ON THE STATUS OF THE GEODESIC LAW IN GENERAL RELATIVITY

by

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Abstract

The geodesic law for test particles is one of the fundamental principles of general relativity and is extensively used. It is thought to be a consequence of the field laws but no rigorous proof exists. This thesis is concerned with a precise formulation of the geodesic law for test particles and with the extent of its validity. It will be shown to be true in certain cases but not in others.

A rigorous version of the Infeld/Schild theorem is presented. Several explicit examples of both geodesic and non-geodesic motion of singularities are given. In the case of a test particle derived from a test body with a regular internal stress-energy tensor, a proof of the geodesic law for an ideal fluid test particle under plausible, explicitly stated conditions is given. It is also shown that the geodesic law is not generally true, even for weak fields and slow motion, unless the stress-energy tensor satisfies certain conditions. An explicit example using post-Newtonian theory is given showing how the geodesic law can be violated if these conditions are not satisfied.

Preface

This study represents original work by the author and has not been submitted in any form to another university. Where use was made of the work of others it has been duly acknowledged in the text.

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Notation

Greek indices α, β etc. run from 0 to 3 and Latin indices i, j etc. run from 1 to 3.
Coordinates of events are typically denoted by $(x^0, x^1, x^2, x^3) = (x) = (x^\alpha)$.
Occasionally we put $\underline{x} = (x^1, x^2, x^3)$.
The Minkowski metric is given by $ds^2 = -(dx^0)^2 + dx^i dx^i = \eta_{\alpha\beta} dx^\alpha dx^\beta$.
The convention as regards the speed of light is not uniform throughout. The reason for this is that most texts use geometric units where $c = 1$ while the post-Newtonian theory as expounded in Damour, Soffel, Xu (1991) uses units where $c \neq 1$. Hence chapters 1 to 3 use geometric units where $c = 1$ and $x^0 = t$ while chapters 4 and 5 use units where $c \neq 1$ and $x^0 = ct$.
A metric tensor is typically denoted by $g_{\alpha\beta}$ and Christoffel symbols by $\Gamma^\alpha_{\beta\delta}$.
Covariant derivatives are denoted by a semi-colon and partial derivatives are denoted by a comma.

Contents

Introduction

The geodesic law may be briefly stated as: "A test particle moves on a geodesic of the background field". This thesis is concerned with a precise formulation of this statement and with the extent of its validity. The work was largely inspired by an article by J. Ehlers (1987) from which I quote here :
".... the problem of deriving the geodesic law for test bodies has not been rigorously solved .... various approximation methods indicate that the law holds approximately .... it still remains a challenge to derive a corresponding limit theorem from the field equations, perhaps by means of Dixon's description of bodies in general relativity or by "rigorising" the Infeld/Schild argument."

The main objectives of this work are:

- To clarify the concept of test particle
- To state and prove the Infeld/Schild theorem in a rigorous fashion
- To find examples of non-geodesic motion
- To investigate the geodesic law in an approximate post-Newtonian sense for weak fields, slow motion with a view to determining

(a) the assumptions on which it rests,

(b) an extension where possible to an exact geodesic law.

The following outlines the structure of the thesis:

- Introduction.
- Chapter 1: Background material including historical notes and a discussion and criticism of various textbook "proofs" of the geodesic law.
- Chapter 2: A definition of test particle and a number of examples in terms of this definition of both geodesic and non-geodesic motion.
- Chapter 3: A new proof of the Infeld/Schild theorem and an investigation of the hypotheses of this theorem. It should be pointed out that most of the material in chapters 2 and 3 has been published in Nevin (1995).
- Chapter 4: The derivation of an equation of motion for a small body in the first post-Newtonian approximation using the potentials of Blanchet/Damour (1989). Its relationship to the geodesic law and some applications.
- Chapter 5: The derivation of an equation of motion for a small body in a more general setting and its relationship to the geodesic law. An exact result for an ideal fluid. Parts of chapters 4 and 5 have been accepted for publication in Nevin (1998).
- Conclusion.

Chapter 1

Background material

1.1 Concepts involved in the statement of the geodesic law

Consider the statement : "A test particle moves on a geodesic of the background field". Intuitively one imagines that a very small test body is introduced into an existing gravitational field, the body being so small that its influence on the field is "negligible". This body then follows a geodesic of the original field which is called the *background field.* The usual textbook derivation of the advance of the perihelion of Mercury due to general relativity uses the geodesic law in this form, taking the background field to be a spherically symmetric Schwarzschild field due to the sun alone. Note that the background field is not the field of the two-body Sun/Mercury system. The latter field will be referred to as the "*total field*". The total field is the real field of the system, the background field is an imaginary concept and in general will need to be defined through a limiting process. Since Mercury is not a test particle, the text-book calculation of the advance of the perihelion using the geodesic law is not fully justified but requires a more complex theory of the motion of extended bodies. A test particle must also be defined through a limiting process. Intuitively it is a limit of a test body as the body shrinks to a point. Technical aspects of these limits will be considered in chapter 2.

1.2 An exact result concerning geodesics

The best known exact result is an extremely simple theorem which shows that if the stress-energy tensor inside the test body is as for dust with $T^{\alpha\beta} = \rho u^\alpha u^\beta$ where u^α is the fluid 4-velocity, then each fluid particle moves on a geodesic of the total field.

In the ideal fluid case where $T^{\alpha\beta} = \rho u^\alpha u^\beta + p g^{\alpha\beta}$ it has been shown that if p is constant on each material curve, then that material curve which has the maximum value for p is a geodesic of the total field. This result was shown by Taub (1962) where he corrected an earlier erroneous result of Thomas (1962). (Some comment on the condition that p be constant along material curves may be found in Taub (1959)). The following is a compact version of the results of Taub and Thomas :

Starting from $T^{\alpha\beta} = \rho u^\alpha u^\beta + p g^{\alpha\beta}$ one finds

$$T^{\alpha\beta}_{;\beta} = \rho u^\beta u^\alpha_{;\beta} + (\rho u^\beta)_{;\beta} u^\alpha + p_{,\beta} g^{\alpha\beta} = 0 \tag{1.2.1}$$

Contracting (1.2.1) with u_α and using $u^\alpha_{;\beta} u_\alpha = 0$ gives

$$(\rho u^\beta)_{;\beta} - p_{,\beta} u^\beta = 0$$

and substituting back into (1.2.1) gives

$$\rho u^\beta u^\alpha_{;\beta} = -p_{,\beta}(u^\alpha u^\beta + g^{\alpha\beta}) \tag{1.2.2}$$

If $p = 0$ as in dust, then (1.2.2) shows that material curves are geodesics. Otherwise let L denote the material curve with maximal p. It is easily seen that $p_{,\beta} = 0$ on L by using coordinates such that proper time on L is x^0.

Thus from (1.2.2), $u^\beta u^\alpha_{;\beta} = 0$ on L showing L to be a geodesic.

The above theorems produce geodesics of the total field which lie inside the test body. To derive the geodesic law one would need to show that geodesics of the total field approach geodesics of the background field as the test body shrinks. However the main reason that these results fall short is the unnatural assumptions made concerning the stress-energy tensor . One would not expect p to be zero or constant along material curves. In fact intuition suggests that a stress-energy tensor as for dust may be inconsistent with the notion of a compact fairly rigid body.

1.3 On the history of the problem of motion

The problem of motion of bodies in general relativity has historically separated roughly into two parts. The first part has centred around the geodesic law for a test particle while the second part has centred around equations of motion for interacting bodies of comparable masses. This thesis is concerned mainly with the first part. It is seen that these two parts do not essentially overlap since an arbitrarily small test body cannot be of comparable mass to the bodies creating the background field. (Throughout this work we make the assumption, dictated by conditions in the solar system, that all bodies are of comparable density).

Equations of motion in the second part are approximate while in the first part an exact limiting law is sought. There are nevertheless some results in the second part which relate to geodesics. Recent work in this area is contained in Damour, Soffel, Xu (1991-1994) hereafter called DSX, where it is claimed that the DSX centre of mass of each of the bodies follows a geodesic (in an approximate post-Newtonian sense) of the DSX background field provided these bodies are monopoles and are well separated. A derivation of this with attendant definitions is given in DSX (1991,1992) where it is used to derive the Einstein, Infeld, Hoffmann (EIH) equations of motion. Review articles dealing with the second part are Goldberg (1962) and Damour (1987).

Havas (1986) contains a detailed account of the early history of the problem of motion. Quoting from this article: " the derivation of the approximate equations of motion of several, slowly moving, particles of comparable masses is generally ascribed to a paper by Einstein, Infeld and Hoffmann (1938), in the following referred to as EIH, and the "exact" derivation of the geodesic law to Infeld and Schild (1949)". Hovever Havas goes on to demonstrate that these attributions of credit are incorect since the EIH equations were first derived by Droste, de Sitter and Lorentz while a considerable body of work on the geodesic law existed before 1940, for example Eddington (1923), Mathisson (1931), Robertson (1936). According to Havas, most people attribute the idea that the geodesic law can be deduced from the field equations to Einstein and Grommer (1927).

These early papers (and in fact most later papers too!) generally suffer from a lack of definition of various quantities, a lack of justification of limiting processes and/or strange assumptions concerning $T^{\alpha\beta}$. Eddington (1923) uses $T^{\alpha\beta}$ as for dust. Mathisson (1931) uses a distributional form for $T^{\alpha\beta}$. As explained in Ehlers (1980), there is no justification for $T^{\alpha\beta}$ of this form. Havas and Goldberg (1962) is a more recent paper based on the methods of Mathisson (1931). Robertson (1936) has an undefined separation of $T^{\alpha\beta}$ into "internal" and "external" parts.

Some later papers which have been cited in various textbooks are discussed in the following section. There is very little recent literature which is relevant to the geodesic law. This is illustrated by the fact that Brumberg (1991) which is intimately concerned with the problem of motion, refers to papers published before 1960 for a proof. It appears that by the 1970's the geodesic law had somehow got accepted as proved although no satisfactory proof existed. This fact was pointed out by Ehlers and others in the late 1970's and 1980's but seems to have been largely ignored. There have however been a number of papers which have extended the EIH equations to higher post-Newtonian orders. For example Grishchuk and Kopejkin (1986) obtain equations of motion for a centre of mass of a spherically symmetric ideal fluid body but do not justify their approximations. Damour and Schafer (1985), Schafer, Wex, Norbert (1993) obtain equations of motion for the 2PN two-body problem. Again the bodies are assumed to be "point particles " characterized by their mass only and it is difficult to gauge the range of validity of these equations.

1.4 Views from the text books

Fock (1959)

Fock (1959, section 6.3) gives a proof of the geodesic law which is similar to that of Eddington (1923). He uses $T^{\alpha\beta} = \rho u^{\alpha} u^{\beta}$ and integrates $T^{\alpha\beta}_{;\beta}$ over the test body on a time slice.

Synge (1960)

Synge presents an argument on page 252 in favour of the geodesic law but makes it quite clear that he does not regard this as sufficient for a proof. It is also clear from the footnote that Synge is not one of the alleged majority who regard Infeld and Schild (1949) as the solution to the geodesic problem!

Hawking and Ellis (1973)

The authors (page 63) refer to the work of Dixon (1970) as follows: "a small isolated body moves approximately on a timelike geodesic curve independent of its internal constitution provided that the energy density of matter in it is non-negative (for an account of the motion of a small body in relativity, see Dixon (1970))"

A possible method of taking Dixon's equations to the test particle limit and deriving the geodesic law was outlined in Ehlers and Rudolph (1977) and is described briefly below, but according to a conversation with J. Ehlers at GR14 in 1995 this programme was never satisfactorily completed and later abandoned. One of the remarkable aspects of Dixon's theory is that his equations are fully covariant and he defined a centre of mass line for an extended body covariantly. The definition is implicit, using properties of certain vector and tensor fields, which makes it rather difficult to handle. The fact that a centre of mass line l_0 satisfying the required conditions actually exists and is unique was proved by Shattner (1978) using fixed point theorems of analysis. There are some conditions for the proof to go through including a "weak field" assumption. Also it is not a simple matter to describe the position of the Dixon centre of mass relative to the body. This is important since the only measurable is the position of the boundary of the body world tube.

One of Dixon's equations of motion has the form

$$\dot{p}_\kappa = R^{*}_{\kappa\alpha\beta\gamma}\dot{z}^\alpha S^\beta u^\gamma + K_\kappa \quad \text{on } l_0 \tag{1.4.1}$$

where $z^\alpha(t)$ is the curve l_0, t is an arbitrary parameter on l_0, dot denotes absolute derivative with respect to t, $R^{*}_{\kappa\alpha\beta\gamma}$ is the right dual of the Riemann tensor of the (total) field $g_{\alpha\beta}$, p_κ, S^β, K^κ are vector fields on l_0 defined (non-locally) in terms of $g_{\alpha\beta}$ and $T^{\alpha\beta}$, u^γ is a unit vector parallel to p^γ. Putting $M = (p^\lambda p_\lambda)^{\frac{1}{2}}$, dividing by M and contracting with the projection tensor $\gamma_{\kappa\lambda} = g_{\kappa\lambda} + u_\kappa u_\lambda$, equation (1.4.1) becomes

$$\dot{u}_\kappa = \gamma^\lambda_\kappa(R^{*}_{\lambda\alpha\beta\gamma}\dot{z}^\alpha s^\beta u^\gamma + k_\lambda) \tag{1.4.2}$$

where $s^\beta = \frac{S^\beta}{M}$, $k_\lambda = \frac{K_\lambda}{M}$.

The method proposed by Ehlers and Rudolph was to show by making various estimates of quantities in (1.4.2), that in the limit as the body shrinks to a point, $\dot{u}^\alpha$ tends to zero and also $\dot{z}^\alpha$ tends to u^α and from this to conclude that l_0 approaches a geodesic of the total field and also of the background field. As mentioned above this has not yet been achieved.

Misner, Thorne, Wheeler (1973)

The treatment here is particularly vague and descriptive, lacking precise definitions. There is a reference to Infeld and Schild (1949) but not to their theorem. The authors are apparently using ideas similar to Infeld and Schild when they say (p. 479) that "the perturbation in the metric is qualitatively of the form $\delta g \sim r^2 R + \frac{m}{r}$ where r is the distance from the geodesic".

The Infeld/Schild method will be discussed in detail in the following chapter.

Stephani (1982)

Stephani (1982, p.88) uses a distributional form for $T^{\alpha\beta}$ defined by means of a δ-function as follows:

$$T^{\alpha\beta}(y^\epsilon) = mc\int \frac{\delta^4(y^\epsilon - x^\epsilon(\tau))}{\sqrt{-g}}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}d\tau$$

This is said to represent a point-like particle of constant rest mass m. The equation $T^{\alpha\beta}_{;\beta} = 0$ is applied to yield the geodesic equation. As stated earlier, this distributional form for the stress-energy tensor is unjustified. Furthermore the mathematical content is minimal and one might say that the use of $T^{\alpha\beta}$ in the above form is almost equivalent to assuming the geodesic law. Nevertheless this form for $T^{\alpha\beta}$ is still in use to the present day. Mannheim (1993) states in his introduction that the above form for the stress-energy tensor is "completely standard"!

Wald (1984)

Wald refers to Geroch and Yang (1975). Geroch and Yang prove a theorem which is stated as follows:

"Let M, $g_{\alpha\beta}$ be a space-time. Let Γ be a curve on M satisfying the following condition: For any neighbourhood U of Γ, there exists a non-zero symmetric conserved tensor field $T^{\alpha\beta}$ on M which satisfies the energy condition and whose support is in U. Then Γ is a timelike geodesic."

This theorem does not seem to have a direct bearing on the geodesic law. Note that in the theorem there is only one $g_{\alpha\beta}$ but a whole family of $T^{\alpha\beta}$. Since any $g_{\alpha\beta}$ determines its own $T^{\alpha\beta}$ uniquely through the field equations it is clear that the $T^{\alpha\beta}$ do not belong to $g_{\alpha\beta}$ in this sense. On the other hand $T^{\alpha\beta}$ is conserved with respect to the metric $g_{\alpha\beta}$. This makes the physical interpretation of $T^{\alpha\beta}$ rather difficult. $T^{\alpha\beta}$ does not represent a test body in the usual sense because it is conserved in the wrong metric. I therefore do not see that the above theorem proves the geodesic law. (Also note that in this paper it is stated incorrectly that Taub (1962) has proved the existence of a timelike geodesic inside the world tube of a body with "isotropic pressure".)

Brumberg (1991)

Brumberg refers the reader to Fock (1959) and to Infeld and Schild (1949) for proof of the geodesic law.

Chapter 2

Test Particles

2.1 Introduction

Firstly, in order to clarify the concepts, it is helpful to look at a family of exact solutions in Newtonian mechanics. Consider two uniform spherical balls of constant density ρ doing circular motion in the x, y plane of an inertial frame about their combined centre of mass situated at the origin. At at any instant the centres of the balls lie in a line through the origin and this line rotates with constant angular velocity ω where $\omega^2 = d^{-3}G(M_1 + M_2)$, M_1 and M_2 are the masses of the balls and d is the constant distance between their centres. This system is an exact solution of the equations of Newtonian mechanics for any choice of M_1, M_2, d, ρ. Let M_1, d, ρ be held fixed and let M_2 be a parameter which tends to zero. Thus M_1 will be regarded as the body creating the background field while M_2 is the test body.

The Newtonian potential for the two-body system at events outside the bodies is given by

$$\begin{aligned} \phi(x, y, z, t; M_2) &= \frac{GM_1}{|(x + R_1 \cos\omega t, y + R_1 \sin\omega t, z)|} \\ &+ \frac{GM_2}{|(x - R_2 \cos\omega t, y - R_2 \sin\omega t, z)|} \end{aligned} \tag{2.1.1}$$

where

$$R_1 = \frac{M_2 d}{M_1 + M_2};\ R_2 = \frac{M_1 d}{M_1 + M_2},\ \omega = (d^{-3}G(M_1 + M_2))^{\frac{1}{2}} \tag{2.1.2}$$

ϕ could also be evaluated inside the bodies if required.

$\phi(x, y, z, t; M_2)$ is a one parameter family of functions of four variables which represents the Newtonian gravitational field of the two-body system. From (2.1.2)

$$\lim_{M_2 \to 0} R_1 = 0, \quad \lim_{M_2 \to 0} R_2 = d, \quad \lim_{M_2 \to 0} \omega = (d^{-3} G M_1)^{\frac{1}{2}} = \omega_0.$$

From (2.1.1) for $(x, y) \neq (d \cos \omega t, d \sin \omega t)$ and $|(x, y, z)| >$ radius M_1

$$\lim_{M_2 \to 0} \phi(x, y, z, t; M_2) = \frac{G M_1}{|(x, y, z)|}$$

As one would expect, this is the potential of M_1 stationary at the origin. In the limit as $M_2 \to 0$, the centre of mass shifts to the centre of M_1 and the vanishingly small particle M_2 does circular motion about M_1 at a distance d with angular velocity ω_0. Thus the limiting path of M_2 is $(x, y, z, t) = (d \cos \omega_0 t, d \sin \omega_0 t, 0, t)$ which will be called $\mathcal{C}$.

When $M_2 \neq 0$, the path of the centre of M_2 is

$$(M_1 (M_1 + M_2)^{-1} d \cos \omega t, \; M_1 (M_1 + M_2)^{-1} d \sin \omega t, 0, t)$$

which converges to $\mathcal{C}$ as $M_2 \to 0$.

The family of functions $\phi(x, y, z, t; M_2)$ defines a "Newtonian test particle moving on the curve $\mathcal{C}$". For convenience this name is given to the entire family of functions although it is mainly the limit as $M_2 \to 0$ which is of interest.

Note that if a tube surrounding $\mathcal{C}$ is taken so that each time slice is a sphere of radius ϵ centred on $\mathcal{C}$, then the world tube of M_2 will lie entirely within this tube if M_2 is small enough.

2.2 A definition for a test particle in general relativity

This definition is modelled on the essential features of the Newtonian example above. The conditions given here are minimal but they may be augmented later with further conditions. The idea of a test particle being defined by a shrinking world tube is certainly not new. It is used for example in Robertson (1936) and in Infeld and Schild (1949). However the precise meaning to be attached to the limit is not always given. The definition given here may differ in some respects from others. The abbreviation $x = (x^0, x^1, x^2, x^3)$ is used, m is the family parameter which does not necessarily represent mass in any form.

Definition: A one parameter family of functions $g_{\alpha\beta}(x; m)$ will be said to constitute (or define) a *test particle* moving on a curve C of a background metric $g_{\alpha\beta}(x)$ if

(1) $g_{\alpha\beta}(x)$ satisfies the vacuum Einstein Field Equations (E.F.E) in a region containing the curve C.

(2) The tube conditon:
On a finite segment of C, for each $m > 0$, there exists a tube U_m centred on C of geodesic radius $\epsilon(m) > 0$ in the background, so that inside U_m, $g_{\alpha\beta}(\underline{x}; m)$ either has singularities or is non-vacuum, while outside U_m, $g_{\alpha\beta}(\underline{x}; m)$ is a non-singular matrix which satisfies the vacuum E.F.E. Furthermore, $\epsilon(m) \to 0$ as $m \to 0$.

(3) For each x not in C, $\lim_{m \to 0} g_{\alpha\beta}(x; m) = g_{\alpha\beta}(x)$. (Note that for m sufficiently small, x will be outside of U_m and hence $g_{\alpha\beta}(x; m)$ will be defined.)

The above conditions are seen to be invariant under a smooth change of coordinates $x \to \bar{x}(x)$. For each value of m, $g_{\alpha\beta}(x; m)$ is regarded as the metric tensor representing a test body whose world tube lies inside U_m. The limit of $g_{\alpha\beta}(x; m)$ as $m \to 0$ is taken pointwise, moving along paths consisting of points (events) which have the same coordinates in each spacetime. As a consequence of (3) one may extend the domain of $g_{\alpha\beta}(x; m)$ by defining $g_{\alpha\beta}(x; 0) = \lim_{m \to 0} g_{\alpha\beta}(x; m) = g_{\alpha\beta}(x)$ for x not on C. This makes the extended function continuous in m at $(x; 0)$. It will be found necessary later to impose further continuity/differentiability conditions on the extended function $g_{\alpha\beta}(x;m)$.

An example of a test particle is given by the Schwarzschild family of metrics which in isotropic coordinates has metric tensor

$$g_{\alpha\beta}(x;m)dx^\alpha dx^\beta = -\frac{\left(1-\frac{m}{2r}\right)^2}{\left(1+\frac{m}{2r}\right)^2}(dx^0)^2 + \left(1+\frac{m}{2r}\right)^4 dx^i dx^i \qquad \text{where } r = (x^i x^i)^{1/2}$$

Here

$$g_{\alpha\beta}(x)dx^\alpha dx^\beta = -(dx^0)^2 + dx^i dx^i$$

so the background metric is Minkowskian.

$\mathcal{C}$ is the curve $x^i = 0$ which is a geodesic in Minkowski space. One may take $\epsilon(m) = \frac{m}{2}$ since r is geodesic distance in Minkowski space from x to $\mathcal{C}$ and $g_{\alpha\beta}(x;m)$ is non-singular for $r > \frac{m}{2}$.

There is a further aspect of the test particle definition which should be mentioned. One may make a different coordinate transformation on $g_{\alpha\beta}(x;m)$ for each m. Thus one may make the transformation $\tilde{x}^\alpha = x^\alpha + mV^\alpha(x)$ where V^α are arbitrary smooth functions. This transformation is the identity when $m = 0$ so x^α and $\tilde{x}^\alpha$ may be identified on the background field. If $\tilde{g}_{\alpha\beta}(\tilde{x};m)$ is the new $g_{\alpha\beta}(x;m)$ thus obtained, it is found that $\tilde{g}_{\alpha\beta}(\tilde{x};m)$ is again a test particle on $\mathcal{C}$ if the V^α are bounded in a neighbourhood of $\mathcal{C}$ (and possibly even if the V^α are not bounded). The spacetimes represented by $g_{\alpha\beta}(x;m)$ are not changed by the transformation but the paths along which the limit is taken are changed.

The following chapter will be concerned with the *linearization* $\gamma_{\alpha\beta}(x)$ of the family of metrics $g_{\alpha\beta}(x;m)$ about the background metric $g_{\alpha\beta}(x)$ which is defined as

$$\gamma_{\alpha\beta}(x) = \lim_{m\to 0}\frac{\partial g_{\alpha\beta}}{\partial m}(x;m) \qquad \text{for } x \text{ not on } \mathcal{C} \tag{2.2.1}$$

If this limit exists $\gamma_{\alpha\beta}(x)$ is a covariant tensor field on the background spacetime excepting $\mathcal{C}$. If it is assumed that the extended function $g_{\alpha\beta}(x;m)$ has continuous first partial derivatives in a neighbourhood of $(x;0)$ where x is not on $\mathcal{C}$, then $\tilde{\gamma}_{\alpha\beta}$ is given by the formula

$$\tilde{\gamma}_{\alpha\beta}(x) = \gamma_{\alpha\beta}(x) - V_{\alpha;\beta}(x) - V_{\beta;\alpha}(x) \qquad \text{for } x \text{ not on } \mathcal{C} \tag{2.2.2}$$

where covariant differentiation and raising/lowering of indices is with respect to the background metric $g_{\alpha\beta}(x)$. The linearization $\gamma_{\alpha\beta}(x)$ is a covariant tensor field on the background spacetime. Equation (2.2.2) is derived in Wald (1984) appendix C2 using Lie derivatives but it may also be found directly by differentiating the tensor transform equation:

$$g_{\alpha\beta}(x;m) = \tilde{g}_{\lambda\mu}(x + mV(x);m)\left(\delta^{\lambda}_{\mu} + m\frac{\partial}{\partial x^{\alpha}}V^{\lambda}(x)\right)\left(\delta^{\mu}_{\beta} + m\frac{\partial}{\partial x^{\beta}}V^{\mu}(x)\right),$$

$$m \geq 0, \ x \text{ not in } \mathcal{C}$$

with respect to m and using

$$\lim_{m\to 0}\tilde{g}_{\lambda\mu}(x + mV(x);m) = g_{\lambda\mu}(x), \qquad \lim_{m\to 0}\frac{\partial\tilde{g}_{\lambda\mu}}{\partial\tilde{x}^{v}}(x + mV(x);m) = \frac{\partial g_{\lambda\mu}}{\partial x^{v}}(x)$$

which follow from the continuity assumptions made.

$\tilde{\gamma}_{\alpha\beta}(x)$ given by (2.2.2) is said to be *gauge equivalent* to $\gamma_{\alpha\beta}$.

It may be verified that $\gamma_{\alpha\beta} = \text{diag}\ \frac{2}{r}$ for the Schwarzschild family given above. If one uses the Schwarzschild metric in the Eddington form

$$ds^2 = -(dx^0)^2 + dx^i dx^i + \frac{2m}{r}\left(\frac{x^i}{r}dx^i + dx^0\right)^2$$

one finds

$$\gamma_{00} = \frac{2}{r}, \quad \gamma_{0i} = \frac{2x^i}{r^2}, \quad \gamma_{ij} = \frac{2x^i x^j}{r^3} \tag{2.2.3}$$

This is gauge equivalent to the diagonal from above using $(V_0, V_i) = -(2\log r, \frac{x^i}{r})$. The background field is Minkowski again.

The Kerr metric in Kerr-Schild coordinates is given by

$$ds^2 = [\eta_{\alpha\beta} + 2Hk_{\alpha}k_{\beta}]\,dx^{\alpha}dx^{\beta}$$

where

$$H = \frac{mr}{r^2 + a^2\left(\frac{z}{r}\right)^2}, \qquad k_\alpha dx^\alpha = \frac{r(x\ dx + y\ dy) + a(x\ dy - y\ dx)}{r^2 + a^2} + \frac{z}{r}\ dz + dt$$

with

$$x^\alpha = (t, x, y, z).$$

This is a two-parameter family of metrics. It may be reduced to a one parameter family by putting $a = km$ with k constant. With this requirement, which corresponds roughly to the idea of fixed angular velocity, $\gamma_{\alpha\beta}$ is found to be exactly the same as (2.2.3) so the rotation of the Kerr metric does not show up in $\gamma_{\alpha\beta}$.

Note that the definition of test particle given here is broad enough to include both black holes and "normal" bodies containing matter. Only the field in the vacuum region outside of the black hole event horizon or outside the matter is used in the calculation of $\gamma_{\alpha\beta}$. One could fill in a central spherical part of the Schwarzschild field with matter (this is done in chapter 4) and it would make no difference to $\gamma_{\alpha\beta}$.

2.3 Examples of test particles on geodesics and on non-geodesics

The examples given in section 2.2 all have Minkowski space for background and they all have $C = (x^0, 0, 0, 0)$ which is a geodesic. Test particle examples are naturally quite scarce since they require whole families of exact solutions of the E.F.E with special properties. All the examples found have spherical or axial symmetry and the latter are mostly derived from the Weyl axisymmetric static metrics.

The existence of examples where C is not a geodesic is known (but not, it would seem, well known). Historically an example of a "self accelerating particle" in general relativity seems to have originated with Bondi (1957). As Bonner and Swaminarayan

(B&S) (1966) have pointed out, this is not entirely a relativistic phenomenon. In Newtonian mechanics one can construct an example, a dipole consisting of two masses $\pm m$, which is self accelerated (provided one allows negative mass, assumes the Newtonian potential $\frac{m}{r}$ also for negative mass and the principle of equivalence asserting that the acceleration of a particle is independent of its mass). The example of Bondi (1957) was extended in B&S(1964), Bonnor (1966) and extended again in Bicak et al (1983)(a), (b) and Bonnor (1988). [Other references to related work by Ernst and others may be found in the bibliography to Bicak et al (a) (1983)]. A good deal of the work in the above-mentioned papers is concerned with the physical interpretation of the fields. None have treated the problem formally in terms of a test particle definition although Bonnor (1966) mentions this aspect briefly. In the first example below I have converted Bonnor's example to a one-parameter family of metrics and fitted it into the test particle definition given in section 2.2. Both the singularities (which are coalescing) are "inside" the test particle. (This was not clear in Bonnor (1966).) The other examples given are found directly from Weyl's axisymmetric static metric. This is similar in principle to the approach of Bicak et al 1983(b) but the method and examples are different. Usually, a metric of the Weyl family will have singularities along whole segments of the axis of symmetry. Special conditions are required in order that singularities occur only at a few isolated points. Test particle examples are constructed by satisfying these conditions.

