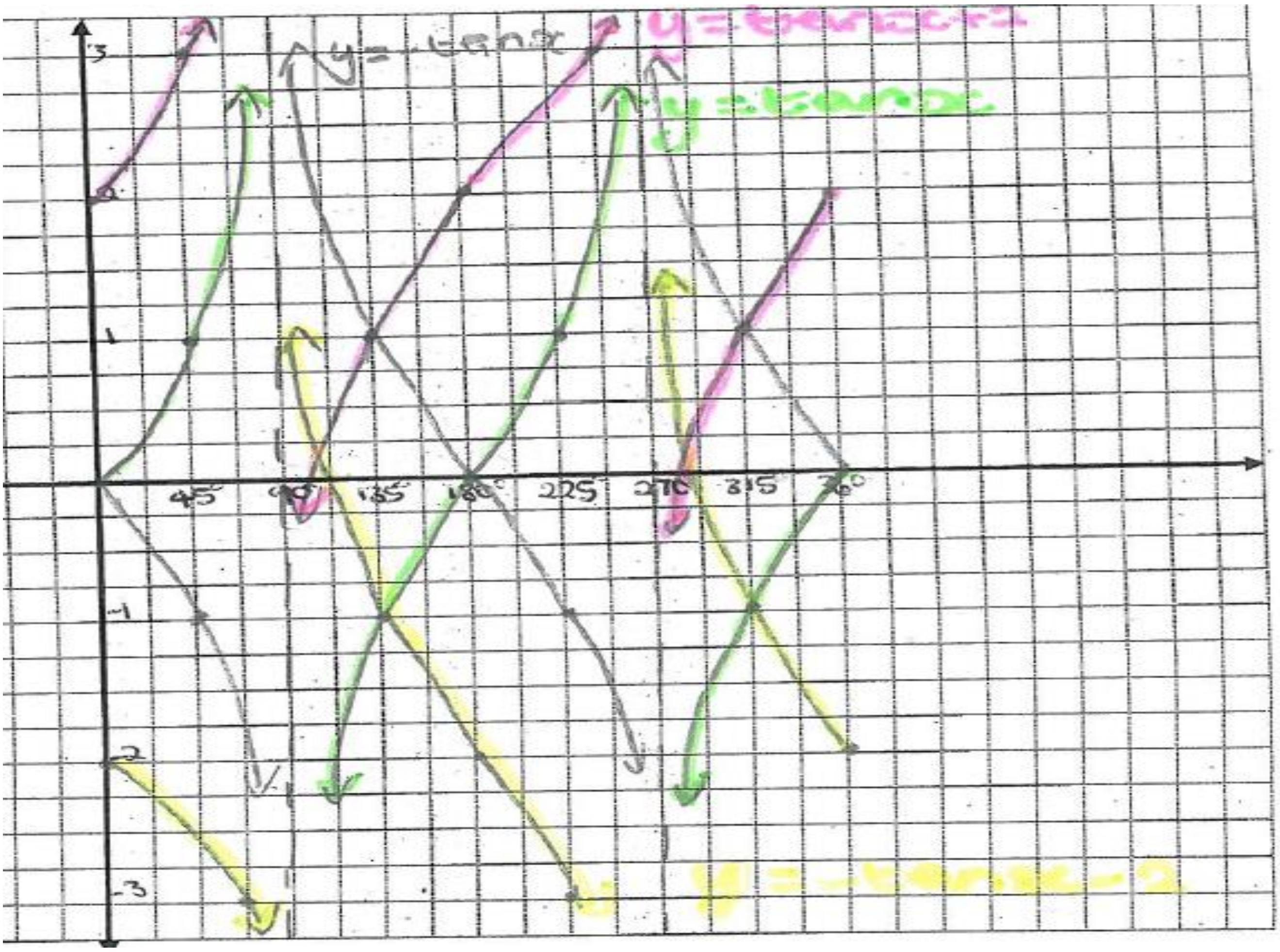


Figure 5. 35: Lindokuhle’s graphical representation of the tangent function under the transformation of ‘*q*’.

5.7 Transformation affected by the parameter *‘q’*

Whereas the first part of this investigation probed participants to ‘describe the transformation that is affected by changing parameter ‘*a*’ in the graphs in 3 trigonometric functions’, the second part probe participants to ‘describe the transformation that is affected by changing parameter ‘*q*’ in the graphs in 3 trigonometric functions’. Phumlani said that ‘since parameter ‘*q*’ is about translation, therefore parameter ‘*q*’ shift the graph vertical upwards/ downwards’. A similar view was shared by Lindokuhle, asserting that ‘the transformation affected by ‘*q*’ is translation, ‘*q*’ translates the graph up and down (in the y-axis)’. Figure 5.36 gives Nonto’s insight on the transformation affected by parameter '*q*'. What lacks in her description is that '*q*' translates the graph up and down. Perhaps in addition to Nontos description, we can use Skhona's words who said that the transformation affected is 'translation because parameter '*q*' has an effect on moving the graph upwards or downwards'.

What transformation is affected by the parameter '*q*' in the graphs above? Explain your answer.

→ Translation : A vertical shift of the graph.
q translates the graph q units up
and also translate

Figure 5. 36: Nonto's answers to Question 6A.

Unlike Nonto who started with the type of transformation (translation) and then described the effect of the transformation, Nomfundo started by describing the effect of the parameter '*q*' she had observed (parameter '*q*' moves the graph '*q*' units vertically) and concluded that the type of transformation is translation. While both answers are correct it is important to acknowledge the inductive and deductive reasoning that these participants used to answer the same question.

q Move the graph by q units Vertically.
there fore the graph of tan x, will Shift q units up or down
⇒ Translation.

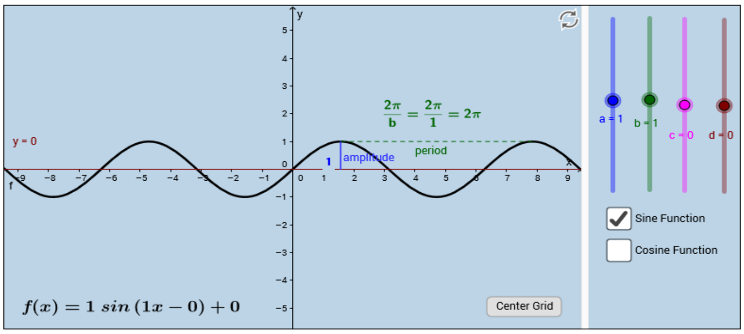
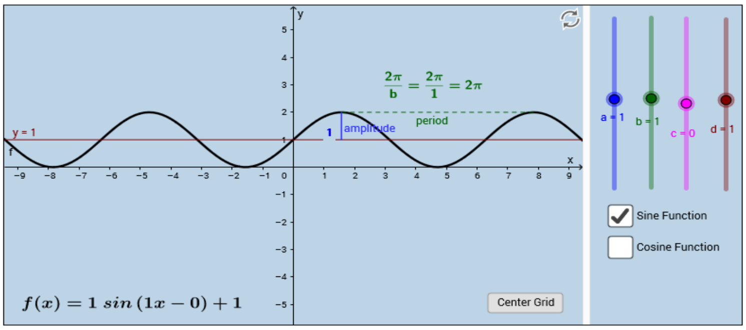
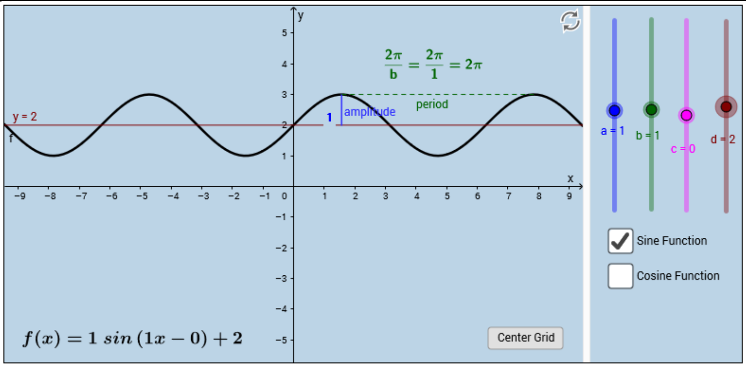
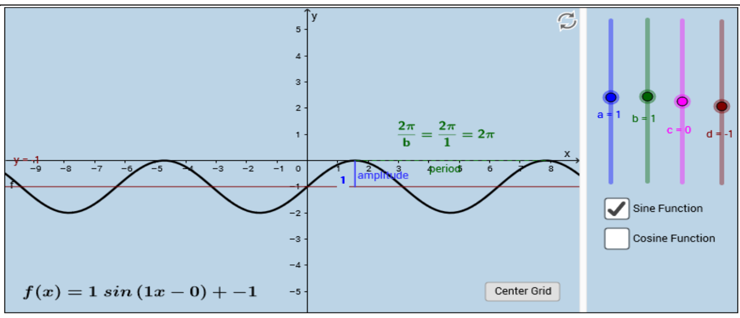
Figure 5. 37:Nomfundo's answers to Question 6A.

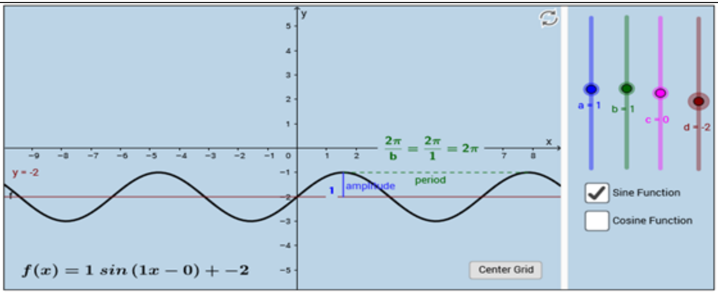
5.8 The use of GeoGebra to investigate the effect of parameter '*q*' in $y = a \sin x + q$ and $y = a \cos x + q$

The focus of this part of the investigation was to allow participants to physically manipulate different parameters in the applet and notice the changes in the graph without having to draw the graph manually. Participants were asked to use the link: http://ggbm.at/fatp4dkx or https://www.geogebra.org/m/RVTxQ6Vm#material/znb4GNk7 to access GeoGebra so they could interact with the software, changing the value of *d* (note that parameter '*d*' is used in the software and not in the equations presented in this theses) and answer questions based on their observations. All participants were able to use a computer, those who were still struggling were far better than how they were in the first investigation (parameter '*a*'). Figure 5.38 shows how

Skhona used the applet to understand the effect of parameter *‘q’*. He kept on adding 1 and subtract 1 on parameter *‘q’*.

5.8.1 Investigating the effect of parameter *‘q’* using GeoGebra

	This is the parent function: *f(x)* = sin *x* + 0
	In the function *f(x)* = sin *x* +1 Here the value of 1 was added to the parameter *‘d’* or *‘q’* in the parental function and the graph shifted 1 unit up.
	In the function *f(x)* = sin *x* +2 Here the value of 2 was added to the parameter *‘d’* or *‘q’* in the parental function and the graph shifted 2 units up.
	In the function *f(x)* = sin *x* -1 The value of 1 was subtracted from the parameter *‘d’* or *‘q’* in the parental function and the graph shifted 1 unit down.



In the function *f(x)* = sin *x* -2

The value of 2 was subtracted from the parameter *'d'* or *'q'* in the parental function and the graph shifted 2 units down.

Figure 5. 38: Skhona's screenshots from his interaction with GeoGebra

After interacting with GeoGebra, participants could see that changing the value of *'q' or 'd'* affects the range of a function. This means that the minimum turning point and the maximum turning point will change with respect to the change in the value of *'d'*. For participants to understand fully the effect of *'d'*, they were given a graph which they had to manipulate and answer questions based on their observations. Figure 5.39 shows different ways in which GeoGebra can be used to understand the effect of *'q'*.

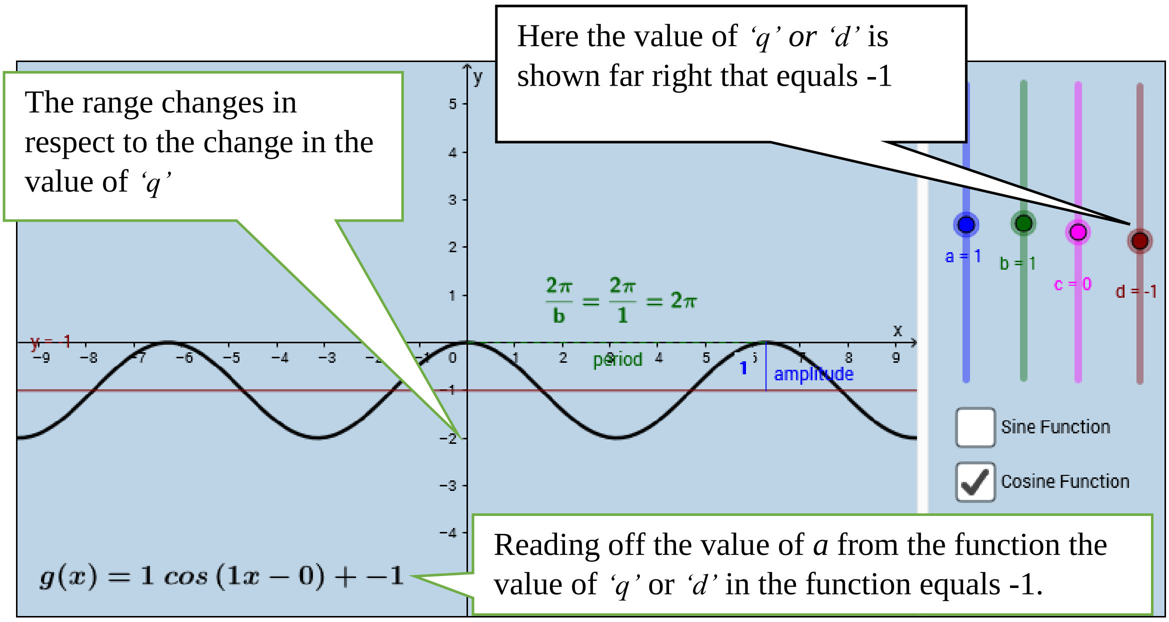


Figure 5. 39: Different ways in which GeoGebra can be used to understand the effect of '*q*'

5.8.2 Interpreting graphs using Geogebra

After participants have identified the values of *'d'* or *'q'* in each trigonometric graph, they were asked to further observe the changes in the range, domain and the transformation that is affected by the effect of *'d'*.

In the first question, 'What is the values of '*d*' for this parent sine function? What is the Range of the graph?' all participants managed to identify the value of *'d'*, in all three trigonometric functions. For this question, participants were looking at the minimum and maximum *y*-value to determine the range of a function. Participants discovered that there is a relationship between the range and the parameter *'d'*. On the parental function the range is [-1 ; 1], and in the function on figure 5.40, the range is '[0, 2]'. This is because, the range is determined by [-1 + *'d'* = min *y*-value ; 1 + *'d'* = max *y*-value]. This implies that the value of *'d'* determines the range.

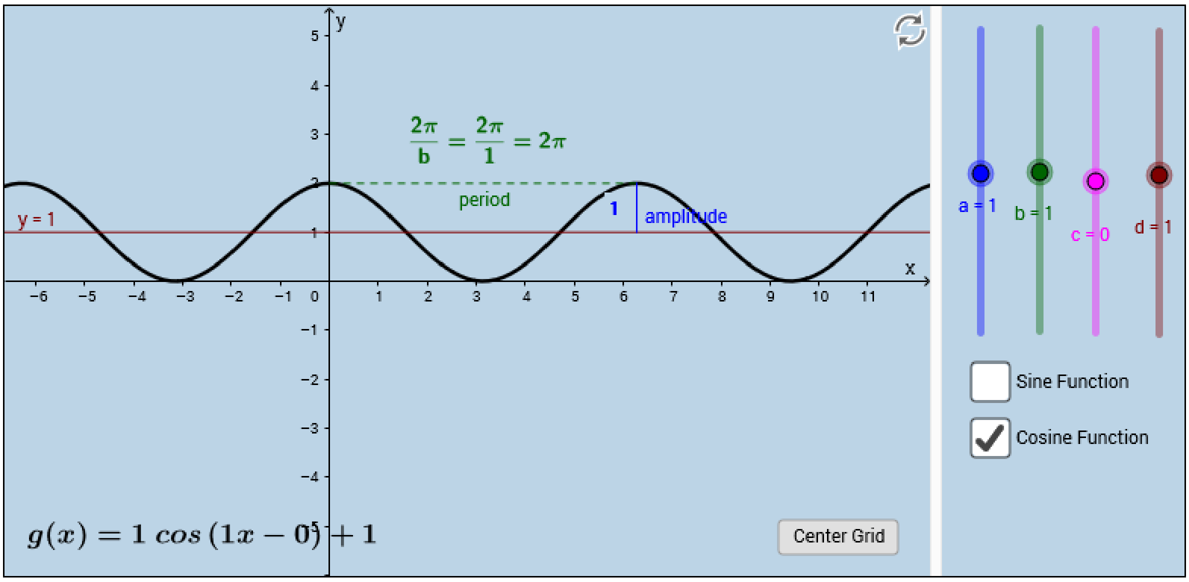


Figure 5. 40:The graph of f(x) = 2 cos (x +0) +1.

Since parameter *'d'* determines the range of a function, there is no relationship between parameter *'d'* and the domain of the function. This is mainly because the domain is concerned with the horizontal shift, while parameter *'d'* is concerned with the vertical shift responding to the question 'Is there any relationship between parameter d and the Domain ? Explain'. Nonto said that '*no, any change in 'd' does not change the domain'*. Skhona said that '*No, parameter 'd' has no effect on the domain because it only moves the graph vertically*'. Similarly, Nomfundo said that '*No, it only affects the vertical shift, the* range'. Phumlani said that '*No, because the domain will never change even if the parameter q has changed*'. In the same way, Lindokuhle said '*No, the domain only focuses on the x-axis of the graph whereas parameter d or q deals with the y-axis*.

Importantly, in question 2.7C where participants were asked to use the link https://www.geogebra.org/m/d5wBD7U8 to access GeoGebra (Figure 5.41) & Interact with the applet changing the value of *'q'*. Unlike the domain of the parental sine and cosine functions which is [0º, 360º], the domain of the parental tangent function is [0º, 360 º]; $x \neq 90º$ and $x \neq 270º$. This means that the graph is bounded between 0º and 360º, however the graph can only get closer and closer (approaches) to the asymptotes and will never touch the indicated *x*-values.

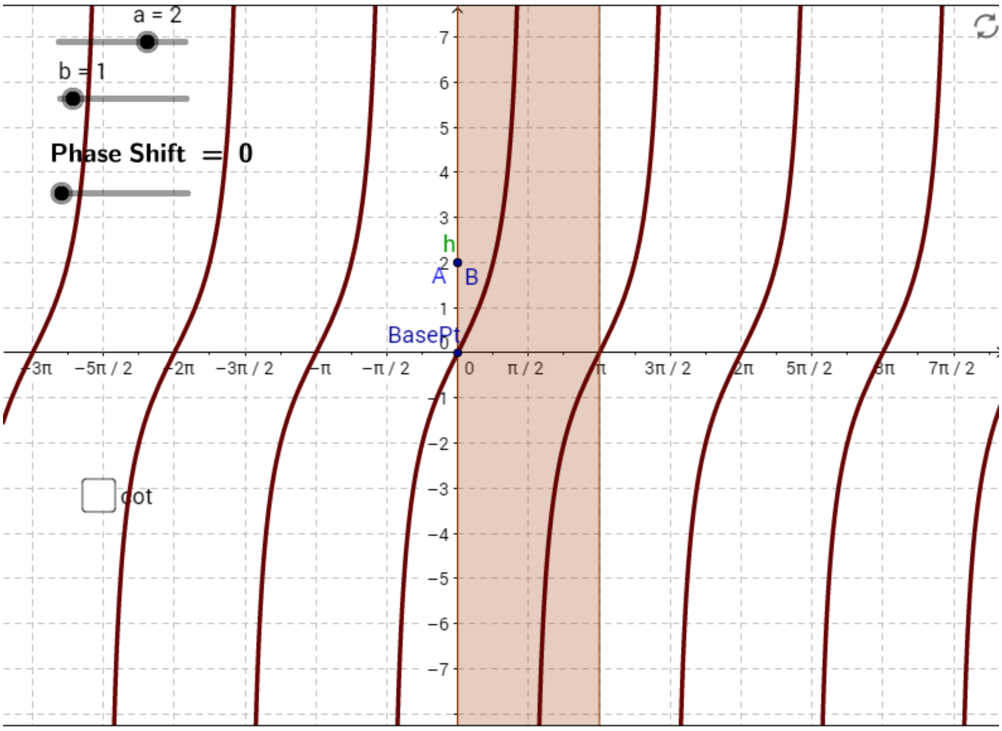


Figure 5. 41: GeoGebra graph of y = 2 tan x

The last question 'is there any difference in domain and range that you noted in the graph of *y = sin x*, *y = cos x* and *y = tan x*? Explain' probed participants to give their summary of what they have investigated. Lindokuhle asserted that *'the domain remains the same as the parental function range in three functions. The range differs with the change in the value of 'd' or 'q', the range of y = tan x is (-∞, ∞), and the domain of y = sin x and y = cos x depends on 'd' or 'q''*. Similarly, Nomfundo said that *'in the graph of y= tan x, the range is from -∞ to ∞. Whereas in the graph if y = sin x and y = cos x the range is determined by the value of 'q' or 'd'. The domain is not affected*'. Nonto asserted that there is no difference in the domain, it remained constant, the range with respect to the change in parameter '*d*'. Skhona shared that *y=sin x and y = cos x has the same domain of* [0º, 360º] *and the same amplitude of 1, but y= tan x has the range of (-∞, ∞).* In the same way Phumlani observed that *the graph of y = sin x and y = cos x has the same domain* [0º, 360º] *but* $x \neq 90º$ or $x \neq 270º$.

5.9 Participants understanding of parameter *'a'* and 'd' in a sine, cosine and tangent function.

The measure of understanding is sometimes unrealistic, taking into consideration the participants concept mastering skills and participants preferred representation of a function. Whereas this is true, this investigation was able to capture participants understanding using different representations of functions. Here the understanding of parameter *'a'* and *'q'* was assessed through graphical, verbal description, tabular, interval notation and mostly through gaining hands-on experience using a mathematical software, GeoGebra. I believe that participants are better assessors of their own knowledge, although they may not be certain with other answers they give. The progression arrow below was used by participants to reflect and indicate their level of understanding.

Figure 5.42 indicates participants 'progressive arrows of understanding'. This was used to describe their current level of understanding in each investigation of parameters in different trigonometric graphs. Participants ranked their understanding on a descriptive scale going from 'I do not understand any of this yet' up to 'I am so confident I could explain this to someone else'.

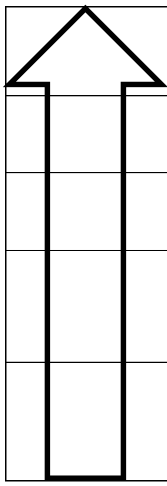
	I am so confident – I could explain this to someone else.
	I can get the right answer but I don't understand well enough to explain it yet.
	I understand some of this but I don't understand all of it yet.
	I tried hard and listened but I am finding this challenging. I will make sure that I get help with this in the next lesson.
	I do not understand any of this yet. There are things I could do to be a better learner next lesson.

Figure 5. 42: Participants understanding of parameter '*a*' and '*q*'

To effectively analyse data, the descriptive scale above was converted to a numeric scale as shown in the table below. The scale starts from one up to five, zero then shows that the learner did not colour in his or her current level of understanding.

Table 5. 25: Progressive arrows of understanding

Descriptive scale	Numeric scale value
I do not understand any of this yet. There are things I could do to be a better learner next lesson.	1
I tried hard and listened but I am finding this challenging. I will make sure that I get help with this in the next lesson	2
I understand some of this but I don't understand all of it yet.	3
I can get the right answer but I don't understand well enough to explain it yet.	4
I am so confident – I could explain this to someone else.	5

Participants understanding was measured using 'progressive arrows of understanding' as indicated in Table 5.25. All participants who were participating in this research had to assess themselves in terms of their ability and confidence in understanding the effect of parameter '*a*' after they have been exposed to the content.

Table 5. 26: Shows summary of participants understanding of the parameter 'a' in different functions

Parameter	Phumlani	Skhona	Lindokuhle	Nomfundo	Nonto	Yandisa
'a' in y= a sin x	3	3	5	4	4	3
'a' in y= a cos x	4	4	5	5	4	3
'a' in y= a tan x	4	5	3	4	4	3

The results in Table 5.26 were best represented in the graph (Figure 5.43). The bar graph then shows how individual participants understand the effect of *'a'* in sine, cosine and tangent graphs.

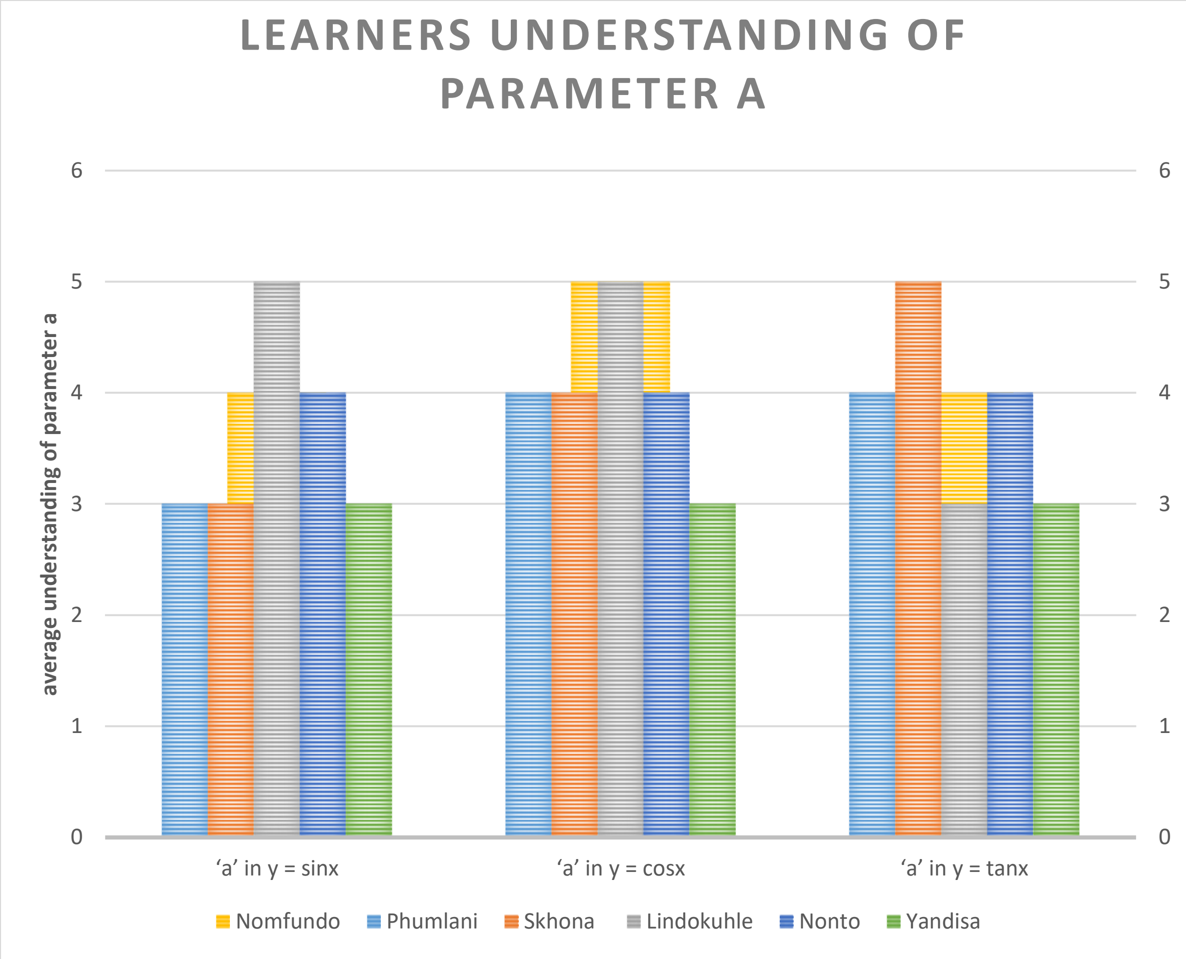


Figure 5. 43: Bar graph showing average participants' understanding of the parameter '*a*'.

It is notable that Yandisa's understands some of the effect of parameter *'a'* but not understand all of it yet'. This then suggests that there is more work that he needs to do so that he can understand all of the content and thus be able to explain it to someone else in his class.

Nonto, Skhona and Phumlani on the other hand had the averages of 4, 4, and 3.7 respectively, which suggests that 'they can get the right answer but they don't understand well enough to explain it yet'. In the same way, there is a room for improvement. Perhaps engaging more on similar activities might have a positive outcome in ensuring that they are more confident with what they do.

Lindokuhle and Nomfundo had the same averages of 4.3, which means that they were between 'I can get the right answer but I don't understand well enough to explain it yet' and 'I am so confident – I could explain this to someone else'. This level of understanding can be said to be the understanding that needs little or no remedial work. Here the learner shows explicit understanding of the content taught. The learner can be able to solve a problem in many different ways and can be able to explain how he or she arrived at the solution.

Table 5. 27: Shows summary of participants understanding of the parameter ‘q’ in different functions

Parameter	Phumlani	Skhona	Lindokuhle	Nomfundo	Nonto	Yandisa
‘q’ in y= sinx + q	5	5	5	5	5	-
‘q’ in y= cosx + q	5	5	5	5	5	-
‘q’ in y= tanx + q	4	5	5	4	4	-

The results in Table 5.27 were best represented in the graph in Figure 5.44. The bar graph then shows how individual participants understand the effect of *‘q’* in sine, cosine and tangent graphs.

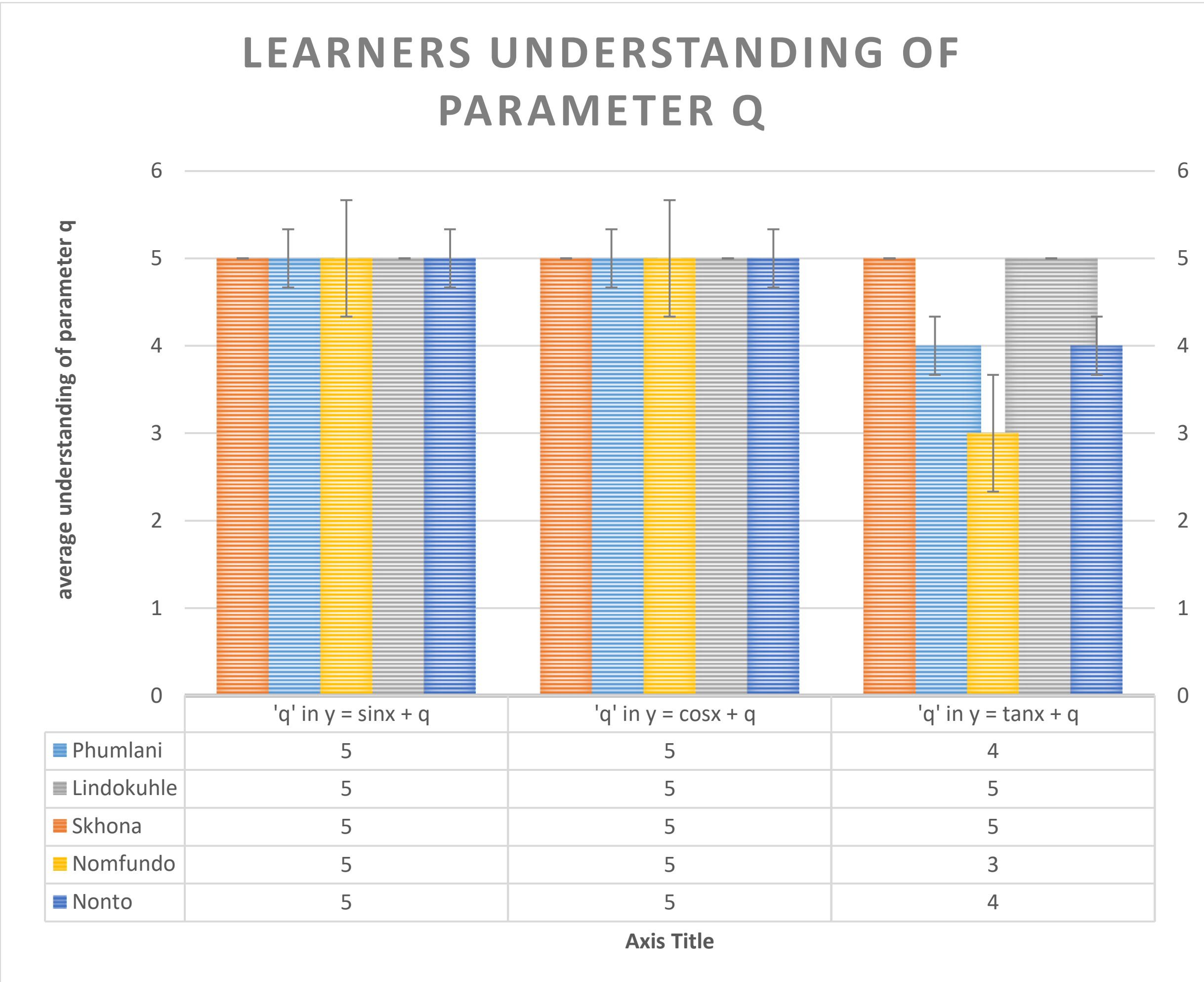


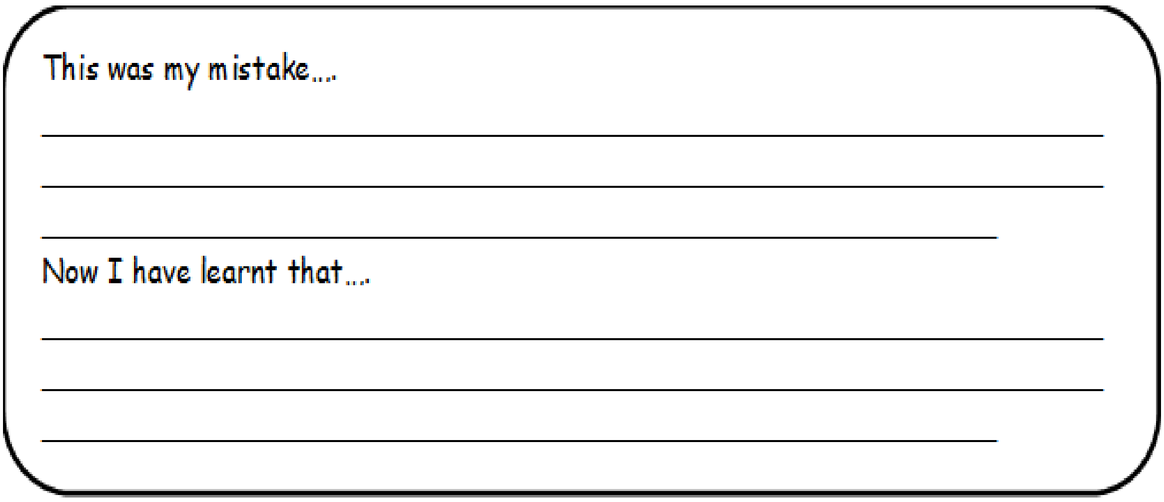
Figure 5. 44: Bar graph showing average participants’ understanding of the parameter ‘*q*’

The results in figure 5.44 imply that the participants understanding has improved as compared to how they were doing in part 1 investigation (investigating the effect of parameter '*a*'. In the second part of the investigation, their performance was much better, except that Yandisa withdrew from the study. Participants that were participating now:

- ✓ Understands how to identify the value of '*a*' and '*q*' from the equation and from the graph.
- ✓ Understands the effect of + '*a'* and '*q*' in sine, cosine and tangent function.
- ✓ Understands the effect of - *'a'* and '*q*' in sine, cosine and tangent function.
- ✓ Know how to sketch the graph of sine, cosine and the tangent showing all intercepts.
- ✓ Can use critical points to draw a rough sketch of a trigonometric function.
- ✓ Can interpret the graph, identifying the minimum and maximum points.
- ✓ Hence, represent the domain, range, amplitude and period of a trigonometric graph.

5.10 A mistake that moved my learning on the effect of '*a*' and '*q*'

Noting from the observations at the end of section 5.9, the list of learning outcomes that participants have to attain for them to understand the content better and be more confident in explaining the same content to their peers. The statements '*this was my mistake*' and '*now I have learnt that*' as shown in Figure 5.45 were designed to assist participants in reflecting on their learning experience. In this way, participants were able to identify their areas of concern, where they made mistakes and to state what they have learnt from those mistakes. Perhaps this may be viewed as a way forward to do better next time they encounter similar problems.


This was my mistake....

Now I have learnt that....

Figure 5. 45: A mistake that moved my learning

Here, Phumlani believes that he understands some aspects of the details around sketching and interpreting graphs with special attention being paid to the parameter *'a'*. Throughout the investigation, it came to light that Phumlani had some difficulties especially with drawing and giving detailed explanations to show that he is able to interpret trigonometric graphs. Part of the reasons for his incompetence in drawing graphs was the assumption that he had about trigonometric graphs, that there are too difficult and so, he chose not to pay attention in class. What could be taken to heart is the fact that he now understands the notion that, 'when you encounter difficulties, it is better to use different strategies'. That is to use different representations to explain one problem.