Example 1:

In Bonnor and Swaminarayan (1964) a 4-parameter exact solution of the vacuum E.F.E. is given as follows

$$ds^2 = -e^{\lambda}dr^2 - r^2e^{-\rho}d\theta^2 + (z^2-t^2)^{-1}\{(z^2e^{\rho}-t^2e^{\lambda})dt^2 - (z^2e^{\lambda}-t^2e^{\rho})dz^2 + 2zt(e^{\lambda}-e^{\rho})dzdt\}$$

where

$$\rho = -\frac{2a_1}{R_1} - \frac{2a_2}{R_2} + \frac{2a_1}{h_1} + \frac{2a_2}{h_2}$$

$$
\begin{aligned}
\lambda &= \frac{a_1 a_2}{(h_1-h_2)^2} f - r^2(z^2-t^2)\left(\frac{a_1^2}{R_1^4}+\frac{a_2^2}{R_2^4}\right)+\frac{2a_1R}{h_1R_1}+\frac{2a_2R}{h_2R_2} \\
R &= \frac{1}{2}(r^2+z^2-t^2)) \\
R_\alpha &= \left\{(R-h_\alpha)^2+2r^2h_\alpha\right\}^{\frac{1}{2}} \qquad \alpha=1,2 \\
f &= 4R_1^{-1}R_2^{-1}\left\{r^2(z^2-t^2)+(R-r^2-h_1)(R-r^2-h_2)-R_1R_2\right\}
\end{aligned}
$$

a_1, a_2, h_1, h_2 are real parameters with $h_1, h_2 > 0$.

Various cases are dealt with in B&S (1964), one of which is

$$a_1 = (h_1-h_2)^2(2h_2)^{-1} \qquad a_2 = -(h_1-h_2)^2(2h_1)^{-1} \tag{2.3.1}$$

We restrict consideration to the region where $z^2 - t^2 > 0$.

It may be shown that the only physical singularities of the above metric subject to condition (2.3.1), are at $r = 0$, $z = (t^2+2h_1)^{\frac{1}{2}}$ and at $r = 0, z = (t^2+2h_2)^{\frac{1}{2}}$ and these are real singularities in the sense that $R^{\alpha\beta\gamma\delta}\ R_{\alpha\beta\gamma\delta}$ is unbounded in their neighbourhood. There are coordinate singularities at other points of the axis but these may be removed by the transformation $x = r\cos\theta$, $y = r\sin\theta$. The metric is interpreted as representing two point particles, one of positive mass a_1 moving on $z = (t^2+2h_1)^{\frac{1}{2}}$ and one of negative mass a_2 moving on $z = (t^2+2h_2)^{\frac{1}{2}}$.

The original 4-parameter family may be reduced to a 1-parameter family with $m = a_1$ by holding $h_1 > 0$ fixed and solving (2.3.1) for h_2, a_2 to obtain:

$$h_2 = h_1 + a_1 + (2a_1h_1 + a_1^2)^{\frac{1}{2}}, \qquad a_2 = -(a_1 + (2a_1h_1+a_1^2)^{\frac{1}{2}})/2h_1$$

As $a_1 \to 0$ through positive values, $a_2 \to 0$ and $h_2 \to h_1$.

As $h_2 \to h_1$, the two particles coalesce forming a "composite" test particle. It will now be shown that the test particle conditions are satisfied.

$\mathcal{C}$ is the curve $r = 0, z = (t^2+2h_1)^{\frac{1}{2}}$. Take x not on $\mathcal{C}$ and consider $\lim_{a_1 \to 0} g_{\alpha\beta}(x; a_1)$.

$R = \frac{1}{2}(r^2 + z^2 - t^2) > 0$ since $z^2 - t^2 > 0$, and R is independent of a_1.

R_1 is also independent of a_1 and $R_1 \neq 0$

$R_2 \to R_1 \quad \text{as} \quad a_1 \to 0$

$\rho \to 0 \qquad \text{as} \quad a_1 \to 0$

$\lambda \to 0 \qquad \text{as} \quad a_1 \to 0.$

Hence the background has metric

$$\begin{aligned}
(ds)^2 &= -dr^2 - r^2 d\theta^2 + (z^2 - t^2)^{-1}\left\{(z^2 - t^2)dt^2 - (z^2 - t^2)dz^2\right\} \\
&= dt^2 - dz^2 - dr^2 - r^2 d\theta^2
\end{aligned}$$

which is a Minkowski space. $\mathcal{C}$ is not a geodesic in this space.

The only singularities of $g_{\alpha\beta}(x; a_1)$ not on $\mathcal{C}$ occur on $r = 0, z = (t^2 + 2h_2)^{\frac{1}{2}}$.

The geodesics in Minkowski space all have linear parametrizations in standard (cartesian) coordinates. To demonstrate the tube condition it is convenient to convert to standard coordinates in Minkowski space so that $\mathcal{C}$ parametrized by u is $(u, 0, 0, (u^2 + 2h_1)^{\frac{1}{2}})$ and the other singularities lie on $(t, 0, 0, (t^2 + 2h_2)^{\frac{1}{2}})$. The foot of the (Minkowskian) perpendicular from $(t, 0, 0, (t^2 + 2h_2)^{\frac{1}{2}})$ to $\mathcal{C}$ is found from the condition that the tangent to $\mathcal{C} : (1, 0, 0, u(u^2 + 2h_1)^{-\frac{1}{2}})$ be perpendicular to the join of the above points.

Surprisingly, this has the simple solution $u^2 = \frac{h_1}{h_2}t^2$.

Hence the geodesic distance in the background Minkowski space from $r = 0, z = (t^2 + 2h_2)^{\frac{1}{2}}$ to $\mathcal{C}$ is found to be:

$$\left\{\left[(t^2 + 2h_2)^{\frac{1}{2}} - \left(\frac{h_1}{h_2}t^2 + 2h_1\right)^{\frac{1}{2}}\right]^2 - t^2\left(1 - \left(\frac{h_1}{h_2}\right)^{\frac{1}{2}}\right)^2\right\}^{\frac{1}{2}}$$

On rationalizing the first term, this becomes

$$\left\{\frac{\left[\frac{t^2}{h_2}(h_2-h_1)+2(h_2-h_1)\right]^2}{\left[(t^2+2h_2)^{\frac{1}{2}}+\left(t^2\frac{h_1}{h_2}+2h_1\right)^{\frac{1}{2}}\right]^2}-t^2\left[1-\left(\frac{h_1}{h_2}\right)^{\frac{1}{2}}\right]^2\right\}^{\frac{1}{2}}.$$

For a fixed t, the numerators both go to zero as $h_2 \to h_1$ and the denominator is bounded below by $h_1^{\frac{1}{2}}$ for all t. If t is restricted to a finite interval $t < T$, it is clear that the tube condition holds.

Examples using Weyl metrics

The Weyl axisymmetric static metrics are given in Weyl's canonical coordinates (T, ρ, ϕ, z) in Kramer et al (1980) on p. 200. The metric has the form

$$ds^2 = e^{-2u}\left[e^{2k}(d\rho^2+dz^2)+\rho^2 d\phi^2\right]-e^{2u}(dT)^2 \tag{2.3.2}$$

where u and k are functions of ρ, z satisfying

$$\frac{\partial^2 u}{\partial z^2}+\frac{\partial^2 u}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial u}{\partial \rho}=0 \tag{2.3.3}$$

and

$$\frac{\partial k}{\partial \rho}=\rho\left(\left(\frac{\partial u}{\partial \rho}\right)^2-\left(\frac{\partial u}{\partial z}\right)^2\right), \qquad \frac{\partial k}{\partial z}=2\rho\,\frac{\partial u}{\partial \rho}\,\frac{\partial u}{\partial z}. \tag{2.3.4}$$

The metric (2.3.2) is singular on the axis where $\rho = 0$. If u, k are C^2 functions of ρ, z satisfying (2.3.3), (2.3.4) in the region where $\rho > 0$, the metric (2.3.2) will satisfy the vacuum E.F.E. in this region. It may be possible to extend the metric (2.3.2) on to parts of the axis $\rho = 0$.

If one applies the transformation $x = \rho\cos\phi$, $y = \rho\sin\phi$ to (2.3.2) it takes the form

$$ds^2 = \left(e^{-2u+2k} + \frac{y^2}{\rho^2}e^{-2u}\left(1-e^{2k}\right)\right)dx^2 + \left[e^{-2u+2k} + \frac{x^2}{\rho^2}e^{-2u}\left(1-e^{2k}\right)\right]dy^2 \quad (2.3.5)$$

$$+e^{-2u+2k}dz^2 - e^{2u}dT^2$$

where ρ, u, k are to be expressed in terms of x, y, z by means of $\rho = (x^2+y^2)^{\frac{1}{2}}$.

The functions u, k will be C^2 functions of x, y, z at all points not on the axis $x = y = 0$ and the metric (2.3.5) will satisfy the vacuum E.F.E. at all such points. If the metric (2.3.5) can be continued to a C^2 metric in a neighbourhood of a point $(0,0,z_1)$ on the axis, then metric (2.3.2) (and metric (2.3.5)) will be said to be *regular* at the point $(0,0,z_1)$.

For regularity at $(0,0,z_1)$ it is clearly necessary that the limits of u, k as $(x,y,z) \to (0,0,z_1)$ should exist. Since $\lim\limits_{(x,y,z)\to(0,0,z_1)} \frac{x^2}{\rho^2}$ does not exist, it is also necessary that

$$\lim_{(\rho,z)\to(0,z_1)} k(\rho,z) = 0. \quad (2.3.6)$$

None of the references in the bibliography define regularity explicitly, but one might infer from appendix II of B& S (1964) that their definition agrees with the above. All the other references state (2.3.6) as "the regularity condition". No proof is offered that (2.3.6) is sufficient. In fact, unless one has some prior knowledge about the form of function k, (2.3.6) does not necessarily imply regularity. For example, if $k = \rho = (x^2+y^2)^{\frac{1}{2}}$ then k satisfies (2.3.6) at all points of the axis but (2.3.5) is not regular on the axis. The metric coefficients in (2.3.5) can be extended onto the axis by continuity but the functions so defined are not C^1 on the axis.

Assuming that u can be continued to a C^2 function on a neighbourhood of $(0,0,z_1)$, one sees from (2.3.5) that a sufficient condition for regularity at $(0,0,z_1)$ is that k has the form

$$k = (x^2+y^2)g(x,y,z) \quad (2.3.7)$$

where $g(x,y,z)$ is C^2 on an entire neighbourhood of $(0,0,z_1)$.

Returning to the problem of finding Weyl metrics, condition (2.3.3) in Cartesian co-ordinates becomes $\nabla^2 u = 0$ so that u is harmonic. Let u be an harmonic function with cylindrical symmetry which is expressed in cylindrical co-ordinates ρ, z. Let

$$A = \rho\left\{\left(\frac{\partial u}{\partial \rho}\right)^2 - \left(\frac{\partial u}{\partial z}\right)^2\right\}, \qquad B = 2\rho\,\frac{\partial u}{\partial \rho}\,\frac{\partial u}{\partial z}, \tag{2.3.8}$$

Then, using equation (2.3.3), one finds $\frac{\partial A}{\partial z} = \frac{\partial B}{\partial \rho}$. This means that equation (2.3.4) is integrable and in the region $\rho > 0$, $k(z,\rho)$ is determined uniquely up to an additive constant by u.

Let us first consider the case where $u(x,y,z)$ is harmonic in a neighbourhood of $(0,0,0)$. Then u has an expansion in this neighbourhood in terms of Legendre polynomials

$$u = \sum_{n=0}^{\alpha} a_n r^n P_n(\cos\theta) \quad \text{where } \rho = r\sin\theta, \quad z = r\cos\theta$$

Consider as an example

$$\begin{aligned} u &= a_1 r P_1(\cos\theta) + a_2 r^2 P_2(\cos\theta) \\ &= a_1 z + a_2(2z^2 - \rho^2) = a_1 z + a_2(2z^2 - (x^2+y^2)). \end{aligned} \tag{2.3.9}$$

Solving (2.3.4) for k gives,with a suitable choice of constant,

$$k = \rho^2\left(-\frac{a_1^2}{2} - 4a_1a_2z + a_2^2\rho^2 - 8a_2^2z^2\right) \tag{2.3.10}$$

which satisfies (2.3.7).

Thus the Weyl metric determined by (2.3.9) is regular everywhere on the axis for any choice of a_1, a_2. A metric of this family will be used as a background metric for a test particle. The test particle will be constructed by adding to u another harmonic function which has an isolated singularity at the origin, to represent the particle.

Hence consider $u = u_b + u_p$ where u_b is given by (2.3.9) above and

$$u_p = m\left(a_3 r^{-1} P_0(\cos\theta) + a_4 r^{-2} P_1(\cos\theta)\right).$$

Thus

$$u = a_1 z + a_2\left(2z^2 - \left(x^2+y^2\right)\right) + m\left(a_3\left(\rho^2+z^2\right)^{-\frac{1}{2}} + a_4 z\left(\rho^2+z^2\right)^{-\frac{3}{2}}\right). \quad (2.3.11)$$

A Weyl metric is determined by (2.3.11) for any set of values of the 5 constants a_1, a_2, a_3, a_4, m. This metric will represent $g_{\alpha\beta}(x,m)$ for a suitable choice of the constants a_1, a_2, a_3, a_4 and it is necessary to determine a_1, a_2, a_3, a_4 so that it is regular at all points in the neighbourhood of the origin excepting the origin itself. The determination of k even in this simple case is lengthy. (Bicak et al (1983) have found a formula for k in a fairly similar situation in terms of prolate spheroidal coordinates and Legendre polynomials but their background metric is not the same as in (2.3.11).) The condition (2.3.6) enables one to find the required condition on the constants by a shorter method without actually finding k (this was done in Nevin (1995)) but since (2.3.6) has not been shown sufficient this does not seem conclusive. The alternative is to determine k and use condition (2.3.7) to show regularity.

By a lengthy process of differentiations and integrations it has been found that k satisfies (2.3.4) where u is given by (2.3.11) if and only if

$$\begin{aligned} k &= \left(\frac{-\rho^2}{2}\right)a_1 + \left(\rho^4 - 8\rho^2 z^2\right)a_2^2 + \left(-\frac{1}{2}\rho^2 r^{-4}\right)m^2 a_3^2 + \left(\frac{-2\rho^2}{r^6} + \frac{9}{4}\frac{\rho^4}{r^8}\right)m^2 a_4^2 \\ &+ \left(-4z\rho^2\right)a_1 a_2 + \left(-2zr^{-1}\right)m a_1 a_3 + \left(2\rho^2 r^{-3}\right)m a_1 a_4 + \left(4\rho^2 r^{-1}\right)m a_2 a_3 \\ &+ \left(-4z\rho^2 r^{-3} - 8z^3 r^{-3}\right)m a_2 a_4 + \left(-2z\rho^2 r^{-6}\right)m^2 a_3 a_4 + C \qquad (2.3.12)\end{aligned}$$

where C is a constant.

In a sufficiently small neighbourhood of any point $(0,0,z_1)$ with $z_1 \neq 0$, r^{-1} is C^2. Hence the only terms in (2.3.12) which do not have the form of (2.3.7) are given by

$$D = 2zr^{-1}ma_1a_3 - 8z^3r^{-3}ma_2a_4$$

$$\begin{array}{rll} \text{for} & z>0, \rho=0 & D = -2ma_1a_3 - 8ma_2a_4 \\ \text{and for} & z<0, \rho=0 & D = 2ma_1a_3 + 8ma_2a_4 \end{array}$$

Assuming $m \neq 0$ and using (2.3.6), a necessary condition for regularity for both $z > 0$ and $z < 0$ is that

$$a_1a_3 + 4a_2a_4 = 0 \tag{2.3.13}$$

Condition (2.3.13) is also sufficient, for if (2.3.13) is satisfied, then

$$\begin{aligned} D &= 8ma_2a_4(zr^{-1} - z^3r^{-3}) \\ &= 8ma_2a_4r^{-3}z\rho^2. \end{aligned}$$

If condition (2.3.13) is satisfied, then (2.3.11) defines a Weyl metric $g_{\alpha\beta}(T,x,y,z;m)$ for all m and $\lim_{m\to 0} g_{\alpha\beta}(T,x,y,z;m)$ is the Weyl metric defined by (2.3.9) which is a regular vacuum metric everywhere. The metric $g_{\alpha\beta}(T,x,y,z;m)$ has singularities at $(T,0,0,0)$. Elsewhere it is a regular vacuum metric. Thus the family $g_{\alpha\beta}(t,x,y,z;m)$ defines a test particle moving (or rather stationary in these co-ordinates) on the curve $\mathcal{C} : (T,0,0,0)$ in the background given by (2.3.9).

The question now arises: Is $\mathcal{C}$ a geodesic of the background metric or not? Using the metric (2.3.5) with u given by (2.3.9),

On $\mathcal{C}$, $-ds^{-2} = g_{00}dT^2$, $g_{00} = -e^{2u}$; $u = 0$ $\therefore ds = dT$

$\frac{dx^\alpha}{dT} = (1,0,0,0)$ so the geodesic equation reduces to $\Gamma^\alpha_{00} = 0$

and this is equivalent to $\frac{\partial g_{00}}{\partial x^i} = 0$. Thus a necessary and sufficient condition for $\mathcal{C}$ to be a geodesic is that $\frac{\partial u}{\partial x^i} = 0$ on $\mathcal{C}$.

$$\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right) = (-2a_2x,\ -2a_2y,\ 2a_2z + a_1)$$

This shows that C is a geodesic of the Weyl metric determined by (2.3.9) if and only if $a_1 = 0$.

There are clearly a lot of solutions of (2.3.13) which have $a_1 \neq 0$ and each of these gives an example of a test particle on a non-geodesic.

Example 2 :

In (2.3.13) let $a_1 = 1,\ a_2 = 0,\ a_3 = 0,\ a_4 = 1$ so that

$$u = z + m\frac{z}{r^3}. \qquad (2.3.14)$$

In this case $u_p = m\frac{z}{r^3}$ and the Weyl metric generated by u_p is sometimes called the "Super Curzon particle". Thus (2.3.14) gives an example of the Super Curzon particle moving on a non-geodesic of a (non-flat) background field.

Example 3 :

In (2.3.13) if $a_3 \neq 0$ and $a_4 = 0$ then $u_p = \frac{m}{r}$ and the Weyl metric generated by u_p is called the Curzon particle.

In this case it follows from (2.3.13) that $a_1 = \frac{-4a_2a_4}{a_3} = 0$ so the Curzon particle has to move on a geodesic (for the restricted family of backgrounds under consideration). Example 3 is a special case of a theorem of Einstein and Grommer (1927) where it is shown that for a general background of the Weyl family, the Curzon particle has to move on a geodesic. This was the first result which showed that certain kinds of singularities had to move on geodesics. The Infeld/Schild theorem of the next chapter is another (more general) result of this kind. Note however, that geodesic motion is not a property of all kinds of singularities. The Super Curzon particle for example does not necessarily move on a geodesic. The Einstein/Grommer theorem is sufficiently interesting to warrant outlining a proof, filling in some of the gaps left in the very sketchy proof of Einstein and Grommer (1927). Another discussion of the Einstein/Grommer theorem may be found in Ehlers (1983).

Let $u = u_b + u_p$ where u_b generates a Weyl metric which is regular in an entire neighbourhood of the origin and u_p generates a Weyl metric which is regular except at the origin.

From (2.3.4), $k(P) - k(Q) = (\Gamma) \int A d\rho + B dz$ where A, B are defined by (2.3.8) and Γ is any path joining P to Q in the region $\rho > 0$ of the ρ, z plane. $\frac{\partial u}{\partial \rho}, \frac{\partial u}{\partial z}$ are bounded in any subregion $0 < \epsilon_1 < \rho < \epsilon_2$ and consequently A and B both tend to zero as $\rho \to 0$ with $z \neq 0$, k can be extended continuously onto the axis except at the origin and $k(P) = k(Q)$ if P, Q are points on the axis which lie on the same side of the origin. However, k may take different values on the axis on either side of the origin (cf the examples above). From (2.3.6) a necessary condition for regularity except at the origin is that k takes the same value on both sides of the origin.
Let Γ be the semicircle $(\rho, z) = (\delta\cos\theta, \delta\sin\theta)$, $-\pi/2 \leq \theta \leq \pi/2$, $\delta > 0$. A necessary condition for regularity of the metric determined by u (except at the origin) is

$$(\Gamma) \int A d\rho + B dz = 0 \tag{2.3.15}$$

Since (2.3.15) is satisfied by u_p and u_b separately one finds on substituting $u = u_p + u_b$ that $(\Gamma) \int A d\rho + B dz =$

$$2(\Gamma) \int \frac{\partial u_b}{\partial z} \rho \left(\frac{\partial u^p}{\partial \rho} dz - \frac{\partial u^p}{\partial z} d\rho \right) + 2(\Gamma) \int \frac{\partial u_b}{\partial \rho} \rho \left(\frac{\partial u^p}{\partial z} dz + \frac{\partial u^p}{\partial \rho} d\rho \right). \tag{2.3.16}$$

If one now specializes by putting $u^p = \frac{m}{r}$, then the second integral is zero since u^p is constant on Γ giving that $\frac{\partial u^p}{\partial z} dz + \frac{\partial u^p}{\partial \rho} d\rho = 0$ on Γ.

On the axis $\frac{\partial u_b}{\partial x} = \frac{\partial u_b}{\partial y} = 0$ because of the cylindrical symmetry, also $\left(\frac{\partial u_b}{\partial z}\right)_\rho = \left(\frac{\partial u_b}{\partial z}\right)_{x,y}$. Hence, by an argument similar to that given previously, $C = (T, 0, 0, 0)$ is a geodesic of the background metric determined by u_b if and only if $\frac{\partial u_b}{\partial z}(T, 0, 0, 0) = 0$.

The value of $(\Gamma) \int A d\rho + B dz$ is independent of δ. By letting $\delta \to 0$ in (2.3.16) one finds that its value is

$$\frac{\partial u_b}{\partial z}(T,0,0,0) \lim_{\delta \to 0} (\Gamma) \int 2\rho \left(\frac{\partial u^p}{\partial \rho}\, dz - \frac{\partial u^p}{\partial z} d\rho \right) = \left(\frac{\partial u_b}{\partial z}(T,0,0,0) \right) (-2m)$$

This shows that if the metric generated by u is to be regular, then C must be a geodesic of the background metric – the Curzon particle must move on a geodesic of the background.

This argument may be applied to other u^p provided two conditions are met:

(i) The second integral in (2.3.16) tends to zero with δ for any u_b

(ii) $\lim_{\delta \to 0} (\Gamma) \int 2\rho \left(\frac{\partial u^p}{\partial \rho} dz - \frac{\partial u^p}{\partial z} d\rho \right) \neq 0.$

The integral occurring in (ii) is usually called the mass of the particle defined by u^p. It is equal to $\pi \oint (grad\ u^p) \cdot \underline{n}\ d\sigma$ over the surface of a sphere centre the origin, radius δ and is consequently independent of δ (so the limit in (ii) could have been omitted).

Condition (ii) then states that the mass of the particle should not be zero. The super Curzon particle fails (ii) since its mass is zero. (It also fails (i).) It is perhaps not unreasonable for a "zero mass" particle to be able to move on a non-geodesic. However one may easily construct an example of a particle of positive mass (in the above sense) which can move on a non-geodesic.

Example 4 :

In (2.3.13) let $a_1 = -4, a_2 = 1, a_3 = 1, a_4 = 1$ then

$$u = -4z + 2z^2 - (x^2 + y^2) + \frac{m}{r} + \frac{mz}{r^3}$$

This defines a test particle which moves on a non geodesic since $a_1 \neq 0$.

Example 4 satisfies (ii) since it has the same mass as the Curzon particle, but it fails (i). Basically, the reason that if fails (i) is because the singularity in u^p is "too strong". It is of the order of $\frac{1}{r^2}$ whereas the singularity for the Curzon particle is of the order of $\frac{1}{r}$. This matter will be considered again in chapter 3 in relation to the Infeld/Schild theorem.

2.4 Comments on the time symmetric two black hole problem

There are no known analytic two-body solutions of the E.F.E. The nearest one might come to such a solution is the time symmetric two black hole problem where analytic initial data is given on a hypersurface of time symmetry. This determines a Cauchy development which may be interpreted physically as representing the head-on approach of two black holes. The Cauchy development must be determined numerically. This problem has been studied in the case where the two black holes have equal mass by Hahn and Lindquist (1964) and by Smarr, Cadez, de Witt, Eppley (1976) although they were concerned with the coalescence of the two black holes whereas here I only want to discuss the motion when the two black holes are well separated.

If a test particle is to be constructed one needs unequal masses with m tending to zero and M fixed. Let

$$\phi(x,y,z) = 1 + \frac{m}{2(x^2+y^2+z^2)^{1/2}} + \frac{M}{2(x-1)^2+y^2+z^2)^{1/2}}.$$

On $\sum : t = 0$ let $g_{\alpha\beta}(x;m) =$ diagonal $(g_{00}(x,y,z;m), \phi^4, \phi^4, \phi^4)$ and $\frac{\partial g_{\alpha\beta}}{\partial t}(x;m) = 0$.

This choice of initial data satisfies the contraints since ϕ is harmonic and ${}^3R = 0$ on $\sum$. (See Hahn & Lindquist p. 310, taking their base metric flat, also Wald p 265.) The test body (black hole) of mass m is situated at the origin at $t = 0$ in each space-time. The lapse and shift functions $g_{00}(x;m)$, $g_{0i}(x;m)$ have to be specified on the whole space to yield a unique solution from the Cauchy data. (The choice of the function $g_{0\alpha}(x;m)$ corresponds to a choice of gauge.) Smarr et al (1976) use a zero shift and maximal slicing which determines the lapse.

Suppose one takes zero shift and chooses the lapse $g_{00}(x;m)$ so that $g_{00}(x;0) = -1$ everywhere, then the background metric $g_{\alpha\beta}(x;0)$ will be a Cauchy development in Gaussian normal coordinates from initial data

$$g_{\alpha\beta}(x,y,z,0;0) = \text{diag}\,(-1,\ \phi_0^4, \phi_0^4, \phi_0^4),\ \frac{\partial g_{\alpha\beta}}{\partial t} = 0$$

where $\phi_0 = 1 + \frac{M}{2(x-1)^2+y^2+z^2)^{1/2}}$. This space can be identified with Schwarzschild corresponding to the single mass M. One may now ask the question whether the above system $g_{\alpha\beta}(x;m)$ defines a test particle on a curve C in the background and if so what is the curve C? Since all the space-times are time symmetric at $t = 0$, the same will hold for the curve C if it exists, hence on C $\left.\frac{dx^i}{dt}\right|_{t=0} = 0$. In Gaussian normal coordinates, the unique geodesic in this direction through the origin is the curve $(t,0,0,0)$ so if one wishes to show that C exists and is a geodesic of the background metric, one must show that C is the curve $(t,0,0,0)$. This amounts to showing that the tube condition holds on $(t,0,0,0)$. The answer to this question depends on the domains of the maximal Cauchy developments and this information does not appear available. However, the answer to the question may be within reach and is of some interest.

Chapter 3

The Infeld/Schild theorem

3.1 Introduction

Let $\gamma_{\alpha\beta}(x) = \lim\limits_{m\to 0} \frac{\partial g_{\alpha\beta}}{\partial m}(x;m)$ as in chapter 2.

In Infeld & Schild (I/S) (1949) their quantity called $b_{\alpha\beta}(x)$ is equivalent to $\gamma_{\alpha\beta}$ and I/S claim to prove that $\mathcal{C}$ is a geodesic if $\gamma_{\alpha\beta}(x)$ is "of the order of $\frac{1}{r}$" where r represents geodesic distance to $\mathcal{C}$ from x in the background metric. Their method is to separate the terms in the linearized vacuum E.F.E. satisfied by $\gamma_{\alpha\beta}(x)$ into various orders in r. Using the terms of orders r^{-3} and r^{-2} only, they deduce that $\mathcal{C}$ is a geodesic. There are a number of aspects of the proof which are unclear. According to I/S a quantity q is said to be of order r^{-n} if $r^n q$ is bounded as $r \to 0$. This condition is not sufficient to allow a unique separation into terms of various orders and yet their method requires that terms of each order in the Einstein tensor be equated to zero separately. Also their choice of coordinate systems is not the most natural and certain equations are assumed without proof to have solutions of a certain type. The treatment to be given here seeks to rectify these and other matters. It is similar in principle to that of I/S but in detail rather different. Note also that in the treatment given here the parameter m is not necessarily related to any kind of mass. On the other hand I/S say that m in their treatment is the mass of their "mass particle". Since no other definition of mass is offered in I/S, this does not seem significant.

3.2 Choice of reference frame

Given a timelike curve $\mathcal{C}$ in a space-time, one may associate with $\mathcal{C}$ a reference frame which is nearly unique. This is referred to in MTW (1971) as the "proper reference frame of an accelerated observer". The definition of this frame depends on the concept of Fermi-Walker transport.