Nonto felt that her mistake was that she initially thought that $y = -2 \cos x$ and $y = 2 \cos x$ will have different amplitudes, meaning that the first one is smaller than the second one. She believe that she has learnt that 'these two have the same magnitude but starting in different sides of the *y*-axis. The graph is only reflected about the *x*-axis'..

Skhona shared the same sentiments, he said that "*in my mind I always had that trigonometry is hard and confusing so I did not put any effort towards solving its problems'*. It is interesting to note that he has learnt that 'there are various methods to attack and solve the problem of trigonometric graphs and also how to use those methods effectively.

Lindokuhle seemed to have a specific reason why he encountered difficulties in understanding the effect of '*a'*. He believes that the mistake he made was, "I thought the parameter *'a'* was the one responsible for shifting the graph up and down' and he has learnt that "parameter '*a'* is only responsible for the change in the amplitude of the graph'. Nomfundo said that at first she was not sure about what parameter means, '*throughout the first investigation I was confused, especially describing the effect of 'a' was a bit challenging'*. Nomfundo believes that when describing the effect of parameter must be based on seen observation. Yandisa did not participate in reflective questions which were specific on the mistake/s that enhanced his learning the effect of '*a*' and '*q*'

5.11 Representation/s that enhance participants understanding of parameter *'a' and 'q'* in trigonometric graphs

Phumlani believe that 'the only thing that helped him to understand trig functions is the use of GeoGebra'. He insisted that it made it so easy for him to understand the relationship between the parameter *'a'* and the amplitude. This is important observation as he can now look at the function (value of '*a*') and be able to determine the amplitude of that function without any hesitation. In the same way, participants can now read off the value of *'a'* from the graph and also determine the amplitude.

Lindokuhle voiced out his opinion about the representation that helped him in his learning curve to eradicate misconceptions (Figure 5.46).

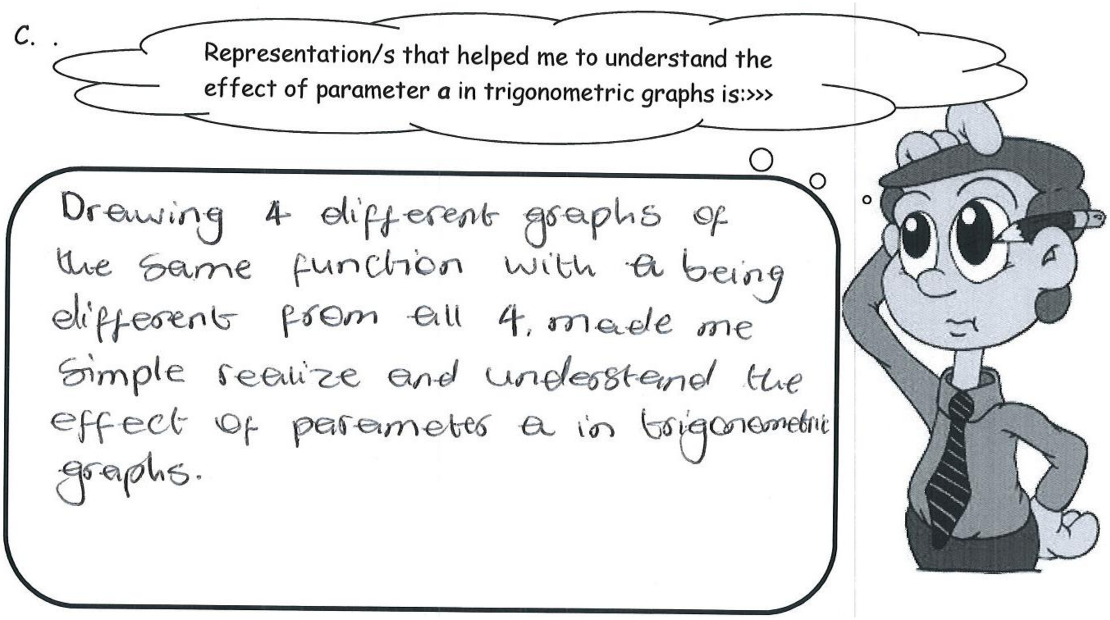


Figure 5. 46: Lindokuhle's reflection on representations that enhanced his understanding

Importantly, he noted that graphical representation made him understand the effect of *'a'*. Repetitively drawing the graphs of $y = \sin x$, $y = 2 \sin x$, $y = - \sin x$ and $y = -2 \sin x$ made him to be able to observe the changes in *'a'* and thus compare the results for him to make conclusions.

Skhona on the other hand believes that engaging with GeoGebra made him purely understand the effect of parameter *'a'* and *'q'*. He noted that point-by-point plotting of trigonometric functions was not helpful as such. Because he is a visual learner, his preference in terms of method of instruction is through the use of pictures that he can visualize. That is exactly what the software GeoGebra did to enhance his understanding. In the same way, Nonto believes that through using GeoGebra she managed to understand the functions as the software quickly interchanged the parameter *'a'* which was positive to the one which was negative.

Figure 5.47 shows Skhona's observation of the representation that helped him to understand the effect of parameter *'a'* in trigonometric graphs.

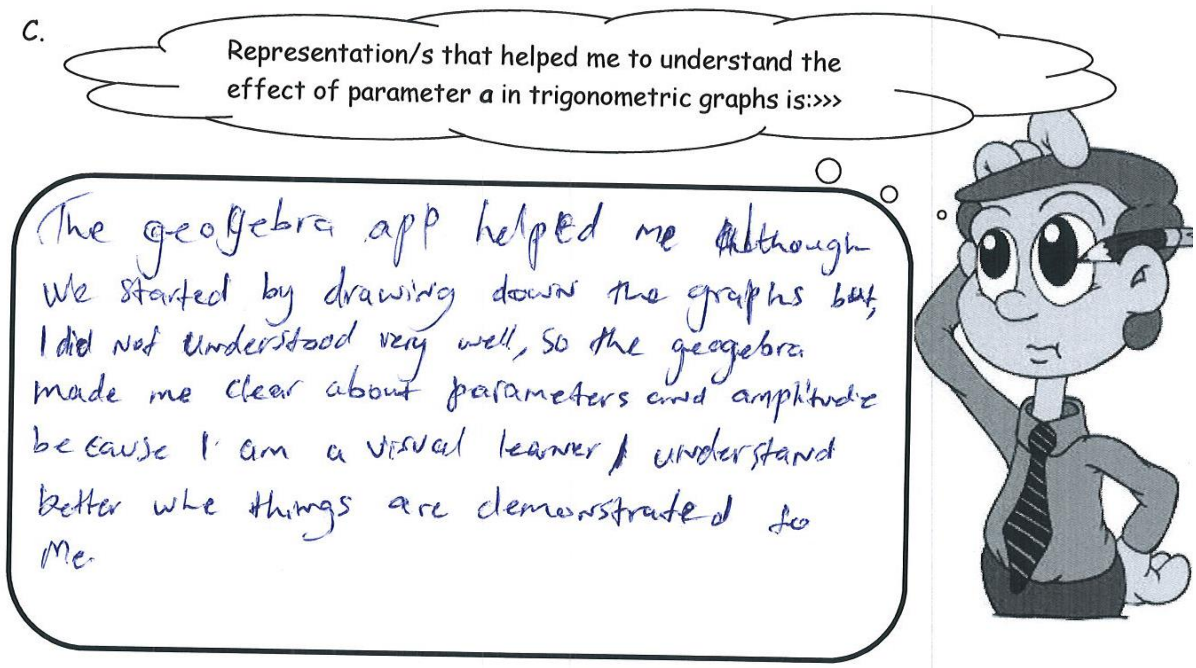


Figure 5. 47: Skhona's reflection on representations that enhanced his understanding

In the same way, Yandisa felt that '*drawing the graphs helped me to understand the effect of 'a', I could see that the graph of y = 2 sin x is one unit more on the y -axis. When I was using a computer, the graphs were very easy to interpret. I could understand even much faster than before*'. Nomfundo felt that the use of computer software (GeoGebra) help him to gain understanding of how the graph shifts with respects to the change in parameters.

In this study, it became evident that making use of different learning strategies does make a difference in participants understanding of complex content. Moreover, advancing teaching and learning aids in Mathematics classrooms provides clearer lenses of the content knowledge. In this study, through the use of GeoGebra as a teaching and learning tool, participants were able to draw, recognize and interprets the graphs of trigonometric functions more easily than when they were required to manually sketch the graphs on a Cartesian plane.

5.12 Teaching using GeoGebra & Learning through GeoGebra

Learning experience can be unpleasant and sometimes fruitful, and it all depends on the eye of the viewer or the one perceiving knowledge. In daily teaching experience, a teacher may view a lesson where all participants are passively sitting back silently listening as a successful one and hence a fruitful lesson, as most teachers prefer such audience. In the eyes of the learner, the same lesson can be perceived as a boring and unproductive lesson. Below are learner's reflections on how they viewed their learning experience when they are taught using GeoGebra 'How they teach it & how I learnt it'. Nonto and Yandisa did not complete the questionnaire and thus no data could be recorded.

In the first question, participants were asked to comment on their learning experience, regarding the different representations that were used in understanding the effect of parameter '*a*' in trigonometric functionns. This was summarized as '*What have you learnt in today's lesson?'*.

Phumlani's learning experience

Phumlani said, '*I have learnt that a problem (trig graphs) can be solved using different ways, such as using GeoGebra to interpret graphs'*. He believed that this method is "too different" from his daily learning experience, in a sense that it reduces time taken to understand the effect of *'a'*. Phumlani believe that GeoGebra makes him understand trigonometric functions easily because he does not have to draw the graph. "All I have to do is to download the app (GeoGebra) and use it to draw and interprets graphs. He urges participants to use GeoGebra as 'it makes life easier'. He asserted that GeoGebra can possible help him to understand trigonometric functions much better. He added that when he manipulates different parameters he can actually see the change immediately on the computer. This is opposed to sketching a graph using point-by-point on a graph paper. It is important to note that these new methods intend not to replace the pen and paper, chalk and talk, but they are learning aids that seek to improve participants understanding.

Skhona's learning experience

Skhona had a thriving experience in learning using different learning aids. He now believes that there are many different ways of solving problems and thus a problem can be easily solved by using any strategy. He added that this learning experience of using computer software is quite different from his normal learning setting of using worksheets. '*It is different in a way that when you are solving problems using these strategies, you don't have to draw a graph and plot some other points. So to reflect the graph, you just have to download the app then use it to solve your problem'*. The point that Skhona is making is that it is easy to work with GeoGebra because the parental function is already drawn for you, thus to see the effect you need to change the parameters and the graphs suddenly adjusted with the new parameters of interest. This is contrary to point by point plotting where you need to complete the table function the critical points then plot the function on the graph paper.

Skhona added that through the use of GeoGebra, he noted that graph interpretation becomes easier in the sense that graphs are already drawn for you and that saves time. This is because the only thing that you have to do is to analyse the given function and manipulate the parameters to see their effects on the graph which is already in the computer software. Skhona believes that using GeoGebra can help him to better understand trigonometric functions *'because it shows you how a graph can be reflected about the x-axis/ y-axis. Its makes you understand better how parameter 'a' or 'q' can have an effect on the graph and what they are basically for'.*

When Skhona was responding to a question of *'whether different methods of solving mathematics problems are helpful or not';* he said that using different strategies gives him an option to choose the appropriate and good method that he understands better and which allows him to solve a problem. He added that this allows him to explore his options and be able to check if his answers or solutions are correct or not by using different methods of solving a problem.

Lindokuhle's learning experience

Lindokuhle's tone on his learning experience was full of hope. He summed up his leaning journey as fruitful in a sense that he learnt that he noted that one problem can be solved in many different ways. He stated that '*there are many different ways a trig graph can be interpreted and solved. I have also learnt that about GeoGebra's function in checking the effect of parameter a, b and c in a graph*'. Lindokuhle felt that this was a new learning experience and argued that '*it is quite different to my normal learning experience because I was only introduced to the methods that were preferable to my teachers*'. What is notable is that, his learning experience was solely dependent on what the teacher had to say. This essentially means that if a teacher know more and can possible explain a concept in different ways, then all participants may be fortunate to grasp one or two things. Contrary to this, if a teacher only knows one method of arriving at the solution, then participants will be disadvantaged.

The degree to which Lindokuhle is likely to recommend the use of GeoGebra as a learning tool was high. He added that the use of GeoGebra enhanced his learning process. Visualising the change in the amplitude as he manipulated the parameter '*a*' and '*q*' on the computer software was interesting. He felt that learning trigonometric functions through this software is the '*best way of learning'*. He added that '*the best ways of learning something is when it is simplified and with less chances of making mistakes'*. Moreover, he noted that knowing how to work with different ways, '*multiple strategies'* is likely to make his life easier when solving mathematics problems in general. He asserted that he would not have to rely on only one way to tackle problems.

Nomfundo's learning experience

More than anything, through the use of GeoGebra and manually sketching the graph using the graph paper, Nomfundo has gained more confidence in working individually. Reflecting on her learning experience, she said, '*I have learnt that using GeoGebra to interpret trigonometric graphs saves time and the software is interesting as I was able to learn how changing the values affects the appearance of the graph. I could correct myself from a word go without waiting for the teacher who is busy with the rest of the class'*.

Nomfundo expressed that this learning experience where there are computers in the mathematics classroom and she is actively making use of the computer to learn mathematics is exciting. She noted that in this lesson, she was able to work on her own to check the effect of parameters. '*I have learnt a lot as the graphs were already on computer, all I had to do was to change the value of 'a' and the computer did the rest, which is drawing the graph accurately'*.

Nomfundo recommends the use of GeoGebra as a learning tool to her fellow grade 10's, '*I would say, it would have taken more time (like in the first activity where I sketched the graphs from the points that I plotted) to interpret the graph if I was not using GeoGebra. I wish most lessons in maths could be taught using this software*'. Furthermore, Nomfundo believes that GeoGebra can enhance her understanding of trigonometric functions better. She felt that she already has a better understanding of the $y= a \sin x$ which is the graph that she has never seen before. She made reference to the effect of '*a'*, that it is now starting to make sense. When she was manipulating other parameters she was told that there are beyond grade 10 scope, however she felt that she already has a clue of their effect through playing along with these parameters.

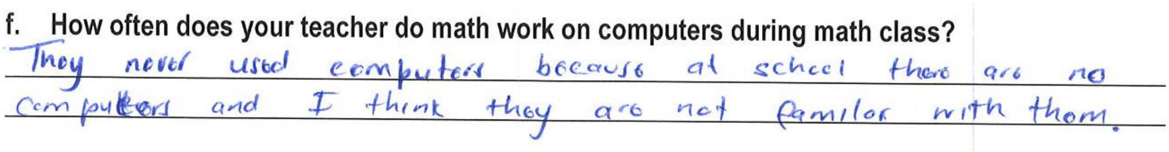
Whilst learning through the use of GeoGebra was an efficient learning tool, Nomfundo's response to whether different ways to solve grade 10 math's problems (i.e. drawing graphs and deducing the effect of parameter '*a*' and '*q*' on the graphs they had to draw and from which they had to deduce the effect of changing the effect of changing parameters on trigonometric functions) helps to understand each problem better or not, she expressed '*yes, definitely. In the phase 1 of the investigation, sir asked us to use calculators to find the y-coordinates for given x-coordinates, then plot different sine graphs. In the second tasks, graphs were in the computer and all we had to do was to investigate the effect of different parameters, and we did. This allowed me not to conclude that this task is difficult but it was through different methods towards the solution that created a barrier in understanding the effect of parameters*'.

5.13 Teacher's use of computers in teaching mathematics

Computers have been used in mathematics classrooms for more than two decades, however there have been varying implementation of its use. Notable, from participants' responses many schools do have computers but there seems to be a restricted access to computer use by both teachers and participants. This is one of the key obstacles in computer usage as a learning tool in mathematics classroom. Either, teachers do not see the value of using computers in class, or they have minimal, or no knowledge of using computers, or they are in schools where there are no computer facilities. Either way, teachers who are using computers in teaching mathematics will know that computers or mathematics software allows participants to work at their own pace and give useful, immediate feedback on learner performance.

When participants were asked “how often does your teacher do math work on a computer during math class?’ It was shocking to note that there is still minimal use of technology in mathematics classrooms. It is the same old methods, just like a century ago. Traditional teaching is what most mathematics classrooms look like. Below is a response from participants about their daily mathematics lessons.

Phumlani maintained that his mathematics teachers have never used a computer in the classroom. He articulated that the underpinning reasons why there was no technology used in mathematics classroom was that there were no computers at his school and he also believes that his mathematics teacher is not familiar with computers. Figure 5.48 shows Phumlani’s response.


f. How often does your teacher do math work on computers during math class?

They never used computers because at school there are no computers and I think they are not familar with them.

Figure 5. 48: Phumlani’s response to Question C

Just like Phumlani, Lindokuhle concured that his mathematics teacher did not use a computer in her teaching. He maintained that the only resource that they normally have access to are the chalkboard notes, textbook and worksheets. Here, the only difference is that Lindokuhle knows that computers are available at his school, however the teacher does not make any effort to teach using computers in class. Figure 5.49 shows Lindokuhle’s response.

f. How often does your teacher do math work on computers during math class?

She does not use one, the only resources we usually use are the chalkboard, textbook and the photocopied material.

Figure 5. 49: Lindokuhle’s response to Question C

In the same way, Skhona, said that often his mathematics teacher relied on the use of the chalkboard when teaching mathematics. Occasionally the teacher would use a computer in class to show other methods of solving the same problem. Figure 5.50 is Skhona’s detailed response on teacher’s usage of computers in mathematics classroom.

f. How often does your teacher do math work on computers during math class?

Mostly my teacher uses chalkboard when teaching maths but sometimes he would use a computer just to show us maybe other strategies he came accross when he was researching different ways to make us understand the concept.

Figure 5. 50: Skhona’s response to Question C

Nomfundo holds a different view that, her teacher does use a computer for teaching, demonstrating some other aspects of the lesson. Nomfundo argued that the teacher uses a computer mostly for his administrative work, either designing assignments or tests. Figure 5.51 shows Nomfundo’s response.

f. How often does your teacher do math work on computers during math class?

He raley use a computer for teaching or demonstrating, however he alway type ~~his~~ our test and assignment and not giving ~~us~~ hand written tests / EXAMS.

Figure 5. 51: Nomfundo’s response to Question C

5.14 Chapter summary

This chapter reported on the responses of the six participants in this study. All participants' insights, their understanding, experiences and areas of concern were shared.