Definition: Let τ be an arclength parameter on $\mathcal{C}$, let u^α be the unit tangent on $\mathcal{C}$, let $a^\alpha = \frac{Du^\alpha}{D\tau}$ be the 4-acceleration on $\mathcal{C}$, let $E(\tau)$ be an event on $\mathcal{C}$, let $\omega^\alpha_{(0)}$ be a vector at $E(\tau_0)$. Then there is one and only one vector field $\omega^\alpha(\tau)$ on $\mathcal{C}$ such that $\frac{D\omega^\alpha}{D\tau} = a^\alpha G(u,\omega) - u^\alpha G(a,\omega)$ and $\omega^\alpha(\tau_0) = \omega^\alpha_0$. Here $G(p,q)$ denotes the inner product $g_{\alpha\beta}p^\alpha q^\beta$. The vector field $\omega^\alpha(\tau)$ is said to be generated from $\omega_{(0)}$ by Fermi-Walker transport on $\mathcal{C}$.

One may show that Fermi-Walker transport preserves the inner product and that the unit tangent to $\mathcal{C}$ is Fermi-Walker transported on $\mathcal{C}$. If $\mathcal{C}$ is a geodesic, Fermi-Walker transport reduces to parallel transport.

The construction of a proper reference frame on $\mathcal{C}$ may be described as follows. Choose an event $E(\tau_0)$ on $\mathcal{C}$ and at this event choose an orthonormal tetrad $u(\tau_0), \omega_0^{(1)}, \omega_0^{(2)}, \omega_0^{(3)}$. Let this orthonormal tetrad be Fermi-Walker transported on $\mathcal{C}$. This generates (uniquely) an orthonormal tetrad $u(\tau), \omega^{(1)}(\tau), \omega^{(2)}(\tau), \omega^{(3)}(\tau)$ at each event $E(\tau)$ on $\mathcal{C}$. Choose a unit vector v at $E(\tau)$ which is orthogonal to $u(\tau)$. Then v is a linear combination of the $\omega^{(i)}$, and $v = \sum_{i=1}^{3} \mu_i \omega^{(i)}(\tau)$. There is a unique geodesic $\mathcal{D}$ through $E(\tau)$ which has v as unit tangent at $E(\tau)$. Let P be the event on $\mathcal{D}$ which is a distance λ along $\mathcal{D}$ from $E(\tau)$ in the direction of v.

Then P is assigned coordinates x^α where $x^0 = \tau, x^i = \lambda\mu_i$. Note that $\sum_{i=1}^{3} x^i x^i = \lambda^2 \sum_{i=1}^{3} \mu_i\mu_i = \lambda^2 G(v,v) = \lambda^2$. Hence $(x^i x^i)^{1/2}$ is the geodesic distance from P to $\mathcal{C}$.

The only indeterminacy in this construction is in the choice of arbitrary constant in the arc-length parameter τ and in the choice of $\omega_0^{(1)}, \omega_0^{(2)}, \omega_0^{(3)}$. A different choice of the latter results in the spatial coordinates x^i being multiplied by a Euclidean orthogonal matrix.

The co-ordinates defined above will be called "proper coordinates on $\mathcal{C}$". (In Nevin (1995) they were called "normal coordinates on $\mathcal{C}$" but it is felt this might be confused with plain "normal coordinates" or with "Fermi normal coordinates" in the present work.)

In the following the notation $t = x^0, x = (t, x^1, x^2, x^3),\ r = (x^i x^i)^{1/2}$ is used.

Let $g_{\alpha\beta}$ be the metric tensor of the space-time in proper coordinates on $\mathcal{C}$ and let $g_{\alpha\beta}$ be analytic near $\mathcal{C}$.

Then $g_{\alpha\beta}(x) = \sum\limits_{i=0}^{\infty} g_{\alpha\beta}^{(i)}(x)$ where $g_{\alpha\beta}^{(i)}(x)$ is the sum of terms of order i in x^1, x^2, x^3 with coefficients functions of t.

It may be shown (MTW p331) that the zero order and the first order terms in $g_{\alpha\beta}$ are

$$g_{\alpha\beta}^{(0)} = \eta_{\alpha\beta}, \quad g_{00}^{(1)} = 2a_i(t)x^i, \quad g_{\alpha\beta}^{(1)} = 0 \ \text{ if } (\alpha, \beta) \neq (0, 0) \tag{3.2.1}$$

where $(0, a_1(t), a_2(t), a_3(t))$ is the 4-acceleration on $\mathcal{C}$ at $(t, 0, 0, 0)$. $\mathcal{C}$ is a geodesic if and only if $a_i(t) = 0,\ i = 1, 2, 3$.

3.3 The definition of "order"

So that one may perform the kind of procedure used in I/S it is necessary to separate expressions in x into terms of various orders such that the following hold:

(i) If P, Q are terms of orders n, m then PQ is a term of order $n + m$.

(ii) If P is a term of order n, then

(a) either $\frac{\partial P}{\partial x^i} = 0$ or $\frac{\partial P}{\partial x^i}$ is a sum of terms of order $n - 1$.

(b) either $\frac{\partial P}{\partial t} = 0$ or $\frac{\partial P}{\partial t}$ is a sum of terms of order n.

(iii) If $\sum_{n=-\infty}^{\infty} P_n = 0$ where P_n is a sum of terms of order n, then $P_n = 0$ for all n.

An example of a system with the above properties is found by taking terms of order n to be of the form $r^n F(x)$ where $F(x) = F(t, x^1, x^2, x^3)$ is differentiable, homogeneous of degree zero in x^1, x^2, x^3 and has no singularities excepting points on C where $x^1 = x^2 = x^3 = 0$.

Definitions:

(i) S will denote the set of functions expressible as infinite series of terms of the type described in the above example, convergent near C, and such that derivatives may be taken term by term.

(ii) S' will denote the larger set with similar properties obtained by including terms of order n which are of the form $(\log r)^m r^n F(x)$.

(iii) $A^{(n)}$ will denote the sum of terms of order n occurring in A, and will be called the component of A of order n.

(iv) Order $A = \min\{n : A$ has a non-zero component of order $n\}$.

Note that S includes all functions analytic in x^i which are expandable as power series in x^i with coefficients functions of t. For such a function f,

$$\begin{aligned} f(x) &= a_0(t) + a_i(t)x^i + a_{ij}(t)x^i x^j + \ldots \\ &= a_0(t) + r\left(a_i(t)\frac{x^i}{r}\right) + r^2\left(a_{ij}(t)\frac{x^i x^j}{r^2}\right) \ldots \end{aligned}$$

and the functions in brackets are homogeneous of degree zero. If $a_0(t) \neq 0, f(x)$ has order zero.

3.4 The order of $\gamma_{\alpha\beta}$ in various examples

Example 1: Consider the example of the Scharzschild test particle in isotropic coordinates as given in section 2.2. The background metric is Minkowskian and C is the curve $(t, 0, 0, 0)$. Proper coordinates on C are the given Minkowskian coordinates and in these coordinates $\gamma_{\alpha\beta} = 0$ when $\alpha \neq \beta$ and $\gamma_{\alpha\beta} = 2/r$ when $\alpha = \beta$. Thus the only non-zero components of $\gamma_{\alpha\beta}$ are of order -1 in proper coordinates on C.

Example 2: Consider example 1 of section 2.3. In order to calculate $\gamma_{\alpha\beta} = \lim\limits_{a_1 \to 0} \frac{\partial g_{\alpha\beta}}{\partial a_1}$, one needs to evaluate $\lim\limits_{a_1 \to 0} \frac{\partial \lambda}{\partial a_1}$ and $\lim\limits_{a_1 \to 0} \frac{\partial \rho}{\partial a_1}$. Both of these limits require evaluation of quantities involving $\frac{\partial h_2}{\partial a_1}$ and $\frac{\partial a_2}{\partial a_1}$ where h_2, a_2 are given by equation (2.3.1) and h_1 is constant.

$$\frac{\partial h_2}{\partial a_1} = 1 + (2a_1h_1 + a_1^2)^{-\frac{1}{2}}(h_1 + a_1) \qquad \text{hence } \frac{\partial h_2}{\partial a_1} \longrightarrow \infty \text{ as } a_1 \to 0.$$

Likewise $\frac{\partial a_2}{\partial a_1} \to \infty$ as $a_1 \to 0$. It is then found that $\lim\limits_{a_1 \to 0} \frac{\partial g_{\alpha\beta}}{\partial a_1}$ does not exist and hence $\gamma_{\alpha\beta}$ is not defined in this case.

Example 3 : Consider example 2 of section 2.3. Since x^α and $r = (x^i x^i)^{1/2}$ will be reserved for proper coordinates, the Weyl coordinates which were previously denoted T, ρ, z, ϕ, and $r = (\rho^2 + z^2)^{1/2}$ are now denoted T, ρ, z, ϕ and $R = (\rho^2 + z^2)^{\frac{1}{2}}$, and components in Weyl coordinates will be denoted by a tilde. With this notation example 2 of section 2.3 is the Weyl metric determined by

$$u = z + \frac{mz}{R^3}$$

Hence $\lim\limits_{m \to 0} u = z$, and $\lim\limits_{m \to 0} \frac{\partial u}{\partial m} = \frac{z}{R^3}$. k was given in section 2.3 and

$$k = \frac{-\rho^2}{2} + 2\rho^2 R^{-3} m + \left(-\frac{2\rho^2}{R^6} + \frac{9}{4}\frac{\rho^4}{R^8} \right) m^2.$$

Hence

$$\lim_{m \to 0} k = \frac{-\rho^2}{2} \quad \text{and} \quad \lim_{m \to 0} \frac{\partial k}{\partial m} = 2\rho^2 R^{-3}.$$

Hence the background metric in Weyl coordinates is

$$\tilde{g}_{\alpha\beta}d(\ )^\alpha d(\ )^\beta = e^{-2z-\rho^2}\,(d\rho^2 + dz^2) + e^{-2z}\rho^2 d\phi^2 - e^{2z}dT^2.$$

and

$$\tilde{\gamma}_{\alpha\beta}d(\ )^\alpha d(\ )^\beta = e^{-2z-\rho^2}\left(\frac{-2z+4\rho^2}{R^3}\right)\,(d\rho^2 + dz^2) + e^{-2z}\left(\frac{-2z}{R^3}\right)\rho^2 d\phi^2$$

$$-e^{2z}\left(\frac{2z}{R^3}\right)dT^2.$$

In order to transform to proper coordinates, the coordinate singularity on the axis is first removed by the transformation, $X^0 = T,\ X^1 = \rho\cos\phi,\ X^2 = \rho\sin\phi,\ X^3 = z$. Tensor components in the X^α frame are denoted by a prime. It is found that

$$g'_{\alpha\beta}(dX)^\alpha(dX)^\beta = e^{-2z}\left(\frac{e^{-\rho^2}-1}{\rho^2}\right)\,(X^1dX^1 + X^2dX^2)^2 + e^{2z+\rho^2}dz^2$$

$$+e^{-2z}((dX^1)^2 + (dX^2)^2) - e^{2z}dT^2.$$

The expression $\frac{e^{-\rho^2}-1}{\rho^2} = -1 + \frac{(X^1)^2+(X^2)^2}{2} - \frac{((X^1)^2+(X^2)^2)^2}{6} + \ldots$ is analytic in $\underline{X} = (X^1, X^2, X^3)$. Hence $g'_{\alpha\beta}$ is analytic in $\underline{X}$. Furthermore,

$$g'_{\alpha\beta} = \eta_{\alpha\beta} + \text{ order } \geq 1 \text{ in } \underline{X} \text{ and } \gamma'_{\alpha\beta} = \frac{1}{R^3}f_{\alpha\beta}(\underline{X}) \tag{3.4.1}$$

where $f_{\alpha\beta}$ is analytic, $f_{\alpha\beta}(\underline{0}) = 0$ and $f_{\alpha\beta}$ has non-zero linear terms.

Let x^α denote proper coordinates. The transformation from X^α to x^α is analytic. On $\mathcal{C}, X^\alpha = (T,0,0,0)$ and $(dx^0)^2 = (e^{2z})_{z=0}(dT)^2$. $g'_{0i} = 0$ and $g'_{\alpha\beta}$ is independent of T. Hence in the X^α frame, $\Gamma^0_{\alpha\beta} = 0$ and on a geodesic $\frac{d^2X^0}{d\tau^2} = 0$.

On a spacelike geodesic orthogonal to $\mathcal{C}$, $\frac{dX^0}{d\tau} = 0$ at the point of intersection with $\mathcal{C}$ and hence X^0 is constant on any spacelike geodesic orthogonal to $\mathcal{C}$. Since $(dX^0)^2 = (dT)^2$ on $\mathcal{C}$ it follows that $X^0 = x^0$ at all events.

On $\mathcal{C}$: $\underline{X} = \underline{0} = \underline{x}$ and $g'_{\alpha\beta} = \eta_{\alpha\beta} = g_{\alpha\beta}$ and $X^i = a_{ij}x^j +$ order ≥ 2 in $\underline{x}$ where $[a_{ij}]_{3\times 3}$ is an orthogonal matrix. Hence $R^2 = (X^iX^i) = r^2 + \phi(x)$ where ϕ is analytic of order ≥ 3 in $\underline{x}$. From this and (3.4.1) $\gamma_{\alpha\beta}$ has order -2.

Example 4: Consider example 3 of section 2.3 taking $a_1 = 0, a_2 = 1, a_3 = 1, a_4 = 0$ in (2.3.13). Then

$$u = 2z^2 - (x^2 + y^2) + \frac{m}{R}, \quad k = (\rho^4 - 8\rho^2 z^2) + (4\rho^2 R^{-1})m + (-\frac{1}{2}\rho^2 R^{-4})m^2.$$

In this case

$$\begin{aligned}
\tilde{\gamma}_{\alpha\beta} d(\ )^\alpha d(\ )^\beta &= e^{-4z^2+2(x^2+y^2)+\rho^4-8\rho^2 z^2} \left(\frac{-2+8\rho^2}{R}\right) (d\rho^2 + dz^2) \\
&+ e^{-4z^2+2(x^2+y^2)} \left(\frac{-2\rho^2}{R}\right) d\phi^2 - e^{4z^2-2(x^2+y^2)} \left(\frac{2}{R}\right) dT^2
\end{aligned}$$

and a similar analysis to the previous example shows that $\gamma_{\alpha\beta}$ is of order -1 in proper coordinates on $\mathcal{C}$.

3.5 Covariant derivatives in proper co-ordinates

Proper coordinates on $\mathcal{C}$ are used throughout this section. If $\gamma_{\alpha\beta}$ is of order ≥ -1, then $\frac{\partial \gamma_{\alpha\beta}}{\partial x^\delta}$ will be of order ≥ -2 and $\frac{\partial^2 \gamma_{\alpha\beta}}{\partial x^\delta \partial x^\epsilon}$ will be of order ≥ -3. In section 3.6 it will be necessary to separate covariant derivatives of various quantities into terms of different orders. To be more specific, it will be necessary to calculate $g^{\alpha\beta}\ \gamma_{\epsilon\delta;\alpha\beta}$ in orders $-3, -2$ and $g^{\beta\delta}\ \gamma_{\epsilon\beta;\delta}$ in orders $-2, -1$ where $\gamma_{\epsilon\delta}$ is of order -1; and to calculate $g^{\beta\delta} V_{\alpha;\beta\delta}$ in orders $-2, -1$ where V_α is of order zero.

Lemma 1: If V_α has order ≥ 0,

$$\begin{aligned}
\left(g^{\beta\delta} V_{\alpha;\beta\delta}\right)^{(-2)} &= V^{(0)}_{\alpha\ ,ii} \\
\left(g^{\beta\delta} V_{0;\beta\delta}\right)^{(-1)} &= -a_i V^{(0)}_{0\ ,i} + V^{(1)}_{0\ ,ii} \\
\left(g^{\beta\delta} V_{i;\beta\delta}\right)^{(-1)} &= a_j V^{(0)}_{i,j} + V^{(1)}_{i,jj}.
\end{aligned}$$

Proof: Inverting the matrix $g_{\alpha\beta}$ one finds from (3.2.1) that for small x^i,

$g^{00} = -1 + 2a_i(t)x^i +$ order ≥ 2 , $g^{\alpha\beta} = \delta_{\alpha\beta} +$ order ≥ 2 if $(\alpha, \beta) \neq (0,0)$.

Also $\Gamma^i_{00} = a_i(t) +$ order ≥ 1, $\Gamma^0_{0i} = \Gamma^0_{i0} = a_i(t) +$ order ≥ 1 and all other Christoffel symbols are of order ≥ 1. Hence

$$V_{\alpha;\beta\delta} = V_{\alpha,\beta\delta} - \Gamma^\kappa_{\alpha\beta} V_{\kappa,\delta} - \Gamma^\kappa_{\alpha\delta} V_{\kappa,\beta} - \Gamma^\kappa_{\beta\delta} V_{\alpha,\kappa} + \text{ order } \geq 0$$

$$g^{\beta\delta} V_{\alpha;\beta\delta} = (-1 + 2a_i x^i)\, V_{\alpha;00} + V_{\alpha;ii} + \text{ order } \geq 0. \tag{3.5.1}$$

$$V_{\alpha;00} = -\Gamma^i_{00} V_{\alpha,i} + \text{ order } \geq 0 \tag{3.5.2}$$

$$V_{\alpha;ii} = V_{\alpha,ii} - 2\Gamma^0_{\alpha i} V_{0,i} + \text{ order } \geq 0 \tag{3.5.3}$$

The Lemma follows from (3.5.1), (3.5.2), (3.5.3).

Lemma 2: If $\gamma_{\alpha\beta}$ is of order -1, then

$$\left(g^{\beta\delta}\gamma_{\alpha\beta;\delta}\right)^{(-2)} = \gamma^{(-1)}_{\alpha i,i}$$

$$\left(g^{\beta\delta}\gamma_{0\beta;\delta}\right)^{(-1)} = -\gamma^{(-1)}_{00,0} + a_i\, \gamma^{(-1)}_{i0} + \gamma^{(0)}_{0i,i}$$

$$\left(g^{\beta\delta}\gamma_{i\beta;\delta}\right)^{(-1)} = -\gamma^{(-1)}_{i0,0} + a_i\gamma^{(-1)}_{00} + a_j\gamma^{(-1)}_{ij} + \gamma^{(0)}_{ij,j}.$$

Proof: The proof is similar to the proof of Lemma 1 and follows from

$$g^{\beta\delta}\gamma_{\alpha\beta;\delta} = (-1 + 2a_i x^i)\gamma_{\alpha 0;0} + \gamma_{\alpha i;i} + \text{ order } \geq 0$$

$$\gamma_{\alpha 0;0} = \gamma_{\alpha 0,0} - \Gamma^\kappa_{\alpha 0}\gamma_{\kappa 0} - \Gamma^\kappa_{00}\gamma_{\alpha\kappa} + \text{ order } \geq 0$$

$$\gamma_{\alpha i;i} = \gamma_{\alpha i,i} - \Gamma^\kappa_{\alpha i}\gamma_{\kappa i} + \text{ order } \geq 0.$$

Lemma 3 : If $\gamma_{\alpha\beta}$ is of order -1, then

$$\left(g^{\delta\epsilon}\gamma_{\alpha\beta;\delta\epsilon}\right)^{(-3)} = \gamma^{(-1)}_{\alpha\beta,ii}$$

$$\left(g^{\delta\epsilon}\gamma_{00;\delta\epsilon}\right)^{(-2)} = \gamma^{(0)}_{00,ii} - 3a_i\gamma^{(-1)}_{00,i}$$

$$\left(g^{\delta\epsilon}\gamma_{0n;\delta\epsilon}\right)^{(-2)} = \gamma^{(0)}_{0n,ii} - a_i\gamma^{(-1)}_{0n,i}$$

$$\left(g^{\delta\epsilon}\gamma_{nm;\delta\epsilon}\right)^{(-2)} = \gamma^{(0)}_{nm,ii} + a_i\gamma^{(-1)}_{nm,i}.$$

Proof: The proof follows from

$$\begin{aligned}\gamma_{\alpha\beta;\delta\epsilon} &= \gamma_{\alpha\beta,\delta\epsilon} - \Gamma^{\kappa}_{\alpha\delta}\gamma_{\kappa\beta,\epsilon} - \Gamma^{\kappa}_{\beta\delta}\gamma_{\alpha\kappa,\epsilon} - \Gamma^{\kappa}_{\alpha\epsilon}\gamma_{\kappa\beta,\delta} - \Gamma^{\kappa}_{\beta\epsilon}\gamma_{\alpha\kappa,\delta}\\ &\quad -\Gamma^{\kappa}_{\delta\epsilon}\gamma_{\alpha\beta,\kappa} + \text{ order } \geq -1\end{aligned}$$

$$g^{\delta\epsilon}\gamma_{\alpha\beta;\delta\epsilon} = (-1 + 2a_ix^i)\gamma_{\alpha\beta;00} + \gamma_{\alpha\beta;ii} + \text{ order } \geq -1$$

$$\gamma_{\alpha\beta;00} = -a_i\gamma_{\alpha\beta,i} + \text{ order } \geq -1$$

$$\gamma_{\alpha\beta;ii} = \gamma_{\alpha\beta,ii} - 2\Gamma^{\kappa}_{\alpha i}\gamma_{\kappa\beta,i} - 2\Gamma^{\kappa}_{\beta i}\gamma_{\alpha\kappa,i} + \text{ order } \geq -1.$$

3.6 The linearized vacuum Einstein field equations

In section 2.2 a definition of a test particle on a curve $\mathcal{C}$ was given in terms of a family of functions $g_{\alpha\beta}(x;m)$ where $g_{\alpha\beta}(x;m)$ satisfies the vacuum E.F.E. outside of a tube U_m which contains $\mathcal{C}$. Hence $R_{\alpha\beta}(x;m) = 0$ outside of U_m where $R_{\alpha\beta}(x;m)$ denotes the Ricci tensor determined by $g_{\alpha\beta}(x;m)$.

Let x be an event not on $\mathcal{C}$ and hold x fixed. There exists a positive number $\delta(x)$ such that x is outside U_m when $0 < m < \delta(x)$. Thus $R_{\alpha\beta}(x;m) = 0$ when $0 < m < \delta(x)$ and hence $\lim\limits_{m\to 0} \frac{\partial}{\partial m}\left(R_{\alpha\beta}(x;m)\right) = 0$.

Provided the extended function $g_{\alpha\beta}(x;m)$ satisfies smoothness conditions at events not on $\mathcal{C}$, which ensure that limits as $m \to 0$ and partial derivatives with respect to

m commute with first and second partial derivatives with respect to x^α, it may be proved (Wald p185) that

$$\lim_{m\to 0}\left(\frac{\partial}{\partial m}R_{\epsilon\delta}(x;m)\right) = -\frac{1}{2}g^{\alpha\beta}\left(\gamma_{\epsilon\delta;\alpha\beta}+\gamma_{\alpha\beta;\epsilon\delta}-\gamma_{\epsilon\alpha;\delta\beta}-\gamma_{\delta\alpha;\epsilon\beta}\right)$$

where $g^{\alpha\beta}$ and covariant derivatives refer to the background metric $g_{\alpha\beta}(x)$. Thus $\gamma_{\alpha\beta}$ satisfies the equation

$$g^{\alpha\beta}\left(\gamma_{\epsilon\delta;\alpha\beta}+\gamma_{\alpha\beta;\epsilon\delta}-\gamma_{\epsilon\alpha;\delta\beta}-\gamma_{\delta\alpha;\epsilon\beta}\right)=0 \qquad (3.6.1)$$

which is the linearized vacuum E.F.E.

3.7 The Infeld/Schild theorem

Considering the various examples which have been given of $\gamma_{\alpha\beta}$, only the Schwarzschild particle, the Kerr particle and Example 4 of section 3.4 have $\gamma_{\alpha\beta}$ which is of order -1 in proper coordinates. All these test particles move on geodesics and they constitute examples of the Infeld/Schild theorem. Roughly speaking, the Infeld/Schild theorem says that if $\gamma_{\alpha\beta}$ is of order -1, then the test particle moves on a geodesic. This theorem is stated more precisely as follows:

The Infeld/Schild theorem : Let $\gamma_{\alpha\beta}(x)$ be defined by (2.2.1) where $g_{\alpha\beta}(x;m)$ is a test particle on $\mathcal{C}$ and (3.6.1) holds. In a proper reference frame on $\mathcal{C}$, suppose that $g_{\alpha\beta}(x)$ is analytic in x^1, x^2, x^3, and $\gamma_{\alpha\beta}(x)\in S$ and the non-zero components of $\gamma_{\alpha\beta}(x)$ are of order ≥ -1.

Then if $\gamma_{00}^{(-1)}(x)\neq 0$, $\mathcal{C}$ is a geodesic of the background metric $g_{\alpha\beta}(x)$. Furthermore if $\mathcal{C}$ is not a geodesic then $\gamma_{\alpha\beta}$ is gauge equivalent to $\omega_{\alpha\beta}\in S'$ where $\omega_{\alpha\beta}$ has all its non-zero components of order ≥ 0.

Proof of the theorem : To make the material more readable, most of the technical details have been placed in an appendix. Proper coordinates are used throughout.

Let

$$\bar{\gamma}_{\alpha\beta}=\gamma_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}\gamma \qquad (3.7.1)$$

where $\gamma = g^{\mu\nu}\gamma_{\mu\nu}$.

It may be shown (appendix Lemma 1) that (3.6.1) is equivalent to

$$g^{\alpha\beta}\left(\gamma_{\epsilon\delta;\alpha\beta} - \bar{\gamma}_{\epsilon\alpha;\beta\delta} - \bar{\gamma}_{\delta\alpha;\beta\epsilon} + 2R_{\epsilon\alpha\delta}{}^{\mu}\gamma_{\mu\beta}\right) = 0 \tag{3.7.2}$$

In the usual perturbation theory where $\gamma_{\alpha\beta}$ does not have singularities, equation (3.7.2) may be simplified by a gauge transformation. One finds $\omega_{\alpha\beta}$ gauge equivalent to $\gamma_{\alpha\beta}$ such that

$$\bar{\omega}_{\alpha\beta;}{}^{\beta} = 0. \tag{3.7.3}$$

This is achieved by solving for V_α the equations

$$g^{\beta\delta}V_{\alpha;\beta\delta} = -\bar{\gamma}_{\alpha\beta;}{}^{\beta} \tag{3.7.4}$$

and defining

$$\omega_{\alpha\beta} = \gamma_{\alpha\beta} + V_{\alpha;\beta} + V_{\beta;\alpha}. \tag{3.7.5}$$

Equation (3.7.3) follows from (3.7.4), (3.7.5). Since $\omega_{\alpha\beta}$ also satisfies (2.2.2) and (3.7.2) one then has from (3.7.2),

$$g^{\alpha\beta}\omega_{\epsilon\delta;\alpha\beta} = -2R_{\epsilon}{}^{\alpha}{}_{\delta}{}^{\mu}\omega_{\mu\alpha}. \tag{3.7.6}$$

Equations (3.7.4) and (3.7.6) are (generalized) wave equations for V_α, $\omega_{\epsilon\delta}$, which are known to have solutions in the non-singular case. We wish to show that a similar procedure may be followed in the singular case.

Consider equation (3.7.4). In the present (singular) case the right hand side of (3.7.4) is in S and a solution is sought in S or in some larger system satisfying the conditions (i), (ii), (iii) of section 3.3. For the purposes of this theorem it is sufficient to solve the simpler problem of satisfying (3.7.4) only in so far as its terms of the two lowest orders are concerned. It is shown in the appendix Lemma 5 that there exists V_α in S' (but not necessarily in S) such that

$$\left(g^{\beta\delta}V_{\alpha;\beta\delta}\right)^{(n)} = \left(-\bar{\gamma}_{\alpha\beta;}{}^{\beta}\right)^{(n)} \quad \text{for } n = -2, -1. \tag{3.7.7}$$

If $\omega_{\alpha\beta}$ is defined by (3.7.5) and (3.7.7) then $\omega_{\alpha\beta} \in S'$ and order $\omega_{\alpha\beta} \geq -1$.

This shows that equation (3.7.4) is satisfied in its two lowest orders.

It is shown in the appendix Lemma 6 that

$$\left(\bar{\omega}_{\epsilon\delta;}{}^{\delta}\right)^{(n)} = 0 \quad \text{for } n = -2, -1 \tag{3.7.8}$$

and

$$\left(g^{\alpha\beta}\bar{\omega}_{\epsilon\delta;\alpha\beta}\right)^{(n)} = 0 \quad \text{for } n = -3, -2. \tag{3.7.9}$$

This shows that equations (3.7.8) and (3.7.9) are satisfied in their two lowest orders. (Note that the right hand side of (3.7.6) has order ≥ -1.)