Furthermore, this chapter gave the presentation and interpretation of the findings resulting from this study. The next chapter will summarise and make general conclusions based on the findings obtained from the data that is discussed in chapter 5. Chapter 6 will provide recommendations for future research, and recommendations for educators, participants and higher education stakeholders.

Chapter 6
Conclusions and Recommendations

6.1 Introduction

Whereas Chapter Five dealt with findings obtained from six cases, this chapter discusses the main findings in respect to the research questions. This chapter focuses on summarised and general conclusions based on the findings obtained from the data is discussed in the previous chapter. Lastly, this chapter will provide recommendations for future research, for educators, participants and higher education stakeholders.

6.2 Conclusions

There are several conclusions that were drawn from the findings. These will be discussed below.

6.2.1 Participants engagement with representations when working with Trigonometric functions

It is concluded that when participants are given education resources that will trigger the use of different mathematical representations, participants are interested to learn and present their answers in different ways. It became evident that participants in grade 10 are able to present one trigonometric function in different ways. In the results, I noted that participants were able to freely translate between different representations. A learner would start by trying to understand a functional notation ($y = a \sin x + q$), identify the value of each parameter, then use a tabular representation of function. That alone required a learner to use a calculator to evaluate trigonometric functions. From that step the learner would make sense of the coordinates emanating from the table, which were plotted point-by-point in the Cartesian plane. Participants then drew the graphs, indicating all the intercepts and critical points. The graphs that were drawn by participants manually needed to be interpreted in terms of the effect that parameters has on each trigonometric function. This exercise equipped participants to use their calculators efficiently, know how to evaluate trigonometric rations and the important skill of rounding off decimals.

The participants who were participating in this study were enthusiastic when they were given computers to work with, to further investigate the effect of parameter '*a*' and '*q*' using GeoGebra. Participants were happy that the graphs that they have drawn manually look much the same with the ones on GeoGebra, which alone gave them quick feedback on the tasks they were doing. Being able to manipulate the trigonometric function using a computer gave them a sense knowledge ownership, not to see mathematics as something there in textbooks. It was life thrilling moment to see mathematics participants enjoying doing mathematics and not just scratching their heads, thinking that they are doing is a difficult boring subject. This view is supported by, Tatar and Zengin (2016) who noted in their study, that the computer-assisted instruction method of using GeoGebra was found to positively contribute to the success of teaching mathematics.

6.2.2 Common representations in a classroom

The outcomes acquired in this study indicate the common representations used by participants in mathematics which amongst others are the equations, diagrams, tabular method, and dynamic graphs. The underpinning reasons why participants used most modes of representations was that the nature of the research tool (investigations) probed participants to engage with different representation modes before they could interpret the graph. In a way, structured questions that enforce participants to give predetermined solutions triggered them to use different representations that they were not going to use if they were only doing the investigation that require only one solution. It was then concluded that the types of activities that are given to participants should trigger them to use or apply different ways of arriving at the solution. Questions like 'in how many ways can you solve this problem', 'use different representations to solve this problem' should be encouraged in mathematics classrooms.

In this study, an intended arrangement of the multiple representations was considered in order to ingress effective learning of trigonometric functions. In support for this systematic pattern is Ainsworth (2008) who maintains that multiple representations bring unique benefits yet they can also inhibit learning should they be in an inappropriate combination. This orderly notion began with an equation representation since the participants have prior knowledge of algebra. The format of an equation encoded information about + and – signs. For instance, the participants determined by themselves that + 'q' shifts the graph vertically up rather than to maintain it will get bigger in size since + intuitively means greater. Hence, participants are able to make connections with the appropriate approach which is translations due to relating the equation and the graph.

Thereafter, the table method representation, which shows the equation in form of a relation between input *x* and output *y*. In this illustration the participants find it easier to see where the function starts and ends in order to answer the questions on domain range, amplitude and period of a function. The same function was then represented as a graph in a coordinate system of *x* and *y*, so participants retained an accurate and clear visual diagram of the equation. This means that the learner acknowledge the type of lines are formed, which are curves. A step further was taken to represent the function as a dynamic graph on the GeoGebra software program. In this manner participants were able to recognize how one parameter affects or does not affect other parts of the graph. Such as stretching parameter 'a' would only affect the range and not the domain nor parameter 'q'. The questions which followed were also appropriate as the participants represented their observations in word descriptions. This is supported by Dreyfus (1991) who contended that learning should happen through 4 stages, where there is use of representation, use of more than 1 representation together, establishing links between parallel representations, and integrates flexibly different representations.

6.2.3 Representations and their effect on understanding

Representations that were used in this study were to develop concrete understanding of trigonometric functions, and met the department of education expectation as stipulated in the CAPS (2011) that participants should be able to "work with relationships between variables in terms of numerical, graphical, verbal and symbolic representations of functions and convert flexibly between these representations; tables, graphs, words and formulae (p. 12).

The type of instruction that these participants that were participating in this study received and the investigation/ tasks that they were engaged in, were effective for developing their understandings of trigonometric functions. It is therefore concluded that when participants are given learning aids that enhance their understanding of different mathematics concepts, they are determined to do well. In this way, representing one problem in multiple ways helped participants in class to understand the abstract concepts. From the emanating findings, I believe that the use of multiple representation can be thought of as hearing the same story from different people. It came to light that participants are different and have their individual learning styles of preference, and it is important that teachers know and use different representations in a classroom to meet participants' education demands.

Ainsworth (1999) expressed that "a common justification for using more than one representation is that this is more likely to capture a learner's interest and, in so doing, play an important role in promoting conditions for effective learning" (p. 131). As stated elsewhere that Gruwell (2007) cited Ignacio Estrada who once said "If a child can't learn the way we teach, maybe we should teach the way they learn." (p. vii). Mathematics teachers need to understand their participants preferred representations so they can integrate those representations in their lessons. This is because participants understood better with the use of representations, Gruwell (2007) asserted that the use of rote memorisation of difficult mathematics concepts may be the strong influence towards participants' failure in mathematics.

Ainstworth (1999) believed that effective learning occurs when participants understands different mathematics' representations and be able to translate within different representations. In the same way, the results emanating from this study suggest that for participants to have concrete understanding of different concepts, they need to know different modes in which they can present their solutions. It is concluded that participants understand better when different mathematics representations are used in the teaching and learning of trigonometric functions. This is consistent with the study by Zengin, Furkan and Tamer Kutluca (2011) who uncovered the positive impact of utilizing mathematical learning software in enhancing learning and understanding of trigonometry. Their study revealed that the effects of dynamic mathematics software GeoGebra on student achievement in teaching of trigonometry was more effective than the ordinary constructivist teaching method. In the same way, Van Voorst's (1992) use of classroom technology helped participants to view mathematics problems better, with a different perspective that aids in enhancing their understanding. He asserted that through technology use in class, "Participants' view mathematics less passively, as a set of procedures, and more actively as reasoning, exploring, solving problems, generating new information, and asking new questions." (p. 2).

6.2.4 Effect of participants use of representations and their performance

The findings emanating from study suggest that there is a strong correlation between participants' use of representations and their performance. Participants feedback suggested that being actively involved in this study made them feel confident in attempting any question that seeks understanding of functions. Some participants who were not familiar with the use of computers attained computer skills and performed well in questions where they had to use GeoGebra.

As a teacher who also evaluated participants understanding, I was impressed with the way in which participants were making observations, using different representations, and switching between different representation modes. Looking at how they were helping each other, the collaborative work that mathematics teachers do not normally witness in mathematics classrooms. These participants showed a great improvement in terms of their conceptual understanding and their approach in knowledge finding. The responses that they gave were meaningful and matches with expected understanding in the 10^{th} grade learning outcomes of trigonometric functions.

From the above findings, it can be said that multiple representations were used to understand trigonometric functions. Participants' reflections and their investigation reports suggests that participants did understand trigonometric functions better than learning without representations. In a similar study by Escuder (2012), he stated that, "As the concepts are introduced with pictorial representations, teachers and their students are able to make the connections between the pictures, the math's concepts, and the symbolic representation" (p. 82). This suggests that the use of mathematical representations help both teachers and participants in terms of content delivery for teachers, and meaning making for participants.

In my 4 year's experience as a mathematics teacher, I have never engaged in the teaching of trigonometric function like I have engaged in this project. The notion of using multiple representations in teaching of trigonometry and other mathematics topics was impractical because the content workshops that are offered by the Department of Education do not emphasize on such important teaching strategies. If mathematics can be learnt the way these participants learnt, it is will be possible for poorly performing the grade 10 participants to improve their understanding and hence their performance in mathematics.

6.3 Recommendations for future study

This study has focused mostly on the essential trigonometry investigations that the South African grade 10 mathematics educators should employ in making sure that their participants attain good understanding of trigonometric functions. Further studies should then focus on the understanding of trigonometric functions in grade 11 and 12. Mostly, the understanding of functions in general may help participants to understand the uniqueness of each function, yet the parameters may have the same effects on different functions. This study recommends that mathematical representations must be the centre of instructions used by mathematics teachers if they want participants to have different ways of solving abstract problems in mathematics. Mathematics educators and participants should strive to incorporate technology use in their classrooms; the use of computers/ laptops, cellphones that can install mathematics' software should be promoted in mathematics classrooms, and not to be seen as source of distraction. It should be noted that teaching using multiple representation does not replace the use of ordinary instruction modes rather it is a way to supplement the path to a better understanding of mathematics.

6.4 Limitations of the study

The researcher's physical involvement in the research may have contributed to the biasness of the results. Participants' behavior may have been different if the researcher was not directly in the research site. The report of findings is subjected to be bias, because I conducted the research while also teaching the group of participating participants. The researcher overcame this limitation by telling the participants that he was there to facilitate and not to judge then or formally assess them as it is done in the classroom. Participants were also given assurance that their identity will not be divulged under any circumstances.

Another limitation is that participants were given questions that probed the use of different representations. In so doing, participants were guided with questions that already have expected representation of answers. There was no room for thinking out of the box, for them to give representations of their own, rather it tested the learner's ability to use the given representations.

The study is limited to one context, one school and to 6 participants who are doing pure mathematics in grade 10. It then neglects different groups of participants who were not sampled in this study. Some schools are underprivileged and the use of computer based technology may be impossible, in a way that some do not even have electricity. It is therefore difficult to make generalization for all South African participants. Nevertheless, this study may be useful when one needs to look at the trends of how participants perform when they are exposed to learning through the use of multiple representations.

6.5 Conclusion

Whereas participants understanding of trigonometric functions using multiple representations has been evident in literature, this study aimed at closing the existing gap in literature where the use of representations has been neglected in most South African schools. Therefore, this study has a potential to inform discussions around the teaching and learning of trigonometry in poorly performing secondary schools, particularly on how representations can be used in different topics in the school curriculum. This study is underpinned by Ignacio Estrada's words that "If a child can't learn the way we teach, maybe we should teach the way they learn." (p. vii). As a teacher I felt that the study has the potential to inform most of the mathematics classroom discussions, in making sure that the classroom becomes a place where participants are actively involved in their learning and not to be passive receivers of information through talk and chalk.

References

Adams, L., Tung, K., Warfield, V., Knaub, K., Mudavanhu, B., & Yong, D. (2000). Middle school mathematics comparisons for Singapore mathematics, Connected Mathematics Program, and Mathematics in Context (including comparisons with the NCTM Principles and Standards 2000). *Unpublished manuscript, Seattle, WA.*

Ainsworth, S. (2006). DeFT: A conceptual framework for considering learning with multiple representations. *Learning and instruction, 16*(3), 183-198.

Ainsworth, S. E., Bibby, P. A., & Wood, D. J. (1998). Analysing the costs and benefits of multi-representational learning environments. *Learning with multiple representations*, 120-134.

Akkus, O., & Cakiroglu, E. (2009a). The effects of multiple representations-based instruction on seventh grade students' algebra performance. *Tersedia: http://ife. enslyon. fr/publications/editionelectronique/cerme6/wg4-01-akkus-cakiroglu. pdf.[11 Maret 2012]*.

Arends, F., Winnaar, L., & Mosimege, M. (2017). Teacher classroom practices and Mathematics performance in South African schools: A reflection on TIMSS 2011. *South African Journal of Education, 37*(3), 1-11.

Bal, A. P. (2014). The examination of representations used by classroom teacher candidates in solving Mathematical problems. *Educational Sciences: Theory and Practice, 14*(6), 2349-2365.

Baxter, P., & Jack, S. (2008). Qualitative case study methodology: Study design and implementation for novice researchers. *The qualitative report, 13*(4), 544-559.

Bertram, C., & Christiansen, I. (2014). *Understanding research: An introduction to reading research*: Van Schaik Publishers.

Bohlmann, C. A., Prince, R. N., & Deacon, A. (2017). Mathematical errors made by high performing candidates writing the National Benchmark Tests. *Pythagoras, 38*(1), 1-10.

Breidenbach, D., Dubinsky, E., Hawks, J., & Nichols, D. (1992). Development of the Process Conception of Function. *Educational Studies in Mathematics, 23*(3), 247-285.

Brijlall, D., Bansilal, S., & Moore-Russo, D. (2012). Exploring teachers' conceptions of representations in mathematics through the lens of positive deliberative interaction: original research. *Pythagoras, 33*(2), 1-8.

Bryman, A., & Bell, E. (2015). *Business research methods*: Oxford University Press, USA.

Byers, P. (2010). Investigating trigonometric representations in the transition to college Mathematics. *College Quarterly, 13*(2), n2.

Cai, J. (2005). US and Chinese teachers' constructing, knowing, and evaluating representations to teach mathematics. *Mathematical Thinking and Learning, 7*(2), 135-169.

Carthy, Y. (2013). B*ringing the NCTM standards to life: Exemplary practices for middle school*. Routledge.

Chahine, I. (2011). The role of translations between and within representations on the conceptual understanding of fraction knowledge: A trans-cultural study. *Journal of Mathematics Education, 4*(1), 47-59.

Chigonga, B. (2016). *Participants's errors when solving Trigonometric equations and suggested interventions from grade 12 Mathematics teachers*.

ÇIKLA, O. A. (2004). T*he effects of multiple representations-based instruction on seventh grade students' algebra performance, attitude toward Mathematics, representations preference.* (Doctor of Philosophy), Middle eEast Technical University.

Cleaves, W. P. (2008). Promoting Mathematics accessibility through Multiple representations: Jigsaws. *Mathematics Teaching in the Middle School, 13*(8), 446-452.

Clement, L. L. (2001). What do students really know about functions? *The Mathematics Teacher, 94*(9), 745.

Cohen, L., Manion, L., & Morrison, K. (2007). *Research methods in education.* London: Routledge.

Cohen, L., Manion, L., & Morrison, K. (2011). Surveys, longitudinal, cross-sectional and trend studies. *Research Methods in Education, 7th edition. Abingdon: Routledge*, 261-264.

Cohen, L., Manion, L., & Morrison, K. (2013). *Research methods in education*. London: Routledge.

Cohen, L. L., & Manion, L., & Morrison, K. (2000). Research methods in education [5th edition] London: Routledge Falmer. *Teaching in Higher Education, 41*. .

Coyne, I. T. (1997). Sampling in qualitative research. Purposeful and theoretical sampling; merging or clear boundaries? *Journal of advanced nursing, 26*(3), 623-630.

Creswell, J. W. (2002). *Educational research: Planning, conducting, and evaluating quantitative*: Prentice Hall Upper Saddle River, NJ.

Creswell, J. W. (2007). Five qualitative approaches to inquiry. *Qualitative inquiry and research design: Choosing among five approaches, 2*, 53-80.

Creswell, J. W. (2013). *Research design: Qualitative, quantitative, and mixed methods approaches*. Thousand Oaks: Sage publications.

Denzin, N. K., & Lincoln, Y.S (2000). The Sage handbook of qualitative research. *Thousand Oaks: Sage publications ua*, 413-427.

Dewey, J. (1985). *The Later Works, 1925-1953: 1932* (Vol. 7): SIU Press.

Dinkelman, M. O. (2013). Using learner-generated examples to support student understanding of functions.

Education, D. o. (2003). *National Curriculum Statement Grades 10-12 (General) Mathematics*: Government Printer Pretoria.

Education, D. o. B. (2011). *Curriculum and Assessment Policy Statement (CAPS)*: Department of Basic Education Pretoria.

Ertsen, M. W. (2014). A poor workman blames his tools or how irrigation systems structure human actions *Objects and Materials* (pp. 58-67): Routledge.

Fennema, E., & Franke, M. L. (1992). Teachers' knowledge and its impact. In D.A Growa (Ed.), *Handbook of research on mathematics teaching and learning* (pp.147-164). New York: Macmillan Publishing Company.

Friendland, A., & Tabach, M. (2001). Promoting multiple representation in álgebra. In A.A. Cuoco & F.R. Curcio (Eds.), *The roles of representation in school mathematics*.Virgina: NCTM.

Gardner, H. (1999). *Intelligence reframed: Multiple intelligences for the 21st century*: Basic books.

Golafshani, N. (2003). Understanding reliability and validity in qualitative research. *The qualitative report, 8*(4), 597-606.

Gonzalez-DeHass, A. R., Furner, J. M., Vásquez-Colina, M. D., & Morris, J. D. (2017). Pre-service elementary teachers' achievement goals and their relationship to math anxiety. *Learning and Individual Differences, 60*, 40-45.

Gruwell, E. (2007). *The Gigantic Book of Teachers' Wisdom*: Skyhorse Publishing Inc.

Guba, E. G., & Lincoln, Y. S. (1994). Competing paradigms in qualitative research. *Handbook of qualitative research, 2*(163-194), 105.

Gulkilik, H., & Arikan, A. (2012). Preservice Secondary Mathematics Teacher's Views about Using Multiple Representations in Mathematics Instruction. *SBSPRO Procedia - Social and Behavioral Sciences, 47*, 1751-1756.

Gur, H. (2009). Trigonometry Learning. *New Horizons in Education, 57*(1), 67-80.

Harris, J., & Bourne, P. A. (2017). Perception of teachers and pupils on factors influencing academic performance in Mathematics among a group of fifth and sixth grades in Jamaica. *International Journal of Transformation in Applied Mathematics & Statistics, 2*(1).

Hartley, J. (2004). Case study research. *Essential guide to qualitative methods in organizational research*, 323-333.

Hohenwarter, M., & Lavicza, Z. (2007). Mathematics teacher development with ICT: towards an International GeoGebra Institute. *Proceedings of the British Society for Research into Learning Mathematics, 27*(3), 49-54.

Janvier, C. (1987). Translation processes in mathematics education. *Problems of representation in the teaching and learning of mathematics, 27*, 32.