Using the results of section 3.5, equations (3.7.8) and (3.7.9) may be written explicitly as equations (1) and (2) respectively below

$$\bar{\omega}^{(-1)}_{\alpha i,i} = 0 \tag{1)(i}$$

$$\bar{\omega}^{(-1)}_{00,0} - a_i\, \bar{\omega}^{(-1)}_{i0} - \bar{\omega}^{(0)}_{0i,i} = 0 \tag{1)(ii}$$

$$\bar{\omega}^{(-1)}_{j0,0} - a_j\bar{\omega}^{(-1)}_{(00)} - a_i\bar{\omega}^{(-1)}_{ji} - \bar{\omega}^{(0)}_{ji,i} = 0 \tag{1)(iii}$$

$$\bar{\omega}^{(-1)}_{\epsilon\delta,ii} = 0 \tag{2)(i}$$

$$\bar{\omega}^{(0)}_{00,ii} = 3a_i\bar{\omega}^{(-1)}_{00,i} \tag{2)(ii}$$

$$\bar{\omega}^{(0)}_{0n,ii} = a_i\bar{\omega}^{(-1)}_{0n,i} \tag{2)(iii}$$

$$\bar{\omega}^{(0)}_{nm,ii} = -a_i\bar{\omega}^{(-1)}_{nm,i}. \tag{2)(iv}$$

The complete solution in S' of (2)(i) is

$$\bar{\omega}^{(-1)}_{\epsilon\delta} = \frac{c_{\epsilon\delta}(t)}{r}. \tag{3.7.10}$$

From (3.7.10) and (1)(i)

$$-\, c_{\alpha i}(t)\frac{x^i}{r^3} = 0. \tag{3.7.11}$$

From (3.7.11)

$$c_{\alpha i}(t) = 0 \quad \text{and from (3.7.10)} \quad \bar{\omega}^{(-1)}_{\alpha i} = 0. \tag{3.7.12}$$

Since $\bar{\omega}_{\alpha\beta}$ is symmetric this leaves $\bar{\omega}_{00}^{-1}$ as the only possible non-zero component of $\bar{\omega}_{\alpha\beta}^{(-1)}$. From (2)($iv$) and (3.7.12)

$$\bar{\omega}_{nm,ii}^{(0)} = 0 \tag{3.7.13}$$

The complete solution of (3.7.13) in S' is

$$\bar{\omega}_{nm}^{(0)} = d_{nm}(t) \tag{3.7.14}$$

From (2)(iii) and (3.7.12)

$$\bar{\omega}_{0n,ii}^{(0)} = 0 \tag{3.7.15}$$

The complete solution of (3.7.15) in S' is

$$\bar{\omega}_{0n}^{(0)} = d_n(t). \tag{3.7.16}$$

From (1)(ii), (3.7.12), (3.7.16)

$$\bar{\omega}_{00,0}^{(-1)} = 0. \tag{3.7.17}$$

From (3.7.17), (3.7.10),

$$\bar{\omega}_{00}^{(-1)} = \frac{c_{00}}{r} \quad \text{with } c_{00} \text{ constant.} \tag{3.7.18}$$

From (1)(iii), (3.7.12), (3.7.14)

$$a_j \bar{\omega}_{00}^{(-1)} = 0. \tag{3.7.19}$$

From (3.7.5)

$$\bar{\omega}_{00}^{(-1)} = \gamma_{00}^{(-1)} \tag{3.7.20}$$

From the appendix Lemma 3

$$\bar{\omega}_{\alpha\beta}^{(-1)} = 0 \text{ for all } \alpha, \beta \text{ if and only if } \omega_{\alpha\beta}^{(-1)} = 0 \text{ for all } \alpha, \beta \tag{3.7.21}$$

Now suppose $\gamma_{00}^{(-1)} \neq 0$, then from (3.7.20) $\omega_{00}^{(-1)} \neq 0$ so by (3.7.21) there exists a non-zero component of $\bar{\omega}_{\alpha\beta}^{(-1)}$. Hence by (3.7.12) $\bar{\omega}_{00}^{(-1)} \neq 0$. Then by (3.7.19) $a_j(t) = 0$ for $j = 1, 2, 3$ showing that C is a geodesic. If C is not a geodesic then (3.7.19) can only be satisfied if $\bar{\omega}_{00}^{(-1)} = 0$ and then by (3.7.12), $\bar{\omega}_{\alpha\beta}^{(-1)} = 0$ and by (3.7.21) $\omega_{\alpha\beta}^{(-1)} = 0$ showing that $\omega_{\alpha\beta}$ has order ≥ 0.

This completes the proof.

Appendix to section 3.7

Lemma 1: Equation (3.7.2) is equivalent to equation (3.6.1) for a vacuum metric.

Proof: From (3.7.1),

$$\bar{\gamma}_{\epsilon\alpha;\beta\delta} = \gamma_{\epsilon\alpha;\beta\delta} - \frac{1}{2}g_{\epsilon\alpha}\gamma_{;\beta\delta} = \gamma_{\epsilon\alpha;\delta\beta} - R_{\beta\delta\epsilon}{}^{\mu}\gamma_{\mu\alpha} - R_{\beta\delta\alpha}{}^{\mu}\gamma_{\epsilon\mu} - \frac{1}{2}g_{\epsilon\alpha}\gamma_{;\beta\delta}$$

Hence

$$g^{\alpha\beta}\bar{\gamma}_{\epsilon\alpha;\beta\delta} = g^{\alpha\beta}\gamma_{\epsilon\alpha;\delta\beta} - g^{\alpha\beta}R_{\beta\delta\epsilon}{}^{\mu}\gamma_{\mu\alpha} - \frac{1}{2}\gamma_{;\epsilon\delta} \qquad (i)$$

since $g^{\alpha\beta}R_{\beta\delta\alpha}{}^{\mu} = 0$ for a vacuum metric. Substituting (i) into (3.7.2) and using the symmetries of $g^{\alpha\beta}, \gamma_{\alpha\beta}, R_{\alpha\beta\delta\epsilon}$ one finds that (3.7.2) reduces to (3.6.1).

Lemma 2: For a vacuum metric, if $\omega_{\alpha\beta} = \gamma_{\alpha\beta} + V_{\alpha;\beta} + V_{\beta;\alpha}$ then

$$\bar{\omega}_{\alpha\beta;}{}^{\beta} = \bar{\gamma}_{\alpha\beta;}{}^{\beta} + g^{\beta\delta}V_{\alpha;\beta\delta}.$$

Proof:

$$\bar{\omega}_{\alpha\beta} = \bar{\gamma}_{\alpha\beta} + V_{\alpha;\beta} + V_{\beta;\alpha} - \tfrac{1}{2}g_{\alpha\beta}g^{\mu\nu}\left(V_{\mu;\nu} + V_{\nu;\mu}\right)$$

$$\bar{\omega}_{\alpha\beta;}{}^{\beta} = \bar{\gamma}_{\alpha\beta;}{}^{\beta} + g^{\kappa\beta}V_{\alpha;\beta\kappa} + g^{\kappa\beta}V_{\beta;\alpha\kappa} - \tfrac{1}{2}g^{\mu\nu}\left(V_{\mu;\nu\alpha} + V_{\nu;\mu\alpha}\right) \qquad (ii)$$

For a vacuum metric $g^{\beta\delta}\left(V_{\beta;\alpha\delta} - V_{\beta;\delta\alpha}\right) = -g^{\beta\delta}R_{\beta\alpha\delta}{}^{\epsilon}V_{\epsilon} = 0$, so the last three terms in (ii) add to zero giving the required result.

Lemma 3: If $\omega_{\alpha\beta}$ is of order ≥ -1 then $\omega_{\alpha\beta}^{(-1)} = 0$ for all α, β if and only if $\bar{\omega}_{\alpha\beta}^{(-1)} = 0$ for all α, β.

Proof: From (3.7.1), $\bar{\omega}_{\alpha\beta}^{(-1)} = \omega_{\alpha\beta}^{(-1)} - \frac{1}{2}\left(g_{\alpha\beta}g^{\mu\nu}\right)^{(0)}\omega_{\mu\nu}^{(-1)}$. Also from (3.7.1), contracting with $g^{\alpha\beta}$, one finds $\bar{\omega} = -\omega$ and hence $\omega_{\alpha\beta}^{(-1)} = \bar{\omega}_{\alpha\beta}^{(-1)} - \frac{1}{2}\left(g_{\alpha\beta}g^{\mu\nu}\right)^{(0)}\bar{\omega}_{\mu\nu}^{(-1)}$. The required result follows from these two equations.

Lemma 4: Let A be a term in S of order k. A may be expressed in terms of spherical harmonics:

$$A = r^k \sum_{n=0}^{\infty} \sum_{\ell=-n}^{n} b_{\ell n} Y_n^\ell$$

Then

(a) If $k = -2$, then all functions ϕ in S' which contain only terms of order zero and which satisfy Poisson's equation $\nabla^2\phi = A$ are given by the formula

$$\phi = c_0 + b_{00} \log r - \sum_{n=1}^{\infty} \sum_{\ell=-n}^{n} \frac{b_{\ell n} Y_n^\ell}{n^2 + n}.$$

(b) If $k = -1$, all functions ϕ in S' which contain only terms of order 1 and which satisfy Poisson's equation $\nabla^2\phi = A$ are given by the formula

$$\phi = c_i x^i + \frac{r \log r}{3} \left(b_{-1,1} Y_1^{-1} + b_{01} Y_1^0 + b_{11} Y_1^1 \right) + r \sum_{n \neq 1} \sum_{\ell=-n}^{n} \frac{b_{\ell n} Y_n^\ell}{(2 - (n^2 + n))}$$

Proof: These formulae may be derived using

$$\nabla^2(r^m Y_n^\ell) = r^{m-2}(m^2 + m - [n^2 + n]) Y_n^\ell \qquad \text{and}$$

$$\nabla^2((\log r) r^m Y_n^\ell) = r^{m-2}(2m + 1 + \log r[m^2 + m - n^2 - n]) Y_n^\ell.$$

Lemma 5: Equation (3.7.7) is valid.

Proof: Using the results of section 3.5 equation (3.7.7) when $n = -2, -1$ is found equivalent to (i) , (ii) below respectively:

$$V_{\alpha,ii}^{(0)} = \bar{\gamma}_{ai,i}^{(-1)} \qquad \text{(i)}$$

$$V_{0,ii}^{(1)} = a_i V_{0,i}^{(0)} - \bar{\gamma}_{00,0}^{(-1)} + a_i \bar{\gamma}_{i0}^{(-1)} + \bar{\gamma}_{0i,i}^{(0)} \qquad \text{(ii)(a)}$$

$$V_{n,ii}^{(1)} = -a_i V_{n,i}^{(0)} - \bar{\gamma}_{n0,0}^{(-1)} + a_n \bar{\gamma}_{00}^{(-1)} + a_i \bar{\gamma}_{ni}^{(-1)} + \bar{\gamma}_{ni,i}^{(0)} \qquad \text{(ii)(b)}$$

$\bar{\gamma}^{(-1)}_{\alpha i,i} \in S$ and is homogeneous of order -2. Hence by Lemma 4(a), there is a solution to (i) in S' for $V^{(0)}_\alpha$. Furthermore $V^{(0)}_{\alpha,i}$ contains no log terms and is a homogeneous function in S of order -1. Thus all terms on the right hand sides of (ii)(a), (ii)(b) are homogeneous functions in S of order -1. By Lemma 4(b) there exist solutions $V^{(1)}_0, V^{(1)}_n$ in S'.

Equation (3.7.7) is satisfied with $V_\alpha = V^{(0)}_\alpha + V^{(1)}_\alpha$.

Lemma 6: Equations (3.7.8) and (3.7.9) are valid.

Proof: Equation (3.7.8) follows immediately from Lemma 2 and equation (3.7.7). $\omega_{\alpha\beta}$ satisfies the linearized vacuum E.F.E. Hence by (3.7.2)

$$g^{\alpha\beta}\omega_{\epsilon\delta;\alpha\beta} = (\bar{\omega}_{\epsilon\alpha;}{}^{\alpha})_{;\delta} + (\bar{\omega}_{\delta\alpha;}{}^{\alpha})_{;\epsilon} - 2g^{\alpha\beta}R_{\epsilon\alpha\delta}{}^{\mu}\omega_{\mu\beta}$$

The first two terms on the right hand side are of order ≥ -1 by equation (3.7.8). Hence $g^{\alpha\beta}\omega_{\epsilon\delta;\alpha\beta}$ is of order ≥ -1.

From (3.7.1), $g^{\alpha\beta}\bar{\omega}_{\epsilon\delta;\alpha\beta} = g^{\alpha\beta}\omega_{\epsilon\delta;\alpha\beta} - \frac{1}{2}g_{\epsilon\delta}g^{\mu\tau}\left(g^{\alpha\beta}\omega_{\mu\tau;\alpha\beta}\right)$. This shows $g^{\alpha\beta}\bar{\omega}_{\epsilon\delta;\alpha\beta}$ is of order ≥ -1.

3.8 Comments on the Infeld/Schild theorem

It is surprising that the Infeld/Schild theorem should be widely regarded as proving the geodesic law while it is quite quite clear that the theorem only proves the geodesic law for test particles having $\gamma_{\alpha\beta}$ of order $\frac{1}{r}$ and also satisfying other conditions. So in fact the Infeld/Schild theorem merely shifts the problem to another problem of showing that the $\frac{1}{r}$ condition holds. Infeld and Schild themselves make no comment on the $\frac{1}{r}$ condition and give no reason why it should hold. They stress the point that their theorem only uses the vacuum field equations and is independent of the nature of the stress-energy tensor of the matter. I believe this latter claim is a misconception. The fact that the $\frac{1}{r}$ condition is required to hold may place restrictions on the nature of $T^{\alpha\beta}$ inside the test body and it is not known what these restrictions are. In Nevin (1995) it was shown that the approximate post-Newtonian metric of DSX (1991) satisfies the $\frac{1}{r}$ condition in the weaker original sense of Infeld/Schild (1949). This result, although suggestive, is not conclusive because the conditions required for the I/S theorem were not all shown to hold and furthermore no mathematical relationship has been shown to exist between $\gamma_{\alpha\beta}$ for the approximate metric and $\gamma_{\alpha\beta}$ for the true metric. Indeed, results obtained in Chapters 4 and 5 of this thesis suggest strongly that the geodesic law is invalid for certain $T^{\alpha\beta}$ in the regime of weak fields with slow motion.

Several authors (e.g. DSX (1991)) have pointed out that derivations of equations of motion may be classified roughly into three types. The first type (to which Infeld/Schild (1949) belongs) makes assumptions about the nature of the field outside of the body, the second type (e.g. The example of section 1.2 and the approach of chapters 4,5) makes assumptions about the nature of the field inside the body and the third type is the method of matched asymptotic expansions which attempts to combine two fields such as a background field and a Schwarzschild field to produce the total field of a "Schwarzschild black hole" moving in a background field. It seems likely that the $\frac{1}{r}$ condition in I/S was inspired by the Schwarzschild test particle on a flat background, so in a certain sense it is related to the third type above.

One of the best known papers of the third type is D'Eath (1975). D'Eath makes an asymptotic expansion of the metric in powers of a parameter M. The zero-th order approximation represents the background field. It does not seem possible to make everything precise in the method of asymptotic expansions. Quoting from the introduction to D'Eath (1975) "A rigorous treatment of the problem is far beyond the scope of this work, and we merely suppose that the true and background universes are sufficiently well behaved in the large that our local considerations are valid." D'Eath shows that the zero-th order world line ℓ_0 is a geodesic in the background metric. I believe that the mechanism which is producing this geodesic result is essentially that of Infeld/Schild. The $\frac{1}{r}$ condition enters because the lowest order part of the internal metric in D'Eath (1975) is the Kerr solution which satisfies the $\frac{1}{r}$ condition (see D'Eath p. 1389). Kates (1980) claims to generalize the work of D'Eath and also obtains a geodesic result by similar methods.

Another paper containing a geodesic result is Hogan and Robinson (1986). The mechanism here appears to be the $\frac{1}{r}$ condition again. It enters in equation (5) p. 457 (without justification).

Any test particle satisfying the hypotheses of the Infeld/Schild theorem must have $\gamma_{\alpha\beta}$ of order -1 in S and $\gamma_{\alpha\beta}$ must satisfy the linearized E.F.E. It is therefore of interest to know whether there are solutions of the linearized .E.F.E. of this type for a test particle on an arbitrary timelike geodesic. This question is considered in section 3.9.

3.9 Initial terms in a solution of the linearized equations

The Infeld/Schild theory raises the obvious problem of whether there exist test particles of the assumed type on arbitrary geodesics. One would like to prove that there exists a test particle, for which $\gamma_{\alpha\beta}$ exists and has order $\frac{1}{r}$, on an arbitrary timelike geodesic of an arbitrary background field. To this end it will be necessary (but not sufficient) to solve equation (3.7.2) in all orders ≥ -3 when $\mathcal{C}$ is a geodesic. For orders $-3, -2$ the equations to be solved involve only $\gamma_{\alpha\beta}^{(-1)}$ and $\gamma_{\alpha\beta}^{(0)}$ and are independent of the background. Thus they may be satisfied with $\gamma_{\alpha\beta}^{(-1)} = \text{diag } \frac{2}{r}$, $\gamma_{\alpha\beta}^{(0)} = 0$ as in Schwarzschild. The curvature of the background enters for the first time at order -1 and an expression for $g_{\alpha\beta}^{(2)}$ in proper coordinates on $\mathcal{C}$ is required. Manasse and Misner (1963) derived the following expressions for $g_{\alpha\beta}^{(2)}$:

$$
\begin{aligned}
g_{00} &= -1 - R^{(0)}_{0\ell o m} x^\ell x^m + \text{ order } \geq 3 \\
g_{oj} &= \frac{-4}{3} R^{(0)}_{0\ell j m} x^\ell x^m + \text{ order } \geq 3
\end{aligned}
$$

$$g_{ij} = \delta_{ij} - \frac{1}{3} R^{(0)}_{i\ell jm} x^\ell x^m + \text{ order } \geq 3$$

where $R^{(0)}_{\alpha\beta\gamma\delta} = R^{(0)}_{\alpha\beta\gamma\delta}(t) =$ Riemann tensor of $g_{\alpha\beta}$ evaluated on $\mathcal{C}$ at $(ct, 0, 0, 0)$.

Inverting the matrix $g_{\alpha\beta}$ one finds

$$\begin{aligned}
g^{00} &= -1 + R^{(0)}_{0\ell om} x^\ell x^m + \text{ order } \geq 3 \\
g^{0j} &= -\frac{4}{3} R^{(0)}_{0\ell jm} x^\ell x^m + \text{ order } \geq 3 \\
g^{ij} &= \delta_{ij} + \frac{1}{3} R^{(0)}_{i\ell jm} x^\ell x^m + \text{ order } \geq 3.
\end{aligned}$$

Also, order $\Gamma^\rho_{\mu v} \geq 1$ and $(\Gamma^0_{00})^{(1)} = 0$, $(\Gamma^0_{0i})^{(1)} = (\Gamma^i_{00})^{(1)} = R^{(0)}_{0m0i} x^m$,
$(\Gamma^0_{ij})^{(1)} = \frac{2}{3}\left(R^{(0)}_{0jim} - R^{(0)}_{0ijm}\right) x^m$, $(\Gamma^i_{0j})^{(1)} = \frac{2}{3}\left(R^{(0)}_{0ijm} - R^{(0)}_{0jim} + 2R^{(0)}_{0mji}\right) x^m$
$(\Gamma^i_{jk})^{(1)} = \frac{1}{3}\left(R^{(0)}_{ijmk} + R^{(0)}_{ikmj}\right) x^m$.

If $\gamma^{(-1)}_{\alpha\beta}$, $\gamma^{(0)}_{\alpha\beta}$ are as in Schwarzschild, one finds $\bar{\gamma}^{(-1)}_{00} = \frac{4}{r}$, $\bar{\gamma}^{(-1)}_{\alpha\beta} = 0$ otherwise and $\bar{\gamma}^{(0)}_{\alpha\beta} = 0$.

One must now attempt to find $\bar{\gamma}^{(1)}_{\alpha\beta}$ so that

$$\left(g^{\alpha\beta} \bar{\gamma}_{\epsilon\alpha;\beta}\right)^{(n)} = 0 \text{ for } n = -2, -1, 0 \tag{3.9.1}$$

and

$$\left[g^{\alpha\beta}\left(\gamma_{\epsilon\delta;\alpha\beta} + 2R_{\epsilon\alpha\delta}{}^\mu \gamma_{\mu\beta}\right)\right]^{(n)} = 0 \text{ for } n = -3. - 2. - 1, \ldots \tag{3.9.2}$$

Any solution of (3.9.1) and (3.9.2) will satisfy (3.7.2) in orders $-3, -2, -1$.

The following Lemma shows that equation (3.9.2) can be written with $\bar{\gamma}_{\alpha\beta}$ in place of $\gamma_{\alpha\beta}$.

Lemma: If the following equation is true, then so is equation (3.9.2):

$$\left[g^{\alpha\beta}\left(\bar{\gamma}_{\epsilon\delta;\alpha\beta} + 2R_{\epsilon\alpha\delta}{}^\mu\, \bar{\gamma}_{\mu\beta}\right)\right]^{(n)} = 0, \text{ for } n = -3, -2, -1. \tag{3.9.3}$$

Proof:

$$g^{\alpha\beta}\gamma_{\epsilon\delta;\alpha\beta} = g^{\alpha\beta}\bar{\gamma}_{\epsilon\delta;\alpha\beta} - \frac{1}{2} g_{\epsilon\delta} g^{\sigma\tau}\left(\bar{\gamma}_{\sigma\tau;\alpha\beta} g^{\alpha\beta}\right)$$

$$= \quad -2g^{\alpha\beta}R_{\epsilon\alpha\delta}{}^{\mu}\bar{\gamma}_{\mu\beta} - \frac{1}{2}g_{\epsilon\delta}g^{\sigma\tau}\left(-2R_{\sigma\alpha\tau}{}^{\mu}\bar{\gamma}_{\mu\beta}g^{\alpha\beta}\right)$$

for orders ≤ 1 from (3.9.3).

In vacuum, $g^{\sigma\tau}R_{\sigma\alpha\tau}{}^{\mu} = 0$ hence the second term above is zero. Substituting $\bar{\gamma}_{\mu\beta} = \gamma_{\mu\beta} - \frac{1}{2}g_{\mu\beta}g^{\sigma\tau}\gamma_{\sigma\tau}$ in the first term and using $R_{\epsilon\mu\delta}{}^{\mu} = 0$, one finds the required result.

Consider equation (3.9.3). When $n = -3$, equation (3.9.3) becomes $\nabla^2\bar{\gamma}^{(-1)}_{\alpha\beta} = 0$ which is satisfied and when $n = -2$, it becomes $\nabla^2\bar{\gamma}^{(0)}_{\alpha\beta} = 0$ which is satisfied. When $n = -1$ equation (3.9.3) gives the following Poisson equation for $\bar{\gamma}^{(1)}_{\alpha\beta}$:

$$\begin{aligned}\nabla^2\bar{\gamma}^{(1)}_{\epsilon\delta} \quad &= \quad -(\Gamma^{i}_{00})^{(1)}\bar{\gamma}^{(-1)}_{\epsilon\delta,i} + (\Gamma^{0}_{\epsilon i})^{(1)}_{,i}\bar{\gamma}^{(-1)}_{0\delta} + (\Gamma^{0}_{\delta i})^{(1)}_{,i}\bar{\gamma}^{(-1)}_{\epsilon 0} + 2(\Gamma^{0}_{\epsilon i})^{(1)}\bar{\gamma}^{(-1)}_{0\delta,i} \\ &\quad + 2(\Gamma^{0}_{\delta i})^{(1)}\bar{\gamma}^{(-1)}_{\epsilon 0,i} + (\Gamma^{j}_{ii})^{(1)}\bar{\gamma}^{(-1)}_{\epsilon\delta,j} - \frac{8}{r}R^{(0)}_{\epsilon 0\delta 0} - \frac{1}{3}R^{(0)}_{i\ell jm}x^{\ell}x^{m}\bar{\gamma}^{(-1)}_{\epsilon\delta,ij}.\end{aligned}$$

A solution to this equation is found to be

$$\bar{\gamma}^{(1)}_{oo} = \frac{10}{3}R^{(0)}_{0i0j}\,\frac{x^ix^j}{r}, \quad \bar{\gamma}^{(1)}_{0n} = \frac{4}{3}R^{(0)}_{0inj}\,\frac{x^ix^j}{r}, \quad \bar{\gamma}^{(1)}_{nm} = -4rR^{(0)}_{0n0m}$$

and all solutions which are homogeneous of order one are obtained by adding linear functions to this.

The values for $\bar{\gamma}^{(n)}_{\alpha\beta}$, $n = -1, 0, 1$ which are given above have been shown to satisfy equation (3.9.3). It may now be verified that they also satisfy equation (3.9.1).

From the values for $\bar{\gamma}^{(n)}_{\alpha\beta}$ one may evaluate $\gamma^{(n)}_{\alpha\beta}$ for $n = -1, 0, 1$ as

$$\begin{aligned}\gamma^{(-1)}_{\alpha\beta} \quad &= \quad \operatorname{diag}\frac{2}{r}, \quad \gamma^{(0)}_{\alpha\beta} = 0 \\ \gamma^{(1)}_{00} \quad &= \quad \frac{5}{3}\,R^{(0)}_{0i0j}\,\frac{x^ix^j}{r}, \quad \gamma^{(1)}_{0n} = -\frac{4}{3}\,R^{(0)}_{0inj}\,\frac{x^ix^j}{r} \\ \gamma^{(1)}_{nm} \quad &= \quad -4R^{(0)}_{0n0m}\,r - \frac{2}{3}\,R^{(0)}_{nimj}\,\frac{x^ix^j}{r} - \frac{1}{3}\delta_{nm}\,R^{(0)}_{0i0j}\,\frac{x^ix^j}{r}.\end{aligned}$$

Note that the fairly lengthy verification of the various results stated above requires frequent use of the symmetries of the Riemann tensor and the fact that $g_{\alpha\beta}$ is a vacuum metric so that the Ricci tensor is zero. It is clear that better methods are required if one attempts to extend the results to higher order terms. This has not yet been achieved.

Chapter 4

Post Newtonian Theory

4.1 Introduction

Blanchet and Damour (1989) defined potentials for a weak, slow motion gravitational field in the near zone at the first post-Newtonian level. These potentials were used in a series of papers by DSX (1991 - 1994) to develop a theory of motion in celestial mechanics. One of the applications of this theory was to give a new derivation of the EIH equations of motion for a monopole which the authors claim to be superior to other derivations. A crucial step in the DSX derivation of the EIH equations is equation (7.17) of DSX (1991). This equation is an equation of motion (to within 1PN accuracy) of the DSX centre of mass of a monopole in terms of potentials of a field defined by the other bodies of the system. This field will be called "the DSX background field" to avoid confusion with the term "background field" as used in chapters 2,3. Equation 7.17 of DSX (1991) is a geodesic equation for the DSX background field.

Equations in DSX are generally of the form: "expression = order c^{-4} ". In most cases the proof that the given expression is actually of order c^{-4} is not given in the paper so many of the equations in DSX are stated rather than proved. Equation 7.17 of DSX (1991) is found from their general equation of motion by making the monopole truncation and dividing by the mass of the monopole. Both the general equation of motion and equation 7.17 have an unspecified error term of order c^{-4}.

It is thus clear that the passage from the general equation of motion to equation 7.17 without explanation is unjustified unless the mass of the monopole is at least of order unity. The latter condition is equivalent to requiring that the mass of the monopole be comparable with the masses of the other bodies of the system. It is interesting to note that this condition has often been required in previous work on the EIH equations (see for example Havas (1986)) but it is not mentioned in DSX.

The main purpose of this chapter is to derive an equation analogous to 7.17 of DSX, which will apply to any body (not necessarily having monopole structure) provided it is sufficiently small. The derivation will use the potentials as defined in Blanchet and Damour (1989) and DSX (1991) and some of the preliminary results of these papers. Those results which are used will be justified as far as possible. The main theory of DSX and their equations of motion will not be used because it is a formidable task to supply the missing proofs for all of this material and to structure the error term in the equation of motion in such a way that one might show that it remains of order c^{-4} after division by the mass of the test body. Even if all this were achieved the result would be extremely lengthy and cumbersome. The object here is to produce a simpler exposition which is reasonably self contained and in which the origin of the geodesic law for test particles is more apparent.

4.2 The Newtonian problem and a system of units

For simplicity a two-body system has been assumed throughout this and subsequent sections. Initially no restricions are placed on the two bodies but in the test particle case body B will be regarded as the body creating the background field while body A will be the test body. In the following M_C, L_C denote the mass and radius respectively of any body C.

In Newtonian mechanics, assuming conservation of mass and the inverse square law of gravitation, one has:

$$M_A \frac{d^2 z^i}{dt^2} = \int_A d^3x\, \rho(x) \frac{\partial w_B}{\partial x^i}(x) \tag{4.2.1}$$

where $x = (ct, x^1, x^2, x^3)$, $d^3x = dx^1 dx^2 dx^3$

$$M_A = \int_A d^3x\, \rho(x) = \text{mass of body A (at time } t) ,$$

$A = A(t)$ is the region of R^3 occupied by body A at time t,

$$w_B(x) = G \int_B d^3x' \frac{\rho(x')}{|x - x'|} = \text{gravitational potential of body B},$$

$$z^i(t) \text{ is defined by} \int_A d^3x \rho(x)(x^i - z^i(t)) = 0$$

and represents the Newtonian centre of mass of body A at time t.

All quantities are measured in a Newtonian "inertial frame" where the centre of mass of the two body system is stationary. In equation (4.2.1) the right hand side represents the total gravitational force on A due to B. The total gravitational force on A due to A is zero. This is a general result for a body with internal forces acting in equal and opposite pairs. It may also be proved directly that

$$\int_A d^3x\, \rho(x) \frac{\partial w_A}{\partial x^i}(x) = 0 \tag{4.2.2}$$

where w_A is the gravitational potential of body A defined in a similar way to w_B.

Since $\rho > 0$ on A, a mean value theorem for integrals (Apostol (1957)) gives that

$$\int_A d^3x\, \rho(x) \frac{\partial w_B}{\partial x^i}(x) = (\int_A d^3x \rho(x)) \frac{\partial w_B}{\partial x^i}(ct, \xi(t)) = M_A \frac{\partial w_B}{\partial x^i}(ct, \xi(t))$$

where $\xi(t) \in A(t)$. Hence from equation (4.2.1) on dividing by M_A,

$$\frac{d^2 z^i}{dt^2} = \frac{\partial w_B}{\partial x^i}(ct, \xi(t)) = \frac{\partial w_B}{\partial x^i}(ct, z(t)) + (\xi^j(t) - z^j(t)) \frac{\partial^2 w_B}{\partial x^j \partial x^i}(ct, \eta(t)) \tag{4.2.3}$$

where $\eta(t)$ is on the join of $\xi(t), z(t)$. From equation (4.2.3),

$$|\frac{d^2 z^i}{dt^2} - \frac{\partial w_B}{\partial x^i}(ct, z(t))| \leq 2L_A K \tag{4.2.4}$$

where L_A is the radius of A and K is a bound for $\frac{\partial^2 w_B}{\partial x^j \partial x^i}(ct, \eta(t))$ which is independent of L_A, depending only on B and the minimum distance from B to A.

Equation (4.2.4) shows how the motion of a small body of arbitrary shape and and composition approximates the motion of a Newtonian "particle" situated at $z(t)$ which has acceleration equal to the gradient of w_B. For an extended body the motion of the centre of mass usually deviates from this path and satisfies an equation of the form

$$\frac{d^2 z^i}{dt^2} = \frac{\partial w_B}{\partial x^i}(ct, z(t)) + \frac{1}{2M_A}\frac{\partial^3 w_B}{\partial x^j \partial x^k \partial x^i}\Omega_{jk}(t) + ... \tag{4.2.5}$$

where Ω_{jk} is the quadrupole moment of A about its centre of mass and terms containing higher moments have been omitted. Equation (4.2.5) may be derived from (4.2.1) by expanding $\frac{\partial w_B}{\partial x^i}(x)$ in (4.2.1) as a power series about $(ct, z(t))$.

There is a special case where the extended body centre of mass has acceleration equal to the gradient of w_B. This is the case of a rigid sphere of uniform density which may be called a monopole since the quadrupole moment and all higher moments in equation (4.2.5) are zero. Alternatively, since w_B and $\frac{\partial w_B}{\partial x^i}$ are harmonic inside A, one may derive this result directly from equation (4.2.1) using the mean value theorem for harmonic functions (see Fritz John (1986) page 99). Hence one finds in this case

$$\frac{d^2 z^i}{dt^2} = \frac{\partial w_B}{\partial x^i}(ct, z(t)) \tag{4.2.6}$$

The post-Newtonian theory of DSX bears a close relationship to the Newtonian Theory presented above. Blanchet/Damour (BD) potentials w_A and w_B will be defined in section 4.3. Equation 7.17 of DSX (1991) is analogous to equation (4.2.6) although it contains additional terms which are of order c^{-2} and an error term of order c^{-4}.

The global reference frame of DSX, described in section 4.3, approximates a Newtonian inertial frame with origin at the centre of mass of the system in the sense that the motion of the bodies in this frame is approximately Newtonian. The system of units used will be characteristic units based on B such that $G = 1, M_B =$

$1, L_B = 1$. In this system of units the dimensionless quantity $\gamma_B = \frac{GM_B}{c^2 L_B} = \frac{1}{c^2}$ so that $O(c^{-4}) = O(\gamma_B^2)$ (See Damour (1987, p 153). For the sun $\gamma_B \sim 2.10^{-6}$, for the earth $\gamma_B \sim 7.10^{-10}$. All quantities including c acquire numerical values in this system of units. DSX on the other hand are not clear as to the meaning of "order c^{-4}" in their papers. I have failed to find any definition of a system of units in DSX. The first occasion on which any meaning is attributed to "order c^{-4}" is in the last paper of the series DSX (1994) p 623, in connection with satellite motion where it appears that in equation 7.17 of DSX (1991), "order c^{-4}" may mean " order $d^{-2}c^{-4}$" in the terminology adopted in this thesis.

In the sequel, following DSX, $O(i)$ means $O(c^{-i})$; $u^{\alpha\beta} = O(a,b,d)$ means $u^{00} = O(a), u^{0i} = O(b), u^{ij} = O(d)$; $u^{\alpha} = O(p,q)$ means $u^0 = O(p), u^i = O(q)$. Also $x^0 = ct$.

4.3 The Blanchet/Damour potentials

The reference frame is chosen to satisfy various conditions. It approximates a Newtonian inertial frame in the sense described in section 4.2 and is post Newtonian so that the following PN assumptions hold

$$g_{00} = -1 + O(2), \quad g_{0i} = O(3), \quad g_{ij} = \delta_{ij} + O(2) \tag{4.3.1}$$

$$T^{00} = O(c^2), \quad T^{0i} = O(c^1), \quad T^{ij} = O(c^0) \tag{4.3.2}$$

where $g_{\alpha\beta}$ is the metric tensor and $T^{\alpha\beta}$ is the stress-energy tensor of the two body system. Furthermore, following the theory of DSX (1991), one may also impose the conditions that the reference frame be both spatially isotropic (equation 4.3.3) and harmonic (equation 4.3.4) at the 1PN level of approximation.

$$g_{ij} = -\frac{\delta_{ij}}{g_{00}} + O(4) \tag{4.3.3}$$

$$\frac{\partial}{\partial x^0}(g_{00} + g_{ii}) - 2\frac{\partial}{\partial x^i}(g_{i0}) = O(5) \tag{4.3.4}$$

A reference frame with all these properties is selected and is used throughout this chapter. In DSX it is called a global harmonic reference frame. All quantities are calculated in this frame using the system of units defined in section 4.2.

The BD potentials w^α, w^α_A, w^α_B and related functions $\tilde{g}_{\alpha\beta}$, $\overline{g}_{\alpha\beta}$ are defined in terms of $T^{\alpha\beta}$ as follows where $x = (ct, \underline{x})$:

$$\sigma = \sigma^0 = \frac{T^{00} + T^{bb}}{c^2}, \; \sigma^i = \frac{T^{0i}}{c}$$

$$w^\alpha(x) = G \int d^3x' \, \frac{\sigma^\alpha(ct - |x - x'|, \underline{x}')}{|x - x'|} \; , \; w = w^0$$

w^α_A , w^α_B are obtained by putting $\sigma^\alpha = 0$

outside of the world tubes of A,B respectively.

$$\tilde{g}_{00} = -\exp(\frac{-2w}{c^2}), \; \tilde{g}_{0i} = \frac{-4w^i}{c^3}, \; \tilde{g}_{ij} = \delta_{ij} \exp(\frac{2w}{c^2})$$

$$\overline{g}_{00} = -\exp(\frac{-2w_B}{c^2}), \; \overline{g}_{0i} = \frac{-4w^i_B}{c^3}, \; \overline{g}_{ij} = \delta_{ij} \exp(\frac{2w_B}{c^2}) \tag{4.3.5}$$

It follows immediately from the above definitions that $w^\alpha = w^\alpha_A + w^\alpha_B$ and w^α satisfies the following non-homogeneous wave equation:

$$\Box \, w^\alpha = -4\pi G \sigma^\alpha \tag{4.3.6}$$

The functions $\tilde{g}_{\alpha\beta}$ define a metric which will be called the "BD approximate metric". An analytical one body example in which $\tilde{g}_{\alpha\beta}$ and $g_{\alpha\beta}$ are calculated explicitly in a reference frame satisfying equations (4.3.1) to (4.3.4) is given in the appendix to this section. It is seen in this example how closely $\tilde{g}_{\alpha\beta}$ and its derivatives approximate $g_{\alpha\beta}$ and its derivatives. A relationship in general between these two metrics is given by equation (4.3.17) below.

The functions $\overline{g}_{\alpha\beta}$ define a metric which is called the "BD background metric". It must be distinguished from the "background metric" as used in a test particle situation in chapters 2 and 3. It depends only on the stress-energy inside B but this depends on A so the BD background is not known a priori.

The field equations are taken in the following form:

$$R^{\alpha\beta} = \frac{8\pi G}{c^4}(T^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}T) \tag{4.3.7}$$

The first step in relating $\tilde{g}_{\alpha\beta}$ to $g_{\alpha\beta}$ is to show that with appropriate assumptions, equation (4.3.7) with $(\alpha, \beta) = (0,0), (0,i)$ respectively is equivalent to the following pair of equations:

$$\Box \ln(-g_{00}) = \frac{8\pi G}{c^2}\sigma + O(6) \tag{4.3.8}$$

$$\Box g_{0i} = \frac{16\pi G}{c^3}\sigma^i + O(5) \tag{4.3.9}$$

These are equations 2.3a,b of BD (1989). Since a derivation of (4.3.8) and (4.3.9) is not shown in BD (1989) or in DSX, it will be given here. The field equations involve second derivatives of $g_{\alpha\beta}$ hence it is found necessary to make assumptions about the first and second derivatives of equations (4.3.1) and (4.3.3) and the first derivative of equation (4.3.4). In fact one needs to assume that the error terms of various orders remain of these same orders after differentiation. These conditions are not mentioned in BD or DSX. They may be verified in the case of the analytical example given in the appendix.

The following definition is introduced for convenience.

Definition: A quantity of O(n) will be said to be $\partial(1)$ or $\partial(2)$ if its first or first and second partial derivatives respectively with respect to x^i, t are O(n).

Theorem : If the error terms appearing in equations (4.3.1), (4.3.3) are $\partial(2)$ and the error terms in (4.3.4) are $\partial(1)$ then equations (4.3.8) and (4.3.9) are valid.

Proof : For notational convenience in this proof quantities W^α are defined by

$$\frac{-2W}{c^2} = \ln(-g_{00}), \quad \frac{-4W^i}{c^3} = g_{0i}$$

(This notation is used only in the present proof. It conflicts with the notation of DSX where W^α has a different meaning.) From (4.3.3)

$$g_{ij} = \delta_{ij}\exp(\frac{2W}{c^2}) + O(4) \tag{4.3.10}$$

From (4.3.1) and the last equation,

$$\frac{-2W}{c^2} \sim \frac{1}{c^2}, \quad \frac{-4W^i}{c^3} \sim \frac{1}{c^3} \tag{4.3.11}$$

Hence $|W^\alpha| < 1$. Since the errors in (4.3.1) are $\partial(2)$, all first and second partials of g_{00} are O(2) and all first and second partials of g_{0i} are O(3). By differentiating (4.3.10) one finds that all first and second partials of W^α with respect to x^i, t have absolute value of the order of unity (or less). Using these bounds on W^α and its partials and the fact that the error in equation (4.3.10) is $\partial(2)$ one may calculate the Christoffel symbols and Riemann tensor components in the following form :

$$\begin{aligned}
\Gamma^0_{00} &= -\frac{1}{c^3}\frac{\partial W}{\partial t} + \frac{4}{c^5}W^i\frac{\partial W}{\partial x^i} + O(7) \\
\Gamma^i_{00} &= -\frac{1}{c^2}\frac{\partial W}{\partial x^i} + \frac{4}{c^4}W\frac{\partial W}{\partial x^i} - \frac{4}{c^4}\frac{\partial W^i}{\partial t} + O(6) \\
\Gamma^0_{i0} &= -\frac{1}{c^2}\frac{\partial W}{\partial x^i} + O(6) \\
\Gamma^i_{j0} &= \delta_{ij}\frac{1}{c^3}\frac{\partial W}{\partial t} - \frac{2}{c^3}\frac{\partial W^i}{\partial x^j} + \frac{2}{c^3}\frac{\partial W^j}{\partial x^i} + O(5) \\
\Gamma^0_{ij} &= \delta_{ij}\frac{1}{c^3}\frac{\partial W}{\partial t} + \frac{2}{c^3}\frac{\partial W^i}{\partial x^j} + \frac{2}{c^3}\frac{\partial W^j}{\partial x^i} + O(5) \\
\Gamma^i_{jk} &= \delta_{ij}\frac{1}{c^2}\frac{\partial W}{\partial x^k} + \delta_{ik}\frac{1}{c^2}\frac{\partial W}{\partial x^j} - \delta_{jk}\frac{1}{c^2}\frac{\partial W}{\partial x^i} + O(4) \\
\Gamma^\nu_{\nu 0} &= \frac{2}{c^3}\frac{\partial W}{\partial t} + O(5) \\
\Gamma^\nu_{\nu i} &= \frac{2}{c^2}\frac{\partial W}{\partial x^i} + O(4) \\
R^{00} &= -\frac{1}{c^2}\nabla^2 W - \frac{3}{c^4}\frac{\partial^2 W}{\partial t^2} - \frac{4}{c^4}\frac{\partial^2 W^i}{\partial t \partial x^i} + O(6) \\
R^{0i} &= -\frac{2}{c^3}\nabla^2 W^i + \frac{2}{c^3}\frac{\partial^2 W^j}{\partial x^i \partial x^j} + \frac{2}{c^3}\frac{\partial^2 W}{\partial t \partial x^i} + O(5) \\
R^{ij} &= -\delta_{ij}\frac{1}{c^2}\nabla^2 W + O(4)
\end{aligned}$$

When equation (4.3.4) is expressed in terms of W^α it becomes

$$\frac{\partial W}{\partial t} + \frac{\partial W^i}{\partial x^i} = O(2)$$

with a $\partial(1)$ error. Consequently

$$\frac{\partial}{\partial t}(\frac{\partial W}{\partial t}+\frac{\partial W^i}{\partial x^i})=O(2),\quad \frac{\partial}{\partial x^j}(\frac{\partial W}{\partial t}+\frac{\partial W^i}{\partial x^i})=O(2) \tag{4.3.12}$$

When (4.3.12) is substituted into R^{00} and R^{0i} one finds

$$R^{00}=-\frac{1}{c^2}\Box W+O(6),\quad R^{0i}=-\frac{2}{c^3}\Box W^i+O(5)$$

After calculating $T=g_{\alpha\beta}T^{\alpha\beta}$ the corresponding field equations are

$$-\frac{1}{c^2}\Box W=\frac{4\pi G}{c^2}\sigma+O(6),\quad -\frac{2}{c^3}\Box W^i=\frac{8\pi G}{c^3}\sigma^i+O(5),$$

which are clearly equivalent to (4.3.8) and (4.3.9) respectively. This completes the proof.

Using (4.3.6), equations (4.3.8) and (4.3.9) may be written in the following form:

$$\Box\ln(-g_{00})=\Box(\frac{-2w}{c^2})+O(6) \tag{4.3.13}$$

$$\Box g_{0i}=\Box(\frac{-4w^i}{c^3})+O(5) \tag{4.3.14}$$

The next step in relating $\tilde{g}_{\alpha\beta}$ to $g_{\alpha\beta}$ is to pass from (4.3.13), (4.3.14) to the following two equations:

$$\ln(-g_{00})=(\frac{-2w}{c^2})+O(6) \tag{4.3.15}$$

$$-g_{0i}=(\frac{-4w^i}{c^3})+O(5) \tag{4.3.16}$$

This step is equivalent (apart from the complication of the unspecified residual terms) to selecting a particular solution to the differential equations (4.3.13), (4.3.14). The validity of this step was assumed in BD (1989) with references to previous work. It will also be assumed here but may be demonstrated explicitly in the example below which is a static case with no radiation. In BD (1989) it was shown that a matching procedure between a near zone metric given by (4.3.15), (4.3.16), (4.3.3) and a wave zone metric results in a radiation formula which agrees with the classical quadrupole formula at the lowest order. (Note that in DSX half advanced/half retarded solutions are used in place of the retarded solutions (4.3.15), (4.3.16) of BD (1989).)

The final step is to use (4.3.15), (4.3.16), (4.3.3) and the definition of $\tilde{g}_{\alpha\beta}$ in terms of w^α to find

$$g_{\alpha\beta} = \tilde{g}_{\alpha\beta} + O(6,5,4) \tag{4.3.17}$$

Equation (4.3.17) is equation 2.10 of BD (1989).

There are no known analytical solutions of the Einstein equations representing two bodies in vacuum on which to test the various assumptions listed above but they may be tested to a limited extent on a one body exact solution with a Schwarzschild exterior as in the analytical example given in the appendix to this section. Since the one body field in this example is the limit of a two-body field with a second test body tending to zero, it is essential that the assumptions should pass this test.

Appendix to section 4.3

Consider the example of a body with a Schwarzschild exterior and a uniform density interior as given in Misner, Thorne, Wheeler (1973) page 609 in Schwarzschild coordinates t, r, θ, ϕ. [To convert from MTW to the present notation: In the metric replace dt, m, M by dx^0, $\frac{Gm}{c^2}$, $\frac{GM}{c^2}$ respectively. Multiply all components of $T^{\alpha\beta}$ by c^2.]

The reference frame must first be changed to a PN reference frame $\overline{x}^\alpha$ in which (4.3.1) to (4.3.4) hold. The internal metric has the form

$$ds^2 = -e^{2\Phi}(dx^0)^2 + \frac{dr^2}{(1 - \frac{2Gm}{c^2 r})} + r^2 d\Omega^2$$

where

$$e^\Phi = \frac{3}{2}(1-2\gamma)^{\frac{1}{2}} - \tfrac{1}{2}(1-2\gamma(\frac{r}{R})^2)^{\frac{1}{2}},$$

$$\gamma = \frac{GM}{c^2 R}, \quad m = \frac{4\pi}{3}\rho_0 r^3, \quad M = \frac{4\pi}{3}\rho_0 R^3$$

This matches the exterior Schwarzschild metric in standard form at the boundary $r = R$.

One may attempt a transformation of the form

$$\overline{x}^0 = x^0,\ \overline{x}^1 = \overline{r}\sin\theta\cos\phi,\ \overline{x}^2 = \overline{r}\sin\theta\sin\phi,\ \overline{x}^3 = \overline{r}\cos\theta$$

where $\overline{r} = f(r)$. The transformed metric now takes the form

$$ds^2 = -e^{2\Phi}(dx^0)^2 + ((\frac{r}{\overline{r}})^2 + \frac{\overline{x}^2}{\overline{r}^2}[(\frac{dr}{d\overline{r}})^2(1-\gamma(\frac{r}{R})^2)^{-1} - (\frac{r}{\overline{r}})^2])d\overline{x}^2 + ... \tag{4.3.18}$$

$$+[(\frac{dr}{d\overline{r}})^2(1-\gamma(\frac{r}{R})^2)^{-1} - (\frac{r}{\overline{r}})^2]2\frac{\overline{x}\overline{y}}{\overline{r}^2}d\overline{x}d\overline{y} + ...$$

Since $\gamma = c^{-2}$ and $\frac{\overline{x}}{\overline{r}} \sim 1$, it is necessary from (4.3.3) considering the off-diagonal terms, to have

$$[(\frac{dr}{d\overline{r}})^2(1-\gamma(\frac{r}{R})^2)^{-1} - (\frac{r}{\overline{r}})^2] = O(\gamma^2) \tag{4.3.19}$$

$$\overline{g}_{00} = -e^{2\Phi} = -(1-\gamma(\frac{3}{4}-\frac{1}{4}(\frac{r}{R})^2) + O(\gamma^2)$$

In order to satisfy (4.3.3) it is necessary to have

$$\overline{g}_{ij} = \delta_{ij}(1+\gamma(\frac{3}{4}-\frac{1}{4}(\frac{r}{R})^2) + O(\gamma^2) \tag{4.3.20}$$

From (4.3.18) and (4.3.19) it follows that

$$\overline{g}_{ij} = (\frac{r}{\overline{r}})^2\delta_{ij} + O(\gamma^2) \tag{4.3.21}$$

On equating (4.3.20) and (4.3.21) one obtains an equation relating r and $\overline{r}$ from which it is found that a suitable transformation for the internal region is

$$\overline{r} = r - \gamma(\frac{3r}{2} - \frac{r^3}{2R^2}) \text{ for } r < R$$

A suitable transformation for the external region is found to be

$$\overline{r} = r - \gamma R \text{ for } r > R$$

and the transformed metrics match on the boundary $r = R$. In the new co-ordinates one finds on the interior ($r < R$), letting $u = \frac{r}{R}$ and writing $g_{\alpha\beta}$ for $\overline{g}_{\alpha\beta}$:

$$g_{00} = -1 + (3-u^2)\gamma + (-\frac{3}{4}+\frac{3}{2}u^2-\frac{3}{4}u^4)\gamma^2 + (-\frac{3}{4}+\frac{3}{4}u^2+\frac{3}{4}u^4-\frac{3}{4}u^6)\gamma^3 + O(\gamma^4)$$

$$g_{0i} = 0$$

$$g_{ij} = \delta_{ij} + \delta_{ij}(3 - u^2)\gamma + (\delta_{ij}(\frac{27}{4} - \frac{9}{2}u^2 + \frac{3}{4}u^4) + \frac{x^i x^j}{R^2}(-3 + 4u^2))\gamma^2 + O(\gamma^3)$$

where $x^i = \frac{r}{\bar{r}}\bar{x}^i$. $T^{\alpha\beta}$ may also be found in the new coordinates and hence w^α and $\tilde{g}_{\alpha\beta}$ may be calculated. The results after a lengthy calculation are as follows:

$$\frac{2w}{c^2} = (3 - u^2)\gamma + (\frac{15}{4} - \frac{3}{2}u^2 - \frac{1}{4}u^4)\gamma^2 + O(\gamma^3), \quad w^i = 0$$

$$g_{00} = \tilde{g}_{00} + (\frac{23}{2} - 12u^2 + \frac{63}{10}u^4 - \frac{11}{7}u^6)\gamma^3 + O(\gamma^4)$$

$$g_{0i} = \tilde{g}_{0i}$$

$$g_{ij} = \tilde{g}_{ij} + (\delta_{ij}(\frac{27}{4} - \frac{9}{2}u^2 + \frac{3}{4}u^4) + \frac{x^i x^j}{R^2}(-3 + 4u^2) - \frac{33}{4} - \frac{9}{2}u^2 - \frac{1}{4}u^4)\gamma^2 + O(\gamma^3)$$

On the exterior ($r > R$) one finds, letting $U = \frac{R}{r}$:

$$g_{00} = -1 + 2U\gamma$$

$$g_{0i} = 0$$

$$g_{ij} = \delta_{ij} + (2U\delta_{ij})\gamma + (3U^2\delta_{ij} + \frac{x^i x^j}{R^2}U^4)\gamma^2 + O(\gamma^3)$$

$$\frac{2w}{c^2} = 2U\gamma + 2U^2\gamma^2 + O(\gamma^3) \ , w^i = 0$$

$$g_{00} = \tilde{g}_{00} + (\frac{78}{35}U + 2U^3)\gamma^3 + O(\gamma^4)$$

$$g_{0i} = \tilde{g}_{0i}$$

$$g_{ij} = \tilde{g}_{ij} + (-U^2\delta_{ij} + \frac{x^i x^j}{R^2}U^4)\gamma^2 + O(\gamma^3)$$

In the above $g_{\alpha\beta}$ and $\tilde{g}_{\alpha\beta}$ have been expanded in powers of γ as far as the leading parts of the error terms in (4.3.17).

The above example may be used to model the field of B alone or the field of A alone. The error terms are given in terms of $u, U,$ and γ. The leading terms suffice when $\gamma < 10^{-6}$ as in the solar system. Derivatives with respect to $\overline{x}^i$ may be calculated using

$$|\frac{\partial u^2}{\partial x^i}| \leq \frac{2u}{R}, \quad |\frac{\partial U}{\partial x^i}| \leq \frac{U^2}{R}$$

$$|\frac{\partial^2 u^2}{\partial x^i \partial x^j}| \leq \frac{2\delta_{ij}}{R^2}, \quad |\frac{\partial^2 U}{\partial x^i \partial x^j}| \leq \frac{2U^3}{R^2}$$

For the field of B alone, using the units defined in section 4.2, $R = 1$ and $\gamma = \gamma_B = \frac{1}{c^2}$.

For the field of A alone, in the same system of units, the derivatives of u^2 and U are unbounded as $R = L_A \to 0$ but this is compensated by the fact that $\gamma_A \to 0$. More precisely, since $\gamma_A = \frac{4\pi G \rho_A L_A^2}{3c^2}$ one has

$$\frac{\gamma_A}{L_A} \to 0 \text{ as } L_A \to 0$$

$$\text{and } \frac{\gamma_A}{L_A^2} = \frac{4\pi G \rho_A}{3c^2} = (\frac{\rho_A}{\rho_B})\gamma_B = (\frac{\rho_A}{\rho_B})c^{-2}$$

It will be assumed that $\frac{\rho_A}{\rho_B} \sim 1$.

All the assumptions of section (4.3) may be shown valid for A and B separately in this model.

4.4 Geodesic Acceleration

The standard form for the geodesic equation with respect to a field $g_{\alpha\beta}$ is

$$\frac{d^2 z^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\nu} \frac{dz^\mu}{d\tau} \frac{dz^\nu}{d\tau} = 0 \tag{4.4.1}$$

Since $\frac{dz^\alpha}{d\tau}$ is a unit vector, only three of the four components of (4.4.1) are independent and (4.4.1) may be written in a more convenient form as :

$$\frac{d^2 z^i}{dt^2} + (\Gamma^i_{\mu\nu} - \frac{1}{c}\Gamma^0_{\mu\nu}\frac{dz^i}{dt})\frac{dz^\mu}{dt}\frac{dz^\nu}{dt} = 0 \tag{4.4.2}$$

For future convenience equation (4.4.2) may be written as

$$\frac{d^2 z^i}{dt^2} = \mathcal{G}^i(g_{\alpha\beta}, z^\gamma) \tag{4.4.3}$$

where the quantity on the right hand side is defined by

$$\mathcal{G}^i(g_{\alpha\beta}, z^\gamma) = -(\Gamma^i_{\mu\nu} - \frac{1}{c}\Gamma^0_{\mu\nu}\frac{dz^i}{dt})\frac{dz^\mu}{dt}\frac{dz^\nu}{dt}$$

with $\Gamma^\alpha_{\beta\gamma}$ evaluated on $z^\alpha(t)$, and will be called "the geodesic acceleration on the curve $z^\alpha(t)$ for the field $g_{\alpha\beta}(x)$".

In previous sections all equations applied to the entire region containing the two bodies but in dealing with geodesic acceleration on a curve within body A one need only calculate Christoffel symbols inside A and consequently the distance d from A to B enters the calculations. Christoffel symbols for the fields $\tilde{g}_{\alpha\beta}$ and $\overline{g}_{\alpha\beta}$ are expressible in terms of w^α and w^α_B respectively and their first derivatives inside A. These rather complicated expressions may be truncated as in the theorem of section 4.3, with terms which are negligible at 1PN order assigned to an error term whose order of magnitude may be given in powers of c, d provided one has estimates for w^α_B and w^α_A and their first derivatives inside A. For a test body in Earth orbit about the sun, $d \sim 10^2$, $c \sim 10^3$ while for the satellite LAGEOS orbiting the Earth, $d \sim 2$, $c \sim 10^5$.

The potentials w^α_B are expressible in terms of T^{00} and T^{0i} inside B. Let V_A, V_B denote approximate speeds of A,B respectively. From Newtonian theory for orbits with small eccentricity and for small L_A, $V_A^2 \sim \frac{1}{d} < 1$ and $V_B << V_A$. It is assumed that $\frac{T^{00}}{c^2} \sim 1$ and that inside B, $|\frac{cT^{0i}}{T^{00}}| \sim V_B < V_A$. These are only rough order of magnitude estimates. It must be stressed that $T^{\alpha\beta}$ is not assumed to have the ideal fluid form or any other particular analytic form. Apart from the retardation effect, w_B is like the Newtonian potential of body B having density $\sigma \sim \frac{T^{00}}{c^2}$ and w^i_B is like the Newtonian potential of body B having "density" $\sigma^i = \frac{T^{0i}}{c}$. It is also assumed that d is fairly large or failing that, B is close to being spherically symmetric. On the basis of these assumptions one may make the following rough order of magnitude estimates for w^α_B inside A. These estimates enable one to derive equation 7.17 of

DSX, which is the geodesic equation for the field $\overline{g}_{\alpha\beta}$.

$$w_B \sim \frac{1}{d}, \quad \frac{\partial w_B}{\partial x^i} \sim \frac{1}{d^2}, \quad \frac{\partial^2 w_B}{\partial x^i \partial x^j} \sim \frac{1}{d^3}$$

$$w_B^i < \frac{V_A}{d}, \quad \frac{\partial w_B^i}{\partial x^j} < \frac{V_A}{d^2}, \quad \frac{\partial^2 w_B^i}{\partial x^j \partial x^k} < \frac{V_A}{d^3}$$

In addition the usual PN assumption is made that inside A, time derivatives are smaller than space derivatives by a factor of the order of V_A.

Calculation of the Christoffel symbols for $\overline{g}_{\alpha\beta}$ inside A is as in the theorem of section 4.3 except that w_B replaces W and the above estimates are used for the error instead of taking all quantities to be of the order of unity. The result is that inside A (putting $V_A \sim d^{-\frac{1}{2}}$):

$$\begin{aligned}
\overline{\Gamma}^0_{00} &= -\frac{1}{c^3}\frac{\partial w_B}{\partial t} + \frac{4}{c^5} w_B^i \frac{\partial w_B}{\partial x^i} + d^{-4.5}O(7) \\
\overline{\Gamma}^i_{00} &= -\frac{1}{c^2}\frac{\partial w_B}{\partial x^i} + \frac{4}{c^4} w_B \frac{\partial w_B}{\partial x^i} - \frac{4}{c^4}\frac{\partial w_B^i}{\partial t} + d^{-4}O(6) \\
\overline{\Gamma}^0_{i0} &= -\frac{1}{c^2}\frac{\partial w_B}{\partial x^i} + d^{-4}O(6) \\
\overline{\Gamma}^i_{j0} &= \delta_{ij}\frac{1}{c^3}\frac{\partial w_B}{\partial t} - \frac{2}{c^3}\frac{\partial w_B^i}{\partial x^j} + \frac{2}{c^3}\frac{\partial w_B^j}{\partial x^i} + d^{-3.5}O(5) \\
\overline{\Gamma}^0_{ij} &= \delta_{ij}\frac{1}{c^3}\frac{\partial w_B}{\partial t} + \frac{2}{c^3}\frac{\partial w_B^i}{\partial x^j} + \frac{2}{c^3}\frac{\partial w_B^j}{\partial x^i} + d^{-3.5}O(5) \\
\overline{\Gamma}^i_{jk} &= \delta_{ij}\frac{1}{c^2}\frac{\partial w_B}{\partial x^k} + \delta_{ik}\frac{1}{c^2}\frac{\partial w_B}{\partial x^j} - \delta_{jk}\frac{1}{c^2}\frac{\partial w_B}{\partial x^i} + d^{-3}O(4)
\end{aligned}$$

Using these values one may now calculate

$$\mathcal{G}^i(\overline{g}_{\alpha\beta}, z^\gamma) = \frac{\partial w_B}{\partial x^i} + \frac{1}{c^2}\left(\frac{dz^k}{dt}\frac{dz^k}{dt}\frac{\partial w_B}{\partial x^i} - 3\frac{dz^i}{dt}\frac{\partial w_B}{\partial t} + 4\frac{\partial w_B^i}{\partial t}\right. \tag{4.4.4}$$

$$\left. -4w_B\frac{\partial w_B}{\partial x^i} + \left(\frac{\partial w_B^j}{\partial x^i} - \frac{\partial w_B^i}{\partial x^j}\right)\frac{dz^j}{dt} - 4\frac{\partial w_B}{\partial x^j}\frac{dz^i}{dt}\frac{dz^j}{dt}\right) + d^{-4}O(4)$$

where it is assumed that $z^\alpha(t)$ is inside A and $\frac{dz^\alpha}{dt} \sim V_A$. If (4.4.4) is substituted into (4.4.3) then (4.4.3) becomes equation 7.17 of DSX (1991).