Kelly, A. E., Lesh, R. A., & Baek, J. Y. (2014). *Handbook of design research methods in education: Innovations in science, technology, engineering, and mathematics learning and teaching*: Routledge.

Kitchen, R. S., DePree, J., Celed, S., & Brinkerhoff, J. (2017). *Mathematics education at highly effective schools that serve the poor: Strategies for change*: Routledge.

Larvor, B. (2016). *Mathematical Cultures: The London Meetings 2012-2014*: Birkhäuser.

Lesh, R., Post, T., & Behr, M. (1987). Representations and translations among representations in mathematics learning and problem solving. *Problems of representation in the teaching and learning of mathematics, 21*, 33-40.

Lincoln, Y. S., & Denzin, N. K. (1994). The fifth moment. *Handbook of qualitative research, 1*, 575-586.

Lincoln, Y. S., & Guba, E. G. (1985). *Naturalistic inquiry* (Vol. 75). Thousand Oaks, CA: Sage Publications.

Loewenberg Ball, D., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of teacher education, 59*(5), 389-407.

Mason, J. (2002). Linking qualitative and quantitative data analysis. In *Analyzing qualitative data* (pp. 103-124). Routledge.

Mathematics, N. C. o. T. o. (2000). *Principles and standards for school mathematics* (Vol. 1): National Council of Teachers of Mathematics Reston, VA: NCTM.

Miller, C. (2017). Quantitative research in. *Muddy Boots and Smart Suits: Researching Asia-Pacific Affairs*, 73.

Mudaly, V. (2010). Thinking with diagrams whilst writing with words. *Pythagoras, 2010*(71), 65-75.

Mudaly, V. (2013). Is Proving a Visual Act? *Online Submission, 3*(3), 36-44.

Mudaly, V., & Rampersad, R. (2010). The role of visualisation in participants' conceptual understanding of graphical functional relationships. *African Journal of Research in Mathematics, Science and Technology Education, 14*(1), 36-48.

Navarro Sada, A., & Maldonado, A. (2007). *Research Methods in Education*. - Cohen, Manion and Morrison: Taylor & Francis.

Ozgun-Koca, S. A. (1998). Students' use of representations in Mathematics education.

Palinkas, L. A., Horwitz, S. M., Green, C. A., Wisdom, J. P., Duan, N., & Hoagwood, K. (2015). Purposeful sampling for qualitative data collection and analysis in mixed method implementation research. *Administration and Policy in Mental Health and Mental Health Services Research, 42*(5), 533-544.

Patten, M. L. (2017). *Understanding research methods: An overview of the essentials*: Taylor & Francis.

Phothongsunan, S. (2015). Interpretive paradigm in educational research.

Rule, P., & John, V. (2011). *Your guide to case study research*: Van Schaik Pretoria.

Schultz, J. E., & Waters, M. S. (2000). Why representations? *The Mathematics Teacher, 93*(6), 448-453.

Schumacher, S., & McMillan, J. (2006). Research in education evidence–based inquiry. *Boston: Pearson Education*.

Seeger, F., Voigt, J., & Waschescio, U. (1998). *The culture of the mathematics classroom*: Cambridge University Press.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational researcher, 15*(2), 4-14.

Sinclair, M. (2007). A guide to understanding theoretical and conceptual frameworks. *Evidence-Based Midwifery, 5*(2), 39-40.

Stake, R. E. (1995). *The art of case study research*: Sage Publications. Starman, A. B. (2013). The case study as a type of qualitative research. *Journal of Contemporary Educational Studies/Sodobna Pedagogika, 64*(1).

Terre Blanche, M., & Durrheim, K. (1999). Histories of the present: Social science research in context. *Research in practice: Applied methods for the social sciences, 2*, 1-17.

Thompson, D. R., & Chappell, M. F. (2007). Communication and Representation as Elements in Mathematical Literacy. *Reading & Writing Quarterly, 23*(2), 179-196. doi:10.1080/10573560601158495

Tripathi, P. N. (2008). Developing mathematical understanding through multiple representations. *Mathematics Teaching in the Middle School, 13*(8), 438-445.

Trochim, W. M. (2006). Qualitative validity. *Research measures knowledge base, 361*.

Turgut, G. (2013). International tests and the US educational reforms: Can success be replicated? *The Clearing House: A Journal of Educational Strategies, Issues and Ideas, 86*(2), 64-73.

Turnuklu, E. B., & Yesildere, S. (2007). The Pedagogical Content Knowledge in Mathematics: Pre-Service Primary Mathematics Teachers' Perspectives in Turkey. *Issues in the Undergraduate Mathematics Preparation of School Teachers, 1*.

Van Dyke, F. (1994). Relating to graphs in introductory algebra. *The Mathematics Teacher, 87*(6), 427.

Weber, K. (2005). Students' understanding of trigonometric functions. *Mathematics Education Research Journal, 17*(3), 91-112.

Yin, R. (2011). Building trustworthiness and credibility into qualitative research. *Qualitative research from start to finish*, 19-21.

Yin, R. K. (2003). Case study research design and methods third edition. *Applied social research methods series, 5*.

Yin, R. K. (2009). Case study research: Design and methods. SAGE publications. *Thousand Oaks*.

Yin, R. K. (2013). *Case study research: Design and methods*: Sage publications. *Thousand Oaks*.

Zainal, Z. (2007). Case study as a research method. *Jurnal Kemanusiaan, 9*.

Zengin, Y., Furkan, H., & Kutluca, T. (2012). The effect of dynamic mathematics software geogebra on student achievement in teaching of trigonometry. *Procedia-Social and Behavioral Sciences*, *31*, 183-187.

Appendices

A1. Approval letter from KZN Department of Education

education

Department:
Education
PROVINCE OF KWAZULU-NATAL

Enquiries: Phindile Duma Tel: 033 392 1063 Ref.:2/4/8/1502

Mr M.M Khoza

Flat 9, Bluegrass
3 kings Road
Pinetown
3610

Dear Mr Khoza

PERMISSION TO CONDUCT RESEARCH IN THE KZN DoE INSTITUTIONS

Your application to conduct research entitled: **"EXPLORING HOW THE MATHEMATICAL REPRESENTATIONS INFLUENCE 10TH GRADE LEARNERS UNDERSTANDING OF TRIGONOMETRY IN A POOR PERFORMING SCHOOL"** in the KwaZulu-Natal Department of Education Institutions has been approved. The conditions of the approval are as follows:

1. The researcher will make all the arrangements concerning the research and interviews.
2. The researcher must ensure that Educator and learning programmes are not interrupted.
3. Interviews are not conducted during the time of writing examinations in schools.
4. Learners, Educators, Schools and Institutions are not identifiable in any way from the results of the research.
5. A copy of this letter is submitted to District Managers, Principals and Heads of Institutions where the Intended research and interviews are to be conducted.
6. The period of investigation is limited to the period from 10 April 2018 to 09 July 2020.
7. Your research and interviews will be limited to the schools you have proposed and approved by the Head of Department. Please note that Principals, Educators, Departmental Officials and Learners are under no obligation to participate or assist you in your investigation.
8. Should you wish to extend the period of your survey at the school(s), please contact Miss Phindile Duma at the contact numbers below.
9. Upon completion of the research, a brief summary of the findings, recommendations or a full report/dissertation/thesis must be submitted to the research office of the Department. Please address it to The Office of the HOD, Private Bag X9137, Pietermaritzburg, 3200.
10. Please note that your research and interviews will be limited to schools and institutions in KwaZulu-Natal Department of Education.

(SEE SCHOOLS LIST ATTACHED)

Nzama

Dr. EV Nzama
Head of Department: Education
Date: 10 April 2018

...Championing Quality Education - Creating and Securing a Brighter Future

KWAZULU-NATAL DEPARTMENT OF EDUCATION
Postal Address: Private Bag X9137 • Pietermaritzburg • 3200 • Republic of South Africa
Physical Address: 247 Burger Street • Anton Lembede Building • Pietermaritzburg • 3201
Tel.: +27 33 392 1063 • **Fax.:** +27 033 392 1203• **Email:**Phindile.Duma@kzndoe.gov.za •**Web:**www.kzneducation.gov.za
Facebook: KZNDOE....Twitter: @DBE_KZN....Instagram: kzn_education....Youtube:kzndoe

education

Department:
Education
PROVINCE OF KWAZULU-NATAL

LIST OF SCHOOLS:

1. [redacted]

...Championing Quality Education - Creating and Securing a Brighter Future

KWAZULU-NATAL DEPARTMENT OF EDUCATION
Postal Address: Private Bag X9137 • Pietermaritzburg • 3200 • Republic of South Africa
Physical Address: 247 Burger Street • Anton Lembede Building • Pietermaritzburg • 3201
Tel.: +27 33 392 1063 • **Fax.:** +27 033 392 1203• **Email:**Phindile.Duma@kzndoe.gov.za •**Web:**www.kzneducation.gov.za
Facebook: KZNDOE....Twitter: @DBE_KZN....Instagram: kzn_education....Youtube:kzndoe

A2. UKZN Ethical Clearance

UNIVERSITY OF KWAZULU-NATAL
INYUVESI YAKWAZULU-NATALI

3 November 2017

Mr Mfundo Mondli Khoza 209536877
School of Education
Edgewood Campus

Dear Mr Khoza

Protocol reference number: HSS/0014/017M
Project title: Exploring how the use of mathematical representations influence 10th Grade learners understanding of Trigonometry in a poor performing school

Full Approval – Expedited Application

In response to your application received on 19 December 2016, the Humanities & Social Sciences Research Ethics Committee has considered the abovementioned application and **FULL APPROVAL** for the protocol has been granted.

Any alteration/s to the approved research protocol i.e. Questionnaire/Interview Schedule, Informed Consent Form, Title of the Project, Location of the Study, Research Approach and Methods must be reviewed and approved through the amendment/modification prior to its implementation. In case you have further queries, please quote the above reference number.

PLEASE NOTE: Research data should be securely stored in the discipline/department for a period of 5 years.

The ethical clearance certificate is only valid for a period of 3 years from the date of issue. Thereafter Recertification must be applied for on an annual basis.

I take this opportunity of wishing you everything of the best with your study.

Yours faithfully

....

Dr Shamila Naidoo (Deputy Chair)
Humanities & Social Sciences Research Ethics Committee

/pm

cc Supervisor: Dr Vimolan Mudaly
cc. Academic Leader Research: Dr SB Khoza
cc. School Administrator: Ms Tyzer Khumalo

Humanities & Social Sciences Research Ethics Committee
Dr Shenuka Singh (Chair)
Westville Campus, Govan Mbeki Building
Postal Address: Private Bag X54001, Durban 4000
Telephone: +27 (0) 31 260 3587/8350/4557 **Facsimile:** +27 (0) 31 260 4609 **Email:** ximbap@ukzn.ac.za / snymanm@ukzn.ac.za / mohunp@ukzn.ac.za
Website: www.ukzn.ac.za

1910 - 2010

UNIVERSITY OF KWAZULU-NATAL
INYUVESI YAKWAZULU-NATALI

The Principal
Request for permission to conduct research at your school

My name is Mfundo Khoza, I am a Master of Educations (Masters) student at the University of KwaZulu-Natal Edgewood campus. I am currently engaged in a research project titled, Exploring the influence of mathematical representations on 10^{th} grade participants' understanding of trigonometry in a poorly performing school.

The purpose of this project is to engage participants in teaching practice in which representations are in use to cater for individual preferences and differences among participants. In this project, participants will be engaged in different tasks where they will investigate the effect of changing the values of parameters of Trigonometric functions. This is important in showing participants that the same change has the same effect on many function and also in helping participants to sketch graphs using the properties of the functions rather than using a tables. Participants will also get an opportunity to be hands-on with Mathematics software (GeoGebra) to draw different Trigonometry graphs and manipulate the software to understand the effects of parameters in trig graphs.The results of this study have the potential to inform discussions around the teaching and learning of trigonometry at the secondary school, particularly on how representations can be used in different topics in the school curriculum.

I hereby request to conduct my research with Grade 10 participants at your school. I would like to collect data from Grade 10 participants using multiple methods of data collection. These include Complete 3 Investigations on effect of changing the values of parameters, where participants are expected to spend at least 60 minutes to work on each investigation; manipulate the software GeoGebra and reflect on their experience/s for a period of 3 days. There will be semi-structured interviews which will help in getting insight of what participants will be reflecting about.

This study is purely for academic purposes and there will be no financial gain involved. The importance of this study is that it is expected that participants in grade 10 will get an insight of how they can use different learning aids to improve their understanding in mathematics. In a broader view, the results of this study have the potential to inform discussions around the teaching and learning of trigonometry in secondary schools, particularly on how representations can be used in different topics in the school curriculum.

You are assured that the findings of this research will not be used for any other purpose other than the Masters dissertation. In this regard, no harm will be caused and the participants participating in this project. This study will ensure that anonymity of both school and participants is ensured by using pseudonyms to protect your school and participants.

The decision to participate in this study is entirely voluntary and you may withdraw your permission for the research without any negative consequences. If you have any further questions about the study, at any time feel free to contact me. Should you have any other concerns about your rights as research participant, you may contact my supervisor, Prof. Vimolan Mudaly.

Thank You

Yours faithfully

Mfundo Mondli Khoza

Student number: 209536877

Email: 209536877@stu.ukzn.ac.za / mfundo.khoza@gmail.com

Supervisor:
Prof. Vimolan Mudaly, Phone: 0312603682; Fax: 0312603697 or Email: mudalyv@ukzn.ac.za

Research Ethics:
Ms Phume Ximba from the Research office may also be contacted. Her details are:

University of KwaZulu-Natal
Research Ethics Offices: HSSREC
Private Bag x54001, Durban, 4000
Telephone +2731 260 3587
Email: ximbap@ukzn.ac.za: email to : ximbap@ukzn.ac.za
Email: HssrecHumanities@ukzn.ac.za / email to: HssrecHumanities@ukzn.ac.za

....

Acknowledgement by the principal

I________________________________,the Principal of _____________________________________ grant permission to Mfundo Mondli Khoza to conduct his research in the above mentioned school.

_______________________________________ _______________________________________

Signature of principal Date

A4. Consent letter to parents/ guardians for each participating learner

UNIVERSITY OF KWAZULU-NATAL
INYUVESI YAKWAZULU-NATALI

PARENTAL CONSENT FORM FOR CHILD PARTICIPATION IN RESEARCH

Title of research: Exploring the influence of mathematical representations on 10th grade participants' understanding of trigonometry in a poorly performing school.

I ….. hereby consent to my child ....
*Parent/Guardian's Name (*please print*) *Child Participant Name (*please print*)

Participating, as requested, in the school for the research project on t*he use of Mathematical representations to enhance understanding of Trigonometric functions in the 10th grade.*

1. I have read the information provided.
2. Details of procedures and any risks have been explained to my satisfaction.
3. I agree to my child's involvement in the research and that my child's information may be used
4. I am aware that I should retain a copy of the Information Sheet and Consent Form for future reference
5. I understand that:
   - ✓ My child may not directly benefit from taking part in this research.
   - ✓ My child is free to withdraw from the project at any time and is free to decline to answer particular questions.
   - ✓ While the information gained in this study will be published as explained, my child will not be identified, and individual information will remain confidential.
   - ✓ Whether my child participates or not, or withdraws after participating, will have no effect on any treatment or service that is being provided to him/her.
   - ✓ Whether my child participates or not, or withdraws after participating, will have no effect on his/her progress in his/her course of study, or results gained.
   - ✓ My child may ask that the interviews / task based activities and observation be stopped at any time, and he/she may withdraw at any time from the session or the research without disadvantage.
6. **I agree/do not agree** to the all information obtained from my child being made available to other researchers who are not members of this research team, but who are judged by the research team to be doing related research, on condition that my identity is not revealed.

Parents/ guardian's signature…………………………… Date…………………........................

I certify that I have explained the study to the participant/s and consider that she/he understands what is involved and freely consents to participation.

Researcher's name: Mfundo Mondli Khoza

Researcher's signature………………………………..... Date………………………………………

7. I, the participant whose signature appears below, have read a transcript of my participation and agree to its use by the researcher as explained.

Participant's signature…………………………………… Date……………………………………………...

A5. Learner assent form to participate in the study

UNIVERSITY OF KWAZULU-NATAL
INYUVESI YAKWAZULU-NATALI

LEARNER ASSENT FORM TO PARTICIPATE IN A RESEARCH PROJECT

Title of research: Exploring the influence of mathematical representations on 10th grade participants' understanding of trigonometry in a poorly performing school.

Researcher : Mr. Mfundo Mondli Khoza

Dear Participant / Learner

My name is Mfundo Khoza, I am a Master of Educations (Masters) student at the University of KwaZulu-Natal Edgewood campus. I am currently engaged in a research project titled, Exploring the influence of mathematical representations on 10th grade participants' understanding of trigonometry in a poorly performing school.

The purpose of this project is to engage you as a learner in teaching practice in which representations are in use to cater for individual preferences and differences among participants. In this project, you will be engaged in different tasks where you will investigate the effect of changing the values of parameters of Trigonometric functions. This is important in understanding that the same change has the same effect on many function and also in helping you to sketch graphs using the properties of the functions rather than using tables. You will also get an opportunity to be hands-on with Mathematics software (GeoGebra) to draw different Trigonometry graphs and manipulate the software to understand the effects of parameters in trigonometric graphs. The results of this study have the potential to inform discussions around the teaching and learning of trigonometry at the secondary school, particularly on how representations can be used in different topics in the school curriculum.

I hereby request you, the learner to participate in my research with other grade 10 participants at your school. I would like to collect data from Grade 10 participants using multiple methods of data collection. These include completing 3 Investigations on effect of changing the values of parameters, where participants are expected to spend at least 60 minutes to work on each investigation; manipulate the software GeoGebra and reflect on their experience/s for a period of 3 days. There will be semi-structured interviews which will help in getting insight of what participants will be reflecting about.

This study is purely for academic purposes and there will be no financial gain involved. The importance of this study is that it is expected that participants in grade 10 will get an insight of how they can use different learning aids to improve their understanding in mathematics.

You are assured that the findings of this research will not be used for any purpose other than the Masters dissertation. In this regard, no harm will be caused to the participants participating in this project. This study will ensure that anonymity of both school and participants by using pseudonyms to protect your school and identity of participants involved.

The decision to participate in this study is entirely voluntary and you may withdraw your permission for the research without any negative consequences. If you have any further questions about the study, at any time feel free to contact me. Should you have any other concerns about your rights as research participant, you may contact my supervisor Prof. Vimolan Mudaly. When we are finished with this study we will write a report about what was learned. This report will not include your name or that you were in the study. You do not have to be in this study if you do not want to be. If you decide to stop after we begin, that's okay too. Your parents know about the study too.

Thank You

Yours faithfully

Mfundo Mondli Khoza
Student number: 209536877
Email: 209536877@stu.ukzn.ac.za / mfundo.khoza@gmail.com

Supervisor:
Prof. Vimolan Mudaly,
Phone: 0312603682 Fax: 0312603697 or
Email: mudalyv@ukzn.ac.za

Research Ethics:
Ms Phume Ximba from the Research office may also be contacted. Her details are:

University of KwaZulu-Natal
Research Ethics Offices: HSSREC
Private Bag x54001, Durban, 4000
Telephone +2731 260 3587
Email: ximbap@ukzn.ac.za: email to : ximbap@ukzn.ac.za
Email: HssrecHumanities@ukzn.ac.za / email to: HssrecHumanities@ukzn.ac.za

………

LEARNER ASSENT TO PARTICIPATE IN THE RESEARCH PROJECT

I ____ (name and surname of learner), aged ______ years (add age), hereby assent/do not assent to participate in the Research project titled 'Exploring the influence of mathematical representations on 10th grade participants' understanding of trigonometry in a poorly performing school'.I understand that participation in this research project is voluntary.

______________________ ______________________________

Print Name and Surname Signature: Learner

A6. Data collection Tool

INVESTIGATING TRIGONOMETRIC FUNCTIONS USING MATHEMATICAL REPRESENTATIONS

Name : ___________

Surname: ___________

Grade/ Division: _____

Content s

1. My learning experience in Mathematics "How they teach it & how I learnt it"

1. What have you learnt in today's lesson

2. How is it similar or different to your normal learning experience in mathematics?

3. How likely are you to recommend or discommend the use of geogebra as a learning tool in mathematics?

4. Do you think using geogebra in your learning can help you to understand Trigonometric functions better? Explain why/why not.

5. Do you think different ways to solve each math problem helps you to understand each problem better? Explain your answer.

6. How often does your teacher do math work on computers during math class?

2. Investigation 1: The effect of parameter $'a'$ in $f(x) = a \sin x; f(x) = a \cos x$ & $f(x) = a \tan x$

2.1 ***Investigation 1A :The effect of parameter $'a'$ in $f(x) = a \sin x$ where $'a'$ is any real number***

1. Complete the following table. Use your calculator to read off the required ratios rounded off to one decimal place.

	0°	30°	60°	90°								330°	360°
$y = \sin x$													
$y = 2 \sin x$													
$y = - \sin x$													
$y = -2 \sin x$													

2. Draw the graphs of the functions $y = \sin x$; $y = 2 \sin x$; $y = - \sin x$; $y = -2 \sin x$ on the same system of axes. The squared paper must be arranged in landscape format. The scale on the $\boldsymbol{y}$ **- axis** should be **4 blocks represent 1 unit** and the scale on the $\boldsymbol{x}$ **-axis** should be **two blocks represents 30**°. Label your graphs clearly.

3. What are the values of $'\boldsymbol{a}'$ in each of these graphs?

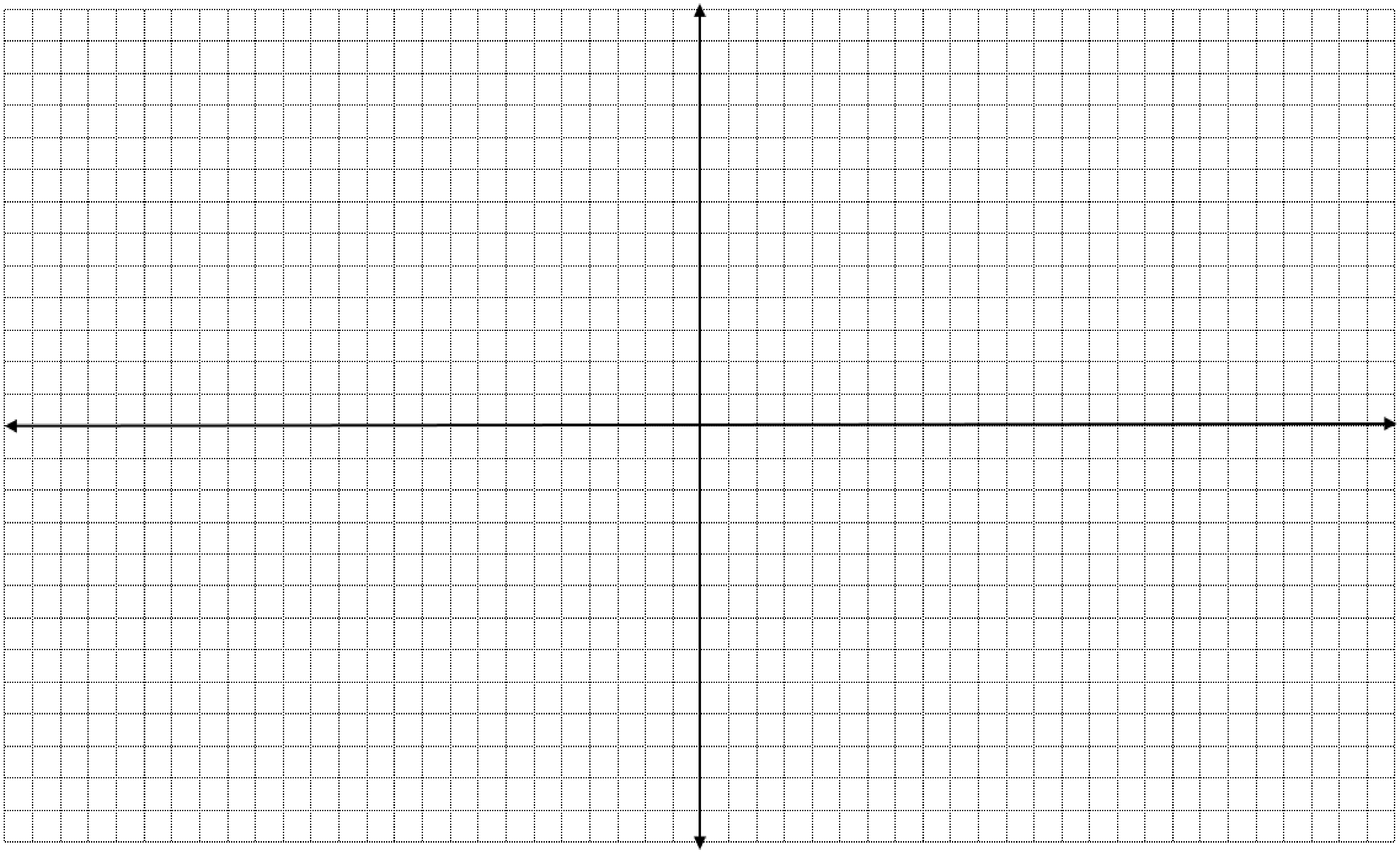
4. Describe in your own words the effect of $'\boldsymbol{a}'$ in the graph of $y = \boldsymbol{a} \sin x$

5. Complete the table below:

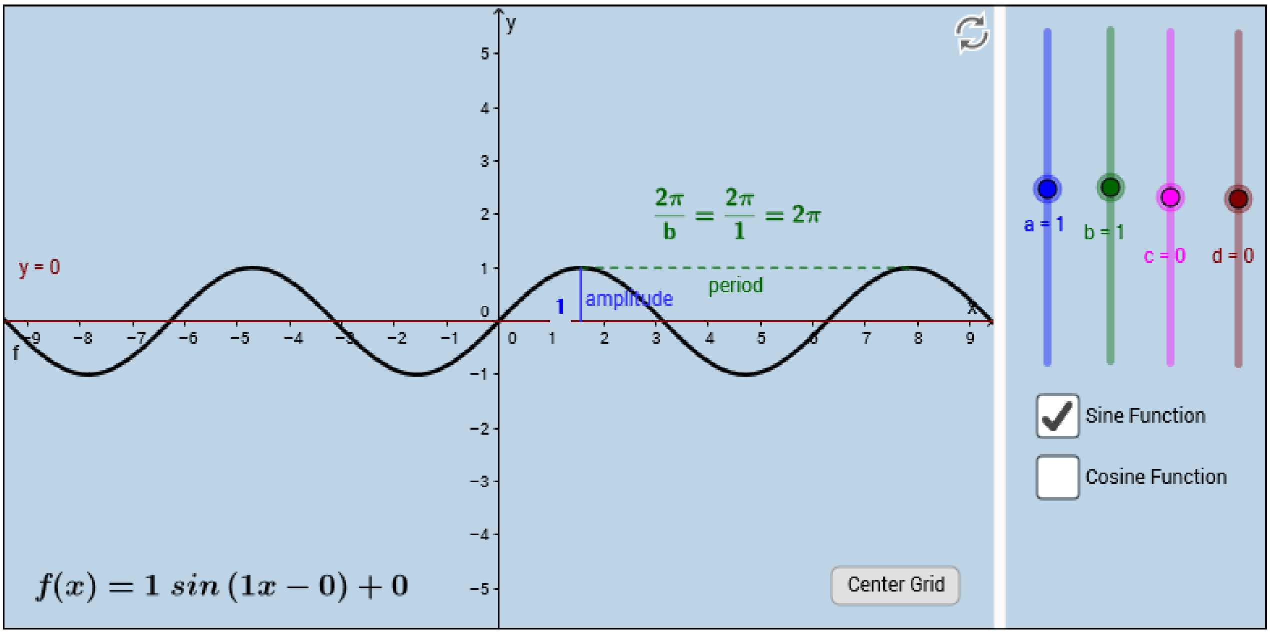
	$y = \sin x$	$y = 2 \sin x$	$y = - \sin x$	$y = -2 \sin x$
Domain				
Range				
Amplitude				
Period				

6. What transformation is affected by the parameter $'\boldsymbol{a}'$ in the graphs above? Explain your answer.

GRAPHS FOR INVESTIGATION 1A



7. Now, use the link below to access GeoGebra & Interact with the applet for a few minutes changing the value of $\boldsymbol{a}$,then answer the questions that follows:
https://www.geogebra.org/m/RVTxQ6Vm#material/znb4GNk7 or http://ggbm.at/fatp4dkx



7.1 What is the value of $\boldsymbol{a}$ for this parent **sine function**? What is the **amplitude** the graph?

7.2 What does the parameter $'\boldsymbol{a}'$ do to the graph of the function $f(x) = \ sin\, x$ under the transformation $y = a^{*}sin(bx - c) + d$? Explain.

7.3 Is there any relationship between parameter $'\boldsymbol{a}'$ and the **amplitude**? Explain.

7.4 What do you think is the effect of $'a'$ in the graph of $y = \ a\ cos\ x$. Use the examples of: $y = 2\ cos\ x$ and $y = \ -2\ cos\ x$ to explain your conjecture.

2.2 ***Investigation 1B: The effect of parameter $'a'$ in $f(x) = a\cos x$ where $'a'$ is any real number.***

1. Complete the following table. Use your calculator to read off the required ratios rounded off to one decimal place.

	0°	30°	60°	90°								330°	360°
$y = \cos x$													
$y = 4\cos x$													
$y = -\cos x$													
$y = -4\cos x$													

2. Draw the graphs of the functions $y = \cos x$; $y = 2\cos x$; $y = -\cos x$; $y = -2\cos x$ on the same system of axes. The squared paper must be arranged in landscape format. The scale on the $\boldsymbol{y}$ **- axis** should be **4 blocks represent 1 unit** and the scale on the $\boldsymbol{x}$ **-axis** should be **two blocks represents 30°**. Label your graphs clearly.

3. What are the values of $'\boldsymbol{a}'$ in each of these graphs?

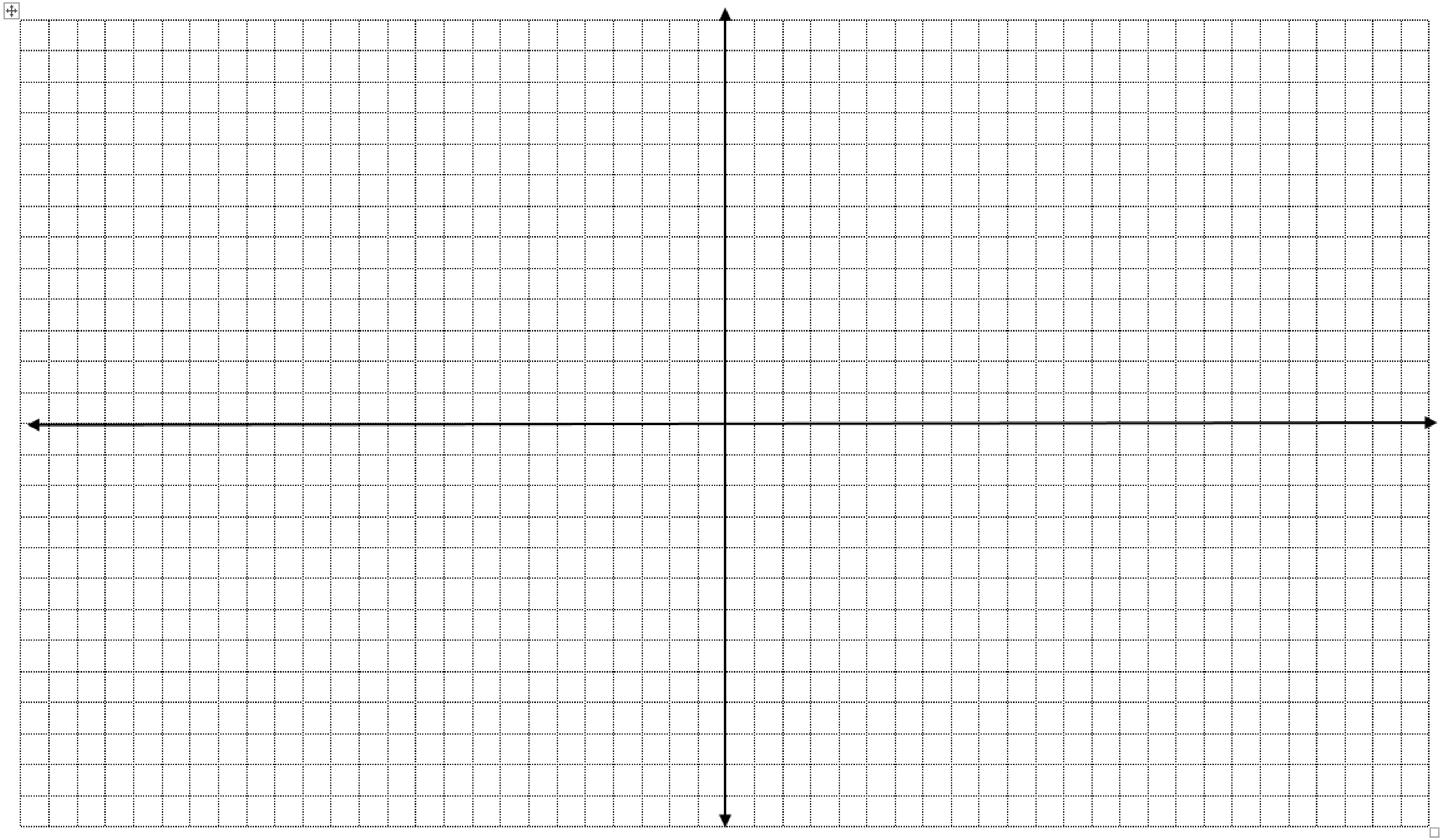
4. Describe in your own words the effect of $'\boldsymbol{a}'$ in the graph of $y = \boldsymbol{a}\cos x$

5. Complete the table below:

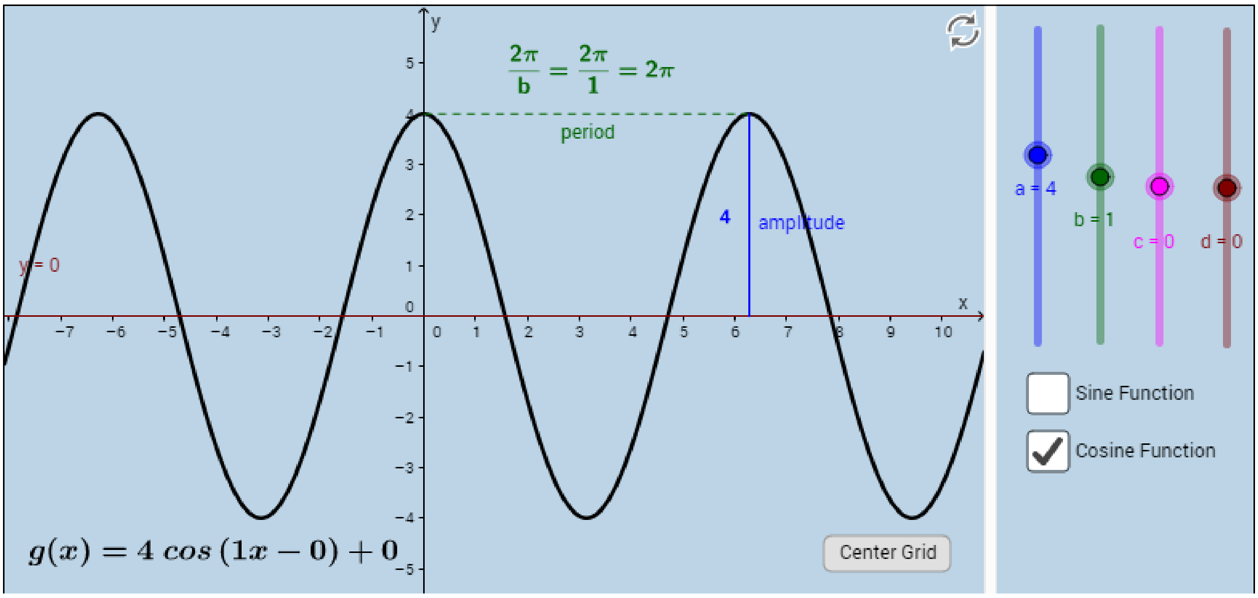
	$y = \cos x$	$y = 4\cos x$	$y = -\cos x$	$y = -4\cos x$
Domain				
Range				
Amplitude				
Period				

6. What transformation is affected by the parameter $'\boldsymbol{a}'$ in the graphs above? Explain your answer.

GRAPHS FOR INVESTIGATION 1B



7. Now, use the link below to access GeoGebra & Interact with the applet for a few minutes changing the value of $'\boldsymbol{a}'$,then answer the questions that follows:
https://www.geogebra.org/m/RVTxQ6Vm#material/znb4GNk7 or http://ggbm.at/fatp4dkx



7.1 What is the value of $'\boldsymbol{a}'$ for this parent **cosine function**? What is the a**mplitude** the graph?

7.2 What does the parameters $'\boldsymbol{a}'$ do to the graph of the function $f(x) = \ cos\ x$ under the transformation $y = a^{*}cos(bx - c) + d$? Explain.