In order to find a formula corresponding to (4.4.4) for $\mathcal{G}^i(\tilde{g}_{\alpha\beta}, z^\gamma)$ one needs estimates for w_A^α and its first derivatives inside A. Here the retardation effect will be truly negligible. Lemma 3(a) p.148 of Kellogg (1929) may be used to show that for the Newtonian potential w_A of section 4.2 at a point inside A, $|w_A| < \rho_{max} 2\pi L_A^2$ and $|\frac{\partial w_A}{\partial x^i}| < \rho_{max} 4\pi L_A$. Thus based on the assumption that inside A $\frac{T^{00}}{c^2} \sim 1$, $|\frac{cT^{0i}}{T^{00}}| \sim V_A$ the following estimates are found:

$$w_A \sim L_A^2, \quad \frac{\partial w_A}{\partial x^i} \sim L_A, \quad \frac{\partial^2 w_A}{\partial x^i \partial x^j} \sim 1$$

$$w_A^i \sim V_A L_A^2, \quad \frac{\partial w_A^i}{\partial x^j} \sim V_A L_A, \quad \frac{\partial^2 w_A^i}{\partial x^j \partial x^k} \sim V_A$$

For time derivatives, the same assumptions are made as before. It is seen from these estimates that if $L_A < d^{-2}$, the estimates for w_A^α and its first derivatives are smaller that the corresponding estimates for w_B^α and its first derivatives. This means that $\tilde{\Gamma}^\alpha_{\beta\gamma}$ bear the same relationship to w^α that $\overline{\Gamma}^\alpha_{\beta\gamma}$ bear to w_B^α when $L_A < d^{-2}$. Explicitly,

$$\begin{aligned}
\tilde{\Gamma}^0_{00} &= -\frac{1}{c^3}\frac{\partial w}{\partial t} + \frac{4}{c^5} w^i \frac{\partial w}{\partial x^i} + d^{-4.5} O(7) \\
\tilde{\Gamma}^i_{00} &= -\frac{1}{c^2}\frac{\partial w}{\partial x^i} + \frac{4}{c^4} w \frac{\partial w}{\partial x^i} - \frac{4}{c^4}\frac{\partial w^i}{\partial t} + d^{-4} O(6) \\
\tilde{\Gamma}^0_{i0} &= -\frac{1}{c^2}\frac{\partial w}{\partial x^i} + d^{-4} O(6) \\
\tilde{\Gamma}^i_{j0} &= \delta_{ij}\frac{1}{c^3}\frac{\partial w}{\partial t} - \frac{2}{c^3}\frac{\partial w^i}{\partial x^j} + \frac{2}{c^3}\frac{\partial w^j}{\partial x^i} + d^{-3.5} O(5) \\
\tilde{\Gamma}^0_{ij} &= \delta_{ij}\frac{1}{c^3}\frac{\partial w}{\partial t} + \frac{2}{c^3}\frac{\partial w^i}{\partial x^j} + \frac{2}{c^3}\frac{\partial w^j}{\partial x^i} + d^{-3.5} O(5) \\
\tilde{\Gamma}^i_{jk} &= \delta_{ij}\frac{1}{c^2}\frac{\partial w}{\partial x^k} + \delta_{ik}\frac{1}{c^2}\frac{\partial w}{\partial x^j} - \delta_{jk}\frac{1}{c^2}\frac{\partial w}{\partial x^i} + d^{-3} O(4)
\end{aligned}$$

Furthermore, w_A^α and first derivatives tend to zero with L_A so that

$$w^\alpha \to w_B^\alpha, \quad \frac{\partial w^\alpha}{\partial x^\beta} \to \frac{\partial w_B^\alpha}{\partial x^\beta}, \quad \tilde{\Gamma}^\alpha_{\beta\gamma} \to \overline{\Gamma}^\alpha_{\beta\gamma} \tag{4.4.5}$$

as $L_A \to 0$. Hence for $L_A < d^{-2}$,

$$\mathcal{G}^i(\tilde{g}_{\alpha\beta}, z^\gamma) = \frac{\partial w}{\partial x^i} + \frac{1}{c^2}(\frac{dz^k}{dt}\frac{dz^k}{dt}\frac{\partial w}{\partial x^i} - 3\frac{dz^i}{dt}\frac{\partial w}{\partial t} + 4\frac{\partial w^i}{\partial t} \tag{4.4.6}$$

$$-4w\frac{\partial w}{\partial x^i} + (\frac{\partial w^j}{\partial x^i} - \frac{\partial w^i}{\partial x^j})\frac{dz^j}{dt} - 4\frac{\partial w}{\partial x^j}\frac{dz^i}{dt}\frac{dz^j}{dt}) + d^{-4} O(4)$$

and

$$|\mathcal{G}^i(\tilde{g}_{\alpha\beta}, z^\gamma) - \mathcal{G}^i(\overline{g}_{\alpha\beta}, z^\gamma)| \to 0 \text{ as } L_A \to 0 \tag{4.4.7}$$

The calculation of Christoffel symbols inside A for the total metric $g_{\alpha\beta}$ presents a different problem since $g_{\alpha\beta}$ is not given by a closed formula in terms of BD potentials. It is related to $\tilde{g}_{\alpha\beta}$ through equations (4.3.17). In the following section an equation of motion for a point inside A is to be obtained using the conservation equation $T^{\alpha\beta}_{;\beta} = 0$ *inside* A. It appears that DSX have calculated the conservation equation using the Christoffel symbols of $\tilde{g}_{\alpha\beta}$ in place of those for $g_{\alpha\beta}$ so to justify their equations the Christoffel symbols for $g_{\alpha\beta}$ need to be related to the Christoffel symbols for $\tilde{g}_{\alpha\beta}$. The latter have been expressed in terms of w^α and its first derivatives. Equation (4.3.17) relates $g_{\alpha\beta}$ and $\tilde{g}_{\alpha\beta}$ but it is also necessary to relate the first derivatives of $g_{\alpha\beta}$, $\tilde{g}_{\alpha\beta}$ and to assume an appropriate dependence on d for the error term inside A. Firstly it is assumed that $L_A < d^{-2}$ so that $\frac{\partial w^\alpha}{\partial x^\beta} \sim \frac{\partial w^\alpha_{B}}{\partial x^\beta}$ inside A. Then it is assumed further that the error terms in (4.3.17) are $\partial(1)$ and that the dependence on d of each first derivative of $g_{\alpha\beta}$ is of the same order as the dependence on d of the corresponding derivative of $\tilde{g}_{\alpha\beta}$. In other words it is assumed that:

$$\frac{\partial g_{00}}{\partial x^i} = \frac{\partial \tilde{g}_{00}}{\partial x^i} + d^{-2}O(6), \frac{\partial g_{00}}{\partial t} = \frac{\partial \tilde{g}_{00}}{\partial t} + d^{-2.5}O(6), \tag{4.4.8}$$

$$\frac{\partial g_{0j}}{\partial x^i} = \frac{\partial \tilde{g}_{0j}}{\partial x^i} + d^{-2.5}O(5), \frac{\partial g_{0j}}{\partial t} = \frac{\partial \tilde{g}_{0j}}{\partial t} + d^{-3}O(5),$$

$$\frac{\partial g_{kj}}{\partial x^i} = \frac{\partial \tilde{g}_{kj}}{\partial x^i} + d^{-2}O(4), \frac{\partial g_{kj}}{\partial t} = \frac{\partial \tilde{g}_{kj}}{\partial t} + d^{-2.5}O(4)$$

These or similar assumptions are necessary if one is to derive the equations 5.6 of DSX (1992). Equation (4.4.8) may be verified in the case of the analytical example and (4.4.8) also serves to establish the following relationship between geodesic accelerations with respect to the total field and the BD approximate field.

$$|\mathcal{G}^i(\tilde{g}_{\alpha\beta}, z^\gamma) - \mathcal{G}^i(g_{\alpha\beta}, z^\gamma)| \sim d^{-2}O(4) \tag{4.4.9}$$

when $L_A < d^{-2}$, z^α is inside A and $\frac{dz^\alpha}{dt} \sim V_A$.

The proof of (4.4.9) is quite lengthy and is outlined in the appendix to this section where it is also shown that (4.4.9) is the best possible estimate of the difference between the two geodesic accelerations.

Appendix to section 4.4

$$\mathcal{G}^i(\tilde{g}_{\alpha\beta}, z^\gamma) - \mathcal{G}^i(g_{\alpha\beta}, z^\gamma) = -(\Delta\Gamma^i_{\mu\nu} - \frac{1}{c}\Delta\Gamma^0_{\mu\nu}\frac{dz^i}{dt})\frac{dz^\mu}{dt}\frac{dz^\nu}{dt} \tag{4.4.10}$$

where $\Delta\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} - \tilde{\Gamma}^\alpha_{\beta\gamma}$.

Applying the Δ-operator in the obvious way to other functions, one may write

$$\Delta\Gamma^\alpha_{\beta\gamma} = \tfrac{1}{2}g^{\alpha\epsilon}\Delta(\frac{\partial g_{\epsilon\beta}}{\partial x^\gamma} + \frac{\partial g_{\epsilon\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\epsilon}) + \tfrac{1}{2}\Delta(g^{\alpha\epsilon})(\frac{\partial \tilde{g}_{\epsilon\beta}}{\partial x^\gamma} + \frac{\partial \tilde{g}_{\epsilon\gamma}}{\partial x^\beta} - \frac{\partial \tilde{g}_{\beta\gamma}}{\partial x^\epsilon}) \tag{4.4.11}$$

From equation (4.4.8) one may read off the values of $\Delta(\frac{\partial g_{\epsilon\beta}}{\partial x^\gamma})$. Thus $\Delta(\frac{\partial g_{00}}{\partial x^i}) = d^{-2}O(6)$ etc. From $g^{\alpha\beta}g_{\beta\gamma} = \tilde{g}^{\alpha\beta}\tilde{g}_{\beta\gamma}$ one finds

$$\Delta g^{00} = O(6),\ \ \Delta g^{0i} = O(5),\ \ \Delta g^{ij} = O(4)$$

The quantities $\frac{\partial \tilde{g}_{\epsilon\beta}}{\partial x^\gamma}$ may be estimated from the estimates for $\frac{\partial w^\alpha}{\partial x^\beta}$. Assuming $L_A < d^{-2}$,

$$\frac{\partial \tilde{g}_{00}}{\partial x^i} \sim \frac{\partial \overline{g}_{00}}{\partial x^i} \sim \frac{1}{c^2}\frac{\partial w_B}{\partial x^i} \sim d^{-2}O(2)$$

Similarly,

$$\begin{aligned}
\frac{\partial \tilde{g}_{00}}{\partial x^i} &\sim d^{-2}O(2)\ \frac{\partial \tilde{g}_{0j}}{\partial x^i} \sim d^{-2.5}O(3)\ \frac{\partial \tilde{g}_{kj}}{\partial x^i} \sim d^{-2}O(2)\\
\frac{\partial \tilde{g}_{00}}{\partial x^0} &\sim d^{-2.5}O(3)\ \frac{\partial \tilde{g}_{0j}}{\partial x^0} \sim d^{-3}O(4)\ \frac{\partial \tilde{g}_{kj}}{\partial x^i} \sim d^{-2.5}O(3)
\end{aligned}$$

Substituting into (4.4.11) one finds the following estimates:

$$\Delta\Gamma^0_{00} \sim d^{-2}O(7),\ \ \Delta\Gamma^i_{00} \sim d^{-2}O(6),\ \ \Delta\Gamma^0_{j0} \sim d^{-2}O(6),$$

$$\Delta\Gamma^i_{j0} \sim d^{-2.5}O(5),\ \ \Delta\Gamma^0_{jk} \sim d^{-2.5}O(5),\ \ \Delta\Gamma^i_{jk} \sim d^{-2}O(4)$$

Substituting these values into (4.4.10) assuming $\frac{dz^\alpha}{dt} \sim V_A$ gives

$$|\mathcal{G}^i(\tilde{g}_{\alpha\beta}, z^\gamma) - \mathcal{G}^i(g_{\alpha\beta}, z^\gamma)| \sim d^{-2}O(4)$$

One may calculate the quantity $|\mathcal{G}^i(\tilde{g}_{\alpha\beta}, z^\gamma) - \mathcal{G}^i(g_{\alpha\beta}, z^\gamma)|$ exactly in the case of the analytical example at a distance $\frac{r}{R} = d$ from the centre of body B. The only terms of O(4) in the expression are $\Delta\Gamma^i_{00}c^2 + \Delta\Gamma^i_{jk}\frac{dz^j}{dt}\frac{dz^k}{dt}$.

$$\Delta\Gamma^i_{00} = [(-\frac{39}{35} - 4U^2)\frac{\partial U}{\partial x^i} + \frac{x^i x^k U^4}{R^2}\frac{\partial U}{\partial x^k}]\gamma^3 + O(8),$$

$U \sim d^{-1}$, $\frac{\partial U}{\partial x^i} \sim d^{-2}$ so this shows that $\Delta\Gamma^i_{00}$ is of the order of $d^{-2}O(4)$ and not smaller. $\Delta\Gamma^i_{jk}$ is also found to be of the order of $d^{-2}O(4)$. On account of the fairly arbitrary factors $\frac{dz^j}{dt}$, the quantity $|\mathcal{G}^i(\tilde{g}_{\alpha\beta}, z^\gamma) - \mathcal{G}^i(g_{\alpha\beta}, z^\gamma)|$ cannot in general be of a lesser order than $d^{-2}O(4)$. This shows that (4.4.9) gives the best possible estimate.

4.5 An equation of motion at the 1PN level

An equation of motion is to be obtained using the conservation equation $T^{\alpha\beta}_{;\beta} = 0$ inside A. (It should be pointed out that another approach to the equation of motion which does not use the BD potentials is given in chapter 5.)

$$T^{\alpha\beta}_{;\beta} = \frac{\partial T^{\alpha\beta}}{\partial x^\beta} + \Gamma^\alpha_{\beta\gamma}T^{\beta\gamma} + \Gamma^\nu_{\beta\nu}T^{\alpha\beta} \tag{4.5.1}$$

By definition $\Gamma^\alpha_{\beta\gamma} = \tilde{\Gamma}^\alpha_{\beta\gamma} + \Delta\Gamma^\alpha_{\beta\gamma}$ where $\tilde{\Gamma}^\alpha_{\beta\gamma}$ may be replaced by the formulae in section 4.4 and $\Delta\Gamma^\alpha_{\beta\gamma}$ may be replaced by the estimates in the appendix to section 4.4 when $L_A < d^{-2}$. Substituting these replacements into (4.5.1), the conservation equation at an event inside A when $L_A < d^{-2}$ is expressed in the following form:

$$\frac{\partial}{\partial t}(\frac{T^{00}}{c^2}) = -\frac{\partial \sigma^a}{\partial x^a} + F + \alpha \tag{4.5.2}$$

$$\frac{\partial \sigma^i}{\partial t} = -\frac{\partial T^{ia}}{\partial x^a} + G^i + \beta^i$$

where

$$F = -\frac{1}{c^2}\sigma\frac{\partial w}{\partial t} \quad ,$$

$$G^i = \frac{\partial w}{\partial x^i}\sigma + \frac{4}{c^2}\frac{\partial w^i}{\partial t}\sigma - \frac{4}{c^2}w\frac{\partial w}{\partial x^i}\sigma - \frac{4}{c^2}\frac{\partial w}{\partial t}\sigma^i + \frac{4}{c^2}\frac{\partial w^i}{\partial x^a}\sigma^a - \frac{4}{c^2}\frac{\partial w^a}{\partial x^i}\sigma^a - \frac{4}{c^2}\frac{\partial w}{\partial x^a}T^{ia}$$

with $\alpha \sim d^{-2.5}O(4)$, $\beta^i \sim d^{-2}O(4)$.

It follows from (4.5.2) that α, β^i have support $A \cup B$.

$z^\alpha(t)$ is defined by

$$\int_A d^3x(x^i - z^i(t))\frac{T^{00}(x)}{c^2} = 0, \quad z^0 = ct \tag{4.5.3}$$

The procedure is to differentiate (4.5.3) twice with respect to t replacing $\frac{\partial}{\partial t}(\frac{T^{00}}{c^2})$ and $\frac{\partial \sigma^i}{\partial t}$ wherever they occur from (4.5.2). The following lemmas will be required.

Lemma 1: If $f(x)$ is zero on the boundary of A then

$$\frac{d}{dt}(\int_A d^3x f(x)) = \int_A d^3x\frac{\partial f(x)}{\partial t}$$

Lemma 2: If ϕ^a is zero on the boundary of A then

$$\int_A d^3x(x^i - z^i(t))(-\frac{\partial \phi^a}{\partial x^a}) = \int_A d^3x\ \phi^i$$

Lemma 2 is a special case of Lemma 4.13 of DSX (1992). Since the latter is not proved in DSX, a proof is outlined here. Consider the term in the integrand with $a = 1$. Let this be converted to a repeated integral and integrated by parts:

$$\int\int dx^2dx^3(\int_{P_1}^{P_2} dx^1(x^i - z^i(t))\frac{\partial \phi^1}{\partial x^1}) = \int\int dx^2dx^3(-\delta^i_1\int_{P_1}^{P_2} dx^1\phi^1)$$

where P_1, P_2 are on the boundary of A at time t. Thus

$$\int_A d^3x(x^i - z^i(t))(-\frac{\partial \phi^1}{\partial x^1}) = \delta^i_1\int_A d^3x\phi^1$$

From this the result follows.

Some further estimates will be required. Field quantities such as $T^{\alpha\beta}$, α, β^i are zero outside of A. For such a quantity f there is the usual PN assumption that $|\frac{\partial f}{\partial t}| \sim V_A|\text{grad } f|$. It will also be assumed that

$$|\int_A d^3x\phi\frac{\partial f}{\partial t}| \sim \frac{V_A|\phi|_{max}|f|_{max}(\text{volume } A)}{L_A} \tag{4.5.4}$$

This estimate is discussed further in the appendix to this section. It amounts to requiring that f does not fluctuate too wildly on A.

A quantity ρ_A is defined by the equation $\int_A d^3x\frac{T^{00}}{c^2} = (\text{volume } A)\rho_A$. For an ideal fluid it corresponds at Newtonian level with the Newtonian average density of A. It is assumed that even in other cases it does not differ much from Newtonian density and as stated earlier, $\rho_A \sim \rho_B \sim \frac{1}{4}$. Thus for any f,

$$\frac{\int_A d^3xf}{\int_A d^3x\frac{T^{00}}{c^2}} = \frac{\text{average} f \text{ on } A}{\rho_A} <\sim 4|f|_{max} \tag{4.5.5}$$

A quantity of O(4) will be interpreted somewhat leniently in the following as meaning a quantity of magnitude less than about $10c^{-4}$.

The first differentiation of (4.5.3) yields

$$\int_A d^3x\frac{dz^i}{dt}\frac{T^{00}(x)}{c^2} - \int_A d^3x(x^i - z^i(t))(-\frac{\partial\sigma^a}{\partial x^a} + F + \alpha) = 0 \tag{4.5.6}$$

Substituting from Lemma 2 into (4.5.6) and differentiating results in

$$\frac{d^2z^i}{dt^2}\int_A d^3x\frac{T^{00}(x)}{c^2} = \int_A d^3x(-2\frac{dz^i}{dt}(F+\alpha)+G^i+\beta^i+(x^i-z^i(t))\frac{\partial}{\partial t}(F+\alpha)) \tag{4.5.7}$$

Equation (4.5.7) is now divided by $\int_A d^3x\frac{T^{00}(x)}{c^2}$. From (4.5.5), the term involving α is $d^{-3}O(4)$ and the term involving β^i is $d^{-2}O(4)$. From (4.5.4) the term involving $\frac{\partial\alpha}{\partial t}$ is $d^{-3}O(4)$. Equation (4.5.7) now becomes

$$\frac{d^2z^i}{dt^2} = \frac{\int_A d^3x(-2\frac{dz^i}{dt}F + G^i + (x^i - z^i(t))\frac{\partial F}{\partial t}}{\int_A d^3x\frac{T^{00}(x)}{c^2}} + d^{-2}O(4) \tag{4.5.8}$$

Equation (4.5.8) has been derived without assumptions as to the size of A apart from the condition $L_A < d^{-2}$ which is satisfied in most applications. (In the case of a body in orbit about the sun at the same distance as the Earth this condition is equivalent to $L_A <\sim 70\,km$)

Section 4.8 will return to equation (4.5.8) in order to make some estimates of deviation from geodesic motion in terms of the size of the test body while the remainder of this section will be devoted to the simpler problem of expressing (4.5.8) in a form which is valid for L_A sufficiently small, and which may be regarded as a limiting form of (4.5.8) as $L_A \to 0$.

From (4.5.2) one finds

$$\frac{\partial F}{\partial t} = -\frac{1}{c^2}(-\frac{\partial \sigma^a}{\partial x^a} + F + \alpha)\frac{\partial w}{\partial t} - \frac{1}{c^4}\frac{\partial T^{bb}}{\partial t}\frac{\partial w}{\partial t} - \frac{1}{c^2}\sigma\frac{\partial^2 w}{\partial t^2} \tag{4.5.9}$$

$\frac{\partial F}{\partial t}$ in (4.5.8) is to be replaced by (4.5.9). The terms in (4.5.9) involving F, α, $\frac{\partial^2 w}{\partial t^2}$ are all bounded and will tend to zero with L_A when multiplied by $(x^i - z^i(t))$. Hence they may be omitted for L_A sufficiently small. Consider the following term in (4.5.8):

$$\frac{\int_A d^3x(x^i - z^i(t))\frac{1}{c^4}\frac{\partial T^{kk}}{\partial t}\frac{\partial w}{\partial t}}{\int_A d^3x\frac{T^{00}}{c^2}} \sim \frac{4V_A^2|T^{kk}|_{max}}{d^2c^4}$$

using (4.5.4) and (4.5.5), this term is $d^{-3}O(4)$.

Now consider the following term in (4.5.8):

$$\frac{\int_A d^3x(x^i - z^i(t))\frac{1}{c^2}\frac{\partial \sigma^k}{\partial x^k}\frac{\partial w}{\partial t}}{\int_A d^3x\frac{T^{00}}{c^2}} \tag{4.5.10}$$

In this term $\frac{\partial w}{\partial t}$ may be expanded:

$$\frac{\partial w}{\partial t}(x) = \frac{\partial w}{\partial t}(ct, z(t)) + (x^i - z^i(t))\frac{\partial^2 w}{\partial x^i \partial t}(\xi)$$

Since $\frac{\partial^2 w}{\partial x^\alpha \partial x^\beta} \sim 1$ and $\int_A d^3x(x^i - z^i(t))^2\frac{1}{c^2}\frac{\partial \sigma^k}{\partial x^k} \sim \frac{L_A|\sigma^k|_{max}(\text{volume } A)}{c^2}$, the second term above may be dropped for L_A sufficiently small. Then using Lemma 2, (4.5.10) becomes for L_A sufficiently small:

$$-\frac{1}{c^2}\frac{\partial w}{\partial t}(ct, z(t))\frac{\int_A d^3x\sigma^i}{\int_A d^3x\frac{T^{00}}{c^2}}$$

The following symbols are defined for convenience:

$$M^i = c\frac{\int_A d^3x T^{0i}}{\int_A d^3x T^{00}} \quad , M^{ij} = c^2\frac{\int_A d^3x T^{ij}}{\int_A d^3x T^{00}}$$

The term (4.5.10) of (4.5.8) then has the form $-\frac{1}{c^2}\frac{\partial w}{\partial t}(ct, z(t))M^i$. In the remaining terms of (4.5.8), F, G^i are replaced by the expressions in (4.5.2) and w^α, $\frac{\partial w^\alpha}{\partial x^\beta}$ are replaced by their values at $(ct, z(t))$ and removed from under the integral sign by the method used for (4.5.10). Furthermore, on account of (4.4.5) w^α may be replaced throughout by w^α_B for L_A sufficiently small. Equation (4.5.8) then takes the form

$$\frac{d^2z^i}{dt^2} = \frac{\partial w_B}{\partial x^i} + \frac{1}{c^2}(M^{bb}\frac{\partial w_B}{\partial x^i} + 2\frac{dz^i}{dt}\frac{\partial w_B}{\partial t} - 5M^i\frac{\partial w_B}{\partial t} + 4\frac{\partial w^i_B}{\partial t} \tag{4.5.11}$$

$$-4w_B\frac{\partial w_B}{\partial x^i} + (\frac{\partial w^j_B}{\partial x^i} - \frac{\partial w^i_B}{\partial x^j})M^j - 4\frac{\partial w_B}{\partial x^j}M^{ij}) + d^{-2}O(4)$$

for L_A sufficiently small, where w^α_B and $\frac{\partial w^\alpha_B}{\partial x^\beta}$ are evaluated at $(ct, z(t))$.

The right hand side of equation (4.5.11) is to be compared with the geodesic acceleration for the BD background field given in (4.4.3). It is immediately seen that these expressions would agree if M^i were replaced by $\frac{dz^i}{dt}$ and M^{ij} were replaced by $\frac{dz^i}{dt}\frac{dz^j}{dt}$. It will be shown in section 4.7 that this substitution can be justified in the case of an ideal fluid. On the other hand, if the expressions for the acceleration in (4.5.11) and (4.4.3) differ by more than $d^{-2}O(4)$ for arbitrarily small L_A, then it is shown in section (4.6) that the geodesic law will be violated.

Appendix to section 4.5

The estimate (4.5.4) is based on the usual PN assumption that for a field quantity f, $|\frac{\partial f}{\partial t}| \sim V_A|\text{grad } f|$ inside A, together with a requirement that f does not have a large variation on A. The justification is as follows:

$$|\int_A d^3x\phi\frac{\partial f}{\partial t}| \;\leq\; \int_A d^3x|\phi||\frac{\partial f}{\partial t}|$$

$$\leq \; |\phi|_{max} \int_A d^3x |\frac{\partial f}{\partial t}|$$
$$\sim \; |\phi|_{max} V_A \int_A d^3x |\text{grad } f|$$

The next (final) step is to say that $\int_A d^3x|\text{grad } f| \sim \frac{|f|_{max}(\text{volume } A)}{L_A}$. This step can be justified if f does not fluctuate too much on A. The situation may be illustrated in a 1-D example where A is the interval [a,b], f is zero on the boundary of A and has successive max/min at $x_1 < x_2 < ... < x_n$. Then

$$\int_a^b |f'(x)|dx = 2\Sigma_{r=1}^n (-1)^{r-1} f(x_r)$$

If $n = 1$ so that f is unimodal,

$$\int_a^b |f'(x)|dx = 2|f|_{max} = \frac{|f|_{max}}{L_A}(\text{length } A)$$

If f has small variations about a unimodal form, this estimate is not much changed but if f has n large swings between say $|f|_{max}$ and -$|f|_{max}$, the estimate must be multiplied by n. This result may immediately be extended by integration to the 3-D case where A is a cube and it may be extended to other cases where A is convex and similar to a cube without much change. In the applications f was either α or T^{kk} and there is no apparent reason why f should fluctuate excessively on A. One should also bear in mind that the estimate $|\int_A d^3x \phi \frac{\partial f}{\partial t}| \sim |\phi|_{max} V_A \int_A d^3x|\text{grad } f|$ was obtained above through a number of crude inequalities and may be much too large so that (4.5.4) could be satisfied even with considerable fluctuations in f.

4.6 Test particles and non-geodesic motion

Consider the theory of section 4.5 in the context of a test particle. All functions in section 4.5 are indexed by m where m goes to zero with L_A and this index is now made explicit. Thus for each m there are functions $g_{\alpha\beta}(x;m)$, $T^{\alpha\beta}(x;m)$, $w^\alpha(x;m)$, $w^\alpha_B(x;m)$, $w^\alpha_A(x;m)$, $\tilde{g}_{\alpha\beta}(x;m)$, $\overline{g}_{\alpha\beta}(x;m)$, $z^\alpha_m(t)$, $M^i(t;m)$, $M^{ij}(t;m)$. We also write $L_m = L_A$.

Functions which are not indexed by m are the background field and associated functions. To avoid conflict with the notation of section 4.5 these will be indexed by (b). Thus $g^{(b)}_{\alpha\beta}(x) = \lim_{m\to 0} g_{\alpha\beta}(x;m)$ and $T^{\alpha\beta}_{(b)}(x)$ is obtained from $g^{(b)}_{\alpha\beta}(x)$ by the field equations. The functions $w^{\alpha}_{(b)}(x)$, $\tilde{g}^{(b)}_{\alpha\beta}(x)$ are defined in the obvious way in terms of $T^{\alpha\beta}_{(b)}(x)$ as in section 4.3. The curve $\mathcal{C}$ defined in chapter 2 will be denoted by $(ct, \zeta(t))$. The point $z^i_m(t)$ is inside A_m and hence $z^i_m(t) \to \zeta^i(t)$ as $m \to 0$. It is assumed further that

$$\frac{dz^i_m}{dt} \to \frac{d\zeta^i}{dt}, \quad \frac{d^2 z^i_m}{dt^2} \to \frac{d^2\zeta^i}{dt^2}$$

as $m \to 0$.