7.3 Is there any relationship between parameter $'\boldsymbol{a}$' and the **amplitude**? Explain.

7.4 What do you think is the effect of $'a'$ in the graph of $y = \ a\ tan\ x$. Use the examples of: $y = 2\ tan\ x$ and $y = \ -2tan\ x$ to explain your conjecture.

2.3 ***Investigation 1C: The effect of parameter*** $'a'$ ***in*** $f(x) = a\tan x$ **.**

1. Complete the following table. Use your calculator to read off the required ratios rounded off to one decimal place.

	0°	45°	90°	135°	180°	225°	270°	315°	360°
$y = tanx\ x$									
$y = 2\ tanx$									
$y = -\ tan\,x$									
$y = -2\ tan\,x$									

2. Draw the graphs of the functions $y = tan\,x$; $y = 2\ tan\,x$; $y = -\ tan\,x$; $y = -2\ tan\,x$ on the same system of axes. The squared paper must be arranged in landscape format. The scale on the $\boldsymbol{y}$ **- axis** should be **4 blocks represent 1 unit** and the scale on the $\boldsymbol{x}$ **-axis** should be **two blocks represents 45**°. Label your graphs clearly.

3. What are the values of $'\boldsymbol{a}'$ in each of these graphs?

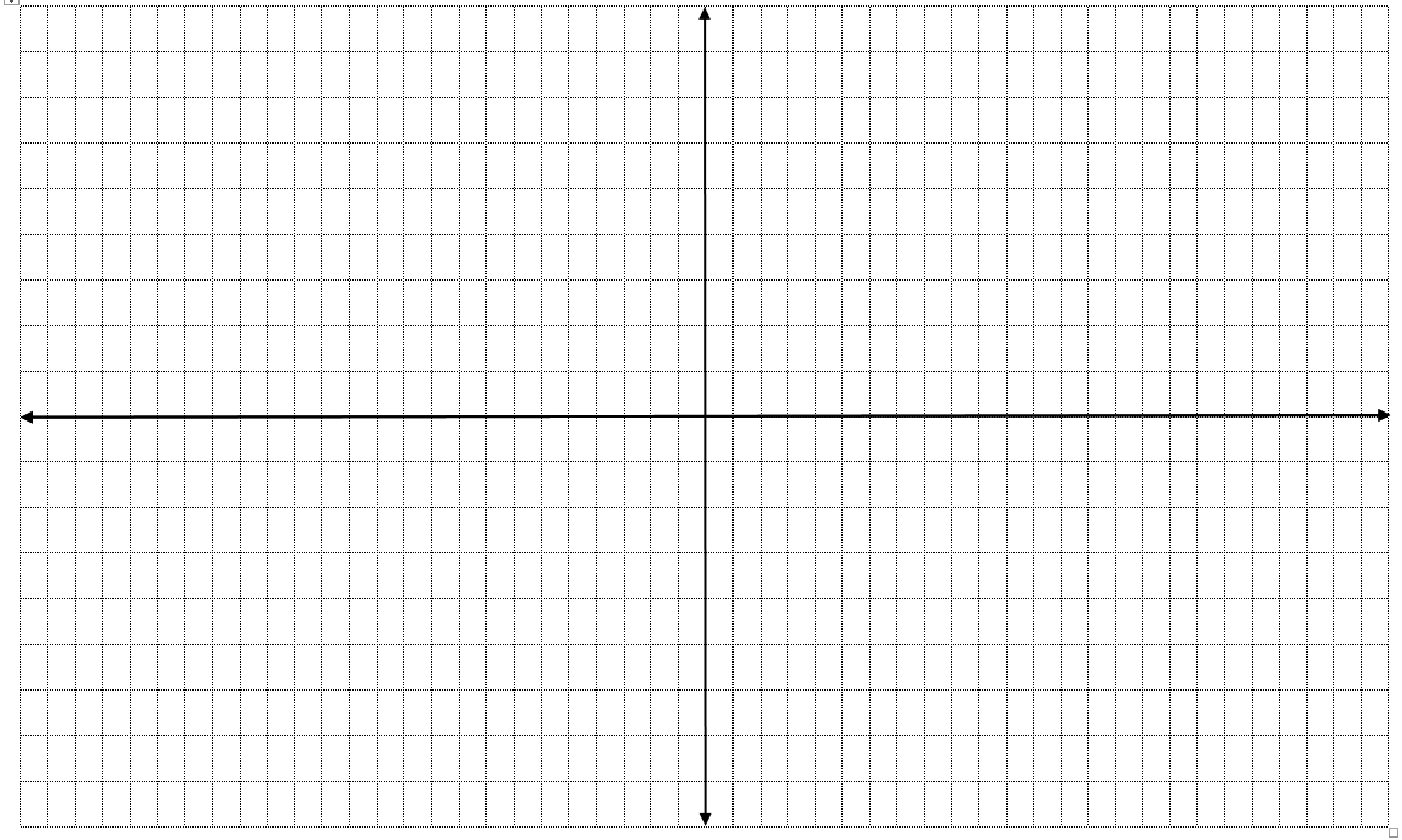
4. Describe in your own words the effect of $'\boldsymbol{a}'$ in the graph of $y = \boldsymbol{a}\ tan\,x$

5. Complete the table below:

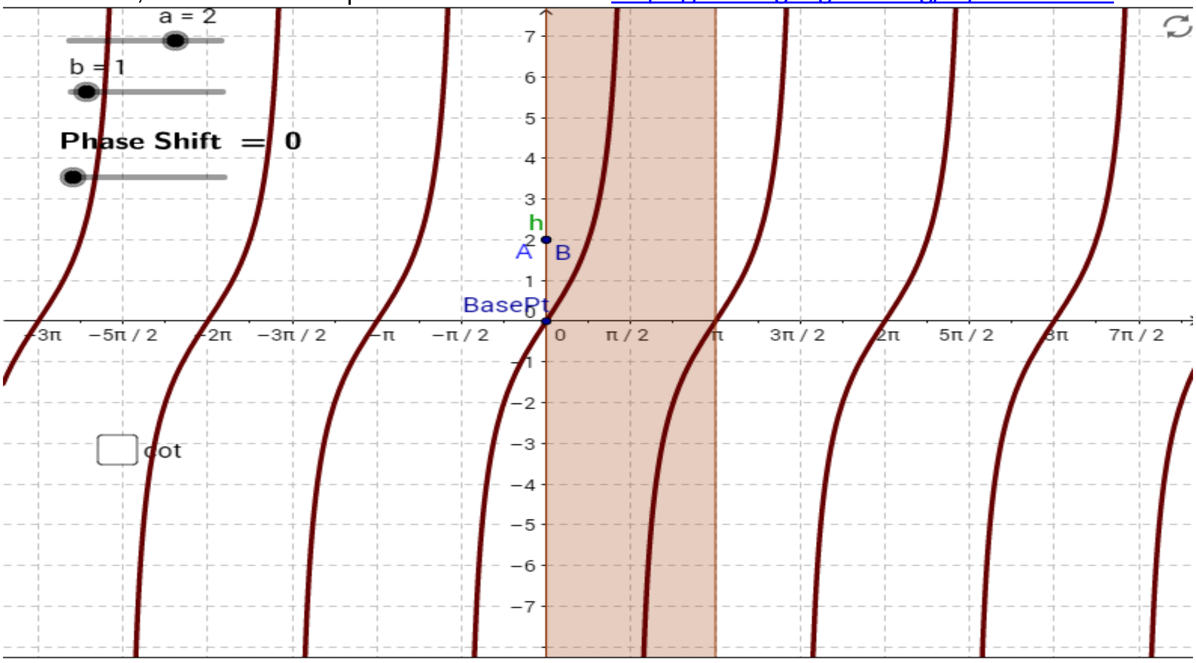
	$y = tan\,x$	$y = 2\ tan\,x$	$y = -\ tan\,x$	$y = -2tan\,x$
Domain				
Range				
Amplitude				
Period				

6. What transformation is affected by the parameter $'\boldsymbol{a}'$ in the graphs above? Explain your answer.

GRAPHS FOR INVESTIGATION 1C



7. Now, use the link below to access GeoGebra & Interact with the applet for a few minutes changing the value of $'\boldsymbol{a}'$,then answer the questions that follows: https://www.geogebra.org/m/d5wBD7U8



7.1 What is the values of $'\boldsymbol{a}'$ for this parent **tangent function**? What is the **amplitude** of the graph?

7.2 What does the parameter $'\boldsymbol{a}'$ do to the graph of the function $f(x) = tan\, x$ under the transformation $y = a^{*}tan(bx - c) + d$? Explain.

7.3 Is there any relationship between parameter $'\boldsymbol{a}'$ and the **amplitude**? Explain.

7.4 What difference can you think is the effect of $'a'$ in the graph of $y = \ a \tan x$. Use the examples of: $y = 2\ tan\ x$ and $y = \ -2tan\ x$ to explain your conjecture compared to effect of a in the graph of $y = a\ cos\ x$.

Investigation 1: Reflection

3. Reflection on my learning about the effect of parameter $'a'$ in trigonometric Functions

3.1 Colour in the arrow, up to the statement which best describes your current understanding.

	I am so confident – I could explain this to someone else.
	I can get the right answer but I don't understand well enough to explain it yet.
	I understand some of this but I don't understand all of it yet.
	I tried hard and listened but I am finding this challenging. I will make sure that I get help with this in the next lesson.
	I do not understand any of this yet. There are things I could do to be a better learner next lesson.

3.2 **A mistake that moved my learning on**

This was my mistake….

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Now I have learnt that….

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3.3. Representation/s that helped me to understand the effect of parameter '**a**' in

4. *Investigation 2: The effect of $'q'$ in $f(x) = \sin x + q$; $f(x) = \cos x + q$; $f(x) = \tan x + q$ where $'q'$ is any real number.*

4.1 Investigation 2A :The effect of parameter $'q'$ in $f(x) = \sin x + q$ where 'q' is any real number

1. Complete the following table. Use your calculator to read off the required ratios rounded off to one decimal place.

	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$y = \sin x$													
$y = \sin x + 1$													
$y = \sin x - 2$													
$y = \sin x - \frac{1}{2}$													

2. Draw the graphs of the functions $y = \sin x$; $y = \sin x + 1$; $y = \sin x - 2$; -2 $y = \sin x - \frac{1}{2}$ on the same system of axes. The squared paper must be arranged in landscape format. The scale on the $\boldsymbol{y}$ **- axis** should be **4 blocks represent 1 unit** and the scale on the $\boldsymbol{x}$ **-axis** should be **two blocks represents 30**°. Label your graphs clearly.

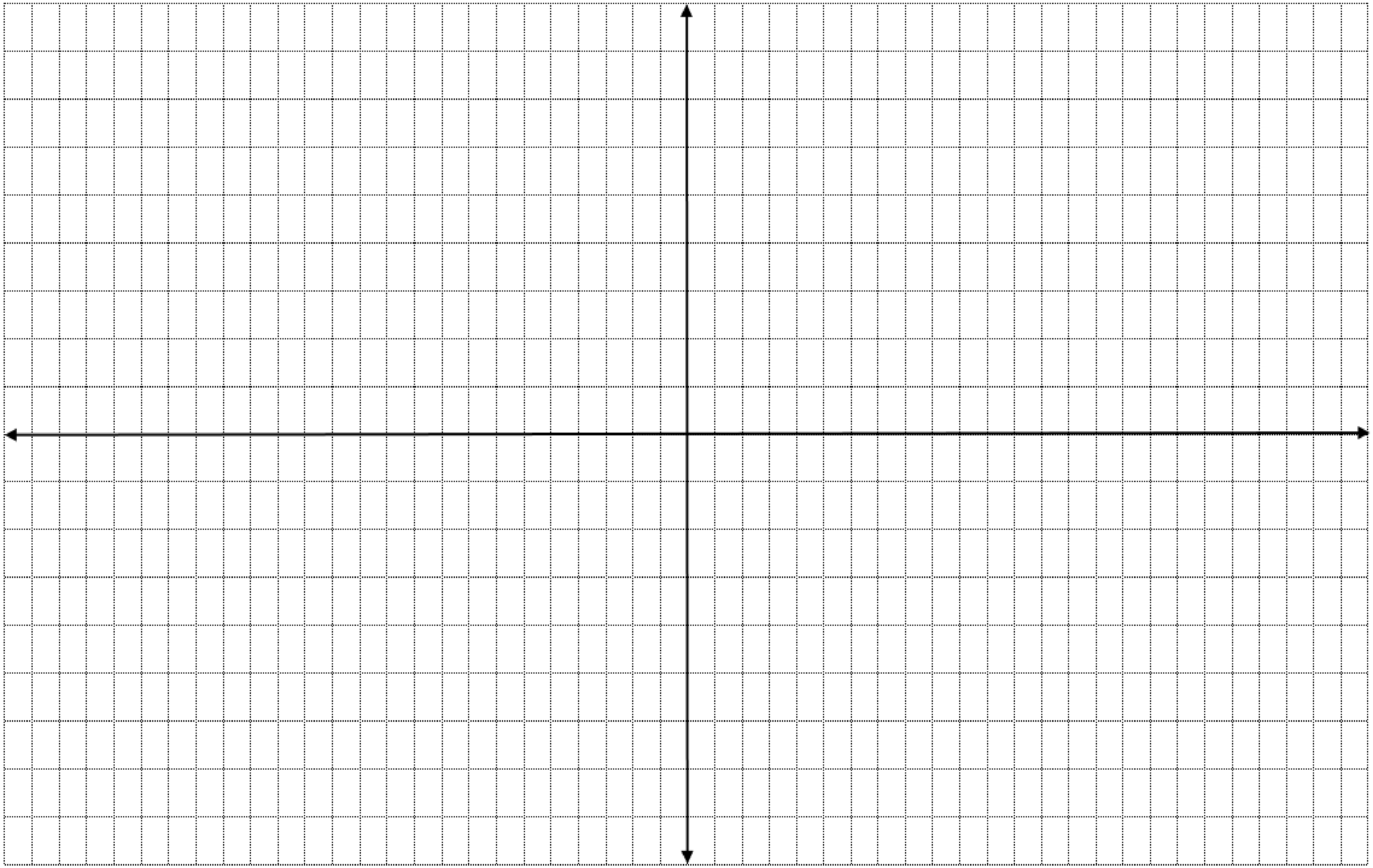
3. What are the values of $'\boldsymbol{q}'$ in each of these graphs? Describe the effect of $'\boldsymbol{q}'$ in the graph of $y = \sin x + q$

4. Complete the table below:

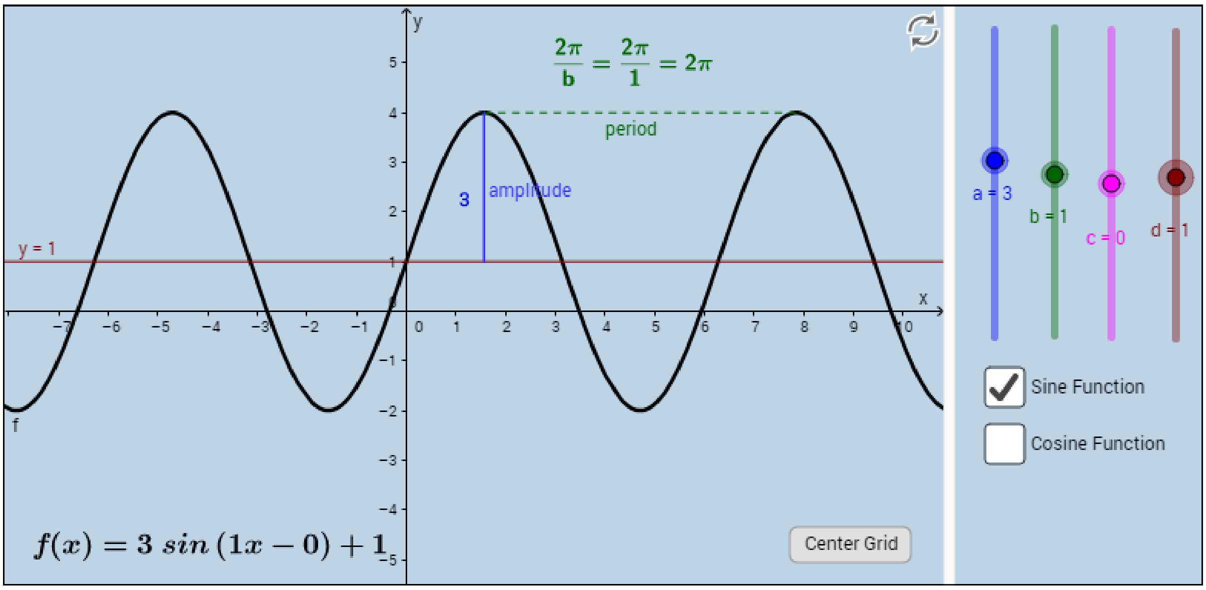
	$y = \sin x$	$y = \sin x + 1$	$y = \sin x - 2$	$y = \sin x - \frac{1}{2}$
Domain				
Range				
Amplitude				
Period				

5. What transformation is affected by the parameter $'\boldsymbol{q}'$ in the graphs above? Explain your answer.

GRAPHS FOR INVESTIGATION 2A



6. Now, use the link below to access GeoGebra & Interact with the applet for a few minutes changing the value of $'\boldsymbol{d}'$,then answer the questions that follows:
https://www.geogebra.org/m/RVTxQ6Vm#material/znb4GNk7 or http://ggbm.at/fatp4dkx



6.1 What is the value of $'d'$ for this parent **sine function**? What is the **Range** of the graph?

6.2 What does the parameter $'\boldsymbol{d}'$ do to the graph of the function $f(x) = sin\, x$ under the transformation $y = asin(bx - c) + d^*$? Explain.

6.3 Is there any relationship between parameter $'\boldsymbol{d}'$ and the **Range** ? Explain.

6.4 Is there any relationship between parameter $'\boldsymbol{d}'$ and the **Domain** ? Explain.

4.2 Investigation 2B: The effect of parameter $'q'$ in $\mathbf{f(x) = \cos x + q}$ where q'' is any real number.

1. Complete the following table. Use your calculator to read off the required ratios rounded off to one decimal place.

	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$y = cos\, x$													
$y = cos\, x + 2$													
$y = cos\, x - 2$													
$y = -cos + 2$													

2. Draw the graphs of the functions $y = cos\, x$; $y = cos\, x + 2$; $y = cos\, x - 2$; $y = -\, cos\, x + 2$ on the same system of axes. The squared paper must be arranged in landscape format. The scale on the $\boldsymbol{y}$ **- axis** should be **4 blocks represent 1 unit** and the scale on the $\boldsymbol{x}$ **-axis** should be **two blocks represents 30**°. Label your graphs clearly.