For convenience, the expression in w_B and $\frac{dz^i}{dt}$ which occurs in equation (4.4.3) is abbreviated to $F^i(w_B; z^\mu)$ so in this notation equation (4.4.3) becomes

$$\mathcal{G}^i(\overline{g}_{\alpha\beta}(m); z^\mu_m) = F^i(w_B(m); z^\mu_m) + d^{-4}O(4) \tag{4.6.1}$$

It is apparent that as $m \to 0$, the influence of A_m on the position of B_m and on the stress-energy tensor inside B_m must go to zero so that so that when x is inside or near B, $T^{\alpha\beta}(x;m)$ and its derivatives with respect to x^α tend respectively to $T^{\alpha\beta}_{(b)}(x)$ and its derivatives with respect to x^α. (The same is not true on the curve $\mathcal{C}$ since inside A_m, $T^{00}(x;m) \sim \frac{c^2}{4}$ while $T^{00}_{(b)} = 0$.) Consequently at all points, $w^\alpha_B(x;m)$ and its derivatives tend respectively to $w^\alpha_{(b)}(x)$ and its derivatives as $m \to 0$. Taking the limit of (4.6.1) as $m \to 0$ gives:

$$\mathcal{G}^i(\tilde{g}^{(b)}_{\alpha\beta}; \zeta^\mu) = F^i(w_{(b)}; \zeta^\mu) + d^{-4}O(4) \tag{4.6.2}$$

Equation (4.5.11) may be rearranged and written as:

$$\frac{d^2 z^i_m}{dt^2} = F^i(w^\alpha_B(m); z^i_m) + \frac{1}{c^2}\Big[\frac{\partial w_B}{\partial x^i}(M^{kk}(t;m) - \frac{dz^k_m}{dt}\frac{dz^k_m}{dt}) - 5\frac{\partial w_B}{\partial t}(M^i(t;m) - \frac{dz^i_m}{dt}) \tag{4.6.3}$$

$$(\frac{\partial w^j_B}{\partial x^i} - \frac{\partial w^i_B}{\partial x^j})(M^j(t;m) - \frac{dz^j_m}{dt}) - 4\frac{\partial w_B}{\partial x^j}(M^{ij}(t;m) - \frac{dz^i_m}{dt}\frac{dz^j_m}{dt})\Big] + d^{-2}O(4)$$

for L_A sufficiently small, where $\frac{\partial w^\alpha_B}{\partial x^\beta} = \frac{\partial w^\alpha_B}{\partial x^\beta}((ct, z^i_m(t)); m)$.

Let the terms with square brackets in (4.6.3) be called

$$Q^i = \frac{1}{c^2}[\frac{\partial w_B}{\partial x^i}(M^{kk}(t;m) - \frac{dz_m^k}{dt}\frac{dz_m^k}{dt}) - 5\frac{\partial w_B}{\partial t}(M^i(t;m) - \frac{dz_m^i}{dt}) + \tag{4.6.4}$$

$$(\frac{\partial w_B^j}{\partial x^i} - \frac{\partial w_B^i}{\partial x^j})(M^j(t;m) - \frac{dz_m^j}{dt}) - 4\frac{\partial w_B}{\partial x^j}(M^{ij}(t;m) - \frac{dz_m^i}{dt}\frac{dz_m^j}{dt})]$$

Using (4.6.2) and (4.4.6) and letting m tend to zero in (4.6.3) one obtains for m sufficiently small:

$$\frac{d^2\zeta^i}{dt^2} = \mathcal{G}^i(g_{\alpha\beta}^{(b)};\zeta^\nu) + Q^i + d^{-2}O(4) \tag{4.6.5}$$

If Q^i is a fair amount greater than $d^{-2}O(4)$ for all sufficiently small m, then it follows from (4.6.5) that $\frac{d^2\zeta^i}{dt^2} \neq \mathcal{G}^i(g_{\alpha\beta}^{(b)};\zeta^\nu)$ and $\mathcal{C}$ is not a geodesic of the background field.

There is a fundamental difficulty in producing a two-body counter-example to geodesic motion - there are no known solutions of the field equations to serve as possible candidates! What will be shown here is that, supposing there exist two-body solutions for which the stress energy tensor of the test body satisfies certain constraints (and there seems to be no a priori reason why this should not be the case) then there do exist two-body counter-examples to geodesic motion.

The example to be constructed is similar to the Newtonian example of Nevin (1995). The background field is taken to be the exterior Schwarzschild field of the analytical example with Earth (regarded as isolated) as the central body B so that $\gamma \sim 7.10^{-10}$ in characteristic units based on the Earth.

Let $g_{\alpha\beta}(x;m)$ define a test particle on the curve $\mathcal{C} : (ct, \zeta(t))$. Since we are concerned only with m sufficiently small, $\frac{\partial w_B^\alpha}{\partial x^\beta}((ct, z_m(t));m)$ in Q^i may be replaced by its limit as m tends to zero which is $\frac{\partial w_{(b)}^\alpha}{\partial x^\beta}((ct,\zeta(t)))$. The potentials $w_{(b)}^\alpha$ are given explicitly in the appendix to section 4.3. Substituting these values into Q^i it is found that for m sufficiently small

$$Q^i = \frac{1}{c^2}\left((-\frac{\zeta^i}{|\zeta|^3} + \mu^i c^{-2})(M^{kk} - \frac{d\zeta^k}{dt}\frac{d\zeta^k}{dt}) + 4(\frac{\zeta^j}{|\zeta|^3} + \nu^i c^{-2})(M^{ij} - \frac{d\zeta^i}{dt}\frac{d\zeta^j}{dt})\right) \tag{4.6.6}$$

where $|\mu^i|, \ |\nu^i| < 2$.

Suppose the test bodies A_m are made of material such that in the Earth's gravitational field, the stress-energy tensor inside A_m satisfies the following inequality for all m sufficiently small:

$$|M^{ij} - (1 + 10^{-2})\frac{dz_m^i}{dt}\frac{dz_m^j}{dt}| < 10^{-12} \tag{4.6.7}$$

For example, (4.6.7) would be true if
$\frac{c^2 T^{ij}}{T^{00}} \rightarrow (1 + 10^{-2})\frac{dz_m^i}{dt}\frac{dz_m^j}{dt}$ as m tends to zero.

Note that (4.6.7) is compatible with post-Newtonian and other restrictions previously placed on $T^{\alpha\beta}$. It may also be possible to satisfy (4.6.7) together with the requirement that there is a flow field u^α which is an eigenvector of $T^{\alpha\beta}$. This is a condition of Carter and Quintana (1972) for a perfectly elastic body. Thus if functions T^{ij}, T^{00}, u^α, u_α are given such that (4.6.7) is satisfied and $u^\alpha u_\alpha = -1$, the eigenvalue equation $T^{\alpha\beta}u_\alpha = \lambda u^\beta$ is found to be a set of four linearly independent equations in T^{0i}, λ and may be solved to yield unique corresponding values for T^{0i}, λ . It is not known whether this procedure can yield a space-time in the sense that there exists $g_{\alpha\beta}$ so that $u_\alpha = g_{\alpha\beta}u^\beta$ and the field equations (4.3.7) are satisfied.

From (4.6.7) for m sufficiently small,

$$M^{ij} - \frac{d\zeta^i}{dt}\frac{d\zeta^j}{dt} = 10^{-2}\frac{d\zeta^i}{dt}\frac{d\zeta^j}{dt} + \nu^{ij}$$

where $|\nu^{ij}| < 10^{-12}$. Substituting this in (4.6.6) gives for m sufficiently small:

$$Q^i = 10^{-2}c^{-2}(4\frac{\zeta^j}{|\zeta|^3}\frac{d\zeta^i}{dt}\frac{d\zeta^j}{dt} - \frac{\zeta^i}{|\zeta|^3}\frac{d\zeta^b}{dt}\frac{d\zeta^b}{dt}) + q^i c^{-4} \tag{4.6.8}$$

where $|q^i| < 1$.

Let initial values of $\zeta^i(t)$, $\frac{d\zeta^i}{dt}$ be as for a circular Newtonian orbit of radius 2 (i.e. two Earth radii from the centre of the Earth), then initially

$$(\zeta^1, \zeta^2, \zeta^3) = (2, 0, 0), \ \ (\frac{d\zeta^1}{dt}, \frac{d\zeta^2}{dt}, \frac{d\zeta^1}{dt}) = (0, 2\omega, 0), \ \ \omega^2 = \frac{1}{8},$$

$$Q^i = 10^{-2}c^{-2}(-\frac{1}{8}, 0, 0) + (q^i)c^{-4}$$

With these values it is seen that initially $Q^1 > 10^{-13}$ for any sufficiently small m while $d^{-2}c^{-4} \sim 10^{-19}$, showing that C is not a geodesic of the background field.

Note that the non-geodesic motion shown here is not related to the result of Papapetrou (1951) who showed (very non-rigorously) that a "spinning" particle does not move on a geodesic. With the assumptions made here as regards $T^{\alpha\beta}$, the quantity spin/mass defined by Papapetrou would go to zero with m and Papapetrou's equation would reduce to the geodesic equation in the test particle limit.

The above example shows that any "proof" of the geodesic law which does not somehow eliminate the possibility of (4.6.7) being true, must be false. In the next section it will be shown that (4.6.7) does not hold for an ideal fluid. Experiment has shown that to a high degree of accuracy, small bodies made of different materials experience the same acceleration in an external gravitational field (the weak equivalence principle). It may be of interest to note that (4.6.7) is actually consistent with this experimental finding. If two sufficiently small bodies both satisfied (4.6.7), they would both experience the same (non-geodesic) acceleration $\mathcal{G}^i(g^{(b)}_{\alpha\beta}; \zeta^\nu) + Q^i + d^{-2}O(4)$ where Q^i is given by (4.6.8).

Finally, one may estimate in a simple case the effect that an assumption like (4.6.7) has on the advance of the perihelion. Assume that

$$\lim_{m \to 0} M^{ij} = k\frac{dz^i}{dt}\frac{dz^j}{dt}$$

with k constant. In the non-geodesic example above one could have $k = 1 + 10^{-2}$ while $k = 1$ would give geodesic motion at the post-Newtonian level.

Assuming that L_A is small enough so that w_B in (4.5.11) may be replaced by w for the exterior Schwarzschild metric as given in the appendix to section 4.3 in DSX coordinates x^α, equation (4.5.11) becomes (putting $z^i = x^i$):

$$\frac{d^2x^i}{dt^2} = \frac{\partial w}{\partial x^i}(1 + \frac{k}{c^2}\frac{dx^j}{dt}\frac{dx^j}{dt} - \frac{4}{c^2}w) - \frac{4k}{c^2}\frac{\partial w}{\partial x^j}\frac{dx^i}{dt}\frac{dx^j}{dt} + O(4) \qquad (4.6.9)$$

where

$$w = \frac{1}{r+\gamma} + \frac{\gamma}{(r+\gamma)^2} = \frac{1}{r} + O(4), \ \gamma = c^{-2} \tag{4.6.10}$$

(There is a slight conflict of notation here with the notation of the appendix to section 4.3 where DSX coordinates were denoted by $\overline{x}^\alpha$ and $\overline{r} = r - \gamma$ denoted radial DSX distance.)

Equation (4.6.9) now becomes, using (4.6.10),

$$\frac{d^2x^i}{dt^2} = -\frac{x^i}{r^3} + \frac{1}{c^2}\left(-\frac{kx^i}{r^3}\frac{dx^j}{dt}\frac{dx^j}{dt} + \frac{4x^i}{r^4} + \frac{4kx^j}{r^3}\frac{dx^i}{dt}\frac{dx^j}{dt}\right) + O(4) \tag{4.6.11}$$

In the geodesic case where $k = 1$, equation (4.6.11) agrees with equation 3.73 of Hestenes (1990) with $\mu = 1$.

The perturbation theory necessary to derive a formula for the secular precession of the perihelion is developed in Hestenes (1990). For an equation of the form

$$\frac{d^2x^i}{dt^2} = -\frac{x^i}{r^3} + f^i$$

where f^i is the perturbing force per unit mass, the following are defined:

$$\underline{v} = \frac{d\underline{x}}{dt}, \ \underline{h} = \underline{r} \times \underline{v}, \ \underline{\epsilon} = (\underline{v} \times \underline{h}) - \underline{\hat{r}}$$

$\underline{\epsilon}$ is the Runge-Lenz vector, $|\underline{\epsilon}| = \epsilon$ is the eccentricity of the osculating orbit and $\underline{\epsilon}$ points in the direction of the major axis of the osculating orbit. The rate of precession of the perihelion is $|\frac{d\hat{\epsilon}}{dt}|$ and from Hestenes equation 2.15 page 530,

$$\left|\frac{d\underline{\hat{\epsilon}}}{dt}\right| = \underline{w} \times \underline{\hat{\epsilon}}, \quad \underline{w} = \frac{\underline{h} \times (\underline{r} \times \underline{f})}{h^2} - \frac{1}{\epsilon^2 h^2}[(\underline{\epsilon}.\underline{v})((\underline{r} \times \underline{f}).\underline{h}) + h^2 \underline{\epsilon}.\underline{f}]\underline{h} \tag{4.6.12}$$

Secular quantities are time averaged over one orbit. To calculate the secular rate of precession one needs to find orbital averages of the terms in (4.6.12). Since $\underline{\epsilon}$ and $\underline{h}$ change by a negligible amount over one orbit they are taken as constant in the orbital averaging procedure. From (4.6.11)

$$\underline{f} = \frac{1}{c^2}\left(-kv^2\frac{\underline{r}}{r^3} + 4\frac{\underline{r}}{r^4} + 4k\left(\frac{\underline{r}.\underline{v}}{r^3}\underline{v}\right)\right) + O(4)$$

and

$$\underline{r} \times \underline{f} = \frac{4k}{c^2}(\underline{v}.\frac{\underline{r}}{r^3})\underline{h}$$

Orbital averages (orbav) may be calculated using tables and other formulae from Hestenes and the following results are found:

$$\text{orbav } \frac{\underline{h} \times (\underline{r} \times \underline{f})}{h^2} = 0$$

$$\text{orbav } \frac{\underline{r}.\underline{v}}{r^3}\underline{v} = -\frac{1}{2b^3}\underline{\epsilon}$$

$$\text{orbav } (\underline{\epsilon}.\underline{v})((\underline{r} \times \underline{f}).\underline{h}) = -\frac{2kh^2}{c^2b^3}\epsilon^2$$

$$\text{orbav } \underline{f} = -\frac{-3k+2}{c^2b^3}\underline{\epsilon}$$

$$\text{orbav } \underline{w} = \frac{5k-2}{c^2b^3}\underline{h}$$

Hence the secular rate of precession is

$$\frac{5k-2}{c^2b^3}\underline{h}. \tag{4.6.13}$$

(For relations between the constants a, b, h, ϵ on the osculating orbit, see Hestenes.)

When $k = 1$, (4.6.13) gives the usual geodesic rate of precession. If $k = 1 + 10^{-2}$ the geodesic precession rate is changed by a factor of $\frac{6}{6.1}$. This would be a change of less than one second of arc per century for Mercury.

The formula (4.6.13) when $k = 1$ is derived in Hestenes but note that although Hestenes has the correct answer he has made two very confusing errors in his derivation. Firstly his statement on page 559 that "the secular torque vanishes. Therefore .... its ($\underline{f}$'s) effect on apse precession is completely determined by its secular average" is incorrect. The secular torque does vanish but the orbital average of $(\underline{\epsilon}.\underline{v})((\underline{r} \times \underline{f}).\underline{h})$ does not vanish. Its value is given above. This term has been omitted in Hestenes but the omission is exactly compensated by another error by Hestenes in the calculation of the orbital average of $\underline{f}$.

4.7 The special case of an ideal fluid

In this section the theory of section 4.6 is applied to the case of an ideal fluid and a 1PN version of the geodesic law is obtained. For an ideal fluid the stress-energy tensor is given by

$$T^{\alpha\beta} = (c^2\rho + p)u^\alpha u^\beta + pg^{\alpha\beta} \tag{4.7.1}$$

Along the path of a fluid particle,
$-c^2d\tau^2 = g_{\alpha\beta}dx^\alpha dx^\beta$, $x^0 = ct$, $u^\alpha = \frac{1}{c}\frac{dx^\alpha}{d\tau}$ and $\frac{dx^i}{dt} = v^i$ is the fluid 3-velocity.

Assume that there is a bound K, independent of L_A, so that inside A

$$|v^i(x) - V^i| < L_A K \text{ where } V^i = \frac{dz^i}{dt} \tag{4.7.2}$$

V^i is the 3-velocity on $z^\alpha(t)$ which was defined in section 4.5. (4.7.2) could only fail to be true under very peculiar circumstances. For example (4.7.2) would be true if A were fairly rigid with approximate angular velocity $< K$ where K can have any value. (4.7.2) forces v^i to approach V^i as L_A tends to zero. Substituting (4.3.1) into $-c^2d\tau^2 = g_{\alpha\beta}dx^\alpha dx^\beta$ one finds that $\frac{d\tau}{dt} = \frac{1}{u^0} = 1 + O(2)$. Let ϵ be the dimensionless quantity $\epsilon = \frac{p}{c^2\rho+p}$. In the analytical example $\epsilon \sim \gamma_A \sim \frac{L_A^2}{c^2}$ in characteristic units based on B, and one may safely assume that in general,

$$\epsilon < \frac{L_A}{c^2} \tag{4.7.3}$$

From (4.7.1)

$$\frac{cT^{0i}}{T^{00}} = v^i + \epsilon O(2) \tag{4.7.4}$$

$$\frac{c^2T^{ij}}{T^{00}} = v^iv^j + c^2\epsilon(1 + O(2)) = v^iv^j + O(L_A)$$

Substituting (4.7.2) and (4.7.3) into (4.7.4) one finds

$$cT^{0i} = (V^i + \epsilon_1)T^{00}, \quad c^2T^{ij} = (V^iV^j + \epsilon_2)T^{00} \tag{4.7.5}$$

where $|\epsilon_1|, |\epsilon_2| < L_AK_1$ and K_1 is independent of L_A. Integrating (4.7.5) over A yields

$$\int_A d^3x cT^{0i} = V^i \int_A d^3x T^{00} + \int_A d^3x \epsilon_1 T^{00}$$

$T^{00} \geq 0$, hence $|\int_A d^3x\epsilon_1 T^{00}| < L_A K_1 \int_A d^3x T^{00}$. Dealing similarly with the second equation of (4.7.5) and dividing by $\int_A d^3x T^{00}$ one finds

$$M^i = V^i + \epsilon_3, \quad M^{ij} = V^i V^j + \epsilon_4 \tag{4.7.6}$$

where $|\epsilon_3|$, $|\epsilon_4| < L_A K_1$.

It follows from (4.7.6) that Q^i (defined in section 4.6) tends to zero with L_A and hence from (4.6.5), for L_A sufficiently small, the acceleration of ζ^α differs from geodesic acceleration in the background field by an amount less than $d^{-2}c^{-4}$. This is a 1PN version of the geodesic law for an ideal fluid. In chapter 5 an exact result will be obtained for an ideal fluid. This is in many respects superior to the 1PN result but in one respect it not superior - it requires an assumption concerning the limit of the total field as m tends to zero which is not required in the 1PN theory. This is the reason that the 1PN result has been included here.

4.8 Ideal fluid test bodies

In this section some estimates of deviation of z^α from geodesic motion with respect to the BD and proper background fields are given in terms of the size of the test body. This is done in an attempt to justify the use of the geodesic law in a practical situation where the test body is of a definite size and not "arbitrarily small". It is assumed here that the test body has the ideal fluid form for $T^{\alpha\beta}$.

Returning to section 4.6, equation (4.5.8) was obtained without assumptions on the size of L_A apart from $L_A < d^{-2}$. It is now necessary to estimate various terms which were previously discarded because they tended to zero with L_A. In (4.5.8) $\frac{\partial F}{\partial t}$ is replaced by (4.5.9). The terms arising from F, α, T^{kk} in $\frac{\partial F}{\partial t}$ may be absorbed into the error term since they are of order $d^{-2}O(4)$. Equation (4.5.8) now becomes:

$$\frac{d^2z^i}{dt^2} = (\int_A d^3x \frac{\partial w}{\partial x^i}\sigma + \frac{2}{c^2}\frac{dz^i}{dt}\frac{\partial w}{\partial t}\sigma + \frac{4}{c^2}\frac{\partial w^i}{\partial t}\sigma - \frac{4}{c^2}w\frac{\partial w}{\partial x^i}\sigma - \frac{4}{c^2}\frac{\partial w}{\partial t}\sigma^i \tag{4.8.1}$$

$$+\frac{4}{c^2}\frac{\partial w^i}{\partial x^a}\sigma^a - \frac{4}{c^2}\frac{\partial w^a}{\partial x^i}\sigma^a - \frac{4}{c^2}\frac{\partial w}{\partial x^a}T^{ia}$$

$$+(x^i - z^i(t))(\frac{1}{c^2}\frac{\partial w}{\partial t}\frac{\partial \sigma^a}{\partial x^a} - \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2}\sigma))/(\int_A d^3x\frac{T^{00}}{c^2}) + d^{-2}O(4)$$

The largest term in (4.8.1) is

$$\frac{\int_A d^3x\frac{\partial w}{\partial x^i}\sigma}{\int_A d^3x\frac{T^{00}}{c^2}} = \frac{\int_A d^3x(\frac{\partial w_A}{\partial x^i} + \frac{\partial w_B}{\partial x^i})\sigma}{\int_A d^3x\frac{T^{00}}{c^2}}$$

The three parts of (4.8.1), namely

$$\frac{\int_A d^3x\frac{\partial w_A}{\partial x^i}\sigma}{\int_A d^3x\frac{T^{00}}{c^2}}, \quad \frac{\int_A d^3x(\frac{\partial w_B}{\partial x^i})\sigma}{\int_A d^3x\frac{T^{00}}{c^2}}, \quad \text{terms containing } c^{-2}$$

will now be treated in turn. Expanding the integrand in the definition of w_A,

$$w_A(x) = \int d^3x'(\frac{\sigma(x)}{|x-x'|} - \frac{1}{c}\frac{\partial \sigma}{\partial t}(ct, \underline{x}') + \frac{|x-x'|}{2c^2}\frac{\partial^2\sigma}{\partial t^2}(ct', \underline{x}')) \qquad (4.8.2)$$

where t' is between t, $t - \frac{|x-x'|}{c}$ and $\sigma = 0$ outside of the world tube of A. Let the three integrals arising from the three terms in the integrand of (4.8.2) be denoted by $w_A^{(1)}(x)$, $w_A^{(2)}(x)$, $w_A^{(3)}(x)$ respectively. Then $w_A^{(1)}(x)$ is the Newtonian potential of body A with density $\sigma(x)$, $w_A^{(2)}(x)$ is independent of x, $w_A^{(3)}(x) \sim \frac{(\text{volume } A)\sigma}{2c^2 L_A} \sim \frac{L_A^2}{c^2}$. From (4.2.2), $\int_A d^3x \frac{\partial w_A^{(1)}}{\partial x^i}\sigma(x) = 0$ Hence

$$\frac{\int_A d^3x\frac{\partial w_A}{\partial x^i}\sigma}{\int_A d^3x\frac{T^{00}}{c^2}} = O(\frac{L_A}{c^2}) \qquad (4.8.3)$$

Expanding

$$\frac{\partial w_B}{\partial x^i}(x) = \frac{\partial w_B}{\partial x^i}(ct, z(t)) + (x^j - z^j(t))\frac{\partial^2 w_B}{\partial x^i \partial x^j}(ct, z(t))$$

$$+\frac{1}{2}(x^j - z^j(t))(x^k - z^k(t))\frac{\partial^3 w_B}{\partial x^i \partial x^j \partial x^k}(ct, \xi(t))$$

$$\int_A d^3x(x^j - z^j(t))\sigma = \int_A d^3x(x^j - z^j(t))\frac{T^{kk}}{c^2} \sim \frac{L_A V_A^2}{c^2}(\text{volume } A),$$

$$\frac{\partial^2 w_B}{\partial x^i \partial x^j} \sim d^{-3}, \quad \frac{\partial^3 w_B}{\partial x^i \partial x^j \partial x^k} \sim d^{-4}$$

Hence

$$\frac{\int_A d^3x(\frac{\partial w_B}{\partial x^i})\sigma}{\int_A d^3x\frac{T^{00}}{c^2}} = \frac{\partial w_B}{\partial x^i}(ct, z(t))(1+\frac{M^{kk}}{c^2})+\text{term of } O(\frac{L_A}{d^3c^2})+\text{term of } O(\frac{L_A^2}{d^4}) \quad (4.8.4)$$

where use has been made of $\int_A d^3x(x^j - z^j(t))\frac{T^{00}}{c^2} = 0$.

Terms containing c^{-2} will now be considered. The term containing $\frac{\partial^2 w}{\partial t^2}$ is of the order of $\frac{L_A}{c^2}$. Replacing w by w_B in the other terms is found to cause an error of the order of $\frac{L_A}{c^2}$. The Taylor expansion of $\frac{\partial w_B}{\partial x^i}$ above shows that replacing $\frac{\partial w_B^\alpha}{\partial x^i}(x)$ by $\frac{\partial w_B^\alpha}{\partial x^i}(ct, z(t))$ creates an error of the order of $d^{-3}c^{-2}L_A$. Equation (4.8.1) may thus be written in the following form, similar to (4.5.11):

$$\frac{d^2z^i}{dt^2} = \frac{\partial w_B}{\partial x^i} + \frac{1}{c^2}(M^{bb}\frac{\partial w_B}{\partial x^i} + 2\frac{dz^i}{dt}\frac{\partial w_B}{\partial t} - 5M^i\frac{\partial w_B}{\partial t} + 4\frac{\partial w_B^i}{\partial t} \quad (4.8.5)$$

$$-4w_B\frac{\partial w_B}{\partial x^i} + (\frac{\partial w_B^j}{\partial x^i} - \frac{\partial w_B^i}{\partial x^j})M^j - 4\frac{\partial w_B}{\partial x^j}M^{ij}) + O(c^{-2}L_A) + O(d^{-4}L_A^2) + d^{-2}O(4)$$

where w_B^α and $\frac{\partial w_B^\alpha}{\partial x^\beta}$ are evaluated at $(ct, z(t))$. Equation (4.8.5) is valid for $L_A < d^{-2}$ whereas (4.5.11) was only valid for L_A sufficiently small. Equation (4.8.5) may be written in the following form similar to (4.6.3):

$$\frac{d^2z^i}{dt^2} = \mathcal{G}^i(\overline{g}_{\alpha\beta}; z^\mu) + Q^i + O(c^{-2}L_A) + O(d^{-4}L_A^2) + d^{-2}O(4) \quad (4.8.6)$$

It will be assumed that A is fairly rigid with appoximate angular velocity Ω and that $\Omega < 1$. (For comparison, the orbital angular velocity of $A \sim d^{-1.5}$.) Then $|v^i - V^i| \sim L_A\Omega$ and $|\epsilon_3|$, $|\epsilon_4|$ in (4.7.6) are $O(L_A\Omega)$. Hence $|Q^i| \sim O(\frac{L_A}{d^2c^2})$ and finally (4.8.6) becomes:

$$\frac{d^2z^i}{dt^2} = \mathcal{G}^i(\overline{g}_{\alpha\beta}; z^\mu) + O(c^{-2}L_A) + O(d^{-4}L_A^2) + d^{-2}O(4) \quad (4.8.7)$$

For the physical application of equation (4.8.7) two matters should be considered. Firstly, what would be the effect of using a different point inside A instead of z^α as defined in (4.5.3)? Consider the effect in Newtonian rigid body mechanics of having two points both moving slowly relative to the body and separated by a distance

$\delta < L_A$. These points would have velocities differing by $O(\delta\Omega)$ and accelerations differing by $O(\delta\Omega^2)$ so one might conjecture that with $\Omega < 1$ the error in (4.8.7) would change by an amount of order δ and if $\delta < c^{-2}L_A$, equation (4.8.7) would be unchanged. The point z^α coincides at Newtonian order with the Newtonian centre of mass of A. Various relativistic definitions of centre of mass such as that of DSX are expected not be further from z^α than $c^{-2}L_A$ although this is not proven. I therefore expect that equation (4.8.7) will be true with z^α replaced by a relativistic centre of mass provided $|\Omega| < 1$.

Secondly one should attempt to quantify deviation from geodesic motion with respect to the true background field in the test particle situation because the BD background field is not known a priori. Let this deviation be defined by

$$\frac{d^2 z_m^i}{dt^2} - \mathcal{G}^i(g_{\alpha\beta}^{(b)}; z_m^\mu)$$

One may use the previously derived estimates:

$$|\mathcal{G}^i(g_{\alpha\beta}^{(b)}; z_m^\mu) - \mathcal{G}^i(\tilde{g}_{\alpha\beta}^{(b)}; z_m^\mu)| \sim d^{-2}O(4)$$

$$|F^i(w_{(b)}^\alpha; z_m^\mu) - \mathcal{G}^i(\tilde{g}_{\alpha\beta}^{(b)}; z_m^\mu)| \sim d^{-4}O(4)$$

It remains therefore to estimate

$$|F^i(w_{(b)}^\alpha; z_m^\mu) - F^i(w_B^\alpha(m); z_m^\mu)| = \Delta$$

$w_B^\alpha(m)$ is computed by integrating an expression in $T^{\alpha\beta}(x;m)$ over B where B is disturbed by the presence of A, while $w_{(b)}^\alpha$ is computed by performing a similar integration in the absence of A. One may only speculate on the difference but consideration of a Newtonian case suggests that Δ may be of the order of $d^{-2}L_A^{-3}$ which is negligible in the examples considered below and that consequently deviation from geodesic motion in the background field is given by the same formula as deviation from geodesic motion in the BD background field in these cases.