3. What are the values of 'q' in each of these graphs?

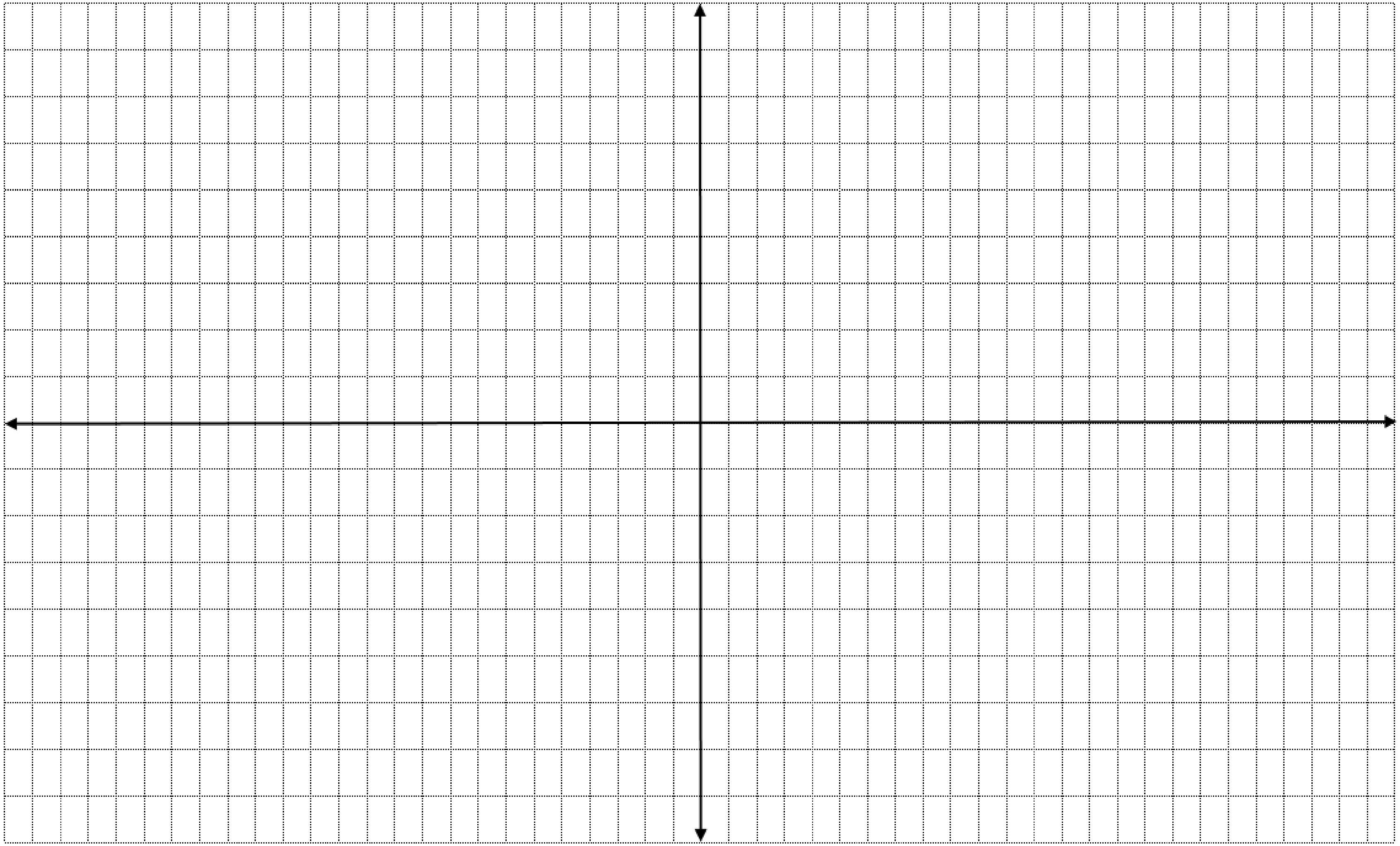
4. Describe in your own words the effect of $'\boldsymbol{q}'$ in the graph of $y = \ cos\, x + \boldsymbol{q}$

5. Complete the table below:

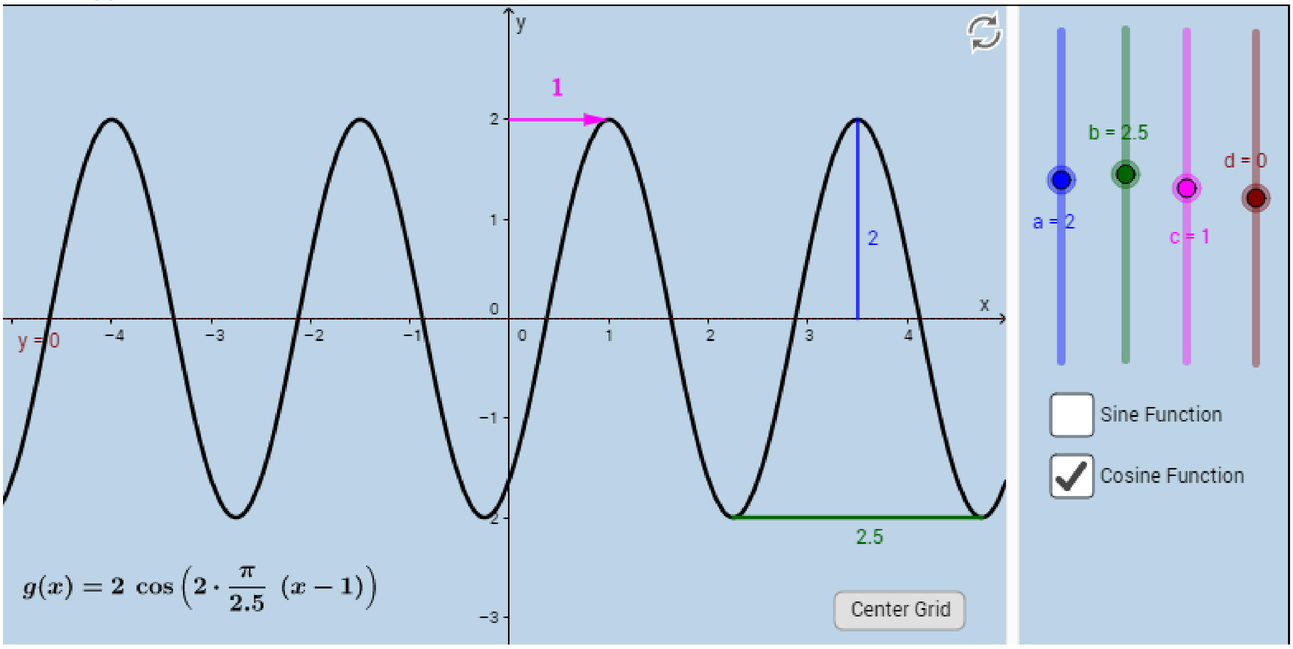
	$y = cos\, x$	$y = cos\, x + 2$	$y = cos\, x - 2$	$y = -cos\, x + 2$
Domain				
Range				
Amplitude				
Period				

6. What transformation is affected by the parameter $'\boldsymbol{q}'$ in the graphs above? Explain your answer.

GRAPHS FOR INVESTIGATION 2B



7. Now, use the link below to access GeoGebra & Interact with the applet for a few minutes changing the value of $'\boldsymbol{d}'$,then answer the questions that follows: https://www.geogebra.org/m/RVTxQ6Vm#material/znb4GNk7 or http://ggbm.at/fatp4dkx



7.1 What is the value of $'\boldsymbol{d}'$ for this parent **cosine function**? What is the **range** of the graph?

7.2 What does the parameters '$\boldsymbol{d}$' do to the graph of the function $f(x) = \ cos\,x$ under the transformation $y = acos(bx - c) + d^*$? Explain.

7.3 Is there any relationship between parameter $'\boldsymbol{d}'$ and the **range**? Explain.

7.4 Is there any relationship between parameter $'\boldsymbol{d}'$ and the **domain** ? Explain.

4.3 Investigation 2C: The effect of parameter 'q' in $f(x) = \tan x + q$ where $'q'$ is any real number.

1. Complete the following table. Use your calculator to read off the required ratios rounded off to one decimal place.

	0°	45°	90°	135°	180°	225°	270°	315°	360°	0°
$y = tanx\ x$										
$y = tanx + 2$										
$y = -\tan x$										
$y = -\tan x - 2$										

2. Draw the graphs of the functions $y = \tan x$; $y = 2\tan x$; $y = -\tan x$; $y = -2\tan x$ on the same system of axes. The squared paper must be arranged in landscape format. The scale on the $\boldsymbol{y}$ **- axis** should be **4 blocks represent 1 unit** and the scale on the $\boldsymbol{x}$ **-axis** should be **two blocks represents 45**°. Label your graphs clearly.

3. What are the values of $'\boldsymbol{q}'$ in each of these graphs?

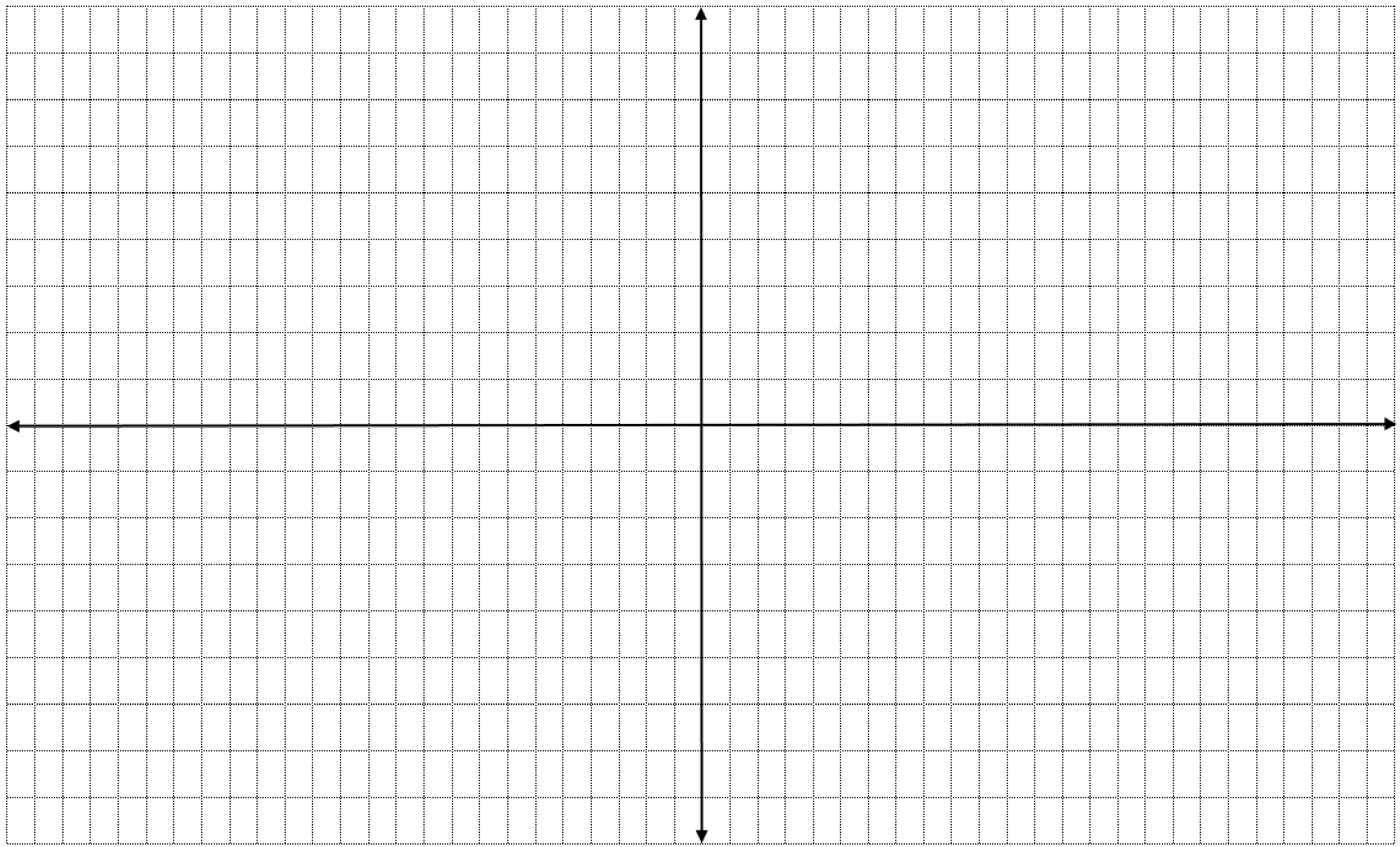
4. Describe in your own words the effect of $'\boldsymbol{q}'$ in the graph of $y = \tan x + q$

5. Complete the table below:

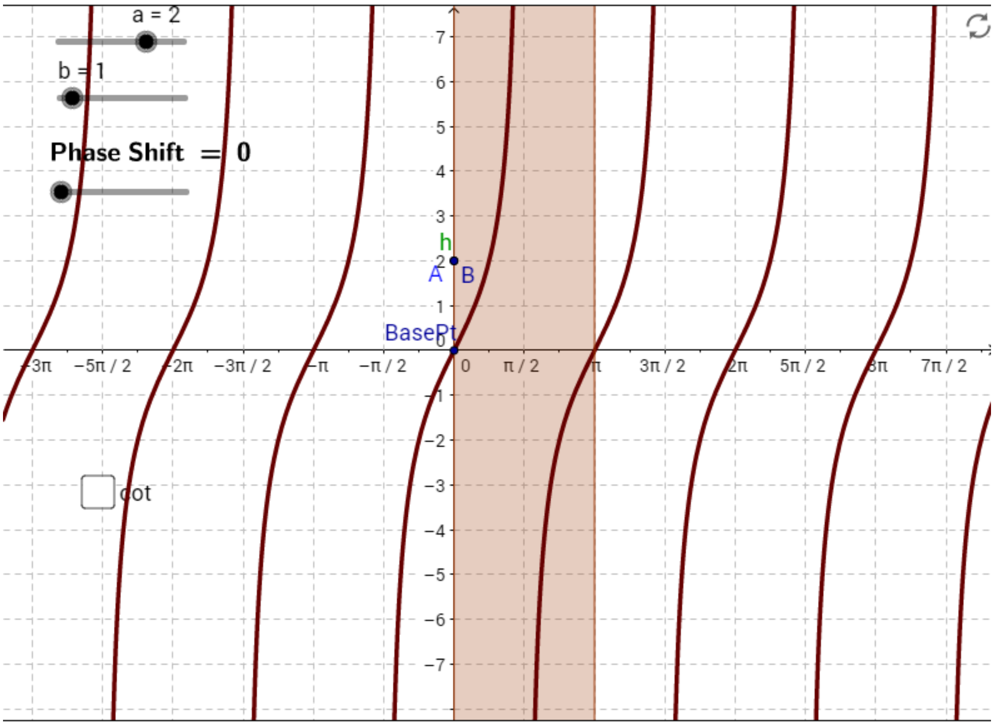
	$y = \tan x$	$y = \tan x + 2$	$y = -\tan x$	$y = -\tan x - 2$
Domain				
Range				
Amplitude				
Period				

6. What transformation is affected by the parameter $'\boldsymbol{q}'$ in the graphs above? Explain your answer.

GRAPHS FOR INVESTIGATION 2C



7. Now, use the link below to access GeoGebra & Interact with the applet for a few minutes changing the value of $'d'$,then answer the questions that follows: https://www.geogebra.org/m/d5wBD7U8



7.1 What is the value of $'\boldsymbol{d}'$ for this parent **tangent function**? What is the **amplitude** of the graph?

7.2 What does the parameters 'd' do to the graph of the function $f(x) = \tan x$ under the transformation $y = a\tan(bx - c) + d^*$? Explain.

7.3 Is the any relationship between parameter $'\boldsymbol{d}'$ and the **range**? Explain.

7.4 Is there any difference in amplitude and range that you noticed in the graph *of y = sin x; y = cos x and y= tan x*? Explain.

Investigation 2: Reflection

5. Reflection on my learning about the effect of *parameter 'q' in trigonometric Functions*

Colour in the arrow, up to the statement which best describes your current understanding.

	I am so confident – I could explain this to someone else.
	I can get the right answer but I don't understand well enough to explain it yet.
	I understand some of this but I don't understand all of it yet.
	I tried hard and listened but I am finding this challenging. I will make sure that I get help with this in the next lesson.
	I do not understand any of this yet. There are things I could do to be a better learner next lesson.

A mistake that moved my learning on

This was my mistake….

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Now I have learnt that….

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Representation/s that helped me to understand the effect of parameter '**q**' in trigonometric graphs is:>>>

A7. Turn-it-in Report

EXPLORING THE INFLUENCE OF MATHEMATICAL REPRESENTATIONS ON 10TH GRADE LEARNERS' UNDERSTANDING OF TRIGONOMETRY IN A POORLY PERFORMING SCHOOL

ORIGINALITY REPORT

6%	%	6%	%
SIMILARITY INDEX	INTERNET SOURCES	PUBLICATIONS	STUDENT PAPERS

PRIMARY SOURCES

1	Charles P. McKeague. "Graphing and Inverse Functions", Elsevier BV, 1984 Publication	1%
2	Advanced mathematics 1, 1980. Publication	<1%
3	Ogbonnaya, Ugorji I., and David Mogari. "The Relationship Between Grade 11 Students' Achievement in Trigonometric Functions and Their Teachers' Content Knowledge", Mediterranean Journal of Social Sciences, 2014. Publication	<1%
4	(HTW Dresden, Informatik, Mathematik). "Proceedings of the tenth international conference Models in developing mathematics education", Saechsische Landesbibliothek-Staats- und Universitaetsbibliothek Dresden, 2012.	<1%

A8. Language Clearance Certificate

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Phone 031 – 7075912
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Fax 031 - 7110458
E-mail:
dr1govender@telkomsa.net
sathsgovender4@gmail.com

A8. Language Clearance Certificate

Dr Saths Govender

4 JUNE 2018

TO WHOM IT MAY CONCERN

LANGUAGE CLEARANCE CERTIFICATE

This serves to inform that I have read the final version of the dissertation titled:

EXPLORING THE INFLUENCE OF MATHEMATICAL REPRESENTATIONS ON 10th GRADE LEARNERS' UNDERSTANDING OF TRIGONOMETRY IN A POORLY PERFORMING SCHOOL by M.M. Khoza, student no. 209536877.

To the best of my knowledge, all the proposed amendments have been effected and the work is free of spelling and grammatical errors. I am of the view that the quality of language used meets generally accepted academic standards.

Yours faithfully

S. Govender

DR S. GOVENDER
B Paed. (Arts), B.A. (Hons), B Ed.
Cambridge Certificate for English Medium Teachers
MPA, D Admin.