Consider the case of a fluid body which is the same size as the satellite LAGEOS (see Soffel (1989), DSX (1994)) orbiting the earth with $d \sim 2$ and $L_A = 30\ cm \sim$

$\frac{1}{2}10^{-7}$ in characteristic units based on the earth (regarded as isolated). In these units $O(2) \sim 7.10^{-10}$, $d^{-2}O(4) \sim 10^{-19}$ $\frac{L_A}{c^2} \sim 3.10^{-17}$ and $\frac{L_A^2}{d^4} \sim 10^{-15}$. The dominant error term here in (4.8.7) is the latter which is of Newtonian origin and for LAGEOS, it is probably a good deal smaller than this estimate on account of symmetry which has not been taken into account. One may infer from (4.8.7) that the acceleration of z^α differs from geodesic acceleration in the BD background (and probably also in the test particle background) by less than about 10^{-15}. This represents a fractional correction to the O(2) terms of about 10^{-5}. Bearing in mind the difficulty of separating out factors like radiation pressure, this shows that the assumption of geodesic motion for z^α or the DSX centre of mass is unlikely to lead to any conflict with experimental measurements. On the other hand it does not fully justify the use of equation 3.2 in DSX (1994) p. 623 since the uncontrolled error here is much larger than $d^{-2}O(4)$. However it must be pointed out that DSX have not justified this equation at all. Their equation of motion for the DSX centre of mass of A has not been shown to apply to a body as small as LAGEOS and even if it were applicable they would still need to justify the monopole approximation.

Consider the same body orbiting the sun at about the distance of earth-orbit. In characteristic units based on the sun, $O(2) \sim 2.10^{-6}$, $d \sim 10^2$, $L_A \sim 4.10^{-10}$, $\frac{L_A}{c^2} \sim 10^{-15}$, $\frac{L_A^2}{d^4} \sim 2.10^{-27}$, $d^{-2}O(4) \sim 10^{-16}$. The dominant error term here is $\frac{L_A}{c^2}$. It represents a fractional correction of about 10^{-3} to the O(2) terms. If the body were ten times larger with a radius of 3 metres, the fractional correction would be of the order of 10^{-2}.

Chapter 5

A general equation of motion

5.1 Introduction

The striking relationship between equations (4.5.11) and (4.4.3) is not fortuitous. In this chapter the problem is approached from a more general point of view. Here the Christoffel symbols are carried through the calculation without being replaced by expressions in terms of potentials. In some respects the theorem of section 5.2 may be regarded as a generalization of (4.5.11). The assumptions of this chapter are less restrictive than those of chapter 4 and do not require the use of the special DSX reference frames of chapter 4. This chapter is concerned solely with the problem of whether the (exact as opposed to 1PN approximate) geodesic law can be deduced in a mathematical sense from the field equations with possible restrictions on $T^{\alpha\beta}$. There is no attempt to measure deviation from geodesic motion for a real body as was done in chapter 4. In section 5.2 an equation of motion is derived which shows in a direct way how the motion of a point within the body is related to geodesic motion as the body tends to a test particle. In section 5.3 it is shown that under reasonable conditions the geodesic law in the limiting sense of a test particle is exactly true for an ideal fluid.

5.2 An equation of motion

As in section 4.6, let $g_{\alpha\beta}(x;m)$ be a family of metric functions with $m \geq 0$ as family parameter. Each metric function represents a compact body A_m of radius L_m in an external field and satisfies the vacuum E.F.E. in a region surrounding the world tube of A_m.

For each value of t, a point $z^\alpha_m(t)$ is defined as before by

$$\int_{A_m} d^3x(x^i - z^i_m(t))T^{00}(x;m) = 0, \quad z^0_m(t) = ct \tag{5.2.1}$$

This is a frame dependent definition but the analysis will be done without change of reference frame. The final results showing that a test particle does or does not move on a geodesic are clearly covariant. The essential property of $z^\alpha_m(t)$ required here is that if $T^{00} \geq 0$, then $z^\alpha_m(t)$ lies within the convex hull of A_m and consequently serves to describe the position of the test particle in the limit as m tends to zero.

It is assumed that there exist positive constants K_i, $i = 1, 2, 3...$, independent of m, such that the following hold inside A:

$$|g_{\alpha\beta}(x;m)| < K_1, \quad |g^{\alpha\beta}(x;m)| < K_2, \quad |\frac{\partial g_{\alpha\beta}(x;m)}{\partial x^\gamma}| < K_3, \quad |\frac{\partial^2 g_{\alpha\beta}(x;m)}{\partial x^\gamma \partial x^\delta}| < K_4,$$

$$|\frac{dz^i_m(t)}{dt}| < K_5, \quad |\int_{A_m} d^3x \frac{T^{00}(x;m)}{c^2}| > K_6(\text{volume}\, A_m),$$

$$|\int_{A_m} d^3x\ \phi(x;m)\frac{\partial T^{\alpha\beta}}{\partial x^\gamma}(x;m)| < K_7 \frac{|\phi|_{max}|T^{\alpha\beta}|_{max}(\text{volume}\, A_m)}{L_m}$$

The above seven inequalities will be referred to as $I_1, I_2, ...I_7$ respectively. They are discussed in an appendix to this section. As a consequence of the inequalities $(I) = (I_1, ...I_7)$ and the field equations, there exist further constants independent of m such that

$$|\Gamma^\alpha_{\beta\gamma}| < K_8, \quad |\frac{\partial}{\partial x^\delta}\Gamma^\alpha_{\beta\gamma}(x;m)| < K_9, \quad |T^{\alpha\beta}(x;m)| < K_{10}$$

Lemmas 1 and 2 of section 4.5 will be required in the proof of the theorem. Note that M^{ij} defined below agrees with the definition of chapter 4 while $M^{0i} = cM^i$, $M^{00} = c^2$.

Theorem: Let

$$M^{\alpha\beta}(t;m) = c^2 \frac{\int_{A_m} d^3x T^{\alpha\beta}(x;m)}{\int_{A_m} d^3x T^{00}(x;m)}$$

and let $z^\alpha_m(t)$ be defined by equation (5.2.1). Then assuming the inequalities I_1 to I_7 are satisfied,

$$\frac{d^2 z^i_m}{dt^2} = \mathcal{G}^i(g_{\alpha\beta}(m); z^\mu_m) + X^i_1 + X^i_2 + \delta^i \tag{5.2.2}$$

where $g_{\alpha\beta}(m) = g_{\alpha\beta}(x;m)$ and writing $z_m = z^\alpha_m(t)$,

$$\begin{aligned} X^i_1 &= (-\Gamma^i_{\mu\nu}(z_m) + \frac{1}{c}\frac{dz^i_m}{dt}\Gamma^0_{\mu\nu}(z_m))(M^{\mu\nu} - \frac{dz^\mu}{dt}\frac{dz^\nu}{dt}) \\ &\quad +(\Gamma^0_{00}(z_m) + 2\Gamma^\nu_{0\nu}(z_m))(\frac{1}{c}\frac{dz^i_m}{dt}M^{00} - M^{0i}) \\ &\quad +(2\Gamma^0_{j0}(z_m) + 2\Gamma^\nu_{j\nu}(z_m))(\frac{1}{c}\frac{dz^i_m}{dt}M^{0j} - M^{ij}), \qquad (5.2.3) \\ X^i_2 &= \frac{1}{c}\Gamma^0_{jk}(z_m)\frac{dz^i_m}{dt}M^{jk} - c\Gamma^0_{jk}(z_m)(\frac{\int_{A_m} d^3x(x^i - z^i_m)\frac{\partial T^{jk}(x;m)}{\partial t}}{\int_{A_m} d^3x T^{00}(x;m)}), \\ |\delta^i| &< L_m K \text{ where } K \text{ is a constant independent of } m. \end{aligned}$$

In the above expressions $M^{\alpha\beta} = M^{\alpha\beta}(t;m)$ and the Christoffel symbols are for the metric $g_{\alpha\beta}(x;m)$.

Proof: The conservation equation $T^{\alpha\beta}_{;\beta} = 0$ may be written:

$$\frac{\partial T^{\alpha 0}}{\partial x^0} = -\frac{\partial T^{\alpha i}}{\partial x^i} - \Gamma^\alpha_{\beta\delta}T^{\beta\delta} - \Gamma^\beta_{\beta\delta}T^{\alpha\delta} \tag{5.2.4}$$

The procedure is, as before, to differentiate (5.2.1) twice with respect to t using Lemmas (1) and (2) and replacing $\frac{\partial T^{00}}{\partial t}$ and $\frac{\partial T^{0i}}{\partial t}$ wherever they occur from (5.2.4). The only partial derivatives of $T^{\alpha\beta}$ then remaining are $\frac{\partial T^{ij}}{\partial t}$ which appear in the final term of X^i_2. Certain terms which arise will be found to be bounded by $L_m K$ where K is a constant independent of m and the sum of these terms is δ^i.

The following abbreviations are used:

$$T^{\alpha\beta} = T^{\alpha\beta}(x;m), \;\; H^\alpha = \Gamma^\alpha_{\beta\delta}(x;m)T^{\beta\delta} + \Gamma^\beta_{\beta\delta}(x;m)T^{\alpha\delta}, \;\; z_m = z^\alpha_m(t).$$

The first differentiation yields the identity

$$\int_{A_m} d^3x \frac{dz_m^i}{dt} T^{00} + c\int_{A_m} d^3x(x^i - z_m^i(t))(\frac{\partial T^{0a}}{\partial x^a} + H^0) = 0$$

By lemma (2), $\int_{A_m} d^3x(x^i - z_m^i(t))\frac{\partial T^{0a}}{\partial x^a} = -\int_{A_m} d^3x T^{0i}$. Hence

$$\int_{A_m} d^3x \frac{dz_m^i}{dt} T^{00} - c\int_{A_m} d^3x T^{0i} + c\int_{A_m} d^3x(x^i - z_m^i(t))H^0 = 0$$

The second differentiation yields the identity

$$\frac{d^2 z_m^i}{dt^2} = P^i + Q^i + R^i + S^i \tag{5.2.5}$$

where

$$\begin{aligned}
P^i &= (\frac{1}{c}\int_{A_m} d^3x\, 2\frac{dz_m^i}{dt} H^0)/(\int_{A_m} d^3x \frac{T^{00}}{c^2}) \\
Q^i &= (-\int_{A_m} d^3x H^i)/(\int_{A_m} d^3x \frac{T^{00}}{c^2}) \\
R^i &= (\frac{1}{c}\int_{A_m} d^3x(x^i - z_m^i(t))(\frac{\partial}{\partial t}\Gamma^0_{\beta\delta} T^{\beta\delta} + \frac{\partial}{\partial t}\Gamma^\beta_{\beta\delta} T^{0\delta})/(\int_{A_m} d^3x \frac{T^{00}}{c^2}) \\
S^i &= (\int_{A_m} d^3x(x^i - z_m^i(t))[\Gamma^0_{00}(\frac{\partial T^{0a}}{\partial x^a} + H^0) + 2\Gamma^0_{0j}(\frac{\partial T^{ja}}{\partial x^a} + H^j) \\
&\quad + \Gamma^\nu_{\beta\nu}(\frac{\partial T^{j\beta}}{\partial x^j} + H^\beta) - \frac{1}{c}\Gamma^0_{jk}\frac{\partial T^{jk}}{\partial t}])/(\int_{A_m} d^3x \frac{T^{00}}{c^2})
\end{aligned}$$

For x in A_m, $|\Gamma^\alpha_{\beta\gamma}(x) - \Gamma^\alpha_{\beta\gamma}(z_m)| < \sup_{A_m}(|\text{grad } \Gamma^\alpha_{\beta\gamma}|)2L_m < 2\sqrt{3}K_9 L_m$

Thus if one of the $\Gamma^\alpha_{\beta\gamma}(x;m)$ in H^0 is replaced by $\Gamma^\alpha_{\beta\gamma}(z_m)$ the error made in P^i is less than $4\sqrt{3}L_m \frac{K_5 K_9 K_{10}}{cK_6}$. Hence

$$P^i = \frac{2}{c}\frac{dz_m^i}{dt}(\Gamma^0_{\beta\gamma}(z_m)M^{\beta\gamma} + \Gamma^\nu_{\beta\nu}(z_m)M^{0\beta}) + \delta(P^i)$$

Where $|\delta(P^i)| < K_{11}L_m$ and $K_{11} = 128\sqrt{3}\frac{K_5 K_9 K_{10}}{cK_6}$. Similarly it may be shown

$$Q^i = -\Gamma^i_{\beta\gamma}(z_m)M^{\beta\gamma} - \Gamma^\nu_{\beta\nu}(z_m)M^{i\beta}) + \delta(Q^i), \quad |R^i| < K_{13}L_m$$

where $|\delta(Q^i)| < K_{12}$ and K_{12}, K_{13} are constants independent of m.

The terms in S^i involving spatial derivatives may be treated as follows:

$$\int_{A_m} d^3x(x^i - z_m^i(t))\Gamma^0_{00}\frac{\partial T^{0a}}{\partial x^a} = \Gamma^0_{00}(z_m)\int_{A_m} d^3x(x^i - z_m^i(t))\frac{\partial T^{0a}}{\partial x^a}$$

$$+\int_{A_m} d^3x(x^i - z_m^i(t))\delta\frac{\partial T^{0a}}{\partial x^a}$$

where

$$|\delta| = |\Gamma^0_{00}(x) - \Gamma^)_{00}(z_m)| < 2L_m\sqrt{3}K_9$$

In I_7, let $\phi = (x^i - z_m^i)\delta$. Then by I_7

$$|\int_{A_m} d^3x(x^i - z_m^i(t))\delta\frac{\partial T^{0a}}{\partial x^a}| < 4\sqrt{3}K_7K_9K_{10}L_m(\text{volume } A_m)$$

Also $\int_{A_m} d^3x(x^i - z_m^i(t))\frac{\partial T^{0a}}{\partial x^a} = -\int_{A_m} d^3xT^{0i}$ by lemma (2). Hence

$$\begin{aligned} S^i &= -\Gamma^0_{00}(z_m)M^{0i} - 2\Gamma^0_{0j}(z_m)M^{ji} - \Gamma^\nu_{\beta\nu}(z_m)M^{\beta i} \\ &- \frac{1}{c}\Gamma^0_{jk}(z_m)\frac{\int_{A_m} d^3x(x^i - z_m^i(t))\frac{\partial T^{jk}}{\partial t}}{(\int_{A_m} d^3x\frac{T^{00}}{c^2})} + \delta(S^i) \end{aligned}$$

where $|\delta(S^i)| < K_{14}L_m$.

The above expressions for P^i, Q^i, R^i, S^i are now substituted into (5.2.5) and terms are rearranged using the following identity:

$$\frac{1}{c}\frac{dz_m^i}{dt}(\Gamma^0_{\beta\gamma}(z_m)M^{\beta\gamma} + 2\Gamma^\nu_{\beta\nu}(z_m)M^{0\beta}) - 2\Gamma^\nu_{\beta\nu}(z_m)M^{i\beta} - \Gamma^0_{00}(z_m)M^{0i} - 2\Gamma^0_{0j}(z_m)M^{ij}$$

$$= (\frac{1}{c}\frac{dz_m^i}{dt}M^{00} - M^{0i})(\Gamma^0_{00}(z_m) + 2\Gamma^\nu_{0\nu}(z_m)) + (\frac{1}{c}\frac{dz_m^i}{dt}M^{0j} - M^{ij})(2\Gamma^0_{0j}(z_m) + 2\Gamma^\nu_{j\nu}(z_m))$$

$$+\frac{1}{c}\frac{dz_m^i}{dt}(\Gamma^0_{jk}(z_m)M^{jk}$$

Equation(5.2.5) then becomes equation (5.2.2) which completes the proof of the theorem.

Equation (5.2.2) is a very general result. It is valid in any reference frame where the inequalities (I) hold. As shown in the appendix, these inequalities are expected to hold for slow motion weak fields in any post-Newtonian reference frame and they appear to hold much more generally. The quantity $X_1^i + X_2^i + \delta^i$ is the difference between the acceleration of the point $z_m^\alpha(t)$ which lies inside the body, and geodesic acceleration of this point with respect to the total field. It will be shown in the next section for an ideal fluid under reasonable conditions, that this quantity tends to zero with m.

Appendix to section 5.2

(1) The system of units has not been specified but the inequalities all contain undetermined constants and their form will be unaltered by a change of scale.

(2) The inequalities I_1 to I_6 hold for slow motion, weak fields in any post-Newtonian reference frame. For example in the DSX reference frame of chapter 4 with units specified there, letting d denote distance from A to B,

$$|g_{\alpha\beta}(x;m)| \sim 1, \ \ g^{\alpha\beta}(x;m) \sim 1, \ \ |\frac{\partial g_{\alpha\beta}}{\partial x^\gamma}| \sim c^{-2}, \ \ |\frac{\partial^2 g_{\alpha\beta}}{\partial x^\gamma \partial x^\epsilon}| \sim c^{-2},$$

$$|\frac{dz_m^i(t)}{dt}| \sim d^{-\frac{1}{2}}, \ \ |\int_{A_m} d^3x \frac{T^{00}}{c^2}| \sim \frac{1}{4}(\mathrm{volume}\, A)$$

It is not necessary to discuss transformation properties of (I) here since it suffices to do the analysis without change of frame but it is evident that (I) will remain of the same form under a fairly wide class of transformations where the Jacobian and its inverse have bounded first and second partial derivatives.

(3) A stronger version of inequality I_7 was discussed in the appendix to section 4.5. Here the presence of the constant K, which could be very large, ensures that I_7 will be satisfied under practically any circumstances provided the total fluctuation of $T^{\alpha\beta}$ in A_m is bounded independently of m and the ratio of time derivatives to space derivatives is also bounded independently of m.

5.3 Application to an ideal fluid

In this section subscripts on K will be omitted. This means that the symbol K does not have the same value throughout but it always represents a constant which is independent of m.

Since $|\delta^i| < L_m K$ in (5.2.2) it is clear that δ^i tends to zero with m. If $\frac{d^2 z^i_m}{dt^2}$ approaches the geodesic acceleration $\mathcal{G}^i(g_{\alpha\beta}(m); z^\mu_m)$ as m tends to zero, then from (5.2.2), $X^i_1 + X^i_2$ must also tend to zero with m.

In the case of dust it was seen in section 1.2 that material particles do follow geodesics of $g_{\alpha\beta}(x;m)$. The point $z^i_m(t)$ is not necessarily a material particle so the situation is a little different here, but one would nevertheless expect that for dust under reasonable conditions $X^i_1 + X^i_2$ should tend to zero with m. This will be shown below to hold more generally for an ideal fluid provided the pressure tends to zero with m and certain other quite weak conditions concerning the internal fluid motion are met. If conditions are as in chapter 4 it is seen that $X^i_2 \sim d^{-2}O(4)$ while it is feasible that X^i_1 is much larger than $d^{-2}O(4)$. This formed the basis of the example of non-geodesic motion which was constructed in section 4.6. Such an example is not possible for an ideal fluid since it will be shown that X^i_1, X^i_2 separately both tend to zero with m. Conditions required to obtain this result are essentially conditions (4.7.2),(4.7.3) and a weakened version of condition (4.5.4).

Written explicitly the conditions are:

C_1 : $\quad |p_m| < L_m K,$

C_2 : $\quad |v^i_m(x) - \dfrac{dz^i_m}{dt}| < L_m K,$

C_3 : $\quad$ Inequality I_7 is valid with $T^{\alpha\beta}$ replaced by other bounded field quantities as required in the proof below

where all K are independent of m , v^i_m is fluid 3-velocity inside A_m and p_m is fluid pressure inside A_m. All these conditions are expected to hold for weak fields, slow motion in post-Newtonian reference frames as in chapter 4 but because of the presence of the unspecified constants K, they appear to hold much more generally.

For an ideal fluid

$$T^{\alpha\beta} = (c^2\rho + p)u^\alpha u^\beta + pg^{\alpha\beta} \tag{5.3.1}$$

As shown in section 4.2, C_1, C_2 and (5.3.1) imply that (4.2.5) and (4.2.6). hold. In the current notation these equations are:

$$|\frac{cT^{0i}(x;m)}{T^{00}(x;m)} - \frac{dz^i_m}{dt}| < L_m K, \quad |\frac{c^2T^{ij}(x;m)}{T^{00}(x;m)} - \frac{dz^i_m}{dt}\frac{dz^j_m}{dt}| < L_m K \tag{5.3.2}$$

and

$$|\frac{M^{0i}}{c} - \frac{dz^i_m}{dt}| < L_m K, \quad |M^{ij} - \frac{dz^i_m}{dt}\frac{dz^j_m}{dt}| < L_m K \tag{5.3.3}$$

It follows from (5.3.3) that $|X^i_1| < L_m K$ and X^i_1 tends to zero with m.

Combining the terms of X^i_2 using lemma(1),

$$X^i_2 = -\frac{1}{c}\Gamma^0_{jk}(\frac{d}{dt}\int_{A_m} d^3x(x^i - z^i_m(t))T^{jk}(x;m))/(\int_{A_m} d^3x\frac{T^{00}}{c^2})$$

From (5.3.2),

$$T^{jk}(x;m) = \frac{dz^j_m}{dt}\frac{dz^k_m}{dt}\frac{T^{00}(x;m)}{c^2} + \mu^{jk}\frac{T^{00}(x;m)}{c^2} \tag{5.3.4}$$

where $|\mu^{jk}| < L_m K$. Using (5.3.4) and (5.2.1),

$$\frac{d}{dt}\int_{A_m} d^3x(x^i - z^i_m(t))T^{jk} = \frac{d}{dt}\int_{A_m} d^3x(x^i - z^i_m(t))\mu^{jk}\frac{T^{00}(x;m)}{c^2} \tag{5.3.5}$$

The integrand in (5.3.5) is less than $(L_m)^2K$ and it is zero on the boundary of A_m.

Taking the differentiation inside the integral by lemma(1) and using C_3 with $\phi = 1$ shows that $|X^i_2| < L_m K$.

Since $|X^i_1| < L_m K$, $|X^i_2| < L_m K$ these terms may be absorbed into δ^i and for an ideal fluid under the given conditions, it has been shown that

$$\frac{d^2z^i_m}{dt^2} = \mathcal{G}^i(g_{\alpha\beta}(m); z^\mu_m) + \delta^i \tag{5.3.6}$$

where $|\delta^i| < L_m K$. In fact it has been shown more generally that (5.3.6) is true if (5.3.2) is satisfied. Equation (5.3.6) shows that for a perfect fluid the acceleration of z_m^i approaches geodesic acceleration with respect to the total metric as m tends to zero.

Suppose now that the family of metrics $g_{\alpha\beta}(x;m)$ constitutes a test particle and let $\mathcal{C} : (ct, \zeta(t))$ be the limiting curve which is the path of the test particle.

If m tends to zero in (5.3.6), then $\delta^i \to 0$ and $z_m^\alpha(t) \to \zeta^\alpha(t)$. It seems reasonable to assume as before that

$$\frac{dz_m^i(t)}{dt} \to \frac{d\zeta^i(t)}{dt}, \frac{d^2 z_m^i(t)}{dt^2} \to \frac{d^2\zeta^i(t)}{dt^2} \tag{5.3.7}$$

Then $\mathcal{C}$ is a geodesic of the background field $g_{\alpha\beta}^{(b)}(x)$ if and only if

$$(\mathcal{G}^i(g_{\alpha\beta}(m); z_m^\mu) - \mathcal{G}^i(g_{\alpha\beta}^{(b)}; \zeta^\mu)) \to 0 \text{ as } m \to 0 \tag{5.3.8}$$

By definition $g_{\alpha\beta}(x;m) \to g_{\alpha\beta}^{(b)}(x)$ as $m \to 0$. A sufficient condition for the validity of (5.3.8) is

$$\frac{\partial g_{\alpha\beta}}{\partial x^\delta}(x;m) \to \frac{\partial g_{\alpha\beta}^{(b)}}{\partial x^\delta}(x) \text{ as } m \to 0 \tag{5.3.9}$$

The condition (5.3.9) may be examined in the case of the analytical example for the field of A alone (using characteristic units based on B). It was seen in section 4.3 that $\frac{\partial g_{\alpha\beta}(x;m)}{\partial x^\gamma} \sim L_m c^{-2}$ and hence $\frac{\partial g_{\alpha\beta}(x;m)}{\partial x^\gamma} \to 0$ as $m \to 0$. For this example the background is flat with $\frac{\partial g_{\alpha\beta}^{(b)}(x)}{\partial x^\delta} = 0$ so in this case the condition (5.3.9) is true. Intuitively one would not expect the curvature of the background to affect the validity of (5.3.9) which should then be generally true under the conditions assumed in this section. However, lacking any two-body solutions, there seems to be no prospect of proving (5.3.9). Historically it seems always to have been assumed, usually implicitly. Note however that in the DSX theory of chapter 4, $(\mathcal{G}^i(g_{\alpha\beta}(m); z_m^\mu) - \mathcal{G}^i(g_{\alpha\beta}^{(b)}; \zeta^\mu)) \sim d^{-2}c^{-4}$ for m sufficiently small. Thus one might say that according to the post-Newtonian theory of DSX, (5.3.8) is satisfied modulo $O(c^{-4})$. Because of this it was possible to produce the counter-example to the geodesic law in section (4.6) without assuming (5.3.8).

To sum up the results of this section; the geodesic law for an ideal fluid test particle has been proved subject to inequalities (I), (C), and equations (5.3.7), (5.3.9) holding in some system of reference frames. All these conditions are expected to hold for weak fields, slow motion in post-Newtonian reference frames and they may hold much more generally. Equation (5.3.9) is possibly the most contentious of the various conditions. Personally I believe that (5.3.9) will hold, but it may be worth stressing that if (5.3.9) does not hold while the other conditions do hold, then the geodesic law is invalid even for an ideal fluid test particle.

Chapter 6

Conclusion

In the formulation of the definition of a test particle a test body may be a body in the usual sense, having internal structure with a well defined internal stress-energy tensor or it may be a singularity so that $T^{\alpha\beta}$ inside the test body is not defined.

In the latter case several examples of non-geodesic motion have been exhibited. These examples are "non-physical" so that they do not exclude the possibility of all "physical" test particles moving on geodesics. They do however show that geodesic motion as a mathematical consequence of the field laws is not generally true for singularities. The famous paper of Einstein and Grommer (1927) showing geodesic motion of a singularity in a special case created a false impression. As demonstrated in section 2.3, had Einstein and Grommer taken a different example from the family of Weyl metrics they would have found non-geodesic motion.

If the test bodies are bodies in the usual sense with an internal stress-energy tensor the situation is different. Since there are no known exact solutions representing two or more bodies in vacuum it is impossible to produce an explicit example of non geodesic motion of this nature. However it was shown in section 4.6 that if $T^{\alpha\beta}$ lies between certain limits which are allowable in post-Newtonian theory, then non-geodesic motion will result if some rather plausible conditions are met. This shows, provided one accepts the conditions, that even in a post-Newtonian regime with weak fields and slow motion, the geodesic law does not generally follow from the field equations. I do not find this result surprising. Indeed it is in accordance with common sense for the following reason: It seems to be generally agreed that the laws of motion are governed by the conservation equation which involves $T^{\alpha\beta}$ and Christoffel symbols of the total metric. The geodesic equation involves velocities and Christoffel symbols of the background metric. One therefore expects that in order to derive the geodesic law some relationship will be required between $T^{\alpha\beta}$ inside A and the velocity of A in the test particle limit. This is in agreement with what has been found. It has been shown that at the post-Newtonian level of approximation it is neccessary that in the test particle limit, the quantity Q^i be zero modulo O(4) and this cannot generally be the case unless $M^i = \frac{dz^i}{dt}$ modulo O(4) and $M^{ij} = \frac{dz^i}{dt}\frac{dz^j}{dt}$ modulo O(4).

On the other hand, the geodesic law has been shown true in certain cases. Firstly it has been shown true if the conditions of the Infeld/Schild theorem hold. The Infeld/Schild theorem accounts for some of the special analytical cases of geodesic motion which appear in the literature and is also connected with cases of geodesic motion (none of which are rigorously proved) resulting from the method of matched asymptotic expansions but as a general proof of geodesic motion it fails because it has not been established that the hypotheses of the theorem hold under general physical conditions. Indeed it has not been established that they hold for any significant class of problems.

Secondly as a consequence of the theorem of section 5.2, the geodesic law has been shown to hold if the test body satisfies (5.3.3) plus some side conditions which are very plausible in the case of weak fields, slow motion and possibly hold much more generally. I believe this proof is more satisfactory than any other which has been offered to date. The assumptions have been stated explicitly. One sees quite clearly how the geodesic law arises and the fact is that it arises essentially because the stress energy tensor approaches that of dust as m tends to zero. Equation (5.3.3) includes the case of ideal fluids.

In view of the fact that the background Γ^0_{ij} can be zero it seems likely that (5.3.3) is also necessary in order to ensure geodesic motion on any background. This is a kind of converse of the theorem of chapter 1 where it was shown that a dust fluid obeys a geodesic law. Here one has that if the geodesic law is to be true, the stress-energy tensor of the test body must approach that of dust in the limit as m tends to zero.

Another important consequence of the theorem of section 5.2 concerns the condition (5.3.9) which requires that Christoffel symbols for the total field approach Christoffel symbols for the background field as $m \to 0$. It has been shown that if this condition is rejected while the other conditions are accepted, then the geodesic law is not valid for an ideal fluid. This means that condition (5.3.9) is essentially a prerequisite for the geodesic law.

Thus far my comments have been concerned solely with the central issue of this thesis which is the problem of whether the geodesic law can be deduced in a mathematical sense from the field equations with possible restrictions on $T^{\alpha\beta}$. The method assumes that the field equations are applicable to *arbitrarily* small bodies. This is patently false. If any practical application is to be possible some measure is needed of deviation from geodesic motion for a real test body which is not "vanishingly small". This aspect was considered in the post-Newtonian theory in section 4.8 . The outcome here was that some results of practical significance could be obtained by the methods used.

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